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POLARIZATION AND ANGULAR DISTRIBUTIONS IN THE CASCADE DECAY $\psi'(3684) \to \gamma\gamma\psi(3095)$

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ABSTRACT

The angular distributions in the cascade decay process $e^+e^- \to \psi'(3684) \to \gamma\chi \to \gamma\gamma\psi(3095) \to \gamma\gamma\mu^+\mu^-$ are considered as a means of determining the spins and radiative couplings of the χ particles. The expression for the full angular distribution is brought into a simple factorized form using an approach suggested by spin diagrams. All moments of the distribution are tabulated for a χ spin ≤ 2 and pure or mixed transitions with photon multipolarity ≤ 2.

I. INTRODUCTION

The recent discovery of a number of high mass even-C states[1,2] in the mass region between the $\psi(3095)$ and $\psi'(3684)$ is a clear indication that a new spectroscopy of hadrons is emerging. A determination of the spins of the "χ" states is of great importance for an

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understanding of the underlying dynamics. In addition, the multipolarity (or equivalently, the helicity structure) of the coupling for the decay

$$\psi' \to \gamma\chi \tag{1}$$

as well as that for

$$\chi \to \gamma\psi \tag{2}$$

(which has been observed for at least one of the χ-particles[1]) may be very revealing.

In this paper we consider the angular dependence of the reaction

$$e^+e^- \to \psi' \to \gamma_1\chi \to \gamma_1\gamma_2\psi \to \gamma_1\gamma_2\mu^+\mu^- \tag{3}$$

as a means of determining the spin and couplings of the χ particles.* The main result will be a compact and convenient formula for the angular dependences of the process (3) in terms of its moments. These moments are in turn expressed as relatively simple functions of the various possible multipole couplings for a given χ spin j. For $j = 0$, 1, and 2, we have tabulated these moments assuming for the latter two cases an abitrary mixture of E1 and M2 couplings.† This allows the full angular distribution or any partially integrated distributions for (3) to be easily and explicitly written down.

* Sequential photon emission processes have of course been extensively studied previously in nuclear physics, for the specific kinematic situation encountered there. For an early calculation see Ref. 3.

† For convenience we adopt the nomenclature appropriate for positive parity χ states. All the results of Section III are independent of the χ parity.

Angular distributions for the case of higher spin and/or higher multipolarity can be constructed, if necessary, using the formulas given here. Previously derived[4,5] expressions for certain angular correlations and particular couplings are obtained as special cases.*,†

II. ANGULAR DISTRIBUTIONS

The experimental situation that is currently relevant to the study of reaction (3) is that in which the e^+ and e^- beams are unpolarized and the polarizations of the outgoing photons and muons are not observed. However, for the approach developed in this section it will cost us no extra effort (in fact the formulas will appear somewhat more symmetric) if we consider the most general case in which the beams carry some arbitrary polarization described by density matrices $\rho(e^+)_{\alpha\alpha'}$ and $\rho(e^-)_{\beta\beta'}$, and the outgoing photons and muons are subject to some polarization measurement. In the next section we discuss more specifically the case in which there is no beam polarization and no polarization measurements. Thus, we consider the decay of a ψ' whose (unnormalized) polarization matrix is

* Note that the results of Ref. 4 were corrected after publication to agree with those of Ref. 5 (and with the results in this paper).

† During the course of this work we learned of another calculation[6] of angular correlations in $\psi' \to \gamma\gamma\psi$. The approach of Ref. 6 is, however, rather different from ours.

$$\rho(\psi')_{\sigma\sigma'} = T\Big(e^+(\alpha)\, e^-(\beta) \to \psi'(\sigma)\Big)$$

$$\times \quad T^*\Big(e^+(\alpha')\, e^-(\beta') \to \psi'(\sigma')\Big)\ \rho(e^+)_{\alpha\alpha'}\ \rho(e^-)_{\beta\beta'} \qquad (4)$$

where, here and throughout, a sum over repeated spin component indices is implied. In Eq. (4), $\rho(e^+)$, $\rho(e^-)$, and $\rho(\psi')$ are all defined in some fixed lab frame coordinate system (A) whose z-axis is along the beam direction. The labels α, β, etc. indicate spin components along the z-axis.

Next we define a rotation R_1 which takes us from the coordinate system A to another coordinate system B whose z-axis is along the direction of the momentum of γ_1. In general, the polarization of χ in the coordinate system B is described by

$$\rho(\chi)_{\alpha\alpha'} = T\Big(\psi'(\beta) \to \gamma_1(\lambda)\ \chi(\alpha)\Big) T\Big(\psi'(\beta') \to \gamma_1(\lambda')\ \chi(\alpha')\Big)$$

$$\times\ D^1_{\beta\sigma}(R_1^{-1})\ \rho(\psi')_{\sigma\sigma'}\ D^1_{\sigma'\beta'}(R_1)\ \rho(\gamma_1)_{\lambda'\lambda} \quad . \qquad (5)$$

Here, $\rho(\gamma_1)$ represents a measurement of the γ_1 photon polarization (by rescattering or some other means), and $\rho(\chi)$ is the polarization matrix of the χ associated with that measurement. If the polarization of γ_1 is not observed, then (5) involves simply a diagonal sum over photon helicities, i.e., $\rho(\gamma_1)_{11} = \rho(\gamma_1)_{-1-1} = 1$ and $\rho(\gamma_1)_{\lambda'\lambda} = 0$ otherwise. In (5) we have made use of the fact that the radiative transition amplitudes are real[7].

Throughout this paper we will neglect the recoil of the χ and the ψ in (3). Thus, we take the χ and the ψ to be essentially at rest in the lab, and hence, the $\mu^+\mu^-$ pair to be essentially collinear. Within this approximation, we can treat the decay $\chi \to \gamma_2\psi$ by the same procedure that led to (5), i.e., rotate $\rho(\chi)$ by R_2 to the direction of the outgoing γ_2 and then calculate the polarization matrix of the ψ in terms of the $\chi \to \gamma_2\psi$ decay helicity amplitudes. Finally, we rotate by R_3 to the $\mu^+\mu^-$ axis and calculate the polarization matrix for the $\mu^+\mu^-$ state. The full angular distribution for (3) is then obtained by tracing this quantity with matrices $\rho(\mu^+)$ and $\rho(\mu^-)$ representing possible measurements of the μ^+ and μ^- polarizations (2×2 diagonal unit matrices if these polarizations are unobserved). Thus, we obtain the angular distribution

$$
\begin{aligned}
f(R_1,R_2,R_3) \;=\; & D_{\kappa\nu}{}^{1}(R_3{}^{-1})\; T\big(\chi(\mu) \to \gamma_2(\lambda_2)\,\psi(\nu)\big)\; D_{\mu\alpha}{}^{j}(R_2{}^{-1}) \\
& \times\; T\big(\psi'(\beta) \to \gamma_1(\lambda_1)\,\chi(\alpha)\big)\; D_{\beta\sigma}{}^{1}(R_1{}^{-1})\; \rho(\psi')_{\sigma\sigma'}\; D^{1}_{\sigma'\beta'}(R_1) \\
& \times\; T\big(\psi'(\beta') \to \gamma_1(\lambda_1')\,\chi(\alpha')\big) D^{j}_{\alpha'\mu'}(R_2)\; T\big(\chi(\mu') \to \gamma_2(\lambda_2')\,\psi(\nu')\big) \\
& \times\; D^{1}_{\nu'\kappa'}(R_3)\; \rho(\psi)_{\kappa'\kappa}\; \rho(\gamma_1)_{\lambda_1'\lambda_1}\; \rho(\gamma_2)_{\lambda_2'\lambda_2} \quad , \qquad (6)
\end{aligned}
$$

where j is the spin of χ, $\rho(\psi')$ is given by (4), and

$$
\begin{aligned}
\rho(\psi)_{\kappa'\kappa} \;=\; & T\big(\psi(\kappa) \to \mu^+(\alpha)\,\mu^-(\beta)\big)\; T\big(\psi(\kappa') \to \mu^+(\alpha')\,\mu^-(\beta')\big) \\
& \times\; \rho(\mu^+)_{\alpha'\alpha}\; \rho(\mu^-)_{\beta'\beta} \quad . \qquad (7)
\end{aligned}
$$

All amplitudes in (6) are defined with spins quantized along the relevant decay axis. The rotations R_1, R_2, and R_3 are defined as transforming from coordinate system A to B, from B to C, and

from C to D, respectively, where the z-axes of coordinate systems A, B, C, and D are along the e^+e^-, γ_1, γ_2, and $\mu^+\mu^-$ directions, respectively. Thus, Eq. (6) depends on nine Euler angles in the most general case. For the case of no beam polarization and no polarization measurements (see Section III), one of the azimuthal rotations around each of the four axes e^+e^-, γ_1, γ_2, and $\mu^+\mu^-$, becomes superfluous, and in that case the distribution is a function of five angles.

The matrices $\rho(\gamma_1)$ and $\rho(\gamma_2)$ in (6) and $\rho(\mu^+)$ and $\rho(\mu^-)$ in (7) represent possible measurements of the photon and muon polarizations. The particular form which Eq. (6) assumes in the case of no beam polarization and no polarization measurements can be most easily stated by defining a matrix

$$
\rho^{0} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} . \qquad (8)
$$

If photon polarizations are not observed, then $\rho(\gamma_1) = \rho(\gamma_2) = 2\rho^{0}$ corresponding to a simple sum over photon helicities. If muon polarizations are not observed, then $\rho(\mu^+)$ and $\rho(\mu^-)$ are unit matrices, and (7) yields (up to overall factors which do not affect the angular distribution) $\rho(\psi) = \rho^{0}$. Finally, if there is no beam polarization, then $\rho(e^+)$ and $\rho(e^-)$ are unit matrices, and (again up to overall factors) $\rho(\psi') = \rho^{0}$ (we recall that the matrices $\rho(\psi')$, $\rho(\psi)$, and $\rho(\gamma)$ are all defined in their appropriate coordinate systems, with the z-axis along the decay momentum).

The reduction of Eq. (6) to a transparent form is facilitated by replacing the helicity amplitudes by the corresponding multipole couplings. In terms of the Wigner 3-j symbols defined in the Appendix, Eqs. (A1) and (A4), we define the ℓth multipole amplitude $T_{\psi'}(\ell,\lambda)$ for $\psi' \to \gamma\chi$ by

$$T_{\psi'}(\ell,\lambda) = (2\ell+1)\begin{pmatrix} \ell & j & \beta \\ \lambda & \alpha & 1 \end{pmatrix} T\Big(\psi'(\beta) \to \gamma_1(\beta-\alpha)\ \chi(\alpha)\Big) \qquad (9)$$

(a sum over repeated helicity indices still is implied). Here we have used the covariant-contravariant 3-j symbol notation of Wigner[8] (see Appendix). Similarly, for $\chi \to \gamma\psi$, we define the multipole amplitudes

$$T_{\chi}(\ell,\lambda) = (2\ell+1)\begin{pmatrix} \ell & 1 & \mu \\ \lambda & \nu & j \end{pmatrix} T\Big(\chi(\mu) \to \gamma_2(\mu-\nu)\ \psi(\nu)\Big) \ . \qquad (10)$$

It is not difficult to show that, as a consequence of parity conservation,

$$T(\ell,\lambda) = \eta_{\chi}(-1)^{\ell+1}\, T(\ell,-\lambda) \qquad (11)$$

for both $T_{\psi'}$ and T_{χ}. Here, η_{χ} is the parity of the χ state. Thus, as is well known, there is only one independent amplitude for each multipolarity. For a χ of spin j, there are in general $(j > 1)$ 3 independent amplitudes, $\ell = j-1,\ j,\ j+1$.

Inverting Eqs. (9) and (10) and substituting into Eq. (6), we obtain the following expression for the angular distribution:

$$f(R_1,R_2,R_3) = \sum_{\substack{\ell_1,\ell_1' \\ \ell_2,\ell_2'}} f(R_1,R_2,R_3;\ \ell_1,\ell_1',\ell_2,\ell_2') \qquad (12)$$

where

$$f(R_1,R_2,R_3;\ \ell_1,\ell_1',\ell_2,\ell_2') = D_{\beta\sigma}^{\ 1}(R_1^{-1})\ \rho(\psi')_{\sigma\sigma'}\ D^1_{\sigma'\beta'}(R_1)$$

$$\times \begin{pmatrix} \ell_1' & j & \beta' \\ \lambda_1' & \alpha' & 1 \end{pmatrix} D^j_{\alpha'\mu'}(R_2) \begin{pmatrix} \ell_2' & 1 & \mu' \\ \lambda_2' & \nu' & j \end{pmatrix} D^1_{\nu'\kappa'}(R_3)\ \rho(\psi)_{\kappa'\kappa}$$

$$\times\ D_{\kappa\nu}^{\ 1}(R_3^{-1}) \begin{pmatrix} \lambda_2 & \nu & j \\ \ell_2 & 1 & \mu \end{pmatrix}$$

$$\times\ D_{\mu\alpha}^{\ j}(R_2^{-1}) \begin{pmatrix} \lambda_1 & \alpha & 1 \\ \ell_1 & j & \beta \end{pmatrix} T_{\psi'}(\ell_1',\lambda_1')\ \rho(\gamma_1)_{\lambda_1'\lambda_1}\ T_{\psi'}(\ell_1,\lambda_1)$$

$$\times\ T_{\chi}(\ell_2',\lambda_2')\ \rho(\gamma_2)_{\lambda_2'\lambda_2}\ T_{\chi}(\ell_2,\lambda_2) \ . \qquad (13)$$

We have found it very convenient to interpret Eq. (13) and the subsequent manipulation of this expression in terms of "spin diagrams" which were developed previously[9] in a different context.* The components of these diagrams which will be needed in this paper

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* The diagrammatic method used here and in Ref. 9 is closely related to earlier proposals[10]. The main advantage of the present approach is that we use the covariant notation of Wigner[8], which has a very natural graphical interpretation.

are depicted and defined in Fig. 1. Some familiar relations (see Appendix) are expressed diagrammatically in Figs. 2 and 3. The diagrammatic interpretation of Eq. (13) is shown in Fig. 4.

The fact that the D-matrices in (13) occur in three pairs, each of which is a function of one of the three rotations R_1, R_2, or R_3, suggests the introduction of a form of the Clebsch-Gordan series:

$$D^{j'}_{\sigma'\beta'}(R)\, D_{\beta\sigma}{}^{j}(R^{-1}) = \sum_J (-1)^{2j'} (2J+1)$$

$$\begin{pmatrix} \sigma' & j & J \\ j' & \sigma & \mu' \end{pmatrix}\begin{pmatrix} j' & \beta & \mu \\ \beta' & j & J \end{pmatrix} D^J_{\mu'\mu}(R) \quad . \tag{14}$$

The diagrammatic form of the Clebsch-Gordan series is shown in Fig. 5. Using this reduction on all three pairs of D-matrices in Fig. 4, we obtain Fig. 6, or algebraically from Eq. (13),

$$f(R_1,R_2,R_3;\, \ell_1,\ell_1',\ell_2,\ell_2') = \sum_{j_1,j_2,j_3} (2j_1+1)(2j_2+1)(2j_3+1)$$

$$\times\; t^{\psi'}_{\mu_1'}(j_1)\, D^{j_1}_{\mu_1'\mu_1}(R_1)\, \Gamma_{\mu_1\mu_2'\lambda_1\lambda_1'}(j_1,j_2;\, \ell_1,\ell_1';\, 1,j)\, T_{\psi'}(\ell_1',\lambda_1')$$

$$\times\; \rho(\gamma_1)_{\lambda_1'\lambda_1}\, T_{\psi'}(\ell_1,\lambda_1)\, D^{j_2}_{\mu_2'\mu_2}(R_2)\, \Gamma_{\mu_2\mu_3'\lambda_2\lambda_2'}(j_2,j_3;\, \ell_2,\ell_2';\, j,1)$$

$$\times\; T_\chi(\ell_2',\lambda_2')\, \rho(\gamma_2)_{\lambda_2'\lambda_2}\, T_\chi(\ell_2,\lambda_2)\, D^{j_3}_{\mu_3'\mu_3}(R_3)\, t_{\mu_3}{}^{\psi}(j_3) \quad . \tag{15}$$

Here, $t^{\psi'}$ and t^{ψ} are the so-called "statistical tensors" or multipole moments of the matrices $\rho(\psi')$ and $\rho(\psi)$, defined by

$$t^{\psi'}_{\mu_1'}(j_1) = \rho(\psi')_{\sigma\sigma'}\begin{pmatrix} \sigma' & 1 & j_1 \\ 1 & \sigma & \mu_1' \end{pmatrix} \tag{16}$$

and

$$t_{\mu_3}{}^{\psi}(j_3) = \begin{pmatrix} 1 & \kappa & \mu_3 \\ \kappa' & 1 & j_3 \end{pmatrix}\rho(\psi)_{\kappa'\kappa} \quad . \tag{17}$$

The four-index quantity Γ in (15) is the quadrilateral contraction of four 3-j symbols shown in Fig. 7:

$$\Gamma_{\mu_1\mu_2'\lambda_1\lambda_1'}(j_1,j_2;\, \ell_1,\ell_1';\, 1,j) =$$

$$\begin{pmatrix} 1 & \beta & \mu_1 \\ \beta' & 1 & j_1 \end{pmatrix}\begin{pmatrix} \ell_1' & j & \beta' \\ \lambda_1' & \alpha' & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & \alpha & 1 \\ \ell_1 & j & \beta \end{pmatrix}\begin{pmatrix} \alpha' & j & j_2 \\ j & \alpha & \mu_2' \end{pmatrix} \tag{18}$$

It is interesting to note how the steps which led to Eq. (15) and Fig. 6 have produced a formula which is segmented into four pieces strung together by single D-matrices, with each of the four pieces relating to a single step of the physical process, i.e., the quantity to the left of $D(R_1)$ in (15) refers only to $e^+e^- \to \psi'$, that between $D(R_1)$ and $D(R_2)$ refers to $\psi' \to \gamma_1\chi$, that between $D(R_2)$ and $D(R_3)$ refers to $\chi \to \gamma_2\psi$, and that to the right of $D(R_3)$ refers to

$\psi \to \mu^+\mu^-$. In addition to being an extremely convenient way of calculating angular distributions (see Sec. III), such a formulation should be of some conceptual value. It is clear that cascade processes involving an arbitrary number of emissions can be treated in a similar way.

The four-sided contraction of 3-j symbols (18) is closely related to the 9-j symbol (see Appendix) and can be written, using Eq. (A10) or Fig. 3b as

$$\Gamma_{\mu_1\mu_2'\lambda_1\lambda_1'}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j) =$$

$$\sum_L (2L+1)\begin{Bmatrix} 1 & j & \ell_1 \\ 1 & j & \ell_1' \\ j_1 & j_2 & L \end{Bmatrix}\begin{pmatrix} \ell_1' & \lambda_1 & \mu \\ \lambda_1' & \ell_1 & L \end{pmatrix}\begin{pmatrix} L & j_2 & \mu_1 \\ \mu & \mu_2' & j_1 \end{pmatrix} . \tag{19}$$

III. DISCUSSION

We will now consider more specifically the angular distributions when there is no beam polarization and none of the polarizations of the outgoing muons and photons are observed. As pointed out in Sec. II, this distribution is obtained (ignoring overall factors which do not affect the angular dependence) by setting $\rho(\psi') = \rho(\psi) = \rho(\gamma_1) = \rho(\gamma_2) = \rho^{\circ}$ in Eq. (15), where ρ° is defined by (8). From the properties of the 3-j symbols in (16) and (17), one finds that the statistical tensors $t_\mu^{\psi'}(j_1)$ and $t_\mu^{\psi}(j_3)$ vanish unless $\mu = 0$, and that

$$\begin{aligned} t_0(j_1) &= \frac{1}{\sqrt{3}} & j_1 &= 0 \\ &= 0 & j_1 &= 1 \\ &= \frac{1}{\sqrt{30}} & j_1 &= 2 \end{aligned} \tag{20}$$

for both t_0^{ψ} and $t_0^{\psi'}$. It is also clear from (16) and (17) that $t_0(j_1)$ vanishes for $j_1 > 2$.

Using $\rho(\gamma_1) = \rho^{\circ}$, and recalling the parity property (11) of the multipole couplings, we can write part of Eq. (15) as follows:

$$\Gamma_{\mu_1\mu_2'\lambda_1\lambda_1'}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j)\ T_{\psi'}(\ell_1',\lambda_1')\ \rho(\gamma_1)_{\lambda_1'\lambda_1}\ T_{\psi'}(\ell_1,\lambda_1)$$

$$\begin{aligned} &= \Gamma_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j)\ T_{\psi'}(\ell_1)\ T_{\psi'}(\ell_1') \quad \text{for} \quad \mu_2' = \mu_1 \\ &= 0 \qquad \text{for} \quad \mu_2' \neq \mu_1 \quad . \end{aligned} \tag{21}$$

Here,

$$T_{\psi'}(\ell_1) \equiv T_{\psi'}(\ell_1,+1) \quad , \tag{22}$$

and

$$\Gamma_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j) = \frac{1}{2}\sum_\lambda \left[\delta_{\lambda,1} + (-1)^{\ell_1+\ell_1'}\ \delta_{\lambda,-1}\right]$$

$$\times \begin{pmatrix} 1 & \beta & \mu_1 \\ \beta' & 1 & j_1 \end{pmatrix}\begin{pmatrix} \ell_1' & j & \beta' \\ \lambda & \alpha' & 1 \end{pmatrix}\begin{pmatrix} \lambda & \alpha & 1 \\ \ell_1 & j & \beta \end{pmatrix}\begin{pmatrix} \alpha' & j & j_2 \\ j & \alpha & \mu_1 \end{pmatrix} \tag{23a}$$

(no sum over μ_1). Alternatively, from Eq. (19),

$$\Gamma_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j) = \sum_L \tfrac{1}{2}\left[1+(-1)^L\right](2L+1)$$

$$\times \begin{Bmatrix} 1 & j & \ell_1 \\ 1 & j & \ell_1' \\ j_1 & j_2 & L \end{Bmatrix} \begin{pmatrix} \ell_1' & +1 & 0 \\ +1 & \ell_1 & L \end{pmatrix} \begin{pmatrix} L & j_2 & \mu_1 \\ 0 & \mu_1 & j_1 \end{pmatrix} \quad \text{(no sum over } \mu_1\text{)} . \qquad (23b)$$

In writing (21) we have also taken note of the fact that $\Gamma_{\mu_1\mu_2'\lambda_1\lambda_1'}$ vanishes unless $\mu_1 - \mu_2' = \lambda_1' - \lambda_1$, as can be seen from its definition (19). Now, since j_1 is an even integer by virtue of (20) and L is an even integer by virtue of the first factor in (23b), the 9-j symbol in (23b) has the property (see Appendix)

$$\begin{Bmatrix} 1 & j & \ell_1 \\ 1 & j & \ell_1' \\ j_1 & j_2 & L \end{Bmatrix} = (-1)^{\ell_1+\ell_1'+j_2} \begin{Bmatrix} 1 & j & \ell_1' \\ 1 & j & \ell_1 \\ j_1 & j_2 & L \end{Bmatrix} . \qquad (24)$$

Also, the first 3-j symbol in (23b) has the property

$$\begin{pmatrix} \ell_1' & +1 & 0 \\ +1 & \ell_1 & L \end{pmatrix} = (-1)^{\ell_1+\ell_1'} \begin{pmatrix} \ell_1 & +1 & 0 \\ +1 & \ell_1' & L \end{pmatrix} . \qquad (25)$$

Combining (24) and (25) gives

$$\Gamma_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j) = (-1)^{j_2}\, \Gamma_{\mu_1}(j_1,j_2;\ \ell_1',\ell_1;\ 1,j) . \qquad (26)$$

Thus, when we sum over ℓ_1 and ℓ_1', only terms having j_2 = even integer contribute to the angular distribution.

The part of Eq. (15) involving the amplitudes for $\chi \to \gamma_2\psi$ can be treated in the same way using $\rho(\gamma_2) = \rho^0$. This gives

$$\Gamma_{\mu_2\mu_3'\lambda_2\lambda_2'}(j_2,j_3;\ \ell_2,\ell_2';\ j,1)\ T_\chi(\ell_2',\lambda_2')\ \rho(\gamma_2)_{\lambda_2'\lambda_2}\ T_\chi(\ell_2,\lambda_2)$$

$$= \Gamma_{\mu_2}(j_2,j_3;\ \ell_2,\ell_2';\ j,1)\ T_\chi(\ell_2)\ T_\chi(\ell_2') \qquad \text{for } \mu_3' = \mu_2$$

$$= 0 \qquad \text{for } \mu_3' \neq \mu_2 . \qquad (27)$$

It will be convenient to rewrite (27) using

$$\Gamma_{\mu_2}(j_2,j_3;\ \ell_2,\ell_2';\ j,1) = (-1)^{\ell_2+\ell_2'}\, \Gamma_{\mu_2}(j_3,j_2;\ \ell_2,\ell_2';\ 1,j) \qquad (28)$$

which follows from symmetries of the 3-j and 9-j symbols in (23b) and from the fact that j_2 and j_3 can be taken as even integers.

Using (21) and (27) in (15), we obtain

$$f(R_1,R_2,R_3;\ \ell_1,\ell_1',\ell_2,\ell_2') = (-1)^{\ell_2+\ell_2'} \sum_{j_1,j_3=0,2}\ \sum_{j_2=0,2,4,\cdots}$$

$$\times\ (2j_1+1)(2j_2+1)(2j_3+1)\ t_0(j_1)\ D^{j_1}_{0\mu_1}(R_1)$$

$$\times\ \Gamma_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ 1,j)\ T_{\psi'}(\ell_1)\ T_{\psi'}(\ell_1')\ D^{j_2}_{\mu_1\mu_2}(R_2)$$

$$\times\ \Gamma_{\mu_2}(j_3,j_2;\ \ell_2,\ell_2';\ 1,j)\ T_\chi(\ell_2)\ T_\chi(\ell_2')\ D^{j_3}_{\mu_2 0}(R_3)\ t_0(j_3) . \qquad (29)$$

To write (29) as an explicit function of the five angles involved, let us refer to frame 1 as the coordinate system in which the z-axis is along the γ_1 direction and γ_2 is moving in the x-z plane with $(p_{\gamma_2})_x > 0$. Frame 2 is obtained from frame 1 by a rotation around the y-axis by $\theta_{\gamma\gamma}$, the angle between the γ_1 and γ_2 directions, with $0 \leq \theta_{\gamma\gamma} \leq \pi$. An event is then completely described by the five angles $\theta_{\gamma\gamma}$, θ_1, ϕ_1, θ_2, and ϕ_2, where θ_1, ϕ_1 are the polar and azimuthal angles of the e^+ beam direction with respect to frame 1, and θ_2, ϕ_2 are the polar and azimuthal angles of the outgoing μ^+ with respect to frame 2. The product of D-matrices in (29) then becomes

$$D^{j_1}_{0\mu_1}(R_1)\, D^{j_2}_{\mu_1\mu_2}(R_2)\, D^{j_3}_{\mu_2 0}(R_3) = d^{j_1}_{0\mu_1}(-\theta_1)\; e^{i\mu_1\phi_1}\; d^{j_2}_{\mu_1\mu_2}(\theta_{\gamma\gamma})$$

$$\times\; e^{-i\mu_2\phi_2}\; d^{j_3}_{\mu_2 0}(\theta_2) \quad . \tag{30}$$

The fact that (29) is a purely real function can be seen to follow from a symmetry of (23b), viz.

$$\Gamma_\mu(\cdots) = \Gamma_{-\mu}(\cdots) \tag{31}$$

(again using the fact that j_1, j_2, and j_3 are all even integers). Thus we can replace $e^{i(\mu_1\phi_1-\mu_2\phi_2)}$ in (30) by $\cos(\mu_1\phi_1 - \mu_2\phi_2)$.

To recapitulate, the full angular distribution is given by

$$f(\theta_1,\phi_1,\theta_2,\phi_2,\theta_{\gamma\gamma}) = \sum_{\substack{\ell_1,\ell_1',\\ \ell_2,\ell_2'}} (-1)^{\ell_2+\ell_2'}\; \hat{T}_{\psi'}(\ell_1)\, \hat{T}_{\psi'}(\ell_1')$$

$$\times\; \hat{T}_\chi(\ell_2)\, \hat{T}_\chi(\ell_2')\; \tilde{f}(\theta_1,\phi_1,\theta_2,\phi_2,\theta_{\gamma\gamma};\ \ell_1,\ell_1',\ell_2,\ell_2') \tag{32a}$$

where (indicating all sums explicitly)

$$\tilde{f}(\theta_1,\phi_1,\theta_2,\phi_2,\theta_{\gamma\gamma};\ \ell_1,\ell_1',\ell_2,\ell_2') = \sum_{j_1,j_3=0,2}\ \sum_{j_2=0,2,\cdots}\ \sum_{\mu_1,\mu_2}$$

$$\times\; C_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ j)\; C_{\mu_2}(j_3,j_2;\ \ell_2,\ell_2';\ j)\; d^{j_1}_{\mu_1 0}(\theta_1)\; d^{j_2}_{\mu_1\mu_2}(\theta_{\gamma\gamma})$$

$$\times\; d^{j_3}_{\mu_2 0}(\theta_2)\, \cos(\mu_1\phi_1 - \mu_2\phi_2) \quad . \tag{32b}$$

Here we have for simplicity scaled the amplitudes by a constant factor,

$$\hat{T}_{\psi'}(\ell_1) = \left[t_0(0)\,\Gamma_0(0,0;\ \ell_1,\ell_1;\ 1,j)\right]^{\frac{1}{2}}\, T_{\psi'}(\ell_1) \tag{33}$$

with an identical relation for $\hat{T}_\chi(\ell_2)$. The coefficients C_μ in Eq. (32b) are then

$$C_\mu(j_a,j_b;\ \ell,\ell';\ j) = (2j_a+1)\sqrt{2j_b+1}\ \frac{t_0(j_a)}{t_0(0)}$$

$$\times\; \frac{\Gamma_\mu(j_a,j_b;\ \ell,\ell';\ 1,j)}{\left[\Gamma_0(0,0;\ \ell,\ell;\ 1,j)\,\Gamma_0(0,0;\ \ell',\ell';\ 1,j)\right]^{\frac{1}{2}}} \tag{34}$$

with $t_0(j_a)$ given in (20) and Γ_μ defined by (23a) or (23b).

By taking angular moments of the distribution (32) one can completely factorize the dependence on the amplitudes describing the decays $\psi' \to \gamma_1 \chi$ and $\chi \to \gamma_2 \psi$. Thus, using the orthogonality of D-functions we have (no sum on repeated helicity indices implied):

$$\int_{-1}^{1} d(\cos\theta_1)\, d(\cos\theta_{\gamma\gamma})\, d(\cos\theta_2) \int_0^{2\pi} d\phi_1 d\phi_2\, f(\theta_1,\phi_1,\theta_2,\phi_2,\theta_{\gamma\gamma})$$

$$\times\ d^{j_1}_{\mu_1 0}(\theta_1)\, d^{j_2}_{\mu_1\mu_2}(\theta_{\gamma\gamma})\, d^{j_3}_{\mu_2 0}(\theta_2)\, \cos(\mu_1\phi_1 - \mu_2\phi_2) \ =$$

$$\frac{32\pi^2}{(2j_1+1)(2j_2+1)(2j_3+1)}$$

$$\times \left[\sum_{\ell_1\ell_1'} \hat{T}_{\psi'}(\ell_1)\, \hat{T}_{\psi'}(\ell_1')\, C_{\mu_1}(j_1,j_2;\ \ell_1,\ell_1';\ j)\right]$$

$$\times \left[\sum_{\ell_2,\ell_2'} (-1)^{\ell_2+\ell_2'}\, \hat{T}_\chi(\ell_2)\, \hat{T}_\chi(\ell_2')\, C_{\mu_2}(j_3,j_2;\ \ell_2,\ell_2';\ j)\right] \quad . \qquad (35)$$

Similarly, one can of course take moments only over some angles.

The coefficients C_μ defined by Eq. (34) can be calculated using the expression (23b) and a table of 9-j symbols, or directly from Eq. (23a). For convenience we have tabulated all the coefficients C_μ needed to obtain angular distributions for χ spin ≤ 2 and photon multipolarity ≤ 2. These coefficients are given in Table I. The numbers listed in this table are also of interest in that they give a fairly direct indication of which moments of the angular distribution

are most sensitive to the spin of χ and which ones are most sensitive to the multipolarity of the photons.

As an example of the use of Table I, let us give the general expression for the $\theta_{\gamma\gamma}$ distribution for a χ of spin 1. When all other angles are integrated over, only terms with $j_1 = j_3 = \mu_1 = \mu_2 = 0$ and $j_2 = 0,2$ contribute in Eq. (32b). Using Table I we get then,

$$f(\theta_{\gamma\gamma}) = \left[\hat{T}_{\psi'}^2(1) + \hat{T}_{\psi'}^2(2)\right]\left[\hat{T}_\chi^2(1) + \hat{T}_\chi^2(2)\right]$$

$$+ \frac{1}{8}\left[\hat{T}_{\psi'}^2(1) + 6\hat{T}_{\psi'}(1)\,\hat{T}_{\psi'}(2) + T_{\psi'}^2(2)\right]$$

$$\times \left[\hat{T}_\chi^2(1) - 6\hat{T}_\chi(1)\,\hat{T}_\chi(2) + \hat{T}_\chi^2(2)\right] d_{00}^2(\theta_{\gamma\gamma}) \quad . \qquad (36)$$

It is clear from Eq. (36) that the $\theta_{\gamma\gamma}$-distribution is invariant under an interchange of couplings: $\hat{T}_{\psi',\chi}(1) \leftrightarrow \hat{T}_{\psi',\chi}(2)$. Thus this distribution does not in itself determine which coupling is dominant. In fact, from the equality of the coefficients C_μ in Table I we can see that only a measurement of the azimuthal (ϕ_1,ϕ_2) angular distributions would distinguish between the two couplings (when the spin of χ is 1).

It is interesting to observe that when all angles except $\theta_{\gamma\gamma}$ are integrated over (as in the above example), the coefficients C_μ entering in Eq. (32b) can be simply expressed in terms of 6-j symbols. This can be seen diagrammatically from Fig. 6, which when $j_1 = j_3 = 0$ reduces to Fig. 8. The two triangles in Fig. 8 are proportional to 6-j symbols according to Fig. 3a. Analytically, using Eq. (A11) we get

$$C_0(0,j_b;\ \ell,\ell';\ j) = (-1)^{j+\ell+\ell'} \left[(2\ell+1)(2\ell'+1)(2j+1)\right]^{\frac{1}{2}}$$

$$\langle \ell\ell',1-1|j_b\ 0\rangle \begin{Bmatrix} j_b & \ell & \ell' \\ 1 & j & j \end{Bmatrix} \quad . \qquad (37)$$

With this formula and Eq. (32) any $\theta_{\gamma\gamma}$-distributions can be easily witten down. In particular, one can verify the expressions previously given[5] in the special case of pure E1 coupling and χ spin $j = 1$ or 2.

The formulas given in this paper are quite general. The best way of analyzing a given set of data will, of course, depend on the acceptance and available statistics. Ideally, an analysis in terms of the angular moments defined by Eq. (35) would give the most direct information on the χ spin and couplings. Alternatively, one could perform a maximum likelihood fit to the angular distribution obtained from Eqs. (32) and Table I, with the multipole couplings $\hat{T}(\ell)$ as free parameters. In any case, the rather compact formulas developed above should be of help in interpreting the experimental distributions.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we give the definitions of 3-j, 6-j, and 9-j symbols and briefly describe some of their symmetry properties. The 3-j symbol is defined in terms of the usual Clebsch-Gordan coefficient by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-\lambda_3}}{\sqrt{2j_3+1}} \langle j_1,j_2;\ \lambda_1,\lambda_2|j_3,-\lambda_3\rangle \quad . \qquad (A1)$$

The diagrammatic 3-j symbol is shown in Fig. 1e. We have found it convenient to employ Wigner's[8] covariant notation for spin indices. Defining the "metric tensor"

$$C_{\lambda\sigma}{}^{j} = (-1)^{j+\lambda}\,\delta_{\lambda,-\sigma} = (-1)^{2j}\,C_{\sigma\lambda}{}^{j} \qquad (A2)$$

$$C_j{}^{\lambda\sigma} = (-1)^{j+\sigma}\,\delta_{\lambda,-\sigma} = (-1)^{2j}\,C_j{}^{\sigma\lambda} \ , \qquad (A3)$$

a 3-j symbol with one or more contravariant indices is obtained by raising those indices from the left, e.g.,

$$\begin{pmatrix} j_1 & j_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & j_3 \end{pmatrix} = C_{j_3}^{\lambda_3\lambda_3'} \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3' \end{pmatrix} =$$

$$(-1)^{j_3-\lambda_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & -\lambda_3 \end{pmatrix} \quad . \qquad (A4)$$

(To avoid confusion due to the changing position of indices, we always use Latin letters j, ℓ, L, etc. for representation labels and lower case Greek letters $\lambda, \sigma, \mu, \nu, \cdots$ for spin component indices. A sum over repeated helicity indices will always be implied.) Among the useful symmetries of the 3-j symbol are: (1) It is symmetric (antisymmetric) under an odd permutation of its columns if $j_1 + j_2 + j_3$ is an even (odd) integer, and (2) A fully covariant 3-j symbol is equal to the fully contravariant one obtained by inverting all three columns. The 3-j symbol satisfies an orthogonality relation

$$\begin{pmatrix} j_1 & j_2 & j \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda' \\ j_1 & j_2 & j' \end{pmatrix} = \frac{\delta_{jj'}\,\delta_{\lambda\lambda'}}{(2j+1)} \qquad \text{(A5)}$$

shown in Fig. 2a, and a completeness relation

$$\sum_j (2j+1)\begin{pmatrix} j_1 & j_2 & j \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix}\begin{pmatrix} \lambda_1' & \lambda_2' & \lambda \\ j_1 & j_2 & j \end{pmatrix} = \delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'} \qquad \text{(A6)}$$

shown in Fig. 2b. Note that, in order to completely specify the meaning of the diagrams, we have indicated the ordering of the lines in a 3-j symbol by a + (-) sign, corresponding to the diagram being read off counter-clockwise (clockwise).

A tetrahedral contraction of 3-j symbols gives a 6-j symbol,

$$\begin{pmatrix} j_1 & \sigma_2 & \ell_3 \\ \lambda_1 & \ell_2 & \sigma_3 \end{pmatrix}\begin{pmatrix} \ell_1 & j_2 & \sigma_3 \\ \sigma_1 & \lambda_2 & \ell_3 \end{pmatrix}\begin{pmatrix} \sigma_1 & \ell_2 & j_3 \\ \ell_1 & \sigma_2 & \lambda_3 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix} . \qquad \text{(A7)}$$

The 6-j symbol often enters a calculation via the identity shown in Fig. 3a,

$$\begin{pmatrix} j_1 & \sigma_2 & \ell_3 \\ \lambda_1 & \ell_2 & \sigma_3 \end{pmatrix}\begin{pmatrix} \ell_1 & j_2 & \sigma_3 \\ \sigma_1 & \lambda_2 & \ell_3 \end{pmatrix}\begin{pmatrix} \sigma_1 & \ell_2 & j_3 \\ \ell_1 & \sigma_2 & \lambda_3 \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} . \qquad \text{(A8)}$$

Since in this paper we have made minimal use of the 6-j symbol, and then only as a special case of a 9-j symbol, we refer to standard texts[8,11] for further discussion of its properties.

A 9-j symbol is obtained by contracting six 3-j symbols together into a closed figure in which there are no triangular contractions:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = \begin{pmatrix} j_1 & j_2 & j_{12} \\ \lambda_1 & \lambda_2 & \lambda_{12} \end{pmatrix}\begin{pmatrix} j_3 & j_4 & j_{34} \\ \lambda_3 & \lambda_4 & \lambda_{34} \end{pmatrix}$$

$$\times \begin{pmatrix} j_{13} & j_{24} & j \\ \lambda_{13} & \lambda_{24} & \lambda \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_3 & \lambda_{13} \\ j_1 & j_3 & j_{13} \end{pmatrix}\begin{pmatrix} \lambda_2 & \lambda_4 & \lambda_{24} \\ j_2 & j_4 & j_{24} \end{pmatrix}$$

$$\times \begin{pmatrix} \lambda_{12} & \lambda_{34} & \lambda \\ j_{12} & j_{34} & j \end{pmatrix} . \qquad \text{(A9)}$$

The identity by which the 9-j symbol enters our discussion of angular distributions is shown in Fig. 3b,

$$\begin{pmatrix} j_1 & j_2 & j_{12} \\ \lambda_1 & \lambda_2 & \lambda_{12} \end{pmatrix}\begin{pmatrix} j_3 & j_4 & j_{34} \\ \lambda_3 & \lambda_4 & \lambda_{34} \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_3 & \lambda_{13} \\ j_1 & j_3 & j_{13} \end{pmatrix}$$

$$\times \begin{pmatrix} \lambda_2 & \lambda_4 & \lambda_{24} \\ j_2 & j_4 & j_{24} \end{pmatrix}$$

$$= \sum_j (2j+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \begin{pmatrix} j_{12} & j_{34} & j \\ \lambda_{12} & \lambda_{34} & \lambda \end{pmatrix}$$

$$\times \begin{pmatrix} \lambda_{13} & \lambda_{24} & \lambda \\ j_{13} & j_{24} & j \end{pmatrix} . \qquad \text{(A10)}$$

From its definition, it is clear that a 9-j symbol vanishes unless the three entries in each of its rows and columns can be related by vector addition. It can also be shown that a 9-j symbol is symmetric (antisymmetric) under interchange of any two of its rows or any two of its columns if the sum of all nine of its entries is an even (odd) integer. In particular, a 9-j symbol with two identical rows (columns) is equal to zero if the sum of the three elements in the remaining row (column) is odd. If a zero appears anywhere in a 9-j symbol, it can be converted to a 6-j symbol by the identity

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_2 & \ell_1 & j_3 \\ \ell_3 & \ell_3 & 0 \end{Bmatrix} = \frac{(-1)^{j_2+j_3+\ell_2+\ell_3}}{\sqrt{2j_3+1}\ \sqrt{2\ell_3+1}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix}$$

(A11)

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Table I. Values of the coefficients $C_\mu(j_a,j_b;\ \ell,\ell';\ j)$ defined by Eq. (34) of the text. All nonzero coefficients needed for a χ spin $j \le 2$ and photon multipolarity $\ell,\ell' \le 2$ are listed. The value of C_μ is independent of the sign of μ according to Eq. (31). Since j_b is an even integer, C_μ is invariant under the interchange $\ell \leftrightarrow \ell'$, according to Eq. (26).

$C_\mu(j_a,j_b;\ \ell,\ell';\ j)$		(j_a,j_b,μ)									
(ℓ,ℓ')	j	(0,0,0)	(0,2,0)	(2,0,0)	(2,2,0)	(2,2,1)	(2,2,2)	(0,4,0)	(2,4,0)	(2,4,1)	(2,4,2)
(1,1)	0	1	0	$\frac{1}{2}$	0	0	0	0	0	0	0
	1	1	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{3}{4\sqrt{2}}$	0	0	0	0	0
	2	1	$\frac{7}{2\sqrt{70}}$	$\frac{1}{20}$	$\frac{2\sqrt{2}}{\sqrt{35}}$	$\frac{15}{4\sqrt{70}}$	$\frac{3}{\sqrt{70}}$	0	$\frac{9}{5\sqrt{14}}$	$\frac{3\sqrt{3}}{2\sqrt{35}}$	$\frac{3\sqrt{3}}{2\sqrt{70}}$
(1,2)	1	0	$-\frac{3}{2\sqrt{2}}$	$\frac{3}{4}$	0	0	0	0	0	0	0
	2	0	$-\frac{7}{2\sqrt{14}}$	$-\frac{3}{2\sqrt{20}}$	$-\frac{1}{\sqrt{14}}$	$-\frac{1}{2\sqrt{14}}$	$\frac{1}{\sqrt{14}}$	0	$\frac{3}{\sqrt{70}}$	$\frac{3}{2\sqrt{21}}$	$\frac{\sqrt{3}}{2\sqrt{14}}$

Table I continued next page

Table I Continued. Values of the coefficients $C_\mu(j_a,j_b;\ell,\ell';j)$ defined by Eq. (34) of the text. All nonzero coefficients needed for a χ spin $j \le 2$ and photon multipolarity $\ell,\ell' \le 2$ are listed. The value of C_μ is independent of the sign of μ according to Eq. (31). Since j_b is an even integer, C_μ is invariant under the interchange $\ell \leftrightarrow \ell'$, according to Eq. (26).

$C_\mu(j_a,j_b;\ell,\ell',j)$		(j_a,j_b,μ)									
(ℓ,ℓ')	j	(0,0,0)	(0,2,0)	(2,0,0)	(2,2,0)	(2,2,1)	(2,2,2)	(0,4,0)	(2,4,0)	(2,4,1)	(2,4,2)
(2,2)	1	1	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{3}{4\sqrt{2}}$	0	0	0	0	0
	2	1	$-\frac{\sqrt{5}}{2\sqrt{14}}$	$\frac{1}{4}$	0	$-\frac{15}{4\sqrt{70}}$	$-\frac{5}{\sqrt{70}}$	$\frac{8}{3\sqrt{14}}$	$\frac{7}{3\sqrt{14}}$	$\frac{5\sqrt{3}}{6\sqrt{35}}$	$-\frac{5\sqrt{3}}{2\sqrt{70}}$

FIGURE CAPTIONS

Fig. 1. Definition of the components in the spin diagrams:

(a) Spin j propagator.

(b) The ψ' polarization matrix defined by Eq. (4).

(c) The multipole amplitude $T_{\psi'}(\ell,\lambda)$, defined by Eq. (9). We use wavy lines for photon propagators.

(d) The diagrammatic representation of a rotation matrix $D^j_{\lambda'\lambda}(R)$.

(e) The 3-j symbol defined by Eq. (A1.) A + (-) sign next to the vertex indicates that the diagram should be read off in a counterclockwise (clockwise) direction. Vertices with one or more outgoing lines have the corresponding helicities in the "up" position in the 3-j symbol (cf. λ_3 in Eq. (A4)).

Fig. 2. (a) The orthogonality relation (A5).

(b) The completeness relation (A6).

Fig. 3. (a) Spin diagram representation of Eq. (A8).

(b) Spin diagram representation of Eq. (A10).

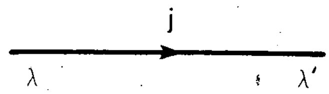
Fig. 4. The expression (13) of the text.

Fig. 5. The Clebsch-Gordan series (14).

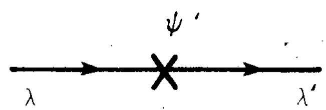
Fig. 6. The expression (15) of the text.

Fig. 7. The symbol $\Gamma_{\mu_1\mu_2'\lambda_1\lambda_1'}(j_1,j_2;\ell_1,\ell_1';1,j)$ defined by Eq. (18).

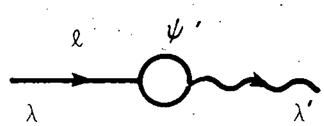
Fig. 8. The special case of Fig. 6 when $j_1 = j_3 = 0$ (the ψ,ψ' polarization matrices are factored off).

(a) 

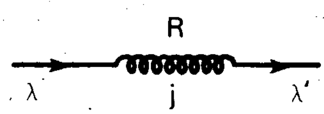
$= \delta_{\lambda'\lambda}$

(b) 

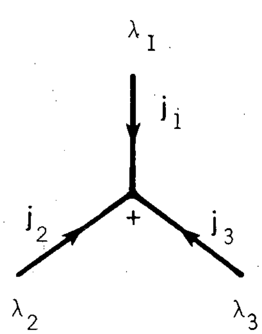
$= \rho(\psi')_{\lambda'\lambda}$

(c) 

$= T_{\psi'}(\ell, \lambda)\delta_{\lambda'\lambda}$

(d) 

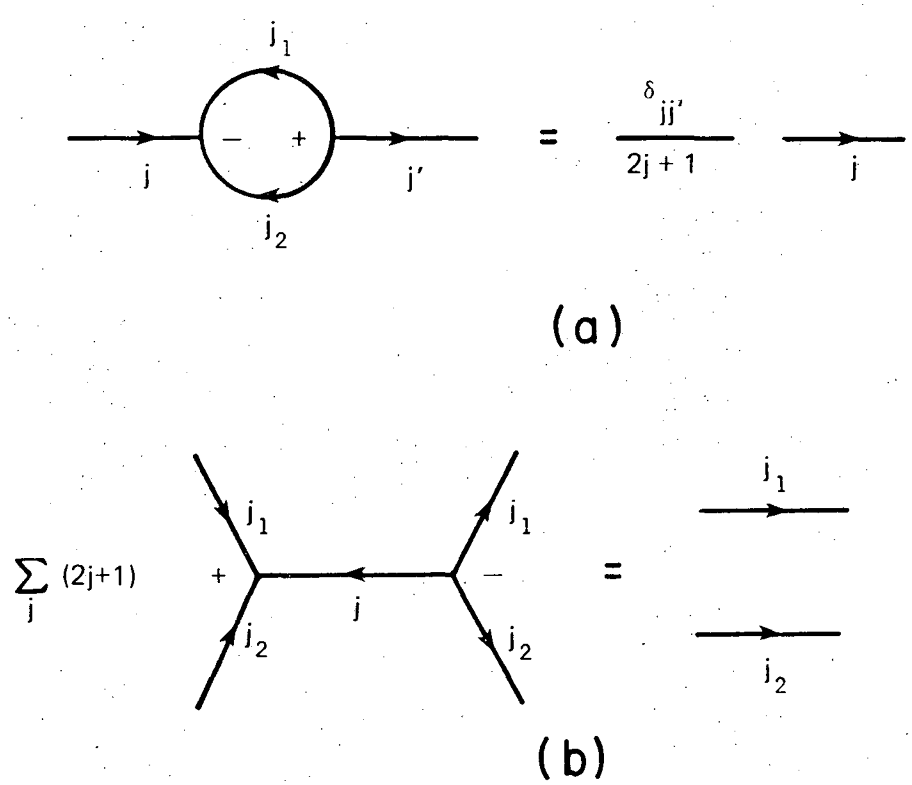
$= D^{j}_{\lambda'\lambda}(R)$

(e) 

$= \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}$

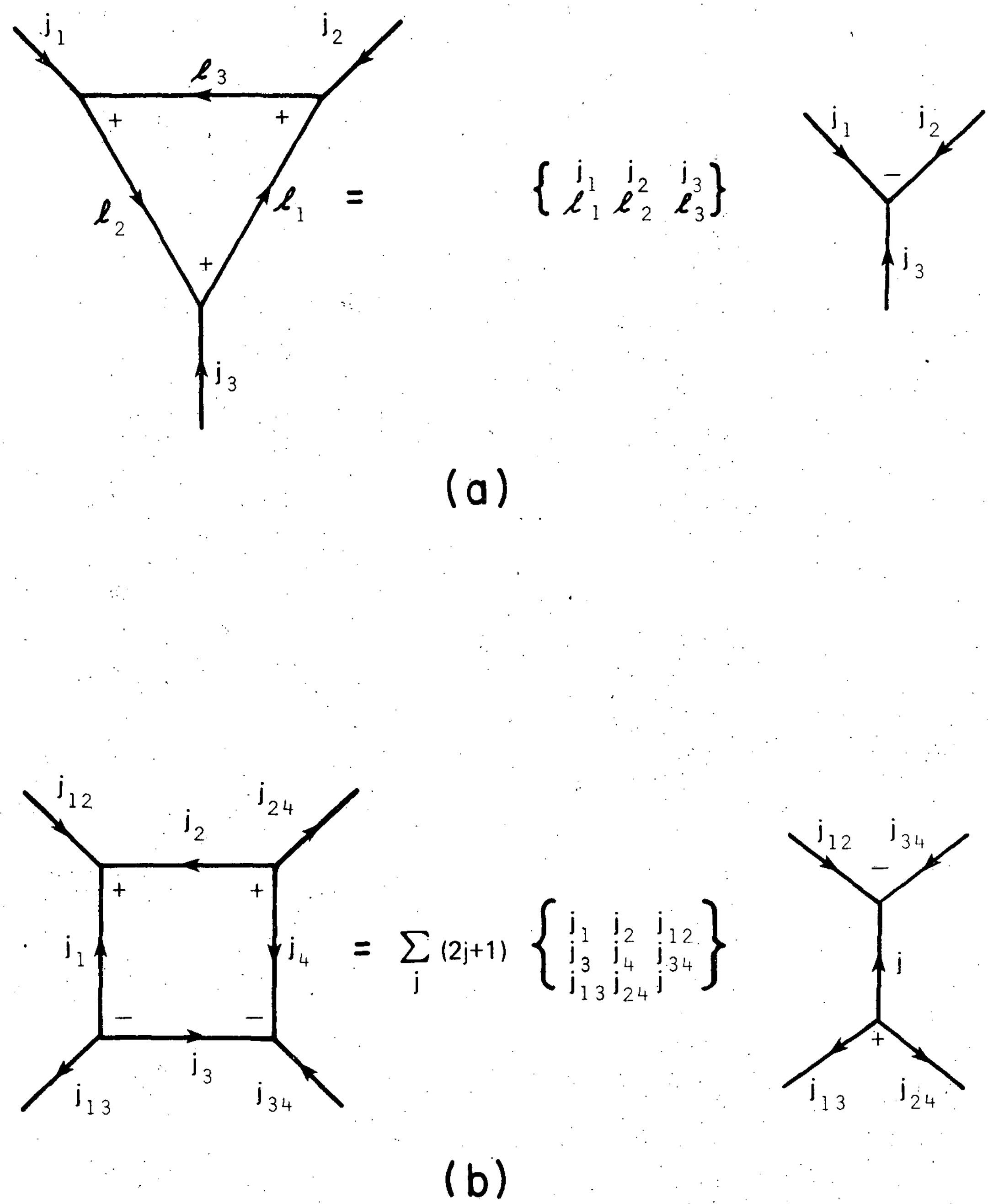
Fig. 1

XBL7511-9540



XBL7511-9541

Fig. 2



XBL7511-9542

Fig. 3

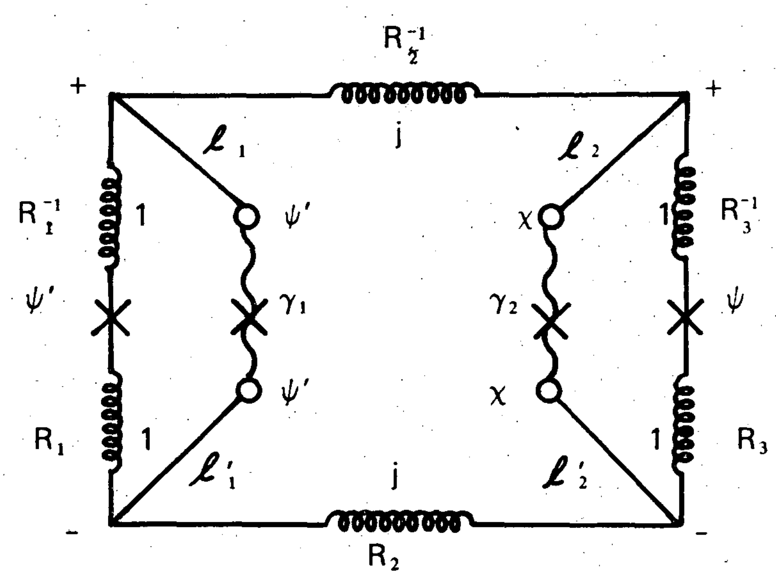


Fig. 4

XBL 7511-9538

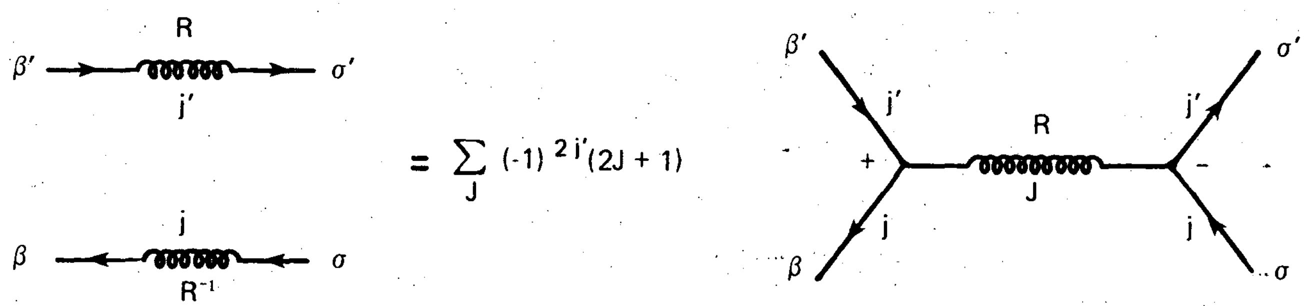


Fig. 5

XBL7511-9537

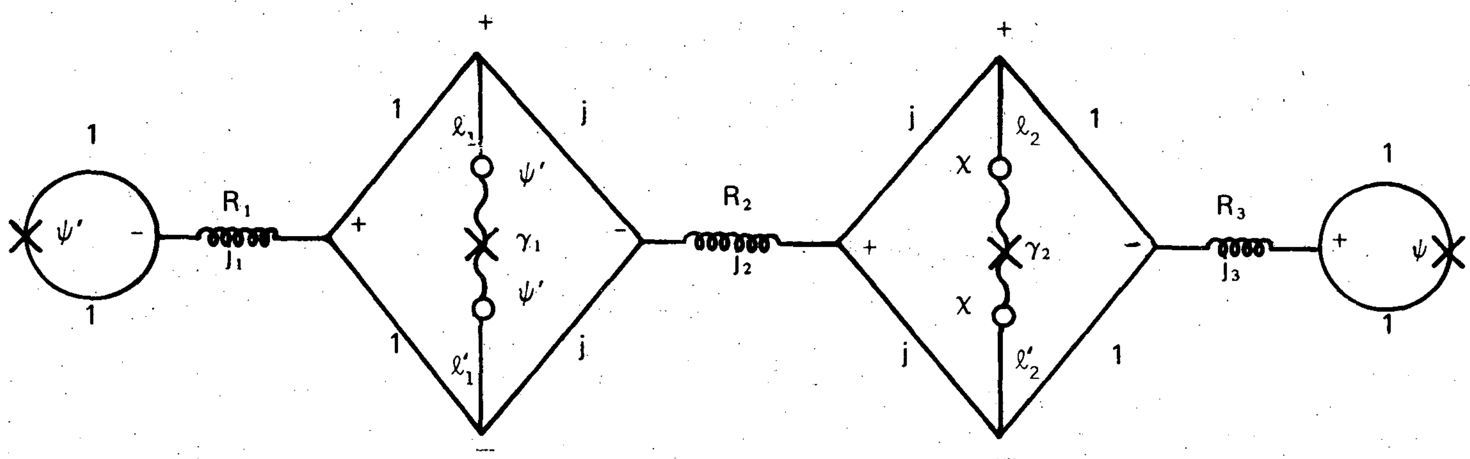


Fig. 6

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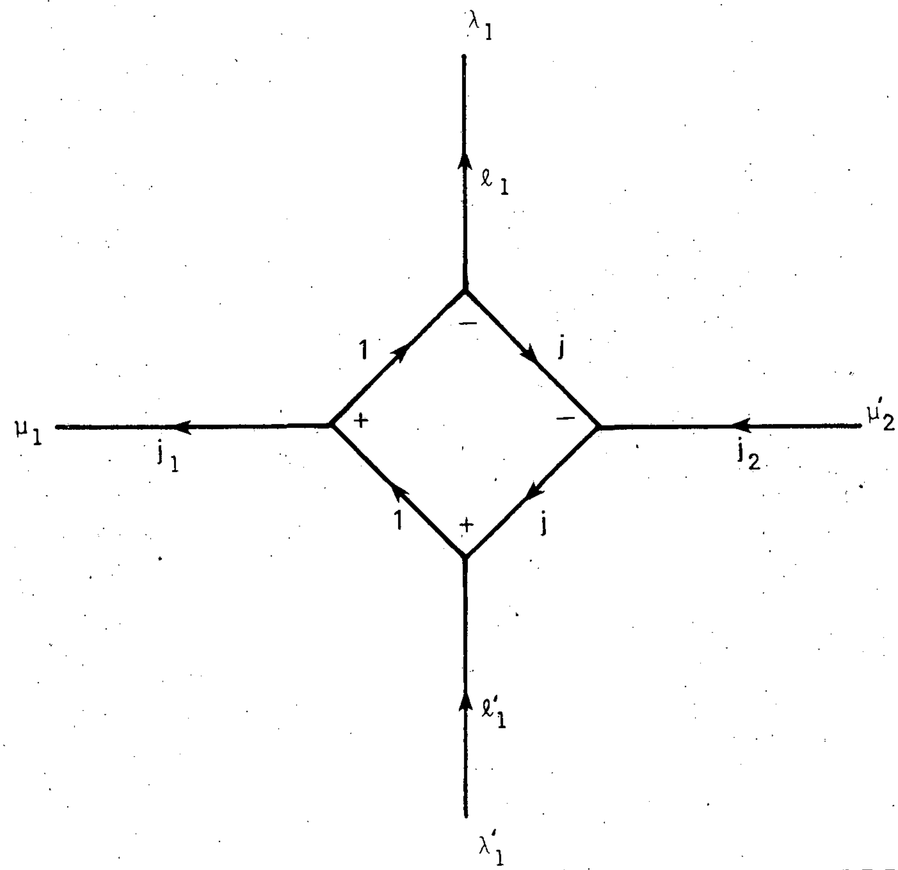
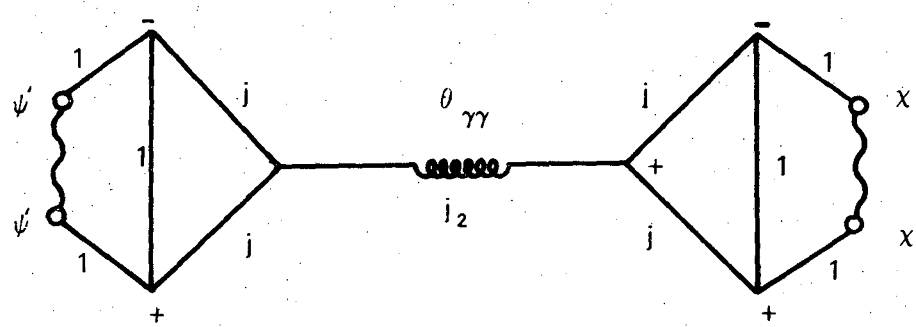


XBL7511-9535

Fig. 7



XBL 7511-9536

Fig. 8

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