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ABSTRACT

A systematic procedure is described for deriving the guiding-center Hamiltonian K, to any order in the ratio of gyroradius to magnetic scale length perpendicular to the magnetic field. This ratio may in principle be arbitrary; it is the ratio of the bounce frequency to the gyrofrequency which must be small for a valid guiding-center theory. An explicit expression is obtained for K, valid to second order in both ratios.

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In some confinement schemes of current interest, for example in mirrors and in the Tormac sheath region, the parameter η, defined as the ratio of a typical gyroradius ρ to the magnetic scale length L perpendicular to the magnetic field $\underline{B}$, may take on appreciable values, as large as 1/3. Since standard guiding-center theories are valid only in the limit $\eta \to 0$, it is desirable to study how finite gyroradius effects modify guiding-center motion.

In most confinement schemes, including the two just mentioned, there is another relevant parameter ε, the ratio of ρ to the magnetic scale length $L_{\parallel}$ along $\underline{B}$ (or equivalently, the ratio of the longitudinal bounce frequency to the gyrofrequency), which is much less than one. The separation of time scales implied by $\varepsilon \ll 1$ allows one to formulate a valid guiding-center theory, even for η comparable to one. In particular, one can find an adiabatic invariant J, the gyroaction, valid for finite η and ε, and reducing to mc/e times the usual magnetic moment $\frac{1}{2}mv_{\perp}^2/B$ in the limit $\eta, \varepsilon \to 0$.

In previous work[1-3] on finite gyroradius guiding-center motion, the assumption was made that the scale lengths $L_{\perp}$ and $L_{\parallel}$ were of the same order, and much greater than ρ, and an expansion in the single small parameter $\varepsilon \equiv \rho/L_{\parallel} \sim \rho/L_{\perp} \equiv \eta$ was performed. Only $\varepsilon \ll 1$ is necessary for a valid guiding-center theory, the assumption $\varepsilon \sim \eta$ being insufficiently flexible for configurations to which the present theory is applicable.

Some progress has previously been made in studying finite gyroradius guiding-center motion. Gardner[1] outlines a prescription by which one could in principle obtain a finite gyroradius guiding center Hamiltonian K, in which one performs a series of canonical transformations on the original Hamiltonian H, to remove the dependence of H on gyrophase Θ to successively higher orders. Stern[2] then carried out this scheme explicitly to first order. Even to this order, the algebra involved becomes cumbersome, and finding the appropriate canonical transformation is an unsystematic process. Yet the first order result is simple; K has the same form as its zero-order limit, but with the variables having a slightly different meaning.

More recently, Northrop and Rome[3] studied guiding-center motion to second order, using a noncanonical framework. In so doing they dispense with the need for canonical transformations, but sacrifice the advantages of a Hamiltonian theory, where all the equations of motion come from a single function, and symmetries and conservation laws are most readily apparent. As a result, these authors find a breakdown in conservation of canonical angular momentum P_ϕ at second order which a canonical formulation automatically avoids.

The findings of Ref. 3 parallel those of Ref. 2; to first order, the equations of motion have the same form as the zero order equations, but with a generalized form of gyroaction. The second order corrections to the perpendicular drifts are also obtained in Ref. 3, but require an amount of algebra which is "truly extensive and presents an almost infinite opportunity for mistakes."

The present paper describes a treatment of the guiding-center motion problem which attempts to minimize the difficulties and shortcomings encountered in the work just described. A Hamiltonian framework is adopted, to take advantage of the elegance and power of canonical methods, but it employs a rather different method from that of Refs. 1 and 2 for canonically transforming H. This method is systematic, and enables one to obtain explicit expressions for the guiding-center Hamiltonian K to arbitrarily high order in both ε and η. The transformation process relies heavily on Lie perturbative methods,[4] which greatly reduce the labor involved in making canonical transformations.

The theory is first phrased in a form which is perturbative in ε, but nonperturbative in η, reflecting the fact this it is only $\varepsilon << 1$ which is necessary for a valid guiding-center theory. To obtain <u>explicit</u> expressions for K, one must then also do an expansion in η. This procedure is carried out to second order in both ε and η.

The present work assumes an axisymmetric magnetic geometry, independent of toroidal angle ϕ. However, the same method should be applicable to a fully three-dimensional geometry, the only essential requirement being $\varepsilon \ll 1$.

The other two coordinates, describing the particle position in the poloidal plane, are α, constant along the poloidal field lines (specifically, α is taken as the poloidal flux enclosed by a poloidal field line), and β, which measures the position along a poloidal field line, chosen so that α and β are orthogonal coordinates. This is in contrast to the Euler potentials α, β used in Refs. 1 and 2, and so the convenient properties of Euler potentials are sacrified, obtaining in exchange a diagonal metric tensor $g^{\mu\nu}(\alpha,\varepsilon\beta)$, with diagonal elements $g^{\alpha} \equiv |\nabla\alpha|^2$, $g^{\beta} \equiv |\nabla\beta|^2$, $g^{\phi} \equiv |\nabla\phi|^2 = R^{-2}(\alpha,\varepsilon\beta)$. The original Hamiltonian H, described in terms of the coordinates $q^{\mu} = (\alpha,\beta,\phi)$ and conjugate momenta $p_{\mu} = (p_{\alpha}, p_{\beta}, p_{\phi})$, is given by $(m = c = 1)$

$$H(\alpha,p_{\alpha};\varepsilon\beta, p_{\beta};p_{\phi}) = \frac{1}{2}g^{\alpha}(\alpha,\varepsilon\beta)p_{\alpha}^2 + U(\alpha|\varepsilon\beta, p_{\beta}, p_{\phi}), \tag{1}$$

where

$$U \equiv \frac{1}{2} g^{\beta}(\alpha,\varepsilon\beta)(p_{\beta} - eA_{\beta})^2 + \frac{1}{2} g^{\phi}(\alpha,\varepsilon\beta)(p_{\phi} + e\alpha)^2 + e\Phi \tag{2}$$

is the effective potential for $\alpha(t)$. The magnetic field has poloidal and toroidal components

$$B_p \equiv \hat{\beta}\cdot\underline{B} = (g^{\alpha}g^{\phi})^{1/2}, \quad \text{and} \quad B_T \equiv \hat{\phi}\cdot\underline{B} = (g^{\alpha}g^{\beta})^{1/2}\partial A_{\beta}(\alpha,\varepsilon\beta)/\partial\alpha .$$

The electric field is

$$\underline{E} = -\underline{\nabla}\Phi(\alpha,\varepsilon\beta) = -\hat{\alpha}(g^{\alpha})^{1/2}\partial\Phi/\partial\alpha - \varepsilon\hat{\beta}(g^{\beta})^{1/2}\partial\Phi/\partial(\varepsilon\beta).$$

The slow variation of the geometry parallel to $\underline{B}_p$ is manifested by the parameter ε multiplying β everywhere it occurs, while the dependence of H on the perpendicular coordinate α is arbitrary, corresponding to arbitrary η.

If the parallel variables $\lambda^0 \equiv (\varepsilon\beta, p_\beta)$ were taken as constant parameters in Eq. (1), H would describe a one-dimensional oscillation in potential well $U(\alpha|\lambda^0)$. [The superscript "0" on λ denotes that it is the "old" or "original" variables, as opposed to the superscripts "m" and "f" on λ, to be used later which refer to "mixed" and "final" variables, respectively.] The position α_0 of the bottom of that well satisfies $\partial U(\alpha|\lambda)/\partial\alpha = 0$, a convenient choice for a particle's guiding-center α position, valid for arbitrary η. Solving this for α_0 gives $\alpha_0 = \alpha_0(\lambda)$. For $\varepsilon \neq 0$, however, λ^0 is <u>not</u> constant, but has an $\mathcal{O}(\varepsilon)$ jitter at the gyrofrequency. This jitter in λ^0 causes the well $U[\alpha|\lambda^0(t)]$, in which a particle oscillates, to itself oscillate at the gyrofrequency. Thus the effective curvature of the well which the particle sees is shifted from its "frozen-λ" value. One therefore expects finite ε to modify the particle's frequency of oscillation, i.e. the gyrofrequency Ω. As the well jitters, the position $\alpha_0(\lambda^0)$ of the well bottom does also, and therefore $\alpha_0(\lambda^0)$ is not a satisfactory definition of a guiding-center coordinate. The transformation procedure to be described yields a final pair $\lambda^f \equiv (\varepsilon b, P_b)$ of parallel variables, which are equal to the original pair λ^0, but with the jitter due to gyration eliminated. Thus $\alpha_0(\lambda^f)$ is an appropriate definition for the guiding-center α-coordinate. Similarly, the guiding-center β coordinate may be taken to be b, and subtracting off the jitter from Hamilton's equation for $\dot{\beta}$, one obtains an approximate physical interpretation for P_b: for mirror-like geometries $(B_T \simeq 0 \simeq A_\beta)$, $P_b \simeq \overline{(\dot{\beta}/g^{\beta})}^{t}$, and so P_b gives the time-averaged parallel velocity, while for Tokamak-like geometries $(B_T >> B_p)$, $P_b \simeq eA_\beta(\alpha_0, \varepsilon b)$, so α_0 is principally determined by P_b.

The transformation from H to the guiding-center Hamiltonian K proceeds in two stages. In stage one a canonical transformation S is performed, from the original variables $(\alpha, p_\alpha; \beta, p_\beta)$ in terms of which H is written, to variables $(\Theta', J'; b', P_b')$, in terms of which the intermediate Hamiltonian K' is written. Here (Θ', J') are action-angle variables of gyration, and (b', P_b') correspond to (β, p_β), with part of the gyration jitter eliminated. [The variables (ϕ, p_ϕ) have been suppressed, since ϕ is ignorable, so that p_ϕ is an exact invariant, which the transformations of both stages leave unchanged.]

The dependence of K' on Θ' is $\mathcal{O}(\varepsilon)$, so that J' is not quite the true adiabatic invariant J, but lies close enough to J that the remaining portion of the transformation process, to $(\Theta, J; b, P_b)$ and K, is perturbative in nature. This transformation (stage two) can thus be accomplished by Lie methods, which greatly reduce the amount of computational labor.

The principal difficulty of the transformation process lies in the transformation S of stage one, i.e. in performing a transformation to action-angle variables (Θ', J') such that (Θ', J') are only perturbatively far from the desired variables (Θ, J), and also such that transformation is canonical. The canonicity of the transformation is ensured by using a generating function S of the mixed-variable type.[5] The requirement that (Θ', J') lie sufficiently close to (Θ, J) is met by constructing S in a manner fully analogous to that which exactly solves the one-dimensional limit $\varepsilon \to 0$ of the full twodimensional problem being studied. For $\varepsilon = 0$, the pair of variables λ^0 are exactly constant in time, and so may be treated simply as parameters. The generating function S solving the $\varepsilon = 0$ problem is chosen as follows. From Eq. 1, the particle energy E is given by $E = H(\alpha, p_\alpha; \lambda^0)$. One solves this for p_α,

$$p_\alpha(\alpha, E, \lambda) \equiv \left\{2[E - U(\alpha,\lambda)]/g^\alpha(\alpha,\lambda)\right\}^{1/2},$$

and defines the functional form of the action J' by $J'(E,\lambda) \equiv (2\pi)^{-1} \oint d\alpha p_\alpha(\alpha,E,\lambda)$. One then inverts this, $J'(E,\lambda) \to E(J',\lambda)$, and finally chooses S as

$$S(\alpha, J'; \lambda^m) \equiv \beta P_b^0 + S_1, \tag{3}$$

$$S_1(\alpha, J'; \lambda^m) \equiv \int^\alpha d\alpha' p_\alpha[\alpha', E(J',\lambda^m),\lambda^m]. \tag{4}$$

It is $E(J'; \lambda^0)$ and not $E(J', \lambda^m)$ which is the particle's true energy. The replacement $\lambda^0 \to \lambda^m$ has been made in constructing S because of the requirement that S be expressed in terms of mixed variables. In the case $\varepsilon = 0$ now considered, S_1 is independent of β, and therefore $p_\beta = \partial S/\partial\beta = p_b$, so that $\lambda^0 = \lambda^m$. The action J' is then exactly the desired gyroinvariant J, and correspondingly $\Theta' = \Theta$, and $K' = K$.

In the case $\varepsilon \neq 0$, λ^0 and λ^m differ by a term of order ε, as do (Θ', J') and (Θ, J). But this is satisfactory; it is only required that (Θ', J') lie near (Θ, J). Though the procedure just described for choosing S was discussed in the context of $\varepsilon = 0$, it is also well-defined and valid for the $\varepsilon \neq 0$ case, ensuring that the transformation is canonical.

The transformation procedure described thus far is valid for arbitrary η, requiring only that $\varepsilon << 1$. However, the procedure given for inducing the transformation S involves doing integrals (to obtain S_1) and functional inversions [to obtain $E(J')$ from $J'(E)$, and to "unmix" the mixed-variable transformation equations], which are in practice very difficult. We now discuss a procedure for circumventing these obstacles, to permit practial expressions for K to be found.

The key to the procedure lies in the observation that the transformation $S: (\alpha, p_\alpha) \to (\Theta', J')$ is formally the same as that of a truly one-degree-of-freedom problem where λ^m serves only as a parameter. One may solve this "frozen-λ" problem either by the generating-function method already described, or much more simply using the Lie method formulation of Ref. 4. The latter method yields the transformation equations $(\alpha, p_\alpha) \to (\Theta', J')$ and the Hamiltonian H governing the frozen-λ problem as expansions in η. The transformations using the Lie procedure are determined by a choice for the desired form of the transformed Hamiltonian. Therefore the problem of functional inversion $J'(E) \to E(J')$ found in the generating-function method is completely avoided using the Lie approach. Moreover, once the transformation equations $\alpha(\Theta', J';\lambda^m)$, $p_\alpha(\Theta', J';\lambda^m)$ are known, one can evaluate S_1 using

$$S_1 \equiv \int^{\alpha} d\alpha p_\alpha = \int^{\Theta'} d\Theta p_\alpha(\Theta, J'; \lambda^m)\partial\alpha(\Theta, J'; \lambda^m)/\partial\Theta . \tag{5}$$

As opposed to the original integral over α called for by the generating-function method, the integral over Θ is trivial. In the $\varepsilon = 0$ problem, where $\lambda^0 = \lambda^m$ really <u>is</u> frozen, the Lie method of solution of the frozen-λ problem yields the solution for the full problem, and the calculation of S via Eq. (5) becomes superfluous. For $\varepsilon \neq 0$, however, one still needs S to ensure that the transformation of stage one is canonical, in the full two-degree-of-freedom phase space.

Because of the expansion in η used in the Lie approach to obtain S, practical expressions for K (and H^0) emerge as expansions in both ε and η:

$$K(J;\lambda^f) = \sum_{\ell,m=0} \varepsilon^\ell \eta^m K_{\ell m}(J;\lambda^f) \tag{6}$$

Carrying out this procedure explicitly through $\mathcal{O}(\varepsilon^2, \varepsilon\eta, \eta^2)$, one obtains $K_{01} = K_{10} = K_{11} = 0$, and

$$K_{00} = U_0 + \Omega_0 J, \qquad \eta^2 K_{02} = C_1 J^2, \qquad K_{20} = C_2 + C_3 J + C_4 J^2. \tag{7}$$

The functions U_0, Ω_0, C_1, C_2, C_3, and C_4 are all functions of only $\lambda^f \equiv (\varepsilon b, P_b)$ which, as already discussed, determines a particle's guiding-center position $[\alpha_0(\lambda^f), b]$. The explicit form of these functions is given elsewhere.[6]

The parameter Ω_0 is the gyrofrequency of a particle with small gyroradius (η, $J \to 0$), somewhat generalized over the usual expression $\Omega_c \equiv eB/mc$, to include modifications due to finite $(v_\parallel / L_s\Omega_c)$, where L_s is the shear scale length. For this reason K_{00} itself is a slight generalization of the usual expressions given for the ($\eta \to 0$) guiding-center Hamiltonian. From (7), one sees that $K = K_{00} + \mathcal{O}(\varepsilon^2, \eta^2)$, consistent with the findings of Refs. 2 and 3. This lowest-order expression K_{00} for K yields the lowest order curvature and grad-B drifts, as well as banana orbits.

The term $\eta^2 K_{02}$ is the lowest-order nonzero correction to K due to finite η; it is present even in the $\varepsilon \to 0$ limit, in which the parameter λ^0 is truly frozen. The term $\varepsilon^2 K_{20}$ comes from the fact that λ^0 is <u>not</u> frozen, but has an $\mathcal{O}(\varepsilon)$ jitter at the gyrofrequency. The correction $\varepsilon^2 \partial K_{20}/\partial J$ to the equation for gyrofrequency, $\Omega \equiv \dot{\Theta} = \partial K/\partial J$, mathematically expresses the shift in Ω due to the oscillation in the well $U(\alpha|\lambda^0)$ caused by the jitter in λ^0, already described. The term $\varepsilon^2 K_{20}$ also produces an $\mathcal{O}(\varepsilon^2)$ correction to the guiding-center drifts, through its presence in the equations for $\dot{b}$ and $\dot{P}_b$.

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