Duality and Infinity

Studies in Possibility Semantics
and Semiconstructive Mathematics

By

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Logic and Methodology of Science

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2024

Abstract

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Many results in logic and mathematics rely on techniques that allow for concrete, often visual, representations of abstract concepts. A primary example of this phenomenon in logic is the distinction between syntax and semantics, itself an example of the more general duality in mathematics between algebra and geometry. Such representations, however, often rely on the existence of certain maximal objects having particular properties such as points, possible worlds or Tarskian first-order structures.

This dissertation explores an alternative to such representations known as *possibility semantics*. Its core idea is to replace maximal objects with ordered systems of partial approximations. Although it originates in the semantics of modal logic and the representation of abstract ordered structures, I argue that it has far-reaching mathematical, foundational and philosophical significance, especially in the context of *semiconstructive mathematics*, a foundational framework that does not assume any fragment of the Axiom of Choice beyond the Axiom of Dependent Choices.

The dissertation is divided in two main parts. The first part explores various applications of the mathematical framework underlying possibility semantics to lattice theory and non-classical propositional logics. A major theme is the development of constructive dualities for various categories of lattices, which are related to standard non-contructive dualities via Vietoris constructions.

The second part of the dissertation explores the alternative foundational setting of semi-constructive mathematics, focusing on three applications of possibility semantics for classical first-order logic to the philosophy of the mathematical infinite. In particular, we introduce *generic powers*, a semi-constructive analogue of ultrapowers in classical model theory, and we explore the merits of these structures from a foundational, conceptual and historical viewpoint.

Contents

Acknowledgements

First and foremost, I want to thank my dissertation co-chairs, Wes Holliday and Paolo Mancosu, for their endless support throughout the writing of this dissertation. I am equally grateful for the guidance they provided and for the freedom they have let me enjoy over the course of my studies in Berkeley. The work in this dissertation not only owes a lot to Wes's work on possibility semantics and to Paolo's writings on the infinite, but it was also profoundly shaped by a desire to some day emulate their enthusiasm, dedication and intellectual rigor.

Many thanks also to John Steel, Tom Scanlon and John MacFarlane, for agreeing to be on my dissertation committee and for helpful comments on some of the material presented here. Several chapters have also benefited from feedback from and conversations with Johan van Benthem, Nick and Guram Bezhanishvili, David Gabelaia, Tomáš Jakl, Mamuka Jibladze, Frederik Lauridsen, Dana Scott and Sean Walsh. Special thanks also to Anna Bellomo, who has kindly allowed me to include our joint work on Bolzano in this dissertation, and has been a valuable friend for many years.

I also wish to thank all those who have been part of my network of friends and colleagues at Berkeley, and especially Raha Ahmadian, Diego Bejarano, Katalin Berlow, Jordan Brown, Alex Burka, David Casey, Ahmee Christensen, Sean Cody, Ethan Dlugie, Gabe Goldberg, Patrick Lutz, Yiping Miao, Scott Mutchnik, Adrian Ommundsen, Adele Padgett, Luca Passi, Robert Schütz, Benny Siskind, Sebastiano Thei, James Walsh, Kentarô Yamamoto, the esteemed members of Bill's Boys and the unwavering audience of the Student Logic Colloquium. Many thanks also to my friends from the Old Continent for sharing their time (and sometimes also their house) with me over the years, and especially to Zoé, Damien, Sarah, Louis, Antoine, Olivier, Romain, Alexandre, Marion, Aymeri, Mehdi, Julien, Bérenger, Laurie, Clément, Marie, Sylvain and Will.

Finally, I want to thank my family for their support, and for granting me the freedom to live far from home while always having the safety of a home from afar.

Introduction

The main topic of this dissertation is *possibility semantics*. Possibility semantics has (at least) two distinct points of origin in logic, a fact that will be reflected in the very structure of this dissertation. The phrase was first coined by Humberstone [140] in the context of the semantics of modal logic. Humberstone's idea was to propose an alternative to the now-standard *Kripke* semantics for modal logic that would take *possibilities* rather than *worlds* as basic. The key difference between the two is that possibilities can be partial, and therefore pairwise compatible, while worlds are maximal and pairwise incompatible objects. Consequently, some possibilities may be more informative than others, and possibilities can therefore be ordered in terms of how complete a description of a world they provide. Humberstone provides a semantics for modal propositional logic that follows this basic intuition. In particular, a conjunction is satisfied by a possibility p if and only if both conjuncts are satisfied at p, but a disjunction may be satisfied at a possibility p without any of its disjuncts being satisfied at p. Indeed, a possibility p could settle a disjunction as true because no possible extension of p could refute *both* disjuncts at the same time. But this does not mean that such a p should *decide* which disjunct is true. We take this feature to be the common denominator to all the conceptual frameworks that we will liken to possibility semantics here. Since Humberstone's work, the idea of defining a concrete semantics for a (possibly non-classical) propositional logic in which points may satisfy a disjunction without satisfying one of the two disjuncts has found applications in many domains. In classical modal logic, it was used to great effect in the study of modal incompleteness [137, 134, 266]. Various possibility semantics were also developed for intuitionistic logic [37, 36, 188], (modal) orthologic [109, 138], and fundamental logic, a weaker logic that generalizes both intuitionistic logic and orthologic [131].

For first-order classical logic, an early work in this tradition is a manuscript by van Benthem [23]. But it would be somewhat misleading to think that possibility semantics for classical logic originated only around that time, or as a special case of Humberstone's ideas. Indeed, the basic idea of possibility semantics, namely to evaluate formulas in a first-order language at points in a partially ordered set, already essentially appeared in Cohen's method of forcing in set theory [59]. In forcing semantics, conditions in a forcing notion can be thought of as approximations of a generic extension of a model of set theory to which they belong. Of course, because it is only a partial approximation, a condition may force a disjunction to hold in a generic extension without forcing any specific disjunct to hold.

Accordingly, the second point of origin of possibility semantics is just the definition of the forcing relation in set theory. Given the influence that forcing has had in many areas of mathematical logic, this means however that possibility semantics has strong ties with a venerable tradition of non-Tarskian approaches to the semantics of first-order logic, including, perhaps most importantly for this dissertation, Boolean-valued models [233, 10] and sheaf semantics [181, 195]. This dual nature of possibility semantics will be reflected in the structure of dissertation, which is composed of two main parts, one focusing on non-classical propositional logics, and the second one focusing on possibility semantics for first-order classical logic. Those two parts also differ in their contributions. The first part is mostly a mathematical investigation of various frameworks related to possibility semantics in the context of duality theory, while the second part focuses of applications of first-order possibility semantics to several philosophical problems about the mathematical infinite broadly construed.

The second theme of this dissertation is *semiconstructive mathematics*. As I will use it here, it refers to a foundational setting that sits between constructive and classical mathematics, namely the kind of mathematics that can be carried out in $ZF + DC$, where ZF stands for Zermelo-Frankel set theory, and DC stands for the Axiom of Dependent Choices, a weak fragment of the axiom of choice.[1] This foundational framework, sometimes also called *quasi*-constructive mathematics [230], is often taken to be a natural setting for most of ordinary mathematics, and in particular it may be thought of as a "paradise" for analysis, as it neither assumes nor rejects some of the more puzzling consequences of the Axiom of Choice. Under some mild set-theoretic assumptions, $ZF + DC$ is consistent with the statement that every set of reals is Lebesgue measurable [242], and under some stronger large cardinal assumptions, it is consistent with the Axiom of Determinacy [263]. A key fact here is that the Boolean Prime Ideal Theorem, a strong fragment of the Axiom of Choice usually proved as an immediate application of Zorn's Lemma, cannot be proved in the semiconstructive setting.

In this dissertation, we will study interactions between possibility semantics and semiconstructive mathematics of the following two kinds. First, in the context of non-classical propositional logics, the development and study of semantics for many logics is grounded in a branch of category theory called duality theory (see [102] for a recent overview of the field). Duality theory studies a certain kind of strong correspondences between ordered algebraic structures, which abstract away from syntax in logic, and concrete geometric structures, which typically provide semantics for various logical systems. Most dualities, including Stone duality between compact zero-dimensional Hausdorff spaces and Boolean algebras, rely on the Boolean Prime Ideal theorem in an essential way, and therefore do not hold in a semiconstructive setting. However, the discrete "forcing duality" between complete Boolean algebras and posets, which is at heart of the basic machinery of forcing and possibility semantics, can be lifted to a full duality for Boolean algebras within the resources of semiconstruc-

[1]Note that the phrase "semi-constructive" or "semi-intuitionistic" [88, 217] may also designate a different framework, in which the background logic is somewhere between intuitionistic and classical logic. By contrast, we will use the law of excluded middle throughout this dissertation.

tive mathematics, as was shown by Bezhanishvili and Holliday in [41]. Much of the work in Part I of this dissertation is devoted to generalizations of this fact beyond the Boolean case. In particular, we identify a pattern in which non-constructive topological dualities can be replaced by semiconstructive ones in which the points of nonconstructive topological spaces are "semipoints", i.e. points in another topological space that approximates them. Moreover, these approximations can always be analysed as a particular kind of topological construction known as *Upper Vietoris hyperspaces*. Accordingly, our motivating slogan will be that a shadow of non-constructive dualities can be found in the semiconstructive setting by "factoring" through Vietoris hyperspaces:

$$\mathsf{Non\text{-}constructive\ Dualities} = \frac{\mathsf{Constructive\ Dualities}}{\mathsf{Upper\ Vietoris\ Hyperspaces}}.$$

The second point of contact between possibility semantics and semiconstructive mathematics that we will investigate is in the use of the former as a semantics for first-order classical logic. Indeed, many concepts in standard, Tarskian model theory rely on the Axiom of Choice, and, specifically, on the Boolean Prime Ideal Theorem. By contrast, possibility structures offer a semantics for first-order logic that is not as entangled with non-constructive assumptions. For example, the proof of the Completeness Theorem for first-order logic requires, for uncountable languages, the Boolean Prime Ideal Theorem, while one can give a choice-free proof for possibility structures [135]. In other words, the added complexity of possibility semantics compared to Tarskian semantics is reflected in the lower complexity of the metatheory it requires, as captured by the following "equation":

$$\mathsf{ZFC} + \mathsf{Tarskian\ Semantics} = \mathsf{ZF} + \mathsf{Possibility\ Semantics}.$$

In Part II of this dissertation, we will focus on a particular kind of model-theoretic constructions whose existence relies on the Boolean Prime Ideal, namely, ultrapowers. The key insight that will we pursue is that ultrapowers can be replaced in the semiconstructive setting with *generic powers*, possibility structures that essentially capture all only those features shared by all ultrapowers of a model modulo an ultrafilter on a given index set. The choice of terminology, of course, suggests that there is a strong connection with forcing. In fact, we will show that, in many cases, classical ultrapowers are merely "one forcing away" from generic powers, an idea summarized by the following slogan:

$$\mathsf{Ultrapowers} = \mathsf{Generic\ Powers} \times \mathsf{Forcing}.$$

Moreover, I will argue that generic powers have technical and conceptual advantages over Tarskian ultrapowers that make them particularly interesting structures to study in connection with several philosophical problems about infinity.

I will now give a more detailed overview of the dissertation and of its main results. As mentioned, the dissertation is divided into two parts, one in which the emphasis is largely on mathematical results about structures related to possibility semantics for propositional logics, and one in which the emphasis is on the application of possibility structures to several philosophical problems.

Part I focuses on possibility semantics for propositional non-classical logics and its relationship to duality theory. Chapter 1 covers some background on lattices and duality theory. In particular, Table 1.1 gives a systematic overview of most of the dualities relevant for our purposes, highlighting which ones are new.

Chapter 2 introduces a discrete duality between the category **cLat** of complete lattices and a category of bi-preordered sets called *b-frames*. The duality is established in Section 2.2, and restricted in various ways to subcategories of **cLat** in Sections 2.3 and 2.4. Two applications to the theory of Heyting algebras conclude the chapter. First, we obtain a decomposition theorem for complete Heyting algebras that generalizes a classical result about complete Boolean algebras (Section 2.5). Second, a possibility semantics for propositional intuitionistic logic is defined on bi-ordered sets, and used to prove the incompleteness of a certain intermediate logic with respect to the class of complete bi-Heyting algebras, thus improving on one of the only known results in the field (Section 2.6).

In Chapter 3, we enter the realm of topological dualities and investigate extensions of the choice-free version of Stone duality to two generalizations of Boolean algebras. Our approach is *semi-pointfree*, meaning that we will interested in the representation of algebraic objects via ordered topological spaces in which the points can intuitively be thought of as rough approximations of maximally precise points. Section 3.3 presents a choice-free duality for *de Vries algebras*, which are complete Boolean algebras endowed with a proximity relation. In Section 3.4, the link between de Vries's original duality and its choice-free version is established via Upper Vietoris constructions, and two applications of the choice-free duality are given. The rest of the chapter focuses on establishing two choice-free dualities for the category of distributive lattices. The first one, established in Section 3.6, is a bitopological duality, while the second one, established in Section 3.7, considers a category of ordered topological space that can be thought of as choice-free versions of Priestley spaces. In both cases, the two dualities can be connected to their non-constructive counterparts via an Upper Vietoris construction, as shown in Section 3.8.

In Chapter 4, we extend the techniques from the previous chapters beyond the distributive setting, and present a duality for the category of all lattices. We first use Vietoris constructions and their algebraic duals to present a small extension of Priestley duality (Section 4.3), which is then itself used to obtain our main result, namely a duality between the category of lattices and a category of Priestley spaces endowed with a relation, which we call FI-spaces (Section 4.4). In Section 4.5, we refine this duality to a duality between a category of lattices augmented with a weak complement operation and a category **FIN** of FI-spaces endowed with an additional relation. Finally, Section 4.6 connects our work in this chapter to the b-frames of Chapter 2, the choice-free dualities for distributive lattices of Chapter 3, and the possibility semantics for Fundamental Logic presented in [131].

Chapter 5 concludes the first part of the dissertation with an application of possibility semantics to a famous problem in the philosophy of time that goes back to Aristotle. After introducing the problem (Section 5.2), I present a new solution, *orthofuturism*, which is based on a non-distributive logic OF. A possibility semantics for OF is presented in Section 5.5, and the philosophical merits of orthofuturism in contrast with its competitors are explored in Sections 5.4 and 5.6.

Part II of the dissertation focuses on possibility semantics for classical first-order logic, and applications of it to several problems in the philosophy of the infinite. Chapter 6 introduces the basics of possibility semantics for first-order logic, and contains several results about embeddings between possibility structures. Generic powers, which are the key technical notion for this part of the dissertation, are introduced in Section 6.2. This section also contains the proofs for the three fundamental results about generic powers that are used throughout Part II: the Structure Lemma, the Truth Lemma and the Genericity Lemma.

In Chapter 7, we explore an application of possibility semantics to the foundations of nonstandard analysis. In Section 7.2, generic powers are used to provide a semiconstructive analogue ${}^{\dagger}\mathcal{R}$ of the non-constructive hyperreal fields obtained as ultrapowers modulo a non-principal ultrafilter on ω, and versions of several basic results of classical NSA are proved in this setting. Sections 7.3 to 7.5 focus on a detailed comparison between the approach via generic powers and other alternative approaches to nonstandard analysis. In particular, I argue that ${}^{\dagger}\mathcal{R}$ is a natural convergence point between reduced powers, sheaves and Boolean-valued models. Finally, I discuss in Section 7.6 several philosophical objections that have been raised against the use of nonstandard methods in analysis because of their alleged lack of purity and canonicity, and I argue that the semiconstructive approach allows us to convincingly answer these objections.

In Chapter 8, we turn to the philosophy of mathematics, and to recent debates surrounding the possibility of a "Euclidean" notion of the infinite. Our discussion focuses on two distinct but related topics. First, the theory of numerosities [18, 19, 16], which is intended as an alternative to the Cantorian theory of size for infinite collections and, second, NAP functions, an application of this theory to some problems in probability theory. After presenting both theories in Section 8.3, I argue that numerosities and NAP functions fail to meet some of the challenges that have been raised against them in the literature (Section 8.4). This prompts me to develop in Section 8.5 an alternative approach to both numerosities and NAP functions via generic powers and possibility semantics which, as I argue in Section 8.6, has some significant advantages over the standard proposals.

Finally, Chapter 9 explores a related theme, but this time in the historical context of Bernard Bolzano's philosophy. Bolzano famously sketched a "calculation of the infinite" in his *Paradoxes of the Infinite* [46] which has mostly been read as a failed attempt at a theory of transfinite sets based on the preservation of part-whole intuitions rather than on the Cantorian notion of cardinality. After introducing the relevant passages in Bolzano's text and the way they have usually been received in the existing literature (Sections 9.2 and 9.3), an alternative interpretation of Bolzano's "calculation of the infinite" as a theory of infinite sums is developed, and the coherence and fruitfulness of his views are established via a formal reconstruction of his theory that uses standard model-theoretic techniques (Sections 9.4 and 9.5). Finally, I introduce in Section 9.7 an alternative formal reconstruction that uses generic powers rather than ultrapowers, and I argue that this approach sheds some new light on some interpretive issues regarding Bolzano's views on the infinite.

Let me conclude this introduction by mentioning the sources for the material included in the dissertation. Some of the content included in each chapter has already been published elsewhere or has been obtained in collaboration with others. Chapter 2 was published at the *Annals of Pure and Applied Logic* under the title "B-frame Duality" [187]. Many ideas for applications of the general framework developed in there came from fruitful conversations with Wes Holliday. The first part of Chapter 3 on de Vries algebras was accepted for publication in *Advances in Modal Logic*, Vol.14. under the title "Choice-Free de Vries Duality". The second part on distributive lattices stems from several conversations with Wes Holliday, Nick Bezhanishvili and especially Tomáš Jakl, whose insights regarding the connection between spaces of filter-ideal pairs and free distributive lattices also influenced the way I wrote Chapter 4. Chapter 5 benefited from fruitful conversations with Wes Holliday and John MacFarlane, and was largely inspired by their work on epistemic modals and the open future, respectively. Chapter 7 has been accepted for publication at the *Australasian Journal of Logic* under the title "A Semiconstructive Approach to the Hyperreal Line". Earlier drafts of the chapter benefited from especially valuable feedback from Wes Holliday, Dana Scott, Johan van Benthem, Sean Walsh and an anonymous referee. Many ideas in Chapter 8 originated from conversations with Wes Holliday and Paolo Mancosu, as well as from a graduate seminar on Probability Theory that they co-organized in Spring 2021, and from Paolo's graduate seminar on Infinity in Spring 2020. Finally, Chapter 9 is adapted from the paper "Bolzano's Mathematical Infinite" [14], co-authored with Anna Bellomo and published at the *Review of Symbolic Logic*. Earlier drafts of the paper benefited from valuable feedback from Paolo Mancosu, Luca Incurvati, Wes Holliday, Arianna Betti and Annapaola Ginammi. I also thank Anna for allowing me to include our joint work in this dissertation. Moreover, the material in Section 9.7 is new and was not part of our joint paper. Some results in that section were made possible thanks to conversations with Gabe Goldberg and Robert Schütz.

Part I

Possibilities and Duality

Chapter 1

Background on Duality Theory

This chapter introduces some background and motivation for the first part of this dissertation. We will first provide some background on ordered structures such posets, lattices and Boolean algebras, before offering a systematic overview of the topics in duality theory that we focus on.

1.1 Background on Lattices

In this section, we will introduce some background results on posets and lattices. As all the results mentioned here are well-known, we will not provide proofs, but we refer the interested reader to standard sources in universal algebra and lattice theory [66, 229, 28, 42]. Recall that a partial order on a set X is a binary relation that is reflexive, transitive and antisymmetric. We first fix the following terminology.

Definition 1.1.1. A map f between posets $\mathcal{P} = (P, \leq_{\mathcal{P}})$ and $\mathcal{Q} = (Q, \leq_{\mathcal{Q}})$ is *monotone* or *order-preserving* if $p \leq_{\mathcal{P}} p'$ implies $f(p) \leq_{\mathcal{Q}} f(p')$ for all $p, p' \in P$. It is *antitone* or *order-reversing* if $p \leq_{\mathcal{P}} p'$ implies $f(p) \geq_{\mathcal{Q}} f(p')$.

Let us now introduce a concept that is ubiquitous in order theory, and originates from Galois theory.

Definition 1.1.2. Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ and $\mathcal{Q} = (Q, \leq_{\mathcal{Q}})$ be posets. A *monotone Galois connection* between $\mathcal{P}$ and $\mathcal{Q}$ is a pair of order-preserving maps $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ satisfying the following condition for any $p \in P, q \in Q$:

$$F(p) \leq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} G(q)$$

In such a monotone Galois connection, F is usually called the *left adjoint* of G, and G is the *right adjoint* of F. An *antitone* Galois connection is a pair of order-reversing maps F and G such that the following condition holds for any $p \in P, q \in Q$:

$$p \leq_{\mathcal{Q}} G(q) \Leftrightarrow q \leq_{\mathcal{P}} F(p).$$

The phrase "Galois connection" is often used ambiguously in the literature, meaning either a monotone or antitone Galois connection. We will follow this convention, making sure to specify however whether a Galois connection is order-preserving or order-reversing when it is not immediately clear from context. Let us now move on to our main structures of interest, lattices.

Definition 1.1.3. A *meet-semilattice* (resp. *join-semilattice*) is a structure $(L, \wedge, \leq)$ (resp. $(L, \vee, \leq)$) such that $\leq_L$ is a partial order on L and any two elements $a, b \in L$ have a $\leq_L$-greatest lower bound $a \wedge b$ (resp. a $\leq_L$-least upper bound $a \vee b$). A *lattice* is a structure $(L, \wedge, \vee, \leq)$ such that $(L, \wedge, \leq)$ is a meet-semilattice and $(L, \vee, \leq)$ is a join-semilattice. It is *bounded* if it has a $\leq_L$-greatest element 1 and a $\leq_l$-least element 0, and it is *complete* if any subset A of L has a $\leq_L$-greatest lower bound $\bigwedge A$ and a $\leq_L$-least upper bound $\bigvee A$.

In what follows, we will always assume that a lattice is bounded, and will therefore always use the phrase "lattice" to mean "bounded lattice".

Definition 1.1.4. A *meet-semilattice homomorphism* (resp. *join-semilattice homomorphism*) is a map $f : L \to M$ between two meet-semilattices (resp. two join-semilattices) L and M such that $f(a \wedge_L b) = f(a) \wedge_M f(b)$ (resp. $f(a \vee_L b) = f(a) \vee_M f(b)$) for any $a, b \in L$. A *lattice homomorphism* is a map between lattices which is both a meet-semilattice homomorphism and a join-semilattice homomorphism. A meet-semilattice (resp. join-semilattice) homomorphism f between lattices L and M is *meet-complete* (resp. *join-complete*) if for any $A \subseteq L$ such that $\bigwedge_L(A)$ (resp. $\bigvee_L A$) exists in L, so does $\bigwedge_M f[A] = \bigwedge_M\{f(a) \mid a \in A\}$ (resp. $\bigvee_M f[A] = \bigwedge_M\{f(a) \mid a \in A\}$), and $f(\bigwedge_L A) = \bigwedge_M f[A]$ (resp. $f(\bigvee_L A) = \bigvee_M f[A]$).

The following is a key result in lattice theory, and it will play a role in some form or other in many different settings.

Theorem 1.1.5 (Adjoint Functor Theorem for Lattices)**.** *Let L and M be lattices, and let $f : L \to M$. Then:*

- *If f has a left adjoint $g : M \to L$, then f is meet-complete.*
- *If f has a right adjoint $g : M \to L$, then f is join-complete*
- *If L and M are complete lattices, then f has a left-adjoint if and only if it is a meet-complete, and f has a right-adjoint if and only if it is join-complete.*

Galois connections are particularly useful in order theory because they provide a canonical way of defining closure operators on posets.

Definition 1.1.6. Let $\mathcal{P} = (P, \leq_P)$ be a poset. A *closure operator* on $\mathcal{P}$ is a map $\kappa : P \to P$ with the following three properties for any $a, b \in P$:

- $a \leq_{\mathcal{P}} b \Rightarrow \kappa(a) \leq_{\mathcal{P}} \kappa(b)$ (monotone);
- $a \leq_{\mathcal{P}} \kappa(a)$ (increasing);

- $\kappa\kappa(a) \leq \kappa(a)$ (idempotent).

A *kernel operator* on $\mathcal{P}$ is a map $\lambda : P \to P$ such that λ is a closure operator on the poset $(P, \geq_{\mathcal{P}})$ dual to $\mathcal{P}$. In other words, λ is monotone, decreasing and idempotent. Given a closure operator κ on $\mathcal{P}$, the *fixpoints* of κ are the elements $a \in \mathcal{P}$ such that $\kappa(a) = a$.

Lemma 1.1.7. *Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ and $\mathcal{Q} = (Q, \leq_{\mathcal{Q}})$ be posets, and suppose $f : \mathcal{P} \to \mathcal{Q}$ has a right adjoint $G : \mathcal{Q} \to \mathcal{P}$. Then the operation $f \circ g : \mathcal{Q} \to \mathcal{Q}$ is a closure operator on $\mathcal{P}$, and the operation $g \circ f : \mathcal{P} \to \mathcal{P}$ is a kernel operator.*

In the case of a lattice L, the fixpoints of a closure operator on L have even more structure.

Theorem 1.1.8. *Let L be a lattice.*

- *If κ a closure operator on L, then the fixpoints of κ form a lattice with meets computed as in L and joins given by $a \sqcup b = \kappa(a \vee_L b)$.*
- *Dually, if λ is a kernel operator on L, then the fixpoints of λ form a lattice with joins computed as in L and meets given by $a \sqcap b = \lambda(a \wedge_M b)$.*
- *If L is a complete lattice, then the lattice of fixpoints of a closure (resp. kernel) operator on L also form a complete lattice.*

Let us now consider classes of lattices with some additional properties.

Definition 1.1.9. A lattice L is distributive if the operations $\wedge_L$ and $\vee_L$ satisfy the following properties for any $a, b, c \in L$:

$$a \wedge_L (b \vee_L c) \leq_L (a \wedge_L b) \vee_L (a \wedge_L c)$$
$$a \vee_L (b \wedge_L c) \leq_L (a \vee_L b) \wedge_L (a \vee_L c).$$

Equivalently, for any $a, b \in L$, if there is $c \in L$ such that $a \wedge_L c \leq_L b$ and $a \leq_L b \vee_L c$, then $a \leq_L b$.

Definition 1.1.10. A *Heyting algebra* is a distributive lattice L such that for any $b \in L$, the map $\cdot \wedge_L b : L \to L$ given by $a \mapsto a \wedge_L b$ has a right adjoint $b \to_L \cdot : L \to L$. Equivalently, there is a binary operation $\to_L$ on L such that for any $a, b, c \in L$:

$$a \wedge_L b \leq_L c \Leftrightarrow a \leq_L b \to_L c.$$

Definition 1.1.11. A *Boolean algebra* is a Heyting algebra L such that the map $\cdot \to_L 0 : L \to L$ is self-adjoint. Equivalently, L is a distributive lattice with a unary operation $\neg_L$ on L satisfying:

$$a \wedge_L \neg_L a = 0$$
$$a \vee_L \neg_L a = 1.$$

Distributive lattices, Heyting algebras and Boolean algebras will play a central role in the next three chapters. The main notion in Chapter 5 will be that of an ortholattice, which we will refrain from defining for now. In the case of Heyting algebras, the correct notion of morphism between them is stronger than that of a mere lattice homomorphism.

Definition 1.1.12. A *Heyting homomorphism* is a lattice homomorphism $f : L \to M$ that also preserves the operation $\to$. A Boolean homomorphism is a Heyting homomorphism between Boolean algebras.

We will also often need to consider elements of distributive lattices with the following key properties.

Definition 1.1.13. Let L be a lattice.

- An element $a \in L$ is *join-prime* (resp. *meet-prime*) if $a \leq_L b \vee_L c$ implies $a \leq_L b$ or $a \leq_L c$ (resp. $b \wedge_L c \leq_L a$ implies $b \leq_L a$ or $c \leq_L a$) for any $b, c \in L$.
- It is *completely join-prime* (resp. *completely meet-prime*) if for every $C \subseteq L$, $a \leq_L \bigvee C$ implies $a \leq_L c$ for some $c \in C$ (resp. $\bigwedge C \leq_L a$ implies $c \leq_L a$ for some $c \in C$).
- Completely join-prime elements in Boolean algebras are often called *atoms*, and completely meet-prime elements are called *co-atoms*.

Finally, we conclude this section by introducing the two dual notions filters and ideals, which are central in lattice theory. As is well known, ideals play a major role in abstract algebra as the kernels of homomorphisms, and filters play a key role both in logic, where they can be viewed as abstract consistent theories, and in general topology, where they can be thought of as ways of partially locating points.

Definition 1.1.14. A *filter* on a lattice L is a non-empty set $F \subseteq L$ with the following two properties for any $a, b \in L$:

- $a \in F$ and $a \leq_L b$ together imply $b \in F$;
- $a, b \in F$ implies $a \wedge_L b \in F$.

A filter F is proper if $F \neq L$.

Dually, an *ideal* on a lattice L is a non-empty set $I \subseteq L$ with the following two properties for any $a, b \in L$:

- $a \in I$ and $b \leq_L a$ together imply $b \in I$;
- $a, b \in I$ implies $a \vee_L b \in I$.

An ideal I is proper if $I \neq L$.

We will also be considering filters and ideals with some specific properties

Definition 1.1.15. Let L be a lattice.

- A proper filter F (resp. a proper ideal I) on L is *principal* if there is $a \in L$ such that $F = \{b \in L \mid a \leq_L b\}$ (resp. $I = \{b \in L \mid b \leq_L a\}$). The principal filter (resp. principal ideal) determined by an element $a \in L$ is often denoted by $\uparrow a$ (resp. $\downarrow a$).

- A proper filter F (resp. a proper ideal I) on a lattice L is *prime* if $a, b \notin F$ implies $a \vee_L b \notin F$ (resp. $a, b \notin I$ implies $a \wedge_L b \notin I$).

- Moreover, if L is a complete lattice, then F (resp. I) is *completely prime* if $\bigvee_L A \in F$ implies $F \cap A \neq \emptyset$ (resp. $\bigwedge_L A \in I$ implies $I \cap A \neq \emptyset$) for any $A \subseteq L$.

- A prime filter on a Boolean algebra B is also called an *ultrafilter*.

It is an easy exercise to verify that in any lattice L, $p \subseteq L$ is a prime filter on L if and only if $L \setminus p$ is a prime ideal on L. Moreover, it is a standard fact that prime filters (resp. prime ideals) coincide with maximal filters (resp. maximal ideals) for Boolean algebras. In the case of distributive lattices, maximal filters (resp. ideals) are always prime, but a prime filter (resp. ideal) may not always be maximal. Finally, maximal filters (resp. ideals) may fail to be prime in the case of arbitrary lattices.

The existence of prime filters or ideals on a given lattice L is a well-known problem that often determines much about the structure of L. In any lattice L, any two elements a, b such that $a \not\leq_L b$ can always be separated by the pair $(\uparrow a, \downarrow b)$. However, the requirement that a and b be separated by a prime filter whenever $a \not\leq b$ cannot be satisfied in arbitrary lattices, and actually follows from the following theorem in the case of distributive lattices.

Theorem 1.1.16 (Prime Filter Theorem)**.** *Let L be a distributive lattice. For any filter F and ideal I on L such that $F \cap I = \emptyset$, there is a prime filter p on L such that $F \subseteq p$ and $p \cap I = \emptyset$.*

The Prime Filter Theorem (PFT form now on) uses the Axiom of Choice in an essential way. In fact, it is often considered as a *fragment* of the full Axiom of Choice, rather than as merely one of its consequences. Of crucial relevance to the semiconstructive setting mentioned in the Introduction is the fact that PFT is independent from $ZF + DC$ [89] (see also Solovay's model [242] of $ZF + DC+$ "All sets of reals are Lebesgue measurable"). Perhaps surprisingly, it is however equivalent to a restricted form of it to Boolean algebras, known in the literature as the Boolean Prime Ideal Theorem (BPI from now on):

Theorem 1.1.17 (Boolean Prime Ideal Theorem)**.** *Let B be a Boolean algebra and $a \neq 0$ an element of B. There is an ultrafilter p on B such that $a \in p$.*

Both PFT and BPI have played a crucial role in the development of modern duality theory. Much of the work in the next three chapters is devoted to the development of alternative dualities that bypass the need to appeal to such non-constructive principles, and can therefore be carried out in $ZF + DC$. For now, let us turn to some background on dualities.

1.2 Discrete and Topological Dualities

In this section, I will review some basic notions from duality theory, and give an overview of some of the results that play an important role in the next three chapters. I will assume some familiarity with the basic terminology of category theory (e.g., categories, functors, isomorphisms and natural transformations) as well as rudiments of general topology. A reader unfamiliar with category theory but more versed in the language of set theory may appeal to the following intuition. A *category* is a structured class of mathematical objects of a given kind (such as sets, posets, groups or topological spaces), in which the structure of the category is determined by a fixed class of maps (called *arrows* or *morphisms*) between the objects it contains. Typically, the maps between the objects of a category are functions that preserve all or part of the internal structure (or lack thereof) of such objects, e.g., functions between sets, monotone maps between posets, groups homomorphisms between groups, or continuous maps between topological spaces. *Functors* are structure-preserving maps between categories, i.e., a functor F maps objects in a category $\mathbf{C}$ to objects in a category $\mathbf{D}$ and maps in $\mathbf{C}$ to maps in $\mathbf{D}$, in such a way that $F(f)$ is a map from $F(C)$ to $F(D)$ whenever f is a map from C to D $\mathbf{C}$., except in a case of *contravariant* functors, which "flip" the direction of the arrows from the source category to the target category (i.e., a contravariant functor F from $\mathbf{C}$ to $\mathbf{D}$ sends a map $f : C \to D$ in $\mathbf{C}$ to a map $F(f) : F(D) \to F(C)$ in $\mathbf{D}$. Finally, a *natural transformation* is a uniform way of passing from a functor $F : \mathbf{C} \to \mathbf{D}$ to a functor $G : \mathbf{C} \to \mathbf{D}$. Formally, a natural transformation $\eta : F \to G$ if given by a family $\{\eta_C\}_C \in \mathbf{C}$ of maps in $\mathbf{D}$ which satisfy the following "naturality" square:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(D) \\ \downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_D} \\ G(C) & \xrightarrow{F(g)} & G(D) \end{array}$$

meaning that composition $F(f)$ with η_D yields the same result as composing η_C with $G(f)$ for any $f : C \to D$ in $\mathbf{C}$. Naturality is often understood as a formal way to capture the informal notion of "canonicity" in mathematical practice, but an equally valid way to interpret this requirement (at least within the purview of this dissertation) is to think of it as requiring that the maps η_C by "uniformly definable" in some sense. We close this short and informal primer on category theory with a reminder of the following notions.

Definition 1.2.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories.

- An *adjunction* between $\mathbf{C}$ and $\mathbf{D}$ is a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that for any objects $C \in \mathbf{C}$ and $D \in \mathbf{D}$, there are bijections $\eta_{C,D} : \mathbf{Hom}_{\mathbf{D}}(F(C), D) \to \mathbf{Hom}_{\mathbf{C}}(C, G(D))$ natural in C and D.

- Given an adjunction (F, G), F is called the *left adjoint* of G, and G is called the *right adjoint* of F. Moreover, the family of maps $\eta_{F(C),F(C)}(1_{F(C)}) : C \to GF(C)$

for every object $C \in \mathbf{C}$ is called the *unit* of the adjunction, and the family of maps $\eta^{-1}_{G(D),G(D)}(1_{G(D)}) : FG(D) \to D$ for every object $D \in \mathbf{D}$ is called the *counit* of the adjunction.

- A pair of functors (F, G) is a *contravariant adjunction* if F and G are contravariant functors and for any two objects $C \in \mathbf{C}$ and $D \in \mathbf{D}$, there is a bijection $\eta_{C,D} : \mathbf{Hom}_{\mathbf{D}}(D, F(C)) \to \mathbf{Hom}_{\mathbf{C}}(C, G(D))$ natural in C and D.

Adjunctions are one of the most pervasive concepts in category theory. In a precise sense, an adjunction of functors between two categories generalizes the notion of a Galois connection of monotone maps between two posets. Just like a Galois connection allows one to move back and forth between the orders of two distinct posets, an adjunction allows one to move back and forth between the morphisms in two distinct categories. A close relative to the concept of an adjunction is that of an equivalence of categories.

Definition 1.2.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An *equivalence* of categories is given by a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\{\eta_C : C \to GF(C)\}_{C \in \mathbf{C}}$ and $\{\epsilon_D : D \to FG(D)\}_{D \in \mathbf{D}}$. A *duality* between $\mathbf{C}$ and $\mathbf{D}$ is an equivalence between $\mathbf{C}$ and $\mathbf{D}^{op}$, i.e., it is given by a pair of contravariant functors.

Informally, the notion equivalence is often taken to be the most fruitful notion of "sameness" between two categories. Indeed, the vast majority of categorical concepts are invariant under equivalence of categories. This means that in general, establishing the existence of an equivalence between two categories allows for the study of the objects in one category via the study of the corresponding objects in the other category. An equivalence of categories also often paves the way for many fruitful interactions between two mathematical domains, and they can be seen as "bridge" theorems that offer a way of translating results or problems in one setting into results or problems in another. In that respect, equivalences between categories of algebraic and geometric objects are often particularly useful. Dualities, in particular, exhibit an additional feature that makes them particularly desirable. Indeed, many categorical concepts come in "dual pairs", where one kind of construction is obtained from his dual by reverting all the arrows involved in its definition. Typical examples of such dual pairs include subalgebras and quotients, products and coproducts, limits and colimits, etc... Dualities allow us to "translate" constructions in one setting into dual constructions in the dual setting, which sometimes make them easier to comprehend. Let us now turn to the specific dualities that will occupy us for the next chapters.

Duality theory focuses on dualities between categories of lattices and categories of geometric structures such as graphs and topological spaces. The cornerstone of the field is arguably Stone's duality between Boolean algebras and Stone spaces [245]. Let us first recall the following definition.

Definition 1.2.3. A *Stone space* is a topological space $\mathscr{X} = (X, \tau)$ which is compact, Hausdorff and 0-dimensional (i.e., it has a basis of clopen sets).

Stone duality is the following theorem.

Theorem 1.2.4 (Stone 1936)**.** *The category of Boolean algebras and Boolean homomorphisms between them is dual to the category of Stone spaces and continuous maps between them.*

Let us quickly describe this duality here, as it provides the original template for many of the techniques that are still in use in duality theory up to this day. Given a Boolean algebra B, its dual Stone space is the topological space $(Spec(B), \sigma)$, where $Spec(B)$ is the set of all prime filters on B (equivalently, all ultrafilters on B), and σ is the topology generated by the sets of the form $\widehat{a} = \{p \in Spec(B) \mid a \in p\}$. Conversely, given a Stone space $\mathscr{X} = (X, \tau)$, its dual Boolean algebra is the Boolean algebra $Clop(\mathscr{X})$ of clopen subsets of X. That $\mathscr{X}$ is homeomorphic to $SpecClop(\mathscr{X})$ is essentially due to the definition of Stone spaces. Indeed, it is a basic fact from general topology that any ultrafilter on the subsets of a topological space $\mathscr{X}$ converges to at least one point if $\mathscr{X}$ is compact, and to at most one point if $\mathscr{X}$ is Hausdorff. Moreover, 0-dimensionality implies that two ultrafilters U and V on $\mathscr{X}$ converge to the same point if and only if $\mathsf{U} \cap Clop(\mathscr{X}) = \mathsf{V} \cap Clop(\mathscr{X})$. Hence the map $\dot{\in} : \mathscr{X} \to SpecClop(\mathscr{X})$ given by $x \mapsto \{U \in Clop(\mathscr{X}) \mid x \in U\}$ is a bijection, and it is easy to see that $\dot{\in}^{-1}[\widehat{U}] = U$ for every $U \in Clop(\mathscr{X})$. The converse direction, however, is the celebrated Stone Representation Theorem:

Theorem 1.2.5 (Stone Representation Theorem)**.** *For any Boolean algebra B, $Spec(B)$ is a Stone space, and the map $\widehat{\cdot} : B \to Clop(Spec(B))$ is a Boolean isomorphism.*

As is well known, Stone's theorem requires (BPI) in essentially two places. First, in establishing that $Spec(B)$ is compact. Second, in making sure that the map $a \mapsto \widehat{a}$ is injective, as one needs to show that if $a \nleq b$ for some $a, b \in B$, there is $p \in Spec(B)$ such that $a \in p$ and $b \notin p$.

Finally, the correspondence between Boolean algebras and Stone spaces is extended to a correspondence between their respective morphisms by taking inverse images (which explains why we obtain a duality instead of an equivalence). Indeed, any Boolean homomorphism $f : B \to C$ induces a continuous map $Spec(f) : Spec(C) \to Spec(B)$ given by the map $p \mapsto f^{-1}[p] = \{a \in B \mid f(a) \in p\}$. Conversely, since the preimage of a clopen set under a continuous map is clopen, and preimages under any function preserve intersections and set-theoretic complements, the inverse image map $Clop(g) : Clop(\mathscr{X}) \to Clop(\mathscr{Y})$ is a Boolean homomorphism for any continuous map $g : \mathscr{Y} \to \mathscr{X}$.

Stone duality was one of the earlier results establishing a tight correspondence between order theory and topology. His duality theorem was soon extended, both on the topological side (e.g., by de Vries [69], who identified a category of lattices dual to the category of compact Hausdorff spaces) and on the algebraic side. Stone himself extended his result to a duality between distributive lattices and spectral spaces [246].

Definition 1.2.6. A *spectral space* is a compact T_0 topological space $\mathscr{X} = (X, \tau)$ such that:

- the compact open sets $\mathbf{CO}(\mathscr{X})$ form a basis for τ and are closed under intersections;

- $\mathscr{X}$ is *sober*, i.e., any completely prime filter on the lattice of open sets of $\mathscr{X}$ converges to a unique point.

A *spectral map* between spectral spaces is a continuous map $f : \mathscr{X} \to \mathscr{Y}$ such that $f^{-1}[U] \in \mathbf{CO}(\mathscr{X})$ for every $U \in \mathbf{CO}(\mathscr{Y})$.

Given a distributive lattice L, Stone considers the space $Spec(L)$ of all prime filters over L, endowed with the same topology as the one he defined on the dual spaces of Boolean algebras, i.e., the topology generated by sets of the form $\widehat{a}$ for $a \in L$. Using PFT, one can show that $Spec(L)$ is compact for any distributive lattice L, and moreover that $\mathbf{CO}(Spec(L))$ is always a distributive lattice into which L embeds via the map $a \mapsto \widehat{a}$. He thus obtained the following:

Theorem 1.2.7 (Spectral Duality)**.** *The category* **DL** *of distributive lattices and lattice homomorphisms between them is dual to the category* **Spec** *of spectral spaces and spectral maps between them.*

Spectral spaces, however, are not as nicely behaved as Stone spaces. This prompted Priestley to modify Stone's definition in order to obtain better behaved dual spaces of distributive lattices [210]. Priestley's idea was to consider the *patch topology* of a spectral space $(\mathscr{X}, \tau)$, i.e., the topology generated by the compact opens in τ together with their complements. In the case of a spectral space, this yields a Stone space. Priestley duality then recovers the structure of the original distributive lattice by adding a partial order on the topological space thus obtained. The Priestley dual of a distributive lattice L is therefore the ordered topological space $(Spec(L), \tau', \subseteq)$, where τ' is generated by the sets of the form $\widehat{a}$ for any $a \in L$ and their complements, and $\subseteq$ is the inclusion ordering on the set of prime filters of L. Using PFT, one can then verify that this space satisfies the following definition.

Definition 1.2.8. A *Priestley space* is a triple $(X, \tau, \leq)$ such that $\mathscr{X} = (X, \tau)$ is a Stone space, $\leq$ is a partial order on X satisfying the Priestley Separation Axiom:

(PSA) For any $x, y \in X$, if $x \nleq y$, then there is a clopen upset $U \subseteq X$ such that $x \in U$ and $y \notin U$.

An *order-continuous* map between Priestley spaces is a continuous map that is also monotone with respect to the two partial orderings.

Given a Priestley space $(X, \tau, \leq)$, the clopen upsets of $\mathscr{X}$ form a distributive lattice $ClopUp(\mathscr{X})$. Moreover, using PFT, one can show that $(Spec(L), \tau', \leq)$ is a Priestley space for every distributive lattice L, and that the map $a \mapsto \widehat{a}$ is a lattice isomorphism between L and $ClopUp(L)$. This yields the following.

Theorem 1.2.9 (Priestley Duality)**.** *The category* **DL** *is dual to the category* **PS** *of Priestley spaces and order-continuous maps between them.*

Working with ideas similar to Priestley's, Esakia [83, 82] developed a duality for Heyting algebras that is an elegant restriction of Priestley's.

Definition 1.2.10. A Priestley space is an *Esakia space* if the downset of every clopen set is clopen. An order-continuous map $f : \mathscr{X} \to \mathscr{Y}$ between Esakia spaces is an *Esakia morphism* if it is a *p-morphism*, i.e., it satisfies the following condition for any $x \in \mathscr{X}, y \in \mathscr{Y}$:

- If $f(x) \leq_{\mathscr{Y}} y$, then there is $x' \in \mathscr{X}$ such that $x \leq_{\mathscr{X}} x'$ and $f(x') = y$.

Theorem 1.2.11 (Esakia Duality)**.** *The category* **HA** *of Heyting algebras and Heyting homomorphisms between them is dual to the category* **ES** *of Esakia spaces and Esakia morphisms between them.*

Beyond distributive lattices and Heyting algebras, however, duality theory often loses some of its steam. Several topological dualities have been proposed for many categories of lattices, but the corresponding topological spaces are often not as well behaved as Stone or Priestley spaces. One of the reasons for this is the lack of an analogue of PFT. Many properties of a topological space $\mathscr{X}$ can indeed be given characterizations in terms of prime filters on the powerset of $\mathscr{X}$ or on its lattice of open sets. In the absence of PFT, it is more difficult to relate the topological properties of a space with the order-theoretic properties of some of its subsets.

Arguably, Stone's key insight, followed by Priestley, Esakia, and many more, was the realization that topology could help in representing a wider variety of algebraic structures than discrete geometric structures such as sets and graphs.[1] In the case of Boolean algebras, his representation theorem generalized another, conceptually simpler one, often attributed to Tarski. Let **CABA** be the category of complete and atomic Boolean algebras, where a Boolean algebra B is atomic if for every $b \in B$ there is an atom $a \in B$ such that $a \leq_B b$. Then there is a one-to-one correspondence between $CABAs$ and sets, mapping each $CABA$ B to its set of atoms $At(B)$, and every set S to its powerset $\mathscr{P}(S)$. This simple observation turned out to play an important role in the development of *discrete* dualities between categories of lattices and categories of relational structures, paving the way for providing some solid algebraic foundations to what would eventually become the most common semantics for modal logic [150, 151]. This discrete approach can also be carried out beyond the Boolean case, to obtain relational semantics for (modal) intuitionistic or positive logic [216, 68, 160], as well as beyond the setting of distributive lattices to substructural logics [2, 98]. One of the most general result in this area, inspired from the old tradition of *polarities* due to Birkhoff, is Gehrke's representation of *perfect lattices* via *generalized Kripke frames* [101].

A distinctive feature of this approach is that the representation of a given lattice L is obtained via very specific filters or ideals, which are both principal and satisfy some maximality constraint such as being completely prime or completely irreducible (a variant of the notion of primality that is equivalent to it in the distributive case). Consequently, this type of approach can only represent complete lattices with some very strong properties, just like Tarski's representation for complete Boolean algebras via powersets can only represent

[1] Perhaps apocryphally, he is often credited for the motto that "One should always topologize!".

complete and atomic Boolean algebras. There is, however, another way of providing discrete representations for Boolean algebras, based on working with principal filters rather than principal filters with maximal properties. This is precisely what we called the *Forcing duality* between complete Boolean algebras and separative and complete posets in the Introduction, and it plays for possibility semantics the role that Tarski's **CABA** duality plays for Kripke-style semantics in non-classical and modal logics. Moreover, this duality can be topologized so as to extend it to all Boolean algebras. This is the choice-free duality via UV-spaces of Holliday and Bezahnishvili [41] (see Section 3.2.3 below). Importantly, this duality relates to Stone duality via the Vietoris hyperspace construction [258], which we briefly describe here, as it will play a key role in the following chapters.

For any compact Hausdorff space $\mathscr{X} = (X, \tau)$, let $\mathcal{K}(\mathscr{X})$ be the set of all compact subsets of $\mathscr{X}$. Given an open set $U \in \tau$, let $\Box U = \{V \in \mathcal{K}(\mathscr{X}) \mid V \subseteq U\}$ and $\Diamond U = \{V \in \mathcal{K}(\mathscr{X}) \mid V \cap U \neq \emptyset\}$. Vietoris considered the following construction on metric spaces, which was then extended to all compact Hausdorff spaces.

Definition 1.2.12. Let $\mathscr{X} = (X, \tau)$ be a compact Hausdorff space.

- The *Upper Vietoris hyperspace* of $\mathscr{X}$ is the topological space $\mathbb{V}_\Box(\mathscr{X}) = (\mathcal{K}(\mathscr{X}), \tau^\Box)$, where $\tau^\Box$ is the topology generated by the set $\{\Box U \mid U \in \tau\}$.
- The *Lower Vietoris hyperspace* of $\mathscr{X}$ is the topological space $\mathbb{V}_\Diamond V(\mathscr{X}) = (\mathcal{K}(\mathscr{X}), \tau_\Diamond)$, where $\tau_\Diamond$ is the topology generated by the set $\{\Diamond U \mid U \in \tau\}$.
- Finally, the *Vietoris hyperspace* of $\mathscr{X}$ is the topological space $\mathbb{V}(\mathscr{X}) = (\mathcal{K}(\mathscr{X}), \tau_\Diamond^\Box$, where $\tau_\Diamond^\Box$ is the join of the topologies $\tau_\Box$ and $\tau_\Diamond$, i.e., the topology generated by the family of sets $\{\Box U, \Diamond U \mid U \in \tau\}$.

Assuming BPI, one can show that the Vietoris hyperspace of a compact Hausdorff space is always compact and Hausdorff, and that the Vietoris hyperspace of a Stone is always a Stone space.[2] The Vietoris hyperspace construction can also be lifted to a functor on the category **KHaus** of compact Hausdorff spaces, which yields a powerful method to reason about topological semantics for modal logic [257]. Upper Vietoris spaces, on the other hand, offer a bridge between Stone duality and its choice-free counterpart:

Theorem 1.2.13 ([41], Thm 7.7)**.** *A topological space $\mathscr{X}$ is a UV-space if and only if $\mathscr{X} = \mathbb{M}_\Box(\mathsf{CO}\mathcal{RO}(\mathscr{X}))$, where $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ is the choice-free dual Boolean algebra of $\mathscr{X}$.*

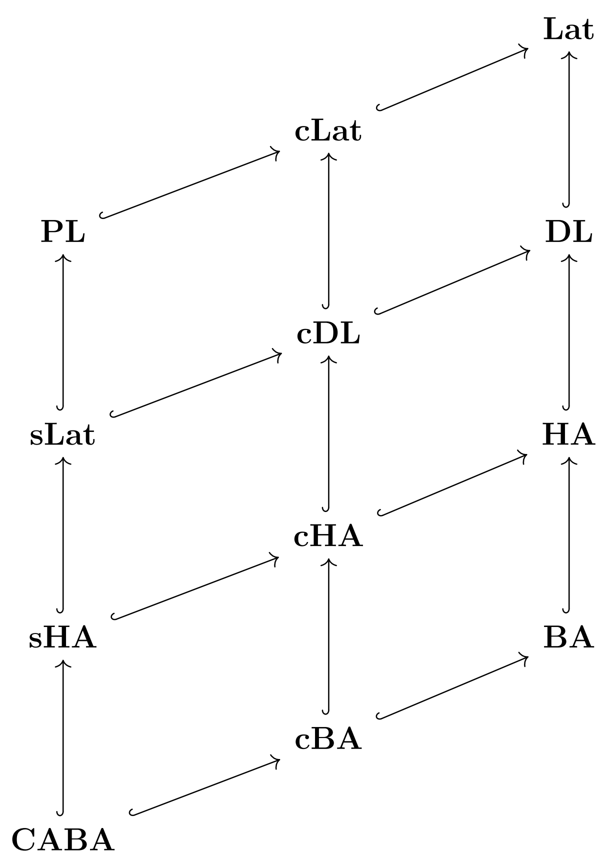
In other words, the choice-free dual space of a Boolean algebra can be obtained from its Stone dual by the upper Vietoris construction.

Much of the work in the next three chapters is devoted to extending the pattern outlined here beyond the setting of Boolean algebras. We conclude this introduction by offering a somewhat systematic picture of the dualities that play a role in the rest of our work. As

[2]This is, in fact, equivalent to BPI, see [146, p. 122].

transpired from the discussion above, dualities can be classified by at least two of their features. The first one is the type of properties they require of the filter / ideal objects they use in their representation of algebraic structures. Discrete dualities typically use principal filters / ideals, while dualities that rely on some form of the Axiom of Choice use filter / ideals with some maximal property like primality. The second feature is their level of generality. Discrete dualities can only be established for categories of complete lattices, while topological dualities can capture larger categories of lattices. Dualities based on filters can be developed for categories of lattices that exhibit strong symmetry properties, like Boolean algebras or distributive lattices, while filter-ideal based dualities can have a broader scope.

In order to help the reader orientate themselves around all these dualities, we conclude with two visual aides. First, the following diagram of categories of lattices, in which arrows indicate inclusion of categories, and each point contains a hyperlink to a part of the dissertation where a duality for the corresponding category of lattices is discussed.



Second, the table below offers a classification of many of the dualities discussed in this dissertation according to their topological/discrete and constructive or non-constructive nature and their level of generality. Each cell contains a hyperlink to a part of this dissertation discussing it. Finally, bold-faced notions are new dualities that are introduced in this dissertation.

Finally, Table 1.1 can be seen as also providing a roadmap for the next three chapters.

	Boolean Algebras	Heyting Algebras	Distributive Lattices	Lattices	
Principal Prime/Maximal Filters	Tarski	de Jongh/Troelstra	Raney	Gehrke	Discrete / Choice-free
Principal Filters	Forcing	**Heyting B-Frames**	**Distributive B-Frames**	**B-Frames**	Discrete / Choice-free
Prime/Maximal Filters	Stone	Esakia	Stone/Priestley	Hartung-Urquhart	Topological / Choice
Filters	UV-Spaces	**Heyting Bispaces**	$\boldsymbol{PUV}$**-spaces** / $\boldsymbol{UVP}$**-Spaces**	Moshier-Jipsen / Dunn-Hartonas-Allwein / $\boldsymbol{FI}$**-Spaces**	Topological / Choice(-free)

Table 1.1: Systematic Overview of Many Dualities

Chapter 2 is devoted to discrete dualities that generalize the Forcing duality to the category of all complete lattices and to some of its subcategories. Chapter 3 focuses on extending the choice-free Stone duality beyond Boolean algebras but still within the realm of distributivity. Finally, Chapter 4 tackles the issue of extending this approach to the most general category considered here, namely the category of all lattices.

Chapter 2

A Discrete Duality for Complete Lattices

2.1 Introduction

Topological dualities have become a standard tool in the representation of lattices and in the semantics of non-classical logics. Stone famously established a duality between Boolean algebras and Stone spaces [245] which he later generalized to a duality between distributive lattices and spectral spaces [246]. Priestley [210] presented an alternative duality between distributive lattices and Priestley spaces, while Esakia's work [81, 83] yields dualities for Heyting and bi-Heyting algebras. In the general case of bounded lattices, several dualities have been proposed. Urquhart [256] gave a topological representation of bounded lattices that directly generalizes Stone and Priestley's theorems and which was later lifted by Hartung [127] to a duality for bounded lattices and surjective lattice morphisms. Other dualities for lattices and various lattice expansions have been proposed by Allwein, Dunn and Hartonas [1, 122, 124, 125], as well as by Jipsen and Moshier [197] and Gehrke and van Gool [105]. Although these topological dualities can be used to give representations of complete lattices, there is also a long tradition of discrete, purely relational representations of complete lattices. This tradition originates with Tarski's duality between sets and complete and atomic Boolean algebras, which was later expanded to Boolean algebras with operators (BAOs) and used to provide a semantics for modal logic. Tarski's duality was also generalized to a duality between posets and superalgebraic locales [68], also known as completely join-prime generated complete lattices [215, 216]. In set theory, an alternative representation of complete Boolean algebras as the regular open sets of a poset has also become a cornerstone of forcing [144, 161] and has recently been used to provide an alternative semantics for modal logic known as possibility semantics [24, 133, 134, 137, 140]. This latest representation of complete Boolean algebras is also related to a more general representation of complete lattices obtained by Allwein and MacCaull in [2].

In this chapter, we lift Allwein and MacCaull's representation theorem to a full duality. This is achieved by establishing first an idempotent adjunction between the category

cLat of all complete lattices and a category **Bos** of bi-preordered sets (*bosets* for short). Bi-ordered sets already played a role in Urquhart's representation theorem, although the Allwein-MacCaull dual bosets we consider differ from Urquhart's, and they have also been discussed in connection with the representation of complete Heyting algebras [36, 37, 188]. As shown in [136], there is also a strong connection between representations of complete lattices via bi-ordered sets and via polarities [42, 62, 101, 125, 126]. We use our b-frame duality to provide discrete representations of various classes of complete lattices and use these alternative characterizations to obtain some results in the theory of complete Heyting algebras and the semantics of intermediate logics.

The chapter is organized as follows. In Section 2.2, we introduce bosets and the relevant notion of morphism between them, and we lift the Allwein-MacCaull representation of complete latties to an idempotent adjunction between the category of bosets and the category of complete lattices. In order to restrict this adjunction to a duality, we generalize the notion of dense embeddings from forcing posets to the setting of bosets, and we use this to characterize the fixpoints of the adjunction. This allows us to define the category **bF** of b-frames, dual to the category **cLat** of complete lattices. We conclude the section by comparing b-frame duality to some existing discrete and topological representations of lattices.

In the following two sections, we develop this framework further by establishing a correspondence between algebraic properties of complete lattices and first-order properties of b-frames. This allows us to obtain alternative representations of complete distributive lattices, complete Heyting algebras and complete Boolean algebras in Section 2.3, while in Section 2.4 the duality obtained for complete Heyting algebras is further restricted to obtain geometric, amalgamation-like characterizations of the duals of spatial and superalgebraic locales.

The last two sections are devoted to new applications of this framework to the theory of complete Heyting algebras and the semantics of intermediate logics. In Section 2.5, the notion of a coproduct of two bosets is defined and used to prove the following decomposition theorem for complete bi-Heyting algebras:

Theorem 2.5.14. *Let L be a complete bi-Heyting algebra. Then L is a complete subdirect product of $L_1 \times L_2$ in* **cLat**, *where L_1 is a completely join-prime generated locale and L_2 is locale with no completely join-prime element.*

A related result in the theory of Boolean algebras [106] states that any complete Boolean algebra is the product of an atomic and an atomless Boolean algebra, although Theorem 2.5.14 is a result about complete bi-Heyting algebras in **cLat**, rather than in the category of complete bi-Heyting algebras and complete bi-Heyting morphisms, which is not a full subcategory of **cLat**.

Finally, Section 2.6 discusses some applications of this framework to the semantics of intuitionistic logic. We introduce boset semantics, a semantics as general as locale semantics for intuitionistic logic and show how semantics that are equivalent to Kripke and topological semantics arise as natural restrictions imposed on boset semantics. As a consequence, boset semantics provides a uniform presentation of most of the semantic hierarchy for intuitionistic logic introduced in [36]. We conclude with an application to the incompleteness problem for intermediate logics:

Theorem 2.6.14. *The intermediate logic SL, originally proved by Shehtman [239] to be Kripke-incomplete, is in fact incomplete with respect to the larger class of all complete bi-Heyting algebras.*

A similar result has recently and independently been obtained by Bezhanishvili, Gabelaia and Jibladze in [35], via Esakia duality and through a fairly intricate argument. By contrast, our proof is a straightforward adaptation of Shehtman's original argument, which we take as evidence that boset semantics can be a fruitful framework for the study of intermediate logics.

2.2 B-frame Duality

In this section, we introduce the category **bF** of b-frames and prove that it is dual to the category of complete lattices **cLat**. This is done in two steps. First, we introduce a category **Bos** of bi-preordered sets and establish an idempotent adjunction between **Bos** and **cLat**. As was already noted by Allwein and McCaul in their representation theorem for complete lattices obtained in [2], all complete lattices are fixpoints of this adjunction. This means that we only need to restrict **Bos** to a full subcategory of fixpoints in order to obtain a category dual to **cLat**. We call such fixpoints *b-frames* and show that they are completely characterized by certain properties of bi-preordered sets. Finally, in Section 2.2.5, we connect this adjunction to well-known discrete dualities for complete lattices, showing in particular how it generalizes Tarski's duality between **CABA** and **Sets**, Raney's duality between superalgebraic lattices and posets and the forcing duality between complete Boolean algebras and separative posets. We also discuss connections with several existing dualities for lattices, including Urquhart-Hartung duality [127, 256], Allwein-Hartonas duality [122] and Hartonas-Dunn duality [124, 125, 126].

2.2.1 Bosets and B-morphisms

Our starting point is the notion of a bi-preordered set, which will be called *bosets* for short. In other words, a boset is a tuple $(X, \leq_1, \leq_2)$ such that $\leq_1$ and $\leq_2$ are preorders on X. Bi-ordered sets have been used before in the representation theory of bounded lattices, in particular by Urquhart [256], Hartung [127] and in various ways by Allwein, MacCaull, Hartonas and Dunn [1, 2, 125, 126]. We refer the reader to Section 2.2.5 for a comparison of our approach to this literature. More recently, bi-ordered sets have also been discussed in connection with the representation of complete Heyting algebras in [36, 37, 188]. This connection will be explored further in Section 2.6. For now, we introduce the notion of morphism between bosets that will be relevant for our purposes:

Definition 2.2.1. Let $(X, \leq_1^X, \leq_2^X)$ and $(Y, \leq_1^Y, \leq_2^Y)$ be two bosets. A map $f : X \to Y$ is a *boset morphism* (b-morphism) if the following are true:

1. f is monotone in both orderings, i.e., for any $x, y \in X$, if $x \leq_i^X y$, then $f(x) \leq_i^Y f(y)$ for $i \in \{1, 2\}$;
2. $\forall x \in X\, \forall y \geq_2^Y f(x)\, \exists z \geq_2^X x : f(z) \geq_1^Y y$;
3. $\forall x \in X\, \forall y \geq_1^Y f(x)\, \exists z \geq_1^X x : f(z) \geq_2^Y y$.

It is straightforward to verify that the composition of two b-morphisms is still a b-morphism.

Lemma 2.2.2. *Let $f : (X, \leq_1^X, \leq_2^X) \to (Y, \leq_1^Y, \leq_2^Y)$ and $g : (Y, \leq_1^Y, \leq_2^Y) \to (Z, \leq_1^Z, \leq_2^Z)$ be two b-morphisms. Then $g \circ f : (X, \leq_1^X, \leq_2^X) \to (Z, \leq_1^Z, \leq_2^Z)$ is a b-morphism.*

Proof. Monotonicity is clear. Suppose $x \in X$ and $z \geq_2^Z gf(x)$. Then since g is a b-morphism there is $y \geq_2^Y f(x)$ such that $g(y) \geq_1^Z z$. But since f is a b-morphism, this implies that there is $x' \geq_2^X x$ such that $f(x') \geq_1^Y y$. Thus $gf(x') \geq_1^Z g(y) \geq_1^Z z$. Hence $g \circ f$ satisfies condition 2. The proof that $g \circ f$ also satisfies condtion 3 is completely similar. □

Therefore bosets and b-morphisms form a category **Bos**. Our main goal is to understand how this category relates to **cLat**, the category of complete lattices and complete lattice morphisms between them. Throughout this chapter, given a poset $(P, \leq)$ and $A \subseteq P$, we will write $\uparrow A$ and $\downarrow A$ for the sets $\{p \in P \mid \exists q \in A : q \leq p\}$ and $\{p \in P \mid \exists q \in A : p \leq q\}$ respectively.

Example 2.2.3. Any preordered set $\mathbb{P} = (P, \leq)$ may be viewed as a boset in two different ways: either as a *Kripke boset* $\mathbb{P}_F = (P, \leq, \geq)$, i.e., by letting the second ordering be the converse of the ordering on P, or as *forcing boset* $\mathbb{P}_B = (P, \geq, \geq)$, obtained by letting the two orderings be the converse ordering.[1] It is straightforward to verify that a b-morphism between Kripke bosets is precisely a monotone map between the underlying preordered sets, while a b-morphism between forcing bosets is a monotone map $f : (P, \leq_P) \to (Q, \leq_Q)$ that is also *weakly dense*, i.e., f is such that $f[\downarrow p]$ is dense (in the sense of the downset topology induced by the ordering) in $\downarrow f[p]$ for every $p \in P$.

Any boset $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ can be regarded as a bi-topological space, by letting τ_1 and τ_2 be the upset topologies induced by the orders $\leq_1$ and $\leq_2$ respectively. We write C_1 and C_2 for the corresponding closure operators. We can then consider the complete lattices O_1

[1] The reason for flipping the order is simply historical: in the forcing literature, one typically works with regular open *downsets*, while Kripke semantics is typically defined in terms of *upsets*. Since we will be working with upsets, yet several notions defined below are generalizations of notions about forcing posets, flipping the order when representing forcing posets as bosets will help avoid any confusion.

and O_2 of open sets in τ_1 and τ_2 respectively and define two antitone maps: $\neg_1 : \mathsf{O}_2 \to \mathsf{O}_1$ and $\neg_2 : \mathsf{O}_1 \to \mathsf{O}_2$ by letting $\neg_i U = X - C_i(U)$ for any $U \in \mathsf{O}_j$ and $i \neq j \in \{1,2\}$. Now clearly for any $U \in \mathsf{O}_1$ and $V \in \mathsf{O}_2$,

$$U \subseteq \neg_1 V \text{ iff } U \subseteq X - V,$$

and

$$V \subseteq \neg_2 U \text{ iff } V \subseteq X - U.$$

So $\neg_1$ and $\neg_2$ form a Galois connection, which means that the composite map $\neg_1\neg_2$ is a closure operator on O_1. A fixpoint of $\neg_1\neg_2$ is called *regular open*. Notice in particular that if $\tau_1 = \tau_2$, this definition coincides with the usual notion of a regular open subset of a topological space. It is useful to observe that a set $U \subseteq X$ is regular open if and only if for any $x \in X$:

$$x \in U \text{ iff } \forall y \geq_1^X x \, \exists z \geq_2^X y : z \in U.$$

As the fixpoints of a closure operator on a complete lattice always form a complete lattice (Theorem 1.1.8, see also [229, Thm. 5.2]), it follows that the fixpoints $\mathsf{RO}_{12}(\mathscr{X})$ form a complete lattice. It is straightforward to verify that for any collection $\{U_i\}_{i\in I}$ of sets in $\mathsf{RO}_{12}(\mathscr{X})$, $\bigwedge_{i\in I} U_i = \bigcap_{i\in I} U_i$ and $\bigvee_{i\in I} U_i = \neg_1\neg_2(\bigcup_{i\in I} U_i)$. Regular open sets in bitopological spaces have been studied before in the context of duality theory for lattices, in particular in the Pairwise Stone duality for distributive lattices developed in [39] and in using Priestley and Esakia duality to give a topological characterization of MacNeille completions of Heyting algebras [119]. The next lemma shows that the inverse image of a b-morphism maps regular opens to regular opens.

Lemma 2.2.4. *Let $f : (X, \leq_1^X, \leq_2^X) \to (Y \leq_1^Y, \leq_2^Y)$ be a b-morphism. Then for any 1-upset $U \subseteq Y$, $f^{-1}[\neg_1\neg_2(U)] = \neg_1\neg_2 f^{-1}[U]$.*

Proof. We claim that $C_i(f^{-1}[U]) = f^{-1}[C_i(U)]$ for any $U \in \mathsf{O}_j$, $i \neq j \in \{1,2\}$. This is clearly enough to establish that $f^{-1}[\neg_1\neg_2(U)] = \neg_1\neg_2 f^{-1}[U]$ for any $U \in \mathsf{O}_1$. For the proof of the claim, suppose $f(y) \in U$ for some $y \geq_i^X x$. Since $f(y) \geq_i^X f(x)$, by the b-morphism conditions, there is some $z \geq_i^X x$ such that $f(z) \geq_j^Y f(y)$, for $j \neq i$. Since U is j-open, $f(z) \in U$, so $f(x) \in C_i(U)$. This shows that $C_i(f^{-1}[U]) \subseteq f^{-1}[C_i(U)]$. Conversely, if $f(x) \leq_i^Y y$ for some $y \in U$, then by the b-morphism conditions again there is some $z \geq_i^X x$ such that $f(z) \geq_j y$. Once again, since U is j-open this implies that $z \in f^{-1}[U]$, and thus $x \in C_i(f^{-1}[U])$. □

This allows us to define a contravariant *regular open functor* $\rho : \mathbf{Bos} \to \mathbf{cLat}$:

- For any boset $\mathscr{X} = (X, \leq_1, \leq_2)$, $\rho(\mathscr{X}) = \mathsf{RO}_{12}(\mathscr{X})$, i.e., the complete lattice of fixpoints of the $\neg_1\neg_2$ closure operator on the 1-upward closed sets of $\mathscr{X}$.
- Given a b-morphism $f : (X, \leq_1^X, \leq_2^X) \to (Y, \leq_1^Y, \leq_2^Y)$, $\rho(f) : \mathsf{RO}_{12}(Y) \to \mathsf{RO}_{12}(X)$ is defined as the restriction to $\mathsf{RO}_{12}(Y)$ of the preimage function f^{-1}.

Since $\mathsf{RO}_{12}(\mathscr{X})$ is a complete lattice for any boset $\mathscr{X}$, ρ is well-defined on objects. To see that it is well defined on morphisms, suppose $f : \mathscr{X} \to \mathscr{Y}$ is a b-morphism. Then by Lemma 2.2.4 $\rho(f)$ is a map from $\mathsf{RO}_{12}(\mathscr{Y}) \to \mathsf{RO}_{12}(\mathscr{X})$. Moreover, let $\{U_i\}_{i\in I}$ be any collection of sets in $\mathsf{RO}_{12}(\mathscr{Y})$. It is routine to check that $\bigcap_{i\in I} f^{-1}[U_i] = f^{-1}[\bigcap_{i\in I} U_i]$, and by Lemma 2.2.4, we have that

$$f^{-1}[\neg_1\neg_2(\bigcup_{i\in I} U_i)] = \neg_1\neg_2(f^{-1}[\bigcup_{i\in I} U_i]) = \neg_1\neg_2(\bigcup_{i\in I} f^{-1}[U_i]).$$

Thus $\rho(f)$ is a complete lattice morphism from $\mathsf{RO}_{12}(\mathscr{Y})$ to $\mathsf{RO}_{12}(\mathscr{X})$.

2.2.2 From Lattices to Bosets

Having constructed the first half of the adjunction, let us now define a functor going from complete lattices to bosets. The construction on objects was already introduced in [2], although Allwein and MacCaull do not extend their representation theorem to a full duality.

Definition 2.2.5. Let L be a complete lattice. The *dual Allwein-MacCaull boset* of L is the boset $(P_L, \leq_1^L, \leq_2^L)$ such that:

- $P_L = \{(a,b) \in L \mid a \not\leq b\}$;
- $(a,b) \leq_1^L (c,d)$ iff $a \geq_L c$;
- $(a,b) \leq_2^L (c,d)$ iff $b \leq_L d$.

Let $f : L \to M$ be a complete lattice homomorphism. By the adjoint functor theorem, f has a left adjoint $\cdot^f$ and a right adjoint $\cdot_f$, where for any $a \in M$, $a^f = \bigwedge\{c \in L \mid f(c) \geq a\}$ and $a_f = \bigvee\{c \in L \mid f(c) \leq a\}$. The following lemma shows how to use the existence of those adjoints to construct a b-morphism from f.

Lemma 2.2.6. *Let $f : L \to M$ be a complete lattice homomorphism. The map $\alpha(f) : (P_M, \leq_1^M, \leq_2^M) \to (P_L, \leq_1^L, \leq_2^L)$ defined by $\alpha(f)(a,b) = (a^f, b_f)$ is a b-morphism.*

Proof.

- Showing that $\alpha(f)$ is well defined amounts to proving that for any $a, b \in M$, $a^f \leq b_f$ implies that $a \leq b$. But clearly as $\cdot^f$ and $\cdot_f$ are left and right adjoint to f respectively, we have that $a \leq f(a^f)$ and $f(b_f) \leq b$, so by monotonicity of f, $a^f \leq b_f$ implies that $a \leq f(a^f) \leq f(b_f) \leq b$.
- Monotonicity of $\alpha(f)$ in the two orderings is straightforward.
- Let $(a,b) \in P_M$ and suppose $\alpha(f)(a,b) \leq_2^L (c,d)$ for some $c \not\leq d \in L$. We claim that the pair $(f(c), b)$ is in P_M. To see this, note that if $f(c) \leq b$, then $c \leq b_f$. But since $\alpha(f)(a,b) = (a^f, b_f) \leq_2^L (c,d)$, we have that $c \leq d$, a contradiction. Thus $f(c) \not\leq b$. Therefore $(a,b) \leq_2^M (f(c), b)$ and $(c,d) \leq_1^L (f(c)^f, b_f)$.

- Let $(a,b) \in P_M$ and suppose $\alpha(f)(a,b) \leq_1^L (c,d)$ for some $c \not\leq d \in L$. We claim that the pair $(a, f(d))$ is in P_M. To see this, note that if $a \leq f(d)$, then $a^f \leq d$. But since $\alpha(f)(a,b) = (a^f, b_f) \leq_1^L (c,d)$, we have that $c \leq d$, a contradiction. Thus $a \not\leq f(d)$. Therefore $(a,b) \leq_1^M (a, f(d))$ and $(c,d) \leq_2^L (a^f, f(d)_f) = \alpha(f)(a, f(d))$. □

The contravariant α : **cLat** $\to$ **Bos** is defined as follows:

- for any complete lattice L, $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$, the dual Allwein-MacCaull boset of L.
- for any complete lattice morphism $f : L \to M$, $\alpha(f) : (P_M, \leq_1^M, \leq_2^M) \to (P_L, \leq_1^L, \leq_2^L)$ is defined as the map $(a,b) \mapsto (a^f, b_f)$.

Remark 2.2.7. A careful look at the definition of the functor α reveals that it could easily be extended to the category of all lattices and morphisms that have both a left and right adjoint. However, in the absence of the adjoint functor theorem, this condition on morphisms is fairly cumbersome. We therefore limit ourselves to discussing morphisms between complete lattices, for which having a left and a right adjoint is equivalent to being a complete lattice homomorphism.

We are now in a position to provide a representation theorem for all complete lattices. As mentioned in the introduction, this result was already obtained in [2, Thm. 4.2.9]. However, Allwein and MacCaull use some notation introduced by Urquhart [256], which differs quite significantly from ours. A more similar proof to the one we give below can be found in [136], although Holliday works with downsets while we work with upsets. Moreover, none of the works mentioned above presents their result from a categorical viewpoint, while we are also in a position to establish the naturality of the isomorphism between L and the regular opens of its dual boset, a key step in proving the idempotent adjunction we are after.

Lemma 2.2.8. *For any complete lattice L, L is isomorphic to $\rho\alpha(L)$ naturally in L.*

Proof. Let L be a complete lattice with dual boset $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$. We claim that the map $\varphi_L : a \mapsto {\uparrow_1}(a,0)$ is a complete lattice isomorphism natural in L between L and $\mathsf{RO}_{12}(P_L)$.

- φ_L is well defined: let $(c,d) \in P_L$ such that $c \not\leq a$. Then the pair (c,a) is in P_L, which implies that $(c,d) \notin \neg_1\neg_2({\uparrow_1}(a,0))$. Thus $\neg_1\neg_2({\uparrow_1}(a,0)) = {\uparrow_1}(a,0)$.
- φ_L is order-preserving and order-reflecting: suppose $a \leq_L b$. Then $(b,0) \leq_1^L (a,0)$, which implies that ${\uparrow_1}(a,0) \subseteq {\uparrow_1}(b,0)$. Conversely, if $a \not\leq b$, then $(b,0) \notin {\uparrow_1}(a,0)$.
- φ_L is surjective: Suppose $U \subseteq P_L$ such that $\neg_1\neg_2(U) = U$. We claim that $U = \varphi(a)$, where $a = \bigvee\{c \mid \varphi(c) \subseteq U\}$. Suppose that $(c,d) \in U$ for some $c \not\leq d \in U$. Then since U is a 1-upset, ${\uparrow_1}(c,d) = {\uparrow_1}(c,0) \subseteq U$, so $c \leq a$. Since φ_L is order-preserving, this implies that ${\uparrow_1}(c,0) = \varphi_L(c) \subseteq \varphi_L(a)$, and therefore $U \subseteq \varphi_L(a)$. For the converse, let $(c,d) \geq_1^L (a,0)$. We claim that there is $b \in L$ such that $\varphi_L(b) \subseteq U$ and $b \not\leq d$. To

see this, note that, otherwise, d is an upper bound of the set $\{b \in L \mid \varphi_L(b) \subseteq U\}$, which implies that $a \leq d$. But $c \leq a$, and therefore $c \leq d$, a contradiction. Thus $(c,d) \leq_2^L (b,d)$ for some b such that $\varphi_L(b) \subseteq U$, and therefore $(a,0) \in \neg_1\neg_2(U) = U$. This completes the proof that φ_L is a complete lattice isomorphism.

- For naturality in L, suppose we have a complete lattice morphism $f : L \to M$. We want to show that $\varphi_M(f)(a) = \rho\alpha(f)(\varphi_L(a))$ for any $a \in L$. Note that $\varphi_M(f)(a) = \uparrow_1(f(a),0)$. Then we compute:

$$\begin{aligned}\rho\alpha(f)(\varphi_L(a)) &= \rho\alpha(f)(\uparrow_1(a,0)) \\ &= \{(c,d) \in \alpha(M) \mid \alpha(f)(c,d) \in \uparrow_1(a,0)\} \\ &= \{(c,d) \in \alpha(M) \mid (c^f, d_f) \geq_1^L (a,0)\} \\ &= \{(c,d) \in \alpha(M) \mid c^f \leq_1^L a\} \\ &= \{(c,d) \in \alpha(M) \mid c \leq_1^L f(a)\} = \uparrow_1(f(a),0).\end{aligned}$$

This completes the proof. □

This result yields a representation of complete lattices as regular opens of some boset. For our purposes however, it also allows us to establish that all complete lattices are fixpoints of a contravariant adjunction. The existence of this adjunction is the main theorem of this section:

Theorem 2.2.9. *The functors $\alpha : \mathbf{cLat} \to \mathbf{Bos}$ and $\rho : \mathbf{Bos} \to \mathbf{cLat}$ form a contravariant adjunction.*

Proof. Let L be a complete lattice and $\mathscr{X} = (X, \leq_1, \leq_2)$ a boset. We will define a family of bijections between $\mathrm{Hom}_{\mathbf{cLat}}(L, \rho(\mathscr{X}))$ and $\mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X}, \alpha(L))$, natural in both L and $\mathscr{X}$.

For any $f : L \to \rho(\mathscr{X})$ and any $x \in X$, let $x^f = \bigwedge\{a \in L \mid x \in f(a)\}$ and $x_f = \bigvee\{b \in L \mid x \in \neg_2(f(b))\}$. Let $\overline{f} : \mathscr{X} \to \alpha(L)$ be defined as $\overline{f}(x) = (x^f, x_f)$. We claim that $\overline{\cdot} : \mathrm{Hom}_{\mathbf{cLat}}(L, \rho(\mathscr{X})) \to \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X}, \alpha(L))$ is an isomorphism natural in L and $\mathscr{X}$.

- $\overline{f}$ is a b-morphism:
  - Note first that $\overline{f}$ is well defined: since f is a complete lattice morphism, for any $x \in X$, $x \in f(x^f)$ and $x \notin f(x_f)$. Thus $\overline{f}(x) \in P_L$.
  - For monotonicity, notice that $x \leq_1 y$ implies that if $x \in f(a)$, then $y \in f(a)$ for any $a \in L$. Therefore $x^f \geq y^f$, and therefore $\overline{f}(x) \leq_1^L \overline{f}(y)$. Similarly, if $x \leq_2 y$, then $x \in \neg_2(f(b))$ implies $y \in \neg_2(f(b)$, and thus $x_f \leq y_f$. Therefore $\overline{f}(x) \leq_2^L \overline{f}(y)$.
  - Suppose that $(c,d) \geq_2^L \overline{f}(x)$ for some $x \in X$, $c,d \in L$. We claim that there is $y \geq_2 x$ such that $y \in f(c)$. Otherwise, $x \in \neg_2(f(c))$, and therefore $c \leq x_f$. But this implies that $c \leq d$, a contradiction. Thus such a $y \geq_2 x$ exists. But since $y \in f(c)$, it follows that $\overline{f}(y) \geq_1^L (c,d)$.

  - Suppose now that $(c,d) \geq_1^L \overline{f}(x)$. We claim that there is $y \geq_1 x$ such that $y \in \neg_2(f(d))$. Otherwise, $x \in \neg_1\neg_2(f(d)) = f(d)$, and thus $c \leq x^f \leq d$, a contradiction. Now since $y \in \neg_2(f(d))$, we have that $d \leq f_y$, and thus $(c,d) \leq_2^L \overline{f}(y)$.

- $\overline{\cdot}$ is injective: let $f_1, f_2 : L \to \rho(\mathscr{X})$ such that $f_1 \neq f_2$. Without loss of generality, there is some $a \in L$ such that $f_1(a) \nsubseteq f_2(a)$. Let $x \in f_1(a)$ such that $x \notin f_2(a)$. Then there is $y \geq_1 x$ such that $y \in f_1(a)$ and $y \in \neg_2(f_2(a))$. This implies that $y^{f_1} \leq a \leq y_{f_2}$. As $y^{f_1} \nleq y_{f_1}$, this means that $y_{f_1} \neq y_{f_2}$, and therefore $\overline{f_1}(y) \neq \overline{f_2}(y)$.

- $\overline{\cdot}$ is surjective: let $g : \mathscr{X} \to \alpha(L)$ and consider the map $f : L \to \rho(\mathscr{X})$ defined by $f(a) = g^{-1}[\uparrow_1(a,0)]$. We claim that $g = \overline{f}$. Indeed, for any $x \in \mathscr{X}$ such that $g(x) = (c,d)$ and any $a \in L$, we have that $g(x) \in f(a)$ iff $c \leq_1 a$, and $g(x) \in \neg_2(f(a))$ iff $a \leq d$. Thus $\overline{f}(x) = (c,d)$.

Finally, it remains to verify that $\sim$ is natural in both L and $\mathscr{X}$. This means that for any $M \in \mathbf{cLat}$, $\mathscr{Y} \in \mathbf{Bos}$, $g : M \to L$ and $h : \mathscr{Y} \to \mathscr{X}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{cLat}}(L,\rho(\mathscr{X})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X},\alpha(L)) \\ \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{cLat}}(g,\rho(h))} & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Bos}}(\alpha(g),h)} \\ \mathrm{Hom}_{\mathbf{cLat}}(M,\rho(\mathscr{Y})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{Y},\alpha(M)) \end{array}$$

i.e., $\overline{\rho(h) \circ f \circ g} = \alpha(g) \circ \overline{f} \circ h$ for any $f : L \to \rho(\mathscr{X})$. Now let $y \in \mathscr{Y}$ and compute that:

$$\begin{aligned} \alpha(g)(\overline{f})(h)(y) &= \alpha(g)((h(y)^f, (h(y))_f)) \\ &= (\bigwedge\{a \in M \mid g(a) \geq (h(y))^f\}, \bigvee\{a \in M \mid g(a) \leq (h(y))_f\}) \end{aligned}$$

and

$$\begin{aligned} \overline{\rho(h) \circ f \circ g}(y) &= (\bigwedge\{a \in M \mid y \in \rho(h)(f)(g(a))\}, \bigvee\{a \in M \mid y \in \neg_2(\rho(h)(f)(g(a)))\}) \\ &= (\bigwedge\{a \in M \mid y \in h^{-1}[f(g(a))]\}, \bigvee\{a \in M \mid y \in \neg_2 h^{-1}[f(g(a))]\}). \end{aligned}$$

Thus it is enough to show for any $a \in M$ that:

$$g(a) \geq (h(y))^f \Leftrightarrow h(y) \in f(g(a)) \tag{2.1a}$$

$$g(a) \leq (h(y))_f \Leftrightarrow y \in \neg_2 h^{-1}[f(g(a))] \tag{2.1b}$$

Now (2.1a) follows directly from the definition of $(h(y))^f$. For (2.1b) on the other hand, note that $g(a) \leq (h(y))_f$ iff $h(y) \in \neg_2 f(g(a))$ iff $y \in h^{-1}[\neg_2 f(g(a))]$. Since h is a b-morphism, we have that $h^{-1}[\neg_2 f(g(a))] = \neg_2 h^{-1}[f(g(a))]$, which completes the proof. □

For the sake of clarity, we will sometimes refer to this adjunction as a *covariant* adjunction between **cLat** and $\mathbf{Bos}^{op}$. If we think of α and ρ as covariant functors, it then follows from the previous theorem that α is left-adjoint to ρ. It therefore makes sense to talk about the unit and counit of this adjunction as natural transformations $\eta : Id_{\mathbf{cLat}} \to \rho\alpha$ and $\epsilon : Id_{\mathbf{bF}} \to \alpha\rho$.

Remark 2.2.10. Closer inspection of the proof of Theorem 2.2.9 shows that the counit of the adjunction is given by the map $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$, defined as $\epsilon_{\mathscr{X}}(x) = (U^x, V_x)$, where for any $x \in \mathscr{X}$, $U^x = \neg_1\neg_2(\uparrow_1 x)$ and $V_x = \{z \mid \neg\exists y : y \geq_2 x \wedge y \geq_1 z\}$. To see that V_x is regular open, note first that it is clearly a 1-upset. Now suppose that $y \notin V_x$. This means that there is $z \in \mathscr{X}$ such that $z \geq_2 x$ and $z \geq_1 y$. But then for any $w \geq_2 z$, $w \geq_2 x$, and therefore $y \notin \neg_1\neg_2(V_x)$. Hence $\neg_1\neg_2(V_x) \subseteq V_x$, which implies that V_x is regular open.

The Allwein-MacCaull representation theorem (Lemma 2.2.8), when coupled with Theorem 2.2.9, implies the corollary mentioned above.

Corollary 2.2.11. *The functors α and ρ form an idempotent contravariant adjunction.*

Indeed, to establish that the adjunction is idempotent, it is enough to show that the unit of the adjunction is a natural isomorphism. Now for any $\mathscr{X} \in \mathbf{Bos}$, $L \in \mathbf{cLat}$, $g \in \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X}, \alpha(L))$, and $a \in L$, $\overline{g}^{-1}(a) = g^{-1}[\uparrow_1(a, 0)] = \rho(g)(\varphi_L(a))$. Thus φ_L is the unit of the adjunction between α and ρ. Moreover, since by Lemma 2.2.6 φ_L is an isomorphism natural in L, $\alpha(\varphi_L)$ is an isomorphism natural in $\alpha(L)$.

It is a general categorical fact that the fixpoints of an idempotent adjunction induce an equivalence of categories. Therefore the following definition is natural.

Definition 2.2.12. A *b-frame* is a boset $\mathscr{X}$ such that $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is an isomorphism. Let **bF** be the full subcategory of **Bos** of all b-frames.

As an immediate consequence of Corollary 2.2.11, we obtain the following:

Theorem 2.2.13. *The categories* **cLat** *and* **bF** *are dually equivalent.*

This result, however, only amounts to an abstract characterization of the duals of complete lattices. A more useful characterization would identify precisely which properties of a boset $\mathscr{X}$ guarantee that $\epsilon_{\mathscr{X}}$ is an isomorphism. In the next part of this section, a special class of b-morphisms, which generalize the notion of a dense embedding in the forcing literature, is introduced. We then show that for any boset $\mathscr{X}$, $\epsilon_{\mathscr{X}}$ is such a dense embedding. Finally, in the last part, we will show that imposing some natural conditions on bosets allows us to strengthen this dense embedding to an isomorphism, thus obtaining a more concrete characterization of b-frames.

2.2.3 Dense Embeddings

We begin by introducing the following notation which will be used extensively:

Definition 2.2.14. Let $\mathscr{X} := (X, \leq_1, \leq_2)$ be a boset. For any $x, y \in X$ and $k, j \in \mathscr{P}(\{1,2\}) - \{\emptyset\}$, we introduce the following notation:

$$x\ {}_j\!\perp_k y \text{ iff } \neg\exists z : y \leq_s z \text{ for all } s \in j \text{ and } x \leq_t z \text{ for all } t \in k.$$

In particular, we say that x is *independent from* y whenever $x\,{}_2\!\perp_1 y$.

It is straightforward to note that for any poset $\mathbb{P} = (P, \leq)$, if we view $\mathbb{P}$ as a Kripke boset $(P, \leq, \geq)$, we have that $x_2\bot_1 y$ iff $y \not\leq x$, while if we view $\mathbb{P}$ as a forcing boset $(P, \geq, \geq)$, we have that $x_2\bot_1 y$ iff $x \bot y$, where $\bot$ is the standard incompatibility relation in the forcing literature. More generally, following the notation introduced in Remark 2.2.10, we have in any boset $\mathscr{X}$ that $x_2\bot_1 y$ iff $x \notin U^y$ iff $y \in V_x$.

In Allwein-MacCaull bosets, i.e., bosets of the form $\alpha(L)$ for some complete lattice L, independence can be seen as a purely graph-theoretic way of capturing the order on L:

Lemma 2.2.15. *Let $(X, \leq_1, \leq_2)$ be $\alpha(L)$ for some complete lattice L. Then for any $x = (f_x, i_x)$ and any $y = (f_y, i_y)$, we have that:*

1. *$x_2\bot_1 y$ iff $f_y \leq i_x$;*
2. *$x\,_{12}\bot_2 y$ iff $f_x \leq i_x \vee i_y$;*
3. *$x\,_{12}\bot_1 y$ iff $f_x \wedge f_y \leq i_y$.*

Proof. All three items follow immediately from the fact that for any $a, b \in L$, $a \not\leq b$ iff the pair $(a, b) \in \alpha(L)$. □

Let us now focus on a specific class of b-morphisms, which generalize in a natural way the notion of a dense embedding between forcing posets.[2]

Definition 2.2.16. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ and $\mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be two bosets and $f : \mathscr{X} \to \mathscr{Y}$ a b-morphism. Then:

- f is *dense* if for any $y \in \mathscr{Y}$ there is $x \in \mathscr{X}$ such that $y \leq_{12}^Y f(x)$.
- f is an *embedding* if for any $x, y \in \mathscr{X}$, we have that $x_1\bot_2 y$ iff $f(x)_1\bot_2 f(y)$.

The next two lemmas show that dense b-morphisms and embeddings are dual to injective and surjective lattice morphisms respectively.

Lemma 2.2.17. *Let $f : \mathscr{X} = (X, \leq_1^X, \leq_2^X) \to \mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be a b-morphism. Then:*

1. *f is dense iff $\rho(f)$ is injective;*
2. *f is an embedding iff $\rho(f)$ is surjective.*

Proof.

1. Suppose f is dense, and let $U, V \in \rho(\mathscr{Y})$ such that $U \neq V$. Without loss of generality, there is $y \in U \cap \neg_2 V$, and since f is dense, there must be $x \in \mathscr{X}$ such that $f(x) \geq_{12}^Y y$. But then $f(x) \in U \cap \neg_2 V$, which means that $x \in \rho(f)(U) - \rho(f)(V)$. Hence $\rho(f)$ is injective.
Conversely, suppose there is $y \in \mathscr{Y}$ such that for all $x \in X$, $f(x) \not\geq_{12}^Y y$. Let $U =$

[2]For an overview of the basic notions and techniques in forcing, see for example [161].

$\neg_1\neg_2(\uparrow_1 y)$ and $V = \{z \mid y_2\bot_1 z\}$. Clearly, $U \nsubseteq V$, but we claim that $f^{-1}[U] \subseteq f^{-1}[V]$. Note that this implies that $U \cap V \neq U$ but $f^{-1}[U] = f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$ and thus that $\rho(f)$ is not injective. For the proof of the claim, suppose towards a contradiction that there is $x \in f^{-1}[U] - f^{-1}[V]$. Since both $f^{-1}[U]$ and $f^{-1}[V]$ are regular open, without loss of generality we may assume that $x \in \neg_2 f^{-1}[V]$. Moreover, note that, since $x \in f^{-1}[U]$, then there is $q \geq_1^Y y$ such that $f(x) \leq_2^Y q$. But this means that there is $z \geq_2^X x$ such that $f(z) \geq_1^Y q \geq_1^Y y$. Now since $x \in \neg_2 f^{-1}[V]$, this means that $f(z) \notin V$, and hence there is $q' \geq_1^Y f(z)$ such that $q' \geq_2^Y y$. Hence there is $z' \geq_1^X z$ such that $f(z') \geq_2^Y q' \geq_2^Y y$. But since also $f(z') \geq_1^Y f(z) \geq_1^Y y$, we have that $f(z') \geq_{12}^Y y$, contradicting our assumption. This completes the proof.

2. Suppose f is an embedding, and let $U \in \rho(\mathscr{X})$. We claim that $f^{-1}f[U] = U$. To see this, assume that $x \in f^{-1}f[U]$. Then $f(x) = f(y)$ for some $y \in U$. Now for any $z \geq_1 x$, this implies that $f(z) \geq_1 f(y)$, and thus $\neg f(y)_1\bot_2 f(z)$. Hence $\neg y_1\bot_2 z$, which implies that $z \in C_2(U)$, and hence $x \in \neg_1\neg_2(U) = U$. Thus $f^{-1}f[U] \subseteq U$, and the converse direction is obvious. Now let $V = \neg_1\neg_2 f[U]$, and note that we have that

$$\rho(f)(V) = f^{-1}(\neg_1\neg_2 f[U]) = \neg_1\neg_2(f^{-1}f[U]) = \neg_1\neg_2(U) = U.$$

Thus $\rho(f)$ is surjective.
Conversely, assume there are $x, y \in \mathscr{X}$ such that $x_1\bot_2 y$ but there is $z \in \mathscr{Y}$ such that $f(x) \leq_1 z$ and $f(y) \leq_2 z$. Note that this implies that there is $y' \geq_2 y$ such that $z \leq_1 f(y')$. We claim that for any $U \in \rho(\mathscr{Y})$, if $f(x) \in U$, then $f(y') \in U$. Since $y' \notin \neg_1\neg_2(\uparrow_1 x)$, this will imply that $\rho(f)$ is not surjective. For the proof of the claim, it is enough to notice that $f(y') \geq_1 z \geq_1 f(x)$, since any $U \in \rho(\mathscr{Y})$ is a 1-upset. This completes the proof. □

Lemma 2.2.18. *Let $f : L \to M$ be a lattice homomorphism. Then:*

1. *f is injective iff $\alpha(f)$ is dense;*

2. *f is surjective iff $\alpha(f)$ is an embedding.*

Proof.

1. Note that by lemma 1.11, we have that f is injective iff $\rho\alpha(f)$ is injective. But by the previous lemma, we have that $\rho\alpha(f)$ is injective iff $\alpha(f)$ is dense.

2. Similarly, we have that f is surjective iff $\rho\alpha(f)$ is surjective, which by the previous lemma is equivalent to $\alpha(f)$ being an embedding. □

Dense embeddings will be of crucial relevance later on, as we will use extensively the fact that a dense embedding between two bosets induces an isomorphism of the dual complete lattices. Once again, this can be seen as a generalization of the well-known result that two posets are forcing equivalent iff there is a dense embedding between them. In particular, if $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ is a boset, then a *dense sub-boset* of $\mathscr{X}$ is a boset $\mathscr{Y} = (Y, \leq_1^X|Y, \leq_2^X|Y)$, where $Y \subseteq X$ and for any $x \in X$ there is $y \in Y$ such that $x \leq_{12} y$. The proof of the following lemma is immediate when one realizes that if $\mathscr{Y}$ is a dense sub-boset of $\mathscr{X}$, then the inclusion map $\iota : \mathscr{Y} \to \mathscr{X}$ is a dense embedding.

Lemma 2.2.19. *Let $\mathscr{Y}$ be a dense sub-boset of $\mathscr{X}$. Then $\rho(\mathscr{X})$ is isomorphic to $\rho(\mathscr{Y})$.*

Moreover, as shown in Lemma 2.2.6, the unit η_L of the adjunction $\alpha \dashv \rho$ is an isomorphism for any complete lattice L. A similar result holds for the counit $\epsilon_{\mathscr{X}}$.

Lemma 2.2.20. *For any boset $\mathscr{X}$, the map $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding.*

Proof. Suppose we have that $x_2 \perp_1 y$. Then $y \in V_x$, which implies that $U^y \subseteq V_x$. Hence $(U^x, V_x)_2 \perp_1 (U^y, V_y)$, which means that $\epsilon_{\mathscr{X}}$ is an embedding. For density, assume $U, V \in \rho(\mathscr{X})$ are such that $U \nsubseteq V$. Then since both U and V are regular open there is $y \in \mathscr{X}$ such that $y \in U \cap \neg_2 V$. But this implies that $U^y \subseteq U$ and that $V \subseteq V_y$, and hence $(U, V) \leq_{12} (U^y, V_y)$. □

However, it is easy to verify that dense embeddings are not isomorphisms in the category of bosets: since b-morphisms are maps sending points to points, any b-morphism with an inverse must be bijective. In order to characterize b-frames, we must therefore impose some extra conditions on a boset $\mathscr{X}$ that guarantee that the dense embedding $\epsilon_{\mathscr{X}}$ is an isomorphism.

2.2.4 Characterizing B-frames

The following definition generalizes the notion of a separative poset in forcing:

Definition 2.2.21. A boset $\mathscr{X} = (X, \leq_1, \leq_2)$ is *separative* if it satisfies the following three properties:

- $\leq_1 \cap \leq_2$ is anti-symmetric;
- for any $x, y \in \mathscr{X}$, $x \leq_1 y \Leftrightarrow \forall z(z_2 \perp_1 x \to z_2 \perp_1 y)$ (1-*separativity*);
- for any $x, y \in \mathscr{X}$, $x \leq_2 y \Leftrightarrow \forall z(x_2 \perp_1 z \to y_2 \perp_1 z)$ (2-*separativity*).

In particular, it is straightforward to verify that any poset $(X, \leq)$ is separative iff the corresponding forcing boset $(X, \geq, \geq)$ is separative.

In order to characterize b-frames, we will also need a second property.

Definition 2.2.22. A boset $\mathscr{X} = (X, \leq_1, \leq_2)$ is *complete* if for any $U, V \in \rho(\mathscr{X})$ such that $U \nsubseteq V$, there is $z \in \mathscr{X}$ such that $U = U^z$ and $V = V_z$.

Unlike separativity, this property requires (monadic) second-order quantification to be expressed. We will show later on (Lemma 2.3.24) that this requirement is necessary, i.e., that there is no possible first-order axiomatization of b-frames.

We can now establish that separativity and completeness entirely characterize b-frames. Let us start by observing that the regular open sets of a complete separative boset $\mathscr{X}$ have a very concrete characterization: they are precisely the principal 1-upsets of $\mathscr{X}$.

Lemma 2.2.23. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a complete separative boset. Then for any non-empty $U \subseteq X$, $U \in \rho(\mathscr{X})$ iff $U = \uparrow_1 x$ for some $x \in X$.*

Proof. We first claim that for any $x \in X$, $\neg_1\neg_2(\uparrow_1 x) = \uparrow_1 x$. To see this, note that it suffices to show the left-to-right direction since $\uparrow_1 x$ is 1-upward closed. By separativity, if $x \nleq_1 y$ for some $y \in X$, then there is $z \in X$ such that $z_2\bot_1 x$ but $\neg z_2\bot_1 y$. Let $z' \geq_1 y$ such that $z' \geq_2 z$. Clearly, $z'_2\bot_1 x$, for otherwise we would have $\neg z_2\bot_1 x$. Hence $z' \in \neg_2(\uparrow_1 x)$, which implies that $y \notin \neg_1\neg_2(\uparrow_1 x)$, which concludes the proof of the claim. Hence for any $x \in X$, $\uparrow_1 x \in \rho(\mathscr{X})$. Now let U be a non-empty subset in $\rho(\mathscr{X})$. Then as $U \nsubseteq \emptyset$, there is some $x \in X$ such that $U = U_x = \neg_1\neg_2(\uparrow_1 x) = \uparrow_1 x$. Thus any non-empty $U \in \rho(\mathscr{X})$ is $\uparrow_1 x$ for some $x \in X$. □

The next two lemmas establish the characterization of b-frames mentioned above.

Lemma 2.2.24. *Every b-frame is separative and complete.*

Proof. It is enough to show that $\alpha(L)$ is separative and complete for any complete lattice L. Note first that it is clear from the definition of $\alpha(L)$ that $\leq_1 \cap \leq_2$ is antisymmetric. For 1-separativity, suppose $(a, b) \nleq_1 (c, d)$ for some $a, b, c, d \in L$. This means that $c \nleq a$, and thus $(c, a) \in \alpha(L)$. But clearly $(c, a)_2\bot_1(a, b)$ yet $\neg(c, a)_2\bot_1(c, d)$. The converse direction is trivial. For 2-separativity, suppose $(a, b) \nleq_2 (c, d)$. Then $b \nleq d$, which means that $(b, d) \in \alpha(L)$. But then $(a, b)_2\bot_1(b, d)$, yet $\neg(c, d)_2\bot_1(b, d)$. Hence $\alpha(L)$ is separative. For completeness, recall first that $\eta_L : L \to \rho\alpha(L)$ is an isomorphism. For any $U \nsubseteq V \in \rho\alpha(L)$, let $a = \eta_L^{-1}(U)$ and $b = \eta_L^{-1}(V)$ be elements of L, and note that we have that $(a, b) \in \alpha(L)$. Since $U = \eta_L(a) = \uparrow_1(a, 0) = \uparrow_1(a, b)$, we have that $U = U^{(a,b)}$. Moreover, for any $(c, d) \in \alpha(L)$, we have that $(a, b)_2\bot_1(c, d)$ iff $c \leq b$ iff $(b, 0) \leq_1 (c, d)$ iff $(c, d) \in \eta_L(b) = V$. Thus $V = V_{(a,b)}$, which completes the proof. □

Coupled with Lemma 2.2.20, this lemma generalizes to bosets the standard result that any poset is forcing equivalent to a separative poset.

Lemma 2.2.25. *Every complete separative boset is a b-frame.*

Proof. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a complete separative boset. We have to show that the map $\epsilon_{\mathscr{X}} = \mathscr{X} \to \alpha\rho(\mathscr{X})$ is an isomorphism, i.e., that it is bijective and reflects both preorders.

- Note first that since $\leq_1 \cap \leq_2$ is antisymmetric, to prove injectivity it is enough to show that both preorders are reflected by $\epsilon_{\mathscr{X}}$. Let $x, y \in \mathscr{X}$, and assume $x \nleq_1 y$. Then $\uparrow_1 y \nsubseteq \uparrow_1 x$, which since $\mathscr{X}$ is separative implies that $U^y \nsubseteq U^x$ and hence that
$$\epsilon_{\mathscr{X}}(x) = (U^x, V_x) \nleq_1 (U^y, V_y) = \epsilon_{\mathscr{X}}(y).$$
Similarly, if $x \nleq_2 y$, by separativity there is $z \in \mathscr{X}$ such that $x_2\bot_1 z$ but $\neg y_2\bot_1 z$. But this implies that $z \in V_x$ yet $z \notin V_y$. Hence $\epsilon_{\mathscr{X}}(x) \nleq_2 \epsilon_{\mathscr{X}}(y)$.
- Finally, surjectivity is an immediate consequence of $\mathscr{X}$ being complete, since points in $\alpha\rho(\mathscr{X})$ are precisely pairs (U, V) of elements of $\rho(\mathscr{X})$ such that $U \nsubseteq V$. □

Putting the previous two lemmas together, we obtain the last result of this section.

Theorem 2.2.26. *A boset is a b-frame iff it is separative and complete.*

2.2.5 B-frame Duality and Lattice Representations

Let us conclude this section by comparing the results obtained above with known results in the literature. There exist, of course, many adjunctions and dualities between categories of lattices and concrete categories, which have various advantages and drawbacks. Representations of lattices as upsets of certain posets typically involve adding some further structure, either in the form of a topology as in Priestley and Esakia duality, or in the form of a second relation. Since our b-frame duality is of the latter kind, we first discuss how it relates to some classical discrete representations in the literature, before comparing it to some well-known dualities of the former kind.

Discrete Representations

As mentioned in the introduction, the duality exposed here can be seen as a generalization both of the duality between posets and completely join-prime generated or superalgebraic locales, which itself generalizes Tarski duality between sets and complete atomic Boolean algebras, and of the representation of complete Boolean algebras as regular open sets of separative posets which lies at the heart of some classical results in forcing. Although these representations are well known, they are not always presented from a categorical perspective. We therefore briefly present them below in some detail and somewhat more systematically than what is commonly found in the literature, as this will illuminate the sense in which the b-frame duality presented here generalizes those results.

Definition 2.2.27. Let **cBA** and **CABA** be the full subcategories of **cLat** whose objects are complete Boolean algebras and complete and atomic Boolean algebras respectively. Let **sLat** be the subcategory of **cLat** whose objects are superalgebraic complete lattices (i.e., completely join-prime generated complete lattices) and whose morphisms are complete lattice homomorphisms.

Thus we obtain the following diagram of inclusions of categories:

$$\begin{array}{ccc} & \mathbf{CABA} & \\ \swarrow & & \searrow \\ \mathbf{sLat} & & \mathbf{cBA} \\ \searrow & & \swarrow \\ & \mathbf{cLat} & \end{array}$$

On the geometric side of these dualities, we have the category of sets and two categories of posets:

Definition 2.2.28. Let **Set** be the category of all sets and functions between them, $\mathbf{Pos_1}$ the category of posets and monotone maps between them and $\mathbf{Pos_2}$ the category of posets and weakly-dense maps between them.

As mentioned in Example 2.2.3 above, a poset $(P, \leq_P)$ can be viewed as the boset $(P, \leq_P, \geq_P)$, or as the boset $(P, \geq_P, \geq_P)$. It is straightforward to verify that both constructions lift to two full embedding functors $\kappa : \mathbf{Pos_1} \to \mathbf{Bos}$ and $\delta : \mathbf{Pos_2} \to \mathbf{Bos}$, from which we obtain the following commuting diagram:

$$\begin{array}{ccccc} & & \mathbf{Set} & & \\ & \swarrow & & \searrow & \\ \mathbf{Pos_1} & & & & \mathbf{Pos_2} \\ & \kappa\searrow & & \swarrow\delta & \\ & & \mathbf{Bos} & & \end{array}$$

We now briefly recall the various correspondences that our result aims to generalize:

Theorem 2.2.29 (Tarski)**.** **CABA** *and* **Set** *are dual to one another:*

- *The functor* $At : \mathbf{CABA} \to \mathbf{Set}$ *maps any complete atomic Boolean algebra to the set of its atoms and any complete Boolean homomorphism* $h : B \to C$ *to the restriction of its left adjoint* $h^* : C \to B$ *to the atoms of* B *and* C*.*
- *The functor* $\mathscr{P} : \mathbf{Set} \to \mathbf{CABA}$ *maps any set* S *to its powerset* $\mathscr{P}(S)$ *and any function* $f : S \to T$ *to the inverse image map* $f^{-1} : \mathscr{P}(T) \to \mathscr{P}(S)$*.*

An early reference for the following result is [216]:

Theorem 2.2.30 (Raney)**.** **sLat** *and* $\mathbf{Pos_1}$ *are dual to one another:*

- *The functor* $\gamma : \mathbf{sLat} \to \mathbf{Pos_1}$ *maps any superalgebraic locale* L *to the poset of its completely join-prime elements with the reverse order on* L *and any complete lattice homomorphism* $h : L \to M$ *maps to the restriction of its left adjoint* $h^* : M \to L$ *to the completely join-prime elements of* M *and* L*.*
- *The functor* $\tau : \mathbf{Pos_1} \to \mathbf{sLat}$ *maps any poset* $(P, \leq_P)$ *to its complete lattice of upsets* $Up(P)$ *and any monotone map* $f : (P, \leq_P) \to (Q, \leq_Q)$ *to the inverse image map* $f^{-1} : Up(Q) \to Up(P)$*.*
- *This* γ*-*τ *duality restricts precisely to Tarski duality between* **CABA** *and* **Set***.*

Let us also note that De Jongh and Troelstra [68] observed that the Raney dual of a complete lattice homomorphism h is a p-morphism if and only if h is also a complete Heyting morphism, meaning that it also preserves the right-adjoint of the meet operation, which exists in any superalgebraic lattice. This yields a restriction of Raney duality to De Jongh-Troesltra duality between the category of superalgebraic locales and Heyting morphisms between them, and the category of posets and p-morphisms, which can also be shown to be generalized by our b-frame duality.

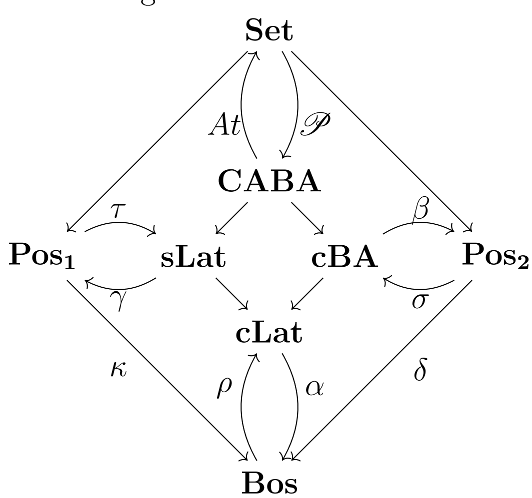
Finally, the following definition is needed in order to express the last one of our theorems:

Definition 2.2.31. A poset $(P, \leq_P)$ is *separative* if for any $x, y \in P$ such that $x \nleq_P y$, there is $z \in P$ such that $z \leq_P x$ and for all $w \in P$, if $w \leq_P z$ then $w \nleq_P y$. A poset $(P, \leq_P)$ is *complete* if for every non-empty regular open subset U of $(P, \leq_P)$ there is $p \in P$ such that $U = {\downarrow}p$.

Theorem 2.2.32 ([144, Chap. 14])**.** *There is an idempotent contravariant adjunction between* **cBA** *and* $\mathbf{Pos_2}$*:*

- *The functor* $\beta : \mathbf{cBA} \to \mathbf{Pos_2}$ *maps any complete Boolean algebra* B *to the poset* $(B_+, \leq_B |B_+)$*, where* $B_+ = B \setminus 0$ *and any complete Boolean homomorphism* $h : B \to C$ *to the restriction of its left-adjoint* $h^* : C_+ \to B_+$*.*
- *The functor* $\sigma : \mathbf{Pos_2} \to \mathbf{cBA}$ *maps any poset to its Boolean algebra of regular open downsets* $RO(P)$ *and any weakly dense map* $f : (P, \leq_P) \to (Q, \leq_Q)$ *to the inverse image map* $f^{-1} : RO(Q) \to RO(P)$*.*
- *The functors* σ *and* β *restrict to a duality between the full subcategories of fixpoints of* $\sigma\beta$ *and* $\beta\sigma$*, i.e., between* **cBA** *and the full subcategory of* $\mathbf{Pos_2}$ *of complete separative posets.*
- *If* B *is a complete atomic Boolean algebra, then* $At(B)$ *with the discrete order is a dense subposet of* $\beta(B)$*. Conversely, if* S *is a set, then* $\sigma(S, \Delta_S)$ *is isomorphic to* $\mathscr{P}(S)$*.*

Combining these results with our adjunction between complete lattices and bosets, we obtain the following diagram of categories:



Note that not all squares in the diagram above commute, not even up to isomorphisms. For example, if B is a complete and atomic Boolean algebra, then $At(B)$ is a discrete poset, while the order on $\beta(B)$ is the restriction of the order on B. Similarly, for a superalgebraic lattice L, the second order on $\kappa(L)$ is the converse of the first one, while this is not the case for $\alpha(L)$. Nonetheless, we have the following result, which gives a precise meaning to the claim that our α-ρ adjunction generalizes both the γ-τ duality and the β-σ adjunction (the obvious inclusion functors have been omitted):

Theorem 2.2.33.

1. *There is a natural isomorphism between the functors $\rho\kappa$ and τ and between the functors $\rho\delta$ and σ.*

2. *There are natural transformations $\eta^1 : \kappa\gamma \to \alpha$ and $\eta^2 : \delta\beta \to \alpha$, such that each component is a dense embedding.*

Proof.

1. Let $\mathbb{P} = (P, \leq_P)$ be a poset. Then $\kappa(\mathbb{P}) = (P, \leq_P, \Delta_P)$. Since the second ordering on P is discrete, the regular opens of $\kappa(\mathbb{P})$ are clearly the upsets of $(\mathbb{P})$, hence $\rho\kappa(\mathbb{P})$ is isomorphic to $\tau(\mathbb{P})$. The naturality condition is straightforward. Similarly, if $\mathbb{Q} = (Q, \leq_Q)$ is a poset, then $\delta(\mathbb{Q}) = (Q, \geq_Q, \geq_Q)$. Clearly, the regular opens of $\delta(\mathbb{Q})$ are precisely the regular open downsets of $\mathbb{Q}$, hence $\rho\delta(\mathbb{Q})$ is isomorphic to $\sigma(\mathbb{Q})$. Again, the naturality condition is straightforward.

2. Let L be superalgebraic, with $\leq$ the order on L. Then $\gamma(L) = (J(L), \geq |J(L))$, where $J(L)$ is the set of completely join-prime elements of L and $\kappa\gamma(L) = (J(L), \geq |J(L), \Delta_{J(L)})$. Now given a completely-join prime element $p \in J(L)$, let $\eta^1_L(p) = (p, p^\delta)$, where $p^\delta = \bigvee\{c \in L \mid p \nleq c\}$. It is straightforward to check that $\eta^1_L : \kappa\gamma(L) \to \alpha(L)$ is well-defined and is a b-morphism. Moreover, for any $(a, b) \in \alpha(L)$, we have that $a \nleq b$ and hence, since L is superalgebraic, there is $p \in J(L)$ such that $p \leq a$ but $p \nleq b$. But this at once implies that $(a, b) \leq_{12} \eta^1_L(p)$, which establishes that η^L_1 is dense. Finally, to see that it is an embedding, note that for any $p, q \in J(L)$, $p_2\bot_1 q$ iff $p \nleq q$ iff $q \leq p^\delta$, from which it follows that $(p, p^\delta)_2\bot_1(q, q^\delta)$. Hence each component of $\eta^1 : \kappa\gamma \to \alpha$ is a dense embedding. The naturality condition on η^1 is left to the reader.
   Similarly, let B be a complete Boolean algebra, with $\leq$ the order on B. Then $\beta(B) = (B_+, \leq |B_+)$ and $\delta\beta(B) = (B_+, \geq |B_+, \geq |B_+)$. Given $b \in B_+$, let $\eta^2_B(b) = (b, \neg b)$. Once again it is straightfoward to check that $\eta^2_B : \delta\beta(B) \to \alpha(B)$ is well-defined and a b-morphism. Moreover, for any $(a, b) \in \alpha(B)$, we have that $a \nleq b$, hence $a \wedge \neg b \in B_+$. But clearly $(a, b) \leq_{12} \eta^2_B(a \wedge \neg b)$, which shows that η^2_B is dense. Finally, to check that it is also an embedding, note that, for any $a, b \in B_+$, $a_2\bot_1 b$ iff $a \wedge b \leq 0$ iff $b \leq \neg a$ iff $\eta^2_B(a)_2\bot_1\eta^2_B(b)$. Hence each component of $\eta^2 : \delta\beta \to \alpha$ is a dense embedding. Once again, the naturality condition is left to the reader.

□

Finally, let us conclude by mentioning once again that the representation of any complete lattice as the regular opens of some boset was already proved in [2]. However, Allwein and MacCaull do not offer a treatment of morphisms, nor do they identify the duals of complete lattices. By contrast, the notion of a dense embedding, which is a generalization of a standard tool in forcing, plays a central role in our characterization of b-frames and will also prove itself very useful in establishing correspondences between lattice equations and first-order properties of bosets.

Topological Dualities

The discrete dualities presented above only offer representations for complete lattices. As is well known, extending such dualities to categories of (possibly incomplete) lattices typically requires one to topologize the dual geometric structures. The celebrated examples are, of course, Stone's duality between Boolean algebras and Stone spaces [245], Priestley's duality between bounded distributive lattices and Priestley spaces [210] and Esakia's duality between Heyting algebras and Esakia spaces [81].

For bounded lattices, several dualities have been developed. Urquhart [256] developed a topological representation for bounded lattices in which the points in the dual space of a bounded lattice are pairs of a filter and an ideal which are maximal with respect to one another. This representation, which appeals to Zorn's Lemma in an essential way, was later lifted to a duality by Hartung [127] and generalizes Stone and Priestley's dualities, in the sense that the restriction to distributive lattices and Boolean algebras yields Priestley and Stone spaces. However, the morphisms covered by the Urquhart-Hartung duality are only the surjective lattice homomorphisms, and the duality is often seen as more cumbersome to work with than Priestley or Stone's. As a consequence, a number of alternative dualities for bounded lattices have been proposed over the years. Gehrke and van Gool [105] have recently developed a duality closely related to Urquhart-Hartung duality, in which however the morphisms between lattices considered are not the usual lattice morphisms. Dualities based on spaces of filters rather than maximal filters have also been offered by Hartonas [122] and Jipsen and Moshier [197]. These dualities do not immediately generalize Stone duality for Boolean algebras or Priestley duality for distributive lattices (even though Jipsen and Moshier's approach is closely related to Stone's duality for distributive lattices via spectral spaces [246]), see also Section 3.2.4 below, but still involve defining only one topology on the space of proper filters of the dual lattice.

Finally, the existing dualities closest to b-frame duality are filter-ideal based dualities, such as the duality between bounded lattices and enhanced L-spaces presented by Allwein and Hartonas in [1] and the duality with L-frames introduced by Hartonas and Dunn in [125] and subsequently developed more recently by Hartonas in [124, 126]. The Allwein-Hartonas representation of a bounded lattice L is obtained by considering pairs of a filter and an ideal on L that do not intersect and using the inclusion orderings on filters and ideals to define order-closed sets that generate a topology. Morphisms are defined as continuous functions that preserve both orderings and satisfy a condition similar to that imposed on b-morphisms. The Hartonas-Dunn duality, by contrast, is inspired from the theory of polarities and has close ties with the theory of generalized Kripke frames of Gehrke [101]. A bounded lattice L is mapped to the triple $(X, \perp, Y)$, where X and Y are the posets of filters and ideals of L endowed with a Stone-like topology, and $\perp$ is a relation on $X \times Y$. Such triples are called L-frames, and a lattice L can then be recovered as the clopen sets of X that are also fixpoints of the Galois connection between $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ induced by $\perp$. Morphisms between two L-frames $(X, \perp, Y)$ and $(X', \perp', Y')$ are pairs of continuous maps between X and X' and between Y and Y' that commute with the closure operators generated by $\perp$ and $\perp'$. Since points in our b-frame representation of a complete lattice L are pairs of elements of the original lattice, and we work with two orderings, it is natural to see this latest duality as giving rise to a "topologized" version of our b-frame duality, just like Stone duality topologizes Tarski duality, or the more recent duality between Boolean algebras and UV-spaces presented in [41] topologizes the duality between complete Boolean algebras and complete separative posets. Instead of being a triple composed of two spaces and a relation between them, the duals of lattices in such a topologized version of b-frame duality would rather be single bitopological spaces of filter-ideal pairs, and fixpoints of the Galois connection induced by the relation $\perp$ would be replaced regular open sets induced by the two topologies. The details of such a duality and of its exact relationship to the Hartonas-Dunn one are left for future work, although we will discuss a similar issue in Chapter 4. For a systematic comparison of the representation of complete lattices via polarities and bi-ordered sets, we refer the reader to [136].

2.3 Correspondence Theory

In the previous section, we established an idempotent adjunction between complete lattices and bosets and showed how to restrict it to a duality between complete lattices and b-frames. In this section, we will see how this duality restricts further to specific classes of complete lattices. The goal is to identify properties of b-frames which correspond to properties of complete lattices, in the precise sense that a b-frame $\mathscr{X}$ has a property P if and only if $\rho(\mathscr{X})$ is in a certain class K of complete lattices. As it will become apparent later on, once we find such a characterizing condition on b-frames, we can always extend our result to a correspondence between *bosets* and complete lattices. In this section, we restrict ourselves to equationally definable classes and focus on characterizing the duals of complete distributive lattices, Heyting algebras and Boolean algebras. Our approach for Heyting algebras is also straightforwardly adapted in Section 2.5 to provide characterizations of complete co-Heyting and bi-Heyting algebras. As mentioned in the previous section, there is a well-established duality theory for such structures, originating with Stone duality for Boolean algebras [245]. The Stone duals of complete Boolean algebras are *extremally disconnected* Stone spaces, in which the closure of every open set is open. Building on this characterization, Priestley [210] identifies the Priestley duals of complete distributive lattices as those Priestley spaces in which the smallest closed upset containing S is open for every open upset S. An equivalent characterization in terms regular opens being clopens also exists in the bitopological duality for distributive lattices of [39, Thm. 6.25]. Finally, the topological representations of MacNeille completions of Heyting, co-Heyting and bi-Heyting algebras via Esakia duality obtained in [119] also yield characterizations of the Esakia duals of complete Heyting, co-Heyting and bi-Heyting algebras.

In the context of the study of semantics for non-classical logics based on complete lattices, we see two advantages of the discrete approach we develop here over the standard topological approach. First, discrete, graph-theoretic semantics allow for simple geometric arguments that are sometimes harder to adapt in a topological setting. Of course, there is always the option to “discretize” a topological representation. For example, one can forget about the topology on the dual Priestley space of a distributive lattice L and focus instead on the lattice of upsets of the resulting poset. But the obvious drawback is that this lattice will not be isomorphic to L, but only to its canonical extension,[3] which is always a superalgebraic locale. Furthermore, all characterizations of particular classes of complete lattices mentioned above require imposing second-order conditions on the dual topological spaces. By contrast, the dual b-frames of the kind of complete lattices considered in this section and the next two can be straightforwardly given first-order, geometrically intuitive characterizations in the language of bosets, even though the corresponding characterization for bosets must be second-order. To sum up, there is a necessary trade-off between generality and concreteness when giving representations of lattices, and we believe that the discrete representation of complete lattices developed here is a suitable equilibrium point for our purposes.

2.3.1 Distributive Lattices

We start by characterizing the duals of distributive lattices. It is well known that the variety of distributive lattices, unlike the varieties of Heyting and Boolean algebras, is not closed under MacNeille completions. A similar phenomenon manifests itself here: the characterization of the dual b-frames of complete distributive lattices is more intricate and uses the duality in an essential way.

We start by identifying a property of b-frames that are the duals of distributive lattices.

Lemma 2.3.1. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a b-frame such that $\rho(\mathscr{X})$ is distributive. Then $\mathscr{X}$ satisfies the following property:*

$$\forall x, y, z((x \,{}_{12}{\perp}_1 y \wedge x \,{}_{12}{\perp}_2 z) \rightarrow \exists w(y \,{}_1{\perp}_2 w \wedge w \,{}_1{\perp}_2 z)). \tag{2.2}$$

Proof. Let $x = (f_x, i_x), y = (f_y, i_y), z = (f_z, i_z)$ such that $x \,{}_{12}{\perp}_1 y$ and $x \,{}_{12}{\perp}_2 z$. Then $f_x \wedge f_y \leq i_x$ and $f_x \leq i_z \vee i_x$. We claim that this implies that $i_z \nleq f_y$. Note that if this is true, then there is $w = (i_z, f_y)$ such that $y \,{}_1{\perp}_2 w$ and $w \,{}_1{\perp}_2 z$. For the proof of the claim, assume towards a contradiction that $i_z \leq f_y$. Then

$$f_x = f_x \wedge (i_z \vee i_x) \leq f_x \wedge (f_y \vee i_x) \leq (f_x \wedge f_y) \vee (f_x \wedge i_x) \leq i_x \vee (f_x \wedge i_x) = i_x,$$

a contradiction. □

It is also straightforward to see that this property is also sufficient for the dual lattice of a b-frame to be distributive:

Lemma 2.3.2. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be the dual b-frame of some complete lattice L. Then if $\mathscr{X}$ satisfies (2.2), L is distributive.*

[3] See [77, 103, 104, 150, 151] for some literature on canonical extensions.

Proof. Recall that a lattice L is distributive iff for any $a, b, c \in L$, $a \wedge c \leq b$ and $a \leq b \vee c$ implies that $a \leq b$. So assume towards a contradiction that there are $a, b, c \in L$ such that $a \nleq b$, but $a \wedge c \leq b$ and $a \leq b \vee c$. Since this implies that $1 \nleq c \nleq 0$, consider the points $x = (a, b)$, $y = (c, 0)$ and $z = (1, c)$. Note that, by assumption, we have that $x\,{}_{12}\!\perp_1 y$ and $x\,{}_{12}\!\perp_2 z$, so since (2.2) holds there is some $w = (f_w, i_w)$ such that $y\,{}_1\!\perp_2 w$ and $w\,{}_1\!\perp_2 z$. But the former implies that $c \leq i_w$ and the latter implies that $f_w \leq c$, and therefore $f_w \leq i_w$, a contradiction. □

In light of the previous two lemmas, we may define a *distributive* boset to be a boset $\mathscr{X}$ satisfying (2.2). Distributive b-frames (i.e., distributive bosets that are also b-frames) and b-morphisms between them form a category **DbF**. The following theorem is an immediate consequence of the two previous lemmas.

Theorem 2.3.3. *The duality between* **cLat** *and* **bF** *restricts to a duality between* **cDL** *and* **DbF**.

We therefore obtain a first-order characterization of the dual b-frames of distributive lattices. Moreover, using results from the previous section, we can also obtain a second-order characterization of bosets $\mathscr{X}$ such that $\rho(\mathscr{X})$ is distributive as follows:

Lemma 2.3.4. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is distributive iff $\mathscr{X}$ densely embeds into a distributive b-frame.*

Proof. For the left to right direction, recall that $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding. Moreover, by the previous lemma, $\alpha\rho(\mathscr{X})$ is distributive if $\rho(\mathscr{X})$ is distributive. For the converse direction, recall that if $f : \mathscr{X} \to \mathscr{Y}$ is a dense embedding, then $\rho(f)$ is an isomorphism. Thus if $\mathscr{X}$ densely embeds into a distributive b-frame, $\rho(\mathscr{X})$ must be distributive. □

2.3.2 Heyting Algebras

Let us now move on to the case of Heyting algebras. We will first isolate a property of certain points in a boset, called Heyting points and show that the existence of enough such points in a boset $\mathscr{X}$ guarantees that $\rho(X)$ is a Heyting algebra. As we will see, for an arbitrary boset $\mathscr{X}$, the existence of enough Heyting points in $\mathscr{X}$ is not necessary for $\rho(\mathscr{X})$ to be a *cHA*, but we will show that it is in the case of b-frames. This will give us a complete, first-order characterization of the dual b-frames of *cHA*'s, which can then be extended to bosets in a straightforward way. A key notion in this characterization is that of a *nucleus* on a complete lattice. Nuclei play an important role in pointfree topology [146, 207], where they provide an algebraic generalization of the notion of subspace of a topological space. Nuclei on complete Heyting algebras have also been used to provide alternative semantics for intuitionistic logic [36, 37]. The connection with nuclear semantics for intuitionistic logic will be further explored in Section 2.6.

Definition 2.3.5. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *Heyting point* of $\mathscr{X}$ is a point x^* such that $\forall y \in \mathscr{X}$, $x^*_{12}\!\perp_1 y$ iff $x^*{}_2\!\perp_1 y$.

Note that, in this definition, the right-to-left direction is satisfied by any point x in a boset $\mathscr{X}$: for any two $x, y \in \mathscr{X}$, if there is no 2-successor of x that is also a 1-successor of y, then in particular there is no 1-and-2-successor of x that is also a 1 successor of y. The converse direction, however, does not hold in general. Thus Heyting points are those for which their independence from any other point is equivalent to a weaker condition.

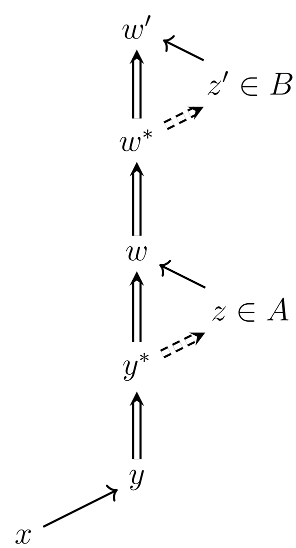
Definition 2.3.6. A *Heyting boset* is a boset $\mathscr{X}$ such that the Heyting points of $\mathscr{X}$ are dense, i.e., the following holds:

$$\forall x \exists x^* \geq_{12} x \forall y (x^*_{12} \perp_1 y \leftrightarrow x^*_2 \perp_1 y). \tag{2.3}$$

Equivalently, Heyting bosets are bosets in which the sub-boset of Heyting points is dense. In that sense, we may think of Heyting bosets as bosets in which there are "enough" Heyting points. The importance of Heyting points is established by the next lemma.

Lemma 2.3.7. *Let $(X, \leq_1, \leq_2)$ be a Heyting boset. Then $\neg_1\neg_2$ is a nucleus on $\mathscr{O}_1$.*

Proof. Recall first that a nucleus on a complete lattice L is a closure operator j such that $j(a \wedge b) = j(a) \wedge j(b)$ for any $a, b \in L$. Since $\neg_1\neg_2$ is always a closure operator on $\mathscr{O}_1$, we only need to check that for any $A, B \in \mathscr{O}_1$, $\neg_1\neg_2(X) \cap \neg_1\neg_2(Y) \subseteq \neg_1\neg_2(X \cap Y)$. Suppose $x \in \neg_1\neg_2(X) \cap \neg_1\neg_2(Y)$, and let $y \geq_1 x$. Fix some Heyting point $y^* \geq_{12} y$, and note that $y^* \geq_1 x$, which means that there is $z \geq_2 y^*$ such that $z \in A$. Since $\neg\ (y^*{}_2\perp_1 z)$, we also have $\neg\ (y^*{}_{12}\perp_1 z)$, so let $w \geq_{12} y^*$ such that $z \leq_1 w$, and fix a Heyting point $w^* \geq_{12} w$. Since A is a 1-upset, we have that $w^* \in A$. Moreover, since $x \leq_1 y \leq_1 y^* \leq_1 w^*$, there is $z' \geq_2 w^*$ such that $z' \in B$. Since $\neg\ (w^*{}_2\perp_1 z')$, we also have $\neg\ (w^*{}_{12}\perp_1 z')$, so let $w' \geq_{12} w^*$ such that $w' \geq_1 z'$. Since both A and B are 1-upsets, we have that $w' \in A \cap B$. Moreover, since $y \leq_2 y^* \leq_2 w^* \leq_2 w'$, it follows that $y \in C_2(A \cap B)$. The entire argument is summarized by the following diagram, where single lines represent the first ordering, dashed double lines the second ordering and full double lines the intersection of the two orderings:[4]



[4]From now on, we will use this convention to denote the various orderings diagrammatically.

Thus $\neg_1\neg_2(A) \cap \neg_1\neg_2(B) \subseteq \neg_1\neg_2(A \cap B)$, which establishes that $\neg_1\neg_2$ is a nucleus. □

The fixpoints of a nucleus on a complete Heyting algebra always form a complete Heyting algebra [76, p. 71]. Thus the previous lemma implies that the regular opens of any Heyting boset always form a cHA. On the other hand, it is easy to see that the converse fails: a boset $\mathscr{X}$ need not be Heyting for $\rho(\mathscr{X})$ to be a cHA.

Example 2.3.8. Suppose $\mathbb{P} = (P, \leq_P)$ is a poset such that $Up(\mathbb{P})$ is not a complete Boolean algebra (for example $P = \omega$ with the usual order). Note that this implies that there must be some $A \in Up(\mathbb{P})$ such that $A \cup I(P - A) \subsetneq P$, where I is the interior operator induced by the upset topology. This in turn means that $P = I(A \cup P{-}A) \nsubseteq A \cup I(P - A)$, so $I(U \cup V) \neq I(U) \cup I(V)$ in general. Taking complements, this means that the topological closure C induced by the upset topology on $\mathbb{P}$ is not a nucleus. However, the downsets of any poset always form a cHA. Thus, if we think of C as a closure operator on the lattice of open sets of P when P is endowed with the discrete topology, this gives us an example of a closure operator k on the lattice of upsets of a poset which is not a nucleus even though the fixpoints of k form a cHA.[5] But it is now easy to turn this into an example of a non-Heyting boset whose dual lattice is a cHA: letting $\mathscr{P} = (P, \Delta_P, \leq_P)$, we have that $\neg_1\neg_2$ on $\mathscr{P}$ is precisely the closure operator C above.

Thus for an arbitrary boset $\mathscr{X}$, the existence of densely many Heyting points is not necessary for $\rho(\mathscr{X})$ to be a cHA. On the other hand, the next lemma shows that the dual b-frame of a cHA is always Heyting.

Lemma 2.3.9. *Let A be a cHA and $\alpha(A) := (X, \leq_1, \leq_2)$ its dual b-frame. Then $\alpha(A)$ is a Heyting b-frame.*

Proof. Let $x = (f_x, i_x) \in X$ and consider the point $x^* = (f_x, f_x \to i_x)$. Clearly, $x \leq_{12} x^*$. Now for any $y \in X$, we have that $x^*\,{}_{12}{\perp}_1 y$ iff $f_x \wedge f_y \leq f_x \to i_x$ iff $f_x \wedge f_y \leq i_x$ iff $f_y \leq f_x \to i_x$ iff $x^*\,{}_2{\perp}_1 y$. □

Moreover, by Lemma 2.3.7 and the fact that $\eta_L : L \to \rho\alpha(L)$ is an isomorphism, the converse also holds:

Lemma 2.3.10. *Let L be a complete lattice such that $\alpha(L)$ is Heyting. Then L is a Heyting algebra.*

As an immediate consequence of the previous results, we obtain the following corollary.

Corollary 2.3.11. *Let L be a lattice. Then L is a Heyting algebra iff $\alpha(L)$ is Heyting.*

Thus we obtain a complete characterization of the dual b-frames of complete Heyting algebras. Once again, using results established in the previous section, we can now give necessary and sufficient conditions for when the regular opens of any boset form a complete Heyting algebra.

[5] More involved examples of such posets are also given in [37] and [76].

Lemma 2.3.12. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is a Heyting algebra iff $\mathscr{X}$ densely embeds into a Heyting b-frame.*

Proof. From left to right, if $\rho(\mathscr{X})$ is a Heyting algebra, then $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding into a Heyting b-frame.
Conversely, if $\mathscr{X}$ densely embeds into a Heyting b-frame $\mathscr{Y}$, then $\rho(\mathscr{X})$ is isomorphic to $\rho(\mathscr{Y})$ and thus is a Heyting algebra by Lemma 2.3.7. □

Finally, recall that morphisms of cHA's are complete lattice homomorphisms which also preserve the Heyting implication. In order to identify the duals of such morphisms, we need the following strenghtening of the definition of a b-morphism:

Definition 2.3.13. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ and $\mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be two bosets. A *Heyting b-morphism* (h-morphism) from $\mathscr{X}$ and $\mathscr{Y}$ is a b-morphism satisfying the following strengthening of condition 3:

3'. $\forall x \in X \, \forall y \geq_1^Y f(x) \, \exists z \geq_1^X x : f(z) \geq_{12}^Y y$.

The next lemma shows that if $f : \mathscr{X} \to \mathscr{Y}$ is an h-morphism of Heyting bosets, then $\rho(f)$ preserves Heyting implications.

Lemma 2.3.14. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a h-morphism. Then for any $A, B \in \mathsf{RO}_{12}(Y)$, we have $f^{-1}[I_1((Y - A) \cup B)] = I_1(f^{-1}[Y - A] \cup f^{-1}[B])$.*

Proof. Note that the left-to-right inclusion is an immediate consequence of f being 1-monotone. For the converse, assume that for all $y \geq_1^X x$, $f(y) \notin A$ or $f(y) \in B$, and let $y \geq_1^Y f(x)$ be in A. We claim that $y \in B$. To see this, let $z \geq_1^Y y$. By condition $3'$ of an h-morphism, there is $z' \geq_1^X x$ such that $z \geq_{12}^Y f(z')$. Since $x \leq_1^X z'$, by assumption we have that $f(z') \notin A$ or $f(z') \in B$. But since $y \leq_1^Y z \leq_{12}^Y f(z')$ and A is a 1-upset, we have that $f(z') \in B$. Thus $y \in \neg_1\neg_2(B) = B$. □

It follows that if $f : \mathscr{X} \to \mathscr{Y}$ is a h-morphism between Heyting bosets, then the dual $\rho(f) : \rho(\mathscr{Y}) \to \rho(\mathscr{X})$ is a complete HA-homomorphism. Conversely:

Lemma 2.3.15. *Let L, M be two complete Heyting algebras, and let $f : L \to M$ be a complete HA-homomorphism. Then $\alpha(f) : \alpha(M) \to \alpha(L)$ is a h-morphism.*

Proof. Recall that for any $(a, b) \in \alpha(M)$,

$$\alpha(f)(a, b) = (a^f, b_f).$$

Since $\alpha(f)$ is a b-morphism, we only have to check that condition $3'$ holds. So assume $(c, d) \geq_1^L \alpha(f)(a, b)$ for some $(a, b) \in \alpha(M)$. We claim that $(a \wedge f(c), f(d)) \in \alpha(M)$. To see this, note that, otherwise, $a \wedge f(c) \leq f(d)$, and hence $a \leq f(c) \to f(d) = f(c \to d)$. But then $c \leq a^f \leq c \to d$, which implies that $c \leq d$, a contradiction. Thus $(a \wedge f(c), f(d)) \in \alpha(M)$ and clearly $(a, b) \leq_1^M (a \wedge f(c), f(d))$. Moreover, since $(a \wedge f(c))^f \leq c$ and $d \leq (f(d))^f$, it follows that $(c, d) \leq_{12}^L \alpha(f)(a \wedge f(c), f(d))$. Thus $\alpha(f)$ is an h-morphism. □

We may therefore form the category **HbF** of Heyting b-frames and h-morphisms between them. The previous results readily imply the following theorem:

Theorem 2.3.16. *The duality between* **cLat** *and* **bF** *restricts to a duality between* **cHA** *and* **HbF**.

2.3.3 Boolean Algebras

Finally, let us consider the case of Boolean algebras. Here we will follow a similar pattern as in the case of Heyting algebras. We start with the definition of a Boolean point.

Definition 2.3.17. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *Boolean point* of $\mathscr{X}$ is a point $x^* \in X$ such that for any $y \in X$, $x^*_1 \bot_{12} y \leftrightarrow x^*_2 \bot_{12} y$.

Similarly to the definition of a Heyting boset as a boset having "enough" Heyting points, we may define a *Boolean boset* as a boset $\mathscr{X}$ such that the Boolean points of $\mathscr{X}$ are dense, i.e., the following holds:

$$\forall x \exists x^* \geq_{12} x \forall y ({x^*}_1 \bot_{12} y \leftrightarrow {x^*}_2 \bot_{12} y)). \tag{2.4}$$

The existence of a dense set of Boolean points in a boset $\mathscr{X}$ has some important consequences for the operator $\neg_1\neg_2$.

Lemma 2.3.18. *Let $(X, \leq_1, \leq_2)$ be a Boolean boset. Then $\neg_1\neg_2$ is the double negation nucleus on $\mathscr{O}_1$.*

Proof. We first show that $\neg_1\neg_2(A) \subseteq \neg_1\neg_1(A)$ for any $A \in \mathscr{O}_1$. Let $y \geq_1 x$ for some $x \in \neg_1\neg_2(A)$. Then since $x \leq_1 y^*$, there is $z \geq_2 y^*$ such that $z \in A$. This implies that $\neg\, {y^*}_2 \bot_{12} z$, and thus also $\neg {y^*}_1 \bot_{12} z$. But this implies that $y \in C_1(A)$. Thus $x \in \neg_1\neg_1(A)$.

We now show the converse, i.e., that $\neg_1\neg_1(A) \subseteq \neg_1\neg_2(A)$. Let $y \geq_1 x$ for some $x \in \neg_1\neg_1(A)$. Since $x \leq_1 y^*$, there is $z \geq_1 y^*$ such that $z \in A$. Since this implies that $\neg {y^*}_1 \bot_{12} z$, it follows that $\neg {y^*}_2 \bot_{12} z$. But this implies that $y \in C_2(A)$, and therefore $x \in \neg_1\neg_2(A)$. □

Since the regular open sets of any topological space always form a complete Boolean algebra, the previous lemma clearly implies:

Lemma 2.3.19. *Let L be a lattice such that $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$ is Boolean. Then L is a Boolean algebra.*

Moreover, the converse holds as well:

Lemma 2.3.20. *Let $\alpha(L) := (X, \leq_1, \leq_2)$ be the dual b-frame of a Boolean algebra L. Then $\alpha(L)$ is Boolean.*

Proof. Given $x = (f_x, i_x)$, let $x^* = (f_x \wedge \neg i_x, \neg f_x \vee i_x)$. Note that

$$(f_x \wedge \neg i_x) \to (\neg f_x \vee i_x) = \neg f_x \vee i_x = f_x \to i_x \neq 1,$$

thus x^* is well defined. Moreover, for any $y = (f_y, i_y)$, we have that $(f_x \wedge \neg i_x) \wedge f_y \leq i_y$ iff $f_y \leq (f_x \wedge \neg i_x) \to i_y$ iff $f_y \leq (\neg f_x \vee i_x) \vee i_y$. □

Corollary 2.3.21. *A complete lattice L is a Boolean algebra iff its dual b-frame is Boolean.*

Note that, once again, this first-order characterization of the b-frames that are dual to a complete Boolean algebra extends to a characterization of bosets $\mathscr{X}$ for which $\rho(\mathscr{X})$ is a Boolean algebra.

Corollary 2.3.22. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is a Boolean algebra iff $\mathscr{X}$ densely embeds into a Boolean b-frame.*

Proof. From left to right, if $\rho(\mathscr{X})$ is a Boolean algebra, then $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding into a Boolean b-frame. Conversely, if $\mathscr{X}$ densely embeds into a Boolean b-frame $\mathscr{Y}$, then $\rho(\mathscr{X})$ is isomorphic to $\rho(\mathscr{Y})$ and thus is a Boolean algebra by Lemma 2.3.19. □

Finally, since complete lattice homomorphisms between complete Boolean algebras are complete Boolean homomorphisms, we obtain the following duality:

Theorem 2.3.23. *Boolean b-frames and b-morphisms form a category* **BbF** *dual to the category* **cBA** *of complete Boolean algebras and complete Boolean homomorphisms.*

Before discussing other classes of complete lattices, let us derive a straightforward application of this characterization of the dual b-frames of Boolean algebras.

Lemma 2.3.24. *The class of b-frames is not first-order definable. In particular, completeness is not first-order definable in the language of bosets.*

Proof. Suppose that completeness is equivalent to some set Φ of first-order formulas in the language of bosets (i.e., pure first-order logic with two relation symbols $\leq_1$ and $\leq_2$). Let $\alpha(C)$ be the dual b-frame of the Cohen algebra C, i.e., the MacNeille completion of the countable atomless Boolean algebra [144, Chap. 30]. Since C has size $2^{\aleph_0}$, and points in $\alpha(C)$ are pairs of elements in C, $\alpha(C)$ also has size continuum. Let M be a countable elementary substructure of $\alpha(C)$, which exists by the downward Löwenheim-Skolem theorem. Now since separativity is a first-order condition and so is completeness by assumption, it follows that M is a b-frame, hence M is isomorphic to $\alpha(L)$ for some complete lattice L. Moreover, since the property of having a dense set of Boolean points is also first-order, M is a Boolean b-frame, and therefore L is a complete Boolean algebra. But M is countable, hence $\alpha(L)$ is also countable. Since there is a surjection $\pi : \alpha(L) \to L \setminus \{0\}$ defined by $(a, 0) \mapsto a$, it follows that L is countable. But there is no countable complete Boolean algebra. Thus the property of completeness is not first-order definable. □

2.4 Spatial and Superalgebraic Locales

In this section, we focus on two classes of complete Heyting algebras that are of particular relevance in the literature on semantics for intuitionistic logic: spatial and superalgebraic locales. Both classes have been extensively studied in the literature. Spatiality is a key notion in pointfree topology [146, 207], as spatial locales are precisely those locales that can be represented as the lattice of open sets of a topological space. Superalgebraic or completely join-prime generated locales on the other hand have long been known to be precisely the lattices that arise as the collection of downward- or upward-closed sets of a poset [68, 216]. Our goal here is to offer alternative representations of both spatial and superalgebraic locales by restricting the duality between Heyting b-frames and complete Heyting algebras obtained in Section 2.3.2. These results are then used in Section 2.6 to provide a unified framework for Kripke, topological and nuclear semantics for intuitionistic logic. We start by recalling the following definitions.

Definition 2.4.1. Let L be a cHA.

- L is *spatial* iff L is isomorphic to the lattice of open sets $\Omega(\mathscr{X})$ for some topological space $\mathscr{X} = (X, \tau)$.
- L is *superalgebraic* iff L is isomorphic to the lattice of upward-closed sets $Up(\mathscr{X})$ of a poset $\mathscr{X} = (X, \leq)$.[6]

Our goal in this section is to characterize b-frames whose dual lattices are spatial and superalgebraic locales. Our strategy will be the same for both classes of cHA's: first, we recall that spatial and superalgebraic locales are characterized by having certain algebraic "separation properties": any two distinct elements of a spatial locale can be separated by a meet-prime element, while any two distinct elements of a superalgebraic locale can be separated by a completely join-prime element. We then translate these algebraic properties into graph-theoretic properties of b-frames and prove that those properties do characterize the duals of spatial and superalgebraic locales. We conclude this section by an immediate application of these results: a new, purely b-frame-theoretic proof that any spatial Boolean locale is also superalgebraic.

2.4.1 Spatial Locales

Recall that, given a lattice L, an element $c \in L$ is *meet-prime* if for any $a, b \in L$, $a \wedge b \leq c$ iff $a \leq c$ or $b \leq c$. It is *completely join-prime* if for any $A \subseteq L$, $c \leq \bigvee A$ iff $c \leq a$ for some $a \in A$. The following is a basic result of pointfree topology.

Lemma 2.4.2 ([207, Prop. II.5.3])**.** *A locale L is spatial iff for any $a \nleq b \in L$, there is a meet-prime element $c \in L$ such that $a \nleq c$ and $b \leq c$.*

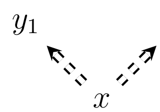
Identifying the points in a b-frame that correspond to meet-prime elements in the dual lattice is therefore an essential step in characterizing the duals of spatial locales. This is the role of the following definition:

Definition 2.4.3. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *spatial point* of $\mathscr{X}$ is a point $x \in X$ such that the following holds:

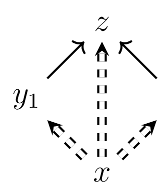
$$\forall y_1 y_2 (x \leq_2 y_1 \wedge x \leq_2 y_2 \to \exists z (y_1 \leq_1 z \wedge y_2 \leq_1 z \wedge x \leq_2 z)).$$

[6]This terminology is used by Picado and Pultr in [207], who first define superalgebraic locales as join-prime generated locales, before proving the equivalence with the definition given here.

Spatial points can be understood as having a certain amalgamation property. Indeed, by simply spelling out the previous definition, we may notice that a point $x \in \mathscr{X}$ is spatial precisely if any diagram of the form



can be completed as follows:



The next two lemmas highlight the relevance of spatial points in identifying the duals of spatial locales.

Lemma 2.4.4. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset such that every point in $\mathscr{X}$ is spatial. Then $\rho(\mathscr{X})$ is a spatial locale.*

Proof. Suppose that every point in $\mathscr{X}$ is spatial. Note that since $\neg_1$ and $\neg_2$ form a Galois connection, the regular opens $\rho(\mathscr{X})$ are isomorphic to the regular closed sets of $\mathscr{X}$, i.e., the lattice of sets $U \subseteq X$ such that $U = C_2 I_1(U)$ or, equivalently, $-U = \neg_2 \neg_1(-U)$. We claim that the regular closed sets of $\mathscr{X}$ form a topology on X. Clearly for any family $\{U_i\}_{i \in I}$ of regular closed sets, we have that $C_2 I_1(U_i) \subseteq C_2 I_1(\bigcup_{i \in I} U_i)$ for any $i \in I$, and therefore

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} C_2 I_1(U_i) \subseteq C_2 I_1(\bigcup_{i \in I} U_i).$$

Since $C_2 I_1$ is a kernel operator on the 2-downsets of $\mathscr{X}$, this implies that the regular closed sets of $\mathscr{X}$ are closed under arbitrary unions. Therefore we only have to check that they are also closed under finite intersection. Suppose U_1, U_2 are regular closed. Clearly $U_1 \cap U_2$ is also a 2-downset, and hence $C_2 I_1(U_1 \cap U_2) \subseteq U_1 \cap U_2$. For the converse, suppose $x \in U_1 \cap U_2$. Since both U_1 and U_2 are regular closed, this means that there is $y_1 \in I_1(U_1)$ and $y_2 \in I_1(U_2)$ such that $x \leq_2 y_1$ and $x \leq_2 y_2$. Since by assumption x is spatial, this means that there is $z \geq_2 x$ such that $z \geq_1 y_1, y_2$. But this implies that $z \in I_1(U_1) \cap I_1(U_2) = I_1(U_1 \cap U_2)$. Hence $x \in C_2 I_1(U_1 \cap U_2)$ and $U_1 \cap U_2$ is regular closed, which completes the proof that the regular closed sets form a topology on X. Therefore $\rho(X)$ is spatial. □

Lemma 2.4.5. *Let L be a spatial locale. Then the set of spatial points of $\alpha(L)$ is dense.*

Proof. Suppose L is spatial and $(a, b) \in \alpha(L)$. Since L is spatial, there is a meet prime $c \in L$ such that $a \nleq c$ and $b \leq c$. Hence the point $(a, c) \in \alpha(L)$, and we have that $(a, b) \leq_{12} (a, c)$. We claim that (a, c) is a spatial point of $\alpha(L)$. To see this, suppose that $(a, c) \leq_2 (x_1, y_1)$

and $(a, c) \leq_2 (x_2, y_2)$. Since $x_i \nleq y_i$ and $c \leq y_i$, we have that $x_i \nleq c$ for $i \in \{1, 2\}$. Since c is meet-prime, this means that $x_1 \wedge x_2 \nleq c$. But then $(x_1 \wedge x_2, c)$ is the required point. □

As a straightforward consequence, we obtain the following characterization of the duals of spatial locales:

Theorem 2.4.6.

1. *A locale L is spatial iff the set of spatial points of $\alpha(L)$ is dense.*

2. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is spatial iff $\mathscr{X}$ densely embeds into a b-frame $\mathscr{Y}$ with densely many spatial points.*

Proof.

1. The left-to-right direction follows from the previous lemma. For the converse, if the spatial points of $\alpha(L)$ are dense, then letting $\mathscr{X}$ be the dense subframe of $\alpha(L)$ induced by its spatial points, we have by Lemma 2.4.4 that $\rho(\mathscr{X})$ is spatial and by Lemma 2.2.19 that L is isomorphic to $\rho(\mathscr{X})$, hence also spatial.

2. This follows directly from the first part. □

Let us now move on to superalgebraic locales, for which we apply a similar method.

2.4.2 Splitting Locales

As mentioned above, superalgebraic locales are precisely those locales in which any two distinct elements can be separated by a completely join-prime one. Our characterization of the dual b-frames of superalgebraic locales essentially uses this fact, but the following property will be easier to work with:

Definition 2.4.7. Let L be a lattice. Given $a, b \in L$ such that $a \nleq b$, a *splitting pair* for the pair (a, b) is a pair (c, d) of elements of L such that $c \nleq d$, $c \leq a$, $b \leq d$ and for any $x \in L$, $c \leq x$ or $x \leq d$.

A locale L is *splitting* if for any $a \nleq b \in L$, there is a splitting pair (c, d) for the pair (a, b).

Splittings in lattices have a long history, going back to Whitman [261]. Splitting locales are a special kind of separated locales, the study of which originates with Raney [216]. While separated locales coincide with supercontinuous locales and are precisely the complete homomorphic images of frames of downsets of posets (or, equivalently, completely distributive complete lattices [207, Prop. VII.8.5.1]), splitting locales coincide with superalgebraic locales, as is well-known.

Lemma 2.4.8. *A locale L is superalgebraic iff L is splitting.*

Proof. For the left-to-right direction, assume without loss of generality that $L = Up(\mathscr{X})$ for some poset $\mathscr{X} = (X, \leq)$. Given $U \nsubseteq V \in \mathscr{X}$, let $x \in U - V$, and let $U' = \uparrow x$ and $V' = X - \downarrow x$. Then clearly $U' \nsubseteq V'$, $U' \subseteq U$ and $V \subseteq V'$, and moreover for any $Y \in Up(\mathscr{X})$, since either $x \in Y$ or $x \notin Y$, we must have that $U' \subseteq Y$ or $Y \subseteq V'$.

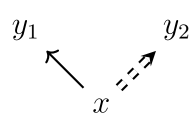
For the converse direction, as superalgebraic locales are precisely the completely join-prime generated locales (see for example [207, Prop. VII.8.3]), it is enough to observe that for any splitting pair $(c, d) \in L$, c is completely join-prime. But this is a well-known argument [192, Remark. 4.1]. □

We now define the boset counterpart of splitting pairs.

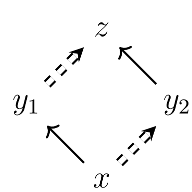
Definition 2.4.9. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *splitting point* of $\mathscr{X}$ is a point $x \in X$ such that the following holds:

$$\forall y_1 y_2 (x \leq_1 y_1 \wedge x \leq_2 y_2 \rightarrow \exists z (y_1 \leq_2 z \wedge y_2 \leq_1 z)).$$

Similarly to spatial points, splitting points exhibit a certain amalgmation property. Indeed, a point $x \in \mathscr{X}$ is splitting precisely if any diagram of the form



can be completed as follows:



The next two lemmas establish the equivalence between separation by splitting pairs in lattices and density of splitting points in bosets:

Lemma 2.4.10. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset such that the splitting points of $\mathscr{X}$ are dense. Then $\rho(\mathscr{X})$ is splitting.*

Proof. Let $U, V \in \rho(\mathscr{X})$ such that $U \nsubseteq V$. This means that there is $x \in \mathscr{X}$ such that $x \in U \cap \neg_2 V$. Let $x' \geq_{12} x$ be a splitting point, and notice that $x' \in U \cap \neg_2 V$. We claim that $(U^{x'}, V_{x'})$ is a splitting pair for (U, V). By Lemma 2.4.8, this implies that $\rho(\mathscr{X})$ is superalgebraic. For the proof of the claim, it is clear that $U^{x'} \subseteq U$, $V \subseteq V_{x'}$ and that $U^{x'} \nsubseteq V_{x'}$. Now let T be any regular open set. If $x' \in T$, then $U^{x'} \subseteq T$. Otherwise, if $x' \notin T$, there is $y_1 \geq_1 x'$ such that $y_1 \in \neg_2 T$. But then for any $w \in T$, if $\neg x'{}_2{\perp}_1 w$, there must be some $y_2 \geq_2 x$ such that $y_2 \geq_1 w$. Since by assumption x' is splitting, there is $z \in \mathscr{X}$ such that $z \geq_2 y_1$ and $z \geq_1 y_2$. But this is a contradiction, since $z \geq_1 y_2 \geq_1 w$ implies that $z \in T$, while $z \geq_2 y_1$ implies that $z \notin T$ since $y_1 \in \neg_2 T$. Hence for any $w \in T$, $x'{}_2{\perp}_1 w$, which means that $T \subseteq V_{x'}$. This completes the proof. □

Lemma 2.4.11. *Let L be superalgebraic. Then the splitting points of $\alpha(L)$ are dense.*

Proof. Let $(a, b) \in \alpha(L)$. Since L is superalgebraic, by Lemma 2.4.2, there is a splitting pair (c, d) for the pair (a, b). Note that by the definition of a splitting pair, we have that $(a, b) \leq_{12} (c, d)$. We claim that (c, d) is a splitting point of $\alpha(L)$. Suppose $(c, d) \leq_1 (x_1, y_1)$ and $(c, d) \leq_2 (x_2, y_2)$. Now $x_1 \leq c$ yet $x_1 \nleq y_1$, which means that $c \nleq y_1$, and hence $y_1 \leq d$ since (c, d) is a splitting pair. Similarly $d \leq y_2$ yet $x_2 \nleq y_2$, which implies that $x_2 \nleq d$, and therefore that $c \leq x_2$. Hence $(x_1, y_1) \leq_2 (c, d)$, and $(x_2, y_2) \leq_1 (c, d)$, which shows that $\neg(x_1, y_1)_2\perp_1(x_2, y_2)$ and establishes that (c, d) is a splitting point. □

As a consequence, we obtain the following characterization of b-frames that are dual to superalgebraic locales:

Theorem 2.4.12.

1. *A locale L is superalgebraic iff the set of splitting points of $\alpha(L)$ is dense.*
2. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is splitting iff $\mathscr{X}$ densely embeds into a b-frame $\mathscr{Y}$ with densely many splitting points.*

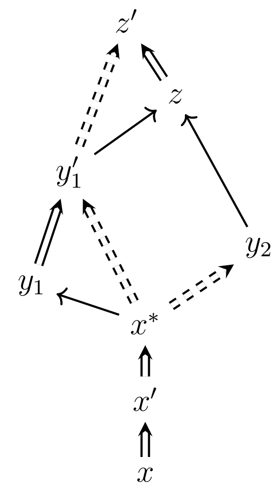
Proof.

1. The left-to-right direction follows from the previous lemma. For the converse, if the splitting points of $\alpha(L)$ are dense, then by Lemma 2.4.10 $\rho\alpha(L)$ is superalgebraic. But since L is isomorphic to $\rho\alpha(L)$, it is also superalgebraic.
2. This follows readily from the first part. □

As an immediate application of the results of this section, we can now use b-frames to prove the following well-known fact about Boolean locales (see [207, Section II.5.4] for a standard proof):

Corollary 2.4.13. *Any spatial Boolean locale is superalgebraic.*

Proof. Let B be a spatial Boolean locale. We claim that $\alpha(B)$ is a splitting b-frame. To see this, let $x \in \alpha(B)$. Since B is spatial, $\alpha(B)$ is a spatial b-frame, which means that there is some spatial point $x' \geq_{12} x$. Since B is also Boolean, there is a Boolean point $x^* \geq_{12} x'$. We claim that x^* is a splitting point. Indeed, suppose $y_1 \geq_1 x^*$ and $y_2 \geq_2 x^*$. Since x^* is Boolean and $\neg y_1\,{}_1\!\perp_{12} x^*$, there is some $y_1' \geq_{12} y_1$ such that $x^* \leq_2 y_1$, and note that we may assume that y_1' is Boolean. Hence we have that $x' \leq_2 y_2$ and $x' \leq_2 y_1'$, so since x' is spatial we have some $z \geq_1 y_2, y_1'$. Now since y_1' is Boolean and $\neg y_1'\,{}_1\!\perp_{12} z$, there must be some $z' \geq_{12} z$ such that $z' \geq_2 y_1'$. Thus $z' \geq_2 y_1' \geq_2 y_1$, and $z' \geq_1 z \geq_1 y_2$. Hence x^* is a splitting point. The argument is summarized by the diagram below:



□

2.5 A Decomposition Theorem for bi-Heyting Algebras

In this section, we apply elements of our b-frame duality to prove a new result regarding complete bi-Heyting algebras. The motivation for our result is the following theorem about complete Boolean algebras:[7]

Lemma 2.5.1. *For any complete Boolean algebra B, there are complete Boolean algebras C_1 and C_2 such that C_1 is atomic, C_2 is atomless, and $B = C_1 \times C_2$.*

In our setting, atomic Boolean algebras must be generalized to superalgebraic locales (notice that Boolean superalgebraic locales are precisely the atomic Boolean algebras). This is in line with the fact that completely join-prime elements are usually taken to be the relevant generalization of atoms for cHA's. Accordingly, we propose as a relevant generalization of atomless Boolean algebras the following definition:

Definition 2.5.2. A complete lattice is *anti-algebraic* if it has no completely join-prime element.

We will use our b-frame duality to show that any complete Heyting algebra is, in the category of complete lattices, a subdirect product of a superalgebraic locale and an anti-algebraic locale. As will be made explicit below, this decomposition theorem holds in the category **cLat** of complete lattices and complete lattice homomorphisms, but not in the category of complete bi-Heyting algebras and complete bi-Heyting homomorphisms between them, which is not a full subcategory of **cLat**. Of course, the issue does not arise in the Boolean case, since **cBA** is a full subcategory of **cLat**.

[7]See for example [106], p. 227.

2.5.1 Coproducts of Bosets

We start by defining the coproduct of two bosets. By duality, this induces a boset representation of the product of two complete lattices. This is an adaptation of the standard correspondence between products and disjoint unions.

Definition 2.5.3. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ and $\mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be two bosets. The *disjoint sum* of $\mathscr{X}$ and $\mathscr{Y}$, written as $\mathscr{X} \sqcup \mathscr{Y}$, is the boset $(Z, \leq_1^Z, \leq_2^Z)$, where $Z = X \sqcup Y$, $\leq_1^Z = \leq_1^X \sqcup \leq_1^Y$, and $\leq_2^Z = \leq_2^X \sqcup \leq_2^Y$.

Lemma 2.5.4. *For any two bosets $\mathscr{X}$ and $\mathscr{Y}$, $\mathscr{X} \sqcup \mathscr{Y}$ is the coproduct of $\mathscr{X}$ and $\mathscr{Y}$ in the category of bosets.*

Proof. Note first that we have two obvious inclusion b-morphisms $\lambda_1 : \mathscr{X} \to \mathscr{X} \sqcup \mathscr{Y}$ and $\lambda_2 : \mathscr{Y} \to \mathscr{X} \sqcup \mathscr{Y}$. Moreover, if $\mathscr{T}$ is any boset such that there are b-morphisms $\tau_1 : \mathscr{X} \to \mathscr{T}$ and $\tau_2 : \mathscr{Y} \to \mathscr{T}$, then it is routine to check that the map $h : \mathscr{X} \sqcup \mathscr{Y} \to \mathscr{T}$ given by $h(z) = \tau_1(z)$ if $z \in \mathscr{X}$, and $h(z) = \tau_2(z)$ if $z \in \mathscr{Y}$ witnesses the universal property of the coproduct. □

Lemma 2.5.5. *Let $\mathscr{X}$, $\mathscr{Y}$ be two bosets. Then $\rho(\mathscr{X}) \times \rho(\mathscr{Y}) = \rho(\mathscr{X} \sqcup \mathscr{Y})$.*

Proof. Recall that, as a covariant functor from $\mathbf{bF}^{op}$ into **cLat**, ρ has a left adjoint α. This means that ρ preserves limits. Since $\mathscr{X} \sqcup \mathscr{Y}$ is the coproduct of $\mathscr{X}$ and $\mathscr{Y}$ in **bF**, it is their product in $\mathbf{bF}^{op}$, and thus $\rho(\mathscr{X} \sqcup \mathscr{Y}) = \rho(\mathscr{X}) \times \rho(\mathscr{Y})$. □

2.5.2 Characterizing Co- and Bi-Heyting Algebras

Next, we extend the characterization of the dual b-frames of Heyting algebras obtained in Section 2.3 to co- and bi-Heyting algebras. Recall that a co-Heyting algebra is a distributive lattice in which the join $\vee$ has a left adjoint $\prec$, and that a bi-Heyting algebra is a Heyting algebra that is also a co-Heyting algebra. Bi-Heyting algebras and their representation theory were extensively studied by Rauszer [218, 219, 220, 221].

Definition 2.5.6. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ be a boset.

- A point $x^* \in \mathscr{X}$ is *co-Heyting* if for all $y \in \mathscr{X}$, $x^* {}_{12}{\perp}_1 y$ iff $x^* {}_1{\perp}_2 y$.
- $\mathscr{X}$ is a *co-Heyting boset* if the co-Heyting points of $\mathscr{X}$ are dense.
- A b-morphism $f : \mathscr{X} \to \mathscr{Y}$ is a *co-Heyting morphism* (denoted coh-morphism) if it satisfies the following strengthening of condition 2:

  2′ $\forall x \in X \, \forall y \geq_2^Y f(x) \, \exists z \geq_2^X x f(x) \geq_{12}^Y y$.

Lemma 2.5.7.

1. *If $\mathscr{X}$ is a co-Heyting boset, then $\neg_2\neg_1$ is a nucleus on $Up_2(\mathscr{X})$, and consequently $\rho(\mathscr{X})$ is a co-Heyting algebra.*

2. *If L is a complete co-Heyting algebra, then $\alpha(L)$ is a co-Heyting b-fame.*

3. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is a co-Heyting algebra iff $\mathscr{X}$ densely embeds into a co-Heyting b-frame.*

Proof.

1 Similar to Lemma 2.3.7.

2 Similar to Lemma 2.3.9. Given a pair $(a, b) \in \alpha(L)$, $a \nleq b$ implies that $a \prec b \neq b$, and thus $(a \prec b, b) \in \alpha(L)$. It is routine to check that this is a co-Heyting point of $\alpha(L)$.

3 Similar to Lemma 2.3.12. □

Lemma 2.5.8. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a b-morphism.*

1. *If f is a coh-morphism, then $\rho(f) : \rho(\mathscr{Y}) \to \rho(\mathscr{X})$ is a co-Heyting homomorphism.*

2. *If $h : L \to M$ is a co-Heyting homomorphism of co-Heyting algebras then the map $\alpha(h) : \alpha(M) \to \alpha(L)$ is a coh-morphism.*

Proof.

1. Similar to Lemma 2.3.14.

2. Similar to Lemma 2.3.15. □

We therefore obtain a description of the dual of the category of co-Heyting algebras and co-Heyting homomorphisms:

Theorem 2.5.9. *Co-Heyting b-frames and coh-morphisms form a category* **coHbF** *dual to the category* **cocHA** *of complete co-Heyting algebras and co-Heyting homomorphisms.*

Standard dualities for complete co-Heyting and bi-Heyting algebras can also be obtained via Esakia duality [32]. In our setting, we also need to identify the dual b-frames of complete bi-Heyting algebras. Given a boset $\mathscr{X}$, let us define a *bi-Heyting point* of $\mathscr{X}$ as a point $x^* \in \mathscr{X}$ that is both a Heyting and a co-Heyting point. Establishing the existence of bi-Heyting points in dual b-frames of bi-Heyting algebras requires a technical lemma.

Lemma 2.5.10. *Let L be a complete bi-Heyting algebra. Then for any $a, b, c, d \in L$:*

1. *$(a \prec b) \wedge c \leq a \to b$ iff $c \leq a \to b$;*

2. *$a \prec b \leq (a \to b) \vee d$ iff $a \prec b \leq d$.*

Proof.

1. Note that:

$$\begin{aligned}
&a{-\!\!<}\, b \wedge c \leq a \to b \\
\text{iff } &(a \wedge c) \wedge (a{-\!\!<}\, b) \leq b \\
\text{iff } &((a \wedge c) \vee b) \wedge ((a{-\!\!<}\, b) \vee b) \leq b \\
\text{iff } &(a \vee b) \wedge (c \vee b) \wedge (a \vee b) \leq b \\
\text{iff } &(a \wedge c) \vee b \leq b \\
\text{iff } &c \leq a \to b.
\end{aligned}$$

2. This follows from 1 applied to L^δ, the dual bi-Heyting algebra to L. □

Theorem 2.5.11. *Let L be a complete lattice. Then L is a bi-Heyting algebra iff the bi-Heyting points of $\alpha(L)$ are dense.*

Proof. The right-to-left direction follows immediately from Lemmas 2.3.9 and 2.5.8. For the left-to-right direction, suppose L is a bi-Heyting algebra and $(a, b) \in \alpha(L)$. We claim that $(a{-\!\!<}\, b, a \to b)$ is a bi-Heyting point of $\alpha(L)$. That $(a{-\!\!<}\, b, a \to b) \in \alpha(L)$ follows from the previous lemma, with $c = 1$. Moreover, for any $(c, d) \in \alpha(L)$ we have by the previous lemma (item 1) that $(a{-\!\!<}\, b, a \to b)_2 \bot_1 (c, d)$ iff $c \leq a \to b$ iff $a{-\!\!<}\, b \wedge c \leq a \to b$ iff $(a{-\!\!<}\, b, a \to b)_{12} \bot_1 (c, d)$. Hence $(a{-\!\!<}\, b, a \to b)$ is a Heyting point. Similarly, by item 2 in the previous lemma, we have that $(a{-\!\!<}\, b, a \to b)_1 \bot_2 (c, d)$ iff $a{-\!\!<}\, b \leq d$ iff $a{-\!\!<}\, b \leq a \to b \vee d$ iff $(a{-\!\!<}\, b, a \to b)_{12} \bot_2 (c, d)$. Hence $(a{-\!\!<}\, b, a \to b)$ is a bi-Heyting point, and it is immediate that $(a, b) \leq_{12} (a{-\!\!<}\, b, a \to b)$. Therefore the bi-Heyting points of $\alpha(L)$ are dense. □

2.5.3 Subdirect Product Representation of bi-HAs

We are now in a position to prove our main result about complete bi-Heyting algebras.

Recall first that if $\{B_i\}_{i \in I}$ is a family of complete lattices, then a complete lattice A is a *subdirect product* of $\{B_i\}_{i \in I}$ if there is an injective homomorphism $e : A \to \Pi_{i \in I} B_i$ such that for any $i \in I$, $\pi_i \circ e$ is surjective.

We start by defining a *maximal* point of a boset $\mathscr{X}$ as a maximal point in the 1-and-2 ordering, that is a point $x \in \mathscr{X}$ such that for any $y \in \mathscr{X}$, $y \geq_{12} x$ implies that $y = x$. If $\mathscr{X}$ is a distributive b-frame, then maximal points in $\mathscr{X}$ correspond to very specific pairs of elements of the dual lattice:

Lemma 2.5.12. *Let L be a complete distributive lattice and $(c, d) \in \alpha(L)$. The following are equivalent:*

1. *(c, d) is maximal;*

2. *(c, d) is a splitting pair of L;*

3. *c is completely join prime, d is completely meet-prime, $d = \bigvee\{f \in L \mid c \not\leq f\}$ and $c = \bigwedge\{e \in L \mid e \not\leq d\}$;*

4. *for any* $(a, b) \in \alpha(L)$*, if* $(c, d) \leq_1 (a, b)$*, then* $(a, b) \leq_2 (c, d)$*, and if* $(c, d) \leq_2 (a, b)$*, then* $(a, b) \leq_1 (c, d)$*.*

Proof.

$1 \Rightarrow 2$ Suppose that there is some $k \in L$ such that $c \not\leq k$ and $k \not\leq d$. Since L is distributive and $c \not\leq d$, either $c \wedge k \not\leq d$ or $c \not\leq k \vee d$. Either way, we have a pair $(c', d') \neq (c, d)$ such that $(c, d) \leq_{12} (c', d')$, contradicting maximality.

$2 \Rightarrow 3$ The equivalence between 2 and 3 is well known [192]. We include the argument for the left-to-right direction for the sake of completeness. Let $F \subseteq L$. If $c \not\leq f$ for all $f \in F$, then, since (c, d) is a splitting pair, $f \leq d$ for all $f \in F$, from which it follows that $\bigvee F \leq d$ and therefore $c \not\leq \bigvee F$. Similarly, if $f \not\leq d$ for all $f \in F$, then $c \leq f$ for all $f \in F$, hence $c \leq \bigwedge F$ and therefore $\bigwedge F \not\leq d$. Thus c and d are completely join-prime and completely meet-prime respectively. Finally, note that for any $f \in L$, $f \leq d$ iff $c \not\leq f$, from which it follows that $\bigvee\{f \mid c \not\leq f\} \leq d$ and $c \leq \bigwedge\{e \mid e \not\leq d\}$. Since $c \not\leq d$, we conclude that $\bigvee\{f \mid c \not\leq f\} = d$ and $c = \bigwedge\{e \mid e \not\leq d\}$.

$3 \Rightarrow 4$ Suppose that $(c, d) \leq_1 (a, b)$. Then $c \not\leq b$, hence $b \leq d$, which implies that $(a, b) \leq_2 (c, d)$. Similarly, if $(c, d) \leq_2 (a, b)$, then $a \not\leq d$, hence $c \leq a$, and $(a, b) \leq_1 (c, d)$.

$4 \Rightarrow 1$ Suppose $(c, d) \leq_{12} (a, b)$. Then since 4 holds we have that $(a, b) \leq_{12} (c, d)$, so $c = a$ and $b = d$. □

The next lemma relates maximal points in separative bosets and anti-algebraic locales:

Lemma 2.5.13. *Let* $\mathscr{X}$ *be a separative boset with no maximal point. Then* $\rho(\mathscr{X})$ *is anti-algebraic.*

Proof. We show that there are no splitting pairs in $\rho(\mathscr{X})$. Let $U \not\subseteq V \in \rho(\mathscr{X})$, and suppose $x \in U \cap \neg_2 V$. Since x is not a maximal point, there is $y \geq_{12} x$ such that $y \not\leq_1 x$ or $y \not\leq_2 x$. We distinguish two cases:

- $y \not\leq_1 x$: By separativity $\uparrow_1 y = U^y \in \rho(\mathscr{X})$, and since $x \notin \uparrow_1 y$ and $y \in \neg_2 V$, we have that $U \not\subseteq U^y$ and $U^y \not\subseteq V$.
- $y \not\leq_2 x$: By separativity there is $z \in \mathscr{X}$ such that $x_2 \perp_1 z$ and $\neg y_2 \perp_1 z$, which implies that $z \in V_y \setminus V$, so that $V_y \not\subseteq V$. On the other hand, since $y \in U$, we have that $U \not\subseteq V_y$.

Hence (U, V) is not a splitting pair. But this in turn implies that U is not completely join-prime and therefore that L is anti-algebraic. □

We can now prove the main theorem of this section. As will become clear below, we are considering bi-Heyting algebras as complete lattices in the category **cLat**, meaning that the morphisms considered here need not preserve the Heyting or co-Heyting implication.

Theorem 2.5.14. *Let* L *be a complete bi-Heyting algebra. Then* L *is a subdirect product of* $L_1 \times L_2$ *in* **cLat***, where* L_1 *is superalgebraic and* L_2 *is anti-algebraic.*

Proof. Let $\mathscr{X}$ be the subframe of $\alpha(L)$ induced by the set of all maximal points in $\alpha(L)$, and let $\mathscr{Y}$ be the subframe of $\alpha(L)$ induced by the set of all bi-Heyting points $y \in \alpha(L)$ such that for any $z \geq_{12} y$, z is not a maximal point. Note that, by duality, it is enough to show that there are embeddings $\nu_1 : \mathscr{X} \to \alpha(L)$ and $\nu_2 : \mathscr{Y} \to \alpha(L)$ such that the induced b-morphism $\nu : \mathscr{X} \sqcup \mathscr{Y} \to \alpha(L)$ is dense, since this will imply that $\rho(\nu) : L \to \rho(\mathscr{X}) \times \rho(\mathscr{Y})$ is injective and that $\rho(\nu_1) = \rho(\nu) \circ \rho(\lambda_1) : L \to \rho(\mathscr{X})$ and $\rho(\nu_2) = \rho(\nu) \circ \rho(\lambda_2) : L \to \rho(\mathscr{Y})$ are surjective.

- For any $(c, d) \in \mathscr{X}$, define $\nu_1(c, d) = (c, d)$. We claim that $\nu_1 : \mathscr{X} \to \alpha(L)$ is an embedding. Monotonicity is clear. If $\nu_1(c, d) \leq_1 (a, b)$ for some $(a, b) \in \alpha(L)$, then, since (c, d) is maximal, we have that $(a, b) \leq_2 \nu_1(c, d)$, which means that ν_1 satisfies condition 2 of a b-morphism. Similarly, if $\nu_1(c, d) \leq_2 (a, b)$, we have that $(a, b) \leq_1 \nu_1(c, d)$, and thus ν_1 is a b-morphism. Finally, to see that it is an embedding, suppose that $\neg \nu_1(c, d)_2 \bot_1 \nu_1(c', d')$. Then there is some $(a, b) \in \alpha(L)$ such that $(c, d) \leq_2 (a, b)$ and $(c', d') \leq_1 (a, b)$. But this in turn implies that $(c', d') \leq_1 (a, b) \leq_1 (c, d)$, so $\neg(c, d)_2 \bot_1 (c', d')$.

- For any $(a, b) \in \mathscr{Y}$, define $\nu_2(a, b) = (a, b)$. We claim that $\nu_2 : \mathscr{X} \to \alpha(L)$ is an embedding. Once again, monotonicity is clear. To see that ν_2 satisfies conditions 2 and 3 of b-morphism, note first that for any $(a, b) \in \mathscr{Y}$ and any bi-Heyting $(a', b') \in \alpha(L)$, if $(a, b) \leq_{12} (a', b')$, then $(a', b') \in \mathscr{Y}$. Now fix some $(a, b) \in \mathscr{Y}$ and assume that $\nu_2(a, b) \leq_1 (c, d)$ for some $(c, d) \in \alpha(L)$. Since (a, b) is bi-Heyting, we have that $(a, b) \leq_{12} (a', b')$ for some bi-Heyting $(a', b') \geq_2 (c, d)$. But then $(a', b') \in \mathscr{Y}$, which shows that ν_2 satisfies property 2. Similarly, assume that $\nu_2(a, b) \leq_2 (c, d)$ for some $(c, d) \in \alpha(L)$. Since (a, b) is bi-Heyting, we have some bi-Heyting $(a', b') \geq_{12} (a, b)$ such that $(a', b') \geq_1 (c, d)$. But then $(a', b') \in \mathscr{Y}$, so ν_2 satisfies property 3 of a b-morphism. Finally, to see that ν_2 is an embedding, assume $\neg \nu_2(a, b)_2 \bot_1 \nu_2(a', b')$ for some $(a, b), (a', b') \in \mathscr{Y}$. Then there is some $(c, d) \in \alpha(L)$ such that $(a, b) \leq_2 (c, d)$ and $(a', b') \leq_1 (c, d)$. As (a, b) is bi-Heyting, there is some bi-Heyting point $(a^*, b^*) \geq_{12} (a, b)$ such that $(c, d) \leq_1 (a^*, b^*)$. But then $(a', b') \leq_1 (a^*, b^*)$ and $(a^*, b^*) \in \mathscr{Y}$, which implies that $\neg(a, b)_2 \bot_1 (a', b')$.

- Finally, by the universal property of the coproduct, the map $\nu : \mathscr{X} \sqcup \mathscr{Y} \to \alpha(L)$, defined by $\nu(a, b) = (a, b)$ for any $(a, b) \in \mathscr{X} \sqcup \mathscr{Y}$, is a b-morphism. Moreover, we claim that it is dense. Suppose $(a, b) \in \alpha(L)$. There are two possible cases:

  - $(a, b) \leq_{12} (c, d)$, for some maximal point (c, d). Then $(c, d) \in \mathscr{X}$.
  - $(a, b) \not\leq_{12} (c, d)$ for any maximal point (c, d). Then since L is a bi-Heyting algebra, $(a, b) \leq_{12} (a', b')$ for some bi-Heyting point (a', b') such that $(a', b') \not\leq_{12} (c, d)$ for any maximal point (c, d), which implies that $(a', b') \in \mathscr{Y}$.

  Hence for any $(a, b) \in \alpha(L)$ there is some $(c, d) \in \mathscr{X} \sqcup \mathscr{Y}$ such that $(a, b) \leq_{12} \nu(c, d)$, and hence ν is dense.

Thus, in **cLat**, L is a subdirect product of $\rho(\mathscr{X})$ and $\rho(\mathscr{Y})$. It remains to be shown that $\mathscr{X}$ is superalgebraic and that $\mathscr{Y}$ is anti-algebraic.

- Since all points in $\mathscr{X}$ are maximal, they are also splitting points: if $(c, d) \leq_1 (c_1, d_1)$ and $(c, d) \leq_2 (c_2, d_2)$, for some $(c, d), (c_1, d_1), (c_2, d_2) \in \mathscr{X}$, then $(c_1, d_1) \leq_2 (c, d)$ and $(c_2, d_2) \leq_1 (c, d)$, and thus $\neg(c_1, d_1)_2\bot_1(c_2, d_2)$. Hence $\rho(\mathscr{X})$ is superalgebraic.
- Clearly, by construction, $\mathscr{Y}$ has no maximal points. So it is enough to show that $\mathscr{Y}$ is separative in order to establish that $\rho(\mathscr{Y})$ is anti-algebraic. Suppose $(a, b) \not\leq_1 (a', b')$ for some $(a, b), (a', b') \in \mathscr{Y}$. Then since $\alpha(L)$ is separative, there is some $(a'', b'') \geq_1 (a', b')$ such that $(a'', b'')_2\bot_1(a, b)$. Since (a', b') is bi-Heyting, there is some $(a^*, b^*) \geq_{12} (a', b')$ such that $(a^*, b^*) \geq_2 (a'', b'')$. But then $(a^*, b^*) \in \mathscr{Y}$ and $(a^*, b^*)_2\bot_1(a, b)$. This shows that $\mathscr{Y}$ is 1-separative. The argument for 2-separativity is completely similar. Thus $\mathscr{Y}$ is separative and has no maximal points, from which it follows that $\rho(\mathscr{Y})$ is anti-algebraic.

This completes the proof of the theorem. □

Let us conclude this section with some remarks on the theorem obtained in this section. First, the proof of this theorem does not simply rely on the Allwein-MacCaull representation of complete lattices, but requires the full power of b-frame duality. Moreover, the main idea of the proof uses the fact that bosets can be "split" in a fairly simple way, because they are discrete structures.[8]

Furthermore, it is worth emphasizing that this result only holds in **cLat**, i.e., the morphisms under consideration here are complete lattice homomorphisms and not Heyting or bi-Heyting homomorphisms. Indeed, as was pointed out by an anonymous referee, the following is an example of a subdirectly irreducible complete bi-Heyting algebra that is neither superalgebraic nor anti-algebraic:

Example 2.5.15. Consider the chain $A = \mathbb{N} \oplus [0, 1] \oplus \top$, where $\mathbb{N}$ and $[0, 1]$ have the usual order. Every element in $\mathbb{N}$ is completely join-prime, while no element of $[0, 1]$ is completely join-prime. Moreover, A is a complete bi-Heyting algebra with a second least and a second greatest element, which means that it is subdirectly irreducible in the category of bi-Heyting algebras and bi-Heyting homomorphisms. But clearly, A is neither superalgebraic nor anti-algebraic and thus cannot be written as a subdirect product of a superalgebraic and an anti-algebraic locales.

However, the standard decomposition result about Boolean algebras follows directly from Theorem 2.5.14, once one recalls that **cBA** is a full subcategory of **cLat** and that join-prime generated elements in Boolean algebras coincide with co-atoms.

[8]I thank an anonymous referee for pointing out that one can also follow a similar strategy and prove this result using the more standard techniques of Priestley and Esakia duality.

Finally, while the definition of anti-algebraic locales does not seem to appear anywhere in the literature, it is arguably a straightforward generalization of the notion of a complete atomless Boolean algebra. Moreover, existentially-closed Heyting algebras (in the sense of model theory) have recently been axiomatized by Darnière in [64] as those Heyting algebras A satisfying the two "strong order" axioms of Density and Splitting, as well as a countable set of formulas expressing the fact that the complete theory of A eliminates quantifiers. Since, as is well known [56, p. 194], atomless Boolean algebras are precisely the existentially-closed Boolean algebras, one may wonder whether the anti-algebraic locales we define here satisfy Darnière's axioms. For now, we leave this as an open problem and move on to discussing applications of bosets to the semantics of intuitionistic logic.

2.6 Semantics for IPC

In this final section, we outline some applications of the results obtained above to the semantics of intuitionistic propositional logic. As shown in [36], the algebraic approach to a semantics $\mathcal{S}$ for IPC associates $\mathcal{S}$ to the class $\mathcal{H}_{\mathcal{S}}$ of Heyting algebras represented by the models of $\mathcal{S}$. Given two semantics $\mathcal{S}$ and $\mathcal{S}'$, $\mathcal{S}$ is *more general* than $\mathcal{S}'$ (denoted $\mathcal{S}' \leq \mathcal{S}$) whenever every Heyting algebra in $\mathcal{H}_{\mathcal{S}'}$ is isomorphic to a Heyting algebra in $\mathcal{H}_{\mathcal{S}}$. Under this ordering, it can be shown that Kripke semantics is strictly less general than topological semantics, which is itself strictly less general than nuclear semantics such as Dragalin [76] and Fairtlough-Mendler [85] semantics. Indeed, the Heyting algebras that arise as the upset of a Kripke frame are precisely superalgebraic locales and those arising as open sets of a topological space are spatial locales. In nuclear semantics, a nucleus is defined on the upset of a poset $(P, \leq)$, for example by endowing this poset with a function $D : P \to \mathscr{P}(\mathscr{P}(P))$ satisfying certain conditions (as is done in Dragalin semantics), or by adding a second ordering $\preccurlyeq$ on P such that $\preccurlyeq \subseteq \leq$ (as is the case in FM semantics). Formulas of IPC are then evaluated as upsets of $(P, \leq)$ that are also fixpoints of the nucleus thus defined. Building on a result of Dragalin [76, pp. 75-76], Bezhanishvili and Holliday [37] proved that any locale arises as the fixpoints of such a nucleus and that both Dragalin and FM-semantics are as general a semantics for intuitionistic logic as locale semantics.

This semantic hierarchy is particularly relevant for the study of the incompleteness phenomenon for intermediate logics. Indeed, if $\mathcal{S}' \leq \mathcal{S}$, then every $\mathcal{S}'$-complete intermediate logic is also $\mathcal{S}$-complete, but the converse may fail to be true. However, in contrast with the situation in modal logic [137], little is known about Kripke, topological or locale incompleteness for intermediate logics. One possible explanation for this phenomenon is the fact that IPC is a much less expressive language than modal propositional logic. Moreover, the standard representation theorems that underlie each of these semantics do not fit neatly in a hierarchy that immediately witnesses the increase of generality between them. Dragalin's representation of any locale as the fixpoints of a nuclear algebra does not restrict to the $\Omega - pt$ representation of spatial locales of pointfree topology, which itself does not restrict to the de Jongh-Troelstra representation of superalgebraic locales.

Our goal in this section is to provide a uniform framework for comparing Kripke, topological and nuclear semantics for intuitionistic logic. We first show how Heyting bosets can be used to provide a semantics for IPC that is as general as nuclear semantics and thus equivalent to FM and Dragalin semantics. We then show how the characterizations of spatial and superalgebraic locales obtained in Section 2.4 allow us to restrict boset semantics to semantics that are equivalent to topological and Kripke semantics. Finally, our main result is a strengthening of one of the only known results regarding Kripke incompleteness of intermediate logics. Using boset semantics, we show that a logic shown in [239] to be Kripke incomplete is in fact incomplete with respect to all complete bi-Heyting algebras. As mentioned in Section 2.1, a similar result has recently been obtained independently by Bezhanishvili, Gabelaia and Jibladze [35], using Esakia duality.

2.6.1 Boset Semantics

As is standard, we let Var be a countable set of propositional variables and Fml be the set of all formulas of IPC over this set of propositional variables and proceed to define valuations inductively. However, it is useful to define a relation of refutation of a formula at a point, on top of the usual definition of satisfaction. Refutation systems for propositional and modal logic have a long history [112], going back to Łukasiewicz [173]. Refutation relations have also recently been used in the context of generalized Kripke semantics for non-classical logics [57, 62, 101, 120, 123]. The introduction of such a relation alongside a satisfaction relation is motivated by the two-sorted nature of these generalized Kripke frames, itself a consequence of the underlying representation of complete lattices via polarity relations of [125] mentioned in Section 2.2.5.

Definition 2.6.1. A *boset model* for IPC is a structure $(X, \leq_1, \leq_2, V)$ in which the underlying domain $\mathscr{X} = (X, \leq_1, \leq_2)$ is a Heyting boset and V is a map from Var to $\rho(\mathscr{X})$.

Definition 2.6.2. Let $(\mathscr{X}, V)$ be a boset model. We define the relations $\Vdash^+$ (*satisfaction*) and $\Vdash^-$ (*refutation*) on $X \times Fml$ inductively as follows:

- $x \Vdash^+ p$ iff $x \in V(p)$;
- $x \Vdash^- p$ iff $x \in \neg_2 V(p)$;
- $x \Vdash^+ \varphi \wedge \psi$ iff $x \Vdash^+ \varphi$ and $x \Vdash^+ \psi$;
- $x \Vdash^- \varphi \wedge \psi$ iff $\forall y \geq_2 x \exists z \geq_1 y : z \Vdash^- \varphi$ or $z \Vdash^- \psi$;
- $x \Vdash^+ \varphi \vee \psi$ iff $\forall y \geq_1 x \exists z \geq_2 y : z \Vdash^+ \varphi$ or $z \Vdash^+ \psi$;
- $x \Vdash^- \varphi \vee \psi$ iff $x \Vdash^- \varphi$ and $x \Vdash^- \psi$;
- $x \Vdash^+ \varphi \to \psi$ iff $\forall y \geq_1 x : y \Vdash^+ \varphi$ implies $y \Vdash^+ \psi$;
- $x \Vdash^- \varphi \to \psi$ iff $\forall y \geq_2 x \exists z \geq_1 y : y \Vdash^+ \varphi$ and $y \Vdash^- \psi$.

For any formula φ, we write the sets $\{x \in X : x \Vdash^+ \varphi\}$ and $\{x \in X : x \Vdash^- \varphi\}$ as $V^+(\varphi)$ and $V^-(\varphi)$ respectively.

This definition ensures that the semantic value of any formula is always a regular open set. Indeed, a simple induction on the complexity of formulas establishes the following:

Lemma 2.6.3. *For any formula φ:*

- $V^-(\varphi) = \neg_2(V^+(\varphi))$, *and* $V^+(\varphi) = \neg_1(V^-(\varphi))$;
- $\neg_1\neg_2 V^+(\varphi) = V^+(\varphi)$ *and* $\neg_2\neg_1 V^-(\varphi) = V^-(\varphi)$.
- V^+ *is a homomorphism from the Lindenbaum-Tarski algebra of IPC into $\rho(\mathscr{X})$.*

Next, we define validity in the standard way:

Definition 2.6.4. Let $\mathscr{X}$ be a Heyting boset. A formula φ is valid on a boset model $(\mathscr{X}, V)$ if $V^+(\varphi) = X$, and φ is valid on $\mathscr{X}$ if it is valid on $(\mathscr{X}, V)$ for any valuation V.

This allows for the following soundness and completeness theorem:

Theorem 2.6.5. *IPC is sound and complete with respect to boset semantics. Moreover, boset semantics is as general as FM and Dragalin semantics.*

Proof. Soundness follows directly from Lemma 2.6.3. For completeness, note first that a model in any semantics for IPC is characterized by a HA-homomorphism from the free Heyting algebra on countably many generators (also called the Lindenbaum-Tarski algebra of IPC) into a Heyting algebra. Thus any isomorphism between Heyting algebras also induces an isomorphism between corresponding models. By Lemma 2.2.8, any complete Heyting algebra can be represented as the regular opens of some Heyting boset. Since the Lindenbaum-Tarski algebra of IPC embeds into its MacNeille completion, the completeness of IPC with respect to boset semantics follows. Moreover, since the regular opens of any Heyting boset always form a cHA, and any cHA can also be represented as the cHA of fixpoints of an FM or Dragalin frame (see [36]), it follows that boset semantics is as general as FM and Dragalin semantics. □

Let us conclude by remarking that the dual b-frame of a locale L is closely related to the *canonical FM-frame* introduced in [36, Def. 4.32], since the latter can be obtained from the former by defining the second ordering as the intersection of the two orderings on $\alpha(L)$. The regular open sets of an FM-frame $(X, \leq, \preccurlyeq)$ are guaranteed to form a complete Heyting algebra because of the requirement that $\preccurlyeq$ be a subrelation of $\leq$. As discussed in Section 2.3.2, this condition is not necessary for the regular opens of a boset to be a complete Heyting algebra, unlike the characterization presented in Lemma 2.3.12.

2.6.2 Spatial and Splitting semantics

Bosets semantics provides a uniform framework for semantics for IPC. Indeed, now that we have established that boset semantics is as general a semantics based on complete lattices as possible, we can also use our characterization of spatial and superalgebraic locales to define more stringent semantics which are easily seen to be equivalent to topological and Kripke semantics respectively.

Definition 2.6.6. Let $(\mathscr{X}, V)$ be a boset model.

- $(\mathscr{X}, V)$ is a *spatial model* if for any $x \in X$ and any formulas φ, ψ, $x \Vdash^- \varphi \wedge \psi$ iff $x \Vdash^- \varphi$ or $\Vdash^- \psi$.
- $(\mathscr{X}, V)$ is a *splitting model* if for any $x \in X$ and any formula φ, $x \Vdash^+ \varphi$ or $x \Vdash^- \varphi$.

Note that every splitting model is also spatial: suppose $(\mathscr{X}, V)$ is splitting and let $x \in \mathscr{X}$ and φ, ψ be two formulas. Then if $x \nVdash^- \varphi$ and $x \nVdash^- \psi$, this implies that $x \Vdash^+ \varphi$ and $x \Vdash^+ \psi$, and thus $x \Vdash^+ \varphi \wedge \psi$.

Next, we show how spatial and splitting models relate to spatial and splitting points in a boset:

Lemma 2.6.7. *Let $\mathscr{X}$ be a Heyting boset.*

1. *A point x in $\mathscr{X}$ is spatial iff for any boset model $(\mathscr{X}, V)$ and any formulas φ and ψ, $x \Vdash^- \varphi \wedge \psi$ iff $x \Vdash^- \varphi$ or $\Vdash^- \psi$.*
2. *A point x in $\mathscr{X}$ is splitting iff for any boset model $(\mathscr{X}, V)$ and any formula φ, either $x \Vdash^+ \varphi$ or $x \Vdash^- \varphi$.*

Proof.

1. For the left-to-right direction, assume $x \nVdash^- \varphi$ and $x \nVdash^- \psi$. Then we have $y_1, y_2 \geq_2 x$ such that $y_1 \Vdash^+ \varphi$ and $y_2 \Vdash^+ \psi$. If x is spatial, we can complete the diagram with a point $z \geq_2 x$ such that $z \geq_1 y_1, y_2$. But this implies that $z \Vdash^+ \varphi$ and $z \Vdash^+ \psi$, so $x \nVdash^- \varphi \wedge \psi$. Thus $x \Vdash^- \varphi \wedge \psi$ implies that $x \Vdash^- \varphi$ or $x \Vdash^- \psi$, and the converse direction is always true.
For the right-to-left direction, suppose x is not spatial and we have $y_1, y_2 \geq_2 x$ such that for any $z \geq_1 y_1, y_2$, $z \ngeq_2 x$. Let $V(p) = U^{y_1}$ and $V(q) = U^{y_2}$. Then $y_1 \Vdash^+ p$ and $y_2 \Vdash^+ q$, which means that $x \nVdash^- p$ and $x \nVdash^- q$. On the other hand, since $\mathscr{X}$ is Heyting, we have that $V^+(p \wedge q) = U^{y_1} \cap U^{y_2} = \neg_1\neg_2(\uparrow_1 y_1) \cap \neg_1\neg_2(\uparrow_1 y_2) = \neg_1\neg_2(\uparrow_1 y_1 \cap \uparrow_1 y_2)$. Since $x \in \neg_2(\uparrow_1 y_1 \cap \uparrow_1 y_2)$, this implies that $x \Vdash^- p \wedge q$.
2. For the left-to-right direction, assume $x \nVdash^+ \varphi$ and $x \nVdash^- \varphi$. Then there are $y_1 \geq_1 x$ and $y_2 \geq_2 x$ such that $y_1 \Vdash^- \varphi$ and $y_2 \Vdash^+ \varphi$. But then, if $z \geq_2 y_1$ and $z \geq_1 y_2$, we have that $z \Vdash^+ \varphi$ and $z \Vdash^- \varphi$, a contradiction. Thus, by contraposition, if x is a splitting point, we have that $x \Vdash^+ \varphi$ or $x \Vdash^- \varphi$.

Conversely, assume x is not a splitting point and let $y_1 \geq_1 x$, $y_2 \geq_2 x$ such that $y_1\, {}_2{\perp}_1 y_2$. Define $V(p) = U^{y_2}$. Then clearly $y_2 \in V^+(p)$ and $y_1 \in V^-(p)$, which in turn implies that $x \nVdash^- \varphi$ and $x \nVdash^+ \varphi$. □

Recall that, as we have shown in Section 2.4, any superalgebraic locale is isomorphic to a boset in which all points are splitting, and any spatial locale is isomorphic to a boset in which all points are spatial. Together with the previous result, this implies the following corollary:

Corollary 2.6.8.

1. *An intermediate logic L is Kripke complete iff it is complete with respect to a class of Heyting bosets $\mathfrak{C}$ such that for any $\mathscr{X} \in \mathfrak{C}$, any model $(\mathscr{X}, V)$ is splitting.*

2. *An intermediate logic L is topologically complete iff it is complete with respect to a class of Heyting bosets $\mathfrak{C}$ such that for any $\mathscr{X} \in \mathfrak{C}$, any model $(\mathscr{X}, V)$ is spatial.*

Finally, let us conclude by showing how spatial and splitting models can be respectively turned into topological and Kripke models on the same set:

Lemma 2.6.9. *Let $\mathscr{X}$ be a Heyting boset and V a spatial valuation on $\mathscr{X}$. Then there is a topology τ on $\mathscr{X}$ and a topological valuation V^* such that for any $x \in \mathscr{X}$, $x, V \nVdash^- \varphi$ iff $x, V^* \vDash \varphi$.*

Proof. Let $(\mathscr{X}, V)$ be a spatial model, and let τ be generated by the sets $\{[\varphi] \mid \varphi \in Fml\}$, where for any formula φ, $[\varphi] = \{x \in X \mid x \nVdash^- \varphi\}$. Note that for any φ, ψ, we have that

$$[\varphi \wedge \psi] = \{x \in X \mid x \nVdash^- \varphi \wedge \psi\} = \{x \in X \mid x \nVdash^- \varphi \text{ and } x \nVdash^- \psi\} = [\varphi] \wedge [\psi],$$

where the third equality holds because $(\mathscr{X}, V)$ is a spatial model. This implies that the sets of the form $[\varphi]$ form a basis for τ. Similarly, we have that

$$[\varphi \vee \psi] = \{x \in X \mid x \nVdash^- \varphi \vee \psi\} = \{x \in X \mid x \nVdash^- \varphi \text{ or } x \nVdash^- \psi\} = [\varphi] \cup [\psi].$$

Moreover, we claim that $[\varphi \to \psi] = I_\tau(-[\varphi] \cup [\psi])$. Assume first that $x \in [\varphi \to \psi]$. Then, $x \nVdash^- \varphi \to \psi$. Now if $x \nVdash^- \varphi$, we have that $x \nVdash^- \varphi \wedge (\varphi \to \psi)$, which implies that $x \nVdash^- \psi$. Thus $x \in -[\varphi] \cup [\psi]$. Conversely, assume $x \in I_\tau(-[\varphi] \cup [\psi])$. Since sets of the form $[\varphi]$ form a basis for τ, this means that $x \in [\chi]$ for some formula χ such that $[\chi] \subseteq -[\varphi] \cup [\psi]$. Now suppose that $x \Vdash^- \varphi \to \psi$. Since $x \nVdash^- \chi$, there is $y \geq_2 x$ such that $y \Vdash^+ \chi$ and $y \Vdash^- \varphi \to \psi$. Thus there is $z \geq_1 y$ such that $z \Vdash^+ \chi$, $z \Vdash^+ \varphi$ and $z \Vdash^- \psi$. But then $z \in [\chi] \cap ([\varphi] - [\psi])$, a contradiction. Thus $x \in [\varphi \to \psi]$, which establishes that $[\cdot]$ defines a valuation V^* on (X, τ). Clearly, for any $x \in \mathscr{X}$, we have that $x, V \nVdash^- \varphi$ iff $x, V^* \vDash \varphi$ for any formula φ. □

Lemma 2.6.10. *Let $(X, \leq_1, \leq_2, V)$ be a splitting model. Then there is a Kripke valuation V^* on $(X, \leq_1)$ such that for any $x \in X$ and any formula φ, $x, V \Vdash^+ \varphi$ iff $x, V^* \vDash \varphi$.*

Proof. Let $(X, \leq_1, \leq_2, V)$ be a splitting model, and consider the Kripke model $(X, \leq_1, V^*)$, where for any formula φ, $V^*(\varphi) = \{x \in X \mid x \Vdash^+ \varphi\}$. Note that it is enough to show that V^* is a well-defined valuation in order to complete the proof. To see this, note that for any $x \in X$ and any formulas φ, ψ, we have that $x \in V^*(\varphi \wedge \psi)$ iff $x \Vdash^+ \varphi \wedge \psi$ iff $x \Vdash^+ \varphi$ and $x \Vdash^+ \psi$ iff $x \in V^*(\varphi) \cap V^*(\psi)$. Similarly, we have that $V^*(\varphi \to \psi) = -{\downarrow}(V^*(\varphi) - V^*(\psi))$. Finally, since $(X, \leq_1, \leq_2, V)$ is splitting, for any x, φ and ψ, we have that

$$\begin{aligned} x \Vdash^+ \varphi \vee \psi &\text{ iff } x \nVdash^- \varphi \vee \psi \\ &\text{ iff } x \nVdash^- \varphi \text{ or } x \nVdash^- \varphi \\ &\text{ iff } x \Vdash^+ \varphi \text{ or } x \Vdash^+ \psi. \end{aligned}$$

But this implies at once that for any formulas φ, ψ, $V^*(\varphi \vee \psi) = V^*(\varphi) \cup V^*(\psi)$, which completes the proof. □

Dragalin [76] showed that the open sets of any topological space are isomorphic to the fixpoint of some Dragalin frame. Similarly, Kripke [160] showed that the upset of any poset are isomorphic to the fixpoints of a certain kind of nuclear frame known as a Beth frame (see [36] for more details on Beth semantics). The previous two lemmas can be seen as relating nuclear semantics to Kripke and topological semantics in a similar fashion, although there are two notable differences. First, Dragalin's and Kripke's results in the literature go from less general to more general semantics, while Lemmas 2.6.9 and 2.6.10 go from boset models to Kripke and topological semantics, so from a more general semantics to less general ones. Moreover, while the results mentioned above show how to turn Heyting algebras arising from some semantics into Heyting algebras arising from another one, our results are in some sense more fine-grained, as they show how to turn valuations into valuations, i.e., how to turn Heyting homomorphisms from the Lindenbaum-Tarski algebra of IPC into a complete Heyting algebra into Heyting homomorphisms that arise as valuations in some alternative semantics.

2.6.3 Complete Bi-Heyting algebras and the Shehtman Logic

Finally, we conclude this section with a generalization of an important result in the literature on intermediate logics. Consider the following inference rule schema, which we call *Litak's Rule*, where ϵ is some uniform substitution:

$$\frac{\begin{array}{c}(1)\ \ (\psi \vee (\psi \to \epsilon(\chi))) \to \chi \\ (2)\ \ \psi \leftrightarrow (\sigma \to \tau) \\ (3)\ \ (\sigma \vee \tau) \to \epsilon(\sigma) \wedge \epsilon(\tau) \\ (4)\ \ \chi \leftrightarrow (\psi \vee \epsilon(\tau))\end{array}}{\chi}$$

Proofs of (variants of) the following theorems can be found in [239] and [172]:

Theorem 2.6.11. *Let L be an intermediate logic in which Litak's Rule is not admissible. Then for every class $\mathfrak{C}$ of Kripke frames adequate for L, there is a frame $F \in \mathfrak{C}$ and a point in F which refutes the Gabbay-de Jongh bounded branching axiom bb_2.*[9]

Theorem 2.6.12. *There exists an intermediate logic SL such that $SL \vdash bb_2$ and Litak's rule R is not admissible in SL.*

As a corollary, the Shehtman logic SL is Kripke-incomplete. We strengthen this result as follows:

Theorem 2.6.13. *The Shehtman logic SL is incomplete with respect to all complete bi-Heyting algebras.*

This is established via the following generalization of Theorem 2.6.11:

Theorem 2.6.14. *Let L be an intermediate logic in which Litak's Rule is not admissible. Then for every class $\mathfrak{C}$ of b-frames dual to complete bi-Heyting algebras, if $\mathfrak{C}$ is adequate for L, then there is a b-frame $\mathscr{X} \in \mathfrak{C}$ such that the Gabbay-de Jongh bounded branching axiom bb_2 is refuted at some point in $\mathscr{X}$.*

The proof will take several lemmas. Suppose first that $\mathfrak{C}$ is a class of b-frames dual to a bi-Heyting locale, and notice that this implies that for any $\mathscr{X} \in \mathfrak{X}$, the bi-Heyting points of $\mathscr{X}$ are dense. Assume that $\mathfrak{C}$ is adequate for L. Then since L is valid on any b-frame in $\mathfrak{C}$, the following holds:

Lemma 2.6.15. *Let $\mathscr{X} \in \mathfrak{C}$ and V be a valuation on $\mathscr{X}$. The following are true for any $x \in X$ and $n \in \omega$:*

1. *$x, V \Vdash^+ \epsilon^n(\sigma \vee \tau)$ implies $x, V \Vdash^+ \epsilon^i(\sigma) \wedge \epsilon^j(\tau)$ for all $i, j \geq n$;*
2. *$x, V \Vdash^+ \epsilon^n(\chi)$ implies $x, V \Vdash^+ \epsilon^i(\chi)$ for any $i \leq n$; moreover, $x, V \Vdash^+ \epsilon^n(\psi)$ implies $x, V \Vdash^+ \epsilon^i(\chi)$ for all $i \leq n$.*
3. *$x, V \Vdash^+ \epsilon^n(\sigma)$ implies $x, V \Vdash^+ \epsilon^m(\chi)$ for all $m \in \omega$;*
4. *$x, V \Vdash^+ \epsilon^n(\sigma)$ and $x, V \Vdash^- \epsilon^n(\tau)$ together imply that $x, V \Vdash^- \epsilon^n(\psi)$.*
5. *$x, V \Vdash^- \epsilon^n(\chi)$ implies that there exists $y, z \geq_1 x$ such that $y, V \Vdash^+ \epsilon^n(\sigma)$, $y, V \Vdash^- \epsilon^n(\tau)$, $z, V \Vdash^+ \epsilon^n(\psi)$ and $z, V \Vdash^- \epsilon^{n+1}(\chi)$.*

Proof.

1. By a repeated use of axiom (3).
2. By a repeated use of axiom (1).

[9] See below for the definition of bb_2.

3. Fix $n, m \in \omega$, and let $k = max\{n, m\}$. By 1 above, $x \Vdash^+ \epsilon^n(\sigma)$ implies $x \Vdash^+ \epsilon^{k+1}(\tau)$. By axiom (4), this in turn implies that $x \Vdash^+ \epsilon^k(\chi)$. But then from 2 above it follows that $x \Vdash^+ \epsilon^m(\chi)$.

4. Assume $y \geq_2 x$. Then $y \Vdash^+ \epsilon^n(\sigma)$ and $y \nVdash^+ \epsilon^n(\tau)$, from which it follows that $y \nVdash^+ \epsilon^n(\psi)$. Hence $x \Vdash^- \epsilon^n(\psi)$.

5. Assume $x \Vdash^- \epsilon^n(\chi)$. Then, by axiom (1), $x \Vdash^- \epsilon^n(\psi)$, hence, (by axiom (2)) we have $x \Vdash^- \epsilon^n(\sigma) \to \epsilon^n(\tau)$ and $x \Vdash^- \epsilon^n(\psi) \to \epsilon^{n+1}(\chi))$. This means that there is $y \geq_1 x$ such that $y \Vdash^+ \epsilon^n(\sigma)$ and $y \Vdash^- \epsilon^n(\tau)$, and there is $z \geq_1 x$ such that $z \Vdash^+ \epsilon^n(\psi)$ and $z \Vdash^- \epsilon^{n+1}(\chi)$. □

Now since Litak's Rule is not admissible in L, there is some $\mathscr{X} = (X, \leq_1, \leq_2) \in \mathfrak{C}$ and a valuation V on $\mathscr{X}$ such that all the premises of Litak's rule are true at all points in X and there is $x \in X$ such that $x \Vdash^- \chi$.

In what follows, for $i \in \{0, 1, 2\}$, we write $i+1$ and $i+2$ for $i+1$ and $i+2$ mod 3 respectively. Recall that bb_2 is the axiom:

$$\bigwedge_{i \in \{0,1,2\}} (p_i \to (p_{i+1} \vee p_{i+2}) \to (p_{i+1} \vee p_{i+2})) \to (p_0 \vee p_1 \vee p_2).$$

Definition 2.6.16. For every $n < \omega$, let $S_n = \bigcap_{m \neq n < \omega} V(\epsilon^m(\psi)) - V(\epsilon^n(\psi))$.

It is easy to see that for any $n < \omega$, $\bigcap_{m<\omega} V(\epsilon^m(\psi)) \cup S_n$ is a $\leq_1$-upset (this is because if $x \leq_1 y$ and $x \in S_n$ for some $n < \omega$, then since $y \in V(\epsilon^n(\psi)) \cup (X - V(\epsilon^n(\psi)))$, we have that either $y \in \bigcap_{n<\omega} V(\epsilon^n(\psi))$ or $y \in S_n$).

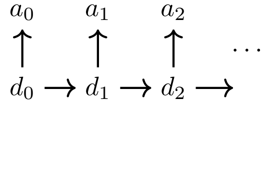
Now let p_0, p_1, p_2 be three fresh propositional variables and define a valuation V' as follows:

- $V'(q) = V(q)$ for any propositional variable $q \in L_{IPC}$ such that $q \neq p_i$ for $i \in \{0, 1, 2\}$;
- $V'(p_i) = \neg_1\neg_2(\bigcap_{n<\omega} V(\epsilon^n(\psi)) \cup \bigcup_{n<\omega} S_{3n+i})$ for $i \in \{0, 1, 2\}$.

We will now need to prove three lemmas that will give the key to the proof. The general idea is the following: We first prove that any x that refutes χ must also refute the disjunction $p_0 \vee p_1 \vee p_2$. We then show that any point that refutes one of the antecedent of bb_2 must be the root of an analogue of the Beth comb[10] in the setting of b-frames. Finally, showing that the teeth of such a Beth comb must satisfy precisely one of $\{p_0, p_1, p_2\}$ will imply, by contradiction, that x must also satisfy the antecedent of bb_2.

We start with the refutation of the consequent of bb_2.

[10] Recall that the Beth comb is the set $\{a_n\}_{n \in \omega} \cup \{d_n\}_{n \in \omega}$ endowed with the following structure:



Lemma 2.6.17. *For all $x \in X$, if $x \Vdash^- \chi$, then $x \Vdash^- p_i$ for $i \in \{0,1,2\}$, which implies that $x \Vdash^- p_0 \vee (p_1 \vee p_2)$.*

Proof. Assume $x \Vdash^- \chi$, and let $y \geq_2 x$. We claim that $y \notin V'(p_i)$ for $i \in \{0,1,2\}$. To see this, let $z \geq_2 y$. Note first that since $z \Vdash^- \chi$, we must have that $z \Vdash^- \epsilon^n(\psi)$ for any $n < \omega$ by Lemma 2.6.15.2. But this implies that for all $z \geq_2 y$, $z \notin \bigcap_{n<\omega} V(\epsilon^n(\psi)) \cup \bigcup_{n<\omega} S_{3n+i}$. Hence $y \notin V'(p_i)$ for $i \in \{0,1,2\}$. From this it follows that $x \Vdash^- p_i$ for $i \in \{0,1,2\}$, and hence $x \Vdash^- p_0 \vee (p_1 \vee p_2)$. □

Let us now move on to the lemmas that will be used to prove that x must also satisfy the antecedent of bb_2.

Lemma 2.6.18. *For any $x \in X$, if $x \Vdash^- p_0 \vee (p_1 \vee p_2)$, then there is $n < \omega$ such that $x \nVdash^+ \epsilon^n(\chi)$.*

Proof. Assume $x_0 \Vdash^+ \epsilon^n(\chi)$ for all $n < \omega$, and $x_0 \Vdash^- p_0 \vee (p_1 \vee p_2)$. Let $x \geq_{12} x_0$ be a bi-Heyting point, and note that this implies that $x \Vdash^+ \epsilon^n(\chi)$ for all $n < \omega$, and $x \Vdash^- p_0 \vee (p_1 \vee p_2)$. This implies that $x \nVdash^+ \epsilon^n(\psi)$ for some $n < \omega$, for otherwise $x \in \bigcap_{n<\omega} V(\epsilon^n(\psi))$, which means that $x \Vdash^+ p_i$ for all $i \in \{0,1,2\}$ (since $\bigcap_{n<\omega} V(\epsilon^n(\psi))$ is a $\leq_1$-upset). Let j be the smallest number n such that $x \nVdash^+ \epsilon^n(\psi)$. Then there is some $y_0 \geq_1 x$ such that $y_0 \Vdash^- \epsilon^j(\psi)$, and since x is bi-Heyting, $y_0 \leq_2 y$ for some bi-Heyting point $y \geq_{12} x$. Now since $x \Vdash^+ \epsilon^j(\chi)$, we also have that $y \Vdash^+ \epsilon^j(\chi)$, so by axiom (4) we have that $y \Vdash^+ \epsilon^j(\psi) \vee \epsilon^{j+1}(\tau)$. But this implies that there is $z_0 \geq_2 y$ such that $z \Vdash^+ \epsilon^{j+1}(\tau)$, and since y is a bi-Heyting point, this implies that there is some $z \geq_1 z_0$ such that $y \leq_{12} z$. Thus $z \Vdash^+ \epsilon^{j+1}(\tau)$, and hence, since j is the smallest number n such that $x_0 \nVdash^+ \epsilon^n(\psi)$, we have that $z \in \bigcap_{n<\omega} V(\epsilon^n(\psi)) \cup S_j$, and therefore $z \Vdash^+ p_i$ for $i = j \ mod\ 3$. But this contradicts the fact that $x \Vdash^- p_0 \vee (p_1 \vee p_2)$, since $x \leq_{12} y \leq_{12} z$. Therefore for any $x \in X$, if $x \Vdash^- p_0 \vee (p_1 \vee p_2)$, then there is $n < \omega$ such that $x \nVdash^+ \epsilon^n(\chi)$. □

The previous lemma used the fact that the bi-Heyting points of $\mathscr{X}$ are dense. It is straightfoward to verify that this is the only place where this fact is used in the proof of Theorem 2.6.14.

Lemma 2.6.19. *For all $x \in X$, if there is $n \in \omega$ such that $x \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi) \wedge \epsilon^n(\sigma)$ and $x \Vdash^- \epsilon^n(\tau)$, then $x \Vdash^+ p_i$ and $x \Vdash^- p_{i+1} \vee p_{i+2}$ for $i \in \{0,1,2\}$ such that $n = i \ mod\ 3$.*

Proof. Assume $x \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi) \wedge \epsilon^n(\sigma)$ and $x \Vdash^- \epsilon^n(\tau)$. Note that this implies that $x \in S_n$, and therefore $x \Vdash^+ p_i$ for $i = n \ mod\ 3$. Moreover, let $y \geq_2 x$. Then we have that $y \nVdash^+ \epsilon^n(\psi)$, and therefore $y \notin \bigcap_{m<\omega} V(\epsilon^m(\psi)) \cup S_k$ for any $k \neq n$. Hence $y \nVdash^+ p_j$ for any $j \neq i \in \{0,1,2\}$. Hence $x \Vdash^- p_j$ for $j \neq i \in \{0,1,2\}$. □

We have now gathered all the ingredients for the proof of Theorem 2.6.14:

Proof. Recall that there is $x \in X$ such that $x \Vdash^- \chi$. We will prove that axiom bb_2 is refuted at x, i.e., we prove that $x \Vdash^+ (p_i \to (p_{i+1} \vee p_{i+2})) \to (p_{i+1} \vee p_{i+2})$ for all $i \in \{0,1,2\}$ and that $x \Vdash^- p_0 \vee (p_1 \vee p_2)$. To see this, note first that the latter follows immediately from

Lemma 2.6.17. Moreover, assume that for some $i \in \{0,1,2\}$, $x \nVdash^+ (p_i \to (p_{i+1} \vee p_{i+2})) \to (p_{i+1} \vee p_{i+2})$. This means that there is $y \geq_1 x$ such that $y \Vdash^+ p_i \to (p_{i+1} \vee p_{i+2})$ and $y \Vdash^- p_{i+1} \vee p_{i+2}$. Note that this implies that $y \Vdash^- p_0 \vee (p_{i+1} \vee p_{i+2})$. Now by Lemma 2.6.18, this means that $y \nVdash^+ \epsilon^n(\chi)$ for some $n \in \omega$ and hence that there is $z \geq_1 y$ such that $z \Vdash^- \epsilon^n(\chi)$. Let n be the smallest number such that $z \Vdash^- \epsilon^n(\chi)$. This means that $z \Vdash^+ \bigwedge_{j<n}(\epsilon^j(\psi) \vee \epsilon^{j+1}(\tau))$. Now since $z \Vdash^+ \epsilon^j(\tau) \to \epsilon^n(\tau)$ for any $j \leq n$, this implies that $z \Vdash^+ \bigwedge_{j<n}(\epsilon^j(\psi) \vee \epsilon^n(\tau))$, i.e., $z \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi) \vee \epsilon^n(\tau)$. This means that there is $z' \geq_2 z$ such that $z' \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi)$ or $z' \Vdash^+ \epsilon^n(\tau)$. But the latter is impossible, since $z \Vdash^- \epsilon^n(\chi)$. Hence $z' \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi)$. But then, by repeated use of Lemma 2.6.15.5, there must be $z'' \geq_1 z' \geq_1 z \geq_1 y$ such that $z'' \Vdash^+ \bigwedge_{j<m} \epsilon^j(\psi) \wedge \epsilon^m \sigma$ and $z'' \Vdash^- \epsilon^m(\tau)$ for $m \geq n$ such that $m = i \ mod\ 3$. By Lemma 2.6.19, this implies that $z'' \Vdash^+ p_i$ and $z'' \Vdash^- p_{i+1} \vee p_{i+2}$, contradicting the fact that $y \Vdash^+ p_i \to (p_{i+1} \vee p_{i+2})$. Hence $x \Vdash^+ (p_i \to (p_{i+1} \vee p_{i+2})) \to (p_{i+1} \vee p_{i+2})$ for all $i \in \{0,1,2\}$, which completes the proof that x refutes bb_2. □

A similar example of an intermediate logic that is incomplete with respect to complete bi-Heyting algebras has recently and independently been obtained by G. Bezhanishvili, D. Gabelaia and M. Jibladze in [35]. It is worth mentioning that the proof presented here is but a minor variation on Litak's proof for Kripke incompleteness, while the proof in [35] requires a significantly different argument. This fact can be seen as an additional reason to believe that boset semantics might offer a generalization of Kripke semantics that still retains many of its attractive features. We should also note that it seems unlikely that the same proof could be generalized any further. Indeed, from an algebraic perspective, the proof appears to be exploiting in a key way the fact that complete bi-Heyting algebras satisfy the Meet Infinite Distributive Law (i.e., arbitrary meets distribute over finite joins). Since complete bi-Heyting algebras are the largest class of cHA's satisfying this law, this can be seen as evidence that we have pushed Shehtman's method to its limits and that new ideas might be needed in order to construct, if at all possible, topologically incomplete logics.

Conclusion

We conclude by outlining some areas for further research.

First of all, we have only presented preliminary results regarding a correspondence theory between lattice equations and b-frame properties. While we have been able to isolate first-order conditions on b-frames that are equivalent to various properties of complete lattices, we are still lacking a systematic procedure for translating lattice equations into b-frame conditions, akin to Sahlqvist correspondence in modal logic.

Moreover, although we focused in our applications on certain classes of complete Heyting algebras, the adjunction we presented holds for all complete lattices. This means in particular that one could use bosets in the study of some categories of enriched lattices, including for example ortholattices, residuated lattices, or lattices expanded with various modal operators. In that respect, the connection with polarity-based semantics for non-classical logics developed in [57, 62, 101, 120, 123] should be explored further.

Finally, the dualities developed here are all discrete dualities between complete lattices and relational structures. This means that we decided to trade off the ability to deal with incomplete lattices for a greater simplicity of the geometric structures we work with. A natural next step would therefore be to topologize the duality presented here and to connect such a generalization both to the Dunn-Hartonas duality for bounded lattices [124, 125] and to the choice-free duality recently developed in [41]. We will now enter the realm of choice-free topological dualities, and our journey there will eventually take us back to this issue in Chapter 4. But first, let us remain within the safer walls of distributivity.

Chapter 3

Constructive Dualities beyond the Boolean Case

3.1 Introduction

Stone's [245] representation of Boolean algebras as clopen sets of compact, Hausdorff and zero-dimensional topological spaces has had a profound influence on the study of interactions between logic, algebra and topology. The realization that some properties of topological spaces could be retrieved by considering the algebraic properties of their lattices of open sets led to the development of *pointfree* topology [146, 147, 207], in which open sets are taken as basic elements of a *frame* rather than defined as sets of points. Stone's representation theorem, and therefore Stone duality, relies on the Boolean Prime Ideal Theorem (BPI), a fragment of the Axiom of Choice. By contrast, the pointfree approach has a more constructive flavor: even in the absence of the Axiom of Choice, the open set functor Ω mapping a topological space to its lattice of open sets has an adjoint functor *pt*, mapping a frame to its set of "points" endowed with a natural Stone-like topology. But the restriction of this adjunction to Stone spaces and compact zero-dimensional frames is only a duality under (BPI). In [41], a choice-free duality between Boolean algebras and a category of UV-spaces has been developed. It is based on the simple but powerful idea that the appeal to (BPI) could be eliminated by working with a partially-ordered set of filters rather than a set of ultrafilters and by viewing these filters as partial approximations of a classical point. This approach has strong ties to both possibility semantics in modal logic [135, 134, 137] and the Vietoris functor on Stone spaces [258] and provides a *semi-pointfree* approach, i.e., both spatial *and* choice-free, to the representation of algebraic objects in *semi-constructive mathematics*, i.e., mathematics carried out in $ZF + DC$ [186, 230].

In [69], de Vries generalized Stone duality to a duality between de Vries algebras (complete Boolean algebras equipped with a *subordination* relation) and compact Hausdorff spaces. Just like Stone, de Vries used (BPI) in his representation of complete compingent algebras as the regular open sets of a compact Hausdorff space. On the pointfree side, Isbell [142] showed that the Ω-*pt* adjunction restricts to a duality between compact Hausdorff spaces and compact regular frames, also under the assumption of (BPI). This leaves open the question of whether a choice-free duality between these algebraic categories and a category of topological spaces can be defined. Similarly, we discussed in Chapter 1 extensions of Stone duality to distributive lattices via spectral and Priestley spaces. Since both generalizations rely on ideas very similar to Stone's original result, one may again wonder whether the main techniques of the UV-duality transfer to the wider setting of distributive lattices.

In this chapter, we show that choice-free dualities for de Vries algebras and distributive lattices can indeed be achieved by generalizing the approach of [41]. For de Vries algebras, we work with a poset of filters rather than with a set of maximal filters, and we define our dual spaces both in terms of their topological properties and in terms of order-theoretic aspects of the induced specialization order. We also show how the spaces we define naturally relate to the Vietoris functor on compact Hausdorff spaces and compact regular frames. For distributive lattices, we work with a special kind of pairs of filters and ideals. We present two choice-free dualities for distributive lattices, one inspired from Priestley duality, and one inspired from a variant of Priestley's result that uses a category of bitopological spaces called *pairwise Stone* spaces. In both cases, we are able to connect our choice-free duality to the standard dualities for **DL** via Upper Vietoris constructions. We take this as evidence of the naturality and fruitfulness of this semi-pointfree approach, in which the basic "points" of our spaces coincide with the closed sets of the standard, non-constructive duality.

The chapter is organized as follows. After reviewing some background on de Vries algebras, compact regular frames, and several categories of topological spaces (Section Section 3.2), we start with de Vries algebras. In Section 3.3, we provide a choice-free duality for de Vries algebras via a category of ordered topological spaces which we call dV-spaces. In Section Section 3.4, we connect our duality to pointfree topology and provide an alternative characterization of dV-spaces via the Vietoris functor on compact regular frames, before listing two straightforward applications of this duality. We then move on to obtaining choice-free dualities for distributive lattices. In Section 3.5, we present a representation theorem for distributive lattices by bitopologizing a set of "relatively maximal" filter-ideal pairs. Those spaces are then axiomatized as *pairwise UV-spaces*, which allows the choice-free representation of distributive lattices to be lifted to a full choice-free duality in Section 3.6. In Section 3.7, we take a slightly different approach and derive a choice-free version of Priestley duality via a certain kind of ordered topological spaces which we call *UVP spaces*. Finally, pairwise UV-spaces and UV Priestley spaces are related in Section 3.8 to the Vietoris functor on pairwise Stone and Priestley spaces respectively, establishing once again that Vietoris constructions provide a canonical way of bridging the gap between constructive and non-constructive dualities.

3.2 Background

In this section, we briefly recall the de Vries and Isbell dualities for compact Hausdorff spaces as well as the choice-free Stone duality between Boolean algebras and UV-spaces presented in [41]. Because they will also play a role later on, we also recall a variation of Priestley duality via pairwise Stone spaces [39] and Moshier and Jipsen's duality for lattices via HMS spaces [197]. We start by fixing some notation that we will use throughout the chapter. Let L be a complete lattice and (X, τ) be a topological space.

1. When no confusion arises, we write $\leq$ to designate the order on L. We designate the meet and join operations on L by $\wedge$ and $\vee$ respectively, and, whenever L is pseudo-complemented, we use $\neg$ for the pseudo-complement operation.

2. We will designate (a subset of) the set of all maximal filters on L by X_L and (a subset of) the set of all filters on L by S_L.

3. By a Stone-like topology on a set Y of filters of L, we mean the topology generated by the sets of the form $\widehat{a} = \{F \in Y \mid a \in F\}$, and we will often designate such a topology by σ. We will also sometimes use the notation $\breve{a}$ to denote sets of the form $\{F \in Y \mid a \notin F\}$. Moreover, when we are considering pairs (F, I) of a filter F and an ideal I on L rather than just filters, we will use the notation a^+ and a^- for the sets $\{(F, I) \mid a \in F\}$ and $\{(F, I) \mid a \in I\}$ respectively.

4. For any $U \subseteq X$, we write $-U$ for $X \setminus U$, $\overline{U}$ for the closure of U and $U^\perp$ for $-\overline{U}$. We write $\mathsf{CO}(\mathscr{X})$ for the set of compact open subsets of X and $\mathsf{RO}(\mathscr{X})$ for the Boolean algebra of regular open subsets of X, i.e., subsets U such that $U^{\perp\perp} = U$.

5. The *specialization preorder* on (X, τ) is represented by the symbol $\leq$ when no confusion arises, and it is defined as $x \leq y$ iff $x \in U$ implies $y \in U$ for every $U \in \tau$.

6. The *upset topology* on X is the topology generated by the set of all upward closed subsets in the specialization preorder. Given $U \subseteq X$, we let $\uparrow U$ be the interior of U in the upset topology, $\downarrow U$ the closure of U, and $\neg U$ the set $-\downarrow U$. We write $\mathcal{RO}(\mathscr{X})$ for the Boolean algebra of order-regular open subsets of X, i.e., subsets U such that $\uparrow\downarrow U = U$, and $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ for $\mathsf{CO}(\mathscr{X}) \cap \mathcal{RO}(\mathscr{X})$.

7. An *ordered topological space* is a tuple $(X, \tau, \preccurlyeq)$ such that (X, τ) is a topological space and $(X, \preccurlyeq)$ is a partial order. We write $Up(\mathscr{X})$ and $Dn(\mathscr{X})$ for the collections of upsets and downsets of $(X, \preccurlyeq)$.

8. Given an ordered topological space $(X, \tau, \preccurlyeq)$, an *open filter* is a subset $U \subseteq X$ such that U is open in (X, τ) and a filter on the poset $(X, \preccurlyeq)$.

9. Finally, given an ordered topological space $\mathscr{X} = (X, \tau, \preccurlyeq)$, we let $\mathsf{OF}(\mathscr{X})$ be the collection of open filters on $\mathscr{X}$, $\mathsf{COF}(\mathscr{X}) = \mathsf{CO}(\mathscr{X}) \cap \mathsf{OF}(\mathscr{X})$, and $\mathsf{CO}\mathcal{RO}\mathsf{F}(\mathscr{X}) = \mathsf{CO}\mathcal{RO}(\mathscr{X}) \cap \mathsf{OF}(\mathscr{X})$.

3.2.1 De Vries Algebras

De Vries algebras were introduced in [69] as an algebraic dual to compact Hausdorff spaces.

Definition 3.2.1. A *compingent algebra* is a pair $(B, \prec)$ such that B is a Boolean algebra with induced order $\leq$, and $\prec$ is a relation on $B \times B$ satisfying the following set of axioms:

(A1) $1 \prec 1$;

(A2) $a \prec b$ implies $a \leq b$;

(A3) $a \leq b \prec c \leq d$ implies $a \prec d$;

(A4) $a \prec b$ and $a \prec c$ together imply $a \prec b \wedge c$;

(A5) $a \prec b$ implies $\neg b \prec \neg a$;

(A6) $a \prec c$ implies that there is $b \in B$ such that $a \prec b \prec c$;

(A7) $a \neq 0$ implies that there is $b \neq 0 \in B$ such that $b \prec a$.

A *de Vries algebra* is a compingent algebra $V = (B, \prec)$ such that B is a complete Boolean algebra. It is *zero-dimensional* if for any $a \prec b \in V$ there is $c \in V$ such that $a \prec c \prec c \prec b$.

Compingent algebras constitute a specific kind of *contact algebras*, Boolean algebras equipped with a binary relation of subordination satisfying (A1)-(A5). One motivation for contact algebras is to develop a region-based theory of space [73, 166], according to which regions of space form a Boolean algebra and a region a is subordinated to a region b precisely if b completely surrounds a. For more on contact and subordination algebras, we refer the reader to [38, 40, 72, 86].

Definition 3.2.2. Let $V = (B, \prec)$ be a de Vries algebra. For any filter F on B, let $\twoheaduparrow F = \{a \in F \mid \exists b \in F : b \prec a\}$. A *concordant filter* on V is a filter F such that $\twoheaduparrow F = F$. An *end* is a maximal concordant filter.

The dual space of a de Vries algebra V is obtained by taking the set X_V of all ends of V and endowing it with the Stone-like topology σ generated by all sets of the form $\{p \in X_V \mid a \in p\}$ for some $p \in V$. Conversely, the dual de Vries algebra of a compact Hausdorff space (X, τ) is the complete Boolean algebra $\mathsf{RO}(\mathscr{X})$ of regular open sets, with the subordination relation $\sqsubset$ given by $U \sqsubset V$ iff $\overline{U} \subseteq V$.

Theorem 3.2.3 ([69], Thm. I.4.3-5)**.** *For any de Vries algebra $V = (B, \prec)$, (X_V, σ) is compact Hausdorff, and $(B, \prec)$ is isomorphic to $(\mathsf{RO}(X_V), \sqsubset)$. Conversely, for any compact Hausdorff space (X, τ), $(\mathsf{RO}(\mathscr{X}), \sqsubset)$ is a de Vries algebra, and (X, τ) is homeomorphic to $(X_{(\mathsf{RO}(\mathscr{X}), \sqsubset)}, \sigma)$.*

We now introduce the relevant notion of morphism between de Vries algebras.

Definition 3.2.4. Let $V_1 = (B_1, \prec_1)$ and $V_2 = (B_2, \prec_2)$ be de Vries algebras. A de Vries morphism from V_1 to V_2 is a function $h : B_1 \to B_2$ satisfying the following set of conditions:

(V1) $h(0) = 0$;

(V2) $h(a \wedge b) = h(a) \wedge h(b)$;

(V3) $a \prec_1 b$ implies $\neg h(\neg a) \prec_2 h(b)$;

(V4) $h(a) = \bigvee\{h(b) \mid b \prec_1 a\}$.

Given two de Vries morphisms $h : V_1 \to V_2$ and $k : V_2 \to V_3$, their composition $k \star h : V_1 \to V_3$ is defined as the map $a \mapsto \bigvee\{kh(b) : b \prec_1 a\}$.

One easily verifies that de Vries morphisms preserve both the order $\leq$ and the subordination relation $\prec$. Given a de Vries morphism $h : V_1 \to V_2$, the map $h^* : X_{V_2} \to X_{V_1}$ given by $h^*(p) = {\Uparrow} h^{-1}[p]$ for any end p on V_2 is a continuous function. Conversely, for any continuous function $f : (X_1, \tau_1) \to (X_2, \tau_2)$, the map $f_* : \mathsf{RO}(\mathscr{X}) \to \mathsf{RO}(\mathscr{Y})$ given by $f_*(U) = (f^{-1}[U])^{\perp\perp}$ for any regular open set U is a de Vries morphism. This allowed de Vries to obtain the following:

Theorem 3.2.5. *The category* **deV** *of de Vries algebras and de Vries morphisms between them is dually equivalent to the category* **KHaus** *of compact Hausdorff spaces and continuous maps between them.*

3.2.2 Compact Regular Frames

Recall that a *frame* is a complete lattice L that satisfies the join-infinite distributive law, i.e., is such that $a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}$ for any $a \in L$ and $B \subseteq L$. Frames are the central object of study of pointfree topology, for which [146, 147, 207] are standard introductions. A frame L is *compact* if $1_L = \bigvee B$ for some $B \subseteq L$ implies that $1_L = \bigvee B'$ for some finite $B' \subseteq B$. A morphism between frames is a map preserving finite meets and arbitrary joins.

Definition 3.2.6. Let L be a frame and $a, b \in L$. Then a is said to be *rather below* b [207, Def. V.5.2], denoted $a \prec b$, if $b \vee \neg a = 1_L$. A *compact regular frame* is a compact frame L such that for any $a \in L$, $a = \bigvee\{b \in L \mid b \prec a\}$.

Given any topological space (X, τ), one can define its frame of open sets $\Omega(\mathscr{X})$. Conversely, given a frame L, one can define a Stone-like topology on the set of completely prime filters $pt(L)$. These constructions give rise to adjoint functors Ω and pt between the categories **Frm** of frames and frame morphisms and **Top** of topological spaces and continuous functions. Assuming (BPI), Isbell [142] showed that this adjunction restricts to a duality in the specific case of compact regular frames:

Theorem 3.2.7. *The category* **KRFrm** *of compact regular frames is dually equivalent to* **KHaus**.

As an immediate consequence of Theorems Theorem 3.2.5 and Theorem 3.2.7, the categories **deV** and **KRFrm** are equivalent. This equivalence has also been given a direct description in [31], which has the advantage of being choice-free. Given a frame L, an element $a \in L$ is *regular* if $\neg\neg a = a$. The *Booleanization* of L [7], denoted $B(L)$, is the subframe of all the regular elements of L. It is straightforward to verify that if L is a compact regular frame, $B(L)$ equipped with the rather below relation $\prec$ is a de Vries algebra. In order to go from de Vries algebras to frames, we need the following definition:

Definition 3.2.8. Let $V = (B, \prec)$ be a de Vries algebra. An ideal on B is *round* if for any $a \in I$, there is $b \in I$ such that $a \prec b$.

It is immediate to see that a proper ideal I on a de Vries algebra V is round if and only if its dual filter $I^\delta = \{\neg a \mid a \in I\}$ is concordant. Given a de Vries algebra V, its set of round ideals ordered by inclusion forms a compact regular frame $\mathfrak{R}(V)$. The equivalence between **KRFrm** and **deV** is then given by the following result:

Theorem 3.2.9. *Any compact regular frame L is isomorphic to $\mathfrak{R}(B(L))$. Conversely, any de Vries algebra V is isomorphic to $B(\mathfrak{R}(V))$, and the maps B and $\mathfrak{R}$ lift to an equivalence between* **KRFrm** *and* **deV**.

3.2.3 UV-spaces

Let us now introduce the choice-free version of Stone duality presented in [41], and mentioned already in Chapter 1.

Definition 3.2.10. A topological space (X, τ) is a *UV-space* if it satisfies the following conditions:

1. (X, τ) is compact and T_0;
2. $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ is closed under $\cap$ and $-\downarrow$ and forms a basis for τ;
3. Any filter on $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ is $\mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(\mathscr{X}) \mid x \in U\}$ for some $x \in X$.

Given a Boolean algebra B, one considers the set S_B of all filters on B, equipped with the usual Stone-like topology σ. It can then be showed without appealing to (BPI) that UV-spaces are the duals of Boolean algebras:

Theorem 3.2.11 ([41], Thm. 5.4)**.** *For any Boolean algebra B, (S_B, σ) is a UV-space, and B is isomorphic to $\mathsf{CO}\mathcal{RO}(S_B)$. Conversely, for any UV-space (X, τ), $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ is a Boolean algebra, and (X, τ) is homeomorphic to $(S_{\mathsf{CO}\mathcal{RO}(\mathscr{X})}, \sigma)$.*

Definition 3.2.12. Given two UV-spaces (X, τ_1) and (Y, τ_2) with induced specialization orders $\leq_1$ and $\leq_2$, a UV-map from (X, τ_1) to (Y, τ_2) is a spectral map $f : X \to Y$ that is also a p-morphism with respect to $\leq_1$ and $\leq_2$, i.e., for any $x \in X$, $y \in Y$, if $f(x) \leq_2 y$, then there is $x' \geq_1 x$ such that $y = f(x')$.

Any UV-map $f : (X, \tau_1) \to (Y, \tau_2)$ gives rise to a Boolean algebra homomorphism $f_* : \mathsf{CO}\mathcal{RO}(\mathscr{Y}) \to \mathsf{CO}\mathcal{RO}(\mathscr{X})$ given by $f_*(U) = f^{-1}[U]$ for any $U \in \mathcal{RO}(\mathscr{Y})$. Conversely, for any Boolean homomorphism $h : B_1 \to B_2$, the map $h^* : (S_{B_2}, \sigma_2) \to (S_{B_1}, \sigma_1)$ given by $h^*(F) = h^{-1}[F]$ for any filter F on B_2 is a UV-map. This yields the following result, which, unlike Stone duality, does not rely on the Axiom of Choice:

Theorem 3.2.13. *The category* **BA** *of Boolean algebras and Boolean homomorphisms between them is dually equivalent to the category* **UV** *of UV-spaces and UV-maps between them.*

3.2.4 Pairwise Stone Spaces and HMS Spaces

We conclude this section by introducing two dualities that will be the inspiration for the two choice-free dualities for distributive lattices presented in Sections 3.6 and 3.7. We start with pairwise Stone spaces, introduced in [39].

Definition 3.2.14. A bitopological space $\mathscr{X} = (X, \tau_1, \tau_2)$ is a *pairwise Stone space* if it satisfies the following conditions:

- $\mathscr{X}$ is *pairwise compact*, i.e., any cover of X by sets in $\tau_1 \cup \tau_2$ has a finite subcover;
- $\mathscr{X}$ is *pairwise Hausdorff*, i.e., for any $x \neq y \in X$, there are disjoints sets $U_i \in \tau_i$ and $V_j \in \tau_j$ for some $i \neq j \in \{1, 2\}$ such that $x \in U_i$ and $y \in V_j$;
- $\mathscr{X}$ is *pairwise* 0*-dimensional*, i.e., the open sets in τ_i that are also closed in τ_j form a basis for τ_i for all $i \neq j \in \{1, 2\}$.

A function between pairwise Stone spaces is *bicontinuous* if it is continuous in both topologies.

Bezhanishvili et al. show in [39] that pairwise Stone spaces coincide with Priestley spaces and spectral spaces in the following way. Recall that Priestley spaces are obtained from spectral spaces by taking the patch topology on a spectral space (X, τ), i.e., by taking the topology generated by the compact open sets in τ and their complements. Similarly, one can turn a spectral space (X, τ) into a pairwise Stone space (X, τ_1, τ_2) by defining $\tau_1 = \tau$ and τ_2 as the topology generated by the complements of the compact open sets in τ_1. Equivalently, the dual pairwise Stone space of a distributive lattice L is obtained by endowing $Spec(L)$ with the topology τ^+ generated by the sets $\{\widehat{a} \mid a \in L\}$ and the topology τ^- generated by the sets $\{\breve{a} \mid a \notin L\}$. Conversely, given a pairwise Stone space $\mathscr{X} = (X, \tau_1, \tau_2)$, its dual distributive lattice is obtained by the lattice $Cl_2Op_1(\mathscr{X})$ of sets that are both open in τ_1 and closed in τ_2. One can then establish the following.

Theorem 3.2.15 (Pairwise Stone Duality)**.** *For any distributive lattice L, the map $\widehat{\cdot} : L \to Cl_2Op_1(Spec(L))$ is an isormophism. Dually, for any pairwise Stone space (X, τ_1, τ_2), the map $\dot{\in} : \mathscr{X} \to Spec(Cl_2Op_1(\mathscr{X}))$ is a bicontinuous homeomorphism. Moreover, Cl_2Op_1 and Spec establish a dual equivalence between* **DL** *and the category* **PStone** *of pairwise Stone spaces and bicontinuous maps between them.*

Finally, we briefly mention a topological duality for meet-semilattices developed by Jipsen and Moshier in [197].

Definition 3.2.16 (Jipsen-Moshier [197])**.** A *HMS space* is a sober space (X, τ) such that $COF(\mathscr{X})$ forms a basis for τ.

Moshier and Jipsen show that for any meet-semilattice L, the set S_L of proper filters of L endowed with the Stone-like topology generated by the sets of the form $\widehat{a}$ is an HMS-space. Moreover, any HMS space $\mathscr{X} = (X, \tau)$ can be viewed as an ordered topological space $(X, \tau, \leq)$, where the order $\leq$ is the specialization preorder induced by τ. The original meet-semilattice structure of L can be retrieved by looking at the lattice $\mathsf{COF}(S_L)$ of compact open filter of S_L. When L is a lattice, however, more work is needed to recover also the joins of L in $\mathsf{COF}(\mathscr{X})$.

Moshier and Jipsen define a closure operator fsat by mapping every $U \subseteq X$ to the intersection of all F-saturated sets that contain U, where a subset of X is *F-saturated* if it is an intersection of open filters in $(X, \tau, \leq)$. They show that when L is a lattice, the fsat operation on its dual HMS-space (S_L, σ) maps open sets to open sets. The joins in L can the be retrieved in $\mathsf{COF}(S_L)$ by taking the fsat-closure of the union of two sets. Finally, Moshier and Jipsen show that the property that fsat maps open sets to open sets completely characterize the dual HMS-spaces of lattices, which they call BL-spaces. Putting things together, they obtain the following duality:

Theorem 3.2.17 ([197], Thm. 5.2 & 5.4)**.** *The category of meet-semilattices and meet-preserving homomorphisms between them is dual to the category of HMS spaces and F-continuous maps between them, where a map f between HMS spaces is F-continuous if the preimage of any compact open filter under f is a compact open filter. Moreover, this duality restricts to a duality between* **Lat** *and the category of BL-spaces and F-stable functions, where a function between BL-spaces if F-stable if $f^{-1}[\mathsf{fsat}(U)] = \mathsf{fsat}(f^{-1}[U])$ for any open set U.*

3.3 Choice-free Duality for de Vries Algebras

The goal of this section is to prove a choice-free analogue of de Vries duality. First, we provide a choice-free representation of any de Vries algebra as the regular open sets of some topological space (Section Section 3.3.1). In Section Section 3.3.2, we then characterize the choice-free duals of de Vries algebras, which we call dV-spaces. Section Section 3.3.3 deals with morphisms and ends with our main result, a choice-free dual equivalence between the category of de Vries algebras and the category of dV-spaces.

3.3.1 A Choice-free Representation for de Vries Algebras

In this section, we complete the first step of the duality by obtaining a choice-free representation of any de Vries algebra as the regular open sets of some topological space. Our approach combines the techniques of Sections Section 3.2.1 and Section 3.2.3 in a natural way.

Definition 3.3.1. Let $V = (B, \prec)$ be a de Vries algebra. The *dual filter space* of V is the topological space (S_V, σ), where:

- S_V is the set of all concordant filters on V;
- σ is the Stone-like topology generated by $\{\widehat{a} = \{F \in S_V \mid a \in F\} \mid a \in V\}$.

The following two lemmas will help us investigate the structure of the space of concordant filters on a de Vries algebra.

Lemma 3.3.2. *Let $V = (B, \prec)$ be a de Vries algebra. Then:*

1. *For any $a \neq 0$, $F = \{c \in V \mid a \prec c\}$ is a concordant filter.*
2. *If F and G are concordant filters and $c \wedge d \neq 0$ for any $c \in F, d \in G$, then the set $H = \{c \wedge d \mid c \in F, d \in G\}$ is a concordant filter.*

Proof. For part (i), by (A3), F is upward-closed, and by (A4), it is downward directed. To verify that $\Uparrow F = F$, note that if $a \prec c$, then by (A6) there is c' such that $a \prec c' \prec c$, so $c \in \Uparrow F$.

For part (ii), let $H = \{c \wedge d \mid c \in F, d \in G\}$. I claim that H is a concordant filter. It is routine to verify that H is a proper filter. To see that $\Uparrow H = H$, take $c \in F$ and $d \in G$. Since F and G are concordant there are $c' \prec c$ in F and $d' \prec d$ in G. Thus $c' \wedge d' \in H$ and $c' \wedge d' \prec c \wedge d$ by (A4), which means that $c \wedge d \in \Uparrow H$. This shows that $H \subseteq \Uparrow H$, and the converse is immediate from (A2). ☐

Lemma 3.3.3. *Let $V = (B, \prec)$ be a de Vries algebra, $a \in V$ and F a concordant filter on V. If $a \notin F$, then there is a concordant filter $G \supseteq F$ such that for any concordant filter $H \supseteq G$, $a \notin H$.*

Proof. Suppose $a \notin F$, and consider the set $G = \{c \wedge d \mid c \in F, \neg a \prec d\}$. I claim that G is a concordant filter. If $c \wedge d = 0$ for some $c \in F$ and d such that $\neg a \prec d$, then $c \leq \neg d \prec \neg\neg a = a$, which contradicts the assumption that $a \notin F$. Thus by Lemma Lemma 3.3.2 G is a concordant filter.

Now suppose H is a concordant filter such that $H \supseteq G$. If $a \in H$, then there is $d \in H$ such that $d \prec a$. But this implies that $\neg a \prec \neg d$, so $\neg d \in G \subseteq H$, a contradiction. ☐

Given a de Vries algebra V with dual space (S_V, σ), we now show that the map $a \mapsto \widehat{a}$ is a Boolean embedding of V into $\mathsf{RO}(S_V)$:

Lemma 3.3.4. *Let $V = (B, \prec)$ be a de Vries algebra with dual filter space (S_V, σ). Then for any $a, b \in V$:*

1. $\widehat{a} \cap \widehat{b} = \widehat{a \wedge b}$*;*

2. *The set* $\{\widehat{a} \mid a \in V\}$ *is a basis for* σ*, and the specialization order on* (S_V, σ) *coincides with the inclusion order;*

3. $\widehat{a} \subseteq \widehat{b}$ *iff* $a \leq b$*;*

4. $\widehat{a}^{\perp} = \widehat{\neg a}$*;*

5. $\updownarrow\downarrow\widehat{a} = \widehat{a} = \widehat{a}^{\perp\perp}$*.*

Proof. Part (i) is a consequence of the fact that the elements of S_V are filters, and part (ii) immediately follows from part (i). For part (iii), the right-to-left direction is immediate, and for the converse, since B is a Boolean algebra it is enough to show that for any $a \neq 0$, there is some concordant filter F such that $a \in F$. To see this, note that, by (A7), if $a \neq 0$ there is $b \neq 0$ such that $b \prec a$. Then $F = \{c \in V \mid b \prec c\}$ is a concordant filter by Lemma Lemma 3.3.2, and $a \in F$.

For part (iv), since the set $\{\widehat{a} \mid a \in V\}$ is a basis for σ by (ii), we have that for any $F \in S_V$, $F \in \overline{\widehat{a}}$ iff for any basic open $\widehat{b}$, $F \in \widehat{b}$ implies $\widehat{a} \cap \widehat{b} \neq \emptyset$. By (i) and (iii), this means that $F \in \overline{\widehat{a}}$ iff $b \wedge a \neq 0$ for all $b \in F$ iff $\neg a \notin F$ iff $F \notin \widehat{\neg a}$. Hence $\widehat{a}^{\perp} = \widehat{\neg a}$.

Finally, for part (v), $\widehat{a} = \widehat{a}^{\perp\perp}$ follows directly from (iv). To show that $\updownarrow\downarrow\widehat{a} = \widehat{a}$, note that the right-to-left inclusion is immediate since $\widehat{a}$ is upward-closed. Since the specialization order on (S_V, σ) coincides with the inclusion ordering, establishing the converse amounts to showing that for any $F \in S_V$, if $a \notin F$, then there is $G \supseteq F$ such that for all $H \supseteq G$, $a \notin H$. But this is precisely Lemma Lemma 3.3.3. □

Corollary 3.3.5. *Let* $V = (B, \prec)$ *be a de Vries algebra with dual filter space* (S_V, σ)*. Then* B *is isomorphic to* $\mathsf{RO}(S_V)$*.*

Proof. Lemma Lemma 3.3.4 implies that the map $a \mapsto \widehat{a}$ is an injective Boolean homomorphism of B into $\mathsf{RO}(S_V)$. Therefore it only remains to show that every regular open subset of S_V is of the form $\widehat{a}$ for some $a \in V$. Let $U = \bigcup_{a \in A} \widehat{a}$ be a regular open set. Recall that $\bigvee A \in B$ since B is a complete Boolean algebra. I claim that $\overline{U} = \overline{\widehat{\bigvee A}}$. Since U is regular open, this will readily imply that $U = \widehat{\bigvee A}$. For the proof of the claim, recall that for any $F \in S_V$, $F \in \overline{\widehat{\bigvee A}}$ iff $\neg \bigvee A \notin F$. Similarly, $F \in \overline{U}$ iff for any $b \in F$ there is $a \in A$ such that $b \nleq \neg a$. But the latter condition is equivalent to $b \nleq \bigwedge\{\neg a \mid a \in A\}$, which is in turn equivalent to $\neg \bigvee A \notin F$. Hence $F \in \overline{U}$ iff $F \in \overline{\widehat{\bigvee A}}$ for any $F \in S_V$, which means that $\overline{U} = \overline{\widehat{\bigvee A}}$. This completes the proof that B is isomorphic to $\mathsf{RO}(S_V)$. □

We now turn to representing the subordination relation on a de Vries algebra. For any topological space (X, τ) and any $U, V \subseteq X$, let $U \ll V$ iff $\overline{U} \subseteq \downarrow V$.

Lemma 3.3.6. *Let* $V = (B, \prec)$ *be a de Vries algebra with dual filter space* (S_V, σ)*. For any* $a, b \in V$*,* $a \prec b$ *iff* $\widehat{a} \ll \widehat{b}$*.*

Proof. For the first direction, suppose that $a \prec b$. Then if F is a concordant filter such that $\neg a \notin F$, by Lemma Lemma 3.3.2 $G = \{c \wedge d \mid c \in F, a \prec d\}$ is a concordant filter extending F and containing b. Now for any concordant filter F, $F \in \overline{\widehat{a}}$ iff $\neg a \notin F$. This shows that $\overline{\widehat{a}} \subseteq \downarrow\widehat{b}$. Conversely, assume that $a \not\prec b$. I claim that there is a concordant filter F such that $\neg a \notin F$ and $b \notin G$ for any concordant filter $G \supseteq F$. Let $F = \{c \wedge d \mid a \prec c, \neg b \prec d\}$. By Lemma Lemma 3.3.2, F is a concordant filter if $c \wedge d \neq 0$ for any $a \prec c$, $\neg b \prec d$. But if $c \wedge d = 0$, then $a \prec c \leq \neg d \prec \neg\neg b = b$, so $a \prec b$ by (A3), contradicting our assumption. Hence F is a concordant filter. Now if $\neg a \in F$ there must be some $e \in F$ such that $e \prec \neg a$. But this means that $a \prec \neg e$ and therefore $\neg e \in F$, a contradiction. Similarly for any concordant $G \supseteq F$, if $b \in G$ there must be some $e \in G$ such that $e \prec b$. But then $\neg b \prec \neg e$ so $\neg e \in F \subseteq G$, a contradiction. Therefore $F \in \overline{\widehat{a}} \setminus \downarrow\widehat{b}$. $\square$

Putting Corollary Corollary 3.3.5 and Lemma Lemma 3.3.6 together yields the desired representation theorem.

Theorem 3.3.7. *Let $V = (B, \prec)$ be a de Vries algebra with dual filter space (S_V, σ). Then V is isomorphic to $(\mathsf{RO}(S_V), \ll)$.*

3.3.2 De Vries Spaces

In this section, we characterize the choice-free duals of de Vries algebras. In other words, we give an axiomatization of topological spaces of the form (S_V, σ) for some de Vries algebra V. In order to do so, we first need to introduce the following separation axioms:

Definition 3.3.8.

1. A topological space (X, τ) is *order-regular* if for any closed set B and any $x \notin \updownarrow B$, there are disjoint open sets U, V such that $x \in U$ and $\updownarrow B \subseteq V$.

2. A topological space (X, τ) is *order-normal* if for any closed set A and any regular closed set B such that A is disjoint from $\updownarrow B$, there are disjoint open sets U and V such that $A \subseteq \downarrow U$ and $\updownarrow B \subseteq V$.

Order-regularity and order-normality are straightforward variations of the usual regularity and normality separation axioms in general topology. Separation axioms for ordered topological spaces have been studied in the past [190, 198, 209], but here we are concerned with a very specific kind of ordered topological spaces, in which the order is determined by the topology. In the case of compact T_1 spaces, these separation properties are essentially equivalent to Hausdorffness:

Lemma 3.3.9. *Let (X, τ) be a compact T_1-space. The following are equivalent:*

1. *(X, τ) is Hausdorff;*

2. *(X, τ) is order-regular;*

3. (X, τ) is order-normal and order-regular.

Proof. Recall that if (X, τ) is T_1, then the specialization preorder on X is just the identity relation. Thus a T_1 order-regular space is regular Hausdorff, which implies that it is also Hausdorff. As compact Hausdorff spaces are also regular, this shows that (i) and (ii) are equivalent. Moreover, (iii) clearly implies (ii), and compact Hausdorff spaces are also normal, which for T_1 spaces implies order-normality, showing that (i) implies (iii). □

As we will now see, for spaces in which the regular opens are also order-regular open, order-normality suffices to establish that they form a de Vries algebras when equipped with the relation $\ll$ defined above.

Lemma 3.3.10. *Let (X, τ) be an order-normal space such that $\mathsf{RO}(\mathscr{X}) \subseteq \mathcal{RO}(\mathscr{X})$. For any $U, V \in \mathsf{RO}(\mathscr{X})$, let $U \ll V$ iff $\overline{U} \subseteq \downarrow V$. Then $(\mathsf{RO}(\mathscr{X}), \ll)$ is a de Vries algebra.*

Proof. Since $\mathsf{RO}(\mathscr{X})$ is a complete Boolean algebra, we only need to verify axioms (A1)-(A7):

(A1) $\boldsymbol{X \ll X}$. Immediate.

(A2) $\boldsymbol{U \ll V}$ **implies** $\boldsymbol{U \subseteq V}$. Suppose $\overline{U} \subseteq \downarrow V$. Taking complements, this yields $-\downarrow V \subseteq \overline{U^{\perp}}$. Because every closed set is a downset, $\downarrow A \subseteq \overline{A}$ for any $A \subseteq X$, so $\downarrow - \downarrow V \subseteq \overline{U^{\perp}}$. Complementing again, we conclude that $U = U^{\perp\perp} \subseteq \uparrow\!\downarrow V = V$.

(A3) $\boldsymbol{U_1 \subseteq U_2 \ll V_1 \subseteq V_2}$ **implies** $\boldsymbol{U_1 \ll V_2}$. We have the following chain on inclusions: $\overline{U_1} \subseteq \overline{U_2} \subseteq \downarrow V_1 \subseteq \downarrow V_2$.

(A4) $\boldsymbol{U \ll V_1}$ **and** $\boldsymbol{U \ll V_2}$ **together imply** $\boldsymbol{U \ll V_1 \cap V_2}$. Suppose both $\overline{U} \subseteq \downarrow V_1$ and $\overline{U} \subseteq \downarrow V_2$. Then since $U, V_1, V_2 \in \mathcal{RO}(\mathscr{X})$, we have that $-\downarrow(U^{\perp}) \subseteq V_1$ and $-\downarrow(U^{\perp}) \subseteq V_2$, hence $\downarrow - \downarrow(U^{\perp}) \subseteq \downarrow(V_1 \cap V_2)$. Now since $U^{\perp} \in \mathcal{RO}(\mathscr{X})$, we have that $\downarrow - \downarrow(U^{\perp}) = -(U^{\perp}) = \overline{U}$, and therefore $\overline{U} \subseteq \downarrow(V_1 \cap V_2)$.

(A5) $\boldsymbol{U \ll V}$ **implies** $\boldsymbol{V^{\perp} \ll U^{\perp}}$. Suppose $\overline{U} \subseteq \downarrow V$. Then $\downarrow - \downarrow V \subseteq \downarrow(U^{\perp})$. Taking complements, we have $-\downarrow(U^{\perp}) \subseteq \uparrow\!\downarrow V = V$ since $V \in \mathcal{RO}(\mathscr{X})$. Now since $V \in \mathsf{RO}(\mathscr{X})$, $-V = \overline{V^{\perp}}$. Therefore, taking complements again, we have that $\overline{V^{\perp}} \subseteq \downarrow(U^{\perp})$, hence $V^{\perp} \ll U^{\perp}$.

(A6) $\boldsymbol{U \ll V}$ **implies that there is** $\boldsymbol{W}$ **such that** $\boldsymbol{U \ll W \ll V}$. Suppose $\overline{U} \subseteq \downarrow V$, and consider the set $X \setminus \downarrow V = \uparrow -V$. As $\overline{U}$ and $\uparrow -V$ are disjoint and $-V$ is regular closed, by order-normality we get some disjoint open sets W_1, W_2 such that $\overline{U} \subseteq \downarrow W_1$ and $\uparrow -V \subseteq W_2$. Note that this implies that $\overline{W_1} \cap \uparrow -V = \emptyset$, and therefore $\overline{W_1} \subseteq \downarrow V$. Letting $W = W_1^{\perp\perp}$, we have that $\overline{U} \subseteq \downarrow W_1 \subseteq \downarrow W$, and $\overline{W} = \overline{W_1} \subseteq \downarrow V$.

(A7) **If** $\boldsymbol{U \neq \emptyset}$ **then there is** $\boldsymbol{V \neq \emptyset}$ **such that** $\boldsymbol{V \ll U}$. Suppose $U \neq \emptyset$ and let $x \in U$. Consider $X \setminus \downarrow U = \uparrow -U$. Note that $\downarrow x$ is disjoint from $\uparrow -U$ and is closed, since $\downarrow x = \bigcap_{x \notin U, U \in \tau} -U$. By order-normality, we have disjoint open sets V_1 and V_2 such that $\downarrow x \subseteq \downarrow V_1$ and $\uparrow -U \subseteq V_2$. Note that this implies that $V_1 \neq \emptyset$ and that $\overline{V_1} \subseteq \downarrow U$. Now letting $V = V_1^{\perp\perp}$, it follows that $V \neq \emptyset$ and $\overline{V} = \overline{V_1} \subseteq \downarrow U$.

Thus $(\mathsf{RO}(\mathscr{X}), \ll)$ is a de Vries algebra. $\square$

We are now in a position to define the choice-free duals of de Vries algebras:

Definition 3.3.11. A *de Vries space* (dV-space for short) is a topological space (X, τ) satisfying the following conditions:

1. (X, τ) is T_0, compact and order-normal;
2. $\mathsf{RO}(\mathscr{X})$ is a basis for τ and $\mathsf{RO}(\mathscr{X}) \subseteq \mathcal{RO}(\mathscr{X})$;
3. For every $x \in X$, $\mathsf{RO}(x) = \{U \in \mathsf{RO}(\mathscr{X}) \mid x \in U\}$ is a concordant filter on $\mathsf{RO}(\mathscr{X})$, and for every filter F on $\mathsf{RO}(\mathscr{X})$, there is $x \in X$ such that $\twoheaduparrow F = \mathsf{RO}(x)$.

Lemma 3.3.12. *Let $V = (B, \prec)$ be a de Vries algebra. Then (S_V, σ) is an order-regular dV-space.*

Proof. Condition (ii) follows from Lemma Lemma 3.3.4, and condition (iii) is immediate from Theorem Theorem 3.3.7, so we only have to check that (S_V, σ) is T_0, compact, order-normal and order-regular. It is routine to verify that (S_V, σ) is T_0. For compactness, note that $\twoheaduparrow\{1\} = \{1\} \in S_V$, so if $S_V \subseteq \bigcup_{a\in A} \widehat{a}$ for some $A \subseteq V$, it follows that $1 \in A$ and thus S_V has a finite subcover.

For order-regularity, let $B = \bigcap_{a\in A} -\widehat{a}$ be a closed set and $F \notin \Uparrow B$. Then $F \in \downarrow -B = \bigcup_{a\in A} \downarrow \widehat{a}$, which means that there is $a \in A$ and $c \prec a$ such that $\neg c \notin F$. By ($A6$) there is some $c' \in V$ such that $c \prec c' \prec a$. Now $F \in -\widehat{\neg c} = \overline{\widehat{c}} \subseteq \downarrow \widehat{c'}$, and $-\widehat{\neg c'} = \overline{c'} \subseteq \downarrow \widehat{a}$, so $\Uparrow B \subseteq \widehat{\neg c'}$. Thus $\widehat{c'}$ and $\widehat{\neg c'}$ are the required open sets.

Finally, for order-normality, fix a closed set $U = \bigcap_{a\in A} -\widehat{a}$ and a regular closed set B such that $\bigcap_{a\in A} -\widehat{a} \subseteq \downarrow -B$. Because B is regular closed it is of the form $-\widehat{b}$ for some $b \in V$. Now consider the concordant filter $F = \{c \in V \mid \neg b \prec c\}$. If there is $G \supseteq F$ such that $G \in \widehat{b}$, then there must be $c \in G$ such that $c \prec b$. But then $\neg c \in F \subseteq G$, and G is not a proper filter, a contradiction. Thus $F \notin \downarrow \widehat{b}$, which means that $F \in \bigcup_{a\in A} \widehat{a}$. Hence there is some $a \in A$ and some $c \in V$ such that $\neg b \prec c \prec a$, which in turn implies that $\neg a \prec \neg c \prec b$. This implies that $-\widehat{a} = \overline{\widehat{\neg a}} \subseteq \downarrow \widehat{\neg c}$, and $-\widehat{c} = \overline{\widehat{\neg c}} \subseteq \downarrow \widehat{b}$, and therefore we have two disjoint open sets, $\widehat{\neg c}$ and $\widehat{c}$, such that $\bigcap_{a\in A} -\widehat{a} \subseteq \downarrow \widehat{\neg c}$ and $\Uparrow -\widehat{b} \subseteq \widehat{c}$. $\square$

Theorem 3.3.13. *Let (X, τ) be a dV-space. Then (X, τ) is homeomorphic to $(S_{(\mathsf{RO}(\mathscr{X}), \ll)}, \sigma)$.*

Proof. Let $f : X \to S_{(\mathsf{RO}(\mathscr{X}), \ll)}$ be given by $f(x) = \mathsf{RO}(x)$. Then f is well-defined and surjective by condition (iii), and it is injective because X is T_0. Moreover, for any $U \in \mathsf{RO}(\mathscr{X})$, we have that $x \in U$ iff $U \in \mathsf{RO}(x)$ iff $U \in f(x)$ iff $f(x) \in \widehat{U}$. By Theorem Theorem 3.3.7 and since $\mathsf{RO}(\mathscr{X})$ is a basis for X, this is enough to conclude that f is open and continuous and therefore a homeomorphism. $\square$

Note that, as a corollary to Lemma Lemma 3.3.12 and Theorem Theorem 3.3.13, we obtain that any dV-space is order-regular.

Let us conclude this section by characterizing UV-spaces as a special kind of dV-spaces. In order to do so, it is convenient to introduce first the following notion.

Definition 3.3.14. Let (X, τ) be a topological space. An open subset of (X, τ) is *well rounded* if for any closed set B such that $B \subseteq {\downarrow}U$, there are disjoint open sets V and W such that $B \subseteq {\downarrow}V$ and $-W \subseteq {\downarrow}U$.

Well-rounded subsets of a dV-space will play an important role later on when connecting our results with some standard notions of pointfree topology. For now, let us note that a topological space in which every open is well-rounded is also order-regular and order-normal. Indeed, order-normality amounts to the requirement that every regular open set be well-rounded, and order-regularity follows from the fact that ${\downarrow}x$ is closed in every topological space. While not every open subset of a dV-space is well rounded, this is true for a special class of those, namely UV-spaces.

Lemma 3.3.15. *A topological space (X, τ) is a UV-space if and only if it is a dV-space such that $(\mathsf{RO}(\mathscr{X}), \ll)$ is zero-dimensional.*

Proof. For the left-to-right direction, suppose (X, τ) is a UV-space. We may therefore view it as (S_B, σ) for some Boolean algebra B. This can be used to show that every open set (X, τ) is well-rounded. Indeed, let $U = \bigcap_{a \in A} -\widehat{a}$ and $V = \bigcap_{c \in C} -\widehat{c}$ for some A, C, subsets of B such that $\bigcap_{a \in A} -\widehat{a} \subseteq {\downarrow} \bigcup_{c \in C} \widehat{c}$. Without loss of generality, we may assume that C is a proper ideal: if $F \in {\downarrow}\widehat{c'}$ for some $c' = c_1 \vee \ldots \vee c_n$ with $c_1, ..., c_n \in C$, then there must be some $i \leq n$ such that $\neg c_i \notin F$, and therefore $F \in {\downarrow}\widehat{c_i}$. So let $F = \{\neg c \mid c \in C\}$ be the dual filter of C. Clearly $F \notin {\downarrow} \bigcup_{c \in C} \widehat{c}$, so $A \cap F \neq \emptyset$. This means that there is some $a \in A$ such that $\neg a \in C$. Thus $U \subseteq -\widehat{a} \subseteq {\downarrow}\widehat{\neg a}$, and $\overline{\widehat{\neg a}} = {\downarrow}\widehat{\neg a} \subseteq {\downarrow}{-V}$. This shows that (X, τ) satisfies condition (i).

By [41, Prop. 4.3.1], $\mathsf{RO}(\mathscr{X}) \subseteq \mathcal{RO}(\mathscr{X})$, so condition (ii) follows from condition (ii) of UV-spaces. Finally, condition (iii) follows from condition (iii) on UV-spaces once we show there is a one-to-one correspondence between concordant filters on $\mathsf{RO}(\mathscr{X})$ and proper filters on B, given by $F \mapsto \{a \in B \mid \widehat{a} \in F\}$. Recall first the observation that for any compact open set U in a UV-space, $\overline{U} = {\downarrow}U$ [41, Prop. 4.1]. This means that $\widehat{a} \ll \widehat{a}$ for any $W \in \mathsf{CO}\mathcal{RO}(\mathscr{X})$. Now assume $U \ll V$ for some $U, V \in \mathsf{RO}(\mathscr{X})$. By [41, Fact 8.2], we may write $U = \bigcup_{a \in A} \widehat{a}$ and $V = \bigcup_{c \in C} \widehat{c}$ for some ideals $A, C \subseteq B$. It is straightforward to see that $\overline{\bigcup_{a \in A} \widehat{a}} \subseteq {\downarrow} \bigcup_{c \in C} \widehat{c}$ implies that there is $c \in C$ such that $a \leq c$ for all $a \in A$, and thus that $\overline{U} \subseteq {\downarrow}\widehat{c}$ for some $c \in C$. Since $\widehat{c} \in \mathsf{CO}\mathcal{RO}(\mathscr{X})$, we also have that $\overline{\widehat{c}} \subseteq {\downarrow}\widehat{c} \subseteq {\downarrow}V$, hence $U \ll \widehat{c} \ll \widehat{c} \ll V$. This shows that $\mathsf{RO}(\mathscr{X})$ is zero-dimensional. Moreover, if F and G are distinct concordant filters on $\mathsf{RO}(\mathscr{X})$, without loss of generality there is $V \in F \setminus G$. But then there is some $U \in F$ such that $U \ll V$, hence $U \ll \widehat{c} \ll V$ for some $c \in B$. This shows that the map $F \mapsto \{c \in B \mid \widehat{c} \in F\}$ is injective. For surjectivity, note that given any proper filter G on B, $G' = \{U \in \mathsf{RO}(\mathscr{X}) \mid \exists c \in G : \widehat{c} \ll U\}$ will be a preimage of G. This completes the proof that X is a dV-space such that $\mathsf{RO}(X, \ll)$ is a zero-dimensional de Vries algebra.

Conversely, suppose that (X, τ) is a dV-space such that $(\mathsf{RO}(\mathscr{X}), \ll)$ is zero-dimensional. Let $B = \{U \in \mathsf{RO}(\mathscr{X}) \mid U \prec U\}$. Clearly, B is a Boolean algebra, so we may consider its dual UV-space $UV(B)$. Since points in X are in one-to-one correspondence with concordant filters on $\mathsf{RO}(\mathscr{X})$, by the same argument as above, there is a one-to-one correspondence between X and $UV(B)$, given by $x \mapsto \{U \in B \mid x \in U\}$. As this map is easily seen to be a homeomorphism, it follows that (X, τ) is a UV-space. □

3.3.3 Morphisms

Having established the object part of our duality, the last step to obtain our duality result is to isolate the adequate notion of morphism between dV-spaces. It turns out to be a natural generalization of UV-maps:

Definition 3.3.16. Let (X, τ_1) and (Y, τ_2) be dV-spaces, and let $\leq_1$ and $\leq_2$ be the specialization orders induced by τ_1 and τ_2 respectively. A *de Vries map* (dV-map for short) $f : X \to Y$ is a continuous function that is also weakly dense, i.e., is such that for any $x \in X$, if $f(x) \leq_2 y$ for some $y \in Y$, then there is $x' \geq_1 x$ such that $y \leq_2 f(x')$.

Let **dVS** be the category of dV-spaces and dV-maps between them. It is straightforward to verify that if $f : (X, \tau_1) \to (Y, \tau_2)$ is weakly dense, then for any upward-closed $V \subseteq Y$, ${\downarrow} f^{-1}[V] = f^{-1}[{\downarrow} V]$. This implies in particular that the preimage of any order-regular open set under a weakly dense map is order-regular open. This fact plays a role in the proof of the following lemma:

Lemma 3.3.17. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a dV-map between dV-spaces. Then $\Phi(f) : (\mathsf{RO}(\mathscr{Y}), \ll_2) \to (\mathsf{RO}(\mathscr{X}), \ll_1)$, given by $\Phi(f)(U) = (f^{-1}[U])^{\perp\perp}$ for any $U \in \mathsf{RO}(\mathscr{Y})$, is a de Vries morphism.*

Proof. We check the four conditions on de Vries morphisms in turn:

(V1) $\boldsymbol{\Phi(f)(\emptyset) = \emptyset}$. Immediate.

(V2) $\boldsymbol{\Phi(f)(U \cap V) = \Phi(f)(U) \cap \Phi(f)(V)}$. Simply compute that:

$$\begin{aligned} \Phi(f)[U] \cap \Phi(f)[V] &= (f^{-1}[U])^{\perp\perp} \cap (f^{-1}[V])^{\perp\perp} \\ &= (f^{-1}[U] \cap f^{-1}[V])^{\perp\perp} \\ &= \Phi(f)(U \cap V). \end{aligned}$$

(V3) $\boldsymbol{U \ll_2 V}$ implies $\boldsymbol{(\Phi(f)(U^{\perp}))^{\perp} \ll_1 \Phi(f)(V)}$. Suppose $\overline{U} \subseteq {\downarrow} V$. This means that $f^{-1}[\overline{U}] \subseteq f^{-1}[{\downarrow} V] = {\downarrow} f^{-1}[V]$, since f is weakly dense. Complementing, this gives us

$$-{\downarrow} f^{-1}[V] \subseteq f^{-1}[U^{\perp}] \subseteq \Phi(f)(U^{\perp}),$$

which, using the fact that $f^{-1}[V]$ is order-regular open, yields

$$-{\downarrow}(\Phi(f)(U^{\perp})) \subseteq f^{-1}[V] \subseteq \Phi(f)(V).$$

Taking order-closure and complements again, this yields

$$-{\downarrow}(\Phi(f)(V)) \subseteq \Phi(f)(U^{\perp}) = (\Phi(f)(U^{\perp}))^{\perp\perp},$$

and therefore

$$\overline{(\Phi(f)(U^{\perp}))^{\perp}} \subseteq {\downarrow}(\Phi(f)(V)).$$

(V4) $\Phi(f)(V) = (\bigcup\{\Phi(U) \mid U \ll_2 V\})^{\perp\perp}$. The right-to-left direction is immediate. For the converse, suppose that $f(x) \in V$, and let $x' \geq_1 x$. Then $f(x') \in V$, which implies that there is some $U \ll_2 V$ such that $f(x') \in {\downarrow}U$. This means that $f(x') \leq_2 y$ for some $y \in U$. Since f is weakly-dense, there is $z \geq_1 x'$ such that $f(z) \geq_2 y$, and therefore $z \in \Phi(f)(U)$. This shows that $f^{-1}[V] \subseteq (\bigcup\{\Phi(U) \mid U \ll_2 V\})^{\perp\perp}$, which clearly implies that $\Phi(f)(V) \subseteq (\bigcup\{\Phi(U) \mid U \ll_2 V\})^{\perp\perp}$.

Therefore $\Phi(f)$ is a de Vries morphism. □

It follows that we may define a contravariant functor $\Phi : \mathbf{dVS} \to \mathbf{deV}$ by letting $\Phi(X,\tau) = (\mathsf{RO}(\mathscr{X}), \ll)$ for any dV-space (X,τ) and mapping any $f : (X,\tau_1) \to (Y,\tau_2)$ to $\Phi(f)$ as in Lemma Lemma 3.3.17. It is straightforward to verify that Φ preserves composition and identity arrows. Going from de Vries algebras to dV-spaces requires the following result:

Lemma 3.3.18. *Let $h : V_1 \to V_2$ be a de Vries morphism. Then the function $\Lambda(h) : (S_{V_2}, \sigma_2) \to (S_{V_1}, \sigma_1)$, given by $\Lambda(h)(F) = {\uparrow}h^{-1}[F]$ for any $F \in S_{V_2}$, is a dV-map.*

Proof. Let us first show that $\Lambda(h)$ is continuous. For any $a \in V_1$ we compute:

$$\begin{aligned}\Lambda(h)^{-1}[\widehat{a}] &= \{F \in S_{V_2} \mid \Lambda(h)(F) \in \widehat{a}\}\\ &= \{F \in S_{V_2} \mid a \in {\uparrow}h^{-1}[F]\}\\ &= \{F \in S_{V_2} \mid \exists c \prec a : h(c) \in F\}\\ &= \bigcup_{c \prec a} \widehat{h(c)}.\end{aligned}$$

Now we check that $\Lambda(h)$ is weakly dense. Let $F \in S_{V_2}$ and $G \in S_{V_1}$ be such that ${\uparrow}h^{-1}[F] \subseteq G$. I claim that

$$H = \{a \in V_2 \mid a \geq \neg h(\neg c) \wedge d \text{ for some } c \in G, d \in F\}$$

is a concordant filter. To see that this is a proper subset of V_2, note that if $h(\neg c) \in F$ for some $c \in G$, then there is $c' \prec c \in G$, which implies that $\neg c \prec \neg c'$ and thus that $\neg c' \in G$, a contradiction. To see that H is a filter, it is enough to verify that for any $c_1, c_2 \in G$, $\neg h(\neg c_1) \wedge \neg h(\neg c_2) \in H$. Since G is concordant, there is $c' \in G$ such that $c' \prec c_1 \wedge c_2$, which implies that

$$\neg h(\neg c) \prec \neg h(\neg(c_1 \wedge c_2)) \leq \neg h(\neg c_1) \wedge \neg h(\neg c_2),$$

and therefore $\neg h(\neg c_1) \wedge \neg h(\neg c_2) \in H$. A similar argument shows that ${\uparrow}H = H$, which completes the proof of the claim.

By construction of H, $F \subseteq H$. Moreover, if $c \in G$, then there are $c_1, c_2 \in G$ such that $c_2 \prec c_1 \prec c$. Then $\neg h(\neg c_2) \prec h(c_1)$, which shows that $c \in \Lambda(h)[H]$, and therefore $G \subseteq \Lambda(h)[H]$. This completes the proof that $\Lambda(h)$ is a dV-map. □

We can therefore construct a functor $\Lambda : \mathbf{deV} \to \mathbf{dVS}$ by mapping any de Vries algebra V to $\Lambda(V) = (S_V, \sigma)$ and any de Vries morphism h to $\Lambda(h)$ as in Lemma Lemma 3.3.18. Again, it is straightforward to verify that Λ preserves composition and identity arrows. We conclude with the main result of this chapter:

Theorem 3.3.19. *The functors* Φ *and* Λ *establish a dual equivalence between the categories* **deV** *and* **dVS**.

Proof. In light of Theorems Theorem 3.3.7 and Theorem 3.3.13, we only need to verify that:

1. for any de Vries morphism $h : V_1 \to V_2$, $\Phi\Lambda(h)(\widehat{a}) = \widehat{h(a)}$ for any $a \in V_1$;
2. for any dV-map $f : (X, \tau_1) \to (Y, \tau_2)$, $\Lambda\Phi(f)(\mathsf{RO}(x)) = \mathsf{RO}(f(x))$ for any $x \in X$.

For (i), it is enough to compute that:

$$\begin{aligned}\Phi\Lambda(h)(\widehat{a}) &= ((\Lambda(h))^{-1}[\widehat{a}])^{\perp\perp}\\ &= (\bigcup\{\widehat{h(b)} \mid b \prec a\})^{\perp\perp}\\ &= \bigvee\{\widehat{h(b) \mid b} \prec a\}\\ &= \widehat{h(a)}.\end{aligned}$$

For (ii), we first compute that:

$$\begin{aligned}\Lambda\Phi(f)(\mathsf{RO}(x)) &= \Uparrow(\Phi(f))^{-1}[\mathsf{RO}(x)]\\ &= \Uparrow\{U \mid \Phi(f)(U) \in \mathsf{RO}(x)\}\\ &= \Uparrow\{U \mid (f^{-1}[U])^{\perp\perp} \in \mathsf{RO}(x)\}.\end{aligned}$$

Now if $V \in \mathsf{RO}(f(x))$, then there is $U \ll V$ such that $U \in \mathsf{RO}(f(x))$, and therefore $x \in f^{-1}[U] \subseteq (f^{-1}[U])^{\perp\perp}$, and hence $V \in \Lambda\Phi(\mathsf{RO}(x))$. For the converse direction, suppose that $x \in (f^{-1}[U])^{\perp\perp}$ and that $U \ll V$ for some $U, V \in \mathsf{RO}(\mathscr{Y})$. I claim that for any $y \geq_2 f(x)$, $y \in \overline{U}$. Since $\overline{U} \subseteq \downarrow V$, this implies that $f(x) \in \uparrow\downarrow V$, and therefore that $v \in \mathsf{RO}(f(x))$. For the proof of the claim, note first that $x \in (f^{-1}[U])^{\perp\perp}$ implies that there is some regular open set $Z \in \mathsf{RO}(x)$ such that for any $x' \in Z$ and any open set W, $x' \in Z \cap W$ implies that $W \cap f^{-1}[U] \neq \emptyset$. Now fix some $y \in Y$ such that $f(x) \leq_2 y$. Since f is weakly dense, there is $x' \geq_1 x$ such that $y \leq_2 f(x')$. The claim is proved if $f(x') \in \overline{U}$. Assume towards a contradiction that this is not the case. Then $x' \in f^{-1}[U^\perp]$, which is open since f is continuous. But $x \leq_1 x'$ implies that $x' \in Z$, so $f^{-1}[U^\perp] \cap f^{-1}[U] \neq \emptyset$, a contradiction. This completes the proof. □

3.4 An Upper Vietoris Perspective and Some Applications

Let us conclude this part on de Vries algebras with an exploration of some of the consequences of our choice-free duality. We will first connect it with the pointfree approach to compact Hausdorff spaces via compact regular frames and to the Vietoris functor on **KHaus**, before discussing two simple applications of our "semi-pointfree" duality.

3.4.1 Pointfree and Hyperspace Approaches

In this section, we relate dV-spaces to compact regular frames. Because both the equivalence between de Vries algebras and compact regular frames on the one hand, and the duality between de Vries algebras and dV-spaces on the other hand, do not rely on the Axiom of Choice, we already know that there is a choice-free duality between compact regular frames and dV-spaces. In order to describe this duality more precisely, we first need the following lemma:

Lemma 3.4.1. *For any de Vries algebra $V = (B, \prec)$, there is an order isomorphism between the poset $w\mathsf{O}\mathcal{RO}(\Lambda(V))$ of well-rounded $\mathsf{O}\mathcal{RO}$ subsets of $\Lambda(V)$ and the round ideals on V.*

Proof. Let $\mathfrak{R}(V)$ be the frame of all round ideals of V and $w\mathsf{O}\mathcal{RO}(\Lambda(V))$ the poset of all well-rounded $\mathsf{O}\mathcal{RO}$ subsets of $\Lambda(V)$ ordered by inclusion. Define $\alpha : \mathfrak{R}(V) \to w\mathsf{O}\mathcal{RO}(\Lambda(V))$ as $I \mapsto \bigcup_{b \in I} \widehat{b}$ and $\beta : w\mathsf{O}\mathcal{RO}(\Lambda(V)) \to \mathfrak{R}(V)$ as $U \mapsto \{b \in B \mid \overline{\widehat{b}} \subseteq {\downarrow}U\}$. I claim that α and β are order preserving and inverses of one another.

First, let us verify that $\alpha(I)$ is a well-rounded $\mathsf{O}\mathcal{RO}$ set for any round ideal I. Clearly, for any round ideal I, $\alpha(I)$ is open. To see that it is order-regular open, suppose $F \notin \alpha(I)$ for some concordant filter F, and consider the set $G = \{c \wedge \neg d \mid c \in F, d \in I\}$. I claim that $G \in \Lambda(V)$. Since I is round, $I^{\delta} = \{\neg d \mid d \in I\}$ is a concordant filter, so by Lemma Lemma 3.3.2 we only need to verify that $c \wedge \neg d \neq 0$ for any $c \in F, d \in I$. But this follows immediately from the assumption that $F \notin \alpha(I)$. Thus $G \in \Lambda(V)$, and clearly we have that $F \subseteq G$ and $G \notin {\downarrow}\alpha(I)$. Thus $F \notin {\updownarrow}{\downarrow}\alpha(I)$, which shows that $\alpha(I) \in \mathcal{RO}(\Lambda(V))$. Finally, let us check that $\alpha(I)$ is well-rounded. Suppose $W \subseteq {\downarrow}\alpha(I)$ is a closed set of the form $\bigcap_{a \in A} -\widehat{a}$ for some $A \subseteq B$. Note that I^{δ} is a concordant filter and clearly $I^{\delta} \notin {\downarrow}\alpha(I)$, so $A \cap I^{\delta} \neq \emptyset$. This means that $\neg a \in I$ for some $a \in A$. But then $\widehat{\neg a}$ and $\widehat{a}$ are the required open sets. This completes the proof that $\alpha(I) \in w\mathsf{O}\mathcal{RO}(\Lambda(V))$.

Conversely, let us show that for any $w\mathsf{O}\mathcal{RO}$ set U, $\beta(U)$ is a round ideal. Clearly, $\beta(U)$ is downward closed. Now suppose we have $a, b \in V$ such that $\overline{\widehat{a}}, \overline{\widehat{b}} \subseteq {\downarrow}U$. Then $\overline{\widehat{a} \cup \widehat{b}} = \overline{\widehat{a \vee b}} \subseteq {\downarrow}U$. Since U is well-rounded, there must be disjoint open sets W_1, W_2 such that $\widehat{\overline{a \cup b}} \subseteq {\downarrow}W_1$ and $-W_2 \subseteq {\downarrow}U$. By Theorem Theorem 3.3.7, $W_1^{\perp\perp} = \widehat{c}$ for some $c \in V$, and it is straightforward to verify that $\overline{\widehat{a \vee b}} \subseteq {\downarrow}\widehat{c}$ and $\overline{\widehat{c}} \subseteq {\downarrow}U$. This shows that $a \vee b \prec c$ and that $c \in \beta(U)$, establishing that $\beta(U)$ is a round ideal.

It is immediate to see that both maps are order preserving, so we only need to show that they are inverses of one another. Let I be a round ideal. If $b \in I$, then $b \prec a$ for some $a \in I$. But then $\overline{\widehat{b}} \subseteq {\downarrow}\widehat{a} \subseteq {\downarrow}\alpha(I)$, so $b \in \beta\alpha(I)$. Conversely, assume $b \notin I$, and let $F = \{c \wedge \neg d \mid b \prec c, d \in I\}$. If $c \wedge \neg d \leq \neg b$ for some $d \in I$ and c such that $b \prec c$, then $b \wedge \neg d \prec c \wedge \neg d \leq \neg b$, hence $b \wedge \neg d \leq b \wedge \neg d \wedge \neg b \leq 0$. But this implies that $b \leq d$ and thus that $b \in I$, contradicting our assumption. Thus $\neg b \notin F$. By Lemma Lemma 3.3.2, this shows that F is a concordant filter and moreover $F \in \overline{\widehat{b}}$ by Lemma Lemma 3.3.4 (iv). But clearly $F \notin {\downarrow}\alpha(I) = \bigcup_{d \in I} {\downarrow}\widehat{d}$. By contraposition, it follows that if $\overline{\widehat{b}} \subseteq {\downarrow}\alpha(I)$, then $b \in I$. This shows that $\beta\alpha(I) = I$ for any round ideal I.

Similarly, if $F \in U$ for $U \in w\mathsf{O}\mathcal{RO}(\Lambda(V))$, then since U is open there must be some $a \in F$ such that $\widehat{a} \subseteq U$. Since F is concordant, there is $b \prec a$ for some $b \in F$. But then $F \in \widehat{b}$ and $\overline{\widehat{b}} \subseteq {\downarrow}\widehat{a} \subseteq {\downarrow}U$, so $F \in \alpha\beta(U)$. Conversely, suppose $F \in \alpha\beta(U)$. Then there is $a \in F$ such that $\overline{\widehat{a}} \subseteq {\downarrow}U$. Since $\overline{\widehat{a}} = -\widehat{\neg a}$ and for any concordant $G \supseteq F$, $\neg a \notin G$, it follows that $F \in \Uparrow{\downarrow}U = U$. This shows that $\alpha\beta(U) = U$, which completes the proof. $\square$

As a consequence, the well-rounded $\mathsf{O}\mathcal{RO}$ subsets of any dV-space form a compact regular frame, and we can lift this correspondence to a functor $w\mathsf{O}\mathcal{RO} : \mathbf{deV} \to \mathbf{KRFrm}$. To go from compact regular frames to dV-spaces, it is enough to recall that the round ideals on a de Vries algebra V are precisely the duals of concordant filters on V. Thus given a compact regular frame L, we may simply define the topological space $\Xi(L) = (L^-, \delta)$, where $L^- = L \setminus 1_L$ and δ is the topology generated by sets of the form $\check{a} = \{b \mid \neg a \prec b\}$ for any $a \in L$. Indeed, since L is isomorphic to $\mathfrak{R}(B(L))$, we may think of any $b \in L$ as a round ideal I_b on the de Vries algebra $(B(L), \prec)$ such that for any $b \in L$ and $c \in B(L)$, $c \prec b$ iff $\neg c \in I_b$. But since $B(L) = \{\neg a \mid a \in L\}$, we therefore have for any $a \in L$:

$$\begin{aligned}
\check{a} &= \{b \in L^- \mid \neg a \prec b\} \\
&= \{b \in L^- \mid \neg\neg a \in I_b\} \\
&= \{b \in L^- \mid \neg a \in (I_b)^\delta\} \\
&= \{b \in L^- \mid (I_b)^\delta \in \widehat{\neg a}\}.
\end{aligned}$$

This shows that the correspondence $b \mapsto (I_b)^\delta$ is a homeomorphism between $\Xi(L)$ and $\Lambda(B(L))$. It follows that Ξ lifts to a contravariant functor from **KRFrm** to **dVS** and that we have the following theorem:

Theorem 3.4.2. *For any compact regular frame L, L is isomorphic to $w\mathsf{O}\mathcal{RO}(\Xi(L))$. Conversely, any dV-space (X, τ) is homeomorphic to $\Xi(w\mathsf{O}\mathcal{RO}(\mathscr{X}))$. Moreover, $w\mathsf{O}\mathcal{RO}$ and Ξ establish a duality between* **KRFrm** *and* **dVS**.

We may think of Theorem Theorem 3.4.2 as establishing a choice-free analogue of Isbell duality. In the presence of (BPI), any compact regular frame is spatial, meaning that any compact regular frame L is isomorphic to $\Omega(pt(L))$, or equivalently that any compact regular frame is the lattice of open sets of some compact Hausdorff space. In our choice-free case, we do not represent L as the open sets of a topological space (since doing so would imply Isbell duality), but only as the well-rounded order-regular open sets of a dV-space. We might however be interested in better understanding the relationship between the Isbell dual of a compact regular frame and its de Vries dual. The answer turns out to involve the upper Vietoris functor on compact regular frames.

Recall that the Vietoris hyperspace of a compact Hausdorff space (X, τ) is obtained by taking as points the closed subsets of X. That a Vietoris-like construction would play a role in our duality is far from surprising. De Vries had already remarked [69, Theorem I.3.12] that there was a dual order-isomorphism between the closed sets of a compact Hausdorff space and the concordant filters on its de Vries algebra of regular open sets. Moreover, assuming

(BPI), the dual UV-space of a Boolean algebra B is homeomorphic to the upper Vietoris hyperspace of the dual Stone space of B [41, Theorem 7.7]. The upper Vietoris construction can also be defined on compact regular locales [41, 146, 149]:

Definition 3.4.3. Let L be a compact regular locale. The *upper Vietoris space* of L is the topological space $UV(L) = (L^-, \tau_\Box)$, where $\tau_\Box$ is the topology generated by the sets $\Box a = \{b \in L^- \mid a \vee b = 1_L\}$ for any $a \in L$.

Lemma 3.4.4. *For any locale L, $\Xi(L)$ is homeomorphic to $UV(L)$.*

Proof. Since $\Xi(L)$ and $UV(L)$ have the same domain, it is enough to show that the two topologies coincide. For any $a \in L$:

$$\begin{aligned}\check{a} &= \{b \in L^- \mid \neg a \prec b\}\\ &= \{b \in L^- \mid \neg\neg a \vee b = 1_L\}\\ &= \Box\neg\neg a,\end{aligned}$$

which shows that $\delta \subseteq \tau_\Box$. Conversely, I claim that for any $a \in L$,

$$\Box a = \bigcup_{b \prec a} \check{b} = \{c \in L \mid \exists b \prec a : \neg b \prec c\}.$$

To see this, notice first that if $\neg b \prec c$ for some $b \prec a$, then $c \vee \neg\neg b = 1_L$ and $\neg\neg b \leq a$, which implies that $a \vee c = 1_L$. This shows the right-to-left inclusion. For the converse, suppose that $a \vee c = 1_L$. Since L is regular, $a = \bigvee\{b \in L \mid b \prec a\}$, and hence $1_L = \bigvee\{b \vee c \mid b \prec a\}$. Since L is also compact, this means that there are $b_1, ..., b_n$ such that $b_1 \vee ... \vee b_n \prec a$ and $c \vee b_1 \vee ... \vee b_n = 1_L$. Letting $b = \neg\neg(b_1 \vee ... \vee b_n)$, it follows that $b \prec a$ and that $\neg b \prec c$. This shows that $\Box a = \bigcup_{b \prec a} \check{b}$, and therefore that $\tau_\Box \subseteq \delta$. □

As an immediate corollary of the previous lemma, we obtain the following characterization of dV-spaces, which can be seen as a generalization of Theorem 7.7 in [41]:

Theorem 3.4.5. *A topological space is a dV-space if and only if it is homeomorphic to the upper Vietoris space of a compact regular locale.*

Finally, let us note that connections between de Vries duality and the Vietoris functor on compact Hausdorff spaces have already been studied in [33, 34]. In particular, the authors define modal de Vries algebras and prove that they are the duals of coalgebras of the Vietoris functor. For lack of space, we leave as an open problem the relationship between modal de Vries algebras and dV-spaces.

3.4.2 Two Applications

We conclude by briefly mentioning two straightforward applications of the duality presented here. The first one is a choice-free version of Tychonoff's Theorem for compact Hausdorff spaces and the second one deals with the topological semantics of the strong implication calculus defined in [41].

The following is a well-known result in pointfree topology [146, 148, 207]:

Lemma 3.4.6. *The category* **KRFrm** *is closed under coproducts.*

By the duality obtained in the previous section, this means that the category of dV-spaces is closed under products. This means that a version of Tychonoff's Theorem for dV-spaces (the product in **dVS** of a family of dV-spaces is compact) holds in a choice-free setting. Moreover, this also motivates the following definition.

Definition 3.4.7. Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of compact Hausdorff spaces. The *choice free product* of this family is the dV-space $\Xi(\bigoplus_{i\in I} \Omega(X_i))$.

As an immediate consequence of the results in the previous section, we get the following choice-free Tychonoff Theorem for compact Hausdorff spaces:

Theorem 3.4.8. *For any family of compact Hausdorff spaces $\{(X_i, \tau_i)\}_{i\in I}$, their choice-free product is compact. Moreover, under (BPI), it is homeomorphic to the upper-Vietoris space of $\prod_{i\in I}(X_i, \tau_i)$.*

It is worth contrasting this result to one that can be obtained using Isbell duality. Since the category of compact regular frames is closed under coproducts, it can be proved without appealing to the Axiom of Choice that the coproduct of the frames of opens of any family $\{(X_i, \tau_i)\}_{i\in I}$ of compact Hausdorff spaces is a compact frame. Under (BPI), this frame is precisely the frame of opens of the product of $\{(X_i, \tau_i)\}_{i\in I}$ in the category of topological spaces. In the absence of (BPI) however, it may fail to be spatial. We may therefore see Theorem Theorem 3.4.8 as a *semi-pointfree* version of Tychonoff's Theorem, that is choice-free yet remains spatial.

Let us now move on to the second application. De Vries duality has been used in [38] to prove that the Symmetric Strong Implication Calculus $\mathsf{S^2IC}$ is sound and complete with respect to the class of compact Hausdorff spaces. This calculus is obtained by adding a binary relation $\rightsquigarrow$ to the language of classical propositional calculus, to be interpreted as a *strong implication* connective. Given a contact algebra $(B, \prec)$, one can interpret the strong implication connective by letting $a \rightsquigarrow b = 1_B$ if $a \prec b$ and $a \rightsquigarrow b = 0$ otherwise. This gives rise to a binary normal and additive operator $\Delta(a, b) := \neg(a \rightsquigarrow \neg b)$, meaning that one may think of the pair $(B, \rightsquigarrow)$ as a BAO. For details on the axiomatization of $\mathsf{S^2IC}$, we refer to [38]. In order to provide a choice-free topological semantics for $\mathsf{S^2IC}$, we introduce the following notion:

Definition 3.4.9. A *de Vries topological model* is a triple (X, τ, V) such that (X, τ) is a dV-space, and V is a valuation such that for any formulas φ, ψ of $\mathsf{S^2IC}$:

- If φ is propositional letter p, then $V(\varphi) \in \mathsf{RO}(\mathscr{X})$;

- $V(\neg\varphi) = V(\varphi)^{\perp}$ and $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$;
- $V(\varphi \rightsquigarrow \psi) = X$ if $\overline{V(\varphi)} \subseteq {\downarrow}V(\psi)$ and $V(\varphi \rightsquigarrow \psi) = \emptyset$ otherwise.

A formula φ is *valid* on a dV-space (X, τ) iff $V(\varphi) = X$ for any de Vries topological model (X, τ, V).

As a consequence of Theorem Theorem 3.3.19, we have the following result, which does not assume the Axiom of Choice:

Theorem 3.4.10. *The system* $\mathsf{S^2IC}$ *is sound and complete with respect to the class of all* dV*-spaces.*

Proof. By Theorem 5.10 and Remark 5.11 in [38], de Vries algebras provide a sound and complete algebraic semantics for $\mathsf{S^2IC}$, and this result can be obtained choice-free. Combining this result with Theorem Theorem 3.3.19, it follows that dV-spaces also provide a choice-free sound and complete semantics for $\mathsf{S^2IC}$. □

Since dV-spaces constitute a choice-free, filter-based representation of de Vries algebras, we may think of our choice-free de Vries duality as providing a possibility semantics for the logic of region-based theories of space, just as the choice-free Stone duality through UV-spaces serves as a foundation for possibility semantics for classical and modal propositional logic [135, 134, 137].

3.5 Choice-Free Representations of Distributive Lattices

Let us now move on to another generalization of Bezhanishvili and Holliday's choice-free duality for Boolean algebras, this time to the category of distributive lattices. We will first provide a choice-free duality via bitopological spaces which we will call pairwise UV-spaces. Then, taking inspiration from the way in which Priestley spaces can be obtained from pairwise Stone spaces by taking the join of the two topologies, we will turn pairwise UV-spaces into ordered topological spaces of a certain kind, which we call UV-Priestley spaces.

3.5.1 Distributive bispaces

Throughout this section, we will be considering bitopological spaces of the form (X, τ_+, τ_-). We start with the following notation:

Notation 3.5.1. Given a bi-topological space $\mathscr{X} = (X, \tau_+, \tau_-)$, let $\leq_+$ and $\leq_-$ be the specialization preorders for τ_+ and τ_- respectively, and π_+ and π_- be the corresponding upset topologies. We define:

- $\neg_- : \pi_+ \to \pi_-$ such that $\neg_- U = -{\downarrow_-} U$;

- $\neg_+ : \pi_- \to \pi_+$ such that $\neg_+ U = -{\downarrow_+} U$.
- $\sim_-: \tau_+ \to \tau_-$ be defined as $\sim_- U = -C_-(U)$
- $\sim_+: \tau_- \to \tau_+$ be defined as $\sim_+ V = -C_+(V)$

The interest of these definitions lies in the following lemma.

Lemma 3.5.2. *The maps $\neg_-$ and $\neg_+$ form a Galois connection, i.e. for any $U \in \pi_+, V \in \pi_-$, $V \subseteq \neg_- U$ iff $U \subseteq \neg_+ V$. Similarly, $\sim_-$ and $\sim_+$ form a Galois connection between τ_+ and τ_-.*

Proof. This is essentially the same proof as in Section 2.2.1, and it is therefore omitted. □

As a consequence of this lemma, we have that the fixpoints $\mathcal{RO}(\mathscr{X})^+$ and $\mathcal{RO}(\mathscr{X})^-$ of the maps $\neg_+\neg_-$ and $\neg_-\neg_+$ respectively are both complete lattices, and that $\neg^+$ and $\neg^-$ are order anti-isomorphisms.

Notation 3.5.3. Given a bi-topological space $\mathscr{X} = (X, \tau_+, \tau_-)$, let σ_+ and σ_- denote the compact opens in τ_+ and τ_- respectively. We write $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$ and $\mathsf{CO}\mathcal{RO}(\mathscr{X})^-$ for the sets $\sigma_+ \cap RO(\mathscr{X})^+$ and $\sigma_- \cap RO(\mathscr{X})^-$ respectively.

Let us now introduce the following definitions.

Definition 3.5.4. A bi-topological space $\mathscr{X} = (X, \tau_+, \tau_-)$ is *pairwise distributive* if the following two conditions hold:

i) For any $U_1, U_2 \in \tau_+$, $U_1 \cap C_-(U_2) \subseteq C_-(U_1 \cap U_2)$

ii) For any $V_1, V_2 \in \tau_-$, $V_1 \cap C_+(V_2) \subseteq C_+(V_1 \cap V_2)$

It is straightforward to verify that any topological space (X, τ) viewed as a bitopological space (X, τ, τ) is pairwise distributive. In fact, this definition generalizes an elementary fact from general topology. In the bitopological setting however, the definition is non-trivial, which motivates the introduction of the following notion.

Definition 3.5.5. A *distributive bispace* is a bi-topological space $\mathscr{X} = (X, \tau_+, \tau_-)$ such that:

1. σ_+ and σ_- are closed under finite intersections;
2. the maps $\neg_+$ and $\neg_-$ restrict to maps between σ_+ and σ_-;
3. (X, π_+, π_-) is pairwise distributive.

Theorem 3.5.6. *Let $\mathscr{X} = (X, \tau_+, \tau_-)$ be a distributive bispace. Then the families of subsets of X $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$ and $\mathsf{CO}\mathcal{RO}(\mathscr{X})^-$ are distributive lattices.*

Proof. I claim that condition 3 implies that the maps $\neg_+\neg_-$ and $\neg_-\neg_+$ are nuclei on π_+ and π_- respectively. This will imply that $RO(\mathscr{X})^+$ and $RO(\mathscr{X})^-$ are cHA, hence distributive lattices. Conditions 1 and 2 then ensure that $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$ and $\mathsf{CO}\mathcal{RO}(\mathscr{X})^-$ are sublattices. For the proof of the claim, it suffices to show that for any $U_1, U_2 \in \pi_+$, $\neg_+\neg_-(U_1) \cap \neg_+\neg_-(U_2) \subseteq \neg_+\neg_-(U_1 \cap U_2)$. By condition 3, we have that $U_1 \cap C_{\pi_-}(U_2) \subseteq C_{\pi_-}(U_1 \cap U_2)$, so by taking complements and closure in π_+ we have that

$$C_{\pi_+}\neg_-(U_1 \cap U_2) \subseteq C_{\pi_+}(-U_1 \cup \neg_-(U_2)).$$

Since $U_1 \in \pi_+$ and closure distributes over unions, by taking complements this yields

$$U_1 \cap \neg_+\neg_- U_2 \subseteq \neg_+\neg_-(U_1 \cap U_2).$$

Substituting $\neg_+\neg_- U_1$ for U_1, and then swapping U_1 and U_2 yields

$$\neg_+\neg_- U_1 \cap \neg_+\neg_- U_2 \subseteq \neg_+\neg_-(\neg_+\neg_- U_1 \cap U_2) \subseteq \neg_+\neg_-\neg_+\neg_-(U_1 \cap U_2),$$

and we're done since $\neg_+\neg_-$ is idempotent.
The proof that $\neg_-\neg_+$ is a nucleus on π_- is completely dual. □

We are now in a position to introduce our first choice-free representation theorem for distributive lattices.

3.5.2 A Representation Theorem for Distributive Lattices

We start with the following definition, which was independently put forward in the context of distributive lattices in [188, 143] (see also [30] for a more recent work in which an equivalent notion seems to have independently appeared again).

Definition 3.5.7. Let L be a distributive lattice. A *pseudo-prime pair* on L is a pair (F, I) such that $F \in \mathcal{F}(L)$, $I \in \mathcal{I}(L)$ satisfying the following properties:

- $F \cap I = \emptyset$;
- $a \in F$ and $a \wedge b \in I$ implies $b \in I$ (RMP);
- $a \vee b \in F$ and $b \in I$ implies $a \in F$ (LJP).

Intuitively, pseudo-prime pairs can be thought of as pairs of a filter and an ideal that are "relatively prime" to one another. Indeed, it is easy to see that the pair $(p, L \setminus p)$ is pseudo-prime whenever p is a prime filter on a distributive lattice L. However, pseudo-prime pairs are much more constructive objects than prime filters, as the following lemma establishes.

Lemma 3.5.8. *Let L be a DL. For any pair (F, I) such that $F \cap I = \emptyset$, there is a pseudo-prime pair (F', I') such that $F \subseteq F'$ and $I \subseteq I'$.*

Proof. Let (F, I) be such that $F \cap I = \emptyset$, and define $F' = \{c \mid a \leq b \vee c \text{ for some } a \in F, b \in I\}$ and $I' = \{d \mid a \wedge d \leq b \text{ for some } a \in F, b \in I\}$.

- Clearly F' is upward closed and I' is downward closed. To verify that F' is a filter, note that if $a_1 \leq c_1 \vee b_1$ and $a_2 \leq c_2 \vee b_2$, then

$$a_1 \wedge a_2 \leq (c_1 \vee (b_1 \vee b_2)) \wedge (c_2 \vee (b_1 \vee b_2)) \leq (c_1 \wedge c_2) \vee (b_1 \vee b_2),$$

  thus $c_1 \wedge c_2 \in F'$. Similarly, if $a_1 \wedge d_1 \leq b_1$ and $a_2 \wedge d_2 \leq b_2$, then

$$(a_1 \wedge a_2) \wedge (d_1 \vee d_2) \leq ((a_1 \wedge a_2) \wedge d_1) \vee ((a_1 \wedge a_2) \wedge d_2) \leq b_1 \vee b_2,$$

  which implies that I' is an ideal.

- Morever, if there is $c \in F' \cap I'$, then we must have $a_1, a_2 \in F$ and $b_1, b_2 \in I$ such that $a_1 \leq b_1 \vee c$ and $a_2 \wedge c \leq b_2$, which implies that $a_1 \wedge a_2 \leq (b_1 \vee b_2) \vee c$ and $(a_1 \wedge a_2) \wedge c \leq b_1 \vee b_2$. But since L is distributive this implies that $a_1 \wedge a_2 \leq b_1 \vee b_2$, contradicting $F \cap I = \emptyset$.

- To check that (F', I') has the RMP, suppose $c \wedge x \in I'$ for some $c \in F$. This means that we have $a_1, a_2 \in F$, $b_1, b_2 \in I$ such that $a_1 \leq c \vee b_1$ and $a_2 \wedge (c \wedge x) \leq b_2$. But this implies that $(a_1 \wedge a_2) \wedge x \leq c \vee (b_1 \vee b_2)$ and $((a_1 \wedge a_2) \wedge x) \wedge c \leq b_1 \vee b_2$. Since L is distributive, this implies that $(a_1 \wedge a_2) \wedge x \leq b_1 \vee b_2$, and thus $x \in I'$.

- Similarly, for the LJP, suppose $d \vee x \in F'$ for some $d \in I'$. Then we have $a_1, a_2 \in F$, $b_1, b_2 \in I$ such that $a_1 \leq (d \vee x) \vee b_1$ and $a_2 \wedge d \leq b_2$. Now this implies that $(a_1 \wedge a_2) \wedge d \leq (b_1 \vee b_2) \vee x$, and that $(a_1 \wedge a_2) \leq d \vee ((b_1 \vee b_2) \vee x)$, which since L is distributive implies that $a_1 \wedge a_2 \leq (b_1 \vee b_2) \vee x$, and hence that $x \in F'$. □

The previous lemma can be thought of as a choice-free version of the Prime Filter Theorem. It will be of crucial relevance when proving our representation theorem for distributive lattices. Let us now turn to the definition of the dual bispace of a distributive lattice.

Definition 3.5.9. Let L be a DL. The *dual distributive bispace* of L is the bispace $\mathbb{B}(L) = (X, \tau_+, \tau_-)$ where:

- X is the set of all pseudo-prime pairs on L;
- τ_+ is generated by the basis $\{a^+ \mid a \in L\}$, where $a^+ = \{(F, I) \mid a \in F\}$;
- τ^- is generated by the basis $\{a^- \mid a \in L\}$, where $a^- = \{(F, I) \mid a \in I\}$.

The following is immediate.

Lemma 3.5.10. *Let L be a DL with dual bispace (X, τ_+, τ_-). For any $(F, I), (F', I') \in X$, we have that $(F, I) \leq_+ (F', I')$ iff $F \subseteq F'$, and $(F, I) \leq_- (F', I')$ iff $I \subseteq I'$.*

As a consequence, we know that the specialization preorders induced by the topologies τ_+ and τ_- coincide with the inclusion orderings on filters and ideals respectively. This gives us an elegant way of relating the operations $\neg_+ / \neg_-$ and $\sim_+ / \sim_-$ defined above.

Lemma 3.5.11. *Let $a, b \in L$. Then:*

1. $\neg_- a^+ = \sim_- a^+ = a^-$;
2. $\neg_+ a^- = \sim_+ a^- = a^+$;
3. $\neg_-(a^+ \cup b^+) = \sim_- (a^+ \cup b^+) = (a \vee b)^-$;
4. $\neg_+(a^- \cup b^-) = \sim_+ (a^- \cup b^-) = (a \wedge b)^+$.

Proof.

1. $(F, I) \in \sim_- a^+$ iff there is $b \in L$ such that $a^+ \cap b^- = \emptyset$ and $b \in I$ iff there is $b \in L$ such that $a \leq b$ and $b \in I$ iff $a \in I$ iff $(F, I) \in a^-$. Moreover since $F \cap I = \emptyset$ we have that $a^- \subseteq -\downarrow_- a^+ = \neg_- a^+$. For the converse, if $a \notin I$, then $F \vee a \cap I = \emptyset$ since (F, I) has the RMP, and therefore $(F, I) \in \downarrow_- a^+$.

2. Similar to 1 above. We need that every pair (F, I) has the LJP to prove that $-a^+ \subseteq \downarrow_+ a^-$.

3. By 1), we have the following:

$$\begin{aligned}\neg_-(a^+ \cup b^+) &= -(C_- a^+ \cup C_- b^+)\\ &= \neg_- a^+ \cap \neg_- b^+\\ &= a^- \cap b^-\\ &= \sim_- a^+ \cap \sim_- b^+\\ &= \sim^- (a^+ \cup b^+)\end{aligned}$$

. But clearly $a^- \cap b^- = (a \vee b)^-$.

4. Similar to 3. ∎

We have now gathered all the necessary components of our representation theorem.

Theorem 3.5.12. *Let L be a DL and $\mathbb{B}(L) = (X, \tau_+, \tau_-)$ its dual bispace. Then $\mathbb{B}(L)$ is a distributive bispace, and L is isomorphic to $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$.*

Proof. Note first that sets in σ_+ and σ_- respectively are sets of the form $a_1^+ \cup ... \cup a_n^+$ and $a_1^- \cup ... \cup a_n^-$ for some $a_1, ..., a_n \in L$. Now given elements $a_1, ..., a_n$ and $b_1, ..., b_m$ in L, we have that $(F, I) \in a_1^+ \cup ... \cup a_n^+ \cap b_1^+ \cup ... \cup b_m^+$ iff there is $i \leq n$ and $j \leq m$ such that $a_i \wedge b_j \in F$ iff $(F, I) \in \bigcup_{i \leq n, j \leq m}(a_i \wedge b_j)^+$, and similarly $(F, I) \in a_1^- \cup ... \cup a_n^- \cap b_1^- \cup ... \cup b_m^-$ iff there is $i \leq n$ and $j \leq m$ such that $a_i \vee b_j \in I$ iff $(F, I) \in \bigcup_{i \leq n, j \leq m}(a_i \vee b_j)^-$. Thus σ_+ and σ_- are closed under finite intersections. Moreover, the previous lemma guarantees that $\neg_- U \in \sigma_-$ and $\neg_+ V \in \sigma_+$ for any $U \in \sigma_+$, $V \in \sigma_-$, and that $\cdot^+ : L \to \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$ is a surjective homomorphism. For injectivity, it suffices to notice that if $a \not\leq b$, then there is $(F, I) \in X$ such that $a \in F$ and $b \in I$, and therefore $a^+ \not\leq b^+$.

Therefore it only remains to be checked that (X, π_+, π_-) is pairwise distributive. I only show that for any $U_1, U_2 \in \pi^+$, $U_1 \cap C_{\pi_-}(U_2) \subseteq C_{\pi_-}(U_1 \cap U_2)$. Suppose $(F, I) \in U_1 \cap C_{\pi_-}(U_2)$. This means that $(F, I) \in U_1$ and there is $(F', I') \in U_2$ such that $I \subseteq I'$. I claim that $F \vee F' \cap I = \emptyset$. To see this, suppose $a \wedge b \in I$ for some $a \in F, b \in F'$. Then since (F, I) has the RMP, $b \in I \subseteq I'$, and therefore $F' \cap I' \neq \emptyset$, a contradiction. Now this implies that there is $(F^*, I^*) \in X$ such that $F \vee F' \subseteq F^*$ and $I \subseteq I^*$. Since both U_1 and U_2 are upsets, we have that $(F^*, I^*) \in U_1 \cap U_2$, and therefore $(F, I) \in C_{\pi_-}(U_1 \cap U_2)$. □

3.5.3 Morphisms

We conclude this section with a treatment of morphisms between distributive bispaces. The following is inspired from the definition of a UV-map, which has both a topological requirement and an order theoretic one.

Definition 3.5.13. Let $\mathscr{X} = (X, \tau_+, \tau_-)$ and $\mathscr{X}' = (X', \tau'_+, \tau'_-)$ be two distributive bispaces. A *bi-distributive map* is a bicontinuous function $f : \mathscr{X} \to \mathscr{X}'$ such that:

1. For any $U \in \mathsf{CO}\mathcal{RO}(\mathscr{X}')^+$, $f^{-1}(U) \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$;
2. For any $V \in \mathsf{CO}\mathcal{RO}(\mathscr{X}')^-$, $f^{-1}(U) \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^-$;
3. For any $U \in \pi'_+$, $f^{-1}C_{\pi'_-}(U) = C_{\pi_-}f^{-1}(U)$;
4. For any $V \in \pi'_-$, $f^{-1}C_{\pi'_+}(V) = C_{\pi_+}f^{-1}(V)$.

The following remark establishes a connection between bi-distributive maps and the b-morphisms from Section 2.2.1.

Remark 3.5.14.

- A map f satisfies condition 3 if it is monotone with respect to $\leq_-$ and $\leq'_-$ and has the following property: for any $x \in X, x' \in X'$, if $f(x) \leq'_- x'$, then there is $y \geq_- x$ such that $f(y) \geq'_+ x'$.
- Similarly, f satisfies condition 4 if it is monotone with respect to $\leq_+$ and $\leq'_+$ and has the following property: for any $x \in X, x' \in X'$, if $f(x) \leq'_+ x'$, then there is $y \geq_+ x$ such that $f(y) \geq'_- x'$.

Finally, the next two lemmas suggest that bidistributive maps are the correct notion of morphism between distributive bispaces for our purposes.

Lemma 3.5.15. *Let $f : \mathscr{X} \to \mathscr{X}'$ be a bi-distributive map. Then f^{-1} restricts to lattice homomorphisms from $\mathsf{CO}\mathcal{RO}(\mathscr{X}')^+$ to $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$ and from $\mathsf{CO}\mathcal{RO}(\mathscr{X}')^-$ to $\mathsf{CO}\mathcal{RO}(\mathscr{X})^-$.*

Proof. We only prove that f^{-1} restricts to a lattice homomorphism from $\mathsf{CO}\mathcal{RO}(\mathscr{X}')^+$ to $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$, as the other case is similar. Note first that conditions 3 and 4 imply that for any $U \in \pi'_+$, we have that

$$f^{-1}(\neg_+\neg_- U) = f^{-1}[-C_{\pi'_+} - C_{\pi'_-}(U)] = -C_{\pi_+} - C_{\pi_-} f^{-1}(U) = \neg_+\neg_- f^{-1}(U).$$

Together with condition 1, this implies that f^{-1} is well-defined. Checking that f^{-1} preserves intersections and unions is routine, and moreover the previous equality implies that for any $U, V \in \mathsf{CO}\mathcal{RO}(\mathscr{X}')^+$,

$$f^{-1}[\neg_+\neg_-(U \cup V)] = \neg_+\neg_- f^{-1}(U \cup V) = \neg_+\neg_-(f^{-1}(U) \cup f^{-1}(V)).$$

This completes the proof. $\square$

Lemma 3.5.16. *Let $h : L \to M$ a lattice homomorphism between two DL L and M. Then $\mathbb{B}(h) : \mathbb{B}(M) \to \mathbb{B}(L)$, defined as $\mathbb{B}(h)(F, I) = (h^{-1}(F), h^{-1}(I))$ is a bi-distributive map.*

Proof. It is routine to check that $\mathbb{B}(h)$ is well-defined. We verify conditions 1 and 3 of a bi-distributive map (conditions 2 and 4 are checked similarly).

1. For any $a \in L$ and (F, I) a pair on M, we have that $\mathbb{B}(h)(F, I) \in a^+$ iff $a \in h^{-1}(F)$ iff $f(a) \in F$ iff $(F, I) \in f(a)^+$.

3. $\mathbb{B}(h)$ is clearly monotone with respect to the ideal inclusion ordering. Moreover, if (F, I) is a pair on M and (G, J) is a pair on L such that $(h^{-1}(F), h^{-1}(I)) \leq_- (G, J)$, then consider the pair $(h[G], I)$, where $h[G] = \uparrow\{h(c) \mid c \in G\}$. Clearly, $h[G] \cap I = \emptyset$, for otherwise there is $c \in G$ such that $h(c) \in I$, hence $c \in h^{-1}(H) \subseteq J$, a contradiction. So there is a pseudo-prime (F', I') such that $h[G] \subseteq F'$ and $I \subseteq I'$, which implies that $G \subseteq h^{-1}(F')$. Thus, $\mathbb{B}(h)$ satisfies property 3 by Remark 3.5.14. $\square$

3.6 Choice-Free Pairwise Stone Duality

In this section, we axiomatize the dual bispaces of distributive lattices, which we call *pairwise UV-spaces* (PUV-spaces for short), and we prove a choice-free duality between pairwise UV-spaces and distributive lattices which we then restrict to a duality between Heyting algebras and a subcategory of pairwise UV-spaces which we call Heyting UV-spaces.

3.6.1 Pairwise UV-spaces

We first recall the following definition.

Definition 3.6.1. A bispace $\mathscr{X} = (X, \tau_+, \tau_-)$ is *pairwise* T_0 if for any $x \neq y \in X$, there is $U \in \tau_+ \cup \tau_-$ such that U contains precisely one of x, y.

Just like the definition of a UV-space appeals to the notion of a filter on the $\mathsf{CO}\mathcal{RO}$ subsets of a topological space, we will need the following version of this notion in a bitopological context.

Definition 3.6.2. Let $\mathscr{X} = (X, \tau_+, \tau_-)$ be a distributive bispace. We write $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ for the set $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+ \cup \mathsf{CO}\mathcal{RO}(\mathscr{X})^-$. A *filter on* $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ is a filter S on the poset $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ ordered by inclusion, i.e.:

- $S \subseteq \mathsf{CO}\mathcal{RO}(\mathscr{X})$ is non-empty;
- for any $U, V \in S$ and $W \in \mathsf{CO}\mathcal{RO}(\mathscr{X})$, $U \cap V \subseteq W$ implies $W \in S$.

The following links the previous definition with pseudo-prime pairs.

Lemma 3.6.3. *Let $\mathscr{X}$ be a distributive bispace. There is a bijection between proper filters on $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ and pseudo-prime pairs on $\mathsf{CO}\mathcal{RO}(X^+)$.*

Proof. Given a proper filter S on $\mathsf{CO}\mathcal{RO}(\mathscr{X})$, let $F_S = \mathsf{CO}\mathcal{RO}(\mathscr{X})^+ \cap F$ and $I_S = \{U \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+ \mid \neg_- U \in F\}$. It is routine to check that F_S is a filter and I_S is an ideal. For the right-meet property, suppose $U \in F_S$ and $U \cap V \in I_S$. Then $U, \neg_-(U \cap V) \in S$, and since $U \cap C_- V \subseteq C_-(U \cap V)$, we have that $U \cap \neg_-(U \cap V) \subseteq \neg_- V$, and therefore $\neg_- V \in S$, which implies that $V \in I_S$. The LJP is proved similarly.

Conversely, if (F, I) is a pseudo-prime pair on $\mathsf{CO}\mathcal{RO}(\mathscr{X})$, let

$$S_{F,I} = \{W \in \mathsf{CO}\mathcal{RO}(\mathscr{X}) \mid U \cap \neg_- V \subseteq W$$

for some $U \in F$, $V \in I\}$. Then clearly $F \cup \{\neg_- V \mid V \in I\} \subseteq S_{F,I}$. Suppose now that there are $W_1, W_2 \in S_{F,I}$ such that $W_1 \cap W_2 \subseteq W$. Then we have $U_1 \cap U_2 \in F$ and $V_1 \cup V_2 \in I$ such that $U_1 \cap U_2 \cap \neg_-(V_1 \cup V_2) \subseteq W$, so $W \in S_{F,I}$. Thus $S_{F,I}$ is a filter on $\mathsf{CO}\mathcal{RO}(\mathscr{X})$.
Moreover, I claim that $S_{F,I} \subseteq F \cup \{\neg_- V \mid V \in I\}$. To see this, suppose $U \cap \neg_- V \subseteq W$ for some $U \in F$, $V \in I$, and $W \in \mathsf{CO}\mathcal{RO}(\mathscr{X})$. We have two cases:

- if $W \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^-$, we have that $W = \neg_- Z$ for some $Z \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$. Thus $U \cap C_- Z \subseteq C_- V$, which by taking complements yields $\neg_- V \subseteq -U \cup \neg_- Z$. Taking closure in π_+ and complements, we obtain that

  $$U \cap Z = U \cap \neg_+ \neg_- Z \subseteq \neg_+ \neg_- V = V.$$

  Thus $U \cap Z \in I$, and since (F, I) has the RMP, it follows that $Z \in I$.

- if $W \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$, then $U \subseteq W \cup C_- V$, which implies that

  $$\neg_-(W \cup V) = \neg_- W \cap \neg_- V \subseteq \neg_- U.$$

  Taking closure in π_+ and complements yields that $U \subseteq \neg_+ \neg_-(W \cup V)$. But this means that $\neg_+ \neg_-(W \cup V) \in F$, and since (F, I) has the LJP, it follows that $W \in F$.

□

We are now in a position to define pairwise UV-spaces. This definition can be seen as a straightforward translation of the definition of a UV-space to the bitopological setting, just like the definition of a pairwise Stone space adapts the definition of a Stone space to bitopological spaces.

Definition 3.6.4. A *pairwise UV-space* is a pairwise T_0 bispace $\mathscr{X} = (X, \tau_+, \tau_-)$ such that:

1. $\mathscr{X}$ is a distributive bispace;
2. $\mathsf{CO}\mathcal{RO}(\mathscr{X})^*$ is a basis for τ_* for $* \in \{+, -\}$;
3. for any $x, y \in X$, $x \not\leq_* y$ implies that there is $U \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^*$ such that $x \in U$ and $y \notin U$, for $* \in \{+, -\}$;
4. Any filter on $\mathsf{CO}\mathcal{RO}(\mathscr{X})$ is $\mathsf{CO}\mathcal{RO}(x)$ for some $x \in X$.

We can now show that this definition axiomatizes the dual bispaces of distributive lattices.

Theorem 3.6.5. *Let $\mathscr{X} = (X, \tau_+, \tau_-)$ be a pairwise UV-space. Then $\mathscr{X}$ is isomorphic to $\mathbb{B}(\mathsf{CO}\mathcal{RO}(\mathscr{X})^+)$.*

Proof. Consider the map $\theta : \mathscr{X} \to \mathbb{B}(\mathsf{CO}\mathcal{RO}(\mathscr{X})^+)$ given by $\theta(x) = (F_{\mathsf{CO}\mathcal{RO}(x)}, I_{\mathsf{CO}\mathcal{RO}(x)})$. Injectivity follows from condition 2 of pairwise UV-spaces and the fact that $\mathscr{X}$ is pairwise T_0. Surjectivity is given by condition 3. So we only have to check that θ and θ^{-1} are bi-distributive maps. We verify conditions 1 and 3.

1. We have isomorphisms $\cdot^+$ between $\mathsf{CO}\mathcal{RO}(\mathscr{X})^+$ and $\mathsf{CO}\mathcal{RO}(\mathbb{B}(\mathsf{CO}\mathcal{RO}(\mathscr{X})^+))^+$, and $\cdot^-$ between $\mathsf{CO}\mathcal{RO}(\mathscr{X})^-$ and $\mathsf{CO}\mathcal{RO}(\mathbb{B}(\mathsf{CO}\mathcal{RO}(\mathscr{X})^+))^-$. But then for any $U \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$, $x \in \mathscr{X}$, we have that
$$\theta(x) \in U^+ \Leftrightarrow U \in F_{\mathsf{CO}\mathcal{RO}(\mathscr{X})} \Leftrightarrow x \in U.$$
3. By condition 2, we have that for any $x, y \in \mathscr{X}$, $x \leq_- y$ iff $I_{\mathsf{CO}\mathcal{RO}(x)} \subseteq I_{\mathsf{CO}\mathcal{RO}(y)}$, hence both θ and θ^{-1} are monotone with respect to $\leq_-$. But this also implies that $\theta(x) \leq_- (F_{\mathsf{CO}\mathcal{RO}(y)}, I_{\mathsf{CO}\mathcal{RO}(y)})$ iff $x \leq_- y$, and thus by Remark 3.5.14 3.2 both θ and θ^{-1} satisfy property 3.

□

We conclude with our main theorem for this section.

Theorem 3.6.6. *The category* **DL** *is dual to the category of pairwise UV-spaces and bi-distributive maps.*

Proof. In light of Theorems 3.5.12 and 3.6.5, we only have to check the naturality condition for $\cdot^+$ and θ. Suppose we have the following diagram:

$$\begin{array}{ccc} L & \xrightarrow{\cdot^+} & \mathsf{CO}\mathcal{RO}(\mathbb{B}(L))^+ \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathbb{B}(h)^{-1}} \\ M & \xrightarrow{\cdot^+} & \mathsf{CO}\mathcal{RO}(\mathbb{B}(M))^+ \end{array}$$

Then for any $a \in L$, we have that

$$(F, I) \in (h(a))^+ \Leftrightarrow a \in h^{-1}(F) \Leftrightarrow \mathbb{B}(h)(F, I) \in a^+ \Leftrightarrow (F, I) \in \mathbb{B}(h)^{-1}(a^+).$$

Thus $h \circ \cdot^+ = \mathbb{B}(h)^{-1}$.

Similarly, suppose that we have the following diagram:

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\theta} & \mathbb{B}(\mathsf{CO}\mathcal{RO}(\mathscr{X}^+)) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbb{B}(f^{-1})} \\ \mathscr{Y} & \xrightarrow{\theta} & \mathbb{B}(\mathsf{CO}\mathcal{RO}(\mathscr{Y}^+)) \end{array}$$

Then for any $x \in \mathscr{X}$,

$$\begin{aligned} \mathbb{B}(f^{-1})(\theta(x)) &= \mathbb{B}(f^{-1}(F_{\mathsf{CO}\mathcal{RO}(x)}, I_{\mathsf{CO}\mathcal{RO}(x)}) \\ &= (U \mid f^{-1}(U) \in \mathsf{CO}\mathcal{RO}(x)\}, \{V \mid \neg_- f^{-1}(V) \in \mathsf{CO}\mathcal{RO}(x)\}) \\ &= (\{U \mid f(x) \in U\}, \{V \mid f(x) \in \neg_- V\}) \\ &= (F_{\mathsf{CO}\mathcal{RO}(f(x))}, I_{\mathsf{CO}\mathcal{RO}(f(x))}) = \theta(f(x)). \end{aligned}$$

This completes the proof. □

3.6.2 Heyting algebras

As a direct application of Theorem 3.6.6, we may restrict the duality obtained above to a choice-free duality for Heyting algebras. We start with the following definition, which identifies the dual bispaces of Heyting algebras.

Definition 3.6.7. A *Heyting UV-space* is a pairwise UV-space $\mathscr{X}$ satisfying the following condition:

- For any $U \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$, $V \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^-$, $\neg_+(U \cap V) \in \mathsf{CO}\mathcal{RO}(\mathscr{X})^+$.

Lemma 3.6.8. *Let L be a distributive lattice. Then L is a Heyting algebra iff $\mathbb{B}(L)$ is Heyting.*

Proof.

- Suppose first that L is a Heyting algebra. Then if $U \in \mathsf{CO}\mathcal{RO}(\mathbb{B}(L))^+$ and $V \in \mathsf{CO}\mathcal{RO}(\mathbb{B}(L))^-$, we have that $U = a^+$ and $V = b^-$ for some $a, b \in L$. I claim that $C_{\pi_+}(a^+ \cap b^-) = C_+(a \to b)^-$. This will imply that $\neg_+(a^+ \cap b^-) = \neg_+\neg_-(a \to b)^+$, and thus that $\mathbb{B}(L)$ is a Heyting UV-space. For the proof of the claim, note that since any pair (F, I) has the RMP, $a \in F$ and $b \in I$ implies that $a \to b \in I$, and thus $C_{\pi_+}(a^+ \cap b^-) \subseteq C_{\pi_+}(a \to b)^-$. Conversely, note that if $a \to b \notin F$, then $F \vee a \cap {\downarrow} b = \emptyset$ (since otherwise there is $c \in F$ such that $c \wedge a \leq b$, and thus $c \leq a \to b$). Hence there is a pseudo-prime pair (F', I') such that $F \vee a \subseteq F'$ and $b \in I'$, which means that $(F, I) \in C_{\pi_+}(a^+ \cap b^-)$. Since $C_+(a \to b)^- = -(a \to b)^+$, this shows that $C_+(a \to b)^- \subseteq C_+(a^+ \cap b^-)$.

- Conversely, I show that if $\mathbb{B}(L)$ is a Heyting UV-space, then $\mathsf{CO}\mathcal{RO}(\mathbb{B}(L))^+$, and therefore also L, is a Heyting algebra. I claim that for any $U, V, W \in \mathsf{CO}\mathcal{RO}(\mathbb{B}(L))$, we have that $U \cap W \subseteq V$ iff $W \subseteq \neg_+(U \cap \neg_- V)$:

  - For the left-to-right direction, note that if $U \cap W \subseteq V$, then

  $$U \cap C_{\pi_-} W \subseteq C_{\pi_-}(U \cap W) \subseteq C_{\pi_-} V,$$

  and hence $U \cap \neg_- V \subseteq \neg_- W$. Taking closure in π_+ and complements yields

  $$W = \neg_+\neg_- W \subseteq \neg_+(U \cap \neg_- V).$$

  - For the right-to-left direction, note that $\neg_- V \subseteq -U \cup (U \cap \neg_- V)$ implies, by taking closure in π_+, that

  $$C_{\pi_+}\neg_- V \subseteq C_{\pi_+}(-U \cup (U \cap \neg_- V)) = -U \cup C_{\pi_+}(U \cap \neg_- V),$$

  and hence, by taking complements,

  $$U \cap W \subseteq U \cap \neg_+(U \cap \neg_- V) \subseteq \neg_+\neg_- V = V.$$

□

In order to obtain our choice-free bitopological duality for Heyting algebras, we must also restrict bi-distributive maps to the duals of Heyting morphisms.

Definition 3.6.9. A bi-distributive map $f : \mathscr{X} \to \mathscr{X}'$ is *Heyting* if for any $U \in \pi'_+, V \in \pi'_-$, $f^{-1}(C_{\pi'_+}(U \cap V)) = C_{\pi_+}(f^{-1}(U) \cap f^{-1}(V))$.

The following connects Heyting bi-distributive maps and Heyting b-morphisms from Section 2.3.2

Remark 3.6.10. Similarly to remark 3.2 above, any map that is monotone with respect to $\leq_+$ and $\leq'_+$ and is such that for any $x \in X, x' \in X'$, if $f(x) \leq'_+ x'$, then there is $y \geq_+ x$ such that $f(y) \geq_+ x'$ and $f(y) \geq_- x'$ is Heyting.

Let us now verify that Heyting bi-distributive maps correspond exactly to homomorphisms of Heyting algebras.

Lemma 3.6.11. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a Heyting bi-distributive map between two Heyting UV-spaces $\mathscr{X}$ and $\mathscr{Y}$. Then $f^{-1} : \mathsf{C}\mathcal{RO}(\mathscr{Y})^+ \to \mathsf{C}\mathcal{RO}(\mathscr{X})^+$ is a Heyting homomorphism.*

Proof. It is enough to prove that for any $U, V \in \mathsf{C}\mathcal{RO}(\mathscr{Y})^+$,

$$f^{-1}(\neg_+(U \cap \neg_- V)) = \neg_+(f^{-1}(U) \cap \neg_- f^{-1}(V)).$$

But clearly f being Heyting and bi-distributive implies that

$$\begin{aligned} f^{-1}(-C_{\pi_+}(U \cap \neg_- V)) = -f^{-1}(C_{\pi_+}(U \cap \neg_- V)) &= \neg_+(f^{-1}(U) \cap f^{-1}(\neg_- V)) \\ &= \neg_+(f^{-1}(U) \cap \neg_- f^{-1}(V)). \end{aligned}$$

This completes the proof. □

Lemma 3.6.12. *Let $h : L \to M$ be a Heyting homomorphism between two Heyting algebras L and M. Then $\mathbb{B}(h) : \mathbb{B}(M) \to \mathbb{B}(L)$ is a Heyting bi-distributive map.*

Proof. By remark 5.4, it is enough to verify that for any $(F, I) \in \mathbb{B}(M)$, $(G, J) \in \mathbb{B}(L)$ for which we have that $(h^{-1}(F), f^{-1}(I)) \leq_+ (G, J)$, we can find $(F', I') \geq_+ (F, I) \in \mathbb{B}(M)$ such that $(G, J) \leq_+ (h^{-1}(F'), h^{-1}(I'))$ and $(G, J) \leq_- (h^{-1}(F'), h^{-1}(I'))$. I claim that $F \vee h[G] \cap h[J] = \emptyset$. Suppose towards a contradiction that there is $d \in F \vee h[G] \cap h[J]$. Then there is $a \in F, c \in G$ and $b \in J$ such that $a \wedge h(c) \leq d \leq h(b)$. Thus $a \leq h(c) \to h(b) = h(c \to b)$, and since $h^{-1}(F) \subseteq G$, we have that $c \wedge c \to b \in G$, contradicting the fact that $G \cap J = \emptyset$. Hence there is a pseudo-prime pair (F', I') such that $F \vee h[G] \subseteq F'$ and $h[J] \subseteq I'$. But then (F', I') is the required pair in $\mathbb{B}(M)$. □

As a consequence, we have the following restriction of Theorem 3.6.6:

Theorem 3.6.13. *The category of Heyting UV-spaces and Heyting bi-distributive maps is dual to the category of Heyting algebras and Heyting homomorphisms.*

3.7 Choice-Free Priestley Duality

After showing how to translate the pairwise Stone space duality to a choice-free version via pairwise UV-spaces, let us now make a similar attempt in the case of Priestley duality. As we shall see below, one can define choice-free analogues of Priestley spaces by combining the two topologies in a pairwise UV-space, but working out the details of this approach requires some significant work. As in the previous section, we start from a choice-free representation theorem for distributive lattices via ordered topological spaces, before identifying conditions on such spaces that characterize the duals of distributive lattices.

3.7.1 A choice-free representation of DL via ordered topological spaces

In this section, we show how to obtain a representation of any distributive lattice as a collection of subsets of an ordered topological space. This can be thought of as a choice-free analogue to Priestley's representation of distributive lattices as clopens of ordered Stone spaces.

Fix a distributive lattice L, and let X be the set of all pseudo-prime pairs on L. We will occasionally write L^δ for the dual lattice of L, and X^δ for the set of all pseudo-prime pairs of L^δ. Let τ be the topology generated by sets of the form a^+ or a^- for some $a \in L$, and $\leq$ the specialization preorder on (X, τ). We will once again use the notation $\downarrow U$ to designate the closure operator in the topology induced by this specialization preorder, and $\neg U$ to designate the set $-\downarrow U$. As a word of caution, we will soon add an order $\preccurlyeq$ to this topological space. We will then reserve the words "upset" and "downset" to the upward and downward closed sets in the order $\preccurlyeq$, but keep the notation $\downarrow U$ for the closure in the topology generated by the *specialization* preorder, hoping that no confusion will arise. We first make the following observations.

Lemma 3.7.1.

1. *$(F, I) \leq (F', I')$ iff $F \subseteq F'$ and $I \subseteq I'$;*
2. *For any $a, b \in L$, $\neg a^+ = a^-$ and $\neg b^- = b^+$;*
3. *Any open subset of X is of the form $\bigcup_{j \in J} a_j^+ \cap b_j^-$ for some $\{a_j, b_j \mid j \in J\} \subseteq L$;*
4. *For any $a, b \in L$, $a^+ \cap b^-$ is $\mathsf{CO}\mathcal{RO}\mathsf{F}$.*
5. *Any COF subset of X is of the form $a^+ \cap b^-$ for some $a, b \in L$.*

Proof.

1. Note first that a basic open in τ is a finite intersection of sets of the form a^+ or b^- for some $a, b \in L$. So let $U = \bigcap_{j \in J} a_j^+ \cap \bigcap_{k \in K} b_k^-$ for some finite sets J and K. Then a point (F, I) is in U iff $a_j \in F$ for all $j \in J$ and $b_k \in I$ for all $k \in K$ iff $\bigwedge a_j \in F$ and $\bigvee b_k \in I$ iff $(F, I) \in (\bigwedge a_j)^+ \cap (\bigvee b_k)^-$. Hence any basic open is of the form $a^+ \cap b^-$ for some $a, b \in L$. Now since $\leq$ is the specialization order on τ, for any two points (F, I) and (F', I'), $(F, I) \leq (F', I')$ iff for any basic open U, $(F, I) \in U$ implies $(F', I') \in U$ iff for any $a, b \in L$, $(F, I) \in a^+ \cap b^-$ implies $(F', I') \in a^+ \cap b^-$ iff $F \subseteq F'$ and $I \subseteq I'$.

2. Recall that for any $(F, I) \in X$ and $a, b \in L$, $(F \vee a, I)$ and $(F, I \vee b)$ extend to pseudo-prime pairs iff $a \notin I$ and $b \notin F$ respectively. Thus for any $(F, I) \in X$ and $a, b \in L$, $(F, I) \in \downarrow a^+$ iff $a \notin I$ iff $(F, I) \notin a^-$, and $(F, I) \in \downarrow b^-$ iff $b \notin F$ iff $(F, I) \notin b^+$. Thus

$$\neg a^+ = X \setminus \downarrow a^+ = X \setminus (X \setminus a^-) = a^-$$

and

$$\neg b^- = X \setminus \downarrow b^- = X \setminus (X \setminus b^+) = b^+.$$

3. Any open set U is a union of basic open sets. Since basic opens in τ are of the form $a^+ \cap b^-$, $U = \bigcup_{j \in J} a_j^+ \cap b_j^-$ for some set J.

4. Clearly $a^+ \cap b^-$ is open, and moreover since $\neg\neg a^+ = a^+$ and $\neg\neg b^- = b^-$, we have that $\neg\neg(a^+ \cap b^-) = \neg\neg(a^+) \cap \neg\neg(b^-) = a^+ \cap b^-$. We check that $a^+ \cap b^-$ is compact. Note that this is trivial if $a \leq b$, so suppose $a \nleq b$ and $a^+ \cap b^- \subseteq \bigcup_{j \in J} c_j^+ \cap d_j^+$. Let (F, I) be the pseudo-prime pair extending $(\uparrow a, \downarrow b)$. Since $(F, I) \in a^+ \cap b^-$, there is $j \in J$ such that $(F, I) \subseteq c_j^+ \cap d_j^-$. Note however that for any pseudo-prime pair $(F', I') \in a^+ \cap b^-$, $F \subseteq F'$ and $I \subseteq I'$, hence $c_j \in F'$ and $d_j \in I'$. Hence $a^+ \cap b^- \subseteq c_j^+ \cap d_j^-$, which establishes that $a^+ \cap b^-$ is compact. Moreover, the pair (F, I) is the $\leq$-least element in $a^+ \cap b^-$, thus $a^+ \cap b^-$ is a filter.

5. Let U be a $\mathsf{CO}\mathcal{RO}\mathsf{F}$ subset of X. By 3., $U = \bigcup_{j \in J} a_j^+ \cap b_j^-$, and since U is compact, we can take J to be finite. We claim that, since U is a filter, there is $j \in J$ such that for all $k \in J$, $a_k^+ \cap b_k^- \subseteq a_j^+ \cap b_j^-$. We may assume without loss of generality that $a_j \nleq b_j$ for all $j \in J$. For any $j \in J$, let (F_j, I_j) be the pseudo-prime pair extending $(\uparrow a_j, \downarrow b_j)$. Since U is a filter, there is $(F, I) \in U$ such that $(F, I) \leq (F_j, I_j)$ for all $j \in J$. This means that $F \subseteq \bigcap_{j \in J} F_j$ and $I \subseteq \bigcap_{j \in J} I_j$, and since $(F, I) \in U$, there is $j \in J$ such that $a_j \in F$ and $b_j \in I$. But this implies that for any $k \in J$, $a_j \in F_k$ and $b_j \in I_k$. Since (F_k, I_k) is the pseudo-prime pair extending $(\uparrow a_k, \downarrow b_k)$, this means that $a_k \leq a_j \vee b_k$ and $a_k \wedge b_j \leq b_k$. But the former implies that $a_k^+ \cap b_k^- \subseteq a_j^+$, and the latter implies that $a_k^+ \cap b_k^- \subseteq b_j^-$. Therefore $a_k^+ \cap b_k^- \subseteq a_j^+ \cap b_j^-$ for any $k \in J$, and $U = a_j^+ \cap b_j^-$. ☐

Note that this lemma implies that COF sets are also regular opens. In order to characterize sets of the form a^+ for some $a \in L$ as COF sets of a certain type, we need to add a partial order to (X, τ). Let $\preccurlyeq$ be defined on X so that $(F, I) \subseteq (F', I')$ iff $F \subseteq F'$ and $I' \subseteq I$. The following technical definition and lemma will be needed below.

Definition 3.7.2. Let (a, b) be a pair of elements of L. A *right-complement* for (a, b) is an element $k \in L$ such that $k \wedge b \leq 0$ and $a \leq b \vee k$, and a *left-complement* for (a, b) is an element $j \in L$ such that $1 \leq a \vee j$ and $a \wedge j \leq b$. A pair (a, b) of elements of L is *right-complement free* (resp. *left-complement free*) if the pair (a, b) has no right-complement (resp. left-complement).

Lemma 3.7.3. *Let $a, b \in L$.*

1. *If (a, b) is a right-complement free pair in L, then there are pseudo-prime pairs (F, I) and $(F', I') \in X$ such that $(F, I) \in a^+ \cap b^-$, $(F, I) \preccurlyeq (F', I')$, and $b \notin I'$.*

2. *If (a, b) is a left-complement free pair in L, then there are pseudo-prime pairs (F, I) and $(F', I') \in X$ such that $(F, I) \in a^+ \cap b^-$, $(F', I') \preccurlyeq (F, I)$, and $a \notin F'$.*

Proof.

1. Note first that $a \nleq b$, for otherwise 0 would be a pseud-complement for (a, b). This means that the pair $(\uparrow a, \downarrow b)$ extends to a pseudo-prime pair (F, I) such that for any $c, d \in L$, $c \in F$ iff $a \leq b \vee c$ and $d \in I$ iff $a \wedge d \leq b$. Now consider the pair $(F \vee b, \downarrow 0)$. We claim that $F \vee b \cap \downarrow 0 = \emptyset$. Indeed, if $c \in F \vee b \cap \downarrow 0$, then $c = k \wedge b$ for some k such that $a \leq b \vee k$, and $k \wedge b \leq 0$. But then k is a pseudo complement for the pair (a, b), contradicting our assumption. Thus let (F', I') be the pseudo-prime pair extending $(F \vee b, 0)$. This means that for any $c, d \in L$, $c \in F'$ iff $k \wedge b \leq c$ for some k such that $a \leq b \vee k$, and $d \in I'$ iff $k \wedge b \wedge d \leq 0$ for some k such that $a \leq b \vee k$. Now we make the following two claims:

   - $(F, I) \preccurlyeq (F', I')$: clearly $F \subseteq F'$. To see that $I' \subseteq I$, suppose d is such that $k \wedge b \wedge d \leq 0$ for some k such that $a \leq b \vee k$. Then

   $$a \wedge (d \wedge b) \leq (b \vee k) \wedge (d \wedge b) \leq (b \wedge d \wedge b) \vee (k \wedge d \wedge b) \leq d \wedge b \leq b,$$

     hence $a \wedge d \leq b$, and $d \in I$.
   - $b \notin I'$: if $b \in I'$, then $k \wedge b \wedge b = k \wedge b \leq 0$ for some k such that $a \leq b \vee k$. But then k is a pseudo-complement for the pair (a, b).

2. Suppose (a, b) is left-complement free, and work in the dual lattice L^δ. Then the pair (b, a) is right-complement free, so there is $(F, I), (F', I') \in X^\delta$ such that $(F, I) \in b^+ \cap a^-$, $a \notin I'$, $F \subseteq F'$ and $I' \subseteq I$. But then, (I, F) and (I', F') are the required pseudo-prime pairs in X, since $(I', F') \preccurlyeq (I, F)$ and $(I, F) \in a^+ \cap b^-$. □

In Priestley duality, the original distributive lattice L in the Priestley space $Spec(L)$ can only be retrieved by adding the inclusion order on prime filters and taking the clopen upsets. We shall see a similar phenomenon occurring here.

Lemma 3.7.4. *Let L be a distributive lattice.*

1. *For any $a, b \in L$, a^+ is an upset, and b^- is a downset;*

2. *Any* COF *upset is of the form a^+ for some $a \in L$, and any* COF *downset is of the form b^- for some $b \in L$.*

Proof.

1. Let $a, b \in L$ and suppose that $(F, I) \preccurlyeq (F', I')$. Then $a \in F$ implies $a \in F'$, and $b \in I'$ implies $b \in I$, since $F \subseteq F'$ and $I' \subseteq I$. Hence a^+ is an upset and b^- is a downset.

2. Let U be a COF set. By the previous lemma, $U = a^+ \cap b^-$ for some $a, b \in L$. Note that if (a, b) is a right-complement free pair, then by Lemma 3.7.3 U cannot be an upset. Similarly, if (a, b) is a left-complement free pair, then U cannot be a downset. Thus if U is an upset there is $k \in L$ such that $b \wedge k \leq 0$ and $a \leq b \vee k$. But this implies at once that $a^+ \cap b^- \subseteq k^+$ and $k^+ \subseteq b^-$, and hence that $a^+ \cap b^- = a^+ \cap k^+ = (a \wedge k)^+$. Similarly, if U is a downset, there is $j \in L$ such that $1 \leq a \vee j$ and $a \wedge j \leq b$. This implies that $a^+ \wedge b^- \subseteq j^-$ and $j^- \subseteq a^+$, hence that $a^+ \cap b^- = b^- \cap j^- = (b \vee j)^-$. □

We may now prove our choice-free representation theorem for distributive lattices via ordered topological spaces.

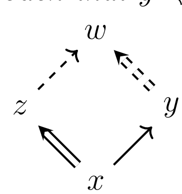
Theorem 3.7.5. *Let L be a distributive lattice and $\mathscr{X} = (X, \tau, \preccurlyeq)$ as above. Then L is isomorphic to the lattice $(\mathsf{COFUP}(\mathscr{X}), \cap, \vee)$, where for any two* COFUP *sets U, V, $U \vee V = \neg\neg(U \cup V)$.*

Proof. By Lemma 3.7.4, any COF upset U is a^+ for some $a \in L$. Thus the map $\cdot^+ : L \to \mathsf{COFUP}(\mathscr{X})$ is surjective, and clearly preserves meets. Moreover, for any $a, b \in L$, $\neg\neg(a^+ \cup b^+) = \neg(\neg a^+ \cap \neg b^+) = \neg(a^- \cap b^-) = \neg(a \vee b)^- = (a \vee b)^+$. Thus $\cdot^+$ is a lattice homomorphism. Finally, if $a \not\leq b$, then there is $(F, I) \in X$ extending the pair $({\uparrow}a, {\downarrow}b)$, thus $\cdot^+$ is injective. Hence L is isomorphic to $\mathsf{COFUP}(\mathscr{X})$. □

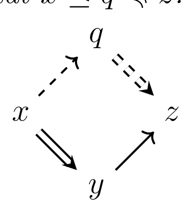
Note that, in $(X, \tau, \preccurlyeq)$, the order $\preccurlyeq$ can be seen to be compatible with the regular open complement $\neg$ in the sense that, for any COF upset U, $\neg U$ is a COF downset, and for any COF downset D, $\neg D$ is a COF upset. In fact, this can be seen as a combination of two distinct properties. The first one is topological in nature, since we impose that $\neg U$ be COF whenever U is a COF upset or downset. The second requirement is order-theoretic, and amounts to the requirement that $\downarrow U$ is an upset whenever U is a COF upset, and a downset whenever U is a COF downset. The following strengthening of this latter condition has a nice characterization in terms of compatibility conditions for $\leq$ and $\preccurlyeq$:

Lemma 3.7.6. *Let $(X, \leq, \preccurlyeq)$ be a bi-preordered set.*

1. *The following are equivalent:*
   a) *For any upset U, $\downarrow U$ is an upset;*
   b) *X satisfies the following Diamond principle (D): for any $x, y, z \in X$ such that $x \leq y$ and $x \preccurlyeq z$, there is $w \in X$ such that $y \preccurlyeq w$ and $z \leq w$:*



2. *The following are equivalent:*
   a) *For any downset D, $\downarrow D$ is a downset;*
   b) *X satisfies the following Exchange principle (E): For any $x, y, z \in X$ such that $x \preccurlyeq y \leq z$, there is $q \in X$ such that $x \leq q \preccurlyeq z$:*



Proof.

1. To see that a) implies b), just apply a) to the upset generated by z. For the converse, assume U is an upset and $x \in \downarrow U$. Then $x \leq y$ for some $y \in U$. Now since (D) holds, for any $z \succcurlyeq x$, there is $w \geq z$ such that $w \succcurlyeq y$. But then $w \in U$ since U is an upset, hence $z \in \downarrow U$. Thus $\downarrow U$ is an upset.

2. To see that a) implies b), apply a) to the downset generated by z. For the converse, assume D is a downset and $x \preccurlyeq y$ for some $y \in \downarrow D$. Then $y \leq z$ for some $z \in D$. Since (E) holds, there is some q such that $x \leq q \succcurlyeq z$. Since D is a downset this implies that $q \in D$, hence $x \in \downarrow D$. □

In fact, we can show that these stronger order-theoretic conditions also hold on the order topological space $\mathscr{X}$ above.

Lemma 3.7.7. *Let U, D be subsets of X.*

1. If U is an upset, then $\downarrow U$ is an upset.

2. If D is a downset, then $\downarrow D$ is a downset.

Proof.

1. We verify that $(X, \leq, \preccurlyeq)$ satisfies the Diamond principle (D) above. Suppose we have $x, y, z \in X$ with $x \leq y$ and $x \preccurlyeq z$. Let $x = (F_x, I_x)$, $y = (F_y, I_y)$ and $z = (F_z, I_z)$. We claim that $F_y \vee F_z \cap I_z = \emptyset$. To see this, notice that if $a \wedge c \in I_z$ for some $a \in F_y, c \in I_z$, then $a \in I_z$. But since $I_z \subseteq I_x \subseteq I_y$, this implies that $F_y \cap I_y \neq \emptyset$, a contradiction. Now let $w = (F_w, I_w)$ be the pseudo-prime pair extending $(F_y \vee F_z, I_z)$. Clearly, $I_z \subseteq I_w$, $F_z \subseteq F_w$ and $F_y \subseteq F_w$. Hence $z \leq w$, and, to verify that $y \preccurlyeq w$, we only need to check that $I_w \subseteq I_y$. So let $d \in I_w$. This means that $a \wedge c \wedge d \in I_z$ for some $a \in F_y$, $c \in F_z$. Thus $a \wedge d \in I_z \subseteq I_y$, from which it follows that $d \in I_y$.

2. We verify that $(X, \leq, \preccurlyeq)$ satisfies the Exchange principle (E) above. Suppose we have $x, y, z \in X$ with $x \preccurlyeq y \leq z$, and let $x = (F_x, I_x)$, $y = (F_y, I_y)$ and $z = (F_z, I_z)$. Observe that $x^\delta = (I_x, F_x)$, $y^\delta = (I_y, F_y)$ and $z^\delta = (F_z, I_z)$ are pseudo-prime pairs on the dual lattice of L, i.e. $x^\delta, y^\delta, z^\delta \in X^\delta$. Moreover, since $x \preccurlyeq y$, we have that $y^\delta \preccurlyeq x^\delta$, and since $y \leq z$, we have that $y^\delta \leq z^\delta$. Thus by the previous result there is $w = (F_w, I_w)$ such that $x^\delta \leq w$ and $z \preccurlyeq w$. But then, letting $q = (I_w, F_w)$, we have that $q \in X$, $x \leq q$, and $q \preccurlyeq z$. □

3.7.2 UVP spaces

In this section, we identify the ordered topological spaces that are isomorphic to the spaces constructed in the previous section. We start by connecting the construction above with the Jipsen-Moshier duality for lattices. Given a DL L, let $\mathbb{D}(L)$ be the ordered topological space $(X, \tau, \preccurlyeq)$ constructed in the previous section.

Lemma 3.7.8. *For any DL L, $\mathbb{D}(L)$ is an HMS space.*

Proof. Recall that any open U can be written as $\bigcup_{j\in J} a_j^+ \cap b_j^-$. Since $a_j^+ \cap b_j^-$ is COF for any $a, b \in L$, it follows that $KOF(\mathscr{X})$ is a basis for τ. Therefore we only have to check that $\mathbb{D}(L)$ is sober, i.e., that every completely prime filter on τ is $O(F, I)$ for some $(F, I) \in X$. So let $\mathscr{F}$ be a completely prime filter, and let $F = \{a \in L \mid a^+ \in \mathscr{F}\}$ and $I = \{b \in L \mid b^- \in \mathscr{I}\}$. The sets F and I are easily seen to be a filter and an ideal on L respectively, and since $\mathscr{F}$ is non-trivial, $F \cap I = \emptyset$. Now for any $a, b \in L$, $a^+ \cap (a \wedge b)^- \subseteq b^-$ and $(a \vee b)^+ \cap b^- \subseteq a^+$, from which it follows that (F, I) is a pseudo-prime pair, and hence $(F, I) \in X$. Finally, let $U = \bigcup_{j\in J} a_j^+ \cap b_j^-$ be an open set in X. Then $(F, I) \in U$ iff $a_j \in F$ and $b_j \in I$ for some $j \in J$ iff $a_j^+ \cap b_j^- \in \mathscr{F}$ for some $j \in J$ iff $U \in \mathscr{F}$. □

Corollary 3.7.9. *For any DL L, $\mathbb{D}(L)$ is a spectral space.*

Proof. This follows from Theorem 2.5 in [197]. □

Thus our choice-free ordered topological duals of distributive lattices will be spectral spaces with some additional properties. Let us also note the following.

Lemma 3.7.10. *Let $h : L \to M$ be a lattice homomorphism between two DL. Then the map $h_* : \mathbb{D}(M) \to \mathbb{D}(L)$ defined by $(F, I) \mapsto (h^{-1}[F], h^{-1}[I])$ is a spectral map. Moreover, for any upset U and downset $D \in \mathbb{D}(L)$, $h_*^{-1}[{\downarrow}U] = {\downarrow}h_*^{-1}[U]$ and $h_*^{-1}[{\downarrow}D] = {\downarrow}h_*^{-1}[D]$.*

Proof. Recall that any compact open subset U of $\mathbb{D}(L)$ is of the from $\bigcup_{j\in J} a_j^+ \cap b_j^-$ for $a_j, b_j \in L$. Clearly, $h_*^{-1}[U] = \bigcup_{j\in J} h_*^{-1}[a_j^+] \cap h_*^{-1}[b_j^-]$, so it is enough to show that $h_*^{-1}[a^+] = h(a)^+$ and $h_*^{-1}[b^-] = h(b)^-$ for any $a, b \in L$. But this was already established in Lemma 3.5.16. Hence h_* is a spectral map. Since this implies that h_* is open, it is also monotone with respect to the specialization ordering $\leq$, so for any subset S of $\mathbb{D}(L)$, then ${\downarrow}h_*^{-1}[S] \subseteq h_*^{-1}[{\downarrow}S]$. Now suppose U is an upset in $\mathbb{D}(L)$, and $x = (F, I) \in h_*^{-1}[{\downarrow}S]$. This means that there is some $(F', I') \in U$ such that $(h^{-1}[F], h^{-1}[I]) \leq (F', I')$. By a standard argument, $F \vee h[F'] \cap I = \emptyset$, so there is $(F^*, I^*) \in \mathbb{D}(M)$ which extends the pair $(F \vee h[F'], I)$ to a pseudo prime pair. Clearly, $(F, I) \leq (F^*, I^*)$, and we claim that $(F', I') \preccurlyeq (h^{-1}[F^*], h^{-1}[I^*])$. Since U is an upset, this will show that $(F, I) \in {\downarrow}h_*^{-1}[U]$. Clearly, if $c \in F'$, then $f(c) \in F^*$, hence $F' \subseteq h^{-1}[F^*]$. Moreover, let $d \in h^{-1}[I^*]$. This means that $h(d) \in I^*$, i.e., there is $a \in F$ and $c \in F'$ such that $a \wedge h(c) \wedge h(d) \in I$. Since h is a homomorphism, this implies that $a \wedge h(c \wedge d) \in I$, and since (F, I) has the RMP, $h(c \wedge d) \in I$. But then $c \wedge d \in h^{-1}[I] \subseteq I'$, and since $c \in F'$ and (F', I') has the RMP it follows that $d \in I'$. Thus $(F', I') \preccurlyeq (F^*, I^*)$. Thus $h_*^{-1}[{\downarrow}U] = {\downarrow}h_*^{-1}[U]$. The corresponding statement for downsets is proved in a similar way. □

Definition 3.7.11. The functor $\mathbb{D} : \mathbf{DL} \to \mathbf{oTop}$ maps any DL L to the ordered topological space $\mathbb{D}(L)$, and any lattice homomorphism $h : L \to M$ to the monotone continuous map $\mathbb{D}(h) := h_* : \mathbb{D}(M) \to \mathbb{D}(L)$.

We can now introduce the main definition of this section:

Definition 3.7.12. A *UVP-space* is a triple $(X, \tau, \preccurlyeq)$ such that:

1. (X, τ) is a T_0-space with specialization order $\leq$, and $\preccurlyeq$ is a partial order on X;
2. A subset S of X is COF if and only if $S = U \cap \neg V$ for some COF upsets U and V;
3. $\mathsf{COF}(\mathscr{X})$ is a basis for (X, τ);
4. $\mathsf{COFUP}(\mathscr{X})$ is a sublattice of $\mathsf{RO}(\mathscr{X})$;
5. For any subset S of X, if S is an upset (resp. downset), then $\downarrow S$ is an upset (resp. downset);
6. Any filter on $\mathsf{COF}(\mathscr{X})$ is $\mathsf{COF}(x)$ for some point $x \in X$;
7. For any $x, y \in X$, if $x \not\preccurlyeq y$, then there is an upset $U \in \mathsf{COF}(\mathscr{X})$ such that $x \in U$ and $y \notin U$, or $x \notin \neg U$ and $y \in \neg U$.

The following is immediate from this definition:

Lemma 3.7.13. *Let $(X, \tau, \preccurlyeq)$ be an UVP space. Then $\mathsf{COFUP}(\mathscr{X})$ is a distributive lattice.*

Proof. By axiom 4, $\mathsf{COFUP}(\mathscr{X})$ is a sublattice of $\mathsf{RO}(\mathscr{X})$. Since $\mathsf{RO}(\mathscr{X})$ is a Boolean algebra, it follows that $\mathsf{COFUP}(\mathscr{X})$ is distributive. □

Let us now define the relevant notion of morphisms between UVP spaces:

Definition 3.7.14. Let $(X, \tau, \preccurlyeq)$ and $(Y, \tau', \preccurlyeq')$ be UVP spaces. A *UVP map* is a function $f : X \to Y$ such that:

1. f is F-continuous, i.e., $f^{-1}[S]$ is COF for any COF subset S of Y;
2. f is monotone: if $x \preccurlyeq x'$, then $f(x) \preccurlyeq' f(x')$;
3. If S is an upset or a downset in Y, then $\downarrow f^{-1}[S] = f^{-1}[\downarrow S]$.

Lemma 3.7.15. *Let $f : (X, \tau, \preccurlyeq) \to (Y, \tau', \preccurlyeq)$ be a UVP map. Then $f^{-1} : \mathsf{COFUP}(\mathscr{Y}) \to \mathsf{COFUP}(\mathscr{X})$ is a lattice-homomorphism.*

Proof. Since f is F-continuous and monotone, it maps COF upsets to COF upsets. Moreover, f^{-1} clearly preserves meets. To see that it preserves joins, suppose U, V are COF upsets in Y. Note that this implies that $\neg U$ and $\neg V$ are downsets, thus that $\neg(U \cup V) = \neg U \cap \neg V$ is also a downset. Hence:

$$f^{-1}[\neg\neg(U \cup V)] = \neg f^{-1}[\neg(U \cup V)] = \neg\neg f^{-1}[U \cup V] = \neg\neg(f^{-1}[U] \cup f^{-1}[V]).$$

This completes the proof. □

Let us now connect the dual ordered topological space of a distributive lattice defined in the previous section and UVP spaces.

Lemma 3.7.16. *For any distributive lattice L, $\mathbb{D}(L)$ is a UVP space.*

Proof. We check that all conditions in Definition 3.7.12 are satisfied by $(X, \tau, \leq)$:

1. This condition is clear from the definition of τ and $\preccurlyeq$;

2. Recall that, by Lemma 3.7.1, if S is a COF subset of X, then $U = a^+ \cap b^-$ for some $a, b \in L$. But a^+, b^+ are COF upsets and $b^- = \neg b^+$.

3. By Lemma 3.7.4, sets of the form $a^+ \cap b^-$ form a basis for τ and are COF.

4. By Lemma 3.7.4, any COF upset is regular open and of the form a^+ for some $a \in L$. Given $a, b \in L$, $a^+ \cap b^+ = (a \wedge b)^+$ is the greatest lower bound of the set $\{a^+, b^+\}$ in COF, and $\neg\neg(a^+ \cup b^+) = (a \vee b)^+$ is its least upper bound. Thus $\mathsf{COFUP}(\mathscr{X})$ is a sublattice of $RO(\mathscr{X})$.

5. This was proved in Lemma 3.7.7.

6. Let K be a filter on $COF(\mathscr{X})$, and let $K^+ = \{a \in L \mid a^+ \in K\}$ and $K^- = \{b \in L \mid b^- \in K\}$. We claim that (K^+, K^-) is a pseudo-prime pair in X, and moreover that for any $a, b \in L$, $(K^+, K^-) \in a^+ \cap b^-$ iff $a^+ \cap b^- \in K$, i.e. $COF(K^+, K^-) = K$. Note first that, since K is a filter, K^+ is clearly a filter on L and K^- is clearly an ideal, and moreover $K^+ \cap K^- = \emptyset$. To see that (K^+, K^-) has the RMP, assume $(a \wedge b)^- \in K$ for some $a, b \in L$ such that $a^+ \in L$. Since all pairs in X have the RMP, $a^+ \cap (a \wedge b)^- \subseteq b^-$, thus $b^- \in K$ and $b \in K^-$. Similarly, suppose that $(a \vee b)^+ \in K$ and $b^- \in K$. Since all pairs in X have the LJP, $(a \vee b)^+ \cap b^- \subseteq a^+$, thus $a \in K^+$, which establishes that (K, K') also has the LJP. Hence $(K, K') \in X$. Finally, note that for any $a, b \in L$, $(K^+, K^-) \in a^+ \cap b^-$ iff $a^+ \in K$ and $b^- \in K$ iff $a^+ \cap b^- \in K$. Thus $K = \mathsf{COF}(K^+, K^-)$.

7. Suppose $x \not\preccurlyeq y$ for some $x, y \in X$. Let $x = (F_x, I_x)$ and $y = (F_y, I_y)$. Then either there is $a \in F_x \setminus F_y$, or there is $b \in I_y \setminus I_x$. In the first case, $x \in a^+$ but $y \notin a^+$, and a^+ is a COF upset, while, in the second case, $x \notin b^-$ but $y \in b^-$, and $b^- = \neg b^+$. □

Moreover, we have the following result for lattice homomorphisms.

Lemma 3.7.17. *Let $h : L \to M$ be a lattice homomorphism between two DL L and M. Then $\mathbb{D}(h)$ is a UVP map.*

Proof. The proof of Lemma 3.7.10 reveals that $\mathbb{D}(h)$ is an F-continous map that satisfies condition 3, and it is clearly monotone with respect to $\preccurlyeq$. □

Thus the functor $\mathbb{D}$ maps **DL** into the category of UVP spaces and UVP maps between them. We now show that this functor is an equivalence.

Theorem 3.7.18. *Let $\mathscr{X} = (X, \tau, \leq)$ be a UVP space. Then $\mathscr{X}$ is isomorphic to $\mathbb{D}(\mathsf{COFUP}(\mathscr{X}))$.*

Proof. For any $x \in X$, let $\mathsf{COFUP}(x)^+ = \{U \in \mathsf{COFUP}(\mathscr{X}) \mid x \in U\}$ and $\mathsf{COFUP}(x)^- = \{V \in \mathsf{COFUP}(\mathscr{X}) \mid x \in \neg V\}$, and let $\theta : \mathscr{X} \to \mathbb{D}(\mathsf{COFUP}(\mathscr{X}))$ be defined such that $\theta(x) = (\mathsf{COFUP}(x)^+, \mathsf{COFUP}(x)^-)$. We first claim the following:

1. θ is well-defined. It is routine to check that $\mathsf{COFUP}(x)^+$ and $\mathsf{COFUP}(x)^-$ are a filter and an ideal on $\mathsf{COFUP}(\mathscr{X})$ with empty intersection. To see that $\theta(x)$ has the RMP, suppose we have COF upsets U, V such that $U \in \mathsf{COFUP}(x)^+$ and $U \cap V \in \mathsf{COFUP}(x)^-$. Then $x \in U \cap \neg(U \cap V)$. Since the regular opens of $\mathscr{X}$ form a Boolean algebra, we have that $U \cap \neg(U \cap V) \subseteq \neg V$, hence $x \in \neg V$ and $V \in \mathsf{COFUP}(x)^-$. Similarly, if $x \in \neg\neg(U \cup V) \cap \neg V$, then since $\neg\neg(U \cup V) \cap \neg V \subseteq U$, it follows that $x \in U$. Thus if $\neg\neg(U \cup V) \in \mathsf{COFUP}(x)^+$ and $V \in \mathsf{COFUP}(x)^-$, we have that $U \in \mathsf{COFUP}(x)^+$, which shows that $theta(x)$ also has the LHP. Hence $\theta(x)$ is a pseudo-prime pair on $\mathsf{COFUP}(\mathscr{X})$, so θ is well-defined.

2. θ is surjective. Let (F, I) be a pseudo-prime pair on $\mathsf{COFUP}(\mathscr{X})$, and consider $K = \{S \in \mathsf{COF}(\mathscr{X}) : \exists U \in F, V \in I(U \cap \neg V \subseteq S)\}$. Clearly, K is a filter on $COF(\mathscr{X})$, so $K = \mathsf{COF}(x)$ for some $x \in X$. Now for any COF upset U, $U \in \mathsf{COFUP}(x)^+$ iff $x \in U$ iff $U \in \mathsf{CORO}(x)$ iff $V_1 \cap \neg V_2 \subseteq U$ for some $V_1 \in F$ and $V_2 \in I$. Now if V_1, V_2 and U are regular opens such that $V_1 \cap \neg V_2 \subseteq U$, it follows that $V_1 \subseteq \neg\neg(V_2 \cup U)$, hence since (F, I) has the LJP $V_1 \in F$ and $V_2 \in I$ implies that $U \in F$. Thus $\mathsf{COFUP}(x)^+ \subseteq F$, and the converse direction is obvious. One proves similarly that $\mathsf{COFUP}(x)^- = I$, and therefore $(F, I) = \theta(x)$.

3. θ preserves and reflects $\preccurlyeq$. Note that this also implies that θ is injective, since $\preccurlyeq$ is a partial order. Suppose $x \preccurlyeq y$. Then every upset containing x also contains y, and every downset containing y also contains x. Thus $\theta(x) \preccurlyeq \theta(y)$. Conversely, suppose $x \not\preccurlyeq y$. Then there is a COF upset U such that either $x \in U$ and $y \notin U$, or $y \in \neg U$ and $x \notin \neg U$. In the first case, $\mathsf{COFUP}(x)^+ \not\subseteq \mathsf{COFUP}(y)^+$, and in the second case $\mathsf{COFUP}(y)^- \not\subseteq \mathsf{COFUP}(x)^-$. Either way, $\theta(x) \not\preccurlyeq \theta(y)$.

Thus θ has an inverse θ^{-1}. Now we claim that both θ and θ^{-1} are UVP morphisms:

1. Any COF set in $\mathbb{D}(\mathsf{COFUP}(\mathscr{X}))$ is of the form $U^+ \cap V^-$ for COF upsets U, V in $\mathscr{X}$. Since $\theta^{-1}[U^+ \cap V^-] = \theta[U^+] \cap \theta[V^-]$, and the COF sets of $\mathscr{X}$ are closed under finite intersections, it is enough to check that $\theta^{-1}[U^+]$ and $\theta^{-1}[V^-]$ is COF to conclude that θ is F-continuous. Now $\theta(x) \in U^+$ iff $U \in \mathsf{COFUP}(x)^+$ iff $x \in U$, and $\theta(x) \in V^-$ iff $V \in \mathsf{COFUP}(x)^-$ iff $x \in \neg V$. Thus $\theta^{-1}[U^+] = U$ and $\theta^{-1}[V^-] = \neg V$, hence θ is spectral.

2. Similarly, any COF set in $\mathscr{X}$ is of the form $U \cap \neg V$ for COF upsets U and V. Now for any $(F, I) \in \mathbb{D}(\mathscr{X})$, $\theta^{-1}(F, I) \in U$ iff $U \in F$ iff $(F, I) \in U^+$, and $\theta^{-1}(F, I) \in \neg V$ iff $V \in I$ iff $(F, I) \in V^-$, so $\theta[U] = U^+$ and $\theta[\neg V] = V^-$. Hence θ^{-1} is F-continuous.

3. Since θ preserves and reflects $\preccurlyeq$, both θ and θ^{-1} are monotone.

4. Finally, since both θ and θ^{-1} are F-continuous, they are also continuous, hence monotone with respect to $\leq$. But then it follows at once that for any subset S of $\mathscr{X}$, $\downarrow\theta[S] = \theta[\downarrow S]$, and for any subset T of $\mathbb{D}(\mathsf{COFUP}(\mathscr{X}))$, $\downarrow\theta^{-1}[T] = \theta^{-1}[\downarrow T]$. □

We conclude with the main theorem of this section, which establishes a choice free duality between **DL** and UVP spaces.

Theorem 3.7.19. *The category* **UVP** *of UVP spaces and UVP maps is dual to the category* **DL** *of distributive lattices and monotone maps.*

Proof. By Theorems 3.7.5 and 3.7.18, the maps $^+$ and θ are natural transformations from $1_{\mathbf{DL}}$ to $\mathsf{COFUP} \circ \mathbb{D}$ and from $1_{\mathbf{UVP}}$ to $\mathbb{D} \circ \mathsf{COFUP}$. Naturality follows from Theorem 3.6.6 and the observation that $\mathbb{B} = \mathbb{D}$ and $\mathsf{CORO} = \mathsf{COFUP}$ on morphisms. □

3.8 Relation to Upper Vietoris Constructions

In this final section, we connect our two choice-free dualities with their non-constructive counterparts via Vietoris constructions. As mentioned before, assuming the Boolean Prime Ideal theorem is enough to show that the dual UV-space of a Boolean algebra is homeomorphic to the Vietoris hyperspace of its dual Stone space endowed with the Upper-Vietoris topology instead of the full Vietoris topology. We show that a similar result holds in the case of the dual pairwise UV-space of a distributive lattice under the assumption of the (equivalent) Prime Filter Theorem. Recall first that the dual pairwise Stone space of L is the bi-topological space (X, τ_1, τ_2), where X is the collection of all prime filters on L, and τ_1 and τ_2 are the topologies generated by the sets $\{\widehat{a} \mid a \in L\}$ and $\{\check{b} \mid b \in L\}$ respectively. We let δ_1 and δ_2 be the closed sets in τ_1 and τ_2 respectively, and $\beta_1 = \tau_1 \cap \delta_2$ and $\beta_2 = \tau_2 \cap \delta_1$.

Theorem 3.8.1 ([39])**.** *Given a DL L and its dual pairwise Stone space (X, τ_1, τ_2) there is an order isomorphism between $(Filt(L), \supseteq)$ and $(\delta_2, \subseteq)$ on the one hand, and $(Idl(L), \subseteq)$ and $(\tau_1, \subseteq)$ on the other hand.*

The Vietoris construction was developed for pairwise Stone spaces by Lauridsen in [169].

Definition 3.8.2. Let (X, τ_1, τ_2) be a pairwise Stone space, and let $K(\mathscr{X}) = \{G_1 \cap G_2 \neq \emptyset \mid G_1 \in \delta_1, G_2 \in \delta_2\}$. The *Vietoris hyperspace* of X is the set $K(\mathscr{X})$ endowed with the topologies generated by the subbases $\{\Diamond U, \Box U\}_{u \in \beta_1}$ and $\{\Diamond U, \Box U\}_{u \in \beta_2}$, where

$$\Diamond U = \{x \in K(\mathscr{X}) \mid x \cap U \neq \emptyset\},$$

and

$$\Box U = \{x \in K(\mathscr{X}) \mid x \subseteq U\}.$$

Moreover, we let $BK(\mathscr{X})$ be the bi-topological space obtained by endowing $K(\mathscr{X})$ with the upper Vietoris topologies π_1 and π_2 generated by $\{\Box U\}_{U \in \beta_1}$ and $\{\Box U\}_{U \in \beta_2}$ respectively.

Lauridsen [169] shows how the previous definition constitutes the object part of an endofunctor on the category on pairwise Stone spaces, which corresponds to the Vietoris functor on the category of Priestley spaces. We show that for any DL L with dual pairwise Stone space (X, τ_1, τ_2), $(K(\mathscr{X}), \pi_1, \pi_2)$ is bi-homeomorphic to $\mathbb{B}(L)$. This result should not be surprising: points in $K(\mathscr{X})$ are determined by a closed set in δ_1 and a closed set in δ_2. Since, by the duality in [39], the complement of the former corresponds to an ideal on L and the latter corresponds to a filter, each point in $K(\mathscr{X})$ determines a pair of a filter and an ideal. We will show that this pair is actually pseudo-prime, and that conversely any pseudo-prime pair determines a point in $K(\mathscr{X})$. From now on, fix a DL L with dual pairwise Stone space (X, τ_1, τ_2) and dual pairwise UV-space (Y, τ_+, τ_-). Recall that:

- basic opens in τ_1 are of the form $\widehat{a} = \{p \in X \mid a \in p\}$;
- basic opens in τ_2 are of the form $\check{b} = \{p \in X \mid b \notin p\}$;
- basic opens in τ_+ are of the form $a^+ = \{(F, I) \in Y \mid a \in F\}$;
- basic opens in τ_- are of the form $b^- = \{(F, I) \in Y \mid b \in I\}$.

Definition 3.8.3. Let $\beta : Y \to K(\mathscr{X})$ be defined such that for any $(F, I) \in Y$, $\beta(\mathscr{Y}) = \bigcap_{a \in F} \widehat{a} \cap \bigcap_{b \in I} \check{b}$.

Since $\widehat{a}$ is closed in τ_2 and $\check{b}$ is closed in τ_1, β is well-defined. Moreover, it is easy to see that that for any pair (F, I), $\beta(F, I) = \{p \in X \mid F \subseteq p \text{ and } I \cap p = \emptyset\}$.

Conversely, we map $K(\mathscr{X})$ into Y as follows:

Definition 3.8.4. Let $x \in K(\mathscr{X})$. Define $F_x = \{a \in L \mid x \subseteq \widehat{a}\}$ and $I_x = \{b \in L \mid x \cap \widehat{b} = \emptyset\}$, and let $\alpha : K(\mathscr{X}) \to Y$ be defined as $\alpha(x) = (F_x, I_x)$.

Lemma 3.8.5. *The map α is well-defined, i.e., for any $x \in X$, (F_x, I_x) is a pseudo-prime pair.*

Proof. It is routine to verify that F_x is a filter and I_x is an ideal, and that $F_x \cap I_x = \emptyset$ if $x \neq \emptyset$. Suppose $a \wedge b \in I_x$ and $a \in F_x$ for some $a, b \in L$. This means that $x \subseteq \widehat{a}$ and $x \cap \widehat{a \wedge b} = \emptyset$. Since $\widehat{a \wedge b} = \widehat{a} \wedge \widehat{b}$, this implies that $x \cap \widehat{b} = \emptyset$, hence $b \in I_x$. Similarly, suppose that $a \vee b \in F_x$ and $b \in I_x$. Then $x \subseteq \widehat{a \vee b}$ and $x \cap \widehat{b} = \emptyset$. Since $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$, this implies that $x \subseteq \widehat{a}$, and thus $a \in F_x$. Therefore for any $x \in K(\mathscr{X})$, (F_x, I_x) is a pseudo-prime pair on L, and α is well-defined. □

Lemma 3.8.6.

1. *For any $(F, I) \in X$, $\alpha\beta(F, I) = (F, I)$.*
2. *For any $x \in K(\mathscr{X})$, $\beta\alpha(x) = x$.*

Proof.

1. Note first that $\alpha\beta(F,I) = (F_{\beta(F,I)}, I_{\beta(F,I)})$. Now:

$$F_{\beta(F,I)} = \{a \in L \mid \beta(F,I) \subseteq \widehat{a}\} = \{a \in L \mid \forall p \in X(F \subseteq p \wedge I \cap p = \emptyset \to a \in p)\}$$

   and:

$$I_{\beta(F,I)} = \{b \in L \mid \beta(F,I) \cap \widehat{b} = \emptyset\} = \{b \in L \mid \forall p \in X(F \subseteq p \wedge I \cap p = \emptyset \to b \notin p)\}.$$

   This immediately implies that if $F \subseteq F_{\beta(F,I)}$ and $I \subseteq F_{\beta(F,I)}$. Now suppose $a \notin F$. Since (F,I) is pseudo prime, $F \cap a \vee I = \emptyset$, so by the Prime Filter Theorem there is $p \in X$ such that $F \subseteq p$ and $p \cap a \vee I = \emptyset$. Hence $a \notin F_{\beta(F,I)}$. This shows that $F = F_{\beta(F,I)}$. Similarly, if $b \notin I$, then $F \vee b \cap I = \emptyset$, from which it follows that there is $p \in X$ such that $F \vee b \subseteq p$ and $p \cap I = \emptyset$, and hence that $b \notin I_{\beta(F,I)}$. This establishes that $I = I_{\beta(F,I)}$, which completes the proof.

2. Let $x \in K(\mathscr{X})$, and note first that $\beta\alpha(x) = \{p \in X \mid F_x \subseteq p \text{ and } p \cap I_x = \emptyset\}$. Now if $p \in x$, then for any $a \in L$ such that $x \subseteq \widehat{a}$, $a \in p$, and for any $b \in L$ such that $x \cap \widehat{b} = \emptyset$, $b \notin p$. But this implies that $F_x \subseteq p$ and $I_x \cap p = \emptyset$, thus $x \subseteq \beta\alpha(x)$. Conversely, note that, since $x \in K(\mathscr{X})$, there are $G, J \subseteq L$ such that $x = \bigcap_{a \in G} \widehat{a} \cap \bigcap_{b \in J} \check{b}$. Now for any $a, b \in L$, if $a \in G$, then $x \subseteq \widehat{a}$, and if $b \in J$, then $x \cap \widehat{b} = \emptyset$, from which it follows that $a \in F_x$ and $b \in I_x$, hence $H \subseteq F_x$ and $J \subseteq I_x$. Hence:

$$\bigcap_{a \in F_x} \widehat{a} \cap \bigcap_{b \in I_x} \check{b} \subseteq \bigcap_{a \in G} \widehat{a} \cap \bigcap_{b \in J} \check{b} \subseteq x,$$

   which concludes the proof that $\beta\alpha(x) = x$. □

Thus $\alpha = \beta^{-1}$. We now show that β and α are bi-continuous maps.

Lemma 3.8.7. *The maps $\beta : \mathbb{B}(L) \to BK(\mathscr{X})$ and $\alpha : BK(\mathscr{X}) \to \mathbb{B}(L)$ are bi-continuous.*

Proof. We claim that for any $a, b \in L$:

1. $\alpha[\Box\widehat{a}] = a^+$ and $\alpha[\Box\check{b}] = b^-$;

2. $\beta[a^+] = \Box\widehat{a}$ and $\beta[b^-] = \Box - \widehat{b}$.

For 1, note that:

$$\begin{aligned}\alpha[\Box\widehat{a}] &= \{\alpha(x) \in \mathbb{B}(L) \mid x \subseteq \widehat{a}\}\\ &= \{(F_x, I_x) \in \mathbb{B}(L) \mid x \subseteq \widehat{a}\}\\ &= \{(F,I) \in \mathbb{B}(L) \mid a \in F\}\\ &= a^+,\end{aligned}$$

and :

$$\begin{aligned}\alpha[\Box\check{b}] \in \mathbb{B}(L) &= \{\alpha(x) \mid x \subseteq \check{b}\}\\ &= \{(F_x, I_x) \in \mathbb{B}(L) \mid x \cap \widehat{b} = \emptyset\}\\ &= \{(F, I) \in \mathbb{B}(L) \mid b \in I\}\\ &= b^-.\end{aligned}$$

For 2, we have that:

$$\begin{aligned}\beta[a^+] &= \{\beta(F, I) \in K(\mathscr{X}) \mid a \in F\}\\ &= \{\bigcap_{a\in F} \widehat{a} \cap \bigcap_{b\in I} \check{b} \in K(\mathscr{X}) \mid a \in F\}\\ &= \{x \in K(\mathscr{X}) \mid x \subseteq \widehat{a}\}\\ &= \Box\widehat{a},\end{aligned}$$

and:

$$\begin{aligned}\beta[b^-] &= \{\beta(F, I) \in K(\mathscr{X}) \mid b \in I\}\\ &= \{\bigcap_{a\in F} \widehat{a} \cap \bigcap_{b\in I} \check{b} \in K(\mathscr{X}) \mid b \in I\}\\ &= \{x \in K(\mathscr{X}) \mid x \cap \widehat{b} = \emptyset\}\\ &= \Box\check{b}.\end{aligned}$$

Thus both α and β maps basic open sets to basic open sets in both topologies, and are therefore bi-continuous. □

Corollary 3.8.8. *The spaces $BK(\mathscr{X})$ and $\mathbb{B}(L)$ are homeomorphic.*

Proof. Immediate from the previous lemma and the fact that α and β are inverses to one another. □

Therefore pairwise UV-space are precisely the upper Vietoris pairwise Stone spaces. One might wonder whether a similar result can be achieved for UVP spaces. Unsurprisingly, this is also the case. In fact, we show that for any DL L, $\mathbb{D}(L)$ is order-homeomorphic to $UV(Pries(L))$, i.e., to the upper Vietoris hyperspace of the dual Priestley space of L.

Definition 3.8.9. Let L be a DL and $Pries(L) = (X, \tau, \leq)$ its dual Priestley space. Let $C(\mathscr{X})$ be the collection of non-empty convex closed subsets of X, and $V(Pries(L)) = (C(\mathscr{X}), \tau^V, \sqsubseteq)$, where τ^V is the Vietoris topology generated by the subbasis

$$\{\Box U, \Diamond U\}_{U\in ClopUp(\mathscr{X})} \cup \{\Box U, \Diamond U\}_{U\in ClopDn(\mathscr{X})},$$

and $\sqsubseteq$ is the Egli-Milner lift of $\leq$, i.e., for any $U, V \in C(\mathscr{X})$, $U \sqsubseteq V$ iff $U \subseteq {\downarrow}V$ and $V \subseteq {\uparrow}U$.

It is straightforward to check that any closed convex set S is equal to $U \cap V$ for some closed upset U and some closed downset V. Since closed upsets in X correspond to filters on L and open upsets correspond to ideals on L, we have that $C(\mathscr{X}) = K(\mathscr{X})$, and thus we may view α and β as inverse maps between $\mathbb{D}(L)$ and $C(\mathscr{X})$. Now let $UV(Pries(L)) = (C(\mathscr{X}), \pi^V, \sqsubseteq)$, where π^V is the upper Vietoris topology generated by sets of the form $\Box U$ for $U \in CpUp(\mathscr{X}) \cup CpDn(\mathscr{X})$.

Lemma 3.8.10. *The maps $\alpha : UV(Pries(L)) \to \mathbb{D}(L)$ and $\beta : \mathbb{D}(L) \to UV(Pries(L))$ are continuous and monotone.*

Proof.

- Recall that clopen upsets and clopen downsets in $Pries(L)$ are of the form $\widehat{a}$ and $\check{b}$ for $a, b \in L$, hence subbasic opens in $UV(Pries(L))$ are of the form $\Box\widehat{a}$ or $\Box - \widehat{b}$. Similarly, subbasic opens in $\mathbb{D}(L)$ are of the form a^+ or b^- for some $a, b \in L$. Thus by *Lemma* 3.8.7 α and β map subbasic opens to subbasic opens, and are therefore continuous since they are inverses of each other. Hence we only have to check that they are both monotone.

- Let $U, V \in C(\mathscr{X})$ such that $U \sqsubseteq V$. We claim that $\alpha(U) = F_U, I_U \leq \alpha(V) = F_V, I_V$, i.e., $F_U \subseteq F_V$ and $I_V \subseteq I_U$. Suppose that $a \in F_U$. Then $U \subseteq \widehat{a}$, and, since $\widehat{a}$ is an upset, $V \subseteq {\uparrow}U \subseteq \widehat{a}$, from which it follows that $a \in F_V$. Similarly, suppose that $b \in I_V$. Then $V \cap \widehat{b} = \emptyset$, i.e., $V \subseteq \check{b}$, and since $\check{b}$ is a downset, we have that $U \subseteq {\downarrow}V \subseteq \check{b}$, and thus $b \in I_U$. Therefore $\alpha(U) \leq \alpha(V)$, which shows that α is monotone.

- Suppose $(F, I) \leq (G, J)$. We claim that $\beta(F, I) \sqsubseteq \beta(G, J)$. Note that $(F, I) \leq (G, J)$ implies that $F \subseteq G$ and $J \subseteq I$. Now assume $p \in \beta(F, I)$, i.e., $F \subseteq p$ and $p \cap I = \emptyset$. We claim that $p \vee F \cap J = \emptyset$. Indeed, if $c \wedge a \in J$ for some $c \in p, a \in J$, since (G, J) has the RMP we have that $c \in J$. But this is a contradiction, as $J \subseteq I$ and $I \cap p = \emptyset$. Thus let q be a prime filter such that $p \vee G \subseteq q$ and $J \cap q = \emptyset$. Then $p \subseteq q$ and $q \in \beta(G, J)$, and hence $\beta(F, I) \subseteq {\downarrow}\beta(G, J)$. Similarly, assume $q \in \beta(G, J)$, i.e., $G \subseteq q$ and $J \cap q = \emptyset$. Let q^δ be the dual prime ideal of q, i.e., $q^\delta = \{d \in L \mid d \notin q\}$. We claim that $F, q^\delta \vee I = \emptyset$. Indeed, suppose that $d \vee b \in F$ for some $d \in q^\delta, b \in I$. Then since (F, I) has the LJP, $d \in F$. But this is a contradiction, as $F \subseteq q$ and $q \cap q^\delta = \emptyset$. Thus let p be a prime filter such that $F \subseteq p$ and $p \cap q^\delta \vee I = \emptyset$. Clearly, $p \in \beta(F, I)$ and $p \subseteq q$, and therefore $\beta(G, J) \subseteq {\uparrow}\beta(F, I)$. Hence $\beta(F, I) \sqsubseteq \beta(G, J)$, which establishes that β is order-preserving. □

Corollary 3.8.11. *Any UVP space is order homeomorphic to the upper Vietoris space of a Priestley space.*

We conclude with a brief summary of our results in this chapter. As we have seen, the methods used in [41] to obtain a choice-free duality for Boolean algebras can be extended in a relatively straightforward way to de Vries algebras, distributive lattices and Heyting algebras. In all three cases, the reliance on the Prime Filter Theorem can be avoided by considering either filters or filter-ideal pairs, and introducing more structure into the geometric duals of the algebraic structures under consideration. Moreover, there is a very tight link between choice-free and non-constructive representations, which can always be made explicit via Upper Vietoris constructions. As we shall see in the next chapter, Vietoris functors and their algebraic duals will play an important although slightly different role in generalizing the dualities presented here to a duality for the category of all lattices.

Chapter 4

A Duality for Lattices and Fundamental Logic

4.1 Introduction

In the previous chapter, we have seen how mild generalizations of Bezhanishvili and Holliday's choice-free version of Stone duality via UV-spaces can be obtained by following a strategy that is essentially similar to the one adopted in Boolean case. Instead of constructing directly the dual space of a given lattice by taking as points a set of filters with some maximality requirements, one approximates such points via a poset of filters. Topologically, this corresponds to a certain kind of Vietoris hyperspace construction, in which points are approximated by closed subsets. The original lattice L can then be recovered as the fixpoints of some closure operator on its dual space. Importantly, this does not yield a representation of L as a subalgebra of a field of sets, because joins are not computed as set-theoretic unions. While this can be seen as a drawback compared to standard representations via the Prime Filter Theorem in the distributive case, this also opens up the possibility of extending these techniques beyond the distributive setting in a straightforward way.

In this chapter, we will present a duality for the category of lattices that is based on similar ideas. The motivation for this is threefold. First, we already developed a discrete duality for complete lattices in Chapter 2, but left open the issue of topologizing this duality in order to represent all lattices. Our first goal is to obtain such a duality. In other words, just like the UV duality topologizes the forcing duality, the duality we present here topologizes b-frame duality. Second, the dual spaces of lattices that we obtain are very familiar structures, and in many ways our duality is very similar to established dualities for lattices that have been mentioned in the first chapter, such as the filter-ideal based dualities of Allwein and Hartonas [1] and of Hartonas and Dunn [125, 124, 126, 121]. The novelty of our approach is that we derive filter-ideal spaces from an embedding of the category of lattices into the category of distributive lattices, and use then a variation of Priestley duality. In other words, one could think of our work here as deriving the Hartonas-Dunn duality from

Priestley duality. This raises the issue of whether this duality could be obtained without the Axiom of Choice, which we will briefly discuss. Finally, the third motivation for the duality we introduce here is that it is closely related to the semantics for Fundamental Logic in terms of compatibility relations presented in [131, 132]. Accordingly, it may be seen as a way of lifting Holliday's correspondence between compatibility frames and lattices to a full duality, obtaining in the process a topological characterization of the duals of lattices. Because the lattices associated with Fundamental Logic are equipped with a unary operation $\neg$ called a weak pseudo-complement, we will also have to extend our basic duality for lattices to one for weakly pseudo-complemented lattices. As we will see, the ideas that play a key role in the duality for lattices will be equally relevant in establishing a duality for weakly pseudo-complemented lattices. Although we will not have the space to do here, this suggests a promising way of proving versions of some cornerstone results in the semantics of non-classical logics, such as Goldblatt-Thomason-style theorems [110], in the setting of compatibility frames for Fundamental Logic. Indeed, duality-theoretic approaches, particularly those that are close to Stone and Priestley duality are often powerful tools in obtaining elegant proofs of Goldblatt-Thomason theorems [162, 114, 61].

The rest of the chapter is organized as follows. In Section 4.2, we review the main ingredients of our approach, which are a certain of free constructions on lattices and Vietoris hyperspaces on Priestley spaces. In Section 4.3, we use these functors to lift Priestley duality to dualities between two categories of maps between distributive lattices that preserve either the meet or join operation, and two categories on relations between Priestley spaces. These dualities are then use in Section 4.4 to obtain a duality between lattices and FI-spaces that is similar the Hartonas-Dunn duality. Finally, a variation of this duality for weakly-pseudo complemented lattices is proved in Section 4.5, while Section 4.6 concludes with some remarks relating our work in this chapter to b-frames and the relational semantics for Fundamental Logic.

4.2 Preliminaries

In this section, we introduce the basic duality-theoretic ingredients that we will use in the rest of this chapter. Most of the material presented here is already known, in some form or other, in the literature, but we give a fairly detailed and systematic presentation here that is tailored to our purposes. Recall that our main goal in this chapter is to use Priestley duality to obtain a duality for the category of all lattices, as well as a method for representing monotone maps between lattices. The first component of our solution will be the definition of two free functors into the category of distributive lattices. We will then define Priestley duals of such free constructions via Vietoris hyperspaces. Because it will simplify several computations later on, we will also introduce a Priestley dual to the "dualizing" endofunctor on the category of lattices.

4.2.1 Free $\Box$ and $\Diamond$ Constructions

We start by introducing the following categories:

Definition 4.2.1. Let $\mathbf{MLat}^*$ be the category of bounded meet-semilattices (i.e., meet-semilattices with a largest element 1) and top-preserving monotone maps between them. We define the following restrictions of this category:

- $\mathbf{MLat}$ is the subcategory of bounded meet-semilattices and meet-preserving maps between them (where a meet-preserving map also preserves the top element 1);
- $\mathbf{DL}_\wedge$ is the subcategory of bounded distributive lattices and meet-preserving maps between them.

Dually, let $\mathbf{JLat}^*$ be the category of bounded join-semilattices (i.e., join-semilattices with a smallest element 0) and bottom-preserving monotone maps between them. We also define the following restrictions of this category:

- $\mathbf{JLat}$ is the subcategory of bounded join-semilattices and join-preserving maps between them (where a join-preserving map also preserves the bottom element 0);
- $\mathbf{DL}_\vee$ is the subcategory of bounded distributive lattices and join-preserving maps between them.

It is easy to see from the previous definition that the category $\mathbf{Lat}^*$ of lattices and monotone maps between them is precisely the intersection of $\mathbf{MLat}^*$ and $\mathbf{JLat}^*$, the category $\mathbf{Lat}$ of lattices and lattice homomorphisms is precisely the intersection of $\mathbf{MLat}$ and $\mathbf{JLat}$, and finally that $\mathbf{DL}$ is the intersection of $\mathbf{DL}_\wedge$ and $\mathbf{DL}_\vee$.

Definition 4.2.2. The functors $\mathbb{M}_\Box : \mathbf{MLat}^* \to \mathbf{DL}_\vee$ and $\mathbb{M}_\Diamond : \mathbf{JLat}^* \to \mathbf{DL}_\wedge$ are defined as follows:

- For any bounded meet-semilattice L, $\mathbb{M}_\Box(L)$ is the free distributive lattice given by the set of generators $\{\Box a \mid a \in L\}$ and the relations $\{\Box a \wedge \Box b = \Box(a \wedge b), \Box 1 = 1, \Box 0 = 0 \mid a, b \in L\}$.
- For any bounded join-semilattice L, $\mathbb{M}_\Diamond(L)$ is the free distributive lattice given by the set of generators $\{\Diamond a \mid a \in L\}$ and the relations $\{\Diamond a \vee \Diamond b = \Diamond(a \vee b), \Diamond 1 = 1, \Diamond 0 = 0 \mid a, b \in L\}$.
- For any bounded meet-semilattices L and M and any monotone map $f : L \to M$, $\mathbb{M}_\Box(f) : \mathbb{M}_\Box(L) \to \mathbb{M}_\Box(M)$ is defined by letting $\mathbb{M}_\Box(f)(\bigvee_{i\in I} \Box a_i) = \bigvee_{i\in I} \Box f(a_i)$ for any $\{a_i \mid i \in I\} \subseteq L$.
- For any bounded join-semilattices L and M and any monotone map $f : L \to M$, $\mathbb{M}_\Diamond(f) : \mathbb{M}_\Diamond(L) \to \mathbb{M}_\Diamond(M)$ is defined by letting $\mathbb{M}_\Diamond(f)(\bigwedge_{j\in J} \Diamond a_j) = \bigwedge_{j\in J} \Diamond f(a_j)$ for any $\{a_j \mid j \in J\} \subseteq L$.

Note that $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$ are well-defined on morphisms because every element in $\mathbb{M}_\Box(L)$ and $\mathbb{M}_\Diamond(L)$ can be written as a join of generators and as a meet of generators respectively. It is also straightforward to verify that $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$ restrict to functors from **MLat** and **JLat** respectively to **DL**. Moreover, $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$ have the following properties:

Lemma 4.2.3. *Let L be a meet-semilattice and M a join-semilattice.*

1. *There is a meet-semilattice embedding of L into $\mathbb{M}_\Box(L)$ given by $a \mapsto \Box a$, and a join-semilattice embedding of M into $\mathbb{M}_\Diamond(M)$ given by $a \mapsto \Diamond a$.*

2. *For any meet-semilattice N and monotone map $f : L \mapsto N$, $\mathbb{M}_\Box(f)$ is the unique join-semilattice homomorphism $g : \mathbb{M}_\Box(L) \to \mathbb{M}_\Box(N)$ such that $g(\Box a) = \Box f(a)$ for any $a \in L$.*

3. *Dually, for any join-semilattice N and monotone map $f : M \mapsto N$, $\mathbb{M}_\Diamond(f)$ is the unique meet-semilattice homomorphism $g : \mathbb{M}_\Diamond(M) \to \mathbb{M}_\Diamond(N)$ such that $g(\Diamond a) = \Diamond f(a)$ for any $a \in L$.*

4. *For any distributive lattice N and any meet-semilattice homomorphism $f : L \to N$, there is a unique lattice homomorphism $\widetilde{f} : \mathbb{M}_\Box(L) \to N$ such that $\widetilde{f}(\Box a) = f(a)$ for any $a \in M$.*

5. *Dually, for any distributive lattice N and any join-semilattice homomorphism $f : M \to N$, there is a unique lattice homomorphism $\underset{\sim}{f} : \mathbb{M}_\Diamond(M) \to N$ such that $\underset{\sim}{f}(\Diamond a) = f(a)$ for any $a \in M$.*

Proof. Part 1 is immediate. For part 2, it is enough to observe that, if $g; \mathbb{M}_\Box(L) \to \mathbb{M}_\Box(M)$ is a join-semilattice map, then $g(\bigvee_{i\in I} \Box a_i) = \bigvee_{i\in I} g(\Box a_i)$ for any $\{a_i \mid i \in I\} \subseteq L$, so that $g(\Box a) = \Box f(a)$ for all $a \in L$ implies that $g = \mathbb{M}_\Box(f)$. Part 3 is proved completely similarly.

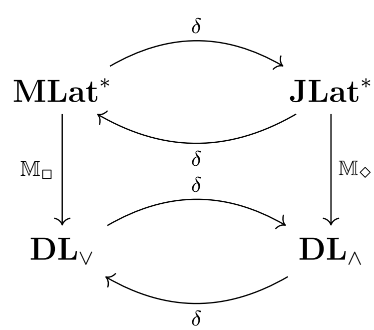
For part 4, let $f : L \to N$ be a meet-semilattice homomorphism, and define $\widetilde{f} : \mathbb{M}_\Box(L) \to N$ by letting $\widetilde{f}(\bigvee_{i\in I} \Box a_i) = \bigvee_{i\in I} \Box f(a_i)$. Clearly, $\widetilde{f}$ is join-preserving. To see that it also preserves meets, note that:

$$\begin{aligned}
\widetilde{f}(\bigvee_{i\in I}\Box a_i)\wedge\widetilde{f}(\bigvee_{j\in J}\Box b_j) &= \bigvee_{i\in I} f(a_i)\wedge\bigvee_{j\in J} f(b_j)\\
&= \bigvee_{i\in I,j\in J}(f(a_i)\wedge f(b_j))\\
&= \bigvee_{i\in I,j\in J} f(a_i\wedge b_j)\\
&= \widetilde{f}(\bigvee_{i\in I,j\in J}\Box(a_i\wedge b_j))\\
&= \widetilde{f}(\bigvee_{i\in I,j\in J}(\Box a_i\wedge\Box b_j))\\
&= \widetilde{f}(\bigvee_{i\in I}\Box a_i\wedge\bigvee_{j\in J}\Box b_j),
\end{aligned}$$

where the second equality holds because N is distributive and the third one because f is meet-preserving. Uniqueness is proved by the same argument as in part 2, and part 5 is a completely dual argument to part 4. □

The following observation makes more precise the sense in which $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$ are dual to one another. Let δ be the functor mapping a poset to its order-dual, and any order-preserving or order-reversing map to itself. Note that δ restricts to functors on **MLat***, **JLat***, **MLat**, **JLat**, $\mathbf{DL}_\wedge$, $\mathbf{DL}_\vee$, **Lat** and **DL** and establishes an isomorphism of categories between **MLat*** and **JLat***, **MLat** and **JLat** and between $\mathbf{DL}_\wedge$ and $\mathbf{DL}_\vee$.

Lemma 4.2.4. *For any meet-semilattice L, $\delta\mathbb{M}_\Box(L)\simeq\mathbb{M}_\Diamond\delta(L)$, and for any join-semilattice M, $\delta\mathbb{M}_\Diamond(M)\simeq\mathbb{M}_\Box\delta(M)$.*

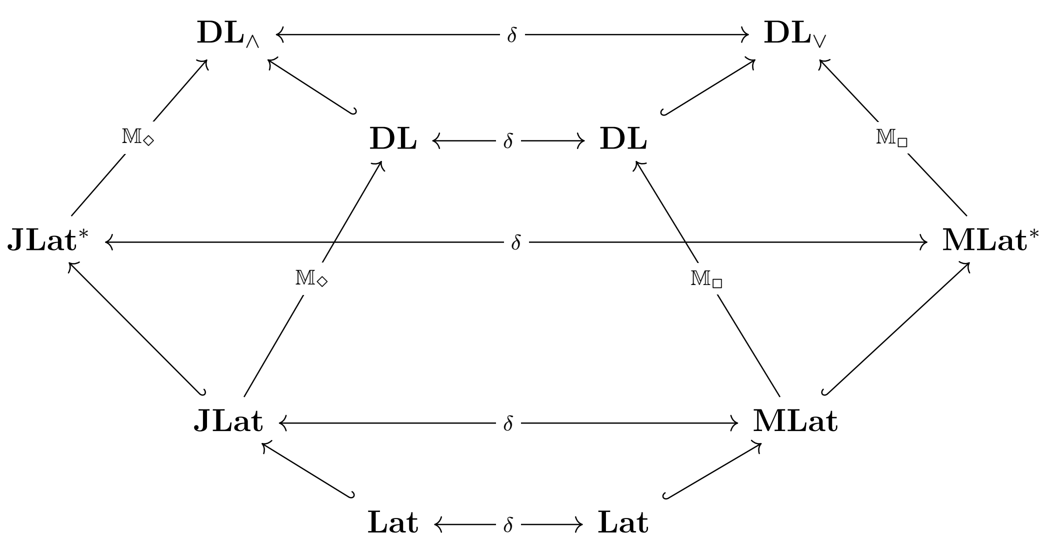


Proof. Since δ establishes isomorphisms between **MLat*** and **JLat*** and between $\mathbf{DL}_\wedge$ and $\mathbf{DL}_\vee$, it is clearly enough to show that $\delta\mathbb{M}_\Box(L)\simeq\mathbb{M}_\Diamond\delta(L)$ for any meet-semilattice L. In what follows, if L is a meet-semilattice (resp. join-semilattice) with meet (resp. join) operation $\wedge$ (resp. $\vee$), we let $\sqcup$ (resp. $\sqcap$) denote the join (resp. meet) operation on its order dual, and if $a\in L$, we write $\boldsymbol{a}$ for the corresponding element in the order dual. Now fix a meet-semilattice L. For any $\{a_i \mid i \in I\}$, let $\zeta_L(\bigvee_{\boldsymbol{i\in I}} \Box \boldsymbol{a_i}) = \bigwedge_{i\in I} \Diamond \boldsymbol{a_i}$. Clearly ζ_L is surjective. Now for any $\{a_i \mid i \in I\}, \{b_j \mid j \in J\} \subseteq L$, one has the following chain of equivalences:

$$\begin{aligned}
\zeta_L(\bigvee_{\boldsymbol{i\in I}} \Box \boldsymbol{a_i}) \leq \zeta_L(\bigvee_{\boldsymbol{j\in J}} \Box \boldsymbol{b_j}) &\Leftrightarrow \bigwedge_{i\in I} \Diamond \boldsymbol{a_i} \leq \bigwedge_{j\in J} \Diamond \boldsymbol{b_j} \\
&\Leftrightarrow \forall j \in J \exists i \in I : \Diamond \boldsymbol{a_i} \leq \Diamond \boldsymbol{b_j} \\
&\Leftrightarrow \forall j \in J \exists i \in I : b_j \leq a_i \\
&\Leftrightarrow \forall j \in J \exists i \in I : \Box b_j \leq \Box a_i \\
&\Leftrightarrow \bigvee_{j\in J} \Box b_j \leq \bigvee_{i\in I} \Box a_i \\
&\Leftrightarrow \bigvee_{\boldsymbol{i\in I}} \Box \boldsymbol{a_i} \leq \bigvee_{\boldsymbol{j\in J}} \Box \boldsymbol{b_j},
\end{aligned}$$

where the third and fourth equivalences hold because $\mathbb{M}_\Box(L)$ and $\mathbb{M}_\Diamond(L)$ are free constructions. This establishes that ζ_L is an order-isomorphism. $\Box$

The relationships between all the functors introduced so far are summed up in the commutative diagram below, where unlabelled arrows are the obvious inclusion maps and we use the same notation for a functor and its restriction to a subcategory.



4.2.2 Two Vietoris Endofunctors

Let us now introduce some well-known Vietoris constructions on Priestley spaces. For some of the results mentioned in this section, we refer the reader to [257, 30].

Definition 4.2.5. The *Upper Vietoris* functor is the functor $\mathbb{V}_\Box : \mathbf{PS} \to \mathbf{PS}$ defined as follows:

- For any Priestley space $(\mathscr{X}, \tau, \leq)$, $\mathbb{V}_\Box(\mathscr{X})$ is the Priestley space given by the set $\uparrow\mathcal{K}(\mathscr{X})$ of all non-empty closed upsets of $\mathscr{X}$, ordered by reverse inclusion and endowed with the topology generated by sets of the form $\Box U = \{C \in \uparrow\mathcal{K}(\mathscr{X}) \mid C \subseteq U\}$ for some $U \in ClopUp(\mathscr{X})$ and their complements.

- For any Priestley map $f : \mathscr{X} \to \mathscr{Y}$, $\mathbb{V}_\Box(f) : \mathbb{V}_\Box(\mathscr{X}) \to \mathbb{V}_\Box(\mathscr{Y})$ maps any $C \in \mathbb{V}_\Box(\mathscr{X})$ to $\uparrow f[C]$.

Dually, the *Lower Vietoris* functor is the functor $\mathbb{V}_\Diamond : \mathbf{PS} \to \mathbf{PS}$ defined as follows:

- For any Priestley space $(\mathscr{X}, \tau, \leq)$, $\mathbb{V}_\Diamond(\mathscr{X})$ is the Priestley space given by the set $\downarrow\mathcal{K}(\mathscr{X})$ of all non-empty closed downsets of $\mathscr{X}$, ordered by inclusion and endowed with the topology generated by the sets $\Diamond U = \{C \in \downarrow\mathcal{K}(\mathscr{X}) \mid C \cap U \neq \emptyset\}$ for some $U \in ClopUp(\mathscr{X})$ and their complements.

- For any Priestley map $f : \mathscr{X} \to \mathscr{Y}$, $\mathbb{V}_\Diamond(f) : \mathbb{V}_\Diamond(\mathscr{X}) \to \mathbb{V}_\Diamond(\mathscr{Y})$ maps any $C \in \mathbb{V}_\Diamond(\mathscr{X})$ to $\downarrow f[C]$.

Of course, one would need to verify that the two functors introduced in Definition 4.2.5 are indeed endofunctors on **PS**, and in particular that $\mathbb{V}_\Box(\mathscr{X})$ and $\mathbb{V}_\Diamond(\mathscr{X})$ are indeed Priestley spaces for any Priestley space $\mathscr{X}$. Instead of checking this directly, however, it is enough to show that $\mathbb{V}_\Box$ and $\mathbb{V}_\Diamond$ are the "topological counterparts" of the restrictions to **DL** of $\mathbb{M}_\Box$ and $\mathbb{V}_\Diamond$ respectively. In order to do this, we start with the following definition, which will play a significant role throughout this chapter.

Definition 4.2.6. Let L be lattice. The *Filter space* of L is the ordered topological space $\mathcal{F}(L) = (Filt(L), \tau_\mathcal{F}, \subseteq)$, where $Filt(L)$ is the set of all proper filters on L and $\tau_\mathcal{F}$ is the topology generated by sets of the form $\widehat{a} = \{F \in \mathcal{F}(L) \mid a \in F\}$ for every $a \in L$, as well as their complements. Dually, the *Ideal space* of L is the ordered topological space $\mathcal{I}(L) = (Idl(L), \tau_\mathcal{I}, \supseteq)$, where $Idl(L)$ is the set of all proper ideals on L and $\tau_\mathcal{I}$ is the topology generated by set of the form $\breve{a} = \{I \in \mathcal{I}(L) \mid a \notin I\}$ and their complements.

The following is well known:

Lemma 4.2.7. *Let L be a distributive lattice. The following are order-preserving homeomorphisms natural in L:*

1. *The map $\eta^L_\Box : Spec\mathbb{M}_\Box(L) \to \mathcal{F}(L)$ given by $\eta^L_\Box(p) = \{a \in L \mid \Box a \in p\}$;*

2. *The map $\epsilon^L_\Box : \mathcal{F}(L) \to \mathbb{V}_\Box Spec(L)$ given by $\epsilon^L_\Box(F) = \bigcap_{a \in F} \widehat{a}$;*

3. *The map $\eta^L_\Diamond : Spec\mathbb{M}_\Diamond(L) \to \mathcal{I}(L)$ given by $\eta^L_\Diamond(p) = \{a \in L \mid \Diamond a \notin p\}$;*

4. *The map $\epsilon^L_\Diamond : \mathcal{I}(L) \to \mathbb{V}_\Diamond Spec(L)$ given by $\epsilon^L_\Diamond(I) = \bigcap_{a \in I} \breve{a}$.*

Proof. It is routine to verify that all maps defined are order-isomorphisms. To check that they are also homeomorphisms, it is enough to observe that each of them and their inverses map basic open sets to basic open sets. We check that this is the case for $\eta^L_\Box$ and $\epsilon^L_\Diamond$, and leave the other cases to the reader.

For $\eta^L_\Box$, it is enough to show that $\eta^L_\Box[\widehat{\Box a}] = \widehat{a}$ for every $a \in L$, since basic opens upsets in $Spec\mathbb{M}_\Box(L)$ are of the form $\widehat{\bigvee_{a\in A} a} = \bigcup_{a\in A} \widehat{a}$ for some finite $A \subseteq L$. But we easily compute:

$$\begin{aligned} \eta^L_\Box(p) \in \widehat{a} &\Leftrightarrow a \in \eta^L_\Box p \\ &\Leftrightarrow \Box a \in p \\ &\Leftrightarrow p \in \widehat{\Box a}, \end{aligned}$$

which shows that ${\eta^L_\Box}^{-1}[\widehat{a}] = \widehat{\Box a}$, and thus that $\widehat{a} = \eta^L_\Box[\widehat{\Box a}]$.

Similarly, for $\epsilon^L_\Diamond$, it is enough to show that $\epsilon^L_\Diamond[\check{a}] = \Diamond\widehat{a}$. But we have:

$$\begin{aligned} \epsilon^L_\Diamond(I) \in \Diamond\widehat{a} &\Leftrightarrow \bigcap_{b\in I} \check{b} \cap \widehat{a} \neq \emptyset \\ &\Leftrightarrow a \notin I \\ &\Leftrightarrow I \in \check{a}, \end{aligned}$$

where the right to left direction in the second equivalence follows from the Prime Filter Theorem. This means that ${\epsilon^L_\Diamond}^{-1}[\Diamond\widehat{a}] = \check{a}$, and thus that $\epsilon^L_\Diamond[\check{a}] = \Diamond\widehat{a}$. $\Box$

As an immediate consequence of Lemma 4.2.7, we have that the maps $\theta^L_\Box : Spec\mathbb{M}_\Box(L) \to \mathbb{V}_\Box Spec(L)$ and $\theta^L_\Diamond : Spec\mathbb{M}_\Diamond(L) \to \mathbb{V}_\Diamond Spec(L)$ are order-homeomorphisms and thus that $\mathbb{V}_\Box Spec(L)$ and $\mathbb{V}_\Diamond Spec(L)$ are Priestley spaces.

$$\begin{array}{ccccc} & & \mathcal{F}(L) & & \\ & \overset{\eta^L_\Box}{\nearrow} & & \overset{\epsilon^L_\Box}{\searrow} & \\ Spec\mathbb{M}_\Box(L) & & \xrightarrow{\ \theta^L_\Box\ } & & \mathbb{V}_\Box Spec(L) \end{array} \qquad \begin{array}{ccccc} & & \mathcal{I}(L) & & \\ & \overset{\eta^L_\Diamond}{\nearrow} & & \overset{\epsilon^L_\Diamond}{\searrow} & \\ Spec\mathbb{M}_\Diamond(L) & & \xrightarrow{\ \theta^L_\Diamond\ } & & \mathbb{V}_\Diamond Spec(L) \end{array}$$

Moreover, we have the following:

Lemma 4.2.8. *Let $f : M \to L$ be a homormorphism betweeen distributive lattices. Then the following diagrams commute:*

$$\begin{array}{ccc} Spec\mathbb{M}_\Box(L) & \xrightarrow{Spec\mathbb{M}_\Box(f)} & Spec\mathbb{M}_\Box(M) \\ \downarrow{\scriptstyle \theta^L_\Box} & & \downarrow{\scriptstyle \theta^M_\Box} \\ \mathbb{V}_\Box Spec(L) & \xrightarrow{\mathbb{V}_\Box Spec(f)} & \mathbb{V}_\Box Spec(L) \end{array} \qquad \begin{array}{ccc} Spec\mathbb{M}_\Diamond(L) & \xrightarrow{Spec\mathbb{M}_\Diamond(f)} & Spec\mathbb{M}_\Diamond(M) \\ \downarrow{\scriptstyle \theta^L_\Diamond} & & \downarrow{\scriptstyle \theta^M_\Diamond} \\ \mathbb{V}_\Diamond Spec(L) & \xrightarrow{\mathbb{V}_\Diamond Spec(f)} & \mathbb{V}_\Diamond Spec(M) \end{array}$$

Proof. We only prove the claim for the diagram on the left. The other case is similar. Let $p \in Spec\mathbb{M}_\Box(L)$. First, we compute that for any $b \in M$, $\Box b \in Spec\mathbb{M}_\Box(f)(p)$ iff $\Box f(b) \in p$, hence $\theta^M_\Box \circ Spec\mathbb{M}_\Box(f) = \bigcap_{\Box f(b)\in p} \widehat{b}$. Moreover, Since $\theta^L_\Box(p) = \bigcap_{\Box a\in p} \widehat{a}$, we have that $\mathbb{V}_\Box Spec(f) \circ \theta^L_\Box = {\uparrow}\{q \mid \exists p' \in \bigcap_{\Box a\in p} \widehat{a} : f^{-1}[p'] = q\}$. Hence it suffices to show that for any $q \in \mathbb{V}_\Box Spec(M)$, $q \in \bigcap_{\Box f(b)\in p} \widehat{b}$ if and only if there is $p' \in \bigcap_{\Box a\in p} \widehat{a}$ such that $q \supseteq f^{-1}[p']$. For the right-to-left inclusion, note that $\Box f(b) \in p$ implies that $\Box f(b) \in p'$ and thus that $b \in f^{-1}[p'] \subseteq q$. For the converse, suppose that $q \in \bigcap_{\Box f(b)\in p} \widehat{b}$. Let $F = \{a \mid \Box a \in p\}$ and $I = \{f(b) \mid b \notin q\}$. Clearly, F is a filter and I is an ideal. Moreover, if there is $c \in F \cap I$, then $\Box f(c) \in p$ and $c \notin q$, contradicting our assumption on q. By the Prime Filter Theorem, there is $p' \in Spec(M)$ such that $F \subseteq p'$ and $p' \cap I = \emptyset$. This shows the left-to-right inclusion, which completes the proof. □

As a consequence, $\mathbb{V}_\Box$ and $\mathbb{V}_\Diamond$ are well-defined endofunctors on **PS**, and the families $\{\theta^L_\Box\}_{L\in\mathbf{DL}}$ and $\{\theta^L_\Diamond\}_{L\in\mathbf{DL}}$ are natural isomorphisms between the functors $Spec\mathbb{M}_\Box$ and $\mathbb{V}_\Box Spec$ and $Spec\mathbb{M}_\Diamond$ and $\mathbb{V}_\Diamond Spec$ respectively. By Priestley duality, we also have natural isomorphisms $\{\kappa^{\mathscr{X}}_\Box\}_{\mathscr{X}\in\mathbf{PS}}$ between the functors $ClopUp\mathbb{V}_\Box$ and $\mathbb{M}_\Box ClopUp$ on one hand, and $\{\kappa^{\mathscr{X}}_\Diamond\}_{\mathscr{X}\in\mathbf{PS}}$ between the functors $ClopUp\mathbb{V}_\Diamond$ and $\mathbb{M}_\Diamond ClopUp$ on the other hand. Although we leave the details to the reader, one can compute that for any Priestley space $\mathscr{X}$ and any set $\{U_i \mid i \in I\}$ of clopen upsets in $\mathscr{X}$, $\kappa^{\mathscr{X}}_\Box(\bigwedge_{i\in I} \Box U_i) = \bigcap_{i\in I} \Box U_i$, and $\kappa^{\mathscr{X}}_\Diamond(\bigvee_{i\in I} \Diamond U_i) = \bigcup_{i\in I} \Diamond U_i$.

Finally, we will use below the observation that the functors $\mathbb{V}_\Box$ and $\mathbb{V}_\Diamond$ are also dual to one another in the same sense that $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$ are also dual. Let us conclude this section by spelling this out in more detail.

4.2.3 Dualization

Definition 4.2.9. For any Priestley space $\mathscr{X} = (X, \tau, \leq)$, let $\gamma(\mathscr{X})$ be the Priestley space $(X, \tau, \geq)$. For any order-continuous map $f : \mathscr{X} \to \mathscr{Y}$ between Priestely spaces, let $\gamma(f) : \gamma(\mathscr{X}) \to \gamma(\mathscr{Y})$ be simply f. This induces a functor $\gamma : \mathbf{PS} \to \mathbf{PS}$.

It is straightforward to verify that γ is well defined. In particular, if $x \not\geq y$ for some $x, y \in \gamma(\mathscr{X})$, then $y \not\leq x$ in $\mathscr{X}$, so by the Priestley Separation Axiom in $\mathscr{X}$ there is a clopen upset U such that $y \in U$ and $x \notin U$. But then $-U$ is a clopen upset in $\gamma(\mathscr{X})$ such that $x \in -U$ and $y \notin -U$. This shows that $\gamma(\mathscr{X})$ satisfies the Priestley Separation Axiom as well.

Lemma 4.2.10. *There are natural isomorphisms $\{\alpha_L\}_{L\in\mathbf{DL}}$ and $\{\beta_{\mathscr{X}}\}_{\mathscr{X}\in\mathbf{PS}}$ respectively between the functors $\gamma\circ Spec$ and $Spec\circ\delta$ and between the functors $ClopUp\circ\gamma$ and $\delta\circ ClopUp$.*

Proof. For any distributive lattice L, let $\alpha_L : \gamma Spec(L) \to Spec\delta(L)$ be the map $p \mapsto L \setminus p$ for any $p \in Spec(L)$. Note that this is well-defined since the complement of a prime filter on L is a prime ideal on L, and thus a prime filter on $\delta(L)$. Clearly, α_L is bijective, monotone and order-reflecting, and $\alpha_L^{-1}[\widehat{\boldsymbol{a}}] = \check{a}$ for any $a \in L$, which is enough to establish that it is in fact a homeomorphism of Priestley spaces.

Similarly, for any Priestley space $\mathscr{X} = (X, \tau, \leq)$, let $\beta_{\mathscr{X}} : ClopUp\gamma(\mathscr{X}) \to \delta ClopUp(\mathscr{X})$ be the map $U \mapsto -U$. Note that this is well defined, since clopen upsets in $\gamma(\mathscr{X})$ are precisely the clopen downsets in $\mathscr{X}$, hence their complements are the clopen upsets of $\mathscr{X}$. Clearly, $\beta_{\mathscr{X}}$ is bijective and order reversing as a map into $(ClopUp\mathscr{X}), \subseteq)$, hence it is an order isomorphism between $ClopUp\gamma(\mathscr{X})$ and $\delta ClopUp(\mathscr{X})$. Finally, the naturality conditions for $\{\alpha_L\}_{L\in\mathbf{DL}}$ and $\{\beta_{\mathscr{X}}\}_{\mathscr{X}\in\mathbf{PS}}$ are easy to check and left to the reader. □

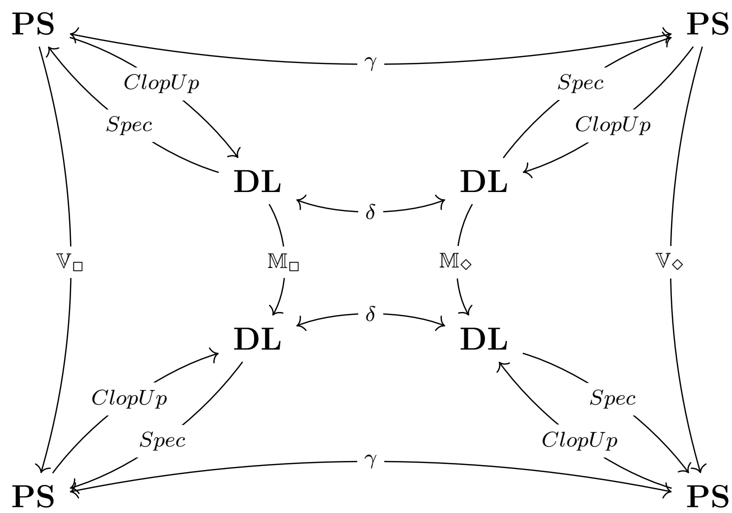
Corollary 4.2.11. *For any Priestley space $\mathscr{X}$, $\gamma\mathbb{V}_\Box(\mathscr{X}) \cong \mathbb{V}_\Diamond\gamma(\mathscr{X})$, and $\mathbb{V}_\Box\gamma(\mathscr{X}) \cong \gamma\mathbb{V}_\Diamond(\mathscr{X})$ naturally in $\mathscr{X}$.*

Proof. Since γ is a self-inverse endofunctor, it is enough to define an order-homeomorphism from $\gamma\mathbb{V}_\Box(\mathscr{X})$ to $\mathbb{V}_\Diamond\gamma(\mathscr{X})$. By Priestley duality, we may assume that $\mathscr{X}$ is $Spec(L)$ for some distributive lattice L. But then we define the order-homeomorphism ξ_L as shown in the following diagram:

$$\begin{array}{ccc}
Spec\delta\mathbb{M}_\Box(L) & \xrightarrow{Spec(\zeta_L)} & Spec\mathbb{M}_\Diamond\delta(L) \\
\Uparrow \alpha_{\mathbb{M}_\Box(L)} & & \Downarrow \theta_\Diamond^{\delta(L)} \\
\gamma Spec\mathbb{M}_\Box(L) & & \mathbb{V}_\Diamond Spec\delta(L) \\
\Uparrow \gamma(\theta_\Box^{L\,-1}) & & \Downarrow \mathbb{V}_\Diamond({\alpha_{\delta(L)}}^{-1}) \\
\gamma\mathbb{V}_\Box Spec(L) & \xrightarrow{\xi_L} & \mathbb{V}_\Diamond\gamma Spec(L)
\end{array}$$

Note that the first and penultimate arrows are order-homeomorphisms by Lemma 4.2.8, the second and last are order-homeomorphisms by Lemma 4.2.10, and the middle one is an order-homeomorphism by Lemma 4.2.4. Again, naturality is left to the reader. □

We conclude this section with a diagram summarizing the various relationships between the endofunctors on **DL** introduced so far. All triangles in the diagram below commute up to isomorphism.



4.3 Lifting Priestley Duality

In this section, we lift Priestley duality to dualities between the arrow categories of $\mathbf{DL}_\vee$ and $\mathbf{DL}_\wedge$ and categories of relations between Priestley spaces. The key idea is inspired from the standard way of lifting Stone Duality to a duality between modal Boolean algebras and Stone spaces with relations. The same approach has also already been extended to distributive lattices and Priestley spaces, in order to obtain a duality-theoretic treatment of positive modal logic. There are, however, two main difference between this approach and the one developed here. First, the modal algebras considered in positive modal logic are distributive lattices expanded with both a meet-preserving and a join-preserving operation. By contrast, our approach in this section treats the two separately. Second, in positive modal logic, $\Box$ and $\Diamond$ are operators from a distributive lattice into itself. As a consequence, the corresponding relations on Priestley spaces are relations on a single Priestley space. By contrast, we will be interested in join-preserving and meet-preserving maps from one distributive lattice into another. Consequently, the relations we will be working with are relations from a Priestley space to another. We start by clarifying the relationship between such closed relations on Priestley spaces and Vietoris hyperspaces.

4.3.1 Closed Relations on Priestley Spaces

We start with the following definitions.

Definition 4.3.1. Let $\mathscr{X} = (X, \tau_X, \leq_X)$ and $\mathscr{Y} = (Y, \tau_Y, \leq_Y)$ be Priestley spaces. A relation $R \subseteq X \times Y$ is a *lower closed relation* if it has the following properties:

1. For any $x \in X$, $R(x) = \{y \in Y \mid xRy\}$ is closed;
2. For any $U \in ClopUp(\mathscr{Y})$, $R^{-1}[U] = \{x \in X \mid R(x) \cap U \neq \emptyset\}$ is a clopen upset;

3. $\geq_Y \circ R \circ \geq_X \subseteq R$. In other words, for any $x, x' \in X$ and $y, y' \in Y$, $x \geq_X x'$, $x'Ry$ and $y \geq_Y y'$ together imply xRy', as shown in the diagram below.

$$\begin{array}{ccc} x & \overset{R}{\dashrightarrow} & y' \\ {\scriptstyle \geq_X}\downarrow & & \uparrow{\scriptstyle \geq_Y} \\ x' & \overset{R}{\longrightarrow} & y \end{array}$$

Dually, a relation $R \subseteq X \times Y$ is an *upper closed relation* if it has the following properties:

1. For any $x \in X$, $R(x) = \{y \in Y \mid xRy\}$ is closed;

2. For any $U \in ClopDn(\mathscr{Y})$, $R^{-1}[U] = \{x \in X \mid R(x) \cap U \neq \emptyset\}$ is a clopen downset;

3. $\leq_Y \circ R \circ \leq_X \subseteq R$. In other words, for any $x, x' \in X$ and $y, y' \in Y$, $x \leq_X x'$, $x'Ry$ and $y \geq_Y y'$ together imply xRy', as shown in the diagram below.

$$\begin{array}{ccc} x & \overset{R}{\dashrightarrow} & y' \\ {\scriptstyle \leq_X}\downarrow & & \uparrow{\scriptstyle \leq_Y} \\ x' & \overset{R}{\longrightarrow} & y \end{array}$$

The following establishes a correspondence between closed relations and morphisms into Vietoris hyperspaces. It is a straightforward generalization of the coalgebraic approach to Kripke semantics for classical modal logic.

Lemma 4.3.2. *For any Priestley spaces $\mathscr{X} = (X, \tau_X, \leq_X)$ and $\mathscr{Y} = (Y, \tau_Y, \leq_Y)$, there is a one-to-one correspondence between the set $R^{\downarrow}(\mathscr{X}, \mathscr{Y})$ of lower closed relations on $X \times Y$ and the set of morphisms in* **PS** *from $\mathscr{X}$ to $\mathbb{V}_{\diamond}(\mathscr{Y})$. Dually, there is a one-to-one correspondence between the set $R^{\uparrow}(\mathscr{X}, \mathscr{Y})$ of upper closed relations on $X \times Y$ and the set of morphisms in* **PS** *from $\mathscr{X}$ to $\mathbb{V}_{\Box}(\mathscr{Y})$.*

Proof. Given an order-continuous morphism $f : \mathscr{X} \to \mathbb{V}_{\diamond}(\mathscr{Y})$, let $R_f \subseteq X \times Y$ be defined by letting xRy iff $y \in f(x)$. It is routine to check that R is a lower closed relation (note in particular that property 2 holds because $R^{-1}[U] = \diamond U \in ClopUp(\mathbb{V}_{\diamond}(\mathscr{Y}))$ for any $U \in ClopUp(\mathscr{Y})$. Conversely, given a lower closed relation R on $X \times Y$, let $f_R : \mathscr{X} \to \mathbb{V}_{\diamond}(\mathscr{Y})$ be the map $x \mapsto R(x)$. Then it is easy to check that properties 1 and 3 of lower closed relations imply that f_R is well defined and monotone, and that property 2 implies that it is continuous, since $f_R^{-1}[\diamond U] = R^{-1}[U]$ for any $U \in ClopUp(\mathscr{Y})$. Finally, we clearly have that the maps $f \mapsto R_f$ and $R \mapsto f_R$ are inverses of one another.

Hence we have a bijection ν between $\mathbf{Hom}_{\mathbf{PS}}(\mathscr{X}, \mathbb{V}_{\diamond}(\mathscr{Y}))$ and $R^{\downarrow}(\mathscr{X}, \mathscr{Y})$, where the latter is the set of all lower closed relations on $X \times Y$. Let us now see that we also have a bijection between $\mathbf{Hom_{PS}}(X, \mathbb{V}_\Box(\mathscr{Y}))$ and $R^\uparrow(\mathscr{X}, \mathscr{Y})$. Notice first that $R \in R^\downarrow(\mathscr{X}, \mathscr{Y})$ iff $R \in R^\uparrow(\gamma(\mathscr{X}), \gamma(\mathscr{Y}))$. Moreover, since γ is self-inverse, we also have that application of γ yields a bijection between $\mathbf{Hom_{PS}}(\mathscr{X}, \mathbb{V}_\Box(\mathscr{Y}))$ and $\mathbf{Hom_{PS}}(\gamma(\mathscr{X}), \gamma\mathbb{V}_\Box(\mathscr{Y}))$. Finally, recall that we have an order-homeomorphism $\xi : \gamma\mathbb{V}_\Box(\mathscr{X}) \to \mathbb{V}_\Diamond\gamma(\mathscr{Y})$, which induces by post-composition with ξ a bijection from $\mathbf{Hom_{PS}}(\gamma(\mathscr{X}), \gamma\mathbb{V}_\Box(\mathscr{Y}))$ to $\mathbf{Hom_{PS}}(\gamma(\mathscr{X}), \mathbb{V}_\Diamond\gamma(\mathscr{Y}))$. Putting things together, this yields the following chain of bijections:

$$
\begin{array}{ccccc}
 & & \mathbf{Hom_{PS}}(\gamma(\mathscr{X}), \mathbb{V}_\Diamond\gamma(\mathscr{Y})) & & \\
 & \overset{\xi\circ -}{\nearrow} & & \overset{\nu}{\searrow} & \\
\mathbf{Hom_{PS}}(\gamma(\mathscr{X}), \gamma\mathbb{V}_\Box(\mathscr{Y})) & & & & R^\downarrow(\gamma(\mathscr{X}), \gamma(\mathscr{Y})) \\
\Uparrow \gamma & & & & \Downarrow = \\
\mathbf{Hom_{PS}}(\mathscr{X}, \mathbb{V}_\Box(\mathscr{Y})) & & \xRightarrow{\quad\mu\quad} & & R^\uparrow(\mathscr{X}, \mathscr{Y})
\end{array}
$$

which yields a bijection μ between $\mathbf{Hom_{PS}}(\mathscr{X}, \mathbb{V}_\Box(\mathscr{Y}))$ and $R^\uparrow(\mathscr{X}, \mathscr{Y})$.

□

Clearly, the maps μ and ν defined in the previous lemma are natural in $\mathscr{X}$ and $\mathscr{Y}$, although we leave the proof of this to the reader. Let us conclude with the following definitions.

Definition 4.3.3. Let $\mathbf{PSR}_\downarrow$ be the category whose objects are lower closed relations between Priestley spaces and where a morphism f from $R : \mathscr{X}_1 \to \mathscr{X}_2$ to $S : \mathscr{Y}_1 \to \mathscr{Y}_2$ is a pair of maps (f_1, f_2) such that $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ for $i = 1, 2$ and the following diagram commutes:

$$
\begin{array}{ccc}
\mathscr{X}_1 & \xrightarrow{\quad\nu(R)\quad} & \mathbb{V}_\Diamond(\mathscr{X}_2) \\
\Big\downarrow f_1 & & \Big\downarrow \mathbb{V}_\Diamond(f_2) \\
\mathscr{Y}_1 & \xrightarrow{\quad\nu(S)\quad} & \mathbb{V}_\Diamond(\mathscr{Y}_2)
\end{array}
$$

Equivalently, for all $x \in \mathscr{X}_1$ and $y \in \mathscr{Y}_2$, $f_1(x)Sy$ iff there is $x' \in \mathscr{X}_2$ such that xRx' and $y \leq_{\mathscr{Y}_2} f_2(x')$.

Dually, $\mathbf{PSR}_\uparrow$ is the category whose objects are upper closed relations between Priestley spaces and where a morphism f from $R : \mathscr{X}_1 \to \mathscr{X}_2$ to $S : \mathscr{Y}_1 \to \mathscr{Y}_2$ is a pair of maps (f_1, f_2) such that $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ for $i = 1, 2$ and the following diagram commutes:

$$\begin{array}{ccc} \mathscr{X}_1 & \xrightarrow{\mu(R)} & \mathbb{V}_{\Box}(\mathscr{X}_2) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathbb{V}_{\Box}(f_2)} \\ \mathscr{Y}_1 & \xrightarrow{\mu(S)} & \mathbb{V}_{\Box}(\mathscr{Y}_2) \end{array}$$

Equivalently, for all $x \in \mathscr{X}_1$ and $y \in \mathscr{Y}_2$, $f_1(x)Sy$ iff there is $x' \in \mathscr{X}_2$ such that xRx' and $y \geq_{\mathscr{Y}_2} f_2(x')$.

4.3.2 Flat and Sharp Functors

We are now in a position to establish a dual equivalence between join-preserving (resp. meet-preserving) maps between distributive lattices and lower (res. upper) closed relations between Priestley spaces. We first introduce the following definitions:

Definition 4.3.4. Let $\mathbf{DL}_{\downarrow}$ (resp. $\mathbf{DL}_{\uparrow}$) be the category whose objects are join-preserving maps (resp. meet-preserving maps) between distributive lattices and where a morphism f from $g : L_1 \to L_2$ to $h : M_1 \to M_2$ is a pair (f_1, f_2) such that $f_i : L_i \to M_i$ is a lattice morphism for $i = 1, 2$, and the following diagram commutes in $\mathbf{DL}_{\vee}$ (resp. in $\mathbf{DL}_{\wedge}$):

$$\begin{array}{ccc} L_1 & \xrightarrow{g} & L_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{h} & M_2 \end{array}$$

Let us now define a functor $S^{\flat} : \mathbf{DL}_{\downarrow} \to \mathbf{PSR}_{\downarrow}$ as follows. Recall first that any join-preserving map $f : L \to M$ between two distributive lattices lifts uniquely to a lattice homomorphism $\underset{\sim}{f} : \mathbb{M}_{\Diamond}(L) \to M$ with a property that $\underset{\sim}{f}(\Diamond a) = f(a)$ for all $a \in L$. By Priestley duality, this yields an order-continuous map $Spec(\underset{\sim}{f}) : Spec(M) \to Spec\mathbb{M}_{\Diamond}(L)$. Moreover, post-composing with the map $\theta^L_{\Diamond}$ yields a map $\theta^L_{\Diamond} \circ Spec(\underset{\sim}{f}) : Spec(M) \to \mathbb{V}_{\Diamond}Spec(L)$. Finally, we define $S^{\flat}(f)$ as the relation $\nu^{-1}(\theta^L_{\Diamond} \circ Spec(\underset{\sim}{f}))$ on $Spec(M) \times Spec(L)$. It is worth describing directly the relation $S^{\flat}(f)$. For any $p \in Spec(M)$ and $q \in Spec(L)$:

$$\begin{aligned} pS^{\flat}(f)q &\Leftrightarrow q \in \theta^{L}_{\diamond} \circ Spec(\underset{\sim}{f})(p) \\ &\Leftrightarrow q \in \theta^{L}_{\diamond}(\underset{\sim}{f}^{-1}[p]) \\ &\Leftrightarrow q \in \bigcap_{\diamond a \notin \underset{\sim}{f}^{-1}[p]} \breve{a} \\ &\Leftrightarrow q \in \bigcap_{f(a) \notin p} \breve{a} \\ &\Leftrightarrow q \subseteq f^{-1}[p]. \end{aligned}$$

Let us now define $S^{\flat}$ on morphisms. Notice first that the pair (f_1, f_2) is a morphism in $\mathbf{DL}_{\downarrow}$ if and only if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{M}_{\diamond}(L_1) & \xrightarrow{\;\underset{\sim}{g}\;} & L_2 \\ \Big\downarrow{\scriptstyle \mathbb{M}_{\diamond}(f_1)} & & \Big\downarrow{\scriptstyle f_2} \\ \mathbb{M}_{\diamond}(M_1) & \xrightarrow{\;\underset{\sim}{h}\;} & M_2 \end{array}$$

Indeed, for any $\bigwedge_{i\in I} \diamond a_i \in \mathbb{V}_{\diamond}(L_1)$,

$$f_2(\underset{\sim}{g}(\bigwedge_{i\in I} \diamond a_i)) = f_2(\bigwedge_{i\in I} g(a_i)) = \bigwedge_{i\in I} f_2(g(a_i)),$$

and

$$\underset{\sim}{h}(\mathbb{M}_{\diamond}(f_1)(\bigwedge_{i\in I} a_i)) = \underset{\sim}{h}(\bigwedge_{i\in I} \diamond f_1(a_i)) = \bigwedge_{i\in I} h(f_1(a_i)).$$

Moreover, in the following diagram:

$$\begin{array}{ccccc} Spec(L_2) & \xrightarrow{\;Spec(\underset{\sim}{g})\;} & Spec\mathbb{M}_{\diamond}(L_1) & \xrightarrow{\;\theta^{L_1}_{\diamond}\;} & \mathbb{V}_{\diamond}Spec(L_1) \\ \Big\uparrow{\scriptstyle Spec(f_2)} & & \Big\uparrow{\scriptstyle Spec\mathbb{M}_{\diamond}(f_1)} & & \Big\uparrow{\scriptstyle \mathbb{V}_{\diamond}Spec(f_1)} \\ Spec(M_2) & \xrightarrow{\;Spec(\underset{\sim}{g})\;} & Spec\mathbb{M}_{\diamond}(M_1) & \xrightarrow{\;\theta^{M_1}_{\diamond}\;} & \mathbb{V}_{\diamond}Spec(M_1) \end{array}$$

the left square commutes by Priestley duality, and the right square commutes by Lemma 4.2.8. Therefore the pair $(Spec(f_2), Spec(f_1))$ is a morphism from $S^{\flat}(g)$ to $S^{\flat}(h)$, so may set $S^{\flat}(f_1, f_2) = (Spec(f_2), Spec(f_1))$.

Let us now define a functor $C^{\flat} : \mathbf{PSR}_{\downarrow} \to \mathbf{DL}_{\downarrow}$ that is dual to $S^{\flat}$. Given a relation $R \in R^{\downarrow}(\mathscr{X}, \mathscr{Y})$, let $C^{\flat}(R) : ClopUp(\mathscr{Y}) \to ClopUp(\mathscr{X})$ be defined as $C^{\flat}(R)(U) = R^{-1}[U]$

for any $U \in ClopUp(\mathscr{Y})$. Note that $C^\flat(R)$ is well defined by property 2 of closed relations. Moreover, we have that $R^{-1}[\emptyset] = \emptyset$, and for any $x \in X$ and $U, V \in ClopUp(\mathscr{Y})$,

$$x \in R^{-1}[U \cup V] \Leftrightarrow R(x) \cap (U \cup V) \neq \emptyset \Leftrightarrow R(x) \cap U \neq \emptyset \text{ or } R(x) \cap V \neq \emptyset \Leftrightarrow x \in R^{-1}[U] \cup R^{-1}[V].$$

Hence $C^\flat(R)$ is an object in $\mathbf{DL}_\downarrow$. Suppose that (f_1, f_2) is a morphism from $R \in R^\downarrow(\mathscr{X}, \mathscr{X})$ to $S \in R^\downarrow(\mathscr{Y}, \mathscr{Y})$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{X}_1 & \xrightarrow{\quad \nu(R) \quad} & \mathbb{V}_\Diamond(\mathscr{X}_2) \\ \big\downarrow {\scriptstyle f_1} & & \big\downarrow {\scriptstyle \mathbb{V}_\Diamond(f_2)} \\ \mathscr{Y}_1 & \xrightarrow{\quad \nu(S) \quad} & \mathbb{V}_\Diamond(\mathscr{Y}_2) \end{array}$$

Consequently, we have the following diagram:

$$\begin{array}{ccccc} \mathbb{M}_\Diamond ClopUp(\mathscr{Y}_2) & \xrightarrow{\quad \kappa_\Diamond^{\mathscr{Y}_2} \quad} & ClopUp\mathbb{V}_\Diamond(\mathscr{Y}_2) & \xrightarrow{\ ClopUp(\nu(S))\ } & ClopUp(\mathscr{Y}_1) \\ \big\downarrow {\scriptstyle \mathbb{M}_\Diamond ClopUp(f_2)} & & \big\downarrow {\scriptstyle ClopUp\mathbb{V}_\Diamond(f_2)} & & \big\downarrow {\scriptstyle ClopUp(f_1)} \\ \mathbb{M}_\Diamond ClopUp(\mathscr{X}_2) & \xrightarrow{\quad \kappa_\Diamond^{\mathscr{X}_1} \quad} & ClopUp\mathbb{V}_\Diamond(\mathscr{X}_2) & \xrightarrow{\ ClopUp(\nu(R))\ } & ClopUp(\mathscr{X}_1) \end{array}$$

where the left square commutes by the duality of $\mathbb{M}_\Diamond$ and $\mathbb{V}_\Diamond$, and the right square commutes by Priestley duality. Moreover, we easily check that $ClopUp(\nu(S)) \circ \kappa_\Diamond^{\mathscr{Y}_2}(\Diamond U) = C^\flat(S)(U)$ for every $U \in ClopUp(\mathscr{Y}_2)$, and that $ClopUp(\nu(R)) \circ \kappa_\Diamond^{\mathscr{X}_2}(\Diamond V) = C^\flat(R)(V)$ for every $V \in ClopUp(\mathscr{X}_2)$, which means that $ClopUp(\nu(S)) \circ \kappa_\Diamond^{\mathscr{Y}_2} = \underset{\sim}{C^\flat(S)}$ and $ClopUp(\nu(R)) \circ \kappa_\Diamond^{\mathscr{X}_2} = \underset{\sim}{C^\flat(R)}$. Hence the following square commutes:

$$\begin{array}{ccc} \mathbb{M}_\Diamond ClopUp(\mathscr{Y}_2) & \xrightarrow{\quad \underset{\sim}{C^\flat(S)} \quad} & ClopUp(\mathscr{Y}_1) \\ \big\downarrow {\scriptstyle \mathbb{M}_\Diamond ClopUp(f_2)} & & \big\downarrow {\scriptstyle ClopUp(f_1)} \\ \mathbb{M}_\Diamond ClopUp(\mathscr{X}_2) & \xrightarrow{\quad \underset{\sim}{C^\flat(R)} \quad} & ClopUp(\mathscr{X}_1) \end{array}$$

By the observation above, this square commutes if and only if the following square also commutes:

$$\begin{array}{ccc} ClopUp(\mathscr{Y}_2) & \xrightarrow{\quad \underset{\sim}{C^\flat(S)} \quad} & ClopUp(\mathscr{Y}_1) \\ \big\downarrow {\scriptstyle ClopUp(f_2)} & & \big\downarrow {\scriptstyle ClopUp(f_1)} \\ ClopUp(\mathscr{X}_2) & \xrightarrow{\quad \underset{\sim}{C^\flat(R)} \quad} & ClopUp(\mathscr{X}_1) \end{array}$$

from which we conclude that $(ClopUp(f_1), ClopUp(f_2)$ is a morphism from $C^\flat(S)$ to $C^\flat(R)$ in $\mathbf{DL}_\downarrow$, and we may therefore set $C^\flat(f_1, f_2) = (ClopUp(f_1), ClopUp(f_2))$.

It remains to show that $S^\flat$ and $C^\flat$ establish a dual equivalence of categories. But this, in fact, almost immediately follows from Priestley duality and the definitions of $S^\flat$ and $C^\flat$. Indeed, recall that the dual equivalence witnessed by *Spec* and *CloUp* is established via the families of natural isomorphisms $\{\widehat{\cdot}_L : L \to ClopUpSpec(L)\}_{L \in \mathbf{DL}}$ and $\{\dot{\in}_{\mathscr{X}} : \mathscr{X} \to SpecClopUp(\mathscr{X})\}_{\mathscr{X} \in \mathbf{PS}}$. Although we leave the details to the reader, we can easily verify that the following diagrams also commute for any $f \in \mathbf{DL}_\downarrow$ and $R \in \mathbf{PSR}_\downarrow$:

$$
\begin{array}{ccc}
L & \xrightarrow{\ \widehat{\cdot}_L\ } & ClopUpSpec(L) \\
\Big\downarrow f & & \Big\downarrow C^\flat S^\flat(f) \\
M & \xrightarrow{\ \widehat{\cdot}_M\ } & ClopUpSpec(M)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\ \dot{\in}_{\mathscr{X}}\ } & SpecClopUp(\mathscr{X}) \\
\Big\downarrow \nu(R) & & \Big\downarrow \nu(S^\flat C^\flat(R)) \\
\mathbb{V}_\diamond(\mathscr{Y}) & \xrightarrow{\ \dot{\in}_{\mathscr{Y}}\ } & \mathbb{V}_\diamond SpecClopUp(\mathscr{Y})
\end{array}
$$

which is essentially enough to establish our dual equivalence.

Finally, we may use this result to also obtain a dual equivalence between $\mathbf{DL}_\uparrow$ and $\mathbf{PSR}_\uparrow$ in a straightforward way. Indeed, using our dualizing functors δ and γ, we may simply use the functors $S^\sharp : \mathbf{DL}_\uparrow \to \mathbf{PSR}_\uparrow$ and $C^\sharp : \mathbf{PSR}_\uparrow \to \mathbf{DL}_\uparrow$ by letting $S^\sharp := \gamma \circ S^\flat \circ \delta$ and $C^\sharp := \delta \circ C^\flat \circ \gamma$. It will be useful later on to have a more direct definition of $S^\sharp$ and $C^\sharp$ on objects. We leave it to the reader to check that this definition is essentially the same as the one above.[1] Given a meet-preserving map $f : L \to M$, $S^\sharp(f)$ is the upper closed relation on $Spec(M) \times Spec(L)$ given by:

$$
\begin{aligned}
pS^\sharp(f)q &\Leftrightarrow q \in \theta^L_\square \circ Spec(\widetilde{f})(p) \\
&\Leftrightarrow q \in \theta^L_\square(\widetilde{f}^{-1}[p]) \\
&\Leftrightarrow q \in \bigcap_{\square a \in \widetilde{f}^{-1}[p]} \widehat{a} \\
&\Leftrightarrow q \in \bigcap_{f(a) \in p} \widehat{a} \\
&\Leftrightarrow f^{-1}[p] \subseteq q.
\end{aligned}
$$

Conversely, given an upper closed relation R on $\mathscr{X} \times \mathscr{Y}$, the meet-preserving map $C^\sharp(R) : ClopUp(\mathscr{Y}) \to ClopUp(\mathscr{X})$ is given by:

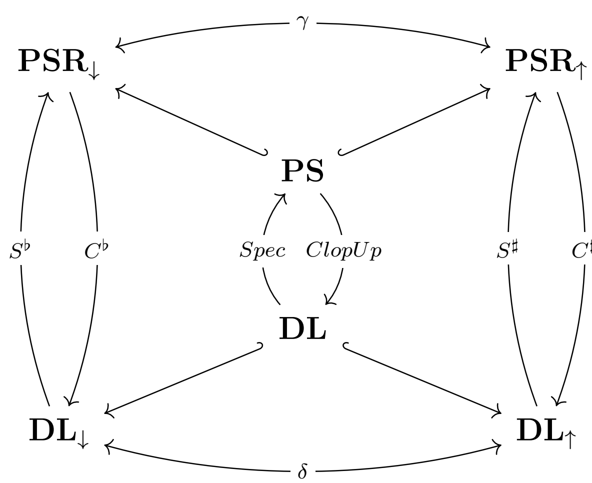
[1] To be precise, the direct and indirect definitions yield the same functors up to isomorphism, which is witnessed by the natural isomorphisms $\{\alpha_L\}_{L \in \mathbf{DL}}$ and $\{\beta_{\mathscr{X}}\}_{\mathscr{X} \in \mathbf{PS}}$ defined above.

$$
\begin{aligned}
C^\sharp(R)(U) &= -C^\flat(R)(-U) \\
&= -R^{-1}[-U] \\
&= \{x \in \mathscr{X} \mid R(x) \cap -U = \emptyset\} \\
&= \{x \in \mathscr{X} \mid R(x) \subseteq U\}.
\end{aligned}
$$

Putting things together, we have the following theorem.

Theorem 4.3.5. *The functors $S^\flat$ and $C^\flat$ establish a dual equivalence between $\mathbf{DL}_\downarrow$ and $\mathbf{PSR}_\downarrow$. Dually, the functors $S^\sharp$ and $C^\sharp$ establish a dual equivalence between $\mathbf{DL}_\uparrow$ and $\mathbf{PSR}_\uparrow$.*

The diagram below summarizes the various relationships between the new functors introduced in this section and those introduced before.



Note that, in this diagram, the various functors commute up to isomorphism. Moreover, the inclusion maps from $\mathbf{DL}$ to $\mathbf{DL}_\downarrow$ and $\mathbf{DL}_\uparrow$ are obtained by mapping a distributive lattice L to the identity on L viewed as a join-preserving or meet-preserving morphism respectively. Similarly, the inclusion maps from $\mathbf{PS}$ to $\mathbf{PSR}_\downarrow$ and $\mathbf{PSR}_\uparrow$ are obtained by mapping a Priestley space $\mathscr{X}$ to the relations $\geq_{\mathscr{X}}$ and $\leq_{\mathscr{X}}$ on $\mathscr{X}$, viewed as lower and upper closed relations respectively.

Let us conclude this section with a straightforward characterization of the duals of lattice morphisms in $\mathbf{PSR}_\downarrow$ and $\mathbf{PSR}_\uparrow$.

Definition 4.3.6. A lower closed relation R between Priestley spaces $\mathscr{X}$ and $\mathscr{Y}$ is *lower functional* if $R(x)$ is a principal downset for every $x \in \mathscr{X}$. Dually, an upper closed relation S between $\mathscr{X}$ and $\mathscr{Y}$ is *upper functional* if $R(x)$ is a principal upset for every $x \in \mathscr{X}$.

Lemma 4.3.7. *For any join-preserving map $f : L \to M$, f is a lattice homomorphism if and only if $S^\flat(R)$ is lower functional. Dually, for any meet-preserving map $g : L \to M$, g is a lattice homormophism if and only if $S^\sharp(R)$ is upper functional. Finally, for any lattice homomorphism $h : L \to M$ and any $p \in Spec(M)$, $S^\flat(R)(p) \cap S^\sharp(R)(p) = \{Spec(f)(p)\}$.*

Proof. We show that a join-preserving map $f : L \to M$ is a lattice homomorphism if and only if $S^\flat(R)$ is lower functional. The case for meet-preserving maps is completely dual and therefore left to the reader. Suppose f is a lattice homomorphism and let $p \in Spec(M)$. Then $f^{-1}[p]$ is a prime filter, and therefore $f^{-1}[p] \in Spec(L)$. But then we have for any $q \in Spec(L)$ that $pS^\flat q$ iff $f^{-1}[p] \subseteq q$, so $S^\flat(p) = {\downarrow} f^{-1}[p]$. This shows that $S^\flat(f)$ is lower functional. Conversely, suppose that $S^\flat(f)$ is lower functional, and fix $U, V \in ClopUpSpec(L)$. I claim that $C^\flat S^\flat(f)(U) \cap C^\flat S^\flat(f)(V) \subseteq C^\flat S^\flat(f)(U \cap V)$. This will show that $C^\flat S^\flat(f)$ is meet-preserving and thus a lattice homomorphism, which by duality implies that f is also a lattice homomorphism. For the proof of the claim, suppose $p \in \mathsf{C}^\flat S^\flat(S)(U) \cap \mathsf{C}^\flat S^\flat(S)(V)$. This means that there are $q_1, q_2 \in S^\flat(p)$ such that $q_1 \in U$ and $q_2 \in V$. Since $S^\flat(p)$ is a principal downset, there is $q \in S^\flat(p)$ such that $q_1, q_2 \subseteq q$. Since U and V are both upsets, this means that $q \in U \cap V$. Therefore $p \in \mathsf{C}^\flat S^\flat(S)(U \cap V)$. This completes the proof of the claim.

Finally, if $h : L \to M$ is a lattice-homomorphism and $p \in Spec(M)$, then $S^\flat(f)(p) = \{q \in Spec(L) \mid q \subseteq f^{-1}[p]\} = {\downarrow} Spec(f)(p)$ and $S^\sharp(f)(p) = \{q \in Spec(L) \mid f^{-1}[p] \subseteq q\} = {\uparrow} Spec(f)(p)$. Hence $S^\flat(f)(p) \cap S^\sharp(f)(p) = \{Spec(f)(p)\}$. □

Intuitively, this lemma shows that, unsurprisingly, some of the information about the lattice structure of a distributive lattice L is lost by the embedding into $\mathbf{DL}_\downarrow$ or $\mathbf{DL}_\uparrow$. However, the full structure of L can always be recovered by combining information from $\mathbf{DL}_\downarrow$ and $\mathbf{DL}_\uparrow$. In the next section, we essentially adopt this strategy in the case of an arbitrary lattice. We define embeddings of the category of lattices into $\mathbf{DL}_\downarrow$ and $\mathbf{DL}_\uparrow$, use our new dualities to partially represent an arbitrary lattice as two closed relation between Priestley spaces, before recovering the full lattice by combining those two representations together.

4.4 The Category of FI-Spaces

In this section, we will establish a duality between **Lat** and a category of relations on Priestley spaces. We proceed as follows. First, we represent every lattice as the fixpoint of some Galois connection between two free distributive lattice. We then use our two dualities for $\mathbf{DL}_\downarrow$ and $\mathbf{DL}_\uparrow$ to translate this representation of an arbitrary lattice to the setting of relations on Priestley spaces. Finally, we axiomatize the dual category of **Lat** that arises from this representation.

4.4.1 A Fixpoint Representation for Arbitrary Lattices

We start with the following definitions.

Definition 4.4.1. Let L be a lattice. We define the maps $\lambda_L : \mathbb{M}_\Box(L) \to \mathbb{M}_\Diamond(L)$ and $\rho_L : \mathbb{M}_\Diamond(L) \to \mathbb{M}_\Box(L)$ as follows:

- $\lambda_L(\bigvee_{i \in I} \Box a_i) = \Diamond \bigvee_{i \in I} a_i$;

- $\rho_L(\bigwedge_{j\in J} \Diamond b_j) = \Box \bigwedge_{j\in J} b_j$.

Whenever the lattice L is clear from context, we will omit the subscript L in λ_L and ρ_L to avoid notational clutter.

Lemma 4.4.2. *Let L be a lattice. The maps λ_L and ρ_L form a Galois connection*

Proof. Fix a lattice L, and two subsets $\{a_i \mid i \in I\}$ and $\{b_j \mid j \in J\}$ of L. We have the following chain of equivalences:

$$\begin{aligned}
\lambda(\bigvee_{i\in I} \Box a_i) \leq \bigwedge_{j\in J} \Diamond b_j &\Leftrightarrow \Diamond \bigvee_{i\in I} a_i \leq \bigwedge_{j\in J} \Diamond b_j \\
&\Leftrightarrow \Diamond \bigvee_{i\in I} a_i \leq \Diamond b_j, \text{ all } j \in J \\
&\Leftrightarrow \bigvee_{i\in I} a_i \leq b_j, \text{ all } j \in J \\
&\Leftrightarrow a_i \leq b_j, \text{ all } i \in I, j \in J \\
&\Leftrightarrow a_i \leq \bigwedge_{j\in J} b_j, \text{ all } i \in I \\
&\Leftrightarrow \Box a_i \leq \Box \bigwedge_{j\in J} b_j, \text{ all } i \in I \\
&\Leftrightarrow \bigvee_{i\in I} \Box a_i \leq \Box \bigwedge_{j\in J} b_j \\
&\Leftrightarrow \bigvee_{i\in I} \Box a_i \leq \rho(\bigwedge_{j\in J} \Diamond b_j),
\end{aligned}$$

where the third and sixth equivalences hold because $\mathbb{M}_\Diamond(L)$ and $\mathbb{M}_\Box(L)$ are free constructions on L respectively. $\Box$

Since lower adjoints in a Galois connection between lattices preserve joins and upper adjoints preserve meets, it follows that λ_L and ρ_L are objects in $\mathbf{DL}_\downarrow$ and $\mathbf{DL}_\uparrow$ respectively. Moreover, we have the following:

Lemma 4.4.3. *For any join-preserving map $f : L \to M$, the pair $(\mathbb{M}_\Box(f), \mathbb{M}_\Diamond(f)) : \lambda_L \to \lambda_M$ is a morphism in $\mathbf{DL}_\downarrow$. Dually, for any meet-preserving map $g : L \to M$, the pair $(\mathbb{M}_\Diamond(f), \mathbb{M}_\Box(f)) : \rho_L \to \rho_M$ is a morphism in $\mathbf{DL}_\uparrow$.*

Proof. We prove the first case, since the second one is completely dual. Fix a join-preserving map $f : L \to M$. We must show that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{M}_\Box(L) & \xrightarrow{\quad \lambda_L \quad} & \mathbb{M}_\Diamond(L) \\
\Big\downarrow{\scriptstyle \mathbb{M}_\Box(f)} & & \Big\downarrow{\scriptstyle \mathbb{M}_\Diamond(f)} \\
\mathbb{M}_\Box(M) & \xrightarrow{\quad \lambda_M \quad} & \mathbb{M}_\Diamond(M)
\end{array}$$

Fix $\bigvee_{i\in I}\Box a_i \in \mathbb{M}_\Box(L)$. We compute:

$$\begin{aligned}
\mathbb{M}_\Diamond(f)\circ\lambda_L(\bigvee_{i\in I}\Box a_i) &= \mathbb{M}_\Diamond(f)(\Diamond\bigvee_{i\in I} a_i)\\
&= \Diamond f(\bigvee_{i\in I} a_i)\\
&= \Diamond\bigvee_{i\in I} f(a_i)\\
&= \lambda_M(\bigvee_{i\in I}\Box f(a_i))\\
&= \lambda_M\circ\mathbb{M}_\Box(f)(\bigvee_{i\in I}\Box a_i),
\end{aligned}$$

where the third equality holds because f is join-preserving. This shows that the pair $(\mathbb{M}_\Box(f), \mathbb{M}_\Diamond(f))$ is a morphism from λ_L to λ_M in $\mathbf{DL}_\downarrow$. ☐

We therefore have embeddings Λ and R of **Lat** into $\mathbf{DL}_\downarrow$ and $\mathbf{DL}_\uparrow$ respectively. Finally, we conclude by the following easy observation:

Lemma 4.4.4. *Let L be a lattice. The maps $a\mapsto\Box a$ and $a\mapsto\Diamond a$ are isomorphisms between L and the fixpoints of the maps $\rho_L\circ\lambda_L : \mathbb{M}_\Box(L)\to\mathbb{M}_\Box(L)$ and $\lambda_L\circ\rho_L : \mathbb{M}_\Diamond(L)\to\mathbb{M}_\Diamond(L)$ respectively.*

Proof. Fix a lattice L. Because λ and ρ form a Galois connection, the fixpoints of $\rho\lambda : \mathbb{M}_\Box(L)\to\mathbb{M}_\Box(L)$ form a lattice with meets given by $\wedge_L$ and joins given by $a\sqcup b = \rho\lambda(a\vee b)$. Clearly, the map $a\mapsto\Box a$ is bijective and preserves meets. Moreover, for any $a, b\in L$, we have that $\Box a\sqcup\Box b = \rho\lambda(\Box a\vee\Box b) = \rho(\Diamond(a\vee b) = \Box(a\vee b)$, which shows that $a\mapsto\Box a$ is a lattice isomorphism. Dually, the fixpoints of $\lambda\rho : \mathbb{M}_\Diamond(L)\to\mathbb{M}_\Diamond(L)$ form a lattice with joins given by $\vee$ and meets given by $a\sqcap b = \lambda\rho(a\wedge b)$. Once again, it is easy to see that the map $a\mapsto\Diamond a$ is an isomorphism, by an argument similar to the one above. ☐

This shows that we are always able to easily retrieve the original lattice L as fixpoints of either $\mathbb{M}_\Box(L)$ or $\mathbb{M}_\Diamond(L)$, provided that we use both free constructions. Moreover, the same is true for lattice morphisms, as shown by the following lemma.

Lemma 4.4.5. *For any two lattices L, M and any join-preserving map $f : \mathbb{M}_\Box(L)\to\mathbb{M}_\Box(M)$, there is a join-preserving map $g : L\to M$ such that $f = \mathbb{M}_\Box(g)$ if and only if $f\circ\rho_L\lambda_L = \rho_M\lambda_M\circ f$. Dually, for any meet-preserving map $f : \mathbb{M}_\Diamond(L)\to\mathbb{M}_\Diamond(M)$, there is a meet-preserving map $g : L\to M$ such that $f = \mathbb{M}_\Diamond(g)$ if and only if $f\circ\lambda_L\rho_L = \lambda_M\rho_M\circ f$.*

Proof. Again, we only prove the first case. Suppose first that $f = \mathbb{M}_\Box(g)$ for some $g : L\to M$. Then we compute for any $\bigvee_{i\in I}\Box a_i$:

$$\begin{aligned}
\rho_M\lambda_M\mathbb{M}_\Box(g)(\bigvee_{i\in I}\Box a_i) &= \rho_M\lambda_M(\bigvee_{i\in I}\Box g(a_i)) \\
&= \Box\bigvee_{i\in I} g(a_i) \\
&= \Box g(\bigvee_{i\in I} a_i) \\
&= \mathbb{M}_\Box(g)(\Box\bigvee_{i\in I} a_i) \\
&= \mathbb{M}_\Box(g)\rho_L\lambda_L(\bigvee_{i\in I}\Box a_i).
\end{aligned}$$

Conversely, suppose that $f\circ\rho_L\lambda_L = \rho_M\lambda_M\circ f$. Note first that this implies that, for any $a\in L$, $f(\Box a) = f\rho_L\lambda_L(\Box a)) = \rho_M\lambda_M f(\Box a)$. Hence $f(\Box a)$ is a fixpoint of $\rho_M\lambda_M$ and is therefore of the form $\Box b$ for some $b\in M$. Let $g : L\to M$ be such that $\Box g(a) = f(\Box a)$ for any $a\in L$. I claim that g is join-preserving. Note that, by Lemma 4.2.3.4, this implies that $f = \mathbb{M}_\Box(g)$. For the proof of the claim, let $a, b\in L$. Because $\mathbb{M}_\Box(L)$ is a free construction, it is enough to show that $\Box g(a\vee b) = \Box(g(a)\vee g(b))$ to establish that $g(a\vee b) = g(a)\vee g(b)$. Now we compute:

$$\begin{aligned}
\Box g(a\vee b) &= f\Box(a\vee b) \\
&= f\rho_L\lambda_L(\Box a\vee\Box b) \\
&= \rho_M\lambda_M f(\Box a\vee\Box b) \\
&= \rho_M\lambda_M(f(\Box a)\vee f(\Box b)) \\
&= \rho_M\lambda_M(\Box g(a)\vee\Box g(b)) \\
&= \Box(g(a)\vee g(b)),
\end{aligned}$$

which completes the proof. □

We have now gathered all the ingredients necessary for our topological representation of arbitrary lattices, to which we now turn.

4.4.2 A Topological Representation of the Category of Lattices

As mentioned in the previous section, our two dualities between $\mathbf{DL}_\downarrow$ and $\mathbf{PSR}_\downarrow$ and between $\mathbf{DL}_\uparrow$ and $\mathbf{PSR}_\uparrow$ respectively both only preserve "one half" of the structure of lattices. Fortunately, we can combine them in a straightforward and elegant way in order to retrieve a full topological representation of arbitrary lattices. The basic idea is the following. We may first represent an arbitrary lattice L by the pair λ_L, ρ_L, where λ_L is an object in $\mathbf{DL}_\downarrow$ and ρ_L is an object in $\mathbf{DL}_\uparrow$. By the results in Section 4.3, this yields a pair $(S^\flat(\lambda_L), S^\sharp(\rho_L))$, where the first term is a lower closed relation on $Spec\mathbb{M}_\diamond(L) \times Spec\mathbb{M}_\square(L)$ and the second one is an upper relation on $Spec\mathbb{M}_\square(L) \times Spec\mathbb{M}_\diamond(L)$. Using the representation of $Spec\mathbb{M}_\square(L)$ as $\mathcal{F}(L)$ and the representation of $Spec\mathbb{M}_\diamond(L)$ as $\mathcal{I}(L)$, this means that we obtain two relations on $\mathcal{I}(L) \times \mathcal{F}(L)$ and $\mathcal{F}(L) \times \mathcal{I}(L)$ respectively. As we will see, those relations are in fact converse to one another so we may focus on only one of them, say the one on $\mathcal{I} \times \mathcal{F}(L)$. Using the fact that Priestley spaces have binary products, we will lift this relation to an upper-closed endo-relation on $\mathcal{F}(L) \times \gamma(\mathcal{I}(L))$. Thus we will send any lattice L to the Priestley space $\mathcal{F}(L) \times \gamma(\mathcal{I}(L))$ endowed with a relation $\lambda^\natural_L$, which, as we will see below, is given by $(F, I)\lambda^\natural_L(G, J)$ iff $G \cap I = \emptyset$. Finally, we will use the relation $\lambda^\natural_L$ to recover L as the fixpoints of a closure operator on $ClopUp(\mathcal{F}(L) \times \gamma(\mathcal{I}(L)))$ that mimicks the $\rho_L\lambda_L$ representation above. Let us now get into the details of this representation. We start with the following observation.

Lemma 4.4.6. *Let L, M be distributive lattices and let $f : L \to M$ and $g : M \to L$ be join-preserving and meet-preserving respectively. Then g is the right adjoint of f iff $S^\sharp(g) = S^\flat(f)^{-1}$, i.e., iff for any $p \in Spec(L)$ and $q \in Spec(M)$ $pS^\sharp(g)q$ if and only if $qS^\flat(f)p$.*

Proof. Fix $f : L \to M$ and $g : M \to L$. Assume first that g is the right adjoint of f. Since f and g form a Galois connection, f is join-preserving and g is meet preserving. Now recall that for any $p \in Spec(L)$ and $q \in Spec(M)$, $qS^\flat(f)p$ iff $p \subseteq f^{-1}[q]$, and $pS^\sharp(g)q$ iff $g^{-1}[p] \subseteq q$. For any $a \in L$, we have that $a \leq gf(a)$, I claim that $p \subseteq f^{-1}[q]$ iff $g^{-1}[p] \subseteq q$, which implies that $S^\sharp(g) = S^\flat(f)^{-1}$. For the proof of the claim, assume first that $p \subseteq f^{-1}[g]$ and let $g(a) \in p$. Then $f(g(a)) \in q$, and since $f(g(a)) \leq a$, it follows that $a \in q$. Conversely, suppose that $g^{-1}[p] \subseteq q$ and let $a \in p$. Since $a \leq g(f(a))$ we have that $g(f(a)) \in p$, hence $f(a) \in q$, and therefore $a \in f^{-1}[q]$. This completes the proof of the claim.

Conversely, assume that $S^\sharp(g) = S^\flat(f)$. I claim that $C^\sharp S^\sharp(g)$ is right adjoint to $C^\flat S^\flat(f)$, which by duality is enough to establish that g is the right adjoint of f. For the proof of the claim, note that we have the following chain of equivalences for any $U \in ClopUpSpec(L)$ and any $C \in ClopUpSpec(M)$:

$$
\begin{aligned}
U \subseteq C^\sharp S^\sharp(g)(V) &\Leftrightarrow \forall p \in U(S^\sharp(g)(p) \subseteq V) \\
&\Leftrightarrow \forall p \in U \forall q \in Spec(M)(pS^\sharp(g)q \to q \in V) \\
&\Leftrightarrow \forall p \in U \forall q \in Spec(M)(qS^\flat(f)p \to q \in V) \\
&\Leftrightarrow \forall q \in Spec(M)(\exists p \in U\, qS^\flat(f)p \to q \in V) \\
&\Leftrightarrow (S^\flat(f))^{-1}[U] \subseteq V \\
&\Leftrightarrow C^\flat S^\flat(f)(U) \subseteq V,
\end{aligned}
$$

which completes the proof. □

The previous lemma shows that we may decide to work only with one of the two relations $S^\flat(\lambda_L), S^\sharp(\rho_L)$, since they are the converse of one another. Now recall that we have two natural order-homeomorphisms $\eta^L_\Box : Spec\mathbb{M}_\Box(L) \to \mathcal{F}(L)$ and $\eta^L_\Diamond : Spec\mathbb{M}_\Diamond(L) \to \mathcal{I}(L)$. This means that we can lift the relations $S^\flat(\lambda_L)$ and $S^\sharp(\lambda_L)$ to two relations $\lambda^* \subseteq \mathcal{I}(L)\times\mathcal{F}(L)$ and $\rho^* \subseteq \mathcal{F}(L) \times \mathcal{I}(L)$ respectively by setting $I\lambda^* F$ iff $\eta^{L^{-1}}_\Diamond(I)S^\flat(\lambda_L)\eta^{L^{-1}}_\Box(F)$ and $F\rho^* I$ iff $\eta^{L^{-1}}_\Box(F)S^\flat(\lambda_L)\eta^{L^{-1}}_\Diamond(I)$. For ease of notation, let us write p_F for $\eta^{L^{-1}}_\Box(F)$ and q_I for $\eta^{L^{-1}}_\Diamond(I)$. We may compute that for any $F \in \mathcal{F}(L)$ and $I \in \mathcal{I}(L)$:

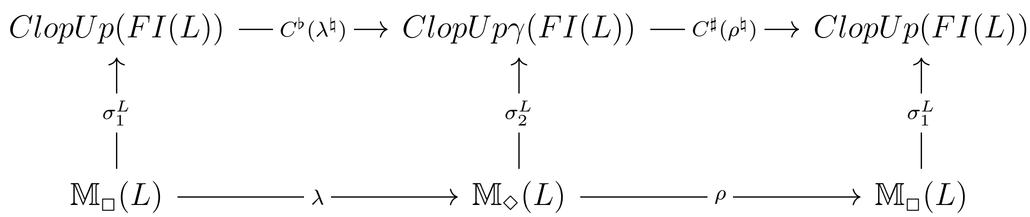
$$\begin{aligned} I\lambda^* F &\Leftrightarrow q_I S^\flat(\lambda_L) p_F \\ &\Leftrightarrow p_F \subseteq \lambda_L^{-1}[q_I] \\ &\Leftrightarrow \bigvee_{i\in I} \Box a_i \in p_F \text{ implies } \lambda_L(\bigvee_{i\in I}\Box a_i) \in q_I, \text{ all } \bigvee_{i\in I}\Box a_i \in \mathbb{M}_\Box(L) \\ &\Leftrightarrow \bigvee_{i\in I} \Box a_i \in p_F \text{ implies } \Diamond\bigvee_{i\in I} a_i \in q_I, \text{ all } \bigvee_{i\in I}\Box a_i \in \mathbb{M}_\Box(L) \\ &\Leftrightarrow a \in F \text{ implies } a \notin I, \text{ all } a \in L \\ &\Leftrightarrow F \cap I = \emptyset. \end{aligned}$$

Note also that, by Lemma 4.4.6, $\rho^* = \lambda^{*-1}$, i.e., $F\rho^* I$ iff $F \cap I = \emptyset$.

Now let $FI(L)$ be the direct product in **DL** of $\mathcal{F}(L)$ and $\gamma(\mathcal{I}(L))$. Using the projections $\pi_1 : FI(L) \to \mathcal{F}(L)$ and $\pi_2 : FI(L) \to \gamma(\mathcal{I}(L))$, we can lift λ^* and ρ^* to relations on $\gamma(FI(L)) \times FI(L)$ and $FI(L) \times \gamma(FI(L))$ respectively. More explicitly, we let $(F, I)\lambda^\natural(G, J)$ iff $G \cap I = \emptyset$, and $(F, I)\rho^\natural(G, J)$ iff $F \cap J = \emptyset$. It is straightforward to verify that $\lambda^\natural$ and $\rho^\natural$ are a lower closed and an upper closed relation on $\gamma(FI(L)) \times FI(L)$ and $FI(L) \times \gamma(FI(L))$ respectively, and that moreover the two following diagrams are morphisms from $\rho^\natural$ to $\rho \in \mathbf{PSR}_\uparrow$ and from $\lambda^\natural$ to λ respectively:

$$\begin{array}{ccc} FI(L) & \xrightarrow{\rho^\natural} & \gamma(FI(L)) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{F}(L) & \xrightarrow{\rho^*} & \mathcal{I}(L) \\ \downarrow{\scriptstyle \eta^{L^{-1}}_\Box} & & \downarrow{\scriptstyle \eta^{L^{-1}}_\Diamond} \\ Spec\mathbb{M}_\Box(L) & \xrightarrow{\rho} & Spec\mathbb{M}_\Diamond(L) \end{array} \qquad \begin{array}{ccc} \gamma(FI(L)) & \xrightarrow{\lambda^\natural} & FI(L) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ \mathcal{I}(L) & \xrightarrow{\lambda^*} & \mathcal{F}(L) \\ \downarrow{\scriptstyle \eta^{L^{-1}}_\Diamond} & & \downarrow{\scriptstyle \eta^{L^{-1}}_\Box} \\ Spec\mathbb{M}_\Diamond(L) & \xrightarrow{\lambda} & Spec\mathbb{M}_\Box(L) \end{array}$$

Dually, this yields the following diagram of distributive lattices, where σ_1^L is the map $\bigvee_{i\in I}\Box a_i \mapsto \bigcup \widehat{a_i} \times \mathcal{I}(L)$ and σ_2^L is the map $\bigwedge_{j\in J} b_j \mapsto \mathcal{F}(L) \times \bigcap_{j\in J} \check{b}_j$. Note that the first square commutes in $\mathbf{DL}_\downarrow$ and the second one commutes in $\mathbf{DL}_\uparrow$.



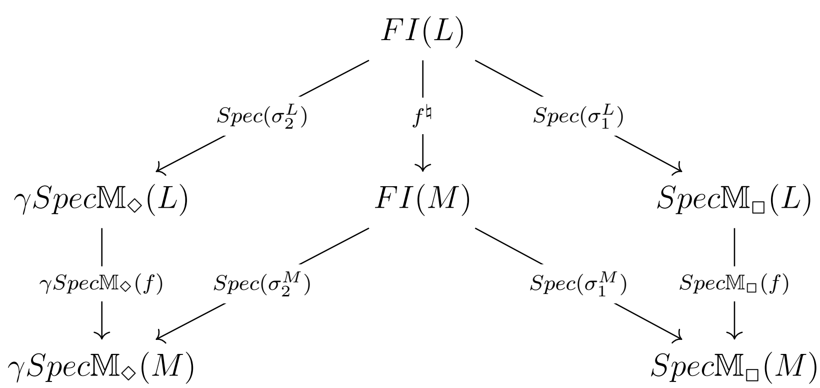
Moreover, for any $U \in ClopUp(\mathcal{F}(L)), V \in ClopUp(\mathcal{I}(L)) \setminus \{\mathcal{I}(L)\}$ and $(F, I) \in FI(L)$, we have that:

$$\begin{aligned}(F, I) \in C^\flat(\lambda^\natural)(U \times -V) &\Leftrightarrow \lambda^\natural(F, I) \cap U \times -V \neq \emptyset \\ &\Leftrightarrow \exists (G, J) \in U \times -V : I \cap J = \emptyset,\end{aligned}$$

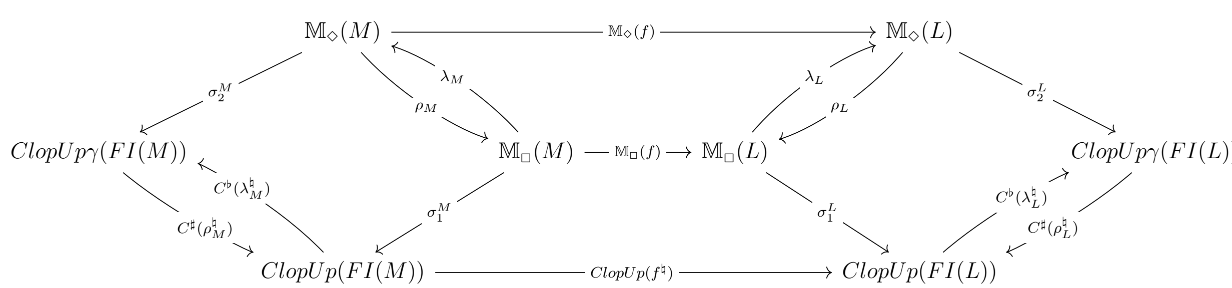
which means that $C^\flat(\lambda^\natural)(U \times -V) = \mathcal{F}(L) \times W$ for some $W \in ClopUp(\mathcal{I}(L))$.

Since any clopen upset in $FI(L)$ is a union of sets of the form $U \times -V$ for some $U \in ClopUp(\mathcal{F}(L)), V \in ClopUp(\mathcal{I}(L))$ and $C^\flat(\lambda^\natural)$ is join-preserving, this implies at once that $ran(C^\flat(\lambda^\natural)) \subseteq ran(ClopUp(\sigma_2^L))$, and hence that the fixpoints of $C^\sharp(\rho^\natural)C^\flat(\lambda^\natural)$ are isomorphic to the fixpoints of $\rho\lambda$. By Lemma 4.4.4, this means that we can represent the original lattice as the fixpoints of $C^\sharp(\rho^\natural)C^\flat(\lambda^\natural)$ via the map $\upsilon_L : a \mapsto \widehat{a} \times \mathcal{I}(L)$.

Moreover, for any lattice morphism $f : M \to L$, the pair $(\mathbb{M}_\Box(f), \mathbb{M}_\Diamond(f))$ induces a unique map $f^\natural : FI(L) \to FI(M)$, as shown in the following diagram:



Moreover, we clearly have that $ClopUp(f^\natural) \circ C^\sharp(\rho^\natural_M)C^\flat(\lambda^\natural_M) = C^\sharp(\rho^\natural_L)C^\flat(\lambda^\natural_L) \circ ClopUp(f^\natural)$, as evidenced by the following diagram (the detailed argument is left to the reader):



But this implies that $\upsilon_L \circ f = ClopUp(f^\natural) \circ \upsilon_M$.

Putting things together, this means that we obtain a functor $S^\natural : \mathbf{Lat} \to \mathbf{PSR}_\downarrow$ mapping any lattice L to the lower closed relation $\lambda^\natural_L \subseteq \gamma(FI(L)) \times FI(L)$, and any lattice homomorphism $f : M \to L$ to the pair $(f^\natural, f^\natural)$, and a natural transformation $\{\upsilon_L\}_{L \in \mathbf{Lat}} : \mathbf{1_{Lat}} \to C^\natural S^\natural$. Let us now turn to the issue of identifying a category corresponding to the range of that functor.

4.4.3 Axiomatizing FI Spaces

We start with the following definition.

Definition 4.4.7. A *FI-space* is a pair $(\mathscr{X}, R_-)$ such that:

1. $\mathscr{X} = (X, \tau, \leq)$ is a Priestley space;
2. $R_- \subseteq X \times X$ is a lower closed relation on $\gamma(\mathscr{X}) \times \mathscr{X}$, and its converse R_+ is an upper closed relation on $\mathscr{X} \times \gamma(\mathscr{X})$.
3. The set $\mathcal{R}^+(\mathscr{X}) \cup \mathcal{R}^-(\mathscr{X})$ is a basis for τ, where:
   - $\mathcal{R}^+(\mathscr{X}) = \{U \in ClopUp(\mathscr{X}) \mid U = C^\sharp(R_+)C^\flat(R_-)(U)\}$;
   - $\mathcal{R}^-(\mathscr{X}) = \{V \in ClopUp(\mathscr{X}) \mid -V = C^\flat(R_-)C^\sharp(R_+)(-V)\}$;
4. For any non-empty $W \in ClopUp(\mathscr{X})$ of the form $\bigcup_{i \in I} U_i \cap \bigcup_{j \in J} V_j$ for some finite $\{U_i\}_{i \in I} \subseteq \mathcal{R}^+(\mathscr{X})$, $\{V_j\}_{j \in J} \subseteq \mathcal{R}^-$, $C^\flat(R_-)(W) = C^\flat(R_-)(\bigcup_{i \in I} U_i)$.

The following establishes that FI-spaces correspond to the dual of lattices under the representation given above.

Theorem 4.4.8. *A pair $(\mathscr{X}, R_-)$ is a FI-space if and only if it is order-homeomorphic to $(FI(L), \lambda^\natural_L)$ for some lattice L.*

Proof. For the right-to-left direction, it is straightforward to verify that the pair $(FI(L), \lambda^\natural_L)$ is a FI-space for every lattice L. Indeed, every $W \in ClopUp(FI(L))$ is a union of sets of the form $\bigcup_{i \in I} \widehat{a_i} \times - \bigcap_{j \in J} \check{b_j}$ for some finite $\{a_i\}_{\{i \in I\}}$, $\{b_j\}_{\{j \in J\}} \subseteq L$, and the fixpoints of $C^\sharp(\rho^\natural_L)C^\flat(\lambda^\natural_L)$ and $C^\sharp(\rho^\natural_L)C^\flat(\lambda^\natural_L)$ are precisely sets of the form $\widehat{a} \times \mathcal{I}(L)$ and $\mathcal{F}(L) \times \check{b}$ respectively. Moreover, one has that

$$\begin{aligned} C^\flat(\lambda^\natural_L)(\bigcup_{i \in I} \widehat{a_i} \times - \bigcap_{j \in J} \check{b_j}) &= \widecheck{\bigvee_{i \in I} a_i} \\ &= C^\flat(\lambda^\natural_L)(\bigcup_{i \in I} \widehat{a_i}) \end{aligned}$$

for any finite $\{a_i\}_{\{i \in I\}}, \{b_j\}_{\{j \in J\}} \subseteq L$.

Conversely, fix a FI-space $(\mathscr{X}, R_-)$. Notice first that, since R_- is a lower closed relation on $\gamma(\mathscr{X}) \times \mathscr{X}$ and R_+ is an upper closed relation $\mathscr{X} \times \gamma(\mathscr{X})$, by Lemma 4.4.6 the maps $C^\flat(R_-)$ and $C^\sharp(R_+)$ form a Galois connection on $ClopUp(\mathscr{X})$. Moreover, $\mathcal{R}^+(\mathscr{X})$ and $\mathcal{R}^-(\mathscr{X}))$ are precisely the lattice of $C^\sharp(R_+)C^\flat(R_-)$-closed subsets of $\mathscr{X}$ and the dual of the lattice of $C^\flat(R_-)C^\sharp(R_+)$-closed subsets of $ClopUp\gamma(\mathscr{X})$ respectively. Let $\omega_{\mathscr{X}} : \mathscr{X} \to FI(\mathcal{R}^+(\mathscr{X}))$ be the map $x \mapsto (F_x, I_x)$, where $F_x = \{U \in \mathcal{R}^+(\mathscr{X}) \mid x \in U\}$ and $I_x = \{V \in \mathcal{R}^-(\mathscr{X}) \mid x \in V\}$. I claim that $\omega_{\mathscr{X}}$ is the required order-homeomorphism. By duality, it is enough to check that $ClopUp(\omega_{\mathscr{X}})$ is an isomorphism and that the following diagram commutes:

$$\begin{array}{ccc} ClopUp(FI(\mathcal{R}^+(\mathscr{X}))) & \xrightarrow{C^\flat(\lambda^\natural_{\mathcal{R}^+(\mathscr{X})})} & ClopUp\gamma(FI(\mathcal{R}^+(\mathscr{X}))) \\ \Big\downarrow{\scriptstyle ClopUp(\omega_{\mathscr{X}})} & & \Big\downarrow{\scriptstyle ClopUp\gamma(\omega_{\mathscr{X}})} \\ ClopUp(\mathscr{X}) & \xrightarrow{C^\flat(R_-)} & ClopUp\gamma(\mathscr{X}) \end{array}$$

Notice that any set in $ClopUp(FI(\mathcal{R}^+(\mathscr{X})))$ is a union of sets of the form $\bigcup_{i\in I} \widehat{U}_i \times -\bigcap_{j\in J} -\check{V}_j$ for some finite $\{U_i\}_{i\in I} \subseteq \mathcal{R}^+(\mathscr{X})$ and $\{V_j\}_{j\in J} \subseteq \mathcal{R}^-(\mathscr{X})$. Moreover, $ClopUp(\omega_{\mathscr{X}})(\bigcup_{i\in I} \widehat{U}_i \times -\bigcap_{j\in J} -\check{V}_j) = \bigcup_{i\in I} U_i \cap \bigcup_{j\in J} V_j$, from which it follows that $ClopUp(\omega_{\mathscr{X}})$ is an order embedding which is also surjective by property 3 of FI-spaces. Finally, for any $\bigcup_{i\in I} \widehat{U}_i \times -\bigcap_{j\in J} -\check{V}_j \in ClopUp(FI(\mathcal{R}^+(\mathscr{X})))$, we have that:

$$\begin{aligned} ClopUp(\omega_{\mathscr{X}}) \circ C^\flat(\lambda^\natural_{\mathcal{R}^+(\mathscr{X})})(\bigcup_{i\in I} \widehat{U}_i \times -\bigcap_{j\in J} -\check{V}_j) &= ClopUp(\omega_{\mathscr{X}})(\bigcup_{i\in I} \widehat{U}_i) \\ &= C^\flat(R_-)(\bigcup_{i\in I} U_i) \\ &= C^\flat(R_-)(\bigcup_{i\in I} U_i \cap \bigcup_{j\in J} V_j) \\ &= C^\flat(R_-) \circ ClopUp(\omega_{\mathscr{X}})(\bigcup_{i\in I} \widehat{U}_i \times -\bigcap_{j\in J} -\check{V}_j), \end{aligned}$$

where the third equality holds by property 4 of FI-spaces. □

Let us now define the correct notion of morphism for our category.

Definition 4.4.9. Let $(\mathscr{X}, R_-)$ and $(\mathscr{Y}, S_-)$ be FI-spaces. A FI-morphism is an order-continuous map $f : \mathscr{X} \to \mathscr{Y}$ with the following properties for any $x \in X$ and $y \in Y$:

1. for all $x' \in X$, xR_-x' implies $f(x)S_-f(x')$;
2. if $f(x)S_-y$, then there is $x' \in X$ such that xR_-x' and $y \leq_{\mathscr{Y}} f(x')$;
3. if $yS_-f(x)$, then there is $x' \in X$ such that $x'R_-x$ and $y \leq_{\mathscr{Y}} f(x')$.

Lemma 4.4.10. *Let $(\mathscr{X}, R_{-})$ and $(\mathscr{Y}, S_{-})$ be FI-spaces with dual lattices L and M respectively. Then a map $f : \mathscr{X} \to \mathscr{Y}$ is a FI-morphism if and only if there is a unique $g : M \to L$ such that $f = g^{\natural}$.*

Proof. Let us first show that $g^{\natural} : FI(L) \to FI(M)$ is a FI-morphism whenever $g : M \to L$ is a lattice morphism. We know already that $g^{\natural}$ is order-continuous. Moreover, an easy computation reveals that $g^{\natural}(F, I) = (g^{-1}[F], g^{-1}[I])$ for any $(F, I) \in FI(L)$. Hence we must check the following for any $(F, I) \in FI(L)$, $(G, J) \in FI(M)$:

1. $(F, I)\lambda^{\natural}_L(F', I')$ implies $(g^{-1}[F], g^{-1}[I])\lambda^{\natural}_M(g^{-1}[F], g^{-1}[I])$;

2. $(g^{-1}[F], g^{-1}[I])\lambda^{\natural}_M(G, J)$ implies that there is $(F', I') \in FI(L)$ such that we have both $(F, I)\lambda^{\natural}_L(F', I')$ and $(G, J) \leq_{FI(M)} (g^{-1}[F'], g^{-1}[I'])$;

3. $(G, J)\lambda^{\natural}_M(g^{-1}[F], g^{-1}[I])$ implies that there is $(F', I') \in FI(L)$ such that we have both $(F', I')\lambda^{\natural}_L(F, I)$ and $(G, J) \leq_{FI(M)} (g^{-1}[F'], g^{-1}[I'])$.

The first item amounts to proving that $F' \cap I = \emptyset$ implies that $g^{-1}[F'] \cap g^{-1}[I] = \emptyset$, which is clear. For the second item, assuming that $G \cap g^{-1}[I] = \emptyset$, we must find (F', I') such that $F' \cap I = \emptyset$, $G \subseteq g^{-1}[F']$ and $J \subseteq g^{-1}[I']$. Let $F' = {\uparrow} g[G]$ and $I' = {\downarrow} g[J]$. Clearly, $G \subseteq g^{-1}[F']$ and $J \subseteq g^{-1}[I']$. Moreover, if $g(a) \leq b$ for some $a \in G$, $b \in I$, then $a \in G \cap g^{-1}[I]$, contradicting our assumption. Hence (F', I') is the required pair. Finally, the third item is proved by the same argument as item 2.

Conversely, let us now assume that $f : (FI(L), \lambda^{\natural}_L) \to (FI(M), \lambda^{\natural}_M)$ is a FI-morphism. It follows from conditions 1 and 2 that the pair $(\gamma(f), f)$ is a morphism between the lower closed relations $\lambda^{\natural}_L$ and $\lambda^{\natural}_M$, and the pair $(f, \gamma(f))$ is a morphism between the upper closed relations $\rho^{\natural}_L$ and $\rho^{\natural}_M$. By duality, this means that we have the following diagram:

$$\begin{array}{ccccccc}
ClopUp(FI(M)) & \xrightarrow{C^{\flat}(\lambda^{\natural}_M)} & ClopUp\gamma(FI(M)) & \xrightarrow{C^{\sharp}(\rho^{\natural}_M)} & ClopUp(FI(L)) & \xrightarrow{C^{\flat}(\lambda^{\natural}_M)} & ClopUp\gamma(FI(M)) \\
\downarrow{\scriptstyle ClopUp(f)} & & \downarrow{\scriptstyle ClopUp\gamma(f)} & & \downarrow{\scriptstyle ClopUp(f)} & & \downarrow{\scriptstyle ClopUp\gamma(f)} \\
ClopUp(FI(L)) & \xrightarrow{C^{\flat}(\lambda^{\natural}_L)} & ClopUp\gamma(FI(L)) & \xrightarrow{C^{\sharp}(\rho^{\natural}_L)} & ClopUp(FI(L)) & \xrightarrow{C^{\flat}(\lambda^{\natural}_L)} & ClopUp\gamma(FI(L))
\end{array}$$

which shows that $ClopUp(f) \circ C^{\sharp}(\rho^{\natural}_M)C^{\flat}(\lambda^{\natural}_M) = C^{\sharp}(\rho^{\natural}_L)C^{\flat}(\lambda^{\natural}_L) \circ ClopUp\gamma(f)$ and that $ClopUp\gamma(f) \circ C^{\flat}(\lambda^{\natural}_M)C^{\sharp}(\rho^{\natural}_M) = C^{\flat}(\lambda^{\natural}_L)C^{\sharp}(\rho^{\natural}_L) \circ ClopUp\gamma(f)$.

Moreover, let $f^{+} : Spec\mathbb{M}_{\Box}(L) \to Spec\mathbb{M}_{\Box}(M)$ map every $p_F \in Spec\mathbb{M}_{\Box}(L)$ to p'_F, where $(F', J) = f(F, {\downarrow}\{0\})$, and let $f^{-} : Spec\mathbb{M}_{\Diamond}(L) \to Spec\mathbb{M}_{\Diamond}(M)$ map every $q_I \in Spec\mathbb{M}_{\Diamond}(L)$ to q'_I, where $(G, I') = f({\uparrow}\{1\}, I)$. It is straightforward to verify that both f^{+} and f^{-} are order-continuous and that the following diagram commutes:

$$\begin{array}{ccccc}
 & & FI(L) & & \\
 & \swarrow^{Spec(\sigma_2^L)} & \downarrow^{f} & \searrow^{Spec(\sigma_1^L)} & \\
\gamma Spec\mathbb{M}_\Diamond(L) & & FI(M) & & Spec\mathbb{M}_\Box(L) \\
\downarrow^{\gamma(f^-)} & \swarrow^{Spec(\sigma_2^M)} & & \searrow^{Spec(\sigma_1^M)} & \downarrow^{f^+} \\
\gamma Spec\mathbb{M}_\Diamond(M) & & & & Spec\mathbb{M}_\Box(M)
\end{array}$$

By Lemma 4.4.5, this means that there is a (necessarily unique) $g : M \to L$ such that $f^+ = Spec\mathbb{M}_\Box(g)$ and $f^- = Spec\mathbb{M}_\Diamond(g)$, and hence that $f = g^\natural$. This completes the proof. ☐

Putting things together, this means that we are able to define a functor $C^\natural$ mapping every FI-space $\mathscr{X}$ to $\mathcal{R}^+(\mathscr{X})$ and every FI-morphism $f : \mathscr{X} \to \mathscr{Y}$ to the unique $g : \mathcal{R}^+(\mathscr{Y}) \to \mathcal{R}^+(\mathscr{X})$ such that $f = g^\natural$. Moreover, this functor finally yields our topological duality for lattices.

Theorem 4.4.11. *Let* **FI** *be the category of FI-spaces and FI-morphisms. There is a dual equivalence between* **Lat** *and* **FI**, *given by the functors $S^\natural$:* **Lat** $\to$ **FI** *and $C^\natural$:* **FI** $\to$ **Lat**.

Proof. By construction of the functors $C^\natural$ and $S^\natural$, the maps $\upsilon_L : L \to C^\natural S^\natural(L)$ for every lattice L and $\omega_{\mathscr{X}} : \mathscr{X} \to S^\natural C^\natural(\mathscr{X})$ for every FI-space $\mathscr{X}$ are isomorphisms. Naturality also straightforwardly follows from the definitions of $C^\natural$ and $S^\natural$, and is left to the reader. ☐

4.5 Application to Fundamental Logic

In this section, we apply our duality for lattices to the category of weakly-pseudo complemented lattices, which provide the algebraic semantics for the Fundamental Logic [131]. As we shall see, the duality we obtain yields spaces that are very close to the relational frames considered by Holliday.

4.5.1 Positive and Negative Projections of FI-Spaces

We start by the following observation about FI-spaces. Since any FI-space is the dual space of a lattice L and thus of the form $FI(L)$, we can view $\mathscr{X}$ as the product of two Priestley spaces, namely $\mathcal{F}(L)$ and $\gamma(\mathcal{I}(L))$. Intuitively, this means that $\mathscr{X}$ can be "split" into two Priestley spaces corresponding to $\mathbb{M}_\Box(L) \simeq \mathbb{M}_\Box(\mathcal{R}^+(L))$ and $\delta(\mathbb{M}_\Diamond(L)) \simeq \mathbb{M}_\Box(\delta(L)) \simeq \mathbb{M}_\Box(\mathcal{R}^-(L))$. respectively. This motivates the following definition:

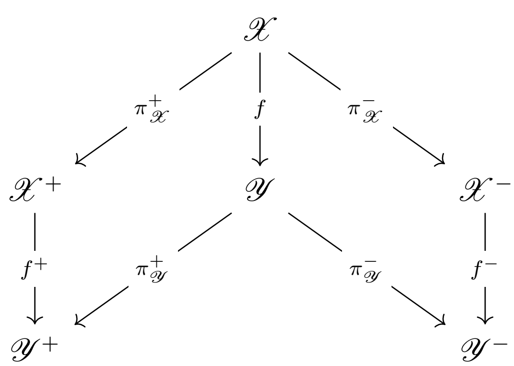
Definition 4.5.1. Let $\mathscr{X}$ be a FI-space. The *positive projection* of $\mathscr{X}$ is the topological space $\mathscr{X}^+ = (X^+, \tau^+_\sim, \leq^+_\sim)$ where:

- τ^+ is the topology on $\mathscr{X}$ generated by $\mathcal{R}^+$, and $\leq^+$ is the induced specialization preorder;
- X^+ is the quotient of X obtained by turning $\leq^+$ into a partial order $\leq^+_\sim$;
- $\tau^+_\sim$ is the quotient topology on X^+ induced by the patch topology of τ^+.

Dually, the negative projection of $\mathscr{X}$ is the topological space $\mathscr{X}^- = (X^-, \tau^-_\sim, \leq^-_\sim)$ where:

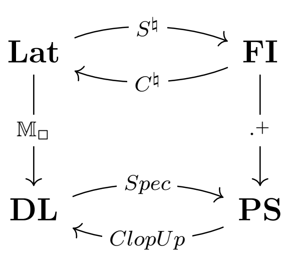
- τ^- is the topology on $\mathscr{X}$ generated by $\mathcal{R}^-$, and $\leq^-$ is the induced specialization preorder;
- X^- is the quotient of X obtained by turning $\leq^-$ into a partial order $\leq^-_\sim$;
- $\tau^-_\sim$ is the quotient topology on X^- induced by the patch topology of τ^-.

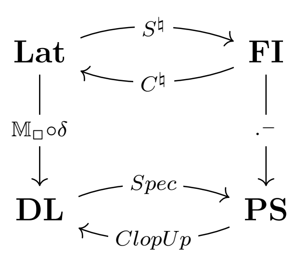
Finally, given a FI-morphism $f : \mathscr{X} \to \mathscr{Y}$, let $f^+ : \mathscr{X}^+ \to \mathscr{Y}^+$ and $f^- : \mathscr{X}^- \to \mathscr{Y}^-$ be the unique order-continuous maps such that the following diagram commutes:



where $\pi^+_{\mathscr{X}}$ and $\pi^-_{\mathscr{X}}$ are the natural quotient maps from $\mathscr{X}$ to $\mathscr{X}^+$ and $\mathscr{X}^-$ respectively, and similarly for $\pi^+_{\mathscr{Y}}$ and $\pi^-_{\mathscr{Y}}$.

Although we will not do so here, it is straightforward to prove that, for any FI-space $\mathscr{X}$, $ClopUp(\mathscr{X}^+) \simeq \mathbb{M}_\Box(\mathcal{R}^+(\mathscr{X}))$ and $ClopUp(\mathscr{X}^-) \simeq \mathbb{M}_\Box(\mathcal{R}^-(\mathscr{X}))$, and for any FI-morphism $f : \mathscr{X} \to \mathscr{Y}$, $f^+ = \mathbb{M}_\Box C^\natural(f)$ and $f^- = \mathbb{M}_\Box C^\natural(\gamma(f))$. In other words, $\cdot^+$ and $\cdot^-$ are functors from **FI** to **PS** such that the following two diagrams commute up to isomorphism:





Using this correspondence, we may now lift our duality between **Lat** and **FI** to a duality between monotone maps between lattices and relations on FI-spaces. We will apply essentially the same strategy as the one we used to lift Priestley duality to a duality between $\mathbf{DL}_\downarrow$ and $\mathbf{PSR}_\downarrow$.

4.5.2 Monotone Maps between Lattices

Let $f : L \to M$ be an order-preserving map between lattices L and M. Then $\mathbb{M}_\Box(f) : \mathbb{M}_\Box(L) \to \mathbb{M}_\Box(M)$ is a join-preserving between distributive lattices, i.e., an object in $\mathbf{DL}_\downarrow$. By Theorem 4.3.5 and Lemma 4.3.7, this means that we may associate to it a lower closed relation $S^\flat(f) : Spec(\mathbb{M}_\Box(M)) \to Spec(\mathbb{M}_\Box(L)$. Since, as we saw in the previous section, $Spec(\mathbb{M}_\Box(L)$ and $Spec(\mathbb{M}_\Box(M))$ are order-homeomorphic to $FI(L)^+$ and $FI(M)^+$ respectively, this means that we may view $S^\flat(f)$ as a relation on $FI(L)^+ \times FI(M)^+$ instead. This motivates the following definition.

Definition 4.5.2. Let $\mathscr{X}$ and $\mathscr{Y}$ be FI-spaces A lower closed relation $R \subseteq \mathscr{X} \times \mathscr{Y}$ is π^+-projective if there is a lower closed relation $R^+ \subseteq \mathscr{X}^+ \times \mathscr{Y}^+$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\quad R \quad} & \mathscr{Y} \\
\downarrow{\scriptstyle \pi^+_{\mathscr{X}}} & & \downarrow{\scriptstyle \pi^+_{\mathscr{Y}}} \\
\mathscr{X}^+ & \xrightarrow{\quad R^+ \quad} & \mathscr{Y}^+
\end{array}$$

in the sense that, for any $x \in \mathscr{X}, y \in \mathscr{Y}$ xRy iff $\pi^+_{\mathscr{X}}(x)R^+\pi^+_{\mathscr{X}}$.

It is easy to see that if R is a lower closed relation on $\mathscr{X} \times \mathscr{Y}$, then the lower closed relation R^+ witnessing that R is π^+-projective, if it exists, is necessarily unique. Let us now see how π^+-projective relations on FI-spaces relate to monotone maps on lattices.

Lemma 4.5.3. *Let $f : M \to L$ be a monotone map between lattices. Then the relation $\Pi^+(f) \subseteq FI(L) \times FI(M)$ given by $(F, I)\Pi^+(f)(G, J)$ iff $G \subseteq f^{-1}[F]$ is a π^+-projective lower closed relation.*

Proof. It is routine to verify that the relation $\Pi^+(f)$ is lower closed. Moreover, I claim that for any $(F, I) \in FI(L)$, $(G, J) \in FI(M)$, $(F, I)\Pi^+(f)(G, J)$ iff $\eta_\Box^{L^{-1}}(\pi^+_L(F, I))S^\flat\eta_\Box^{M^{-1}}(\pi^+_M(G, J))$. If true, this means that there is a lower closed relation $\Pi^+(f)^+$ on $\mathcal{F}(L) \times \mathcal{F}(M)$, as shown in the following diagram:

$$\begin{array}{ccc}
FI(L) & \xrightarrow{\quad \Pi^+(f) \quad} & FI(M) \\
\downarrow{\scriptstyle \pi^+_{FI(L)}} & & \downarrow{\scriptstyle \pi^+_{FI(M)}} \\
\mathcal{F}(L) & \xrightarrow{\quad \Pi^+(f)^+ \quad} & \mathcal{F}(M) \\
\downarrow{\scriptstyle \eta_\Box^{L^{-1}}} & & \downarrow{\scriptstyle \eta_\Box^{M^{-1}}} \\
Spec\mathbb{M}_\Box(L) & \xrightarrow{\quad S^\flat\mathbb{M}_\Box(f) \quad} & Spec\mathbb{M}_\Box(M)
\end{array}$$

and thus that $\Pi^+(f)$ is π^+-projective. For the proof of the claim, fix $(F, I) \in FI(L)$ and $(G, J) \in FI(M)$, and let $p_F = \eta_\Box^{L^{-1}}(F)$ and $p_G = \eta_\Box^{M^{-1}}(G)$. By the definition of the functor $S^\flat$, we have that $p_F S^\flat(\mathbb{M}_\Box(f))p_G$ iff $p_G \subseteq \mathbb{M}_\Box(f)^{-1}[p_F]$, i.e., $\bigvee_{i\in I} \Box a_i \in p_G$ implies $\mathbb{M}_\Box(f)(\bigvee_{i\in I} \Box a_i) = \bigvee_{i\in I} \Box f(a_i) \in p_F$. But this is easily seen to be equivalent to the condition that $G \subseteq f^{-1}[F]$ by the definition of the maps $F \mapsto p_F$ and $G \mapsto p_G$. This completes the proof. $\Box$

Hence any monotone map $f : L \to M$ induces a π^+-projective relation $\Pi^+(f)$ from $FI(L) \to FI(M)$. Let us now go in the converse direction.

Lemma 4.5.4. *Let $\mathscr{X}$, $\mathscr{Y}$ be FI-spaces. Any π^+-projective relation $R : \mathscr{X} \to \mathscr{Y}$ induces a monotone map $\Sigma^+(R) : \mathcal{R}^+(\mathscr{Y}) \to \mathcal{R}^+(\mathscr{X})$.*

Proof. Fix a π^+-projective relation $R \subseteq \mathscr{X} \times \mathscr{Y}$. This means that we have a lower closed relation $R^+ : \mathscr{X}^+ \to \mathscr{Y}^+$. Recall that we also have two order-homeomorphisms $\sigma^+_{\mathscr{X}} : \mathscr{X}^+ \to Spec\mathbb{M}_\Box(\mathcal{R}^+(\mathscr{X}))$ and $\sigma^+_{\mathscr{Y}} : \mathscr{Y}^+ \to Spec\mathbb{M}_\Box((RR^+(\mathscr{Y}))$. Thus we have the following commuting diagram of lower closed relations:

$$\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\quad R \quad} & \mathscr{Y} \\
\big\downarrow{\scriptstyle \pi^+_{\mathscr{X}}} & & \big\downarrow{\scriptstyle \pi^+_{\mathscr{Y}}} \\
\mathscr{X}^+ & \xrightarrow{\quad R^+ \quad} & \mathscr{Y}^+ \\
\big\downarrow{\scriptstyle \sigma^+_{\mathscr{X}}} & & \big\downarrow{\scriptstyle \sigma^+_{\mathscr{Y}}} \\
Spec\mathbb{M}_\Box(\mathcal{R}^+(\mathscr{X})) & \xrightarrow{\quad R^+_\sigma \quad} & Spec\mathbb{M}_\Box(\mathcal{R}^+(\mathscr{Y}))
\end{array}$$

By the duality between lower closed relations and join-preserving morphisms between distributive lattices, this induces the following commuting diagram in **Lat** (treating $ClopUp$ and $Spec$ as inverses of one another for simplicity):

$$\begin{array}{ccc}
\mathcal{R}^+(\mathscr{Y}) & \xrightarrow{\quad \Sigma^+(R) \quad} & \mathcal{R}^+(\mathscr{X}) \\
\big\downarrow{\scriptstyle \Box} & & \big\downarrow{\scriptstyle \Box} \\
\mathbb{M}_\Box(\mathcal{R}^+(\mathscr{Y})) & \xrightarrow{\quad C^\flat(R^+_\sigma) \quad} & \mathbb{M}_\Box(\mathcal{R}^+(\mathscr{X})) \\
\big\downarrow{\scriptstyle ClopUp(\sigma^+_{\mathscr{Y}})} & & \big\downarrow{\scriptstyle ClopUp(\sigma^+_{\mathscr{X}})} \\
\mathscr{Y}^+ & \xrightarrow{\quad C^\flat(R^+) \quad} & \mathscr{X}^+
\end{array}$$

where $\Sigma^+(R)$ is the map given by $U \mapsto C^\flat(R^+)(U)$ for any $U \in \mathcal{R}^+$ and is the unique map $f : \mathsf{R}^+(\mathscr{Y}) \to \mathsf{R}^+(\mathscr{X})$ such that $\Box f(U) = C^\flat(R^+_\sigma)$ for all $U \in \mathcal{R}^+(\mathscr{Y})$. $\Box$

Finally, let us quickly introduce the correct notion of morphisms between π^+-projective lower closed relations:

Definition 4.5.5. Let $R : \mathscr{X}_1 \to \mathscr{Y}_1$ and $S : \mathscr{X}_2 \to \mathscr{Y}_2$ be π^+-projective lower closed relations. A π^+-*projective morphism* is a pair of order-continuous maps (f_1, f_2) such that (f_1^+, f_2^+) is a morphism from R^+ to S^+ in $\mathbf{PSR}_\downarrow$.

We can now lift our duality between lattices and FI-spaces to a duality between monotone maps between lattices and π^+-projective relations on FI-spaces.

Theorem 4.5.6. *Let* $\mathbf{Lat}_I$ *be the category whose objects are monotone maps between lattices and morphisms from* $f_1 : L_1 \to L_2$ *to* $f_2 : M_1 \to M_2$ *are pairs* (g_1, g_2) *such that* $g_i : L_i \to M_i$ *for* $i = 1, 2$ *and* $g_2 \circ f_1 = f_2 \circ g_1$. *Let* $\mathbf{FI}_{\Pi^+}$ *be the category of* π^+*-projective relations and* π^+*-projective morphisms between them. Then the maps* $f \mapsto \Pi^+(f)$ *and* $R \mapsto \Sigma^+(R)$ *lift to functors* $\Pi^+ : \mathbf{Lat}_I \to \mathbf{FI}_{\Pi^+}$ *and* $\Sigma^+ : \mathbf{FI}_{\Pi^+} \to \mathbf{Lat}_I$ *which establish a dual equivalence between the two categories.*

Proof. By Lemmas 4.5.3 and 4.5.4, we know already that the maps Π^+ and Σ^+ map monotone maps to π^+-projective relations and π^+-projective relations to monotone maps. Moreover, it is straightforward to verify that the following diagrams commute for any $f : L \to M \in \mathbf{Lat}_I$, and $R \subseteq \mathscr{X} \times \mathscr{Y} \in \mathbf{FI}_{\Pi^+}$:

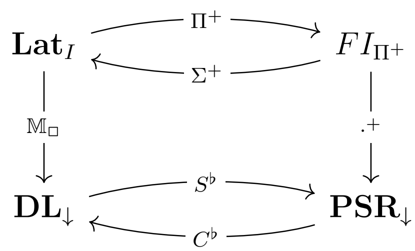
$$\begin{array}{ccc} L & \xrightarrow{\quad f \quad} & M \\ \downarrow{\scriptstyle \upsilon_L} & & \downarrow{\scriptstyle \upsilon_M} \\ \mathcal{R}^+(FI(L)) & \xrightarrow{\Sigma^+\Pi^+(f)} & \mathcal{R}^+(FI(M)) \end{array} \qquad \begin{array}{ccc} \mathscr{X} & \xrightarrow{\quad R \quad} & \mathscr{Y} \\ \downarrow{\scriptstyle \omega_{\mathscr{X}}} & & \downarrow{\scriptstyle \omega_{\mathscr{Y}}} \\ FI(\mathcal{R}^+(\mathscr{X})) & \xrightarrow{\Pi^+\Sigma^+(f)} & FI(\mathcal{R}^+(\mathscr{Y})) \end{array}$$

For any morphism $(g_1, g_2) \in \mathbf{Lat}_I$, we let $\Pi^+(g_1, g_2)$ be the pair $(S^\flat\mathbb{M}_\Box(g_1), S^\flat\mathbb{M}_\Box(g_2))$. Note that this is well-defined since $(\mathbb{M}_\Box(g_1), \mathbb{M}_\Box(g_2))$ is a morphism in $\mathbf{DL}_\downarrow$, hence $\Pi^+(g_1, g_2)$ is a morphism in $\mathbf{PSR}_\downarrow$. Conversely, for any π^+-projective morphism (f_1, f_2) from $R_1 \subseteq \mathscr{X}_1 \times \mathscr{X}_2$ to $R_2 \subseteq \mathscr{Y}_1 \times \mathscr{Y}_2$, we let $\Sigma^+(f_1, f_2) = (h_1, h_2)$, where h_i for $i = 1, 2$ is the unique monotone map such that the following diagram commutes:

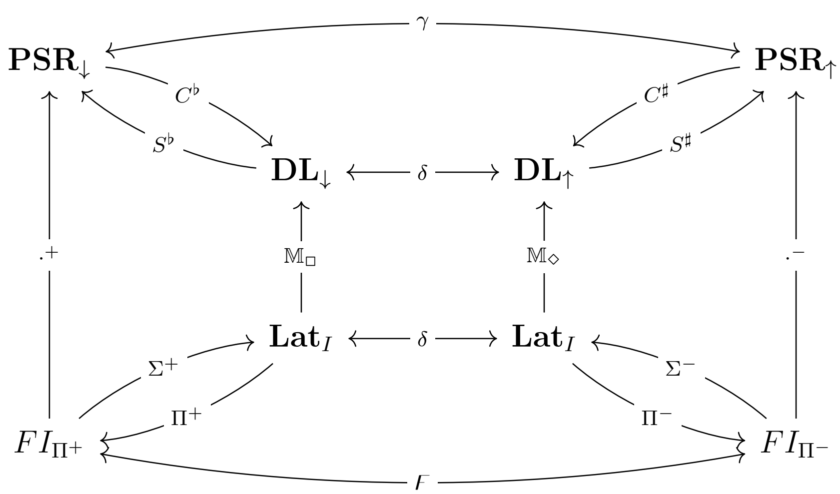
$$\begin{array}{ccc} \mathcal{R}^+(\mathscr{X}_i) & \xrightarrow{\quad h_i \quad} & \mathcal{R}^+(\mathscr{Y}_i) \\ \downarrow{\scriptstyle \Box} & & \downarrow{\scriptstyle \Box} \\ \mathbb{M}_\Box\mathcal{R}^+(\mathscr{X}) & \xrightarrow{ClopUp(f_i^+)} & \mathbb{M}_\Box\mathcal{R}^+(\mathscr{Y})) \end{array}$$

The rest of the proof is straightforward and left to the reader. $\Box$

We conclude by observing that the Π^+/Σ^+ duality presented in this section essentially lifts the duality between $\mathbf{DL}_\downarrow$ and $\mathbf{PSR}_\downarrow$ from Section 4.3, in the sense that the following diagram of functors commutes up to isomorphisms:



Using the functors $\mathbb{M}_\diamond$ and $\cdot^-$ and the duality between $\mathbf{DL}_\uparrow$ and $\mathbf{PSR}_\downarrow$ instead, we could just as well obtain a duality witnessed by functors Π^- and Σ^-, defined so that the following diagram:



commutes up to isomorphism, where ꟻ $: \mathbf{FI} \to \mathbf{FI}$ maps any FI-space $(\mathscr{X}, \lambda^\natural)$ to $(\mathscr{X}, \rho^\natural)$ (i.e., ꟻ merely swaps the projections $\mathscr{X}^+$ and $\mathscr{X}^-$), and FI_{Π^-} is the image of FI_{Π^+} under ꟻ.

4.5.3 Duality for Weakly Pseudo-complemented Lattices

We are now finally ready to reach the second goal of this chapter. Using our new duality between monotone maps and π^+-projective relations, we will now be able to derive a duality for weakly pseudo-complemented lattices. Note that one of the main advantages of our framework is that, in principle, it allows us to obtain a duality for any category of lattices augmented by unary monotone or antitone operations. Thus there could be many more applications of this general than the one we will focus on here. We start by with the following definition.

Definition 4.5.7. A *weak involution* on a lattice L is a map $\neg : L \to L$ such that $a \leq_L \neg b$

iff $b \leq_L \neg a$ for all $a, b \in L$. A *weakly pseudo-complemented lattice* is a pair $(L, \neg)$ such that $\neg$ is a weak involution on L satisfying the additional condition that $a \wedge \neg a = 0$ for all $a \in L$.

Equivalently, a weak involution on a lattice L is a self-adjoint map from L to L. In particular, it is antitone, and turns joins into meets. This means that we may think of a weak involution $\neg : L \to L$ as a monotone map from L to $\delta(L)$ with a right adjoint going from $\delta(L)$ to L. Moreover, there is a clear one-to-one correspondence between pairs $(L, \neg)$ where $\neg$ is a weak involution on L, and monotone maps $\neg : L \to \delta(L)$ that have a right adjoint. Hence our strategy for obtaining a duality for weakly-pseudo complemented lattices will be the following. Using our Π^+/Σ^+ duality, we will first identify the duals of weak involutions and represent those via relations on FI-spaces. Then we will identify a condition on such FI-spaces that corresponds to the weak involution being a weak pseudo-complement. Let us start with the following characterization of weak involutions.

Lemma 4.5.8. *Let $f : L \to \delta(L)$ be a monotone map. Then f is a weak involution if and only if $\Pi^+(f)$, viewed as a relation on $FI(L) \times FI(L)$, has the following properties:*

1. *For any $U \in \mathcal{R}^+(FI(L))$, $C^\flat(\Pi^+(f))(U) \in \mathcal{R}^-(FI(L))$;*
2. *$\Pi^-(\delta(f)) = \Pi^+(f)^{-1}$;*
3. *For any $V \in \mathcal{R}^-(FI(L))$, $C^\sharp(\Pi^-(\delta(f))^{-1})(V) \in \mathcal{R}^+(FI(L))$.*

Proof. Suppose first that f is a weak involution. Viewing f as a monotone map from $L \to \delta(L)$, we have that $\Pi^+(f)$ is a π^+-projective lower closed relation on $FI(\delta(L)) \times FI(L)$. Note that $FI(\delta(L)) = \mathcal{F}(\delta(L)) \times \gamma(\mathcal{I}(L)) = \gamma(\mathcal{I}(L)) \times \mathcal{F}(L) = \digamma(FI(L))$. Hence we may think of $FI(L^\delta)$ as $FI(L)$, but with its projections swapped. In other words, $(FI(L^\delta), \lambda^\natural_{L^\delta}) = (FI(L), \rho^\natural_L)$, and we have that $\mathcal{R}^+(FI(L^\delta)) = \mathcal{R}^-(FI(L))$. By duality, we have that $\Sigma^+\Pi^+(f)(\upsilon_L(a)) = \upsilon_{\delta(L)}(f(a)) \in \mathcal{R}^+(FI(L^\delta))$. But this implies at once that $C^\flat(\Pi^+(f))(U) \in \mathcal{R}^-(FI(L))$.

Now let us consider $\delta(f) : L^\delta \to L$. Using the Σ^-/Π^- duality mentioned at the end of the previous section, it follows from a straightforward diagram chasing argument that $C^\sharp(\Pi^-(\delta(f))^{-1})(V) \in \mathcal{R}^+(FI(L))$ for any $V \in \mathcal{R}^-(FI(L))$. Moreover, I claim that $\Pi^-(\delta(f)) = \Pi^+(f)$. Again, a simple diagram chasing argument shows that for any $(F, I), (G, J) \in FI(L)$, $(F, I)\Pi^-(f)(G, J)$ iff $f^{-1}[F] \subseteq J$. Since we have that $(G, J)\Pi^+(f)(F, I)$ iff $F \subseteq f^{-1}[J]$, we only have to show that $F \subseteq f^{-1}[J]$ iff $f^{-1}[F] \subseteq J$ to establish properties 2 and 3. But this is a standard fact for weak involutions. For the left to right direction, suppose that $f^{-1}[F] \subseteq J$, and let $a \in F$. Since f is a weak involution, $a \leq f(f(a))$, hence $f(f(a)) \in F$. By assumption, this implies that $f(a) \in J$. Conversely, assume that $F \subseteq f^{-1}[J]$, and let $f(a) \in F$. By assumption $f(f(a)) \in J$. Since f is a weak involution, we have that $a \leq f(f(a))$, hence $a \in J$.

Conversely, let us now assume that $\Pi^+(f)$ has all the properties listed in the statement of the lemma. From properties 1 and 3, we have maps $\Sigma^+\Pi^+(f) : \mathcal{R}^+(FI(L)) \to \mathcal{R}^-(FI(L))$ and $\Sigma^-\Pi^-(\delta(f)) : \mathcal{R}^-(FI(L)) \to \mathcal{R}^+(FI(L))$. By duality, we have that $\Sigma^+\Pi^+(f)(\upsilon_L(a)) =$

$\upsilon_{\delta(L)}(\boldsymbol{f(a)})$ and similarly $\Sigma^-\Pi^-(\delta(f))(\upsilon_{\delta(L)}(\boldsymbol{a}) = f(a)$, so we only need to verify that $\Sigma^+\Pi^+(f)$ is left-adjoint to $\Sigma^-\Pi^-(\delta(f))$. Using property 2, we compute for any $U \in \mathcal{R}^+(FI(L))$, $V \in \mathcal{R}^-(FI(L))$:

$$\begin{aligned}
\Sigma^+\Pi^+(f)(U) \subseteq V &\Leftrightarrow C^\flat\Pi^+(f)(U) \subseteq V \\
&\Leftrightarrow \forall(F,I) : (\exists(G,J) \in V : (F,I)\Pi^+(f)(G,J)) \Rightarrow (F,I) \in V \\
&\Leftrightarrow \forall(F,I),(G,J) : ((G,J) \in U \& (G,J)\Pi^+(f)^{-1}(F,I)) \Rightarrow (F,I) \in V \\
&\Leftrightarrow \forall(F,I),(G,J) : ((G,J) \in U \& (G,J)\Pi^-(\delta(f))^{-1}(F,I)) \Rightarrow (F,I) \in V \\
&\Leftrightarrow \forall(G,J) \in U : \Pi^-(\delta(f))^{-1}(G,J) \subseteq V \\
&\Leftrightarrow U \subseteq C^\sharp\Pi^-(\delta(f))(V) \\
&\Leftrightarrow U \subseteq \Sigma^-\Pi^-(\delta(f))(V).
\end{aligned}$$

This completes the proof. □

Using the lemma above, we can therefore represent every weak involution $\neg$ on a lattice L as a relation on $FI(L)$. Moreover, it is straightforward to check that, for any weak involution $\neg : L \to L$ and any $a \in L$, $C^\sharp(\rho^\natural)(-C^\flat(\Pi^+(\neg))(\upsilon_L(a))) = \upsilon_L(\neg a)$. Indeed, by the Σ^+/Π^+ duality, we know that $C^\flat(\Pi^+(\neg))(\upsilon_L(a)) = \mathcal{F}(L)\times\widehat{\neg a}$. Moreover, $-(\mathcal{F}(L)\times\widehat{\neg a}) = \mathcal{F}(L)\times\widecheck{\neg a}$, and $C^\sharp(\rho^\natural)(\mathcal{F}(L) \times \widecheck{\neg a}) = \widehat{\neg a} \times \mathcal{I}(L)$, as shown in Section 4.4. Thus we may conclude that

$$C^\sharp(\rho^\natural)(-C^\flat(\Pi^+(\neg))(\upsilon_L(a))) = C^\sharp(\rho^\natural)(\mathcal{F}(L) \times \widecheck{\neg a}) = \widehat{\neg a} \times \mathcal{I}(L) = \upsilon_L(\neg a)$$

for any $a \in L$. Described explicitly, this defines a function $\neg_{\Pi^+} : \mathcal{R}^+(FI(L)) \to \mathcal{R}^+(FI(L))$ given by

$$\neg_{\Pi^+}(U) = \{(F,I) \mid \forall(G,J)\lambda_L^\natural(F,I)\forall(F',I') : (G,J)\Pi^+(\neg)(F',I') \Rightarrow (F',I') \notin U\}.$$

Let us now identify a condition on Π^+ corresponding to the weak pseudo-complement property.

Lemma 4.5.9. *For any weak involution $\neg : L \to L$, $\neg$ is a weak pseudo complement if and only if $\Pi^+(\neg) \circ \rho^\natural$ is reflexive.*

Proof. Suppose first that $a \wedge \neg a = 0$ for any $a \in L$, and let $(F,I) \in FI(L)$. We must find (G,J) such that $(F,I)\rho^\natural(G,J)$ and $(G,J)\Pi^+(\neg)(F,I)$. In other words, we must find (G,J) such that $F \cap J = \emptyset$ and $F \subseteq \neg^{-1}[J]$. Let $G = \{1\}$ and $J = \downarrow\{\neg a \mid a \in F\}$. Note that J is an ideal, since $\neg a \vee \neg b \leq \neg(a \wedge b)$ follows from the fact that $\neg$ is a weak involution. Moreover, we have that $F \cap J = \emptyset$, since otherwise there is $a \in F$ such that $\neg a \in F$, which implies that $a \wedge \neg a = 0 \in F$ and thus that F is not proper.

Conversely, suppose now that $\Pi^+(\neg)\circ\rho^\natural$ is reflexive. It is enough to show that $\neg(U)\cap U = \emptyset$ for any $U \in \mathcal{R}^+(FI(L))$. Suppose $(F,I) \in FI(L)$. By assumption, we have (G,J) such that $(F,I)\rho^\natural(G,J)$ and $(G,J)\Pi^+(\neg)(F,I)$. But then $(F,I) \in \neg_{\Pi^+}(U)$ implies $(F,I) \notin U$. □

Finally, we can define the duals of weakly pseudocomplemented lattices.

Definition 4.5.10. A FIN-space is a triple $(\mathscr{X}, \lambda^\natural, R_\neg) =$ such that $(\mathscr{X}, \lambda^\natural)$ is a FI-space and $R_\neg$ is a relation on $X \times X$ satisfying the following conditions:

1. $R_\neg$ is a π^+-projective lower-closed relation from $(\mathscr{X}, \rho^\natural)$ to $(\mathscr{X}, \lambda^\natural)$;
2. $R_\neg^{-1}$ is a π^+-projective lower-closed relation from $(\mathscr{X}, \lambda^\natural)$ to $(\mathscr{X}, \rho^\natural)$;
3. The relation $R_\neg \circ \rho^\natural$ is reflexive.

A FIN-morphism between two FIN-spaces $(\mathscr{X}, \lambda^\natural_{\mathscr{X}}, R_\neg)$ and $(\mathscr{Y}, \lambda^\natural_{\mathscr{Y}} S_\neg)$ is an order-continuous map $f : \mathscr{X} \to \mathscr{Y}$ such that the pair $(\digamma(f)^+, f^+)$ is a morphism from $R_\neg^+ \subseteq (\mathscr{X}, \rho^\natural)^+ \times (\mathscr{X}, \lambda^\natural)^+$ to $S_\neg^+ \subseteq (\mathscr{Y}, \rho^\natural)^+ \times (\mathscr{Y}, \lambda^\natural)^+$ in $\mathbf{PSR}_\downarrow$.

By the lemmas above together with the duality from Section 4.5.2, we finally obtain the following:

Theorem 4.5.11. *The category* $\mathbf{Lat}_\neg$ *of weakly pseudo-complemented lattices and weak pseudo-complement-preserving lattice homomorphisms between them is dual to the category* $\mathbf{FIN}$ *of* FIN*-spaces and* π^+*-projective morphisms between them.*

Proof. Note first that $\mathbf{Lat}_\neg$ is isomorphic to the category of monotone maps $\neg : L \to L^\delta$ such that $\delta(\neg) : L \to L^\delta$ is the right-adjoint of $\neg$, and $a \wedge \delta(\neg)(\boldsymbol{a}) = 0$ for all $a \in L$, and morphisms $(f, \delta(f)) : \neg \to \delta(\neg)$. By Lemmas 4.5.8 and 4.5.9, the π^+ projective relations dual to such monotone maps are in a natural one-to-one correspondence with FIN-spaces, and the morphisms between them correspond precisely to FIN-morphisms by the Π^+/Σ^+ duality. Hence $\mathbf{Lat}_\neg$ is dual to $\mathbf{FIN}$. □

4.6 Concluding Remarks

We conclude this chapter with some remarks briefly relating FI and FIN spaces to other representations of complete lattices.

4.6.1 FI-Spaces and B-frames

We start by connecting FI-spaces with the b-frames of Chapter 2. As mentioned in Section 4.1, one of our motivations for developing our duality in the first place was to extend b-frame duality to a topological duality for the category of lattices. *Prima facie*, it is clear that the representation of arbitrary lattices via FI-spaces shares some resemblance with the representation of complete lattices via b-frames. In both cases, the points in the dual space of a lattice L are pairs in which the first and second components capture, informally speaking, "positive" and "negative" information respectively. Moreover, in both cases, the original lattice is recovered as a (sub)algebra of fixpoints of some closure operator on the powerset of the dual space. However, the way such a closure operator is defined is, at least superficially, different in both cases. In the case of b-frames, the closure operator is given by the composition of the closure operation in the upset topology induced by the second ordering and the interior operation in the upset topology induced by the first ordering. In the case of FI-spaces, one uses a relation $\lambda^\natural$ and its converse $\rho^\natural$, and the operations $C^\flat$ and $C^\sharp$. However, there is a standard way of passing from FI-spaces to bosets, which essentially coincides with the correspondence between bi-ordered structures and compatibility relations introduced in [136]. Given a boset $(X, \leq_1, \leq_2)$, we may let $\top$ the complement of the relation ${}_2\bot_1$, i.e, for any $x, y \in X$, $x \top y$ iff there is $z \in X$ such that $z \leq_2 x$ and $z \leq_1 y$. Conversely, given a FI-space $\mathscr{X} = (X, \tau, \leq, \lambda^\natural)$, recall that the "projections" $\mathscr{X}^+$ and $\mathscr{X}^-$ are induced by orderings $\leq^+$ and $\leq^-$ on $\mathscr{X}$. If we assume that $\mathscr{X} = (FI(L)\lambda^\natural)$ for some lattice L, it is easy to check that $\leq^+$ and $\leq^-$ corresponding to the filter and ideal orderings on $FI(L)$ respectively. However, the compatibility relation induced by $\leq_1$ and $\leq_2$ is the universal relation on $\mathcal{F}(L) \times \gamma(\mathcal{I}(L))$, since for any $F \in \mathcal{F}(L), I \in \mathcal{I}(L)$, the pair (F, I) is in $FI(L)$. However, we may consider the following subspace of $FI(L)$.

Lemma 4.6.1. *Let L be a lattice and $FI(L)_R$ the subspace of $FI(L)$ induced by the set $\{(F, I) \in FI(L) \mid (F, I)\lambda^\natural(F, I)\}$. Then the compatibility relation on $FI(L)_R$ induced by the restrictions of the orders $\leq^+$, $\leq^-$ to $FI(L)_R$ coincides with $\lambda^\natural_R = \lambda^\natural | FI(L)^*$. Moreover, for any $U \subseteq FI(L)_R$, one has that $C^\sharp(\rho^\natural_R)C^\flat(\lambda^\natural_R)(U) \subseteq I_+C_-(U)$, where I_+ and C_- are the interior and closure operations on $FI(L)_R$ induced by $\leq^+$ and $\leq^-$ respectively, and that the converse holds if U is a $\leq^+$-upset.*

Proof. Let us first show that $\lambda^\natural_R$ is the compatibility relation induced by $\leq^+$ and $\leq^-$. Fix $(F, I), (G, J) \in FI(L)_R$. Then we have:

$$\begin{aligned}
(F, I)\top(G, J) &\Leftrightarrow \exists (F', I') \in FI(L)_R : (F, I) \leq^- (F', I') \& (G, J) \leq^+ (F', I') \\
&\Leftrightarrow \exists F' \in \mathcal{F}(L), I' \in \mathcal{I}(L) : F' \cap I' = \emptyset \& G \subseteq F' \& I \subseteq I' \\
&\Leftrightarrow G \cap I = \emptyset \\
&\Leftrightarrow (F, I)\lambda^\natural_R(G, J)
\end{aligned}$$

Moreover, fix $U \subseteq FI(L)_R$. First we compute that:

$$I_+C_-(U) = \{(F, I) \mid \forall (G, J) \geq^+ (F, I) \exists (F', I') \geq^- (G, J) : (F', I') \in U\},$$

and that

$$C^\sharp(\rho^\natural_R)C^\flat(\lambda^\natural_R)(U) = \{(F, I) \mid \forall (G, J)\lambda^\natural_R(F, I) \exists (F', I')\rho^\natural_R(F, I) : (F', I') \in U\}.$$

Let us first show the left-to-right inclusion. Suppose that $(F, I) \in I_+C_-(U)$, and let $(G, J)\lambda^\natural_R(F, I)$. Then $F \cap J = \emptyset$, so the pair $(F, J) \in FI(L)_R$. Hence there is $(F', I') \in U$ such that $J \subseteq I'$ and $I' \cap F' = \emptyset$. But then $(F', I')\rho^\natural_R(G, J)$. Hence $(F, I) \in C^\sharp(\rho^\natural_R)C^\flat(\lambda^\natural_R)(U)$. For the converse, suppose that $(F, I) \in C^\sharp(\rho^\natural_R)C^\flat(\lambda^\natural_R)(U)$, and let $(G, J) \in FI(L)_R$ such that $F \subseteq J$. Since $G \cap J = \emptyset$, we have that $F \cap J = \emptyset$, hence $(G, J)\lambda^\natural_R(F, I)$. By assumption, there is $(F', I') \in U$ such that $(F', I')\rho^\natural_R(G, J)$ i.e., $F' \cap J = \emptyset$. Since U is a $\leq^+$-upset, the pair $(F', J) \in U$, and we clearly have that $(G, J) \leq^- (F', J)$. Hence $(F, I) \in I_+C_-(U)$. $\square$

This lemma shows that, when restricting $FI(L)$ to a subset on which the relation $\lambda^\natural$ is reflexive, working with the relation $\lambda^\natural$ becomes essentially equivalent to working with the two orders $\leq^+$ and $\leq^-$, and the representation of the original lattice L via the closure operator induced by λ^+ coincides with the representation via the operation I_+C_-. Of course, the resulting subspace is not itself an FI-space, but it is nonetheless a Priestley space. Indeed, since $FI(L)$ is defined as the product of $\mathcal{F}(L)$ and $\gamma(\mathcal{I}(L))$, which are themselves the Priestley duals of $\mathbb{M}_\Box(L)$ and $\delta(\mathbb{M}_\Diamond(L))$, we may think of the Priestley dual of $FI(L)$ as the coproduct in **DL** of $\mathbb{M}_\Box(L)$ and $\delta(\mathbb{M}_\Diamond(L)) = \mathbb{M}_\Box(L^\delta)$. Since this is a coproduct of free constructions, we may equivalently view it as the distributive lattice $\mathbb{M}_\Box^\blacksquare(L)$ freely generated by elements of the form $\{\Box a, \blacksquare \boldsymbol{a} \mid a \in L\}$ and subject to the relations $\{\Box a \wedge \Box b = \Box a \wedge b, \blacksquare \boldsymbol{a} \wedge \blacksquare \boldsymbol{b} = \blacksquare(\boldsymbol{a} \vee \boldsymbol{b}), \Box 1 = \blacksquare \mathbf{0} = 1, \Box 0 = \blacksquare \mathbf{1} = 0\}$. We can then easily establish a one-to-one correspondence between prime filters on $\mathbb{M}_\Box^\blacksquare(L)$ and points in (F, I) by mapping any prime filter p to the pair (F_p, I_p), where $F_p = \{a \in L \mid \Box a \in p\}$ and $I_p = \{a \in L \mid \blacksquare \boldsymbol{a} \in p\}$. By duality reasoning, selecting a subset of $FI(L)$ corresponds to quotienting $\mathbb{M}_\Box^\blacksquare(L)$ by adding more relations. In the case of $FI(L)_R$, we can easily compute that the additional relations should be $\{\Box a \wedge \blacksquare \boldsymbol{a} = 0 \mid a \in L\}$. As we shall see, a similar strategy can be applied to the comparison between FI-spaces and the choice-free duals of distributive lattices, as well as to the comparison between FIN-spaces and Holliday's compatibility frames for Fundamental Logic.

4.6.2 FI-Spaces and Topological Dualities

In the previous chapter, we presented two choice-free dualities for distributive lattices based of filter-ideal pairs, and we also briefly described Mosher and Jipsen's duality between lattices and BL-spaces. Let us now briefly compare all three with the topological duality for lattices presented in Section 4.4.

Let us start with a comparison with Moshier and Jipsen's BL-duality. In short, one might see a rather striking resemblance between BL-spaces and our overall strategy in this section. Moshier and Jipsen establish first a duality for meet-semilattices via *HMS* spaces, which are a specific kind of spectral spaces. They then restrict this duality to BL-spaces by defining a closure operator fsat, and show that this operator restricts to a closure operator on open sets if and only if the compact open filters of an HMS space form a lattice. Moreover, the dual BL-space of a lattice L is constructed as the set of all proper filters on L, endowed with the Stone topology generated by sets of the form $\widehat{a}$ for any $a \in L$. Equivalently, the dual BL-space of a lattice is the dual *spectral* space of $\mathbb{M}_\Box(L)$. Accordingly, we may summarize the similarities and distinctions between the BL-duality and our FI-spaces as follows. Both dualities essentially rely on an embedding of **Lat** into **DL**, and on an existing topological duality for distributive lattices. But the Moshier-Jipsen duality passes through Stone duality for distributive lattices and through the free construction $\mathbb{M}_\Box$ exclusively, while our construction of FI-spaces relies on Priestley duality and on the two free constructions $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$. As a consequence of the latter, Jipsen and Moshier arguably have more work to do to recover the join-structure of a lattice L inside its dual BL-space. Intuitively speaking, working with their notion of F-saturated sets (which are intersections of open filters) allows them to represent the ideals of L. By contrast, our construction through filter-ideal pairs allows us to work directly with ideals, thus giving an arguably more concrete flavor to our topological representation of lattices.

Let us also quickly address the difference between using spectral spaces and Priestley spaces. In the case of distributive lattices, Priestley spaces are often seen as more convenient structures to work with than spectral spaces, as their topological behavior closely resembles that of Stone spaces. In our case however, it is straightforward to verify that we did not use any specific features about Priestley spaces beyond the fact that they are dual to distributive lattices. In fact, our main reason for using Priestley spaces over spectral spaces was one of convenience. As mentioned in Section 4.2.3, the functor γ, which is the topological counterpart of the dualizing functor δ on lattices, has a very simple definition for Priestley spaces, since $\gamma(\mathscr{X})$ is simply the space $\mathscr{X}$ with its order reversed. But this follows in an essential way from the fact that the topology on a Priestley space is the patch topology of the topology on the corresponding spectral space. If we wanted to define γ on spectral spaces instead, we would essentially need to take the *de Groot* dual of the spectral space $\mathscr{X}$, i.e. the topology generated by the intersections of compact open sets. Given the importance of γ and many constructions above, this would certainly make some arguments more cumbersome than they already are. There is, however, a very good reason to do so, having to do with the essentially choice-free nature of our FI-duality. *Prima facie*, because our duality passes through Priestley duality, it cannot be carried out in a semi-constructive setting. However, it is worth remarking that the actual use we make of the resources of Priestley duality is very limited. This is because all instances of the duality we use are about free constructions of the form $\mathbb{M}_\Box$ or $\mathbb{M}_\Diamond$ and their Priestley duals, which are spaces of filters or ideals. In such free constructions, prime filters are in one-to-one correspondence with filters in another algebraic structure, a fact we used repeatedly. But this means in particular that any such free construction satisfies the following weak form of the Prime Filter Theorem.

Lemma 4.6.2 (Weak PFT)**.** *Let $M = \mathbb{M}_\Box(L)$ for some lattice L, and let $a, b \in M$ be such that $a \not\leq b$. Then there is a prime filter p on M such that $a \in p$ and $b \notin p$.*

Proof. Let us write a as $\bigvee_{i\in I} \Box a_i$ and b as $\bigvee_{j\in J} \Box b_j$ for some finite $\{a_i \mid a \in I\}, \{b_j \mid j \in J\} \subseteq L$. Then $a \not\leq b$ implies that there is $i \in I$ such that $a_i \not\leq b_j$ for all $j \in J$. But then, letting $p = {\eta_\Box^L}^{-1}(\uparrow a)$, we have that $a \in p$ and $b \notin p$. □

Consequently, the injectivity of the map $\widehat{\cdot} : \mathbb{M}_\Box(L) \to ClopUpSpec(\mathbb{M}_\Box(L))$ can be proved without appealing to PFT for any lattice L. However, we run into some issues when trying to prove without PFT that $Spec(\mathbb{M}_\Box(L))$ is compact. Indeed, proving that any open cover in the patch topology contains a finite subcover requires either using PFT or appealing to Alexander's Subbasis Lemma, which is itself equivalent to PFT. Importantly, this issue only arises because we declare as open both sets of the form $\widehat{a}$ and sets of the form $\check{a}$. If we were to consider only the spectral topology generated by the sets $\widehat{a}$, then proving that $Spec(\mathbb{M}_\Box(L))$

is spectral could be achieved within ZF. Indeed, the specialization preorder on $Spec(L)$ induced by the spectral topology coincides with the inclusion order on the set of prime filters, and therefore has a least element p_0, namely ${\eta_{\Box}^{L}}^{-1}(\{1_L\}$. But then any cover of $Spec(\mathbb{M}_{\Box}(L))$ must contain an open set U such that $p_0 \in U$, which implies that $Spec(\mathbb{M}_{\Box}(L)) \subseteq U$. We can therefore conclude from this discussion that our FI-duality is *essentially* choice-free, in the sense that a merely notational variant of it that uses spectral spaces instead of Priestley spaces holds in a semi-constructive setting.[2].

Finally, let us briefly address the relationship between FI-spaces and the choice free duals of distributive lattices introduced in Chapter 3. Here, there are two observations worth making. First, if L is a distributive lattice, it will not follow that $FI(L)$ is UVP-space, nor that $(FI(L), \tau^+, \tau^-)$ is a pairwise UV-space. Intuitively, the main reason for this is that the specialization order on $FI(L)$ is the inclusion order on both filters and ideals, while the representation for distributive lattices in the previous chapter was based on the inclusion ordering on filters and the *reverse* inclusion on ideals. Admittedly, this mismatch between the two constructions is somewhat of a mystery (at least to me). On the one hand, the fact that both pairwise UV-spaces and UVP-spaces arise as upper Vietoris hyperspaces on pairwise Stone spaces and Priestley spaces respectively suggests that the reverse ordering on ideals is the natural choice in this setting. However, setting the dual FI-space of a lattice L to be the Priestley space $\mathcal{F}(L) \times \mathcal{I}(L)$ introduces some obstacles to the functoriality of $S^{\natural}$. More precisely, given a lattice homomorphism f, we would in general lose the ability to define $S^{\natural}(f)$ as the map $(F, I) \mapsto (f^{-1}[F], f^{-1}[I])$, as we would then not be able to prove that it is an FI-morphism. The issue could be avoided by mapping f to the *pair* of maps $(F, I) \mapsto (f^{-1}[F], \{0\})$, $(F, I) \mapsto (\{1\}, f^{-1}[I])$, but this approach is clearly less elegant.

Finally, let us conclude with an observation similar to the one we made regarding b-frames. In the construction of the choice-free duals of distributive lattices, we restricted ourselves to sets of pseudo-prime pairs. It is a straightforward exercise to verify that we this restriction corresponds in the setting of FI-spaces to quotienting the free construction $\mathbb{M}_{\Box}^{\blacksquare}(L)$ by the additional relations

$$\{\Box(a \vee b) \wedge \blacksquare\boldsymbol{b} \leq \Box a,\, \Box a \wedge \blacksquare(\boldsymbol{a} \wedge \boldsymbol{b}) \leq \blacksquare\boldsymbol{b} \mid a, b \in L\}.$$

Constructing this lattice as the coproduct of $\mathbb{M}_{\Box}(L)$ and $\mathbb{M}_{\Diamond}(L)$ instead, these relations translate to

$$\{\Box(a \vee b) \leq \Box a \vee \Diamond b,\, \Box a \wedge \Diamond b \leq \Diamond(a \wedge b)\,\Box,$$

which play a key role in positive modal logic [143, 257].

4.6.3 FIN-Spaces and Compatibility Frames

Compatibility frames provide a semantics for Fundamental Logic that is based on a discrete representation for weakly pseudo-complemented lattices. As shown in [131], given a relation

[2] In that sense, the situation is similar to the choice-free version of Goldblatt's duality for ortholattices that was developed in [191]

$\lhd$ on a set X, one can define closure operator $C_\lhd$ on $\mathscr{P}(X)$ by letting

$$c_\lhd(U) = \{x \in X \mid \forall y \lhd x \exists z \rhd y : z \in U\},$$

where $\rhd$ is the converse of the relation $\lhd$. Moreover, one can also define an operation $\neg_\lhd$ on the lattice $\mathfrak{L}(X)$ of $c_\lhd$-fixpoints of X by letting

$$\neg_\lhd(U) = \{x \in X \mid \forall y \lhd x : y \notin U\}.$$

Holliday shows that every weakly pseudo-complemented lattice can be represented as a sublattice of the complete lattice $(\mathfrak{L}(X), \neg_\lhd)$ for some set X and relation $\lhd$ on some X satisfying some conditions. Let us now quickly compare the representation of weakly pseudo-complemented lattices via compatibility frames and via FIN-spaces.

Note first that, given a lattice L, $\mathcal{R}^+(FI(L))$ is the intersection of $ClopUp(FI(L))$ with the fixpoints of the closure operator $C^\sharp(\rho^\natural)C^\flat(\lambda^\natural)$. Moreover, as we computed above, we have that

$$C^\sharp(\rho^\natural_R)C^\flat(\lambda^\natural_R)(U) = \{(F,I) \mid \forall (G,J)\lambda^\natural_R(F,I)\exists (F',I')\rho^\natural_R(F,I) : (F',I') \in U\}$$

for any $U \subseteq FI(L)$. But this means that we may view $FI(L)$ as a compatibility frame with the relation $\lhd$ being precisely $\lambda^\natural$. Moreover, as shown in the previous section, a weak involution on L is represented in $FI(L)$ by the operation $\neg_{\Pi^+}$ on $\mathcal{R}^+$, given by:

$$\neg_{\Pi^+}(U) = \{(F,I) \mid \forall (G,J)\lambda^\natural_L(F,I)\forall (F',I') : (G,J)\Pi^+(\neg)(F',I') \Rightarrow (F',I') \notin U\}.$$

Hence the representation of the weak pseudo-complement in $FI(L)$ differs from its definition in a compatibility frame. However, just like FI-spaces and b-frames can be brought closer by taking subspaces, a similar strategy applies in this case.

Lemma 4.6.3. *Let $(L,\neg)$ be a weak involution with dual FIN-space $(FI(L), \lambda^\natural, \Pi^+(\neg)$. Let $FI(L)^Q$ be a subspace of $FI(L)$ determined by some subset Q, and $\neg^Q_{\lambda^\natural}$ and $\neg^Q_{\Pi^+}$ the operations on $\mathscr{P}(Q)$ induced by $\neg_{\lambda^\natural}$ and $\neg_{\Pi^+}$ respectively. Then:*

1. *If $Q \subseteq \{(F,I) \mid I \subseteq \neg^{-1}[F]\}$, then $\neg^Q_{\lambda^\natural}(U) \subseteq \neg^Q_{\Pi^+}(U)$ for all $U \subseteq Q$;*
2. *If $Q \subseteq \{(F,I) \mid \neg^{-1}[F] \subseteq I\}$, then $\neg^Q_{\Pi^+}(U) \subseteq \neg^Q_{\lambda^\natural}(U)$ for all $U \subseteq Q$.*

Proof. We prove both items in turn.

1. Suppose first that $I \subseteq {\downarrow}\neg^{-1}[F]$ for all $F, I \in Q$. I claim that $(\Pi^+(\neg) \circ \rho^\natural) \cap Q \subseteq \rho^\natural \cap Q$. Clearly, this will imply that $\neg^Q_{\lambda^\natural}(U) \subseteq \neg^Q_{\Pi^+}(U)$ for all $U \subseteq Q$. For the proof of the claim, suppose that we have $(F,I), (G,J), (F',I') \in Q$ such that $(F,I)\rho^\natural(G,J)\Pi^+(\neg)(F',I')$. Then we have that $F \cap J = \emptyset$ and that $F' \subseteq \neg^{-1}[J]$. Since $\neg$ is a weak involution, the latter is equivalent to $\neg^{-1}[F'] \subseteq J$. But by assumption $I' \subseteq \neg^{-1}[F']$. Hence $I' \cap F = \emptyset$, which implies that $(F,I)\rho^\natural(F',I')$.

2. Suppose now that $\neg^{-1}[F] \subseteq I$ for all $(F,I) \in Q$. Since $\neg$ is a weak involution, this is equivalent to $F \subseteq \neg^{-1}[I]$, and hence to $(F,I)\rho^\natural(F,I)$ for all $(F,I) \in Q$. But it clearly follows from the reflexivity of $\rho^\natural$ on Q that $\neg^Q_{\Pi^+}(U) \subseteq \neg^Q_{\lambda^\natural}(U)$ for all $U \subseteq Q$. □

Hence the two operations coincide if we restrict $FI(L)$ to the set of pairs (F,I) such that $\neg^{-1}[F] = I$. Note also that this does not imperil the representation of L. Indeed, the map $a \mapsto \widehat{a} \times \mathcal{I}(L)$ is still injective, since the pair $(\uparrow a, \downarrow\neg a)$ has the required property. Algebraically, restricting $FI(L)$ to such pairs corresponds once again to quotient the distributive lattice $\mathbb{M}^{\blacksquare}_{\Box}(L)$ by additional relations. This time, it is also straightforward to see that the additional relation is $\{\Box f(a) = \blacksquare \boldsymbol{a} \mid a \in L\}$. It is worth mentioning that, if we wanted to add the extra condition $\{\Box a \wedge \blacksquare \boldsymbol{a} = 0 \mid a \in L\}$ in order to recover the equivalence with the boset representation of L, this would immediately imply that our distributive lattice satisfies $\Box a \wedge \Box f(a) = 0$. One can indeed check also geometrically that the conditions that $\lambda^\natural$ and $\Pi^+(\neg)$ be symmetric immediately imply that $\Pi^+(\neg) \circ \rho^\natural$ is reflexive, which implies that $\neg_{\Pi^+}$ is a weak pseudo complement by Lemma 4.5.9.

It follows from this last observation that one can combine a boset representation of a lattice L with a compatibility representation of a weak involution $\neg$ on L if and only if $\neg$ is weak pseudo-complement. In a sense, this provides an independent motivation for weak pseudo-complements, as they appear to arise as natural algebraic structures that can have an elegant representation. But one should also take this example as a cautionary tale. Competing desiderata on the representation of operations on lattices may sometimes be incompatible unless such operations satisfy some additional properties. At the same time, our results here seem to indicate a promising way of addressing such issues. Our FI duality provides, so to speak, a canonical way of representing lattices and monotone maps on them. Given such a representation, one can then take subspaces of the dual FI-space of a lattice L in order to obtain simpler, more elegant representations. Finally, using the duality between $FI(L)$ and $\mathbb{M}^{\blacksquare}_{\Box}(L)$, one can control algebraically that such restrictions do not imperil the representation of the original lattice, by checking that the map $a \mapsto \Box a$ is still an embedding. We leave a deeper exploration of the full possibilities afforded by such a conceptual framework for future work.

Chapter 5

Orthologic and the Open Future

5.1 Introduction

The intuition that the future is open in ways that the present and the past are not is the source of famously difficult problems at the intersection of logic, semantics and metaphysics. On the one hand, if an assertion made today about a sea battle occurring tomorrow is true today, is it genuinely possible for the sea battle not to happen? If I claim that a sea battle may or may not happen tomorrow, can I also assert that the sea battle will in fact happen without contradicting myself? If the future is genuinely open, then it seems that statements about the future may merely *turn out* to be true or false, not that they are already true or false when uttered. On the other hand, it seems that such an intuition about the openness of the future commits us to the thesis that a statement cannot be true without being inevitably true and that truth coincides with settled truth. This, however, takes us dangerously close to classical arguments for logical determinism, which purport to give purely *a priori* proofs that the future is entirely determined. Can't a statement be true without being inevitably true?

My goal in this chapter is to flesh out a position that vindicates the intuition of an open future without erasing the distinction between truth and settled truth. The key move of this position, which I call *orthofuturism*, is to argue that the correct logic of the open future is not classical logic, but rather orthologic. As we will see below, this allows one to hold that statements about the future cannot be true and contingent at the same time, without having to admit that truth entails settled truth. In this way, the solution I propose resembles the treatment of epistemic modals recently developed by Holliday and Mandelkern in [138]. In their account based on orthologic, propositions such as "It's raining and it might not be raining" are contradictions, yet "It's raining" does not entail "It must be raining". Similarly, the orthologic of the open future that I present here allows one to hold that "There will be a sea battle tomorrow but there might not be one" is contradictory, even though "There will be a sea battle tomorrow" does not entail "Inevitably, there will be a sea battle tomorrow".

The chapter is organized as follows. In Section 5.2, I recall the famous sea battle problem and present a slight variant of it that will help me highlight the specificity of orthofuturism. In Section 5.3, I introduce orthofuturism, first by outlining a strategy for refuting the two arguments presented in Section 5.2 and then by defining a bimodal orthologic OF that provides a formal solution to the sea battle problem. In Section 5.4, I compare this new approach to some well-established solutions based on branching-time semantics and argue that it avoids some of the problems that these competing alternatives have. In Section 5.5, I sketch an intuitive conception of the flow of time for the orthofuturist by providing a concrete, Kripke-like semantics for OF which I call *fragment semantics*. This semantics is in the spirit of possibility semantics [135, 140], and its frames are collections of partial descriptions of moments in time rather than instants that form branching timelines. This allows me to discuss in Section 5.6 the relationship between orthofuturism and MacFarlane's relativist solution to the sea battle problem [178, 179]. Finally, I conclude with some further directions for orthofuturism. The chapter also includes a technical appendix with some details regarding ortholattices and fragment semantics.

5.2 The Sea Battle and the Narrow Pass

Consider the following situation, inspired by a famous passage in Aristotle [6, *De Interpretatione*, Chap. IX]. In the summer of 480 BC, the Athenian general Themistocles prepares for an encounter with a large Persian fleet off the coast of Artemisium, on the Greek island of Euboea. Themistocles has convinced the other members of the Greek alliance that this is where the Greek fleet should attempt to stop the naval forces of King Xerxes. On the evening of the first day of the battle, after superior tactics delivered the Greeks a significant victory, Themistocles wonders whether there will be another battle on the next day. Although the Persian fleet still vastly outnumbers his, an incoming storm together with the losses of the day may convince the Persian generals to avoid a direct confrontation with the Greek fleet, and the Greeks themselves could decide to engage the invader's fleet or to merely try to deter them from attacking again. Although it seems that whether a sea battle will happen on the next day is yet to be decided by meteorological and military considerations, the following argument convinces Themistocles that, in fact, it is already settled one way or the other.

(A1) There will be a sea battle tomorrow or there won't be a sea battle tomorrow.

(A2) Suppose it is the case that there will be a sea battle tomorrow. Then it is already settled that there will be a sea battle tomorrow.

(A3) Suppose it is not the case that there will be a sea battle tomorrow. Then it is already settled that there won't be a sea battle tomorrow.

(A4) Therefore, if either there will be a sea battle tomorrow or there won't be one, then it is already settled that there will be a sea battle tomorrow or it is already settled that there won't be a sea battle tomorrow.

(A5) Therefore, it is already settled that there will be a sea battle tomorrow or it is already settled that there won't be a sea battle tomorrow.

This, of course, is a version of the famous sea battle argument. It is commonly assumed to rely on two premises. According to the first one, often called the Future Excluded Middle, either a sea battle will happen tomorrow or a sea battle won't happen tomorrow. According to the second premise, a true statement is also inevitably true. Variations of the fatalist argument sometimes establish this second premise by invoking the fixity of the past [201, Section 2]: whatever happened cannot be changed, and if a statement, even one about the future, was true in the past, then it is now impossible for that statement not to have been true. But such a detour through the past is unnecessary if one holds, as is commonly admitted, that the present is just as fixed as the past: although the future might still be open, it is too late to change either the past or the present. This asymmetry therefore entails a collapse of modalities: if something is or was the case, it is now settled that it is or was the case. Accordingly, I will call the thesis that truth implies settled truth the Modal Collapse thesis. Statements **(A2)** and **(A3)** are then instantiations of the Modal Collapse thesis in the case of the statements "There will be a sea battle tomorrow" and "There won't be a sea battle tomorrow", respectively. If one accepts the validity of the argument, then one of the two premises must therefore be rejected. But before discussing this argument in more detail, let us consider a second, slightly different situation.

While Themistocles wonders about the sea battle to come, a few miles to the west, the Spartan king Leonidas holds the narrow pass of Thermopylae against the vast terrestrial forces of King Xerxes. Because the Greeks' defensive strategy against the Persian invasion requires both Themistocles's forces to hold the Persian fleet at Artemisium and Leonidas's forces to block the advance of the Persian army at Thermopylae, messages are constantly exchanged between the two positions. After a gruesome first day of fighting during which his men, although outnumbered by a factor of more than ten to one, manage to hold the pass, Leonidas ponders what message to send Themistocles. Will they or won't they manage to hold on to the pass on the next day? It certainly seems that this is not yet settled. Perhaps Leonidas will manage to exhort his men to keep their positions for one more day, or perhaps (as would famously come to happen on the third day of the battle) the Persians will find a way to outflank the Greeks. Although he believes that the fate of the next day's battle is yet to be decided, Leonidas also wants to give a reply to his ally that is as detailed as possible and considers the following argument:

(B1) We may hold the pass tomorrow and we may not hold the pass tomorrow.

(B2) Either we will hold the pass tomorrow or we won't hold the pass tomorrow.

(B3) Suppose we will hold the pass tomorrow. Then we will hold the pass tomorrow and yet we may not hold it.

(B4) Suppose we won't hold the pass tomorrow. Then we won't hold the pass tomorrow and yet we may hold it.

(B5) Therefore, we will hold the pass tomorrow and yet we may not hold it, or we won't hold the pass tomorrow and yet we may hold it.

Swayed by the seemingly flawless logic of the argument, Leonidas concludes that there are only two messages that he could send to Themistocles: either "We will hold the pass tomorrow but we may not hold it", or "We won't hold the pass tomorrow but we may hold it". Although he may not know which message to send, Leonidas also has the distinct impression that either message would profoundly confuse Themistocles. Because the Greeks' defensive strategy relies on them stopping both Xerxes's fleet and his army, Themistocles will retreat if he knows that the Persians have taken Thermopylae, and he will keep fighting if he knows that Leonidas's men are holding firm. But what should he conclude if Leonidas's message is "We will hold the pass tomorrow but we may not hold it"? Intuitively, the second part of Leonidas's message seems to contradict the first part. If Themistocles receives only the first part, then he can confidently tell his men to get ready to hold the Persian fleet at bay for the next day. But if he reads the second part, he will need to decide whether to hope that the bravery of Leonidas's men will keep the Persian army from crossing the pass or to start planning a strategic retreat. Of course, a similar problem would arise if the message that Leonidas sends is "We won't hold the pass, but we may hold it". Both messages sound contradictory and would deeply puzzle his ally. Faced with such a dilemma, Leonidas might conclude as follows:

(B6) It is contradictory to say that we will hold the pass tomorrow and yet we may not hold it, and it is contradictory to say that we won't hold the pass and yet we may hold it.

(B7) Therefore, it is contradictory to say that we may hold the pass tomorrow and that we may not hold it.

Leonidas's dilemma therefore reaches the same ending as Themistocles's meditation: despite all appearances, the future is already settled. The argument above shares one premise with the first one, namely (**B2**), the thesis that any future event either will or won't happen. The other premise, however, is slightly different in Leonidas's argument. Rather than asserting directly that truth implies settled truth, it states that a statement about the future may not at the same time be true and contingent. In that sense, this premise is reminiscent of another of Aristotle's examples, that of a cloak that may be cut up yet won't be, because it will wear out instead.[1] I will call the thesis that the truth of a statement about the future cannot be consistent with its contingent status the Open Future intuition. In Leonidas's argument above, (**B6**) is the conjunction of two instances of the Open Future intuition. Of course, classically, the Open Future intuition is equivalent to the Modal Collapse thesis mentioned in the first argument. If the truth of a statement is inconsistent with it being contingent, then it implies that that statement is also necessary. But, as I will argue below,

[1] For a detailed discussion of this passage and of whether Aristotle's position is consistent with his views on the sea battle problem, see for example [100, Chap. 7].

there is room to deny one thesis while accepting the other, provided one is willing to give up some of the inference rules of classical logic.

In order to analyse further the two arguments presented above, let us introduce a simple propositional language $\mathscr{L}$ given by the following grammar:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathcal{T}\varphi$$

where p belongs to a countably infinite set of propositional variables. Intuitively, $\Box\varphi$ should be interpreted as "It is settled that φ" and $\mathcal{T}\varphi$ as "Tomorrow, it will be the case that φ". As usual, we will define disjunction as the dual of conjunction, i.e., $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, truth as the dual of falsum, i.e., $\top := \neg\bot$, and possibility as the dual of necessity, i.e., $\Diamond\varphi := \neg\Box\neg\varphi$.

In order to formalize the two arguments above, let us isolate the following principles that one may impose on our logic, understood as a consequence relation $\vdash$ between a finite (possibly empty) set of $\mathscr{L}$-formulas Γ and an $\mathscr{L}$-formula φ:

(*F*) $\vdash \mathcal{T}\varphi \vee \mathcal{T}\neg\varphi$ (the Future Excluded Middle);

(*M*) $\mathcal{T}\varphi \vdash \Box\mathcal{T}\varphi$ (the Modal Collapse thesis);

(*O*) $\mathcal{T}\varphi, \Diamond\mathcal{T}\neg\varphi \vdash \bot$ (the Open Future intuition);

(*C*) $\varphi \vdash \chi$ and $\psi \vdash \chi$ together imply $\varphi \vee \psi \vdash \chi$ (Reasoning by Cases);

(*D*) $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$ together imply $\Gamma, \varphi \vee \psi \vdash \chi$ (Reasoning by Cases with Side Assumptions).

The first three of these principles are theses that we have encountered before in describing Themistocles's and Leonidas's thoughts about the future. The last two are inference patterns that are classically valid. A fully rigorous formalization of the arguments also requires some intermediary steps relying on other principles that will not play a major role in what follows and that I list here:

(L_1) $\Gamma, \varphi \vdash \psi$ and $\Delta \vdash \varphi$ together imply $\Gamma, \Delta \vdash \psi$;

(L_2) $\varphi \wedge \psi \vdash \varphi$, $\varphi \wedge \psi \vdash \psi$;

(L_3) $\varphi \vdash \varphi \vee \psi$, $\psi \vdash \varphi \vee \psi$.

We may now offer the following formal reconstruction of Themistocles's argument A ($\alpha_1 - \alpha_7$) and of Leonidas's argument B ($\beta_1 - \beta_7$):

$$
\begin{array}{llr}
(\alpha_1) & \vdash \mathcal{T}p \vee \mathcal{T}\neg p & \boldsymbol{(F)} \\
(\alpha_2) & \mathcal{T}p \vdash \Box\mathcal{T}p & \boldsymbol{(M)} \\
(\alpha_3) & \mathcal{T}\neg p \vdash \Box\mathcal{T}\neg p & \boldsymbol{(M)} \\
(\alpha_4) & \mathcal{T}p \vdash \Box\mathcal{T}p \vee \Box\mathcal{T}\neg p & \boldsymbol{(L_1)}, \boldsymbol{(L_3)}, \alpha_2 \\
(\alpha_5) & \mathcal{T}\neg p \vdash \Box\mathcal{T}p \vee \Box\mathcal{T}\neg p & \boldsymbol{(L_1)}, \boldsymbol{(L_3)}, \alpha_3 \\
(\alpha_6) & \mathcal{T}p \vee \mathcal{T}\neg p \vdash \Box\mathcal{T}p \vee \Box\mathcal{T}\neg p & \boldsymbol{(C)}, \alpha_4, \alpha_5 \\
(\alpha_7) & \vdash \Box\mathcal{T}p \vee \Box\mathcal{T}\neg p & \boldsymbol{(L_1)}, \alpha_1, \alpha_6
\end{array}
$$

$$
\begin{array}{llr}
(\beta_1) & \vdash \mathcal{T}p \vee \mathcal{T}\neg p & \boldsymbol{(F)} \\
(\beta_2) & \mathcal{T}p, \Diamond\mathcal{T}\neg p \vdash \bot & \boldsymbol{(O)} \\
(\beta_3) & \mathcal{T}\neg p, \Diamond\mathcal{T}p \vdash \bot & \boldsymbol{(O)} \\
(\beta_4) & \mathcal{T}p, \Diamond\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p \vdash \bot & \boldsymbol{(L_1)}, \boldsymbol{(L_2)}, \beta_2 \\
(\beta_5) & \mathcal{T}\neg p, \Diamond\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p \vdash \bot & \boldsymbol{(L_1)}, \boldsymbol{(L_2)}, \beta_3 \\
(\beta_6) & \mathcal{T}p \vee \mathcal{T}\neg p, \Diamond\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p \vdash \bot & \boldsymbol{(D)}, \beta_4, \beta_5 \\
(\beta_7) & \Diamond\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p \vdash \bot & \boldsymbol{(L_1)}, \beta_1, \beta_6
\end{array}
$$

This formalization highlights the apparent symmetry between the two arguments. As I will now argue, there is however an appealing way of rejecting both arguments that does not treat them as symmetric versions of one another.

5.3 Orthofuturism

In this section, I introduce a new solution to the sea battle problem that preserves the Open Future intuition without collapsing modalities. I will first informally describe this position, which I call *orthofuturism*, before making my proposal formal by introducing the logic OF.

5.3.1 Open Future without Modal Collapse

The core idea of orthofuturism is that the two arguments above should not be rejected for the same reason. For the orthofuturist, Themistocles's argument A is valid, but one of its premises is false, namely the Modal Collapse thesis, i.e., the thesis that truth coincides with settled truth. By contrast, she thinks that the premises in Leonidas's argument B are both true but that the argument is invalid, because reasoning by cases with side assumptions is not a valid logical principle when reasoning about the future. In other words, the orthofuturist believes that one may not assert at the same time that some proposition about the future is both true and contingent on pain of contradicting oneself, even though the truth of such a statement does not imply that it is settled. Although the future is open in the sense that one may not make true statements about it when what will happen is not yet settled, there is no modal collapse, and settled truth is genuinely distinct from plain truth. This leads her to accept $(\boldsymbol{O})$ and to reject $(\boldsymbol{M})$.

In order to consistently do so, the orthofuturist must therefore reject the inference from $\mathcal{T}p, \Diamond\mathcal{T}\neg p \vdash \bot$ to $\mathcal{T}p \vdash \Box\mathcal{T}p$. Because she accepts the duality between $\Diamond$ and $\Box$, this means that she must reject the inference from

$$\mathcal{T}p, \neg\Box\neg\mathcal{T}\neg p \vdash \bot \tag{a}$$

to $\mathcal{T}p \vdash \Box\mathcal{T}p$. Now if the orthofuturist accepts $(\boldsymbol{F})$, then she also accepts

$$\neg\mathcal{T}p, \neg\mathcal{T}\neg p \vdash \bot \tag{b}$$

. But if she accepts in general that one can conclude $\varphi \vdash \neg\psi$ from $\varphi, \psi \vdash \bot$, then accepting (a) (together with double negation elimination) means that she should also accept

$$\mathcal{T}p \vdash \Box\neg\mathcal{T}\neg p, \tag{a'}$$

and accepting (b) (again assuming double negation elimination) means that she should also accept

$$\neg\mathcal{T}\neg p \vdash \mathcal{T}p. \tag{b'}$$

But (a') and (b'), together with some basic modal reasoning, directly entail $(\boldsymbol{M})$.

A possible option for the orthofuturist would be to simply reject the Future Excluded Middle. While this option exists in the literature ([174, 254], see also the Piercean view below), the Future Excluded Middle seems to have a pretty strong intuitive pull. A more appealing option for the orthofuturist is therefore to deny that, in general, one may infer $\varphi \vdash \neg\psi$ from $\varphi, \psi \vdash \bot$. In particular, the orthofuturist argues that whenever $\psi := \Diamond\mathcal{T}\neg\varphi$, then $\mathcal{T}\varphi \wedge \Diamond\mathcal{T}\neg\varphi \vdash \bot$ is true but $\mathcal{T}\varphi \vdash \Box\mathcal{T}\varphi$ may be false.[2]

The orthofuturist therefore thinks that the future tense and openness modalities share some aspects with epistemic modals. As philosophers of language have argued [113, 265], statements like

(1)* It is raining and it might not be raining.

are contradictory, even though the proposition "It is raining" does not intuitively imply "It must be raining". In light of these phenomena, Holliday and Mandelkern have recently developed [138] an account of epistemic modals that relies on orthologic rather than classical logic. As I will show below, a similar account can be offered for the interplay of the future tense and openness modalities in a way that fleshes out the orthofuturist position in detail.

[2]Note that this move also allows her to keep $(\boldsymbol{F})$ and double negation elimination, without having to endorse that $\neg\mathcal{T}p \vdash \mathcal{T}\neg p$.

This is not to say, of course, that we should interpret "it is open that $\mathcal{T}p$" epistemically; nor am I claiming that native speakers of English use the phrases "it is settled that" or "It will be the case that" in the same way as they use the epistemic modals "must" or "might". Rather, the point of the parallel is to argue that, just as one can in a consistent way model the fact that statements like "It is raining and it might not be raining" are contradictory in natural language without collapsing epistemic modals, one may also do justice to the Open Future intuition that statements of the form "We will hold the pass tomorrow and it is open that we won't hold the pass tomorrow" are contradictory without endorsing the Modal Collapse thesis that settled truth coincides with truth.

As explained in detail in [138], orthologic differs from classical logic in that it rejects the distributive laws:

$$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$
$$(\varphi \vee \psi) \wedge (\varphi \vee \chi) \vdash \varphi \vee (\psi \wedge \chi).$$

At the same time, negation behaves almost classically in orthologic, meaning that the following principles remain valid:

Complementation $\varphi \wedge \neg\varphi \vdash \bot$, and $\vdash \varphi \vee \neg\varphi$;

Contraposition $\varphi \vdash \psi$ implies $\neg\psi \vdash \neg\varphi$;

Reductio $\neg\neg\varphi \vdash \varphi$;

De Morgan $\varphi \wedge \psi \dashv\vdash \neg(\neg\varphi \vee \neg\psi)$, $\varphi \vee \psi \dashv\vdash \neg(\neg\varphi \wedge \neg\psi)$.

Let us now see why the orthofuturist must reject the distributive laws if she wants to hold ***(O)*** as valid without also admitting ***(M)***. In fact, the problem arguably runs even deeper than distributivity. Consider the following weaker form of distributivity, known as the modular law:

$$\varphi \vdash \psi \Rightarrow (\varphi \vee \chi) \wedge \psi \vdash \varphi \vee (\chi \wedge \psi)$$

If one admits that truth implies possibility, which in the case of the openness modality is uncontroversial, then the modular law implies that "There will be a sea battle tomorrow or there won't be a sea battle tomorrow, and there may be a sea battle tomorrow" ($(\mathcal{T}p \vee \mathcal{T}\neg p) \wedge \Diamond\mathcal{T}p$) entails "There will be a sea battle tomorrow, or there won't be a sea battle tomorrow but there may be one" ($\mathcal{T}p \vee (\mathcal{T}\neg p \wedge \Diamond\mathcal{T}p)$). But ***(F)*** implies that the first proposition is equivalent to "There may be a sea battle tomorrow", while ***(O)*** implies that the second proposition is equivalent to "There will be a sea battle tomorrow". Thus ***(M)*** is an immediate consequence of ***(F)***, ***(O)*** and the modular law. As the modular law is a consequence of the distributive laws, this gives the orthofuturist a strong reason to adopt orthologic rather than classical logic.

Moreover, the rejection of the distributive laws also allows the orthofuturist to object to the use of inference pattern $(\boldsymbol{D})$ in Leonidas's argument B. Indeed, the first distributive law can be easily derived from reasoning by cases with side assumptions:

$$\begin{array}{r}\varphi, \psi \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \\ \varphi, \chi \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \\ \hline \varphi, \psi \vee \chi \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \\ \hline \varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi).\end{array} \qquad (\boldsymbol{D})$$

In fact, the specific instance of $(\boldsymbol{D})$ that appears at step β_6 above is also directly seen as invalid for the orthofuturist. Just because two formulas φ and ψ are inconsistent with χ does not mean that the disjunction $\varphi \vee \psi$ is also inconsistent with χ, unless one already assumes some form of distributivity. Rejecting the distributive laws of classical logic therefore allows the orthofuturist to make sense of the Open Future intuition that statements about the future cannot be contingent and true at the same time, without collapsing settled truth onto truth. Giving up distributivity allows her to refute both Themistocles's and Leonidas's arguments above at once, but for different reasons. Let me now make my proposal more formal by defining a logic that captures the orthofuturist's solution as I have sketched it so far.

5.3.2 The Logic OF

As is customary in orthologic [109], let us start by defining our logic as a consequence relation on the set of $\mathscr{L}$-formulas.

Definition 5.3.1. The logic OF is the smallest relation $\vdash$ on $\mathscr{L}$ satisfying the following conditions for any formulas φ, ψ and χ:

- Order:

    1. $\varphi \vdash \varphi$;
    2. $\varphi \vdash \psi$ and $\psi \vdash \chi$ together imply $\varphi \vdash \chi$;

- Connectives:

    3. $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$;
    4. $\varphi \vdash \psi$ and $\varphi \vdash \chi$ together imply $\varphi \vdash \psi \wedge \chi$;
    5. $\varphi \wedge \neg\varphi \vdash \bot$, and $\bot \vdash \psi$;
    6. $\varphi \vdash \psi$ implies $\neg\psi \vdash \neg\varphi$;
    7. $\varphi \vdash \neg\neg\varphi$, and $\neg\neg\varphi \vdash \varphi$;

- Modalities:

    8. $\varphi \vdash \psi$ implies $\Box\varphi \vdash \Box\psi$ and $\mathcal{T}\varphi \vdash \mathcal{T}\psi$;
    9. $\neg\bot \vdash \Box\neg\bot$, and $\neg\bot \vdash \mathcal{T}\neg\bot$;
    10. $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$;
    11. $\mathcal{T}\varphi \wedge \mathcal{T}\psi \vdash \mathcal{T}(\varphi \wedge \psi)$, and $\neg\mathcal{T}\neg\varphi \wedge \neg\mathcal{T}\neg\psi \vdash \neg\mathcal{T}\neg(\varphi \wedge \psi)$;
    12. $\Box\varphi \vdash \varphi$;
    13. $\mathcal{T}\neg\varphi \vdash \neg\mathcal{T}\varphi$;
    14. $\mathcal{T}\varphi \wedge \neg\Box\neg\mathcal{T}\neg\varphi \vdash \bot$.

A formula φ is a *theorem* of OF if $\neg\bot \vdash \varphi$.

The logic OF is closely related to the logic EO of epistemic modals introduced in [138]. If we recall the abbreviations $\top := \neg\bot$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ and $\Diamond\varphi := \neg\Box\neg\varphi$, we can verify in a straightforward way that $\boldsymbol{(O)}$ and $\boldsymbol{(F)}$ are theorems of OF.

Lemma 5.3.2. *For any $\mathscr{L}$-formula φ, $\neg(\mathcal{T}\varphi \wedge \Diamond\mathcal{T}\neg\varphi)$ and $\mathcal{T}\varphi \vee \mathcal{T}\neg\varphi$ are theorems of* OF.

Proof. To show that $\boldsymbol{(O)}$ is valid, it is enough to observe that it is equivalent to condition 14, since $\mathcal{T}\varphi \wedge \Diamond\mathcal{T}\neg\varphi := \mathcal{T}\varphi \wedge \neg\Box\neg\mathcal{T}\neg\varphi$. But then by condition 6 we also have that $\neg(\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p)$ is a theorem of OF.

For $\boldsymbol{(F)}$, note first that $\mathcal{T}\varphi \vee \mathcal{T}\neg\varphi := \neg(\neg\mathcal{T}\varphi \wedge \neg\mathcal{T}\neg\varphi)$ and hence by condition 6 it is enough to show that $\neg\mathcal{T}\varphi \wedge \neg\mathcal{T}\neg\varphi \vdash \bot$. Moreover, since φ and $\neg\neg\varphi$ are equivalent by condition 7, it is in fact enough to show that $\neg\mathcal{T}\neg\neg\varphi \wedge \neg\mathcal{T}\neg\varphi \vdash \bot$. By condition 11, we have that $\neg\mathcal{T}\neg\neg\varphi \wedge \neg\mathcal{T}\neg\varphi \vdash \neg\mathcal{T}\neg(\neg\varphi \wedge \varphi)$. By condition 5, $\neg\varphi \wedge \varphi \vdash \bot$, so by conditions 6 and 8, we have $\neg\mathcal{T}\neg(\neg\varphi \wedge \varphi) \vdash \neg\mathcal{T}\neg\bot$. But by condition 9 together with conditions 6 and 7 we get that $\neg\mathcal{T}\neg\bot \vdash \bot$, so by condition 2 we can conclude that $\neg\mathcal{T}\neg(\neg\varphi \wedge \varphi) \vdash \bot$. This completes the proof that $\boldsymbol{(F)}$ is a theorem of OF. □

It is also straightforward to verify that the inference pattern $\boldsymbol{(C)}$ is valid in OF, i.e., that for any formulas φ, ψ and χ, $\varphi \vdash \chi$ and $\psi \vdash \chi$ together imply $\varphi \vee \psi \vdash \chi$.[3] To establish that OF is an adequate logic for the orthofuturist, it therefore only remains to show that $\Box\mathcal{T}\varphi \vee \Box\mathcal{T}\neg\varphi$ is not a theorem of OF. Note that this will also imply that the Modal Collapse thesis $\boldsymbol{(M)}$ is not a theorem of OF and that $\boldsymbol{(D)}$ is not a valid inference pattern. An elegant way of doing so is to provide a sound an complete algebraic semantics for OF based on ortholattices, before giving an example of a valuation V on an ortholattice L that makes $\boldsymbol{(M)}$ invalid. In order to make my proposal as accessible as possible, I will only sketch such an approach here and postpone most of the technical details to Section 5.8.1.

[3] For a proof of this, we refer the reader to the remark before Theorem 3.13 in [138].

Intuitively, one may think of a lattice as an abstract algebra of propositions that can be assigned to the formulas of a propositional language, ordered by the entailment relation. The propositions we are interested in must be closed under conjunctions, negations, and the "Inevitably" and "Tomorrow" modal operators. We therefore need our algebras of propositions to be partially ordered sets with a binary operation $\wedge$ (usually called "meet") and three unary relations $\neg$ (complementation), $\Box$ ("box") and $\mathcal{T}$ ("Tomorrow"). Given such an algebra L, any function assigning a proposition in L to every propositional variable can then be recursively lifted to a valuation on $\mathscr{L}$-formulas in the obvious way. Given two $\mathscr{L}$-formulas φ and ψ and such an algebra L with order $\leq_L$, we may then write that ψ is a logical consequence of φ relative to L, denoted $\varphi \models_L \psi$, if $V(\varphi) \leq_L V(\psi)$ for every valuation V defined on L.[4] Of course, if we want OF to be sound with respect to this semantics, we must impose some conditions on the kind of algebras we may use to evaluate $\mathscr{L}$-formulas. These conditions follow naturally from the conditions imposed on the logic OF in Definition 5.3.1 and determine the class $\boldsymbol{C}$ of OF lattices (see Definition 5.8.2 in Section 5.8.1).

For any two $\mathscr{L}$-formulas φ and ψ, let $\varphi \models_{\boldsymbol{C}} \psi$ if $\varphi \models_L \psi$ for every OF lattice L. The proof of the following theorem uses standard techniques from algebraic logic [229] and can be found in Section 5.8.1.

Theorem 5.3.3. *The logic* OF *is sound and complete with respect to the consequence relation* $\models_{\boldsymbol{C}}$*: for any two* $\mathscr{L}$*-formulas* φ *and* ψ*,* $\varphi \vdash \psi$ *if and only if* $\varphi \models_{\boldsymbol{C}} \psi$*.*

Figure 5.1 depicts some of the entailment relations that the orthofuturist admits between propositions regarding a sea battle to come, with a proposition φ directly entailing another proposition ψ if and only if there is an edge from φ to ψ with φ below ψ. One can verify that such a configuration of propositions arises from a valuation on an OF lattice, namely the valuation $v(p) = b$ on the OF lattice $\boldsymbol{O_{10}}$ shown in Figure 5.6 in Section 5.8.1. Notice in particular that in this instance $\Box\mathcal{T}p$ entails $\mathcal{T}p$ but that the converse does not hold. Nonetheless, the only proposition that entails both $\mathcal{T}p$ and $\Diamond\mathcal{T}\neg p$ is the contradictory proposition $\bot$, as required by the Open Future intuition. Moreover, the only proposition entailed by both $\mathcal{T}p$ and $\mathcal{T}\neg p$ is the tautology $\top$, in accordance with the Future Excluded Middle. By contrast, the proposition expressing that the sea battle is already settled, i.e., $\Box\mathcal{T}p \vee \Box\mathcal{T}\neg p$, is strictly below $\top$. Together with Theorem 5.3.3, this shows that $\Box\mathcal{T}p \vee \Box\mathcal{T}\neg p$ is not a theorem of OF.

In Section 5.5, I will present a more concrete, Kripke-style semantics for OF. For now, let me just conclude this section with two remarks. First, the lattice-theoretic perspective allows one to highlight in a particularly sharp way the connection between orthofuturism and the failure of the modular law mentioned above. Indeed, if L is any algebra of propositions such that the principles $\varphi \vdash \Diamond\varphi$, $(\boldsymbol{F})$ and $(\boldsymbol{O})$ are valid, but $(\boldsymbol{M})$ is not, then the configuration of propositions depicted in fig. 5.2 must appear in L. By a celebrated result in lattice theory due to Dedekind (see [66, Thm. 4.10] for a proof), the lattice $\boldsymbol{N_5}$ embeds in an arbitrary

[4] For the reader familiar with Kripke semantics for classical modal logic, an ortholattice is the analogue of the algebra of propositions given by the powerset of a set of possible worlds of a Kripke frame F, and a valuation on such an ortholattice corresponds to a model defined on F.

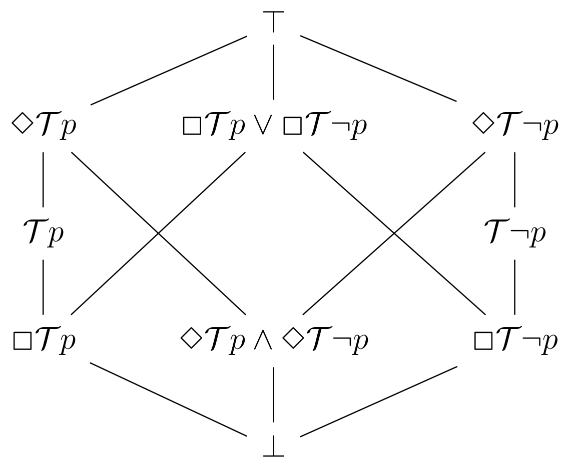


Figure 5.1: Entailment relations between future-tensed propositions according to the orthofuturist.

lattice L if and only if L is not modular, i.e., does not satisfy the lattice-theoretic equation corresponding to the modular law. This shows that the failure of the modular law, and thus also of the distributive laws, is not merely a convenient way of blocking the fatalist's arguments but a core feature of the orthofuturist's position.

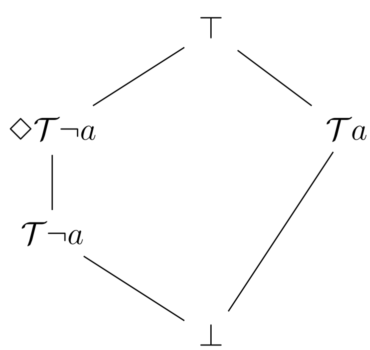


Figure 5.2: The lattice $\boldsymbol{N_5}$

The second remark concerns the strength of the logic OF. I do not claim here that OF is the strongest logic that fits the orthofuturist's position. Rather OF should be thought of as a "basic logic" when reasoning about the open future from an orthofuturist's perspective. In particular, one might consider strengthening conditions 13 and 14 respectively to the following:

13'. $\mathcal{T}\neg\varphi \vdash \neg\mathcal{T}\varphi$, and $\neg\mathcal{T}\varphi \vdash \mathcal{T}\neg\varphi$.

14'. $\varphi \wedge \Diamond\neg\varphi \vdash \bot$.

Condition 13' would make the operator $\mathcal{T}$ commute with negation, while Condition 14' is directly lifted from the treatment of epistemic modals in [138]. Notice that the lattice in Figure 5.6 satisfies both conditions, which shows that we could add them to OF without collapsing modalities. There might be however, other reasons for the orthofuturist to refrain from endorsing conditions 13' and 14'. For one, one might argue that although $\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p \vdash \bot$ is a plausible way of capturing the Open Future intuition, the intuitive status of the closely related $\mathcal{T}p \wedge \neg\Box\mathcal{T}p \vdash \bot$ is more debatable (compare for example "There will be a sea battle tomorrow and it's open that there won't be one" with "There will be a sea battle tomorrow, but it's not settled that there will be one"). But one can easily derive the second principle from condition 14'. Similarly, one can show that, over the standard rules of orthologic, condition 13' and 14 together entail that $\varphi \vee \Box\neg\varphi$ is a theorem for any formula φ. But certainly "There will be a sea battle tomorrow or it's settled that there won't be one" ($\mathcal{T}p \vee \Box\mathcal{T}\neg p$) does not sound intuitively valid. This suggests that the orthofuturist must tread carefully when reasoning about the principles governing the interaction between the openness modality, the tense operator and negation.

5.4 Branching-Time Solutions

In this section, I compare orthofuturism to some classical solutions to the sea battle problem based on branching-time semantics, and I argue that it is a more appealing position than any of them. Since Prior's work on tense logic [211, 212], branching-time semantics has appeared as a promising way of solving the sea battle problem [63, 201]. Intuitively, the future is open at a moment t in time because there is more than one way in which the world could evolve at that time, more than one possible timeline to which t belongs. This basic intuition of branching-time semantics, however, allows for more than one possible theory. Prior himself favored what he called the *Piercean* approach, but he also developed an *Ockhamist* solution, although it has been argued that Ockham's own views are closer to what is usually called the *true futurist* or *Thin Red Line* approach [15, 67, 201]. On the other hand, Thomason's *supervaluationism* [252] offers an alternative solution to the problem that deviates from classical logic. Finally, a recent proposal based on branching time has been offered by MacFarlane [178, 179]. However, since his relativist approach is less suitable to a direct comparison with the orthofuturist solution as I have presented it so far, I will delay such a comparison until Section 5.6 and focus for now on the Piercean, supervaluationist and true futurist approaches.

The starting point of all branching-time solutions is the definition of satisfaction of a formula relative not only to a moment in time but also to a timeline, a maximal linearly ordered set of moments in time. A branching-time model is determined by a partially ordered set $(T, \leq)$ that has a tree-like structure (meaning that the past of any moment in time looks like a line or, formally, $y \leq x$ and $z \leq x$ together imply $y \leq z$ or $z \leq y$) and a valuation function mapping each propositional variable to a set of moments in time. The diagram in Figure 5.3 depicts an elementary model in which the moment t belongs to two distinct timelines $C_0 = \{t, t_0\}$ and $C_1 = \{t, t_1\}$.

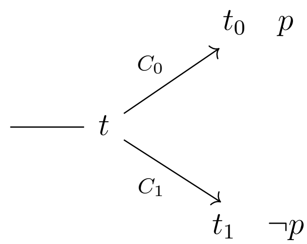


Figure 5.3: A simple branching model

Pierceanism, supervaluationism and true futurism all agree about the following satisfaction conditions of a formula given a valuation V, a moment in time t and a timeline C such that $t \in C$:

$V, t, C \models_{PST} p$ iff $t \in V(p)$; $V, t, C \models_{PST} \bot$ never;

$V, t, C \models_{PST} \neg\varphi$ iff $V, t, C \not\models_{PST} \varphi$;

$V, t, C \models_{PST} \varphi \wedge \psi$ iff $V, t, C \models_{PST} \varphi$ and $V, t, C \models_{PST} \psi$;

$V, t, C \models_{PST} \Box\varphi$ iff $V, t, C' \models_{PST} \varphi$ for every timeline C' such that $t \in C'$.

A first difference appears between Pierceanism and the other two theories when one considers the semantics of the future tense operator $\mathcal{T}$. If we assume a discrete model for the sake of interpreting our language $\mathscr{L}$, given a moment t and a timeline C to which t belongs, there is exactly one moment $F(t, C)$ that immediately follows t and belongs to C. According to the Piercean, "It will be the case that p" means "Inevitably, it will be the case that p", and the semantic clause for the $\mathcal{T}$ operator should therefore be the following:

$V, t, C \models_P \mathcal{T}\varphi$ iff $V, F(t, C'), C' \models_P \varphi$ for every timeline C' such that $t \in C'$.

By contrast, the supervaluationist and the true futurist hold that whether $\mathcal{T}\varphi$ is satisfied at a moment t with respect to a timeline C is completely determined by whether p is satisfied at $F(t, C)$ with respect to C:

$V, t, C \models_{ST} \mathcal{T}\varphi$ iff $V, F(t, C), C \models_{ST} \varphi$.

Moreover, the Piercean, the true futurist and the supervaluationist differ when it comes to defining truth at a moment *simpliciter*, i.e., defining a notion of satisfaction that is not relative to a timeline. For the Piercean, it is straightforward to verify that satisfaction of a proposition at a moment in time does not actually depend on a timeline, in the sense that for any valuation V, any moment t and any two timelines C, C' such that $t \in C \cap C'$, $V, t, C \models_P \varphi$ iff $V, t, C' \models_P \varphi$. The Piercean can therefore define truth at a moment in time t as truth at t relative to some (or every) timeline C to which t belongs:

$V, t \models_P \varphi$ iff $V, t, C \models_P \varphi$ for some timeline C such that $t \in C$.

The supervaluationist, on the other hand, holds that truth at a moment t means *supertruth* at t, i.e., truth in every timeline passing through t:

$V, t \models_S \varphi$ iff $V, t, C \models_{ST} \varphi$ for every timeline C such that $t \in C$.

Finally, the true futurist holds that to any moment in time corresponds an actual future, one timeline singled out from the other possible ones as the actual one by a "thin red line" [15, 67]. Accordingly, truth at a moment t is defined as truth at t relative to the actual timeline of t, $C(t)$:

$V, t \models_T \varphi$ iff $V, t, C(t) \models_{ST} \varphi$.

As an example, we may consider again the model presented in Figure 5.3. If we assume that, for the true futurist, C_0 is the actual timeline of t, then we can see that the three approaches each give a different truth value to the formula $\mathcal{T}p$ at t:

$V, t \models_P \neg\mathcal{T}p$, although $V, t \not\models_P \mathcal{T}\neg p$;

$V, t \not\models_S \mathcal{T}p$ and $V, t \not\models_S \mathcal{T}\neg p$, although $t \models_S \mathcal{T}p \vee \mathcal{T}\neg p$;

$V, t \models_T \mathcal{T}p$, although $V, t \models_T \neg\Box\mathcal{T}p$.

How does each proposal block the two arguments discussed in Section 5.2? For the Piercean, truth implies settled truth. In other words, principles $(\boldsymbol{O})$ and $(\boldsymbol{M})$ are both valid. Because the Piercean's logic is fully classical, contingent propositions about the future are therefore false: $\Diamond\mathcal{T}\neg p \models_P \neg\mathcal{T}p$. If it is still open whether a sea battle may happen tomorrow, then for the Piercean the statements "There will be a sea battle tomorrow" and "There will not be a sea battle tomorrow" are both false. As a consequence, the Piercean blocks both Themistocles's and Leonidas's argument by denying $(\boldsymbol{F})$, the Future Excluded Middle: $\not\models_P \mathcal{T}\varphi \vee \mathcal{T}\neg\varphi$, even though every instance of the Excluded Middle, including $\mathcal{T}\varphi \vee \neg\mathcal{T}\varphi$, remains valid.

By contrast, the true futurist makes a clear distinction between truth and settled truth. For the true futurist, true propositions about the future can also be contingent, and there is no contradiction is asserting: "We will hold the pass tomorrow, yet we may not hold it." As a consequence, neither $(\boldsymbol{O})$ nor $(\boldsymbol{M})$ are valid, which allows the true futurist to block both Themistocles's and Leonidas's argument by denying one of its premises, while preserving classical logic and the validity of $(\boldsymbol{F})$.

Finally, the supervaluationist accepts both the Future Excluded Middle $(\boldsymbol{F})$ and that truth coincides with settled truth, as $\varphi \models_S \Box\varphi$. The supervaluationist also accepts the validity of $(\boldsymbol{O})$, although one has to be careful here: a formula of the form $\varphi \wedge \Diamond\neg\varphi$ may never be supertrue or true at a moment *simpliciter*, but it may be satisfied relative to a moment in time and a timeline. Although it preserves all the validities of classical logic, the supervaluationist blocks Themistocles's and Leonidas's arguments by rejecting both classical inference patterns $(\boldsymbol{C})$ and $(\boldsymbol{D})$.

	$\models_P$	$\models_S$	$\models_T$	$\models_O$
$(\boldsymbol{F}) \vdash \mathcal{T}\varphi \vee \mathcal{T}\neg\varphi$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$(\boldsymbol{M})\ \mathcal{T}\varphi \vdash \Box\mathcal{T}\varphi$	$\checkmark$	$\checkmark$	$\times$	$\times$
$(\boldsymbol{O})\ \mathcal{T}\varphi, \Diamond\mathcal{T}\neg\varphi \vdash \bot$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$
$(\boldsymbol{C})\ \varphi \vdash \chi\ \&\ \psi \vdash \chi \Rightarrow \varphi \vee \psi \vdash \chi$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$
$(\boldsymbol{D})\ \Gamma, \varphi \vdash \chi\ \&\ \Gamma, \psi \vdash \chi \Rightarrow \Gamma, \varphi \vee \psi \vdash \chi$	$\checkmark$	$\times$	$\checkmark$	$\times$

Table 5.1: Orthofuturism vs. Branching-Time Solutions

As we can see summarized in Table 5.1, the orthofuturist's position is the only one that offers a different diagnosis for each one of the arguments from Section 5.2. As a consequence, the orthofuturist's position can be seen as a rather nuanced view that agrees in some respect with each of the three branching-time positions. Against the Piercean and the supervaluationist, the orthofuturist agrees with the true futurist that truth is distinct from settled truth and that the openness modality should not collapse. For this reason, she rejects the unrestricted validity of $(\boldsymbol{M})$. Against the Piercean, she also agrees with the supervaluationist and the true futurist that the Future Excluded Middle is valid, although she might resist going one step further and holding that the tense operator $\mathcal{T}$ commutes with negation. On the other hand, the orthofuturist agrees with the Piercean's intuition that one may not assert at the same time that Leonidas's men will hold the pass and that they may not hold it. Against the true futurist, she thinks that $\mathcal{T}\varphi \wedge \Diamond\mathcal{T}\neg\varphi$ is contradictory, and, by contrast with the supervaluationist, she even goes as far as to agree with the Piercean that its negation $\neg(\mathcal{T}\varphi \wedge \Diamond\mathcal{T}\neg\varphi)$ is a theorem. Finally, she agrees with the supervaluationist that some classical inference patterns are not valid when reasoning about an open future. But while the supervaluationist objects to both reasoning by cases $(\boldsymbol{C})$ and reasoning by cases with side assumptions $(\boldsymbol{D})$, the orthofuturist needs only to reject the second inference pattern. She does not argue that our classical understanding of disjunction is flawed but rather that the logic of the open future fails to be distributive.

As a consequence, orthofuturism emerges as a strong candidate for addressing the sea battle problem. Indeed, there are strong reasons to reject the Modal Collapse thesis $(\boldsymbol{M})$. First, since the converse principle, namely that settled truth implies truth, seems uncontroversial, principle $(\boldsymbol{M})$ therefore amounts to an identification of truth with settled truth. Intuitively, however, there seems to be a difference between asserting that a certain event E will happen and asserting that E will happen *inevitably.* The case can also be made quite strikingly if one considers the credences that rational agents may have about the future. Leonidas and his men, knowing that the Persian army vastly outnumbers their forces, may very well consider it quite likely that they won't be able to hold the Thermopylae pass for a second day but very unlikely that the issue is already settled. In other words, their credence in "We won't hold the pass tomorrow" may be high without their credence in "Inevitably, we won't hold the pass tomorrow" being at least as high. But if we hold that credences should respect logical entailment, as is commonly admitted, then it follows that "We won't hold the pass tomorrow" cannot entail "Inevitably, we won't hold the pass tomorrow" and that we must reject $(\boldsymbol{M})$.

A second argument against the Modal Collapse thesis comes from a more careful analysis of the openness modality. As mentioned in Section 5.2, the notion of necessity involved is meant to capture the intuition of an asymmetry between the past and the present on one hand, and the future on the other hand. Consequently, the motivation for $(\boldsymbol{M})$ given above was to appeal to the fixity of the past and the present. If φ is true now, then it is too late to change that fact; φ is now inevitable, and therefore $\Box\varphi$ is also true. But if the modality $\Box$ is meant to capture the particular way in which the past and the present are fixed, by contrast with the future, then the unrestricted validity of $(\boldsymbol{M})$ is untenable. This is essentially the point famously made by Ockham [200], when he distinguishes statements that are about the past and the present *de re*, like "John is in heaven now", from those that are truly about the future, and about the past or the present merely *de dicto*, such as "John is predestined" or "God knows today that John will enter heaven tomorrow". For Ockham, the fallacy in the fatalist's argument resides precisely in the unrestricted application of the Modal Collapse thesis to statements that may not be genuinely about the past or the present. Regardless of whether one thinks that Ockhamist solutions to the sea battle problem are satisfactory, the mismatch between the intuitive justification of $(\boldsymbol{M})$ and its unrestricted application puts great pressure on the Modal Collapse thesis.

Lastly, one may consider the dual notion of possibility that arises from endorsing $(\boldsymbol{M})$. If one assumes contraposition, then an instance of $(\boldsymbol{M})$ of the form $\mathcal{T}p \vdash \Box\mathcal{T}p$ entails $\Diamond\neg\mathcal{T}p \vdash \neg\mathcal{T}p$, which, assuming that the operator $\mathcal{T}$ commutes with negations, entails $\Diamond\neg\mathcal{T}p \vdash \mathcal{T}\neg p$. But clearly that last inference is unacceptable. From the mere fact that there may not be a sea battle tomorrow, it does not follow that there won't in fact be a sea battle tomorrow. Of course, both the Piercean and the supervaluationist admit that such an inference would yield too strong a notion of possibility, but their endorsement of $(\boldsymbol{M})$ means that they can only do so at a high cost. In order to avoid that the tense operator $\mathcal{T}$ commute with negation, the Piercean must reject the Future Excluded Middle, in spite of its intuitive validity [55]. The supervaluationist, meanwhile, is forced to reject that contraposition is a valid inference pattern, on top of rejecting reasoning by cases. By contrast, the orthofuturist's rejection of $(\boldsymbol{M})$ allows her to maintain both the Future Excluded Middle and contraposition as a valid principle and inference pattern respectively.

At the same time, any branching-time solution that rejects $(\boldsymbol{O})$ arguably comes short of genuinely securing the openness of the future. If it is not settled whether there will be a sea battle tomorrow or not, then introducing an asymmetry between the two possible outcomes seems illegitimate. If time is genuinely branching at any given moment, then no branch should receive a special ontological status as the one that will *actually* happen. The question whether the true-futurist solution to the sea battle problem is compatible with indeterminism raises difficult and intricate metaphysical issues [8, 15, 67, 178, 225] that are largely beyond the scope of this chapter. For now, I will limit myself to pointing out that rejecting the Open Future intuition yields a notion of possibility that is arguably too weak.

Suppose Leonidas only tells his ally Themistocles that the Greeks may or may not hold the pass for a second day. Frustrated with Leonidas's reply, Themistocles hastily consults the oracle in Delphi, which gives him the following answer: "Leonidas spoke truthfully, yet, in fact, his men won't be able to hold the pass for another day". If Themistocles believes that the oracle is never wrong, then should the oracle's reply not be enough of a reason for Themistocles to retreat with the Greek fleet? For if he chooses instead to try and keep the Persian fleet at bay for another day, he certainly *may* be making the right choice but knows that he will, in fact, be making the wrong one. In other words, once Themistocles knows that the Greeks will not stop the Persian army at Thermopylae, Leonidas's (true!) statement that it is still possible for his men to do so becomes entirely irrelevant.

Of course, the true-futurist could reply that, although true, a future contingent can never be known in advance, so that Themistocles could never be in a position to know whether Leonidas and his men would or wouldn't hold the pass. Although there is a fact of the matter regarding the Persian army's victory or defeat on the second day, there is no *determinate* fact of the matter, and so the outcome of the battle to come cannot be known in advance. But clearly the openness of the future that one has in mind when describing the situation at Thermopylae is not merely epistemic. What prompts Leonidas to tell his ally that they may hold the pass and they may not hold it is not a mere ignorance of the full situation, but rather the conviction that some of the facts that will decide the outcome of the battle have yet to obtain. So the true-futurist must now account for a notion of *determinateness* that doesn't coincide with knowability, as well as argue that the openness of the future is best understood as the lack of a *determinate* fact of the matter rather than the lack of a mere fact of the matter. In that respect, it seems that the true-futurist is no better position than the "classical indeterminacy theorist" discussed by Field in [92] in the context of vagueness.

To sum up, accepting the Modal Collapse thesis for the openness modality yields a notion of *necessity* that is too weak to account for the asymmetry between the past and the future, while rejecting the Open Future intuition, as I argued, threatens to yield a notion of *possibility* that is too weak to play any significant role in our reasoning about the future. Yet because the two principles are classically equivalent, a classical logician must either accept both the Modal Collapse thesis and the Open Future intuition, or reject both. But this is precisely where the orthofuturist has an advantage over all the branching-time solutions discussed here. Because she does not accept the distributive laws, she is able to hold that the Modal Collapse thesis $(\boldsymbol{M})$ is false without giving up the Open Future intuition $(\boldsymbol{O})$. In doing so, she can allow for a notion of necessity that is strong enough not to be confused with mere truth and for a dual notion of possibility that is strong enough to secure a genuinely open future.

Let me conclude this sections by briefly discussing how the orthofuturist can address two objections that are often raised against any view that upholds a version of the Modal Collapse thesis or of the Open Future intuition. The first objection raises the *assertion problem*: if a view predicts that a contingent statement about the future can never be assertable, then how come we do seem to routinely make such statements? The second objection raises the *credence problem*: if it is contradictory to hold that statements about the future can be both true and contingent at the same time, then how come one can have credence 1 that the future is open regarding whether a sea battle will happen tomorrow and still have positive credence that the sea battle will in fact occur?

To answer the first challenge, the orthofuturist can essentially follow the strategy outlined by MacFarlane in [177, Section 9.9], and simply argue that assertions about an open future are only correct when made by way of ellipsis. Whenever one makes an assertion about a future contingent, one is merely expressing an intention ("I will come at noon tomorrow") or what one takes to be overwhelmingly likely ("This summer will be hotter than the previous one"). Under that view, asserting future contingents is technically incorrect, but pragmatically justifiable. But the orthofuturist can in fact go one step further. Because she does not accept Modal Collapse, there is a sense in which she might view an assertion about a future contingent valuable even if it cannot be ascribed a classical truth-value. Indeed, unlike the Piercean, she does not view future contingent statements as false, and, unlike the supervaluationist, she also does not view them as having an undefined truth-value. Rather, because she rejects the principle of bivalence, she thinks that statements about the future may have semantic values that go beyond truth and falsity. This opens up the possibility for the orthofuturist of developing a theory of assertion in which asserting statements that are not classically true can be justifiable, provided that they are also not classically false and have a determined semantic value.

Regarding the credence problem, there are at least two ways that the orthofuturist could reply. First, recall that although $\neg(\mathcal{T}p \wedge \Diamond T\neg p)$ is a theorem of OF, we do not have that $\mathcal{T}p \wedge \neg\Box\mathcal{T}p \vdash_{\mathsf{OF}} \bot$. In other words, although the orthofuturist thinks the sentence "There will be a sea battle tomorrow and it's open that there won't be one" is contradictory (and thus should always receive credence 0), this does not prevent her from thinking that the sentence "There will be a sea battle tomorrow but it's not settled that there will be one" is not contradictory. This means that she could answer the credence problem by arguing that we do sometimes have positive credence in future contingent propositions, provided that by this we mean propositions that we take to be true but not settled as true, rather than propositions that we take to be true even though their negation is open. Once again, it is worth mentioning that this option is only available to the orthofuturist because, unlike the Piercean or the supervaluationist, she does not take Modal Collapse to be a valid principle. Finally, the second option for the orthofuturist is to simply adopt the answer offered by Holliday and Mandelkern regarding a similar problem for credences in propositions involving epistemic modals. In a nutshell, the idea is that the revision of classical logic advocated for in the case of epistemic modals calls for a similar revision of classical probability theory. In particular, one should not assume that having credence 1 in a proposition φ entails having equal credence in $\varphi \wedge \psi$ and in ψ. It is easy to see how this could be used to address the credence problem. Whenever $\mathcal{T}p$ is a future contingent proposition, one could have credence 1 in $\Diamond\mathcal{T}\neg p$ and credence .5 in $\mathcal{T}p$, but still have credence 0 in the proposition $\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p$.[5]

[5] See [138, Section 5] for a more comprehensive treatment of this approach.

5.5 Fragment Semantics

So far, I have presented orthofuturism only from an axiomatic perspective, by laying out which principles and inference patterns the orthofuturist holds as valid and which ones she rejects, and I have defined the logic OF as a logic of the open future that captures the orthofuturist's position. In this section, I present a concrete, Kripke-like semantics for this logic. Unlike the branching-time semantics discussed in the previous section, the basic objects of this semantics are not fully determined instants organized in a tree-like structure of splitting timelines, but rather temporal *fragments*, i.e., partial descriptions of the world at an instant that may or may not be compatible with one another.[6] My goal here is not so much to make a metaphysical claim about what the actual structure of moments in time may be like if orthofuturism is correct but rather to provide a fairly intuitive picture of how an orthofuturist may conceive of the flow of time.

Because OF is not a classical modal logic, standard possible-world semantics is not a suitable technical framework for our purposes. Instead, I will present a possibility semantics for OF that shares many aspects with the semantics for epistemic modals developed by Holliday and Mandelkern in [138]. Unlike in possible-world semantics, the basic objects of possibility semantics [135, 140] are not maximal objects such as possible worlds but rather possibilities, which can be thought of as partial descriptions of possible worlds. Models for OF will consist of a set S of fragments endowed with a reflexive and symmetric relation $\vartriangle$ of *compatibility*, an accessibility relation R of *openness*, and a *transition* function $\tau : S \to S$. Intuitively, one may think of the compatibility relation $\vartriangle$ as determining when two fragments do not rule one another out (which is not to say, as will become apparent below, that the two fragments could be combined into a single one). The openness relation R holds from a fragment s to a fragment s' whenever everything that is settled at s also holds at s', while the transition function τ describes what the future looks like from the point of view of a fragment s. Formally, the compatibility relation on S allows us to model the orthocomplementation operation of OF, while the openness relation R and the transition function τ will be used to handle the necessity operator $\Box$ and the tense operator $\mathcal{T}$ respectively.

In possibility semantics, propositions are not evaluated as sets of possible worlds, but rather as subsets of the domain S of possibilities of a certain kind. Let us consider first the following definition.

Definition 5.5.1. Let $\vartriangle$ be a reflexive and symmetric relation on a set S. A $\vartriangle$-*fixpoint* of S is a subset A of S such that for any $s \in S$,

$$s \in A \Leftrightarrow \forall s' \vartriangle s \, \exists s'' \vartriangle s' : s'' \in A.$$

The set of $\vartriangle$-fixpoints of S is denoted $\mathcal{F}_{\vartriangle}(S)$.

[6] The fragment semantics presented here should not be confused with Fine's *fragmentalism* about time and reality [93]. Fine's fragments are maximally coherent collections of facts, while the fragments considered are only partial descriptions of a moment in time.

Because fragments are only partial descriptions, a fragment s may fail to satisfy a proposition φ without outright refuting it. In fact, it seems intuitive to say that s refutes φ precisely when no fragment t compatible with s satisfies φ. Similarly, s may fail to refute φ without satisfying φ. But if φ is not satisfied at s, then it should at least be possible to refute it, meaning that there should be a fragment s compatible with t that refutes φ. But this amounts to requiring that the set of fragments that satisfy φ be a $\triangle$-fixpoint of S. In what follows, we will therefore always make sure that the set of fragments satisfying a proposition is a $\triangle$-fixpoint.

In order to ensure the soundness of OF with respect to our fragment semantics, we must impose some conditions on the interaction between the relations $\triangle$ and R the function τ. Following [138], let us first introduce the following definition.

Definition 5.5.2. Let S be a set, $\triangle$ be a reflexive and symmetric relation on S, R another relation on S and τ a function. Then for any $s, s' \in S$:

- s *refines* s', denoted $s \sqsubseteq s'$, if $\forall z \in S(z \triangle s \to z \triangle s')$;
- s is *compatibly open* to s', denoted $s \triangle_R s'$, if $\exists z : sRz \wedge z \triangle s'$;
- s *anticipates* s', denoted $s \triangle_T s'$, if $\forall z \in S(z \triangle s' \to \exists x(s \triangle x \wedge \tau(x) \sqsubseteq z))$.

Intuitively, a possibility s refines another possibility s' if s' only rules out possibilities that are also ruled out by s. In other words, s imposes more stringent conditions on what the actual world is like than s' does. It is straightforward to see that whenever $s \sqsubseteq s'$ and $A \in \mathcal{F}_\triangle(S)$, $s' \in A$ implies that $s \in A$, i.e., every proposition true at s' is also true at s. The other two notions have a merely technical interest. We are now in a position to define the adequate frames for our logic:

Definition 5.5.3. A *fragment frame* is a tuple $\mathcal{S} = (S, \triangle, R, \tau)$ such that:

1. $\triangle$ is a reflexive and symmetric relation on S, R is a relation on S and τ is a function from S to S;
2. For any $s \in S$ and any $s' \triangle_R s$, $\exists z \triangle s \forall z' \triangle z : s' \triangle_R z'$ (Openness propositions are propositions);
3. For any $s \in S$ and any $s' \triangle \tau(s)$, $\exists z \triangle s \forall z' \triangle z : s' \triangle \tau(z')$ (Future-tensed propositions are propositions).
4. $\forall s \in S : sRs$ (Everything that is settled at s is also true at s);
5. $\forall s \in S\, \exists z \in S : (s \triangle z \wedge \forall x(z \triangle_R x \to \tau(x) \triangle \tau(s)))$ (It is compatible with s that everything that is true at $\tau(s)$ is also settled);
6. $\forall s, s' \in S : s \triangle s' \to \tau(s) \triangle \tau(z)$ (Two compatible fragments evolve in compatible ways);

7. For any $s \in S$ and any $s'\triangle s$, $\exists z : s\triangle_T z \wedge z\triangle\tau(s')$ (Two compatible fragments can compatibly anticipate each other's future).

Intuitively, conditions 2 and 3 guarantee that the set of fragments satisfying formulas of the form $\Box\varphi$ and $\mathcal{T}\varphi$ respectively are $\triangle$-fixpoints and thus genuine propositions. Conditions 4 and 5 are imposed on the openness relation to ensure that settled truth implies truth, and that no fragment can both make $\mathcal{T}\varphi$ true and make $\mathcal{T}\neg\varphi$ possible. Finally, condition 6 ensures that a fragment satisfies a formula of the form $\neg\mathcal{T}\varphi$ if it also satisfies the formula $\mathcal{T}\neg\varphi$, and condition 7 ensures that the complex operator $\neg\mathcal{T}\neg$ preserves conjunctions. Let us now see how to interpret $\mathscr{L}$-formulas in fragment frames:

Definition 5.5.4. A *fragment model* $\mathcal{M}$ is a tuple $(S, \triangle, R, \tau, V)$ such that $(S, \triangle, R, \tau)$ is a fragment frame and V is a propositional valuation such that $V(p) \in \mathcal{F}_\triangle(S)$ for any propositional variable p. The satisfaction relation on a fragment model $(S, \triangle, R, \tau, V)$ is defined inductively for any $s \in S$ and any formula φ:

- $\mathcal{M}, s \models p$ iff $s \in V(p)$;
- $\mathcal{M}, s \models \varphi \wedge \psi$ iff $s \models \varphi$ and $s \models \psi$;
- $\mathcal{M}, s \models \neg\varphi$ iff $s' \not\models \varphi$ for any $s'\triangle s$;
- $\mathcal{M}, s \models \Box\varphi$ iff $s' \models \varphi$ for any s' such that sRs';
- $\mathcal{M}, s \models \mathcal{T}\varphi$ iff $\tau(s) \models \varphi$.

Let $\mathbf{F}$ be the class of all fragment frames. We define the relation of logical consequence in the standard way: for any two formulas φ, ψ, ψ is a logical consequence of φ with respect to $\mathbf{F}$, denoted $\varphi \models_{\mathbf{F}} \psi$, if for any fragment frame $\mathcal{S} = (S, \triangle, R, \tau)$, any fragment model $\mathcal{M}$ based on $\mathcal{S}$ and any $s \in S$, $\mathcal{M}, s \models \varphi$ implies $\mathcal{M}, s \models \psi$. One can show in a straightforward way that the logic OF is sound and complete with respect to this semantics (see Theorem 5.8.7 in Section 5.8.2 for a proof).

To get a feel for fragment semantics, let us consider in some detail the fragment model $\mathcal{M}$ presented in Figure 5.4, where p stands for "There is a sea battle". The compatibility relation $\triangle$ is represented by black lines and the relation R arrows (reflexive lines being omitted for both), while the function τ is represented by blue arrows. The model is determined by the valuation mapping p to the black dot in Figure 5.4, i.e., the valuation V such that $V(p) = \{x_1\}$. Intuitively, the bottom row, which is structurally similar to the *Epistemic Scale* of [138], represents all the present possibilities regarding the sea battle to come, while the top row represents three possible futures: one in which the sea battle is settled as happening, one in which the sea battle is settled as not happening, and one in which the issue is still open.[7] It is straightforward to verify that $\mathcal{M}$ is a fragment model. Note in

[7]Note that the transition function is reflexive on the fragments in the top row. Consequently, the model describes a high-stakes situation: whether there is a sea battle tomorrow will determine whether the world will be in a state of eternal war or eternal peace. This choice is in no way forced upon us, but one advantage is that this makes our toy model finite (unlike the model in Figure 5.5 below).

particular that $\{x_1\} \in \mathcal{F}_\triangle(\mathcal{S})$, since any s distinct from x_1 is compatible with some s' that is incompatible with x_1. Our semantic clauses imply that $\mathcal{M}, s_1 \models \mathcal{T}p$, since $\tau(s_1) = x_1$. At the same time, $\mathcal{M}, s_1 \models \Box\mathcal{T}p$ if and only if $\mathcal{M}, \tau(s) \models p$ for all s such that s_1Rs. But s_1Rr and $\tau(r) = z$, so in fact $\mathcal{M}, s_1 \not\models \Box\mathcal{T}p$. Intuitively, because the future fragment from the point of view of s_1, which is x_1, differs from the future fragment of the accessible fragment r, which is z, s_1 is a fragment in which a sea battle will happen tomorrow, but not in a settled way. At the same time, we may also check that $\mathcal{M}, s_1 \not\models \Diamond\mathcal{T}\neg p$. Given the equivalence between $\Diamond\mathcal{T}\neg p$ and $\neg\Box\neg\mathcal{T}\neg p$ in our semantics, it is enough to show that there is $s'\triangle s_1$ such that $\mathcal{M}, s' \models \Box\neg\mathcal{T}\neg p$. Clearly, s_2 is such a point.

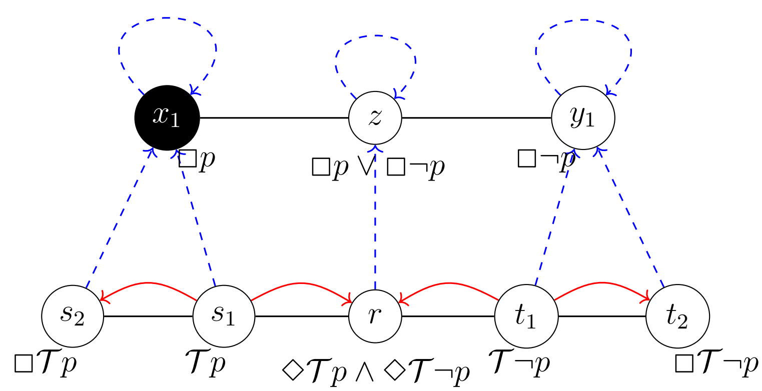


Figure 5.4: The fragment model $\mathcal{M}$.

Similarly, we may consider the fragment r. It is straightforward to verify that $\mathcal{M}, z \not\models p$ and $\mathcal{M}, z \not\models \neg p$, from which it follows that $\mathcal{M}, r \not\models \mathcal{T}p$ and $\mathcal{M}, r \not\models \mathcal{T}\neg p$. However, any fragment s compatible with r is compatible with a fragment s' that models either $\mathcal{T}p$ or $\mathcal{T}\neg p$, from which it follows that $\mathcal{M}, r \models \mathcal{T}p \vee \mathcal{T}\neg p$. In fact, r satisfies the even stronger condition $\mathcal{T}\Box p \vee \mathcal{T}\Box\neg p$. At the same time, because the only fragment satisfying $\Box\mathcal{T}p$ and $\Box\mathcal{T}\neg p$ are s_2 and t_2 respectively, we have that $\mathcal{M}, r \models \Diamond\mathcal{T}p \wedge \Diamond\mathcal{T}\neg p$. In other words, from the point of view r, the sea battle tomorrow is completely undetermined: it is possible that it will happen, and, in fact, r does not rule out that it is already true that it will happen. But it is equally possible that it won't happen, and r does not rule out that it is already true that it won't happen, since $r\triangle t_1$.

Of course, that the possibility of a sea battle not happening tomorrow is compatible with the fact that a sea battle will in fact happen does not mean that both facts can be realized by one and the same fragment. In other words, just because s_1 and r are *compatible*, this does not mean that they are *compossible*, i.e., that there is a fragment that refines both of them. Indeed, as noted by Holliday and Mandelkern in [138], the logic of a possibility frame is classical exactly when compatibility implies compossibility, i.e., when any two compatible possibilities have a common refinement.

Let me conclude this section by briefly comparing fragment semantics with branching-time semantics. A first contrast between the two models is the way in which the openness of the future is secured. While branching-time semantics is built on the idea that a single moment in time may have several distinct futures, every fragment in a fragment frame has a unique future fragment, determined by the transition function τ. But unlike instants in a branching model, fragments are not maximally consistent descriptions of ways the world could be, but only partial descriptions, able to be extended in various incompatible ways. In other words, for the orthofuturist, the openness of the future does not follow from the fact that there could be many mutually incompatible and completely determined futures lying ahead but rather from the fact that, as of now, the unique future lying ahead of us is not completely determined. Moreover, this incompleteness is cashed out by the fact that an open future does not rule out alternative fragments with which it is nonetheless not compossible. From today's point of view, the future does not rule out tomorrow's sea battle, nor does it rule out the absence of such a sea battle. But obviously no future can be a fragment in which the sea battle happens, does not happen, and may or may not happen all at once.

A second difference between the two kinds of models is the way in which they account for the flow of time. A branching-time model provides a purely *external* perspective on time: as time passes by, we move from one point to the next on the tree of moments in time according to one of the possible timelines determined by the model. By contrast, fragment models adopt an *internal* perspective. As time passes by, the moment that best describes the actual world is not entirely determined by the moment that best described it a moment ago and the transition function τ. If it is still open whether there will be a sea battle tomorrow, then, from today's point of view, a sea battle is neither happening nor not happening tomorrow. But come tomorrow, the fragment in which the sea battle is neither happening nor not happening will not be an optimal description of the actual world. Rather, it will be another fragment, compatible with that one, that settles the sea battle as happening or as not happening. In other words, what the future looks like today is not what it will look like when it becomes the present. If the future is genuinely open, this seems like a rather obvious observation, but it is a key feature of the fragment semantics presented here and, as such, it will play a significant role in the next section.

5.6 Orthofuturism and Relativism

In this section, I turn to discussing the relationship between orthofuturism and MacFarlane's recent relativist solution [178, 179] to the sea battle problem. MacFarlane argues that there are two competing intuitions regarding statements about the future. According to the *indeterminacy intuition*, some statements about the future such as "There will be a sea battle tomorrow" are neither true nor false, because at the time that they are uttered the future could still unfold in different ways. On the other hand, according to the *determinacy intuition*, once tomorrow has arrived and the sea battle has or has not happened, the statement uttered the day before retrospectively seems to have had a definite truth value. Once Leonidas's men have managed to hold off the Persians for one more day, they rejoice and agree that, after all, it was true that they would hold the pass. MacFarlane argues that both intuitions are correct and should therefore be preserved, despite their apparent inconsistency. His proposal is to relativize the truth of a statement not only to a context of utterance (the moment at which the statement is uttered), but also to a context of assessment (the moment from which the truth-value of the statement is assessed). Thus as assessed on the first evening by Themistocles, the statement "There will be a sea battle tomorrow" is neither true nor false. But assessed from the second evening, after which his fleet destroyed a small patrol of isolated Persian ships, Themistocles can now consider that that statement was true after all. In other words, MacFarlane agrees with the supervaluationist that a statement about the future cannot be contingent, i.e., true in some timelines and false in some others, and yet true, *as long as* its truth-value is assessed at a moment that belongs to such disagreeing timelines. But since timelines must split eventually, MacFarlane argues that, in the long run, the true-futurist is right, but only *retrospectively*: some statements were true when they were uttered, even though they were only contingently so at that time.

Because MacFarlane's determinacy intuition involves looking into the past in an essential way, it is worth discussing first how to add a backwards-looking modality to OF. Just as temporal logic has both forward looking and backward looking operators, we may add a "Yesterday" connective $\mathcal{Y}$ as a counterpart to our "Tomorrow" connective $\mathcal{T}$. We can therefore define the language $\mathscr{L}_{\mathcal{Y}}$ generated by the following grammar:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathcal{T}\varphi \mid \mathcal{Y}\varphi$$

where again p belongs to a countably infinite set of propositional variables. What conditions should we impose on the interplay between the two temporal modalities? A simple and attractive option is to treat $\mathcal{Y}$ as an inverse of $\mathcal{T}$, in the sense that for any formula φ, we should have that $\mathcal{T}\mathcal{Y}\varphi$ is equivalent to φ and φ is equivalent to $\mathcal{Y}\mathcal{T}\varphi$. A convenient way to spell out this requirement is to impose the following conditions on our logic:

***L*1** $\mathcal{Y}\varphi \vdash \psi$ implies $\varphi \vdash \mathcal{T}\psi$;

***L*2** $\varphi \vdash \mathcal{T}\psi$ implies $\mathcal{Y}\varphi \vdash \psi$;

***R*1** $\mathcal{T}\varphi \vdash \psi$ implies $\varphi \vdash \mathcal{Y}\psi$;

***R*2** $\varphi \vdash \mathcal{Y}\psi$ implies $\mathcal{T}\varphi \vdash \psi$.

One can verify that, over OF, the conjunction of ***L*1** and ***L*2** is equivalent to the conditions $\varphi \vdash \mathcal{T}\mathcal{Y}\varphi$ and $\mathcal{Y}\mathcal{T}\varphi \vdash \varphi$, together with the monotonicity of $\mathcal{Y}$, i.e., the condition that $\varphi \vdash \psi$ implies $\mathcal{Y}\varphi \vdash \mathcal{Y}\psi$. Similarly, the conjunction of ***R*1** and ***R*2** is equivalent to the conditions $\varphi \vdash \mathcal{Y}\mathcal{T}\varphi$ and $\mathcal{T}\mathcal{Y}\varphi \vdash \varphi$, together with the monotonicity of $\mathcal{Y}$.

It is worth noting that ***L*1** and ***L*2** are conditions that intuitively follow the flow of time, while ***R*1** and ***R*2** are conditions that "go against" the flow of time. Intuitively, ***L*1** and ***L*2** assert that the way in which the past affects the present is the same as the way in which the present affects the future. A similar intuition applies to the equivalent principles $\varphi \vdash \mathcal{T}\mathcal{Y}\varphi$ ("There is a sea battle occurring today; therefore tomorrow it will be the case that there was a sea battle occurring the day before") and $\mathcal{Y}\mathcal{T}\varphi \vdash \varphi$ ("Yesterday it was the case that there would be a sea battle the next day; therefore there is a sea battle today"): in both cases, the truth of a proposition at a certain time entails the truth of another proposition at a later time. By contrast, $\boldsymbol{R1}$ and $\boldsymbol{R2}$ go in the opposite direction, as they assert that the way in which the future affects the present is the same as the way in which the present affects the past and are equivalent to the principles $\varphi \vdash \mathcal{Y}\mathcal{T}\varphi$ ("There is a sea battle occurring today; therefore yesterday it was the case that there would be a sea battle the next day") and $\mathcal{T}\mathcal{Y}\varphi \vdash \varphi$ ("Tomorrow it will be the case that there was a sea battle occurring the day before; therefore there is a sea battle occurring today"). One might therefore be tempted to introduce two distinct operators $\mathcal{Y}$ and $\mathcal{W}$, satisfying conditions $\boldsymbol{L1}$-$\boldsymbol{L2}$ and $\boldsymbol{R1}$-$\boldsymbol{R2}$ respectively. Although we cannot pursue this option here, this seems like a promising way of formalizing Plantinga's distinction between *soft* and *hard* facts about the past [67, 208], i.e., between facts regarding the past that depend in some way on the future and facts that do not. One could indeed argue for the intuitive validity of the "forward-looking" rules $\mathcal{Y}\mathcal{T}\varphi \vdash \varphi$ and $\varphi \vdash \mathcal{T}\mathcal{Y}\varphi$ when $\mathcal{Y}$ is interpreted as "It was a hard fact yesterday that", and for the validity of the "backward-looking" rules $\mathcal{T}\mathcal{W}\varphi \vdash \varphi$ and $\varphi \vdash \mathcal{W}\mathcal{T}\varphi$ when $\mathcal{W}$ is interpreted as "It was a soft fact yesterday that". For now, let us define the logic OFY as the smallest relation $\vdash$ on $\mathscr{L}_Y$ satisfying conditions 1-11, $\boldsymbol{L1}$, $\boldsymbol{L2}$, $\boldsymbol{R1}$ and $\boldsymbol{R2}$.

Fragment frames can be modified in a straightforward way to provide a semantics for OFY.

Definition 5.6.1. A *linear fragment frame* is a fragment frame $(S, \triangle, R, \tau)$ such that τ is a bijection and for any $s \in S$ and any $s' \triangle \tau^{-1}(s)$, $\exists z \triangle s \forall z' \triangle z : s' \triangle \tau^{-1}(z')$.

By requiring the transition function τ to be a bijection, we can therefore define the past of any fragment s as $\tau^{-1}(s)$, i.e., as the unique fragment s' such that $\tau(s') = s$. A OFY model $\mathcal{M}$ is defined in the natural way, by letting $\mathcal{M}, s \models \mathcal{Y}\varphi$ iff $\mathcal{M}, \tau^{-1}(s) \models \varphi$. The proof of the following theorem is postponed to Section 5.8.3.

Theorem 5.6.2. *The logic* OFY *is sound and complete with respect to linear fragment frames.*

Let us now return to MacFarlane's relativism. How does the orthofuturist position compare to MacFarlane's? Fragment semantics can certainly account for the indeterminacy intuition. If we consider again the model $\mathcal{M}$ depicted in Figure 5.4, we have that $\mathcal{M}, z_1 \not\models \mathcal{T}p$ and $\mathcal{M}, z_1 \not\models \mathcal{T}\neg p$, which one can interpret as the truth-value of the statement $\mathcal{T}p$ not being determined at z_1. At the same time, if the orthofuturist endorses $\boldsymbol{R1}$, then $\varphi \vdash \mathcal{Y}\mathcal{T}\varphi$ becomes a theorem of her logic, which seems to establish that she can account for the retrospective determinacy intuition as well. However, there is a major difference between orthofuturism and MacFarlane's relativism. For the relativist, an utterance can be both true and contingent in the right context of assessment, while this is a contradiction for the orthofuturist. Indeed, if one considers again the branching time model presented in Figure 5.3, then the

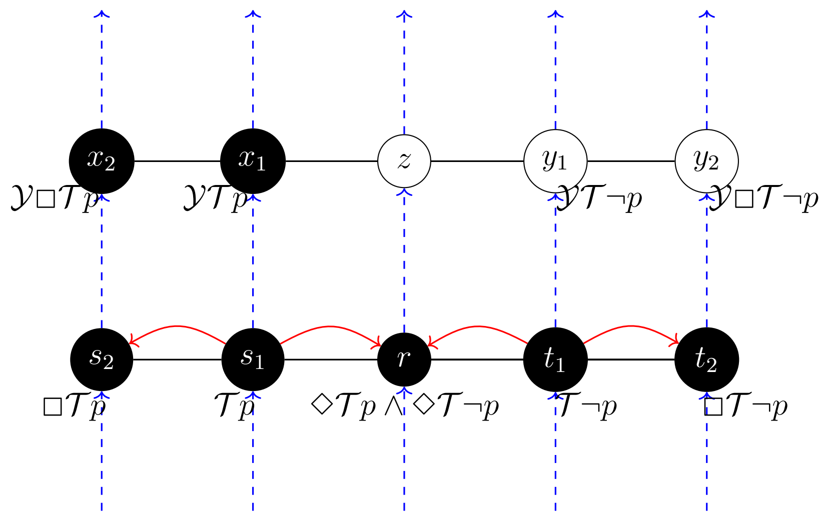


Figure 5.5: The model $\mathcal{N}$

relativist claims that, relative to a context of assessment whose moment is t_0, $\mathcal{T}p$ is true as uttered at t, because it is true at t relative to timeline C for every timeline C to which t_0 belongs, which in this case is only the timeline $\{t, t_0\}$. At the same time, $\Diamond\mathcal{T}\neg p$ is also true when uttered at t relative to a context of assessment whose moment is t_0, since $\Diamond\mathcal{T}\neg p$ is true at t relative to timeline $\{t, t_0\}$, as $\mathcal{T}\neg p$ is true at t relative to timeline $\{t, t_1\}$. Unlike the orthofuturist, the relativist therefore rejects the Open Future intuition **(*O*)**, even though only in some specific contexts. Nonetheless, this does not mean that the orthofuturist must outright reject MacFarlane's point.

Indeed, the orthofuturist may agree with MacFarlane that the determinacy and indeterminacy intuitions are not incompatible, while at the same time maintaining that they are not compossible. This point can made in a particularly clear way by appealing to fragment semantics and its internal approach to the flow of time mentioned in the previous section. For example, one may consider the fragment model $\mathcal{N}$ partially depicted in Figure 5.5. Once again, black lines represent the compatibility relation, red ones the openness relation (with reflexive loops omitted in both cases), and dotted blue lines represent the transition function. The underlying frame consists of infinitely many copies of the top row in Figure 5.5 above infinitely many copies of the bottom row. Since every fragment has both a predecessor and a successor according to the transition function, τ is a bijection and the past of a fragment s is precisely the fragment $\tau^{-1}(s)$. The model $\mathcal{N}$ is determined by the valuation mapping p to the black fragments in Figure 5.5.

Intuitively, r describes the moment at which Themistocles, after having led his fleet through the first day of the battle, wonders whether there will be another sea battle on the next day. In r, the sea battle to come is still a contingent event, and it is neither true that there will be one nor true that there won't be one, since the accurate description of the future at this point is the fragment z, and z satisfies neither p nor $\neg p$. On the second day however, since there is indeed a sea battle occurring between the Persians and the Greeks, the accurate description of the present is not z anymore, but rather x_1. We can check that $\mathcal{N}, x_1 \models \mathcal{YT}p$; indeed $\tau^{-1}(x_1) = s_1$ and $\mathcal{N}, s_1 \models \mathcal{T}p$ because $\tau(s_1) = x_1$. Because OFY is valid in $\mathcal{N}$, it follows that $\mathcal{N}, x_1 \not\models \mathcal{Y}\Diamond\mathcal{T}\neg p$, which we can verify easily by noticing that $\mathcal{N}, s_1 \not\models \Diamond\mathcal{T}\neg p$ since $s_1 \triangle s_2$. In other words, although, yesterday (i.e., in r), it was true that the sea battle could have failed to happen, today (i.e., in x_1), it is not the case anymore that the sea battle could have failed to happen. At the same time, this does not mean that it was already settled yesterday that today's sea battle would occur. Indeed, $\mathcal{N}, s_1 \not\models \Box\mathcal{T}p$, since $\mathcal{N}, r \not\models \mathcal{T}p$, and therefore $\mathcal{N}, x_1 \not\models \mathcal{Y}\Box\mathcal{T}p$. Although it is now settled that the sea battle would occur, it was not settled yesterday that it would. Crucially, what the future looked like on the eve of the second day (i.e., z), is not what the present looks like on the second day (i.e., x_1).

The internal perspective on the flow of time adopted by fragment semantics therefore allows the orthofuturist to account for both the determinacy and the indeterminacy intuitions, without having to give up the Open Future intuition. In fact, one may argue that the relativist's rejection of ***(O)*** stems from a superposition of the internal and the external perspective. Once we look back at the day before the sea battle happened, the way in which it could have failed to happen (back then, the present looked like there may not be a sea battle on the next day) is different from the way in which it was true that it would happen (now that we look into the day before, we can see that the sea battle was going to happen). In the latter case, we are simply considering *internally* what the past looks like from today's viewpoint, while, in the former case, we need to adopt an *external* perspective on what the past looked like when it was the present. At the same time, even from the internal viewpoint, a shadow of the way things used to be persists: although it is not true anymore that the sea battle could have failed to happen (i.e., $\mathcal{Y}\Diamond\mathcal{T}\neg p$ does not hold), it is still not true that it was bound to happen (i.e., $\mathcal{Y}\Box\mathcal{T}p$ does not hold either).

To sum up, the relativist thinks that we can solve the sea battle problem from the external viewpoint on the flow of time, provided that we establish a distinction between context of utterance and context of assertion. In doing so, the relativist develops a two-dimensional theory of meaning in order to resolve the antinomy between the determinacy and the indeterminacy intuitions. In that sense, her solution is essentially (post)-semantic. By contrast, the orthofuturist concludes from the fact that the determinacy and indeterminacy intuitions are never true at the same time that there is no need to reject the Open Future intuition, even retrospectively. But the price that she has to pay is to give up on the possibility of an external, all-encompassing view on the flow of time, and this is reflected in the fact that she must abandon classical logic for orthologic. In that sense, her solution to the sea battle is logical rather than semantic.

5.7 Conclusion

Orthofuturism offers a novel approach to the sea battle problem that has the advantage of making sense both of the intuition of a genuinely open future and of the distinction between truth and settled truth. As I have argued, the orthofuturist holds a nuanced view in the debate between those who think that there is one true future and those who think that there are many equally possible futures ahead of us. Indeed, she thinks that there is only one future but understands its openness as the fact that it is compatible with other ways that the world may turn out to be. Unlike the proponent of branching-time semantics, however, she does not think that all those possible futures coexist as splitting timelines. Rather, she thinks that compatibility does not imply compossibility, and this is ultimately what leads her to reject the distributive laws of classical logic and to favor orthologic instead.

Although orthologic is not classical logic, it is a well-established and tractable formal system, and the algebraic and relational semantics presented here provide powerful tools for investigating the interactions between tense and modality beyond the distributive world. At the same time, there is still much left to explore. On the axiomatic side, a natural next step would be to investigate the addition of a conditional connective to the logic OFY. The interaction of tense with conditionals is a subtle problem [253], and one must be particularly careful in the case of orthologic, as adding a conditional satisfying the deduction theorem would collapse the logic to classical logic. Nonetheless, a modular approach based on modalities [138] or on weak implication connectives [131] seems promising. Moreover, I do not claim that the logic I have introduced is the correct logic of the open future. Rather, I believe that the work presented here motivates further research on logics between pure orthologic and classical logic, and on extensions of orthologic with operations such as modalities, tense operators and conditionals. On the semantic side, further exploration of models for OFY is needed. In particular, a reconciliation of the internal and external perspectives on the flow of time could be a useful enhancement of fragment semantics. One would also hope that this could grow into a conception of the orthofuturist's metaphysics of time that could truly rival branching time semantics. On both accounts, I suspect that a two-dimensional approach, similar to Cariani's recent investigation on classical possibility semantics for the open future [54], could be fruitful. Whether this will help us come to a fully satisfactory answer to a century-old problem, however, is a question that is left for future work.

5.8 Technical Appendix

5.8.1 Appendix A

This appendix contains some technical details regarding the algebraic semantics for OF mentioned in Section 5.3.2. We start with the following definition:

Definition 5.8.1. A *bimodal ortholattice* is a structure $(L, \leq, \wedge, \vee, \neg, 0, 1, \Box, \mathcal{T})$ satisfying the following axioms:

- $(L, \leq, \wedge, \vee, 0, 1)$ is a *bounded lattice*:
    1. $\leq$ is a partial order on L, i.e., a reflexive, transitive and antisymmetric relation;
    2. $\wedge$ and $\vee$ are functions from $L \times L$ to L mapping any pair of elements a, b of L to their *greatest lower bound* $a \wedge b$ and *least upper bound* $a \vee b$ in $(L, \leq)$ respectively;
    3. 0 and 1 are the smallest and greatest elements of L respectively.
- $\neg$ is an *orthocomplementation* on L, i.e., a function from L to L such that for any $a, b \in L$:
    4. $a \leq b$ implies $\neg b \leq \neg a$;
    5. $a = \neg\neg a$;
    6. $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.
- $\Box$ and $\mathcal{T}$ are *modal operators* on L, i.e., functions from L to L such that for any $a, b \in L$:
    7. $\Box 1 = 1$ and $\mathcal{T}1 = 1$;
    8. $\Box(a \wedge b) = \Box a \wedge \Box b$, and $\mathcal{T}(a \wedge b) = \mathcal{T}a \wedge \mathcal{T}b$.

The reader may consult [138, Section 3] for a detailed description of modal ortholattices and their relationship to Boolean algebras with operators, which provide the standard algebraic semantics for classical modal logic. In order to isolate the adequate bimodal ortholattices for OF, we need to impose some additional conditions on the modal operators, which correspond in a straightforward way to conditions 11-14 in Definition 5.3.1.

Definition 5.8.2. An OF lattice is a bimodal ortholattice $(L, \leq, \wedge, \vee, \neg, 0, 1, \Box, \mathcal{T})$ such that for any $a \in L$:

9. $\Box a \leq a$;
10. $\neg\mathcal{T}\neg$ is also a modal operator on L;
11. $\mathcal{T}a \wedge \neg\Box\neg\mathcal{T}\neg a \leq 0$;
12. $\mathcal{T}\neg a \leq \neg\mathcal{T}a$;

Figure 5.6 presents Hasse diagrams for two simple OF lattices. As is customary in Hasse diagrams, a line from an element x to an element y pictured above x means that $x \leq y$ and that there is no distinct element z such that $x \leq z \leq y$. The modal operator $\Box$ is represented by red full arrows, and the tense operator $\mathcal{T}$ is represented by dashed blue arrows.

We may now define rigorously the notion of a valuation on a OF lattice L.

Definition 5.8.3. Let $(L, \leq, \wedge, \vee, \neg, 0, 1, \Box, \mathcal{T})$ be an OF lattice. A *valuation* on L is a function v mapping $\mathscr{L}$-formulas to elements of L so that for any two formulas φ, ψ:

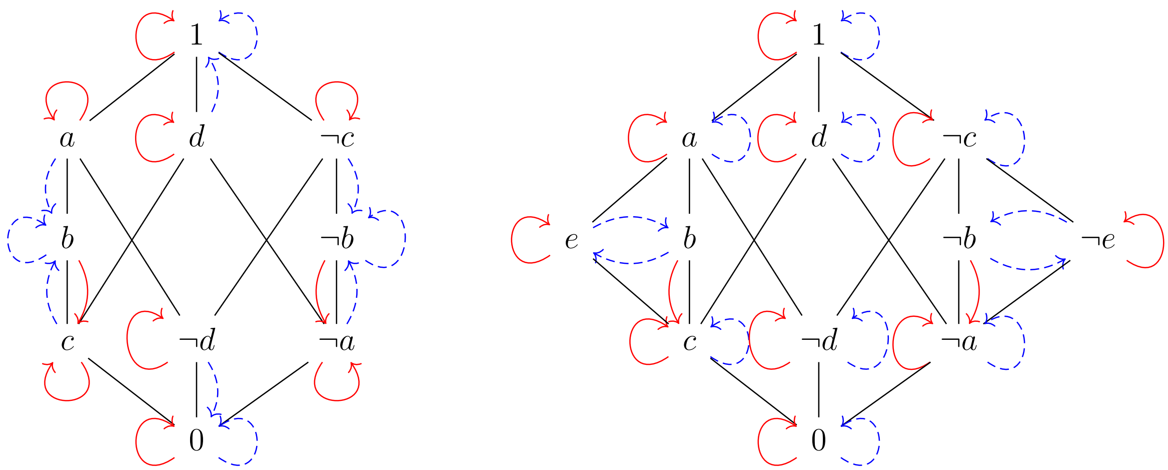


Figure 5.6: The OF lattices $\boldsymbol{O_{10}}$ and $\boldsymbol{O_{12}}$

- $v(\bot) = 0$;
- $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$;
- $v(\neg\varphi) = \neg v(\varphi)$;
- $v(\Box\varphi) = \Box v(\varphi)$;
- $v(\mathcal{T}\varphi) = \mathcal{T} v(\varphi)$.

A formula φ is *valid* on L if for any valuation v on L, $v(\varphi) = 1$. For any two formulas φ and ψ, ψ is a *logical consequence* of φ on L, denoted $\varphi \models_L \psi$, if $v(\varphi) \leq v(\psi)$ for any valuation v on L.

Finally, we have the following soundness and completeness theorem.

Theorem 5.8.4. *The logic* OF *is sound and complete with respect to the consequence relation* $\models_{\boldsymbol{C}}$*: for any two* $\mathscr{L}$*-formulas* φ *and* ψ*,* $\varphi \vdash \psi$ *if and only if* $\varphi \models_{\boldsymbol{C}} \psi$*.*

Proof. For soundness, one can check in a straightforward way that the relation $\models_{\boldsymbol{C}}$ satisfies conditions 1-11 in Definition 5.3.1. Since by definition $\vdash$ is the smallest relation satisfying those conditions, it follows that $\varphi \vdash \psi$ implies $\varphi \models_{\boldsymbol{C}} \psi$ for any two $\mathscr{L}$-formulas φ, ψ.

For completeness, assume that $\varphi \nvdash \psi$. I claim that there is an OF lattice L and a valuation v on L such that $v(\varphi) \nleq v(\psi)$. This lattice is the *Lindenbaum-Tarski* algebra $\mathcal{L}_{\mathsf{OF}}$ of the logic OF, defined as follows. We start by defining an equivalence relation on the set of $\mathscr{L}$-formulas, letting $\varphi \sim_{\mathsf{OF}} \psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$. That $\sim_{\mathsf{OF}}$ is an equivalence relation follows from conditions 1 and 2 in Definition 5.3.1. Given a formula φ, its $\sim_{\mathsf{OF}}$ equivalence class is denoted φ^*. We may then take as elements of $\mathcal{L}_{\mathsf{OF}}$ the set L of all equivalence classes φ^* for some formula φ and define the following relations and operations for any two formulas φ, ψ:

- $\varphi^* \leq \psi^*$ iff $\varphi \vdash \psi$;
- $\varphi^* \wedge \psi^* = (\varphi \wedge \psi)^*$;
- $\neg\varphi^* = (\neg\varphi)^*$;
- $\varphi^* \vee \psi^* = \neg(\neg\varphi^* \wedge \neg\psi^*)$;
- $0 = \bot^*$, $1 = \top^*$;
- $\Box\varphi^* = (\Box\varphi)^*$;
- $\mathcal{T}\varphi^* = (\mathcal{T}\varphi)^*$.

It is a tedious but simple exercise to verify that these operations are well defined and that $\mathcal{L}_{\mathsf{OF}} = (L, \leq, \wedge, \vee, \neg, 0, 1, \Box, \mathcal{T})$ is an OF lattice. Now if $\varphi \nvdash \psi$, then by construction $\varphi^* \nleq \psi^*$. But since the map $\chi \mapsto \chi^*$ for any formula χ is clearly a valuation on $\mathcal{L}_{\mathsf{OF}}$, this implies that there exists an OF lattice L and a valuation v on L such that $v(\varphi) \nleq_L v(\psi)$ and therefore that $\varphi \not\models_{\boldsymbol{C}} \psi$. □

5.8.2 Appendix B

This appendix contains some details regarding the fragment semantics for OF presented in Section 5.5. Recall first that a *complete lattice* is a partially ordered set L such that any subset of L has both a least upper bound and a greatest lower bound. The powerset of any set W of possible worlds ordered by inclusion is a complete lattice, which is precisely what allows one to think of propositions in classical modal logic as sets of possible worlds. More generally, given a reflexive and symmetric relation $\vartriangle$ on a set S, the map:

$$A \mapsto \{s \in S \mid \forall s' \vartriangle s \exists s'' \vartriangle s' : s'' \in A\}$$

is a closure operator on the complete lattice of subsets of S. This ensures that the $\vartriangle$-fixpoints of S always form a complete lattice. In fact, more is true:

Theorem 5.8.5 ([42],§§32-4)**.** *Let $\vartriangle$ be a reflexive and symmetric relation on a set S, Then the $\vartriangle$-fixpoints of S form a complete ortholattice $\mathcal{F}_{\vartriangle}(S)$, with the meet given by intersection and the orthocomplementation given by $\neg_{\vartriangle} A = \{s \in S \mid \forall s' \vartriangle s : s' \notin A\}$.*

. Let us now prove that $\mathcal{F}_{\vartriangle}(S)$ is a complete OF lattice whenever S is a fragment frame.

Lemma 5.8.6. *Let $(S, \vartriangle, R, \tau)$ be a fragment frame. Then $\mathcal{F}_{\vartriangle}(S)$ is a complete* OF *lattice.*

Proof. In light of [138, Prop. 4.28], we only need to verify properties 10-12 in Definition 5.8.2. For any $A \subseteq S$, we let $\mathcal{T}A = \{s \in S \mid \tau(s) \in A\}$ and $\Box_R A = \{s \in S \mid \forall s' \in S : sRs' \to s' \in A\}$. It is straightforward to verify that conditions 2 and 3 in Definition 5.5.4 ensure that $\mathcal{T}$ and $\Box$ map $\vartriangle$-fixpoints to $\vartriangle$-fixpoints.

For condition 10, note first that given a map $f : \mathcal{F}_\vartriangle(S) \to \mathcal{F}_\vartriangle(S)$ and a relation Q such that $f(A) = \{s \in S \mid \forall s' \in S : sRs' \to s' \in A\}$ for any $A \in \mathcal{F}_\vartriangle(S)$, one easily verifies that f is a normal modal operator on $\mathcal{F}_\vartriangle(S)$. Since $\neg_\vartriangle \mathcal{T} \neg_\vartriangle$ is a composition of maps from $\vartriangle$-fixpoints to $\vartriangle$-fixpoints, it also maps $\vartriangle$-fixpoints to $\vartriangle$-fixpoints. Therefore, is enough to check that for any $\vartriangle$-fixpoint A, $\neg_\vartriangle \mathcal{T} \neg_\vartriangle A = \{s \in S \mid \forall s' \in S : s\vartriangle_T s' \to s' \in A\}$. Fix a $\vartriangle$-fixpoint A, and assume first that $s \in \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$. This means that for any $x \vartriangle s$, $\tau(x) \notin \neg_\vartriangle A$. Now assume that $s \vartriangle_T s'$. We want to show that $s' \in A$. Since A is a $\vartriangle$-fixpoint, it is enough to show that there is $y' \vartriangle y$ with $y' \in A$ for any $y \vartriangle s'$. Fix such a y. By the definition of the relation $\vartriangle_T$, there is z such that $s \vartriangle z$ and $\tau(z) \sqsubseteq y$. By assumption on s, we have that $\tau(z) \notin \neg_\vartriangle A$, so there is $y' \in A$ such that $\tau(z) \vartriangle y'$. But from $\tau(z) \sqsubseteq y$ it follows that $y \vartriangle y'$. This shows that $s' \in A$, and thus that $\neg_\vartriangle \mathcal{T} \neg_\vartriangle A \subseteq \{s \in S \mid \forall s' \in S : s \vartriangle_T s' \to s' \in A\}$. For the converse, assume that $s \notin \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$. This means that there is $s' \vartriangle s$ such that $\tau(s') \in \neg_\vartriangle A$. By condition 7 in Definition 5.5.4, there is $z \in Z$ such that $s \delta_T z$ and $z \delta \tau(s')$. But this means that $z \notin A$, and hence that $s \notin \{s \in S \mid \forall s' \in S : s \vartriangle_T s' \to s' \in A\}$. This completes the proof of condition 10.

For condition 11, fix a $\vartriangle$-fixpoint A and assume that $s \in \mathcal{T}A$. We need to show that $s \notin \neg_\vartriangle \Box_R \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$. By condition 5 in Definition 5.5.4, there is $z \vartriangle s$ such that for any x such that $x \vartriangle_R z$, we have that $\tau(x) \vartriangle \tau(s)$. I claim that $z \in \Box_R \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$, which will show that $s \notin \neg_\vartriangle \Box_R \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$. To see this, note that if zRy and $x \vartriangle y$, then $z \vartriangle_R x$, so $\tau(x) \vartriangle \tau(s)$. Since $s \in \mathcal{T}A$, $\tau(s) \in A$, so $x \notin \mathcal{T} \neg_\vartriangle A$. This shows that for any y such that zRy, $y \in \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$, and hence that $z \in \Box_R \neg_\vartriangle \mathcal{T} \neg_\vartriangle A$.

Finally, we show that $\mathcal{T} \neg_\vartriangle A \leq \neg_\vartriangle \mathcal{T} A$ for any $A \in \mathcal{F}_\vartriangle(S)$. To see this, assume that $s \in \mathcal{T} \neg_\vartriangle A$. This means that $\tau(s) \in \neg_\vartriangle A$. Let $s' \vartriangle s$. Then by condition 6 in Definition 5.5.4, $\tau(s') \vartriangle \tau(s)$, so $\tau(s') \notin A$ since $\tau(s) \in \neg_\vartriangle A$. This shows that $s' \notin \mathcal{T}A$ for any $s' \vartriangle s$ and thus that $s \in \neg_\vartriangle \mathcal{T} A$. $\Box$

This result allows us to prove a soundness and completeness theorem for fragment semantics. The proof is a straightforward adaptation of the soundness and completeness theorem for the logic **EO** obtained in [138].

Theorem 5.8.7. *The logic* OF *is sound and complete with respect to* OF *frames. In other words, for any two formulas* φ *and* ψ, $\varphi \vdash_{\mathsf{OF}} \psi$ *iff* $\varphi \models_{\mathbf{S}} \psi$.

Proof. For soundness, given a fragment model $\mathcal{M}$ based on an OF frame $(S, \vartriangle, R, \tau)$, let $[\varphi]_\mathcal{M} = \{s \in S \mid \mathcal{M}, s \models \varphi\}$. A straightforward induction shows that $[\varphi]_\mathcal{M} \in \mathcal{F}_\vartriangle(S)$ for any formula φ, and that the map $\varphi \mapsto [\varphi]_\mathcal{M}$ is a valuation on $\mathcal{F}_\vartriangle(S)$. But it follows from this that the consequence relation $\models_\mathbf{C}$ is a subrelation of $\models_\mathbf{S}$, i.e., we have that $\varphi \models_\mathbf{C} \psi$ implies $\varphi \models_\mathbf{S} \psi$ for any $\mathscr{L}$ formulas φ, ψ. But then soundness follows immediately from the soundness part of Theorem 5.8.4.

For completeness, recall from the proof of Theorem 5.8.4 that $\mathcal{L}_{\mathsf{OF}}$ is the Lindenbaum-Tarski algebra of OF. We consider the frame $(\mathcal{F}, \vartriangle, R, \tau)$, where:

- $\mathcal{F}$ is the set of all proper filters F over $\mathcal{L}_{\mathsf{OF}}$;

- for any two filters $F, G \in \mathcal{F}$, $F \triangle G$ iff there is no $a \in F$ such that $\neg a \in G$, and FRG iff $\Box a \in F$ implies $a \in G$;
- for any filter F, $\tau(F) = \{a \in \mathcal{L}_{\mathsf{OF}} \mid \mathcal{T} a \in F\}$.

Next, I claim that $(\mathcal{F}, \triangle, R, \tau)$ is an OF frame and that the map $a \mapsto \widehat{a} = \{F \in \mathcal{F} \mid a \in F\}$ is an injective OF embedding of $\mathcal{L}_{\mathsf{OF}}$ into $\mathcal{F}_{\triangle}(\mathcal{F})$. In light of the proof of Theorem 4.34 in [138], it is enough to check that conditions 5, 6 and 7 in Definition 5.5.3 are satisfied and that $\widehat{\mathcal{T}a} = \mathcal{T}\widehat{a}$ for any $a \in \mathcal{L}_{\mathsf{OF}}$.

Let us start by making the following observations for any $F, G \in \mathcal{F}$:

1. $F \sqsubseteq G$ iff $G \supseteq F$;
2. $F \triangle_R G$ iff $\Box a \in F$ implies $\neg a \notin G$;
3. $F \triangle_T G$ iff $\neg\mathcal{T}\neg a \in F$ implies $a \in G$.

The first two items are seen from the proof of Theorem 4.34 in [138], so I will only prove the third item. Assume first that $F \triangle_T G$, and assume that $\neg\mathcal{T}\neg a \in F$. Note that if $a \notin G$, then $G \triangle {\uparrow}\neg a$, where ${\uparrow}\neg a$ is the principal filter generated by $\neg a$. Hence it is enough to show that $\neg a \notin H$ for any $H \triangle G$. So suppose $G \triangle H$. Since $F \triangle_T G$, there is K such that $F \triangle K$ and $\tau(K) \sqsubseteq H$. Now if $\neg a \in H$, then $\neg a \in \tau(K)$, so $\mathcal{T}\neg A \in K$. But this is contradiction, since $\neg\mathcal{T}\neg a \in F$ and $F \triangle K$. Conversely, suppose that $\neg\mathcal{T}\neg a \in F$ implies $a \in G$, and let H be such that $G \triangle H$. Let $K = {\uparrow}\{\mathcal{T}a \mid a \in H\}$. It is straightforward to see that K is a filter and that $H \subseteq \tau(K)$. Moreover, if there is $b \in F$ such that $\neg b \in K$, then there is $a \in H$ such that $Ta \leq \neg b$. But this means that $b \leq \neg\mathcal{T}a$, hence that $\neg\mathcal{T}a \in F$. By assumption on F and G, it follows that $\neg a \in G$, but this contradicts $G \triangle H$. Hence $F \triangle K$, which establishes that $F \triangle_T G$.

Let us now show that conditions 5-7 in Definition 5.5.4 hold, starting with condition 5. Fix some $F \in \mathcal{F}$, and let $G = {\uparrow}\{\Box\neg\mathcal{T}\neg a \mid \mathcal{T}a \in F\}$. Note that for any a, b, $\Box\neg\mathcal{T}\neg a \wedge \Box\neg\mathcal{T}\neg b = \Box(\neg\mathcal{T}\neg a \wedge \neg\mathcal{T}\neg b) = \Box\neg\mathcal{T}\neg(a \wedge b)$ by conditions 8 and 10 in the definition of a OF lattice, so G is a filter. To see that $F \triangle G$, suppose there is $b \in F$ such that $\Box\neg\mathcal{T}\neg a \leq \neg b$ for some $\mathcal{T}a \in F$. Then $b \leq \neg\Box\mathcal{T}\neg a$, so $Ta \wedge \neg\Box\neg\mathcal{T}\neg a \in F$, contradicting condition 11 in Definition 5.8.2. Moreover, suppose $G \triangle_R H$, and $a \in \tau(F)$. Then $\mathcal{T}a \in F$, so $\Box\neg\mathcal{T}\neg a \in G$. Since $G \triangle_R H$, this means that $\mathcal{T}\neg a \notin H$, and hence $\neg a \notin \tau(H)$. This shows that $\tau(F) \triangle \tau(H)$, which completes the proof that condition 5 holds.

To show that condition 6 holds, let $F \triangle G \in \mathcal{F}$. Suppose towards a contradiction that there is $a \in \tau(F)$ such that $\neg a \in \tau(G)$. Then $\mathcal{T}a \in F$ and $\mathcal{T}\neg a \in G$. But $\mathcal{T}\neg a = \neg\mathcal{T}a$, which contradicts $F \triangle G$. Hence $\tau(F) \triangle \tau(G)$.

For condition 7, suppose that $F \triangle G$, and let $H = \{a \mid \neg\mathcal{T}\neg a \in F\}$. Note that H is a filter by condition 10 in Definition 5.5.4. Clearly, $F \triangle_T H$. Moreover, if $\neg a \in \tau(G)$, then $\mathcal{T}\neg a \in G$, so $\neg\mathcal{T}\neg a \notin F$. But this means that $a \notin H$, which shows that $H \triangle \tau(G)$ and completes the proof that condition 7 holds.

Lastly, observe that for any $a \in \mathcal{L}_{\mathsf{OF}}$ and any $F \in \mathcal{F}$,

$$F \in \widehat{\mathcal{T}a} \Leftrightarrow \mathcal{T}a \in F \Leftrightarrow a \in \tau(F) \Leftrightarrow \tau(F) \in \widehat{a} \Leftrightarrow F \in \mathcal{T}\widehat{a}.$$

To conclude the proof of completeness, let φ and ψ be formulas such that $\varphi \nvdash_{\mathsf{OF}} \psi$. Then $\varphi^* \nleq_{\mathcal{L}_{\mathsf{OF}}} \psi^*$, which implies that $\widehat{\varphi^*} \nsubseteq \widehat{\psi^*}$. Letting $\mathcal{M}$ be the model based on $(\mathcal{F}, \triangle, R, \tau)$ and determined by the propositional valuation $p \mapsto \widehat{p^*}$, it follows that there is $F \in \mathcal{F}$ such that $\mathcal{M}, F \models \varphi$ but $\mathcal{M}, F \not\models \psi$. Hence $\varphi \not\models_{\mathbf{S}} \psi$. □

5.8.3 Appendix C

This appendix discusses semantics for the logic OFY introduced in Section 5.6. We first adopt an algebraic approach. In order to add a past tense operator to our OF lattices, let us recall the following standard definition in order theory:

Definition 5.8.8. Let F be a monotone operation on a partial order $(P, \leq)$. A *left adjoint* of F is a map $G : P \to P$ such that for any $x, y \in P$:

$$Gx \leq y \Leftrightarrow x \leq Fy.$$

Dually, a *right adjoint* of F is a map $H : P \to P$ satisfying:

$$Fx \leq y \Leftrightarrow x \leq Hy.$$

This motivates the following definition.

Definition 5.8.9. A OFY *lattice* is a tuple $(L, \wedge, \vee, \neg, 0, 1, \Box, \mathcal{T}, \mathcal{Y})$ such that $(L, \wedge, \vee, \neg, 0, 1, \Box, \mathcal{T})$ is an OF lattice and $\mathcal{Y}$ is a both a left- and right-adjoint of $\mathcal{T}$.

Valuations for $\mathscr{L}_P$ formulas on an OF lattice equipped with an additional operator $\mathcal{Y}$ are defined in the obvious way. A moment's reflection shows that OFY lattices are precisely the OF lattices in which conditions $\boldsymbol{L1}$, $\boldsymbol{L2}$, $\boldsymbol{R1}$ and $\boldsymbol{R2}$ are valid. As a consequence, we have the following theorem, which is proved in a completely similar way as Theorem 5.8.4.

Theorem 5.8.10. *The logic* OFY *is sound and complete with respect to the class of all* OFY *lattices.*

One also verifies easily that an OF lattice L induces a OFY lattice if and only if the operator $\mathcal{T}$ is invertible, i.e., there is a monotone map $g : L \to L$ such that $g(\mathcal{T}a) = \mathcal{T}g(a) = a$ for any $a \in L$. This in turns is equivalent to $\mathcal{T}$ being a bijection that reflects the order, i.e., is such that for any $a, b \in L$, $\mathcal{T}a \leq \mathcal{T}b$ implies $a \leq b$. The lattice $\boldsymbol{O_{12}}$ depicted in Figure 5.6 is such an example. Together with the valuation described in Figure 5.1, this shows that $\Box\mathcal{T}p \vee \Box\mathcal{T}\neg p$ is not a theorem of OFY.

We now move on to fragment semantics for OFY. The following is an analogue of Lemma 5.8.6.

Lemma 5.8.11. *For any linear fragment frame* $(S, \triangle, R, \tau)$*,* $\mathcal{F}_{\triangle}(S)$ *is a* OFY *lattice.*

Proof. Define the operation $\mathcal{Y} : \mathscr{P}(S) \to \mathscr{P}(S)$ by letting $\mathcal{Y}A = \{s \in S \mid \tau^{-1}(s) \in A\}$. It is easy to verify that the condition on linear fragment frames in Definition 5.6.1 implies that $\mathcal{Y}$ maps $\triangle$-fixpoints to $\triangle$-fixpoints. By definition of $\mathcal{Y}$, it is then straightforward to see that $\mathcal{Y}\mathcal{T}A = A = \mathcal{T}\mathcal{Y}A$ for any $A \in \mathcal{F}_{\triangle}(S)$ and thus that $\mathcal{F}_{\triangle}(S)$ is a OFY lattice. □

This lemma allows us to give a soundness and completeness proof which closely resembles the proof of Theorem 5.8.7.

Theorem 5.8.12. *The logic* OFY *is sound and complete with respect to linear fragment frames.*

Proof. Soundness follows directly from Lemma 5.8.11. For completeness, we consider the frame defined from the Lindenbaum-Tarski algebra $\mathcal{L}_{\mathsf{OFY}}$ of OFY in the same way as the frame defined in the proof of Theorem 5.8.7. Since $\mathcal{Y}$ has a left adjoint, it preserves meets, meaning that for any proper filter F on $\mathcal{L}_{\mathsf{OFY}}$, we have that $\upsilon(F) = \{a \in \mathcal{L}_{\mathsf{OFY}} \mid \mathcal{Y}a \in F\}$ is a proper filter such that $\tau(\upsilon(F)) = \upsilon(\tau(F)) = F$. This shows that τ is invertible. Moreover, suppose that $\upsilon(F)\triangle G$, and let $H = \{\neg\mathcal{Y}\neg a \mid a \in G\}$. I claim that H is a filter. To see this, observe that, since $\mathcal{Y}$ has a right adjoint, it preserves all joins. But this implies at once that $\neg\mathcal{Y}\neg$ preserves all meets, which is enough to show that H is a filter. Now if $b \in H$, then $b = \neg\mathcal{Y}\neg a$ for some $a \in G$. But then $\neg a \notin \upsilon(F)$, which means that $\neg b = \mathcal{Y}\neg a \notin F$. Hence $H\triangle F$. Moreover, suppose that $K\triangle H$. I claim that $G\triangle\upsilon(H)$. To see this, assume $a \in G$. Then $\neg\mathcal{Y}\neg a \in H$, so $\mathcal{Y}\neg a \notin K$. But this means that $\neg a \notin \upsilon(H)$.

This shows that the condition from Definition 5.6.1 is satisfied and thus that the fragment frame of filters on $\mathcal{L}_{\mathsf{OFY}}$ is a B-OF frame. Finally, one easily checks that $\mathcal{Y}\widehat{a} = \widehat{\mathcal{Y}a}$ for any $a \in \mathcal{L}_{\mathsf{OFY}}$, which completes the completeness proof. □

Part II

Infinity and Semiconstructive Mathematics

Chapter 6

Possibility Semantics for First-Order Logic

The second part of this dissertation will focus on a different aspect of the research program in possibility semantics. While in the first part we were concerned with possibility semantics for various kinds of non-classical propositional logics, from now on we will be exploring some applications of possibility semantics for classical first-order logic. Moreover, while the applications of possibility semantics we explored in the first part were mostly technical, here we will be mostly concerned with philosophical and foundational applications.

Possibility semantics for first-order languages is arguably less developed than its propositional counterpart. The seminal work on the topic is a manuscript by van Benthem [23], and a detailed presentation can be found in [135]. From a mathematical perspective, it shares a lot with the technique of forcing in set theory, especially the correspondence between forcing extensions and Boolean-valued models [10], and with a specific kind of sheaf semantics in topos theory [181]. These connections will be explored in more detail in Sections 7.4 and 7.5 below. From a philosophical perspective, possibility semantics can be seen as a generalization of Kripke semantics for first-order modal logic with a constant domain. In Kripke semantics, models are collections of possible worlds, which can be identified with maximal sets of consistent formulas. By contrast, the points in a possibility structure are "partial" worlds, i.e., they correspond to consistent sets of formulas that may contain neither a formula nor its negation. Points in a possibility structure are naturally ordered by how informative they are, where a point is more informative than another if it satisfies more formulas. In that respect, possibility semantics can also be seen as a variation on Kripke semantics for first-order intuitionistic logic, again with constant domains. The crucial difference however lies in the semantic clauses for disjunction: in possibility semantics, a disjunction may be satisfied at a point p without any of the disjuncts being satisfied at p. This is what allows the law of excluded middle to always be true at a partial world, even when that world is not maximally determined.

An attractive feature of first-order possibility semantics is the existence of a fully constructive completeness theorem for first-order logic, while the usual completeness theorem for first-order logic with respect to Tarskian semantics is known to require the Axiom of

Choice (see [11], p. 140). This makes possibility semantics a very natural option for the development of model-theoretic techniques in a semi-constructive setting. In fact, the common thread among the next three chapters will be the notion of a *generic power*. Generic powers are a special kind of possibility structures which are meant to be an analogue of ultrapowers in classical model theory. Because non-trivial ultrapowers require the existence of non-principal ultrafilters, their use is highly non-constructive. In the next three chapters, we will see how generic powers can be used to avoid this problem in a constructive and semi-constructive setting. More generally, this will also give us a good opportunity to test the potential of possibility semantics for reproducing important pieces of classical mathematics in a semi-constructive setting.

The first chapter will be concerned with the foundations of nonstandard analysis. I will show that one can provide a rigorous and elegant foundation for nonstandard analysis *à la* Robinson in a semi-constructive setting, and I will argue that this approach solves a number of issues regarding the foundational status of the hyperreal line and whether the methods of nonstandard analysis can legitimately be applied to ordinary mathematics.

In the second chapter, we will be interested in a recent debate in the philosophy of the infinite regarding an alternative to the Cantorian notion of size known as the theory of numerosities, and a related issue in the philosophy of probability theory having to do with the recent development of Non-Archimedean Probability theory. Both proposals are heavily influenced by nonstandard analysis, and rely heavily on ultrapowers. I will argue that an alternative approach to numeorisities and Non-Archimedean Probability theory based on possibility structures and generic powers is well-equipped to address the most serious objections that have been raised against both theories, and that possibility structures are uniquely suited to model the *Euclidean* infinite, an alternative to the Cantorian infinite that preserves Euclid's common notion that the whole is always strictly greater than any of its proper parts.

Finally, the last chapter will also be concerned with a non-Cantorian conception of the infinite, but our focus will be more historical. I will argue that possibility structures can be a powerful tool in providing a rich and coherent formal reconstruction of ninteenth-century mathematician Bernard Bolzano's views on infinite sums and infinitely large quantities.

In the rest of this introductory chapter, I will present the basics of possibility semantics for classical first-order logic, before introducing generic powers and proving three results about those that will play a crucial role throught the next three chapters.

6.1 Possibility Structures

In what follows, I will introduce some basics about possibility semantics for first-order logic, starting from the definition of satisfaction in a possibility structure. Let me mention first that I will define first-order possibility structures in a slightly different fashion than in [135], by introducing a minor alteration to the way Holliday defines the interpretation of functions symbols. The motivation for this choice is purely technical. As Holliday remarks in [135, fn. 20], this modification is of no real significance for the constructive completeness theorem obtained in [135, Theorem 4.3.8].

6.1.1 Forcing Semantics

Definition 6.1.1 (Possibility structure)**.** Let $\mathscr{L}$ be a first-order language. A possibility structure is a tuple $(\mathfrak{P}, D, \mathscr{I})$ such that:

- $\mathfrak{P} = (P, \leq)$ is a partially ordered set (*poset* for short) of *viewpoints*;
- D is a set of *guises*;
- $\mathscr{I}$ is a function mapping any $p \in P$ and any n-ary relation symbol R (including the equality symbol $=$) in $\mathscr{L}$ to a subset of D^n, and any n-ary function symbol f in $\mathscr{L}$ to a function from D^n into D, so that for any $p, q \in P$, any n-ary relation R in $\mathscr{L}$, any n-ary function symbol f and any n-tuple $\overline{a}$, the following conditions hold:

  Persistence If $\overline{a} \in \mathscr{I}(p, R)$ and $q \leq p$, then $\overline{a} \in \mathscr{I}(q, R)$;

  Refinability If $\overline{a} \notin \mathscr{I}(p, R)$, then there is $q \leq p$ such that for all $r \leq q$, $\overline{a} \notin (r, R)$;

  Equality-as-equivalence $\mathscr{I}(p, =)$ is an equivalence relation on $D \times D$;

  Equality-as-congruence if $\overline{a} = (a_1, ..., a_n)$ and $\overline{b} = (b_1, ..., b_n)$ is an n-tuple such that $(a_i, b_i) \in \mathscr{I}(p, =)$ for all $i \leq n$, then $(\mathscr{I}(f)(\overline{a}), \mathscr{I}(f)(\overline{b})) \in \mathscr{I}(p, =)$, and $\overline{a} \in \mathscr{I}(p, R)$ iff $\overline{b} \in \mathscr{I}(p, R)$.

Intuitively, any point in a possibility structure provides us with a "partial viewpoint" on how the model actually looks. In particular, the "guises" in the domain D are not objects themselves, but merely distinct ways of presenting objects. Two different guises may actually correspond to one and the same object from one viewpoint, hence the need for the relation symbol for equality to be interpreted as an equivalence relation, rather than as strict equality. As we move from less informative viewpoints to more informative ones, more guises are identified with one another, and more information is gained regarding the relations that hold between the objects that the guises designate. The persistence condition encapsulates the idea that the partial order on the poset does indeed capture the increase of information between viewpoints: no information is lost when we move from a viewpoint to a stronger viewpoint.[1] The refinability condition, by contrast, ensures that our information states,

[1]For a reader unfamiliar with forcing, it might seem counterintuitive that a "smaller" viewpoint is also a "stronger" one. But if one take a viewpoint p to stand in for "all the ways compatible with p in which the model could actually be" and the relation $\leq$ to indicate containment, then it is straightforward to see that $p \leq q$ precisely when "all the ways compatible with p in which the model could actually be" are also "ways compatible with q in which the model could actually be", thus meaning that p imposes stronger conditions than q on what the model could actually be. It is worth mentioning that nothing really hinges upon this choice of defining "less than" as "stronger than", rather than "weaker than", although, as it will become more apparent later on, doing so underscores the tight connection between possibility semantics, forcing, and sheaf semantics.

while partial, are as informative as they could be. More precisely, it is the contrapositive of the following "sure thing" principle: if no further refinement of our current viewpoint could make sure that the tuple $\overline{a}$ does not stand in relation R, then we might as well conclude already that the tuple $\overline{a}$ does stand in relation R. This refinability condition is what sets apart possibility semantics from Kripke semantics for first-order intuitionistic logic. It also appears in the inductive definition of satisfaction of a formula, to which I now turn.

Definition 6.1.2 (Forcing relation)**.** Let $\mathfrak{M} = (\mathfrak{P}, D, \mathscr{I})$ be a $\mathscr{L}$-possibility structure for a first-order language $\mathscr{L}$. The forcing relation $\Vdash$ is inductively defined for any $p \in P$, any $\mathscr{L}$-formula $\varphi(\overline{x})$ with n free variables, and any n-tuple $\overline{a}$ of elements of D as follows:

- If $\varphi := R(t_1(\overline{x_1}), ..., t_j(\overline{x_j}))$, where $t_1, ..., t_j$ are $\mathscr{L}$-terms of arity $n_1, ..., n_j$ summing up to n and R is a j-ary relation symbol, then

  $$p \Vdash R(t_1(\overline{a_1}), ..., t_j(\overline{a_j})) \text{ iff } (\mathscr{I}(t_1)(\overline{a_1}), ..., \mathscr{I}(t_j)(\overline{a_j})) \in \mathscr{I}(p, R),$$

  where $\overline{a} = \overline{a_1}...\overline{a_j}$, $\overline{a_i}$ is an n_i-ary tuple for any $i \leq j$, and the interpretation of an $\mathscr{L}$-term is inductively defined from the interpretation of function symbols as usual;
- If $\varphi := \neg\psi$, then $p \Vdash \varphi$ iff for all $q \leq p$, $q \nVdash \psi$;
- If $\varphi := \psi \wedge \chi$, then $p \Vdash \varphi$ iff $p \Vdash \psi$ and $p \Vdash \chi$;
- If $\varphi := \psi \vee \chi$, then $p \Vdash \varphi$ iff for all $q \leq p$ there is $r \leq q$ such that $r \Vdash \psi$ or $r \Vdash \chi$;
- If $\varphi := \psi \to \chi$, then $p \Vdash \varphi$ iff for all $q \leq p$, $q \Vdash \psi$ implies $q \Vdash \chi$;
- If $\varphi := \forall x \psi$, then $p \Vdash \varphi$ iff $p \Vdash \psi(a)$ for every $a \in D$;
- If $\varphi := \exists x \psi$, then $p \Vdash \varphi$ iff for all $q \leq p$ there is $r \leq q$ such that $r \Vdash \varphi(a)$ for some $a \in D$.

Given a $\mathscr{L}$-formula φ and an n-tuple $\overline{a}$, $\varphi(\overline{a})$ is *valid* in $\mathfrak{M}$ (denoted $\mathfrak{M} \models \varphi(\overline{a})$) if $p \Vdash \varphi(\overline{a})$ for all $p \in P$.

The forcing clauses introduced above are of course reminiscent of both Kripke semantics and the forcing relation in set theory. The refinability condition appears in the clauses for disjunctions and existentials, which can be straightforwardly derived from the clauses for negations, conjunctions and universals and De Morgan's laws. From an algebraic perspective, they also ensure that every $\mathscr{L}$-sentence is given a value in a Boolean algebra of subsets of the poset P. More precisely, we may introduce the following notation:

Notation 6.1.3. Given an $\mathscr{L}$-formula $\varphi(\overline{x})$ with n free variables and a n-tuple $\overline{a}$, the *r-value* of $\varphi(\overline{a})$ is the set $[\![\varphi(\overline{a})]\!] = \{p \in P \mid p \Vdash \varphi(\overline{a})\}$.

Then a simple induction on the complexity of formulas shows that $[\![\varphi(\overline{a})]\!]$ is always a *regular-open* subset of P, i.e., that for any $p \in P$:

$$p \in [\![\varphi(\overline{a})]\!] \Leftrightarrow \forall q \leq p \, \exists r \leq q : r \in [\![\varphi(\overline{a})]\!].$$

Since the regular open sets of any poset always form a Boolean algebra [41, 106], one may therefore think of a possibility structure as a generalized Tarskian model in which sentences are Boolean-valued rather than 2-valued (the relationship with Boolean-valued models will be discussed in more detail in Section 7.5). In fact, Tarskian structures are precisely those possibility structures $\mathfrak{M} = (\mathfrak{P}, D, \mathscr{I})$ in which $\mathfrak{P}$ is a single element poset, and the equality symbol is interpreted as the identity relation. Moreover, first-order logic is still sound with respect to this larger class of models, but the proof that it is also complete can now be carried out even in the absence of the Axiom of Choice:

Theorem 6.1.4 ([135], ZF)**.** *For any first-order language $\mathscr{L}$, first-order logic is sound and complete with respect to the class of $\mathscr{L}$-possibility structures.*

6.1.2 Possibility Embeddings

Let us now introduce some basic results about embeddings between possibility structures. My goal here is not to develop a full-fledged analogue of basic model-theoretic tools for possibility structures (see [23] for some results in this direction) but merely to prove some lemmas that will come in handy in the next chapters when exploring the properties of several possibility structures. I start by recalling the following definition.

Definition 6.1.5. Let $\mathfrak{P} = (P, \leq_{\mathfrak{P}})$ and $\mathfrak{Q} = (Q, \leq_{\mathfrak{Q}})$ be posets. An order-preserving map $\pi : \mathfrak{P} \to \mathfrak{Q}$ is *weakly dense* if for any $p \in P$ and any $q \in Q$ such that $q \leq_{\mathfrak{Q}} \pi(p)$, there is $p' \leq_{\mathfrak{P}} p \in \mathcal{P}$ such that $\pi(p') \leq_{\mathfrak{Q}} q$. The map π is *dense* if for any $q \in \mathfrak{Q}$ there is $p \in \mathfrak{P}$ such that $\pi(p) \leq_{\mathfrak{Q}} q$.

Weakly dense maps are the correct notion of r-value-preserving maps, since any weakly dense map $\pi : \mathfrak{P} \to \mathfrak{Q}$ induces a complete Boolean-homomorphism $\pi_* : \mathsf{RO}(\mathfrak{Q}) \to \mathsf{RO}(\mathfrak{P})$ given by the inverse image function. If π is also dense, then π_* will be injective. Possibility structures differ from Tarskian structures in having an order-theoretic structure on top of a domain of individuals. Consequently, a natural way to adapt the notion of embedding between Tarskian structures to the setting of possibility semantics is to consider a pair of maps where the first map is a weakly dense map between the underlying posets and the second one is a function between the underlying domains. Perhaps surprisingly, the most fruitful notion is actually one in which the domains and codomains of the two maps are "crossed".

Definition 6.1.6. Let $\mathcal{P} = (\mathfrak{P}, D, \mathscr{I})$ and $\mathcal{Q} = (\mathfrak{Q}, E, \mathscr{J})$ be two possibility structures in the same language $\mathscr{L}$. A *possibility embedding* (p.e. for short) is a pair (π, α) such that $\pi : \mathfrak{Q} \to \mathfrak{P}$ is weakly dense, and $\alpha : D \to E$ has the following properties for any $q \in \mathfrak{Q}$, $\overline{a} \in D$, function symbol $f \in \mathscr{L}$ and relation symbol $R \in \mathscr{L}$:

- $\mathcal{Q} \models f(\overline{\alpha(a)}) = \alpha(\mathscr{I}(f, \overline{a}))$;
- $\pi(q) \Vdash R(\overline{a}) \Leftrightarrow q \Vdash R(\overline{\alpha(a)})$.

A p.e. (π, α) is *dense* if π is a dense map and *elementary* (e.p.e. for short) if $\pi(q) \Vdash \varphi(\overline{a})$ iff $q \Vdash \varphi(\overline{\alpha(a)})$ for every $q \in \mathfrak{Q}$, $\overline{a} \in D$ and $\mathscr{L}$-formula $\varphi(\overline{x})$.

Just like in classical model theory, elementary possibility embeddings preserve all the first-order properties of their domain and can be characterized by a criterion for existential formulas, as shown by the Density Lemma.

Lemma 6.1.7 (Density Lemma). *Let $\mathcal{M} = (\mathfrak{P}, D, \mathscr{I})$ and $\mathcal{N} = (\mathfrak{Q}, E, \mathscr{J})$ be two possibility structures in the same signature $\mathscr{L}$, and assume that $(\pi, \alpha) : \mathcal{M} \to \mathcal{N}$ is a p.e. Then:*

1. *(π, α) is elementary iff it satisfies the following "Tarski-Vaught" criterion:*

   (TV) *For any $q \in \mathfrak{Q}$, $\overline{a} \in D$ and $\varphi(\overline{x}, y) \in \mathscr{L}$, if $q \Vdash \varphi(\overline{\alpha(a)}, c)$ for some $c \in E$, then there is $p \leq_{\mathfrak{P}} \pi(q)$ and $b \in D$ such that $p \Vdash \varphi(\overline{\alpha(a)}, \alpha(b))$.*

2. *If (π, α) is dense and elementary, then $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(\overline{\alpha(a)})$ for any $\mathscr{L}$-formula $\varphi(\overline{x})$ and any $\overline{a} \in D$.*

Proof. Let us start by proving item 1. Suppose first that (π, α) is elementary, and assume that $q \Vdash \varphi(\overline{\alpha(a)}, c)$ for some $c \in E$. Then $\pi(q) \Vdash \exists y \varphi(\overline{a}, y)$. Since (π, α) is elementary, this means that $p \Vdash \exists y \varphi(\overline{a}, y)$. Hence there is $p \leq_{\mathfrak{P}} \pi(q)$ and $b \in D$ such that $p \Vdash \varphi(\overline{a}, b)$. By elementarity of (π, α) again, it follows that $\pi(p) \Vdash \varphi(\overline{\alpha(a)}, \alpha(b))$.

Conversely, suppose now that (π, α) satisfies **(TV)**. Note first that the statement that (π, α) is elementary is equivalent to the statement that $\pi_*([\![\varphi(\overline{a})]\!]_{\mathfrak{P}}) = [\![\varphi(\overline{\alpha(a)})]\!]_{\mathfrak{Q}}$ for any formula $\varphi(\overline{x})$ and any tuple $\overline{a} \in D$. We show the latter by induction on the complexity of the formula $\varphi(\overline{x})$. By assumption, we have that $\pi_*([\![\varphi(\overline{a})]\!]_{\mathfrak{P}}) = [\![\varphi(\overline{\alpha(a)})]\!]_{\mathfrak{Q}}$ for any atomic formula $\varphi(\overline{x})$ and any $\overline{a} \in D$. Moreover, the Boolean cases of the inductive step immediately follow from the fact that π_* is a complete Boolean homomorphism, which is true since π is weakly dense. Hence we only need to verify the existential step. I claim that we have the following chain of identities:

$$\begin{aligned}
\pi_*([\![\exists x \varphi(\overline{a}, x)]\!]_{\mathfrak{P}}) &= \pi_*(\bigvee_{b \in D} [\![\varphi(\overline{a}, b)]\!]_{\mathfrak{P}}) \\
&= \bigvee_{b \in D} \pi_*([\![\varphi(\overline{a}, b)]\!]_{\mathfrak{P}}) \\
&= \bigvee_{b \in D} [\![\varphi(\overline{\alpha(a)}, \alpha(b))]\!]_{\mathfrak{Q}} \\
&= \bigvee_{c \in E} [\![\varphi(\overline{\alpha(a)}, c)]\!]_{\mathfrak{Q}} \\
&= [\![\exists x \varphi(\overline{\alpha(a)}, x)]\!]_{\mathfrak{Q}}.
\end{aligned}$$

The first and last identities hold by the semantic clauses of possibility semantics, while the second and third hold because π_* is a complete Boolean homomorphism and by induction hypothesis respectively. Hence we only need to verify the fourth identity. The left-to-right inclusion is trivial, and the right-to-left inclusion follows in a straightforward way from **(TV)**. This completes the proof of item 1.

For item 2, suppose that (π, α) is elementary. Since π is dense, π_* is an injective Boolean homomorphism and hence $\pi_*(U) = \mathfrak{Q}$ iff $U = \mathfrak{P}$ for any $U \in \mathsf{RO}(\mathfrak{P})$. But this means that we have the following chain of equivalences for any $\mathscr{L}$-formula $\varphi(\overline{x})$ and $\overline{a} \in D$:

$$\begin{aligned}\mathcal{M} \models \varphi(\overline{a}) &\Leftrightarrow [\![\varphi(\overline{a})]\!]_{\mathfrak{P}} = \mathfrak{P} \\ &\Leftrightarrow \pi_*([\![\varphi(\overline{a})]\!]_{\mathfrak{P}}) = \pi_*(\mathfrak{P}) \\ &\Leftrightarrow [\![\varphi(\overline{\alpha(a)})]\!]_{\mathfrak{Q}} = \mathfrak{Q} \\ &\Leftrightarrow \mathcal{N} \models \varphi(\overline{\alpha(a)}).\end{aligned}$$

This completes the proof. □

6.1.3 The Colimit Construction

Let us now briefly see how one can use possibility embeddings to define the colimit of a directed system of possibility structures. This construction will be particularly useful in Chapter 9. Recall first that a monotone map $f : \mathfrak{P} \to \mathfrak{Q}$ between posets is a *p-morphism* if whenever $f(p) \leq_{\mathfrak{Q}} q$, there is $p' \leq_{\mathfrak{P}} p$ such that $f(p') = q$.

Definition 6.1.8. Let $\mathcal{I} = (I, \leq_{\mathsf{I}})$ be a directed poset. A *tight inverse system* of posets over I is a family $(\{\mathfrak{P}_i\}_{i \in \mathsf{I}}, \{\pi_{ij}\}_{i \leq_{\mathsf{I}} j})$ with the following properties:

- For any $i \in \mathsf{I}$, $\mathfrak{P}_i$ is a poset;
- For any $i \leq_{\mathsf{I}} j$, $\pi_{ij} : \mathfrak{P}_i \to \mathfrak{P}_j$ is a p-morphism;
- π_{ii} is the identity map for any $i \in \mathsf{I}$, and for any $i \leq_{\mathsf{I}} j \leq_{\mathsf{I}} k$, $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$;
- Whenever $\{i_\beta\}_{\beta<\lambda}$ is an increasing chain of elements of I for some ordinal λ and $\{p_\beta\}_{\beta<\lambda}$ is a sequence such that $p_\beta \in \mathfrak{P}_{i_\beta}$ for any $\beta < \lambda$ and $\pi_{i_\beta i_\gamma}(p_\gamma) = \pi_\beta$ for any $\beta < \gamma < \lambda$, then for any $q \in \mathfrak{P}_\lambda$ such that $p_\beta \leq_{\mathfrak{P}_{i_\beta}} \pi_{i_\beta i_\lambda}(q)$ for any $\beta < \lambda$, there is $p_\lambda \leq_{\mathfrak{P}_{i_\lambda}} q$ such that $\pi_{i_\beta i_\lambda}(p_\lambda) = \pi_\beta$ for any $\beta < \lambda$.

The first three conditions in the definition of a tight inverse system are not surprising, although we do need to strengthen the notion of a weakly-dense map to that of a p-morphism for our purposes here. The last condition simply makes sure that the system of p-morphisms is well behaved at "limit stages". Of course, it is trivially satisfied whenever I has no infinite bounded chains. Let us now introduce the notion of a directed system of possibility structures.

Definition 6.1.9. Fix a language $\mathscr{L}$. Let I be a directed poset. A *directed system* of possibility embeddings over I is a tuple $(\{\mathcal{P}_i\}_{i\in\mathsf{I}}, \{\epsilon_{ij}\}_{i\leq_{\mathsf{I}} j})$ satisfying the following conditions:

- For any $i \in \mathsf{I}$, $\mathcal{P}_i = (\mathfrak{P}_i, D_i, \mathscr{I}_i)$ is a $\mathscr{L}$-possibility structure;
- For any $i, j \in \mathsf{I}$ such that $i \leq_{\mathsf{I}} j$, $\epsilon_{ij} = (\pi_{ij}, \alpha_{ij}) : \mathcal{P}_i \to \mathcal{P}_j$ is a possibility embedding;
- $(\{\mathfrak{P}_i\}_{i\in\mathsf{I}}, \{\pi_{ij}\}_{i\leq_{\mathsf{I}} j})$ is a tight inverse system over I;
- For any $i \in \mathsf{I}$, α_{ii} is the identity map, and for any $i, j, k \in \mathsf{I}$ such that $i \leq_{\mathsf{I}} j \leq_{\mathsf{I}} k$, $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$.

A directed system $(\{\mathcal{P}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i\leq_{\mathsf{I}} j})$ is *elementary* if ϵ_{ij} is elementary for any $i \leq_{\mathsf{I}} j$.

Given a directed system of possibility embeddings over an index set I, our goal is to define an analogue of the direct limit or colimit of a directed system of Tarskian structures. Accordingly, the domain of such a structure will be similar to what one would expect, namely a disjoint union of all the domains of the possibility structures in the directed system. However, the dense map components of the possibility embeddings in a directed system form an inverse system, meaning that their limit should be an inverse limit rather than a colimit. This motivates the following definition.

Definition 6.1.10. Let I be a directed poset and $(\{\mathcal{P}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i\leq_{\mathsf{I}} j})$ a directed system over I. The *colimit* of the directed system $(\{\mathcal{P}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i\leq_{\mathsf{I}} j})$ is the possibility structure $\overrightarrow{\mathcal{P}_{\mathsf{I}}} = (\bigotimes_{\mathsf{I}} \mathfrak{P}_i, \bigoplus_{\mathsf{I}} D_i, \mathscr{I})$, where:

- $\bigotimes_{\mathsf{I}} \mathfrak{P}_i$ is the poset of all functions f from I into the disjoint union of the posets $\mathfrak{P}_i$ for $i \in \mathsf{I}$ such that:
  - $f(i) \in \mathfrak{P}_i$ for any $i \in \mathsf{I}$, and
  - $\epsilon_{ij}(f(j)) = f(i)$ whenever $i \leq_{\mathsf{I}} j$,

  with the order defined pointwise, i.e., $f \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} g$ iff $f(i) \leq_{\mathfrak{P}_i} g(i)$ for all $i \in \mathsf{I}$;
- $\bigoplus_{\mathsf{I}} D_i$ is the disjoint union of the domains D_i for $i \in \mathsf{I}$;
- For any function symbol $f(x_1, ..., x_k)$ and any $a_1, ..., a_k$ such that $a_n \in D_{i_n}$ for all $n \in \{1, ..., k\}$, there is $j \geq_{\mathsf{I}} i_1, ..., i_k$ such that $\mathscr{I}(f, a_1, ..., a_k) = \mathscr{I}_j(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$;
- For any relation symbol $R(x_1, ..., x_k)$, any $a_1, ..., a_k$ with $a_n \in D_{i_n}$ for all $n \in \{1, ..., k\}$ and any $f \in \bigotimes_{\mathsf{I}} \mathfrak{P}_i$, $(a_1, ..., a_k) \in \mathscr{I}(f, R)$ iff there is $j \geq_{\mathsf{I}} i_1, ..., i_k$ such that $(\alpha_{i_i j}(a_1), \alpha_{i_k j}(a_k)) \in \mathscr{I}_j(f(j), R)$.

For convenience, I will assume in what follows that the domains D_i in a directed system are always disjoint, so that we may identity $\bigoplus_{\mathsf{I}} D_i$ with the union $\bigcup_{i\in\mathsf{I}} D_i$. Our goal is to show that the structure defined above is a proper notion of colimit for directed systems of possibility embeddings. This will be established by the Colimit Lemma below. Before proving this lemma, we need the following technical observation.

Lemma 6.1.11 ($DC_{|\mathsf{I}|}$)**.** *Let* I *be a directed poset and* $(\{\mathcal{P}_i\}i \in \mathsf{I}, \{\pi_{ij} : \mathfrak{P}_j \to \mathfrak{P}_i\}_{i \leq_{\mathsf{I}}})$ *a tight inverse system over* I*. Then for any* $i \in \mathcal{I}$*, any* $f \in \bigotimes_{\mathsf{I}} \mathfrak{P}_i$ *and any* $q \leq_{\mathfrak{P}_i} f(i)$*, there is* $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$ *such that* $g(i) = q$*.*

Proof. Fix f, i and q, and let $\{i_\beta\}_{\beta<\lambda}$ be a cofinal chain of elements of I with $i = i_0$ (note that such a chain can be constructed using $DC_{|\mathsf{I}|}$). We define inductively a sequence of elements $\{q_\beta\}_{\beta<\lambda}$ as follows:

- $q_0 = q$;
- Assuming that q_β is defined such that $q_\beta \leq_{\mathfrak{P}_{i_\beta}} p(\beta)$, note that $\pi_{i_\beta, i_{\beta+1}}(p(\beta+1)) = p(\beta)$. Since $\pi_{i_\beta, i_{\beta+1}}$ is a p-morphism, there is $q' \leq_{\mathfrak{P}_{i_{\beta+1}}} p(\beta+1)$ such that $\pi_{i_\beta, i_{\beta+1}}(q') = q_\beta$. So we set $q_{\beta+1} = q'$.
- For $\nu < \lambda$ a limit ordinal, we have a sequence $\{q_\beta\}_{\beta<\nu}$ such that $\pi_{i_\beta i_\gamma}(q_\gamma) = q_\beta$ whenever $\beta < \gamma < \nu$ and $q_\beta \leq_{\mathfrak{P}_{i_\beta}} \pi_{i_\beta i_\nu}(p(\nu)$ for any $\beta < \nu$. Since $(\{\mathcal{P}_i\}i \in \mathsf{I}, \{\pi_{ij} : \mathfrak{P}_j \to \mathfrak{P}_i\}_{i \leq_{\mathsf{I}}})$ is a tight inverse system, we have $q' \in \mathfrak{P}_{i_\nu}$ such that $q' \leq p(\nu)$ and $\pi_{i_\beta i_\nu}(q') = q_\beta$ for all $\beta < \nu$. So we set $q_\nu = q'$.

Note that the existence of such a sequence is guaranteed by $DC_{|\mathsf{I}|}$ again. Finally, we defined $g \in \bigotimes_{\mathsf{I}} \mathfrak{P}_i$ by letting $g(i) = q_\beta$ whenever $i = i_\beta$ for some $\beta < \lambda$, and $g(i) = \pi_{ii_\beta}(g(i_\beta))$ for some i_β such that $i \leq_{\mathsf{I}} i_\beta$ otherwise. It is routine to check that this is well-defined and that $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$. □

Lemma 6.1.12 (First Colimit Lemma)**.** *Let* I *be a directed poset and* $(\{\mathcal{P}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i \leq_{\mathsf{I}} j})$ *a directed system over* I*. Then:*

1. $\overrightarrow{\mathcal{P}_{\mathsf{I}}}$ *is a possibility structure;*
2. *There is a system of maps* $\{\epsilon_i := (\pi_i, \alpha_i)\}_{i\in\mathsf{I}}$ *such that:*
   - $\pi_i : \bigotimes_{\mathsf{I}} \mathfrak{P}_i \to \mathfrak{P}_i$ *is a p-morphism for any* $i \in \mathsf{I}$*, and* $\alpha_i : D_i \to \bigoplus_{\mathsf{I}} D_i$ *is a function,*
   - *whenever* $i \leq_{\mathsf{I}} j$*,* $\pi_i = \pi_{ij} \circ \pi_j$ *and* $\overrightarrow{\mathcal{P}_{\mathsf{I}}} \models \alpha_i(a) = \alpha_j \circ \alpha_{ij}(a)$ *for any* $a \in D_i$*, and*
   - $\epsilon_i : \mathcal{P}_i \to \overrightarrow{\mathcal{P}_{\mathsf{I}}}$ *is a possibility embedding;*
3. *If* $(\{\mathcal{P}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i \leq_{\mathsf{I}} j})$ *is elementary, then each* ϵ_i *is elementary, and for any* $f \in \bigotimes \mathfrak{P}_i$*, any formula* $\varphi(\overline{x})$ *and tuple* $a_1, ..., a_k \in \bigoplus_{\mathsf{I}} D_i$ *with* $a_n \in D_{i_n}$ *for all* $n \in \{1, ..., k\}$*,* $f \Vdash \varphi(a_1, ..., a_k)$ *iff there is* $j \geq_{\mathsf{I}} i_1, ..., i_k$ *such that* $f(j) \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$*.*

Proof. Fix a directed poset I and a directed system $(\{\mathcal{P}_i\}_{i\in I}, \{\epsilon_{ij}\}_{i \leq_{\mathsf{I}} j})$ over I.

1. We check all four conditions in turn.

Persistence Suppose that $g \leq f$ and $f \Vdash R(a_1, ...a_k)$, with $a_n \in D_{i_n}$ for all $n \in \{1, ..., k\}$. Then there is $j \in \mathsf{I}$ such that $f(j) \Vdash R(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. Since $g \leq f$, we have that $g(j) \leq_{\mathfrak{P}_j} f(j)$, and hence $g(j) \Vdash R(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. But this implies that $g \Vdash R(a_1, ..., a_k)$.

Refinability Suppose that $g \leq f$ and $f \nVdash R(a_1, ...a_k)$, with $a_n \in D_{i_n}$ for all $n \in \{1, ..., k\}$. Let $j \geq_{\mathsf{I}} i_1, ..., i_k$. Note that $f(j) \nVdash R(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. By persistence, there is $q \leq_{\mathfrak{P}_j} f(j)$ such that for all $r \leq_{\mathfrak{P}_j} q$, $r \nVdash R(\alpha_{i_1 j}(a_1), ...\alpha_{i_k j}(a_k))$. By Lemma 6.1.11, there is $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$ such that $g(j) = q$. Now I claim that for any $h \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} g$, $h \nVdash R(a_1, ..., a_k)$. Indeed, let $h \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} g$ and $j' \geq_{\mathsf{I}} i_1, ..., i_k$. Let j^* be such that $j, j' \leq_{\mathsf{I}} j^*$. Since $h(j) \leq_{\mathfrak{P}_j} g(j)$, we have the following:

$$\begin{aligned}
h(j) \nVdash R(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k)) &\Leftrightarrow \pi_{jj^*}(h(j^*)) \nVdash R(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k)) \\
&\Leftrightarrow h(j^*) \nVdash R(\alpha_{jj^*} \circ \alpha_{i_1 j}(a_1), ..., \alpha_{jj^*} \circ \alpha_{i_k j}(a_k)) \\
&\Leftrightarrow h(j^*) \nVdash R(\alpha_{i_1 j^*}(a_1), ..., \alpha_{i_k j^*}(a_k)) \\
&\Leftrightarrow h(j^*) \nVdash R(\alpha_{j' j^*} \circ \alpha_{i_1 j'}(a_1), ..., \alpha_{j' j^*} \circ \alpha_{i_k j'}(a_k)) \\
&\Leftrightarrow \pi_{j' j^*}(h(j^*)) \nVdash R(\alpha_{i_1 j'}(a_1), ..., \alpha_{i_k j'}(a_k)) \\
&\Leftrightarrow h(j') \nVdash R(\alpha_{i_1 j'}(a_1), ..., \alpha_{i_k j'}(a_k)).
\end{aligned}$$

This completes the proof of refinability.

Equality-as-equivalence Reflexivity and symmetry are clear, so we only check transitivity. Suppose $f(j) \Vdash \alpha_{i_1 j}(a_1) = \alpha_{i_2 j}(a_2)$ and $f(j') \Vdash \alpha_{i_2 j'}(a_2) = \alpha_{i_3 j'}(a_3)$. Let $j^* \geq_{\mathsf{I}} j, j'$. Then we have that $\pi_{jj^*}(f(j^*)) = f(j) \Vdash \alpha_{i_1 j}(a_1) = \alpha_{i_2 j}(a_2)$, so $f(j^*) \Vdash \alpha_{jj^*} \circ \alpha_{i_1 j}(a_1) = \alpha_{jj^*} \circ \alpha_{i_2 j}(a_2)$, and $\pi_{j' j^*}(f(j^*)) = f(j') \Vdash \alpha_{i_2 j'}(a_2) = \alpha_{i_3 j'}(a_3)$, so $f(j^*) \Vdash \alpha_{j' j^*} \circ \alpha_{i_2 j'}(a_2) = \alpha_{j' j^*} \circ \alpha_{i_3 j'}(a_3)$. Hence $f(j^*) \Vdash \alpha_{i_1 j^*}(a_1) = \alpha_{i_2 j^*}(a_2) = \alpha_{i_3 j^*}(a_3)$, from which it follows that $f(j^*) \Vdash \alpha_{i_1 j^*}(a_1) = \alpha_{i_3 j^*}(a_3)$. Hence $f \Vdash a_1 = a_3$.

Equality-as-congruence The proof is similar to the proof of **Equality-as-equivalence** above. Fix a k-ary relation symbol R and a k-ary function symbol g. Given tuples $\overline{a_1, ...a_k}$ and $\overline{b_1, ...b_k}$, fix j large enough so that $f(j) \Vdash \alpha_{i_n j}(a_n) = \alpha_{i_n j}(b_n)$ for all $n \in \{1, ..., k\}$. Using directedness again if necessary, it is then easy to find j' such that $f(j') \Vdash R(\alpha_{i_1 j'}(b_1), ..., \alpha_{i_k j'}(b_k))$ and j^* such that $f(j^*) \Vdash g(\alpha_{i_1 j'}(a_1), ..., \alpha_{i_k j'}(a_k)) = g(\alpha_{i_1 j'}(b_1), ..., \alpha_{i_k j'}(b_k))$.

2. For any $i \in \mathsf{I}$, let $\pi_i : \bigotimes_{\mathsf{I}} \mathfrak{P}_i \to \mathfrak{P}_i$ and $\alpha_i : D_i \to \bigoplus_{\mathsf{I}} D_i$ be the maps $f \mapsto f(i)$ and $a \mapsto a$. It is clear that, whenever $i \leq_{\mathsf{I}} j$, we have that $\pi_i = \pi_{ij} \circ \pi_j$ and that $\overrightarrow{\mathcal{P}_{\mathsf{I}}} \models \alpha_i(a) = \alpha_j \circ \alpha_{ij}(a)$ for any $a \in D_i$, since for any $f \in \bigotimes_{\mathsf{I}} \mathfrak{P}_i$, $f(j) \Vdash \alpha_{ij}(a) = \alpha_{ij}(a)$. Moreover, each π_i is clearly order-preserving, and if $q \leq_{i \in \mathsf{I}} \pi_i f$, then by Lemma 6.1.11 there is $g \leq f$ such that $\pi_i(g) = g(i) = q$, which shows that each π_i is a p-morphism.

   Now let us show that for any $i \in \mathsf{I}$ tuple $\overline{a} \in D_i$ and relation symbol R, $\pi_i(f) \Vdash R(\overline{a})$ iff $f \Vdash R(\alpha_i(\overline{a}))$. The left-to-right direction is obvious. For the right-to-left direction, suppose that $f \Vdash R(\alpha_i(\overline{a}))$. Then there is $j \geq_{\mathsf{I}} i$ such that $f(j) \Vdash R(\alpha_{ij}(\overline{a})$. But then $f(i) = \pi_{ij}(f(j) \Vdash R(\overline{a})$. Finally, let us check that $\overrightarrow{\mathcal{P}_{\mathsf{I}}} \models h(\overline{\alpha_i(a)}) = \alpha_i(\mathscr{I}_i(h, \overline{a}))$ for any $\overline{a} \in D_i$ and any function symbol h. Note that $\mathscr{I}(h, \overline{a}) = \mathscr{I}_j(h, \overline{\alpha_i j(a)})$ for some $j \geq_{\mathsf{I}} i$. But then for any $f \in \bigotimes_{\mathsf{I}} \mathfrak{P}_i$, $f(j) \Vdash h(\overline{\alpha_{ij}(a)}) = \alpha_{ij}(\mathscr{I}_i(h, \overline{a}))$. Hence $f \Vdash h(\overline{a}) = \mathscr{I}_i(h, \overline{a})$. This shows that ϵ_i is a possibility embedding for every $i \in \mathsf{I}$.

3. Assume that $(\{\mathcal{P}_i\}_{i \in I}, \{\epsilon_{ij}\}_{i \leq_{\mathsf{I}} j})$ is elementary. Let us first show that for any $f \in \bigotimes \mathfrak{P}_i$, any formula $\varphi(\overline{x})$ and tuple $a_1, ..., a_k \in \bigoplus_{\mathsf{I}} D_i$ with $a_n \in D_{i_n}$ for all $n \in \{1, ..., k\}$, $f \Vdash \varphi(a_1, ..., a_k)$ iff there is $j \geq_{\mathsf{I}} i_1, ..., i_k$ such that $f(j) \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. We prove this by induction on the complexity of φ. The atomic case is clear.

   - Suppose that $f \Vdash \neg\varphi(a_1, ..., a_k)$, and let $j \geq_{\mathsf{I}} i_1, ..., i_k$. Fix $q \leq_{\mathfrak{P}_j} f(j)$, and note that since π_j is a p-morphism, there is $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$ such that $g(j) = q$. By the induction hypothesis, this means that $q \nVdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$, for otherwise $g \Vdash \varphi(a_1, ..., a_k)$, contradicting our assumption on f. But then it follows that $f(j) \Vdash \neg\varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. Conversely, suppose that $f \nVdash \varphi(a_1, ...a_k)$. Then there is $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$ such that $g \Vdash \varphi(a_1, ..., a_k)$. Now suppose that $j \geq_{\mathsf{I}} i_1, ..., i_k$. By the induction hypothesis, there is $j' \in \mathsf{I}$ such that $g(j') \Vdash \varphi(\alpha_{i_1 j'}(a_1), ..., \alpha_{i_k j'}(a_k))$. Without loss of generality, assume that $j' \geq_{\mathsf{I}} j$. Then $g(j) \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. But this implies that $f(j) \nVdash \neg\varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$
   - Suppose that $f \Vdash \varphi(a_1, ..., a_k) \wedge \psi(a_1, ..., a_k)$. Then $f \Vdash \varphi(a_1, ..., a_k)$ and $f \Vdash \psi(a_1, ..., a_k)$. By the induction hypothesis, there are $j, j' \in \mathsf{I}$ such that $f(j) \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$ and $f(j') \Vdash \psi(\alpha_{i_1 j'}(a_1), ..., \alpha_{i_k j'}(a_k))$. Let $j^* \geq_{\mathsf{I}} j, j'$. Then $f(j^*) \Vdash \varphi(\alpha_{i_1 j^*}(a_1), ..., \alpha_{i_k j^*}(a_k))$ and $f(j^*) \Vdash \psi(\alpha_{i_1 j^*}(a_1), ..., \alpha_{i_k j^*}(a_k))$, so $f(j^*) \Vdash \varphi(\alpha_{i_1 j^*}(a_1), ..., \alpha_{i_k j^*}(a_k)) \wedge \psi(\alpha_{i_1 j^*}(a_1), ..., \alpha_{i_k j^*}(a_k))$. Conversely, suppose that there is $j \in \mathsf{I}$ such that $f(j) \Vdash \varphi \wedge \psi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$. Then $f(j) \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$ and $f(j) \Vdash \psi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$, so by the induction hypothesis $f \Vdash \varphi(a_1, ..., a_k)$ and $f \Vdash \psi(a_1, ..., a_k)$. Hence $f \vdash \varphi(a_1, ..., a_k) \wedge \psi(a_1, ..., a_k)$.
   - Suppose that $f \Vdash \exists x \varphi(a_1, ..., a_k, x)$. Let $j \geq_{\mathsf{I}} i_1, ..., i_k$, and let $q \leq_{\mathfrak{P}_j} f(j)$. Since π_j is a p-morphism, there is $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$ such that $g(j) = q$. Since $f \Vdash \exists x \varphi(a_1, ..., a_k, x)$, there is $g' \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} g$ and $c \in D_{i'}$ for some $i' \in \mathsf{I}$ such that $g' \Vdash \varphi(a_1, ..., a_k, c)$. By the induction hypothesis, there is $j' \geq_{\mathsf{I}} i_1, ..., i_k, i'$ such that $g'(j') \Vdash \varphi(\alpha_{i_1 j'}(a_1), ..., \alpha_{i_k j'}(a_k), \alpha_{i' j'}(c))$. Now let $j^* \geq_{\mathsf{I}} j, j'$, and note that $g'(j^*) \Vdash \varphi(\alpha_{i_1 j^*}(a_1), ..., \alpha_{i_k j^*}(a_k), \alpha_{i' j^*}(c))$ by elementarity of $\epsilon_{j' j^*}$. Moreover, since $\epsilon_{j j^*}$ is also elementary, by the Tarski-Vaught criterion there is $q' \leq_{\mathfrak{P}_j} g'(j) \leq_{\mathfrak{P}_j} q$ and $b \in D_j$ such that $q' \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k), b)$. But this means that $f(j) \Vdash \exists x \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k), x)$.

     For the converse, suppose that we have some $j \geq_{\mathsf{I}} i_1, ..., i_k$ such that $f(j) \Vdash \exists x \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k), x)$, and let $g \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} f$. Then there is $q \leq_{\mathfrak{P}_j} g(j)$ and $c \in D_j$ such that $q \Vdash \varphi(\alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k), c)$. Since π_j is a p-morphism, there is $g' \leq_{\bigotimes_{\mathsf{I}} \mathfrak{P}_i} g$ such that $g'(j) = q$. By the induction hypothesis, we have that $g' \Vdash \varphi(a_1, ..., a_k, c)$. But this shows that $f \Vdash \exists x \varphi(a_1, ..., a_k, x)$.

We may now prove that each $\epsilon_i : \mathcal{P}_i \to \overrightarrow{\mathcal{P}_\mathsf{I}}$ is elementary. By the Density Lemma, we only need to check the Tarski-Vaught criterion for ϵ_i. Suppose $f \Vdash \varphi(\overline{\alpha_i(a)}, c)$ for some $\overline{a} \in D_i$ and $c \in D_{i'}$ for some $i' \in \mathsf{I}$. By the result above, there $j \geq_\mathsf{I} i, i'$ such that $f(j) \Vdash \varphi(\overline{\alpha_{ij}(a)}, \alpha_{i'j}(c))$. Since ϵ_{ij} is elementary, by the Tarski-Vaught criterion there is $b \in D_i$ and $q \leq \pi_{ij}(f(j)) = f(i)$ such that $q \Vdash \varphi(\overline{a}, b)$. But this means that ϵ_i satisfies the Tarski-Vaught criterion and is therefore elementary. This completes the proof. □

Let us conclude this section with a small remark about the Colimit Lemma. In classical model theory, the embeddings from each model A_i into the colimit $\overrightarrow{A_\mathsf{I}}$ that one obtains by the standard colimit construction all commute with the embeddings $\{\alpha_{ij}\}_{i \leq_\mathsf{I} j}$ between the models in the system. By contrast, the embeddings one obtains in the colimit of a directed system of possibility structures only commute "internally" with the embeddings of the form α_{ij}, in the sense of item 2 in the Colimit Lemma. Of course, we could modify the definition of the domain of the colimit $\overrightarrow{\mathcal{P}_\mathsf{I}}$ to make the embeddings commute "externally", by taking equivalence classes over the disjoint union of the domains rather than taking the disjoint union itself. The point, however, is that taking equivalence classes is not needed in the case of possibility structures, since the equality is interpreted as a mere equivalence class. Indeed, colimits still enjoy the following universal mapping property.

Lemma 6.1.13 (Second Colimit Lemma)**.** *Let* I *be a directed poset and* $(\{\mathcal{P}_i\}_{i \in I}, \{\epsilon_{ij}\}_{i \leq_\mathsf{I} j})$ *a directed system over* I*. Suppose that there is a possibility structure* $\mathcal{Q}$ *and a system* $\{\eta_i = \sigma_i, \beta_i\}$ *such that each* $\eta_i : \mathcal{P}_i \to \mathcal{Q}$ *is a possibility embedding and for any* $i \leq_\mathsf{I} j$ *and any* $a \in D_i$*,* $\mathcal{Q} \models \beta_i(a) = \beta_j \circ \alpha_{ij}(a)$*. Then there is a possibility embedding* $\eta = (\sigma, \beta) : \overrightarrow{\mathcal{P}_\mathsf{I}} \to \mathcal{Q}$ *such that* $\eta \circ \epsilon_i = \eta_i$ *for all* $i \in \mathsf{I}$*. Moreover, if every* η_i *is elementary, then so is* η*.*

Proof. Define $\eta = (\sigma, \beta) : \overrightarrow{\mathcal{P}_\mathsf{I}} \to \mathcal{Q}$ by letting $\sigma(q)(i) = \sigma_i(q)$ for any $q \in \mathfrak{Q}$ and $i \in \mathsf{I}$, and $\eta(a) = \eta_i(a)$ for any $a \in D_i$. It is clear that $\sigma \circ \pi_i = \sigma_i$ and that $\beta \circ \alpha_i = \beta_i$, and hence that $\eta \circ \epsilon_i = \eta_i$ for any $i \in \mathsf{I}$. Moreover, checking that η is a (possibly elementary if each η_i is elementary) possibility embedding is routine, except possibly for the first condition on possibility embeddings, which we now show. Fix an k-ary function symbol f and a tuple $a_1, ..., a_k$ with $a_n \in D_{i_n}$ for $n \in \{1, ..., k\}$. I claim that $\mathcal{Q} \models f(\beta(a_1), ..., \beta(a_k)) = \beta(\mathscr{I}(f, a_1, ..., a_k))$. By definition, $\mathscr{I}(f, a_1, ..., a_k) = \mathscr{I}_j(f, \alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))$ for some $j \geq_\mathsf{I} i_1, ..., i_k$. Hence $\beta(\mathscr{I}(f, a_1, ..., a_k)) = \beta_j(\mathscr{I}_j(f, \alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k)))$. Using the fact that $\mathcal{Q} \models \beta_i(a) = \beta_j \circ \alpha_{ij}(a)$ whenever $i \mathsf{I} j$ and $a \in D_i$ and β_j is a possibility embedding, we have that $\mathcal{Q} \models \beta_j(\mathscr{I}_j(f, \alpha_{i_1 j}(a_1), ..., \alpha_{i_k j}(a_k))) = f(\beta_{i_1}(a_1), ..., \beta_{i_k}(a_k))$. But from this it follows at once that $\mathcal{Q} \models f(\beta(a_1), ..., \beta(a_k)) = \beta(\mathscr{I}(f, a_1, ..., a_k))$. This completes the proof. □

6.2 Generic Powers

The rest of this introductory chapter is devoted to a specific kind of possibility structures which I will call *generic powers*. These structures will play a crucial role in the next three chapters, and their main features will follow from three key results about them, which I will refer to as the Structure Lemma, the Truth Lemma and the Genericity Lemma. Let me start with the following definitions.

Definition 6.2.1. Let I be a set. A *rich family* is a collection $\mathcal{E}$ of filters on I such that for any $A \subseteq I$ and $F \in \mathcal{E}$, if $A \notin F$, then there is $G \supseteq F$ such that $I \setminus A \in G$.

Definition 6.2.2. The $\mathfrak{E}$-*generic power* of a first-order structure $\mathbb{M}$ in a language $\mathcal{L}$ is given by the tuple $\mathbb{M}^{\mathfrak{E}} = (\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$, where:

- $\mathfrak{E}$ is the poset $(\mathcal{E}, \supseteq)$, where $\mathcal{E}$ is a rich family.
- $\mathbb{M}^I$ is the set of all functions $a : I \to \mathbb{M}$;
- for any function symbol $f \in \mathcal{L}$ and any tuple $\overline{a}$ of elements of $\mathbb{M}^I$, $\mathscr{I}(f)(\overline{a})(i) = \overline{a(i)^{\mathbb{M}}}$ for any $i \in I$;
- for any relation symbol $R \in \mathcal{L}$ (including equality), any $F \in \mathcal{E}$ and any tuple $\overline{a} \in \mathbb{M}^I$, $\overline{a} \in \mathscr{I}(F, R)$ iff $\{i \in I \mid \mathbb{M} \models R(\overline{a(i)}\} \in F$.

Intuitively, the $\mathfrak{E}$-generic power of $\mathbb{M}$ can be thought of as a collection of partial approximations of what a classical ultrapower of $\mathbb{M}$ modulo a non-principal ultrafilter U on I might look like. Elements in such an ultrapower are equivalence classes of functions in M^I, where two functions f and g are considered equivalent if they agree on a U-large set, meaning that $\{i \in I \mid f(i) = g(i)\} \in \mathsf{U}$. By contrast, in $(\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$, a function f is a mere guise for what its equivalence class would be in a Tarskian ultrapower, and consequently, f is identified at some $F \in \mathfrak{E}$ with another function g precisely when f and g agree on a large enough set from the viewpoint of F, i.e., whenever $\{i \in I \mid f(i) = g(i)\} \in F$. This identification of viewpoints in $\mathfrak{F}$ with partial approximations of an ultrapower will be made more precise below. For now, we will focus on establishing the sense in which generic powers can be seen as an analogue of ultrapowers in classical model theory. There are two key results about classical ultrapowers of first-order structures: they are themselves first-order structures, and their first-order properties are entirely determined by the filter one uses to define them. Let us now see that generic powers have similar properties by proving the Structure and Truth Lemmas respectively.

6.2.1 The Structure Lemma

Our first key result about generic powers shows that they are possibility structures. In order to establish this, we introduce the following notation, which is standard in the literature on ultrapowers.

Notation 6.2.3. Let $\mathbb{M}$ be a Tarskian $\mathscr{L}$-structure and I a set. Given a n-tuple $\overline{f} = (f_1, ...f_n)$ of functions in M^I, let $\overline{f(i)}$ be the n-tuple of elements $(f_1(i), ..., f_n(i)) \in M^n$ for any $i \in I$. Given any $\varphi(\overline{x})$ an $\mathscr{L}$-formula in n-variables, and any n-tuple $\overline{f}$, let $||\varphi(\overline{f})||_I = \{i \in I \mid \mathbb{M} \models \varphi(\overline{f(i)})\}$.

Given an $\mathscr{L}$-sentence φ, one should distinguish its *I-value* $||\varphi||_I$, which is a subset of I, from its *r-value* $[\![\varphi]\!]$, introduced in Notation 6.1.3, which is a subset of $\mathfrak{F}$. As we will see below, there is however a tight connection between the two sets. Using the notation just introduced, the interpretation function $\mathscr{I}$ in $(\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$ can be conveniently rephrased. Given a viewpoint $F \in \mathcal{E}$, an n-ary relation symbol R and an n-tuple $\overline{a}$ in M^I, $\overline{a} \in \mathscr{I}(F, R)$ iff $||R(\overline{a})||_I \in F$.

Lemma 6.2.4 (Structure Lemma)**.** *For any first-order structure* $\mathbb{M}$ *and any rich family* $\mathcal{E}$*,* $\mathbb{M}^{\mathfrak{E}}$ *is a possibility structure.*

Proof. Fix filters $F, G \in \mathcal{E}$, a tuple $\overline{a}$ of elements in $\mathbb{M}^I$, a relation symbol R and a function symbol f. We check the four conditions on $\mathscr{I}$ in turn.

Persistence Suppose $G \supseteq F$ and $\overline{a} \in \mathscr{I}(F, R)$. Then $||R(\overline{a})||_I \in F$, and therefore $||R(\overline{a})||_I \in G$, which implies that $\overline{a} \in \mathscr{I}(G, R)$.

Refinability Assume $\overline{a} \notin \mathscr{I}(F, R)$. Then $||R(\overline{a})||_I \notin F$. Since $\mathcal{E}$ is a rich family, this implies that there is a filter $G \supseteq F \cup \{I \setminus ||R(\overline{a})||_I\}$. But then clearly $||R(\overline{a})||_I \notin H$ for any $H \supseteq G$, and therefore $\overline{a} \notin \mathscr{I}(H, R)$ for any $H \supseteq G$.

Equality-as-equivalence Note first that for any $a \in M^I$, $||a = a||_I = I$. Moreover, for any $a, b, c \in M^I$, $||a = b||_I \subseteq ||b = a||_I$, and $||a = b||_I \cap ||b = c||_I \subseteq ||a = c||_I$. This shows that for any filter F, $||a = a||_I \in F$, $||a = b||_I \in F$ implies $||b = a||_I \in F$, and $||a = b||_I$ and $||b = c||_I \in F$ together imply $||a = c||_I \in F$. Thus $\mathscr{I}(F, =)$ is an equivalence relation on M^I for any $F \in \mathcal{E}$.

Equality as congruence Suppose that $(a_i, b_i) \in \mathscr{I}(F, =)$ for any $i \leq n$, fix a function symbol f and a relation symbol R, and let $\overline{a} = (a_1, ..., a_n)$ and $\overline{b} = (b_1, ..., b_n)$. Note that $\bigcap_{i \leq n} ||a_i = b_i||_I \subseteq ||f(\overline{a}) = f(\overline{b})||_I$, hence that $||f(\overline{a}) = f(\overline{b})||_I \in F$ and thus that $(f(\overline{a}), f(\overline{b})) \in \mathscr{I}(F, =)$. Moreover, assume that $\overline{a} \in \mathscr{I}(F, R)$. Note that $\bigcap_{i \leq n} ||a_i = b_i||_I \cap ||R(\overline{a})||_I \subseteq ||R(\overline{b})||_I$, which implies that $||R(\overline{b})||_I \in F$ and hence that $\overline{b} \in \mathscr{I}(F, R)$. ☐

6.2.2 The Truth Lemma

As we shall see, Łoś's Theorem generalizes in a natural way to generic powers and enables us to understand the forcing relation in such structures very concretely. Note that a version of the Truth Lemma was already obtained by Van Benthem in [23] for his closely related notion of *filter product*.

Definition 6.2.5. Let $\mathcal{L}$ be a first-order language and $\mathbb{M}^{\mathfrak{E}} = (\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$ the $\mathfrak{E}$-generic power of $\mathbb{M}$. For any $\mathcal{L}$-formula $\varphi(\overline{x})$ and tuple $\overline{a}$, the *I-value* of $\varphi(\overline{a})$, denoted $||\varphi||_I$ is the set $\{i \in I \mid \mathbb{M} \models \varphi(\overline{a(i)})\}$.

Lemma 6.2.6 (Truth Lemma)**.** *Assume* $AC_{||}$*. For any first-order* $\mathcal{L}$*-structure* $\mathbb{M}$*, any rich family* $\mathcal{E}$*, any* $F \in \mathcal{E}$*, any* $\mathcal{L}$*-formula* $\varphi(\overline{x})$ *and any tuple* $\overline{a} \in \mathbb{M}^I$*,* $F \Vdash \varphi(\overline{a})$ *iff* $||\varphi(\overline{a})||_I \in F$*.*

Proof. The proof proceeds by induction on the complexity of formulas. The atomic case follows immediately from the definition of the interpretation function $\mathscr{I}$. For the inductive case, I only treat the cases of negation, conjunction and existential quantification, since the other Boolean connectives and quantifiers are definable from this set.

- Suppose $\varphi := \neg\psi$. Then $F \Vdash \varphi(\overline{a})$ iff for all $G \supseteq F \in \mathcal{E}$, $G \nVdash \psi(\overline{a})$. By the induction hypothesis, the latter is equivalent to the claim that for all $G \supseteq F \in \mathcal{E}$, $||\psi(\overline{a})||_I \notin G$. Since $\mathcal{E}$ is rich, this means that $I \setminus ||\psi(\overline{a})||_I \in F$. Clearly, $I \setminus ||\psi(\overline{a})||_I = ||\neg\psi(\overline{a})||_I$, from which it follows that $F \Vdash \varphi(\overline{a})$ iff $||\varphi(\overline{a})||_I \in F$.

- Suppose $\varphi := \psi_1 \wedge \psi_2$. Then $F \Vdash \varphi(\overline{a})$ iff $F \Vdash \psi_1(\overline{a})$ and $F \Vdash \psi_2(\overline{a})$. By the induction hypothesis, this is equivalent to $||\psi_1(\overline{a})||_I \in F$ and $||\psi_2(\overline{a})||_I \in F$, which in turn is equivalent to $||\psi_1(\overline{a})||_I \cap ||\psi_2(\overline{a})||_I \in F$ as F is a filter. But the latter is clearly equivalent to $||\varphi(\overline{a})||_I \in F$.

- Suppose $\varphi := \exists x\psi(x)$. If $F \Vdash \exists x\psi(x,\overline{a})$, then for any $G \supseteq F$, there is $b \in \mathbb{M}^\omega$ and $H \supseteq G$ such that $H \Vdash \psi(b,\overline{a})$. By the induction hypothesis, this means that for all $G \supseteq F$, there is $H \supseteq G$ and $b \in \mathbb{M}^\omega$ such that $||\psi(b,\overline{a})||_I \in H$. Since, for any $b \in \mathbb{M}^\omega$, $||\psi(b,\overline{a}))||_I \subseteq ||\exists x\psi(x,\overline{a})||_I$, it follows that for any $G \supseteq F$, there is $H \supseteq G$ such that $||\exists x\psi(x,\overline{a})||_I \in H$. This in turn implies that for any $G \supseteq F$, $I \setminus ||\exists x\psi(x,\overline{a})||_I \notin G$, from which it follows that $||\exists x\psi(x,\overline{a})||_I \in F$ since $\mathcal{E}$ is a rich family.
Conversely, suppose $||\exists x\psi(x,\overline{a})||_I \in F$. Using $AC_{|I|}$, define $b : I \to \mathbb{M}$ such that $\mathbb{M} \models \psi(b(i),\overline{a}(i))$ whenever $i \in ||\exists x\psi(x,\overline{a})||_I$, and $b(i)$ is arbitrary otherwise. Then $||\psi(b,\overline{a})||_I \supseteq ||\exists x\psi(x,\overline{a})||_I$, so $||\psi(b,\overline{a})||_I \in F$. By the induction hypothesis, this means that $F \Vdash \varphi(b,\overline{a})$ and hence $F \Vdash \exists x\psi(x,\overline{a})$, which completes the proof. □

A few remarks can be made regarding the proof of Theorem 7.2.5:

Remark 6.2.7.

1. Using the notation in 6.1.3 and 6.2.3, Theorem 7.2.5 can be succinctly phrased as follows: for any $\mathscr{L}$-formula $\varphi(\overline{x})$ with n free variables and any n-tuple $\overline{a} \in \mathbb{M}^{\mathfrak{E}}$,
$$[\![\varphi(\overline{a})]\!] = \{F \in \mathfrak{F} \mid ||\varphi(\overline{a})|| \in F\}.$$

2. Closer inspection of the existential case of the proof reveals that $\mathbb{M}^{\mathfrak{E}}$ is a *full* model in the following sense: for any formula $\varphi(x,\overline{y})$, any tuple $\overline{a}$ and any $F \in \mathfrak{F}$, $F \Vdash \exists x\varphi(x,\overline{a})$ iff there is $b \in \mathbb{M}^{\mathfrak{E}}$ such that $F \Vdash \varphi(b,\overline{a})$. In other words, although standard possibility semantics allows for the witness to an existential sentence to vary across refinements, a stronger condition actually holds in the case of generic powers.

3. The right-to-left direction of the existential case is the only part of the theorem that requires some fragment of the Axiom of Choice, namely $AC_{|I|}$. This means in particular that the Truth Lemma holds in $ZF + DC$ whenever I is a countable set.

6.2.3 The Genericity Lemma

Finally, our last key result about generic powers connects them with ultrapowers in a powerful way. The core idea of the Genericity Lemma is to establish that satisfaction in a generic power $(\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$ coincides with truth in every ultrapower that $(\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$ "stands for" or "approximates". The precise way to state this in its full generality requires some terminology from forcing in set theory, but the proofs themselves are fairly simple. For some background on forcing, we refer the reader to [161]. We first need the following definitions.

Definition 6.2.8. Let $\mathfrak{P}$ be a poset. A subset $D \subseteq \mathfrak{P}$ is *dense* if the inclusion map $\iota : D \to \mathfrak{P}$ is dense. Two elements $p, q \in \mathfrak{P}$ are *incompatible* (denoted $p \perp q$) if there is no $r \in \mathfrak{P}$ such that $r \leq_{\mathfrak{P}} p, q$.

Definition 6.2.9. Let $\mathcal{P} = (\mathfrak{P}, D, \mathscr{I})$ be a possibility structure in a first-order language $\mathscr{L}$. A *definable subset* of $\mathfrak{P}$ is a subset of the form $[\![\varphi(\overline{a})]\!] \cup [\![\neg\varphi(\overline{a})]\!]$ for some $\mathscr{L}$-formula $\varphi(\overline{x})$ and some tuple $\overline{a} \in D$. A *Henkin subset* of $\mathfrak{P}$ is a subset of the form $\mathfrak{P} \setminus [\![\exists y \varphi(\overline{a}, y)]\!] \cup \bigcup_{b \in D} [\![\varphi(\overline{a}, b)]\!]$ for some $\mathscr{L}$-formula $\varphi(\overline{x}, y)$ and some tuple $\overline{a} \in D$.

Definition 6.2.10. Let $\mathcal{P} = (\mathfrak{P}, D, \mathscr{I})$ be a possibility structure in a first-order language $\mathscr{L}$. A subset G of $\mathfrak{P}$ is a $\mathcal{P}$-*generic* filter if it satisfies the following conditions:

1. G is downward directed and upward closed;

2. $\mathsf{G} \cap D \neq \emptyset$ for any definable subset D;

3. $\mathsf{G} \cap H \neq \emptyset$ for any Henkin subset H.

4. For any $p \in \mathfrak{P}$, $p \in \mathsf{G}$ or there is $q \in \mathsf{G}$ such that $p \perp q$.

This definition can be seen as a weakening of the notion of a generic filter on a poset in forcing. A generic filter on a poset $\mathfrak{P}$ meets every dense subset of $\mathfrak{P}$, and it follows from the semantic clauses of possibility semantics that any subset that is either definable of Henkin in a possibility structure $\mathcal{P}$ is dense. Moreover, for any $p \in \mathfrak{P}$, the set $\{q \in \mathfrak{P} \mid q \leq_{\mathfrak{P}} p \text{ or } p \perp q\}$ is also dense.

A result very similar to the following lemma was already proved by van Benthem [23], who defines a notion of *generic branches* similar to our notion of a $\mathcal{P}$-*generic* filter.

Lemma 6.2.11. *Let $\mathcal{P} = (\mathfrak{P}, D, \mathscr{I})$ be a possibility structure in a first-order language $\mathscr{L}$ and G a $\mathcal{P}$-generic filter. Then there is a Tarskian model $\mathcal{P}_{\mathsf{G}} = (D_{\mathsf{G}}, \mathscr{I}_{\mathsf{G}})$ and a map $\cdot_{\mathsf{G}} : D \to D_{\mathsf{G}}$ such that for any formula $\varphi(\overline{x})$, $\mathcal{P}_{\mathsf{G}} \models \varphi(\overline{a_{\mathsf{G}}})$ iff there is $p \in \mathsf{G}$ such that $p \Vdash \varphi(\overline{a})$.*

Proof. Define an equivalence relation on D by letting $a \sim_{\mathsf{G}} b$ iff there is $p \in \mathsf{G}$ such that $p \Vdash a = b$. Let $\mathcal{P}_{\mathsf{G}}$ be the Tarskian model $(D_{\mathsf{G}}, \mathscr{I}_{\mathsf{G}})$ where $D_{\mathsf{G}} = \{a_{\mathsf{G}} \mid a \in D\}$ is the set of all equivalence classes under the relation $\sim_{\mathsf{G}}$, and for any $\overline{a} \in D$, $\overline{a_{\mathsf{G}}} \in \mathscr{I}_{\mathsf{G}}(R)$ iff there is $p \in \mathsf{G}$ such that $p \Vdash R(\overline{a})$. It is routine to verify that this is a well-defined Tarskian structure.

Moreover, a straightforward induction on the complexity of formulas establishes that for any $\mathscr{L}$-formula $\varphi(\overline{x})$ and any tuple $\overline{a} \in D$, we have

$$\mathcal{P}_{\mathsf{G}} \models \varphi(\overline{a_{\mathsf{G}}}) \Leftrightarrow \exists p \in \mathsf{G} : p \Vdash \varphi(\overline{a}).$$

The base case follows from the definition of $\mathscr{I}_{\mathsf{G}}$. For the inductive step, the case of conjunctions follows from basic properties of filters, the case for negation follows from the fact that G meets every definable subset of $\mathfrak{P}$, and the case for existential formulas follows from the fact that G meets every Henkin subset. □

We will follow van Benthem in calling such Tarskian structures generic models. They are of particular interest for the following kind of possibility structures:

Definition 6.2.12. A possibility structure $\mathcal{P} = (\mathfrak{P}, D, \mathscr{I})$ is *normal* if for any $p \in \mathfrak{P}$ there is a $\mathcal{P}$-generic filter G such that $p \in \mathsf{G}$.

The following result establishes the sense in which points in a possibility structure approximate Tarskian models. Again, it is related to van Benthem's remarks on generic branches in [23]:

Lemma 6.2.13. *Let $\mathcal{P} = (\mathfrak{P}, D, \mathscr{I})$ be a normal possibility structure in a language $\mathscr{L}$. Then for any $p \in \mathfrak{P}$, any $\mathscr{L}$-formula $\varphi(\overline{x})$ and any tuple $\overline{a} \in D$, $p \Vdash \varphi(\overline{a})$ iff $\mathcal{P}_{\mathsf{G}} \models \varphi(\overline{a_{\mathsf{G}}})$ for every $\mathcal{P}$-generic filter G such that $p \in \mathsf{G}$.*

Proof. Suppose first that $p \Vdash \varphi(\overline{a})$. Then, by Lemma 6.2.11, $\mathcal{P}_{\mathsf{G}} \models \varphi(\overline{a_{\mathsf{G}}})$ whenever $p \in \mathsf{G}$. Conversely, suppose that $p \nVdash \varphi(\overline{a})$. Then there is $q \leq_{\mathfrak{P}} p$ such that $q \Vdash \neg\varphi(\overline{a})$. Since $\mathcal{P}$ is normal, there is a $\mathcal{P}$-generic filter G such that $q \in \mathcal{P}$. But then we have that $p \in \mathsf{G}$ and $\mathcal{P}_{\mathsf{G}} \models \neg\varphi(\overline{a_{\mathsf{G}}})$. □

Finally, let us now turn to the specific case of generic powers. The Genericity Lemma will be a special case of Lemma 6.2.13 which connects the forcing relation on the generic power of a Tarskian model $\mathbb{M}$ with ultrapowers of $\mathbb{M}$. We start with the following definition.

Definition 6.2.14. Let $\mathcal{E}$ be a rich family on a set I. For any ultrafilter U on I, let $\alpha(\mathsf{U}) = \{F \in \mathcal{E} \mid F \subseteq \mathsf{U}\}$. An ultrafilter U on I is $\mathfrak{E}$-*generic* if $\mathsf{U} = \bigcup \alpha(\mathsf{U})$.

Lemma 6.2.15 (Genericity Lemma)**.** *Let $\mathbb{M}^{\mathfrak{E}} = (\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$ be the $\mathfrak{E}$-generic power of a first-order $\mathcal{L}$-structure $\mathbb{M}$.*

1. *For any $\mathfrak{E}$-generic ultrafilter U, $\alpha(\mathsf{U})$ is a $\mathbb{M}^{\mathfrak{E}}$-generic filter, and $\mathbb{M}^{\mathfrak{E}}_{\alpha(\mathsf{U})}$ is isomorphic to $\mathbb{M}^I/\mathsf{U}$, the ultrapower of $\mathbb{M}$ modulo U.*

2. *Moreover, if for any $F \in \mathcal{E}$ there is a $\mathfrak{E}$-generic ultrafilter U such that $F \subseteq \mathsf{U}$, then $\mathbb{M}^{\mathfrak{E}}$ is normal and for any $\mathscr{L}$-formula $\varphi(\overline{x})$, any tuple $\overline{a}$ of elements of $\mathbb{M}^I$ and any $F \in \mathcal{E}$, $F \Vdash \varphi(\overline{a})$ iff for any $\mathfrak{E}$-generic ultrafilter U such that $F \subseteq U$, $\mathbb{M}^I/\mathsf{U} \models \varphi(\overline{a_{\mathsf{U}}})$.*

Proof. Note first that item 2 follows immediately from item 1. To prove item 1, assume that U is $\mathfrak{E}$-generic, and let us first show that $\alpha(\mathsf{U})$ is $\mathbb{M}^{\mathfrak{E}}$-generic. Clearly, $\alpha(\mathsf{U})$ is upward closed and downward directed. Moreover, let D be a definable subset of $\mathfrak{E}$. Then $D = [\![\varphi(\overline{a})]\!] \cup [\![\neg\varphi(\overline{a})]\!]$ for some $\mathscr{L}$-formula $\varphi(\overline{x})$ and tuple $\overline{a} \in D$. Since U is an ultrafilter, either $||\varphi(\overline{a})||_I$ or $I \setminus ||\varphi(\overline{a})||_I = ||\neg\varphi(\overline{a})||_I$ belongs to U. By assumption, $\mathsf{U} = \bigcup \alpha(\mathsf{U})$, so there is $F \in \alpha(\mathsf{U})$ such that $||\varphi(\overline{a})|| \in F$ or $||\neg\varphi(\overline{a})|| \in F$. But then it follows from the Truth Lemma that $F \in [\![\varphi(\overline{a})]\!] \cup [\![\neg\varphi(\overline{a})]\!]$ and hence $\alpha(\mathsf{U}) \cap F \neq \emptyset$. Next, note that, because $\mathbb{M}^{\mathfrak{E}}$ is *full* in the sense of Remark 6.2.7.2, we have that any Henkin set is actually $\mathfrak{E}$ and thus trivially has non-empty intersection with $\alpha(\mathsf{U})$. Finally, suppose $F \notin \alpha(\mathsf{U})$. Then there is $A \in F$ such that $A \notin \mathsf{U}$. Since U is an ultrafilter, this means that $I \setminus A \in \mathsf{U}$, and therefore, since $\mathsf{U} = \bigcup \alpha(\mathsf{U})$, there is $F' \in \alpha(U)$ such that $I \setminus A \in F'$. But clearly $F \perp F'$, which completes the proof that $\alpha(\mathsf{U})$ is $\mathbb{M}^{\mathfrak{E}}$-generic.

Moreover, for any $a \in \mathbb{M}^I$, let a_{U} be the equivalence class of a modulo the U. Note that, for any atomic $\mathscr{L}$-relation symbol $R(\overline{x})$ (including the equality), we have:

$$\begin{aligned}\mathbb{M}^I/\mathsf{U} \models R(\overline{a_{\mathsf{U}}}) &\Leftrightarrow ||R(\overline{a})|| \in \mathsf{U}\\ &\Leftrightarrow \exists F \in \alpha(\mathsf{U}) : ||R(\overline{a})|| \in F\\ &\Leftrightarrow \mathbb{M}^{\mathfrak{E}}_{\alpha(\mathsf{U})} \models R(\overline{a_{\alpha(\mathsf{U})}}),\end{aligned}$$

where the first equivalence holds from Łoś's Theorem, the second from the fact that U is $\mathcal{E}$-generic, and the last one from Lemma 6.2.11. But this shows that the map $a_{\mathsf{U}} \mapsto a_{\alpha(\mathsf{U})}$ is an isomomorphism between $\mathbb{M}^I/\mathsf{U}$ and $\mathbb{M}^{\mathfrak{E}}_{\alpha(\mathsf{U})}$. □

The Genericity Lemma shows that ultrapowers can often be represented as generic models over some possibility structure. Interestingly, the notion of genericity that we introduced for ultrafilters is not dependent on the language $\mathscr{L}$, while the notion of genericity we introduced for generic models does depend on the language. This explains why, in general, although every ultrapower can be represented as a generic model, the converse may not hold, as the language $\mathscr{L}$ may be "too weak" for every generic model to be isomorphic to an ultrapower. Let us conclude this discussion of the Genericity Lemma by giving sufficient conditions on the language $\mathscr{L}$ for the converse to hold for a specific kind of rich families of filters.

Definition 6.2.16. A rich family $\mathcal{E}$ of filters on a set I is *dense* if for any $F, F' \in \mathcal{E}$, if there is a proper filter G on I such that $F, F' \subseteq G$, then there is $G' \in \mathcal{E}$ such that $F, F' \subseteq G'$.

Lemma 6.2.17. *Let $\mathbb{M}^{\mathfrak{E}} = (\mathfrak{E}, \mathbb{M}^I, \mathscr{I})$ be the $\mathfrak{E}$-generic power of a first-order $\mathcal{L}$-structure $\mathbb{M}$ determined by a dense family $\mathcal{E}$, and suppose that there is a predicate symbol A in $\mathscr{L}$ for every $A \subseteq \mathbb{M}$. Then every generic model $\mathbb{M}^{\mathfrak{E}}_{\mathsf{G}}$ is isomorphic to an ultrapower of $\mathbb{M}$ modulo a $\mathfrak{E}$-generic ultrafilter.*

Proof. Suppose that there is a predicate symbol A in $\mathscr{L}$ for every $A \subseteq \mathbb{M}$. Let $f \in \mathbb{M}^I$ be any function, and let $\mathsf{U} = \{f^{-1}[A] \mid \mathbb{M}^{\mathfrak{E}}_{\mathsf{G}} \models \mathsf{A}(f)\}$. I claim that U is a $\mathfrak{E}$-generic ultrafilter and moreover that $\alpha(\mathsf{U}) = \mathsf{G}$. Note that this will conclude the proof by the Genericity Lemma. It is routine to verify that U is an ultrafilter on I, as this follows from the fact that

$f^{-1} : \mathscr{P}(\mathbb{M}) \to \mathscr{P}(I)$ is a Boolean homomorphism and the fact that $\mathbb{M}$ elementarily embeds into $\mathbb{M}^{\mathfrak{E}}_{\mathsf{G}}$ via the map sending any $b \in \mathbb{M}$ to c^b_{G}, where c^b is the function with range $\{b\}$. To see that U is $\mathfrak{E}$-generic, notice that, for any $A \subseteq \mathbb{M}$,

$$\begin{aligned}\mathbb{M}^{\mathfrak{E}}_{\mathsf{G}} \models \mathsf{A}(f) &\Leftrightarrow \exists F \in \mathsf{G} : F \Vdash \mathsf{A}(f)\\ &\Leftrightarrow \exists F \in \mathsf{G} : ||\mathsf{A}(f)|| \in F\\ &\Leftrightarrow \exists F \in \mathsf{G} : f^{-1}[A] \in F,\end{aligned}$$

where the first equivalence follows from Lemma 6.2.11 and the second one from the Truth Lemma. Next, let us verify that $\alpha(\mathsf{U}) = \mathsf{G}$. Let us first show that $\mathsf{G} \subseteq \alpha(\mathsf{U})$. Suppose $F \in \mathsf{G}$. We want to show that $F \subseteq \mathsf{U}$. Let $X \in F$, and let $A \subseteq \mathbb{M}$ be the set $\{f(i) \mid i \in X\}$. Clearly we have that $X = f^{-1}[A]$ which, by the chain of equivalences above, also implies that $X \in \mathsf{U}$. Hence $F \subseteq \mathsf{U}$. Conversely, suppose that $F \notin \mathsf{G}$. Then since G is $\mathcal{M}^{\mathfrak{E}}$-generic, there is $F' \in \mathsf{G}$ such that $F \perp F'$. Because $\mathcal{E}$ is a dense family, this implies that there is $X \subseteq I$ such that $X \in F$ and $I \setminus X \in F'$. But then, letting $A = \{f(i) \mid i \in X\}$, it follows that $f^{-1}[A] \in \mathsf{G}$ and yet $\mathbb{M}^{\mathfrak{E}}_{\mathsf{G}} \models \neg\mathsf{A}(f)$. Hence $F \nsubseteq \mathsf{U}$. This completes the proof. □

Chapter 7

The Fréchet Hyperreals

7.1 Introduction

Non-standard analysis is a branch of mathematical logic which focuses on the application of powerful metamathematical methods to ordinary mathematics, following the groundbreaking work of Abraham Robinson [223] in the 60s. Although it has now developed into a diverse field interested in more than one structure, it originated from the development of a system of hyperreal numbers which is powerful and suggestive enough to serve as an alternative foundation to classical analysis. The common way of introducing such numbers is via the hyperreal line ${}^*\mathcal{R}$, constructed as some ultrapower of the standard real line modulo a free ultrafilter on the set ω of the natural numbers [155]. The main advantage of this ultrapower construction is that it can easily be seen to satisfy two fundamental principles: the *Transfer Principle*, which guarantees that the hyperreal line is *standard enough* to have the same first-order theory as the real line, and the (Countable) *Saturation Principle*, which allows one to derive the existence in ${}^*\mathcal{R}$ of many *non-standard objects* that can play a key role in simplifying many classical arguments in analysis. The existence of a first-order structure extending the reals in which every finitary relation A on the reals has a non-standard extension A^* and which satisfies both the Transfer and Saturation Principle is enough to deliver most applications of nonstandard methods to ordinary mathematics [111].

The existence of free ultrafilters on ω, however, is a set-theoretic result that exceeds the resources of semi- or quasi-constructive mathematics as described in [230], i.e., classical mathematics that can be carried out in ZF with the addition of the Axiom of Dependent Choices (DC). As Schechter writes ([230, Chap. 14]), $ZF + DC$ is a natural setting for analysis, as it neither assumes nor rejects some of the more counter-intuitive consequences of the Axiom of Choice in analysis, such as the existence of a well-ordering of the reals or of non-Lebesgue measurable sets. Free ultrafilters on infinite Boolean algebras exist in the presence of the Ultrafilter Lemma, a fragment of the Axiom of Choice which is known to be independent from $ZF + DC$ [89]. As one can straightforwardly define non-Lebesgue measurable sets from a free ultrafilter on ω, such objects go beyond the resources of semi-constructive mathematics, and they form what Schechter calls *intangibles*, objects whose existence can be proved in classical mathematics even though they cannot themselves be *explicitly* constructed.

On the other hand, in a strictly semi-constructive setting, preserving both the Transfer and Saturation Principles seems hopeless. A straightforward argument shows that the existence of an elementary embedding $e : \mathcal{R} \to \mathcal{M}$ in a sufficiently rich language $\mathscr{L}$ implies the existence of a free ultrafilter on ω. Indeed, the Countable Saturation Principle implies the existence of at least one infinite hypernatural integer N, larger than any finite number n. If $\mathcal{M}$ contains a nonstandard extension A^* for every finitary relation A on $\mathcal{R}$, then one may consider the set $\mathcal{U} = \{A \subseteq \omega \mid \mathcal{M} \models A^*(N)\}$. By the Transfer Principle and the truth-conditions of Tarskian semantics, $\mathcal{U}$ is then easily seen to be a free ultrafilter on ω. We are therefore faced with the following trilemma: we must either give up the Transfer Principle, the Saturation Principle, or Tarskian semantics. It is this third option that I would like to pursue here, by appealing to an alternative semantics for classical logic known as possibility semantics. As we will see, possibility semantics allows for the construction of a first-order structure that shares many features with the classical hyperreal line, including versions of the Transfer and Saturation Principles, yet is entirely independent of the Ultrafilter Lemma. An alternative solution to the problem has recently been independently developed by Hrbáček and Katz in [139]. The problem they discuss is very similar to the one I mentioned here, but the solution they propose is syntactic (i.e., to work in a weaker, semi-constructively acceptable axiomatic theory rather than in Nelson's Internal Set Theory) rather than semantic.[1] By contrast, the key idea of the Fréchet hyperreals is to replace one single, static first-order model determined by an ultrafilter with a system of *viewpoints*, partial approximations of such a model each determined by a filter. This idea itself is not new, although, as I argue below, it can be presented in a particularly powerful, simple and concrete way thanks to the machinery of possibility semantics. Not only does the resulting structure have some technical advantages over some other alternative approaches to nonstandard analysis, it also has some methodological and conceptual advantages over the classical hyperreal line obtained via ultrapowers. In particular, I argue that it is well-suited to address concerns that have been raised in the literature regarding the application of nonstandard methods to standard mathematics, including issues about the purity (in the sense of [4, 5]) of nonstandard methods and the canonicity of the hyperreal line.

The chapter is organized as follows. In Section 7.2, I introduce a specific kind of possibility structures, *Fréchet powers*, which constitute an alternative to classical ultrapowers modulo a free ultrafilter on ω. This allows me to define the Fréchet hyperreals ${}^{\dagger}\mathcal{R}$ as a Fréchet power of the reals and to explore some of its mathematical properties. In particular, I show that versions of the Transfer and Saturation Principles of classical nonstandard analysis hold on ${}^{\dagger}\mathcal{R}$ and that the Fréchet hyperreals enjoy many features of the classical hyperreal line, including a natural characterization of continuous real-valued functions and a robust theory of internal sets.

[1] I will come back to their proposal in Section 7.6 below.

Sections 7.3 to 7.5 are then devoted to assessing the technical merits of ${}^\dagger\mathcal{R}$ over other alternatives to classical nonstandard analysis. As we will see, the central idea behind the Fréchet hyperreals, i.e., to avoid relying on a single free ultrafilter by working with a system of filters instead, can be traced back to several distinct mathematical endeavors. I distinguish three kinds of alternative approaches to nonstandard analysis and argue that ${}^\dagger\mathcal{R}$ constitutes a natural meeting point for all three approaches. Indeed, it can be seen as a suitable framework to develop the natural and historically influential idea that the properties of infinite sequences "in the limit" are determined by the properties of all but finitely many of their values, thus continuing some historically-minded work by Laugwitz and offering a formal counterpart to a more recent proposal of Tao (Section 7.3). On the other hand, the dynamic aspect of possibility semantics, and its connections with sheaf semantics and topos theory, also allows one to think of ${}^\dagger\mathcal{R}$ as a "varying" reduced power of the reals (Section 7.4). Moreover, ${}^\dagger\mathcal{R}$ can be viewed as a Boolean-valued model of analysis in which viewpoints are partial approximations of classical ultrapowers, which then arise as generic constructions in the precise sense of forcing (Section 7.5).

Finally, in Section 7.6, I turn to some philosophical objections to the application of nonstandard methods in ordinary mathematics and argue that the Fréchet hyperreals offer a novel way to tackle the problems raised by these objections. In particular, the fact that ${}^\dagger\mathcal{R}$ can be constructed in a semi-constructive setting means that it is better suited to address purity concerns regarding the use of nonstandard methods in analysis, and the use of a system of filters in lieu of a single ultrafilter turns it into a canonical structure.

7.2 The Fréchet Hyperreals

In this section, I introduce the Fréchet hyperreals as a concrete possibility structure. Because the points in a possibility structure are partial possibilities, instead of maximal possible worlds, the construction of possibility structures is typically more constructive than the construction of standard, possible worlds models. In possibility semantics, the role usually played by maximal, non-constructive objects (possible worlds, ultrafilters), is taken up by partial, constructive objects (partial worlds, filters). As we will see below, it is precisely this feature that will allow for a construction of the hyperreals that bypasses the usual reliance on a free ultrafilter on ω. I start by introducing *Fréchet powers* as a particular kind of generic powers, before defining the Fréchet hyperreals ${}^\dagger\mathcal{R}$ as a Fréchet power of the real line and investigating their main properties.

7.2.1 Fréchet Powers

In the setting of possibility semantics, Fréchet powers are an alternative to the ultrapowers modulo a free ultrafilter on ω of Tarskian semantics, which I will call *Luxemburg ultrapowers* from now on, as, according to Keisler in [155], their relevance to nonstandard analysis was first identified by Luxemburg in [175]. As mentioned in the previous chapter, an analogue of classical ultraproducts in the setting of possibility semantics was already discussed in [23].

However, van Benthem is interested in defining a general notion of product for all possibility structures, while I will only be interested in defining Fréchet powers of Tarskian models, which simplifies van Benthem's construction quite significantly. It is also worth mentioning that Fréchet powers can be defined constructively, and that only the Axiom of Countable Choices, which is a theorem of $ZF + DC$, will be required to prove the Transfer Principle in Section 7.2.2.

Let us start with the definition of the poset that will play a crucial role throughout this chapter. Recall first that a filter F on ω is *free* if the set of all cofinite subsets of ω is a subset of F. Throughout this chapter, I will assume that a free filter F is always proper, meaning that $F \neq \mathscr{P}(\omega)$.

Definition 7.2.1 (Fréchet Poset)**.** The *Fréchet poset* is the poset $\mathfrak{F} = (\mathcal{F}, \supseteq)$, where $\mathcal{F}$ is the set of all free filters on ω, ordered by reverse inclusion.

The Fréchet poset takes its name from the fact that any element in $\mathfrak{F}$ extends the filter of all cofinite subsets of ω, which is usually called the *Fréchet filter*, and that I will denote by F_0. One quickly verifies that $\mathcal{F}$ is a dense family: for any $A \subseteq \omega$ and any $F \in \mathcal{F}$, if $A \notin F$, then the set $G = \{\omega \setminus A \cap B \mid B \in F\}$ is a proper filter extending F and containing $\omega \setminus A$. The second condition on dense families is clear. This motivates the following definition.

Definition 7.2.2 (Fréchet Power)**.** Let $\mathbb{M}$ be a Tarskian model of a first-order language $\mathscr{L}$. The *Fréchet power* ${}^{\dagger}\mathcal{M} = (\mathfrak{F}, M^{\omega}, \mathscr{I})$ of $\mathbb{M}$ is the $\mathscr{L}$-possibility structure determined by the following data:

- $(\mathfrak{F}, \supseteq)$ is the Fréchet poset;
- M^{ω} is the set of all functions from ω into the domain of $\mathbb{M}$;
- For any n-ary relation symbol $R \in \mathscr{L}$ and any n-tuple $\overline{a}$, $\overline{a} \in \mathscr{I}(F, R)$ iff $\{i \in \omega \mid \mathbb{M} \models R(\overline{a}(i))\} \in F$;
- For any n-ary function symbol $f(\overline{x}) \in \mathscr{L}$ and any $a_1, ..., a_n \in M^{\omega}$, $\mathscr{I}(f)(a_1, ..., a_n)$ is the function $g : \omega \to M$ such that for any $i \in \omega$, $g(i) = f(a_1(i), ..., a_n(i))$.

The observation that $\mathcal{F}$ is a dense family, together with the Structure Lemma, immediately yields the following result:

Lemma 7.2.3. *Let $\mathbb{M}$ be a Tarskian model of a first-order language $\mathscr{L}$. The Fréchet power ${}^{\dagger}\mathcal{M} = (\mathfrak{F}, M^{\omega}, \mathscr{I})$ of $\mathbb{M}$ is a possibility structure.*

We can now introduce the main construction of this chapter, the Fréchet hyperreals, as the Fréchet power of the reals in a significantly rich first-order language.

Definition 7.2.4 (Fréchet hyperreals)**.** Let $\mathscr{L}$ be a first-order language with a relation symbol for every finitary relation on $\mathbb{R}$ and a function symbol for every finitary function on $\mathbb{R}$, and let $\mathcal{R}$ be the Tarskian $\mathscr{L}$-structure with domain $\mathbb{R}$ in which every relation or function symbol is interpreted as the relation or function on $\mathcal{R}$ it corresponds to. The *Fréchet hyperreal line* is the Fréchet power of $\mathcal{R}$, i.e., the $\mathscr{L}$-possibility structure ${}^{\dagger}\mathcal{R} = (\mathfrak{F}, \mathcal{R}^{\omega}, \mathscr{I})$, where for any n-tuple $\overline{a} \in \mathcal{R}^{\omega}$, $\mathscr{I}(f)(\overline{a})(i) = f(\overline{a}(i))$ for any n-ary function symbol $f \in \mathscr{L}$ and any $i \in \omega$, and $\overline{a} \in \mathscr{I}(F, R)$ iff $||R(\overline{a})|| \in F$ for any n-ary relation symbol R in $\mathscr{L}$ and any $F \in \mathfrak{F}$.

Having reached a definition of the Fréchet hyperreals, we may now investigate the properties of this structure more closely. As my main interest is to highlight its similarity with the classical hyperreal line as presented for example in [108], I will be focusing on some of the most well-known features of Robinsonian nonstandard analysis, such as the definition of continuity and the theory of internal sets. I show first that a version of the Transfer Principle holds on ${}^{\dagger}\mathcal{R}$, which allows me to establish that ${}^{\dagger}\mathcal{R}$ contains some infinitesimals and to give a natural characterization of continuity which mirrors the classical nonstandard one. I then show that a notion of internal set can be fruitfully developed on ${}^{\dagger}\mathcal{R}$ for which several properties of classical internal sets hold, including a version of the Countable Saturation Principle. This establishes that possibility semantics offers a way out of the trilemma presented in the introduction: we can preserve the Transfer and Saturation Principles in a semi-constructive setting, if we are willing to move away from Tarskian semantics.

7.2.2 Transfer Principle and Continuity

As we shall see in the next two sections, the Fréchet hyperreals are sufficiently well-behaved to share many features with the classical hyperreal line. In particular, natural versions of two of the most fundamental tools of nonstandard analysis, the Transfer and Saturation Principles, hold in ${}^{\dagger}\mathcal{R}$. That classical ultrapowers of the reals satisfy the Transfer Principle is a direct consequence of Łoś's Theorem [11]. In our setting, the following is an immediate application of the Truth Lemma:

Theorem 7.2.5 (Łoś's Theorem for ${}^{\dagger}\mathcal{R}$). *For any $\mathscr{L}$-formula $\varphi(\overline{x})$ with n free variables, any n-tuple $\overline{a} \in {}^{\dagger}\mathcal{R}$ and any $F \in \mathfrak{F}$, $F \Vdash \varphi(\overline{a})$ iff $||\varphi(\overline{a})|| \in F$.*

The Transfer Principle for ${}^{\dagger}\mathcal{R}$ can then be obtained as an easy corollary.

Corollary 7.2.6 (Transfer Principle). *There exists a function $\delta : \mathcal{R} \to {}^{\dagger}\mathcal{R}$ such that for any $\mathscr{L}$-sentence $\varphi(\overline{x})$ and any tuple of reals $\overline{r}$, $\mathcal{R} \models \varphi(\overline{r})$ iff the Fréchet filter $F_0 \Vdash \varphi(\overline{\delta(r)})$ iff ${}^{\dagger}\mathcal{R} \models \varphi(\overline{\delta(r)})$. In particular, for any $\mathscr{L}$-sentence φ, $\mathcal{R} \models \varphi$ iff ${}^{\dagger}\mathcal{R} \models \varphi$ iff $F_0 \Vdash \varphi$.*

Proof. For any $r \in \mathcal{R}$, let $\delta(r)(i) = r$ for all $i \in \omega$. Then for any $\mathscr{L}$-sentence $\varphi(\overline{x})$, any tuple of reals $\overline{r}$ and any $i \in \omega$, $\mathcal{R} \models \varphi(\overline{r})$ iff $i \in ||\varphi(\overline{\delta(r)})||$. In particular, this means that $||\varphi(\overline{\delta(r)})||$ is either $\emptyset$ or ω, meaning that $F \Vdash \varphi(\overline{\delta(r)})$ iff $||\varphi(\overline{\delta(r)})|| = \omega$ iff $F_0 \Vdash \varphi(\overline{\delta(r)})$. □

It is worth emphasizing that this version of the Transfer Principle only holds because we are using possibility semantics rather than the more standard Kripke semantics. In a sense, this is not surprising: if the Transfer Principle is to hold between $\mathbb{R}$ and ${}^{\dagger}\mathcal{R}$, then in particular all the classical validities must be valid on ${}^\dagger\mathcal{R}$. Using the recursive clauses of Kripke semantics would only deliver intuitionistic logic, and would therefore make the Transfer Principle fail. The issue, in a nutshell, can be traced back to the semantics of disjunction. By contrast with the case of the existential quantifier mentioned in Remark 6.2.7.2, a disjunction maybe be forced at a viewpoint F without any of the disjunct being forced at F.[2] If we were to adopt Kripke semantics, such disjunctions would not be satisfied at every viewpoint in ${}^\dagger\mathcal{R}$, and the Transfer Principle would therefore fail.

Łoś's Theorem and the Transfer Principle for ${}^\dagger\mathcal{R}$ allows us to draw two important consequences. First, there is a rather robust sense in which $\mathcal{R}$ elementarily embeds into ${}^\dagger\mathcal{R}$: even though ${}^\dagger\mathcal{R}$ is not a classical Tarskian structure, if one thinks of every viewpoint in $\mathfrak{F}$ as a partial approximation of what such a Tarskian structure could look like, then, by the Transfer Principle, any such approximation makes sure that this Tarskian structure would have the same first-order theory as $\mathcal{R}$ and would also contain a copy of the reals. By contrast, if one prefers to think of every point in $\mathfrak{F}$ as a "local" viewpoint on a non-constant, ever-changing hyperreal line, then the Transfer Principle guarantees that the reals are fixed, unmovable points of this line. The second consequence that one can draw is that validity on ${}^\dagger\mathcal{R}$ coincides with satisfaction at F_0: a formula will be satisfied at any $F \in \mathfrak{F}$ precisely if it is forced by F_0. Whether one treats points in $\mathfrak{F}$ as approximations of a Tarskian model, or as snapshots of a changing model, this guarantees that the formulas that are true "for sure" or "always", are precisely those formula $\varphi(\overline{a})$ that are true in $\mathcal{R}$ "cofinitely often", i.e., all formulas such that $||\varphi(\overline{a})||$ is a cofinite subset of ω. F_0 can therefore be thought of as a "generic" viewpoint on the Fréchet hyperreals: it forces precisely what must be true of it and nothing else. I will come back to the significance of this fact in Section 7.6 below.

Just as in Luxemburg ultrapowers, we may now define the nonstandard extension of an n-ary relation R on $\mathbb{R}$ as an n-ary relation ${}^\dagger R$ on ${}^\dagger\mathcal{R}$:

Definition 7.2.7 (Nonstandard Extension)**.** Let R be an n-ary relation on $\mathbb{R}$. The *F-nonstandard extension of R*, noted ${}^\dagger R_F$, is the set $\{\overline{a} \in ({}^\dagger\mathcal{R})^n \mid F \Vdash R(\overline{a})\} = \mathscr{I}(F, R)$. Similarly, if f is an n-ary function on $\mathbb{R}$, we let ${}^\dagger f_F$ be the n-ary function $\mathscr{I}(f)$.

Note that, because of our choice of defining functions on ${}^\dagger\mathcal{R}$, the nonstandard extension of a function f on $\mathbb{R}$ is a *bona fide* function that is not relativized to a filter F. Nonetheless, whether some $b \in {}^\dagger\mathcal{R}$ is the image of a tuple $\overline{a}$ under ${}^\dagger f$ is relative to a filter F, since equality is interpreted as an equivalence relation on each filter. As is customary in nonstandard analysis, I will identify a function f with its nonstandard extension ${}^\dagger f$ whenever no ambiguity arises. Let us also introduce the following definitions, which mirror usual notions in nonstandard models:

Definition 7.2.8. For any $a \in {}^\dagger\mathcal{R}$ and any $F \in \mathfrak{F}$, let $a_F = \{b \in {}^\dagger\mathcal{R} \mid F \Vdash a = b\}$, and let the *$F$-halo* of a be the set $(a)_F = \{b \in {}^\dagger\mathcal{R} \mid F \Vdash |a - b| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$.

[2] A simple example of this is the statement $\alpha = 0 \vee \alpha = 1$, for $\alpha(i) = j$ iff $i \equiv j \mod 2$, which is satisfied at F_0 even though neither $\alpha = 0$ nor $\alpha = 1$ are forced at F_0.

In other words, for any $a \in {}^\dagger\mathcal{R}$ and any $F \in \mathfrak{F}$, a_F is the collection of all guises that are identified from the viewpoint F with the same hyperreal as the one a designates, while $(a)_F$ is the set of all guises which are infinitesimally close to a from the viewpoint F. Obviously, $a_F \subseteq (a)_F$, but the converse is never true:

Lemma 7.2.9. *For any $a \in {}^\dagger\mathcal{R}$ and any $F \in \mathfrak{F}$, there is $b \in (a)_F \setminus a_F$.*

Proof. Fix $a \in {}^\dagger\mathcal{R}$ and $F \in \mathfrak{F}$, and let $b : \omega \to \mathbb{R}$ be defined as $b(i) = a(i) + \frac{1}{i}$ for all $i \in \omega$. Then note that $||a = b|| = \emptyset$, which means that $F \Vdash a \neq b$ and hence that $b \notin a_F$. However, $|||a - b| < \frac{1}{n}|| = \{i \in \omega \mid i > n\}$ for any $n \in \mathbb{N}$, which means that $|||a - b| < \frac{1}{n}|| \in F$, since F is free. Hence, by Theorem 7.2.5, $b \in (a)_F$. □

This lemma guarantees that we can meaningfully talk about infinitesimals in the Fréchet hyperreals. Of course, whether two elements $a, b \in {}^\dagger\mathcal{R}$ are infinitesimally close to one another may vary with F. Nonetheless, this still allows us to give a straightforward adaptation of the usual definition of continuity on the hyperreal line.

Definition 7.2.10 (F-continuity)**.** A function $f : {}^\dagger\mathcal{R} \to {}^\dagger\mathcal{R}$ is F-continuous at a point $c \in {}^\dagger\mathcal{R}$ if for any $x \in (c)_F$, $f(x) \in (f(c))_F$.

Just as in Luxemburg ultrapowers, this definition aims to capture in an intuitive way the idea that continuous functions are those functions f for which a small change in the argument of f will only entail a small change in the value of f. Since we want to use our notion of infinite closeness to characterize the idea of a "small change", we must relativize the notion of continuity on the ${}^\dagger\mathcal{R}$ to a viewpoint F. We nonetheless easily obtain the following theorem, which shows that our notion of F-continuity corresponds to the Weierstrass definition of standard analysis.

Theorem 7.2.11. *A function $f : \mathcal{R} \to \mathcal{R}$ is continuous at a point c if and only if ${}^\dagger f$ is F-continuous at c for any $F \in \mathfrak{F}$.*

Proof. Assume first that $f : \mathcal{R} \to \mathcal{R}$ is continuous, and let $F \in \mathfrak{F}$, $c \in \mathcal{R}$, and $x \in (c)_F$. Since f is continuous at c, for any $n \in \omega$, there is a positive $\delta \in \mathcal{R}$ such that for any $a \in \mathcal{R}$, $|c - a| < \delta$ implies $|f(c) - f(a)| < \frac{1}{n}$. Fix $n \in \omega$ and choose such a $\delta \in \mathcal{R}$. Note that $F \Vdash |c-x| < \delta$, and thus $|||c-x| < \delta|| \in F$. By choice of δ, for any $i \in |||c-x| < \delta||$, we have that $\mathcal{R} \models |f(c(i)) - f(x(i))| < \frac{1}{n}$. Thus $|||f(c) - f(x)| < \frac{1}{n}|| \in F$, and $F \Vdash |f(c) - f(x)| < \frac{1}{n}$. Since n was chosen arbitrarily, it follows that $f(x) \in (f(c))_F$.

Conversely, suppose that there is some $F \in \mathfrak{F}$ such that $x \in (c)_F$ implies $f(x) \in (f(c))_F$. Note that this means that for any positive $\epsilon \in \mathcal{R}$, and any positive $\delta \in (0)_F$, $F \Vdash \forall x(|c-x| < \delta \implies |f(c) - f(x)| < \epsilon)$. Since $(0)_F$ is non-empty, we have that $F \Vdash \exists\delta(\delta > 0 \wedge \forall x(|c-x| < \delta \implies |f(c) - f(x)| < \epsilon))$. By Corollary 7.2.6, since this first-order sentence only has a real parameter ϵ, it is also true in $\mathcal{R}$. But then there must be a positive $\delta \in \mathcal{R}$ such that for any $a \in \mathcal{R}$, $|c - a| < \delta$ implies $|f(c) - f(a)| < \epsilon$, thus establishing that f is continuous at c. □

This establishes that one of the most appealing features of classical nonstandard analysis, an intuitive characterization of continuity that uses infinitesimals in a coherent and powerful way, is still available in our semi-constructive setting. In fact, as I will argue in Section 7.6.2, there is a sense in which our non-constructive definition constitutes a better alternative to the Weierstrassian definition of continuity than the classical nonstandard one.

How much infinitesimal calculus could ${}^{\dagger}\mathcal{R}$ provide a foundation for? In the *Epilogue* of his textbook introducing calculus via the infinitesimal method [154], Keisler writes that all the results obtained in the textbook can be derived from the axioms of a complete ordered field for $\mathbb{R}$ and two axioms for the hyperreal line $\mathbb{B}^*$, the *Extension axiom* and the *Transfer axiom*. From the results obtained so far, it is easy to verify that the Extension axiom is valid on ${}^{\dagger}\mathcal{R}$, while the Transfer axiom is a weaker version of Corollary 7.2.6. This means that ${}^{\dagger}\mathcal{R}$ could be used as an alternative foundation for Keisler's textbook. Because it is a possibility structure, however, some caution is in order, and some arguments need to be slightly altered. For instance, Keisler proves the following from his axiom system for $\mathbb{R}$ and $\mathbb{R}^*$:

Standard Part Principle For any *finite* hyperreal number b (i.e., any $b \in \mathbb{R}^*$ such that $|b| < r$ for some $r \in \mathbb{R}$), there is exactly one real number r infinitely close to b.

Of course, this fundamental result of nonstandard analysis is what allows for the definition of the *standard part* function on the finite hyperreals, a key tool in deriving results of classical analysis. It is not true, however, that for any $F \in \mathfrak{F}$ and any $a \in {}^{\dagger}\mathcal{R}$ such that $F \Vdash |a| < \delta(r)$ for some real number r, we have $a \in (\delta(r))_F$ for some unique real r. For example, letting $a \in \mathcal{R}^\omega$ be given by $a(i) = \frac{(-1)^i}{2}$, we have that $F_0 \Vdash -1 < a < 1$, yet clearly $a \notin (\delta(r))_{F_0}$ for any real number r. Moreover, there does not seem to be a principled way of determining what the standard part of a should be at F_0, since, letting G_1 and G_2 be the filters in $\mathfrak{F}$ generated by adding the set of even and odd natural numbers to F_0, respectively, one would clearly want the G_1-standard part of a to be $\frac{-1}{2}$ and the G_2-standard part of a to be $\frac{1}{2}$.

The way out of this conundrum is to "internalize" the Standard Part Principle, i.e., to turn it into a first-order statement in an extended language and show that it is valid on ${}^{\dagger}\mathcal{R}$. The argument, which becomes a straightforward adaptation of Keisler's, can be found in Section 7.8.1. The need for such a detour is essentially due to the fact that the standard part function, just like the standardness predicate, are *external* objects in nonstandard analysis. Once the Standard Part Principle has been internalized, however, ${}^{\dagger}\mathcal{R}$ can play the same foundational role for Keisler's infinitesimal calculus as Luxemburg ultrapowers. Moreover, as we will see in the next section, a straightforward theory of *internal* objects can also be developed for ${}^{\dagger}\mathcal{R}$.

7.2.3 Internal Sets and Countable Saturation

Internal sets play a central role in classical nonstandard analysis, as they form a collection of "well-behaved" subsets of a nonstandard extension of the real line. As we shall see, we can also define in a natural way the notion of an internal subset of ${}^{\dagger}\mathcal{R}$, although this must first be relativized to a point $F \in \mathfrak{F}$.

Definition 7.2.12 (F-internal set)**.** Let $\{A_n\}_{n\in\omega}$ be a family of subsets of $\mathcal{R}$, and let $A_F = \{a \mid \{i \mid \mathcal{R} \models A_i(a(i))\} \in F\}$. An *$F$-internal set* is a subset of D such that $D = A_F$ for some family $\{A_n\}_{n\in\omega}$ of subsets of $\mathcal{R}$.

This definition mirrors the usual definition of an internal set in an ultrapower of the reals, as it can be found for example in [108, Chap. 11]. Since ${}^{\dagger}\mathcal{R}$ is a possibility structure, however, each filter in $\mathfrak{F}$ gives us a different viewpoint on what an internal set determined by a countable sequence $\{A_n\}_{n\in\omega}$ looks like. To keep track of how F-internal sets relate to one another across different filters, it is useful to expand the language $\mathscr{L}$, adding a predicate symbol (which I will call an *internal predicate*) A for each sequence $\{A_n\}_{n\in\omega}$ of subsets of $\mathcal{R}$. Let $\mathscr{L}^+$ be the new language. ${}^{\dagger}\mathcal{R}$ can be seen as an $\mathscr{L}^+$-structure by letting $a \in \mathscr{I}(F, A)$ iff $||A(a)|| = \{i \in \omega \mid \mathcal{R} \models A_i(a(i))\} \in F$ for any internal predicate A, any $a \in {}^{\dagger}\mathcal{R}$ and any $F \in \mathfrak{F}$. The following lemma is straightforward and ensures that this interpretation satisfies the conditions of Definition 6.1.1.

Lemma 7.2.13. *Let $\{A_n\}_{n\in\omega}$ be a sequence of subsets of $\mathcal{R}$ with corresponding internal predicate A, and $a \in {}^{\dagger}\mathcal{R}$. Then for any $F, G \in \mathfrak{F}$:*

Persistence *if $G \supseteq F$ and $a \in \mathscr{I}(F, A)$, then $a \in \mathscr{I}(G, A)$;*

Refinement *if $a \notin \mathscr{I}(F, A)$, then there is $G \supseteq F$ such that for all $H \supseteq G$, $a \notin \mathscr{I}(H, A)$;*

Equality as congruence *if $a \in \mathscr{I}(F, A)$ and $(a, b) \in \mathscr{I}(F, =)$ for some $b \in {}^{\dagger}\mathcal{R}$, then $b \in \mathscr{I}(F, A)$.*

Proof. Fix a countable sequence $\{A_n\}_{n\in\omega}$ of subsets of $\mathcal{R}$ with associated internal predicate A, $a, b \in {}^{\dagger}\mathcal{R}$ and $F \in \mathfrak{F}$.

Persistence Note that $a \in \mathscr{I}(F, A)$ iff $||A(a)|| \in F \subseteq G$, so $a \in \mathscr{I}(G, A)$.

Refinement If $a \notin \mathscr{I}(F, A)$, then there is $G \in \mathfrak{F}$ such that $G \supseteq F \cup (\omega \setminus ||A(a)||)$. Clearly, for any $H \supseteq G$, $||A(a)|| \notin H$, from which it follows that $a \notin \mathscr{I}(H, A)$.

Equality as congruence Note that $||a = b|| \cap ||A(a)|| = \{i \in \omega\mathcal{R} \models a(i) = b(i) \wedge A_i(a(i)\} \subseteq ||A(b)||$. Hence $(a, b) \in \mathscr{I}(F, =)$ and $a \in \mathscr{I}(F, A)$ together imply that $b \in \mathscr{I}(F, A)$. □

More generally, for any $\mathscr{L}^+$-formula $\varphi(A^1, ..., A^K, \overline{x})$ with n free variables where $A^1, ..., A^k$ are the internal predicates appearing in φ, and the corresponding sequences of subsets of $\mathcal{R}$ are $\{A_i^1\}_{i\in\omega}, ..., \{A_i^k\}_{i\in\omega}$, let $||\varphi(A^1, ..., A^k, \overline{a})|| = \{i \in \omega \mid \mathcal{R} \models \varphi(A_i^1, ..., A_i^k, \overline{a(i)})\}$ for any n-tuple $\overline{a} \in {}^{\dagger}\mathcal{R}$. Note that this is well defined since $\varphi(A_i^1, ...A_i^k, \overline{x})$ is an $\mathscr{L}$-formula for any $i \in \omega$. By adapting the proof of Theorem 7.2.5 in a straightforward way, we also obtain the following corollary.

Corollary 7.2.14. *For any $\mathscr{L}^+$ formula $\varphi(\overline{x})$ with n free variables and any n-tuple $\overline{a}$ of elements of ${}^\dagger\mathcal{R}$, we have that $F \Vdash \varphi(\overline{a})$ iff $||\varphi(\overline{a})|| \in F$ for any $F \in \mathfrak{F}$.*

This result allows us to describe the F-internal subsets of ${}^\dagger\mathcal{R}$ "internally", that is, using the satisfaction relation at F. In particular, we can define the following:

Theorem 7.2.15 (F-Internal Set Definition Principle)**.** *Let $\varphi(\overline{A}, \overline{x}, y)$ be an $\mathscr{L}^+$ formula with $n+1$ variables and where $\overline{A}$ is a tuple of internal predicates. Then for any n-tuple $\overline{a} \in {}^\dagger\mathcal{R}$ and any $F \in \mathfrak{F}$, $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \varphi(\overline{A}, \overline{a}, b)\}$ is an F-internal subset of ${}^\dagger\mathcal{R}$.*

Proof. By induction on the complexity of φ. Note that this is trivial if $\varphi(x) = A(x)$ for some internal predicate A.

- If R is an $n+1$-ary $\mathscr{L}$ relation symbol, then let $\{R_i\}_{i\in\omega}$ be such that $R_i = \{b \in \mathcal{R} \mid R(\overline{a}(i), b)\}$, and let ${}^\dagger R^{\overline{a}}$ be the corresponding internal predicate in $\mathscr{L}^+$. Then clearly for any $F \in \mathfrak{F}$ $F \Vdash {}^\dagger R(\overline{a}, b)$ iff $||R(\overline{a}, b)|| \in F$ iff $||\,{}^\dagger R^{\overline{a}}(b)|| \in F$ iff $b \in {}^\dagger R^{\overline{a}}_F$.

- If $\varphi := \neg\psi(\overline{A}, \overline{x}, y)$, then by induction hypothesis we have a countable sequence $\{S^\psi_i\}_{i\in\omega}$ such that $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \psi(\overline{A}, \overline{a}, b)\} = S^\psi_F$ for any $F \in \mathfrak{F}$. Define $\{S^\varphi_i\}_{i\in\omega}$ by letting $S^\varphi_i = \mathcal{R} \setminus S^\psi_i$ for any $i \in \omega$. Then, for any $b \in {}^\dagger\mathcal{R}$, we have that $||S^\varphi(b)|| = \omega \setminus ||S^\psi(b)||$, from which it follows that $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \varphi(\overline{A}, \overline{a}, b)\} = S^\varphi_F$.

- If $\varphi := \psi \wedge \chi(\overline{A}, \overline{x}, y)$, then by induction hypothesis we have countable sequences $\{S^\psi_i\}_{i\in\omega}$ and $\{S^\chi_i\}_{i\in\omega}$ such that $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \psi(\overline{A}, \overline{a}, b)\} = S^\psi_F$ and $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \chi(\overline{A}, \overline{a}, b)\} = S^\chi_F$ for any $F \in \mathfrak{F}$. Define $\{S^\varphi_i\}_{i\in\omega}$ by letting $S^\varphi_i = S^\psi_i \cap S^\chi_i$ for any $i \in \omega$, and note that $||S^\varphi(b)|| = ||S^\psi(b)|| \cap ||S^\chi(b)||$ for any $b \in {}^\dagger\mathcal{R}$, thus showing that $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \varphi(\overline{A}, \overline{a}, b)\} = S^\varphi_F$.

- If $\varphi := \exists z\psi(\overline{A}, \overline{x}, y, z)$, then by induction hypothesis we have countable sequences $\{S^{\psi(c)}_i\}_{i\in\omega}$ such that $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \psi(\overline{A}, \overline{a}, b, c)\} = S^{\psi(c)}_F$ for any $c \in {}^\dagger\mathcal{R}$. For any $i \in \omega$, let $S^\varphi_i = \bigcup_{c\in{}^\dagger\mathcal{R}} S^{\psi(c)}_i$. I claim that $||\exists z\psi(\overline{A}, \overline{a}, b, z)|| = ||S^\varphi(b)||$ for any $b \in {}^\dagger\mathcal{R}$, which is enough to show that $\{b \in {}^\dagger\mathcal{R} \mid F \Vdash \varphi(b)\} = S^\varphi_F$. For the proof of the claim, note first that, using the Axiom of Countable Choices, we can define $c \in R^\omega$ such that $||\exists z\psi(\overline{A}, \overline{a}, b, z|| \subseteq ||\psi(\overline{A}, \overline{a}, b, c|| = ||S^{\psi(c)}(b)|| \subseteq \underline{||S^\varphi(b)||}$. Finally, the converse direction is straightforward: if $i \in \underline{||S^\varphi(b)}||$, then $\mathcal{R} \models \psi(\overline{A_i}, \underline{a(i)}, b(i), \underline{c(i)})$ for some $c \in {}^\dagger\mathcal{R}$, which implies that $\mathcal{R} \models \exists z\varphi(\overline{A_i}, \underline{a(i)}, b(i), z)$, and hence $i \in ||\exists x\varphi(\overline{A}, \overline{a}, b, c)||$. $\square$

As an immediate corollary of Theorem 7.2.15, we obtain the following result on the structure of F-internal sets.

Corollary 7.2.16. *Let $F \in \mathfrak{F}$. Every nonstandard extension of a subset of the reals is an F-internal set. Moreover, the F-internal sets form a Boolean algebra.*

Proof. Recall that if $S \subseteq \mathbb{R}$, then ${}^\dagger S_F = \{b \in {}^\dagger\mathcal{R} \mid F \Vdash S(b)\}$. By Theorem 7.2.15, it follows at once that ${}^\dagger S_F$ is F-internal for any $F \in \mathfrak{F}$. Moreover, for any internal predicates A, B, let $\neg_F A_F = \{b \in {}^\dagger\mathcal{R} \mid F \Vdash \neg A(b)\}$, $A_F \wedge_F B_F = \{b \in {}^\dagger\mathcal{R} \mid F \Vdash A(b) \wedge B(b)\}$, and

$A_F \vee_F B_F = \{b \in {}^\dagger\mathcal{R} \mid F \Vdash A(b) \vee B(b)\}$. By Theorem 7.2.15, the operations $\neg_F$, $\wedge_F$ and $\vee_F$ map F-internal sets to F-internal sets, and, using Corollary 7.2.14, it is straightforward to verify that $\mathfrak{I}_F = (\{A_F\}_{A \in \mathscr{L}^+ \setminus \mathscr{L}}, \wedge_F, \vee_F, \neg_F, {}^\dagger\mathcal{R}, \emptyset)$ is a Boolean algebra. □

The Definition Principle above therefore gives us a powerful way of defining F-internal sets and ensures that many subsets of ${}^\dagger\mathcal{R}$ are F-internal for any F. Note however that, because we are working in a possibility structure, the extension of an internal subset of ${}^\dagger\mathcal{R}$ must always be relativized to a filter F. In other words, the Definition Principle ensures that *the same* internal subsets, understood *intensionally*, exist at any point $F \in \mathfrak{F}$. But whether an element in ${}^\dagger\mathcal{R}$ belongs to a given internal set S will vary with the filter F, i.e., the *extension* of internal sets is relative to the viewpoints in $\mathfrak{F}$. To illustrate this point, consider the internal predicate ${}^\dagger N$, corresponding to the sequence $\{N_i\}_{i \in \omega}$ where N_i is the set of all natural numbers for any $i \in \omega$. In other words, ${}^\dagger N$ represents the nonstandard extension of the natural numbers, i.e., the internal set of hypernatural numbers. For any $F \in \mathfrak{F}$, the hypernatural-numbers-at-F are an F-internal set, namely ${}^\dagger N_F$. But this does not mean that the set of hypernatural numbers is stable across points in $\mathfrak{F}$. As a matter of fact, if we consider the sequence $a \in {}^\dagger\mathcal{R}$ defined by $a(2k) = k$ and $a(2k+1) = \pi$ for all $k \in \omega$, then $a \in {}^\dagger N_F$ precisely when $\{2k \mid k \in \omega\} \in F$. From now on, I will adopt the "intensional" perspective on internal sets whenever possible, meaning that I will use the phrase "internal set" to designate the varying extension of an internal predicate S.

In Luxemburg ultrapowers, a crucial feature of internal sets is their countable saturation property: any countable decreasing sequence of non-empty internal sets has a non-empty intersection. As we shall see, the same property, once relativized in a natural way to viewpoints in $\mathfrak{F}$, can also be proved for the internal sets on ${}^\dagger\mathcal{R}$.

Notation 7.2.17. Let A and B be internal predicates corresponding to countable sequences $\{A_i\}_{i \in \omega}$ and $\{B_i\}_{i \in \omega}$ respectively. Then $||A \subseteq B||$ is defined as the set $\{i \in \omega \mid A_i \subseteq B_i\}$, and $||A \neq \emptyset||$ as the set $\{i \in \omega \mid A_i \neq \emptyset\}$.

By Corollary 7.2.14, it is straightforward to verify that for any $F \in \mathfrak{F}$, and any internal predicates A and B, $||A \subseteq B|| \in F$ iff $F \Vdash \forall x(A(x) \to B(x))$, and $||A \neq \emptyset|| \in F$ iff $F \Vdash \exists x \in A$. We are now in a position to prove the second important feature of ${}^\dagger\mathcal{R}$:

Theorem 7.2.18 (Countable F-saturation Principle)**.** *Let $F \in \mathfrak{F}$ and let $\{X^i\}_{i \in \omega}$ be a family of F-internal sets such that for any $k \in \omega$, $F \Vdash \exists x X^k(x)$ and $F \Vdash \forall x(X^{k+1}(x) \to X^k(x))$. Then there is $a \in {}^\dagger\mathcal{R}$ such that $F \Vdash X^k(a)$ for all $k \in \omega$.*

Proof. The proof mirrors the usual proof for Luxemburg ultrapowers as it can be found for example in [108, Theorem 11.10.1]. For any $k \in \omega$, let $\{A^k_n\}_{n \in \omega}$ be the countable sequence of subsets of $\mathbb{R}$ such that $X^k = A^k_F$. Since $F \Vdash \exists x X^k(x)$, this means that $||A^k \neq \emptyset|| \in F$ for each $k \in \omega$. Similarly, since $F \Vdash \forall x(X^{k+1}(x) \to X^k(x))$ for any $k \in \omega$, this means that $||A^{k+1} \subseteq A^k|| \in F$ for all k. Hence for any $k \in \omega$, letting $J^k = \cap_{i<k} ||A^{i+1} \subseteq A^i|| \cap ||A^k \neq \emptyset||$, we have that $J^k \in F$ and $J^i \subseteq J^k$ for any $i \geq k$. Now if we construct $a \in {}^\dagger\mathcal{R}$ such that for

any $k \in \omega$ and any $n \geq k$, if $n \in J^k$, then $a(n) \in A^k_n$, then as $\{n \in \omega \mid k \leq n\} \cap J^k \in F$ since F extends the Fréchet filter on ω, we will have that $a \in A^k_F$ for all $a \in F$. We define a in the standard way: for any $n \in J^1$, let $k_n = max\{k \mid k \leq n \text{ and } n \in J^k\}$. Clearly, $n \in J^{k_n}$, so by definition $n \in ||A^{k_n} \neq \emptyset||$, i.e., $\mathbb{R} \models \exists x A^{k_n}_n(x)$ and moreover, $A^{j+1}_n \subseteq A^j_n$ for any $j < k$. So let $a(n) \in A^{k_n}$ if $n \in J^1$, and let $a(n)$ be arbitrary otherwise. Finally, fix $k \in \omega$, and note that, if $n \geq k$ and $n \in J^k$, then $k \leq k_n$, hence $a(n) \in A^{k_n}_n \subseteq A^k_n$. Thus $a \in \bigcap_{k \in \omega}(X^k)_F$. □

Let us conclude this section with some results on the cardinality of F-internal sets. Of course, since our internal notion of equality is an equivalence relation relative to a point $F \in \mathfrak{F}$, we must be careful in defining internally what finiteness, countability and uncountability mean for internal sets. For example, we would clearly want the set of natural numbers less than some positive natural number n to be finite. But it is easy to see that for any $F \in \mathfrak{F}$, there are at least countably many $a \in {}^\dagger\mathcal{R}$ for which $F \Vdash a < n$, since for any $k \in \omega$, the function $a_k : \omega \to \mathcal{R}$ given by $a_k(i) = 0$ if $i < k$ and $a_k(i) = n - 1$ otherwise will be one such element of ${}^\dagger\mathcal{R}$. This motivates the following definitions:

Definition 7.2.19. Let $F \in \mathfrak{F}$. A subset A of ${}^\dagger\mathcal{R}$ is *F-finite* if $A_F = \bigcup_{i \in I} a^i_F$ for some finite set I with $a^i \in {}^\dagger\mathcal{R}$ for all $i \in I$, *F-countable* if there is a countable sequence $\{a^i\}_{i \in \omega}$ of elements of ${}^\dagger\mathcal{R}$ such that $A = \bigcup_{i \in \omega} a^i_F$, and *$F$-uncountable* otherwise.

It is clear from the previous definition that sets such as the set of all natural numbers below some positive number n are F-finite for any $F \in \mathfrak{F}$, just as the set of standard natural numbers is F-countable for any F. However, the following lemma also shows that our relativized notions of finiteness and countability interact with the notion of an internal set in a meaningful way.

Lemma 7.2.20.

1. *Let A be a subset of ${}^\dagger\mathcal{R}$ and $F \in \mathfrak{F}$ be such that for all $a \in {}^\dagger\mathcal{R}$, $a \in A$ iff there is $r \in \mathcal{R}$ such that $a \in \delta(r)_F$. Then A is F-internal iff it is F-finite.*

2. *Any internal subset of ${}^\dagger\mathcal{R}$ is either finite or uncountable.*

Proof.

1. Let $B \subseteq \mathcal{R}$ be the set $\{r \in \mathcal{R} \mid \delta(r) \in A\}$. By assumption, we have that $A = \bigcup_{r \in B} \delta(r)_F$. Note that A is F-finite iff B is finite. Clearly, if $B = \{r_1, ..., r_n\}$ is finite, then for all $a \in {}^\dagger\mathcal{R}$, $a \in A$ iff $F \Vdash a = \delta(r_1) \vee \cdots \vee a = \delta(r_n)$, hence A is F-internal by Theorem 7.2.15. Conversely, suppose that A is F-internal, i.e., that there is a sequence $\{B_i\}_{i \in \omega}$ such that $B_F = A$. Assume B is infinite, and let $\{r^i\}_{i \in \omega}$ be a countable sequence of elements of B.[3] Let J be the set of all $i \in \omega$ such that B_i is infinite, and relabel J and $\omega \setminus J$ as $\{n_i\}_{i \in \omega}$ and $\{m_i\}_{i \in \omega}$ respectively. Define $a : \omega \to \mathcal{R}$ as follows. For any $i \in \omega$, $a(m_i) = r^m$ for m largest such that $r_m \in B_{m_i}$ (or some fixed $r \notin B$ if

[3]Note that we are using the Axiom of Countable Choices here to ensure that the infinite set B is also Dedekind-infinite.

B_{m_i} is empty), and $a(n_i) = r^n$ for n smallest such that $r_n \in B_{n_i} \setminus \{a(n_j) \mid j < i\}$. By construction, $i \in ||B_i(a)||$ whenever $B_i \neq \emptyset$, hence $a \in B_F$. Moreover, for any $i \in \omega$, $|||a = r^i|| \cap J| \leq 1$, and therefore $\omega \setminus (||a = r^i|| \cap J) \in F$. Note also that $j \in ||a = r^i|| \setminus J$ implies that $|B_j| \leq i$. But since $F \Vdash \exists x_1, ..., x_{i+1} \bigwedge_{k \neq l \leq i+1} x_k \neq x_l \wedge \bigwedge_{k \leq i+1} B(x_k)$, it follows that $||a = r^i|| \setminus J \notin F$. But this implies that $||a = r^i|| \notin F$ for any $i \in \omega$, and hence that $a \in B_F \setminus A$, contradicting our hypothesis. Hence if A is F-internal, it must also be finite.

2. Let A be an internal subset of ${}^{\dagger}\mathcal{R}$ with $\{A_i\}_{i \in \omega}$ its associated countable sequence of subsets of $\mathcal{R}$, and suppose that A is F-countable. This means that we have a sequence $\{a^i\}_{i \in \omega}$ of elements of ${}^{\dagger}\mathcal{R}$ such that $A_F = \bigcup_{i \in \omega} a^i_F$. By the Definition Principle, for any $n \in \omega$, the set $A^i = A \setminus \{a^1, \ldots .a^i\}$ is F-internal. But if $F \Vdash \exists x A^i(x)$ for all $i \in \omega$, then by Theorem 7.2.18 there is $a \in {}^{\dagger}\mathcal{R}$ such that $F \Vdash A^i(a)$ for all $i \in \omega$, a contradiction. Therefore A^k_F must be empty for some $k \in \omega$, which implies that $A_F = \bigcup_{i \leq k} a^k_F$, and thus that A is F-finite. Hence any F-internal set is either F-finite or uncountable. □

The previous lemma echoes a well-known result for Luxemburg ultrapowers, namely that the only internal subsets of $\mathcal{R}$ on the hyperreal line are the finite ones and that any internal subset of the hyperreal line is either finite or uncountable [108, Sections 11.7 and 11.12]. As an immediate consequence, we have that the set of all standard natural numbers $\mathcal{N}$ and the set of all standard real numbers $\mathcal{R}$ are always F-external.

Let us sum up our progress so far. As we have seen in this section, the Fréchet hyperreals ${}^{\dagger}\mathcal{R}$ form a structure that enjoys many important features of the classical hyperreal line, including versions of the Transfer and Saturation Principles, infinitesimals, and a robust notion of internal set. The crucial difference though is that, while the classical hyperreal is a Tarskian structure whose existence requires strong non-constructive hypotheses like the Ultrafilter Lemma, ${}^{\dagger}\mathcal{R}$ is a possibility structure which can be defined in a semi-constructive setting, assuming only $ZF + DC$. As mentioned in Section 7.1, the idea of proposing more constructive frameworks for nonstandard analysis is far from new and is in fact arguably as old as Robinsonian nonstandard analysis itself [231]. As we will see in the next three sections, ${}^{\dagger}\mathcal{R}$ can be thought of as an "intersection point" of three such research programs: the use of the Fréchet filter in place of a free ultrafilter on ω, the use of sheaves instead of sets as the carrier of a model, and the use of Boolean-valued instead of classical, two-valued, Tarskian models. As we will see, the comparison of ${}^{\dagger}\mathcal{R}$ with each such proposal in the literature also allows for different perspectives on ${}^{\dagger}\mathcal{R}$ and on its relationship with the classical hyperreal line.

7.3 The Asymptotic Approach

The first alternative approach to the hyperreal line that I will discuss aims to do away with the non-constructive aspect of Robinsonian nonstandard analysis by substituting the Fréchet filter to a free ultrafilter on ω. Although constructive, the structure thus obtained, known as a reduced power, has some significant drawbacks, most importantly the lack of a Transfer Principle. On the other hand, according to its proponents, this idea has the advantage of not introducing in analysis an "exotic" object like a free ultrafilter—an *intangible*, to use Schechter's terminology in [230]—and appears to be closer both to ordinary mathematical practice and to historical work that involves reasoning about mathematical objects "in the limit". I will first present Laugwitz's work in [168], before discussing a recent proposal by Tao in [251], which can be seen as a refinement of Laugwitz's idea, and comparing each proposal to ${}^{\dagger}\mathcal{R}$.

7.3.1 Laugwitz and Reduced Powers

Laugwitz's starting point is slightly different from what we have discussed so far. Indeed, while we have been taking the Cantor-Dedekind real line for granted and exploring how to expand it to a hyperreal line, Laugwitz is first interested in exploring the kind of structures that can be obtained from sequences of rationals. Drawing a contrast with Cantor's construction of the reals as equivalence classes of Cauchy-converging sequences of rationals, he writes:

> The two steps of the sequential approach to the real numbers are (i) to restrict the set of admissible rational sequences to "fundamental" sequences, and (ii) to furnish this set of fundamental sequences with an equivalence relation. Since our extended number system is expected to contain "more" numbers than $\mathbf{R}$ itself, we have to change these assumptions, and there are two possibilities to achieve this aim: (i) to admit more sequences, (ii) to relax the equivalence relation. In a radical way, we shall (i) admit all sequences, and (ii) we shall identify only those sequences which are equal for almost all numbers n, $a \equiv b$ iff the complement of $\{n \in \mathbf{N} | a_n = b_n\}$ is finite. [168, p. 11]

Formally, Laugwitz works with a reduced power of a field $(K, +, \cdot, 0, 1)$ modulo the Fréchet filter F_0. The domain of such a structure, noted ${}^{\Omega}K$ by Laugwitz, is given by equivalence classes of sequences $a : \omega \to K$, where two such sequences a and b are equivalent if and only if they agree on cofinitely many values of n. Operations on K can be extended to ${}^{\Omega}K$ pointwise: for example, given two sequences a, b with equivalence classes a_{F_0} and b_{F_0}, $a_{F_0} + b_{F_0}$ is the equivalence class of the function $c : \omega \to K$ defined by $c(n) = a(n) + b(n)$ for every $n \in \omega$. The properties of a filter are enough to guarantee that $({}^{\Omega}K, +, \cdot)$ is a well-defined first-order structure. But, as Laugwitz notes, working with a reduced power has some important drawbacks:

> It is an easy task to show that $({}^{\Omega}K, +, \cdot)$ is a commutative ring with unit element [1] [...]. Unfortunately ${}^{\Omega}K$ fails to be a field. Consider the sequences
>
> $$a_n = 1 + (-1)^n, \quad b_n = 1 - (-1)^n \tag{3}$$

> then $\alpha \neq 0$ and $\beta \neq 0$ since neither the set of odd numbers nor the set of even numbers is cofinite. But, obviously, $\alpha \cdot \beta = 0$. So $({}^{\Omega}K, +, \cdot)$ is a ring with divisors of zero and certainly not a field. Moreover, the generalized numbers α, β defined by (3) are neither positive nor negative, which means that our ring is only partially ordered by $<$. [p. 13]

Of course, this is an immediate consequence of the use of the Fréchet filter rather than a free ultrafilter: while a weaker version of Łoś's Theorem holds for a fairly large class of first-order sentences that includes equations,[4] it does not hold for all first-order sentences, particularly sentences involving disjunctions and negations. This means that the "expanded real line" that Laugwitz is considering does not have the structure of a field and is not even a linear order. Interestingly, Laugwitz argues that this feature is not necessarily an issue, by drawing a parallel to the fact that the complex numbers are a domain extension of the ordered field of reals that is not itself an ordered field:

> Now this situation is not surprising. We are used to the fact that not all good properties of a number system extend automatically to a larger system. For instance, the order structure of **R** cannot be extended to the complex numbers. But nevertheless we reach the essential goal of the complex numbers, to solve algebraic equations. Since we aimed at introducing infinitely large numbers as well as infinitesimals we shall look for elements of ${}^{\Omega}K$ enjoying these properties. [p. 13]

Laugwitz's argument seems to be that working in a structure that is neither a field nor a linear order is an acceptable price to pay in order to have infinitesimal and infinitely large numbers, just like working in a non linearly-ordered field is an acceptable price to pay to obtain a field that is algebraically closed. Of course, much could be said about this analogy. Although no algebraically closed field can be linearly ordered, since it follows from the ordered field axioms that $x^2 > 0$ whenever $x < 0$ or $x > 0$, it is not true that a linearly ordered field must always be Archimedean: the hyperreal line is a counterexample, but so are Conway's surreal numbers, or the Levi-Civita field [79]. As we have seen in the previous section, ${}^{\dagger}\mathcal{R}$ is also a possibility structure on which all the axioms of a field are valid.

Nevertheless, Laugwitz goes on to show how his framework allows him to rigorously reproduce several historical arguments from Leibniz, Euler, and Cauchy, among others. He argues, quite convincingly, that the reduced power construction captures a powerful and historically significant intuition that the behavior of sequences "in the limit" is determined by their behavior "almost everywhere", i.e., on cofinitely many of their values:

> I should like to emphasize that our rather simple definitions already permit correct interpretations of large parts of the history of infinitesimal mathematics as well as of the use of divergent expressions. [...] The proofs invariably run as follows: Write n instead of ${}^{\Omega}$ and show that everything holds for almost all $n \in \mathbf{N}$! [p. 14]

[4] In fact, Horn sentences true in the base structure remain true in the reduced power, see [56]

Finally, Laugwitz shows how a system isomorphic to the Cantor-Dedekind real numbers can be obtained as a quotient of the structure ${}^{\Omega}\mathbb{Q}$. His notation is at times a bit obscure, but the main idea is to define a *cut number* in ${}^{\Omega}\mathbb{Q}$ as a sequence $a : \omega \to \mathbb{Q}$ such that the absolute value of $a(n)$ is bounded by some rational q cofinitely often, and for any rational number r, $|a(n) - r|$ is smaller than any fraction $\frac{1}{m}$ for any $m \in \omega$ for some large enough n (in which case a is infinitesimally close to r), or $a(n) < r$ cofinitely often, or $a(n) > r$ cofinitely often. In other words, a cut number is a bounded element a_{F_0} of the reduced power of $\mathbb{Q}$ modulo the Fréchet filter for which a form of the trichotomy law holds with respect to the rationals: for all $q \in \mathbb{Q}$, either ${}^{\Omega}\mathbb{Q} \models a_{F_0} < q \vee a_{F_0} > q$, or a_{F_0} is in the *monad* of q (denoted mon q) i.e., ${}^{\Omega}\mathbb{Q} \models |a_{F_0} - q| < \frac{1}{m}$ for every natural number m.[5] Laugwitz goes on to prove that the cut numbers form a subring C of the bounded numbers of ${}^{\Omega}\mathbb{Q}$, and that mon 0, i.e., the set of numbers infinitesimally close to 0, is a maximal ideal C. Taking the quotient $C/\text{mon } 0$ thus yields a field, which can then be shown to be isomorphic to the Cantor-Dedekind reals. This construction of the reals from the hyperrational numbers can be carried out in a similar fashion in classical nonstandard analysis, by just quotienting the ring of bounded hyperrationals by the relation of infinitesimal closeness. Laugwitz's setting however requires slightly more work, since the ring ${}^{\Omega}\mathbb{Q}$ is not totally ordered. Laugwitz uses this result to give an intuitive interpretation of the cut numbers, comparing once more the role they can play in calculus to the role that the complex numbers play in algebra:

> We are now in a position to give some intuitive interpretations of our concepts. Though the cut numbers themselves are not a totally ordered set their equivalence classes or monads enjoy this property. We can imagine them as clusters like pearls on a string. Of course between each two of these clusters there are others.[...] The elements of these clusters can be regarded as representatives of the fine structure of real numbers. [...] Though these numbers lack a representation in the geometrical continuum we use them as calculation devices just as everybody uses complex numbers to get real results more easily. [p. 18]

To sum up, Laugwitz works with a reduced power of a field K modulo the Fréchet filter in order to introduce new infinite and infinitesimal numbers in K. He shows how many classical results about infinite series and calculus can be derived in an elegant fashion in this framework and how the reduced power of $\mathbb{Q}$ allows for an alternative construction of the Cantor-Dedekind reals. The major drawback of this approach, of course, is the lack of a Transfer Principle between the field K and the reduced power ${}^{\Omega}K$, which fails to be a field and is not totally ordered. It is worth mentioning that, in some precise sense, the Fréchet hyperreals ${}^{\dagger}\mathcal{R}$ contain a copy of the reduced power of $\mathcal{R}$ modulo the Fréchet filter F_0. Indeed, taking the quotient of $\mathcal{R}^{\omega}$ modulo the relation $\sim_{F_0} = \{(a, b) \in \mathcal{R}^{\omega} \mid F_0 \Vdash a = b\}$, and interpreting any $\mathscr{L}$-relation symbol $R(\overline{x})$ by $\{\overline{a_{F_0}} \in (\mathcal{R}^{\omega})_{/\sim_{F_0}} \mid F_0 \Vdash R(\overline{a})\}$, one obtains exactly Laugwitz's ${}^{\Omega}\mathcal{R}$. Note however that this does not imply in any way that the forcing relation at F_0 coincides with satisfaction in ${}^{\Omega}\mathcal{R}$, since the Transfer Principle guarantees that

[5] Laugwitz's *monads* correspond to what we called *halos* in Definition 7.2.8, following [108].

the axioms of an ordered field are forced at any $F \in \mathfrak{F}$. The difference comes down to the recursive clauses of the forcing conditions in possibility semantics, which, unlike the recursive clauses in Tarskian semantics, allow us to consider not only satisfaction of atomic formulas at a reduced power of $\mathcal{R}$ modulo the Fréchet filter but also at any other reduced power of $\mathcal{R}$ modulo a free ultrafilter on ω.

Since validity on ${}^{\dagger}\mathcal{R}$ coincides with the forcing relation at F_0, we may therefore argue that ${}^{\dagger}\mathcal{R}$, just like Laugwitz's framework, formalizes the idea that many classical arguments from calculus rely on the intuition that the behavior of mathematical objects "in the limit" is determined by how often they exhibit such a behavior in the finite and that this intuition can be given a rigorous and elegant presentation in a structure that contains infinitely large and infinitely small elements. But where Laugwitz weakens the requirements on the axioms that such an enlarged structure should satisfy (requiring it to be merely a ring rather than an ordered field), we weaken the requirement that this enlarged structure be a classical, Tarskian structure and construct it instead as a possibility structure that satisfies all the axioms of ordered fields. This trade-off between standard semantics and powerful axiomatics appears also in Tao's proposal of a "cheap" nonstandard analysis, which shares many features with Laugwitz's approach, but can also be thought of as an informal version of ${}^{\dagger}\mathcal{R}$.

7.3.2 Tao and "Cheap" Nonstandard Analysis

In [251], Tao introduces what he calls "cheap" nonstandard analysis as an informal framework that allows one to apply some of the elegant reasoning of nonstandard analysis, without relying on the existence of free ultrafilters and the construction of ultrapowers. Tao's motivation seems both foundational, as the use of infinitary methods to obtain "finitary" results seems to violate some purity concern, as well as purely technical:

> On the other hand, nonprincipal ultrafilters do have some unappealing features. [...O]ne cannot actually write down any explicit example of a nonprincipal ultrafilter, but must instead rely on nonconstructive tools such as Zorn's lemma, [...] or the [Ultrafilter Lemma] to locate one. As such, ultrafilters definitely belong to the "infinitary" side of mathematics, and one may feel that it is inappropriate to use such tools for "finitary" mathematical applications, such as those which arise in hard analysis. From a more practical viewpoint, because of the presence of the infinitary ultrafilter, it can be quite difficult [...] to take a finitary result proven via nonstandard analysis and coax an effective quantitative bound from it.

Tao's suggestion is very close to Laugwitz's idea, as it involves working with the Fréchet filter rather than with a free ultrafilter on ω. Tao also seems to think that, unlike free ultrafilters, the Fréchet filter is not an "exotic" mathematical object but rather already plays a key role in any kind of ordinary mathematical reasoning that involves passing to a limit:

> There is however a "cheap" version of nonstandard analysis which is less powerful than the full version, but is not as infinitary in that it is constructive [...]. It is obtained by replacing the nonprincipal ultrafilter in fully nonstandard analysis with the more classical Fréchet filter of cofinite subsets of the natural numbers, which is the filter that implicitly underlies the concept of the classical limit $\lim_{\mathbf{n}\to\infty} a_{\mathbf{n}}$ of a sequence when the underlying asymptotic parameter $\mathbf{n}$ goes off to infinity. As such, "cheap nonstandard analysis" aligns very well with traditional mathematics [...].

More precisely, the starting point of Tao's proposal is to consider mathematical objects that may vary along some parameter $\mathbf{n}$, which ranges over ω. Such a mathematical object $x_{\mathbf{n}}$ can therefore be thought of as a countable sequence of classical objects. The mathematical universe of Tao's cheap nonstandard analysis is therefore two-sorted: there are *standard* or classical mathematical objects, which do not depend on the parameter $\mathbf{n}$, and *nonstandard* ones, which may vary with $\mathbf{n}$. Of course, we may always think of standard objects as nonstandard ones that happen to not vary with $\mathbf{n}$. Two nonstandard objects $x_{\mathbf{n}}$ and $y_{\mathbf{n}}$ are considered equal if they differ on only finitely many values of the parameter $\mathbf{n}$. More generally, a tuple of nonstandard objects $\overline{x_{\mathbf{n}}}$ stand in a relation R if and only if they stand in relation R for cofinitely many values of the parameter $\mathbf{n}$. From this, it follows pretty clearly that Tao's approach amounts to working with equivalence classes of the countable sequences of objects modulo the Fréchet filter and is thus very similar to Laugwitz's proposal. As Tao himself remarks, it has similar shortcomings:

> The catch is that the Fréchet filter is merely a filter and not an ultrafilter, and as such some of the key features of fully nonstandard analysis are lost. Most notably, the law of the excluded middle does not transfer over perfectly from standard analysis to cheap nonstandard analysis [...]. The loss of such a fundamental law of mathematical reasoning may seem like a major disadvantage for cheap nonstandard analysis, and it does indeed make cheap nonstandard analysis somewhat weaker than fully nonstandard analysis.

Tao's framework therefore not only does not satisfy the Transfer Principle but does not even allow for classical reasoning. Tao argues that the problem can often be bypassed by reasoning intuitionistically and that a large class of formulas transfer from the standard universe to the nonstandard universe, although he does not precisely isolate a fragment of the first-order language for which the Transfer Principle would remain valid. Moreover, he also discusses a way of retrieving some form of the law of excluded middle, by using a technique that he calls "passing to a subsequence":

> Furthermore, the law of the excluded middle can be recovered by adopting the freedom to pass to subsequences with regards to the asymptotic parameter $\mathbf{n}$; this technique is already in widespread use in the analysis of partial differential equations, although it is generally referred to by names such as "the compactness method" rather than as "cheap nonstandard analysis".

What does Tao mean by "passing to a subsequence" here? His treatment of the notion remains informal, but the idea is that a given mathematical object $x_{\mathbf{n}}$, indexed by some parameter $\mathbf{n}$ that ranges over ω, may be identified with another mathematical object $x'_{\mathbf{n}}$, in which the parameter $\mathbf{n}$ now only ranges over some proper infinite subset $\Sigma \subseteq \omega$, and $x'(n) \neq x(n)$ for only finitely many $n \in \Sigma$. The example he gives is that of two parametrized reals $x_{\mathbf{n}}$ and $y_{\mathbf{n}}$ such that $x_{\mathbf{n}}y_{\mathbf{n}} = 0$. Clearly, this does not imply that $x_{\mathbf{n}} = 0$ or $y_{\mathbf{n}} = 0$, but it implies that there must be some infinite $\Sigma \subseteq \omega$ such that $x(n) = 0$ for all $n \in \Sigma$ or $y(n) = 0$ for all $n \in \Sigma$, since $\{n \in \omega : x(n)y(n) = 0\} = \{n \in \omega \mid x(n) = 0\} \cup \{n \in \omega \mid y(n) = 0\}$ is infinite. Therefore, by "passing to the subsequence" parametrized by n ranging over Σ, it follows that $x_{\mathbf{n}} = 0$ or $y_{\mathbf{n}} = 0$. Now let F_Σ be the filter on ω extending the Fréchet filter and generated by Σ. Clearly, "passing to the subsequence" determined by Σ amounts to declaring that $x_{\mathbf{n}}$ or $y_{\mathbf{n}}$ is equivalent to 0, provided we weaken the notion of equivalence between two parametrized objects from being equal on a cofinite set to being equivalent on a set in F_Σ. Thus Tao's technique of passing to a subsequence in cheap nonstandard analysis can be identified with our use in ${}^\dagger\mathcal{R}$ of the entire poset $\mathfrak{F}$ of free filters on ω, since reducing the range of the parameter $\mathbf{n}$ amounts to identifying together more parametrized objects, just like more elements of ${}^\dagger\mathcal{R}$ are forced to be equal at a filter G than at a filter F when $G \supseteq F$. More evidence that Tao's cheap nonstandard analysis can be meaningfully interpreted as an informal version of ${}^\dagger\mathcal{R}$ comes from his remark that "in cheap nonstandard analysis one only works with statements which remain valid under the operation of restricting the underlying domain of the asymptotic parameter", which is analogous to the *persistence* condition imposed on formulas in possibility semantics. Moreover, Tao also discusses what he calls a version of the Saturation Principle:

> Let us call a nonstandard property $P(x)$ pertaining to a nonstandard object x satisfiable if, no matter how one restricts the domain Σ from its current value, one can find an x for which $P(x)$ becomes true, possibly after a further restriction of the domain Σ. The countable saturation property is then the assertion that if one has a countable sequence $P_1, P_2, \dots$ of properties, such that $P_1(x), \dots, P_n(x)$ is jointly satisfiable for any finite n, then the entire sequence $P_1(x), P_2(x), \dots$ is simultaneously satisfiable.

Once again, the analogy between a parameter space Σ and the free filter F_Σ on ω generated by the finite set Σ on one hand, and between passing to a subsequence Σ' and extending the filter to one containing Σ', allows us to see the similarity between Tao's principle and Theorem 7.2.18, as Tao's notion of satisfiability of a property $P(x)$ on a parameter space Σ becomes quite transparently analogous to the formula $\exists x P(x)$ being forced at F_Σ.

To sum up, Tao's cheap nonstandard analysis has the same starting point as Laugwitz's proposal, namely considering objects that vary along a parameter $\mathbf{n}$ and whose properties are exactly those properties that these varying objects have on cofinitely many of their values. The main drawback of such a framework is that it does not satisfy the law of excluded middle and therefore lacks the powerful Transfer Principle of classical nonstandard analysis. As we have argued, ${}^{\dagger}\mathcal{R}$ stems from a similar idea but solves the Transfer Principle problem thanks to the machinery of possibility semantics, which also allows one to reason classically within the structure. Although it remains informal, Tao's proposal of having the freedom to pass to a subsequence by varying the range of the parameter $\mathbf{n}$ can conceivably be seen as analogous to the freedom to vary the free filter on ω that is afforded by possibility semantics. In this sense, ${}^{\dagger}\mathcal{R}$ can a seen as formal way to capture Tao's proposal, with the added benefit of having a powerful version of the Transfer Principle. Interestingly, Tao himself argues that "[t]he dynamic nature of the parameter space Σ makes it a little tricky to properly formalise cheap nonstandard analysis in the usual static framework of mathematical logic". This seems to suggest that the central feature of ${}^{\dagger}\mathcal{R}$ that allows it to formalize Tao's idea lies in its more "dynamic" nature, i.e., in the fact that it is able to reason across various classical models rather than just within one. As we shall see in the next section, this "dynamic" character of ${}^{\dagger}\mathcal{R}$ can be captured quite precisely, by comparing it to another constructive version of nonstandard analysis that uses sheaf semantics and topos theory.

7.4 The Dynamic Approach

The second kind of work I will be considering approaches nonstandard analysis from a constructive and intuitionistic perspective. There are, of course, many different constructive or quasi-constructive approaches to analysis, which differ more or less dramatically from classical analysis, such as Bishop's constructive analysis [43] or Weyl's predicative analysis [260]. The development of topos theory and sheaf semantics has also allowed for the construction of several rich models of the intuitionistic continuum [9, 96, 235, 236]. In a series of articles [202, 203, 204], Erik Palmgren has developed a constructive version of nonstandard analysis which shares many aspects of ${}^{\dagger}\mathcal{R}$.[6] Palmgren's model, however, uses sheaf semantics, and the nonstandard extensions of a set he defines are not themselves sets but rather set-valued functors on a large category of filters. As such, his version of the hyperreal line can be thought of as a dynamic or varying model, a model whose exact domain varies along each object in the category of filters. As we shall see below, Palmgren's construction can be dramatically simplified in a semi-constructive setting, which yields a structure that is essentially equivalent to ${}^{\dagger}\mathcal{R}$.

7.4.1 Palmgren's Sheaf Model

Palmgren's starting point is the observation that, from a constructive perspective, the existence of a nonstandard extension of the real line on which the Transfer and Saturation Principle hold seems to pose some insurmountable problems. Indeed, if $\varphi(x)$ is any statement about the natural numbers, then the following is constructively valid:

$$\mathbb{R} \models \forall n \in \mathbb{N}(\forall m \leq n(m \in \mathbb{N} \to (\varphi(m) \vee \neg\varphi(m)))).$$

[6] A more syntactic approach building in part on Palmgren's work was also developed in [25, 115]

Thus if there is a nonstandard extension $\mathbb{R}^*$ of $\mathbb{R}$ for which the Transfer Principle holds, it follows that:

$$\mathbb{R}^* \models \forall n \in \mathbb{N}^*(\forall m \leq n(m \in \mathbb{N}^* \rightarrow (\varphi(m) \vee \neg\varphi(m)))),$$

But if $\mathbb{R}^*$ satisfies the Saturation Principle, it contains some infinitely large hypernatural number N, which implies that:

$$\mathbb{R}^* \models \forall m \leq N(m \in \mathbb{N}^* \rightarrow (\varphi(m) \vee \neg\varphi(m))).$$

Thus for any natural number m,

$$\mathbb{R}^* \models \varphi(m) \text{ or } \mathbb{R}^* \models \neg\varphi(m),$$

which by Transfer implies that

$$\mathbb{R} \models \varphi(m) \text{ or } \mathbb{R} \models \neg\varphi(m),$$

which is certainly a problematic conclusion for the constructivist, as it would imply that the Law of Excluded Middle holds on the natural number domain. Thus, just like the existence of a Tarskian structure satisfying the Transfer and Saturation Principle implies in a semi-constructive setting the existence of a free ultrafilter on ω, it also implies in a constructive setting the Law of Excluded Middle on the domain of the natural numbers. Building on some previous work by Moerdijk [194], Palmgren's solution is similar to the solution offered by $^\dagger\mathcal{R}$ in that the model of constructive nonstandard analysis he develops uses sheaf semantics rather than a constructive version of Tarskian semantics. As a consequence, the nonstandard extension of the reals that Palmgren obtains is a sheaf living in a Grothendieck topos, rather than a Tarskian model in the universe of sets. In what follows, I assume some elementary knowledge of category theory (objects and arrows in a category, functors between categories, natural transformations) and limit myself to a description of Grothendieck toposes and sheaf semantics that is precise enough to give the reader a sense of the relationship between Palmgren's model and $^\dagger\mathcal{R}$. A detailed exposition of topos theory is beyond the scope of this chapter but can be found in [181].

Given a category $\mathbf{C}$, a *sieve* S on an object A in $\mathbf{C}$ is a collection of arrows in C with codomain A and closed under precomposition, i.e., such that for any arrow $f : B \rightarrow A$ in S and any arrow $g : C \rightarrow B$, their composition $f \circ g : C \rightarrow A$ is in S. A Grothendieck topology J on $\mathbf{C}$ assigns to every object A in $\mathbf{C}$ a collection $J(A)$ of *covering sieves* on A in a "compatible" way. The pair $(\mathbf{C}, J)$, where J is a Grothendieck topology, is called a *site*. A *presheaf* on $\mathbf{C}$ is a contravariant functor $\alpha : \mathbf{C} \rightarrow \mathbf{Sets}$, meaning that $\alpha(A)$ is a set for any object A in $\mathbf{C}$, and $\alpha(f)$ is a function from $\alpha(B)$ to $\alpha(A)$ whenever $f : A \rightarrow B$ is an arrow in $\mathbf{C}$. Given a site $(\mathbf{C}, J)$, a presheaf α on $\mathbf{C}$ and a sieve $S = \{f_i : B_i \rightarrow A\}_{i \in I}$ covering some object A in $\mathbf{C}$, a *matching family* for S is a family $\{a_i\}_{i \in I}$ such that $a_i \in \alpha(B_i)$ for every $i \in I$, and for any arrow in $\mathbf{C}$ $g : D \rightarrow B_j$ for some $j \in I$, $\alpha(g)(a_j) = a_k$, where $g \circ f_j = f_k$. Finally, a *sheaf* on a site $(\mathbf{C}, J)$ is a presheaf on $\mathbf{C}$ such that for any matching family $\{a_i\}_{i \in I}$ for some covering sieve $S = \{f_i : B_i \rightarrow A\}_{i \in I}$, there is a unique $a \in \alpha(A)$ such that $\alpha(f_i)(a) = a_i$ for any $f_i \in S$. The category of all sheaves on a site $(\mathbf{C}, J)$ and natural transformations between them, usually written $\mathscr{S}(\mathbf{C}, J)$, is a *Grothendieck topos*.

Informally, one can think of a Grothendieck topos as a universe of *varying sets*. Just like the category **Sets** of sets and functions between them, it has enough structure and closure properties to support a significant amount of set-theoretic reasoning. But unlike in **Sets** the objects in a Grothendieck topos $\mathscr{S}(\mathbf{C}, J)$ are sheaves, i.e., contravariant functors from the base category $\mathbf{C}$ to the category of sets. This means that an object α in a Grothendieck topos may be thought of as a $\mathbf{C}$-indexed set, meaning that α designates a set $\alpha(A)$ that may vary along the objects in $\mathbf{C}$. Just like a Tarskian model is a set endowed with an interpretation for the non-logical vocabulary of some first-order language, one can turn sheaves in a Grothendieck topos into models of a first-order language by defining an interpretation for non-logical vocabulary and some recursive clauses for first-order formulas. Because a sheaf α is a varying set, the interpretation and recursive clauses must be defined on every possible value that α can take, and some requirements must be imposed in order to guarantee the validity of certain logical laws. Such a framework is precisely the one provided by sheaf semantics. Formally, given a sheaf α on a site $(\mathbf{C}, J)$ and a first-order language $\mathscr{L}$, one defines first a forcing relation between a set $\alpha(A)$ for some object A in $\mathbf{C}$, elements of $\alpha(A)$ and atomic $\mathscr{L}$-formulas that must satisfy the following two conditions:

Monotonicity For any n-ary $\mathscr{L}$-relation symbol R, any n-tuple $\overline{a}$ of elements of $\alpha(A)$, and any $f : B \to A$, if $\alpha(A) \Vdash R(\overline{a})$, then $\alpha(B) \Vdash R(\alpha(f)(\overline{a}))$;

Local Character For any covering sieve $S = \{f_i : B_i \to A_i\}_{i \in I}$ and any n-tuple $\overline{a}$ of elements of $\alpha(A)$, if $\alpha(B_i) \Vdash R(\alpha(f_i)(\overline{a}))$ for every $i \in I$, then $\alpha(A) \Vdash R(\overline{a})$.

This forcing relation is then extended to first-order formulas by the following inductive clauses:

- If $\varphi := \neg\psi$, then $\alpha(A) \Vdash \varphi(\overline{a})$ iff for all $f : B \to A$, $\alpha(B) \nVdash \psi(\alpha(f)(\overline{a}))$;
- If $\varphi := \psi \wedge \chi$, then $\alpha(A) \Vdash \varphi$ iff $\alpha(A) \Vdash \psi$ and $\alpha(A) \Vdash \chi$;
- If $\varphi := \psi \vee \chi$, then $\alpha(A) \Vdash \varphi(\overline{a})$ iff there is a covering sieve $S = \{f_i : B_i \to A\}_{i \in I}$ of A such that, for any $i \in I$, $\alpha(B_i) \Vdash \psi(\alpha(f_i)(\overline{a}))$ or $\alpha(B_i) \Vdash \chi(\alpha(f_i)(\overline{a}))$;
- If $\varphi := \psi \to \chi$, then $\alpha(A) \Vdash \varphi(\overline{a})$ iff for all $f : B \to A$, $\alpha(B) \Vdash \psi(\alpha(f)(\overline{a}))$ implies $\alpha(B) \Vdash \chi(\alpha(f)(\overline{a}))$;
- If $\varphi := \forall x\psi$, then $\alpha(A) \Vdash \varphi(\overline{a})$ iff for all $f : B \to A$, $\alpha(B) \Vdash \psi(\alpha(f)(\overline{a}), b)$ for every $b \in \alpha(B)$;
- If $\varphi := \exists x\psi$, then $\alpha(A) \Vdash \varphi(\overline{a})$ iff there is a covering sieve $S = \{f_i : B_i \to A\}_{i \in I}$ of A such that, for any $i \in I$, $\alpha(B_i) \Vdash \psi(\alpha(f_i)(\overline{a}), b)$ for some $b \in \alpha(B_i)$.

As the reader probably noticed, these clauses are strikingly similar to the inductive clauses of possibility semantics as they were defined in Definition 6.1.2. The next section will expand on the relationship between the two semantics, but, for now, we must note an important discrepancy between them. Indeed, these inductive clauses only guarantee that the axioms and rules of first-order intuitionistic logic are valid on any sheaf model, whereas the clauses of possibility semantics guarantee the validity of first-order classical logic on any possibility structure. This is the reason why the "internal logic" of Grothendieck toposes, described by its sheaf semantics, is always intuitionistic, but rarely classical, a feature that is particularly appealing to constructivists like Palmgren.

We are now in a position to describe Palmgren's constructive model of nonstandard analysis.[7] This involves first describing the base category $\mathbb{B}$ of filter bases and its Grothendieck topology, and then identifying the sheaves on this site that play the role of the nonstandard extension of standard sets.

Definition 7.4.1. A *filter base* $\mathscr{F}$ is a pair $(S_F, \{F_i\}_{i\in I})$ such that S_F is a set and $\{F_i\}_{i\in I}$ is a collection of subsets of S_F such that $\{U \subseteq S_F \mid F_i \subseteq U \text{ for some } i \in I\}$ is a filter on S_F. A *continuous morphism* between two filter bases $\mathscr{F} = (S_F, \{F_i\}_{i\in I})$ and $\mathscr{G} = (S_G, \{G_j\}_{j\in J})$ is a map $\alpha : S_F \to S_G$ such that for any $j \in J$ there is $i \in I$ such that $F_i \subseteq \alpha^{-1}[G_j]$. Filter bases and continuous morphisms form a category $\mathbb{B}$.

Note that, by its definition, $\mathbb{B}$ is a large category, since any set induces a distinct set of filter bases in $\mathbb{B}$, which implies that the collection of objects in $\mathbb{B}$ is a proper class. It is however easy to see that $\mathbb{B}$ is locally small, meaning that there are only set many arrows betwen any two objects. This implies in particular that the contravariant representable functors $\mathbf{y}_A$, i.e., functors mapping any object B to the collection of all arrows in $\mathbb{B}$ from B to some fixed object A, are set-valued and therefore presheaves. Defining sheaves requires the notion of a cover of a filter base:

Definition 7.4.2. For any filter base $\mathscr{F} = (S_F, \{F_i\}_{i\in I})$, a *cover* of $\mathscr{F}$ is a finite set of maps $\{\alpha_k\}_{k\leq n}$ with $\alpha_k : (S_{G_k}, \{G_j\}_{j\in J_k}) \to \mathscr{F}$ for each $k \leq n$, and such that for any $G_{j_1}, ..., G_{j_n}$ with $G_{j_k} \in \{G_j\}_{j\in J_k}$ for each $k \leq n$, there is $i \in I$ such that $F_i \subseteq \bigcup_{k\leq n} \alpha_k[G_{j_k}]$.

Palmgren goes on to prove that this notion of covering generates a well-behaved Grothendieck topology on $\mathbb{B}$.

Theorem 7.4.3. *The covers on filter bases in $\mathbb{B}$ form a subcanonical Grothendieck topology $\mathscr{J}$ on $\mathbb{B}$. As a consequence, every representable presheaf on $\mathbb{B}$ is a sheaf.*

Given a set A, Palmgren considers the filter base $\mathscr{A} = (A, \{A\})$ and defines the nonstandard extension of A as the representable presheaf $\mathbf{y}_{\mathscr{A}}$. More concretely, for a filter base $\mathscr{F} = \{S_F, \{F_i\}_{i\in I}\}$, $\mathbf{y}_{\mathscr{A}}(\mathscr{F})$ is the reduced power of A modulo the filter on S_F generated

[7] Palmgren's background theory in [204] is Martin-Löf's constructive type theory, but here I will work in $ZF + DC$ for the sake of simplicity.

by $\{F_i\}_{i\in I}$. Since any representable presheaf is a sheaf by Theorem 7.4.3, the Grothendieck topos $\mathscr{S}(\mathbb{B}, \mathscr{J})$ provides a model for a constructive version of Nelson's Internal Set Theory [199], in which any standard set has a nonstandard extension, and versions of the Transfer and Saturation Principles hold. Palmgren also proves constructively that analogues of some classical theorems of analysis such as the Heine-Borel Theorem hold in this model. Palmgren's framework has therefore the advantage of being entirely constructive, but quite powerful, which he takes to be an advantage of constructive nonstandard analysis over some other frameworks like Bishop's constructive analysis. From a semi-constructive perspective however, the choice of a Grothendieck topos as a model of nonstandard analysis comes at quite a significant cost, as the logic of such a topos is intuitionistic, but not classical, and sheaf semantics is significantly more involved than Tarskian semantics. One way of overcoming both problems is to modify the base category $\mathbb{B}$ so that presheaves on that category become easier to describe, and the recursive clauses of sheaf semantics become tractable enough to guarantee the validity of classical logic. As we shall see, this can be achieved by choosing $\mathfrak{F}$ as our base category, and this construction will yield that $^\dagger\mathcal{R}$ is essentially a presheaf on $\mathfrak{F}$, but not a sheaf.

7.4.2 Presheaves on the Fréchet Category

The first step in our simplification of Palmgren's approach is to turn the poset $\mathfrak{F}$ into a site. Equivalently, this amounts to *decategorifying* Palmgren's base category $\mathbb{B}$, by replacing it with a poset, which are typically more concrete and tractable objects.

Definition 7.4.4. The Fréchet Category is the poset category $\mathfrak{F}$, whose objects are all proper filters on ω extending the Fréchet filter of cofinite subsets, and for any two such filters F, F', there is a unique arrow $F' \to F$ iff $F \subseteq F'$. Two filters $F, F' \in \mathfrak{F}$ are *incompatible* (noted $F \perp F'$) if there is no $H \in \mathfrak{F}$ such that $H \supseteq F, F'$.

Definition 7.4.5. A collection of filters $\{H_i\}_{i\in I}$ *covers* a filter $F \in \mathfrak{F}$ if it satisfies the following three properties:

- For any $i \in I$, $H_i \supseteq F$;
- For any $i \in I$ and $H \in \mathfrak{F}$, if $H \supseteq H_i$, then $H = H_j$ for some $j \in I$;
- For any $G \supseteq F$, there is $i \in I$ such that $H_i \supseteq G$.

For any filter $F \in \mathfrak{F}$, we let $\mathscr{C}(F)$ be the collection of all covers on F.

Note that it follows straightforwardly from the definition of a cover that for any $F \in \mathfrak{F}$, a collection of filters $\{H_i\}_{i\in I}$ covers F if and only if it is a dense open subset of the downset of F in the downset topology on $\mathfrak{F}$. As a consequence, we have the following result:

Lemma 7.4.6. *The covers $\{\mathscr{C}(F)\}_{F\in\mathfrak{F}}$ form a Grothendieck topology on $\mathfrak{F}$.*

This result is a special case of a well-known result in topos theory, namely that the dense topology on any poset category is a Grothendieck topology [181, Section III.2].

Let us now fix an $\mathscr{L}$-structure $\mathcal{A}$ with domain A.

Definition 7.4.7. The nonstandard extension of $\mathcal{A}$ is the presheaf ${}^*\mathcal{A} : \mathfrak{F} \to \mathbf{Sets}$ defined as follows:

- Given $F \in \mathfrak{F}$, ${}^*\mathcal{A}(F)$ is the reduced power of A modulo F, that is, the set of equivalence classes of functions $a : \omega \to A$ under the equivalence relation:

$$a \sim_F a' \Leftrightarrow \{i \in \omega : a(i) = a'(i)\} \in F.$$

- Given $f : F' \supseteq F \in \mathfrak{F}$, ${}^*\mathcal{A}(f) : {}^*\mathcal{A}(F) \to {}^*\mathcal{A}(F')$ is defined by ${}^*\mathcal{A}(f)(a_F) = a_{F'}$, where a_F and $a_{F'}$ are the equivalence classes of the function $a : \omega \to A$ under $\sim_F$ and $\sim_{F'}$ respectively.

Note that if $F \subseteq F'$, then $a \sim_F a'$ implies $a \sim_{F'} a'$, and hence the map ${}^*\mathcal{A}(f) : {}^*\mathcal{A}(F) \to {}^*\mathcal{A}(F')$ is well-defined. In order to turn ${}^*\mathcal{A}$ into a structure for some language $\mathscr{L}$, we need to define a forcing relation for atomic formulas. By analogy with Palmgren's model, the natural choice here is to use classical satisfaction in the reduced powers ${}^*\mathcal{A}(F)$ for any $F \in \mathfrak{F}$. Using the notation introduced in Notation 6.2.3, for any $F \in \mathfrak{F}$, any n-ary $\mathscr{L}$-relation symbol R and any n-tuple $\overline{a}$, we let ${}^*\mathcal{A}(F) \models R(\overline{a_F})$ iff $||R(\overline{a})|| \in F$. We can then show that this relation satisfies the monotonicity and local character condition of sheaf semantics:

Lemma 7.4.8. *The forcing relation $\Vdash$ defined as ${}^*\mathcal{A}(F) \Vdash R(\overline{a_F})$ iff $||R(\overline{a})|| \in F$ for any $F \in \mathfrak{F}$, any n-ary $\mathscr{L}$-relation symbol R and any n-tuple $\overline{a}$ of elements of A^ω satisfies the monotonicity and local character conditions of sheaf semantics, namely:*

1. *Monotonicity: For any $f : F' \supseteq F$, ${}^*\mathcal{A}(F) \Vdash R(\overline{a_F})$ implies ${}^*\mathcal{A}(F') \Vdash R({}^*\mathcal{A}(f)(\overline{a_F}))$;*

2. *Local Character: For any cover $\{f_i : F_i \supseteq F\}_{i \in I}$ of some $F \in \mathfrak{F}$, if ${}^*\mathcal{A}(F_i) \Vdash R({}^*\mathcal{A}(f_i)(\overline{a_F}))$ for all $i \in I$, then ${}^*\mathcal{A}(F) \Vdash R(\overline{a_F})$.*

Proof. First, let us recall that for any morphism $f : F' \supseteq F$ in $\mathfrak{F}$ and any $a \in A^\omega$, ${}^*\mathcal{A}(f)(a_F) = a_{F'}$. Now we prove that the forcing relation satisfies monotonicity and local character in turn:

1. Suppose $f : F' \supseteq F$ and ${}^*\mathcal{A}(F) \Vdash R(\overline{a_F})$. This means that $||R(\overline{a})|| \in F \subseteq F'$, so ${}^*\mathcal{A}(F') \Vdash R(\overline{a_{F'}})$.

2. Let $\{f_i : F_i \supseteq F\}_{i \in I}$ be a cover of F, and suppose that ${}^*\mathcal{A}(F) \nVdash R(\overline{a_F})$. It is enough to show that there is some $i \in I$ such that ${}^*\mathcal{A}(F_i) \nVdash R(\overline{a_{F_i}})$. By definition of the forcing relation, $||R(\overline{a})|| \notin F$. Let H be the filter generated by $F \cup -||R(\overline{a})||$. Since $\{f_i\}_{i \in I}$ covers F, there is $i \in I$ such that $F_i \supseteq H$. But then it follows that $||R(\overline{a})|| \notin F_i$, so ${}^*\mathcal{A}(F_i) \nVdash R(\overline{a_{F_i}})$. $\square$

We can then extend this forcing relation to any first-order formula φ using the inductive clauses of sheaf semantics. If $\mathscr{A} = \mathcal{R}$, we therefore obtain a presheaf ${}^{*}\mathcal{R}$ which can be thought of as a "varying reduced power". As a matter of fact, we obtain a structure that is equivalent to ${}^{\dagger}\mathcal{R}$ in the following sense:

Theorem 7.4.9. *There is a system of functions $\{\pi_F\}_{F\in\mathfrak{F}}$ such that for any $F, G \in \mathfrak{F}$:*

- $\pi_F : \mathcal{R}^\omega \to {}^{*}\mathcal{R}(F)$;
- *if* $f : G \supseteq F$, *then* ${}^{*}\mathcal{R}(f) \circ \pi_F = \pi_G$;
- *for any $\mathscr{L}$-formula $\varphi(x)$ and any tuple $\overline{a}$ of elements of $\mathcal{R}^\omega$, ${}^{*}\mathcal{R}(F) \Vdash \varphi(\pi_F(\overline{a}))$ iff ${}^{\dagger}\mathcal{R}, F \Vdash \varphi(\overline{a})$.*

Proof. For any $F \in \mathfrak{F}$, let π_F map any $a \in \mathcal{R}^\omega$ to its equivalence class a_F in the reduced power ${}^{*}\mathcal{R}(F)$. The rest of the proof is a straightforward induction on the complexity of formulas which exploits the similarity between possibility semantics and sheaf semantics. In particular, in the atomic case, the persistence and refinability conditions correspond to the monotonicity and local character conditions of sheaf semantics respectively, and in the inductive step for disjunctions and existential quantification, one uses the fact that covers on a filter F coincide with dense sets of filters extending F. □

Theorem 7.4.9 highlights the similarities between Palmgren's approach and ours. His nonstandard extensions are varying reduced powers of a classical model modulo a proper class of filters in which satisfaction is defined according to sheaf semantics, while we may think of ${}^{\dagger}\mathcal{R}$ as a "dynamic" reduced power of $\mathcal{R}$ modulo the free filters on ω, in which satisfaction is defined locally and amalgamated in a coherent way by possibility semantics. However, taking the Grothendieck topos of sheaves over the site $(\mathfrak{F}, \{\mathscr{C}(F)\}_{F\in\mathfrak{F}})$ does not yield a nonstandard universe of sets, as many presheaves, including the presheaf corresponding to ${}^{\dagger}\mathcal{R}$, fail to be sheaves.

Fact 7.4.10. Let $\mathscr{A}$ be an $\mathscr{L}$-structure with a domain A such that there is a injection $\pi : 2^\omega \to A$. Then ${}^{*}\mathcal{A}$ is not a sheaf.

Proof. Recall that for any $F \in \mathfrak{F}$ and any cover $\{F_i\}_{i\in I}$ of F, a *matching family* is a family $\{a_i\}_{i\in I}$ such that $a_i \in {}^{*}\mathcal{A}(F_i)$ for any $i \in I$, and for any $i, j \in I$ with $f : F_j \supseteq F_i$, ${}^{*}\mathcal{A}(f)(a_i) = a_j$. To show that ${}^{*}\mathcal{A}$ is not a sheaf, it is enough to find a cover $\{F_i\}_{i\in I}$ of a some $F \in \mathfrak{F}$ and a matching family $\{a_i\}_{i\in I}$ such that for all $a \in A^\omega$ there is $i \in I$ with $a \notin a_i$.
Let F be the Fréchet filter, and recall that one can prove in $ZF + DC$ that there is an almost disjoint family $\{A_f\}_{f\in 2^\omega}$ of infinite subsets of ω indexed by functions from ω to $\{0, 1\}$ [144, p. 118]. For any $f : \omega \to 2$, let F_f be the filter generated by $F \cup A_f$. Note that, whenever $f \neq g$, F_f and F_g have no common extension since $|A_f \cap A_g| < \aleph_0$. Let $\{H_j\}_{j\in J}$ be the collection of all $H \supseteq F$ such that $H \perp F_f$ for every $f \in 2^\omega$, and let $D = \bigcup_{f\in 2^\omega}\{H \in \mathfrak{F} \mid H \supseteq F_f\} \cup \{H_j\}_{j\in J}$. By construction D is a cover of F. Now for any $f \in 2^\omega$, let $a^f : \omega \to A$ be the constant function with range $\{\pi(f)\}$, and let $\alpha : \omega \to A$ be such that $\alpha(i) = \pi(\chi_{\{i\}})$ for

any $i \in \omega$, where $\chi_{\{i\}}$ is the characteristic function of the singleton set $\{i\}$. Now consider the family $C = \bigcup_{f \in 2^\omega} \{a^f_H \mid H \supseteq F_f\} \cup \{\alpha_{H_j}\}_{j \in J}$. Clearly, since on the one hand we selected the H_j-equivalence class of α for any $j \in J$ and the H-equivalence class of a^f for any $f \in 2^\omega$ and any H extending F_f, and since on the other hand for any $f \in 2^\omega$, F_f is incompatible with H_j for any $j \in J$ and with any $H \supseteq F_g$ for any $g \neq f \in 2^\omega$, C is a matching family. However, for any $a : \omega \to A$ and any distinct $f, g : \omega \to 2$, observe that $||a = a^f|| \cap ||a = a^g|| = \emptyset$. Moreover, since $\omega \supseteq \bigcup_{f \in 2^\omega} ||a = a^f||$, it must be the case that $||a = a^f|| = \emptyset$ for some $f \in 2^\omega$ (otherwise, one could define a injection from 2^ω into ω, contradicting Cantor's theorem). Hence for any $a : \omega \to A$ there is $f \in 2^\omega$ such that $a \notin a^f_{F_f}$, which shows that ${}^*\mathcal{A}$ is not a sheaf. □

Let us note that, if we assume the Axiom of Choice, the argument above can be easily be modified to show that ${}^*\mathcal{A}$ is not a sheaf for any uncountable structure $\mathscr{A}$. Whether this is a necessary condition on $\mathscr{A}$ for ${}^*\mathcal{A}$ not to be a sheaf is left as an open problem. In any case, since we can define in $ZF + DC$ an injection from 2^ω into $\mathcal{R}$, it follows that ${}^*\mathcal{R}$ is not a sheaf. Since the nonstandard extensions we defined are not sheaves in general, we cannot work in the Grothendieck topos of sheaves over $\mathfrak{F}$ and straightforwardly apply the machinery of sheaf semantics. Fortunately, possibility semantics still allows us to have a notion of a model which, as we have seen in the first section, is robust enough to develop a significant part of nonstandard analysis *à la* Robinson. In other words, the "decategorification" of the category of filters in Palmgren's approach, while allowing for a drastic simplification of many technical details, comes at the cost of not providing a nonstandard "ambient universe", like the one postulated in Nelson's Internal Set Theory or offered by the Grothendieck topos of sheaves over $\mathbb{B}$. In that sense, ${}^\dagger\mathcal{R}$ can be seen as a watered down version of Palmgren's constructive nonstandard analysis: because we work in a semi-constructive setting, we do not need the full generality of sheaf semantics nor the full power of topos theory, and we can work instead in a possibility structure which resembles more closely Tarskian semantics. Additionally, the fact that ${}^\dagger\mathcal{R}$ can be presented as a presheaf on a poset category endowed with the dense topology is, in itself, not surprising in the slightest. Indeed, sheaf semantics was originally conceived as a generalization of Cohen forcing, which corresponds precisely in topos theory to working in a topos of sheaves over a poset category endowed with the dense topology [181, Section 6.2]. As we will now see, ${}^\dagger\mathcal{R}$ itself is tightly connected to forcing and to Boolean-valued models of analysis.

7.5 The Generic Approach

The third and last alternative approach to classical nonstandard analysis that I consider here involves using Boolean-valued models rather than two-valued models. Unlike the previous two, it does not stem from a desire to make nonstandard analysis more constructive but instead from an interest in viewing the ultrapower construction as a special case of a more general kind of algebraic construction. This approach is also the one with the tightest relationship to set-theoretic forcing, and, as we will see, will motivate our last perspective on ${}^{\dagger}\mathcal{R}$ as a generic approximation of a Luxemburg ultrapower.

7.5.1 Boolean-Valued Analysis

In [234], Scott remarks that Boolean-valued models, even though they appeared originally in the context of forcing in set theory, can also be seen as a generalization of ultraproducts:

> The idea of constructing Boolean-valued models could have been (but was not) discovered as a generalization of the ultraproduct method used now so often to obtain nonstandard models for ordinary analysis. Roughly, we can say that ultraproducts use the *standard* Boolean algebras (the power-set algebras) to obtain models elementarily equivalent to the standard model, whereas the Boolean methods allows the nonstandard complete algebras (such as the *Lebesgue* algebra of measurable sets modulo sets of measure zero or the *Baire* algebra of Borel sets modulo sets of the first category). Thus the Boolean method leads to *nonstandard* nonstandard models that are not only not isomorphic to the standard model but are not even equivalent. Nevertheless, they do satisfy all the usual axioms and deserve to be called models of analysis. [234, pp. 87-88]

Scott goes on to explain how ultraproducts can be seen as a special case of quotients of a Boolean-valued model. Given a family of Tarskian $\mathscr{L}$-structures $\{\mathbb{A}_i\}_{i\in I}$, one may consider the Boolean-valued model whose domain is the direct product $A = \Pi_{i\in I}A_i$ of the domains of the models $\{\mathbb{A}_i\}_{i\in I}$, the Boolean algebra of truth-values is the powerset of the index set $\mathscr{P}(I)$, and for any tuple $\overline{a}$ of elements of A, and any $\mathscr{L}$-formula $\varphi(\overline{x})$, the truth-value of $\varphi(\overline{a})$ is $||\varphi(\overline{a})|| = \{i \in \omega \mid \mathbb{A}_i \models \varphi(\overline{a}(i))\}$. A straightforward induction on the complexity of formulas shows that this indeed correctly defines a Boolean-valued model. An ultraproduct of the models $\{\mathbb{A}_i\}_{i\in I}$ can then be obtained by taking an ultrafilter on $\mathscr{P}(I)$, which is the same as a Boolean homomorphism $f_U : \mathscr{P}(I) \to \{0,1\}$, quotienting the domain A by the equivalence relation $a \sim_U b$ iff $f_U(||a = b||) = 1$, and defining a Tarskian model on this domain by letting $\varphi(\overline{a_U})$ be true in the model iff $f_U(||\varphi(\overline{a})||) = 1$ for $\varphi(\overline{x})$ a $\mathscr{L}$-formula and $\overline{a_U}$ the tuple of equivalence classses of a tuple $\overline{a}$ of elements of A. By Łoś's Theorem, the model thus obtained is precisely the ultraproduct of the family of models $\{\mathbb{A}_i\}_{i\in I}$ by the ultrafilter U. As Scott concludes:

> In short, we have divided the ultraproduct construction into two stages: *product* followed by *ultra*. It is the generalization of the product part we wish to emphasize. [p. 89]

Scott goes on to show how to define a B-valued model of analysis in a similar fashion when B is a complete Boolean algebra, but not necessarily the powerset set of some set I, and that one may still retrieve Tarskian models as quotients of such Boolean-valued models modulo an ultrafilter on B. In other words, the Boolean-valued models of analysis so constructed are "one ultrafilter away" from being Tarskian, nonstandard models of analysis.

This Boolean-valued approach was picked up by Takeuti in a series of articles [247, 248, 249, 250], and continues to bear fruits in the contemporary Russian School of Boolean-valued analysis [163, 164, 165].

Let us now briefly see how ${}^\dagger\mathcal{R}$ connects to Boolean-valued models. As mentioned in Chapter 6, given an $\mathscr{L}$-possibility structure $(\mathfrak{P}, D, \mathscr{I})$, an $\mathscr{L}$-formula $\varphi(\overline{a})$ and a tuple $\overline{a}$ of elements of D, the set $[\![\varphi(\overline{a})]\!] = \{p \in P \mid p \Vdash \varphi(\overline{a})\}$ is always a regular-open subset of $\mathfrak{P}$ and the regular opens of any poset $\mathfrak{P}$ always form a complete Boolean algebra $\mathsf{RO}(\mathfrak{P})$. We may therefore think of any possibility structure $(\mathfrak{P}, D, \mathscr{I})$ as a Boolean-valued model with domain D and algebra of truth values $\mathsf{RO}(\mathfrak{P})$. Since any complete Boolean algebra is isomorphic to $\mathsf{RO}(\mathfrak{P})$ for some poset $\mathfrak{P}$, this means that possibility semantics is as general as the Boolean-valued models discussed by Scott.

What about the algebra of truth-values of ${}^\dagger\mathcal{R}$ specifically? The free filters on $\mathscr{P}(\omega)$ are in one-to-one correspondence with the proper filters on $\mathscr{P}(\omega)^*$, the Boolean algebra obtained by quotienting $\mathscr{P}(\omega)$ by the ideal of finite sets. Moreover, it follows from [134, Theorem 5.49] that the algebra of regular open sets of the poset of all proper filters on a Boolean algebra B is isomorphic to the *canonical extension* of B, usually written B^δ.[8] Thus ${}^\dagger\mathcal{R}$ can be thought of as a $(\mathscr{P}(\omega)^*)^\delta$-valued Boolean valued model of analysis. Since the Ultrafilter Lemma is equivalent to the statement that the canonical extension of any Boolean algebra B is always isomorphic to the powerset of some set (in fact, of the set X_B of all ultrafilters on B), it holds for $\mathscr{P}(\omega)$ if and only if ${}^\dagger\mathcal{R}$ is a "standard" nonstandard Boolean-valued model of analysis, to use Scott's terminology. As we will see in the next section, there is indeed a strong sense in which ${}^\dagger\mathcal{R}$ is "one ultrafilter away" from a Luxemburg ultrapower.

7.5.2 Generic Models and Luxemburg ultrapowers

As the comparison with Scott's Boolean-valued models of analysis in the previous section showed, we may think of ${}^\dagger\mathcal{R}$ as a Boolean-valued model that captures the semi-constructive part of the standard ultrapower construction, i.e., the "product" part. If we think of a Boolean-valued model as a "fuzzy" Tarskian model, in which the classical truth-value of every formula is not always settled, we may therefore think that ${}^\dagger\mathcal{R}$ is a partial approximation of a Luxemburg ultrapower, the best thing we can get in the absence of the Axiom of Choice. Similarly, viewpoints in $\mathfrak{F}$ can also be interpreted as approximations of classical Luxemburg ultrapowers in a fairly strong sense. A simple way to flesh out the details of the idea is to appeal to the Genericity Lemma. Let us recall once again the definition of a generic filter on a poset.

Definition 7.5.1. A subset D of $\mathfrak{F}$ is *dense* if for any $F \in \mathfrak{F}$ there is $F' \in D$ such that $F \subseteq F'$. A *generic filter* on $\mathfrak{F}$ is a directed subset G of $\mathfrak{F}$ (meaning that for any $F, F' \in \mathsf{G}$ there is $H \in \mathsf{G}$ such that $H \supseteq F, F'$) that is upward closed (meaning that $F \subseteq F' \in \mathsf{G}$ implies $F \in \mathsf{G}$) and has non-empty intersection with every dense subset of $\mathfrak{F}$.

[8] See [77] for more on canonical extensions.

A generic filter on a forcing poset $\mathbb{P}$ can usually be thought as a coherent way of choosing a maximal set of conditions in that poset. As mentioned in the previous chapter, any generic filter on $\mathfrak{F}$ is also a ${}^{\dagger}\mathcal{R}$-generic filter. This means in particular that every generic filter will induce a generic model of ${}^{\dagger}\mathcal{R}$. Recall that generic models are the Tarskian models that are being approximated by viewpoints in a possibility structure. As we shall see below, in the case ${}^{\dagger}\mathcal{R}$ they are tightly connected to Luxemburg ultrapowers. The next lemma establishes a one-to-one correspondence between generic filters on $\mathfrak{F}$ and free ultrafilters on ω, which can then be used to show that every generic model is isomorphic to a Luxemburg ultrapower, and vice-versa.

Lemma 7.5.2. *The generic filters on $\mathfrak{F}$ are in one-to-one correspondence with the free ultrafilters on ω.*

Proof. Given G a generic filter on $\mathfrak{F}$, let $\beta(\mathsf{G}) = \bigcup \mathsf{G}$. To see that this is well defined, observe first that, since G is a filter on $\mathfrak{F}$, it is a directed family of filters on ω, and thus its union is also a filter on ω. More over, for any $A \subseteq \omega$, the set $D_A = \{F \in \mathfrak{F} \mid A \in F \text{ or } -A \in F\}$ is dense. Since G is generic, this means that $D_A \cap \mathsf{G} \neq \emptyset$, hence $A \in \bigcup \mathsf{G}$ or $-A \in \bigcup \mathsf{G}$. This shows that $\beta(\mathsf{G})$ is an ultrafilter on ω, and since $\beta(\mathsf{G})$ contains the Fréchet filter, it is clearly free.

Conversely, given a free ultrafilter U on ω, let $\alpha(U) = \{F \in \mathfrak{F} \mid F \subseteq \mathsf{U}\}$. Since $\alpha(\mathsf{U})$ is the principal upset generated by U in $\mathfrak{F}$, it is clearly a filter. Moreover, if U is an ultrafilter, then it is an atom in $\mathfrak{F}$, meaning that $F' = \mathsf{U}$ for any $F' \supseteq \mathsf{U}$. Hence $\mathsf{U} \in D$ for any dense subset D of $\mathfrak{F}$, from which it follows that $\alpha(\mathsf{U})$ is generic.

Next, we check that for any generic G, $\alpha(\beta(G)) = \{F \in \mathfrak{F} \mid F \subseteq \bigcup \mathsf{G}\} = \mathsf{G}$. The right-to-left inclusion is immediate. For the converse, suppose $F \notin G$, and consider the set $D_F = \{H \in \mathfrak{F} \mid H \supseteq F \text{ or } H \perp F\}$. Clearly, D_F is dense, so there is $H \in D_F \cap \mathsf{G}$. Moreover, since $F \notin \mathsf{G}$, it must be the case that $F \perp H$. Hence there is $B \in H$ such that $\neg B \in F$. Since $H \subseteq \bigcup \mathsf{G}$ and $\bigcup \mathsf{G}$ is a filter on ω, this means that $F \nsubseteq \bigcup \mathsf{G}$.

Similarly, I claim that for any ultrafilter U, $\beta(\alpha(\mathsf{U})) = \bigcup \{F \in \mathfrak{F} \mid F \subseteq \mathsf{U}\} = \mathsf{U}$. The left-to-right inclusion is clear, and the converse follow from the fact that $\beta(\alpha(\mathsf{U}))$ is an ultrafilter, hence maximal. □

Let us note that, assuming the Ultrafilter Lemma, there is also an elegant proof of the result above that uses the theory of Boolean algebras. By the observation made at the end of Section 7.5.1, the algebra $\mathsf{RO}(\mathfrak{F})$ of regular open subsets of $\mathfrak{F}$ is isomorphic to the canonical extension of $\mathscr{P}(\omega)^*$, which, by Stone duality, is itself isomorphic to $\mathscr{P}(X_{\mathscr{P}(\omega)^*})$, the powerset of the set of ultrafilters on $\mathscr{P}(\omega)^*$. Moreover, it is a well-known result in the forcing literature [145, p. 156] that a generic filter on a poset $\mathbb{P}$ corresponds to a complete free ultrafilter on $\mathsf{RO}(\mathbb{P})$, where an ultrafilter U on a complete Boolean algebra B is complete if $\bigwedge S \in U$ whenever $S \subseteq U$, for any $S \subseteq B$. This means that any generic filter on $\mathfrak{F}$ is essentially a principal ultrafilter on $\mathscr{P}(X_{\mathscr{P}(\omega)^*})$, i.e., it is induced by a point in $X_{\mathscr{P}(\omega)^*}$. But points in $\mathscr{P}(X_{\mathscr{P}(\omega)^*})$ are ultrafilters on $\mathscr{P}(\omega)^*$, which are in one-to-one correspondence with free ultrafilters on ω. However, the direct construction of the one-to-one correspondence between generic filters on $\mathfrak{F}$ and free ultrafilters on ω described in the proof of Lemma 7.5.2 also allows us to use the genericity results from the previous chapter to establish an equivalence between generic filters, ${}^{\dagger}\mathcal{R}$-generic filters and free ultrafilters on ω.

Theorem 7.5.3. *A subset* G *of* $\mathfrak{F}$ *is a* ${}^{\dagger}\mathcal{R}$*-generic filter iff it is generic over* $\mathfrak{F}$*, and an ultrafilter on* ω *is* $\mathfrak{F}$*-generic iff it is free. Moreover, if* ${}^{\dagger}\mathcal{R}$ *is normal, then for any* $F \in \mathfrak{F}$*, any* $\mathscr{L}$*-formula* $\varphi(\overline{x})$ *and any tuple* $\overline{a}$*,* $F \Vdash \varphi(\overline{a})$ *iff for any free ultrafilter* U *on* ω*,* $\mathcal{R}^{\omega}/\mathsf{U} \models \varphi(\overline{a_{\mathsf{U}}})$*.*

Proof. Let us first show that G is ${}^{\dagger}\mathcal{R}$-generic iff it is generic. The right-to-left direction is clear. For the left-to-right direction, suppose that G is ${}^{\dagger}\mathcal{R}$-generic and note that, since there is a predicate symbol in $\mathscr{L}$ for every subset of $\mathbb{R}$, ${}^{\dagger}\mathcal{R}$ satisfies the conditions of Lemma 6.2.17. But this means that $\bigcup \mathsf{G}$ is a $\mathfrak{F}$-generic ultrafilter on ω, which clearly implies that it is also free, since no $F \in \mathfrak{F}$ contains a finite set. But this in turn means that $\alpha(\bigcup \mathsf{G})$ is a generic filter, and moreover $\alpha(\bigcup(\mathsf{G})) = \mathsf{G}$ by the proof of Lemma 6.2.17. Hence G is generic. Next, let us see check that every free ultrafilter on ω is $\mathfrak{F}$-generic since, as mentioned above, the converse is clear. Suppose U is free, and let $A \in \mathsf{U}$. Then the filter $F = \{A \cap B \mid B \in F_0\}$ extends F_0 and contains A, and clearly we have that $F \subseteq \mathsf{U}$. Hence $\mathsf{U} = \bigcup\{F \in \mathfrak{F} \mid F \subseteq \mathsf{U}\}$.

By the Genericity Lemma and Lemma 6.2.17, it follows that every Luxemburg ultrapower of $\mathcal{R}$ is isomorphic to a generic model ${}^{\dagger}\mathcal{R}_{\mathsf{G}}$, and conversely every generic model over ${}^{\dagger}\mathcal{R}$ is isomorphic to a Luxemburg ultrapower $\mathcal{R}^{\omega}/\mathsf{U}$. The rest of the theorem follows from the Genericity Lemma. □

As a consequence of Theorem 7.5.3, we may now think of points in $\mathfrak{F}$ as partial approximations of a Luxemburg ultrapower in a very precise way. It is easy to see that, assuming the Ultrafilter Lemma for $\mathscr{P}(\omega)^*$, or, equivalently, that $\mathfrak{F}$ is a normal family, satisfaction of a formula φ at a viewpoint F corresponds to truth in all ultrapowers induced by an ultrafilter extending F. In the presence of a strong enough fragment of the Axiom of Choice like the Ultrafilter Lemma, this means that one can take any $F \in \mathfrak{F}$ for a representative of the set of all Luxemburg ultrapowers induced by an ultrafilter extending F. As satisfaction at F coincides with truth in all ultrapowers induced by an ultrafilter extending F, this means that any filter F can be thought of as providing a partial viewpoint on ${}^{\dagger}\mathcal{R}$, in the sense that the satisfaction relation on F captures exactly how much information we would have about a classical hyperreal line if we knew only that F is in the ultrafilter. This also means that F_0, the Fréchet filter, truly stands for a "generic" ultrapower, since F_0 forces precisely those formulas that would be satisfied by any Luxemburg ultrapower.

7.6 Fréchet hyperreals and Objections to the Hyperreals

In the previous sections, I have explored some of the mathematical properties of ${}^{\dagger}\mathcal{R}$ (Section 7.2) and argued that it arises naturally as a point of convergence for several distinct mathematical endeavors (Sections 7.3 to 7.5). Along the way, I have also compared ${}^{\dagger}\mathcal{R}$ with these alternative approaches to nonstandard analysis and argued that it has several technical advantages over its competitors, as it is a more powerful version of Laugwitz's and

Tao's approach with reduced powers, a simpler framework than Palmgren's sheaf-theoretic approach, and a more concrete version of the Boolean-valued approach of Scott. In this final section, I will relate the Fréchet hyperreals to the classical Robinsonian hyperreal line from a philosophical and methodological perspective. Robinson's work and its significance for the role that infinitesimals can play both in mathematics and in the empirical sciences has sparked many debates over the years. Here, I want to focus on objections that have been raised against the applicability of nonstandard analysis to mathematics itself, rather than the broader debate of its applicability to other sciences. In particular, I will examine in turn two distinct worries about Robinsonian hyperreals, namely, that they introduce impure methods in analysis and that they do not form a canonical structure. In both cases, I will argue that our semi-constructive Fréchet hyperreals fare better than the classical hyperreal line with respect to these arguments.

7.6.1 The Argument from Purity and Definability

There is a longstanding tradition of criticisms of Robinsonian nonstandard analysis. As we have seen in Section 7.3.2, Tao expressed some uneasiness towards the fact that classical nonstandard analysis uses infinitary, highly non-constructive objects like free ultrafilters to derive results about ordinary mathematical objects such as sets of real numbers. Dauben [65] recalls that Bishop, who proposed an alternative, constructive approach to analysis, criticized the use of notions from nonstandard analysis to teach elementary calculus in harsh terms, claiming that "[i]t is difficult to believe that debasement of meaning could be carried so far" [44, pp. 513-514]. Dauben also argues that Robinson's interest in the history of infinitesimal calculus was, at least in part, motivated by the desire to show that the tools and methods of nonstandard analysis provided in fact a natural conceptual framework for standard analysis:

> History could serve the mathematician as propaganda. Robinson was apparently concerned that many mathematicians were prepared to adopt a "so what" attitude toward nonstandard analysis because of the more familiar reduction that was always possible to classical foundations [...]. But, as Robinson also began to argue with increasing frequency and in greater detail, *historically* the concept of infinitesimals has always seemed natural and intuitively preferable to more convoluted and less intuitive sorts of rigor. [65, p. 184]

It seems that the debate between Robinson and his critics here can be conveniently phrased in terms of purity of methods, in particular in terms of *topical* and *elemental* purity. As discussed in detail in [4, 5], topically pure proofs of a theorem are proofs whose content does not exceed the content of the theorem, i.e., proofs that do not involve objects or concepts that are foreign to the topic of the theorem. By contrast, elementally pure proofs of a theorem are proofs whose proof-theoretic or foundational resources do not exceed those of the theorem. Both kinds of concerns seem to be routinely raised against the methods of nonstandard analysis. Bishop's criticism that the use of nonstandard concepts in elementary calculus is a “debasement of meaning” can be understood as the complaint that such proofs disregard the meaning of elementary notions of calculus like that of a limit, while Tao's uneasiness with free ultrafilters and the infinitary aspect of nonstandard methods speaks to a desire for elementally pure proofs.

How does ${}^{\dagger}\mathcal{R}$ compare to the classical hyperreal line in that respect? I think we may argue that it performs strictly better than Robinson's hyperreal line both from the topical point of view, i.e., regarding the continuity with historical and contemporary mathematical practice, and from the foundational or elemental viewpoint. Indeed, the asymptotic perspective on ${}^{\dagger}\mathcal{R}$ presented in Section 7.3 highlights its continuity with both historical developments in infinitesimal calculus, as evidenced by the fact that ${}^{\dagger}\mathcal{R}$ is a strengthening of Laugwitz's approach with reduced powers, and with contemporary mathematical practice, since it can be seen as a formal counterpart to Tao's cheap nonstandard analysis. As a simplified version of Palmgren's approach, it might even seem more appealing to constructivists than the classical hyperreal line. Moreover, the fact that ${}^{\dagger}\mathcal{R}$ can be constructed in $ZF + DC$ when the Robinsonian hyperreals cannot is an advantage from the point of view of Arana and Detlefsen's analysis of the epistemic merit of pure proofs over impure proofs [5]. They argue that the key advantage of pure proofs of a theorem is their *stability under retraction*. According to them, one may think of a mathematical proof as providing a solution to a given problem. Both the problem and the solution offered by a proof, however, may rely on some background assumptions or *commitments* that one takes for granted when entertaining the problem as an interesting one or accepting the proof as a valid solution to the problem (p. 9). When one gives an impure proof of a theorem, one runs the risk of making some commitments that are necessary for the proof to count as a valid proof, but not for the problem to remain a problem worth investigating. In that case, should we come to later revise our background assumptions and to retract our some of our commitments, an impure proof may cease to provide a answer to a problem that would remain nonetheless a pressing one. By contrast, the very nature of topically pure proofs makes them immune to this sort of worry: indeed, retracting one of the commitments that is necessary for the proof to be admitted as a valid one would also imply retracting one of the commitments that make the problem worth raising in the first place (i.e., pure proofs provide what Arana and Detlefsen call a *cofinal solution* to the problem, p. 10). Here, it seems that the analysis by Arana and Detlefsen captures well the distinction between ${}^{\dagger}\mathcal{R}$ and the classical hyperreal line. Indeed, one may argue that any nonstandard proof of a theorem of analysis that relies on the existence of a classical hyperreal line is impure, because one may retract some commitments that are necessary for such a structure to exist, such as the Ultrafilter Lemma, without altering much of the structure of the reals or analysis as a mathematical field. By contrast, retracting some of the commitments that are necessary for the construction of ${}^{\dagger}\mathcal{R}$ would most likely dramatically alter the practice of analysis as we know it.

This later point also relates to the discussion of the elemental purity of nonstandard methods. From the foundational perspective, I believe indeed that the fact that the existence of ${}^{\dagger}\mathcal{R}$ only requires a semi-constructive setting makes it clearly closer to the usual conceptual resources of analysis. One might think that the reliance on *some* amount of choice puts ${}^{\dagger}\mathcal{R}$ in a similar situation as the classical hyperreal line, but I think that argument can be resisted. Indeed, many central results from nineteenth-century analysis rely heavily on the Axiom of Dependent Choices, while the original reception of Zermelo's full Axiom of Choice among analysts was lukewarm at best [84, Chap. 23]. As Moschovakis writes in [196]:

> We have remarked that before it was formulated precisely by Zermelo, the Axiom of Choice had been used many times "silently" in classical mathematics, and in particular in analysis. *These classical applications, however, can all be justified on the basis of the Axiom of Dependent Choices*—in fact most of them need only the weaker Countable Principle of Choice. [...] This difference between the choice principles needed for classical mathematics and those required by Cantor's new theory of sets explains in part the strident reaction to the axioms of Zermelo by the distinguished analysts of his time (including the great Borel), who had used choice principles routinely in their work—and continued using them, as they denounced general set theory and called it an illusion: in the context of 19th century classical analysis, the Axiom of Dependent Choices is natural and necessary, while the full Axiom of Choice is unnecessary and even has some counterintuitive consequences, including certainly the Wellordering Theorem. [196, pp. 116-117]

We may therefore argue that the proof-theoretic resources required by ${}^{\dagger}\mathcal{R}$ fit squarely within the natural resources required by classical analysis. Unlike in Robinsonian nonstandard analysis, the fragment of the Axiom of Choice that belongs to the semi-constructive setting does not yield counter-intuitive consequences from the point of view of the theory of the real numbers, like a well-ordering of the reals or the existence of a non-Lebesgue measurable set. In fact, this latter point also allows us to draw a sharp contrast between ${}^{\dagger}\mathcal{R}$ and the classical hyperreal line with respect to an influential criticism of nonstandard analysis recently voiced by Alain Connes.

In a famous passage in the first chapter of [60], Connes, discussing the relationship between "logic and reality", raises the following objection to the use of nonstandard analysis in ordinary mathematics:

> But in the final analysis, I became aware of an absolutely major flaw in this theory, an irremediable defect. It is this: in nonstandard analysis, one is supposed to manipulate infinitesimals; yet, if such an infinitesimal is given, starting from any given nonstandard number, a subset of the interval $[0, 1]$ automatically arises which is not measurable in the sense of Lebesgue. [...] The conclusion I drew was that no one will ever be able to show me a nonstandard number. [60, p. 16]

This leads him to argue that nonstandard analysis does not describe the "primordial mathematical reality" that he claims is the true subject of mathematics:

> What conclusion can we draw about nonstandard analysis? This means that, since no one will ever be able to name a nonstandard number, the theory remains virtual [...]. [p. 17]

Connes doesn't specify exactly what he means by "naming" or "showing" a non-standard number, but his argument seems to go along the following lines. Non-measurable sets of real numbers are highly abstract objects, which tend not to appear in ordinary mathematical practice. Connes cites here Solovay's model of $ZF + DC +$ "All sets of reals are Lebesgue measurable" in [242] as evidence that no non-measurable set of reals would ever be encountered in ordinary mathematics. However, one may use a free ultrafilter on ω to define a non-measurable set of reals. As we have seen before, such ultrafilters can moreover be defined themselves from hyperfinite natural numbers in a sufficiently saturated elementary extension of the reals. Consequently, such hyperfinite natural numbers must be as abstract and ineffable as nonmeasurable sets of reals. Although Connes does not explicitly phrase his argument in terms of purity considerations, it seems nonetheless possible to understand his worry as such and stemming from similar considerations as Schechter's distinction between quasi-constructive mathematics and *intangibles*. The crucial problem of nonstandard analysis is that it uses resources beyond the universe of ordinary mathematics, "virtual" mathematical objects that cannot be "named" or explicitly constructed, in order to derive results about what Connes calls the primordial mathematical reality.

As noted by Kanovei et al. in [152], the issue of whether hyperreal numbers can be "named" or "defined" is a delicate problem. One the one hand, Solovay showed that it is consistent with ZFC that no nonmeasurable set of the reals is ordinal definable (as the ordinal definable sets in this model form precisely Solovay's model of $ZF + DC+$ "All sets of reals are Lebesgue measurable"), which yields a model in which no free ultrafilter on ω is ordinal definable. On the other hand, Gödel's constructible universe L provides a well-known example of a model of ZFC in which nonmeasurable sets appear very low in the projective hierarchy and are not only ordinal definable but in fact constructible. Kanovei et al. also cite a result in [153] in which the existence of an ordinal definable hyperreal extension is proved in ZFC.

I do not wish to adjudicate here the debate between Connes and his critics regarding whether Robinsonian hyperreals can be named or not, but I would rather like to point out that the debate does not arise in the case of the Fréchet hyperreals. Indeed, since ${}^{\dagger}\mathcal{R}$ is constructed assuming only $ZF + DC$, it can be constructed even in Solovay's model. Moreover, elements in ${}^{\dagger}\mathcal{R}$ can easily be named and are very familiar objects, since they are merely countable sequences of real numbers. Exhibiting an F-infinitesimal for any $F \in \mathfrak{F}$ is exceedingly easy, since any F_0-infinitesimal is an F-infinitesimal for any $F \in \mathfrak{F}$, and such infinitesimals are easily defined. Of course, determining *whether* a given element $a \in {}^{\dagger}\mathcal{R}$ is an F-infinitesimal for some $F \in \mathfrak{F}$ is a much more complex task, just like determining whether two elements $a, b \in \mathfrak{F}$ are identified at F. One might be tempted to conclude from this that not much is truly gained by moving away from a Luxemburg ultrapower to ${}^{\dagger}\mathcal{R}$. Indeed, one might argue that one could slightly modify the ultrapower construction to take countable sequences of reals as elements, and define the semantics of the equality predicate as the equivalence relation on this domain induced by the ultrafilter, rather than as *bona fide* equality. Clearly, in such a construction, one could "name" infinitesimals just as easily as in ${}^{\dagger}\mathcal{R}$. But there is a significant difference between the two constructions that I think is worth emphasizing. As noted in Section 7.2, validity in ${}^{\dagger}\mathcal{R}$ coincides with the forcing relation at F_0, which is itself *absolute* from the existence of ultrafilters in a rather strong sense. By Corollary 7.2.14, determining whether a first-order sentence is forced at F_0 only requires examining whether the set of natural numbers it determines is cofinite and does not depend on the existence of any free ultrafilter on ω. By contrast, even with the modification proposed above, Luxemburg ultrapowers still require the existence of free ultrafilters to be defined, and Łoś's Theorem shows that satisfaction in such a model is truly determined by the ultrafilter. Thus the real gain from taking ${}^{\dagger}\mathcal{R}$ as our hyperreal line comes from the fact that satisfaction in the model becomes more tractable, without losing the power of the Transfer and Saturation Principles.

Therefore, in contrast with the classical hyperreal line, ${}^{\dagger}\mathcal{R}$ seems to be an acceptable structure even from Connes's viewpoint. In fact, the generic perspective on ${}^{\dagger}\mathcal{R}$ seems to align quite well with Connes's claim that the theory of Luxemburg ultrapowers "remains virtual", since it establishes in a semi-constructive setting that Luxemburg ultrapowers are essentially obtained by forcing over $\mathfrak{F}$. At the same time, it allows us to disentangle a theory of hyperreals from the theory of ultrapowers and to identify only the latter as "virtual", because of its reliance on the Ultrafilter Lemma. To sum up, I have argued that, regarding criticisms against nonstandard analysis that are rooted in purity concerns broadly understood, the semi-constructive approach fares better than the classical approach. The reasons to argue for a topical continuity between nonstandard methods and historical and contemporary mathematical practice remain valid, while from the foundational or elemental viewpoint, the fact that the Axiom of Dependent Choices is enough for the semantics of ${}^{\dagger}\mathcal{R}$ to behave in a tractable way is a clear advantage over the classical hyperreal line.

7.6.2 The Argument from Canonicity

I will now discuss a second kind of argument against the nonstandard approach, which points out a certain kind of arbitrariness of the hyperreal line. By contrast with the first kind of argument, which targets the methods of nonstandard analysis and in particular its reliance on the Axiom of Choice, this second line of argument targets the very structure of the hyperreal line, arguing that it lacks the kind of mathematical properties enjoyed by other standard number systems. An eloquent proponent of this view is Machover in [180]. Reflecting on the developments in nonstandard analysis in the first thirty years after Robinson's seminal work, Machover observes that the mathematical community at large has not embraced nonstandard methods as quickly as some had hoped and offers a explanation for this phenomenon. According to him, a central problem with nonstandard analysis lies in the fact that the structures it studies, enlargements of the standard universe, fail to be *canonical* in the way that the natural numbers or the reals are:

> The point is that whereas the classical number systems (the integers, the rationals the reals etc.) are canonical, there is no such thing as *the* canonical system of *integers, *rationals or *reals. The former can be characterized (informally or within set theory) uniquely up to isomorphism by virtue of their *mathematical* properties: for example, the field of rationals is the smallest field containing the integers, and the field of reals is the completion of the field of rationals. But there is no such thing as *the* [enlarged] field of *reals. The *reals (and in particular the infinitesimal *reals) we happen to deal with in a given nonstandard discourse depend on the enlargement chosen. There is no known way of singling out a particular enlargement that can plausibly be regarded as canonical, nor is there any reason to be sure that a method for obtaining a canonical enlargement will necessarily be invented. [180, pp. 207-208]

Setting aside the broader context of enlargements, the issue that Machover discusses arises already at the level of Luxemburg ultrapowers, since, if the Continuum Hypothesis fails, two ultrapowers of the reals may fail to be isomorphic. Machover's point is then that speaking of *the* hyperreal line is an abuse of language and that nothing in the practice of nonstandard analysis guarantees that the structure that is being studied is fixed precisely enough, that is, up to isomorphism. Here, the fact that Luxemburg ultrapowers rely on the Ultrafilter Lemma is relevant once again: since free ultrafilters are abstract objects which can be proved to exist but cannot be explicilty constructed, it is not possible to fix one particular ultrafilter as a *canonical* one, and consequently no Luxemburg ultrapower can claim to be the privileged hyperreal line. In other words, fixing one ultrapower as the hyperreal line seems both necessary, due to the consistency of the existence of non-isomorphic ultrapowers, and impossible without introducing an element of arbitrariness. In [152], Kanovei et al. note that Machover's point should be nuanced by two existing results. First, the existence of an ordinal definable model of the hyperreals due to Kanovei and Shelah mentioned above, and second, a result of Morley and Vaught establishing that ultrapowers induced by ultrafilters on a cardinal κ such that $2^{<\kappa} = \kappa$ are all isomorphic. I am not certain, however, that this entirely addresses Machover's criticism, as I believe that there is more to the notion of canonicity than the simple fact that a structure is specified up to isomorphism or "definable" in a very narrow set-theoretic sense. Indeed, if the only reason to declare the Kanovei-Shelah model or one of the ultrapowers shown by Morley and Vaught to be unique up to isomorphism as *the* hyperreal line is to rebuke Machover's criticism, then this hardly eliminates the charge of arbitrariness that is at the heart of the canonicity objection. Why should we choose the Kanovei-Shelah model, or one of the Morley-Vaught ones, over one another, as the privileged hyperreal line?

According to Machover, the lack of a canonical hyperreal line also means that the well-definedness of many concepts from nonstandard analysis ultimately relies on the existence of their standard counterpart. He takes as a paradigmatic example of this the nonstandard definition of continuity of a real-valued function f at a real number r. As mentioned in Section 7.2, one of the most appealing features of nonstandard analysis is that infinitesimals can be used in a rigorous way to give an intuitive definition of continuity at a real number r: f is continuous at r if and only if for any *real x, $f(x)$ is infinitesimally close to $f(r)$ whenever x is infinitesimally close to r. In other words, the nonstandard halo of r must be mapped by f to the nonstandard halo of $f(r)$. But, as Machover observes:

> [W]hat we want to define here is a binary relation between two standard objects, f and r; in order to legitimize [the nonstandard definition of continuity] as a definition of this relation, we must make sure that it is independent of the choice of the enlargement. (Otherwise, what is being defined would be a *ternary* relation between f, r and the enlargement.) The easiest way—in fact, the only practicable way, as far as I know—to prove this invariance of [the nonstandard definition] is to show that it is equivalent to the standard $\delta - \epsilon$ definition. Therefore, [the nonstandard definition] cannot displace the old standard definition altogether, if one's aim is to achieve proper rigour and methodological correctness. [p. 208]

Because the halo of a real number is dependent on the ambient hyperreal line or nonstandard universe, the nonstandard definition of continuity must be supplemented with a proof that it is actually independent from the choice of an enlargement. According to Machover, such a proof can in practice only be given by showing the equivalence with the standard definition of continuity. Hence the nonstandard definition owes a conceptual debt to the standard definition, which, according to Machover, means that nonstandard analysis cannot aim at rigorously replacing classical analysis.

Machover's example of the definition of continuity is a particularly good point to compare the classical hyperreal line with $^\dagger\mathcal{R}$. As we have seen in Theorem 7.2.11, a real-valued function f is continuous at a point r if and only if f is F-continuous at r for any $F \in \mathfrak{F}$, i.e., $x \in (r)_F$ implies $f(x) \in (f(r))_F$ for any $x \in \mathcal{R}^\omega$. It seems at first sight that the situation is comparable to the nonstandard one: we first define continuity as a ternary relation between a real-valued function, a real number and a filter F, before showing that the relation actually holds independently of the filter F by showing that it is equivalent to the standard definition of continuity. However, in our case, there is a simple argument that allows us to show that a function f is either F-continuous at some real number r for every $F \in \mathfrak{F}$ or for no $F \in \mathfrak{F}$. Indeed, we may expand our language $\mathscr{L}$ to a language $\mathscr{L}^*$ with a new binary relation symbol $\simeq$ to represent infinitesimal closeness of two Fréchet hyperreals. Formally, this means that we interpret this new symbol $\simeq$ so that for any $a, b \in \mathcal{R}^\omega$ and any $F \in \mathfrak{F}$, $F \Vdash a \simeq b$ iff $|||a - b| < \frac{1}{n}|| \in F$ for every $n \in \omega$. It is straightforward to verify that this interpretation of $\simeq$ satisfies the persistence and refinability conditions of possibility semantics. Hence we may now view $^\dagger\mathcal{R}$ as an $\mathscr{L}^*$-possibility structure, in which F-continuity of a function f at some real r is equivalent to $F \Vdash \forall x(x \simeq r \rightarrow f(x) \simeq f(r))$. Similarly to Corollary 7.2.14, Łoś's Theorem holds for all $\mathscr{L}^*$-formulas. I now claim the following:

Lemma 7.6.1. *For any function $f : \mathcal{R} \rightarrow \mathcal{R}$ and any $r \in \mathcal{R}$, $F_0 \Vdash \forall x(x \simeq r \rightarrow f(x) \simeq f(r))$, or $F_0 \Vdash \exists x(x \simeq r \wedge \neg(f(x) \simeq f(r))$. As a consequence, f is either F-continuous at r for every $F \in \mathfrak{F}$ or for no $F \in \mathfrak{F}$.*

Proof. By persistence, it is immediate to see that if F_0 forces that f is continuous at r or that F is not continuous at r, then f must either be F-continuous at r for every $F \in \mathfrak{F}$, or F-discontinuous at r for any $F \in \mathfrak{F}$ respectively. So let us show that either $F_0 \Vdash \forall x(x \simeq r \to f(x) \simeq f(r))$ or $F_0 \Vdash \exists x(x \simeq r \wedge \neg f(x) \simeq f(r))$. Suppose that $F_0 \nVdash \forall x(x \simeq r \to f(x) \simeq f(r))$. Then there is some $F \in \mathfrak{F}$ and some $a \in \mathcal{R}^\omega$ such that $F \Vdash a \simeq r$ and $F \Vdash \neg f(a) \simeq f(r)$. This means that $|||a - r| < \frac{1}{n}|| \in F$ for all $n \in \omega$, and that there is some $m \in \omega$ such that $|||f(a) - f(r)| \geq \frac{1}{m}|| \in F$. Since F is a free filter on ω, this means that for any $i \in \omega$, the set $A_i = \bigcap_{k<i} |||a - r| < \frac{1}{k}|| \cap |||f(a) - f(r)| \geq \frac{1}{m}||$ is infinite. Let $b : \omega \to \mathcal{R}$ be defined by letting $b(i) = a(j)$, where j is the least $n \in A_i$ such that $n > i$. By construction, we have that $i \in |||b - r| < \frac{1}{n}||$ whenever $n < i$, which means that $F_0 \Vdash b \simeq r$. Moreover, $|||f(b) - f(r)| \geq \frac{1}{m}|| = \omega$, hence $F_0 \Vdash \neg f(b) \simeq f(r)$. This completes the proof. □

Note that the proof above does not mention the standard definition of continuity in any way. Indeed, the core of the argument is to show that any counterexample to the F-continuity of f for *some* free filter F can be turned into a counterexample to the F_0-continuity of f. As a consequence, F-continuity of a function is a property that is entirely determined by the Fréchet filter, and we could therefore substitute F-continuity for the standard definition of continuity without any risk of ambiguity. Interestingly, the argument above also gives a way out of Machover's criticism of the nonstandard definition of continuity. Indeed, as the comparison between ${}^\dagger\mathcal{R}$ and the generic approach to nonstandard analysis has made apparent, Luxemburg ultrapowers of $\mathcal{R}$ are precisely generic models over $\mathfrak{F}$. It is straightforward to verify that the results in Section 7.5.2 would extend to ${}^\dagger\mathcal{R}$ considered as an $\mathscr{L}^*$-structure. In particular, this means that for any ultrafilter U on ω, a function f will be continuous at a real number r according to the ultrapower $\mathcal{R}_{/U}$ if and only if it is F-continuous for some $U \supseteq F \in \mathfrak{F}$. But this, in connection with Lemma 7.6.1, straightforwardly implies that the non-standard definition of continuity is independent of the choice of the ultrapower $\mathcal{R}_{/U}$. This shows that, if one takes Machover's challenge seriously, ${}^\dagger\mathcal{R}$ has some significant foundational consequences for nonstandard analysis even in the presence of the Ultrafilter Lemma.

What about Machover's broader point regarding the lack of canonicity of the classical hyperreal line? From the purely mathematical perspective, there is a precise way in which ${}^\dagger\mathcal{R}$ is canonical in the sense of being characterized up to isomorphism. As outlined in Section 7.5, ${}^\dagger\mathcal{R}$ can be seen as a Boolean-valued model with domain $\mathcal{R}^\omega$ and algebra of truth-values $\mathsf{RO}(\mathfrak{F})$, where the interpretation of relation symbols in the language $\mathscr{L}$ is given by the map $R(\overline{a}) = [\![R(\overline{a})]\!])$ for any n-ary $\mathscr{L}$ symbol R and any n-tuple $\overline{a}$ of elements of $\mathcal{R}^\omega$. In fact, one can show that ${}^\dagger\mathcal{R}$ is, up to isomorphism, the unique Boolean valued-model satisfying certain properties. The argument is straightforward but slightly tedious, and can be found in Section 7.8.2.

Moreover, as I have mentioned above, there is probably more to the informal notion of canonicity than a mere characterization up to isomorphism. In the case of ${}^\dagger\mathcal{R}$, I think it is also possible to argue that the structure we obtain is a *natural* mathematical object to investigate. Indeed, as I have argued at length in the previous sections, ${}^\dagger\mathcal{R}$ seems to arise as a common ground for many distinct mathematical projects. It is a strengthening of

Laugwitz's attempt to provide rigorous foundations to Leibniz and Cauchy's calculus and to Euler's work with infinite sequences, and at the same time it is a formalization of Tao's attempt to bring nonstandard analysis closer to ordinary mathematical practice. It is a simplification of Palmgren's topos-theoretic approach to constructive nonstandard analysis, as well as a more concrete approach to Boolean-valued models of analysis. Finally, it brings a generic perspective to classical nonstandard analysis, allowing Luxemburg ultrapowers to be seen as generic models in the sense of forcing.

As such, ${}^{\dagger}\mathcal{R}$ can also be seen as providing both a diagnosis and a cure to the canonicity problem identified by Machover. If we agree with Laugwitz that the core idea of the infinitesimal method is that the properties of sequences "in the limit" are determined by the properties of all but finitely many of their values, then only the cofinite sets of ω can be deemed as large enough to determine truth in the hyperreal line. As Tarskian semantics requires an ultrafilter for the Transfer Principle to apply, Robinsonian hyperreals force us to select some infinite and co-infinite sets to count as "large enough" as well, thus introducing an element of seemingly unavoidable arbitrariness. By contrast, the dynamic character of possibility semantics allows us to consider all possible such choices at once and to restore the Transfer Principle at the only canonical stage, that of the Fréchet filter.

Finally, let me conclude this section by comparing the theoretical virtues that I have argued ${}^{\dagger}\mathcal{R}$ enjoys with the axiomatic approach recently developed independently by Hrbáček and Katz in [139]. The authors investigate several fragments of Nelson's system IST and Kanovei's variant BST. The most relevant system for a direct comparison with ${}^{\dagger}\mathcal{R}$ is the system that they call $SCOT$, as they show that $SCOT$ is a conservative extension of $ZF+DC$ in which a nonstandard extension of the reals satisfying a version of the Transfer and Saturation principles hold. Interestingly, their proof of this result involves a construction that is remarkably close to that of ${}^{\dagger}\mathcal{R}$: working in an arbitrary model $\mathcal{M}$ of $ZF+DC$, they show how to build a forcing extension $\mathcal{M}^*$ that models $SCOT$ and is such that $\mathcal{M}$ coincides with the standard sets in $\mathcal{M}^*$ (Sections 4 and 5). I believe, however, that their axiomatic perspective is less likely to convincly address the objections to the use of nonstandard methods that I have considered in this section than the semantic approach based on possibility semantics. For one, it is not clear that the axiomatic approach would help with the canonicity problem raised by Machover. Katz and Hrbáyek do not discuss the issue, but it does not seem that $SCOT$ can prove that there is a structure resembling the Robinsonian hyperreals that can be characterized uniquely up to isomorphism in the way that ${}^{\dagger}\mathcal{R}$ can be characterized in $ZF+DC$. Moreover, I think there are also reasons to doubt that Hrbáček and Katz's axiomatic approach convincingly addresses the issues discussed in Section 7.6.1, although the issue is more subtle. From a strictly formalist or instrumentalist viewpoint, one could certainly be satisfied with the conservativity result proved by Hrbáček and Katz. Indeed, if one views hyperreals as mere tools that one appeals to in order to obtain results about real numbers (or even just finite objects), and if one also considers that this is, in a sense, all there is to the practice of mathematics, then one could happily adopt the axiomatic approach to nonstandard analysis that Hrbáček and Katz develop, knowing that any theorem one could derive about the reals in $SCOT$ could in principle be proved in $ZF+DC$. But it is not clear that this would achieve much towards convincing the critics of nonstandard analysis whom I have mentioned in Section 7.6.1, since few of them would consider themselves formalists or instrumentalists. Formalists, and instrumentalists in general, are not likely to be bothered by the introduction of ideal numbers to prove more theorems of analysis, especially when the new structure introduced satisfies something like the Transfer Principle.[9] By contrast, it seems to me that the most vocal critics of nonstandard methods have a more substantial notion of reality that the practice of mathematics is meant to reveal, such as Connes's "primordial mathematical reality". Here, I think that the conservativity results in [139] would likely fall short of convincing anyone who thinks that any existence claim regarding Robinsonian infinitesimals is simply false. Without going too far into debates regarding the relationship between conservativity, interpretability and mathematical truth, it is enough to point out that one could consistently argue that all the axioms of $ZF + DC$ are true in Connes' "primordial mathematical reality", even though some of axioms of $SCOT$, namely those asserting the existence of nonstandard entities, happen to be false. In short, although the conservativity results should reassure Connes that no false consequence about the reals could be derived from $SCOT$, the fact that $SCOT$ holds in a forcing extension of any model of $ZF + DC$ may fail to completely dispel his misgivings about nonstandard methods.

7.7 Conclusion

In this chapter, I have used possibility semantics for first-order logic to define the Fréchet hyperreal line ${}^{\dagger}\mathcal{R}$ as an alternative to the classical ultrapower approach to nonstandard analysis. As I have argued, ${}^{\dagger}\mathcal{R}$ shares many of the technical advantages of the classical hyperreal line, arguably coming closer to it than the asymptotic, dynamic and generic approaches I have discussed. At the same time, its more constructive character makes it an attractive option from a foundational and methodological standpoint. I therefore hope to have convinced the reader that the semi-constructive approach to nonstandard analysis I have sketched here is a mathematically natural and philosophically rich alternative to explore.

The work presented here does not exhaust the ways in which possibility semantics may interact with nonstandard analysis. For one, I have only discussed an alternative to the use of countably saturated ultrapowers, but nonstandard analysis is a much wider field. In particular, it routinely studies a larger class of structures, such as enlargements of the standard universe of sets, and it remains to be seen whether one could also develop a satisfactory alternative to such structures in a semi-constructive setting. Similarly, I have only touched on the idea that the Fréchet hyperreals are a straightforward attempt at capturing an intuition about the properties of sequences being determined by their values "almost everywhere". Whether this idea is as historically significant as hinted by Laugwitz and whether it could motivate a proper *conception* of the continuum, in the sense of [87], will have to be explored in future work.

[9] It is perhaps worth recalling here that Robinson himself was a self-avowed formalist [222].

7.8 Appendix

7.8.1 The Standard Part Function

In this appendix, I show how to define an analogue of the standard part function on ${}^{\dagger}\mathcal{R}$. As mentioned in Section 7.2, it is not possible to define an actual function $f : \mathcal{R}^{\omega} \to \mathcal{R}$ that would map any finite Fréchet hyperreal to its standard part, on pain of arbitrariness or inconsistency. Nonetheless, it is possible to define such a function *internally*, meaning that one may extend the language of ${}^{\dagger}\mathcal{R}$ so as to include a binary relation symbol $st(x, y)$, to be interpreted as "x is the standard part of y". One can then show that any viewpoint $F \in \mathfrak{F}$ forces that any finite Fréchet hyperreal has a unique standard part. The first step is to extend the language introduced in Section 7.6.2 to include a *standardness* predicate:

Definition 7.8.1. Let $\mathscr{L}^{\dagger}$ be the language $\mathscr{L}^{*}$ augmented with a unary predicate symbol $S(x)$ and a binary relation symbol $st(x, y)$. We extend the interpretation function $\mathscr{I}$ to $\mathscr{L}^{\dagger}$ as follows:

- For any $a \in {}^{\dagger}\mathcal{R}$ and any $F \in \mathfrak{F}$, $a \in \mathscr{I}(F, S)$ iff $||\bigvee_{r \in S} a = \delta(r)|| \in F$ for some finite subset S of $\mathcal{R}$.
- For any $a, b \in {}^{\dagger}\mathcal{R}$ and any $F \in \mathfrak{F}$, $(a, b) \in \mathscr{I}(F, st)$ iff $F \Vdash S(a) \wedge a \simeq b$.

It is mostly straightforward to verify that the interpretation of $S(x)$ and $st(x, y)$ satisfies the conditions of Definition 6.1.1. We only show that refinability condition holds, and leave the rest as an exercise to the reader. Suppose first that $a \notin \mathscr{I}(F, S)$, and let G be the filter generated by $F \cup \{||a \neq \delta(r)|| \mid r \in \mathbb{R}\}$. I claim that G is a proper filter, and thus that $G \in \mathfrak{F}$. Suppose towards a contradiction that G is not proper. Then for some finite $R \subseteq \mathbb{R}$, we have that $\omega \setminus \bigcap_{r \in R} ||a \neq \delta(r)|| \in F$. But $\omega \setminus \bigcap_{r \in R} ||a \neq \delta(r)|| = \bigcup_{r \in R} ||a = \delta(r)|| = ||\bigvee_{r \in R} a = \delta(r)||$, so this contradicts our assumption on F. Hence G is a proper filter extending F. Clearly, $\omega \setminus ||\bigvee_{r \in R} a = \delta(r)|| \in G$ for any finite subset R of $\mathcal{R}$, so $a \notin \mathscr{I}(H, S)$ for any $H \supseteq G$. This shows that the interpretation of $S(x)$ satisfies the refinability condition. To show that the interpretation of $st(x, y)$ also satisfies it, suppose $F \nVdash st(a, b)$. Then $F \nvDash S(a)$ or $F \nVdash a \simeq b$. Hence either there is $G \supseteq F$ such that $H \nVdash S(a)$ for any $H \supseteq G$, or there is $G \supseteq F$ such that $H \nVdash a \simeq b$ for every $H \supseteq G$. Either way, there is $G \supseteq F$ such that $H \nVdash st(a, b)$ for every $H \supseteq G$, which shows that the interpretation of $st(x, y)$ also satisfies the refinability condition.

We may now show that $st(x, y)$ is interpreted as a total function on finite Fréchet hyperreals in ${}^{\dagger}\mathcal{R}$:

Lemma 7.8.2. *For any $F \in \mathfrak{F}$,*

$$F \Vdash \forall y(\exists z_1 \exists z_2 (S(z_1) \wedge S(z_2) \wedge z_1 < y < z_2) \to \exists! x(st(x, y)).$$

Proof. Fix some $F \in \mathfrak{F}$ and some $b \in \mathcal{R}^{\omega}$ such that $F \Vdash z_1 < b < z_2$ for some F-standard z_1, z_2 which we may assume to be real numbers. Let us first establish that $F \Vdash \exists x(st(x, b))$.

By Definition 6.1.2, we must show that for any $G \supseteq F$, there is $H \supseteq G$ and $a \in \mathcal{R}^\omega$ such that $H \Vdash st(a, b)$. Fix such a filter G. We consider the following two cases:

- Case 1: For some $r \in \mathcal{R}$, $||b \neq \delta(r)|| \notin G$. Then letting H be the filter generated by $G \cup ||b = \delta(r)||$, we have that $H \Vdash b = \delta(r)$. Since $\delta(r)$ is clearly H-standard, it follows that $H \Vdash st(\delta(r), b)$.

- Case 2: For all $r \in \mathcal{R}$, $||b \neq \delta(r)|| \in G$. Then let A be the set $\{r \in \mathcal{R} \mid G \nVdash b \leq \delta(r)\}$. Note that $z_1 \in A$ and that z_2 is an upper bound of A, so by the completeness of the real line A has a least upper bound r. Now we distinguish again two cases:

  – Case 2.1: $G \Vdash b < r$. Since r is the least upper bound of the set A, we must have that $G \nVdash b \leq \delta(r - \frac{1}{n})$ for any $n \in \mathbb{N}$, since otherwise $r - \frac{1}{n}$ would be an upper bound of A. But this means that $||b \leq \delta(r - \frac{1}{n})|| \notin G$ for any $n \in \mathbb{N}$, and thus the set $G \cup \{||\delta(r) - b < \frac{1}{n}|| \mid n \in \mathbb{N}\}$ generates a filter H. By construction, $H \Vdash \delta(r) \simeq b$, and therefore $H \Vdash st(\delta(r), b)$.

  – Case 2.2: $G \nVdash b < r$. For any $n \in \mathbb{N}$, let $B_n = ||0 < b - \delta(r) < \frac{1}{n}||$. I claim that the set $G \cup \{B_n \mid n \in \mathbb{N}\}$ generates a filter. To show this, it is enough to show that $\omega \setminus B_n \notin G$ for any $n \in \mathbb{N}$. Suppose towards a contradiction that $\omega \setminus B_n \in G$ for some $n \in \mathbb{N}$. This means that $G \Vdash b \leq \delta(r) \vee \delta(r + \frac{1}{n}) \leq b$. Since r is an upper bound of A, $r + \frac{1}{n} \notin A$, which means that $G \Vdash b \leq \delta(r + \frac{1}{n})$. By distributivity, this means that

  $$G \Vdash (b \leq \delta(r) \wedge b \leq \delta(r + \frac{1}{n})) \vee (\delta(r + \frac{1}{n}) \leq b \wedge b \leq \delta(r + \frac{1}{n})),$$

  which is equivalent to

  $$G \Vdash b < \delta(r) \vee b = \delta(r) \vee b = \delta(r + \frac{1}{n}).$$

  Since $||b \neq \delta(s)|| \in G$ for all $s \in \mathcal{R}$, this implies that $G \Vdash b < \delta(r)$, a contradiction. Thus $G \cup \{B_n \mid n \in \mathbb{N}\}$ generates a filter $H \in \mathfrak{F}$. By construction, $H \Vdash \delta(r) \simeq b$, and therefore $H \Vdash st(\delta(r), b)$.

This completes the proof that $F \Vdash \exists x(st(x, b))$. For uniqueness, it suffices to show that $F \Vdash st(a_1, b) \wedge st(a_2, b)$ implies $F \Vdash a_1 = a_2$ for any $a_1, a_2 \in \mathcal{R}^\omega$ and $F \in \mathfrak{F}$. Let $F' \supseteq F$. From $F' \Vdash st(a_1, b)$, we have that $||a_1 = \delta(r_1)|| \in G$ for some $r_1 \in \mathcal{R}$ and $G \supseteq F'$ and that $G \Vdash a_1 \simeq b$. But this means that $|||\delta(r_1) - b| < \frac{1}{n}|| \in G$ for any $n \in \mathbb{N}$. Similarly, there is $r_2 \in \mathcal{R}$ and $H \supseteq G$ such that $|||\delta(r_2) - b| < \frac{1}{n}|| \in H$ for any $n \in \mathbb{N}$. By the triangle inequality, $|\delta(r_1)(i) - \delta(r_2)(i)| \leq |\delta(r_1)(i) - y(i)| + |\delta(r_2)(i) - y(i)|$ for any $i \in \omega$, hence for any $n \in \mathbb{N}$, $|||\delta(r_1) - y| < \frac{1}{2n}|| \cap |||\delta(r_2) - y| < \frac{1}{2n}|| \subseteq |||\delta(r_1) - \delta(r_2)| < \frac{1}{n}||$, which implies that $|||\delta(r_1) - \delta(r_2)| < \frac{1}{n}|| \in H$ and is thus non-empty. But this implies at once that $|r_1 - r_2| < \frac{1}{n}$ for any $n \in \mathbb{N}$, and therefore $r_1 = r_2$. Thus $||a_1 = \delta(r_1)|| \cap ||a_2 = \delta(r_2)|| \subseteq ||a_1 = a_2||$, and $H \Vdash a_1 = a_2$. Thus we've shown that for any $F' \supseteq F$ there is $H \supseteq F'$ such that $H \Vdash a_1 = a_2$, from which we conclude that $F \Vdash a_1 = a_2$. □

7.8.2 A Characterization of the Fréchet Hyperreals

In this appendix, I give a characterization of ${}^{\dagger}\mathcal{R}$ as a Boolean-valued model that is unique up to isomorphism. Recall first that Boolean-valued models are given by a triple $(D, \mathbb{B}, \mathscr{I})$, where D is a set, $\mathbb{B}$ is (usually complete) Boolean algebra, and $\mathscr{I}$ maps any n-ary relation symbol in the language to a function with domain D^n and co-domain $\mathbb{B}$.[10] Given two Boolean-valued models $\mathcal{M}_1 = (D, \mathbb{B}, \mathscr{I})$ and $\mathcal{M}_2 = (E, \mathbb{C}, \mathcal{J})$, an isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$ is given by a pair (η, θ) such that $\eta : D \to E$ is a bijection, $\theta : \mathbb{B} \to \mathbb{C}$ is a Boolean isomorphism, and $\eta(\mathscr{I}(R)(a_1, ..., a_n)) = \mathcal{J}(R)(\theta(a_1), ..., \theta(a_n))$. As discussed in Section 7.5.1, the Boolean algebra $\mathsf{RO}(\mathfrak{F})$ of truth-values in ${}^{\dagger}\mathcal{R}$ is the canonical extension of $\mathscr{P}(\omega)^*$. Since the canonical extension of a Boolean algebra B can always be characterized purely algebraically up to a unique isomorphism fixing B (see for instance [103]), characterizing $\mathsf{RO}(\mathfrak{F})$ uniquely up to isomorphism reduces to characterizing up to isomorphism the Boolean algebra $\mathscr{P}(\omega)^*$. This turns out to be a slightly more involved matter. In the presence of the Continuum Hypothesis, this can be done directly, as shown in [193]. Without assuming CH, one can still argue in an indirect way as follows. By the Lindenbaum-Tarski-Jónsson duality between complete atomic Boolean algebras and sets (see [146, Chap. 6]), $\mathscr{P}(\omega)$ is the unique complete Boolean algebra generated by a countable set of atoms. Moreover, a *compact* element in a complete Boolean algebra B is some element $b \in B$ such that for any $X \subseteq B$, $b \leq \bigvee X$ implies $b \leq \bigvee X'$ for some finite $X' \subseteq X$. It is straightforward to verify that the compact elements of a Boolean algebra B always form an ideal on B, and that if $f : B \to \mathscr{P}(\omega)$ is a Boolean isomorphism, then b is a compact element of B iff $f(b)$ is a finite subset of ω. Thus, up to isomorphism, $\mathsf{RO}(\mathfrak{F})$ is the unique canonical extension of the quotient of the unique complete Boolean algebra generated by a countable set of atoms modulo its ideal of compact elements. We may now offer the following characterization of ${}^{\dagger}\mathcal{R}$:

Lemma 7.8.3. *The Fréchet hyperreals* ${}^{\dagger}\mathcal{R} = (\mathcal{R}^{\omega}, \mathsf{RO}(\mathfrak{F}), [\![\cdot]\!])$ *is, up to isomorphism, the unique Boolean-valued* $\mathscr{L}$*-structure* $(D, \mathbb{B}, \mathscr{I})$ *such that:*

- *there is a countable set* U *and a Boolean isomorphism* χ *from* $\mathbb{B}$ *into* $(\mathscr{P}(U)/Fin)^{\delta}$*, the canonical extension of the quotient* $\mathscr{P}(U)/Fin$ *of* $\mathscr{P}(U)$ *modulo its ideal of compact elements;*
- *there is a complete ordered field* $\mathscr{X}$ *and a bijection* f *from* D *onto the set of functions with domain* U *and codomain* $\mathscr{X}$ *such that, for any* n*-ary relation* R *and tuple* $\overline{a} \in D^n$*,* $\mathscr{I}(R)(\overline{a}) = \epsilon(\{u \in U \mid \mathbb{X} \models R(\overline{f(a)(u)})\})$*, where* ϵ *is the Boolean homomorphism obtained by composing the canonical map from* $\mathscr{P}(U)$ *to* $(\mathscr{P}(U)/Fin)^{\delta}$ *with* χ*.*

Proof. Let us prove that a Boolean-valued $\mathscr{L}$-structure $\mathcal{M} = (D, \mathbb{B}, \mathscr{I})$ satisfies the condition of the lemma if and only if it is isomorphic to ${}^{\dagger}\mathcal{R}$. Clearly, if (η, θ) is an isomorphism from $\mathcal{M}$ to ${}^{\dagger}\mathcal{R}$, then ω is the required set U, $\mathcal{R}$ is the required field $\mathscr{X}$, η is the required

[10] Function and constant symbols can be treated as special relation symbols mapped by $\mathscr{I}$ to a function with codomain $\{0_{\mathbb{B}}, 1_{\mathbb{B}}\}$ for all relevant purposes here.

isomorphism χ and θ is the required bijection f. This shows the right-to-left direction of the biconditional. For the converse, suppose now that we have U, $\mathscr{X}$, χ and f as in the statement of the lemma. Before defining the maps η and θ, we introduce the following maps:

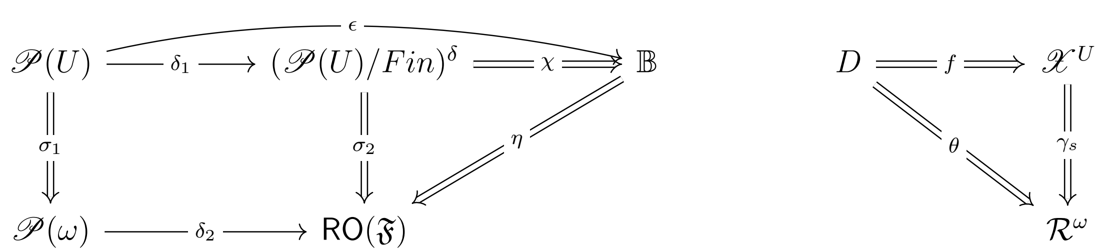
- We fix a bijection $s : \omega \to U$. This maps induces a Boolean isomorphism $\sigma_1 : \mathscr{P}(U) \to \mathscr{P}(\omega)$ given by $\sigma_1(S) = \{i \in \omega \mid s(i) \in S\}$.

- The canonical extension of $\mathscr{P}(U)/Fin$, denoted $(\mathscr{P}(U)/Fin)^\delta$, can be constructed as the Boolean algebra of regular open sets of the poset of filters on $\mathscr{P}(U)$ that extend the filter of all cofinite subsets of U, ordered by reverse inclusion. The map $\delta_1 : \mathscr{P}(U) \to (\mathscr{P}(U)/Fin)^\delta$ given by $S \mapsto \{F \mid S \in F\}$ is the canonical embedding of $\mathscr{P}(U)$ into $(\mathscr{P}(U)/Fin)^\delta$, so $\epsilon = \chi \circ \delta_1$.

- Similarly, let $\delta_2 : \mathscr{P}(\omega) \to \mathsf{RO}(\mathfrak{F})$ be given by $\delta_2(S) = \{F \in \mathfrak{F} \mid S \in F\}$. The Boolean isomorphism σ_1 can be lifted along δ_1 and δ_2 to a Boolean isomorphism $\sigma_2 : (\mathscr{P}(U)/Fin)^\delta \to \mathsf{RO}(\mathfrak{F})$.

- Finally, we fix an isomorphism $\gamma : \mathscr{X} \to \mathcal{R}$. Combined with $s : \omega \to U$, this induces a bijection $\gamma_s : \mathscr{X}^U \to \mathcal{R}^\omega$ where, for any $a : U \to \mathscr{X}$, $\gamma_s(a)(i) = \gamma(a(s(i)))$.

We now let $\eta : \mathbb{B} \to \mathsf{RO}(\mathfrak{F})$ be the map $\sigma_2 \circ \chi^{-1}$ and $\theta : D \to \mathcal{R}^\omega$ as the map $\gamma_s \circ f$. All definitions above are summed up in the two commutative diagrams below. The arrows in the leftmost diagram are Boolean homomorphisms, and functions between sets in the rightmost one. In both diagrams, double arrows are maps that are invertible (i.e., Boolean isomorphisms in the leftmost diagram and bijections in the rightmost one). Note in particular that $\delta_2 \circ \sigma_1 = \sigma_2 \circ \delta_1 = \sigma_2 \circ \chi^{-1} \circ \chi \circ \delta_1 = \eta \circ \epsilon$.



Clearly, η is a Boolean isomorphism and θ is a bijection. So we only need to check that for any n-ary relation symbol R and any tuple $d_1, ..., d_n$ of elements of D, $\eta(\mathscr{I}(R))(d_1, ..., d_n)) = [\![R(\theta(d_1), ..., \theta(d_n))]\!]$. By Theorem 7.2.5, we have that $[\![R(\theta(d_1), ..., \theta(d_n))]\!] = \{F \in \mathfrak{F} \mid ||R(\theta(d_1), ..., \theta(d_n)|| \in F\} = \delta_2(||R(\theta(d_1), ..., \theta(d_n)||)$. Now we compute:

$$
\begin{aligned}
[\![R(\theta(d_1),...,\theta(d_n))]\!] &= \delta_2(||R(\theta(d_1),...,\theta(d_n))||) \\
&= \delta_2(\{i \in \omega \mid \mathcal{R} \models R(\theta(d_1)(i),...,\theta(d_n)(i))\}) \\
&= \delta_2(\{i \in \omega \mid \mathcal{R} \models R(\gamma_s(f(d_1))(i),...,\gamma_s(f(d_n))(i))\}) \\
&= \delta_2(\{i \in \omega \mid \mathcal{R} \models R(\gamma(f(d_1))(s(i)),...,\gamma(f(d_n))(s(i)))\}) \\
&= \delta_2(\{i \in \omega \mid \mathscr{X} \models R(f(d_1)(s(i)),...,f(d_n)(s(i)))\}) \\
&= \sigma_2(\sigma_1(\{u \in U \mid \mathscr{X} \models R(f(d_1)(u),...,f(d_n)(u))\})) \\
&= \eta(\epsilon(\{u \in U \mid \mathscr{X} \models R(f(d_1)(u),...,f(d_n)(u))\})) \\
&= \eta(\mathscr{I}(R)(d_1,...,d_n)),
\end{aligned}
$$

where the last equality follows from the assumption on ϵ and f. This concludes the proof. □

Chapter 8

Possibility Semantics and the Euclidean Infinite

8.1 Introduction

In this chapter, I explore another application of generic powers and possibility semantics in the philosophy of the mathematical infinite. Just as nonstandard analysis aims to provide a powerful mathematical framework for an alternative conception of the continuum, techniques inspired from NSA have recently been used to develop a sophisticated mathematical theory of the size of infinite sets that vastly differs from the Cantorian one. The key point is that the modern notion of cardinality, as is well known, does not preserve the pre-theoretic intuition that a whole should always be greater than any of its proper parts. Because this principle can be traced back to the Common Notions in Euclid's Elements, theories of size of infinite sets that preserve it are often called Euclidean. The interest in Euclidean theories of size has recently been revived by the theory of numerosities, developed primarily by Benci and di Nasso [18, 19, 16]. Originally, numerosity structures were introduced as nonstandard extensions of the natural numbers and usually constructed as ultrapowers of $\mathbb{N}$. Their key feature is that one could assign a numerosity to every subset of the natural numbers in a manner that would preserve the part-whole intuition. More recently, numerosities have also been applied to address some issues regarding probabilistic scenarios on an infinite sample space. Just as one may have the pre-theoretic intuition that the whole is always bigger than any of its proper parts, one may also have the pre-theoretic intuition that events that are strictly more likely than others should be assigned a strictly greater probability of occurring. But this intuition is famously not preserved by standard probability theory, since possible events often receive probability 0 of occurring whenever the space of outcomes is infinite. In a series of recent papers, Benci, Horsten and Wenmackers [21, 20, 259] have used techniques similar to numerosities to propose a novel approach to probability theory on infinite sample spaces. Their theory, Non-Archimedean Probability (NAP) theory, has the advantage of allowing one to define probability measures on infinite sample spaces that preserve the intuition that only impossible events should be assigned probability 0 of occurring.

My goal in this chapter is to use possibility semantics to explore an alternative approach to numerosities and NAP theory. Just as the Fréchet Hyperreals offer an alternative way of constructing the hyperreals that has some foundational benefits over Luxemburg ultrapowers, I will argue that possibility structures can be used to construct numerosity and NAP functions that have several conceptual advantages over their Tarskian counterparts. While we investigated applications of possibility semantics in analysis in the previous chapter, we will this time be concerned with basic results of algebra regarding partially ordered (semi)rings and fields. Since algebra is notoriously an area of classical mathematics in which non-constructive principles such as Zorn's Lemma are routinely used, this will provide us with a good opportunity to test the robustness and limits of possibility semantics for the development of classical mathematics in a semi-constructive setting. To keep the discussion contained, I will be focusing for most of the chapter on two important problems, Galileo's Paradox and fair lotteries on infinite sets, as both offer a simple and concrete way to understand the issues that Euclidean theories of the infinite typically face.

The chapter is organized as follows. In Section 8.2, I briefly recall the aspects of the current debate on the Euclidean infinite that I will be discussing. In particular, I emphasize the significance of numerosities and NAP functions for debates in the philosophy of mathematics and the philosophy of probability, respectively. In Section 8.3, I give some background on numerosities and NAP functions, with the aim to provide an intuitive and accessible presentation on the main technical achievements of both. In Section 8.4, I discuss some objections that have been raised against Euclidean theories in the literature, focusing in particular of the claim that numerosity assignments and NAP functions do not capture robust notions of size and probability respectively, because many of their features are neither invariant under permutations of their domains nor conceptually well-motivated. This prompts me to develop in Section 8.5 what I call a *generic* approach to numerosities and NAP functions based on possibility structures, before arguing in Section 8.6 that generic numerosities and NAP functions do not face the same issues as their Tarskian counterparts.

8.2 Galileo's Paradox and De Finetti's Lottery

In this section, I introduce the two problems in the philosophy of the mathematical infinite that will occupy us in this chapter. In both cases, what can be seen as the "mathematically orthodox" solution to the problem can be contrasted with an alternative, "heterodox" solution that has nonetheless some intuitive pull and historical predecessors. I will present each problem in turn and briefly recall the broader philosophical context that makes each of them relevant.

8.2.1 Infinite Sizes and the Part-Whole Principle

The first problem involves extending the concepts of counting and size from finite to infinite collections. Galileo offered a vivid presentation of it in his *Dialogue of the Two New Sciences*

[99], although, as thoroughly established by Mancosu in [184], the problem itself has a much longer and very rich history. On the first day of their dialogue, Salviati (Galileo's stand-in the Dialogue) is challenged by a perplexed Simplicio who asks the following question: how is it possible that a line, containing infinitely many points, may however be a proper part of another line, who must therefore contain a greater infinity of points? Salviati replies:

> This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. [99, p. 30]

Salviati argues for his position by giving another example. By a series of questions to Simplicio about the collection of all square numbers and the collection of all numbers, Salviati manages to make Simplicio contradict himself. On the one hand, since every square is a natural number, but the converse is false, there must be stricly more natural numbers than squares. On the other hand, since every square number can be mapped uniquely to its root, and every natural number is the root of some square, there must in fact be as many squares as there are roots of squares, and thus as many square numbers as there are natural numbers. Salviati draws the following conclusion from the situation:

> So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal", "greater", and "less" are not applicable to infinite, but only to finite, quantities. [p. 31-32]

Galileo's solution to the paradox is therefore to retreat: it is simply impossible to extend coherently the concept of size in a way that would allow us to meaningfully compare infinite collections. There seems to be two competing intuitions, which, following [184], I will call the *Bijection* and *Part-Whole* principles (**(BP)** and **(PW)** for short). If one takes the notions of collection and parthood among them as primitive, both principles can then be stated as follows for any two collections A, B:

(BP) $size(A) = size(B)$ iff there is a bijection $f : A \to B$;

(PW) if A is a proper part of B, then $size(A) < size(B)$.

If one only considers finite collections, then the two principles are not only compatible but true, assuming that one takes the size of a collection to be the number of its elements. But if one understands the notion of a collection as being synonymous with that of a set, and that of being a part of a collection as being synonymous with being a subset of a set, then as soon as one allows for the existence of a Dedekind-infinite set, the two principles come apart. Indeed, if B is a Dedekind-infinite set, then there is a proper subset A of B and a bijection $f : B \to A$. But then, by **(BP)**, we should conclude that $size(A) = size(B)$, while **(PW)** implies that $size(A) < size(B)$. Galileo's Paradox is simply an instance of this phenomenon in which one takes B to be the set of all natural numbers and A the set of all square numbers.

Cantor famously solved the issue by endorsing **(BP)** and rejecting **(PW)**. In doing so, he developed the modern notion of cardinality, which is often taken to be an extension of the notion of size to infinite sets. In short, one may view the Cantorian notion of an ordinal as an extension of the concept of *counting* into the transfinite. Ordinals, however, turn out to be too fine-grained to also deliver a satisfactory notion of size: distinct ordinals may be bijectable with one another, and ordinal arithmetic fails to be commutative. Cardinals, by contrast, defined either as equivalence classes of equinumerous ordinals or as canonical representatives thereof (i.e., as least ordinals in such equivalence classes) form a rich and well-behaved structure that satisfies **(BP)** and on which the natural notions of addition and multiplication are commutative.

The success of the Cantorian approach has led to a widespread belief that choosing **(BP)** over **(PW)** was essentially the *only way* to extend the concept of size into the infinite. Gödel [107] famously argued for such a position when discussing the status of the Continuum Hypothesis in set theory:

> Cantor's continuum problem is simply the question: How many points are there on a straight line in Euclidean space? In other terms, the question is: How many different sets of integers do there exist?
> This question, of course, could arise only after the concept of "number" had been extended to infinite sets; hence it might be doubted if this extension can be effected in a uniquely determined manner and if, therefore, the statement of the problem in the simple terms used above is justified. Closer examination, however, shows that Cantor's definition of infinite numbers really has this character of uniqueness, and that in a very striking manner. For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g., their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. [107, p. 515]

Such a view, however, is challenged by the development of alternative theories of size for infinite sets. Mancosu [184] lists several historical attempts to develop an arithmetic of the infinite based on the Part-Whole Principle rather than on the Bijection Principle. A prominent figure among such attempts in the Bohemian polymath Bernard Bolzano, who tried to develop such a "Calculation of the Infinite" in his *Paradoxes of the Infinite* [14, 227]. More recently, modern model-theoretic tools have been used to develop the theory of *numerosities* [18, 19, 16], which assigns sizes to infinite sets (and in particular to sets of natural numbers) in a way that is consistent with **(PW)**. A distinctive feature of numerosities is that the existence of a bijection is a necessary but not sufficient condition for the equinumerosity of two sets. As such, numerosities are a vastly more fine-grained notion of size than cardinality: while all infinite subsets of the natural numbers have the same cardinality, they have infinitely many distinct numerosities. Despite their sophistication, several arguments have been raised against the claim that the theory of numerosities offers a genuine alternative to the Cantorian path (see Section 8.4 below).

The issue around the existence of an alternative way of extending the concept of size into the infinite is significant for several other debates in the philosophy of mathematics. First, the original purpose of Gödel's argument is to show that the continuum hypothesis, which is strictly speaking an issue about cardinality, can also be understood as a basic question about the size of sets of real numbers. For Gödel, the fact that the continuum hypothesis can be presented as such a basic problem is evidence that it must, in fact, have a definite answer, since it asks an elementary question about our concept of set. Should cardinality, however, be one among several ways of extending our pre-theoretic notion of size into the infinite, this fact would therefore also be significant for the debate in the philosophy of set theory regarding whether the continuum hypothesis is a definite problem [90, 118]. Moreover, the status of the Bijection Principle is also relevant in the debate on neologicism as a foundation for mathematics. One of the central tenets of the neologicist program [264] is the claim that Hume's Principle, from which the Dedekind-Peano axioms can be derived in second-order logic, is analytic. Hume's Principle states that the number of objects falling under a concept F is the same as the number of objects falling under another concept G if and only if there is a bijection between the Fs and the Gs. Neologicists hold that this is a conceptual truth, self-evident for anyone who grasps the concept of number. This view is consistent with the idea that **(BP)** is the one principle that one must follow in extending the concept of size (and the closely related concept of number of elements) into the transfinite. However, as remarked in [128, 183], if one thinks that there are alternative ways of extending the concept of number from the finite to the infinite, including some that are consistent with **(PW)** rather than with **(BP)**, then the neologicist's claims become much harder to maintain. In particular, this leads to the "Good Company" objection to the neologicist program: since there are consistent principles based on **(PW)** that can deliver the Dedekind-Peano axioms yet are incompatible with Hume's Principle, on what grounds could the claim that the latter is a conceptual truth about the concept of number be based? Finally, the existence of notions of size for infinite sets that preserve part-whole intuitions is also relevant for a debate in the philosophy of probability regarding regularity. Let me now turn to this issue in more detail.

8.2.2 Regularity and Probability

Kolmogorov's axioms [158] form the backbone of modern probability theory. In Kolmogorov's framework, a probability measure is given by a set of outcomes Ω, together with a σ-algebra of events $\mathbb{B}$ (i.e., a Boolean subalgebra of $\mathscr{P}(\Omega)$ closed under countable unions and intersections) and a function $\mu : \mathbb{B} \to \mathbb{R}$ satisfying the following conditions:

(K1) $\mu(A) \geq 0$ for all $A \in \mathbb{B}$ (Non-negativity);

(K2) $\mu(\Omega) = 1$ (Normality);

(K3) whenever $\{A_i\}_{i\in\omega}$ is a countable sequence of pairwise disjoint sets in $\mathbb{B}$, $\mu(\bigcup_{i\in\omega} A_i = \sum_{i\in\omega} \mu(A_i)$ (Countable Additivity).

It is well known, however, that standard probability theory faces some serious issues when dealing with uniform probability measures on infinite sample spaces. A famous example is *de Finetti's Lottery* [94] (also called "God's Lottery" in [189]), in which a natural number is chosen at random. It is easy to see that no probability function defined on all singletons could model such a situation. Indeed, such a function μ should assign the same probability ϵ to every singleton $\{i\} \in \omega$, and, by Axiom **(K1)**, $\epsilon \geq 0$. One the one hand, if $\epsilon = 0$, then by Axiom **(K3)** $\mu(\omega) = 0$, contradicting Axiom **(K2)**. On the other hand, if $\epsilon > 0$, then by the Archimedeanity of the reals there must be some $n \in \mathbb{N}$ such that $\epsilon > \frac{1}{n}$, and thus $\mu(\omega) > 1$ by **(K3)**, once again contradicting **(K2)**. De Finetti's own solution was to give up countable additivity in favor of finite additivity and to argue that each finite subset of ω has probability 0 of containing the winning ticket. Such a solution, however, is counter-intuitive, as it forces us to admit that some event that has probability 0 of occurring will in fact occur. Our pre-theoretic intuition of quantitative probability arguably respects the constraint that any possible event should receive a positive probability of occurring. Within the set-theoretic framework of Kolmogorov probability theory, this gives rise to the following constraint:

Regularity For any $A \in \mathbb{B}$, $A \neq 0$ implies $\mu(A) > 0$.[1]

A example related to de Finetti's lottery is the case of an infinitely thin dart thrown at the $[0, 1]$ interval of the real line at random. Here, the issue of how to accurately model this situation is tied to Lebesgue's *measure problem* [58]. Can one define a function μ from the powerset of the reals into the reals that is 1) countably additive 2) translation invariant 3) assigns to every interval $[a, b]$ its length $b - a$? Lebesgue famously defined a algebra of events for which this was possible (the Lebesgue-measurable sets), and Vitaly showed that, under the Axiom of Choice, not every set of reals was Lebesgue measurable. The connection between the measure problem and the dart thrown at the real line is almost immediate. Given a subset of $[0, 1]$, we would intuitively want the probability that the dart lands on a point contained in that subset to be the "length" of that set, given by its Lebesgue measure. But here we face two problems. First, the Lebesgue measure is not total under the Axiom of Choice, so we would be forced to admit that some possible events do not get assigned any probability at all. Second, any singleton gets assigned measure 0 under the Lebesgue measure, and for a good reason: by countable additivity together with the Archimedeanity

[1]Some authors such as Hájek [116] have a more stringent definition of regularity, according to which any subset of the sample space must be assigned a positive probability (i.e., regularity fails if a non-empty subset of the domain has probability 0 or is not measurable).

of $\mathbb{R}$, this is the only way for the whole interval $[0, 1]$ to have finite measure. In this case again, the regularity constraint is incompatible with the Kolmogorovian setup.

There is an extensive debate in the literature regarding whether regularity should be a constraint on probabilities understood both as chances [130, 262, 213] and as credences [171, 241, 116, 78]. On the one hand, regularity has some intuitive pull, and moreover the fact that possible events may get assigned probability 0 raises a number of issues for orthodox bayesianism, especially regarding how one may learn that such an event happened or how one may conditionalize on such an event. On the other hand, accepting the regularity constraint would force us to revise standard probability theory, and in particular we would have to generalize the notion of a probability function to functions whose codomain is a non-Archimedean field. However, because the Dedekind-completeness of a field implies that it is also Archimedean, any probability theory developed over a non-Archimedean field will have to abandon countable additivity, since the existence of limits of bounded increasing series is not guaranteed anymore.

Advocates of regularity as a constraint on probabilities have appealed to the existence of fields containing infinitesimals such as the hyperreals. In particular, Lewis [171] suggested that the issue could be solved by considering results in nonstandard analysis [29] establishing the existence of a hyperreal-valued total measure defined on a nonstandard unit interval. More recently, an alternative probability theory has been put forward by the proponents of Non-Archimedean Probability theory (NAP) [20, 21, 259]. Their theory has strong ties with the theory of numerosities. Just as numerosities are an attempt at developing an arithmetic of infinite collections that preserves many aspects of finite arithmetic, NAP presents itself as a probability theory that closely resembles probability theory on finite sample spaces, but also applies to infinite sample spaces. More broadly, the connection between regularity in probability theory and part-whole reasoning in comparing the sizes of infinite sets can be easily captured by the following observation.

Fact 8.2.1. Let $\mathbb{B}$ be a field of sets and $(V, +, 0, <)$ an ordered Abelian group. Then for every finitely additive function $f : \mathbb{B} \to V$, f satisfies **Regularity** if and only if f satisfies **(PW)**.

Proof. Note first that f being finitely additive implies that $f(\emptyset) = 0$. Now if f is regular, then for any $A, B \in \mathbb{B}$ such that $A \subsetneq B$, $f(B \setminus A) > 0$ so $f(B) = f(A) + f(B \setminus A) > f(A)$ by finite additivity, which shows that f satisfies **(PW)**. Conversely, if f satisfies (PW), then for any $A \in \mathbb{B}$ such that $A \neq \emptyset$, we have that $0 = f(\emptyset) < f(A)$, which shows that f is regular. □

NAP can therefore be seen as a generalization of the theory of numerosities from size to probability. In the case of de Finetti's lottery, the connection to Galileo's Paradox is even more immediate. If one has a way of ascribing a size $\nu(A)$ to every set of natural numbers in a way that obeys **(PW)**, then one can simply set the probability that a set A contains the winning ticket on the lottery as $\frac{\nu(A)}{\nu(\mathbb{N})}$, by analogy with fair lotteries on a finite sample space.

Let me now describe in more detail the version of the theory of numerosities that I will be interested in this chapter as well as its relationship to NAP.

8.3 The Euclidean Infinite

In this section, I briefly present the theory of numerosities and its application to probability theory in NAP. There have been several presentations of the main idea over the years [18, 19, 17], with the most recent and more comprehensive treatment being [16]. My goal here is not to present all the aspects of the theory in detail. Rather, I will focus on the way in which the theory deals with arguably the two simplest cases, namely Galileo's Paradox and de Finetti's Lottery. This is because most of the conceptually interesting aspects of the theory happen already at the level of sets of natural numbers, and there are some rather strong pre-theoretic intuitions that we can use to assess the merits of the proposal. In what follows, I will try to motivate the construction both from a technical and intuitive viewpoint. Moreover, although I believe that my presentation is faithful to the spirit of the original proposal, I do not wish to claim that it would be wholeheartedly endorsed by the proponents of numerosities themselves.

8.3.1 Numerosities of Sets of Natural Numbers

The theory of numerosities for sets of natural numbers was developed by Benci and di Nasso in [18]. The starting point is the idea that one would like to construct a number structure $\mathcal{N}$ and a function $\mathfrak{n} : \mathscr{P}(\mathbb{N}) \to \mathcal{N}$ satisfying the following desiderata:

1. $\mathcal{N}$ can be endowed with two operations $\oplus$ and $\otimes$ and an order relation $<$ such that the resulting structure is a discrete, linearly-ordered positive semiring (numerosities resemble the natural numbers);

2. For any $B \in \mathscr{P}(\mathbb{N})$ and $\alpha \in \mathcal{N}$, $\alpha < \mathfrak{n}(B)$ iff $\alpha = \mathfrak{n}(A)$ for some $A \subsetneq B$ (numerosities preserve **(PW)**)

3. For any $A, B \in \mathscr{P}(\mathbb{N})$ with $A \cap B = \emptyset$, $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) \oplus \mathfrak{n}(B)$ and $\mathfrak{n}(A \times B) = \mathfrak{n}(A) \otimes \mathfrak{n}(B)$ ($\mathfrak{n}$ respects disjoint unions and cartesian products).[2]

The central idea of the construction of numerosities is that the numerosity of any set of natural numbers A can be approximated with increasing precision by considering the size of the finite sets $A_n = A \cap \{i \in \omega \mid i < n\}$. Formally, one may associate to any set A of natural numbers its *approximating sequence* $\sigma_A : \omega \to \mathbb{N}$ given by $n \mapsto |A_n|$. Whenever A is a finite set, σ_A is eventually constant, and outputs the cardinality of A cofinitely often. When A

[2] Benci and di Nasso's original requirement is stronger than this, since they don't include the condition that A and B be disjoint. However, if A and B are not disjoint then their disjoint union cannot be identified with a subset of $\mathbb{N}$. This is not a problem in their setting since they define the numerosity function of what they call *labelled sets*, but I am simplifying their account here.

is infinite however, σ_A is an unbounded non-decreasing function. Clearly, the function that "grows" the slowest is $\sigma_\emptyset$, which is constantly 0, and the function that grows the fastest is $\sigma_\mathbb{N}$, which is also the identity function $i \mapsto i$. Moreover, we can make the following observation:

Fact 8.3.1. Let $(I, \leq)$ be the set of all non-decreasing functions $\alpha : \omega \to \mathbb{N}$ satisfying $\alpha(i+1) \leq \alpha(i) + 1$ for all $i \in \omega$ and ordered pointwise. Then $\sigma : (\mathscr{P}(\mathbb{N}), \subseteq) \to (I, <)$ is an order-isomorphism.

Proof. Let us show first that $\sigma_A \neq \sigma_B$ whenever $A \neq B$. Suppose $A \neq B$, and let $i \in \omega$ be the smallest element in their symmetric difference $A \Delta B$, i.e., in the set $(A \setminus B) \cup (B \setminus A)$. Then it follows that $|A_i| \neq |B_i|$, hence $\sigma_A(i) \neq \sigma_B(i)$. This shows that $\sigma : A \mapsto \sigma_A$ is injective. For surjectivity, fix a non-decreasing sequence $\alpha : \omega \to \mathbb{N}$ such that $\alpha(i+1) \leq \alpha + 1$ for all $i \in \omega$, and let A_α be such that $i \in A_\alpha$ iff $\alpha(i) < \alpha(i+1)$. Then one verifies quickly that $\sigma_{A_\alpha} = \alpha$. Finally, to verify that σ preserves and reflects the order, note first that for any $A, B \subseteq \mathbb{N}$, we have that $A \subsetneq B$, iff $|A_i| \leq |B_i|$ for all $i \in \omega$ iff $\sigma_A(i) \leq \sigma_B(i)$ for all $i \in \omega$ iff $\sigma_A \leq \sigma_B$. □

Non-decreasing countable sequences of natural numbers ordered pointwise therefore contain a "copy" of the order-structure of $\mathscr{P}(\mathbb{N})$. But this structure is too fine-grained to be the required structure of numerosities. Intuitively, equinumerous finite sets should have the same numerosity, while $\sigma_A \neq \sigma_B$ whenever $A \neq B$, even when both A and B are finite sets of the same cardinality. A related issue is the fact that $(I, <)$ is only a partial order, while we would want numerosities to be linearly ordered if our arithmetic of infinite sets is to bear any resemblance to finite arithmetic. Benci and di Nasso's solution is inspired from the construction of hyperreal fields in nonstandard analysis and more generally of ultrapowers in model theory. Specifically, they work with an ultrapower of the natural numbers modulo a Ramsey ultrafilter, which is a specific kind of ultrafilter on ω.

Definition 8.3.2. For any set X and any cardinal $\kappa \leq |X|$, we write $[X]^\kappa$ for the set of all subsets of X of size κ. A *Ramsey ultrafilter* on ω is a non-principal ultrafilter U satisfying any of the following equivalent conditions:

1. For any coloring $c : [\omega]^2 \to 0, 1$, there is $S \in \mathsf{U}$ such that $c(x) = c(y)$ for any $x, y \in [S]^2$.
2. For any $A \subseteq [\omega]^2$, there is $S \in \mathsf{U}$ such that $[S]^2 \subseteq A$ or $[S]^2 \cap A = \emptyset$.
3. For any $f : \omega \to \mathbb{N}$, there is $S \in \mathsf{U}$ such that $f|S$ is non-decreasing.

The following can be given as an intuitive motivation for the choice of an ultrapower of the natural numbers modulo a Ramsey ultrafilter. As mentioned above, the cardinality of a finite set is the value that its approximating sequence outputs cofinitely often. Consequently, any two equinumerous finite sets will have the same approximating sequence up to finitely many indices. This is a good indication that we could represent numerosities as equivalence classes of non-decreasing countable sequences of natural numbers. However, whenever A is an infinite set, then no finite point in the approximating sequence of A gives us a complete viewpoint on the numerosity of A as if the process of counting all the elements in A had been terminated. One can think of a non-principal ultrafilter on ω as providing such a viewpoint as it abstractly describes, so to speak, a "natural number at infinity" by listing all the properties that such a number would have (i.e., by determining which subsets of ω it would belong to). A non-principal ultrafilter on ω therefore offers us a vantage point on the completed structure of numerosities, allowing us to determine order relationships between them as well as when two sequences of natural numbers approximate the same numerosity. Finally, the Ramsey property of the ultrafilter guarantees that our "vantage point at infinity" does not make too many distinctions: every function is equivalent to a non-decreasing one or, equivalently, every numerosity below $\mathfrak{n}(\mathbb{N})$ is the numerosity of some set of natural numbers.

Definition 8.3.3. Let U be a Ramsey ultrafilter on ω. The *ring of numerosities* $(\mathcal{N}, \oplus, \otimes, <)$ is defined as the ultrapower of $(\mathbb{N}, +, \times, <)$ modulo U. More precisely:

- Elements of $\mathcal{N}$ are equivalence classes α_{U} of functions $\alpha : \omega \to \mathbb{N}$, where $\beta \in \alpha_{\mathsf{U}}$ iff $||\alpha = \beta|| = \{i \in \omega \mid \alpha(i) = \beta(i)\} \in \mathsf{U}$;
- Operations and relations are defined pointwise, i.e., $\alpha_{\mathsf{U}} \oplus \beta_{\mathsf{U}} = (\alpha + \beta)_{\mathsf{U}}$, $\alpha_{\mathsf{U}} \otimes \beta_{\mathsf{U}} = (\alpha \times \beta)_{\mathsf{U}}$, and $\alpha_{\mathsf{U}} < \beta_{\mathsf{U}}$ iff $||\alpha < \beta|| = \{i \in \omega \mid \alpha(i) < \beta(i)\} \in \mathsf{U}$ for any $\alpha, \beta : \omega \to \mathbb{N}$.

Moreover, the *numerosity function* is the map $\mathfrak{n} : \mathscr{P}(\mathbb{N}) \to \mathcal{N}$ given by $\mathfrak{n}(A) = (\sigma_A)_{\mathsf{U}}$ for any $A \subseteq \omega$.

Theorem 8.3.4 (Benci and di Nasso). *The structure $(\mathcal{N}, \oplus, \otimes, <)$ has the following properties:*

1. *There is an elementary embedding $\iota : \mathbb{N} \to \mathcal{N}$. In particular, $\mathcal{N}$ has the same first-order theory as $\mathbb{N}$ and is a discrete, linearly-ordered positive semiring;*
2. *For any $A, B \subseteq \mathbb{N}$ with $A \subsetneq B$, $\mathfrak{n}(A) < \mathfrak{n}(B)$;*
3. *For any finite $A \subseteq \mathbb{N}$, $\mathfrak{n}(A) = \iota(|A|)$;*
4. *Whenever $A \cap B = \emptyset$, $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) \oplus \mathfrak{n}(B)$ and $\mathfrak{n}(A) \otimes \mathfrak{n}(B) = \mathfrak{n}(A \times B)$;*
5. *For any $\alpha : \omega \to \mathbb{N}$, there is a non-decreasing function $\beta : \omega \to \mathbb{N}$ such that $\alpha \in \beta^{\mathsf{U}}$.*

The ultrapower construction therefore delivers all the desiderata that Benci and di Nasso wanted for numerosities. It is worth however pointing out which properties are used in establishing items 1-5 in Theorem 8.3.4. In order to establish properties 2-4, it would be enough to consider equivalence classes of functions from ω into $\mathbb{N}$ modulo the equivalence relation $\alpha \sim \beta$ iff $||\alpha = \beta||_{\omega} = \{i \in \omega \mid \alpha(i) = \beta(i)\}$ is cofinite. This would be equivalent to working with the reduced power of $\mathbb{N}$ modulo the Fréchet filter on ω. However, such a structure would not be elementarily equivalent to the natural numbers, and in particular it would not be linearly ordered. Requiring that U be a non-principal ultrafilter instead of the Fréchet filter is what guarantees that 1 holds as well. Finally, property 5 immediately follows from taking U to be a Ramsey ultrafilter.

8.3.2 Non-Archimedean Probability Theory

Having provided some technical and intuitive background on numerosities, let me now discuss their application to probability theory and de Finetti's Lottery. It is well-known that the standard resources of Kolmogorov probability theory allow for a straightforward representation of any *finite* fair lottery. Given a finite set Ω and a set $A \subseteq \Omega$, one may define the probability that a randomly selected element x of Ω belongs to A as $\frac{|A|}{|\Omega|}$. The key idea of NAP is that this approach can be extended to the case of an infinite sample space by conditionalizing on finite sets. Given a finite set $A \subseteq \Omega$ and an event $U \subseteq \Omega$, we can define the probability of U conditional on A as the real number $P(U|A) = \frac{|A \cap U|}{|A|}$. This gives us a partial approximation of the probability of U, indexed by the finite set A. In other words, we obtain a function $\chi_U : \mathscr{P}_{Fin}(\Omega) \to \mathbb{R}$, where $\mathscr{P}_{Fin}(\Omega)$ is the set of all finite subsets of Ω, defined by $\chi_U(A) = P(U|A)$ for any finite $A \subseteq \Omega$.

The second step of the approach developed by NAP theory is to define for each function χ_U that approximates the unconditional probability of U a limit to which it converges. Here, a natural choice is to use the function χ_U itself to define the limit, just like one takes the limit of a Cauchy sequence of rationals to be represented by that sequence itself in Cantor's construction of the real numbers. Concretely, this means that one should consider the ring $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ of all functions from $\mathscr{P}_{Fin}(\Omega)$ into $\mathbb{R}$. When one defines the ring operations $+$ and $\cdot$ and the order $\leq$ on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ pointwise, $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ becomes a partially ordered non-Archimedean ring in which the reals embed via the map sending every real r to the constant function $\overline{r}$ defined by $\overline{r}(A) = r$ for all $A \in \mathscr{P}_{Fin}(\Omega)$. But $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ is not an adequate structure, as it is in some sense too fine-grained. Indeed, it is easy to see that for any two distinct sets U, V, $\chi_U \neq \chi_V$: simply pick some $x \in U \Delta V$ (the symmetric difference of U and V), and note that it must be the case that $\chi_U(\{x\}) \neq \chi_V(\{x\})$. This is a most unwelcome consequence if we want to have a uniform distribution on an infinite sample space. Indeed, in the case of a fair lottery on an infinite set Ω, we would want that $\mu(\{x\}) = \mu(\{y\})$ for any $x, y \in \Omega$. But this fails if we take $\mu(U)$ to be χ_U and the codomain of our function to be $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$. In order to solve this issue, we must find a way to identify some of our functions with one another. A standard way to do this is to quotient $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ by an ideal. Here, a useful idea is to identify two functions together whenever they coincide up to some finite error. Concretely, this means that we should identify two functions $\alpha, \beta \in \mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ whenever $\alpha - \beta(A) = 0$ for "almost all" $A \in \mathscr{P}_{Fin}(\Omega)$. The formal definition we need is the following.

Definition 8.3.5. A function $\alpha \in \mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ *vanishes almost everywhere* if there is $A \in \mathscr{P}_{Fin}(\Omega)$ such that $\forall B \supseteq A$, $\alpha(B) = 0$. The *fine ideal* I_0 is the ideal of all almost-everywhere vanishing functions.

Quotienting the ring $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ by the fine ideal I_0 solves the problem mentioned above. Indeed, if $A, B \subseteq \Omega$ are two finite sets such that $|A| = |B|$, then it is easy to see that, for any $C \supseteq A \cup B$, $\chi_A(C) = \chi_B(C)$, and hence that $\chi_A - \chi_B \in I_0$. But $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ fails to be an adequate range of values for another reason, which is not addressed by merely quotienting it by I_0. Indeed, it is easy to see that $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ fails to be an ordered field, because it has zero divisors and incomparable elements. Indeed, consider the functions ϵ and o, defined by $\epsilon(A) = 1$ if $|A|$ is odd, $\epsilon(A) = 0$ if $|A|$ is even, and $o(A) = 1 - \epsilon(A)$. Clearly, for any finite set A, there is $B, C \supseteq A$ such that $\epsilon(B) \neq 0$ and $o(C) \neq 0$, so neither B nor C are identified with $\overline{0}$ in the quotient ring $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})/I_0$. However, it is easy to see that the pointwise product $\epsilon \cdot o$ is the constantly 0 function $\overline{0}$, which shows that $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})/I_0$ is not a field as it contains zero divisors. One can similarly show that for any $\alpha \in I_0$, we have that $\epsilon \nleq o + \alpha$ and $o \nleq \epsilon + \alpha$, which shows that the order on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})/I_0$ induced by the pointwise order on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ is not linear.

This result is not surprising. Indeed, by an elementary result in abstract algebra, a quotient ring R/I is a field if and only if I is a maximal ideal of R. In order to turn $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ into a field, the NAP theorist therefore extends I_0 to a *maximal fine ideal* J. Let $\mathcal{F}/J$ be the field obtained by quotienting $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ by such a maximal fine ideal J, and $P_J : \mathscr{P}(\Omega) \to \mathcal{F}/J$ be given by $P_J(U) = [\chi_U]_J$, which we will refer to as the *fair NAP function* on Ω induced by J. It turns out that $\mathcal{F}/J$ is a totally ordered field, and that P_J is a well-behaved probability function. Benci, Horsten and Wenmackers prove the following theorem.

Theorem 8.3.6. *The function P_J has the following properties:*

1. $P_J[\mathscr{P}(\Omega)] \subseteq [0, 1]$, $P_J(\Omega) = 1$.
2. P_J *is finitely additive, i.e.,* $P_J(U \cup V) = P_J(U) + P_J(V)$ *whenever* $U \cap V = \emptyset$.
3. $P_J(\{x\}) = P_J(\{y\})$ *for any* $x, y \in \Omega$.
4. $P_J(U) = 0$ *implies* $U = \emptyset$ *for any* $U \subseteq \Omega$.

Let us conclude this section by making explicit the link between NAP functions and numerosity functions. There is an immediate analogy between the two constructions. In both cases, an ordered (semi)ring structure is obtained by first taking the set of all functions from a countable set into a (semi)ring, before quotienting by a maximal object such as a Ramsey ultrafilter or a maximal ideal. Moreover, from a purely mathematical perspective, there is a very tight connection between the construction of numerosities as ultrapowers of the natural numbers and the definition of the range of NAP functions as quotients of functions modulo a maximal ideal. Indeed, whenever I is a maximal ideal of $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$, the quotient field $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})/I$ is isomorphic to an ultrapower of $\mathbb{R}$ modulo an ultrafilter U_I on $\mathscr{P}(\mathscr{P}_{Fin}(\Omega))$. In that sense, NAP functions can be seen as a generalization of numerosities. Note however that the semiring of numerosities is defined as an ultrapower of $\mathbb{N}$ modulo a Ramsey ultrafilter on ω, while the range of a NAP function is an ultrapower of $\mathbb{R}$ modulo a fine ultrafilter on a set of finite subsets ordered by inclusion. When discussing the connection between numerosity functions for sets of natural numbers and NAP functions (e.g., in [21, Section 5.1-2]), Benci, Horsten and Wenmackers often mention that one could also define a more general notion of NAP functions for a sample space Ω in which one replaces $\mathscr{P}_{Fin}(\Omega)$

with a subset Λ satisfying some properties.[3] In the case of a lottery on $\mathbb{N}$, the codomain of the NAP function becomes isomorphic to an ultrapower of $\mathbb{N}$ modulo a non-principal ultrafilter on ω whenever one takes Λ to be the set of all initial segments of $\mathbb{N}$. We will return to the significance of the correspondence between ranges of NAP functions and ultrapowers in Section 8.5 and to the distinction between working with $\mathscr{P}_{Fin}(\Omega)$ and working with a subset Λ in Section 8.6.

8.4 Objections to the Euclidean Infinite

In this section, I introduce some of the criticisms that have been raised against numerosities and NAP theories as solutions to Galileo's Paradox and the problem of fair lotteries on infinite sets respectively. Since the introduction of numerosities in [18], and particularly since their relevance for the philosophy of the infinite was pointed out in [184] and their possible application to de Finetti's Lottery was investigated in [259, 21], there has been a rich literature on whether numerosities constitute a genuine alternative to the Cantorian transfinite, and whether NAP can be a credible alternative to Kolmogorovian probability theory. Arguments against numerosities and NAP can be roughly divided in two categories: those targeting numerosities and NAP specifically, and those targeting any theory of size preserving the part-whole principle, or any probability theory satisfying the regularity constraint. Although most opponents of the Euclidean infinite often appeal to the two kinds of arguments, I will be most interested in discussing arguments of the first kind, i.e., arguments to the effect that numerosities/NAP are flawed implementations of a nonetheless possible alternative to the standard view on the infinite. The reason for this is twofold: first, I will argue in the next section that possibility structures allow for the development of a Euclidean approach to Galileo's Paradox and infinite fair lotteries that fares better than numerosities and NAP functions with respect to objections of this kind. Second, as we will see below, many arguments of the second kind arguably presuppose a certain affinity towards the Cantorian infinite, which makes them less relevant to my project in this chapter. In other words, I will not focus much on arguments that make claims that are incompatible with either (**PW**) or **Regularity**. Rather, I will discuss arguments that point out flaws of numerosities and NAP functions *without* being hostile to the Euclidean infinite itself. With respect to numerosities, I will particularly focus on some arguments in Parker's paper on the part-whole principle [205], and, with respect to NAP functions, on some arguments made by Easwaran regarding regularity and hyperreal-valued probability functions [78]. I will also briefly discuss some other work by Parker [206], Pruss [213, 214] and Williamson [262]. Finally, the arguments that I will focus on roughly fall within two categories: claims that numerosities and NAP functions fail to satisfy certain invariance conditions and claims that numerosities and NAP functions are too fine-grained and arbitrary. I will discuss each type of argument in turn.

[3] More specifically, Λ should be a directed subset of $\mathscr{P}_{Fin}(\Omega)$ (meaning that for any $A, B \in \Lambda$ there is $C \in \Lambda$ such that $A \cup B \subseteq C$), and Λ should also be such that $\bigcup \Lambda = \Omega$.

8.4.1 The Invariance Problem

Let us start with claims to the effect that numerosities and NAP functions fail to deliver a satisfactory theory of size and probability respectively, because they do not satisfy certain invariance conditions. In the case of numerosities, this is one of the major points that Parker argues for in [205]. According to Parker, any theory of size that preserves the part-whole principle (which Parker calls a *Euclidean theory*) must be "either too weak or arbitary". By "weak", Parker means a theory that would either have very limited applicability or that would not be informative enough. By "arbitrary", he means a theory that makes fine-grained predictions that do not seem conceptually or pragmatically motivated:

> "Arbitrary" can mean many things, but the main thing I mean by it here is that Euclidean size assignments are unmotivated in many of their specific details. Particular sizes could be chosen differently without any significant loss of utility or elegance. [205, p. 9]

Parker's main argument to establish the arbitrariness of Euclidean theories of size amount to showing that size assignments that preserve the part-whole principle fail to be invariant under certain functions. Parker's examples are in the setting of metric spaces, and the functions he considers are rigid transformations such as translations, i.e., bijections that preserve the metric structure. I think however that the choice of metric spaces makes Parker's point more obscure (I will return to this below). Accordingly, the examples I will discuss in this section are simply sets of natural numbers. In this context, a translation becomes simply a permutation on $\mathbb{N}$, i.e., a bijection $\pi : \mathbb{N} \to \mathbb{N}$. Any permutation π induces a function $\pi_* : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$, given by $\pi_*(A) = \{\pi(a) \mid a \in A\}$ for every $A \subseteq \mathbb{N}$. Parker considers two kinds of invariance criteria for a size assignment ν, which in the setting of size assignments to sets of natural numbers can be presented as follows:

Absolute Invariance Criterion For any permutation π and any $A \subseteq \mathbb{N}$, $\nu(A) = \nu(\pi_*(A))$.

Relative Invariance Criterion For any permutation π and any $A, B \subseteq \mathbb{N}$, $\nu(A) \leq \nu(B)$ implies that $\nu(\pi_*(A)) \leq \nu(\pi_*(B))$.

It is straightforward to see that the Absolute Invariance Criterion implies the Relative Invariance Criterion, but that the converse need not be the case. In fact, I will argue that one should agree with Parker that the Relative Invariance Criterion should be satisfied by a good Euclidean theory of size, while one should disagree with Parker regarding the Absolute Invariance Criterion. Before I make this point, however, let us see how, according to Parker, any Euclidean theory of size must fail to satisfy either criterion. For the first criterion, suppose that $\nu : \mathscr{P}(\mathbb{N}) \to \mathcal{N}$ is a function into a partially ordered set $\mathcal{N}$ satisfying (**PW**). Recall that, given a permutation $\pi : \mathbb{N} \to \mathbb{N}$, the *orbit* $O(i)$ of some $i \in \mathbb{N}$ is the set of all elements in $\mathbb{N}$ that can be reached by successively applying π, starting from i. Let π be a permutation and $i \in \mathbb{N}$ such that $O(i)$ is infinite (for example, let π be defined by $\pi(0) = 1$, $\pi(2(n+1)) = 2n$ and $\pi(2n+1) = 2(n+1)+1$, and let $i = 0$). Then it is easy to see that $O(\pi(i)) \subsetneq O(i)$, hence, by (**PW**), $\nu(O(\pi(i))) < \nu(O(i))$. At the same time, $\pi_*(O(i)) = O(\pi(i))$, so this shows that ν does not satisfy the Absolute Invariance Criterion. In the specific case of numerosities, one doesn't even need to appeal to a permutation with an infinite orbit. Indeed, let $\mathfrak{n}$ be a numerosity function determined by some Ramsey ultrafilter U, and let $2\mathbb{N}$ and $2\mathbb{N}+1$ be the set of even and odd numbers respectively. An easy argument shows that $\mathfrak{n}(2\mathbb{N}) = \mathfrak{n}(2\mathbb{N}+1)$ if the set $\{2i \mid i \in \omega\} \in \mathsf{U}$, and $\mathfrak{n}(2\mathbb{N}) > \mathfrak{n}(2\mathbb{N}+1)$ if $\{2i+1 \mid i \in \omega\} \in \mathsf{U}$. In the latter case, the function π swapping the parity of every natural number (i.e., $\pi(2n) = 2n+1$ and $\pi(2n+1) = 2n$) is such that $\nu(\pi_*(2\mathbb{N})) = \mathfrak{n}(2\mathbb{N}+1) < \mathfrak{n}(2\mathbb{N})$, while in the former case, letting π' be the function swapping the parity of every positive natural number while letting 0 fixed (i.e., $\pi'(2(n+1)) = 2n$ and $\pi'(2n+1) = 2(n+1)$), we have that $\pi'_*(2\mathbb{N}+1) \subsetneq (2\mathbb{N})$, and hence that $\mathfrak{n}(\pi'_*(2\mathbb{N}+1)) < \mathfrak{n}(2\mathbb{N}) = \mathfrak{n}(2\mathbb{N}+1)$. Either way, numerosities must therefore violate the Absolute Invariance Criterion. Notice that the same argument shows that they also violate the Relative Invariance Criterion. Indeed, if $\mathfrak{n}(2\mathbb{N}) = \mathfrak{n}(2\mathbb{N}+1)$, then $\mathfrak{n}(2\mathbb{N}) \leq \mathfrak{n}(2\mathbb{N}+1)$ yet $\mathfrak{n}(\pi'_*(2\mathbb{N}+1) < \mathfrak{n}(2\mathbb{N}) = \mathfrak{n}(2\mathbb{N}+1) < \mathfrak{n}(\pi'_*(2\mathbb{N}))$, and if $\mathfrak{n}(2\mathbb{N}+1) < \mathfrak{n}(2\mathbb{N})$, then $\mathfrak{n}(\pi_*(2\mathbb{N}) = \mathfrak{n}(2\mathbb{N}+1) < \mathfrak{n}(2\mathbb{N}) = \mathfrak{n}(\pi_*(2\mathbb{N}+1)$. Parker shows that the failure of the Relative Invariance Criterion is more general and applies to any Euclidean theory in which sizes satisfy certain conditions.

It seems to me, however, that a proponent of the Euclidean infinite could resist Parker's claim that a good theory of size should satisfy the Absolute Invariance Criterion. Arguing for his view, Parker writes that the failure of the Absolute Invariance Criterion implies that "[t]wo sets [...] can be entirely alike in structure, and yet, due only to which particular elements are contained within this structure and where they happen to be, unequal in size"[205, p. 14]. The idea here is that the rigid transformations that Parker considers preserve all the "structure" of mathematical objects and thus should also preserve their size. I think however that Parker is too quick here and that ultimately his point rests on an ambiguity about the notion of "structure". To make that point clear, let me start with a somewhat crude picture of mathematical objects. Most mathematical objects are typically represented as structures in the sense of model theory, i.e., tuples of sets including a domain of individuals and various relations or operations on that domain. One may understand the domain of individuals as the "matter" which the mathematical object is made of, while the tuple of relations and operations correspond to its "form", i.e., the structure that is imposed on the domain. For lack of a better term, call this view about mathematical objects the *hylomorphic* view. For the hylomorphist, the structure of a mathematical object is entirely given by the relations and operations defined on it, and not by its domain, which is a set, i.e., the prototype of an amorphous, *unstructured* mathematical entity. Accordingly, a "structure-preserving" map will be a map from the domain of a mathematical object into the domain of another of the same type that commutes with all the relations and operations defined on both of them. Under that view, then, the "size" of a mathematical object need not be preserved by a structure-preserving map, just like two busts of Caesar may be identical in shape but not in size. One may argue that, as dense linear orders without endpoints, $\mathbb{Q}$ and $\mathbb{R}$ have the same structure even though they differ in (Cantorian) cardinality, or that all finite groups of prime order have the same structure, because they are all cyclic groups, even though they have distinct finite sizes. According to such a view, the notion of a homomorphism is strong enough to indicate preservation of structure. On the other hand, one may instead consider that the "size" of a mathematical object is in fact part of its structure and that "structure-preserving" maps should also preserve size. But someone advocating for such a view will then not be satisfied with a definition of a structure-preserving map as a map that commutes with all relations and operations. Instead, they will want to impose more conditions on such a map, depending on what they consider a good criterion of size-preservation. Typically, they will require the map to be a bijection on top of a homomorphism.

My point here is not to discuss whether a "structure-preserving" map between mathematical objects should preserve its size, as this strikes me as a mostly verbal dispute. Rather, my goal is to point out that there can be some genuine ambiguity regarding whether a map is "structure-preserving" or not and that, regardless of how one defines "structure-preserving", the fact that a map commutes with the relations and operations defined on the domain of a mathematical object gives us no more indication that the "size" of a mathematical object is being preserved than the fact that a golf ball and a tennis ball have the same spherical shape is an indication that they have the same size. It seems to me, however, that Parker relies precisely on this ambiguity between two notions of structure-preservation in his argument that Euclidean theories of size are flawed because they violate the Absolute Invariance Criterion. Indeed, Parker claims at the same time that "structure-preserving maps" should preserve size and that maps such as rigid transformations are "structure-preserving". Under the first sense of "structure" mentioned above, Parker is correct that rigid transformations preserve structure, since they commute with relations and operations. But only under the second sense of "structure" are "structure-preserving" maps supposed to also preserve size. Parker would need to provide an independent argument as per why the rigid transformations he considers are "structure-preserving" in that stronger sense. Here however, there does not seem to be a way of arguing that bijections preserve size without appealing to (**BP**) and to the Cantorian notion of size. Parker anticipates the objection of circularity and answers it as such:

> Even if we are prepared to drop [(**BP**)], it would be useful if the size of a set indicated some antecedent fact about its structure. So preferably, transformations that preserve structure or a great deal of structure ought to preserve size. In general, one-to-one correspondence preserves very little structure, while a translation preserves as much as possible without necessarily preserving the internal structures of the individual elements. So it is one thing to say that Euclidean sizes violate [(**BP**)], and quite another to say that they violate principles like [AIC]. Violating [AIC] suggests that set size is not determined by structure at all. [p. 16]

Again, I think that Parker's point here rests on an ambiguity between the two notions of structure discussed above. Under the hylomorphic view, there is nothing wrong with the suggestion that "set size is not determined by structure at all". In fact, one might even argue that set size should *not* be determined by structure at all. Since sets are *unstructured* entities and yet have a size, how could their (lack of) structure determine their size? On the other hand, if one thinks that the size of a mathematical object is part of its structure, then Parker has not given us any reason to believe that rigid transformations or translations preserve size along with *other* structural features of the mathematical objects he considers.

Ultimately, it therefore seems to me that Parker's claim that a good theory of size should satisfy the Absolute Invariance Criterion presupposes that one adheres to (**BP**) in the first place and can thus be dismissed by the proponents of the Euclidean infinite. Note that a similar discussion arises in the context of NAP. Williamson [262], Parker [206] and Pruss [214] all argue that probability functions should be invariant under certain symmetries of the sample space. Williamson considers the example of a fair coin being tossed infinitely many times and argues that the probability that the coin always lands on heads should be equal to the probability that it always lands on heads from the second flip onward, because the two events are "isomorphic". According to Williamson, the probability of the first event should also be half of the probability of the second one since the coin flips are independent, and therefore both events must receive probability 0, thus violating **Regularity**. Pruss and Parker, on the other hand, independently discuss the case of a dart thrown at the unit circle and show that a rotation-invariant probability function cannot be regular. In all such cases, the core of the argument seems to rely on the intuition that two events modelled by sets that can be mapped onto one another by a "structure-preserving" function should have the same probability, because the corresponding sets must have the same size. Once again, it seems to me that the intuition driving these arguments ultimately rests on an ambiguity regarding the notion of "structure" implicitly assumed.

The situation, however, is much different with respect to the Relative Invariance Criterion. Let me start by remarking that, unlike the Absolute Invariance Criterion, the Relative Invariance Criterion does immediately seem to contradict (**PW**). Indeed, if $\pi : X \to X$ is a permutation of the elements of a set X and $A \subsetneq B \subseteq X$, then clearly also $\pi_*(A) \subsetneq \pi_*(B)$. Strict part-whole relations are preserved by permutations. As a consequence, whenever a strict size comparison between two sets holds in virtue of the part-whole principle, the same relation also holds between the images of those two sets under π. Thus there is no prima facie incompatibility between (**PW**) and the Relative Invariance Criterion. I would also argue that there is an independent reason to hold on to the Relative Invariance Criterion.

According to either of the Cantorian and Euclidean picture of the infinite, relative size assignments track part-whole relations between sets as captured by the subset relation. To put the point differently, both Cantorian and Euclidean size assignments determine a (possibly partial) order on the powerset of a set that extends the inclusion order. As such, they can be thought of as more coarse-grained relations on powersets than the inclusion order. But if size relations between subsets of a set are more coarse-grained than the inclusion order, then it follows that they should be preserved by permutations, since, for any permutation $\pi : X \to X$, $\pi_* : \mathscr{P}(X) \to \mathscr{P}(X)$ preserves the inclusion order. Therefore, a size assignment that does not satisfy the Relative Invariance Criterion must be tracking relational properties between sets that go beyond the inclusion order and, in the case of pure sets, it is hard to see what such properties could be (although I will come back in more detail to this point in Section 8.6). For now, let me conclude by emphasizing the distinction between this justification of the Relative Invariance Criterion and Parker's argument for the Absolute Invariance Criterion. Parker argues that rigid transformations on a set X induce maps on $\mathscr{P}(X)$ that map every subset of X to a structurally-identical set and that therefore size is a property that should be preserved under such maps. By contrast, my claim is that every permutation π on a set X induces an order isomorphism π_* on $\mathscr{P}(X)$ endowed with the inclusion order and that, since the ordering induced on $\mathscr{P}(X)$ by a size assignment is coarser than the inclusion ordering, it should also be preserved by π_*. In other words, Parker claims that rigid transformations (or permutations) preserve the local structure of individual sets, while my point is that permutations preserve the global structure of the set of subsets. This is why I think that Parker's claim is ultimately dubious, because it relies on an ambiguity regarding whether size is a "structural" property of sets, while my argument for the Relative Invariance Criterion does not have this issue. Since the "structure" under consideration here is that of the ordered set $(\mathscr{P}(X), \subseteq)$, π_* is always a structure-preserving map, regardless of whether one thinks of the size of a set as part of its individual structure.

8.4.2 Arbitrariness and Non-Uniqueness

The second type of arguments against numerosity and NAP functions intends to establish that they deliver a defective notion of size and probability because many of their specific features are not well-motivated. Here, the main thrust of these arguments does not lie in the fact that a given numerosity or NAP function is not invariant under certain structure-preserving maps on its underlying domain, but rather in the fact that such functions are not uniquely specified. This is a direct consequence of the fact that semirings of numerosities and codomains of NAP functions are always defined relative to a maximal object such as a Ramsey ultrafilter or a maximal fine ideal. Consequently, specifying a particular numerosity function (resp. a NAP function) requires choosing a particular Ramsey ultrafilter (resp. a maximal fine ideal). But since the existence of such objects is only guaranteed under the Axiom of Choice, there is no principled way of choosing one such object as a *canonical* one, and one must therefore allow for the existence of infinitely many functions.

Moreover, not only will such functions assign different values to the same set, but also they often contradict one another in determining relative size relationships. Let us illustrate this with a simple example in the case of numerosities. For $i \in \{0, 1, 2, 3\}$, let $U_i = \{4n + i \mid n \in \mathbb{N}\}$, and let $V_0 = U_0 \cup U_3$ and $V_1 = U_1 \cup U_2$. A simple computation shows that for any numerosity function $\mathfrak{n}$ determined by a Ramsey ultrafilter U, $\mathfrak{n}(V_0) < \mathfrak{n}(V_1)$ if $U_1 \in \mathsf{U}$, $\mathfrak{n}(V_0) = \mathfrak{n}(V_1)$ if $U_0 \cup U_2 \in \mathsf{U}$, and $\mathfrak{n}(V_0) > \mathfrak{n}(V_1)$ if $U_3 \in \mathsf{U}$. Since $U_0, ..., U_4$ partition ω, exactly one of them must be in U. Since the existence of one Ramsey ultrafilter guarantees the existence of Ramsey ultrafilters containing each of $U_0, ..., U_4$ (see Lemma 8.5.6 below), this means that having just one numerosity function implies having also distinct numerosities assigning all possible size relationships to V_0 and V_1. Of course, a proponent of numerosities could answer that some Ramsey ultrafilters should be preferred to others. In the example above, one could argue that, in fact, all of U_0,...,U_3 should have the same numerosity, which can be achieved by making sure that $U_0 \in \mathsf{U}$. In fact, numerosity theorists often point out that one could require that every set of the form $n\mathbb{N} = \{nm \mid m \in \mathbb{N}\}$ belong to the Ramsey ultrafilter U, with the consequence that any partition of $\mathbb{N}$ determined by Euclidean division modulo any natural number m will yield cells of equal numerosity. Nonetheless, there are reasons to think that this approach doesn't entirely solve the issue at stake here. For example, given a natural number n, let $b(n)$ be the number of digits of the binary expansion of n. As before, define sets $X_0, ..., X_3$ by $X_i = \{n \mid b(n) = i \mod 4\}$, and let $Y_0 = X_0 \cup X_3$ and $Y_1 = X_1 \cup X_2$. Again, one might want to impose that $\mathfrak{n}(Y_0) = \mathfrak{n}(Y_1)$. But it is easy to see that the set $\{i \in \omega \mid \sigma_{Y_0}(i) = \sigma_{Y_1}(i)\}$ is infinite but does not contain any set of the form $n\mathbb{N}$, meaning that one would need to impose further conditions on the Ramsey ultrafilters under consideration to ensure that $\mathfrak{n}(Y_0) = \mathfrak{n}(Y_1)$.

Arguably, this sense of arbitrariness is made worse by the fact that semirings of numerosities and ranges of NAP functions are linearly ordered. Indeed, this means that the numerosities of any two sets, or the probability of any two events in an infinite fair lottery, must always be comparable. But there are many instances for which it seems that such order relationships, whatever they maybe, would appear to be poorly motivated. For example, Kremer [159] shows that there exists a set of natural numbers S_∞ such that for any rational number $q \in (0, 1)$, there is a NAP function μ according to which the probability that the winning ticket in a fair lottery on $\mathbb{N}$ belongs to S_∞ is exactly q. Kremer argues that this shows that the probability of the event corresponding to this set is simply *indeterminate* according to NAP theory and that this might be a desirable feature:

> As for the set S_∞ constructed below, we might welcome the fact that its probability is indeterminate, since we have no fixed intuitions about it. Maybe this indeterminacy is a feature, not a bug.[159, p. 1759]

The underlying assumption in Kremer's reasoning here is that one should take NAP theory to be modelling a fair lottery on an infinite set Ω with a set of probability functions, all determined by a distinct maximal fine ideal on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$, rather than with a single probability function. Whenever all NAP functions agree on a certain value for the probability of an event U or an order relationship between the probabilities of two events U and V, one should take this to be a genuine prediction of the theory. Whenever NAP functions disagree, however, one should simply conclude that the theory doesn't give a determinate answer. Horsten and Wenmackers make a similar point when discussing an early version of NAP as a way of modelling de Finetti's Lottery:

> The problem stated as "Find a probability measure on all of $\mathbb{N}$ that satisfies FAIR, ALL and SUM" is highly underdetermined: there are as many different ways to draw a random number from N in a fair way as there are free ultrafilters,

> and the probability function [given] should be seen as a whole family of solutions, all of which are, as far as we can presently see, equally relevant.[259, p. 19]

Together with Benci, they make a similar point when arguing that the fact that NAP functions are determined by a maximal fine ideal (or, equivalently, by a fine ultrafilter) does not put the NAP theorist at a disadvantage against the classical probability theorist:

> [I]n order to model a given conceptually possible probabilistic situation, three choices need to be made. The first two choices [(a sample space Ω and a weight function)] are familiar from classical probability theory. If one believes in the existence of objective probabilities, then these choices can be constrained: one can require Ω to be a subset of the 'universal sample space' of physically possible point events, and one can take the weight function to be physically determined. In the classical setting, a third choice has to be made (in uncountably infinite sample spaces): one has to pick a σ-algebra of events. The defender of NAP also has a third choice to make: the choice of an ultrafilter. In both approaches, this third choice may involve arbitrariness. [...] [O]n both the classical approach and in NAP, probability is partially arbitrary, in the sense that it involves a choice that is not empirically accessible. Once a choice has been made (for a particular σ-algebra of events or an ultrafilter, respectively), the probability function is unique (relative to this choice).[20, pp. 541-542]

The proponents of NAP theory therefore seem to acknowledge that the fact that their solution relies on the choice of a maximal, non-constructive object such as a non-principal ultrafilter ultimately introduces an element of arbitrariness to the way in which they model de Finetti's Lottery. According to them, the arbitrariness introduced by the choice of ultrafilter is on a par with the arbitrariness introduced by the choice of a σ-algebra of events in classical probability theory. But there are reasons to push back on this. The arbitrariness of the choice of a σ-algebra of events means that some events will not be measurable and thus will not receive any probability. The choice of an ultrafilter, by contrast, will influence the particular values that the corresponding events receive, both in an absolute way (the range of a NAP function is defined modulo the ultrafilter, so two distinct ultrafilters will yield entirely different sets as ranges) and in a relative way (two events may receive equal probability, or one may have smaller probability than the other, depending on the ultrafilter). As such, it is not entirely clear that NAP theory manages to model a genuinely fair lottery on $\mathbb{N}$. Depending on the choice of ultrafilter, the set $2\mathbb{N}$ of even numbers will either receive probability equal to or strictly greater (by an infinitesimal) than the set $2\mathbb{N}+1$ of odd numbers. One could therefore be tempted to argue that, in the first case, the choice of the ultrafilter favors the odds, and in the latter case, it favors the evens. In other words, it might be the case that de Finetti's Lottery is genuinely indeterminate; but that does not mean that NAP functions are all equally valid ways of adequately modelling it, rather than equally correct ways of modelling different, unfair lotteries.

The arbitrariness of NAP functions has prompted some opponents of **Regularity** such as Easwaran [78] to argue that they introduce too much structure among the probabilities of events in a fair lottery on an infinite set. Easwaran argues that the intuition that an event A represented by a proper subset of the set representing an event B should receive smaller probability than B can be captured via a qualitative, partial ordering on events that extends both the inclusion order on the set of events and the order induced by a classical probability distribution. In the case of de Finetti's Lottery, we could take μ to be a real-valued finitely additive probability function assigning probability 0 to every finite subset of $\mathbb{N}$, and define the relation $\prec$ on $\mathscr{P}(\mathbb{N})$ by letting $U \prec V$ iff $\mu(U) < \mu(V)$ or $U \subsetneq V$. This yields a *qualitative* (i.e., non-numerical) probability function on $\mathscr{P}(N)$ that satisfies the regularity constraint. According to Easwaran, this solution is based on the observation that the entire structure of the classical probabilistic setup can be used to decide, for example, whether one should bet on the winning ticket being even or being even or prime. Requiring that relevant data such as the inclusion order on events be reflected by the probability assignment itself amounts to what Easwaran calls a "numerical fallacy". There are reasons to think, however, that Easwaran's proposal is still lacking. For example, suppose that a bookmaker offers me a choice between betting that the winning number will be even, or betting that the winning number will be even and above 4, or strictly positive. In the second case, my winning set is $U = 2\mathbb{N} \setminus \{0\} \cup \{1, 3\}$. Intuitively, giving up on the chance of winning if the ticket turns out to be 0 for the chance of winning if the ticket is either 1 or 3 seems like a bet I should take. But under a real-valued finitely additive probability distribution, both $2\mathbb{N}$ and U have the same probability, and neither is a subset of the other, so we do not have that $2\mathbb{N} \prec U$. Moreover, one could also argue that numerical relationships between events, rather than mere ordinal ones, should also be available. Suppose that I have a bet on $2\mathbb{N}$, and that I am willing to pay a non-0 (but possibly infinitesimal) price ϵ to swap my bet on $2\mathbb{N}$ to a bet on $2\mathbb{N} \cup \{1\}$. Imagine now that the bookmaker offers to swap my bet on $2\mathbb{N}$ for a bet on $V = 2\mathbb{N} \cup \{1, 3\}$ for a price of 3ϵ. Should I take this bet on V or swap for $2\mathbb{N} \cup \{1\}$ for a price of ϵ instead? Intuitively, it seems that I should choose to swap my bet on $2\mathbb{N}$ to a bet on $2\mathbb{N} \cup \{1\}$, but not to a bet on V. Although V is more likely to happen than $2\mathbb{N} \cup \{1\}$, it is so exactly to the extent that $2\mathbb{N} \cup \{1\}$ is more likely to happen than $2\mathbb{N}$. Thus what I should recognize as a fair price to swap my bet on $2\mathbb{N}$ for a bet on V is 2ϵ, and not 3ϵ. Clearly, this reasoning requires me to be able to determine not only that some events are more likely than some others, but also to have a sense of when the probability of a event A is closer to the probability of an event B or to that of an event C. To use the terminology of measurement theory [244], probability assignments should at least be on an *interval* scale, not merely on an *ordinal* scale. Parker [205] makes a similar point when discussing the possibility of partially ordered Euclidean sizes. According to Parker, defining an order on the powerset of the natural number by letting $U < V$ iff $U \subsetneq V$ or U, V are finite and $|U| < |V|$ does not yield an interesting notion of size, and only obscures the conceptual distinction between being a proper subset and being smaller in cardinality, which one can clearly state in set-theoretical terms. Unless one can endow such an order with richer properties such as totality or numerical operations, there is little value in exploring this relation.

Consequently, it seems that, both in the case of Euclidean notions of sizes of sets and in the case of regular probability distributions on infinite sample spaces, we are faced with a dilemma. On the one hand, a genuine alternative to cardinalities or classical probabilities requires us to construct a rich and robust mathematical structure on which various kinds of computations can be performed. On the other hand, the guiding principles and intuitions on which we aim to ground our theory seem too weak to motivate such a richly detailed structure, and the proposals we have discussed seem to introduce such a structure artificially and in an arbitrary way. This is exactly where, as I will now argue, possibility structures can play a major role. By giving up on the explicit assumption that the structure we need must be Tarskian and by allowing ourselves to consider generic powers, we will be in a position to reach a natural equilibrium point between the need for a strong theory and the threat of arbitrariness.

8.5 The Euclidean Infinite via Possibility Structures

In this section, I will use possibility structures to address some of the criticisms raised against the Euclidean infinite in the previous section. I will treat Galileo's Paradox and de Finetti's Lottery one after the other, but the strategy in both cases will be very similar. Instead of defining a Tarskian structure (a semiring of numerosities, or a field of values for NAP functions) using a maximal object (Ramsey ultrafilter, maximal fine ideal), I will instead construct a possibility structure in which the goal of the poset structure is to approximate such a maximal object. Once such a possibility structure has been defined, it is easy to verify that it satisfies, via a genericity argument, the first-order properties that are common to all corresponding Tarskian structures.

8.5.1 Generic Numerosities

Let us start with the following definition.

Definition 8.5.1. For any $A, B \subseteq \omega$, A is *almost contained* in B, denoted $A \subseteq_* B$, if $A \setminus B$ is finite.

The relation of almost containment is clearly reflexive and transitive, and therefore it induces an equivalence relation $\sim_*$ on $\mathscr{P}(\omega)$ and a partial order on the set of equivalence classes.

Definition 8.5.2. The poset $\mathfrak{G}$ is given by the set $\{A^* \mid A \in \mathscr{P}(\omega)\} \setminus \{\emptyset^*\}$ of equivalence classes under the relation $\sim_*$, ordered by almost containment.

This poset is well known in the literature on infinite combinatorics. Indeed, it is easy to see that it is essentially $\mathscr{P}(\omega)^*_+$, i.e., the non-zero elements in the quotient of $\mathscr{P}(\omega)$ by the ideal of finite sets. As the set of infinite subsets of ω preordered by the relation $\subseteq_*$, it was topologized by Ellentuck [80], yielding results in Ramsey theory that have been generalized further [74, 75].

We can now give the main definition of this section:

Definition 8.5.3. Let $\mathcal{L}$ be the language of ordered semirings. The semiring of generic numerosities is the possibility structure $\mathcal{G} = (\mathfrak{G}, \mathbb{N}^\omega, \mathscr{I})$, where:

- $\mathscr{I}(0)$ and $\mathscr{I}(1)$ are defined as the constant functions $\overline{0}$ and $\overline{1}$ respectively;
- $\mathscr{I}$ is defined pointwise for the operations $+$ and $\times$;
- for any $A^* \in \mathcal{G}$, $R \in \{=, \leq\}$ and $\alpha, \beta \in \mathbb{N}^\omega$, $(\alpha, \beta) \in \mathscr{I}(A^*, R)$ iff $A^* \subseteq_* ||\alpha R \beta||$, where $||\alpha R \beta|| = \{i \in \omega \mid \mathbb{N} \models \alpha(i) R \beta(i)\}$.

An intuitive motivation for the definition of $\mathcal{G}$ can be given along the following lines. Just like in the standard construction of numerosities, we aim to construct a semiring of numerosities out of the set of all ω-sequences of natural numbers, by identifying together two sequences whenever they approximate the same numerosity from the viewpoint of an ideal "natural number at infinity" ι. However, by contrast with the standard construction, we do not take a Ramsey ultrafilter to be a stand-in for ι; rather, we approximate this viewpoint via the poset structure. Each infinite set A acts as a partial description of ι, in the sense that, according to A, ι has all the properties that correspond to sets in which A is almost included. As is standard in possibility semantics, going down along the ordering $\leq$ on $\mathcal{G}$ therefore means gaining some information, since the smaller the infinite set A gets, the more precise a description of ι it yields.

The next two lemmas establish that $\mathcal{G}$ is the correct structure for our purposes, as it behaves like a "generic" semiring of numerosities.

Lemma 8.5.4. *The generic numerosities $\mathcal{G}$ form a possibility structure. Moreover, for any $\mathcal{L}$-formula $\varphi(\overline{x})$, any tuple $\overline{a} \in \mathbb{N}^\omega$ and any $A^* \in \mathfrak{G}$, $A^* \Vdash \varphi(\overline{a})$ if and only if $||\varphi(\overline{a})|| \supseteq_* A$.*

Proof. For any infinite set $A \subseteq \omega$, let F_A be the filter $\{B \in \omega \mid \exists C \subseteq \omega$ cofinite such that $A \cap C \subseteq B\}$. Note that for any two infinite sets $A, B \subseteq \omega$, $B \in C_A$ if and only if $A \setminus B$ is finite. Let $\mathcal{H} = \{F_A \mid A \subseteq \omega$ is infinite$\}$. I first claim that $\mathcal{H}$ is a dense family. Indeed, if $B \notin F_A$ for some infinite set A, then $A \setminus B$ is infinite. But clearly $F_{A \setminus B} \supseteq F_A$ and $\omega \setminus B \in F_{A \setminus B}$. Moreover, it is easy to see that for any two infinite sets $A, B \subseteq \omega$, there exists a proper filter F such that $F \supseteq F_A, F_B$ if and only if $A \cap B$ is infinite. But, in that case, we have that $F_{A \cap B} \in \mathcal{H}$ and $F_A, F_B \subseteq F_{A \cap B}$, which shows that $\mathcal{H}$ is dense. Moreover, the map $A^* \mapsto F_A$ is clearly an order-isomorphism between $\mathfrak{G}$ and $\mathfrak{H}$. This induces an isomorphism between $\mathcal{G}$ and the $\mathfrak{H}$-generic power of $\mathbb{N}$. The rest of the lemma follows directly from the Structure and Truth Lemmas. □

Corollary 8.5.5. *Let $\mathfrak{n}$ be the map $A \mapsto \sigma_A$. The generic numerosities $\mathcal{G}$ have the following properties:*

1. *$\mathcal{G}$ has the same first-order theory as $\mathbb{N}$. Moreover, the map $\iota : \mathbb{N} \to \mathbb{N}^\omega$ is such that $\mathbb{N} \models \varphi(\overline{n})$ iff $\mathcal{G} \models \varphi(\overline{\iota(n)})$ for any $\mathcal{L}$ formula φ and tuple of natural numbers $\overline{n}$;*
2. *The map $\mathfrak{n}$ satisfies **(PW)**, in the sense that $\mathcal{G} \models \mathfrak{n}(A) < \mathfrak{n}(B)$ whenever $A \subsetneq B$;*

3. *For any finite* $A \subseteq \mathbb{N}$, $\mathcal{G} \models \mathfrak{n}(A) = \iota(|A|)$;

4. *Whenever* $A \cap B = \emptyset$, $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) \oplus \mathfrak{n}(B)$ *and* $\mathfrak{n}(A) \otimes \mathfrak{n}(B) = \mathfrak{n}(A \times B)$;

5. *For any* $\alpha \in \mathbb{N}^\omega$ *and any infinite* $A \subseteq \omega$, *there is* $B \subseteq_* A$ *and* $\beta \in \mathbb{N}^\omega$ *non-decreasing such that* $B \Vdash \alpha = \beta$.

Proof. Items 1-4 follow directly from Lemma 8.5.4. Indeed, for any sentence φ true in $\mathbb{N}$, $||\varphi|| = \omega$, so $A \subseteq_* ||\varphi||$ for any $A \subseteq \omega$ and thus $A^* \Vdash \varphi$ for every $A^* \in \mathcal{G}$. Similarly, if $A \subsetneq B$, then $||\sigma_A < \sigma_B||$ is cofinite, hence $A \subseteq_* ||\sigma_A < \sigma_B||$, and therefore $A^* \Vdash \sigma_A < \sigma_B$. The rest is proved similarly.

Item 5 requires a slightly longer albeit standard argument. Fix $\alpha \in \mathbb{N}^\omega$ and an infinite $A \subseteq \omega$, and let $\pi : A \to \omega$ be an order-preserving bijection. Define a coloring $c : [\omega]^2 \to \{0,1\}$ by letting $c(\{i,j\}) = 1$ iff $\alpha(\pi^{-1}(i)) \leq \alpha(\pi^{-1}(j))$ whenever $i < j$. By the infinite Ramsey Theorem, there is a homogeneous infinite $X \subseteq \omega$. This mean that we have either $\alpha(\pi^{-1}(i)) \leq \alpha(\pi^{-1}(j))$ whenever $i < j \in X$, or $\alpha(\pi^{-1}(i)) > \alpha(\pi^{-1}(j))$ whenever $i < j \in X$. But the latter is impossible, since otherwise we would have an infinite descending sequence of natural numbers. Now let $B \subseteq A$ be the inverse image of X under π. By construction, for any $i < j \in B$, we have that $\pi(i) < \pi(j) \in X$, so $\alpha(\pi^{-1}\pi(i)) \leq \alpha(\pi^{-1}\pi(j))$, and therefore $\alpha(i) < \alpha(j)$ whenever $i < j \in B$. Now let $\beta : \omega \to \mathbb{N}$ be defined by $\beta(i) = \alpha(j_i)$, where j_i is the least $j \in B$ such that $i \leq j$. Then it follows that β is non-decreasing, and moreover $B \subseteq ||\alpha = \beta||$, which completes the proof of item 5. □

Modulo adapting some notions to the setting of possibility structures, this shows that $\mathcal{G}$ enjoys the same features as the semirings of numerosities constructed by Benci and di Nasso. Showing that it is indeed a "generic" semiring of numerosities involves some set-theoretic assumptions that go beyond ZFC. In particular, we will need the assumption that Ramsey ultrafilters exist. For the latter, note that assuming the existence of a single Ramsey ultrafilter is enough to derive the existence of many, as is well known.

Lemma 8.5.6. *Assume that there exists a Ramsey ultrafilter. Then every infinite subset of* ω *belongs to some Ramsey ultrafilter.*

Proof. Let U be a Ramsey ultrafilter. It is straightforward to verify that, if $\varphi : \omega \to \omega$ is a permutation on ω, then the lift $\varphi_* : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ preserves the Ramsey property of U, meaning that $\varphi_*[\mathsf{U}] = \{\varphi_*(A) \mid A \in \mathsf{U}\}$ is a Ramsey ultrafilter. Now let $A \subseteq \omega$ be infinite. If $A \in \mathsf{U}$, we are done. Otherwise, $\omega \setminus A \in \mathsf{U}$. Let φ be a bijection between A and $\omega \setminus A$, and notice that φ induces a permutation on ω such that $\varphi_*(A) = \omega \setminus A$. But then $A \in \varphi_*(\mathsf{U})$, hence A belongs to some Ramsey ultrafilter. □

Lemma 8.5.7. *Assume that there is a Ramsey ultrafilter. For any* $A^* \in \mathfrak{G}$, *any* $\mathcal{L}$*-formula* $\varphi(\overline{x})$ *and any tuple* $\overline{a} \in \mathbb{N}^\omega$, $A^* \Vdash \varphi(\overline{a})$ *iff* $\mathbb{N}^\omega/\mathsf{U} \models \varphi(\overline{a_\mathsf{U}})$ *for every Ramsey ultrafilter* U *such that* $A \in \mathsf{U}$.

Proof. Recall first the observation in the proof of Lemma 8.5.4 that $\mathfrak{G}$ is order-isomorphic to $\mathfrak{H}$ for the dense family $\mathcal{H}$ of principal proper filters extending the Fréchet filter on ω. It is easy to see that, if U is a non-principal ultrafilter, it is also $\mathfrak{H}$-generic. Since every Ramsey ultrafilter is non-principal, by the first part of the Genericity Lemma together with Lemma 8.5.6, this means that $\mathfrak{G}$ satisfies a strong version of the condition of the second part of the Genericity Lemma, with the property of being $\mathfrak{G}$-generic replaced with the property of being Ramsey. Therefore we have that for any infinite $A \subseteq \omega$, any $\varphi(\overline{x})$ and any tuple $\overline{a}$, $A^* \Vdash \varphi(\overline{a})$ iff $\mathbb{N}/\mathsf{U} \models \varphi(\overline{a}/\mathsf{U})$ for every Ramsey ultrafilter U such that $A \in \mathsf{U}$. □

The result above raises an interesting question regarding the relationship between various notions of genericity on $\mathfrak{G}$. The weakest notion is that of $\mathcal{G}$-genericity, which is relative to the language of the possibility structure $\mathcal{G}$. Being $\mathfrak{G}$-generic is a stronger condition, which is equivalent to being non-principal for an ultrafilter on ω. However, as remarked in Lemma 6.2.17, the two notions would coincide if we were to add a predicate in $\mathscr{L}$ for every subset of $\mathbb{N}$. One could therefore wonder whether we could strengthen the notion of $\mathcal{G}$-genericity in order to make it equivalent to the Ramsey property. Some results in infinitary combinatorics suggest that this might however be quite a difficult issue. Since $\mathfrak{G}$ is equivalent to $\mathscr{P}(\omega)^*_+$, we can indeed quote some results in the literature on forcing over the latter. Since this poset is not atomic, it does not have any generic filter, but one can restrict the notion of genericity to an inner model $\mathfrak{M}$ of ZFC and obtain some interesting results. It is well known that if $\mathfrak{M}$ is an inner model of ZFC containing $\mathscr{P}(\omega)$, then any $\mathfrak{M}$-generic filter over $\mathscr{P}(\omega)^*_+$ is Ramsey. Moreover, by a result of Todorčević (see [156, Thm. 2.1] for a proof), under the assumption that there exists infinitely many Woodin cardinals and a measurable above them (a strong but standard large cardinal assumption in modern set theory), an ultrafilter U on ω is $L(\mathbb{R})$-generic over $\mathscr{P}(\omega)^*_+$ if and only if U is Ramsey. Thus the strengthening of $\mathcal{G}$-genericity that we would be looking for should be somewhat in the vicinity of $L(\mathbb{R})$-genericity, at least under some strong set-theoretic assumptions.

For now, we conclude with a remark on the significance of Lemma 8.5.7. It immediately follows from it that truth on the semiring of generic numerosities coincides with truth in every semiring of standard numerosities. This means in particular that, although $\mathcal{G}$ satisfies the statement that the numerosities of any two sets are comparable, it doesn't decide which set has greater numerosity than another whenever two distinct constructions of numerosities would yield different answers. In that sense, $\mathcal{G}$ encapsulates a view reminiscent of Galileo's: sometimes, size relationships between infinite sets simply cannot be determined. It is worth pointing out however, that $\mathcal{G}$ does give a definite answer in the case of Galileo's Paradox itself. Because $\mathcal{G}$ satisfies **(PW)**, we have indeed that $\mathcal{G} \models \mathfrak{n}(\mathbb{N}^2) < \mathfrak{n}(\mathbb{N})$, where $\mathbb{N}^2$ is the set of all squares. In other words, generic numerosities offer a solution to Galileo's Paradox that both upholds the part-whole principle and preserves some aspects of Galileo's own solution. Let us now move on to the case of de Finetti's Lottery.

8.5.2 Non-Archimedean Possibility Fields

Recall that the starting point of the NAP solution to the issues raised by fair lotteries on infinite sets is to consider the partially ordered ring $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ of all functions from $\mathscr{P}_{Fin}(\Omega)$ to $\mathbb{R}$, with the ring operations and ordering defined pointwise. In order to turn this partially ordered ring into a field, we have two tasks to perform. First, we must find a way to identify elements of $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ so that every non-zero element becomes invertible. Second, we need to extend the partial ordering of $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ to a total order. Classically, one typically relies on the existence of maximal objects and thus on Zorn's Lemma to accomplish both tasks. It turns out however that we can bypass the reliance on maximal objects by appealing to possibility structures. We start by presenting a general way of turning a ring into a possibility structure satisfying the axioms of an ordered field, and a partially ordered ring into a possibility structure satisfying the axioms of a totally ordered ring. Although we will ultimately be more interested in combining the two approaches in the case of function rings, this detour via a more general perspective also allows us to explore the potential of possibility structures for obtaining semi-constructive analogues of classical, non-constructive techniques in elementary abstract algebra. For the rest of this chapter, every ring will be assumed to be commutative, and given an ideal J on a ring $\mathcal{R}$, I will write a_J for the image of an element $a \in \mathcal{R}$ under the canonical quotient map from $\mathcal{R}$ to $\mathcal{R}/J$.

Definition 8.5.8. Let $\mathcal{R}$ be a ring and I an ideal on $\mathcal{R}$. The *possibility quotient ring of* $\mathcal{R}$ *by* I, denoted $\mathcal{Q}(\mathcal{R}, I)$ is given by the possibility structure $(\mathfrak{I}, \mathcal{R}, \mathscr{I})$, where:

- $\mathfrak{I}$ is the poset of all proper ideals extending I, ordered by inclusion;
- $\mathscr{I}$ interprets the ring operations as in $\mathcal{R}$;
- for any $a, b \in \mathcal{R}$ and any $J \in \mathfrak{I}$, $(a, b) \in \mathscr{I}(J, =)$ if and only if, for any $K \supseteq J$, there is $K' \supseteq K$ such that $a_{K'} = b_{K'}$.

Lemma 8.5.9. *For any ring* $\mathcal{R}$ *and any ideal* I *on* $\mathcal{R}$, $\mathcal{Q}(\mathcal{R}, I)$ *is a possibility structure satisfying the axioms of an ordered field.*

Proof. First, it is straightforward to verify that $\mathcal{Q}(\mathcal{R}, I)$ is a possibility structure. Indeed, the definition of $\mathscr{I}$ guarantees that atomic formulas are mapped to regular open subsets of $\mathfrak{I}$, and it is routine to verify that the equality conditions hold.

Similarly, since $\mathcal{R}$ is a ring and the ring operations on $\mathcal{Q}(\mathcal{R}, I)$ are defined as in $\mathcal{R}$, one easily checks that the ring axioms are forced by any viewpoint $J \supseteq I$.

Finally, let us verify that for any $J \in \mathfrak{I}$ and any $a \in \mathcal{R}$, $J \Vdash a = 0 \vee \exists x(xa = 1)$. Let $K \supseteq J$, and consider the ideal K' generated by the set $K \cup \{a\}$. We distinguish two cases. First, $K' = R$. Since K' is generated by $K \cup \{a\}$, this means that there are $b, r_1, ..., r_n \in \mathcal{R}$ and $a_1, ..., a_n \in K$ such that $1 = ab + a_1 r_1 + \cdots + a_n r_n$. Since $a_i r_i \in K$ for every $i \leq n$, this means that $a_K b_K = 1_K$. In the second case, K' is a proper ideal, so $K' \in \mathfrak{I}$, and by construction $K' \Vdash a = 0$. Hence for any $K \supseteq J$ there is $K' \supseteq K$ such that $K' \Vdash a = 0$ or $K' \Vdash \exists x(xa = 1)$. This shows that $J \Vdash a = 0 \vee \exists x(xa = 1)$. ☐

The previous lemma establishes that the possibility quotient ring of a ring $\mathcal{R}$ by an ideal I can play a role analogous to the quotient field of $\mathcal{R}$ modulo a maximal ideal extending I. Let us move on to discussing how to turn a partially ordered ring into a possibility structure satisfying the axioms of a totally ordered ring.

Definition 8.5.10. Let $\mathcal{R}$ be a ring. Given sets $A, B \subseteq \mathcal{R}$, let $-A$, $A+B$ and AB be the sets $\{-a \mid a \in A\}$, $\{a+b \mid a \in A, b \in B\}$ and $\{ab \mid a \in A, b \in B\}$. Given an ideal I on $\mathcal{R}$, a set $P \subseteq \mathcal{R}$ is an I-*cone* if $P \cap -P = I$, $P + P \subseteq P$ and $PP \subseteq P$. A *cone* is simply a $\{0\}$-cone. Given a cone P on $\mathcal{R}$, an ideal I is P-*convex* if for any $a, b \in I$ and any $c \in \mathcal{R}$, $c - a, b - c \in P$ implies $c \in I$.

It is well known that any partial order $\leq$ on a ring $\mathcal{R}$ determines a cone $P_{\leq} = \{a \in \mathcal{R} \mid 0 \leq a\}$. Conversely, any cone P on $\mathcal{R}$ induces a partial order $\leq_P$ defined by letting $a \leq_P b$ iff $b - a \in P$ for any $a, b \in \mathcal{R}$. Clearly, a partial order on a ring $\mathcal{R}$ is total iff $P_{\leq} \cup -P_{\leq} = \mathcal{R}$. This motivates the following construction.

Lemma 8.5.11. *Let $\mathcal{R}$ be a partially ordered ring with associated cone P_0, and let $\mathcal{O}(\mathcal{R}, \mathfrak{P})$ be a possibility structure $(\mathfrak{P}, \mathcal{R}, \mathscr{I})$ such that:*

- *$\mathfrak{P}$ is a set of cones on $\mathcal{R}$, ordered by reverse inclusion, such that for every $P \in \mathfrak{P}$, $P_0 \subseteq P$ and for any $a \in \mathcal{R}$ there is $P' \supseteq P$ such that $a \in P' \cup -P'$;*
- *$(a, b) \in \mathscr{I}(P, =)$ iff $a = b$, and $(a, b) \in \mathscr{I}(P, \leq)$ iff for every $Q \supseteq P$ there is $Q' \supseteq Q$ such that $b - a \in Q'$.*

Then $\mathcal{O}(\mathcal{R}, \mathfrak{P})$ satisfies the axioms of a totally ordered ring.

Proof. Once again it is straightforward to verify that $\mathcal{O}(\mathcal{R}, \mathfrak{P})$ is a possibility structure and that it satisfies all the axioms of a partially ordered ring. As an example, let us check that $P \Vdash \forall x \forall y \forall z (x \leq y \to x + z \leq y + z)$. It is enough to show that for any $a, b, c \in \mathcal{R}$ and any $P \in \mathfrak{P}$, $P \Vdash a \leq b$ implies $P \Vdash a + c \leq b + c$. So fix a, b, c and P, and assume $P \Vdash a \leq b$. By definition, this means that $b - a \in P'$ for any $P' \supseteq P$. Since $b - a = (b + c) - (a + c)$, we also have that $(b + c) - (a + c) \in P'$ for any $P' \supseteq P$. But this implies at once that $P \Vdash a + c \leq b + c$. Finally, let us check that $\mathcal{O}(\mathcal{R}, \mathfrak{P}) \models \forall x \forall y (x \leq y \vee y \leq x)$. Since $\mathcal{O}(\mathcal{R}, \mathfrak{P})$ satisfies the ring axioms, it is enough to show that $\mathcal{O}(\mathcal{R}, \mathfrak{P}) \models \forall x (x \leq 0 \vee 0 \leq x)$. So fix $P \in \mathfrak{P}$, $Q \supseteq P$ and $a \in \mathcal{R}$. By assumption, there is $Q' \supseteq Q$ such that $a \in Q \cup -Q$. Clearly, this means that for any $Q^* \supseteq Q'$, $a \in Q^*$ or $-a \in Q^*$, from which it follows that either $Q' \Vdash 0 \leq a$ or $Q' \Vdash a \leq 0$. Hence for any $Q \supseteq P$ there is $Q' \supseteq Q$ such that $Q' \Vdash a \leq 0$ or $Q' \Vdash 0 \leq a$, which shows that $P \Vdash a \leq 0 \vee 0 \leq a$. This completes the proof. □

As a corollary, we obtain the following characterization of partially ordered rings that can be turned into a possibility structure satisfying the axioms of a totally ordered ring.

Corollary 8.5.12. *Let $\mathcal{R}$ be a partially ordered ring with positive cone P_0. Then there is a possibility structure $\mathcal{S}$ with domain $\mathcal{R}$, satisfying the axioms of a totally ordered ring, and such that $S \models 0 \leq a$ for any $a \in P_0$ if and only if P_0 satisfies the following condition:*

(∗) *For any* $a_1, ..., a_n \in \mathcal{R}$*, there are* $\epsilon_1, ..., \epsilon_n \in \{1, -1\}$ *such that the semiring generated by the set* $P_0 \cup \{\epsilon_i a_i \mid i \leq n\}$ *is a cone.*

Proof. For the left-to-right direction, suppose we have such a possibility structure $\mathcal{S}$, and let p be a viewpoint of $\mathcal{S}$. Since $\mathcal{S}$ satisfies that $\mathcal{R}$ is totally ordered, we have that $p \Vdash 0 \leq a_i \vee a_i \leq 0$ for every $i \leq n$. This implies that, by taking successive refinements of p, we can find a viewpoint $q \leq p$ such that $q \Vdash 0 \leq a_i$ or $q \Vdash a_i \leq 0$ for every $i \leq n$. Let $\epsilon_i = 1$ if $q \Vdash 0 \leq a_i$ and $\epsilon_i = -1$ otherwise, and let $P_q = \{a \in \mathcal{R} \mid q \Vdash 0 \leq a\}$. Then since the axioms of a partially ordering ring are valid on $\mathcal{S}$ and $\mathcal{S} \models 0 \leq a$ for every $a \in P_0$, one easily verifies that P_q is a cone containing the semiring generated by the set $P_0 \cup \{\epsilon_i a_i \mid i \leq n\}$, which implies that the latter is also a cone. Conversely, let $\mathfrak{P}$ be the set of all cones extending P_0 and satisfying condition (∗), ordered by reverse inclusion. It is a standard result about partially ordered rings [97, Chapt. 7] that this set has the property that for any $P \in \mathfrak{P}$ and any $a \in \mathcal{R}$, there is $P' \supseteq P \in \mathfrak{P}$ such that $a \in P' \cup -P'$. But then, by Lemma 8.5.11, the corresponding possibility structure $\mathcal{O}(\mathcal{R}, \mathfrak{P})$ satisfies the axioms of a totally ordered ring. □

This result can be seen as a semi-constructive analogue of a standard result about ordered rings, according to which a partial order on a ring $\mathcal{R}$ with cone P can be extended to a total order on $\mathcal{R}$ if and only if P has property **(*)** above (see for example [97, Chap. 7, Thm. 1]).

Let us now move on to the issue of turning a partially ordered ring into a possibility structure satisfying the axioms of an ordered field. In general, one would need to combine the techniques of Lemmas 8.5.9 and 8.5.11 and consider a poset composed of pairs (P, J) such that P is a J-cone and J is a P-convex ideal and satisfying some properties similar to the ones imposed on the poset $\mathfrak{P}$ in Lemma 8.5.9. In the case of rings of functions from a set into a totally ordered field, however, the situation turns out to be simpler, as every ideal induces a canonical cone. We start from the following definition and observation.

Definition 8.5.13. Let I be a set, $\mathcal{R}$ a field and $\mathcal{F}(I, \mathcal{R})$ the ring of all functions from I to $\mathcal{R}$. For any $\alpha \in \mathcal{F}(I, \mathcal{R})$, the 0*-set of* α is the set $\alpha_0 = \{i \in I \mid \alpha(i) = 0\}$.

Lemma 8.5.14. *Let* I *be a set,* $\mathcal{R}$ *a field and* $\mathcal{F}(I, \mathcal{R})$ *the ring of all functions from* I *to* $\mathcal{R}$*. For any filter* F *on* I*, let* $J_F = \{a \in \mathcal{F}(I, \mathcal{R}) \mid a_0 \in F\}$ *and for any ideal* J *on* $\mathcal{F}(I, \mathcal{R})$*, let* $F_J = \{A \subseteq I \mid \exists a \in J : a_0 \subseteq A\}$*. The maps* $F \mapsto F_J$ *and* $J \mapsto J_F$ *establish an order isomorphism between the proper filters on* I *and the proper ideals on* $\mathcal{F}(I, \mathcal{R})$*, with both sets ordered by reverse inclusion.*

Proof. Let us first verify that J_F is an ideal for every filter F and that F_J is a filter for every ideal J. Given $a, b \in \mathcal{F}(I, \mathcal{R})$, note that $(a \cdot b)_0 \supseteq a_0$ and that $(a + b)_0 \supseteq a_0 \cap b_0$. From this it follows that $\varphi(F)$ is an ideal whenever F is a filter. To see that J_F is proper whenever F is proper, it suffices to notice that $1_0 = \emptyset$.

Conversely, suppose that J is an ideal. Clearly, F_J is upward closed, so we only need to check that $A \in B \in F_J$ whenever $A, B \in F_J$. Suppose there are $a, b \in J$ such that

$a_0 \subseteq A$ and $b_0 \subseteq B$. Let $k : I \to \mathcal{R}$ be defined as $k(i) = -1$ if $a(i) = -b(i)$ and $k(i) = 1$ otherwise. Since J is an ideal, we have that $ka + b \in J$. But for any $i \in I$, $(ka + b)(i) = 0$ implies that $ka(i) = -b(i)$, which by choice of k implies that $a(i) = -a(i) = b(i) = 0$. Hence $(ka + b)_0 \subseteq a_0 \cap b_0$, which shows that $a_0 \cap b_0 \in F_J$. Finally, to show that F_J is proper whenever F is proper, suppose $\emptyset \in F_J$. Then there is $a \in J$ such that $a_0 = \emptyset$. Define $b : I \to \mathcal{R}$ by letting $b(i) = \frac{1}{a(i)}$ for every $i \in I$, and note that this is well-defined since $\mathcal{R}$ is a field and $a(i) \neq 0$ for every $i \in I$. But then $ab \in I$ and clearly $ab = 1$, which shows that J is not proper.

Thus $F \mapsto J_F$ and $J \mapsto F_J$ are well-defined maps between the poset of all filters on I and the poset of all ideals of $\mathcal{F}(I, \mathcal{R})$, both ordered by reverse inclusion. Clearly, both maps are order-preserving, so we only need to verify that they are inverses of one another. Fix a filter F. Clearly, $F \subseteq F_{J_F}$. To show the converse, suppose $A \in F_{J_F}$. Then there is $a \in J_F$ such that $a_0 \subseteq A$. But $a \in J_F$ implies that $a_0 \in F$, and hence $A \in F$ since F is upward-closed. Similarly, fix an ideal J. Again, it is clear that $J \subseteq J_{F_J}$, so suppose $a \in J_{F_J}$. This means that there is $b \in J$ such that $b_0 \subseteq a_0$. Define $k : I \to \mathcal{R}$ such that $k(i) = \frac{a(i)}{b(i)}$ whenever $b(i) \neq 0$, and $k(i) = a(i)$ otherwise. Then it follows that $a(i) = b(i)k(i)$ for every $i \in I$, and thus $a = bk$. Since J is an ideal, this implies that $a \in J$, which completes the proof. □

We may now see how Lemma 8.5.14 yields a natural choice for the canonical cone associated to an ideal on a ring of functions from a set into an ordered field.

Lemma 8.5.15. *Let I be a set and $\mathcal{R}$ be an ordered field. For any ideal J on $\mathcal{F}(I, \mathcal{R})$, let $P_J = \{a \in \mathcal{F}(I, \mathcal{R}) \mid ||0 \leq a||_I \in F_J\}$. Then P_J is a J-cone, and J is P_J-convex.*

Proof. Fix $a, b \in \mathcal{F}(I, \mathcal{R})$. Note that $||0 \leq a||_I \cap ||0 \leq b||_I \subseteq ||0 \leq a+b||_I \cap ||0 \leq ab||_I$, which shows that $P_J + P_J \subseteq P_J$ and $P_J P_J \subseteq P_J$ for any ideal J. Moreover, since $a \in J$ implies $||a = 0||_I \in F_J$ and $||a = 0||_J = || - a = 0||_J$, we have that $J \subseteq P_J \cap -P_J$. For the converse, note that $||0 \leq a||_I \cap ||0 \leq -a||_I \subseteq ||a = 0||_I$, which means that $||a = 0||_I \in F_J$ and thus that $a \in J_{F_J} = J$. Hence P_J is a J-cone. Finally, to see that J is P_J-convex, note that $||0 \leq b - a||_I = ||a \leq b||_I$, and that $||0 = a||_I \cap ||a \leq c||_I \cap ||c \leq b||_I \cap ||b - 0||_I \subseteq ||c = 0||_I$. Hence $a - c, b - c \in P_J$ and $a, b \in J$ together imply that $c \in J$, which shows that J is P_J-convex. □

We have now gathered all the ingredient for the definition of the semi-constructive analogue of the ranges of NAP functions.

Definition 8.5.16. Let I be a set, $\mathcal{R}$ an ordered field, $\mathcal{F}(I, \mathcal{R})$ the partially ordered ring of all functions, and J_0 an ideal on $\mathcal{F}(I, \mathcal{R})$. The *generic ordered field* of $\mathcal{F}(I, \mathcal{R})$ induced by J_0 is the possibility structure $\mathcal{H}(\mathfrak{J}, \mathcal{R}) = (\mathfrak{J}, \mathcal{R}^I, \mathscr{I})$, where:

- $\mathfrak{J}$ is the set of all ideals on $\mathcal{F}(I, \mathcal{R})$ extending J_0, ordered by reverse inclusion;
- function symbols are interpreted pointwise, and for any $a, b \in \mathcal{R}^I$ and $J \in \mathfrak{J}$:
- $(a, b) \in \mathscr{I}(J, =)$ iff $a - b \in J$, and

- $(a, b) \in \mathscr{I}(J, \leq)$ iff $b - a \in P_J$.

Theorem 8.5.17. *For any set I, ordered field $\mathcal{R}$ and ideal J_0 on $\mathcal{F}(I, \mathcal{R})$, the corresponding generic ordered field $\mathcal{H}(\mathfrak{J}, \mathcal{R})$ satisfies the axioms of the ordered field.*

Proof. Let us start with the following two claims.

- Claim 1: For any $a, b \in \mathcal{R}^I$ and $J \in \mathfrak{J}$, $J \Vdash a = b$ iff for all $K \supseteq J$ there is $J' \supseteq K$ such that $K' \Vdash a = b$ iff $a_J = b_J$.

- Claim 2: For any $a, b \in \mathcal{R}^I$ and $J \in \mathfrak{J}$, $J \Vdash a \leq b$ iff for all $K \supseteq J$ there is $J' \supseteq K$ such that $K' \Vdash a \leq b$ iff $a_J \leq_{P_J} b_J$.

For the proof of the first claim, it is enough to show that if $a - b \notin J$, then there is $K \supseteq J$ such that for all $K' \supseteq K$, $a - b \notin K'$. Suppose $a - b \notin J$. Then $||0 = a - b||_I \notin F_J$, so let G be the filter generated by the set $F_J \cup \{||0 \neq a - b||_I\}$. Clearly, G is a proper filter, so $J_G \in \mathfrak{J}$. Moreover, for any $K' \supseteq K$, $F_{K'}$ is a proper filter extending G, so $||0 = a - b|| \notin F_{K'}$ and hence $a - b \notin K'$. The proof of the second claim is entirely similar. Note that this also establishes that $\mathfrak{H}$ is a possibility structure. Indeed, the refinability condition immediately follows from the two claims, and the two equality conditions are also straightforward. Similarly, it follows from an argument similar to the one given here, using the fact that P_J is a J-cone and that J is P_J-convex, that the axioms of a partially ordered ring are satisfied at every $J \in \mathfrak{J}$.

Moreover, the $\{\leq\}$-free reduct of $\mathcal{H}(\mathfrak{J}, \mathcal{R})$ is simply the possibility quotient ring $\mathcal{Q}(\mathcal{R}, J_0)$, which by Lemma 8.5.9 implies that it satisfies the axioms of a field. Finally, an argument similar to the one in the proof of Lemma 8.5.11 shows that $\mathcal{H}(\mathfrak{J}, \mathcal{R})$ also satisfies that $\leq$ is a total order. First, let us establish that for any $J \in \mathfrak{J}$ and any $a \in \mathcal{R}^I$, there is $K \supseteq J$ such that $a \in P_K \cup -P_K$. To show this, suppose $a \notin P_J$. Then $||0 \leq a||_I \notin F_J$, so the filter G generated by $F_J \cup \{||0 \not\leq a||_I\}$ is proper. Since $\mathcal{R}$ is totally ordered, we have that $||0 \not\leq a||_I \subseteq ||a \leq 0||_I \subseteq ||0 \leq -a||_I$, so $a \in -P_{J_G}$. This shows that for any $J \in \mathfrak{J}$ and any $a \in \mathcal{R}^I$, there is $K \supseteq J$ such that $a \in P_K \cup -P_K$. But then this implies that $J \Vdash 0 \leq a \vee a \leq 0$ for any $J \in \mathfrak{J}$ and $a \in \mathcal{R}^I$, which concludes the proof that $\mathcal{H}(\mathfrak{J}, \mathcal{R})$ satisfies the axioms of an ordered field. □

As a consequence, we many now define a semi-constructive alternative to NAP functions.

Definition 8.5.18. Let Ω be a set, J_0 be the ideal

$$\{\alpha \in \mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R}) \mid \exists A \in \mathscr{P}_{Fin}(\Omega)\, \forall B \supseteq A : \alpha(B) = 0\}$$

and $\mathfrak{J}$ the poset of all ideals on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ extending J_0. The *generic NAP function* modelling a fair lottery on Ω is the map $\pi : \mathscr{P}(\Omega) \to \mathcal{H}(\mathfrak{J}, \mathbb{R})$ defined by $\pi(U) = \chi_U$ for every $U \in \mathscr{P}(\Omega)$.

Let us verify that generic NAP-functions have all the desirable properties of NAP functions.

Theorem 8.5.19. *For any set Ω, the generic NAP function modelling a fair lottery on Ω has the following properties:*

1. $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models 0 \leq \pi(U) \leq 1$ *for any* $U \in \mathscr{P}(\Omega)$*, and* $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \pi(\Omega) = 1$.

2. π *is finitely additive, i.e.,* $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \pi(U \cup V) = \pi(U) + \pi(V)$ *whenever* $U \cap V = \emptyset$.

3. $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \pi(\{x\}) = \pi(\{y\})$ *for any* $x, y \in \Omega$.

4. $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \pi(U) \neq 0$ *iff* $U \neq \emptyset$ *for any* $U \subseteq \Omega$.

Moreover, under $AC_{\mathscr{P}(|\Omega|)}$*,* π *also has the following property:*

5. *For any first-order formula* $\varphi(x_1, ..., x_n)$ *and any* n*-tuple of sets* $U_1, ..., U_n \subseteq \Omega$*,* $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \varphi(\pi(U_1), ..., \pi(U_n))$ *iff* $\varphi(P_J(U_1), ..., P_J(U_n))$ *is true for every maximal fine ideal* J *on* $\mathcal{F}(\mathscr{P}_{Fin}(\Omega),\mathbb{R})$ *and any fair NAP function* P_J *on* Ω *induced by* J.

Proof. For items 1, 2 and 3, it is enough to show that each statement is forced at J_0. For item 1, note that for any $U \subseteq \Omega$ and $A \in \mathscr{P}_{Fin}(\Omega)$, $0 \leq \chi_U(A) \leq 1$, so $||0 \leq \pi(U)||_{\mathscr{P}_{Fin}(\Omega)} = ||\pi(U) \leq 1||_{\mathscr{P}_{Fin}(\Omega)} = \Omega$, hence $J_0 \Vdash 0 \leq \pi(U)$ and $J_0 \Vdash \pi(U) \leq 1$. Similarly, since $\chi_\Omega(A) = 1$ for every $A \in \mathscr{P}_{Fin}(\Omega)$, we have that $\pi(A) - 1 = 0 \in J_0$, hence $J_0 \Vdash \pi(\Omega) = 1$. Similarly, note that, if $U \cap V = \emptyset$, then $\chi_{U \cup V}(A) = \chi_U(A) + \chi_V(A)$ for every $A \in \mathscr{P}_{Fin}(\Omega)$, so $\pi(U \cup V) - (\pi(U) + \pi(V) = 0 \in J_0)$, and hence $J_0 \Vdash \pi(U \cup V) = \pi(U) + \pi(V)$, which proves item 2. For item 3, let $x, y \in \Omega$. Note that for any $A \supseteq \{x, y\}$, $\chi_{\{x\}}(A) = \chi_{\{y\}}(A) = \frac{1}{|A|}$, from which it follows that $(\chi_{\{x\}} - \chi_{\{y\}})(A) = 0$ for any $A \supseteq \{x, y\}$. Hence $\pi(\{x\}) - \pi(\{y\}) = 0 \in J_0$, which implies that $J_0 \Vdash \pi(\{x\}) = \pi(\{y\})$.

For item 4, it is clear that $\pi(\emptyset) = 0$ and hence that $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \pi(\emptyset) = 0$. Moreover, let U be a non-empty subset of Ω. For any $A \in \mathscr{P}_{Fin}(\Omega)$ such that $A \cap U \neq \emptyset$, $\chi_U(A) > 0$. This means that $\mathscr{P}_{Fin}(\Omega) \setminus (\pi(U))_0 \in J_0$, and thus $J \nVdash \pi(U) = 0$ for any $J \in \mathfrak{J}$. Hence $\mathcal{H}(\mathfrak{J},\mathbb{R}) \models \pi(U) \neq 0$.

Finally, assume $AC_{\mathscr{P}(|\Omega|)}$. By Lemma 8.5.14, $\mathfrak{J}$ is order-isomorphic to the poset $\mathfrak{E} = (\mathcal{E}, \supseteq)$ of all filters extending F_{J_0}. Notice that since $\mathcal{E}$ is a dense family, we may consider $(\mathfrak{E}, \mathbb{R}^{\mathscr{P}_{Fin}(\Omega)}, \mathscr{I})$, the $\mathfrak{E}$-generic power of $\mathbb{R}$. Moreover, the $\mathfrak{E}$-generic ultrafilters coincide with the maximal points in $\mathfrak{E}$, i.e., with the ultrafilters on $\mathscr{P}_{Fin}(\Omega)$ extending F_{J_0}. Since $AC_{|\mathscr{P}(\Omega)|}$ holds, this means that every $F \in \mathcal{E}$ belongs to some $\mathfrak{E}$-generic ultrafilter and thus that $(\mathfrak{E}, \mathbb{R}^{\mathscr{P}_{Fin}(\Omega)}, \mathscr{I})$ satisfies the conditions of the Genericity Lemma. Finally, the map $F \mapsto J_F$ is a dense map, and $J \in [\![R(\overline{a})]\!]_{\mathfrak{J}}$ iff $F_J \in [\![R(\overline{a})]\!]_{\mathfrak{E}}$ for any $J \in \mathfrak{J}$, atomic formula $\varphi(\overline{x})$ and tuple $\overline{a} \in \mathbb{R}^{\mathscr{P}_{Fin}(\Omega)}$. By the Density and Genericity Lemmas, it follows that $J \Vdash \varphi(\overline{a})$ for any first-order formula $\varphi(\overline{x})$ iff $\mathcal{F}(\mathscr{P}_{Fin}(\Omega),\mathbb{R})/\mathsf{U} \models \varphi(\overline{a_{\mathsf{U}}})$ for every ultrafilter extending F_{J_0}. Since every such ultrafilter is of the form F_J for a maximal fine ideal J, $\mathcal{F}(\mathscr{P}_{Fin}(\Omega),\mathbb{R})/J$ is isomorphic to $\mathbb{R}^{\mathscr{P}_{Fin}(\Omega)}/F_J$ for every maximal fine ideal J and $\pi(U)_{F_J} = P_J(U)$ for any

$U \subseteq \Omega$, it follows at once that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \varphi(\pi(U_1), ..., \pi(U_n))$ iff $\varphi(P_J(U_1), ..., P_J(U_n))$ is true for every maximal fine ideal J on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ and any fair NAP function P_J on Ω induced by J. This completes the proof. □

The last item of Theorem 8.5.19 explains the sense in which generic NAP-functions are indeed generic: given any sample space Ω, the standard NAP functions for Ω can all be obtained by forcing over the poset $\mathfrak{J}$ of all ideals extending the smallest fine ideal on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$. Moreover, generic NAP-functions possess exactly those first-order properties that are shared by every NAP-function. It is also worth mentioning that one could have proved directly that generic ordered fields of function rings $\mathcal{F}(I, \mathcal{R})$ satisfy the axioms of ordered fields by using the fact they are isomorphic as possibility structures to $\mathfrak{E}$-generic powers of $\mathcal{R}$ for $\mathfrak{E}$ the poset of all proper filters on $\mathscr{P}(\mathscr{P}_{Fin}(\Omega))$ extending the filter $F_0 = \{U \subseteq \mathscr{P}_{Fin}(\Omega) \mid \exists A \in \mathscr{P}_{Fin}(\Omega) \forall B \supseteq A : B \in U\}$, before appealing to the Truth and Genericity Lemma. But this argument requires some amount of the Axiom of Choice, while the proof of Theorem 8.5.17 can be carried out entirely in ZF.

8.6 Upshot

In this final section, I return to the objections against numerosities and NAP functions discussed in Section 8.4. I will argue that the generic possibility structures introduced in the previous section allow one to solve the issues pointed out in Section 8.4 in a convincing fashion. I will first discuss the invariance problem, before addressing the strength vs. arbitrariness dilemma. Finally, I will also argue that the semiconstructive features of generic structures allow them to escape the scope of the complexity argument raised by Easwaran in [78].

8.6.1 The Invariance Problem

Recall that, in Section 8.4, I introduced two criteria which, according to Parker [205], any theory of size should respect, namely the Absolute Invariance and Relative Invariance Criteria. I also argued at length against the Absolute Invariance Criterion, but in favor of the Relative Invariance Criterion. Both numerosities and NAP functions fail to satisfy the Relative Invariance Criterion. As we shall now see, the situation is different for generic numerosities and generic NAP functions. We start with the following lemma.

Lemma 8.6.1. *Let Ω be an infinite set. For any $U, V \subseteq \Omega$, $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(U) \leq \pi(V)$ iff there is a finite set $C \subseteq V \setminus U$ such that $|U \setminus V| \leq |C|$.*

Proof. Suppose that $U, V \subseteq \Omega$ are such that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(U) \leq \pi(V)$. Since the smallest fine ideal J_0 is the root of the poset $\mathfrak{J}$, this is equivalent to $J_0 \Vdash \pi(U) \leq \pi(V)$, which by definition is equivalent to $||\pi(U) \leq \pi(V)||_{\mathscr{P}_{Fin}(\Omega)} \in F_{J_0}$. Hence we have to show that $\exists A \in \mathscr{P}_{Fin}(\Omega) \forall B \supseteq A : B \in ||\pi(U) \leq \pi(V)||_{\mathscr{P}_{Fin}(\Omega)}$ iff there is a finite set $C \subseteq V \setminus U$ such that $|U \setminus V| \leq |C|$. For the left-to-right direction, suppose that C is a finite set such that

$C \subseteq V \setminus U$ and $|U \setminus V| \leq |C|$. Now for any $B \supseteq C$ finite, we have that $U \setminus V \cap (B \cap (U \cap V)) = \emptyset$, and $B \cap U \subseteq U \setminus V \cup B \cap (U \cap V)$. Moreover, we have that $C \cap (B \cap (U \cap V)) = \emptyset$ and that $C \cup (B \cap (U \cap V)) \subseteq B \cap V$. Hence we have the following inequalities:

$$\begin{aligned} |B \cap U| &\leq |U \setminus V| + |B \cap (U \cap V)| \\ &\leq |C| + |B \cap (U \cap V)| \\ &\leq |B \cap V|. \end{aligned}$$

But then it follows that $\chi_U(B) \leq \chi_V(B)$ for all $B \supseteq C$ finite and thus that $||\pi(U) \leq \pi(V)||_{\mathscr{P}_{Fin}(\Omega)} \in F_{J_0}$.

For the converse direction, assume that there is $A \in \mathscr{P}_{Fin}(\Omega)$ such that for all $B \supseteq A$ finite, $\chi_U(B) \leq \chi_V(B)$. Note that this means that $|B \cap U| \leq |B \cap V|$ for all $B \supseteq A$ finite. Let $C = A \cap V \setminus U$. I claim that $|U \setminus V| \leq |C|$. Assume towards a contradiction that this is not the case, and let $D \subseteq U \setminus V$ be a finite set such that $|D| > |C|$. Let $B = D \cup A$. Note that $D \cap (A \cap (U \cap V)) = \emptyset$, and that $D \cup (A \cap (U \cap V)) \subseteq B \cap U$. Similarly, $C \cap (A \cap (U \cap V)) = \emptyset$, and $B \cap V \subseteq C \cup (A \cap (U \cap V)$. Hence we have the following inequalities:

$$\begin{aligned} |B \cap V| &\leq |C| + |A \cap (U \cap V)| \\ &< |D| + |A \cap (U \cap V)| \\ &\leq |B \cap U|, \end{aligned}$$

which implies that $\chi_U(B) > \chi_V(B)$, contradicting our assumption on A. Hence C is the required set. ☐

Lemma 8.6.1 gives us a complete characterization of which order relationships hold between probabilities of events according to the generic NAP function. An immediate consequence of this is the following.

Corollary 8.6.2. *Let Ω be an infinite set and $U, V \subseteq \Omega$. For any permutation $\alpha : \Omega \to \Omega$, $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(U) \leq \pi(V)$ iff $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(\alpha_*(U)) \leq \pi(\alpha_*(V))$.*

Proof. Since the inverse of a permutation is a permutation, it is enough to show that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(U) \leq \pi(V)$ implies that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(\alpha_*(U)) \leq \pi(\alpha_*(V))$. So suppose $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(U) \leq \pi(V)$. By Lemma 8.6.1, there is a finite set $C \subseteq V \setminus U$ such that $|U \setminus V| \leq |C|$. But then $\alpha_*(C) \subseteq \alpha_*(V) \setminus \alpha_*(U)$, and since $\alpha_*(U \setminus V) = \alpha_*(U) \setminus \alpha_*(V)$, we also have that $|\alpha_*(U) \setminus \alpha_*(V)| = |\alpha_*(U \setminus V)| \leq |\alpha_*(C)|$. But then by Lemma 8.6.1 again it follows that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(\alpha_*(U)) \leq \pi(\alpha_*(V))$. ☐

As a consequence, the generic NAP function for Ω satisfies the Relative Invariance Criterion. This is a major difference with standard NAP functions, and, if Parker and I are correct that the Relative Invariance Criterion should be satisfied by any theory of size or probability, it is a significant advantage of the generic NAP function over standard NAP functions. Of course, the key difference here is the fact that order relationships in the range of the generic NAP function are entirely determined by order-theoretic facts about $\mathscr{P}(\Omega)$, and are therefore preserved by permutations of Ω. By eliminating the need for a maximal ultrafilter in order to define our NAP function π, we also eliminated some of the properties of π that are not invariant under permutations of the sample space. We should note however that the Absolute Invariance Criterion is not satisfied by π, since π satisfies **Regularity**. But, as I argued in Section 8.4, this is too strong a criterion to impose anyway.

Generic NAP functions therefore pass the invariance test. Perhaps surprisingly, the same cannot be said of generic numerosities in the way I have defined them so far. A simple example will make that point clear. Let $2^{\mathbb{N}}$ be the set of all natural numbers that are powers of 2 and recall that $\mathbb{N}^2$ is the set of all squares. It is easy to verify that for any $n+1 \in \mathbb{N}$, $\sigma_{2^{\mathbb{N}}}(n+1) = \lfloor \log_2(n) \rfloor$ and $\sigma_{\mathbb{N}^2}(n+1) = \lfloor \sqrt{n} \rfloor$, and thus that $||\mathfrak{n}(2^{\mathbb{N}}) < \mathfrak{n}(\mathbb{N}^2)||_\omega$ is a cofinite set. But this implies that $\mathcal{G} \models \mathfrak{n}(2^{\mathbb{N}}) < \mathfrak{n}(\mathbb{N}^2)$. Since one can easily define a permutation α of $\mathbb{N}$ such that $\alpha_*(2^{\mathbb{N}}) = \mathbb{N}^2$ and $\alpha_*(\mathbb{N}^2) = 2^{\mathbb{N}}$, this shows that the generic numerosities do not satisfy the Relative Invariance Criterion. There is, however, an easy way to fix the issue, which I think reveals that the standard numerosities actually aim at capturing two distinct intuitions about size relationships between infinite collections. I will now elaborate on this point.

8.6.2 Part-Whole Principle and Density Intuition

Consider the following adjustment to the possibility structure of numerosities $\mathcal{G}$, inspired from the construction of the generic NAP function. Instead of taking the generic power of $\mathbb{N}$ determined by the set $\mathfrak{G}$ of all infinite sets of natural numbers or, equivalently, the set of all principal filters on ω, we could take the generic power of $\mathbb{N}$ determined by the set of all filters on $\mathscr{P}_{Fin}(\mathbb{N})$ extending the filter $G_0 = \{X \subseteq \mathscr{P}_{Fin}(\mathbb{N}) \mid \exists A \in \mathscr{P}_{Fin}(\mathbb{N}) \, \forall B \supseteq A : B \in X\}$, and define the numerosity of a set U as the map $\tau_U : \mathscr{P}_{Fin}(\mathbb{N}) \to \mathbb{N}$ given by $\tau_U(A) = |A \cap U|$ for any finite set A. In other words, instead of approximating the numerosity of an infinite set U by considering the sequence of approximations obtained by truncating U by increasingly large initial segments of the natural numbers, we would approximate the numerosity of U by considering its size when intersected with *any* finite set. It is easy to see the parallel with the construction of NAP functions, in which the probability of a set U is determined by conditionalizing on every finite subset of the sample space Ω.[4] Let $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N}))$ be the possibility structure thus obtained. For reasons that I will explain below, I will call this structure the semiring of *Euclidean numerosities*. We have the following theorem.

Theorem 8.6.3. *Let* $\mathfrak{m} : \mathscr{P}(\mathbb{N}) \to \mathbb{N}^{\mathscr{P}_{Fin}(\mathbb{N})}$ *be the map* $U \mapsto \tau_U$*. The Euclidean numerosities have the following properties:*

1. $\mathcal{G}$ *has the same first-order theory as* $\mathbb{N}$*. Moreover, the map* $\iota : \mathbb{N} \to \mathbb{N}^{\mathscr{P}_{Fin}(\mathbb{N})}$ *is such that* $\mathbb{N} \models \varphi(\overline{n})$ *iff* $\mathcal{G} \models \varphi(\overline{\iota(n)})$ *for any* $\mathcal{L}$ *formula* φ *and tuple of natural numbers* $\overline{n}$*.*

[4] In fact, this can be seen as the converse of Benci, Horsten and Wenmackers's choice to model a fair lottery on $\mathbb{N}$ using as Λ the set of all initial segments of $\mathbb{N}$ instead of $\mathscr{P}_{Fin}(\mathbb{N})$. In the case of de Finetti's Lottery, they choose to align NAP theory on Benci and di Nasso's original construction of numerosities, while I am describing a structure of numerosities that is aligned on the general setup of NAP functions.

2. *The map* $\mathfrak{m}$ *satisfies* ***(PW)***, *in the sense that* $\mathcal{G} \models \mathfrak{m}(U) < \mathfrak{m}(V)$ *whenever* $U \subsetneq V$;

3. *For any finite* $A \subseteq \mathbb{N}$, $\mathcal{G} \models \mathfrak{m}(A) = \iota(|A|)$;

4. *Whenever* $U \cap V = \emptyset$, $\mathfrak{m}(U \cup V) = \mathfrak{m}(U) \oplus \mathfrak{m}(V)$ *and* $\mathfrak{m}(U) \otimes \mathfrak{m}(V) = \mathfrak{m}(U \times V)$;

5. *For any permutation* $\alpha : \mathbb{N} \to \mathbb{N}$ *and any* $U, V \subseteq \mathbb{N}$, $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N})) \models \mathfrak{m}(U) \leq \mathfrak{m}(V)$ *iff* $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N})) \models \mathfrak{m}(\alpha_*(U)) \leq \mathfrak{m}(\alpha_*(V))$.

Items 1-4 follow from the fact that $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N}))$ is a generic power, and thus we can apply the Truth Lemma and the Genericity Lemma.[5] Item 5, by contrast, is established by observing that $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N})) \models \mathfrak{m}(U) \leq \mathfrak{m}(V)$ iff there exists a finite $C \subseteq V \setminus U$ such that $|C| > |U \setminus V|$, which is proved exactly like Lemma 8.6.1. Importantly, this means that, going back the example above, we do not have that $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N})) \models \mathfrak{m}(2^{\mathbb{N}}) < \mathfrak{m}(\mathbb{N}^2)$. It is easy to see why this is so. If it were the case that $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N})) \models \mathfrak{m}(2^{\mathbb{N}}) < \mathfrak{m}(\mathbb{N}^2)$, this would be because there is a finite set A such that, whenever $B \supseteq A$, $|B \cap 2^{\mathbb{N}}| < |B \cap \mathbb{N}^2|$. But since A is finite and $2^{\mathbb{N}} \setminus \mathbb{N}^2$ is infinite, we could keep adding elements of $2^{\mathbb{N}} \setminus \mathbb{N}^2$ to A until we have more elements of $2^{\mathbb{N}}$ than elements of $\mathbb{N}^2$. But this is only possible because we are including arbitrary finite subsets of ω in our approximation of the numerosity of sets. By contrast, when we define the numerosity of a set U via its approximating sequence $\sigma_U : \mathbb{N} \to \mathbb{N}$, we are only taking into consideration initial segments of the natural numbers.

It seems to me that these two ways of constructing numerosities indicate that there are actually two non-Cantorian conceptions of the size of infinite sets of natural numbers here that need to be disentangled. The first one is based entirely on the part-whole principle and, as such, is relatively invariant under permutations. I would argue that this is the conception that properly deserves to be called Euclidean and that it is faithfully modelled by the semiring of Euclidean numerosities. According to the Euclidean conception, size assignments should respect the part-whole principle. This is enough to determine size relationships between any two finite sets and between any two sets U and V for which the comparison between $U \setminus V$ and $V \setminus U$ can be reduced to a comparison between finite sets, or between one finite set and one infinite set. This is precisely captured in the Euclidean numerosities by the fact that $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N})) \models \mathfrak{m}(U) \leq \mathfrak{m}(V)$ iff there is a finite $C \subseteq V \setminus U$ such that $|U \setminus V| \leq |C|$. As shown by Theorem 8.6.3, the resulting structure has the rich algebraic properties of a semiring and is linearly ordered, provided of course that it is understood as a possibility structure rather than a Tarskian one.

The second conception of the size of infinite sets of natural numbers, by contrast, is based on what I will call the *Density Intuition*. According to the Density Intuition, the size of an

[5] It is worth mentioning that, compared to Corollary 8.5.5, there is no such property like item 5 holding for the Euclidean numerosities. There are two reasons for this. The first one is that, since we are not working with functions from $\mathbb{N}$ to $\mathbb{N}$, we would need to have a different notion of non-decreasing function to have an analogue of item 5. The second, more important reason, is that we are working here with all fine filters on $\mathscr{P}_{Fin}(\mathbb{N})$, rather than with infinite subsets ordered by almost inclusion. There is no principled reason to do this other than simplicity, so one could work with the latter instead of the former, and try to prove an analogue of item 5 for the corresponding possibility structure.

infinite set U should be a reflection of "how often" elements of U appear in the sequence of all natural numbers. A classical way of capturing this intuition is to use the standard notion of natural or asymptotic density, where the asymptotic density of a set $U \subseteq \mathbb{N}$ is defined as $A(U) = \lim_{n\to\infty} \frac{|U\cap\{0,\ldots,n-1\}|}{n}$. As is well known, not every set of natural numbers has an asymptotic density, and asymptotic density does not preserve the part-whole principle, since every finite set and even some infinite sets such as the set of prime numbers have density 0. But it also has some intuitive properties, such as the density of $n\mathbb{N}$ being $\frac{1}{n}$ for every natural number n. Importantly, it is also not permutation invariant: many permutations of the order of the natural numbers will induce a variation in the asymptotic density of many sets. This is not surprising, and in fact, one could even argue that this is a welcome feature from the viewpoint of the Density Intuition. Indeed, if the size of a set of natural numbers is determined by the distribution of its elements in the sequence of natural numbers, then a major disturbance of this sequence, such as the one induced by permutations that swap infinitely many natural numbers, will entail a change in the density of many infinite sets.

Standard numerosity functions, like those constructed by Benci and di Nasso, seem to me to adhere to both the Density Intuition and the Part-Whole Principle. From this perspective, it is natural to consider that the numerosity of a set U should be approximated not by considering the intersection of U with every finite set, as this would only guarantee that numerosities respect (**PW**), but by considering instead the intersection of U with every initial segment of $\mathbb{N}$, since those are the finite subsets of $\mathbb{N}$ that also contain information about the distribution of the elements of U in the sequence of natural numbers. That the proponents of numerosities often suggest to add conditions on Ramsey ultrafilters so that $\mathfrak{n}(\mathbb{N})$ becomes a multiple of $\mathfrak{n}(n\mathbb{N})$ for any natural number n is only more evidence that they are trying to preserve the Density Intuition on top of (**PW**). Because the semiring $\mathcal{G}$ of generic numerosities is a generic power of the standard numerosities, it has the properties that every Tarskian semiring of numerosities has. As such, it is also meant to preserve both (**PW**) and the Density Intuition, and this explains why it does not satisfy the Relative Invariance Criterion. It is worth mentioning that the argument that I gave for the Relative Invariance Criterion in Section 8.4 can be rebuked on the basis of the Density Intuition. For a proponent of the Density Intuition, there is more to the size relationships between sets of natural numbers than what can be inferred from the inclusion order on $\mathscr{P}(\mathbb{N})$. Indeed, the distribution of the elements of a set U along the sequence of natural numbers is not a feature of U that is captured by the relation of subset inclusion. Accordingly, the fact that a permutation α on $\mathbb{N}$ can be lifted to an order-isomorphism on $\mathscr{P}(\mathbb{N})$ is not enough to conclude that α_* should preserve size relationships between sets of natural numbers since, as mentioned before, α itself could disturb quite considerably the sequence of natural numbers. This suggests that numerosities that obey the Density Intuition should satisfy a weaker form of the Relative Invariance Criterion, according to which relative size relationships between sets should be invariant under permutations that also preserve the order on the underlying set. Because the identity is the only order-preserving permutation on the natural numbers, this weaker criterion is trivially satisfied both by standard numerosity functions and by generic numerosities. But let me quickly show how the generic approach can make a difference over standard numerosities in a slightly more general setting.

Suppose that we are interested in assigning sizes to sets of integers, rather than sets of natural numbers. If we want to respect both the Density Intuition and the Part-Whole Principle, it seems natural to consider that we should approximate the numerosity of a set $U \subseteq \mathbb{Z}$ by intersecting U with finite intervals of $\mathbb{Z}$ and considering the cardinality of the set $U \cap (n, m)$ for any $n < m \in \mathbb{Z}$. This means that we should be working with functions from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{N}$ and mapping every U to the function υ_U given by $\upsilon_U(n, m) = |U \cap (n, m)|$. Suppose now that, following the standard approach to numerosities, we take an ultrapower of $\mathbb{N}$ modulo a non-principal ultrafilter on $\mathscr{P}(\mathbb{Z} \times \mathbb{Z})$ and obtain a numerosity function $\mathfrak{n}$. The resulting structure, as observed by Parker [205, p. 18], will not satisfy even the weaker version of the Relative Invariance Criterion. Indeed, one can show that regardless or what size relationships $\mathfrak{n}$ assigns to $2\mathbb{N}$ and $\mathbb{N} \setminus 2\mathbb{N}$, the order-preserving permutation $n \mapsto n - 1$ will not preserve those size relationships. By contrast, suppose we take $\mathcal{E}$ to be the generic power of $\mathbb{N}$ determined by the poset of all filters on $\mathscr{P}(\mathbb{Z} \times \mathbb{Z})$ extending the filter of all cofinite subsets. By the Truth Lemma, we have that for any $U, V \subseteq \mathbb{Z}$, $\mathcal{E} \models \mathfrak{n}(U) \leq \mathfrak{n}(V)$ iff the set of pairs of integers (n, m) such that $|U \cap (n, m)| \leq |V \cap (n, m)|$ is cofinite. Now let $\alpha : \mathbb{Z} \to \mathbb{Z}$ be any order-preserving permutation. Since α is order-preserving, we have for any $X \subseteq \mathbb{Z}$ and any $n, m \in \mathbb{Z}$ that $|X \cap (n, m)| = |\alpha_*(X) \cap (\alpha(n), \alpha(m))|$. It follows that the set of integers n, m such that $|\alpha_*(U) \cap (n, m)| \leq |\alpha_*(V) \cap (n, m)|$ is also cofinite and hence that $\mathcal{E} \models \mathfrak{n}(\alpha_*(U)) \leq \mathfrak{n}(\alpha_*(V))$. Thus, unlike the standard numerosities approach via ultraproducts, the generic approach satisfies the version of the Relative Invariance Criterion which, as I have argued, is appropriate for a theory of size that wants to preserve both the Part-Whole Principle and the Density Intuition.

Finally, let me conclude on this topic by returning to Parker's arguments against Euclidean theories of size. Parker [205] shows that any Euclidean theory of size satisfying two conditions that he calls Totality and Discreteness must violate the Relative Invariance Criterion. However, these conditions are satisfied both by the Euclidean numerosities $\mathcal{G}(\mathscr{P}_{Fin}(\mathbb{N}))$ and the generic power $\mathcal{E}$ defined above, even though both also satisfy a version of the Relative Invariance Criterion. This is not in contradiction with Parker's result, but there is a subtelty worth highlithing here. Parker's result applies to Tarskian structures, while we are working with possibility structures. The possibility structures we define will satisfy the statement that the numerosities of any two sets are comparable. But, in general, they will not settle which set is bigger than the other, which is what Parker needs for his proof to go through. This shows that the resources afforded to us by considering possibility structures instead of Tarskian structures allow us to escape some established impossibility results such as Parker's. Of course there is a price to pay in abandoning the simplicity of Tarskian semantics, but the incompatibility of the Relative Invariance Criterion with a Tarskian understanding of disjunctions in the statement of the linearity property is arguably a convincing reason to do so.

8.6.3 Strength and Canonicity

The second main objection to numerosity and NAP functions that was discussed in Section 8.4 was that these functions had several arbitrary features, due in part to the need to rely on non-constructive maximal objects to define them. Here, the fact that all the possibility structures that I have presented so far can be defined constructively seems like a good indication that generic numerosity and NAP functions do not face the same issue. In both cases, because we approximate a maximal object such as a Ramsey ultrafilter or a maximal fine ideal by the poset of viewpoints, we avoid the arbitrariness coming from the choice of a specific maximal object. This is reflected by the fact that satisfaction on the generic structure coincides with satisfaction in *every* corresponding Tarskian structure, a direct consequence of the Genericity Lemma. In other words, the size relationships determined by the generic numerosity function are precisely those size relationships that hold according to every standard numerosity function, and the same holds for generic NAP functions and standard NAP functions regarding assessments about the relative likelihood of events in an infinite lottery. In the latter case, I would also argue that generic NAP functions offer a more faithful representation of genuinely fair lotteries on infinite sets than their Tarskian counterparts. Indeed, as I argued in Section 8.4 already, the choice of a particular maximal fine ideal on $\mathcal{F}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$, or, equivalently, of an ultrafilter on $\mathscr{P}_{Fin}(\Omega)$, arguably entails choosing some subsets of $\mathscr{P}(\Omega)$ over some others. As we have seen, the proponents of NAP functions agree that the choice of such an ultrafilter introduces an element of arbitrariness but argue that the ultrafilter should be thought of as an extra parameter and that, accordingly, the arbitrariness induced by the choice of an ultrafilter is no worse than the arbitrariness of the choice of a σ-algebra of events in classical probability theory. But the approach to infinite fair lotteries via generic NAP functions requires neither fixing a σ-algebra of events nor choosing a particular ultrafilter. Consequently, there is no extra parameter that we need to fix when modelling a fair lottery on an infinite set. The situation can instead be modelled by a generic NAP function that is only determined by the infinite set Ω.

Of course, the price to pay for this is that the codomain $\mathcal{H}(\mathfrak{J}, \mathbb{R})$ of a generic NAP function is a possibility ordered field, rather than a Tarskian ordered field. As such, it is a field and it is linearly ordered, but not in the usual sense. The ordinary computations that one performs with fields can still be carried out on $\mathcal{H}(\mathfrak{J}, \mathbb{R})$, but one must exert some caution in doing so, as we have to reason *internally*, i.e., always relative to a specific viewpoint. But there are reasons to argue that this is in fact a welcome feature of generic NAP functions (and the same could be said about generic numerosities), because it solves in an original and satisfactory way the dilemma between strength and arbitrariness. Recall that Easwaran's suggestion of working with a qualitative (i.e., non-numerical) ordering of events that would extend both the "strictly less than in probability" and the "strictly including in" relations on the powerset of an sample space Ω faced the issue that the order thus proposed was not enough to justify the preferences that any rational agents should have regarding certain bets on an infinite lottery. It is easy to see that we do not suffer from such issues when we consider the generic NAP function for Ω. Consider again the case of a lottery on the natural numbers and a choice being offered by a bookmaker between betting that the winning ticket will be even or that it will be in the set $U = 2\mathbb{N} \setminus \{0\} \cup \{1, 3\}$. Using Lemma 8.6.1, we can easily see that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(2\mathbb{N}) < \pi(U)$. Moreover, since we are working with a structure on which addition and subtraction are defined, we can establish that a fair price to switch a bet on a set X for a bet on a set Y is precisely $\pi(Y) - \pi(X)$. Recall that the other example we discussed was between paying some price ϵ to switch from a bet on $2\mathbb{N}$ to a bet on U, and paying a price of 3ϵ to switch from a bet on $2\mathbb{N}$ to a bet on $V = 2\mathbb{N} \cup \{1\}$. Using the Truth Lemma for generic powers, it is easy to see that $\mathcal{H}(\mathfrak{J}, \mathbb{R}) \models \pi(V) - \pi(2\mathbb{N}) = 2(\pi(U) - \pi(2\mathbb{N}))$ and thus to conclude that a rational agent should prefer to switch her bet on $2\mathbb{N}$ for a bet on U at a price ϵ over switching for a bet on V at a price of 3ϵ. Arguably, the ranges of generic NAP functions therefore have enough structure to allow for more decision-theoretic arguments than the mere qualitative approach advocated by Easwaran.

At the same time, the fact that they are possibility structures means that we can escape the threat of arbitrariness. We have seen already, in the context of numerosities, that the fact that Euclidean and generic numerosities were totally ordered only in the sense of possibility semantics allowed us to escape Parker's incompatibility results between linearity, preservation of the Part-Whole Principle and Relative Invariance. A similar point could be made regarding generic NAP functions. Although possibility fields satisfy linearity in the sense that $a \leq b \vee b \leq a$ is always satisfied for any two elements a, b, they do not in general decide which disjunct holds. This allows us to model a fair lottery on an infinite set in such a way that the probability of any two events is always comparable, even though which event is more likely than another might not be determined. In other words, this opens up the possibility that there might be some questions regarding an infinite fair lottery that do not have a determinate answer, even when such a lottery is faithfully and exhaustively represented. Whether the winning ticket being odd is equally as likely as it being even is such an example. What is however, determinately true about such a situation, is that probabilities are always linearly ordered, and that any event has a probability of happening.

To put the point differently, one could argue that the concepts of a Euclidean notion of size for sets of natural numbers, or of a fair lottery on an infinite set Ω, entirely determine a possibility structure, but not a specific Tarskian structure. In the former case, it is part of the concept of a Euclidean theory of size that any set has a size, that sizes are linearly ordered, and that all sets have size strictly greater than their proper subsets. One could even go further and argue that sizes come with a natural algebraic structure that satisfies the axioms of a linearly ordered semiring. But none of this commits us to the view that the Euclidean notion of size entails that there must be as many evens as odds, strictly more squares than powers of 2, or twice as many multiples of 4 as there are multiples of 2, although all of these arguably follow from the Density Intuition. Again, it is consistent to hold that our concept of a Euclidean theory of size entails that any two sets must have comparable sizes but not, in general, what the exact size relationship is, just like the concept of a bear cub entails that the cub is either male or female, but does not entail one of the two in particular. Consequently, a mathematical representation of Euclidean sizes that is both faithful and exhaustive will satisfy some disjunctions without satisfying any of the disjuncts; and this is precisely where the resources of possibility semantics become relevant. A similar point can be made about fair lotteries on infinite sets. A full and faithful representation of such a situation must involve a function whose domain is an ordered field, because it is part of our concept of probability that probabilities should have enough algebraic structure to enable the kind of reasoning that involves adding, multiplying and ordering them in a similar way as we do with real numbers. At the same time, every event in a probability space must be assigned a probability, but these two facts alone are not enough to conclude that the specific numerical or order-theoretic relationships between the probabilities of any two events must always be determined. Here again, possibility structures allow us to escape the dilemma between strength and arbitrariness. In order to work with a structure that has the resources of an ordered field, we do not need to impose some extra arbitrary structure on a fair lottery in the form of a non-principal ultrafilter. We can instead work with a possibility structure rather than a Tarskian one.

As I have argued, the fact that the semiring of generic numerosities and the ordered field of generic NAP functions are possibility structures allows us to construct numerosity and probability functions that enjoy some robust mathematical properties without having features that cannot be motivated by the concepts of a Euclidean theory of size or of a fair lottery on an infinite set respectively. There is however, an alternative that the proponent of Tarskian numerosities and NAP functions could consider, namely to argue that their "real" solution involves quantifying over *all* the structures they define. Benci, Horsten and Wenmackers seem to adopt such a move when pushing back against Easwaran's claim that hyperreal-valued probability functions such as NAP functions introduce some arbitrary structure:

> One might think that ignoring relevant existing structure (a sin of omission) is not as grave as adding structure (a sin of misinformation). However, it has to be borne in mind that one can always consider the entire family of NAP functions modelling a given situation, rather than an–arbitrary–representative of it (see also Wenmackers and Horsten [2013]). Such a family is the set of all NAP functions that meet a common specification, such as 'a fair lottery on $\mathbb{R}$', which fixed the sample space and the weight function, and possibly additional constraints on the directed set. As a whole, the family shows us how much the probabilities of a given event, and the order of probabilities of multiple events, can vary (dependent on the choice of ultrafilter).
>
> To put it differently, there may be multiple, equally good ways to model the same situation, corresponding to different choices of the ultrafilter. What matters is what is true (or false) on all ways of making these arbitrary choices–what is supertrue (or superfalse)–as well as the spread of possible assignments. We need not project these arbitrary choices onto what is being modelled. [20, p. 544]

Parker also discusses a similar option in the context of numerosities:

> Another possible objection is that the Euclidean theories on offer *don't* determine sizes arbitrarily; they leave sizes indeterminate where well motivated principles do not decide them, and this is just what they should do. Indeed, the [theory of numerosities] and others do not determine the sizes of all sets, or even all sets of whole numbers. However, those theories are *about* Euclidean assignments that *are* total over some broad class of sets. [205, p.10]

The idea here is that the various Tarskian structures that one considers, such as numerosity and NAP functions, should not be taken to be the actual proposal of the advocate of numerosities and NAP theory respectively. Rather, one should supervaluate over all acceptable numerosity and probability assignments, and this is the theory whose merits should be judged. I will call this view the *supervaluational strategy*. It seems to me that this view is getting really close to the *generic approach* that I have advocated for so far but that there are nonetheless significant differences that tilt the scale in favor of possibility structures.

First, let me start by noting an obvious connection between generic structures and the supervaluational approach. In the case of numerosities as well as NAP functions, satisfaction on the generic structure coincides with satisfaction in every corresponding Tarskian structure. Consequently, as *theories*, i.e., sets of first-order sentences, the generic approach agrees entirely with the supervaluational strategy. But there are, however, at least three significant differences worth mentioning. The first one is that supervaluations admit of only three possible semantic values for sentences: (super)truth, (super)falsity, and "neither" or "indeterminate" truth-value. By contrast, possibility structures are Boolean-valued: they admit a lattice of semantic values that is as rich as the Boolean algebra of regular open subsets of the underlying poset of viewpoints. One could, of course, do the same thing for supervaluations, and associate to every first-order sentence the set of all Tarskian structures that satisfy it as its "semantic value". But supervaluations in themselves lack the machinery to exploit this more fine-grained notion of semantic value. By contrast, the Boolean algebra of semantic values is naturally built into the possibility structure and represented by the regular open subsets of the poset of viewpoints. As such, we can always move between the "local" and "global" viewpoints on a possibility structure, i.e., between satisfaction at a specific viewpoint and satisfaction at every viewpoint.

A second difference between the generic approach and the supervaluational strategy is that the first one offers a *bona fide* structure, while the second one only offers a theory. Faced with a question like "What is the probability that a randomly picked ticket in a lottery on the natural numbers is even", the advocate of the generic NAP function can give a specific answer, namely, $\pi(2\mathbb{N})$. She may point out that not every question regarding such a value can be answered with a definite yes or no, but for any number of such statements, she will in principle always be able to tell whether they are consistent, whether they exhaust the space of possibilities, and which ones entail which others. By contrast, the advocate of the supervaluational strategy cannot answer the question above by pointing to a specific mathematical object. She might either reply that the answer depends on which ultrafilter one uses to model such an infinite lottery or that she can give a range of possibilities like $\frac{1}{2} \pm \epsilon$ for some infinitesimal ϵ (and, there again, she cannot point to what specifically such an infinitesimal is, as those according to her are equivalence classes modulo an ultrafilter that needs to be specified), or that, although she cannot say what the probability that the number is even is, she can, in principle, assess for any given first-order statement about that probability whether it is true, false or indeterminate. It seems to me that the advocate of the supervaluational strategy here is simply not offering enough. In the approach via possibility structures, one works with well-defined mathematical objects. Of course, one must be careful to manipulate them according to the specific rules of possibility semantics, but one can manipulate them nonetheless. In the supervaluational approach, one is left with a cluster of objects, none of which, according to the theory itself, accurately represents the situation one intends to model, and a rudimentary way of talking about the properties that all such objects share.

Finally, a third difference between the generic approach and the supervaluational strategy is that the supervaluational strategy relies on the existence of non-constructive entities such as the ultrafilters needed to define semirings of Tarskian numerosities and the maximal fine ideals needed to obtains the codomains of NAP functions, while the generic is much more parsimonious in its use of the Axiom of Choice. As we have seen before, the non-constructive character of Tarskian numerosity and NAP functions plays a role in their exhibiting arbitrary features, and the point of the supervaluational approach is precisely to "wash out" all such arbitrary features by quantifying over all the relevant functions. However, the nonconstructive nature of NAP functions creates another problem for them, in the form of the complexity argument put forward by Easwaran in [78]. I will conclude this section by rehashing Easwaran's argument, and argue that, regardless of its merits, the fact that possibility structures allow us to escape its grip is another virtue of the generic approach over the supervaluational strategy.

8.6.4 Easwaran's Complexity Argument

Eswaran's argument is intended as an objection against any probability theory that appeals to hyperreal-valued probability functions. By a hyperreal structure, Easwaran means a non-Archimedean ordered field into which the reals embed. Since the ranges of NAP functions have these properties, NAP theory falls within the scope of the argument, even though Easwaran himself doesn't mention NAP explicitly in [78]. The goal of the argument is to establish that the actual credences of physical agents cannot be faithfully represented by hyperreal-valued probability functions, because such functions are too complex and must therefore introduce more mathematical structure than the actual credence function of physical agents can possibly have. Here is how Easwaran presents the argument:

> The premises and conclusion of the argument are as follows:
>
> 1. Credences supervene on the physical, in the sense that there is a function that takes as input a complete mathematical description of the physical

world, and a specification of an agent and a proposition, and returns as output the number representing the credence of the agent in that proposition.

2. The function relating credences to the physical is not so complex that its existence is independent of Zermelo-Fraenkel set theory (ZF).
3. All physical quantities can be entirely parameterized using the standard real numbers.
4. The existence of a function with standard real number inputs and hyperreal outputs is independent of ZF.
5. Therefore, credences in ordinary propositions (ones expressible without mention of hyperreals or closely related notions) do not have hyperreal values. [78, p. 29]

Easwaran admits that premises 1 and 3 may be controversial but argues that they are at least plausible. Accordingly, rejecting either premise cannot be done on the sole basis of a commitment to the thesis that credence functions should be regular, but would rather require "doing serious physics, or philosophy of mind"(p. 30). He also points out (footnote 31, pp. 31-32) that, even if premise 3 were false and some physical quantities do in fact need to be parametrized using hyperreal structures, the existence of regular credence functions representing some probabilistic scenarios about such quantities would still require the existence of finer mathematical structures, i.e., "hyper-hyperreals", which might mean that a variant of his argument could still go through.[6] Premise 2, on the other hand, is meant to be understood as a weak version of the "physical Church-Turing thesis", according to which any physical process could be in principle simulated by a Turing machine. Easwaran's premise 2 is much weaker, since it only amounts to the thesis that any physical process could be in principle simulated by a mathematical object whose existence can be proved in ZF, i.e., without using the Axiom of Choice.

Regardless of how controversial one might think the first three premises of Easwaran's argument are, I think that the overall argument makes the fairly convincing point that, when it comes to credences, the complexity of the credence function of a rational agent should not exceed the complexity of the probabilistic scenarios that they consider. The argument aims to show that it is impossible for actual physical agents to have hyperreal-valued credences, because hyperreal-valued functions are too complex to be the sort of processes that can happen in nature. Importantly, Easwaran makes a distinction between the added complexity of hyperreal-valued structures and the idealization that is common in Bayesianism as well as in many scientific theories:

[6] There are reasons to doubt, however, that this would actually be the case. As Easwaran notices, the existence of a hyperreal structure is enough to derive the existence of a non-principal ultrafilter over ZF, which in turn is enough to construct nontrivial ultrapowers of any first-order structure. In other words, there could still be a "mismatch" in this situation between the complexity of the mathematical structures arising in nature and the complexity of regular crendence functions about those structures, but ZF would already be strong enough to derive the existence of the latter from that of the former.

> Although Bayesianism concerns itself with idealized rational agents, and not the imperfect physical beings we encounter in our daily life, I claim that the essentially nonphysical nature of agents with hyperreal credences makes them irrelevant for the epistemology of physical agents. The other idealizations, of logical omniscience and the like, are not physically impossible, and we can make sense of a way in which actual imperfect agents might become more and more like these idealized agents. These idealizations are like the ones from physics involving frictionless surfaces, and infinitely deep water for waves to travel on. But where these idealizations involve the removal of some limitation, the hyperreals involve the addition of nonphysical structure. [78, pp. 28-29]

I think that Easwaran's distinction makes sense in this context, particularly for someone who adheres to the supervaluational strategy mentioned above. For the supervaluationally-minded NAP theorist, NAP functions do not represent the credences of an ideal rational agent. The credences of an ideal rational agent, instead, should correspond to what is true according to *all* NAP functions of a certain kind. Thus, the supervaluationally-minded NAP theorist cannot reply to Easwaran's argument that the complexity of NAP functions is just a consequence of the fact that we are only considering ideal rational agents. Rather, she would probably concede to Easwaran that NAP functions are a flawed representation of the credences of rational agents and that the additional structure imposed on their codomains does not track a genuine feature of the credences of neither actual nor ideal physical agents.

By contrast, the generic approach allows one to resist Easwaran's complexity argument by rejecting premise 4. As such, unlike any reply that would reject one of the first three premises, it is a purely *mathematical* reply to Easwaran's challenge. Of course, Easwaran's premise 4 is correct if by a "hyperreal-valued function", one means a function whose codomain is a Tarskian hyperreal field. As noted in the previous chapter, the existence of a Tarskian non-Archimedean field into which the reals embed is enough to derive the existence of a nonprincipal ultrafilter on ω. But this construction breaks down if one is instead working with a possibility field. In fact, one can define the codomain of a generic NAP function for a sample set Ω entirely constructively, and similarly for the function itself.

There is, however, one subtlety on this issue that deserves to be discussed in more detail. As mentioned at the end of Section 8.5, given a sample space Ω, some of the properties of the possibility field $\mathcal{H}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ can only be proved to hold under some amount of choice. In particular, proving that $\mathbb{R}$ elementarily embeds into $\mathcal{H}(\mathscr{P}_{Fin}(\Omega), \mathbb{R})$ requires $AC_{|\Omega|}$ to hold. But there is an essential difference here between this feature of generic NAP functions and the nonconstructiveness of standard NAP functions. In the case of standard NAP functions, defining them requires constructing first their codomain, the existence of which requires one to appeal to the Axiom of Choice. In the case of generic NAP functions, their codomains can be proved to exist without any appeal to the Axiom of Choice. As established by Theorem 8.5.17, one does not need any amount of choice to show that the generic ordered field induced by an ideal on a ring of functions does indeed satisfy the axioms of an on ordered field. In other words, the *existence* of the possibility field can be secured constructively, and it is only some *properties* of that structure that require some amount of choice to be proved. But this means that generic NAP functions do not fall prey to Easwaran's complexity argument any more than real-valued probability functions. Indeed, it is well known that the reals can be a very pathological structure for measure theory in the absence of the Axiom of Choice. For example, if $AC_{\aleph_0}$ fails, $\mathbb{R}$ could be a countable union of countable sets. For this reason, classical analysis and measure theory typically assume some amount of choice, usually in the vicinity of the Axiom of Dependent Choices. But Easwaran certainly would not take his argument to establish that the credences of rational agents cannot be real-valued, because the existence of many real-valued functions can be proved constructively, even though probability theory typically assumes that their codomain, i.e., the reals, has some properties that can only proved using some amount of choice. For this reason, the fact that the codomains of generic NAP functions are better behaved under some amount of choice does not mean that generic NAP functions are too complex to adequately represent the credences of actual physical agents.

One can therefore conclude that the generic approach to NAP functions allows for a much more convincing rebuke of Easwaran's argument that the supervaluational strategy. Indeed, the proponent of the generic approach can fully agree with Easwaran that credence functions whose existence requires the Axiom of Choice are a fundamentally flawed way of representing the credences of actual physical agents, without giving up the claim that the credence function of a rational agent is (or should be) regular. Interestingly, one could argue that the generic approach also fares better on this issue than the non-regular approach to de Finetti's lottery. Indeed, recall that de Finetti argued that the credences of a rational agent regarding a lottery on the natural number should determine a real-valued function that assigns probability 0 to every finite set of natural numbers. However, Lauwers [170] showed that no such finitely additive probability measure defined on $\mathscr{P}(\mathbb{N})$ can be constructively proved to exist. Hence, Easwaran's argument, if correct, would establish that any probability function defined on the whole of $\mathscr{P}(\mathbb{N})$ and assigning probability 0 to every finite set of natural numbers is too complex to represent the rational credences of physical agents. Consequently, a proponent of real-valued probability functions would be faced with an uneasy dilemma regarding the credences of a rational agent about de Finetti's lottery, for they must either argue that such an agent would either not assign any probability of occurring to many events, or assign strictly positive probability of being the winning ticket to some numbers, and strictly lower probability to some others. But the latter option does not seem viable, since the lottery is assumed to be fair, while the first option entails that not even an *ideal* rational agent could assign a probability of containing the winning ticket to every set of natural numbers. Here again, the constructive features of possibility structures seem to be a significant advantage of the generic approach over its non-constructive competitors.

8.7 Conclusion

Let me conclude this chapter by summarizing its main contributions. Both in the case of the debate between Cantorian and Euclidean notions of size of infinite sets and the debate between non-Archimedean probability theory and real-valued probability theory, I have argued that the generic approach to numerosities and NAP functions respectively allowed us to evade the invariance problem and the dilemma between strength and arbitrariness. Because their domains are possibility structures, generic numerosity and NAP functions have *just enough* properties to be fruitful mathematical implementations of the pre-theoretic intuition that the whole should be greater than any of its proper parts without exhibiting arbitrary or non-robust features. As such, I would argue that this shows that one can develop meaningful alternatives to the Cantorian notion of size and classical probability theory, provided that one is willing to work with possibility structures. At least in the case of probabilities defined on infinite sample spaces, possibility structures seem to be a more powerful and flexible tool than supervaluations over Tarskian structures. This suggests that one could also investigate applications of possibility semantics as a way to model imprecise probabilities in other contexts.

Moreover, the possibility structures introduced here play a signficant role in distinguishing several non-Cantorian conceptions of the infinite. As argued in the previous section, Benci and di Nasso's numerosities are best understood as a hybrid between two distinct intuitions, the properly Euclidean idea that the whole should always be strictly greater than its proper parts on the one hand and the Density Intuition that the size of sets of natural numbers somehow reflects the distribution of their elements in the sequence of natural numbers. A convenient way of disentangling the two conceptions is to consider the invariance constraints on size assignment that one can justify on the basis of each conception. Tarskian numerosities fail to satisfy these invariant constraints, but generic and Euclidean numerosities do, and, as such, can be seen as genuinely distinct options for an alternative notion of size of infinite sets. Although I have not explored this point in detail in this chapter, it seems to me that the strictly Euclidean approach is also a better candidate for such an alternative to the Cantorian notion. One obvious issue with the Density Intuition is that it only applies to ordered sets, while the strictly Euclidean approach isn't as limited in its scope. Moreover, it seems that more work would be needed to clarify exactly what properties of numerosity functions could be justified on the basis of the Density Intuition. By contrast, it seems to me that, at least in the case of countable sets, Lemma 8.6.1 offers a clear characterization of the order relationship between sets that is induced by the Euclidean conception and that this arguably makes it closer to being on a par with the Cantorian one. I will conclude this chapter by elaborating slightly on this point. As mentioned before, a key difference between the Bijection Principle and the Part-Whole Principle is that the former can be used to derive necessary and sufficient conditions for size relationships between sets: For any two sets A and B, $|A| \leq |B|$ iff there is an injection from A to B. As is well known, the statement that the order thus defined is linear is equivalent to the Axiom of Choice. I think one could draw a similar picture for the Euclidean infinite and the numerosity of countable sets. Indeed, in light of Lemma 8.6.1, one can define an order on the subsets of a countable set by letting $A \prec B$ iff there exists a finite $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |C|$. In this case, the linearity of the order thus defined does not depend on the Axiom of Choice but rather on interpreting the underlying structure as a possibility structure rather than a Tarskian one. Whether this idea could be developed further into a full-fledged Euclidean conception of the infinite beyond the realm of the countable will be left for future work.

Chapter 9

Bolzano's Mathematical Infinite

9.1 Introduction

Bernard Bolzano (1781-1848) was a Bohemian priest with eclectic interests ranging from logic and mathematics to political and moral philosophy. One of his more famous writings is a booklet his pupil Příhonský published under the title *Paradoxien des Unendlichen* (from now on PU for short), Paradoxes of the Infinite. Likely contributing to its fame, this booklet was read and referred to by both Cantor and Dedekind. Perhaps because of this association, the booklet is also routinely interpreted as a text anticipating several ideas of Cantor's transfinite set theory (*cf.* [26, 27, 238, 226]), especially in sections §§29-33, in which Bolzano sketches a "calculation of the infinite". As a consequence, appraisal of the PU is almost exclusively conducted in terms of how much Bolzano's work on the infinite agrees with later developments in set theory. In particular, many shortcomings of Bolzano's calculation of the infinite are attributed to his adherence to the *Part-Whole Principle*:

PW For any sets A, B, if $A \subsetneq B$, then $size(A) < size(B)$.

As we discussed extensively in the previous chapter, the privileged status of the Bijection Principle has started to be scrutinized in recent years thanks to a renewed interest in potential alternatives to Cantor's theory of the mathematical infinite. In particular, Mancosu [184] shows that there is a long historical tradition of thinkers and mathematicians who favored **PW** over the bijection principle. Together with the recent development of mathematical tools that allow for precise formalizations of such alternatives, this suggests that a reappraisal of alternative theories that until recently had been dismissed as essentially misguided or inconsistent might be a valuable endeavor.

Our main goal is to offer such a reappraisal of Bolzano's mature theory of the mathematical infinite. In particular, we propose an interpretation of Bolzano's calculation of the infinite in §§29-33 of the PU which stresses its conceptual and mathematical independence from set theory proper, and argue that Bolzano is more interested in developing a theory of infinite sums rather than a way of measuring the sizes of infinite collections. This leads us to reassess the role that part-whole reasoning plays in Bolzano's computations and to provide formal reconstructions of his position that underscore its coherence and originality, and offer overall a more charitable appreciation of Bolzano's ideas on the infinite. In particular, we show that Bolzanian sums in our interpretation form a non-commutative ordered ring, a well-behaved algebraic structure that nonetheless vastly differs from Cantorian cardinalities. In order to keep the conceptual difficulties of our endeavor separate from technical complexities, we will start with a first approximation of such a reconstruction via standar model-theoretic techniques. This formalization of Bolzano's theory via Tarskian ultrapowers will be essentially enough to make the main points we intend to make. However, we will also highlight several issues with this first approach, and offer a second one using generic powers.

We proceed as follows. In Section 9.2 we discuss several sources of what we call the received view of the *PU*, and introduce enough background to set the stage for our novel interpretation. In Section 9.3 we focus on Bolzano's calculation of the infinite and argue that his work is best understood as a theory of infinite sums. This leads in Sections 9.4 and 9.5 to the first formal reconstruction of Bolzano's computations with infinite quantities, which aims to establish both the consistency and the originality of his position. In Section 9.6, we recap the main points of our formalization and discuss its implications for the interpretation of the *PU*. Finally, in Section 9.7, we address investigate further some of the issues that arise with the first formalization via Tarskian structures, and we show how generic powers permit to address those issues and to offer an alternative formalization of Bolzano's calculation of the infinite that is arguably more faithful to his writings.

9.2 The Received View on the *PU*

Bolzano's *PU* is a short yet ambitious booklet in which the author aims to show that, when properly defined and handled, the concept of the infinite is not intrinsically contradictory, and many paradoxes having to do with the infinite in mathematics (but also in physics and metaphysics) can actually be solved. In the course of addressing the paradoxes of the infinite in mathematics, Bolzano develops what looks like a theory of transfinite quantities (§§28-29, 32-33), which is what commentators tend to focus on when appraising the contents of the *PU*.

One such commentator is, as is known [238, 226, 91], Georg Cantor. Cantor [53] introduces Bolzano as a proponent of actual infinity, and specifically actually infinite numbers in mathematics, in contrast to Leibniz's arguments against infinite numbers:

> Still, the actual infinite such as we confront for example in the well-defined point sets or in the constitution of bodies out of point-like atoms [...] has found its most authoritative defender in Bernard Bolzano, one of the most perceptive philosophers and mathematicians of our century, who has developed his views on the topic in the beautiful and rich script *Paradoxes of the Infinite*, Leipzig 1851. The aim is to prove how the contradictions of the infinite sought for by the sceptics and peripatetics of all times do not exist at all, as soon as one makes

> the not always quite easy effort of taking into account the concepts of the infinite according to their true content. ([53] in [0, p. 179])[1]

And still:

> Bolzano is perhaps the only one who confers a certain status to actually infinite numbers, or at least they are often mentioned [by him]; nevertheless I completely and wholly disagree with the way in which he handles them, not being able to formulate a proper definition thereof, and I consider for instance §§29-33 of that book as untenable and wrong. For a genuine definition of actually infinite numbers, the author is lacking both the general concept of *power*, and the accurate concept of *number*. It is true that the seeds of both notions appear in a few places in the form of special cases, but it seems to me he does not work his way through to full clarity and distinction, and this explains several contradictions and even a few mistakes of this worthwhile script. (*ibid.*, p. 180)[2]

Cantor's comments in many ways set the tone of how the *PU* are mainly perceived even today, namely as a rich and interesting essay that nevertheless displays some serious shortcomings. Cantor diagnoses Bolzano's mistakes as being fundamentally due to an imprecise characterization of *power* and *number*. Without entering a discussion on Cantorian powers, it is useful for us to notice how Cantor is readily reinterpreting Bolzano's text in the light of his own research. The concept and terminology of *powers* was Cantor's own, which he introduced starting from 1878 in his papers. What Cantor means is that Bolzano did not have the right notion of size for infinite sets, the right notion being Cantor's own powers, and this shortcoming causes Bolzano to go astray in §§29-33. Another aspect of Cantor's comments on the *PU* which we want to stress is that Cantor straightforwardly presents Bolzano's "calculation of the infinite" (*Rechnung des Unendlichen*, §28) as a version of his own transfinite arithmetic, albeit imprecise and imperfect.

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[1] In this and all other cases for which a published English translation is not cited, the translations are the authors'. Original German:
Doch den entschiedensten Verteidiger hat das Eigentlich-unendliche, wie es uns beispielsweise in den wohldefinierten Punktmengen oder in der Konstitution der Körper aus punktuellen Atomen [...] entgegentritt, in einem höchst scharfsinnigen Philosophen und Mathematiker unseres Jahrhunderts, in Bernard Bolzano gefunden, der seine betreffenden Ansichten namentlich in der schönen und gehaltreichen Schrift: ,,Paradoxien des Unendlichen, Leipzig 1851" entwickelt hat, deren Zweck es ist, nachzuweisen, wie die von Skeptikern und Peripatetikern *aller Zeiten* im Unendlichen gesuchten Widersprüche gar nicht vorhanden sind, sobald man sich nur die freilich nicht immer ganz leichte Mühe nimmt, die Unendlichkeitsbegriffe allen Ernstes ihrem wahren Inhalte nach in sich aufzunehmen.

[2] Bolzano ist vielleicht der einzige, bei dem die eigentlich-unendlichen Zahlen zu einem gewissen Rechte kommen, wenigstens ist von ihnen vielfach die Rede; doch stimme ich gerade in der Art, wie er mit ihnen umgeht, ohne eine rechte Definition von ihnen aufstellen zu können, ganz und gar *nicht* mit ihm überein und sehe beispielsweise die §§29-33 jenes Buches als haltlos und irrig an. Es fehlt dem Autor zur wirklichen Begriffsfassung bestimmt-unendlicher Zahlen sowohl der allgemeine *Mächtigkeitsbegriff*, wie auch der präzise *Anzahlbegriff*. Beide treten zwar an einzelnen Stellen ihrem Keime nach in Form von Spezialitäten bei ihm auf, er arbeitet sich aber dabei zu der vollen Klarheit und Bestimmtheit, wie mir scheint, *nicht* durch, und daraus erklären sich viele Inkonsequenzen und selbst manche Irrtümer dieser wertvollen Schrift.

All commentaries on the *PU* we were able to find seem to follow suit from Cantor in that they evaluate and interpret the *PU*, and §§29-33 in particular, against the backdrop of the development of set theory. Thus Bolzano's *PU* are about infinite sets according to editors and translators of Bolzano's text (e.g., Hans Hahn in [47], Donald Steele in [0]), as well as scholars such as Berg [27, 26], Šebestík [238, 237], Lapointe [167], Ferreirós [91] and Rusnock [226]. We now examine the most informative of these interpretations in some detail.

Among Bolzano scholars, Jan Berg is perhaps the one that embraces a set theoretic reading of Bolzano with the most conviction. Berg [26, p. 176] writes:

> In *PU* [...] Bolzano repudiates the notion of equivalence as sufficient condition for the identity of powers of infinite sets. [...] As a result, a number of statements follow which do not correspond to Cantor's view on this subject. E.g., if "N_0" denotes the number of natural numbers (PU 45) [§29; Berg refers to the page of the 1851 edition], then in the series: $N_0, {N_0}^2, {N_0}^3, \ldots$ each ${N_0}^m$ is said to "exceed infinitely" the preceding term ${N_0}^{m-1}$ (PU 46) [§29]. But Bolzano's comparison of the powers of infinite sets is impossible to understand, since nowhere does he offer any clear sufficient condition for the equinumerousness of infinite sets.

Berg makes the same points as Cantor, namely that Bolzano's writings in *PU* are about the *powers* of infinite *sets*, and that his reasoning is impossible to follow as he does not offer sufficient conditions for the equality of size of sets. However Berg [see, for instance, his 26, p. 177] remains convinced that a letter[3] written by Bolzano in the last year of his life witnesses a change of heart regarding how infinite sets should be compared, moving from his rejection of one-to-one correspondence to an acceptance of it as a sufficient criterion for size equality.

On the heels of this interpretation, Berg [27, pp. 42-43] sketches what he takes to be Bolzano's theory of the infinite. In a nutshell, Berg believes that any two infinite sets of natural numbers are of the same size according to Bolzano just in case "the members are related to each other by finitely many rational operations (addition, multiplication and their inverses)" [27]. Even though Berg does not use this terminology, his interpretation seems to suggest that $\mathbb{N}$ is equinumerous with an infinite subset $S \subseteq \mathbb{N}$ whenever the bijection $f : S \to \mathbb{N}$ is primitive recursive. This is an interesting suggestion, but it would imply that, for example, $\mathbb{N} - \{1\}$ and $\mathbb{N}$ are equinumerous, while this seems to contradict Bolzano's reasoning in *PU* §29 (see Section 9.3 below). Moreover, Berg's interpretation of the letter is far from uncontroversial (see [226, pp. 194-195], [238, pp. 469-470], to be discussed below, and [184, 182]), so his interpretation of this aspect of Bolzano's work is not a foregone conclusion. We will not engage with it any more than what we have already done as the controversy has less to do with *PU* and more to do with what views about the infinite Bolzano held at the

[3] This letter, dated 9 March 1848 and intended for Bolzano's former pupil Robert Zimmermann, has been published in [51, pp. 187-189]. Berg is the editor for the volume and his editorial notes to the letter are a reiteration of his interpretation of Bolzano having changed his mind regarding part-whole and one-to-one correspondence for infinite collections.

very moment of his death.

A more nuanced view is offered by Šebestík [238, pp. 435-473]. When presenting the contribution of Bolzano's *PU*, Šebestík summarizes it thus:

> For the first time, the actual infinite, whose properties cease to be contradictory to simply become paradoxical, is admitted in mathematics as a well-defined concept, having a referent and only attaching to those objects capable of enumeration or measurement, that is, to sets and quantities.[4] [238, p. 435]

Šebestík also interprets the *PU* as about sets and their being infinite. Even though at p. 445 he more faithfully writes that "the infinite is first and foremost a property of *pluralities* [our emphasis]",[5] on p. 462 he then reverts to set talk at a crucial point, namely when giving his interpretation of *PU* §33:

> [Referring to §33] It is the first and last time within the *Paradoxes of the Infinite* that Bolzano deduces from the reflexivity of the set of natural numbers to the equality of number between a set and one of its proper subsets.[6]

According to Šebestík's interpretation then, and unlike Berg's, it is not quite the case that Bolzano changed his mind regarding what criterion to use to compare the size of infinite sets *after* the *PU* and just before his death. Rather, Bolzano's views in the *PU* itself are already inconsistent, because at various points in the text Bolzano either implicitly or explicitly endorses the following views:

1. The part-whole principle, that is, the whole is greater than any of its proper parts.
2. All infinite sets can be put in one-to-one correspondence with any of their infinite subsets.
3. Every set has a definite size.
4. If two sets are in one-to-one correspondence then they have the same plurality.

It is quite telling that for 1, 3, and 4 Šebestík [238, pp. 463-464] feels the need to add set theoretic glosses, so that 1 becomes "$card(A) < card(B)$ iff A is equivalent to a proper part of B" ('$card(A) < card(B)$ si et seulement si A est équivalent à une partie propre de B'), 3 is Every set has a "unique cardinal number" ('nombre cardinal unique') and 4 If two sets

[4] Original French: Pour la première fois, l'infini actuel dont les propriétés cessent d'être cotradictoires pour devenir simplement paradoxales, est admis en mathématiques à titre de concept défini, ayant une référence et attaché aux seuls objets susceptibles de dénombrement ou de mesure, c'est-à-dire aux ensembles et aux grandeurs.

[5] Original: "L'infini est d'abord et avant tout une propriété des multitudes".

[6] C'est pour la première et dernière fois que, dans les *Paradoxes de l'Infini*, Bolzano conclut de la réflexivité de l'ensemble des nombres naturels à l'égalité numérique entre un ensemble et l'un de ses sous-ensembles propres.

are in one-to-one correspondence then they have “the same cardinal number” (‘ont le même nombre cardinal’).

Thus formulated, 1-4 do indeed yield a contradiction. Consider any two infinite sets A and B such that A is a proper part of B. By 3, they each have a unique cardinality, and by 1 $card(A) < card(B)$. But also, since A and B can be put into one-to-one correspondence (by 2), they have the same cardinality, by 4, so $card(A) = card(B)$, contradicting our earlier deduction that $card(A) < card(B)$. We will give our argument as per why Šebestík's contradiction does not go through in Section 9.3, where we highlight that a crucial ingredient in this family of counterexamples to Bolzano's claim to internal consistency in the *PU* is largely due to the set theoretic interpretation of 4.

The last interpretation we want to consider in detail is Rusnock's (2000). Rusnock [226, p. 193] writes that in §§21-22 Bolzano “apparently based this opinion [of the insufficiency of one-to-one correspondence for equality of size] on considerations involving parts and wholes, assuming perhaps that the multiplicity of the whole must be greater than those of its parts. (Rusnock translates with “multiplicity” what we, following [227], translate as “plurality”, namely *Vielheit.*) Rusnock [226, *ibid.*] then continues:

> But this seems to be a mistake, even in Bolzano's own terms. For his sets (*Mengen*) are by definition invariant under rearrangements of their members, and thus the appeal to the “mode of determination” seems to be illegitimate in this context.

Rusnock then produces an example to show why Bolzano is mistaken by his own lights when embracing “considerations of parts and whole”. Consider the straight line abc, where a is to the left of b and b is to the left of c; call A the set of points between a and b, B the set of points between a and c. Then it is possible to map each point of A to a point of B via a translation map that is also a one-to-one correspondence. Since a translation map only “rearranges” points from one region of space to another, then B is just a rearrangement of A. Thus, A and B should be the same “set”, since Bolzano's definition of “set” (*Menge*) entails that something considered as a “set” is invariant under rearrangement of parts. Yet, because A is a proper part of B, A should be strictly smaller than B, in virtue of what from now on we call “the part-whole principle”: The whole is greater than any of its proper parts. This principle then is inconsistent with Bolzano's own definition of multitude.

It is not warranted however that an example such as Rusnock's really counts as a rearrangement of parts on Bolzano's terms, essentially because it relies on a metaphorical use of the term “rearrangement” in a geometric context. This metaphorical use in turn suggests conceiving of geometric figures (points and lines) as objects that move through the two-dimensional (Euclidean) space. Yet Bolzano famously rejected metaphorical talk of motion in mathematical contexts [49, Introduction], and lacking that, we are not sure there is a way of rephrasing Rusnock's example so that it really counts as a rearrangement of parts on Bolzano's terms.

On the basis of our overview, we can now distil the received view about the *PU* into two theses:

(Sets) In §§29-33, Bolzano is concerned with determining size relationships between infinite sets.

(Set-PW) Bolzano's computations in §§29-33 are, at least partially, motivated by the part-whole principle for sets.

As we have seen above, the combination of these two theses motivates a reading of Bolzano's calculation of the infinite as a pre-Cantorian transfinite arithmetic that is either mistaken or downright inconsistent because of its adherence to the part-whole principle. As it will soon become apparent, we believe however that both theses incorrectly describe §§29-33 of the *PU*. Our main claim is that the standard view's identification of Bolzanian collections with the modern notion of set, and of all instances of part-whole reasoning in the *PU* to **PW**, is too quick. Discussing the standard interpretation of Bolzano's calculation of the infinite therefore requires a clarification of the status of collections in the *PU*, and an assessment of the role that part-whole reasoning plays in Bolzano's arguments. We will take those two issues in turn. First, we briefly recap the various notions of collections that Bolzano introduces at the beginning of the *PU*, and explain the role they play in his definition of the infinite. Second, we review sections §§20-24 of the *PU*, in which Bolzano is usually interpreted as rejecting the bijection principle in favor of something like **PW**. We believe this will provide the reader with the necessary background for our in-depth discussion of §§29-33 in Section 9.3.

9.2.1 Bolzano's Collections, Multitudes, and Sums

Bolzano's first goal in the *PU* is to arrive at a rigorous definition of the infinite. To that end, he relies on his logical system first developed in his *Wissenschaftslehre* (*Theory of Science*, [52] for short). In particular, Bolzano devotes the first section of the *PU* to defining several distinct notions of collection. Without going into too much detail, we summarize here the most important definitions.

Collection The concept of collection (*Inbegriff*) applies to any and all objects which are made of parts, i.e. that are not simple. In that sense, [collection] is the most general concept as it applies to any composite object. Collections, as opposed to units (*Einheiten*, sometimes also translated as unity/unities), can be decomposed into simpler parts. Anything that is made of at least two parts is a collection. (see [46] §3)

Multitude The concept of multitude (*Menge*) is best illustrated with Bolzano's own example of a drinking glass ([46] §6). Consider the glass as intact, and then as shattered into pieces. What changes between these two states of the glass is the arrangement (*Anordnung*) of the pieces, although the amount of glass is the same before and after. When we consider the glass as that which remains unchanged before and after the breakage, we are considering it as a multitude. "A collection which we put under a concept so that the arrangement of its parts is unimportant (in which therefore nothing essential changes for us if we merely change this arrangement) I call a multitude." ([46] §4)[7]

Plurality When the parts of a multitude all fall under the same concept A and are therefore considered as units of kind A (i.e. simple objects of kind A), that multitude is called a plurality (*Vielheit*) of kind A. (*ibid.*)

Sum A sum (*Summe*) is a collection such that (a) its parts can also be collections, and (b) the parts of its parts can be considered as parts of the whole sum, without the sum itself having changed ([46] §5). Consider the glass example again. Suppose we break our glass G and it shatters in exactly three pieces, a, b and c. Then suppose a breaks also into two pieces a_1 and a_2. Then our glass G, considered as a sum, is still the same: $G = a + b + c = a_1 + a_2 + b + c$.

Quantity Bolzano defines a quantity (*Größe*) as an object that can be considered of a kind A such that any two objects M, N of kind A satisfy a certain law of trichotomy (not Bolzano's expression): either they are equal to one another ($M = N$) or "one of them presents itself as a sum which includes a part equal to the other one" ([46] §6), that is to say, $M = N + \nu$ or $N = M + \mu$. The remaining parts μ, ν themselves also need to satisfy the condition that, for any other X of kind A, either $X = \mu$ ($X = \nu$, respectively) or one of them can be presented as a sum of which the other is just a part.

To avoid any confusion, it should be noted that the concepts of multitudes, pluralities, sums and quantities are specifications of the concept of collections, and the same object can be conceptualized as more than one kind of collection at once. Quantities are a great example. From their definition, it is clear that anything that is a quantity is also a plurality, because a quantity is a multitude (of a certain kind, say A) whose parts are also objects of kind A. At the same time, the way Bolzano expresses the trichotomy law holding of relationships between quantities suggests that a quantity is also a sum, namely, an object such that the parts of its immediate parts are also parts of the object itself, and nothing about the object changes if we consider it as made of the parts of its parts, instead of just of its own immediate parts.

Moreover, the existence of various notions of collections in Bolzano's framework is at odds with the thesis **(Sets)** of the received view, according to which Bolzano tries to develop an arithmetic of infinite sets. Indeed, it is far from clear that any of the notions described above can be straightforwardly mapped onto the modern notion of a set. Following Incurvati [141, p. 11], we consider the concept of set as used in (philosophy of) mathematics contexts to be sufficiently individuated by the three criteria:

(Unity) A set is a single entity over and above its elements.

[7] Translations of Bolzano's *PU* are always from [227].

(Decomposition) A set can be decomposed in a unique way into its elements.

(Extensionality) Sets are identical if and only if they have exactly the same elements.

Bolzano's own definitions do not imply that his multitudes, or pluralities, or sums satisfy all three criteria at once. Since multitudes, pluralities and sums are the infinite collections Bolzano concerns himself with, the identification of his infinite collections with Cantorian infinite sets is unwarranted and far from obvious. For more on Bolzano's multitudes and sets, see Simons [240].

Nevertheless, **(Sets)** might gain some traction from the fact that Bolzano's definition of the infinite only applies to collections, or, more precisely, to pluralities:

> [...] I shall call a plurality which is greater than every finite one, i.e., a plurality which has the property that every finite multitude represents only a part of it, an *infinite plurality.* ([46] §9)

However, the choice of defining an infinite *plurality* as opposed to simply infinity is justified in §10, where Bolzano argues that in the use made by mathematicians, "the infinite" is always an infinite plurality:

> Therefore it [is] only a question of whether through a mere definition of what is called an infinite plurality we are in a position to determine what is [the nature of] the infinite in general. This would be the case if it should prove that, strictly speaking, there is nothing other than pluralities to which the concept of infinity may be applied in its true meaning, i.e., if it should prove that infinity is really only a property of a plurality or that everything which we have defined as infinite is only called so because, and in so far as, we discover a property in it which can be regarded as an infinite plurality. Now it seems to me that is really the case. The mathematician obviously never uses this word in any other sense. For generally it is nearly always quantities with whose determination he is occupied and for which he makes use of the assumption of one of those of the same kind for the unit, and then of the concept of a number. ([46] §10)

Bolzano's target when defining infinity solely as the attribute of certain collections are the imprecise definitions of infinity given by some philosophers (Hegel and his followers are cited explicitly here) who consider the mathematical infinity Bolzano talks about to be the "bad" kind ([46] §11), while the one true infinity is God's absolute infinity. The strategy to push against this *qualitative infinite* of the philosophers is to show that, even in the case of God, who is the unity *par excellence*, when we assign infinity to Him as one of His attributes, what we are really saying is that some other attribute of His has an infinite multitude as a component.

> What I do not concede is merely that the philosopher may know an object on which he is justified in conferring the predicate of being infinite without first

> having identified in some respect an infinite magnitude [*Größe*] or plurality in this object. If I can prove that even in God as that being which we consider as the most perfect unity, viewpoints can be identified from which we see in him an infinite plurality, and that it is only from these viewpoints that we attribute infinity to him, then it will hardly be necessary to demonstrate further that similar considerations underlie all other cases where the concept of infinity is well justified. Now I say we call God infinite because we concede to him powers of more than one kind that have an infinite magnitude. Thus we must attribute to him a power of knowledge that is true omniscience, that therefore comprehends an infinite multitude of truths because all truths in general etc. ([46] §11)

With that, Bolzano considers himself to have exhaustively argued for his definition of mathematical infinity as being inextricable from the concepts of plurality and quantity and inapplicable to the one-ness of any unity, even God. Thus, we conclude that Bolzano's insistence on defining only an infinite plurality does not lend particular credence to **(Sets)** after all. Bolzano's definition unequivocally makes of infinity a quantifying attribute which, as such, can only apply to pluralities and quantities. But his insistence on discussing only infinite pluralities should be understood as in contrast with the Hegelian infinite as an attribute of a single infinite being. Talking about infinite collections, for Bolzano, is a way of clearly setting apart the *quantitative* infinite he is interested in from the qualitative infinite of the hegelians.

9.2.2 Bolzano's Commitment to Part-Whole in the *PU*

As the discussion of the received view on the *PU* made clear, one point of contention in interpreting Bolzano's work on the infinite is whether (and to what extent) the principles that guide his computations with infinite quantities mirror those later used by Cantor. While part-whole considerations play an important role in Bolzano's [52] (in particular, §102 therein; cf. [182, pp. 130-131], [184, pp. 624-625]), the discussion in Berg and Šebestík's interpretations has brought to light the issue of whether, on the whole, Bolzano's treatment of infinite quantities in the *PU* obeys the part-whole principle or not. Setting aside the issue of whether an adoption of one-to-one correspondence is implicit in Bolzano's §33 (something we will come back to in Section 9.3), here we review §§20-24, which are usually taken to be Bolzano's discussion of one-to-one correspondence as an insufficient criterion for size equality of infinite collections on the grounds of part-whole considerations.

Let us note first that some form of part-whole reasoning seems to be present in the very notion of "being greater/smaller than" employed in the *PU*, as this passage from §19 witnesses:

> Even with the examples of the infinite considered so far it could not escape our notice that not all infinite multitudes are to be regarded as equal to one another in respect of their plurality, but that some of them are greater (or smaller) than others, i.e., another multitude is contained as a part in one multitude (or on the contrary one multitude occurs in another as a mere part). ([46] §19)

Here, Bolzano glosses the claim that some multitudes are greater than others as some containing others as a part. A similar use of the part-whole principle is to be found in §20, when Bolzano compares the size of the collection of quantities smaller than 5 and the size of the collection of those smaller than 12:

> If we take two arbitrary (abstract) quantities, e.g. 5 and 12, then it is clear that the multitude of quantities which there are between zero and 5 (or which are smaller than 5) is infinite, likewise also the multitude of quantities which are smaller than 12 is infinite. And equally certainly the latter multitude is greater since the former is indisputably only a part of it. ([46] §20)

This suggests that Bolzano's writings commit him to upholding the part-whole principle even when it comes to the comparison of infinite quantities, because the principle is part and parcel of the definition of the order relation among quantities.

Having thus established Bolzano's commitment to part-whole, let us also show his explicit rejection of what nowadays we call one-to-one correspondence as a sufficient criterion for equality of size for infinite collections:

> I claim that two multitudes, that are both infinite, can stand in such a relationship to each other that, on the one hand, it is possible to combine each thing belonging to one multitude, with a thing of the other multitude, into a pair, with the result that no single thing in both multitudes remains without connection to a pair, and no single thing appears in two or more pairs, and also, on the other hand it is possible that one of these multitudes contains the other in itself as a mere part, so that the pluralities which they represent if we consider the members of them all as equal, i.e., as units, have the most varied relationships to one another. ([46] §20)

In the quote above, Bolzano remarks that it is possible for two infinite multitudes to both be in a one-to-one correspondence with each other and be related as a part to its whole. This state of affairs can have the appearance of a paradox, because in the finite case checking whether two multitudes can be put into one-to-one correspondence suffices to determine whether they have the same number of terms, whereas the part-whole relation implies that one multitude must be greater than the other. Bolzano insists that the part-whole relation is what determines the greater-than relation, too:

> Therefore merely for the reason that two multitudes A and B stand in such a relation to one another that to every part a occurring in one of them A, we can seek out according to a certain rule, a part b occurring in B, with the result that all the pairs $(a + b)$ which we form in this way contain everything which occurs in A or B and contains each thing only once—merely from this circumstance we can—as we see—in no way conclude that these two multitudes are equal to one another if they are infinite with respect to the plurality of their parts (i.e., if

> we disregard all differences between them). But rather they are able, in spite of that relationship between them that is the same for both of them, to have a relationship of inequality in their plurality, so that one of them can be presented as a whole, of which the other is a part. ([46] §21)

This consideration is illustrated in the preceding §20 by way of two examples, or, two versions of the same example, which considers the two intervals $(0, 5)$ and $(0, 12)$ on the real line and concludes that, since $(0, 5)$ is only a part of $(0, 12)$, $(0, 12)$ contains more quantities (or more points) than $(0, 5)$.

The reason why one has to drop the apparently successful one-to-one correspondence criterion when considering infinite quantities is that what makes one-to-one correspondence work in the finite case is precisely that one has to do with finite collections; hence at some point the process of pairing off each element from the collection with a natural number stops, whereas in the infinite case there is no last element, so the pairing-off never ends. Hence the need for a different criterion for size comparison (*PU* §22). Bolzano gives a brief explanation of how one-to-one correspondence does not suffice to reach conclusions regarding comparisons of infinite sums in §24:

> [From the proposition of §20] follows as the next consequence of it that we may not immediately put equal to one another, two sums of quantities which are equal to one another pair-wise (i.e., every one from one with every one from the other), if their multitude is infinite, unless we have convinced ourselves that the infinite plurality of these quantities in both sums is the same. That the summands determine their sums, and that therefore equal summands also give equal sums, is indeed completely indisputable, and holds not only if the multitude of these summands is finite but also if it is infinite. But because there are different infinite multitudes, in the latter case it must also be proved that the infinite multitude of these summands in the one sum is exactly the same as in the other. But by our proposition it is in no way sufficient, to be able to conclude this, if in some way one can discover for every term occurring in one sum, another equal to it in the other sum. Instead this can only be concluded with certainty if both multitudes have the same basis for their determination. ([46] §24)

Bolzano considers here the case of a one-to-one correspondence between the terms of two infinite sums S_1 and S_2 that would map each term in S_1 to an equal term in S_2. Since the existence of a one-to-one correspondence is not enough to guarantee that S_1 and S_2 have the same number of terms, one cannot conclude that S_1 and S_2 are equal, unless the two sums also have the same "basis for their determination". This phrase does not have, to our knowledge, a standard interpretation in Bolzanian scholarship. Šebestík [238, p. 460] does attempt an explanation of what the "determining elements" (*bestimmende Stücke*) of an object can be, according to Bolzano. However, we are not convinced that the explanation offered there extends to a notion of determination for mathematical entities. For now, we simply draw the reader's attention to the fact that Bolzano concludes his discussion of the one-to-one correspondence criterion with a methodological point about infinite *sums* which plays a crucial role in §32 and §33 (see Section 9.3.2 and Section 9.3.3 below).

To sum up, in this section we have presented what we take to be the received view on Bolzano's calculation of the infinite, and shown that it relies on the two theses **(Sets)** and **(Set-PW)**. We have argued that the existence of various notions of collections in Bolzano's framework puts some pressure on **(Sets)**, as it does not seem obvious that any of Bolzano's notions closely matches our modern notion of set. Regarding **(Set-PW)** we have shown how Bolzano appeals in §§20-24 to part-whole reasoning in the context of determining size relationships between certain infinite collections. However, we also noted that, by §24, Bolzano has pivoted from discussing sufficient criteria for the equality of size of two infinite collections to discussing sufficient criteria for the equality of two infinite sums. As we will argue in the next section, this is a crucial shift in perspective that is missed by the standard interpretation of Bolzano's calculation of the infinite. We now turn to a close analysis of the text and to our arguments in favour of a different reading of [46] §§29-33.

9.3 Bolzano's Calculation of the Infinite

As discussed in the previous section, by §24 Bolzano has established the following facts about infinite multitudes and pluralities:

1. Some infinite multitudes are greater than others "with respect to their plurality" (§19).
2. Two infinite multitudes can both be related as part and whole and be in a one-to-one correspondence (§20).
3. One-to-one correspondence is not sufficient to determine equality of infinite multitudes (§§21-22).
4. In the case of comparing two infinite sums, if one wants to conclude that they are equal, one needs to make sure both that there are as many summands in one as there are in the other and that each term from one sum is equal to the corresponding one in the other sum (§24).

These are the "basic rules" (*Grundregeln*, [46] §28) which govern a proper handling of the infinite in mathematics. Bolzano is aware however that his readers might still be skeptical towards the possibility of computing with the infinite, so he explains what he means by "calculation of the infinite" in the following passage:

> Even the *concept* of a *calculation of the infinite* has, I admit, the appearance of being self-contradictory. To want to *calculate* something means to attempt a *determination of something* through numbers. But how can one determine the infinite through numbers—that infinite which according to our own definition must always be something which we can consider as a multitude consisting of

> infinitely many parts, i.e., as a multitude which is greater than every number, which therefore cannot possibly be determined by the statement of a mere number? But this doubtfulness disappears if we take into account that a calculation of the infinite done correctly does not aim at a calculation of that which is determinable through no number, namely not a calculation of the infinite plurality in itself, but only a determination of the *relationship* of one infinity to another. This is a matter which is feasible, in certain cases at any rate, as we shall show by several examples. ([46] §28)

Bolzano's calculation of the infinite is minimal. He does not purport to have extended the concept of number so as to introduce infinite *numbers* (*pace* Cantor—see Section 9.2 above),[8] but he aims to study the relationship—that is, the ratios as well as the "greater than" relation—between two infinities whenever this can be done in a sound way, that is, in accordance with the principles he has argued for in the preceding portion of the *PU*. Armed with such principles, Bolzano can show his reader how to properly handle some apparently paradoxical results in mathematics, starting from the general theory of quantity.

9.3.1 Computing with Infinite Sums

The first computations with infinite quantities are found in earnest in §29; as we will see, these quantities are always introduced and treated as *sums*.

Bolzano introduces the symbol $\overset{0}{N}$ through a symbolic equation—that is, an equation which establishes that the reference of two signs is the same (cf. definition in *Grössenlehre*, [45, pp. 131-132]) —to stand for the *Menge* of all natural numbers. He then introduces $\overset{n}{N}$ to stand for the *Menge* of all natural numbers strictly greater than $n \in \mathbb{N}$. $\overset{1}{S}$, on the other hand (which is first introduced as $\overset{0}{S}$), is the symbol for the sum of all natural numbers.

In Bolzano's words:

> [...] if we denote the series of natural numbers by
>
> $$1, 2, 3, 4, ..., n, n+1, \dots \ \textit{in inf.}$$
>
> then the expression
>
> $$1 + 2 + 3 + 4 + \cdots + n + (n+1) + \dots \ \textit{in inf.}$$
>
> will be the sum of these natural numbers, and the following expression
>
> $$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n+1)^0 + \dots \ \textit{in inf.}$$

[8]As Mancosu [182, p. 163] notes, this refusal to admit infinite numbers was not unique to Bolzano's position but was shared also by Dedekind [70] and perhaps Schröder [232].

in which the single summands, $1^0, 2^0, 3^0, \ldots$ all represent mere units, represents just the number [*Menge*] of all natural numbers. If we designate this by $\overset{0}{N}$ and therefore form the merely symbolic equation

$$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n+1)^0 + \ldots \textit{ in inf.} = \overset{0}{N} \tag{1}$$

and in the same way we designate the number [*Menge*] of natural numbers from $(n+1)$ by $\overset{n}{N}$, and therefore form the equation

$$(n+1)^0 + (n+2)^0 + (n+3)^0 + \ldots \textit{ in inf.} = \overset{n}{N} \tag{2}$$

Then we obtain by subtraction the certain and quite unobjectionable equation

$$1^0 + 2^0 + 3^0 + \ldots + n^0 = n = \overset{0}{N} - \overset{n}{N} \tag{3}$$

This passage mentions several notions that will be central to the remainder of our analysis of Bolzano's *PU*, hence we will briefly go over them now.

First is the notion of "series" (*Reihe*), which Bolzano defines ([46] §7) as a collection of "terms" (*Glieder*) $a, b, c, d, \ldots$ such that for each term c there is exactly another term d such that, by using the same rule for any pair c, d we can obtain (determine, *bestimmen*) c by applying said rule to d, or the inverse rule to c to obtain d instead. The natural numbers, that is, the "whole numbers" (*ganze Zahlen*) are defined as a series of objects of a certain kind A where the first term is a unit of kind A and the subsequent terms are sums obtained by adding one unit to their immediate predecessor.

The second concept we want to introduce is that of *Gliedermenge* (alternatively expressed by Bolzano as *Gliedermenge*, *Menge von Gliedern* or *Menge der Glieder*). As one can infer from [46] §9, Bolzano considers any number series to have a *Gliedermenge*. Because a *Gliedermenge* is said to be sometimes greater, sometimes smaller, it seems reasonable to assume that this *Gliedermenge* is, if not a quantity properly said, at least something that can be quantified, i.e., treated as a quantity. In the passage we quote from §29, Bolzano introduces first the series of all natural numbers, then their sum and the *Menge* of such a sum. Given what was just said about series and *Gliedermenge* thereof, this occurrence of the word *Menge* should be read as a shorthand for *Gliedermenge* or one of its synonyms.

This occurrence of *Menge* is therefore at odds with any interpretation of Bolzano's definition of "multitude"(*Menge*) that sees it as (almost) synonymous with "set" in the modern sense. If the concept of multitude is virtually identical with that of set, then the multitude of $1+2+3+4+\ldots$ *in inf.* should be just $1, 2, 3, 4, \ldots$ *in inf.* and not $1+1+1+1+\ldots$ *in inf.* For the sake of preserving coherence in Bolzano's work in [46] §§29-33 it is therefore sensible to insist that "*Gliedermenge*" is a quantitative concept. As a consequence, since we believe that translating *Menge* here as "set", like Steele [0], or "multitude", as we would have to if we were to translate *Menge* rigidly, obfuscates this quantitative aspect of the concept of "*Gliedermenge*", we prefer to respect Russ's (2004) choice and translate *Menge* as "number" when it seems to be short for *Menge der Glieder* or similar. As long as it is clear that we do not think Bolzano is introducing here genuine infinite numbers (in the sense of the German *Zahlen*), we will translate *Menge* as "number" in these contexts.

$\overset{0}{N}$ thus denotes the number (*Menge*) of all natural numbers, and for any natural number n, $\overset{n}{N}$ represents the size of the collection of all natural numbers strictly greater than n. This is all written as follows:

$$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n+1)^0 = \overset{0}{N} \tag{4}$$

$$(n+1)^0 + (n+2)^0 + \cdots = \overset{n}{N} \tag{5}$$

The 0^{th} power works in the standard way here, meaning $n^0 = 1$ for any natural number n. So for instance the size of the set of all natural numbers up to n is $1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 = 1 + 1 + 1 \cdots + 1 = n$.

Having defined $\overset{0}{N}$ and $\overset{n}{N}$, Bolzano proceeds to show how they can be added or multiplied with one another thanks to distributivity. One then obtains a hierarchy of infinite quantities of ever-increasing *order*:

$$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \ldots \ \textit{in inf.} = (\overset{0}{N})^2$$
$$1^0.(\overset{0}{N})^2 + 2^0.(\overset{0}{N})^2 + 3^0.(\overset{0}{N})^2 + \ldots \ \textit{in inf.} = (\overset{0}{N})^3$$
$$\text{etc.}$$

The notion of quantities being of different orders of infinity does not start with Bolzano and already existed in the context of infinitesimal calculus.[9] However, we will argue in Section 9.5 that Bolzano's computation of the product of infinite quantities is in fact very original and hence very significant for a comparison with Cantor's theory of the infinite (which we carry out in Section 9.6).

Having looked carefully at Bolzano's first computations with infinite sums, we now proceed to our next piece of evidence for interpreting Bolzano as primarily interested in infinite sums, namely, §32 of the *PU*.

9.3.2 Grandi's Series

In [46] §32, Bolzano criticizes a report by a certain M.R.S. in *Gergonne's Annales* [176] which purports to prove that the infinite sum

$$a - a + a - a + a \ldots \ (1)$$

[9] See for example the debate between Leibniz and Nieuwentijt on the existence of such higher-order infinitesimal, as presented in [185, pp. 160-164].

has value $\frac{a}{2}$.

The series Bolzano focuses on is sometimes called Grandi's series after the Italian 18th century monk who first tried to compute a value for this infinite sum. Kline [157] reports that this series was an object of great interest for mathematicians throughout the 19th century, that "caused endless dispute"[157, pp. 307-308]. It is not necessary for our summary of Bolzano's views to rehash the whole debate surrounding Grandi's series (and other divergent series) in great detail, though it is perhaps worth mentioning that Grandi's opinion, that the value of this series should be $\frac{a}{2}$, was shared also by Leibniz [157, p. 307]. Kline also reports that Leibniz's argument—which differed from Grandi's—was accepted by the Bernoulli brothers. This acceptance notwithstanding, by the time Bolzano is active there is still no clear consensus on how to treat what we would now consider divergent series. For Bolzano and his contemporaries, the question of how to assign a value to infinite sums such as Grandi's series was still a live question, one which would later lead some mathematicians (e.g., the Italian Cesàro) to define different sorts of summation.

It is therefore not surprising that one should come across a piece of writing such as M.R.S.'s. M.R.S. purports to prove that the value of Grandi's series is $\frac{a}{2}$ via an algebraic reasoning, as opposed to Leibniz's more "probabilistic" (per Kline) approach—and presumably, as opposed to Grandi's geometric approach, too. Here we quote M.R.S.'s own exposition of his proof:

> The summation of the terms of a geometric progression decreasing into the infinite can be easily deduced from the above; in fact, if one has
>
> $$x = a + aq + aq^2 + aq^3 + aq^4 + \dots,$$
>
> one can then write
>
> $$x = a + q(a + aq + aq^2 + aq^3 + \dots),$$
>
> then $x = a + qx$ or $(1 - q)x = a$, hence $x = \frac{a}{1-q}$. As per the remarks in (5), the equation
>
> $$x = a - a + a - a + a - a + \dots$$
>
> could not help in the approximation of x, as it successively gives the approximate values $a, 0, a, 0, a, 0, \dots$ among which the differences are constant; but, without resorting to Leibniz's subtle reasoning, one can immediately see that this equation comes to
>
> $$x = a - x,$$
>
> hence $x = \frac{1}{2}a$.[10] [176, pp. 363-364]

[10]Original French: La sommation des termes d'une prógression géométrique décroissante à l'infini se déduit bien simplement de ce qui précède; si en effet on a

$$x = a + aq + aq^2 + aq^3 + aq^4 + \dots,$$

As the text shows, M.R.S.'s treatment of Grandi's series has the virtue of treating it uniformly with other (converging) geometric series. Bolzano however is not impressed with M.R.S.'s algebraic manipulations and sees two mistakes in them. Bolzano spells out M.R.S.'s argument as follows. First, he sets

$$x = a - a + a - a + a - \dots \textit{ in inf.} \tag{1}$$

Then, one can rewrite (1) as

$$a - (a - a + a - a + \dots \textit{ in inf.}) \tag{2}$$

This yields $x = a - x$ and therefore $x = \frac{a}{2}$. Bolzano points out that while x is defined as $a - a + a - a + a - \dots$ *in inf.*, the expression in (2) is not identical with it, because it does not have the same *Gliedermenge* as $a - a + a - a + a - \dots$ *in inf.* in (1). The first a is missing so that the correct substitution ought to be the tautological $x = a + (x - a)$.

Even though Bolzano does not pause to point this out to the reader, M.R.S. is making exactly one of those mistakes Bolzano was cautioning against in §24: he has assumed equality of two quantities arising from summing up two series without checking that the two series have the same *Gliedermenge*. Note that here again Bolzano seems to be using *Menge* in a way that is closer to the meaning of "number" than to that of "set", and Russ's (2004) translation accordingly translates the term as "number". While again one should not take the translation literally, we agree with the attempt to capture a more quantitative use of *Menge* in this kind of context.

The second criticism Bolzano levels at M.R.S.'s argument is that it presupposes that $a - a + a - a + a \dots$ refers to an actual quantity, whereas Bolzano argues that it does not. The argument Bolzano gives for this position is an example of Bolzano putting to (mathematical) use his logico-philosophical apparatus: Grandi's infinite sum is a spurious one because it does not display the sum property ([46] §31)

$$(A + B) + C = A + (B + C) = (A + C) + B.$$

If one tries to rewrite Grandi's sum according to Bolzano's equations, the left-hand side becomes $(a - a) + (a - a) + \dots$ *in inf.*, which according to Bolzano equals 0, whereas if one rearranges the parentheses as $a + (-a + a) + (-a + a) + (-a + a) + \dots$ *in inf.*, one obtains a as a result. Thus indeed Grandi's expression does not satisfy Bolzano's definition of sum. Tapp [48, p. 193] notes here that Bolzano's criterion is quite similar to Riemann's result [3, p. 197] which states that every infinite series is absolutely convergent if and only if it is preserved under permutation (an absolutely convergent series is one in which the series of the absolute values of its terms also converges). It is unfortunate though that Bolzano's criterion taken literally is too strong, as it seems to be also implying that $\overset{0}{N}$ does not designate an actual

on pourra d'abord écrire

$$x = a + q(a + aq + aq^2 + aq^3 + \dots),$$

puis $x = a + qx$ ou $(1 - q)x = a$ d'où $x = \frac{a}{1-q}$.

quantity (see Section 9.4 below).

We take this section of the *PU* as helping our case that Bolzano's work in §§29-33 should not be read as an imperfect set theory. Indeed, §32 is an example of Bolzano's principles for the computations of the infinite at work: a result published by a fellow mathematician about the computation of infinite *sums* is rejected on the basis of a violation of one of these principles. However most other commentators do not devote particular attention to §32. One notable exception is Steele, who thus summarizes §32: "Some errors in the pretended summation of $\Sigma(-1)^n a$, which is a symbol not expressing any true quantity at all" [0, p. 66]. Even more intriguingly, he mentions Grandi's series and the whole controversy surrounding it when introducing the historical context of the *PU* [0, pp. 3-4]. Yet it is as if this does not leave a trace when giving an overall appraisal of the contributions of the *PU*, or of Bolzano's contributions to mathematics and its philosophy. Bolzano is still presented as someone who *almost* anticipated Cantorian set theory, except he did not.

9.3.3 The Sum of all Squares

In the previous section, we argued that some passages of the *PU* offer textual evidence for the claim that Bolzano's work on the sizes of infinite collections should be understood as about sizes of infinite sums, that is, infinite series in modern terminology, rather than as about sizes of infinite countable sets. We now make a theoretical case as per why this interpretation is also the most charitable one.

Just following the discussion of §32, Bolzano writes that

> [...] if we wish to avoid getting onto the wrong track in our calculations with the infinite then we may never allow ourselves to declare two infinitely large quantities, which originated from the summation of the terms of two infinite series, as equal, or one to be greater or smaller than the other, because every term in the one is either equal to one in the other series, or greater or smaller than it. ([46] §33)

So, for two infinite sums α and β, it is not the case that, say, $\alpha > \beta$ if for every term of α there is one in β that is strictly smaller.

He then continues:

> We may just as little declare such a sum as the greater just because it includes all the terms of the other and in addition many, even infinitely many, terms (which are all positive), which are absent in the other.

As an example of this principle in action, Bolzano asks us to consider the two series

$$1 + 4 + 9 + 16 + \ldots \textit{ in inf.} = \overset{2}{S}$$

and

$$1 + 2 + 3 + 4 + 5 + \ldots \textit{ in inf.} = \overset{1}{S}.$$

According to Bolzano, "no one can deny that every term of the series of all *squares*"—that is, $\overset{2}{S}$—"because it is also a natural number, also appears in the series of first powers of the natural numbers and likewise in the latter series $\overset{1}{S}$, together with all the terms of $\overset{2}{S}$ there appear many (even infinitely many) terms which are missing from $\overset{2}{S}$ because they are not square numbers." [46, §33] So, the series $\overset{1}{S}$ and $\overset{2}{S}$ are such that the terms of the latter all appear in the former, and the former also includes infinitely many terms that the second series does not include. The next step in Bolzano's argument is to claim the following:

> Nevertheless $\overset{2}{S}$, the sum of all square numbers, is not smaller but is indisputably greater than $\overset{1}{S}$, the sum of the first powers of all numbers. [46, §33]

Bolzano argues for this point by claiming two things: first, that "in spite of all appearance to the contrary, the *multitude of terms* [*Gliedermenge*] in both series (not considered as sums, and therefore not divisible into arbitrary multitudes of parts) is certainly the same." Second, that with the exclusion of the first term, all terms of $\overset{2}{S}$ are greater than the corresponding term in $\overset{1}{S}$. Since then the two series have the same amount of terms, but the terms of $\overset{2}{S}$ are greater than all but one of the terms in $\overset{1}{S}$, Bolzano concludes that $\overset{2}{S}$ is greater than $\overset{1}{S}$, because it is possible to termwise subtract $\overset{1}{S}$ from $\overset{2}{S}$ and one would still have a positive remainder as a result:

> But if the multitude of terms [*Menge der Glieder*] in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the first term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, [...] [46, §33]

As we can see, in §33 Bolzano repeats twice the idea that $\overset{1}{S}$ and $\overset{2}{S}$ have the same *Gliedermenge* (translated by Russ as "multitude of terms"). He is committed then to the claim

(Terms) The *Gliedermenge* in series $\overset{1}{S}$ and $\overset{2}{S}$ is the same.

This is often [see, e.g., 26, 27, 238] interpreted as a sign that Bolzano was using here one-to-one correspondence to compare the size of the sets corresponding to $\overset{1}{S}$ and $\overset{2}{S}$, namely $\mathbb{N}$, the set of all natural numbers, and $\mathbb{N}^{(2)}$, the set of all squares, respectively. But if this is the case, then Bolzano is essentially violating part-whole as applied to sets, the way Šebestík suggests (cf. Section 9.2).

§29 and §33 taken together raise the question of how, if at all, Bolzano envisioned to generalize his notion of *Gliedermenge* from the collection of all natural numbers to any infinite subcollection thereof—or what would be a "Bolzanian enough" way of doing this.

Let us take a step back and reconsider what Bolzano does in §29. Recall that $\overset{0}{N} = 1^0 + 2^0 + 3^0 + 4^0 + \dots$ *in inf.*, where each n^0 is one unit, as Bolzano reminds us. Assuming that $\overset{0}{N}$ is what Bolzano intended to be the *size* of $\mathbb{N}$ just in the same way as cardinals are considered to capture set size in modern set theory, the question is how to extend Bolzano's notion of size of $\mathbb{N}$ to infinite (proper) subsets of $\mathbb{N}$. Given the importance that the example of squares has in Bolzano scholarship (see our Section 9.2), let us try to answer the question for $\mathbb{N}^{(2)}$, specifically.

Per §29, the procedure to obtain the *Menge* (of terms, *von Gliedern*) of a series $\alpha := \alpha_1, \alpha_2, \alpha_3 \dots$ is to first consider it as a sum

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots \textit{ in inf.},$$

and then raise each term to the power of 0. The *number* of terms in α is then identified with the value of the infinite sum $\overset{\alpha}{N} = \alpha_1^0 + \alpha_2^0 + \alpha_3^0 + \dots$ *in inf.* This means that if we list all square numbers as $sq := 1, 4, 9, 16, 25, 36, \dots$, the number of terms (hence the number of square numbers) should be identified with

$$\overset{sq}{N} = 1^0 + 4^0 + 9^0 + 16^0 + 25^0 + 36^0 + \dots \textit{ in inf.}$$

Now notice that if we apply the same procedure to the series of terms of $\overset{2}{S}$, we obtain exactly the same. Since $\overset{2}{S}$ as a sum is $\overset{2}{S}$ itself, i.e.,

$$1 + 4 + 9 + 16 + 25 + 36 + \dots \textit{ in inf.},$$

raising each term to the power of 0 yields

$$1^0 + 4^0 + 9^0 + 16^0 + 25^0 + 36^0 + \dots \textit{ in inf.} = \overset{sq}{N}.$$

Thus the number of square numbers is the same as the number of terms in $\overset{2}{S}$. But since Bolzano endorses **(Terms)**, the number of terms in $\overset{2}{S}$ is equal to the number of terms in $\overset{1}{S}$, which is itself computed as $1^0 + 2^0 + 3^0 + \dots$ *in inf.* $= \overset{0}{N}$. From this it immediately follows that $\overset{sq}{N}$ and $\overset{0}{N}$ have the same *Gliedermenge*. Moreover, since any term in each sum is regarded as a unit, both sums also have equal terms. Now by Bolzano's remark (*PU* §24) that "equal summands also give equal sums", we must therefore conclude that $\overset{sq}{N} = \overset{0}{N}$. But if the first one is the number of squares and the second one is the number of natural numbers, then under the standard (set theoretic) interpretation those two *sets* have the same *size*, which directly contradicts the part-whole principle. So it seems that we have reached a contradiction similar to the one highlighted by Šebestík [238, pp. 463-464].

The first reaction would be of course to bite the bullet and accept that perhaps Bolzano did not realize that §29 and §33 would lead to a contradiction, and what is more, to a violation of part-whole. This seems to be the line that a set theoretic interpretation forces upon the reader. For, if $\overset{0}{N}$, being the *Gliedermenge* of $\overset{1}{S}$, is somehow also the size of $\mathbb{N}$, and the *Gliedermenge* of $\overset{2}{S}$ is also the size of $\mathbb{N}^{(2)}$, then of course Bolzano's remark in §33 that $\overset{1}{S}$ and $\overset{2}{S}$ have the same *Gliedermenge* cannot be reconciled with part-whole as applied to sets (**PW**).

A second option would be to reject the generalization of the procedure of §29 to arrive at $\overset{0}{N}$ and argue that there is no analogue to $\overset{0}{N}$ for $\overset{2}{S}$. One could defend this position by pointing out that, in §28, Bolzano only commits to be able to *sometimes* compute with the infinite—not always. In particular, he does not commit to be able to determine the size of every subset of $\mathbb{N}$. We believe however that this answer is not entirely satisfactory. For one, this solution might feel *ad hoc*, because even though Bolzano may have not intended for the procedure of §29 to be applied indiscriminately to any set composed only of natural numbers, there is nothing intrinsic to the procedure itself that bars such a generalization from being carried out. Moreover, while §29 does not explicitly mention a general procedure for determining the *Gliedermenge* of an infinite sum, determining when two sums have the same *Gliedermenge* is necessary to determine whether one is greater than another, as Bolzano himself notes (see §§24 and 32). Since *Gliedermengen* are *Mengen*, multitudes, it is natural to ask whether part-whole reasoning applies to, or is even compatible with, the procedure of determining when the *Gliedermengen* of two sums are equal. In a way, then, this second option does not solve the theoretical problem raised by Bolzano's work so much as skirt around it via a "monster-barring" move.

There is a third option though, which hinges upon a closer reading of §29. Indeed, when computing quantities of the form $\overset{n}{N}$, which for him corresponds to the number of natural numbers greater than n, Bolzano does seem to apply the procedure sketched above, namely writing down the sum $(n+1)+(n+2)+(n+3)+\dots$ *in inf.*, and then raising each term to the 0^{th} power, thus obtaining the sum $(n+1)^0+(n+2)^0+(n+3)^0+\dots$ *in inf.*. However, if, as evidenced again in §33, the difference of two infinite sums is computed termwise, $\overset{0}{N}-\overset{n}{N}$ should be computed as:

$$(1^0-(n+1)^0)+(2^0-(n+2)^0)+\dots \textit{ in inf.}$$

But each term in this sum is the difference of a unit and a unit, so it equals 0. Hence Bolzano should conclude $\overset{0}{N}-\overset{n}{N}=0$. Instead, Bolzano writes that

$$\overset{0}{N}-\overset{n}{N}=1^0+2^0+\dots+n^0,$$

which strongly suggests that Bolzano thinks that $\overset{0}{N} - \overset{n}{N}$ is equal to the infinite sum

$$(1^0) + (2^0) + \ldots + (n^0) + ((n+1)^0 - (n+1)^0) + ((n+2)^0 - (n+2)^0) + \ldots \textit{ in inf.}$$

But this in turn suggests that a more accurate way of representing $\overset{n}{N}$ is in fact as

$$\underbrace{+ \quad + \ldots +}_{n \text{ times}} + (n+1)^0 + (n+2)^0 + \ldots \textit{ in inf.}$$

In other words, $\overset{n}{N}$ is not obtained by listing all the numbers above n in an infinite sum and raising each of them to the power of 0, but is instead obtained by erasing the first n terms from the sum corresponding to $\overset{0}{N}$. This procedure clearly changes the number of terms in the resulting sum. In order to compare $\overset{n}{N}$ to $\overset{0}{N}$, we must therefore make sure first that the two sums have the same *Gliedermenge*, which implies adding n terms to $\overset{n}{N}$ which act, quite literally, as the "ghosts of departed quantities".

This reading of Bolzano's text now gives a way out of the problem of the sum of all squares presented above. Let us consider again the example of $\mathbb{N}^{(2)}$. If we want to compute its size as a subset of $\mathbb{N}$, the way to obtain said size is first to compute that of $\mathbb{N}$, namely, $\overset{0}{N}$. We then remove from $\overset{0}{N}$ the elements whose base is not an element of $\mathbb{N}^{(2)}$, thus obtaining

$$\overset{SQ}{N} = 1^0 + \quad + \quad + \quad + 4^0 + \quad + \quad + \quad + \quad + 9^0 + \quad + \quad + \quad + \quad + \quad + \quad + 16^0 + \ldots \textit{ in inf.}$$

The difference between $\overset{sq}{N}$ and $\overset{SQ}{N}$ is that, in the former, 4^0 is the second term of the sum, while it is the fourth term in $\overset{SQ}{N}$—and so on. The idea would be then that such an erasure procedure *does* change the number of elements from one set to the other, because $\mathbb{N}^{(2)}$ considered as a subset of $\mathbb{N}$ has a different size from when considered as the set underlying the sum $\overset{2}{S}$. Note that this distinction between $\overset{sq}{N}$ and $\overset{SQ}{N}$ is not available to a proponent of the received view: if $\overset{sq}{N}$ and $\overset{SQ}{N}$ are *sets*, i.e. entirely determined by their elements, then as the two sums clearly have the same terms, they should also be equal to one another. By contrast, the difference between the two sums is easy to express in our interpretation of Bolzano's computations (see next section), because $\overset{sq}{N}$ would correspond to a countable sequence with graph $\{\langle 1, 1^0\rangle, \langle 2, 4^0\rangle, \langle 3, 9^0\rangle, \ldots\}$ whereas $\overset{SQ}{N}$ has graph $\{\langle 1, 1^0\rangle, \langle 2, 0\rangle, \langle 3, 0\rangle, \langle 4, 4^0\rangle, \ldots\}$. Incidentally, Tapp [48, p. 191] suggests a similar idea for the interpretation of §29, raising the question whether such an interpretation can actually lead to a fully-fledged coherent reading of the *PU*. Our next two sections address that question.

9.4 An Ultrapower Construction Modelling Bolzano's Arithmetic of the Infinite

Our goal in this section is to offer a model of Bolzano's computations with infinite sums. More precisely, we interpret Bolzano's talk of infinite sums and operations between them as statements about a certain model and show that all of Bolzano's positive results as summarized in the previous section also hold in our model. Additionally, we argue that our model accurately represents Bolzano's *reasoning*, in that several of the proofs we provide closely match Bolzano's own arguments in the *PU*.

Our main idea is to associate to each infinite sum a corresponding infinite quantity. Our proposal here is closely related to the theory of numerosities ([18]; [more recently 22, Ch. 17]), in which the numerosity of a set of natural numbers is defined as an element in an ultrapower of $\mathbb{N}$. However, since our focus is on assigning infinite quantities to certain infinite sums of integers, and not on assigning numerosities to sets of natural numbers, our proposal will be slightly different. Part of our model is in fact closer to the construction presented by Trlifajová [255, pp. 20-24], which we will discuss in Section 9.4.4. In order to do that, we first need to outline our own proposal.

9.4.1 The Basic Framework

We start by representing Bolzano's infinite sums of integers as countable sequences of integers. Formally, we write ω^+ for the set of positive natural numbers and Z for the set of all integers, and we consider functions from $\omega^+ \to Z$. To any infinite sum $a_1+a_2+a_3+\dots$ *in inf.*, we associate the function $f : i \mapsto a_i$, i.e., the function that maps each positive natural number i to the i^{th} summand of the infinite sum. As is customary, we will often identify a function $f : \omega^+ \to Z$ with the countable sequence of integers $(f(1), f(2), f(3), \dots)$. In the case of a Bolzanian sum α which has a different *Gliedermenge* because it has been obtained from another sum by erasing certain terms, we treat the erased terms as 0 and obtain the function associated to α accordingly. For example, since the sequence associated to $\overset{0}{N}$ is $(1, 1, 1, \dots)$, the sequence associated to $\overset{2}{N}$ is $(0, 0, 1, 1, \dots)$.

We consider the structure $\mathbb{Z} := (Z, +, -, 0, 1, <)$ of integers with their usual ordering and addition operation, and take an ultrapower $\mathbb{Z}_{\mathsf{U}}$ of this structure by a non-principal ultrafilter on ω^+ (i.e., a non-empty collection $\mathscr{U}$ of infinite subsets of ω^+ closed under supersets and finite intersections and such that for any $A \subseteq \omega^+$, precisely one of $A, \omega^+ \setminus A$ belongs to $\mathscr{U}$). Ultrapowers are standard constructions in mathematical logic, and a detailed presentation of their theory is beyond the scope of this paper. Instead, we refer the reader to Bell and Slomson [11, Chs. 5, 6] for a standard introduction to ultrapowers and ultraproducts, and simply list some crucial facts below:

Lemma 9.4.1.

1. *Elements in the ultrapower* $\mathbb{Z}_{\mathsf{U}}$ *are equivalence classes of functions from* ω^+ *to* Z*. For any* $f : \omega^+ \to Z$*, we write its corresponding equivalence class as* f^**. For any* $f, g : \omega^+ \to Z$*,* $g^* = f^*$ *if and only if* f *and* g *are equal for* U*-many elements in* ω^+*, i.e.,* $\{i \in \omega^+ : f(i) = g(i)\} \in \mathsf{U}$*.*

2. *There is a canonical elementary embedding of* $\mathbb{Z}$ *into* $\mathbb{Z}_{\mathsf{U}}$*, obtained by mapping any integer* z *to the equivalence class of the constant function* $e_z : \omega^+ \to Z$ *sending any* $i \in \omega^+$ *to* z*. It is customary to identify* z *with* e_z^* *and to view* $\mathbb{Z}$ *as an elementary substructure of* $\mathbb{Z}_{\mathsf{U}}$*.*

3. *Addition and subtraction are defined in* $\mathbb{Z}_{\mathsf{U}}$*. Given* $f, g : \omega^+ \to Z$*,* $f^* + g^*$ *is the equivalence class of the function* $h : \omega^+ \to Z$ *such that* $h(i) = f(i) + g(i)$ *for any* $i \in \omega^+$*. Similarly,* $f^* - g^*$ *is the equivalence class of the function* $h : \omega^+ \to Z$ *such that* $h(i) = f(i) - g(i)$ *for any* $i \in \omega^+$*.*

4. *Elements in* $\mathbb{Z}_{\mathsf{U}}$ *are linearly ordered. More precisely, given any* $f, g : \omega^+ \to Z$*, we have that* $\mathbb{Z}_{\mathsf{U}} \models f < g$ *if and only if* $\{i \in \omega^+ : \mathbb{Z} \models f(i) < g(i)\} \in \mathsf{U}$*.*

5. *Given any first-order formula* $\varphi(x_1, \ldots, x_n)$ *and any functions* $f_1, \ldots, f_n : \omega^+ \to Z$*, we write* $||\varphi(f_1^*, \ldots, f_n^*)||$ *for the set* $\{i \in \omega^+ : \mathbb{Z} \models \varphi(f_1(i), \ldots, f_n(i))\}$*. Łoś's Theorem states that for any* $\varphi(x_1, \ldots, x_n)$ *and any functions* $f_1, \ldots, f_n$*,*

$$\mathbb{Z}_{\mathsf{U}} \models \varphi(f_1^*, \ldots, f_n^*) \textit{ iff } ||\varphi(f_1^*, \ldots, f_n^*)|| \in \mathsf{U}.$$

6. *As a direct consequence of Łoś's Theorem,* $\mathbb{Z}$ *and* $\mathbb{Z}_{\mathsf{U}}$ *are elementarily equivalent.*

An intuitive motivation for our use of an ultrapower of $\mathbb{Z}$ can be provided along the following lines. As we have argued, we take Bolzanian infinite quantities to be infinite sums. Given an infinite sum α, we may decompose α into a sequence of partial sums $\{\alpha_n\}_{n\in\omega^+}$, where, for any positive integer n, α_n is the sum of the first n terms in α. Any such sum can be seen as providing some partial information about α, and if α were a *finite* sum with n terms, then α_n would be α itself. However, since α is infinite, there is no last term of α and no partial sum that would give us total information about α. In order to overcome this difficulty, we must try to organize the partial information given by each partial sum of the first n terms of α into a coherent whole. This is precisely the role that a non-principal ultrafilter U on ω^+ will play for us. One may think of U as a collection of properties of positive integers that describe a natural number "at infinity", distinct from all finite numbers, and providing a vantage point from which all the partial sums of α form a coherent picture. We therefore encourage the reader who may not be familiar with ultrapowers to keep the following two principles in mind:

- Properties of an infinite sum α are those that are shared by "most" partial sums of the form α_n;

- What "most" partial sums means is determined by U. Given a set of positive integers A, the set $\{\alpha_n : n \in A\}$ contains "most" partial sums of α if and only if $A \in \mathsf{U}$.

Given a function $f : \omega^+ \to Z$, we define the *approximating sequence* of f to be the function $\sigma(f) : \omega^+ \to Z$ defined by $\sigma(f)(i) = \sum_{j=1}^{i} f(j)$ for any $i \in \omega^+$. In the case of a function f representing a Bolzanian sum α, the approximating sequence of f is simply the sequence of partial sums $(\alpha^1, \alpha^2, \dots)$ mentioned above. Our proposal consists in identifying the (possibly infinite) quantity designated by a Bolzanian sum f with $\sigma(f)^*$, i.e., with the equivalence class of its approximating sequence. To simplify notation, we will write $\mathbf{f}$ for the element $\sigma(f)^*$ in $\mathbb{Z}_\mathsf{U}$, but we will sometimes abuse notation and write $\mathbf{f}(i)$ for $\sigma(f)(i)$.

We are now able to represent all infinite sums and infinite quantities discussed by Bolzano, except products of infinite quantities, which we will discuss in Section 9.5. As outlined above, the procedure consists in turning a Bolzanian infinite sum into a countable sequence of integers, to which (the equivalence class of) an approximating sequence is then associated. Additions and order relations between infinite sums are then determined by the ultrapower. As an example, the infinite sum $1^0 + 2^0 + 3^0 + \dots$ *in inf.* is represented by the sequence $\overset{0}{N} := (1, 1, 1, \dots)$, since, according to Bolzano, each summand of this sum is a unit. Consequently, the approximating sequence of $\overset{0}{N}$ is the sequence $\sigma(\overset{0}{N}) = (1, 2, 3, \dots)$, which corresponds to the identity function on ω^+, and $\overset{\mathbf{0}}{\mathbf{N}}$ is the equivalence class of the sequence $(1, 2, 3, \dots)$. Similarly, infinite sums of the form $(n + 1)^0 + (n + 2)^0 + \dots$ *in inf.*, which Bolzano writes as $\overset{n}{N}$, are sums that according to him have n fewer terms than $\overset{0}{N}$. We therefore propose to model $\overset{n}{N}$ as a countable sequence in which the first n summands are 0, i.e., by the sequence $(\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 1, \dots)$. The corresponding approximating sequence $\sigma(\overset{n}{N})$ is $(\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, \dots)$. Equivalently, for any $i \in \omega^+$, $\sigma(\overset{n}{N})(i) = i \dot{-} n$, where $i \dot{-} n = 0$ if $i \leq n$ and $i - n$ otherwise.

A similar approach can be applied to represent the sums $\overset{1}{S}$ and $\overset{n}{S}$, as well as Grandi's series of the form $G_a = a - a + a - a + \dots$ *in inf.* For clarity's sake, we have collected the representation of $\overset{0}{N}, \overset{n}{N}, \overset{1}{S}, \overset{n}{S}$, and G_a in the table below:

Bolzanian Infinite Sum	Sequence Representation	approximating sequence	Corresponding Function	Infinite Quantity
$1^0 + 2^0 + \dots$ *in inf.*	$\overset{0}{N} = (1, 1, 1, 1, \dots)$	$\sigma(\overset{0}{N}) = (1, 2, 3, 4, \dots)$	$\sigma(\overset{0}{N})(i) = i$	$\overset{\mathbf{0}}{\mathbf{N}} = \sigma(\overset{0}{N})^*$
$(n+1)^0 + (n+2)^0 + \dots$ *in inf.*	$\overset{n}{N} = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 1, \dots)$	$\sigma(\overset{n}{N}) = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, 3, \dots)$	$\sigma(\overset{n}{N})(i) = i \dot{-} n$	$\overset{\mathbf{n}}{\mathbf{N}} = \sigma(\overset{n}{N})^*$
$1 + 2 + 3 + \dots$ *in inf.*	$\overset{1}{S} = (1, 2, 3, 4, \dots)$	$\sigma(\overset{1}{S}) = (1, 3, 6, 10, \dots)$	$\sigma(\overset{1}{S})(i) = \sum_{j=1}^{i} j$	$\overset{\mathbf{1}}{\mathbf{S}} = \sigma(\overset{1}{S})^*$
$1^n + 2^n + 3^n \dots$ *in inf.*	$\overset{n}{S} = (1^n, 2^n, 3^n, 4^n, \dots)$	$\sigma(\overset{n}{S}) = (1^n, (1^n + 2^n), \dots)$	$\sigma(\overset{n}{S})(i) = \sum_{j=1}^{i} j^n$	$\overset{\mathbf{n}}{\mathbf{S}} = \sigma(\overset{n}{S})^*$
$a - a + a - a + \dots$ *in inf.*	$G_a = (a, -a, a, -a, \dots)$	$\sigma(G_a) = (a, 0, a, 0, \dots)$	$\sigma(G_a)(i) = \begin{cases} a & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$	$\mathbf{G_a} = \sigma(G_a)^*$

Table 9.1: Representation of Bolzanian sums in $\mathbb{Z}_\mathsf{U}$

9.4.2 Modelling Bolzano's Results about Infinite Sums

We now establish some results that echo Bolzano's own computations. We will first give proofs in our framework, then argue that those proofs are very close in spirit to Bolzano's arguments. We start with results about infinite sums of the form $\overset{\mathbf{n}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{S}}$:

Lemma 9.4.2.

1. *For any natural numbers* i, n, $\mathbb{Z}_{\mathsf{U}} \models i < \overset{\mathbf{n}}{\mathbf{N}}$.

2. *For any natural number* n, $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n$.

3. *For any natural number* i, $\mathbb{Z}_{\mathsf{U}} \models i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}$.

4. *For any natural numbers* i, n, $\mathbb{Z}_{\mathsf{U}} \models i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$.

The first result asserts that all sums of the form $\overset{\mathbf{n}}{\mathbf{N}}$ are infinite, in the sense that they are greater than any finite number. The second shows that our model preserves Bolzano's part-whole intuition that certain infinite sums might have fewer terms than some others and that, as a consequence, two infinite quantities might differ by a finite quantity. Finally, the last two correspond to Bolzano's claim that some infinite quantities might be infinitely greater than some others. Note that we write $n\alpha$ as a shorthand for the sum of α with itself n times, which is defined in the ultrapower.

The proofs for all four items are all similar and can be thought of as "arguments by cofiniteness". In all cases, we show that $\mathbb{Z}_{\mathsf{U}}$ satisfies a formula φ by showing that $||\varphi||$ is a cofinite subset of ω^+ and must therefore belong to U (since U is non-principal, it contains no finite set, so it must contain all cofinite sets).

Proof.

1. Recall that, in $\mathbb{Z}_{\mathsf{U}}$, the natural number i corresponds to (the equivalence class of) the function $e_i : m \mapsto i$. Moreover, for any natural number n, $\overset{\mathbf{n}}{\mathbf{N}}(i) = i \dot{-} n$. Thus $||i < \overset{\mathbf{n}}{\mathbf{N}}|| = \{j \in \omega^+ : i < j \dot{-} n\} = \{j \in \omega^+ : i + n < j\}$. Hence $||i < \overset{\mathbf{n}}{\mathbf{N}}||$ is a cofinite subset of ω^+ and belongs to U, from which it follows that $\mathbb{Z}_{\mathsf{U}} \models i < \overset{\mathbf{n}}{\mathbf{N}}$.

2. Again, in $\mathbb{Z}_{\mathsf{U}}$, n is (the equivalence class of) the function $e_n : m \mapsto n$. Moreover, $\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}}$ is (the equivalence class of) the function $f : \omega^+ \to Z$ such that

$$f(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) - \overset{\mathbf{n}}{\mathbf{N}}(i) = i - (i \dot{-} n)$$

for any $i \in \omega^+$. Hence $||\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n|| = \{i \in \omega^+ : i - (i \dot{-} n) = n\} = \{i \in \omega^+ : i \geq n\}$. Hence $||\overset{\mathbf{n}}{\mathbf{N}} = \overset{\mathbf{0}}{\mathbf{N}} - n||$ is a cofinite subset of ω^+, and $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{n}}{\mathbf{N}} = \overset{\mathbf{0}}{\mathbf{N}} - n$.

3. Since $i\overset{\mathbf{0}}{\mathbf{N}} = \underbrace{\overset{\mathbf{0}}{\mathbf{N}} + \ldots + \overset{\mathbf{0}}{\mathbf{N}}}_{i \text{ times}}$, we have that $i\overset{\mathbf{0}}{\mathbf{N}}(j) = i \times j$ for any $j \in \omega^+$. On the other hand, $\overset{\mathbf{1}}{\mathbf{S}}(j) = \sum_{k=1}^{j} k$ which, by Gauss's summation theorem, is equal to $\frac{j(j+1)}{2}$. Hence

$$||i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}|| = \{j \in \omega^+ : i \times j < \sum_{k=1}^{j} k\} = \{j \in \omega^+ : i \times j < \frac{j(j+1)}{2}\}$$
$$= \{j \in \omega^+ : i < \frac{j+1}{2}\}.$$

   Hence $||i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}||$ is cofinite, and $\mathbb{Z}_{\mathsf{U}} \models i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}$.

4. The argument is a simple generalization of the one above. Fix some natural numbers i and n. Then for any $k \in \omega^+$, $i\overset{\mathbf{n}}{\mathbf{S}}(k) = i\sum_{j=1}^{k} j^n$, and $\overset{\mathbf{n+1}}{\mathbf{S}}(k) = \sum_{j=1}^{k} j^{n+1}$. This means that $(\overset{\mathbf{n+1}}{\mathbf{S}} - i\overset{\mathbf{n}}{\mathbf{S}})(k) = \overset{\mathbf{n+1}}{\mathbf{S}}(k) - i\overset{\mathbf{n}}{\mathbf{S}}(k) = \sum_{j=1}^{k}(j^{n+1} - ij^n)$ for any $k \in \omega^+$. Now since $(j^{n+1} - ij^n)$ is positive for any $j > i$ and in fact assumes arbitrarily large positive values, it follows that $(\overset{\mathbf{n+1}}{\mathbf{S}} - i\overset{\mathbf{n}}{\mathbf{S}})(k)$ is positive for any large enough k. Thus $||\overset{\mathbf{n+1}}{\mathbf{S}} - \overset{\mathbf{n}}{\mathbf{S}} > 0||$ is a cofinite subset of ω^+. Now since $\mathbb{Z} \models \forall x \forall y (x - y > 0 \to y < x)$, by Łoś's Theorem we have that $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{n+1}}{\mathbf{S}} - i\overset{\mathbf{n}}{\mathbf{S}} > 0 \to i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$. Hence $\mathbb{Z}_{\mathsf{U}} \models i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$ for any natural numbers i and n.[11] □

Let us now compare the proofs above with Bolzano's arguments in sections 29 and 33 of *PU*. Bolzano does not explicitly argue for results (1) and (3): in §29, he seems to take for granted that sums of the form $\overset{\mathbf{0}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{N}}$ designate infinite quantities, and he simply writes that $\overset{\mathbf{1}}{\mathbf{S}}$ is "far greater than $\overset{0}{N}$". However, the same section contains the following argument for (2):

> If we designate [the number of all natural numbers] by $\overset{0}{N}$ and therefore form the merely symbolic equation
>
> $$1^0 + 2^0 + 3^0 + \ldots + n^0 + (n+1)^0 + \ldots \textit{ in inf.} = \overset{0}{N} \quad (1)$$

[11] A more direct proof of this result can also be given using more advanced resources from number theory. It is a standard number-theoretic fact (using for example Faulhaber's formula) that for any natural numbers k, n, $\sum_{j=1}^{k} j^n$ is a polynomial of degree $n+1$ in k, with leading term $\frac{1}{n+1}k^{n+1}$. Thus $i\overset{\mathbf{n}}{\mathbf{S}}(k)$ is a polynomial in k of degree $n+1$ with leading term $\frac{i}{n+1}k^{n+1}$, while $\overset{\mathbf{n+1}}{\mathbf{S}}(k)$ is a polynomial in k of degree $n+2$ with leading term $\frac{1}{n+2}k^{n+2}$. This means that $i\overset{\mathbf{n}}{\mathbf{S}}(k) < \overset{\mathbf{n+1}}{\mathbf{S}}(k)$ for k sufficiently large, and thus $||i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}||$ is a cofinite subset of ω^+.

> and in the same way we designate the number of natural numbers from $(n+1)$ $\overset{n}{N}$, and therefore form the equation
>
> $$(n+1)^0 + (n+2)^0 + (n+3)^0 + \dots \textit{ in inf.} = \overset{n}{N}, \qquad (2)$$
>
> then we obtain by subtraction the certain and quite unobjectionable equation
>
> $$1^0 + 2^0 + 3^0 + \dots + n^0 = n = \overset{0}{N} - \overset{n}{N} \qquad (3)$$
>
> from which we therefore see how two infinite quantities $\overset{0}{N}$ and $\overset{n}{N}$ sometimes have a completely definite finite difference.

As mentioned in Section 9.3, we read Bolzano as arguing that subtracting $\overset{n}{N}$ from $\overset{0}{N}$ amounts to subtracting from each term i^0 after the n^{th} summand in $\overset{0}{N}$ the corresponding term i^0 in $\overset{n}{N}$. The only terms left in $\overset{0}{N} - \overset{n}{N}$ after this procedure are the first n summands in $\overset{0}{N}$, from which it follows that $\overset{0}{N} - \overset{n}{N} = n$. In our setting, $\overset{\mathbf{0}}{\mathbf{N}}$ is represented by (the equivalence class of) the sequence $(1, 2, 3, \dots)$, while $\overset{\mathbf{n}}{\mathbf{N}}$ is represented by the sequence $(\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, 3 \dots)$, and $\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}}$ is the sequence obtained by subtracting $\overset{\mathbf{0}}{\mathbf{N}}$ from $\overset{\mathbf{n}}{\mathbf{N}}$ componentwise, i.e., the sequence $(1, 2, 3 \dots, n, n, n, \dots)$, which over U is equivalent to n. Similarly to Bolzano's argument, the difference between the two infinite sums $\overset{\mathbf{0}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{N}}$ is determined by the difference between matching summands (i.e., the difference is computed componentwise) and is precisely n.

Finally, Bolzano does not explicitly argue for (4) in its full generality. In a very revealing passage in §33, however, he gives a detailed argument for the $n = 1$ instance of (4) when arguing that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$:

> But if the multitude of terms [*Menge der Glieder*] in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the *first* term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms [*Gliederzahl*] as $\overset{1}{S}$, namely:
>
> $$0, 2, 6, 12, 20, 30, 42, 56, \dots, n(n-1), \dots \textit{ in inf.},$$

> in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. If we therefore subtract from this remainder the series $\overset{1}{S}$ for the second time, then we obtain as the *second* remainder a series of the same number of terms [*Gliedermenge*]
>
> $$-1, 0, 3, 8, 15, 24, 35, 48, \ldots, n(n-2), \ldots \textit{ in inf.}$$
>
> in which, with the exception of the first *three* terms all the following terms are greater than the corresponding ones in $\overset{1}{S}$, so that also this third remainder is without contradiction greater than $\overset{1}{S}$. Now since these arguments can be continued without end it is clear that the sum $\overset{2}{S}$ is infinitely greater than the sum $\overset{1}{S}$, while in general we have
>
> $$\overset{2}{S} - m\overset{1}{S} = (1-m)+(2^2-2m)+(3^2-3m)+(4^2-4m)+\ldots+(m^2-m^2)+\ldots+n(n-m)+\ldots \textit{ in inf.}$$
>
> In this series only a finite multitude of terms [*Menge von Gliedern*], namely the first $m-1$ are negative and the m^{th} is 0, but all succeeding ones are positive and increase indefinitely.

Let us note two features of Bolzano's argument that are shared by our interpretation. First, when determining whether one infinite sum is greater than another one, Bolzano considers which terms in the first sum are greater than the corresponding terms in the second one: this is reminiscent of the way relations between (equivalence classes of) functions are determined in an ultrapower. Moreover, Bolzano's reason to claim that $\overset{2}{S}$ is greater than $\overset{1}{S}$, $2\overset{1}{S}$, $3\overset{1}{S}$, and so on, is that in all such cases, all but finitely many terms in $\overset{2}{S}$ are strictly greater than the corresponding terms in any finite multiple of $\overset{1}{S}$. This seems very similar to the "argument by cofiniteness" that we presented above: even though the first terms of the sum $m\overset{1}{S}$ might be greater than the first terms of the sum $\overset{2}{S}$, the terms in the second sum become greater than the corresponding terms in the first one from some point onwards. In our setting, we prove that $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{2}}{\mathbf{S}} > m\overset{\mathbf{1}}{\mathbf{S}}$ by showing that $\overset{\mathbf{2}}{\mathbf{S}}(i) - m\overset{\mathbf{1}}{\mathbf{S}}(i) > 0$ for cofinitely many natural numbers i. To establish this, it is enough to observe, like Bolzano, that $i^2 - mi$ is positive for any $i > m$, as this implies that the sum $\sum_{j=1}^{i}(j^2 - mj)$ must be positive for i large enough. It is worth mentioning that, unlike in Bolzano's argument, our "tipping point", i.e., the value i at which $\overset{\mathbf{2}}{\mathbf{S}}(i)$ becomes strictly greater than $m\overset{\mathbf{1}}{\mathbf{S}}(i)$ is not $m+1$. This is because $\overset{\mathbf{2}}{\mathbf{S}}$ and $\overset{\mathbf{1}}{\mathbf{S}}$ are the approximating sequences of the sequences $(1, 2, 3, \ldots)$ and $(1, 4, 9, \ldots)$ respectively, while Bolzano is reasoning with the sequences of terms themselves. We therefore conclude that the general proof given for 4 closely matches Bolzano's own reasoning. In particular, our use of an ultrapower construction enables us to lift the following criterion for the inequality of two integers:

$$\forall m, n(m < n \leftrightarrow n - m > 0) \tag{9.1}$$

to a criterion for the inequality of two infinite sums:

$$\forall \boldsymbol{\alpha}, \boldsymbol{\beta}(\boldsymbol{\alpha} < \boldsymbol{\beta} \leftrightarrow \{i \in \omega^+ : (\boldsymbol{\beta} - \boldsymbol{\alpha})(i) > 0\} \in \mathsf{U}). \tag{9.2}$$

In other words, in our formalism, in order to determine whether an infinite sum $\boldsymbol{\alpha}$ is greater than another infinite sum $\boldsymbol{\beta}$, it is enough to compute their difference $\boldsymbol{\beta} - \boldsymbol{\alpha}$, which is defined termwise, and then determine whether the sum of the first i terms of $\boldsymbol{\beta} - \boldsymbol{\alpha}$ is positive for U-many i. Our claim is that this reasoning is very close to the one displayed by Bolzano in §33. Moreover, let us note that when he argues that $\overset{2}{S}$ is greater than $m\overset{1}{S}$ for any m, because all but finitely many terms in the infinite sum $\overset{2}{S} - m\overset{1}{S}$ are positive, Bolzano can be seen as implicitly displaying a form of part-whole reasoning about *sums*, rather than sets: $m\overset{1}{S}$ is smaller than $\overset{2}{S}$ because it is contained "as a part". This is established by showing that the difference $\overset{2}{S} - m\overset{1}{S}$ is positive, and this latter fact is established in turn by noticing that all but finitely many terms in $\overset{2}{S} - m\overset{1}{S}$ are positive. Thus Bolzano can be read here as providing a criterion for when the quantity designated by a sum α is a proper part of the quantity designated by another sum β. We will come back to this point in Section 9.6, and we will discuss its implication for the role that part-whole reasoning plays in Bolzano's computations with the infinitely large.

9.4.3 Grandi's Series

Finally, let us address some of Bolzano's remarks on Grandi's series. As noted above, Bolzano disagrees with the claim (attributed to M.R.S.) that the infinite sum

$$x = a - a + a - a + \dots \textit{ in inf.}$$

designates the quantity $\frac{a}{2}$. In particular, Bolzano claims that the mistake in M.R.S.'s proof is to treat the sum obtained by discarding the first term of x as $-x$. In our setting, x designates the quantity $\mathbf{G_a}$, i.e., the equivalence class of the sequence $(a, 0, a, 0, \dots)$. On the other hand, following the strategy adopted for "truncated" infinite sums like $\overset{\mathbf{n}}{\mathbf{N}}$, it seems that the infinite sum obtained by discarding the first term in x should be interpreted as the countable sequence $(0, -a, a, -a, a, \dots)$. If we write this sequence as $\overset{1}{G}_a$, we then have that $\overset{\mathbf{1}}{\mathbf{G}}_\mathbf{a}$ is the equivalence class of the sequence $(0, -a, 0, -a, \dots)$. But then, it follows that

$$\mathbb{Z}_\mathsf{U} \models \mathbf{G_a} - \overset{\mathbf{1}}{\mathbf{G}}_\mathbf{a} = a.$$

Indeed, for any $i \in \omega^+$, $\mathbf{G_a}(i) = a$ if i is even and 0 if i is odd, while $\overset{\mathbf{1}}{\mathbf{G}}_{\mathbf{a}}(i) = 0$ if i is even and $-a$ if i is odd. Thus $\mathbf{G_a}(i) - \overset{\mathbf{1}}{\mathbf{G}}_{\mathbf{a}}(i) = a$ for any i. Hence our interpretation agrees with Bolzano's diagnostic of the fallacy in M.R.S.'s proof:

> The series in the brackets obviously does not have the same multitude of terms [*Gliedermenge*] as the one put $= x$ at first, rather it is lacking the first a. Therefore its value, supposing it could actually be stated, would have to be denoted by $x - a$. But this would have given the identical equation
>
> $$x = a + x - a.$$

Moreover, recall that Bolzano raises a second, deeper argument against M.R.S.'s conclusion: the infinite sum x cannot designate an "actual quantity", since different ways of parsing this infinite sum yield different conclusions regarding which quantity it allegedly designates. According to Bolzano, the infinite sum

$$a - a + a - a + \dots \textit{ in inf.}$$

represents the same quantity as the sums

$$(a - a) + (a - a) + (a - a) + \dots \textit{ in inf.}$$

and

$$a + (-a + a) + (-a + a) + (-a + a) + \dots \textit{ in inf.}$$

But the first expression simplifies as

$$0 + 0 + 0 + \dots \textit{ in inf.},$$

while the second one simplifies as

$$a + 0 + 0 + 0 + \dots \textit{ in inf.}.$$

Therefore, if it were a real quantity, x should be equal to both 0 and a, which is a contradiction.

What does this argument become in our interpretation? At first sight, it seems that we cannot make sense of Bolzano's claim that Grandi's series does not represent any actual quantity, since we attributed to this series the element $\mathbf{G_a}$ in $\mathbb{Z}_{\mathsf{U}}$. However, it is straightforward to verify that, depending on which subsets of ω^+ are in U, $\mathbf{G_a}$ is computed differently in the ultrapower. Indeed, since $\mathbf{G_a}$ is the (equivalence class of) the sequence $(a, 0, a, 0, \dots)$, we have that $||\mathbf{G_a} = a|| = \{2i - 1 : i \in \omega^+\}$, while $||\mathbf{G_a} = 0|| = \{2i : i \in \omega^+\}$. Now since U is an ultrafilter, exactly one of $||\mathbf{G_a} = a||$ or $||\mathbf{G_a} = 0||$ belongs to U. This implies that $\mathbb{Z}_{\mathsf{U}} \models \mathbf{G_a} = a \vee \mathbf{G_a} = 0$ regardless of our choice of ultrafilter, but the choice of U determines whether $\mathbb{Z}_{\mathsf{U}} \models \mathbf{G_a} = a$ or $\mathbb{Z}_{\mathsf{U}} \models \mathbf{G_a} = 0$. Thus we seem to recover at last part of Bolzano's intuition that the quantity designated by the sum $a-a+a-a+\dots$ *in inf.* is indeterminate, as it can be computed to be equal to 0 or to a.

Bolzano also argues that the sum $a-a+a-a+\dots$ *in inf.* should represent the same quantity as the sum

$$-a+(a-a)+(a-a)+\dots \textit{ in inf.},$$

which simplifies to

$$-a+0+0+\dots \textit{ in inf.},$$

and should therefore designate the quantity $-a$. His argument is that one may first compute Grandi's series as

$$(a-a)+(a-a)+\dots \textit{ in inf.}$$

Using commutativity of addition an infinite number of times, swap each pair of terms in order to obtain the series

$$(-a+a)+(-a+a)+\dots \textit{ in inf.},$$

which, by associativity is then equivalent to

$$-a+(a-a)+(a-a)+\dots \textit{ in inf.}$$

In our setting, the infinite sum $-a+a-a+a-\dots$ *in inf.* is represented by its approximating sequence $(-a,0,-a,0,\dots)$. As a consequence, the infinite sums $a-a+a-a+\dots$ *in inf.* and $-a+a-a+a-\dots$ *in inf.* will be identified in $\mathbb{Z}_{\mathsf{U}}$ precisely if $\{2i : i\in\omega^+\}\in\mathsf{U}$. In fact, as shown above, in such a case both series will be identified with 0.

In light of the remarks above, it might be tempting to conclude that Bolzano's criterion for an infinite sum to represent an actual quantity, namely that the order in which the terms are summed do not change the result of the summation, could be interpreted in our framework as some kind of absoluteness of the corresponding sequences under the choice of a non-principal ultrafilter U. However, it is straightforward to observe that Bolzano's own criterion is too strong for his purposes. Indeed, let us consider again the infinite sum $\overset{0}{N}=1^0+2^0+3^0+\dots$ *in inf.* If we interpret, as we have done so far, n^0 as equal to 1 for any natural number n, then this infinite sum may actually be written as $1+1+1+\dots$ *in inf.*, which is a special case of a geometric series of the form $\sum_{n=0}^{\infty}ar^n$ where $a=r=1$. Similarly to Bolzano's argument for Grandi's series, we may now rewrite $\overset{0}{N}$ as

$$\begin{aligned}(1+1)+(1+1)+(1+1)+\dots \textit{ in inf.} &= 2+2+2\dots \textit{ in inf.}\\ &= 2(1+1+1\dots \textit{ in inf.})\\ &= 2\overset{0}{N},\end{aligned}$$

from which we would be forced to conclude that $\overset{0}{N}-\overset{0}{N}=\overset{0}{N}$, implying that $\overset{0}{N}=0$. Thus $\overset{0}{N}$ does not designate any infinite quantity after all, since it is equal to 0. This means that the order in which the terms in $\overset{0}{N}$ are summed determine which quantity the sum designates, which, by Bolzano's own criterion, is impossible. Of course, a Bolzanian could reply to that argument that there is a fallacy in deriving this equality, because the sum between parenthesis on the second line above does not have the same *Gliedermenge* as the original $1+1+1+\ldots$ *in inf.* Note that this response implies that changing the order in which terms are summed together, although it does not change the quantity designated by the sum, does change its *Gliedermenge*. Moreover, this answer is not entirely satisfactory. Indeed, if we assume that the right-hand side of the first equation above does not have the same *Gliedermenge* as $\overset{0}{N}$, we may therefore represent the two sums $(1+1)+(1+1)+(1+1)+\ldots$ *in inf.* and $1+(1+1)+(1+1)+\ldots$ *in inf.* by the sequences $A_1 := (0,2,0,2,0,2,\ldots)$ and $A_2 := (1,0,2,0,2,0,\ldots)$ respectively. Since both sums correspond to different ways of writing $\overset{0}{N}$, we should expect that $\boldsymbol{A_1} = \boldsymbol{A_2} = \overset{\mathbf{0}}{\mathbf{N}}$. However, one quickly notices that $\sigma(A_1)(i) < \sigma(A_2)(i)$ whenever i is odd, and $\sigma(A_2)(i) < \sigma(A_1)(i)$ whenever i is even. But this immediately implies that $\mathbb{Z}_{\mathsf{U}} \models \boldsymbol{A_1} \neq \boldsymbol{A_2}$. In other words, if we interpret the two infinite sums $(1+1)+(1+1)+(1+1)+\ldots$ *in inf.* and $1+(1+1)+(1+1)+\ldots$ *in inf.* by A_1 and A_2, then in order to satisfy Bolzano's requirement that infinite associativity holds, we would need both the set of even numbers and the set of odd numbers to be in U, which is not possible. Note however that this has little to do with our formalization: Bolzano himself seems committed to the following equalities:

$$(1+1)+(1+1)+(1+1)+\ldots \textit{ in inf.} = \overset{0}{N} = 1+(1+1)+(1+1)+\ldots \textit{ in inf.}$$
$$2+2+2+\ldots \textit{ in inf.} = 1+2+2+\ldots \textit{ in inf.},$$

but there does not seem to be any reasonable way of establishing directly the latter equality. However, let us conclude this section by noting that a weaker requirement could be imposed on infinite sums which designate actual quantities, namely that any *finite* permutation of the terms or of the order in which such terms are summed does not change the value of the sum. However, it is straightforward to verify that all sums in our formalization satisfy this criterion: any two infinite sums that differ from one another only by a finite permutation of their terms or by finitely many rearrangements of the order in which those terms are summed are represented by approximating sequences which agree on a cofinite set and are therefore identified in $\mathbb{Z}_{\mathsf{U}}$. Thus this alternative criterion is too weak to rule out Grandi's series. In short, while Bolzano's first argument against M.R.S can easily be translated in our framework, his second argument seems to prove either too much, or too little, for his purposes.

9.4.4 Comparisons with Related Work

Our central proposal is to model Bolzano's computations inside an ultrapower of the integers, and to identify the quantities designated by Bolzanian infinite sums with equivalence classes of functions from the positive integers to the integers. This idea is very close to a proposal made by Trlifajová [255], although there are a few important differences that we must remark on. First, Trlifajová seems to be primarily interested in connecting Bolzano's ideas with some modern approaches to non-standard analysis, while we are more interested in a close reading of Bolzano's arguments and in establishing the consistency of our interpretation. Second, Trlifajová works mainly with equivalence classes of functions from ω to the real numbers. By contrast, we work with countable sequences of integers. Indeed, we believe that determining whether Bolzano's notion of a real number corresponds to our modern notion is a difficult problem. Bolzano, of course, made some significant contributions to the foundations of analysis. In particular, he developed a theory of *measurable* numbers [50, Part VII] which is often seen as an attempt to define the real numbers [see e.g. 224, 243, 228]. Trying to model Bolzano's computations with real numbers would require us to provide a detailed discussion of Bolzano's theory of measurable numbers. Since we are primarily interested in challenging the received view according to which Bolzano's computations should be read as a flawed attempt to develop an arithmetic of the transfinite, we believe that addressing this issue would take us too far astray. Just as Bolzano's measurable numbers are beyond the scope of our goals for this paper, so are Bolzano's arguments in PU involving infinitely small quantities or infinitesimal calculus. Third, let us note that, in Trlifajová's framework, two sequences are identified if they agree on a cofinite set of natural numbers. Formally, this means that she works with a reduced power of $\mathbb{R}$ rather than an ultrapower. While we do see the appeal of using only the Fréchet filter on the natural numbers instead of a non-principal ultrafilter, we have several reasons to believe that our framework is more suitable to our purposes.

For one, only a weaker version of Łoś's theorem holds for reduced powers [see 129, p. 445], which means that the resulting structure will not be as well-behaved as the ultrapower construction we are using. While this does not create significant technical issues at this stage, we will argue in the next section that the most accurate way of modelling Bolzano's views on the product of infinite sums is to conceive of it as an interated infinite summation. This means that one will have to work with either iterated ultrapowers, or iterated reduced powers, the general theory of which is much less developed.

Moreover, we believe that the use of a non-principal ultrafilter rather than the Fréchet filter can also be justified on interpretive grounds. Indeed, it is straightforward to verify that the reduced power $\mathbb{Z}_{\mathscr{F}}$, in which two sequences α and β are identified only if $||\alpha = \beta||$ is cofinite, does not satisfy trichotomy. For example, for the sequence $\alpha = (1, 0, 1, 0, \dots)$, we have that none of $||\alpha = 0||$, $||\alpha < 0||$ or $||0 < \alpha||$ is a cofinite set, and thus $\mathbb{Z}_{\mathscr{F}} \models \neg(\alpha < 0) \wedge \neg(\alpha > 0) \wedge \neg(\alpha = 0)$. One might argue that this is a desirable feature of a formal reconstruction of Bolzano's ideas about the infinite, since, in §28, Bolzano writes that "a determination of the relationship of one infinity to one another [...] is feasible, in certain cases at any rate ...". Nonetheless, we think that Bolzano should not be read in this passage as claiming that trichotomy may not hold in the case of infinite quantities. Indeed, as mentioned in Section 9.2.1, it is part of Bolzano's very definition of a quantity that it must obey the law of trichotomy. All things considered, then, we believe that a formalisation that preserves trichotomy–such as ours, using ultrapowers–is more faithful to the text than a formalization that preempts the very possibility of trichotomy for infinite quantities, such as one using the Fréchet filter.[12]

A second related work is the recently proposed theory of numerosities [18], which we have discussed extensively in the previous chapter. Numerosities share some features with our interpretation of Bolzano's computations, in particular regarding the way sums of infinite quantities are defined. However, a central motivation for the numerosity framework is to develop a theory of the size of sets of natural numbers that is consistent with what we called the set-theoretic part-whole principle **PW**. As we will argue in Section 9.6, we take Bolzano's arithmetic of the infinite to be compatible with the set-theoretic part-whole principle but not motivated by it, as we do not believe that Bolzano is primarily concerned with counting sets of natural numbers but rather with developing a theory of infinite sums.

9.5 Higher-order Infinities

9.5.1 The Product of two Infinite Quantities

So far, we have shown how to interpret Bolzano's computations regarding infinite sums of the form $\overset{\mathbf{n}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{S}}$, as well as Grandi's series. We have, however, refrained from giving an interpretation of Bolzano's computations involving products of two infinite quantities. Although our treatment of Bolzano's computations so far closely matches Trlifajová's and is consistent with numerosities, our account of Bolzanian products of infinite quantities will be quite different. Indeed, it seems at first sight that there is a natural way to define the product of two quantities in $\mathbb{Z}_{\mathsf{U}}$. Similarly to the way addition is defined, we could define the product componentwise. Formally, for any $f, g : \omega^+ \to Z$, letting $f \cdot g : \omega^+ \to Z$ be the function mapping any $i \in \omega^+$ to $f(i) \times g(i)$, we may define $f^* \cdot g^*$ as $(f \cdot g)^*$. This is the definition adopted by Benci and Di Nasso [18] and Trlifajová [255], and it is straightforward to check that, under this definition, the structure $(\mathbb{Z}_{\mathsf{U}}, +, \cdot, 0, 1, <)$ is an ordered commutative ring. However, we believe that this definition of the product does not satisfactorily account for Bolzano's ideas as exposed in *PU*. We will first lay out our textual evidence for this claim and then explain how our interpretation works.

Bolzano gives explicit computations of the product of two infinite quantities in only one passage towards the end of §29:

> The purely symbolic equation [(1)][13] underlying all this will surely allow the

[12] The perceptive reader will have remarked that the dialectic between ultrapowers and reduced powers arising here is quite familiar. We have encountered similar issues in Chapter 7 when discussing the more adequate way of constructing the hyperreals, and in Chapter 8 when discussing both numerosities and NAP functions. In both cases, we have argued that generic powers offered a way out of the dilemmas presented there. Perhaps unsurprisingly, generic powers will also offer us a way out of the debate here, as we will show in Section 9.7 below.

[13] The German version of the text reads (4) here, but the context clearly suggests that this is a mistake.

> derivation, through successive multiplication of both sides by $\overset{0}{N}$, of the following equations:
>
> $$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \ldots \textit{ in inf.} = (\overset{0}{N})^2$$
> $$1^0.\overset{0}{N}^2 + 2^0.\overset{0}{N}^2 + 3^0.\overset{0}{N}^2 + \ldots \textit{ in inf.} = (\overset{0}{N})^3 \qquad \text{etc.}$$
>
> from which we are convinced that there [are] also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times. But it also certainly follows from this [that] there are infinite quantities which have every arbitrary rational, as well as irrational, ratio $\alpha : \beta$ to one another, because, as long as $\overset{0}{N}$ denotes some infinite quantity which always remains the same, $\alpha.\overset{0}{N}$ and $\beta.\overset{0}{N}$ are likewise a pair of infinite quantities which are in the ratio $\alpha : \beta$.

Bolzano defines the product of the quantity $\overset{0}{N}$ with itself, noted $(\overset{0}{N})^2$, as the result of summing $\overset{0}{N}$ with itself $\overset{0}{N}$ many times. The equation

$$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \ldots \textit{ in inf.} = (\overset{0}{N})^2$$

is obtained from the equation

$$1^0 + 2^0 + 3^0 + \ldots n^0 + (n+1)^0 + \ldots \textit{ in inf.} = \overset{0}{N}$$

by multiplying by $\overset{0}{N}$ on both sides. This seems to suggest that Bolzano assumes some form of distributivity of multiplication over infinite summation, which allows him to equate $(1^0 + 2^0 + 3^0 + \ldots \textit{ in inf.}).\overset{0}{N}$ with $1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} \ldots \textit{ in inf.}$ on the left-hand side of the equality symbol. Understood as such, $(\overset{0}{N})^2$ is an infinite sum in which all terms are infinite quantities. Quantities of the form $(\overset{0}{N})^n$ are the only example in Bolzano's text of quantities defined explicitly as infinite sums of infinite quantities. It is also worth mentioning that, even though Bolzano discusses other examples of infinite quantities being infinitely smaller or larger than one another, this is the only case in §§29-33 where some infinite quantities are explicitly referred to as being "of higher order" than some others.[14]

[14]The authors thank an anonymous referee for noting that an alternative interpretation of §29 is also plausible. When introducing $(\overset{0}{N})^2$ and $(\overset{0}{N})^3$, Bolzano writes that this "convinc[es us] that there are also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times." This can be read as meaning that whenever an infinite quantity A exceeds an infinite quantity B infinitely many

If we were to interpret $(\overset{0}{N})^2$ in a similar fashion as Trlifajová and Benci and Di Nasso, we would have to define the quantity $(\overset{\mathbf{0}}{\mathbf{N}})^2$ in such a way that $(\overset{\mathbf{0}}{\mathbf{N}})^2(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) \cdot \overset{\mathbf{0}}{\mathbf{N}}(i) = i^2$ for all $i \in \omega^+$. However, due to the well-known fact that the sum of the first n odd numbers is always equal to n^2, the infinite sum $\overset{Odds}{S} := 1+3+5+7\ldots$ *in inf.* is also represented by (the equivalence class of) the sequence $(1, 4, 9, 16, ...)$. It would therefore follow that $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{Odds}}{\mathbf{S}} = (\overset{\mathbf{0}}{\mathbf{N}})^2$. We should conclude that the two infinite sums $1+3+5+\ldots$ *in inf.* and $\overset{0}{N}+\overset{0}{N}+\overset{0}{N}+\ldots$ *in inf.* actually designate the same quantity. But this seems a clear violation of Bolzano's treatment of order relationships between infinite sums. Indeed, we saw above that, in showing that $\overset{2}{S}$ was infinitely greater than $\overset{1}{S}$, Bolzano reached his conclusion by showing that the difference between matching summands in $\overset{2}{S}$ and in any finite multiple of $\overset{1}{S}$ is always positive for all but finitely many summands. In this case too, since $\overset{Odds}{S}$ and $(\overset{0}{N})^2$ have the same number of terms, we could also argue along Bolzanian lines that, for any natural number i, the difference $(\overset{0}{N})^2 - i\overset{Odds}{S}$ is given by the sum $(\overset{0}{N} - i) + (\overset{0}{N} - 3i) + (\overset{0}{N} - 5i) + \ldots$ *in inf.*, in which all summands are positive (and in fact infinite). As we have argued in Section 9.4.2, one can extract from Bolzano's writings a sufficient criterion for one sum α to be strictly greater than another sum β, namely when all but finitely many terms in the sum $\alpha - \beta$ are positive. We will come back to this issue at greater length in Section 9.6. For now, let us note that, if our interpretation is correct, we must conclude in the present case that $(\overset{0}{N})^2$ is greater than any finite multiple of $\overset{Odds}{S}$, and thus that $(\overset{0}{N})^2 \neq \overset{Odds}{S}$. The componentwise definition of the product of two quantities is therefore incompatible with Bolzano's own criterion for comparing infinite sums.

Moreover, another passage from §29 seems to explicitly contradict the "componentwise" interpretation of the product of two infinite quantities. Indeed, when introducing the sum of all natural numbers $\overset{1}{S}$, Bolzano writes:

> On the other hand if we designate the quantity which represents the *sum* of all

times, then A is an infinite of higher order with respect to B. In other words, the definition of infinities of higher order is infinities that exceed smaller infinities by an infinitely large factor. This understanding of "higher order" is problematic, however, for at least two reasons. First, if "of higher order" simply meant "infinitely larger or smaller", then the introduction of $\overset{1}{S}$ in §29 should have sufficed to establish the existence of infinite quantities of higher-order, since Bolzano has already noted by that point that $\overset{1}{S}$ is "far greater than" $\overset{0}{N}$. Second, in the definition of "infinite" (§10), Bolzano presents the concept of infinitely smaller and infinitely greater quantities of higher order as quantities derived from, but not identical with, infinitely small and infinitely large quantities. The referee's interpretation, by contrast, would collapse the notion of infinities of higher order into that of infinities *simpliciter*, per Bolzano's definition.

natural numbers by $[\overset{1}{S}]$, or assert the merely symbolic equation

$$1 + 2 + 3 + \ldots + n + (n+1) + \ldots \text{ in inf.} = [\overset{1}{S}] \tag{4}$$

then we will certainly realize that $[\overset{1}{S}]$ must be far greater than $\overset{0}{N}$. But it is not so easy to determine precisely the difference between these two infinite quantities or even their (geometrical) *ratio* to one another. For if, as some people have done, we wanted to form the equation

$$[\overset{1}{S}] = \frac{\overset{0}{N} \cdot (\overset{0}{N} + 1)}{2}$$

then we could hardly justify it on any other ground than that for every finite multitude of terms [*Menge von Gliedern*] the equation

$$1 + 2 + 3 + \ldots + n = \frac{n.(n+1)}{2}$$

holds, from which it appears to follow that for the complete infinite multitude of numbers n just becomes $\overset{0}{N}$. However it is in fact not so, because with an infinite series it is absurd to speak of a last term which has the value $\overset{0}{N}$.

Bolzano's point here seems to be that one cannot infer from the validity of Gauss's summation theorem for finite numbers that an "infinitary" version of the summation theorem also holds for infinite quantities. His rejection of the infinite summation theorem can be given two readings, one stronger, and one weaker. On the stronger reading, Bolzano is arguing that the infinite summation theorem is false, because the only way of justifying it, namely, through an inference from the finite to the infinite, leads to a false consequence. On the weaker reading, by contrast, Bolzano is not asserting the falsity of the infinite summation theorem, but he is merely refraining from asserting its truth, because what is ostensibly the only argument to prove its truth is a defective argument.

Under the first reading, which we tend to find more natural, the componentwise definition of the product *à la* Trlifajová [255] and Benci and Di Nasso [18] is simply inconsistent with Bolzano's own views, as the infinite summation theorem is true in the structure $(\mathbb{Z}_{\mathsf{U}}, +, \cdot, 0, 1, <)$:

Lemma 9.5.1. *Let* $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)$ *be such that* $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i)$ *for any* $i \in \omega^+$*. Then* $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}$.

Proof. By definition, $||\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}|| = \{i \in \omega^+ : (\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1))(i) = 2\overset{\mathbf{1}}{\mathbf{S}}(i)\}$. Now for any $i \in \omega^+$, $2\overset{\mathbf{1}}{\mathbf{S}}(i) = 2 \times \frac{i(i+1)}{2} = i(i+1)$ by Gauss's summation theorem. On the other hand, $(\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1))(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i) = i \times (i+1)$. Thus $||\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}|| = \omega^+$, and therefore is contained in U. □

Since we are interested in establishing at least the consistency of Bolzano's calculation of the infinite, the stronger reading of this passage of the infinite summation theorem compels us to provide an alternative definition of the product of two Bolzanian quantities.

Moreover, we find that this conclusion also follows from the second, weaker reading mentioned above. Indeed, even if Bolzano is merely punting here on the truth of the infinite summation theorem, we find it quite revealing that he would object to the infinite summation theorem being a direct consequence of Gauss's summation theorem. Indeed, this passing from the finite to the infinite is very similar to the various "arguments by cofiniteness" that Bolzano appeals to in §§29 and 32, and which we discussed at length in the previous section. As we have noticed above, the formal setting of ultrapowers, in which operations can be defined componentwise, allows for a straightforward reconstruction of such arguments by cofiniteness, with the help of Łoś's theorem. In fact, the proof of Lemma 9.5.1 above proceeds precisely in the same way as the inference rejected by Bolzano: since the summation theorem holds for any $i \in \omega^+$, it transfers to the infinite quantities $\overset{1}{\mathbf{S}}$ and $\overset{0}{\mathbf{N}}$. Bolzano therefore seems to have two distinct attitudes with regard to these inferences from the finite to the infinite: while he uses arguments by cofiniteness when establishing results about sums and differences of infinite sums, he explicitly rejects this style of reasoning when discussing ratios of infinite sums, i.e., results about products of infinite sums. If we were to model such products componentwise, we would be allowing in our formal setting precisely the type of inference that Bolzano objects to. This seems cause enough to us to propose an alternative definition of the products of two Bolzanian sums.

9.5.2 Second-Order Infinities via an Iterated Ultrapower

As shown above, the componentwise interpretation of the product adopted both by Trlifajová and Benci and Di Nasso has unfortunate consequences for our project. If we want to model Bolzanian computations with the infinite as accurately as possible, we must therefore propose an alternative interpretation. Our solution springs from the observation above that the product $(\overset{0}{N})^2$ is written by Bolzano as an infinite sum in which the summands themselves are infinite quantities. Since we decided to model infinite sums of integers as functions from an index set ω^+ into the integers, we should therefore model infinite sums of possibly infinite quantities as functions from ω^+ into a structure that contains those infinite quantities, i.e., into $\mathbb{Z}_{\mathsf{U}}$.

Formally, this means that we should now work in an ultrapower of $\mathbb{Z}_{\mathsf{U}}$, i.e., in an iterated ultrapower. Letting $(\mathbb{Z}_{\mathsf{U}})^2$ denote this ultrapower, we have a straightforward embedding $\iota : \mathbb{Z}_{\mathsf{U}} \to (\mathbb{Z}_{\mathsf{U}})^2$, induced by the map sending any $f : \omega^+ \to Z$ to the map $i \mapsto \overline{f(i)}$, where $\overline{f(i)}$ is the constant function returning $f(i)$ for any $j \in \omega^+$. Given an infinite sum of (possibly infinite) quantities in $\mathbb{Z}_{\mathsf{U}}$, say $\alpha_1 + \alpha_2 + \alpha_3 + \dots$ *in inf.*, we proceed as before by identifying this sum with the countable sequence $\alpha := (\alpha_1, \alpha_2, \alpha_3, \dots)$, and determining its quantity α as the equivalence class in the iterated ultrapower $(\mathbb{Z}_{\mathsf{U}})^2$ of the sequence $(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots)$, where the partial sums of the first n terms in α are computed inside $\mathbb{Z}_{\mathsf{U}}$. In the case of $(\overset{0}{N})^2$, this means that we identify the infinite sum with the sequence $(\overset{0}{N})^2 := (\overset{\mathbf{0}}{\mathbf{N}}, \overset{\mathbf{0}}{\mathbf{N}}, \overset{\mathbf{0}}{\mathbf{N}}, \dots)$. The corresponding approximating sequence is then $(\overset{\mathbf{0}}{\mathbf{N}}, 2\overset{\mathbf{0}}{\mathbf{N}}, 3\overset{\mathbf{0}}{\mathbf{N}}, \dots)$, which means that $(\overset{\mathbf{0}}{\mathbf{N}})^2$ is the equivalence class of the function assigning to each $i \in \omega^+$ the quantity $i\overset{\mathbf{0}}{\mathbf{N}}$. Similarly, we could form the infinite sum $\overset{1}{S}+\overset{1}{S}+\overset{1}{S}+\dots$ *in inf.*, which corresponds to summing the quantity $\overset{1}{S}$ $\overset{0}{N}$-many times to itself. This sum is interpreted as the series $\overset{0}{N}.\overset{1}{S} := (\overset{\mathbf{1}}{\mathbf{S}}, \overset{\mathbf{1}}{\mathbf{S}}, \overset{\mathbf{1}}{\mathbf{S}}, \dots)$, with approximating sequence $(\overset{\mathbf{1}}{\mathbf{S}}, 2\overset{\mathbf{1}}{\mathbf{S}}, 3\overset{\mathbf{1}}{\mathbf{S}}, \dots)$, so $\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}}$ is the equivalence class of the function assigning $i\overset{\mathbf{1}}{\mathbf{S}}$ to each $i \in \omega^+$.

Going one step further, we could also wonder how the product $\overset{1}{S}.\overset{0}{N}$, i.e., summing $\overset{1}{S}$-many times the quantity $\overset{0}{N}$, should be interpreted. Just as we computed $(\overset{0}{N})^2$ by taking $\overset{0}{N}$ as a unit in our summation instead of 1, it seems that, in computing $\overset{1}{S}.\overset{0}{N}$, we should take $\overset{0}{N}$ as a unit in the summation $1 + 2 + 3 + \dots$ *in inf.* which yields $\overset{1}{S}$. This suggests that summing $\overset{0}{N}$ $\overset{1}{S}$-many times with itself yields the infinite sum

$$\overset{0}{N} + 2\overset{0}{N} + 3\overset{0}{N} + \dots \textit{ in inf.}$$

According to our interpretation, this sum is represented by the sequence

$$\overset{1}{S}.\overset{0}{N} := (\overset{0}{N}, 2\overset{0}{N}, 3\overset{0}{N}, \dots),$$

whose approximating sequence is $(\overset{0}{N}, 3\overset{0}{N}, 6\overset{0}{N}, \dots)$. Hence $\overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}}$ is the equivalence class of the function that assigns $\overset{\mathbf{1}}{\mathbf{S}}(i)\overset{\mathbf{0}}{\mathbf{N}} = \frac{i(i+1)}{2}\overset{\mathbf{0}}{\mathbf{N}}$ to any $i < \omega$. More generally, given any two infinite quantities $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathbb{Z}_{\mathsf{U}}$, we may define the product $\boldsymbol{\alpha}.\boldsymbol{\beta} \in (\mathbb{Z}_{\mathsf{U}})^2$ as the equivalence class of the function mapping any $i < \omega^+$ to $\boldsymbol{\alpha}(i) \times \boldsymbol{\beta}$, where $\boldsymbol{\alpha}(i) \times \boldsymbol{\beta} = \underbrace{\boldsymbol{\beta} + \boldsymbol{\beta} + \dots + \boldsymbol{\beta}}_{\boldsymbol{\alpha}(i) \text{ times}}$. The relevant definitions are summarized in the table below:

Bolzanian Infinite Sum	Sequence Representation	approximating sequence	Corresponding Function	Infinite Quantity
$\overset{0}{N} + \overset{0}{N} + \overset{0}{N} + \dots$ *in inf.*	$(\overset{0}{N})^2 = (\overset{\mathbf{0}}{\mathbf{N}}, \overset{\mathbf{0}}{\mathbf{N}}, \overset{\mathbf{0}}{\mathbf{N}}, \dots)$	$\sigma(\overset{0}{N})^2) = (\overset{\mathbf{0}}{\mathbf{N}}, 2\overset{\mathbf{0}}{\mathbf{N}}, 3\overset{\mathbf{0}}{\mathbf{N}}, \dots)$	$\sigma((\overset{0}{N})^2)(i) = i\overset{\mathbf{0}}{\mathbf{N}}$	$(\overset{\mathbf{0}}{\mathbf{N}})^2 = \sigma((\overset{0}{N})^2)^*$
$\overset{1}{S} + \overset{1}{S} + \overset{1}{S} + \dots$ *in inf.*	$\overset{0}{N}.\overset{1}{S} = (\overset{\mathbf{1}}{\mathbf{S}}, \overset{\mathbf{1}}{\mathbf{S}}, \overset{\mathbf{1}}{\mathbf{S}} \dots)$	$\sigma(\overset{0}{N}.\overset{1}{S}) = (\overset{\mathbf{1}}{\mathbf{S}}, 2\overset{\mathbf{1}}{\mathbf{S}}, 3\overset{\mathbf{1}}{\mathbf{S}} \dots)$	$\sigma(\overset{0}{N}.\overset{1}{S})(i) = i\overset{\mathbf{1}}{\mathbf{S}}$	$\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}} = \sigma(\overset{0}{N}.\overset{1}{S})^*$
$1\overset{0}{N} + 2\overset{0}{N} + 3\overset{0}{N} + \dots$ *in inf.*	$\overset{1}{S}.\overset{0}{N} = (1\overset{\mathbf{0}}{\mathbf{N}}, 2\overset{\mathbf{0}}{\mathbf{N}}, 3\overset{\mathbf{0}}{\mathbf{N}}, 4\overset{\mathbf{0}}{\mathbf{N}}, \dots)$	$\sigma(\overset{1}{S}.\overset{0}{N}) = (1\overset{\mathbf{0}}{\mathbf{N}}, 3\overset{\mathbf{0}}{\mathbf{N}}, 6\overset{\mathbf{0}}{\mathbf{N}}, 10\overset{\mathbf{0}}{\mathbf{N}}, \dots)$	$\sigma(\overset{1}{S}.\overset{0}{N})(i) = \sum_{j=1}^{i} j\overset{\mathbf{0}}{\mathbf{N}}$	$\overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}} = \sigma(\overset{1}{S}.\overset{0}{N})^*$
$\alpha(1)\beta + \alpha(2)\beta + \dots$ *in inf.*	$\alpha.\beta = (\alpha(1)\boldsymbol{\beta}, \alpha(2)\boldsymbol{\beta}, \dots)$	$\sigma(\alpha.\beta) = (\sigma(\alpha)(1)\boldsymbol{\beta}, \sigma(\alpha)(2)\boldsymbol{\beta}, \dots)$	$\sigma(\alpha.\beta)(i) = \sigma(\alpha)(i)\boldsymbol{\beta}$	$\boldsymbol{\alpha}.\boldsymbol{\beta} = \sigma(\alpha.\beta)^*$

Table 9.2: Representation of Bolzanian products in $(\mathbb{Z}_{\mathsf{U}})^2$

This definition of the product of two infinite quantities has three important consequences. First, as evidenced already by the examples of $\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}}$ and $\overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}}$ above, the product operation will in general not be commutative. Although this might seem as a highly non-Bolzanian feature of our setup, we remark that this does not directly contradict any of Bolzano's computations in *PU*. Moreover, contrary to the associativity and commutativity of addition, which he sees as rooted in the concept of sum and therefore a feature of the general theory of quantity, associativity and commutativity of multiplication of integers are introduced as theorems in Part III of his *Reine Zahlenlehre*, §§19-20, [50, pp. 62-63], instead of being part of the definition of a product. Moreover, we think that the non-commutativity of the product of two infinite quantities is itself motivated by Bolzanian considerations. Indeed, if one agrees that the correct interpretation for $\overset{0}{N}.\overset{1}{S}$ and $\overset{1}{S}.\overset{0}{N}$ are the infinite sums $\overset{1}{S}+\overset{1}{S}+\overset{1}{S}+\dots$ *in inf.* and $\overset{0}{N}+2\overset{0}{N}+3\overset{0}{N}+\dots$ *in inf.* respectively, then the Bolzanian strategy for comparing two infinite sums, namely computing their difference term by term, yields that $\overset{0}{N}.\overset{1}{S}-\overset{1}{S}.\overset{0}{N}=(\overset{1}{S}-\overset{0}{N})+(\overset{1}{S}-2\overset{0}{N})+(\overset{1}{S}-3\overset{0}{N})+\dots$ *in inf.* is itself an infinite sum of positive quantities. It is therefore positive, which means that $\overset{0}{N}.\overset{1}{S}$ should be stricly greater than $\overset{1}{S}.\overset{0}{N}$.

Second, it is easy to verify that, under this definition of the product, the summation theorem does not hold in the infinite case. Indeed, in our interpretation, $\overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}}+1)$ is the function mapping any $i \in \omega^+$ to $\overset{\mathbf{0}}{\mathbf{N}}(i).(\overset{\mathbf{0}}{\mathbf{N}}+1)$. Now since $\overset{\mathbf{0}}{\mathbf{N}}+1$ is (the equivalence class of) the function mapping any $j \in \omega^+$ to $j+1$, it follows that $\overset{\mathbf{0}}{\mathbf{N}}(i).(\overset{\mathbf{0}}{\mathbf{N}}+1) = i \times (\overset{\mathbf{0}}{\mathbf{N}}+1)$ maps any $j \in \omega^+$ to $i(j+1)$. On the other hand, in $(\mathbb{Z}_{\mathsf{U}})^2$, $2\overset{\mathbf{1}}{\mathbf{S}}$ maps any $i \in \omega^+$ to $2\overset{\mathbf{1}}{\mathbf{S}}(i) = \overline{i(i+1)}$. Hence $||2\overset{\mathbf{1}}{\mathbf{S}} = \overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}}+1)|| = \{i \in \omega^+ : \mathbb{Z}_{\mathsf{U}} \models \overline{i(i+1)} = i \times (\overset{\mathbf{0}}{\mathbf{N}}+1)\}$. Now for any $i, j \in \omega^+$, $\overline{i(i+1)}(j) = i(i+1)$, while $(i \times (\overset{\mathbf{0}}{\mathbf{N}}+1))(j) = i(j+1)$, hence $\mathbb{Z}_{\mathsf{U}} \models \overline{i(i+1)} < i \times (\overset{\mathbf{0}}{\mathbf{N}}+1)$ for all $i < \omega^+$. Therefore $(\mathbb{Z}_{\mathsf{U}})^2 \models 2\overset{\mathbf{1}}{\mathbf{S}} \neq \overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}}+1)$.

Finally, we argue that this definition of the product gives a better interpretation of Bolzano's remark that quantities like $(\overset{0}{N})^2$ are infinities of a "higher order". Indeed, our construction introduces a clear stratification between integers, infinite quantities of the first order (i.e., elements introduced in the first ultrapower $\mathbb{Z}_{\mathsf{U}}$), and infinite quantities of the second order (i.e., elements introduced in the second ultrapower $(\mathbb{Z}_{\mathsf{U}})^2$). In fact, in our interpretation, genuine second-order infinite positive quantities are always larger than any first-order infinite quantity:

Lemma 9.5.2. *Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{Z}_{\mathsf{U}}$ are such that $\mathbb{Z}_{\mathsf{U}} \models \boldsymbol{\alpha} > m \wedge \boldsymbol{\beta} > m$ for any integer m. Then $(\mathbb{Z}_{\mathsf{U}})^2 \models \boldsymbol{\alpha}.\boldsymbol{\beta} > \boldsymbol{\gamma}$.*

Proof. We claim that $||\boldsymbol{\alpha}.\boldsymbol{\beta} > \boldsymbol{\gamma}|| \in \mathsf{U}$. This amounts to showing that, for U-many $j \in \omega^+$, $||\boldsymbol{\alpha}(j) \times \boldsymbol{\beta} > \overline{\boldsymbol{\gamma}(j)}|| \in \mathsf{U}$. Now suppose $\boldsymbol{\alpha}(j) > 0$ (which is true for U-many $j \in \omega^+$). Then $k \in ||\boldsymbol{\alpha}(j) \times \boldsymbol{\beta} > \overline{\boldsymbol{\gamma}(j)}||$ if and only if $\boldsymbol{\beta}(k) > \frac{\boldsymbol{\gamma}(j)}{\boldsymbol{\alpha}(j)}$, which is true for U-many k since, letting m be the smallest integer greater than $\frac{\boldsymbol{\gamma}(j)}{\boldsymbol{\alpha}(j)}$, we have that $\mathbb{Z}_{\mathsf{U}} \models \boldsymbol{\beta} > m$. □

However, an obvious drawback of modelling second order infinite quantities by iterating the ultrapower construction is that we must repeat this procedure again in order to account for third-order infinite quantities, and so on. In fact, provided we want to make sense of quantities of the form $(\overset{0}{N})^n$ for any natural number n, we must iterate our ultrapower construction countably many times. This requires us to construct models of the form $(\mathbb{Z}_{\mathsf{U}})^n$ for any n, with embeddings from each $(\mathbb{Z}_{\mathsf{U}})^n$ into $(\mathbb{Z}_{\mathsf{U}})^{n+1}$:

$$\mathbb{Z} \xrightarrow[\iota_0]{} \mathbb{Z}_{\mathsf{U}} \xrightarrow[\iota_1]{} (\mathbb{Z}_{\mathsf{U}})^2 \xrightarrow[\iota_2]{} (\mathbb{Z}_{\mathsf{U}})^3 \xrightarrow[\iota_3]{} \dots$$

Limits of iterated ultrapowers are a standard tool in mathematical logic. The direct limit $\mathbb{B}$ of this chain of ultrapowers contains quantities of arbitrarily large orders of infinity, and allows for a rigorous definition of the product $\boldsymbol{\alpha.\beta}$ of two infinite quantities α and β. In fact, we obtain a particularly well-behaved structure:

Theorem 9.5.3. *The structure* $\mathbb{B} = (B, +, -, 0, 1, <, .)$ *is a non-commutative ordered ring.*

We refer the interested reader to Section 9.8 for a proof of this theorem as well as details about the structure $\mathbb{B}$. For now, let us simply conclude that this formal result establishes that our interpretation of Bolzanian sums yields a rich and original structure which nonetheless shares many properties with the integers.

9.6 Reassessing the *PU*

In the preceding sections we have touched on the following three issues:

1. Whether Bolzano's work truly was about (something like) the sets of set theory, or not. We argued that Bolzano's work in [46] §§29-33 is best understood as being an attempt at giving solid foundations to the handling of infinite series (which correspond to Bolzano's infinite *sums*).

2. Whether part-whole reasoning plays an important role or not in Bolzano's computations. We argue that a form of part-whole reasoning about infinite *sums*, not about infinite *sets*, plays a central role in Bolzano's argument, even though Bolzano's argument does not contradict set-theoretic part-whole (**PW** in the Introduction).

3. Whether Bolzano's relation to what we may call the "first generation" of set theorists (specifically Cantor) needs to be reassessed. We think it does.

In this section, we discuss in detail where we stand on each point in turn, making use of the formalization from Section 9.4 and Section 9.5 whenever necessary.

9.6.1 A Theory of Infinite Sums

We have argued that Bolzano's primary interest in [46] §§29-33 is in infinite sums of integers, rather than sets and their sizes. To be more specific, we wanted to illustrate that by interpreting these sections as trying (and largely failing) to anticipate Cantorian inventions, one would fundamentally misrepresent Bolzano's work. Instead of there being one notion that, like Cantor's cardinals (or powers) captures the quantitative aspect of a collection, Bolzano has rather two quantity notions associated to each of his infinite sums: the *Gliedermenge* of the corresponding series of summands, and the sum itself (which for us would be the *value*, or the result of performing the infinite addition—Bolzano's notion of sum does not allow for a distinction between a sum and its value).

These infinite sums (or the underlying series) can undergo certain transformations, which may induce a change in the *Gliedermenge*, a change in the value of the sum, or both. We saw in Bolzano's work three examples of such operations:

1. raising all the terms in a sum to the same power;

2. "erasing" some of the terms in a sum;

3. permuting terms in a sum or computing summands in a different order.

§§29 and 33 suggest that raising all natural numbers at once to the same power does not change the *Gliedermenge* of an infinite sum, but it does change its value. Indeed, $\overset{0}{N}$ and $\overset{2}{S}$ are obtained from the infinite sum $1 + 2 + 3 + \ldots$ *in inf.* (i.e., $\overset{1}{S}$) by raising all terms in this sum to the 0^{th} and 2^{nd} power, respectively. Bolzano explicitly states in §33 that this operation does not change the *Gliedermenge* of the corresponding sum, which is why he is able to determine that $\overset{0}{N} < \overset{1}{S} < \overset{2}{S}$. On the other hand, the second operation, which consists in erasing some of the terms in an infinite sum, does change the *Gliedermenge* of the infinite sum in such a way that also induces a change in the overall value of the sum. Bolzano's clearest examples of this are quantities of the form $\overset{n}{N}$, which vary from $\overset{0}{N}$ only in that the first n terms of the sum are removed. Nonetheless, as we have seen above, this reasoning also appears in §32, where it plays a crucial role in Bolzano's rejection of M.R.S's identification of the infinite sum $a - a + a - a + \ldots$ *in inf.* with the sum within brackets in $a - (a - a + a - a + \ldots$ *in inf.*). Finally, regarding the third operation, Bolzano seems to adhere to the idea that because the laws of commutativity and associativity should always hold for addition, this operation should not change the value of the sum if the sum designates any value at all. As we have shown above, Bolzano uses this criterion to argue that Grandi's series does not designate any actual quantity, but seems unaware of the fact that his argument also creates difficulties for infinite sums like $1 + 1 + 1 + 1 + \ldots$ *in inf.* We have also argued that those issues should commit Bolzano to the thesis that changing the order in which terms are summed in an infinite sum also changes its *Gliedermenge*, although he does not explicitly make this point.

In our formalization of Bolzano's computations, we treat all infinite sums as countable sequences of integers, to which we associate a countable sequence of partial sums. For infinite sums which have the same *Gliedermenge* as $\overset{0}{N}$, this can be done in a straightforward way by identifying an infinite sum with its sequence of partial sums, and our ultrapower construction allows us to assign different values to such sums. For infinite sums which have a different *Gliedermenge*, like $\overset{n}{N}$ or $\overset{1}{G}_a$, we only need to make some natural choices in the way we represent them to retrieve Bolzano's results. We therefore believe to have established that Bolzano's computations in PU form a consistent theory of divergent infinite sums, which paint a picture of the arithmetic of the infinite largely different from our modern, set-theoretic, conception. In particular, interpreting Bolzano as developing a theory of infinite sums allows us to reassess the role that part-whole considerations play in his theory.

9.6.2 Part-whole Reasoning in Bolzano's Computations

As we have mentioned above, we do not think, *pace* Berg and Šebestík, that Bolzano's computations in [46] §§29-33 are incompatible with his use of part-whole reasoning in §§17-24. In fact, we argue that part-whole reasoning plays a central role in Bolzano's determination of the relationship between infinite quantities. However, since, as we have argued, Bolzano is developing in §§29-33 a theory of infinite sums and not a theory of infinite (set-like) collections, we must exert caution in determining how we should understand the principle that "the whole is always greater than its proper parts". The more common interpretation of this principle [see, e.g., 184] is set-theoretic:

PW For any sets A, B, if $A \subsetneq B$, then $size(A) < size(B)$.

This formulation of the part-whole principle is, by and large, the one satisfied for labelled sets of natural numbers by numerosities as defined by Benci and Di Nasso [18]. In particular, in the numerosity structure $\langle \mathscr{N}, \leq \rangle$ constructed by Benci and di Nasso, the following holds:

Num For any (labelled) set of natural numbers A and any numerosity ξ, $\xi < num(A)$ if and only if there is a (labelled) set $B \subsetneq A$ such that $num(B) = \xi$.

However, a more general version of the part-whole principle, which avoids set-theoretic parlance entirely, is given by Bolzano in his *Größenlehre*. This is to be found in the definition of "greater than", which we transcribe here together with the immediately following remark, which shows that Bolzano is aware of the difficulty his definition of "less/ greater than" creates for determining relationships between quantities which may be infinitely large or infinitely small, but adopts it nonetheless:

> §27 Def. If the quantity N lets itself be considered as a whole, which includes in itself the quantity M or one that is equivalent to it as part, then we say that N is greater than M, and M is smaller than N and we write it as $N > M$ or $M < N$. Should this much be established, that M is not greater or not smaller

> than N; then we write in the first case $M \not> N$ and in the second case $M \not< N$.
> §28 Remark. What I here pick as definition, that each whole must be greater than its part, and the part smaller than the whole (as long as they are both quantities) some, namely already Gregory of St. Vincent and in more recent times also Schultz (in his Foundations of the pure Mathesis), do not want to concede, because of quantities which are infinitely large or infinitely small. If M is infinitely large, but m is finite, or M is finite, but m infinitely small, then people say that the whole $(M+m)$ composed from the parts m and M isn't to be truly called greater than the part M. [...][15] [45, p. 237]

The quote above clearly indicates both that Bolzano sees himself as employing some version of the part-whole principle as the criterion for size comparison between quantities, and that two quantities A and B are related as whole and part, respectively, if and only if there is a positive (non-negative, non-zero) quantity C such that $A = B+C$. Then Bolzano's definition of less-than ($<$) can be formulated as follows:

PW2 For any two quantities A, B, $A < B$ if and only if there is some positive quantity C such that $A + C = B$.

This latter principle can indeed be seen as preserving the part-whole intuition: if A is a proper part of B, then the part C of B obtained by removing A from B is non-null, and clearly its sum with A yields back B. In particular, if the operation of taking the sum of two quantities has an inverse (removing a part from a whole), then **PW2** can be rephrased as follows:

PW3 For any two quantities A, B, $A < B$ if and only if $B - A$ is positive.

Our claim is that Bolzano is endorsing **PW3** when determining order relations between infinite sums. Note that for **PW3** to apply to infinite sums, one needs first to define two things:

a) the difference $\alpha - \beta$ of two infinite sums α and β;

b) when an infinite sum α is positive.

[15] §.27 Erkl. Wenn sich die Größe N als ein Ganzes ansehen läßt, welches die Größe M oder eine ihr gleichkommende als ein Theil in sich schließt; so sagen wir, N sey größer als M, M aber kleiner als N und schreiben dieß $N > M$ oder $M < N$. Wenn um so viel bestimmt werden soll, daß M nicht größer oder nicht, kleiner als N sey; so schreiben wir im ersten Falle $M \not> N$ oder im zweyten $M \not< N$.
§28 Anm. Was ich hier als Erklärung annehme, daß jedes Ganze größer als sein Theil, und der Theil kleiner als das Ganze seyn müsse, (so fern beyde Größen sind), haben Einige, nahmentlich schon Gregor v. St.Vincenz und in neuerer Zeit auch wieder Schultz (in seinen Anfangsgr. d. rei. Mathesis) in Hinsicht solcher Größen, die unendlich groß oder klein sind, nicht zugestehen wollen. Wenn M unendlich groß, m aber endlich ist, oder wenn M endlich, m aber unendlich klein ist; so behauptet man daß aus den Theilen M und m zusammengesetzte Ganze $(M + m)$ sey nicht wirklich größer als der Theil M zu nennen. [...]

As we have argued above, Bolzano solves those two issues in his calculation of the infinite as follows:

a) For two infinite sums α and β having the same *Gliedermenge*, their difference $\alpha - \beta$ is computed termwise: $\alpha - \beta$ is the infinite sum in which the i^{th} term is $\alpha_i - \beta_i$, i.e., the difference of the i^{th} terms of α and β respectively;

b) An infinite sum α is positive if all but finitely many of its terms are positive.

Bolzano is thus able to derive from **PW3** a *sufficient* criterion for order relationships between infinite sums:

PW4 For any two infinite sums α, β, $\alpha < \beta$ if all but finitely many terms in $\beta - \alpha$ are positive.

It is worth noting once again that this criterion is exactly the version of **PW3** at play in Bolzano's proof that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$ in §33. Moreover, Bolzano explains his reasoning in terms of part-whole relationships between sums:

> So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, namely:
>
> $$0, 2, 6, 12, 20, 30, 42, 56, \ldots, n(n-1), \ldots \textit{ in inf.},$$
>
> in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. [46, §33]

We therefore conclude that the part-whole principle plays an important role in Bolzano's computations, but also that, in his calculation of the infinite, Bolzano's text should not be interpreted as displaying some instances of part-whole reasoning about sets and their proper subsets. Rather, in deriving those results, part-whole reasoning is applied to infinite sums in the precise sense of **PW4**.[16] In our formalization of Bolzano's computations, we have shown

[16] This does not mean that **PW4** is the correct interpretation of Bolzano's part-whole reasoning throughout the *PU*. As we have noted in Section 9.2, Bolzano is clearly committed to a form of part-whole reasoning about collections in §§19 − 24. We also thank an anonymous referee for pointing out that in the following passage from §29, Bolzano seems to endorse a form of set-theoretic part-whole principle about continuous magnitudes:

> the whole *multitude* (plurality) of quantities which lie between two given quantities, e.g. 7 and 8, although it is equal to an *infinite* [multitude] and therefore cannot be determined by any number however great, depends solely on the magnitude of the distance of those two boundary quantities from one another, i.e. on the quantity 8 − 7, and therefore must be an equal [multitude] whenever this distance is equal.

This suggests that a more fine-grained analysis might be required in order to fully assess the role that part-whole reasoning plays in the *PU* as a whole.

that computations with infinite sums based on **PW4** could be carried out in a consistent fashion. In fact, as a simple consequence of the fact that our structure $\mathbb{B}$ is elementarily equivalent to the integers, we have that $\mathbb{B} \models \forall \alpha, \beta(\alpha < \beta \leftrightarrow \beta - \alpha > 0)$. Moreover, we have also argued that Bolzano's criterion could also be applied in a productive way to determine order relations between infinities of higher order. As a consequence, we showed how a Bolzanian product of infinite quantities could be interpreted as a non-commutative monoidal operation, i.e., a well-behaved operation which is nonetheless considerably different from the product of Cantorian cardinalities or even the product of numerosities.

Finally, let us note that, although we have argued that the correct way to interpret Bolzano's part-whole reasoning does not commit him to the set-theoretic part-whole principle (**PW**), we nonetheless believe that **PW** is *compatible* with Bolzano's arguments. In fact, we are now in a position to fully describe a way out for Bolzano from the apparent contradiction of §33 (cf. Section 9.3) that we believe is satisfactory even from a modern standpoint. Indeed, following the position sketched in Section 9.3, we may argue that the number (*Menge*) of natural squares is not equal to the *Gliedermenge* of the infinite sum $\overset{2}{S}$ but that it must be computed, in relation with $\overset{0}{N}$, as the *value* of the sum $\overset{SQ}{N} = 1^0 + \quad + \quad + 4^0 + \dots$ *in inf.* The approximating sequence of this sum is $(1, 1, 1, 2, 2, \dots)$, and it is therefore straightforward to verify that, in our model, $\mathbb{B} \models \overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{SQ}}{\mathbf{N}} > 0$. In other words, this interpretation avoids making Bolzano's computations inconsistent with his adherence to the principle that the whole is always greater than its proper parts. The price to pay is to argue that the existence of a one-to-one correspondence between natural numbers and squares does not imply that the two sets have the same size, even though, in the specific case of $\overset{1}{S}$ and $\overset{2}{S}$, it is instrumental in establishing that the two sums have the same *Gliedermenge*. In fact, this strategy can be generalized to any set of natural numbers. Indeed, if $A \subseteq \omega^+$, let $\chi_A : \omega^+ \to \{0, 1\}$ be the characteristic function of A, i.e., for any $n \in \omega^+$, $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \notin A$. We may then consider the infinite sum $\tau_A = \sum_{i=1}^{\infty} \chi_A(i)$ and identify the *number* of elements in A with $\boldsymbol{\tau_A}$. It is then straightforward to verify the following fact:

PW5 For any two $A, B \subseteq \omega^+$, if $A \subsetneq B$, then $\mathbb{B} \models \boldsymbol{\tau_A} < \boldsymbol{\tau_B}$.

Indeed, if $A \subsetneq B$, let n be the smallest number in $B \setminus A$, and observe that, for any $j \geq n$, $\boldsymbol{\tau_A}(j) = \sum_{i=1}^{j} \chi_A(j) < \sum_{i=1}^{j} \chi_B(j) = \boldsymbol{\tau_B}(j)$. Thus $||\boldsymbol{\tau_A} < \boldsymbol{\tau_B}||$ is cofinite, so $\mathbb{B} \models \boldsymbol{\tau_A} < \boldsymbol{\tau_B}$. In fact, this "Bolzanian" way of assigning quantities to sets of natural numbers completely coincides with how a set of natural numbers is assigned a numerosity when the structure is constructed out of an ultrapower of the natural numbers, as in Benci and Di Nasso [18].

We stop short of arguing that this was Bolzano's position, as we do not believe that there is enough evidence in the text of *PU* to make this claim; nor are we convinced that Bolzano had a notion of sets of natural numbers and of their sizes that would allow him to conceive of the problem in those terms. Our point, however, is that Bolzano's computations with infinite sums, and his attempts to develop a general theory of a calculation of the infinite, do not, as our formalization makes clear, commit him to a rejection of the part-whole principle for sets of natural numbers.

9.6.3 Bolzano and Early Set Theory

Even though our interpretation sees Bolzano as not necessarily concerned with sets and their cardinalities, this should not be seen as a claim that Bolzano's work is completely separate from, and irrelevant for, the historical development of set theory. We believe that ours is just a more cautious evaluation of the interactions between the PU and the early development of set theory as seen mainly in Cantor's work.

What follows is not an exhaustive comparison between Bolzano's §§29-33 and Cantorian set theory but a selective comparison on just a couple of points: the status of infinite quantities in Bolzano's and Cantor's work and the arithmetic of the infinite, respectively.

Insofar as the actual infinite in mathematics is concerned, Bolzano and Cantor are both advocates for its existence. In addition to defending the existence of the actual infinite, Bolzano provides specific examples of infinite multitudes of mathematical objects such as the multitude of all natural numbers, which is an infinitely large quantity [46, §16]. Infinitely large quantities exist, and they are fully legitimate objects for mathematics, meaning their relationships to one another can be computed. Although Bolzano asserts this in [46] §28, he also makes it clear that he is not claiming to be able to express the infinite quantities themselves through numbers. The symbols $\overset{0}{N}, \overset{n}{N}, \overset{1}{S}, \overset{2}{S}$ are just shorthand for the infinite sum expressions Bolzano concludes with "... *in inf.*"—they are not separate entities, like cardinals (and ordinals) with respect to sets.[17]

Indeed, in modern set theory, ordinals are defined as canonical representatives of order types of well-ordered sets, while cardinals are canonical representatives of equivalence classes of equipollent sets (i.e., sets that can be bijected with one another). Thus, while cardinals are sets and each cardinal is the cardinal of itself, in general a set and its cardinal are two distinct entities. Whether or not Cantor himself held precisely such a view at some point during his lifetime is a complex issue that depends on how one understands the role that Cantor assigns to abstraction in his original construction of the transfinite numbers. Cantor defines the cardinal number or power of a set M to be the result of a "double act of abstraction" performed on M: first, to abstract from the nature of each individual element of M, and second, to abstract from the order of the elements relative to one another. A detailed discussion of the correct interpretation of Cantor's abstraction is beyond the scope of this paper, and we therefore refer the interested reader to Hallett [117, pp. 119-128] and Mancosu [182, pp. 52-59].

[17] Florio and Leach-Krouse [95] have recently proposed a non-objectual interpretation of ordinals. Provided an analogous treatment can be extended to cardinals, the objectuality of cardinals as a *conceptual* difference between contemporary set theory and Bolzano's approach to infinite collections might appear less significant than what it seems to be right now. However, it would still be the case that a Cantorian definition by abstraction for cardinals certainly lends itself to a straightforward objectual interpretation, and thus our point regarding the difference in conception between Cantor and Bolzano would still hold true.

For our purposes, it suffices to stress that the definition of cardinal Cantor gives is such that any set, in principle, can be abstracted from twice and hence give rise to its own cardinal. Thus for instance the cardinal $\aleph_0$ can be obtained from the set of natural numbers $\mathbb{N}$ by abstracting first from the nature of each single natural number and then from the order of $\mathbb{N}$ as a whole. But one fundamental consequence of Cantor's double abstraction definition is that any set has a cardinal.[18] For Bolzano instead not all infinite strings of integers can give rise to a sum, as the case of Grandi's series witnesses, and determining which such expressions do correspond to a sum is one of the problems he tries to solve.

A second point of comparison between Cantor's and Bolzano's treatments of the infinite is the computations they perform with infinite quantities. They both strive to give a meaningful account of arithmetical operations (addition and multiplication, but also subtraction and division, or "ratios" in Bolzano's case) between transfinite cardinals and infinite sums. What this means and how they achieve it is however very different for each of them.

Cardinal multiplication is defined as taking the cardinal of the product of two sets A, B, and addition is defined as the cardinality of the disjoint union of two sets (according to Hallett [117, p. 82] this was already Cantor's own definition). In the presence of the axiom of choice, it is an elementary fact of cardinal arithmetic that for any two infinite cardinals κ, λ, $\kappa \cdot \lambda = \kappa + \lambda = \max\{\kappa, \lambda\}$. This was already proved in the early 20th century by Hessenberg and Jourdain, who were able to generalize Cantor's result that $\aleph_0^2 = \aleph_0$ to $\aleph_\alpha \cdot \aleph_\beta = \aleph_{\max\{\alpha,\beta\}}$ (cf. [117, pp. 79, 82]). They were also able to show that for addition the same holds, namely $\aleph_\alpha + \aleph_\beta = \aleph_{\max\{\alpha,\beta\}}$. This collapse of addition and multiplication into taking the greatest of the addends in the addition case, or factors in the multiplication case, is very far from Bolzano's approach to computing with the infinite.

One important similarity between Cantor and Bolzano is that, for both of them, an actually infinite quantity, like $\overset{0}{N}$ for Bolzano or ω for Cantor, can be obtained by iterating a finite operation (adding units for Bolzano, taking successor ordinals for Cantor) on finite quantities. But they seem to conceive of this process of infinitary addition in different terms, as evidenced by the role subtraction plays in their respective systems. Cantor does not define subtraction of infinite cardinals, while, as we have seen, for Bolzano the ability to compute the difference between two infinite sums is an essential tool in determining order relationships between infinite quantities. Moreover, no two infinite cardinals can have a finite difference, in the sense that for any two infinite cardinals κ, λ, if $\kappa < \lambda$ and there is a cardinal μ such that $\kappa + \mu = \lambda$, then μ must be infinite (in fact $\mu = \lambda$). Here again Bolzano's infinities behave vastly differently, since one of his most basic results is that two infinite sums such as $\overset{0}{N}$ and $\overset{n}{N}$ have a strictly finite difference, namely n.

Similarly, in §29 we see Bolzano generate new infinities, infinities of higher order, as he

[18]Note however that if one reads Cantor as associating to any set not only its equipollence class but also a canonical well-ordered representative for it, this is actually equivalent to the Well-Ordering Principle according to which any set can be well-ordered. Therefore, if one were to reject the Well-Ordering Principle, not all sets would have a Cantorian cardinal.

claims, simply by multiplying $\overset{0}{N}$ by itself, so that $\overset{0}{N} < (\overset{0}{N})^2 < (\overset{0}{N})^3$. This is in stark contrast with Cantor's result that $\aleph_0^n = \aleph_0$, mentioned above. Moreover, we have argued that a faithful interpretation of Bolzano's criterion for inequality between infinite sums implies that the Bolzanian product of two infinite sums should be non-commutative. In fact, according to us, Bolzanian products are significantly different from products of cardinals. Bolzano does not conceive of multiplying quantities as akin to taking Cartesian products of sets. He rather seems to be extending the definition of multiplication of natural numbers that he had in his *Reine Zahlenlehre* [50, p. 57], without introducing infinite numbers. Just like the product of two finite numbers $m \times n$ is defined as $\underbrace{n + \ldots + n}_{m \text{ times}}$, i.e., as obtained from the sum $m = \underbrace{1 + \ldots + 1}_{m \text{ times}}$ by replacing each unit by n, the product of two infinite quantities $\alpha.\beta$ may be obtained by writing the corresponding infinite sum for α and replacing each unit by β, as in the case of $(\overset{0}{N})^2$. Perhaps surprisingly, this latter feature of Bolzano's computation may in fact be seen as the most modern one, especially under our interpretation of the Bolzanian product. Indeed, by allowing not only his finitary operations, but also his infinitary operations (like infinite summation) to range over both finite and infinite quantities, Bolzano, just as Cantor, is able to generate a hierarchy of infinities of ever increasing order.

9.7 Coda: Bolzanian Quantities via Possibility Structures

In this final section, I will offer an alternative reconstruction of Bolzano's Calculation of the Infinite that takes into account the objections raised in the previous section. The main motivation for doing so will be given in the next subsection. I will argue that the formalization offered in Section 9.4 suffers from two main issues: first, the choice of representing Bolzanian sums as countable sequences of integers committed us to the idea that Bolzano shared what I will call the Rate Intuition regarding infinite sums. Second, our representation via ultrapowers limits our ability to account for Bolzano's views regarding permutations of infinite sums. The solution I present in this section will have two main features. First, we will not be representing basic Bolzanian sums as countable sequences, but rather as functions from the set of all finite subsets of ω^+ into the integers. Second, instead of using ultrafilters, we will be considering a poset of filters. I will first define a possibility structure for infinite quantities of the first-order, before adapting the construction of iterated ultrapowers to the setting of possibility semantics.

9.7.1 The Rate Intuition and the Permutation Problem

Recall that, in Section 9.6, we outlined a strategy for assigning sizes to sets of natural numbers along Bolzanian lines, and we remarked that this approach would come really close to numerosities as defined in [18]. As I have argued in the previous chapter, there are however reasons to object to the implementation of **(PW)** via Tarskian structures, having to do without their lack of invariance under permutations that do not preserve the order of the natural number sequence. I think a similar issue comes up in the case of infinite sums and that there is enough evidence in Bolzano's writings to suggest that his notion of infinite sums should exhibit more symmetry properties than what ultrapowers modulo a free ultrafilter on ω^+ can offer.

Let us start by making precise the notion of invariance at stake. For simplicity, I will only consider infinite sums "of the first-order", i.e., sums in which every term is a finite number. Recall that, to every Bolzanian sum $\alpha_1 + \alpha_2 + \ldots$ *in inf.*, we may associate a function $\alpha : \omega^+ \to \mathbb{Z}$ given by $\alpha(i) = \alpha_i$. Given a permutation π of ω^+, it is natural to consider the Bolzanian sum $\alpha_{\pi^{-1}(1)} + \alpha_{\pi^{-1}(2)} + \ldots$ *in inf.*, in which the terms in α are "shuffled around" according to the permutation π. I will write $\pi(\alpha)$ for the sequence associated with that sum. Just like in the previous chapter, we may consider imposing the following two invariance requirements on Bolzanian sums:

Relative Invariance For any Bolzanian sums α, β, if $\alpha \leq \beta$ and π is a permutation, then $\pi(\alpha) \leq \pi(\beta)$.

Absolute Invariance For any Bolzanian sum α and any permutation π, $\alpha = \pi(\alpha)$.

Just like in the case of numeorsities and NAP functions, I will argue that we should adopt Relative Invariance as a constraint in our formal reconstruction of Bolzano's Calculation of the Infinite, but that the situation is more complex with Absolute Invariance. Let me start by mentioning that Bolzano himself seems to make a difference between absolute and relative determinations of the quantities designated by infinite sums in §28, which immediately precedes the beginning of the Calculation of the Infinite:

> [A] calculation of the infinite done correctly does not aim at a calculation of that which is determinable through no number, namely not a calculation of the infinite plurality in itself, but only a determination of the relationship of one infinity to another. This is a matter which is feasible, in certain cases at any rate, as we shall show by several examples.

In other words, Bolzano makes a distinction between computing the value of an infinite quantity and determining the order relationship between two infinite quantities. He argues that only the latter is possible, since achieving the former would require one to assign a (necessarily finite) number to an infinite quantity, which is impossible. But this, to me, seems like evidence that Bolzano would accept **Relative Invariance** as a constraint on Bolzanian sums. For if one can determine unequivocally the relationship between the infinite quantities designated by two Bolzanian sums, shouldn't such a determination be invariant under changing the order in which the terms are summed? Here again, we find in an already quoted passage of §32 reasons to believe that Bolzano would hold such a view:

> In particular a series, if we want to consider it only as a quantity, namely only as the sum of its terms must, by virtue of the concept of a sum (which belongs to multitudes, i.e. to those totalities for which no attention is paid to the order of their parts) have such a nature that it undergoes no change in value when we make a change in the order of its terms. With quantities especially it must be that:
> $$(A + B) + C = A + (B + C) = (A + C) + B.$$

Putting things together, this suggests that changing the order of terms in two Bolzanian sums should not change the order relationships between the quantities they designate. However, it is easy to see that the formal reconstruction of Bolzano's Calculation of the Infinite via the structure $\mathbb{Z}_{\mathsf{U}}$ described in Section 9.4 does not satisfy **Relative Invariance**. One could provide many examples depending on the choice of the ultrafilter U needed to construct $\mathbb{Z}_{\mathsf{U}}$. However, the following is a more interesting example, because it applies regardless of the choice of U. Consider the following permutation of ω^+:

$$\pi^{-1}(i) = \begin{cases} 2n & \text{if } i = 3n \\ 4n+1 & \text{if } i = 3n+1 \\ 4n+3 & \text{if } i = 3n+2 \end{cases}$$

It is easy to see that $\pi(\overset{1}{\mathbf{S}})$ is the sequence corresponding to the Bolzanian sum:

$$1 + 3 + 2 + 5 + 7 + 4 + 9 + 11 + \dots \text{ in inf.}$$

Once we compute the difference $\overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}})$ termwise, this yields the following sum:

$$\begin{aligned} \overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}}) &= (1-1) + (2-3) + (3-2) + (4-5) + (5-7) + (6-4) + (7-9) + \dots \text{ in inf.} \\ &= 0 - 1 + 1 - 1 - 2 + 2 - 2 + \dots \text{ in inf.} \end{aligned}$$

It is easy to see that, for cofinitely many $i \in \omega^+$, the sum of the first i terms of $\overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}})$ is negative. According to the formalization introduced in Section 9.4, it follows that $\mathbb{Z}_{\mathsf{U}} \models \neg(\overset{1}{\mathbf{S}} \geq \pi(\overset{1}{\mathbf{S}}))$. However, letting π^{-1} be the inverse permutation of π, we clearly have that $\mathbb{Z}_{\mathsf{U}} \models \pi^{-1}(\overset{1}{\mathbf{S}}) \geq \overset{1}{\mathbf{S}}$. Since $\pi^{-1}(\pi(\overset{1}{\mathbf{S}})) = \overset{1}{\mathbf{S}}$, it follows that $\mathbb{Z}_{\mathsf{U}} \models \pi^{-1}(\overset{1}{\mathbf{S}}) \geq \pi^{-1}(\pi(\overset{1}{\mathbf{S}}))$. This shows that $\mathbb{Z}_{\mathsf{U}}$ does not satisfy **Relative Invariance**.

It is worth reflecting on an interesting feature of the example just given, which I think gives an intuitive explanation of the formal result just obtained. When computing the Bolzanian sum $\overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}})$, we can see that positive and negative terms appear infinitely many times. By contrast, when we compute the partial sums of the first i terms for $i \in \omega^+$, we see that the result is negative for cofinitely many i. But this is only the case because we consider initial segments of the infinite sum $\overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}})$. Indeed, if we were to consider instead partial sums of the form $\sum_{i \leq n}(\overset{1}{\mathbf{S}}(3i) - \pi(\overset{1}{\mathbf{S}})(3i))$ for some $n \in \omega^+$, we would have that

$$\sum_{i \leq n}(\overset{1}{\mathbf{S}}(3i) - \pi(\overset{1}{\mathbf{S}})(3i)) = \sum_{i \leq n}(3i - \pi^{-1}(3i)) = \frac{n(n+1)}{2}.$$

In other words, the formalization of Bolzanian sums in $\mathbb{Z}_{\mathsf{U}}$ seems to follow the intuition that the order of the terms in a Bolzanian sum must be kept fixed when computing partial approximations of the value of the sum. Intuitively, although there are infinitely many positive terms in $\overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}})$, those positive terms appear "less frequently" than the negative ones, and, consequently, the sequence of partial approximations of the infinite sum $\overset{1}{\mathbf{S}} - \pi(\overset{1}{\mathbf{S}})$ "converges" to an infinitely large negative quantity. One can now see the connection between this aspect of the formal reconstruction presented in Section 9.4 and the Density Intuition discussed in the previous chapter. Just like, according to the Density Intuition, the distribution of the elements of a set U along the standard progression of the natural numbers determines the size of the set U, the distribution of positive and negative terms in an infinite sum ultimately determines the sign of the sum. One could also make the same point differently by considering the "rate" at which the two sums $\overset{1}{\mathbf{S}}$ and $\pi(\overset{1}{\mathbf{S}})$ seem to increase indefinitely. When one compares initial segments of the two sums:

$$\overset{1}{\mathbf{S}} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \dots \textit{ in inf.}$$
$$\pi(\overset{1}{\mathbf{S}}) = 1 + 3 + 2 + 5 + 7 + 4 + 9 + 11 + \dots \textit{ in inf.},$$

it is easy to notice that larger numbers seem come "earlier" twice more often in $\pi(\overset{1}{\mathbf{S}})$ than in $\overset{1}{\mathbf{S}}$, whence the intuition that $\pi(\overset{1}{\mathbf{S}})$ seems to "grow" at a faster rate than $\overset{1}{\mathbf{S}}$.

It seems to me that, although there might be a coherent story to be given in favor of what I will call the "Rate Intuition" about infinite sums, it is largely absent from Bolzano's writings, and, in fact, arguably inconsistent with his own views. We have seen several times before that, according to Bolzano, it is part of the very concept of a "sum" that the quantity corresponding to a sum is not determined by the order in which its terms are summed. Moreover, I think that rejecting the Rate Intuition can also be helpful in offering a novel insight into a notoriously difficult passage in Bolzano's writings. Let me now briefly expand on this point.

In a famous passage of his Theory of Science [52, §102], Bolzano gives a example of a sequence of concepts in which the extension of each concept is infinitely greater than the extension of the following one:

> Let us abbreviate the concept of any arbitrary integer by the letter n. Then the numbers n,n^2,n^4,n^8,n^{16},n^{32},... express concepts each of which includes infinitely

> many objects (namely, infinitely many numbers). Furthermore, it is clear that any object that stands under one of the concepts following n, e.g., n^{16}, also stands under the predecessor of that concept, n^8. It is also clear that very many objects that stand under the preceding (n^8) do not stand under the following (n^{16}). Thus of the concepts n, n^2, n^4, n^8, n^{16}, n^{32},..., each is subordinated to its predecessor. It is, furthermore, undeniable that the width of any of these concepts is infinitely larger than the width of the concept immediately following it. (And this holds even more for concepts that follow later in the sequence.) For, if we assume that the largest of all numbers to which we want to extend our computation is N, then the largest number that can be represented by the concept n^{16} is N and thus the number of objects that it includes equals or is smaller than $N^{\frac{1}{16}}$ and likewise the number of objects that stand under the concept n^8 equals or is smaller than $N^{\frac{1}{8}}$. Hence the relation between the width of the concept n^8 and that of the concept n^{16} is $N^{\frac{1}{16}} : N^{\frac{1}{8}} = N^{\frac{1}{16}} : 1$. Since $N^{\frac{1}{16}}$ can become larger than any given quantity, if N is large enough, and since we can take N as large as we please, and since we can come closer and closer to the true relation between the widths of the concepts n^8 and n^{16}, the larger we take N, it follows that the width of the concept n^8 surpasses infinitely many times that of the concept n^{16}. Since the sequence n, n^2, n^4, n^8, n^{16}, n^{32},... can be continued indefinitely, this sequence itself gives us an example of an infinite sequence of concepts each of which is of infinitely greater width than the following.

Much later in his life, however, Bolzano seems to have taken issue with this argument, as evidenced by this passage from a letter to his pupil Zimmermann [51, pp.187–188][19]:

> Wissenschaftslehre vol. I, p. 473. The matter is not only obscurely presented, but also, as I just began to recognize, quite wrong. If one designates by n the concept of every arbitrary whole number, or to say it better, if by n every arbitrary whole number would be represented, then with this it is already decided which (infinite) set of objects this sign represents. This will not change the least, if we by means of addition of an exponent like n^2, n^4, n^8, n^{16}, ... require that each of these numbers now must be raised to the second, now to the fourth, ... power. The set of these objects which is represented by n is still exactly the same as before, although the objects themselves, which are represented by n^2 are not the same as those represented by n. The wrong result was due to an unwarranted inference from a finite set of numbers, namely those not exceeding the number N, to all of them."

As mentioned before, several commentators (including [26]) have taken this passage as evidence that Bolzano ultimately abandoned the part-whole principle for infinite collections. I do not wish to weigh in on the subtleties of this complex debate here (see [13] and [12, Chap. 3] for a thorough discussion of the issue), and I will only limit myself to pointing out

[19] I am reproducing here the translation given in [13].

a similarity between this passage in the letter to Zimmermann and a passage from §29 of the *PU* quoted above in which Bolzano discusses an infinitary version of Gauss's summation theorem:

> [I]f, as some people have done, we wanted to form the equation
>
> $$[\overset{1}{S}] = \frac{\overset{0}{N} \cdot (\overset{0}{N} + 1)}{2}$$
>
> then we could hardly justify it on any other ground than that for every finite multitude of terms [*Menge von Gliedern*] the equation
>
> $$1 + 2 + 3 + \ldots + n = \frac{n.(n+1)}{2}$$
>
> holds, from which it appears to follow that for the complete infinite multitude of numbers n just becomes $\overset{0}{N}$. However it is in fact not so, because with an infinite series it is absurd to speak of a last term which has the value $\overset{0}{N}$.

In both passages, Bolzano seems to be objecting to a form of argument that one might call "passing from the finite to the infinite". In the first case, one argues that the ratio between the size of the collection of all numbers of the form n^8 and the collection of those of the form n^{16} is best approximated by the sequence of ratios between initial segments of the two sets. In the second case, one argues that the ratio between $\overset{1}{\mathbf{S}}$ and $\overset{0}{\mathbf{N}}$ can be determined exactly by considering initial segments of the two sums and applying Gauss's summation theorem. My hypothesis here is that Bolzano came to believe that this type of argument was too quick. Although some information about order relationships between infinite collections can sometimes be gained by considering finite approximations of those, drawing such conclusions based on the sole consideration of initial segments could lead to error. To use some more Bolzanian terminology, the mistake lies in using a *sequence* (Reihen) representation of an infinite quantity (like the collection of all natural numbers or their sum) to argue that it has a given property on the basis that every quantity represented by an initial segment of that sequence has that same property. Indeed, this would be forgetting the distinction between sums and sequences, and, although this might not be an issue for finite objects, one has to exert more caution when dealing with infinite ones. At the same time, this does not have to mean that Bolzano gives up on any form of argument that is motivated by part-whole considerations or that he adopts the Cantorian Bijection Principle. As I will show below, one can offer a formal reconstruction of Bolzano's computations in the Calculation of the Infinite that aligns with his arguments in the text of the *PU* but does not validate the Rate Intuition. The two key moves of this formalization are, first, to consider all partial approximations of an infinite sum in the determination of the quantity it designates instead of considering only initial segments of the sum and, second, to use generic powers instead of ultrapowers.

Before getting into the details of this alternative formalization, let me briefly discuss the second invariance condition mentioned above, namely, **Absolute Invariance**. Here, we quickly run into some difficult issues having to do with Bolzano's views on Grandi's series. Consider for example the following permutation τ:

$$\tau(i) = \begin{cases} 2 & \text{if i = 1} \\ i-2 & \text{if i = 2n+1, n} \geq 1 \\ i+2 & \text{if i = 2n, n} \geq 1 \end{cases}$$

The Bolzanian sum $\tau(\overset{1}{\mathbf{S}})$ is then the following sum:

$$3 + 1 + 5 + 2 + 7 + 4 + \dots \textit{ in inf.}$$

It is then straightforward to verify the following:

$$\begin{aligned} \tau(\overset{1}{\mathbf{S}}) - \overset{1}{\mathbf{S}} &= 2 - 1 + 2 - 2 + 2 - 2 + \dots \textit{ in inf.} \\ &= \mathbf{G_2} - 1. \end{aligned}$$

Now if we were to accept **Absolute Invariance**, it would follow that $\tau(\overset{1}{\mathbf{S}}) - \overset{1}{\mathbf{S}} = 0$ and thus that $\mathbf{G_2} = 1$. But this is precisely the value assigned by M.R.S's method to Grandi's series when $a = 2$! Hence Bolzano's views on divergent series like Grandi's seem to directly clash with **Absolute Invariance**.

It seems to me that one could possibly solve the issue by appealing to Bolzano's notion of *Gliedermenge* here. Recall that, in the passage from *PU* §24 quoted above, Bolzano argued that the existence of a bijection between two infinite sums mapping equal terms to equal terms was not enough to conclude that they had the same value, because such a bijection did not guarantee that the two sums had the same number of terms. I think that a promising strategy for explaining what goes wrong in the example above is to argue on Bolzano's behalf that the permutation τ defined above may not preserve the *Gliedermenge* of $\overset{1}{\mathbf{S}}$. At the same time, some permutations must preserve the *Gliedermenge* of their sums, otherwise Bolzano's claim that sums should be invariant under the order in which their terms are summed would be meaningless. One is therefore quickly faced with the issue of determining which permutations preserve the *Gliedermenge* of infinite sums, and which don't. It is not clear that one can give a completely satisfactory answer to this question, but there is at least one important remark that one can make.

Recall that, given a element x of a set S and a permutation π of S, the *orbit* of x under S is the smallest subset $O(x)$ of S containing x and closed under the map π, meaning that $\pi(y) \in O(x)$ whenever $y \in O(x)$. Let us say that a permutation on ω^+ is *of finite order* whenever $O(i)$ is finite for every $i \in \omega^+$. One easily verifies that the permutation π defined above is of finite order, just like the permutation swapping the signs in every term of $\mathbf{G_a}$ that Bolzano uses in §32. By contrast, the permutation τ defined above has a single infinite orbit. The suggestion, therefore, is that permutations with infinite orbits do not preserve the *Gliedermenge* of infinite sums, and that for this reason **Absolute Invariance** does not apply to them. Intuitively, this seems in line with Bolzano's rejection of M.R.S's computations for Grandi's series. Indeed, Bolzano argues that the error in the reasoning leading to the conclusion that $\mathbf{G_a} = \frac{a}{2}$ is that M.R.S identifies two sums that do not have the same *Gliedermenge*. Similarly, one could argue that applying **Absolute Invariance** in the case of the permutation τ leads one to identify two infinite sums with a different *Gliedermenge*, and that it is therefore essentially the same mistake that leads to the conclusion that $\mathbf{G_2} = 1$. It is also worth mentioning a connection here with the previous chapter, where we discussed the fact that permutations with infinite orbits immediately established the inconsistency between the Absolute Invariance Criterion and (PW). It is therefore not surprising to find a similar incompatibility arising again here, and that we would be forced to conclude that **Absolute Invariance** does not apply in the case of permutations with infinite orbits. What about permutations of finite order? Here again, the situation is quite subtle. As we shall see below, we might not be able to guarantee that **Absolute Invariance** is always satisfied when π is a permutation of finite order, but we can ensure that it is never refuted. Here again, the flexibility of generic powers, and in particular the fact that not every statement must be satisfied or refuted in a possibility structure, will allow us to offer quite a nuanced picture of the issue.

9.7.2 A Generic Power Construction for Infinite Sums

Recall that the main idea of the formal reconstruction of Bolzano's Calculation of the Infinite in Section 9.4 was to associate to each Bolzanian sum a countable sequence of partial approximations of the value of that sum. I will follow a similar strategy here, except that I will represent a Bolzanian sum of the form $a_1 + a_2 + \ldots$ *in inf.* by a function mapping each finite subset A of ω^+ to $\sum_{i \in A} a_i$. The intuition here is that our formal representation of an infinite sum will take into account all possible partial approximations of its value, instead of considering only the partial sums obtained by taking the sum of all the values in some initial segment. As we shall see below, this is what will allow us to obtain a structure in which Bolzanian sums behave better under permutation of their terms. Let us start with the following definitions.

Definition 9.7.1. We let Λ_1 be the set of all finite subsets of ω^+, ordered by inclusion. A function $\alpha : \Lambda_1 \to \mathbb{Z}$ is *Bolzanian* if for any $A \in \Lambda_1$, $\alpha(A) = \sum_{i \in A} \alpha(\{i\})$. We denote the subset of $\mathbb{Z}^{\Lambda_1}$ of all Bolzanian functions by B_1. Finally, we let $\mathcal{B}_1$ be the generic power $(\mathfrak{B}, ZZ^{\Lambda_1}, \mathscr{I})$ of $\mathbb{Z}$ in the language $\mathscr{L}' = \{0, 1, +, -, <\}$ determined by the poset $\mathfrak{B}$ of all fine filters on Λ_1 ordered by reverse inclusion.

Clearly, any Bolzanian function is determined by its value on singletons. Thus, when defining an element in $\alpha \in B_1$ as in the next definition, it will be enough to specify what $\alpha(\{i\})$ is for every $i \in \omega^+$. We will often abuse notation and write $\alpha(i)$ instead of $\alpha(\{i\})$. Moreover, one easily notices that B_1 is closed under addition and subtraction.

Definition 9.7.2. We introduce the following Bolzanian functions for any $n \in \omega$ and $a \in \mathbb{Z}$:

- $\overset{\mathbf{n}}{\mathbf{N}}$, given by $\overset{\mathbf{n}}{\mathbf{N}}(i) = 1$ if $i > n$, and $\overset{\mathbf{n}}{\mathbf{N}}(i) = 0$ otherwise.
- $\overset{\mathbf{n}}{\mathbf{S}}$, given by $\overset{\mathbf{n}}{\mathbf{S}}(i) = i^n$ for all $i \in \omega^+$.
- $\mathbf{G_a}$, given by $\mathbf{G_a}(i) = a$ if i is odd and $\mathbf{G_a}(i) = -a$ if i is even.

By the Truth Lemma, $\mathcal{B}_1$ is elementarily equivalent to $\mathbb{Z}$, and thus satisfies the axioms of a totally ordered discrete Abelian group. For any $a \in \mathbb{Z}^{\Lambda_1}$ and $i \in \omega^+$, we may therefore write ia for the sum of a with itself i times as computed in $\mathcal{B}$. Moreover, there is a canonical embedding $i \mapsto \bar{i}$ of $\mathbb{Z}$ into $\mathcal{B}_1$, given by constant functions. Note that for any non-negative $i \in \mathbb{Z}$, there is $a_i \in B_1$ such that $\mathcal{B}_1 \models \bar{i} = a_i$, given by $a_i(j) = 1$ if $j \leq i$, and $a_i(j) = 0$ otherwise. Thus we may view all integers as Bolzanian functions. In what follows, we will use the representation of integers as constant functions and as Bolzanian functions interchangeably, depending on convenience. Let us now prove an analogue of Lemma 9.4.2:

Lemma 9.7.3.

1. *For any natural numbers i, n, $\mathcal{B}_1 \models i < \overset{\mathbf{n}}{\mathbf{N}}$.*
2. *For any natural number n, $\mathcal{B}_1 \models \overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n$.*
3. *For any natural number i, $\mathcal{B}_1 \models i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}$.*
4. *For any natural numbers i, n, $\mathcal{B}_1 \models i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$.*
5. *For any $a, b \in \mathbb{Z}$, there is $G \in \mathfrak{B}$ such that $G \Vdash \mathbf{G_a} = ab$.*

Proof. By the Truth Lemma, it is enough to show for each of items 1-5 that $||\varphi||_{\Lambda_1} \in F$ for every fine filter on Λ_1, where φ is the formula corresponding to each item. This in turn is equivalent to showing that there is $B \in \Lambda_1$ such that for all $A \supseteq B$, $A \in ||\varphi||_{\Lambda_1}$. We show this in turn.

1. Note first that for any $A \in \Lambda_1$, $A \in ||i < \overset{\mathbf{n}}{\mathbf{N}}||_{\Lambda_1}$ iff $|A \cap \{m \in \omega^+ \mid n < m\}| > i$. Hence for any $A \supseteq \{1, ..., n + i + 1\}$, we have that $i(A) = i$ and $\overset{\mathbf{n}}{\mathbf{N}}(A) > i$. Therefore $||i < \overset{\mathbf{n}}{\mathbf{N}}||_{\Lambda_1}$ is in every fine filter on Λ_1.

2. Suppose $A \supseteq \{1, ..., n\}$. Then we have that $\overset{\mathbf{0}}{\mathbf{N}}(A) = \overset{\mathbf{n}}{\mathbf{N}}(A) + n$, hence $A \in ||\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n||_{\Lambda_1}$. This shows that $||\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n||_{\Lambda_1}$ is in every fine filter on Λ_1.

3. For any nonempty $A \in \Lambda_1$, let $w(A) = \frac{\sum_{i \in A} i}{|A|}$. Note first that whenever $w(A) > i$, then $i|A| < \sum_{i \in A} i$, and thus $A \in ||i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}||_{\Lambda_1}$. Moreover, whenever $B \supseteq A$, we have that $w(A) \leq w(B)$. Hence it is enough to find A such that $w(A) > i$. Set $A = \{1, ..., 2i\}$.

By Gauss's summation theorem, we have that $w(A) = \frac{(2i)(2i+1)}{2(2i)} = \frac{2(i+1)}{2} > i$. This shows that $||i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}||_{\Lambda_1}$ is in every fine filter on Λ_1.

4. For any nonempty $A \in \Lambda_1$, let $w_n(A) = \frac{\sum_{i \in A} i^n}{\sum_{i \in A} i^{n+1}}$. Again, we have that $A \in ||i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}||_{\Lambda_1}$ whenever $w_n(A) > i$. Clearly, $\lim_{|A| \to \infty} w_n(A) = +\infty$, so there must be $B \in \Lambda_1$ such that $w_n(A) > i$ for all $A \supseteq B$. This shows that $||i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}||_{\Lambda_1}$ is in every fine filter on Λ_1.

5. Finally, fix $a \in \mathbb{Z}$, and denote the sets of odd and even numbers in ω^+ by $2\omega^+ + 1$ and $2\omega^+$ respectively. For any $A \in \Lambda_1$, let $g_a(A) = |A \cap 2\omega^+ + 1| - |A \cap 2\omega^+|$. Note that for any $A \in \Lambda_1$ and any $b \in \mathbb{Z}$, we have that $A \in ||\mathbf{G_a} = ab||_{\Lambda_1}$ iff $g_a(A) = b$ and moreover that there is $B \supseteq A$ such that $g_a(B) = b$ (since both $2\omega^+ + 1$ and $2\omega^+$ are infinite). Now for any $b \in \mathbb{Z}$, let G be a fine filter containing the set $\{B \in \Lambda_1 \mid g_a(B) = b\}$. By the second observation above, there is such a filter in $\mathfrak{B}$. By the first observation above together with the Truth Lemma, we have that $G \Vdash \mathbf{G_a} = ab$. This completes the proof. □

At this point, it is worth emphasizing a difference between the formalization of Bolzano's computations in possibility semantics and the previous formalization via an ultrapower. In the case of Grandi's series, one can show that the quantity assigned to $\mathbf{G_a}$ in the ultrapower $\mathbb{Z}_{\mathsf{U}}$ must be either 0 or a. By contrast, Lemma 9.7.3.5 shows that the quantity assigned to $\mathbf{G_a}$ in $\mathcal{B}_1$ is much more undetermined, in the sense that $\mathcal{B}_1 \not\models \mathbf{G_a} \neq ab$ for *any* $b \in \mathbb{Z}$. In particular, this means that $\mathcal{B}_1 \not\models \bigvee_{b \in S} \mathbf{G_a} = b$ for any finite set S of integers. Note however that, by the proof of item 5 above, for any $A \in \Lambda_1$, there is $b \in \mathbb{Z}$ such that $A \in ||\mathbf{G_a} = ab||_{\Lambda_1}$. Hence $||\exists x \mathbf{G_a} = ax||_{\Lambda_1} = \Lambda_1$, which means that $\mathcal{B}_1 \models \exists x \mathbf{G_a} = ax$. Hence we still preserve Bolzano's view that M.R.S's result for Grandi's series is incorrect.

Intuitively, the previous observation suggests that we made some progress towards obtaining some invariance under permutations for Bolzanian sums. Indeed, in the case of $\mathbf{G_a}$, it is easy to see that the intuitive motivation for thinking that $\mathbf{G_a}$ must be equal to either 0 or a collapses once one rearranges the terms in $\mathbf{G_a}$ according to a permutation of finite order (like Bolzano does when showing that $\mathbf{G_a} = -a$). For example, we might be inclined to think that $\mathbf{G_a} = 2a$ if we were to rearrange the terms of the series as follows:

$$a + a + a - a + a + a + a - a + \dots \textit{ in inf.},$$

which can clearly be done with a permutation of finite order. As we shall see, the assignment of quantities in B_1 to Bolzanian sums does are indeed more stable under permutations of the terms of a sum than the assignment of quantities in an ultrapower. However, we first need to gain a deeper understanding of the conditions under which the order relationships between two Bolzanian functions is determined in $\mathcal{B}$.

Definition 9.7.4. Let $\mathbb{N}_\infty$ be the set containing $\mathbb{N}$ and the formal symbol ∞. We define the relation $\preccurlyeq$ on $\mathbb{N}_\infty$ by $a \preccurlyeq b$ iff $a \in \mathbb{N}$ and either $b = \infty$ or $b \in \mathbb{N}$ and $a \leq b$. We let $a \approx b$ iff $a \preccurlyeq b$ and $b \preccurlyeq a$, and $a \prec b$ iff $a \preccurlyeq b$ and $a \not\approx b$. For any $a \in \mathbb{Z}^{\text{-1}}$, let $a^+ \in \mathbb{N}_\infty$ be the smallest natural number that is an upper bound of the set $\{a(\overline{A}) \mid \overline{A} \in \Lambda_1\}$, if it exists, and ∞ otherwise. Similarly, let a^- be the smallest natural number that is an upper bound of the set $\max(\{-a(\overline{A}) \mid \overline{A} \in \Lambda_1\}$, it is exists, and ∞ otherwise.

The following lemma gives a precise characterization of when a given Bolzanian function α is forced to be positive, negative or equal to 0 in $\mathcal{B}$.

Lemma 9.7.5. *For any $\alpha \in B_1$, $\mathcal{B}_1 \models \alpha \leq 0$ iff $\alpha^+ \preccurlyeq \alpha^-$ $\mathcal{B}_1 \models \alpha \geq 0$ iff $\alpha^+ \succcurlyeq \alpha^-$, and $\mathcal{B}_1 \models \alpha = 0$ iff $\alpha^+ \approx \alpha^-$ iff there is a finite set B such that $\alpha(B) = 0$ and $\alpha(i) = 0$ for all $i \in \omega^+ \setminus B$.*

Proof. Note first that $\mathcal{B}_1 \models \alpha \leq 0$ iff $||\alpha \leq 0||_{\Lambda_1} \in F_0$ iff there is $B \in \Lambda_1$ such that for all $C \supseteq B$, $\alpha(C) \leq 0$. Similarly, $\mathcal{B}_1 \models \alpha = 0$ iff there is $B \in \Lambda_1$ such that $\alpha(C) = 0$ for all $C \supseteq B$. Now I claim the following:

- $\alpha^+ \preccurlyeq \alpha^-$ iff there is $B \in \Lambda_1$ such that $\alpha(C) \leq 0$ for all $C \supseteq B$;
- $\alpha^+ \approx \alpha^-$ iff there is $B \in \Lambda_1$ such that $\alpha(C) = 0$ for all $C \supseteq B$ iff there is a finite set B such that $\alpha(B) = 0$ and $\alpha(i) = 0$ for all $i \in \omega^+ \setminus B$.

For the proof of the first claim, suppose first that $\alpha^+ \not\preccurlyeq \alpha^-$. Then either $\alpha^+ = \infty$ or both $\alpha^-, \alpha^+ \in \mathbb{N}$ and $\alpha^- < \alpha^+$. Let $B \in \Lambda_1$. I claim that there is $C \supseteq B$ such that $\alpha(C) > 0$. Without loss of generality, we may assume that $\alpha(i) \leq 0$ for every $i \in B$. By definition, $-\alpha(B) \leq \alpha^-$. By assumption, there is $C \in \Lambda_1$ such that $\alpha(C) > \alpha^-$, and clearly we may assume that $\alpha(i) > 0$ for all $i \in C$, which implies that $B \cap C = \emptyset$. But then it follows that $\alpha(B \cup C) = \alpha(B) + \alpha(C) > \alpha(B) - \alpha(B) = 0$. This shows the right-to-left implication. For the converse, suppose that $\alpha^+ \preccurlyeq \alpha^-$. This means that $\alpha^+ \in \mathbb{N}$, so fix $B \in \Lambda_1$ such that $\alpha(B) = \alpha^+$, and note that we may assume that $\alpha(i) > 0$ for all $i \in B$. By assumption, there is $C \in \Lambda_1$ such that $-\alpha(C) \geq \alpha(B)$, and, again, we may assume that $\alpha(i) < 0$ for all $i \in C$, and thus that $B \cap C = \emptyset$. Let $A = B \cup C$, and note that $\alpha(A) = \alpha(B) + \alpha(C) \leq \alpha(C) - \alpha(C) = 0$. Moreover, for any $D \supseteq A$, we have that $\alpha(D) \leq \alpha(A)$, for otherwise $\alpha(D \setminus C) > \alpha(B)$, which is impossible since $\alpha(B) = \alpha^+$. This completes the proof of the first claim.

For the proof of the second claim, suppose first that $\alpha^+ \approx \alpha^-$. Note that this means that $\alpha^+, \alpha^- \in \mathbb{N}$ and $\alpha^+ = \alpha^-$. So we may fix sets $B, C \in \Lambda_1$ such that $\alpha(B) = \alpha^+$, $\alpha(C) = -\alpha^-$, $\alpha(i) > 0$ for all $i \in B$ and $\alpha(i) < 0$ for all $i \in C$. Note that for $\alpha(B \cup C) = \alpha^+ - \alpha^- = 0$. Moreover, for any $A \supseteq B \cup C$, we have that $\alpha(A) \leq \alpha(B \cup C) \leq \alpha(A)$, for we would have $-\alpha(A \setminus B) > \alpha^-$ or $\alpha(A \setminus C) > \alpha^+$ if the first or second inequality respectively were false. This shows the left-to-right direction of the first equivalence. For the converse, suppose that there is $B \in \Lambda_1$ such that $\alpha(C) = 0$ for all $C \supseteq B$. Let $B_1 = \{i \in B \mid \alpha(i) > 0\}$ and $B_2 = \{i \in B \mid \alpha(i) < 0\}$. I claim that $\alpha^+ = \alpha(B_1)$ and that $\alpha^- = \alpha(B_2)$. Note that this will imply that $\alpha^+ = \alpha^-$ and that both are natural numbers, hence that $\alpha^+ \approx \alpha^-$. For the proof of the claim, it is enough to observe that, for any $i \notin B$, $\alpha(i) = 0$, for otherwise we would have that $\alpha(B \cup \{i\}) \neq 0$. But this clearly implies that $\alpha^+ = \alpha(B_1)$ and that $\alpha^- = \alpha(B)$. This completes the proof of the first equivalence. Finally, note that the left-to-right direction of the second equivalence was precisely established by the argument just given, and that the right-to-left equivalence is immediate.

From the two claims above, it immediately follows that $\mathcal{B}_1 \models \alpha \leq 0$ iff $\alpha^+ \preccurlyeq \alpha^-$ and that $\mathcal{B}_1 \models \alpha = 0$ iff $\alpha^+ \approx \alpha^-$. Finally, the last equivalence follows from the following chain of equivalences:

$$\begin{aligned} \mathcal{B}_1 \models \alpha \geq 0 &\Leftrightarrow \mathcal{B}_1 \models -\alpha \leq 0 \\ &\Leftrightarrow (-\alpha)^+ \preccurlyeq (-\alpha)^- \\ &\Leftrightarrow \alpha^- \preccurlyeq \alpha^+, \end{aligned}$$

where the last equivalence follows from the observation that $(-\alpha)^+ = \alpha^-$ and that $(-\alpha)^- = \alpha^+$. □

We are now in a position to establish that the assignment of Bolzanian functions in B_1 to Bolzanian sums satisfies some invariance conditions under some permutations of the set of the terms. Note first that, if we rearrange the terms of a Bolzanian sum $a_1 + a_2 + \ldots$ *in inf.* according to a permutation π of ω^+, we should expect to obtain the Bolzanian sum $a_{\pi^{-1}(1)} + a_{\pi^{-1}(2)} = \ldots$ *in inf.*. This motivates the following definition.

Definition 9.7.6. Let π be a permutation of ω^+. For any $\alpha \in B_1$, let $\pi_*(\alpha)(A) = \alpha(\pi^{-1}[A])$.

It is clear that $\pi_*(\alpha) \in B_1$ for any $\alpha \in B_1$. Let us now prove our invariance result for $\mathcal{B}$:

Lemma 9.7.7. *Let π be a permutation of ω^+.*

1. *For any $\alpha, \beta \in B_1$, $\mathcal{B}_1 \models \alpha \leq \beta$ iff $\mathcal{B}_1 \models \pi_*(\alpha) \leq \pi_*(\beta)$.*
2. *If π is a finite permutation, then $\mathcal{B}_1 \models \alpha = \pi_*(\alpha)$ for any $\alpha \in B_1$.*
3. *If π is of finite order, then for any $\alpha \in B_1$, there is $F \in \mathfrak{B}$ such that $F \Vdash \alpha = \pi_*(\alpha)$.*

Proof. Fix a permutation π.

1. Notice first that, since any permutation has an inverse π^{-1}, it is enough to show that $\mathcal{B}_1 \models \alpha \leq \beta$ implies $\mathcal{B}_1 \models \pi_*(\alpha) \leq \pi_*(\beta)$ for all $\alpha, \beta \in B_1$. Moreover, for any $\alpha, \beta \in B_1$, we have that $\mathcal{B}_1 \models \alpha \leq \beta$ iff $\mathcal{B}_1 \models \beta - \alpha \leq 0$. Since $\pi_*(\beta - \alpha) = \pi_*(\beta) - \pi_*(\alpha)$, this means that it is enough to show that $\mathcal{B}_1 \models \alpha \leq 0$ implies $\mathcal{B}_1 \models \pi_*(\alpha) \leq 0$ for any $\alpha \in B_1$. Now suppose that $\mathcal{B}_1 \models \alpha \leq 0$. By Lemma 9.7.5, this means that $\alpha^+ \preccurlyeq \alpha^-$. Note that $\alpha^+ = (\pi_*(\alpha))^+$ and that $\alpha^- = (\pi_*(\alpha))^-$, since $\pi_*(\alpha)(B) = \alpha(\pi^{-1}[B])$ for any $B \in \Lambda_1$. Hence $(\pi_*(\alpha))^+ \preccurlyeq (\pi_*(\alpha))^-$, which by Lemma 9.7.5 again implies that $\mathcal{B}_1 \models \pi_*(\alpha) \leq 0$.
2. If π is a finite permutation, then the set $A = \{i \mid \pi(i) \neq i\}$ is finite and thus belongs to Λ_1. Clearly, for any $B \supseteq A$ we have that $\pi^{-1}[B] = B$, hence $B \in ||\alpha = \pi_*(\alpha)||_{\Lambda_1}$. But this shows that $\mathcal{B}_1 \models \alpha = \pi_*(\alpha)$.

3. Suppose that π has no infinite orbit. For any $i \in \omega^+$, let $O(i)$ be the orbit of i, and for any $B \in \Lambda_1$, let $O(B) = \bigcup\{O(i) \mid i \in B\}$. Note that $O(B)$ is finite for any $B \in \Lambda_1$. Moreover, for any $B \in \Lambda_1$, we have that $O(O(B) \cup B) = O(B) \cup B$. This shows that for any $B \in \Lambda_1$, there is $C \supseteq B$ such that $O(C) = C$. Now let F be the fine filter generated by the set $U = \{B \in \Lambda_1 \mid B = O(B)\}$. As just shown, $F \in \mathfrak{B}$, since for all $B \in \Lambda_1$ there is $C \supseteq B$ such that $C \in U$. Notice however that, if $B \in U$, then $\pi^{-1}[U] = U$. This implies that $\pi_*(\alpha(B)) = \alpha(\pi^{-1}[B]) = \alpha(B)$. Hence $||\alpha = \pi_*(\alpha)||_{\Lambda_1} \in F$, which implies that $F \Vdash \alpha = \pi_*(\alpha)$. □

Lemma 9.7.7 shows that our new formalization of Bolzano's computations satisfies what we identified in Section 9.7.1 as reasonable conditions to impose regarding the stability of Bolzanian sums under permutations. Indeed, the first part of the lemma shows that relative order relationships between the quantities associated to any two Bolzanian sums are preserved under any permutation of the terms of both sums. Meanwhile, the last part of the lemma shows that any Bolzanian function can always be forced to be invariant under a permutation of finite order, or, equivalently, that $\mathcal{B}_1 \not\models \alpha \neq \pi_*(\alpha)$ for any Bolzanian sum α and permutation π of finite order. We therefore have three degrees of absolute invariance of Bolzanian sums under permutation of their terms, depending on the complexity of a given permutation π. If π is a finite permutation, then any Bolzanian sum α is unequivocally invariant under π. If π is an infinite permutation of finite order, then α and its permutation are still conceivably equal, i.e., their equality is not refuted in $\mathcal{B}_1$. Finally, if π is an infinite permutation of infinite order, the equality of α and $\pi_*(\alpha)$ may be refuted in $\mathcal{B}$.

Let us conclude by returning to the issue of formalizing the product of two Bolzanian sums. It is worth mentioning that not all of the reasons mentioned in Section 9.5 for rejecting the pointwise definition for the product operation on Bolzanian sums still apply to the formalization via the generic power $\mathcal{B}_1$. Indeed, recall that Gauss's summation theorem for finite numbers immediately implied that $\mathbb{Z}_{\mathsf{U}} \models \overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}$, a result that Bolzano explicitly rejects, where $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)$ is defined as the pointwise product, i.e., such that $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(n) = n \times (n+1)$ for any n. But it is easy to see that, if we now view $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)$ as the Bolzanian function determined by $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(A) = \sum_{i \in A} i \times (i+1)$, we have that $\mathcal{B}_1 \not\models \overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}$. Indeed, by Lemma 9.7.5, we have that $\mathcal{B}_1 \models \overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) - 2\overset{\mathbf{1}}{\mathbf{S}}$ iff there is a finite set B such that $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(B) = 2\overset{\mathbf{1}}{\mathbf{S}}(B)$ and $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i) = 2\overset{\mathbf{1}}{\mathbf{S}}(i)$ for all $i \notin B$. But $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i) = 2\overset{\mathbf{1}}{\mathbf{S}}(i)$ iff $i \times (i+1) = 2i$ iff $i = 1$. Hence the pointwise definition of the product of two Bolzanian sums is compatible with the refutation of the infinite summation theorem, provided that one models Bolzanian sums as sequences of partial sums indexed by Λ_1 rather than by ω^+. Note that this result is very much in line with the diagnosis of the issues with the ultrapower model that we gave in Section 9.7.1. Under the pointwise interpretation of the product, the infinite summation theorem is a consequence of the Rate Intuition. But if, as I have argued, Bolzano is committed to the rejection of the Rate Intuition, it is therefore no surprise that he would also object to the infinite summation theorem and that adoption a permutation-invariant formalization of Bolzanian sums would also allow us to refute the infinite summation theorem.

At the same time, some of the other reasons for rejecting the pointwise definition of the product of two Bolzanian sums still stand. Indeed, according to the strong reading of what Bolzano means by products being infinite quantities of *higher order*, we would still want $(\overset{\mathbf{0}}{\mathbf{N}})^2$ to be greater than the sum of all odd numbers $\overset{Odds}{S}$. Under the pointwise definition of the product, we can however find some $F \in \mathfrak{B}$ such that $F \Vdash (\overset{\mathbf{0}}{\mathbf{N}})^2 = \overset{Odds}{S}$. Indeed, it is easy to see that all the consequences of the Rate Intuition are forced by the fine filter F containing the set $\{\{1, ..., n\} \mid n \in \omega^+\}$. Moreover, given our understanding of the product as an iterated infinite sum, we still have that $\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}} - \overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}} = (\overset{\mathbf{1}}{\mathbf{S}} - \overset{\mathbf{0}}{\mathbf{N}}) + (\overset{\mathbf{1}}{\mathbf{S}} - 2\overset{\mathbf{0}}{\mathbf{N}}) + (\overset{\mathbf{1}}{\mathbf{S}} - 3\overset{\mathbf{0}}{\mathbf{N}}) + \dots$ *in inf.*. Hence we would want that $\mathcal{B}_1 \models \overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}} > \overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}}$. But $\mathcal{B}$ clearly makes the pointwise product operation commutative. In other words, although moving away from the Rate Intuition allows the pointwise definition of the product to not outright contradict Bolzano's claims, the interpretation of Bolzanian products as iterated Bolzanian sums still require us to introduce a difference construction for products of Bolzanian functions. In order to do so, we will essentially need to adapt the iterated ultrapower technique to possibility structures.

9.7.3 Iterated Generic Powers

Let us start by introducing the following notation and definitions.

Notation 9.7.8. We introduce the following notation.

- For any $i \leq \omega$, Λ_i is the poset of all sequences of elements of Λ_1 of length i, with the order defined pointwise. Elements in Λ_i for some $i \leq \omega$ will usually be denoted $\overline{A}, \overline{B}$, etc...
- For any set $i \in \omega^+$, we will denote the set of all fine filters over Λ_1 by $Filt(\Lambda_i)$.
- Given a set $U \subseteq \Lambda_{m+n}$ for $m, n \in \omega^+$ and $\overline{A} \in \Lambda_n$, let $U|\overline{A} = \{\overline{B} \in \Lambda_m \mid \overline{AB} \in U\}$.
- Given some $\overline{A} \in \Lambda_\beta$ and $\alpha \leq \beta \leq \omega$, we let $\overline{A}|^\beta_\alpha$ be the initial subsequence of $\overline{A}$ of length α.

Definition 9.7.9. Let $\alpha \leq \beta \leq \omega$.

- Let $\pi^\beta_\alpha : \mathscr{P}(\Lambda_\beta) \to \mathscr{P}(\Lambda_\alpha)$ and $\lambda^\alpha_\beta : \mathscr{P}(\Lambda_\alpha) \to \mathscr{P}(\_\beta)$ be the direct image and inverse image lifts respectively of the map $\overline{A} \mapsto \overline{A}|^\beta_\alpha$. More concretely, for any $U \subseteq \Lambda_\beta$, $\pi^\beta_\alpha(U) = \{\overline{A}|^\beta_\alpha \mid \overline{A} \in U\}$ and for any $V \subseteq \Lambda_\alpha$ $\lambda^\alpha_\beta(V) = \{\overline{A} \in \Lambda_\beta \mid \overline{A}|^\beta_\alpha \in V\}$. Moreover, for any $n \in \omega^+$, let κ_n be the closure operator on $\mathscr{P}(\Lambda_\omega)$ given by the composition $\lambda^n_\omega \circ \pi^\omega_n$.

- A set $U \subseteq \Lambda_\omega$ is *n-stable* if $\kappa_n(U) = U$, and *stable* if it is n-stable for some $n \in \omega^+$.

Definition 9.7.10. An *iterating family* is a family $\{F^i \mid F \in Filt(\Lambda_1), i \in \omega^+\}$ with the following properties:

1. For any $i \in \omega^+$ and $F \in Filt(\Lambda_1)$, $F^i \in Filt(\Lambda_i)$;

2. For any $i \in \omega^+$ and $F, G \in Fil(\Lambda_1)$, $F \subseteq G$ implies $F^i \subseteq G^i$.

3. For any $i \in \omega^+$, $\mathcal{B}_i = (\{F^i \mid F \in Filt(\Lambda_1)\}$ is a rich family;

4. For any $F \in Filt(\Lambda_1)$, any $i < j \in \omega^+$ and any $U \in \Lambda_i$, $U \in F^i$ iff $\lambda^i_j(U) \in F^j$;

5. For any $F \in Filt(\Lambda_1)$, any $i, j \in \omega^+$ and any $U \in \Lambda_{i+j}$, if $\{\overline{A} \in \Lambda_i \mid U|\overline{A} \in F^j\} \in F^i$, then $U \in F^{i+j}$.

Given an iterating family $\{F^i \mid F \in Filt(\Lambda_1), i \in \omega^+\}$ and any $F \in Filt(\Lambda_1)$, we let $F^\omega = \bigcup_{i \in \omega^+} (\lambda^i_\omega)^{-1}[F^i]$, i.e., $U \in F^\omega$ iff there is $i \in \omega^+$ and $V \in F^i$ such that $U = \lambda^i_\omega(V)$.

The existence of an iterating family will be established in Section 9.9. For now, we will simply fix such a family $\{F^i | F \in Filt(\Lambda_1), i \in \omega^+\}$, and therefore also the set $\{F^\omega \mid F \in Filt(\Lambda_1)\}$. In fact, such a family can be constructed quite canonically, but the simplest way to describe it and to show that it has the required properties seems to require one to use a forcing argument. Now let us investigate further the properties of sets of the form F^ω.

Lemma 9.7.11.

1. *The set $\mathcal{S}$ of all stable subsets of Λ_ω is a Boolean subalgebra of $\mathscr{P}(\Lambda_\omega)$;*

2. *For any filter $F \in Filt(\Lambda_1)$, F^ω is a filter on $\mathcal{S}$.*

3. *For any filter $F \in Filt(\Lambda_1)$ and any $U \in \mathcal{S}$, if $U \notin F^\omega$, then there is $G \supseteq F$ such that $\Lambda_\omega \setminus U \in G^\omega$.*

Proof. Fix a filter F.

1. Let us now show that $\mathcal{S}$ is a Boolean subalgebra of $\mathscr{P}(\Lambda_\omega)$. Note first that if $i \leq j$, then it is easy to see that $\kappa_j(U) \subseteq \kappa_i(U)$ for any $U \subseteq \Lambda_\omega$. Hence in particular any i-stable set is also j-stable, since $\kappa_j(U) \subseteq \kappa_i(U) \subseteq U$. Now suppose that U is i-stable. I claim that $U' = \Lambda_\omega \setminus U$ is also i-stable. To see this, it is enough to show that $\overline{A} \in \kappa_i(U')$ implies $\overline{A} \in U'$. So suppose $\overline{A} \in \kappa_i(U')$. Then there is $\overline{B} \in U'$ such that $\overline{A}|i = \overline{B}|i$. But then if $\overline{A} \in U$ we also have that $\overline{B} \in U$, since U is i-stable. But this contradicts the assumption that $\overline{B} \in U'$. Hence $\overline{A} \in U'$. Moreover, suppose that $U, V \in \mathcal{S}$. Note that we may assume that both U and V are i-stable for some $i \in \omega^+$ by the remark above. Now I claim that $U \cap V$ is i-stable. For this it is enough to show that $\kappa_i(U \cap V) \subseteq U \cap V$. So suppose $\overline{A} \in \kappa_i(U \cap V)$. Then there is $\overline{B} \in U \cap V$ such that $\overline{A}|i = \overline{B}|i$. But then $\overline{A} \in \kappa_i(U) = U$, and $\overline{A} \in \kappa_i(V) = V$, hence $\overline{A} \in U \cap V$.

2. Let us show that F^ω is a filter on $\mathcal{S}$. Suppose first that $U \subseteq V$ and that $U \in F^\omega$. Then there are $i, j \in \omega$ such that U is i-stable, $\pi_i^\omega \in F^i$ and V is j-stable. Let us distinguish two cases. First, if $j \leq i$, then V is i-stable. Moreover, $\pi_i^\omega(U) \subseteq \pi_i^\omega(V)$ since $U \subseteq V$, so $\pi_i^\omega(V) \in F^i$, which shows that $V \in F^\omega$. Now consider the case $i < j$, and suppose that $j = i + k$. Note that in this case we also have that U is j-stable. I claim that $\pi_j^\omega(U) \in F^j$. Since $\pi_j^\omega(V) \supseteq \pi_j^\omega(U)$, this will be enough to show that $V \in F^\omega$. For the proof of the claim, note that, by item 1, $\pi_j^\omega(U) \in F^j$ iff $\{\overline{A} \in \Lambda_i \mid \pi_j^\omega(U)|\overline{A} \in F^k\} \in F^i$. Observe that, since U is i-stable, whenever $\overline{A} \in \pi_i^\omega(U)$, we have that $\pi_j^\omega(U)|\overline{A} = \Lambda_k$. Hence $\pi_j^\omega(U)|\overline{A} \in F^k$ whenever $\overline{A} \in \pi_i^\omega(U)$, which implies that $\pi_j^\omega(U) \in F^j$ since $\pi_i^\omega(U) \in F^i$. This completes the proof that $V \in F^\omega$.

   Finally, suppose that $U, V \in F^\omega$. An argument similar to the one above shows that there is $i \in \omega^+$ such that both U and V are i-stable and both $\pi_i^\omega(U)$ and $\pi_i^\omega(V) \in F^i$. Since F^i is a filter, it follows that $\pi_i^\omega(U) \cap \pi_i^\omega(V) \in F^i$. Moreover, since U and V are i-stable, we have that $\pi_i^\omega(U) \cap \pi_i^\omega(V) = \pi_i^\omega(U \cap V)$. Moreover, as $U \cap V$ is i-stable, this implies that $U \cap V \in F^\omega$. This completes the proof.

3. Let $F \in Filt(\Lambda_1)$, $U \in \mathcal{S}$, $i \in \omega^+$ and $V \in \mathscr{P}(\Lambda_i)$ such that $U = \lambda_\omega^i(V)$. If $U \notin F^\omega$, this means that $V \notin F^i$. Since $\mathcal{B}_n$ is a rich family, there is $G \in Filt(l_1)$ such that $V' = \Lambda_i \setminus V \in G^i$. But then $\Lambda_\omega \setminus U = \lambda_\omega^i(V') \in G^\omega$. □

Definition 9.7.12. A function $a : \Lambda_\omega \to \mathbb{Z}$ has *support* α if for any $\overline{A}, \overline{B} \in \Lambda_\omega$, $\overline{A}|\alpha = \overline{B}|\alpha$ implies $f(\overline{A}) = f(\overline{B})$. The *support* of a function $a : \Lambda_\omega \to \mathbb{Z}$, denoted $sup(a)$, is the least $\alpha \leq \omega$ such that a has support α. Given a finite sequence $\overline{a} = a_1, ..., a_n$ of functions from Λ_ω into $\mathbb{Z}$ with finite support, we write $sup(\overline{a})$ for the natural number $\max(sup(a_1), ..., sup(a_n))$.

Definition 9.7.13. The *possibility ring of Bolzanian quantities* is the possibility structure $\mathcal{B}$ in the language $\mathscr{L} = \mathscr{L}' \cup \{.\}$, where $\mathscr{L}'$ is the language $\{0, 1, +, -, <\}$, determined by the triple $(\mathfrak{B}, \mathbb{Z}^{\Lambda_\omega}, \mathscr{I})$, where:

- $\mathfrak{B} = \{Filt(\Lambda_1), \supseteq\}$;
- $\mathbb{Z}^{\Lambda_\omega}$ is the set of all functions from Λ_ω into $\mathbb{Z}$ with finite support;
- $\mathscr{I}(0)(\overline{A}) = 0$ and $\mathscr{I}(1)(\overline{A}) = 1$ for all $\overline{A} \in \Lambda_\omega$, and the operations $+$ and $-$ are interpreted pointwise;
- For any $\overline{A}$ and $a, b \in \mathbb{Z}^{\Lambda_\omega}$, $a.b(\overline{A}) = a(\overline{A}) \times b(\overline{A'})$ where $\overline{A'}$ is the sequence obtained by removing $\overline{A}|sup(a)$ from $\overline{A}$;
- For any $F \in \mathcal{F}$ and $a, b \in \mathbb{Z}^{\Lambda_\omega}$, $(a, b) \in \mathscr{I}(F, =)$ (resp.$\mathscr{I}(F, <)$) if $\{\overline{A} \in \Lambda_\omega \mid \mathbb{Z} \models a(\overline{A}) = b(\overline{A})\} \in F^\omega$ (resp. $\{\overline{A} \in \Lambda_\omega \mid \mathbb{Z} \models a(\overline{A}) < b(\overline{A})\} \in F^\omega$).

Definition 9.7.14. For any $\mathscr{L}$-formula $\varphi(x_1...x_n)$ any $a_1, ..., a_n \in \mathbb{Z}^{\Lambda_\omega}$ and any $\overline{A} \in \Lambda_\omega$, let $\varphi_{\overline{A}}(\overline{a})$ be the first-order formula in the language of arithmetic obtained by replacing every occurrence of a term of the form $x_1 + x_j$, $x_i - x_j$ and $x_i.x_j$ with $\mathscr{I}(a_i + a_j)(\overline{A})$, $\mathscr{I}(a_i - a_j)(\overline{A})$ and $\mathscr{I}(a_i.a_j)(\overline{A})$ respectively. Moreover, let $||\varphi(\overline{a})||_{\Lambda_\omega} = \{\overline{A} \in \Lambda_\omega \mid \mathbb{Z} \models \varphi_{\overline{A}}(\overline{a})\}$.

Definition 9.7.15. For any $\mathscr{L}$-formula $\varphi(\overline{x}, \overline{y})$ and any tuple $\overline{a} \in \Lambda_\omega$ of the same length as $\overline{x}$, we define the *support* of $\varphi(\overline{a}, \overline{y})$, denoted $s(\varphi(\overline{a}, \overline{y}))$, recursively as follows:

- For terms:
  - $s(x) = s(0) = s(1) = 0$ for any variable x;
  - $s(a) = sup(a)$ for any $a \in \mathbb{Z}^{\Lambda_\omega}$;
  - $s(t + t') = s(t - t') = \max(s(t), s(t'))$;
  - $s(t.t') = s(t) + s(t')$.
- For formulas:
  - $s(t = t') = s(t < t') = \max(s(t), s(t'))$ for any terms t, t';
  - $s(\varphi \wedge \psi) = \max(s(\varphi), s(\psi))$;
  - $s(\neg\varphi) = s(\exists x\varphi) = s(\varphi)$.

Note that $s(\varphi(\overline{a}, \overline{y}))$ is always finite, since every $a \in \mathbb{Z}^{\Lambda_\omega}$ has finite support.

Lemma 9.7.16. *For any $\mathscr{L}$-formula $\varphi(\overline{x})$ and any tuple $\overline{a} \in \mathbb{Z}^{\Lambda_\omega}$, $||\varphi(\overline{a})||_{\Lambda_\omega}$ is $s(\varphi(\overline{a}))$-stable. Moreover, if $\varphi(\overline{x})$ is a formula in the language $\mathscr{L}'$ and $\overline{a}$, then $||\varphi(\overline{a})||_{\Lambda_\omega}$ is $sup(\overline{a})$-stable.*

Proof. A simple induction on the complexity of formulas (with an equally straightforward induction on the complexity of terms needed for the base case) establishes that for any $\overline{A}, \overline{B} \in \Lambda_\omega$, if $\overline{A}|s(\varphi(\overline{a})) = \overline{B}|s(\varphi(\overline{a}))$, then $\overline{A} \in ||\varphi(\overline{a})||_{\Lambda_\omega}$ iff $\overline{B} \in ||\varphi(\overline{a})||_{\Lambda_\omega}$. But from this it follows at once that $||\varphi(\overline{a})||_{\Lambda_\omega}$ is $s(\varphi(\overline{a}))$-stable. Similarly, an simple induction on the complexity of formulas in the language $\mathscr{L}'$ establishes that $s(\varphi(\overline{a}, \overline{y})) = sup(\overline{a})$ for any tuple $\overline{a} \in \mathbb{Z}^{\Lambda_\omega}$ and any $\varphi(\overline{a}, \overline{y}) \in \mathscr{L} \setminus \{.\}$. Hence for any such formula $\varphi(\overline{x})$ and any $\overline{a} \in \mathbb{Z}^{\Lambda_\omega}$, $||\varphi(\overline{a})||_{\Lambda_\omega}$ is $sup(\overline{a})$-stable. $\square$

Lemma 9.7.17. *The structure $\mathcal{B} = (\mathfrak{B}, \mathbb{Z}^{\Lambda_\omega}, \mathscr{I})$ is a possibility structure.*

Proof. Note that we have that for any $F \in \mathfrak{B}$ and any $a, b \in \mathbb{Z}^{\Lambda_\omega}$, $F \Vdash a = b$ (resp. $F \Vdash a < b$) iff $||a = b||_{\Lambda_\omega} \in F^\omega$ (resp. $||a < b||_{\Lambda_\omega} \in F^\omega$). But then **Persistence** is obvious, **Refinability** follows from Lemma 9.7.11.3, and **Equality-as-Equivalence** and **Equality-as-Congruence** directly follow from Lemma 9.7.11.2, except for the case of the function symbol ., which we discuss in more detail. Suppose that we have $a, a', b, b' \in \mathbb{Z}^{\Lambda_\omega}$ and $F \in \mathfrak{B}$ such that $F \Vdash a = a'$ and $F \Vdash b = b'$. We need to show that $F \Vdash a.b = a'.b'$. Let $i = sup(a, a')$ and $j = sup(b, b')$, and note that, without loss of generality, $s(a.b = a'.b') = i + j$. Now for any $\overline{A} \in \Lambda_i$ and $\overline{B} \in \Lambda_j$, if $\overline{A} \in \pi^\omega_i(||a = a'||_{\Lambda_\omega})$ and $\overline{B} \in \pi^\omega_j(||b = b'||_{\Lambda_\omega})$, then $\overline{AB} \in \pi^\omega_{i+j}(||a.b = a'.b'||_{\Lambda_\omega})$. Let $U = \{\overline{AB} \mid \overline{A} \in \pi^\omega_i(||a = a'||_{\Lambda_\omega})$ and $\overline{B} \in \pi^\omega_j(||b = b||_{\Lambda_\omega})\}$. By the previous reasoning, we have that $\pi^\omega_{i+j}(||a.b = a'.b'||_{\Lambda_\omega}) \supseteq U$. I claim that $U \in F^{i+1}$, which, if true, completes the proof that $F \Vdash a.b = a'.b'$. For the proof of the claim, notice first that for any $\overline{A} \in \Lambda_i$, $U|\overline{A} = \pi^\omega_j(||b = b'||_{\Lambda_\omega})$ if $\overline{A} \in \pi^\omega_i(||a = a'||_{\Lambda_\omega})$, and is empty otherwise. Moreover, since $F \Vdash b = b'$, we have that $\pi^{\omega}_{j}(||b = b'||_{\Lambda_\omega}) \in F^j$. Hence $\{\overline{A} \mid U|A \in F^j\} = \pi^{\omega}_{i}(||a = a'||_{\Lambda_\omega}) \in F^i$, since $F \Vdash a = a'$. By property 5 of iterating families, it follows that $U \in F^{i+j}$, which completes the proof. □

In fact, much more can be said about $\mathcal{B}$. In what follows, we will show that the reduct of $\mathcal{B}$ to the language $\mathscr{L}'$ is in some precise sense isomorphic to a colimit of generic powers, and thus can arguably be described as an "iterated generic power".

Definition 9.7.18. For any $i \in \omega^+$, let $\mathbb{Z}^{\mathfrak{B}_i}$ be the generic power of $\mathbb{Z}$ in the language $\mathscr{L}'$ determined by the poset $\mathfrak{B}_i = (\mathcal{B}_i, \supseteq)$. For any $i \leq j \in \omega^+$, let ϵ_{ij} be the pair $(\pi_{ij}, \overline{\alpha_{ij}})$, where $\pi_{ij} : \mathfrak{B}_j \to \mathfrak{B}_i$ is the map $F^j \mapsto F^i$ and $\alpha_{ij} : \mathbb{Z}^{\Lambda}_i \to \mathbb{Z}^{\Lambda}_j$ is such that $\alpha_{ij}(a)(\overline{A}) = a(\overline{A|^j_i})$ for any $a \in \mathbb{Z}^{\Lambda_i}$ and $\overline{A} \in \Lambda_j$.

Lemma 9.7.19. *The family* $(\{\mathbb{Z}^{\mathfrak{B}_i}\}_{i\in\omega^+}, \{\epsilon_{ij}\}_{i\leq j})$ *is an elementary directed system over* ω.

Proof. Let us first show that $(\{\mathfrak{B}_i\}_{i\in\omega^+}, \{\pi_{ij}\}_{i\leq j})$ is a tight inverse system. Clearly, every bounded chain in ω is finite and the maps $\{\pi_{ij}\}_{i\leq j}$ commute, so we only need to check that π_{ij} is a b-morphism whenever $i \leq j$. Since π_{ij} is surjective, it is enough to show that it is monotone, i.e., that $F^j \supseteq G^j$ implies $F^i \supseteq G^i$. Assume that $F^i \not\supseteq G^i$. Then there is $A \subseteq \Lambda_i$ such that $A \in G^i \setminus F^i$. But then, by property 4 of iterating families, we have that $\lambda^i_j(A) \in G^j \setminus F^j$, and hence $F^j \not\supseteq G^j$.

Moreover, it is clear that the maps $\{\alpha_{ij}\}_{i\leq j}$ commute. So we only need to verify that (π_{ij}, α_{ij}) is an elementary embedding whenever $i \leq j$. Fix a $\mathscr{L}'$-formula $\varphi(\overline{x})$ and a tuple $\overline{a} \in \Lambda_i$. Then I claim that, for any $F \in Filt(\Lambda_1)$, we have the following chain of equivalences:

$$\begin{aligned}
\pi_{ij}(F^j) \Vdash \varphi(\overline{a}) &\Leftrightarrow F^i \Vdash \varphi(\overline{a}) \\
&\Leftrightarrow ||\varphi(\overline{a})||_{\Lambda_i} \in F^i \\
&\Leftrightarrow \lambda^i_j(||\varphi(\overline{a})||_{\Lambda_i}) \in F^j \\
&\Leftrightarrow ||\varphi(\overline{\alpha_{ij}(a)})||_{\Lambda_j} \in F^j \\
&\Leftrightarrow F^j \Vdash \varphi(\overline{\alpha_{ij}(a)}).
\end{aligned}$$

The first equivalence holds by definition of π_{ij}, while the second and fifth equivalences hold by the Truth Lemma for generic powers, and the third equivalence hold by property 4 of iterating families. Therefore we only need to check that for any $\overline{A} \in \Lambda_j$, $\overline{A} \in \lambda^i_j(||\varphi(\overline{a})||_{\Lambda_i})$ iff $\overline{A} \in ||\varphi(\overline{\alpha_{ij}(a)})||_{\Lambda_j}$. But this in turn is easily established by the following chain of equivalences:

$$\begin{aligned}
\overline{A} \in \lambda^i_j(||\varphi(\overline{a})||_{\Lambda_i}) &\Leftrightarrow \mathbb{Z} \models \varphi(\overline{a}(\overline{A}|^j_i)) \\
&\Leftrightarrow \mathbb{Z} \models \varphi(\overline{\alpha_{ij}(a)}(\overline{A})) \\
&\Leftrightarrow \overline{A} \in ||\varphi(\overline{\alpha_{ij}(a)})||_{\Lambda_j},
\end{aligned}$$

where the second equivalence holds by the definition of α_{ij}. This completes the proof. □

Theorem 9.7.20. *Let $\overrightarrow{\mathcal{B}_\omega} = (\bigotimes_\omega \mathfrak{B}_i, \bigoplus_\omega \mathbb{Z}^i, \mathscr{J})$ be the colimit of the elementary directed system $(\{\mathbb{Z}^{\mathfrak{B}_i}\}_{i\in\omega^+}, \{\epsilon_{ij}\}_{i\leq j})$. Then there is a pair (σ, β) such that:*

1. *$(\sigma, \beta) : \overrightarrow{\mathcal{B}_\omega} \to \mathcal{B}$ is an elementary embedding;*
2. *$\sigma : \mathfrak{B} \to \bigotimes_\omega \mathfrak{B}_i$ is an order-isomorphism;*
3. *$\beta : \bigoplus_\omega \mathbb{Z}^i \to \mathbb{Z}^{\Lambda_\omega}$ is surjective and "internally injective", meaning that for any $a, b \in \bigoplus_\omega \mathbb{Z}^i$, $\beta(a) = \beta(b)$ iff $\overrightarrow{\mathcal{B}_\omega} \models a = b$.*

Proof. Note first that, by the universal mapping property of colimits, it is enough to define elementary possibility embeddings $\eta_i = (\sigma_i, \beta_i)$ for any $i \in \omega^+$ in order to obtain an elementary possibility embedding $\eta = (\sigma, \beta) : \overrightarrow{\mathcal{B}_\omega} \to \mathcal{B}$. For any $i \in \omega^+$, define $\sigma_i : \mathfrak{B} \to \mathcal{B}_i$ by $\sigma_i(F) = F^i$, and $\beta_i : \mathbb{Z}^{\Lambda_i} \to \mathbb{Z}^{\Lambda_\omega}$ by $\beta_i(a)(\overline{A}) = a(\overline{A}|_i^\omega)$. By property 2 of iterating families, each σ_i is monotone and thus a weakly dense map, since it is also clearly surjective. Moreover, for any $i \in \omega^+$ and $a, b \in ZZ_i^\Lambda$, we clearly have that $\mathscr{I}_i(+, (a, b)) = \mathscr{I}(+, (\beta_i(a), \beta_i(b)))$ and $\mathscr{I}_i(-, (a, b)) = \mathscr{I}(-, (\beta_i(a), \beta_i(b)))$, since those operations are interpreted pointwise. Moreover, for any $F \in \mathfrak{B}$, any $\mathscr{L}'$-formula $\varphi(\overline{x})$ and any $\overline{a} \in \Lambda_i$, we have that

$$\begin{aligned}\sigma_i(F) \Vdash a = b &\Leftrightarrow F^i \Vdash \varphi(\overline{a})\\ &\Leftrightarrow ||\varphi(\overline{a})||_{\Lambda_i} \in F^i\\ &\Leftrightarrow ||\varphi(\overline{\beta_i(a)})||_{\Lambda_\omega} \in F^\omega\\ &\Leftrightarrow F \Vdash \varphi(\overline{\beta_i(a)}),\end{aligned}$$

where the third equivalence follows from the fact that $||\varphi(\overline{\beta_i(a)}||_{\Lambda_\omega} = \lambda_\omega^i(||\varphi(\overline{a})||_{\Lambda_i})$. Hence each η_i is an elementary possibility embedding. By the Second Colimit Lemma, it follows that $\eta = (\sigma, \beta) : \overrightarrow{\mathcal{B}_\omega} \to \mathcal{B}$, defined by $\sigma(F)(i) = F^i$ for any $F \in \mathfrak{B}$ and $\beta(a) = \beta_i(a)$ for any $a \in D_i$ is also an elementary possibility embedding. Moreover, for any $x \in \bigotimes_\omega \mathfrak{B}_i$, there is $F \in \mathfrak{B}$ such that $x(i) = F^i$ for all $i \in \omega^+$, and hence σ is surjective. Since $F \supseteq G$ implies $F^i \supseteq G^i$ for all $i \in \omega^+$ by property 2 of iterating families, this means that σ is an order-isomorphism. Finally, for any $a \in \mathbb{Z}^{\Lambda_i}$, $b \in \mathbb{Z}^{\Lambda_j}$ for some $i \leq j \in \omega^+$, we have that $\beta(a) = \beta(b)$ iff $\lambda_j^i(a) = b$ iff $\overrightarrow{\mathcal{B}_\omega} \models a = b$. □

Corollary 9.7.21. *The $\mathscr{L}'$-reduct of $\mathcal{B}$ satisfies the axioms of a totally ordered Abelian group. Moreover, for any $\overline{a} \in \mathbb{Z}^{\Lambda_\omega}$, any $\mathscr{L}'$-formula $\varphi(\overline{x})$, and any $F \in \mathfrak{B}$, $F \Vdash \varphi(\overline{a})$ iff $\pi^\omega_{sup(\overline{a})}(||\varphi(\overline{a})||_{\Lambda_\omega}) \in F^{sup(\overline{a})}$.*

Proof. By the Truth Lemma, every possibility structure $\mathcal{B}_i$ is elementarily equivalence to $\mathbb{Z}$, and hence satisfies the axioms of a totally ordered Abelian group. By the First Colimit Lemma, every $\mathcal{B}_i$ elementarily embeds into $\overrightarrow{\mathcal{B}_\omega}$, which itself elementarily embeds into $\mathcal{B}$. Hence the $\mathscr{L}'$-reduct $\mathcal{B}$ also satisfies the axioms of a totally ordered Abelian group. Moreover, fix a la'-formula $\varphi(\overline{x})$ and a tuple $\overline{a} = a_1, ...a_k \in \mathbb{Z}^{\Lambda_\omega}$. Then there are $b_1, ..., b_k \in \mathbb{Z}^{\Lambda_{sup(\overline{a})}}$ such that $\beta(b_n) = a_n$ for all $n \in \{1, ..., k\}$. But then we have the following chain of equivalences:

$$\begin{aligned} F \Vdash \varphi(\overline{a}) &\Leftrightarrow \sigma(F) \Vdash \varphi(b_1, ..., b_k) \\ &\Leftrightarrow \sigma(F)(sup(\overline{a})) \Vdash \varphi(b_1, ..., b_k) \\ &\Leftrightarrow ||\varphi(b_1, ..., b_k)||_{\Lambda_{sup(\overline{a})}} \in F^{sup(\overline{a})} \\ &\Leftrightarrow \pi^{\omega}_{sup(\overline{a})}(||\varphi(\overline{a}||_{\Lambda_\omega}) \in F^{sup(\overline{a})}, \end{aligned}$$

where the first equivalence holds because (σ, β) is elementary, the second one holds because the possibility embedding $(\pi_{sup(\overline{a})}, \alpha_{sup(\overline{a})}) : \mathcal{B}_{sup(\overline{a})} \to \overrightarrow{\mathcal{B}'_\omega}$ is elementary, the third one follows from the Truth Lemma for $\mathcal{B}_{sup(\overline{a})}$, and the last one follows from the fact that $\pi^{\omega}_{sup(\overline{a})}(||\varphi(\overline{a}||_{\Lambda_\omega}) = ||\varphi(b_1, ..., b_k)||_{\Lambda_{sup(\overline{a})}}$. □

We are now in a position to prove our main theorem about the possibility structure of Bolzanian quantities.

Theorem 9.7.22. *The structure $\mathcal{B} = (\mathfrak{B}, \mathbb{Z}^{\Lambda_\omega}, \mathscr{I})$ satisfies the axioms of a (non-commutative) totally ordered ring.*

Proof. In light of Corollary 9.7.21, we only need to prove that the operation . is an associative operation with unit 1 that distributes over addition and satisfies the order axiom. Fix $a, b, c \in \mathbb{Z}^{\Lambda_\omega}$ with support i, j, k respectively, and let $a' = a|^{\omega}_{i}$, $b' = b|^{\omega}_{j}$, and $c' = c|^{\omega}_{k}$.

- **Associativity**: Fix $\overline{A}$, $\overline{B}$ and $\overline{C}$ in Λ_i, Λ_j and Λ_k respectively. Then we compute:

$$\begin{aligned} a'.(b'.c')(\overline{ABC}) &= a'(\overline{A}) \times (b'.c'(\overline{BC})) \\ &= a'(\overline{A}) \times (b'(\overline{B}) \times c'(\overline{C})) \\ &= (a'(\overline{A}) \times b'(\overline{B})) \times c'(\overline{C}) \qquad \text{(by associativity of } \times \text{ in } \mathbb{Z}) \\ &= (a'.b'(\overline{AB})) \times c'(\overline{C}) \\ &= (a'.b').c'(\overline{ABC}). \end{aligned}$$

  But this implies that $\mathcal{B} \models a.(b.c) = (a.b).c$.

- **Multiplicative identity**: Note that any integer z is represented in $\mathbb{B}$ by a function $\boldsymbol{z} \in \mathbb{Z}^{\Lambda_\omega}$ with range $\{z\}$ which is constant and therefore has support 0. Thus for any $\overline{A} \in \Lambda_i$, $a'.\boldsymbol{z}(\overline{A}) = a'(\overline{A}) \times z$ and $\boldsymbol{z}.a'(\overline{A}) = z \times a'(\overline{A})$. Thus $a'.\boldsymbol{z} = \boldsymbol{z}.a' = \underbrace{a' + ... + a'}_{z \text{ times}}$.
  Hence $\mathcal{B} \models a.\boldsymbol{z} = \boldsymbol{z}.a = \underbrace{a + ... + a}_{z \text{ times}}$, and in particular $\mathcal{B} \models \mathbf{1}.a = a.\mathbf{1} = a$.

- **Left-distributivity**: Without loss of generality, assume that $j = k$, which implies that $sup(b + c) = j$. Fix $\overline{A} \in \Lambda_i$ and $\overline{B} \in \Lambda_j$. Then:

$$\begin{aligned} a'.(b' + c')(\overline{AB}) &= a'(\overline{A}) \times (b' + c'(\overline{B})) \\ &= a'(\overline{A}) \times (b'(\overline{B}) + c'(\overline{B})) \\ &= (a'(\overline{A}) \times b'(\overline{B})) + (a'(\overline{A}) \times c'(\overline{B})) \quad \text{(by left-distributivity in } \mathbb{Z}\text{)} \\ &= (a'.b'(\overline{AB})) + (a'.b'(\overline{AB})) \\ &= (a'.b') + (a'.c')(\overline{AB}). \end{aligned}$$

  This implies that $\mathcal{B} \models a.(b + c) = (a.b) + (a.c)$

- **Right-distributivity**: Again, without loss of generality, assume that $i = j = sup(a + b)$, and let $\overline{A} \in \Lambda_i$ and $\overline{C} \in \Lambda_k$. Then:

$$\begin{aligned} (a' + b').c'(\overline{AC}) &= (a' + b'(\overline{A})) \times c'(\overline{C}) \\ &= (a'(\overline{A}) + b'(\overline{A})) \times c'(\overline{C}) \\ &= (a'(\overline{A}) \times c'(\overline{C})) + (b'(\overline{A}) \times c'(\overline{C})) \quad \text{(by right-distributivity in } \mathbb{Z}\text{)} \\ &= (a'.c'(\overline{AC})) + (b'.c'(\overline{AC})) \\ &= (a'.c') + (b'.c')(\overline{AC}). \end{aligned}$$

  This implies that $\mathcal{B} \models (a + b).c = (a.c) + (b.c)$.

- **Order axiom**: Fix $F \in \mathfrak{B}$, and suppose that $F \Vdash 0 < a$ and $F \Vdash 0 < b$. By Corollary 9.7.21, this means that $||0 < a'||_{\Lambda_i} \in F^i$ and $||0 < b'||_{\Lambda_j} \in F^j$. Note that for any $\overline{A} \in \Lambda_i$ and $\overline{B} \in \Lambda_j$, $\overline{A} \in ||0 < a'||_{\Lambda_i}$ and $\overline{B} \in ||0 < b'||_{\Lambda_j}$ together imply that $\overline{AB} \in ||0 < a'.b'||_{\Lambda_{i+j}}$. Thus $||0 < a'.b'||_{\Lambda_i} \supseteq U = \{\overline{AB} \in \Lambda_{i+j} \mid \overline{A} \in ||0 < a'||_{\Lambda_i}$ and $\overline{B} \in ||0 < b'||_{\Lambda_j}\}$. Moreover, by assumption, we have that $\{\overline{A} \in \Lambda_i \mid U|\overline{A} \in F^j\} \in F^i$, which implies that $U \in F^{i+j}$ by property 5 of iterating families. Hence $||0 < a'.b'||_{\Lambda_{i+j}} \in F^{i+j}$, which by Corollary 9.7.21 again implies that $F \Vdash 0 < a.b$. This completes the proof. □

Let us conclude by investigating some properties of the product in our setting. First of all, we can verify that it is not commutative by checking that $\mathcal{B} \models \overset{0}{\mathbf{N}}.\overset{1}{\mathbf{S}} > \overset{1}{\mathbf{S}}.\overset{0}{\mathbf{N}}$. To see this, note first that $\overset{0}{\mathbf{N}}.\overset{1}{\mathbf{S}}(AB\overline{C}) = |A| \times \sum_{j \in B} j$ and that $\overset{1}{\mathbf{S}}.\overset{0}{\mathbf{N}}(AB\overline{C}) = \sum_{i \in A} i \times |B|$ for any $AB\overline{C} \in \Lambda_\omega$. I claim that $U = \{AB \in \Lambda_2 \mid |A| \times \sum_{j \in B} j > \sum_{i \in A} i \times |B|\} \in F^2$ for any $F \in \mathcal{B}$. By Corollary 9.7.21, this will suffice to show that $\mathcal{B} \models \overset{0}{\mathbf{N}}.\overset{1}{\mathbf{S}} > \overset{1}{\mathbf{S}}.\overset{0}{\mathbf{N}}$. For the proof of the claim, fix some $F \in \mathfrak{B}$ and recall first the notation $w(A) = \frac{\sum_{i \in A} i}{|A|}$ introduced in the proof of Lemma 9.7.3.3 for any nonempty $A \subseteq \Lambda_1$. Note that whenever $w(B) > w(A)$, we have that $AB \in U$. Moreover, for any non-empty $A \in \Lambda_1$, letting $B = \{1, ..., 2w(A)\}$, by Gauss's summation theorem we have that $w(B) = \frac{2w(A)(2w(A)+1)}{4w(A)} = \frac{2w(A)+1}{2} > w(A)$, and moreover $w(C) \geq w(B)$ for any $C \supseteq B$. Hence $C \in U|A$ for any $C \supseteq B$, which shows that $U|A \in F$, since F is fine. Hence $U|A \in F$ for any nonempty $A \in \Lambda_1$, from which we conclude that $U \in F^2$ by property 5 of iterating families. This completes the proof of the claim. Therefore $\mathcal{B} \models \overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}} > \overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}}$. Again, this is in line with our interpretation of products as iterated Bolzanian sums, since, under this interpretation, the Bolzanian sum $\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}} - \overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}} = (\overset{\mathbf{1}}{\mathbf{S}} - \overset{\mathbf{0}}{\mathbf{N}}) + (\overset{\mathbf{1}}{\mathbf{S}} - 2\overset{\mathbf{0}}{\mathbf{N}}) + (\overset{\mathbf{1}}{\mathbf{S}} - 3\overset{\mathbf{0}}{\mathbf{N}}) + \ldots$ *in inf.* is an infinite sum of positive quantities.

Finally, we may also verify that the products of infinitely large quantities yield quantities of "higher order" in a strong sense, as evidence by the following lemma.

Lemma 9.7.23. *Let $F \in \mathfrak{B}$ and $a, b \in \mathbb{Z}^{\Lambda_\omega}$ such that $sup(a) = i$, $sup(b) = j$, $F \Vdash a > 0$ and there is $k < j$ such that $F \Vdash b > c$ for all c such that $sup(c) \leq k$. Then $F \Vdash a.b > c$ for every c such that $sup(c) \leq i + k$.*

Proof. Fix F, a, b and k as in the statement of the lemma, and let $c \in \mathbb{Z}^{\Lambda_\omega}$ be such that $sup(c) = i + l \leq i + k$. Let $a' \in ZZ^{\Lambda_i}$, $b' \in \mathbb{Z}^{\Lambda_j}$ and $c' \in \mathbb{Z}^{\Lambda_l}$ such that $\beta_i(a') = a$, $\beta_j(b') = b$ and $\beta_l(c') = c$. Moreover, we will write $a'.b'$ for some $d \in ZZ^{\Lambda_{i+j}}$ such that $\beta_{i+j}(d) = a.b$. By Corollary 9.7.21, it is enough to show that $F^{i+j} \Vdash a'.b' > \epsilon_{l(i+j)}(c')$. Let $U \subseteq \Lambda_{i+j}$ be the set

$$\{\overline{ABC} \mid \overline{A} \in \Lambda_1, \overline{B} \in \Lambda_l, \overline{C} \in \Lambda_{j-l} \text{ and } a'(\overline{A}) \times b'(\overline{BC}) > \epsilon_{l,(i+j)}(c')(\overline{ABC})\}.$$

Note first that we may rewrite U as $\{\overline{ABC} \mid a'(\overline{A}) \times b'(\overline{BC}) > c'(\overline{AB})\}$. Moreover, letting

$$V = \{\overline{ABC} \mid a'(\overline{A}) > 0 \text{ and } b'(\overline{BC}) > \frac{c'(\overline{AB})}{a'(\overline{A})}\},$$

we have that $V \subseteq U$. Now fix $\overline{A} \in \Lambda_1$ such that $\overline{A} > 0$, and let $d \in \mathbb{Z}^{\Lambda_l}$ be given by $d(\overline{B}) = \frac{c'(\overline{AB})}{a'(\overline{A})}$ for any $\overline{B} \in \Lambda_j$. By assumption, we have that $F \Vdash b > \epsilon_l(d)$, and thus that $\{\overline{BC} \in \Lambda_j \mid b'(\overline{BC}) > \frac{c'(\overline{AB})}{a'(\overline{A})}\} \in F^j$. Since $\{\overline{A} \in \Lambda_1 \mid a'(\overline{A}) > 0\} \in F^i$ by assumption, it follows that $\{\overline{A} \mid V|\overline{A} \in F^j\} \in F^i$, and thus that $V \in F^{i+j}$. We may therefore conclude that $U \in F^{i+j}$, and hence that $F^{i+j} \Vdash a'.b' > \epsilon_{l(i+j)}(c')$. This completes the proof. $\square$

As a special consequence of Lemma 9.7.23, we immediately get that $\mathcal{B} \models (\overset{\mathbf{0}}{\mathbf{N}})^2 > \overset{Odds}{S}$ and that $\mathcal{B} \models \overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}} + 1) > \overset{1}{S}$. Thus we are able to recover the strong sense in which products of infinite quantities yield quantities of higher order even in our formal reconstruction using possibility semantics.

Conclusion

Our goal was to provide a faithful interpretation of the *PU* and especially of Bolzano's calculation of the infinite as presented in §§29-33. We argued that Bolzano's computations should not be judged as failed attempts at anticipating Cantor's transfinite arithmetic, and that Bolzano's primary interest was not in measuring the sizes of infinite collections of natural numbers, but in developing an arithmetic of infinite sums of integers. As a consequence, one should not read Bolzano as failing to anticipate Cantor's work because of his commitment to a set-theoretic version of the part-whole principle but rather as developing from part-whole considerations an original and productive way of reasoning about infinite sums. Moreover, far from shutting Bolzano out of future historiographies of set theory, this new interpretation clarifies where Bolzano's approach to the infinite stands within that history. The intentions and methods of Bolzano when computing with the infinitely large are radically different from Cantor's, yet, as we have shown, amenable to a consistent mathematical interpretation. In particular, we have seen that several of Bolzano's ideas about the infinite resonate quite harmoniously with the novel perspective on mathematical objects afforded by possibility semantics. Let us therefore hope that the present work may mark only the beginning of deeper scholarly engagement with Bolzano's mathematical infinite.

9.8 Appendix A

In this appendix, we describe in more detail the ring of Bolzanian quantities $\mathbb{B}$ mentioned in Section 9.5.2. In particular, we show how to construct $\mathbb{B}$ as a direct limit of iterated ultrapowers, define rigorously the product of two infinite quantities, and prove Theorem 9.5.3.

Let us first note that a standard presentation of our construction would require us to take a direct limit of the structures:

$$\mathbb{Z} \xrightarrow[\iota_0]{} \mathbb{Z}_{\mathsf{U}} \xrightarrow[\iota_1]{} (\mathbb{Z}_{\mathsf{U}})^2 \xrightarrow[\iota_2]{} (\mathbb{Z}_{\mathsf{U}})^3 \xrightarrow[\iota_3]{} \dots$$

where for any natural number n, $(\mathbb{Z}_{\mathsf{U}})^{n+1}$ is the ultrapower of $(\mathbb{Z}_{\mathsf{U}})^n$ by U, and each ι_{n+1} : $(\mathbb{Z}_{\mathsf{U}})^{n+1} \to (\mathbb{Z}_{\mathsf{U}})^{n+2}$ maps (any equivalence class of) a function $f : \omega^+ \to \mathbb{Z}^n_{\mathsf{U}}$ to the function mapping any i to $\overline{f(i)}$. The inconvenience of this approach is that it requires us to introduce elements of increasing complexity in our structure, i.e., functions from ω^+ into the integers, functions from ω^+ into functions from ω^+ into the integers, and so on. However, we may present our construction differently, by drawing on the well known fact that for any sets A, B and C, there is a canonical bijection φ between functions from A into C^B and functions from $A \times B$ into C: given any $f : A \to C^B$, the function $\varphi(f) : A \times B$ is such that $\varphi(f)(a, b) = f(a)(b)$ for any $a \in A$ and $b \in B$. Instead of working with functions of higher and higher complexity, we may therefore simply work with functions of finite arity, or, equivalently functions from finite sequences of elements in ω^+ into Z. However, since we still need to identify functions using an ultrafilter U, we also need to generalize our definition of when two n-ary functions are equivalent according to U. This requires the following definition.

Definition 9.8.1. Let U be a non-principal ultrafilter on ω^+. For any natural number n, we define U^n by induction as follows:

- $\mathsf{U}^0 = \{(\omega^+)^0\}$
- U^{n+1} is a collection of subsets of $(\omega^+)^{n+1}$ such that for any $X \subseteq (\omega^+)^{n+1}$, $X \in \mathsf{U}^{n+1}$ if and only if $\{i \in \omega^+ : X|i \in \mathsf{U}^n\} \in \mathsf{U}$, where for any $i \in \omega^+$, $X|i$ is the set of n-tuples $\overline{j}$ in $(\omega^+)^n$ such that the $n+1$-tuples $i\overline{j} \in X$.

Note that $(\omega^+)^0$ is the set of all 0-ary sequences of elements of ω^+, i.e., contains only the empty sequence. It is also straightforward to see that, given the previous definition, $\mathsf{U}^1 = \mathsf{U}$. The following lemma will be useful later on, and is established by a straightforward induction on the natural numbers.

Lemma 9.8.2.

- *For any natural number n, U^n is an ultrafilter on $(\omega^+)^n$ which is non-principal if $n > 0$.*
- *Let m, n be two natural numbers and $X \subseteq (\omega^+)^{m+n}$. Then $X \in \mathsf{U}^{n+m}$ if and only if*

$$\{\overline{i} \in (\omega^+)^m : \{\overline{j} \in (\omega^+)^n : \overline{ij} \in X\} \in \mathsf{U}^n\} \in \mathsf{U}^m.$$

We can then define the following structures:

Definition 9.8.3. Let n be a natural number. We let $\mathbb{Z}_{\mathsf{U}^n} := (Z_{\mathsf{U}^n}, +, -, 0, 1)$ be the ultrapower of $\mathbb{Z}$ by U^n. More precisely, elements in $\mathbb{Z}_{\mathsf{U}^n}$ are equivalence classes of functions from $(\omega^+)^n$ to $\mathbb{Z}$, where for any two functions $f, g : \omega^+ \to \mathbb{Z}$:

- $f^* = g^*$ iff $\{\overline{i} \in \omega^+ : f(\overline{i}) = g(\overline{i})\} \in \mathsf{U}^n$;
- $(f+g)^* = f^* + g^*$, $(f-g)^* = f^* - g^*$;
- $f^* < g^*$ iff $\{\overline{i} \in \omega^+ : f(\overline{i}) < g(\overline{i})\} \in \mathsf{U}^n$.

In particular, it is straightforward to verify that $\mathbb{Z}_{\mathsf{U}^0}$ is isomorphic to $\mathbb{Z}$.

Since U^n is an ultrafilter on $(\omega^+)^n$ for any natural number n, the previous definition is a generalization of the original construction of $\mathbb{Z}_{\mathsf{U}}$. Moreover, we have natural embeddings $\boldsymbol{\lambda_n} : \mathbb{Z}_{\mathsf{U}^n} \to \mathbb{Z}_{\mathsf{U}^{n+1}}$. In fact, those embeddings are always elementary:

Lemma 9.8.4. *For any $f : (\omega^+)^n \to Z$, let $\lambda_n(f) : (\omega^+)^{n+1} \to Z$ be such that for any n-tuple $\overline{i}$ and any $j \in \omega^+$, $\lambda_n(f)(\overline{i}j) = f(\overline{i})$. Then the function $\boldsymbol{\lambda_n} : \mathbb{Z}_{\mathsf{U}^n} \to \mathbb{Z}_{\mathsf{U}^{n+1}}$ defined by $\boldsymbol{\lambda_n}(f^*) = \lambda_n(f)^*$ is an elementary embedding.*

The proof of this lemma is a simple application of the Tarski-Vaught test of elementary substructures. For any natural numbers $m \leq n$, we let $\lambda_{m,n}$ be the composition of the embeddings $\boldsymbol{\lambda_{n-1}} \circ \boldsymbol{\lambda_{n+2}} \circ \ldots \circ \boldsymbol{\lambda_{m+1}} \circ \boldsymbol{\lambda_m}$. We can then define the structure $(B, +, -, 0, 1, <)$ as the direct limit of the system

$$\mathbb{Z}_{\mathsf{U}^0} \xrightarrow[\boldsymbol{\lambda_0}]{} \mathbb{Z}_{\mathsf{U}^1} \xrightarrow[\boldsymbol{\lambda_1}]{} \mathbb{Z}_{\mathsf{U}^2} \xrightarrow[\boldsymbol{\lambda_2}]{} \ldots$$

We will refer to elements in B as *quantities*. By definition of the direct limit of a directed system, quantities are equivalence classes of elements in some $\mathbb{Z}_{\mathsf{U}^n}$, where for any $m \leq n$ and any two equivalence classes $f^* \in \mathbb{Z}_{\mathsf{U}^n}, g^* \in \mathbb{Z}_{\mathsf{U}^m}$, f^* and g^* are identified if and only if $\mathbb{Z}_{\mathsf{U}^n} \models \lambda_{m,n}(f^*) = g^*$. For any quantity $\boldsymbol{\alpha}$, we let the *order* of $\boldsymbol{\alpha}$ be the smallest natural number n such that there is some $f^* \in \boldsymbol{\alpha}$ such that $f^* \in \mathbb{Z}_{\mathsf{U}^n}$. Clearly, any $\boldsymbol{\alpha} \in B$ has a finite order n, and moreover, if $\boldsymbol{\alpha}$ has order n witnessed by some f^*, then for any natural number m, any $g^* \in \mathbb{Z}_{\mathsf{U}^{n+m}}$, and any tuples $\bar{i}$ and $\bar{j}$ of length n and m respectively, $f(\bar{i}) = g(\overline{ij})$. We may therefore abuse notation and view $\boldsymbol{\alpha}$ as a function from m-tuples of elements in ω^+ into Z for any $m \geq n$.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two quantities of order m and n respectively, represented by the functions $f_{\boldsymbol{\alpha}}$ and $f_{\boldsymbol{\beta}}$ of arity m and n respectively. We define the *product* $\boldsymbol{\alpha}.\boldsymbol{\beta}$ as (the equivalence class of) the function $f_{\boldsymbol{\alpha}.\boldsymbol{\beta}} : (\omega^+)^{m+n} \to Z$ such that for any tuples $\bar{i}$ and $\bar{j}$ of length m and n respectively, $f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = f_{\boldsymbol{\alpha}}(\bar{i}) \times f_{\boldsymbol{\beta}}(\bar{j})$, i.e., $\underbrace{f_{\boldsymbol{\beta}}(\bar{j}) + \ldots + f_{\boldsymbol{\beta}}(\bar{j})}_{f_{\boldsymbol{\alpha}}(\bar{i}) \text{ times}}$. It is straightforward to verify that this operation is well-defined. Indeed, suppose $g_{\boldsymbol{\alpha}} \in \boldsymbol{\alpha}$ and $g_{\boldsymbol{\beta}} \in \boldsymbol{\beta}$ are functions of arity m and n respectively. Clearly for any m-tuple $\bar{i}$ and any n-tuple $\bar{j}$, if $f_{\boldsymbol{\alpha}}(\bar{i}) = g_{\boldsymbol{\alpha}}(\bar{i})$ and $f_{\boldsymbol{\beta}}(\bar{j}) = g_{\boldsymbol{\beta}}(\bar{j})$, then $g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})$. Moreover, since $f_{\boldsymbol{\alpha}}$ and $g_{\boldsymbol{\alpha}}$ are U^m equivalent, and $f_{\boldsymbol{\beta}}$ and $g_{\boldsymbol{\beta}}$ are U^n equivalent, it follows that for U^m-many $\bar{i}$ there are U^n-many $\bar{j}$ such that $f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})$. Equivalently,

$$\{\bar{i} \in (\omega^+)^m : \{\bar{j} \in (\omega^+)^n : f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})\} \in \mathsf{U}^n\} \in \mathsf{U}^m,$$

which by Lemma 9.8.2 implies that $\{\overline{ij} \in (\omega^+)^{m+n} : f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})\} \in \mathsf{U}^{m+n}$, and therefore $f^*_{\boldsymbol{\alpha}.\boldsymbol{\beta}} = g^*_{\boldsymbol{\alpha}.\boldsymbol{\beta}}$.

The next lemma establishes that the product of two quantities of order m and n is of order $m + n$. The proof is a simple application of Łoś's theorem.

Lemma 9.8.5. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two quantities of order m and n respectively, and let $\boldsymbol{\gamma}$ be a quantity of order $l < m + n$. Then $\mathbb{B} \models \boldsymbol{\alpha}.\boldsymbol{\beta} \neq \boldsymbol{\gamma}$.*

Finally, we can now prove Theorem 9.5.3 and establish that Bolzanian sums and products form a non-commutative ordered ring.

Theorem 9.8.6. *The structure $\mathbb{B} = (B, +, -, 0, 1, <, .)$ is a non-commutative ordered ring.*

Proof. Note first that by construction, we have an elementary embedding from $\mathbb{Z}$ into the reduct $(B, +, -, 0, 1, <)$, which immediately implies that $\mathbb{B}$ is an ordered additive group. We therefore only need to verify the following properties:

- **Associativity**: Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be three quantities of order l, m and n respectively.

Then for any tuples $\bar{i}$, $\bar{j}$ and $\bar{k}$ of arity l, m and n respectively, we have that:

$$\begin{aligned}\boldsymbol{\alpha}.(\boldsymbol{\beta}.\boldsymbol{\gamma})(\overline{ijk}) &= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta}.\boldsymbol{\gamma}(\overline{jk})) \\ &= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta}(\bar{j}) \times \boldsymbol{\gamma}(\bar{k})) \\ &= (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\beta}(\bar{j})) \times \boldsymbol{\gamma}(\bar{k}) && \text{(by associativity of } \times \text{ in } \mathbb{Z}) \\ &= (\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})) \times \boldsymbol{\gamma}(\bar{k}) \\ &= (\boldsymbol{\alpha}.\boldsymbol{\beta}).\boldsymbol{\gamma}(\overline{ijk}).\end{aligned}$$

- **Multiplicative identity**: Note that any integer z is represented in $\mathbb{B}$ by a quantity $\boldsymbol{z}$ of order 0, which corresponds to the set of all constant functions from finite sequences of elements in ω^+ into Z with range $\{z\}$. For any quantity $\boldsymbol{\alpha}$ of order l, we therefore have that $\boldsymbol{\alpha}.\boldsymbol{z}$ and $\boldsymbol{z}.\boldsymbol{\alpha}$ are quantities of order n such that for any l-tuple $\bar{i}$, $\boldsymbol{\alpha}.\boldsymbol{z}(\bar{i}) = \boldsymbol{\alpha}(\bar{i}) \times z$ and $\boldsymbol{z}.\boldsymbol{\alpha}(\bar{i}) = z \times \boldsymbol{\alpha}(\bar{i})$. Thus $\boldsymbol{\alpha}.\boldsymbol{z} = \boldsymbol{z}.\boldsymbol{\alpha} = \underbrace{\boldsymbol{\alpha} + ... + \boldsymbol{\alpha}}_{z \text{ times}}$. Hence in particular $\mathbf{1}.\boldsymbol{\alpha} = \boldsymbol{\alpha}.\mathbf{1} = \boldsymbol{\alpha}$.

- **Left-distributivity**: Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be as above. Without loss of generality, assume that the order m of $\boldsymbol{\beta}$ is greater than or equal to the order n of $\boldsymbol{\gamma}$, which implies that $\boldsymbol{\beta}+\boldsymbol{\gamma}$ is also of order m. Fix an l-tuple $\bar{i}$ and an n-tuple $\bar{j}$. Note that even though $\boldsymbol{\gamma}$ is of lower order, we may still write $\boldsymbol{\gamma}(\bar{j})$. Then:

$$\begin{aligned}\boldsymbol{\alpha}.(\boldsymbol{\beta}+\boldsymbol{\gamma})(\overline{ij}) &= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta} + \boldsymbol{\gamma}(\bar{j})) \\ &= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta}(\bar{j}) + \boldsymbol{\gamma}(\bar{j})) \\ &= (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\beta}(\bar{j})) + (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\gamma}(\bar{j})) && \text{(by left-distributivity of } \times \text{ over } + \text{ in } \mathbb{Z}) \\ &= (\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})) + (\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})) \\ &= (\boldsymbol{\alpha}.\boldsymbol{\beta}) + (\boldsymbol{\alpha}.\boldsymbol{\gamma})(\overline{ij}).\end{aligned}$$

- **Right-distributivity**: Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ as above, and assume the order l of $\boldsymbol{\alpha}$ is greater than or equal to the order m of $\boldsymbol{\beta}$. Let $\bar{i}$ be an l-tuple and $\bar{k}$ a n-tuple. Then:

$$\begin{aligned}(\boldsymbol{\alpha}+\boldsymbol{\beta}).\boldsymbol{\gamma}(\overline{ik}) &= (\boldsymbol{\alpha} + \boldsymbol{\beta}(\bar{i})) \times \boldsymbol{\gamma}(\bar{k}) \\ &= (\boldsymbol{\alpha}(\bar{i}) + \boldsymbol{\beta}(\bar{i})) \times \boldsymbol{\gamma}(\bar{k}) \\ &= (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\gamma}(\bar{k})) + (\boldsymbol{\beta}(\bar{i}) \times \boldsymbol{\gamma}(\bar{k})) && \text{(by right-distributivity of } \times \text{ over } + \text{ in } \mathbb{Z}) \\ &= (\boldsymbol{\alpha}.\boldsymbol{\gamma}(\overline{ik})) + (\boldsymbol{\beta}.\boldsymbol{\gamma}(\overline{ik})) \\ &= (\boldsymbol{\alpha}.\boldsymbol{\gamma}) + (\boldsymbol{\beta}.\boldsymbol{\gamma})(\overline{ik}).\end{aligned}$$

- **Order axiom**: Suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two quantities of order l and m respectively and are such that $\mathbb{B} \models 0 < \boldsymbol{\alpha}$ and $\mathbb{B} \models 0 < \boldsymbol{\beta}$. We claim that $\mathbb{B} \models 0 < \boldsymbol{\alpha}.\boldsymbol{\beta}$. Indeed, since $\mathbb{B} \models 0 < \boldsymbol{\alpha}$, we have that $\{\bar{i} \in (\omega^+)^l : 0 < \boldsymbol{\alpha}(\bar{i})\} \in \mathsf{U}^l$, while it follows from $\mathbb{B} \models 0 < \boldsymbol{\beta}$ that $\{\bar{j} \in (\omega^+)^m : 0 < \boldsymbol{\beta}(\bar{j})\} \in \mathsf{U}^m$. Now clearly for any l-tuple $\bar{i}$ such that $0 < \boldsymbol{\alpha}(\bar{i})$, if $\bar{j}$ is an m-tuple such that $0 < \boldsymbol{\beta}(\bar{j})$, then $0 < \boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\beta}(\bar{j})$, i.e., $0 < \boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})$. Thus

$$\{\bar{i} \in (\omega^+)^l : \{\bar{j} \in (\omega^+)^m : 0 < \boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})\} \in \mathsf{U}^m\} \in \mathsf{U}^l,$$

which by Lemma 9.8.2 implies that $\{\overline{ij} \in (\omega^+)^{l+m} : 0 < \boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})\} \in \mathsf{U}^{l+m}$, and hence $\mathbb{B} \models 0 < \boldsymbol{\alpha}.\boldsymbol{\beta}$.

□

Let us conclude this appendix with a few remarks regarding the Bolzanian ring of infinite quantities $\mathbb{B}$. First, our formalization only allows us to represent infinite quantities of a finite order, i.e., infinite sums of the form $\boldsymbol{\alpha}(1) + \boldsymbol{\alpha}(2) + \boldsymbol{\alpha}(3) + \cdots$ for which there is an $n < \omega^+$ such that for all $m \geq n$, the order of $\boldsymbol{\alpha}(m)$ is less than or equal to the order of $\boldsymbol{\alpha}(n)$. For example, the following infinite sum is not represented by any element in $\mathbb{B}$:

$$\overset{0}{N} + (\overset{0}{N})^2 + (\overset{0}{N})^3 + \ldots \textit{ in inf.}$$

Of course, if we wanted to include this sum in our model, we would have to take an ultrapower of $\mathbb{B}$ by U and construct another countable sequence of ultrapowers. In fact, if we wanted to close our domain of infinite quantities under taking infinite sums, we would need to keep iterating the ultrapower until the first ordinal with uncountable cofinality, i.e., until ω_1. Our structure $\mathbb{B}$, however, is more than enough to account for Bolzano's examples, and we certainly do not want to claim that the consistency of Bolzano's system requires anything like uncountable ordinals.

Second, it is quite straightforward to observe that the situation described in Lemma 9.5.2 generalizes to the full structure $\mathbb{B}$. Indeed, for any n, the product of any n^{th} order quantity with at least a first-order infinite quantity is always greater than or smaller than any quantity of strictly lower order. Thus, in accordance with Bolzano's original claims, multiplying infinite quantities together yields new quantities that are infinitely larger or infinitely smaller than the previous ones in a very strong sense.

9.9 Appendix B

In this appendix, we show how to construct an iterating family as in Definition 9.7.10. We start from the following definition.

Definition 9.9.1. Given sets S and T, an element $s \in S$ and a set $U \subseteq S \times T$, we let $U|s = \{t \in T \mid (s,t) \in U\}$. Moreover, let F and G be filters on sets S and T respectively. Then the *iterate* of G by F is the set $F \times G \subseteq \mathscr{P}(S \times T)$ such that for any $U \subseteq S \times T$, $U \in F \times G$ iff $\{s \in S \mid U|s \in F\} \in F$. Finally, we define inductively the *iterating sequence* $\{F_i \mid i \in \omega^+\}$ by letting $F_1 = F$ and $F_{i+1} = F_i \times F$ for any $i \in \omega^+$.

In what follows, we will be particularly interested in iterating fine filters on Λ_1.

Lemma 9.9.2. *Let F be a fine filter on Λ_1 and $\{F_i \mid i \in \omega^+\}$ the iterating sequence determined by F.*

- *For any $i, j \in \omega^+$, $F_{i+j} = \{U \subseteq \Lambda_{i+j} \mid \{\overline{A} \subseteq \Lambda_i \mid U|\overline{A} \in F_j\} \in F_i\}$;*

- *For any $i \in \omega^+$, F_i is a fine filter on Λ_i;*

Proof. Fix a fine filter F.

- We prove this by induction on i. The base case for $i = 0$ is immediate. Assuming that the claim holds for i, suppose $U \subseteq \Lambda_{i+1+j}$ and that the set X of sequences $B\overline{A} \in \Lambda_{i+1}$ such that $U|B\overline{A}$ is in F_j is in $F_i + 1$. By definition, this means that for F-many $B \in \Lambda_1$, the set $X|B \in F_i$. Now let B be such that $U|B \in F_i$, and set $V_B = U|B$. Then by assumption we have that the set $V_B|\overline{A} = U|B\overline{A} \in F_j$ for F_i-many $\overline{A} \in \Lambda_i$, so by induction hypothesis it follows that $V_B \in F_{i+j}$. Hence for F-many B the set $V_B = U|B \in F_{i+j}$, so $U \in F_{i+j+1} = F_{i+1+j}$. This shows the right-to-left inclusion.

  For the left-to-right inclusion, suppose $U \in F_{i+1+j} = F_{i+j+1}$. By definition, we have that for F-many $B \in \Lambda_1$, $U|B \in F_{i+j}$. By induction hypothesis, this means that for F_i-many $\overline{A} \in F_i$, the set $U|B\overline{A} \in F_j$. But then we have that for F-many B, the set $X|B = \{\overline{A} \in \Lambda_i \mid U|B\overline{A} \in F_j\}$ is in F_i, hence the set $X = \{B\overline{A} \in \Lambda_{i+1} \mid U|B\overline{A} \in F_j\} \in F_{i+1}$. This completes the proof.

- We prove this by induction on i. Assume that F_i is a fine filter. If $U \in F_{i+1}$, then by definition $U|A \in F_i$ for F-many $A \in \Lambda_1$. But clearly if $U \subseteq V$ then $U|A \subseteq V|A$ for any $A \in \Lambda_1$, so $V|A \in F_i$ for F-many A since F_i is a filter. Hence $V \in F_{i+1}$. Similarly, note that $(U \cap V)|A = U|A \cap V|A$, so if $U, V \in F_{i+1}$, then $(U \cap V)|A \in F_i$ for F-many $A \in \Lambda_1$, hence $U \cap V \in F_{i+1}$. Finally, let us show that F_{i+1} is fine. Let $\overline{A} \in \Lambda_{i+1}$. We need to show that $A^* = \{\overline{B} \in \Lambda_{i+1} \mid \overline{A} \leq \overline{B}\} \in F_{i+1}$, where the ordering on l_{i+1} is the pointwise inclusion ordering. Let $\overline{A} = A_1A_2...A_{i+1}$, and write $\overline{A'}$ for the sequence $A_2...A_{i+1}$. Note that if $B \in \Lambda_1$ is such that $A_1 \subseteq B$, then $A^*|B = \{\overline{B} \mid \overline{A} \leq \overline{B}\}$, hence $A^*|B \in F_i$ since F_i is fine. But since $\{B \mid A \subseteq B\} \in F$ because F is fine, it follows that $A^* \in F_{i+1}$.

□

Let us now construct an iterating family $\{F^i \mid i \in \omega^+, F \in Filt(\Lambda_1)\}$. Recall first that we may view $Filt(\Lambda_1)$ as a forcing notion $\mathbb{P}$, and consider the forcing relation $\Vdash$ defined between conditions in $\mathbb{P}$, i.e., fine filters on Λ_1, and formulas in the forcing language $\mathscr{L}_{\mathbb{P}}$. For our purposes, the ground model M of our forcing is not really relevant, as long as M contains $\mathbb{P}$ and the Forcing Theorem holds, in the sense that we have the following:

Forcing Theorem For any $\mathscr{L}_{\mathbb{P}}$-formula $\varphi(\dot{a}_1, ..., \dot{a}_n)$ and any $F \in \mathbb{P}$, $F \Vdash \varphi(\dot{a}_1, ..., \dot{a}_n)$ iff for any generic filter G over $\mathbb{P}$, $F \in \mathsf{G}$ implies $M[\mathsf{G}] \models \varphi(\dot{a}_{1\mathsf{G}}, ..., \dot{a}_{n\mathsf{G}})$.

For example, M could be the whole set-theoretic universe V, or the inner model $L(\mathbb{R})$. Now clearly, any generic filter G over $\mathbb{P}$ determines a fine ultrafilter U on Λ_1 in $M[\mathsf{G}]$. Similarly, for any $i \in \omega^+$, we may compute in $M[\mathsf{G}]$ the i-th iterate U_i of U. Let us fix names $\dot{\mathsf{U}}$ and $\{\dot{\mathsf{U}}_i\}_{i \in \omega^+}$ such that, for any $F \in \mathbb{P}$, $F \Vdash$ "$\dot{\mathsf{U}}$ is a fine ultrafilter on $\breve{\Lambda}_1$" and $F \Vdash$ "$\dot{\mathsf{U}}_i$ is the i-th iterate of $\dot{\mathsf{U}}$" for any $i \in \omega^+$.

Definition 9.9.3. For any $i \in \omega^+$ and any $F \in Filt(\Lambda_1)$, let

$$F^i = \{U \subseteq \Lambda_i \mid F \Vdash \check{U} \in \dot{\mathsf{U}}_i\}.$$

Lemma 9.9.4. *The family* $\{F^i \mid F \in Filt(\Lambda_1), i \in \omega^+\}$ *is an iterating family.*

Proof. We check the five properties of iterating families in turn. Throughout the proof, we routinely appeal to the forcing theorem, and the fact that the forcing relation is absolute between M and $M[\mathsf{G}]$. Note also that $\mathbb{P}$ is ω-closed, meaning that $\mathscr{P}(\Lambda_i)^{M[\mathsf{G}]} = \mathscr{P}(\Lambda_i)^M$ for any $i \in \omega^+$.

1. Fix F and i. Clearly, $F^i \subseteq \mathscr{P}(\Lambda_i)$, but we must check that it is indeed a fine filter. Clearly $\Lambda_i \in F^i$, and moreover for any $U, V \subseteq \Lambda_i$ we have:

$$\begin{aligned} U, V \in F^i &\Leftrightarrow F \Vdash \check{U} \in \dot{\mathsf{U}}_i \text{ and } F \Vdash \check{V} \in \dot{\mathsf{U}}_i \\ &\Leftrightarrow F \Vdash \check{U} \in \dot{\mathsf{U}}_i \wedge \check{V} \in \dot{\mathsf{U}}_i \\ &\Leftrightarrow F \Vdash \widecheck{U \cap V} \in \dot{\mathsf{U}}_i \\ &\Leftrightarrow U \cap V \in F^i, \end{aligned}$$

   which is enough to conclude that F is a filter. Moreover, for any set U of the form $\{\overline{B} \in \Lambda_i \mid \overline{A} \subseteq \overline{B}\}$ for some $\overline{A} \in \Lambda_i$, we have that $F \Vdash \check{U} \in \dot{\mathsf{U}}_i$, since $F \Vdash$ "$\dot{\mathsf{U}}_i$ is a fine filter". Hence $U \in F^i$, which means that F^i is fine.

2. Fix $i \in \omega^+$, and assume that $F \subseteq G$ and that $U \in F^i$. Then $F \Vdash \check{U} \in \dot{\mathsf{U}}_i$, hence also $G \Vdash \check{U} \in \dot{\mathsf{U}}_i$. But this means that $U \in G^i$, and therefore $F^i \subseteq G^i$.

3. Fix $i \in \omega^+$, and suppose that $U \notin F^i$ for some $U \subseteq \Lambda_i$ and $F \in Filt(\Lambda_1)$. Then $F \nVdash \check{U} \in \dot{\mathsf{U}}_i$, which means that there is $G \supseteq F$ such that $G \Vdash \check{U} \notin \dot{\mathsf{U}}_i$. Since we also have that $G \Vdash \dot{\mathsf{U}}_i$ "is an ultrafilter on $\check{\Lambda}_i$", this means that $\Vdash \widecheck{\Lambda_i \setminus U} \in \dot{\mathsf{U}}_i$. Hence there is $G^i \supseteq F^i$ such that $\Lambda_i \setminus U \in G^i$, which shows that $\mathcal{B}_i$ is a rich family.

4. Fix F and i and let $j = i + k$ for some k. Note that for any $U \subseteq \Lambda_i$ and any $\overline{A} \in \Lambda_i$ $\lambda^i_j(U)|\overline{A} = \Lambda_k$ if $\overline{A} \in U$, and $\emptyset$ otherwise. Now by Lemma 9.9.2, we have for any $U \subseteq \Lambda_j$ that $F \Vdash \check{U} \in \dot{\mathsf{U}}_i \leftrightarrow \{\overline{A} \in \check{\Lambda}_i \mid \widecheck{U}|\overline{A} \in \dot{\mathsf{U}}_k\} \in \dot{\mathsf{U}}_i$. By the observation above, we also have that $F \Vdash \forall \overline{A} \in \check{\Lambda}_i(\widecheck{\lambda^i_j(U)}|\overline{A} \in \dot{\mathsf{U}}_k \leftrightarrow \overline{A} \in \check{U})$ for any $U \subseteq \Lambda_i$. Hence $F \Vdash \{\overline{A} \in \check{\Lambda}_i \mid \check{U}|\overline{A} \in \dot{\mathsf{U}}_k\} = \check{U}$, and therefore we have that $F \Vdash \widecheck{\lambda^i_j(U)} \in \dot{\mathsf{U}}_j \leftrightarrow \check{U} \in \dot{\mathsf{U}}_i$. But then, for any $U \subseteq \Lambda_i$, we have the following chain of equivalences:

$$\begin{aligned} U \in F^i &\Leftrightarrow F \Vdash \check{U} \in \dot{\mathsf{U}}_i \\ &\Leftrightarrow F \Vdash \widecheck{\lambda^i_j(U)} \in \dot{\mathsf{U}}^j \\ &\Leftrightarrow \lambda^i_j(U) \in F^j. \end{aligned}$$

5. Finally, fix $F \in Filt(\Lambda_1)$. I claim first that for any $i \in \omega^+$, $F \Vdash \breve{F^i} \subseteq \dot{\mathsf{U}}_i$. To see this, suppose G is M-generic over $\mathbb{P}$, and recall that the forcing notion $\mathbb{P}$ is absolute between M and $M[\mathsf{G}]$, and that $\mathscr{P}(\Lambda_i)^{M[\mathsf{G}]} = \mathscr{P}(\Lambda_i)^M$. Working in $M[\mathsf{G}]$, we know that for any $U \in F^i$, we have that $F \Vdash \check{U} \in \dot{\mathsf{U}}_i$. Since $F \in \mathsf{G}$, it follows from the Forcing Theorem (in $M[\mathsf{G}]$) that $\check{U}_{\mathsf{G}} \in \dot{\mathsf{U}}_{i\mathsf{G}}$. Hence $F^i \subseteq \dot{\mathsf{U}}_{i\mathsf{G}}$. By the Forcing Theorem again, this time in M, it follows that $F \Vdash \breve{F^i} \subseteq \dot{\mathsf{U}}_i$.

   Now fix $i, j \in \omega^+$ and $U \subseteq \Lambda_{i+j}$ such that $\{\overline{A} \in \Lambda_i \mid U|\overline{A} \in F^j\} \in F^i$. By the claim above, this means that $F \Vdash \{\overline{A} \in \check{\Lambda}_i \mid \check{U}|\overline{A} \in \dot{\mathsf{U}}_j\} \supseteq \{\overline{A} \in \check{\Lambda}_i \mid \check{U}|\overline{A} \in \breve{F^j}\} \in \breve{F^i} \subseteq \dot{\mathsf{U}}_i$. Hence $F \Vdash \check{U} \in \dot{\mathsf{U}}_{i+j}$, and therefore $U \in F^{i+j}$. □

Let us conclude by observing that, although $F^1 = F$ for any $F \in Filt(\Lambda_1)$, it is not true in general that $F^i = F_i$ for every $i \in \omega^+$. In fact, this already fails for $i = 2$ and F the smallest fine filter on Λ_1, as the following example, essentially due to Gabe Goldberg, shows. Fix a bijection ν between Λ_1 and the set of prime natural numbers. For any $B \in \Lambda_1$, let $A_B = \{C \in \Lambda_1 \mid |C| = \nu(B)^i \text{ for some } i \in \mathbb{N}\}$, and let $U = \{(B_1, B_2) \mid B_2 \in A_{B_1}\}$. Finally, let $A' = \Lambda_1 \setminus A$. Let us first see that $A' \notin F_2$. To show this, it is enough to show that $\{B \mid A'|B \in F\} \notin F$. Now for any $B \in \Lambda_1$, $A'|B$ is the set of all finite subsets of ω^+ such that their cardinality is not a multiple of $\nu(B)$. Clearly, for any $C \in \Lambda_1$ there is $D \supseteq C$ such that $|D| = \nu(B)^i$ for some positive integer i, so $A'|B \notin F$ for all $B \in \Lambda_1$. But this implies that $A' \notin F_2$. However, we have that $F \Vdash \check{A}' \in \dot{\mathsf{U}}_2$. Indeed, fix a generic filter G such that $F \in \mathsf{G}$ and let $\mathsf{U} = \dot{\mathsf{U}}_{\mathsf{G}}$. Suppose towards a contradiction that $A' \notin \mathsf{U}_2$. Since U_2 is an ultrafilter, this means that $A \in \mathsf{U}_2$. But then it follows that $\{B \in \Lambda_1 \mid A|B \in \mathsf{U}\} \in \mathsf{U}$. Note however that for any $B, B' \in \Lambda_1$, $A_B \cap A_{B'} \neq \emptyset$ implies $B = B'$. Hence $\{B \in \Lambda_1 \mid A|B\} = \{B\}$ for some $B \in \Lambda_1$. But this is impossible, since $\Lambda_1 \setminus B \in F$ and $F \subseteq \mathsf{U}$. Hence $A' \in \mathsf{U}$. Since G was chosen arbitrarily, it follows that $F \Vdash \check{A}' \in \dot{\mathsf{U}}_2$, which completes the proof that $F^2 \nsubseteq F_2$.

Conclusion

Throughout this dissertation, we hope to have demonstrated that possibility semantics can be both a mathematically viable and philosophically rich formal framework. We conclude by highlighting how the main results obtained here relate to the two "slogans" mentioned in the Introduction, and by suggesting avenues for future research.

Recall first our first slogan, relating nonconstructive and choice-free dualities:

$$\mathsf{Non\text{-}constructive\ Dualities} = \frac{\mathsf{Constructive\ Dualities}}{\mathsf{Upper\ Vietoris\ Hyperspaces}}.$$

By now, we hope to have convinced the reader of the robustness of this approach, in particular thanks to the results in Chapters 3 and 4. Indeed, in the case of both de Vries algebras and distributive lattices, we provided choice-free dualities than can be thought of as combining a non-constructive duality with an Upper Vietoris hyperspace construction (Theorem 3.4.5, Corollaries 3.8.8 and 3.8.11). Moreover, we have seen that Upper Vietoris functors and their algebraic counterparts played a central role in developing a general duality for lattices (Theorems 4.4.11 and 4.5.11). In the context of lattices however, there is no non-constructive duality that the choice-free ones we developed approximate. In a sense, we may therefore view the algebraic duals of Vietoris functors used to embed lattices into distributive lattices are creating "fictional" non-constructive dual spaces, whose "Upper Vietoris hyperspaces" are the constructive duals we define. At any rate, this shows that the techniques used here can be applied beyond the distributive realm to consider much wider categories of lattices. We hope that this helps in the development of a rich theory of non-distributive logics which, as we have argued in Chapter 5, seem like a promising way of tackling difficult problems in philosophical logic.

Our second slogan from in the Introduction related Tarskian semantics and possibility semantics for first-order logic as follows:

$$\mathsf{Ultrapowers} = \mathsf{Generic\ Powers} \times \mathsf{Forcing}.$$

Our mains results about generic powers, and in particular the Truth and Genericity Lemma, show that generic powers share many of the key properties with Tarskian ultrapowers, and that the latter can often be retried from the former via forcing. Moreover, we hope to have convinced the reader that generic powers are interesting structures in their own right and that they can have meaningful applications in formal philosophy. In Chapter 7, Chapters 8 and 9, we have argued that they can be fruitfully used to tackle foundational, conceptual and historical problems respectively. Their semiconstructive nature makes them canonical structures that are tamer than the Tarskian counterparts they approximate, and they are also arguably more natural formal models for philosophical concepts that may be inherently underdetermined. Although we focused on issues surrounding the mathematical infinite in this dissertation, the latter observation suggests that there may be more areas of formal philosophy in which possibility structures in general could prove a more adequate formal framework than classical, Tarskian first-order structures.

Finally, we conclude with an observation that calls for further work. Throughout this dissertation, we have seen that the same set of basic ideas could be used to eliminate the reliance on the Boolean Prime Ideal in many proofs. In particular, we have seen instances of this phenomenon in lattice theory (Chapter 3), (nonstandard) analysis (Chapter 7), and basic abstract algebra (Chapters 8 and 9). In all such cases, there is a rather modest price to pay, which amounts to working either with mathematical objects with more structure, or with a semantics for logical connectives that is more involved than the standard one. This raises the question of whether there could be some general semantic way of eliminating choice-principles from mathematical theories with minimal disturbance to mathematical practice, and of what philosophical significance, if any, such a result could have.

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