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Orbital stability of the high-current betatron

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Orbital stability of the high-current betatron is analyzed. In this modified betatron a toroidal magnetic field is added to the conventional betatron magnetic field. This increases substantially the space charge limit during injection. It gives rise, however, to new problems during acceleration, such as ring stability, Fermi drift, and orbital resonances. A detailed analysis is presented showing that two serious problems may arise: (i) The beam becomes highly unstable when the net focusing force on it becomes zero. The dominant instability in this region is the Fermi drift. This may, however, be utilized in extracting the beam. (ii) Errors in the vertical (betatron) magnetic field result in driven resonances that must be crossed during acceleration. The amplitude growth of the transverse betatron oscillations due to these resonances is negligible up to very high γ (low n resonances). In practice, it may become very difficult to cross the $n = 1$ resonance. Thus, extracting the beam again making use of the Fermi drift while $n > 1$ is preferable.

I. INTRODUCTION

The largest conventional betatron was completed about 1950.¹ The energy was approximately 300 MeV and the beam current about 100 mA. The maximum beam current was limited by space charge during injection. Electrons from a 100 keV electron gun were injected into an orbit of radius $R \approx 1$ meter and a betatron magnetic field of $B_y \approx 10$ G. The space charge limit is proportional to B_y^2 , and for the Kerst Betatron¹ the density was approximately 10^7 cm^{-3} . The space charge limit is also proportional to γ , thus it is greatly increased after acceleration. In a conventional betatron it is the space charge limit at injection which determines the maximum current.

To eliminate the space charge problem the plasma betatron was proposed by Budker.² After many studies³ were carried out, the maximum current reached approximately 10 A, much less than expected. The current is probably instability limited. The precise instability has not been identified, although the negative mass instability is mentioned frequently.⁴

By increasing the injection energy and decreasing the orbit radius in a conventional betatron, the space charge limit can be increased ($n \propto \beta^2 \gamma^3 / R^2$). Small "ironless" betatrons have been developed with an electron energy of 100 MeV and electron currents of approximately 90 A.⁵ Electrons were injected into a 3.9 cm orbit radius with energy of 500 keV. The space charge limit was increased by a factor of approximately 10^4 compared to the Kerst Betatron. The total number of electrons was increased, however, only by a factor of approximately 40.

Controlling space charge of electrons by adding a toroidal magnetic field to the usual betatron field was first investigated theoretically⁶ in 1973 and the first experiments were reported⁷ in 1976. A similar proposal for ions was made in 1978.⁸

The combined toroidal and vertical (betatron) fields increase substantially the space charge limit and the number of electrons that can be trapped and accelerated. Injecting and

trapping into such combined fields require new techniques. Several methods were proposed lately.^{9,10} The first method⁹ has already been demonstrated in the studies of the collective focusing ion accelerator.¹¹ The method is called inductive charging and was first used in the HIPAC accelerator.¹² Electrons are injected from thermionic injectors and trapped by means of a rising toroidal magnetic field. A betatron with this type of injection is illustrated in Fig. 1. Electrons move in an orbit around the torus. They are not drifted in the toroidal field because of the self-field forces and the interaction with the surrounding metallic boundaries. There are no restrictions on B_z as there are on B_y in a conventional betatron. As the space charge limit depends on B_z^2 during injection, a substantial increase in the density can be achieved. With the existing injectors, the current after acceleration will be approximately 1 kA.¹¹ Better injectors may increase this current further by one to two orders of magnitude.

The toroidal magnetic field solves the problem of increasing the space charge limit during injection. However, it creates new problems during acceleration. The purpose of this paper is to investigate the regions of instability of this high current betatron and to suggest an optimum mode of operation. We shall assume throughout this paper that in-

FIG. 1. Schematic design of the high-current betatron with inductive charging.

TABLE I. Standard set of parameters for the high-current betatron (used throughout this paper).

Parameter	Units
Major radius (R)	100 cm
Minor radius (a)	5 cm
B_z^0	10 kG
Line density (N)	$2 \times 10^{11} \text{ cm}^{-1}$
Beam radius (r_b)	1 cm
Ω_z^0	$2 \times 10^{11} \text{ sec}^{-1}$
ω_p^0	$1.4 \times 10^{10} \text{ sec}^{-1}$
$B_0/\beta\gamma$	16 G
$d\gamma/dt$	$1 \mu\text{sec}^{-1}$

ductive charging is used. The parameter set we shall use as a standard reference is given in Table I. We shall discuss only orbital instabilities. Collective instabilities, such as the negative mass instability, or resistive wall instabilities have not been treated in this paper, but it is known¹³ that toroidal field decreases their importance.

II. SINGLE PARTICLE ORBITS

A. Equations of motion

In order to describe the particle orbit in a toroidal electron beam, consider the local coordinates (x, y) or (r, θ) as illustrated in Fig. 2. The toroidal direction is indicated by z . If $\Omega_y = eB_y/\gamma mc$ electrons will move on a radius $R = v_z/\Omega_y$. Furthermore, if B_y is changed in time so that

$$B_y(R) = \frac{1}{R^2} \int_0^R B_y(r) r dr, \quad (1)$$

the electrons will move on a constant radius. Due to the beam self fields and the distribution of the initial conditions, electrons will also move in the transverse directions. This motion determines the maximum current that can be obtained and the quality of the beam. The equations of motion, up to first order in (x/R) , where x is the deviation from the betatron equilibrium orbit, are:

$$m\gamma \left(\ddot{x} + \frac{\dot{\gamma}}{\gamma} \dot{x} + \frac{v_z^2}{R-x} \right) = -e \left((E_r - \beta B_\theta) \frac{x}{r} + \frac{1}{c} (v_y B_z - v_z B_y) \right), \quad (2)$$

$$m\gamma \left(\ddot{y} + \frac{\dot{\gamma}}{\gamma} \dot{y} \right) = -e \left((E_r - \beta B_\theta) \frac{y}{r} + \frac{1}{c} (v_z B_x - v_x B_z) \right).$$

B_z and B_y are the applied toroidal and vertical fields, E_r and B_θ are the self-fields. Interaction with the surrounding wall was neglected here and will be considered separately later. The equilibrium orbit is determined by the relation $B_y(R) = \frac{1}{2} \bar{B}_y(R)$. $\bar{B}_y(R)$ is influenced by the beam current. For a high-current betatron this influence is not negligible, especially for the early stage of acceleration. External coils should then be added to minimize the influence of the self-fields on the equilibrium orbit. A simple decoupling of the toroidal and transverse motions can be easily done only for $\gamma > 1$. We shall deal here only with the case $\gamma > 1$. (The case $\gamma \simeq 1$ will be investigated in a later paper.)

FIG. 2. Coordinates for a toroidal electron beam.

For a cylindrically symmetric beam the self-fields are easily calculated:

$$\frac{-e(E_r - \beta B_\theta)}{m\gamma r} = \frac{\bar{\omega}_p^2}{2\gamma^2} \frac{n(r)}{n},$$

where $\bar{\omega}_p^2 = 4\pi ne^2/\gamma m$, n is the average density of the beam, and $n(r)$ is the average density up to radius r . If the beam is uniform, $n(r)/n = 1$ and the self-force is harmonic.

The betatron field is expanded around the equilibrium orbit;

$$B_y = B_0 \left(\frac{R}{R-x} \right)^s \simeq B_0 \left(1 + \frac{sx}{R} \right),$$

$$B_x \simeq sB_0 y/R. \quad (3)$$

Thus we get the following equations of motion:

$$\ddot{x} + \dot{x} \frac{\dot{\gamma}}{\gamma} + \Omega_z \dot{y} + [(1-s)\Omega_y^2 - \Omega^2]x = 0, \quad (4)$$

$$\ddot{y} + \dot{y} \frac{\dot{\gamma}}{\gamma} - \Omega_z \dot{x} + [s\Omega_y^2 - \Omega^2]y = 0,$$

where $\Omega_z = eB_z/\gamma mc$, $\Omega_y = eB_0/\gamma mc$, $\Omega^2 = \omega_p^2/2\gamma^2$. The toroidal field was assumed constant in space and time. Higher-order corrections will be discussed later.

B. Space charge limits

Define

$$\omega_x^2 = (1-s)\Omega_y^2 - \Omega^2,$$

$$\omega_y^2 = s\Omega_y^2 - \Omega^2, \quad (5)$$

and assume that s , γ , Ω_z , Ω_y and Ω are time-independent. The general solution of Eqs. (4) and (5) can be expressed as a sum of two oscillations having frequencies

$$\omega_{1,2}^2 = \frac{1}{2} \{ \Omega_z^2 + \omega_x^2 + \omega_y^2 \pm [(\Omega_z^2 + \omega_x^2 + \omega_y^2)^2 - 4\omega_x^2 \omega_y^2]^{1/2} \}. \quad (6)$$

Orbits are stable if and only if:

$$(a) \omega_x^2 + \omega_y^2 + \Omega_z^2 > 4\omega_x^2 \omega_y^2,$$

$$(b) \omega_x^2 \omega_y^2 > 0,$$

$$(c) \omega_x^2 + \omega_y^2 + \Omega_z^2 > 0. \quad (7)$$

Denoting $s = (1/2)(1 + \delta)$ these conditions can be written as:

FIG. 3. Graphical representation of the stability conditions for the modified betatron [see Eq. (9)]. The transition takes place when γ is increased.

$$\begin{aligned} (a) & (\Omega_z^2 + 2\Omega_y^2 - 4\Omega^2)\Omega_z^2 + \delta^2\Omega_y^4 > 0, \\ (b) & [(\Omega/\Omega_y)^2 - 1/2]^2 > \frac{1}{4}\delta^2, \\ (c) & \Omega_z^2 + \Omega_y^2 - 2\Omega^2 > 0. \end{aligned} \quad (8)$$

Condition (b) involves only Ω and Ω_y . For $\delta = 0$ it is always satisfied. When $\delta \neq 0$ it is satisfied in two disconnected regions:

$$\left(\frac{\Omega}{\Omega_y}\right)^2 < \frac{1-|\delta|}{2}, \quad \text{or} \quad \left(\frac{\Omega}{\Omega_y}\right)^2 > \frac{1+|\delta|}{2}. \quad (9)$$

The first is the "low-density" region in which conventional betatrons are operated ($\omega_x^2, \omega_y^2 > 0$). The second is the "high-density" region ($\omega_x^2, \omega_y^2 < 0$). In the high-current betatron we intend to start in the high-density region, but as γ rises we eventually have to cross the unstable region (see Fig. 3). Here, the beam will start to expand, but as it expands Ω^2 will drop and a transition to the lower stable region will occur. An upper-limit estimate of the density drop is $|\Delta n/n| < 2|\delta|$, thus for $|\delta| \approx 0.25$ the beam will expand by less than 25% in radius.

This effect can be decreased by crossing the unstable region quickly enough (e.g., by a rapid change of γ).

Conditions (a) and (c) are satisfied for every δ if they are satisfied for $\delta = 0$ ($s = 1/2$). Condition (a) is then stronger than (c). Thus we get:

$$\Omega^2 < \Omega_z^2/4 + \Omega_y^2/2,$$

or

$$n < \gamma \left(\frac{B_y^2}{4\pi mc^2} + \frac{B_z^2}{8\pi mc^2} \right). \quad (10)$$

For the inductive charging, $B_y \ll B_z$ at injection, thus $n_{\max} \propto \gamma B_z^2$. In experiments at the University of California, Irvine, an electron ring of line density approximately $2 \times 10^{11} \text{ cm}^{-1}$ was injected and trapped.¹¹ This corresponds to a current after acceleration of about 1 kA, and is larger by a factor of 10^4 compared to the Kerst Betatron. This is still two orders of magnitude less than the current that can be contained by a 10 kG toroidal field. We expect to increase the density of the trapped electron ring by using higher injection

voltage which is presently 10 kV. In experiments at Maxwell Laboratories⁷ a density of 2×10^{12} was observed with an injector voltage of 30 kV.

During acceleration γB_y^2 rises, thus if Eq. (10) is satisfied during injection it also will be satisfied during acceleration.

The transition from the high Ω/Ω_y region to the low Ω/Ω_y region occurs at

$$\gamma^3 \beta^2 \approx 4NR^2 r_c / r_b^2, \quad (11)$$

where r_c is the electron classical radius and r_b is the beam radius. For our standard example (Table I) this means $\gamma \approx 13$ and $B_y \approx 210 \text{ G}$.

III. RING ORBITS

Assume that the electron ring is shifted away from the axis of the torus. Electric and magnetic images at the wall will push the beam further onto the wall. Together with the toroidal magnetic field a slow rotation ($\vec{F} \times \vec{B}$ drift) around the toroidal axis will start. We shall investigate this transverse motion and find its stability regions. We shall consider only the z -independent mode, and expand the image forces to the lowest order in a/R (where a and R are the minor and major radii, respectively).

To the lowest order in a/R , the image forces may be considered as created by an infinite cylinder of radius a . If the beam is off center by X along the x axis, the image will be a line with the same line density N located at $x = a^2/X$. The electric field at X produced by the image charge is

$$E_x = - \frac{2Ne}{|a^2/X - X|} \approx - \frac{2NeX}{a^2}, \quad (12)$$

and the force exerted on an electron at X will be:

$$F = \gamma m \Omega_b^2 X, \quad (13)$$

where

$$\Omega_b^2 = \frac{2Ne^2}{\gamma m a^2} = \left(\frac{r_b}{a}\right)^2 \frac{\omega_p^2}{2},$$

and r_b is the beam radius.

The magnetic field depends on the image currents. If the wall is an ideal conductor the magnetic force is opposite to the electric force and reduces it by a factor of γ^2 . We shall assume the other extreme case, i.e., immediate decay. This assumption is a good approximation for the high-current betatron now under construction at the University of California, Irvine. The equations of motion for the beam center are:

$$\ddot{X} + \frac{\dot{\gamma}}{\gamma} \dot{X} + \frac{v_z^2}{R - X} = \Omega_b^2 X - \Omega_z \dot{Y} + \frac{eB_y}{\gamma mc} v_z, \quad (14)$$

$$\ddot{Y} + \frac{\dot{\gamma}}{\gamma} \dot{Y} = \Omega_b^2 Y + \Omega_z \dot{X} - \frac{eB_x}{\gamma mc} v_z.$$

X and Y are measured relative to the toroidal axis.

For inductive acceleration on a constant radius R_0 , the betatron condition should hold, namely:

$$B_y(R_0) = \frac{\Omega_y(R_0)}{2\Omega_0(R_0)} \bar{B}_y(R_0), \quad (15)$$

where $\Omega_0(R_0) = v_z/R_0$ and $\bar{B}_y(R_0)$ is the average field inside

a circle with radius R_0 . For a low-current betatron the image fields are negligible and $\Omega_y = \Omega_0$. When the image fields are strong the equilibrium orbit should be determined by the relation

$$\frac{\Omega_y}{\Omega_0} = 1 - \left(\frac{\Omega_b}{\Omega_0} \right)^2 \frac{X}{R_0}. \quad (16)$$

The influence of the image fields is maximum during the initial stage of acceleration. It is thus important to trap the beam close to the toroidal axis.

If X and Y are measured relative to the equilibrium orbit R_0 , and if R_0 does not change in time (this is a good assumption for $\gamma > 1$), Eq. (14) can be written as:

$$\ddot{X} + (\dot{\gamma}/\gamma)\dot{X} + \Omega_z\dot{Y} + \omega_x^2 X = 0, \quad (17)$$

$$\ddot{Y} + (\dot{\gamma}/\gamma)\dot{Y} - \Omega_z\dot{X} + \omega_y^2 Y = 0,$$

where

$$\omega_x^2 = (1-s)\Omega_z^2 - \Omega_b^2, \quad (18)$$

$$\omega_y^2 = s\Omega_z^2 - \Omega_b^2.$$

Equations (17) are similar to Eq. (4). For $s = 1/2$ the motion is stable if

$$\Omega_b^2 < \frac{\Omega_z^2}{4} + \frac{\Omega_y^2}{2}, \quad (19)$$

or

$$n < \frac{1}{\gamma} \left(\frac{a}{r_b} \right)^2 \left(\frac{B_z^2}{8\pi mc^2} + \frac{B_y^2}{4\pi mc^2} \right).$$

If this condition is satisfied the beam will precess around the equilibrium orbit. If it does not hold the radius of the beam center will increase until it will hit the wall. Note that γ appears in the denominator [compare to Eq. (10)]. This is due to the assumption of an immediate decay of the image currents.

For our standard set of parameters Eq. (19) can be written as:

$$N < [(4 \times 10^{14})/\gamma] + 2 \times 10^9 \gamma, \quad (20)$$

where N is the line density and the parameters of Table I were used. If $N \leq 1.8 \times 10^{12} \text{ cm}^{-1}$ the beam is stable for every γ . However, for $N \geq 1.8 \times 10^{12} \text{ cm}^{-1}$ there is a region of instability. For $N = 10^{13} \text{ cm}^{-1}$ this region starts already at $\gamma \approx 40$. To overcome it either B_z should be raised or the currents in the wall should persist longer.

Equations (17) have for $s \neq 1/2$ a stability problem similar to that of Eq. (9). The unstable region now is around

$$\Omega_b^2 \approx \frac{1}{2}\Omega_z^2, \quad \text{or} \quad \gamma\beta^2 \approx 4N(R/a)^2 r_c, \quad (21)$$

where r_c is the electron classical radius. For our standard parameters this will occur at $\gamma \approx 90$. In contrast to the single particle case, the self field depends here only on the line density, which does not change as the instability develops. It is thus necessary to cross this region quickly. An estimate of the growth rate can be given at the maximum; i.e., at $\Omega_b^2 = \frac{1}{2}\Omega_z^2$. Here the e-folding time is

$$\frac{2\Omega_z}{|s - \frac{1}{2}|\Omega_z^2} \approx \frac{0.045}{|s - \frac{1}{2}|} \mu\text{sec},$$

which is much less than one μsec unless $|s - \frac{1}{2}| \ll 1$.

Increasing NR^2 will increase the γ where the transition occurs. Thus the instability may be avoided (if NR^2 is large enough) or it may be used as a means to extract the beam.

In concluding this section a remark should be made concerning the influence of the beam motion on the single particle dynamics. In Eqs. (4) we neglected the image forces. Moreover, we assumed that the beam center was located at the betatron equilibrium orbit. If the beam center is rotating around the equilibrium orbit inhomogeneous terms should be added to Eqs. (4). These terms may influence the single particle dynamics when the two motions are either in resonance or when the radial focusing force on a single particle vanishes. In both cases the beam radius will grow and the instabilities will be self-stabilized. A numerical analysis is now underway. The results will be published in a later paper.

IV. BEAM EMITTANCE

The quality of a beam is sometimes described by its emittance, defined as

$$\epsilon/\pi = \theta\rho = \bar{p}_1\rho/p_z, \quad (22)$$

where ρ is the beam radius and θ is the angular divergence. Emittance is determined mainly by the method of injection. For inductive charging $\bar{p}_1\rho = mv_{10}a = \text{const}$, where $a \approx 5 \text{ cm}$ (the beam radius during injection), and $v_{10}/c \leq 0.2$ corresponding to a 10 keV injector. When the toroidal field reaches its maximum of approximately 10 kG the beam radius is of the order of 1 cm and the transverse kinetic energy is approximately 500 keV.¹¹ During acceleration $\bar{p}_1 = \text{const}$, but ρ decreases due to the B_y increase and the self-pinching effect. For simplicity assume $s = \frac{1}{2}$ and consider Eq. (4). If B_z is changing in time, the induced field E_θ should be included. Transforming then to the rotating Larmor frame we get:

$$\ddot{u} + \dot{u}(\dot{\gamma}/\gamma) + \omega^2 u = 0, \quad (23)$$

where

$$x + iy = \exp\left(\frac{1}{2} \int_0^t \Omega_z(t') dt'\right) u(t),$$

and

$$\omega^2 = \frac{1}{4} \left[\Omega_z^2 + 2\Omega_y^2 - 4\Omega^2 - \left(\frac{\dot{\gamma}}{\gamma} \right)^2 - 2 \frac{d}{dt} \left(\frac{\dot{\gamma}}{\gamma} \right) \right] \\ \approx \frac{1}{4} [\Omega_z^2 + 2\Omega_y^2 - 4\Omega^2].$$

This is correct for any change of Ω_z ; either changing γ or B_z .

The mixed term $\dot{u}(\dot{\gamma}/\gamma)$ can be eliminated by the transformation

$$u = \gamma^{-1/2} z, \quad (24)$$

yielding

$$\ddot{z} + \omega^2 z = 0.$$

Equation (24), for the real and imaginary parts of z , can be solved using the WKB approximation. The general solution can be written as

$$z = \frac{c_i}{\sqrt{\omega}} \cos(\varphi(t) + \varphi_i), \quad \varphi(t) = \int_0^t \omega(s) ds, \quad (25)$$

where c_i and φ_i are determined by the initial conditions, thus

$$r^2 = \gamma^{-1} |z|^2 = (1/\gamma\omega)[A^2 + B^2 \sin(2\varphi(t) + \varphi_0)], \quad (26)$$

where A^2 , B^2 and φ_0 are functions of the initial conditions only. The maximum value of r , and therefore the beam radius, changes in time as $(\gamma\omega)^{-1/2}$, or:

$$\begin{aligned} \frac{r(t)}{r(0)} &= \left| \frac{u(t)}{u(0)} \right| = \left| \frac{\gamma^{-1/2}(t) z(t)}{z(0)} \right| \\ &\simeq \left(\frac{(\Omega_z^2 + 2\Omega_y^2 - 4\Omega_z^2)_0}{\gamma^2(t)(\Omega_z^2 + 2\Omega_y^2 - 4\Omega_z^2)_t} \right)^{1/4}. \end{aligned} \quad (27)$$

During inductive charging $\Omega_y \simeq 0$ and $\gamma \simeq 1$. If $\Omega \simeq 0$, we get the well-known adiabatic condition

$$r^2(t) B_z(t) = \text{const.} \quad (28)$$

This is equivalent to $v_1^2/B = \text{const.}$

As acceleration takes place Ω_y rises and Ω^2 drops down (like γ^3). At high γ the self-term is always negligible due to cancellation of the electric and magnetic forces. The beam radius then changes as

$$r(t)(B_z^2 + 2B_y^2)^{1/4} = \text{const.} \quad (29)$$

Thus the beam is further compressed as it is accelerated. If, after B_y reaches its maximum, B_z is turned off, the final beam radius will be $r_f/r_0 = (B_{z0}^2/2B_{yf}^2)^{1/4}$, for $B_{z0} = 10$ kG, $B_{yf} = 5$ kG, and $r_f/r_0 \simeq 1.2$.

At a final energy of 300 MeV, $\epsilon/\pi \simeq \bar{v}_{10} 1.2 r_0 / \gamma_c \simeq 2 \times 10^{-3}$ rad cm. Instabilities may increase this number. As the emittance depends mainly on the injection method, a full understanding of this process will be necessary to further increase the beam quality.

The ring dynamics during acceleration is very similar to the single particle dynamics. Here, however, the electrostatic interaction with the walls does not reduce during acceleration and according to Eq. (25) the amplitude increases as:

$$r(t) \left[1 + \frac{2B_y^2}{B_z^2} - \left(\frac{2\Omega_b}{\Omega_z} \right)^2 \right]^{1/4} = \text{const.} \quad (30)$$

For the standard parameters the change in r is negligible (even for $\gamma \simeq 600$). If, however, the line density is increased by an order of magnitude, the beam will become unstable at $\gamma \simeq 200$. This can be remedied by the increasing of the toroidal field or by the persisting image currents in the walls [compare to Eq. (19)].

V. FERMI DRIFT

A charged particle moving along a toroidal magnetic field drifts with a constant speed in a direction perpendicular to the toroidal plane. This drift is known as the Fermi (sometimes also $\bar{v}B$) drift. It is due to the nonconstancy of the magnetic field (necessary for $\bar{v} \times \bar{B} = 0$) as a function of the radius. This effect is of second order in (a/R) and was neglected in the previous discussion. In the betatron we describe, which has both toroidal and vertical magnetic fields, the strong self-fields weaken the Fermi drift motion substantially.

Let us start with the simple case of an electron moving perpendicular to a toroidal field. The problem of electrons in

the high-current betatron will be presented as a simple generalization. The equations of motion in the transverse directions are:

$$\begin{aligned} \ddot{x} + \hat{\Omega}_z(x) \dot{y} &= 0, \\ \ddot{y} - \hat{\Omega}_z(x) \dot{x} &= 0, \end{aligned} \quad (31)$$

where

$$\hat{\Omega}_z = \frac{\Omega_z R}{(R-x)} \simeq \Omega_z \left(1 + \frac{x}{R} \right), \quad (32)$$

up to first order in (x/R) . Equations (31) are nonlinear. To use the perturbation technique let us first write them in the following way:

$$\begin{aligned} \ddot{x} + \Omega_z \dot{y} &\simeq -\Omega_z (x/R) \dot{y}, \\ \ddot{y} - \Omega_z \dot{x} &\simeq \Omega_z (x/R) \dot{x}. \end{aligned} \quad (33)$$

Writing $x = x^{(1)} + x^{(2)} + \dots$, $y = y^{(1)} + y^{(2)} + \dots$, where

$$\begin{aligned} \ddot{x}^{(1)} + \Omega_z \dot{y}^{(1)} &= 0, \\ \ddot{y}^{(1)} - \Omega_z \dot{x}^{(1)} &= 0, \end{aligned} \quad (34)$$

with the given initial conditions, and

$$\begin{aligned} \ddot{x}^{(2)} + \Omega_z \dot{y}^{(2)} &= -\Omega_z \frac{(x \cdot \dot{y})^{(2)}}{R} = -\Omega_z \frac{x^{(1)} \dot{y}^{(1)}}{R}, \\ \ddot{y}^{(2)} - \Omega_z \dot{x}^{(2)} &= \Omega_z \left(\frac{x \cdot \dot{x}}{R} \right)^{(2)} = \Omega_z \frac{x^{(1)} \dot{x}^{(1)}}{R}, \end{aligned} \quad (35)$$

with the initial conditions $x^{(2)} = y^{(2)} = 0$ and $\dot{x}^{(2)} = \dot{y}^{(2)} = 0$. Higher order terms will be neglected, consistent with Eq. (32).

The solution to Eq. (34) can be written without any loss of generality as:

$$\begin{aligned} x^{(1)} &= x_0 \cos \Omega_z t, \\ y^{(1)} &= (\dot{y}_0 / \Omega_z) \sin \Omega_z t, \end{aligned} \quad (36)$$

where

$$\dot{y}_0 = \Omega_z x_0.$$

Inserting this into Eq. (35) we get the inhomogeneous equations

$$\begin{aligned} \ddot{x}^{(2)} + \Omega_z \dot{y}^{(2)} &= -(\dot{y}_0^2 / 2R) (1 + \cos 2\Omega_z t), \\ \ddot{y}^{(2)} - \Omega_z \dot{x}^{(2)} &= -(\dot{y}_0^2 / 2R) \sin 2\Omega_z t, \end{aligned} \quad (37)$$

with initial conditions $x_{(0)}^{(2)} = y_{(0)}^{(2)} = 0$, $\dot{x}_{(0)}^{(2)} = \dot{y}_{(0)}^{(2)} = 0$.

The solution is immediate. It consists of a small shift in the direction (Δx is of the order of x_0^2/R which is thus negligible) plus a small amplitude oscillation with a doubled cyclotron frequency ($2\Omega_z$). In addition there is the drift term in the y direction with a speed:

$$\dot{y}_D = -(\dot{y}_0^2 / 2R\Omega_z). \quad (38)$$

In conclusion, up to second order we found that the electron rotates around its guiding center which drifts in the $-y$ direction with constant speed.

Now let us return to the modified betatron case. For simplicity assume $s = 1/2$. The equations of motion up to second order are (leaving only the second order terms which lead to drift motion):

$$\begin{aligned} \ddot{x} + \Omega_z \dot{y} + \omega^2 x &= -\Omega_z (x/R) \dot{y}, \\ \ddot{y} - \Omega_z \dot{x} + \omega^2 y &= \Omega_z (x/R) \dot{x}, \end{aligned} \quad (39)$$

where, from Eq. (5)

$$\omega^2 = \frac{1}{2}\Omega_z^2 - \Omega^2.$$

The first-order solution is given by a combination of the modes:

$$\omega^\pm = \frac{1}{2}[\Omega_z \pm (\Omega_z^2 + 4\omega^2)^{1/2}]. \quad (40)$$

The general form of the solution is:

$$\begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} 1 \\ i\omega^+ \\ i \\ \omega^+ \end{pmatrix} e^{i\omega^+ t} + B \begin{pmatrix} 1 \\ i\omega^- \\ -i \\ \omega^- \end{pmatrix} e^{i\omega^- t} + C \begin{pmatrix} 1 \\ -i\omega^+ \\ i \\ \omega^+ \end{pmatrix} e^{-i\omega^+ t} + D \begin{pmatrix} 1 \\ -i\omega^- \\ i \\ \omega^- \end{pmatrix} e^{-i\omega^- t}, \quad (41)$$

where the constants A, B, C , and D should be determined by the four initial conditions.

It will be shown that the Fermi drift is important here only when $\omega \ll \Omega_z$, thus

$$\begin{aligned} \omega^+ &\simeq \Omega_z, \\ \omega^- &\simeq -\omega^2/\Omega_z. \end{aligned} \quad (42)$$

The equations for the second order corrections are

$$\ddot{x}^{(2)} + \Omega_z \dot{y}^{(2)} + \omega^2 x^{(2)} = -\Omega_z (x/R)^{(1)} \dot{y}^{(1)} \equiv g_x(t), \quad (43)$$

$$\dot{y}^{(2)} - \Omega_z \dot{x}^{(2)} + \omega^2 y^{(2)} = \Omega_z (x/R)^{(1)} \dot{x}^{(1)} \equiv g_y(t).$$

The solutions to these equations, with initial conditions $x^{(2)}(0) = y^{(2)}(0) = 0$, are:

$$\begin{aligned} x^{(2)} = & \frac{1}{\omega^+ - \omega^-} \left(\int_0^t g_x(s) [\sin \omega^+(t-s) - \sin \omega^-(t-s)] ds \right. \\ & \left. + \int_0^t g_y(s) [\cos \omega^+(t-s) - \cos \omega^-(t-s)] ds \right), \end{aligned} \quad (44)$$

$$\begin{aligned} y^{(2)} = & \frac{1}{\omega^+ - \omega^-} \left(\int_0^t g_x(s) [\cos \omega^-(t-s) - \cos \omega^+(t-s)] ds \right. \\ & \left. + \int_0^t g_y(s) [\sin \omega^+(t-s) - \sin \omega^-(t-s)] ds \right). \end{aligned}$$

The dominant terms in the limit $\omega^- \rightarrow 0$ are

$$\begin{aligned} x^{(2)} &= -\frac{1}{\omega^+ - \omega^-} \int_0^t g_x(s) \sin \omega^-(t-s) ds, \\ y^{(2)} &= \frac{1}{\omega^+ - \omega^-} \int_0^t g_x(s) \cos \omega^-(t-s) ds. \end{aligned} \quad (45)$$

Moreover, it is only the constant part of $g_x(s)$ which has a significant contribution. It can be shown by a straightforward calculation that

$$\begin{aligned} y^{(2)} = & -\frac{\Omega_z}{2R(\omega^+ - \omega^-)^3} [\Omega_z(\dot{x}_0^2 + \dot{y}_0^2) \\ & - \omega^2 \Omega_z(x_0^2 + y_0^2) + 2\omega^2(x_0 \dot{y}_0 - y_0 \dot{x}_0)] \frac{\sin \omega^- t}{\omega^-}. \end{aligned} \quad (46)$$

There is a similar expression for $x^{(2)}$. Problems may arise

when $\omega^- \rightarrow 0$. The dominant term in the sum is then $\Omega_z(\dot{x}_0^2 + \dot{y}_0^2)$, thus we get

$$\begin{aligned} x^{(2)} &\simeq -\frac{v_1^2}{2R\Omega_z} \left(\frac{\cos \omega^- t}{\omega^-} - \frac{1}{\omega^-} \right), \\ y^{(2)} &\simeq -\frac{v_1^2}{2R\Omega_z} \frac{\sin \omega^- t}{\omega^-}. \end{aligned} \quad (47)$$

This means that the particles move on a shifted circle (the shift depends on the sign of ω^-) with a radius Δ (the shift is also Δ);

$$\Delta = \frac{v_1^2}{2R\Omega_z|\omega^-|} \simeq \frac{1}{2R} \left| \frac{v_1}{\omega} \right|^2. \quad (48)$$

This result can be obtained also by using the drift approximation.¹⁴

If $|\Delta| \ll r_b$, the Fermi drift can be neglected.

For $\gamma > 1$ (but not too large) $|\omega^2| \simeq \Omega^2 = \omega_p^2/2$, thus the above criterion is

$$(v_1/\omega_p)^2 \ll r_b R, \quad \text{or} \quad \beta_{01}^2 \ll 2\pi n r_b R r_c, \quad (49)$$

where n is the electron density and r_c is the electron classical radius. For $n = 10^{11} \text{ cm}^{-3}$, $\beta_{01}^2 = 0.5$, $r_b = 1 \text{ cm}$ and $R = 100 \text{ cm}$, inequality (49) is satisfied.

When γ is large $\frac{1}{2}\Omega_z^2 > \Omega^2$, and the Fermi drift still can be negligible due to the focusing effect of the betatron field. The condition for this is $(v_1/v_{||})^2 \ll r_b/R$, but since $v_1 = v_1^0/\gamma$ it can be written as:

$$(v_1^0/v_{||})^2 \ll (r_b/R)\gamma^2. \quad (50)$$

Assuming $v_{||} \simeq c$ and $(v_1^0/c)^2 = \frac{1}{2}$, Eq. (50) yields $\gamma^2 \gg 50$.

A problem exists only when $\frac{1}{2}\Omega_z^2 \simeq \Omega^2$. This occurs at $\gamma \simeq 13$ [see Eq. (11)]. At this energy the focusing forces inside the beam vanish and all the particles start drifting. The beam expands as well as drifts rigidly. As it expands the beam density fall down, ω^- changes sign, electrons start precessing in the opposite direction around the beam center, and stability is regained. The rigid motion is also taken care of automatically by the interaction with the surrounding walls. As seen earlier a displacement of the beam induces non-negligible electric image forces which stabilize the beam motion. A full description of the beam dynamics is difficult as the single particle and the beam center motions are strongly coupled. Approximate equations of motion for the beam center can be obtained by averaging Eq. (39) over all the electrons. Let us denote the coordinates of the beam center by X and Y and the single particle coordinates by x and y :

$$\begin{aligned} \ddot{X} + \Omega_z \dot{Y} + \omega_b^2 X &= -(\Omega_z/R) \langle x\dot{y} \rangle, \\ \ddot{Y} - \Omega_z \dot{X} + \omega_b^2 Y &= (\Omega_z/R) \langle x\dot{x} \rangle, \end{aligned} \quad (51)$$

where $\omega_b^2 = \frac{1}{2}\Omega_z^2 - \Omega^2$. The average terms can be calculated immediately:

$$\langle x\dot{x} \rangle = \langle (X + \delta x)(\dot{X} + \dot{\delta x}) \rangle = X\dot{X} + \frac{1}{2} \langle \delta \dot{x}^2 \rangle \simeq X\dot{X},$$

where δx is the x coordinate of a single electron relative to the beam center. Here $\langle \delta \dot{x}^2 \rangle = 0$ is a good approximation only if the beam shape does not change as it precesses:

$$\langle x\dot{y} \rangle = X\dot{Y} + \langle \delta x \delta \dot{y} \rangle \simeq X\dot{Y} + (\langle v_1^2 \rangle / 2\Omega_z),$$

where $\langle \delta x \delta \dot{y} \rangle$ was estimated in a way similar to Eq. (46).

The equations of motion of the beam center are:

$$\begin{aligned}\ddot{X} + \Omega_z Y + \omega_b^2 X &= -(\Omega_z/R)X\dot{Y} - (\langle v_z^2 \rangle / 2R), \\ \ddot{Y} - \Omega_z \dot{X} + \omega_b^2 Y &= X\dot{X}.\end{aligned}\quad (52)$$

The constant term on the right-hand side results in a small shifting of the beam in the $-x$ direction (outside) unless $\omega_b \simeq 0$. If $\omega_b = 0$ a drift in the y direction with a speed $-\langle v_z^2 \rangle / 2R\Omega_z$ will start.

In comparing Eq. (52) to Eq. (39) it is evident that the beam and the single particle drift instabilities occur at different γ [compare Eqs. (21) and (11)]. Thus when the focusing inside the beam vanishes the beam center is stable and it will oscillate with a small amplitude. In order to estimate this beam center drift assume that at $t = 0$ the beam is located at $X = Y = 0$, and it starts drifting with $\dot{X}_0 = 0$, $\dot{Y}_0 = \langle v_z^2 \rangle / 2R\Omega_z$. Following the same steps as for the single particle case, we find that the beam will move on an inwardly shifted circle with radius Δ ;

$$|\Delta| = \frac{\dot{Y}_0^2}{2R\Omega_z|\omega^-|}, \quad \text{where} \quad |\omega^-| = \left| \frac{\omega_b^2}{\Omega_z} \right|,$$

or

$$|\Delta| = \frac{1}{2R} \left| \frac{\dot{Y}_0}{\omega_b} \right|^2.$$

Thus, the criterion for the effect of the single particle Fermi drift on the beam motion to be negligible is

$$|\dot{Y}_0/\omega_b| \ll (2Ra)^{1/2}. \quad (53)$$

ω_b can be estimated if we assume that at the transition $\frac{1}{2}\Omega_z^2 \ll \Omega_b^2$, then we get

$$\langle \beta_{01}^2 \rangle \ll \frac{2\Omega_z^0 \omega_p^0}{c^2} \left(\frac{r_b}{a} \right) (Ra)^{1/2} R. \quad (54)$$

In our standard example, this inequality is obviously satisfied. Thus as the beam internal focusing forces vanish only a small emittance increase is expected.

As γ increases further, the beam will become unstable for $\omega_b \simeq 0$ and will quickly drift to the wall (see Sec. III for other instabilities which occur at this region). The drift motion may be enhanced by the centrifugal drift (occurring when the electrons don't satisfy the exact betatron condition, i.e., when there is spread in the longitudinal velocities). We did not add this term because in our scheme of injection the spread in the transverse velocities dominates.

VI. RESONANCE INSTABILITIES

In the previous discussion we assumed that all the fields were ideal, i.e., they did not depend upon the coordinate z . Errors in these fields give rise to new types of phenomena, known as orbital resonances or parametric instabilities. These instabilities might very easily destroy the beam quality, and it is our purpose here to give quantitative restrictions on the fields in order to assure good beam quality. We shall treat the equations of motion only in the linear approximation. Higher orders are important only when the beam quality is so poor that they are of no interest anyway.

In the previous discussion, the fields were expanded around the equilibrium radius R and the expansion param-

eters were assumed constant. Now we let them be functions of z . If the electrons move with constant v_z , these parameters are also periodic functions of time, with a period T ($T = 2\pi R/v_z$). When T is an integral (sometimes a half integral) multiple of the transverse oscillation time, a resonance occurs. We shall examine some of the interesting cases and find the restrictions they impose on the external fields.

It is common to distinguish between two types of orbital resonances.¹⁵ The first includes driven resonances, which distort the equilibrium orbit and are generally very dangerous. The other type does not affect the beam center. It affects, however, the transverse motion of electrons relative to the beam center, thus changing the emittance. For the high current betatron both cases are significant at most for $\gamma \gg 1$, thus self fields are negligible. Moreover, instabilities occur only in very narrow bands of γ , thus the small terms proportional to $\dot{\gamma}/\gamma$ are negligible.

We shall start with the first type by perturbing the betatron field B_y . This necessarily perturbs the equilibrium radius R . Electrons will be deflected in the x direction. In certain conditions they will resonant with the $\delta B_y(z)$ field. We shall derive these conditions and determine their influence on the operation of the high current betatron. $\delta B_y(z)$ induces an additional component $\delta B_z(y)$ ($\nabla \times \vec{B} = 0$). This will, however, add only second-order terms to the equations of motion and thus will be neglected.

Assuming $\delta B_x = 0$ and expanding the betatron field around the average equilibrium orbit:

$$\begin{aligned}B_y &\simeq B_0(1 + sx/R) + \delta B_y(z), \\ B_x &\simeq B_0(sx/R),\end{aligned}\quad (55)$$

and the equations of motion:

$$\begin{aligned}\ddot{x} + \Omega_z \dot{y} + (1-s)\Omega_z^2 x &= \Omega_z^2 R [\delta B_y(z)/B_0], \\ \ddot{y} - \Omega_z \dot{x} + s\Omega_z^2 y &= 0.\end{aligned}\quad (56)$$

In discussing Eq. (56) let us start with the simpler equation:

$$\ddot{x} + \omega^2 x = \Omega_0^2 R \epsilon(t), \quad (57)$$

where ω^2 changes slowly in time. Here $\epsilon(t)$ is periodic in t with period $T = 2\pi/\Omega_0$, and thus can be decomposed in a Fourier series:

$$\epsilon(t) = \sum_{n=1}^{\infty} (\alpha_n \cos n\Omega_0 t + \beta_n \sin n\Omega_0 t). \quad (58)$$

Equations (56) will be treated as a simple generalization of Eq. (57).

If ω is constant the solution to Eq. (57) with the given initial condition is:

$$x = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t + \frac{1}{\omega} \int_0^t \Omega_0^2 R \epsilon(s) \sin \omega(t-s) ds. \quad (59)$$

Substituting Eq. (58) for $\epsilon(s)$, the integral can be calculated explicitly. A special case is for $\omega = n\Omega_0$. The integral then is

$$\frac{\Omega_0 R}{2n} (\alpha_n \sin n\Omega_0 t - \beta_n \cos n\Omega_0 t) + \sum_{k \neq n}$$

(nonsecular terms). Thus the amplitude grows linearly with time.

For large n , resonances are crossed so quickly that no growth (or at least a very small growth) in the amplitude is expected. When the time Δt_n [necessary to go from $\omega = (n + \frac{1}{2})\Omega_0$ to $\omega = (n - \frac{1}{2})\Omega_0$] is comparable to T , a real growth in the amplitude might occur. Δt_n is given by:

$$\left| \frac{\partial \omega}{\partial t} \right|_{\omega = n\Omega_0} \cdot \Delta t_n = \Omega_0. \quad (60)$$

Take, for simplicity, only the $\cos n\Omega_0 t$ terms of the Fourier series and denote

$$\delta\omega = \omega - n\Omega_0, \\ \bar{\omega} = (\omega + n\Omega_0)/2,$$

then the resonant term will be

$$X_{\text{res}} = \frac{\Omega_0^2 R \alpha_n}{2 \bar{\omega}} \cdot \frac{\sin \delta\omega t}{\delta\omega} \sin \bar{\omega} t. \quad (61)$$

Thus, the amplitude for $\delta\omega \neq 0$ behaves like $(\delta\omega)^{-1}$. This is the reason that real growth occurs only very close to resonance, and each resonance can be taken separately. Thus, let us look only on the n resonance as it is crossed.

If $(1/\omega^2)(\partial\omega/\partial t) \ll 1$, the asymptotic expansion WKB can be employed. The resonance term is:

$$X_{\text{res}} = \frac{\Omega_0^2 R \alpha_n}{2} \left(\int_0^t \frac{\cos[\varphi(s) - n\Omega_0 s]}{\omega(s)} ds \right) \sin \varphi(t), \quad (62)$$

where $\varphi(t) = \int_0^t \omega(t') dt'$. X_{res} can be approximated by:

$$\begin{aligned} & \frac{1}{\omega} \int_{-\Delta t_n/2}^{\Delta t_n/2} \cos \left[\frac{1}{2} \left(\frac{\partial \omega}{\partial t} \right)_{\omega = n\Omega_0} s^2 \right] ds \\ &= \frac{2\sqrt{2}}{\bar{\omega}(\partial\omega/\partial t)^{1/2}} \int_0^{(\frac{1}{2}(\partial\omega/\partial t)^{1/2}(\Delta t_n/2))} \cos t^2 dt \\ &= \frac{2}{\sqrt{\pi}} \frac{T(N_n)^{1/2}}{\bar{\omega}} \int_0^{(\pi/2)^{1/2} N_n} \cos t^2 dt, \end{aligned} \quad (63)$$

where N_n is the number of turns the electron does when it goes from $t = 0$ to $t = \Delta t_n$. We assumed that $N_n > 1$. Fresnel's integral is asymptotically $\sqrt{\pi/8}$ (see Ref. 16). This value is reached for t of the order of one. Using it for estimating the integral (63) we get $\sqrt{2}\pi(\sqrt{N_n}/n\Omega_0^2)$ and substituting back in (62) we get the change in the amplitude as we cross the n resonance:

$$X_{\text{res}} = \frac{\pi}{\sqrt{2}} \frac{R \alpha_n}{n} \sqrt{N_n}. \quad (64)$$

To be specific assume that the error is constant (ΔB) in an arc L , then:

$$\Delta X_{\text{res}} \simeq \left(\frac{\pi}{8} \right)^{1/2} \left(\frac{\Delta B}{B_0} \right) L \cdot \frac{(N_n)^{1/2}}{n}. \quad (65)$$

To minimize ΔX_{res} , $\Delta B \cdot L$ has to be smaller, n larger, and the time needed to cross the resonance smaller.

Now let us return to the modified betatron case in detail. Proceeding in the same way as we did for Eq. (57), the change of the solution due to the inhomogeneous term in Eq. (56) is given by the integrals [see Eq. (44)]:

$$x(t) = \frac{\Omega_y^2 R}{(\omega^+ - \omega^-)} \int_0^t \epsilon(s) [\sin \omega^+(t-s) - \sin \omega^-(t-s)] ds, \quad (66)$$

$$y(t) = \frac{\Omega_y^2 R}{(\omega^+ - \omega^-)} \int_0^t \epsilon(s) [\cos \omega^-(t-s) - \cos \omega^+(t-s)] ds,$$

where $\omega^\pm$ are given by Eq. (40). Resonances occur for $\omega^\pm = m\Omega_0$. However, $|\omega^-| \simeq \Omega_y^2/2\Omega_z < \Omega_0$, thus only ω^+ resonances have to be considered (especially the low n resonances).

Assuming $\omega_{(0)}^+ \simeq 2 \times 10^{11} \text{ sec}^{-1}$ ($B_z^{(0)} = 10^4 \text{ G}$) and $d\gamma/dt \simeq 10^6 \text{ sec}^{-1}$, we obtain

$$\frac{1}{\omega^2} \left| \frac{d\omega}{dt} \right| \simeq \frac{1}{\omega \gamma} \left| \frac{d\gamma}{dt} \right| \simeq 5 \times 10^{-6} \ll 1, \quad (67)$$

and WKB can be employed. The results of the one dimensional case are still valid except for an additional factor $\sim (\Omega_y/\omega^+)^2$. Moreover, the resonance is now also observed in the y direction. In our standard case $(\Omega_y/\omega^+)^2 \ll 1$ almost down to the lowest resonance, thus reducing the growth rates predicted earlier. For example, if $n = 2$, $\gamma \simeq 340$, $B_y \simeq 5 \text{ kG}$, and $(\Omega_y/\omega^+)^2 \simeq 0.25$.

To be specific, let us assume that ΔB is constant in an arc L , then:

$$\Delta X_{\text{res}}^{(n)} \simeq \left(\frac{\pi}{8} \right)^{1/2} \left(\frac{\Delta B}{B_0} \right) L \left(\frac{\Omega_y}{\omega^+} \right)^2 \frac{(N_n)^{1/2}}{n},$$

but

$$N_n = \frac{\Omega_0}{\omega_n^+ T (1/\gamma_n)(d\gamma/dt)_n}.$$

Thus we get the total growth in the amplitude:

$$\Delta X_{\text{res}} = \left(\frac{\pi}{8} \right)^{1/2} \left(\frac{\Delta B}{B_0} \right) L \left(\frac{\Omega_y}{2\pi} \frac{d\gamma}{dt} \right)^{1/2} \sum \frac{1}{n^4}. \quad (68)$$

If $d\gamma/dt \simeq 10^6 \text{ sec}^{-1}$ and $\Omega_z^0 = 2 \times 10^{11} \text{ sec}^{-1}$, $\Delta X_{\text{res}} \simeq 100(\Delta B/B_0)L$. Most of this comes from the $n = 1$ resonance which takes a very long time to cross. Thus it will be useful either to cross the lowest resonances very fast (by turning off B_z in $t \simeq 10 \mu\text{sec}$) or to extract the beam while $n > 1$ (e.g., by using the drift motion of the beam—see Secs. III and V).

Before discussing the general case let us work out another simple case which represents the second type of resonances. Assume that B_z is perturbed along the torus. According to $\nabla \cdot \vec{B} = 0$, $(\partial/\partial x)\delta B_x + (\partial/\partial y)\delta B_y \neq 0$ and additional first-order terms should be included in the equations of motion. If $(\partial/\partial x)\delta B_x = (\partial/\partial y)\delta B_y$, and if $\delta B_x(0,0) = \delta B_y(0,0) = 0$ the equations of motion are:

$$\begin{aligned} \ddot{x} + \Omega_z \dot{y} + \frac{1}{2} \dot{\Omega}_z y + (\frac{1}{2} \Omega_z^2) x &= 0, \\ y - \Omega_z \dot{x} - \frac{1}{2} \dot{\Omega}_z x + (\frac{1}{2} \Omega_z^2) y &= 0. \end{aligned} \quad (69)$$

Using the transformation

$$x + iy = \exp \left(\frac{i}{2} \int_0^t \Omega_z(t') dt' \right) u(t), \quad (70)$$

we get

$$\ddot{u} + \omega^2(z)u = 0, \quad (71)$$

where

$$\omega = \frac{1}{2}(\Omega_z^2 + 2\Omega_y^2 - 4\Omega_z^2)^{1/2}.$$

When ω is constant the general solution is

$$u = A \cos \omega t + B \sin \omega t = u_x + i u_y, \quad (72)$$

and $u_x = \text{Re } u$, $u_y = \text{Im } u$. The four initial conditions are:

$$\begin{aligned} u_x(0) &= x(0), & u_y(0) &= y(0), \\ \dot{u}_x(0) &= \dot{x}(0) + (\Omega_z/2)y(0), & \dot{u}_y(0) &= \dot{y}(0) - (\Omega_z/2)x(0). \end{aligned} \quad (73)$$

Unlike Eqs. (69), Eqs. (71) for u_x and u_y are not coupled, even when ω is not constant.

For the stability analysis it is enough to examine $|u| = |x + iy|$. The amplitude growth rate is determined by the two real equations

$$\ddot{u}_i + \omega^2(z)u_i = 0, \quad (74)$$

where

$$i = x, y.$$

The two independent solutions to each of these equations can be written according to Floquet's theorem as:

$$u(t) = e^{i\mu t} g_\mu(t), \quad (75)$$

where $g_\mu(t+T) = g_\mu(t)$ and the constant μ is either real or pure imaginary. For ω constant it is readily verified that $\mu = \pm \omega_0$. However, when a small perturbation is added to ω_0 , there will be regions of ω_0 for which μ will become pure imaginary and one of the two independent solutions will be unstable.

To show this explicitly let us take the following example, which can be solved analytically. Assume

$$\omega(t) = \begin{cases} \omega_0(1+Q), & 0 < t < a = T/M, \\ \omega_0, & a < t < T, \end{cases} \quad (76)$$

$b = T - a$, and $\omega(t+T) = \omega(t)$. It can be shown that:

$$\begin{aligned} \cos \mu T &= \cos[\omega_0(T+aQ)] \\ &\quad - [Q^2/2(1+Q)] \sin[\omega_0 a(1+Q)] \sin(\omega_0 b). \end{aligned} \quad (77)$$

As $Q \rightarrow 0$ the right-hand side becomes $\cos \omega_0 T$. Resonances are expected around $\omega_0 T = n\pi$. If $Q \neq 0$ the right-hand side might be greater than one (in absolute value) in some intervals. For these intervals μ will be imaginary and the orbital amplitude will grow indefinitely in time.

In order to find the locations and widths of these intervals define

$$X = \omega_0(T+aQ). \quad (78)$$

The right-hand side of Eq. (77) can be written as

$$\cos X(1+G \sin^2 \omega_0 b) - (G/2) \sin 2\omega_0 b \sin X, \quad (79)$$

where

$$G = Q^2/[2(1+Q)].$$

If $|G| \ll 1$ the right-hand side can be ± 1 only when $\cos X \simeq \pm 1$ or $X \simeq n\pi$, or

$$\omega_0^{(n)} = \frac{(n/2)\Omega_0}{(1+Qa/T)}. \quad (80)$$

Expanding the right-hand side around $\omega_0 = \omega_0^{(n)}$ we get

$$\begin{aligned} (-)^n [1 + G \sin^2 \omega_0^{(n)} b] \\ - (G/2) \sin 2\omega_0^{(n)} b (\delta X) - \frac{1}{2} (\delta X)^2. \end{aligned} \quad (81)$$

The maximum of this expression is obtained at

$$\delta X = -(G/2) \sin 2\omega_0^{(n)} b,$$

and is $1 + G \sin^2 \omega_0^{(n)} b$ to first order in G . The width of the n th unstable region is

$$\delta \omega_0^{(n)} \simeq \frac{\sqrt{8G}}{T} |\sin \omega_0^{(n)} b| \simeq \frac{2Q}{T} |\sin \omega_0^{(n)} b|. \quad (82)$$

For small n resonances ($n \ll M$)

$$\delta \omega_0^{(n)} \simeq Q(n/M)\Omega_0, \quad (83)$$

and the e -folding time t_n at the maximum growth rate is

$$t_n \simeq \frac{1}{\pi Q} \cdot \frac{M}{n} \cdot T. \quad (84)$$

In the high-current betatron ω_0 changes in time, due to the change in γ or B_z . At the beginning B_z is held constant (~ 10 kG) and γ rises. In our standard example $n\gamma \simeq 670$ and, unless γ is very high, $n \gg 1$. For $n \gtrsim M$ the number of turns that an electron executes while it crosses the n th resonance is $\leq 10^4 (Q/n^2)$. Thus until n is very low this number is much smaller than one and no real growth is expected. As n becomes of the order of one ($n \ll M$) the number of turns the electron stays in the n th resonance is approximately $3 \times 10^4 (Q/nM)$. Assuming $Q = 10^{-2}$, $M = 10$, and $n = 1$, this number is approximately 30. The amplitude growth will be less than 10%. Such growth in the emittance is reasonable [compare to Eq. (68)].

As we pointed out earlier, the missing of the half integer resonances in the last case was due to the high symmetry of the problem. In order to see the general nature of the motion as B_z is perturbed assume that $\partial \delta B_x / \partial x = 0$. Thus $\partial \delta B_y / \partial y = -\partial \delta B_z / \partial z$, and the equations of motion become

$$\begin{aligned} \ddot{x} + (\Omega_z \dot{y}) &= 0, \\ \ddot{y} - \Omega_z \dot{x} &= 0. \end{aligned} \quad (85)$$

Integrating the first equation and substituting in the second we get

$$\begin{aligned} \dot{x} + \Omega_z y &= \dot{x}^{(0)} + (\Omega_z y)^{(0)} = A, \\ \ddot{y} + \Omega_z^2 y &= A \Omega_z = A(\hat{\Omega}_z + \delta \Omega_z), \end{aligned} \quad (86)$$

where $\hat{\Omega}_z$ is the average along the torus and A is a constant which depends upon the initial conditions. The second equation is a nonhomogeneous Hill's equation and it has integral and half-integral resonances. The symmetry in this case is obviously lower than in the previous case.

Let us now consider the most general perturbations in the fields for a high-current betatron. These fields may be expanded around the average equilibrium radius $R = v_z / \Omega_0$. Using Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$, and keeping terms only up to first order in the equations of motion, we get:

$$\begin{aligned} \ddot{x} + \Omega_z \dot{y} + \left(\frac{1}{2} \Omega_z^2 - \Omega^2 \right) x \\ = \Omega_z^2 R \frac{\delta B_0}{B_0} + \Omega_z^2 \left(\frac{\delta B_0}{2B_0} + \delta s \right) x \\ - \Omega_z \left(\frac{\delta B_z}{B_z} \dot{y} + (1-\alpha) \frac{\delta \dot{B}_z}{B_z} y \right) \equiv f(t), \end{aligned}$$

$$\begin{aligned} \ddot{y} - \Omega_z \dot{x} + \left(\frac{1}{2} \Omega_y^2 - \Omega^2 \right) y \\ = -\Omega_y^2 R \frac{\delta B_x}{B_0} - \Omega_y^2 \left(\frac{\delta B_0}{2B_0} + \delta s \right) y \\ + \Omega_z \left(\frac{\delta B_z}{B_z} \dot{x} + \alpha \frac{\delta \dot{B}_z}{B_z} x \right) \equiv g(t), \end{aligned} \quad (87)$$

where we assumed $s = 1/2$, and:

$$\begin{aligned} B_z &= \hat{B}_z + \delta B_z(z), \\ B_y &= B_0 \left(1 + \frac{sx}{R} \right) + \delta B_0(z) + \left(\frac{\delta B_0(z)}{B_0} + \frac{\delta s(z)}{s} \right) B_0 \frac{sx}{R} \\ &\quad + [\alpha(z) - 1] \frac{\partial \delta B_z}{\partial z} y, \\ B_x &= B_0 \frac{sy}{R} + \delta B_x(z) + \left(\frac{\delta B_0}{B_0} + \frac{\delta s}{s} \right) B_0 \frac{sy}{R} \\ &\quad - \alpha(z) \frac{\partial \delta B_z}{\partial z} x. \end{aligned} \quad (88)$$

$\alpha(z)$ is defined by:

$$\frac{\partial \delta B_x}{\partial x} = -\alpha(z) \frac{\partial \delta B_z}{\partial z}.$$

We wrote the equations of motion so that all the perturbation appears on the right-hand side. However, these terms are not of the same order. The dominant terms are the first. These terms distort the equilibrium orbit and are the most destructive. If $\delta B_0/B_0$ and $\delta B_x/B_0$ are given, their contribution can be calculated using Eq. (66).

The first terms result in resonances for $\omega^+ = n\Omega_0$ ($\Omega_z > \Omega_y$). These resonances were considered in detail [see Eqs. (55)–(68)]. The only difference is that here $B_x(z)$ is perturbed also.

The growth rate at the resonance is determined mainly by the larger of δB_x and δB_y . Thus to minimize their effect on the shape of the beam center both δB_x and δB_y should be decreased.

The second largest terms on the right-hand sides of Eqs. (87) for $\Omega_z > \Omega_y$ are the Ω_z terms. We have seen earlier [see Eqs. (69)–(84)] that for $\alpha = 1/2$ these terms give rise to parametric instabilities with $(\Omega_z^2 + 2\Omega_y^2 - 4\Omega^2)^{1/2} = n\Omega_0$. For $\alpha \neq 1/2$, new resonances appear ($|\omega^\pm| = \frac{1}{2}n\Omega_0$, $\Omega_z = n\Omega_0$). These new modes disappear as $\alpha \rightarrow \frac{1}{2}$. As $\omega^- \rightarrow 0$ and $\omega^+ \rightarrow \Omega_z$ (only B_z is left) we recover the $\Omega_z = (n/2)\Omega_0$ resonances [see Eqs. (85) and (86)].

The other terms (proportional to $\delta B_0/2B_0 + \delta s$) give rise to resonances at $\omega = n\Omega_0$.

All the instabilities we have just considered start growing linearly. Comparing the terms of the right-hand sides of Eqs. (87) it is obvious that the first terms are greater by approximately R/r_b than the second. This makes the first terms much more destructive. The remaining Ω_z terms were investigated carefully and it was found that they are at least one order of magnitude less dangerous than the first terms.

VII. DISCUSSION

In this paper we investigated some aspects of the stability of a high-current betatron. Specifically, we discussed (1)

space charge limits of both single particles and of the electron ring, (2) the Fermi drift, (3) resonance instabilities, and (4) beam emittance.

Two important questions have not been addressed, namely (i) stability during initial acceleration (injection and trapping and initiating acceleration will be discussed separately), (ii) collective modes, such as that of negative mass and resistive walls.¹³ These modes have been discussed extensively by many authors, yet there is no conclusive evidence that they might cause destructive damage to the high current betatron.

Let us summarize the main results:

(1) The space charge limit during injection is increased compared to a conventional betatron by $(B_z^2/2B_y^2)$.

(2) Single particle orbits become unstable (for $s \neq \frac{1}{2}$) at $\gamma^3 \approx 4NR^2 r_c/r_b^2$. This instability, however, is self-stabilized resulting in only a small increase of the beam radius.

(3) The stability of the beam is influenced by the surrounding walls. Ideal conducting walls enhance the stability of the beam relative to the single particle. In the opposite case, when the image currents decay instantaneously the beam may become unstable due to the space charge limit as γ increases, even for the $s = \frac{1}{2}$ case. This occurs only if the line density is high enough.

(4) For $s \neq \frac{1}{2}$ the ring becomes unstable at $\gamma \approx 4N(R/a)^2 r_c$ [see Eq. (21)]. In this region the total focusing force on the beam becomes zero and various phenomena occur which make it practically impossible to hold the beam in the torus.

(5) The Fermi drift is negligible except in the region where the total focusing forces on the beam become zero (this is true even for $s = \frac{1}{2}$). Then the beam drifts quickly to the wall. This drift may be utilized to extract the beam.

(6) The beam emittance changes only slightly during acceleration (neglecting instabilities). It is, thus, determined by the injection method and by the enhancement of instabilities.

(7) Errors in the external fields may cause driven as well as parametric resonances. The first are caused by errors in $B_y(z)$ or $B_x(z)$ which are not zero at the equilibrium orbit ("monopole-like"). These are the most dangerous resonances and put severe restrictions on those fields unless the beam is extracted before they develop ($n > 1$). Parametric resonances ("dipole-like") caused by errors in $B_z(z)$, $B_0(z)$, or $s(z)$ are much less destructive. Higher-order resonances are expected to be negligible.

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