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Thick Market Externality and Concentration of 'Money'*

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Abstract

A thick market external effect is applied to a trading post model of $N \geq 3$ commodities with transaction costs and distinct bid and ask prices. An existence theorem for general equilibrium with external effects in the trading post model is stated and proved. Media of exchange occur endogenously as liquid commodities, characterized by a narrow bid/ask price spread. The thick market externality can lead to concentration of the endogenously determined media of exchange towards an equilibrium with a single medium. In a class of examples, we show that if the households have sufficiently heterogeneous tastes relative to the size of the economy, the monetary equilibrium leads to higher consumption than the barter equilibrium.

Keywords: Arrow-Debreu, barter, bid / ask spread, commodity money, convex, general equilibrium, fiat money, Kakutani Fixed Point Theorem, upper hemicontinuity, lower hemicontinuity, thick market externality, trading post, transaction cost

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1 Why is there money?

Can we derive a theory of money as an outcome of price theory? This question has been a puzzle for general equilibrium theorists for half a century, Kareken & Wallace (1980). Consider four commonplace observations on the character of trade in virtually all modern economies:

(i) Trade is monetary. One side of almost all transactions is the economy's common medium of exchange.

(ii) Money is (locally) unique. Though each economy has a 'money' and the 'money' differs among economies, almost all the transactions in most places most of the time use a single common medium of exchange, or instruments denominated in that medium.

(iii) 'Money' is government-issued fiat money, trading at a positive value though it conveys directly no utility or production.

(iv) Transactions displaying a double coincidence of wants are transacted with money not barter (Examples are presented in section 3, below).

Where economic behavior displays such uniformity, a general fundamental economic theory should be able to account for the universal usages. The most powerful model of the economy, taking account of interactions among goods and exchange (for which money is essential in actual economies), is the Arrow-Debreu general equilibrium. But the Arrow & Debreu (1954) model cannot accommodate money, and observations (i), (ii), and (iii). Point (iv) contradicts the conventional view of the role of money; that it is used to overcome the absence of a double coincidence of wants.

Hahn (1982) writes

> "The...challenge that...money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu."

Debreu (1959) concurs,

> "[an] important ... and difficult question [is] not answered by the approach taken here: the integration of money in the theory of value."

Tobin (1980) contends that general equilibrium theory cannot resolve these issues.

> "Social institutions like money are public goods. Models of general equilibrium, competitive markets and individual optimizing agents, are not well adapted to explaining the existence and quantity of public goods... The use of a particular language or a particular money by one individual increases its value to other actual or potential users. Increasing returns to scale, in this sense, limits the number of languages or moneys in a society and indeed explains the tendency for one basic language or money to monopolize the field. "

Clower (1965) observes, correctly, that

> "money buys goods and goods buy money, but goods do not buy goods."

For Clower, this proposition is an assumption. This essay derives it as a conclusion — and contrary to Tobin's skepticism — from the general equilibrium theory.

We will reconcile these issues in an extended Arrow-Debreu general equilibrium model. $N \geq 3$ commodities trade at $\frac{1}{2}N(N-1)$ commodity pairwise trading posts, each with a budget constraint. Commodity money and (alternatively) fiat money are carriers of purchasing power, on behalf of households, between trading posts. A theory of exchange focuses on transaction costs embodied in a bid/ask spread; goods acquired at one post and disbursed at another are commodity money. A government-issued fiduciary instrument, similarly transacted, is fiat money. We formalize an external effect, a thick markets externality; the most frequently traded instruments have low endogenously determined transaction costs, Rey (2001). That leads to concentration on a single medium (or small number of media) of exchange. Thus, uniqueness of money, the medium of exchange, is a result in equilibrium of an external effect, comparable to Tobin (1980)'s characterization as a public good. Government, as a large economic actor creates high trading volume in its fiat instrument. The resulting endogenously determined low transaction cost assures that its high volume instrument will be the whole economy's medium of exchange.

This paper is organized as follows: Section 2 discusses the theory of money and its interaction with price theory and general equilibrium theory. This issue was very actively discussed in Kareken & Wallace (1980). It is particularly pleasant to have James Tobin's insightful and skeptical comments there. Menger (1892) emphasizes — as does this paper — transaction cost. Section 3 discusses transactions where there is a double coincidence of wants, like Adam Smith's example of an ill-clad baker and a hungry tailor. Section 4 discusses the literature on government's role in maintaining the value of fiat money. Section 5 sets out the trading post general equilibrium model, Section 6 discusses the key concept of Bid/Ask spread in commodity prices. Section 7 revisits the general equilibrium result on a trading post model. Section 8 discusses the role of firms. Section 9 discusses the role of external effects. Section 10 discusses attainable transactions. Section 11 discusses households' behavior. Section 12 presents the general equilibrium result under general conditions. Section 13 extends the general equilibrium model to the case of external effects, It develops sufficient conditions for existence of equilibrium with externalities following the approach of Arrow & Hahn (1971). Section 14 presents examples of barter, monetary, and fiat money equilibria. Section 15 discusses approaches alternative to general equilibrium and Section 16 briefly concludes.

2 Money and General Equilibrium Theory

Tobin (1980) has an extended discussion that sets out the relevant issues, from a very analytic viewpoint, with an engaging impatience with pure mathematical theory. We are tempted to quote at length.

> Why does fiat money, consisting of intrinsically worthless tokens, have positive value? What determines its value? These are classical questions of monetary theory... Starting from the presumption that fiat money should have neither value nor real consequence but confronting the fact that it does, some theorists have been grasping for straws.
>
> The traditional explanation of money is the division of labor, the daily recurring need to exchange specialized endowments or products for diversified consumption goods and services. Long, long ago our precursors pointed out that the use of a common medium of payment facilitates multilateral trade among members of an economy. Barter, in contrast, would restrict transactions to "double coincidences of wants," Jevons' famous phrase... I must say in all irreverent candor that as yet I do not feel significantly better enlightened than by the traditional insight.
>
> Indeed, there is, as the language analogy suggests, arbitrariness and circularity in acceptability. Dollar bills and coins are acceptable because they are acceptable; of course, the state has a lot to do with making them acceptable, by defining them as acceptable for settlement of private contracts and for tax payments... Credible promises to pay those dollars, or to convey other such promises, also serve as generally acceptable media, or as widely acceptable media... General equilibrium theory is not going to explain the institution of a monetary unit of account associated with a basic common means of payment.
>
> If you are stranded in a strange town, it is unlikely that the taxi driver or innkeeper happens to want a lecture or offprint on general equilibrium. It is also unlikely that you previously had or seized the opportunity to contract for the delivery of their services in the precise contingency... Failure of the conditions necessary for Arrow-Debreu equilibrium is one way to describe the reasons societies adopt, use, and value money — a contorted and contrived way, to be sure, but one that comes naturally to economic theorists. So a monetary economy will not achieve such an equilibrium. Neither will barter, given the costs of commodity exchange markets and bilateral transactions. A monetary economy reaches a different second best, presumably a better second best, than a barter economy.

Separately, Tobin & Golub (1997) comment

> Why are some assets selected by a society as generally acceptable media of exchange while others are not? This is not an easy question because the selection is self-justifying.

> If my creditors will take marbles in settlement of my debts to them, why should not I in turn take marbles from my debtors? . . . Anthropology testifies to the variety of assets that various peoples around the world through the ages have chosen as media of exchange.
>
> general acceptability in exchange is one of those phenomena — like language, rules of the road, fashion in dress — where the fact of social consensus is much more important and much more predictable than the content.

Our treatment will respond to Hahn's and Tobin's comments. In response to Hahn, we will create an Arrow-Debreu style general equilibrium model where the existence and designation of 'money' is well defined and endogenous. We hope thereby to refute Tobin's skepticism on the power of general equilibrium theory to explain the existence and function of a medium of exchange. That's Theorem 2 below. Indeed, any economic model that endogenously generates a medium of exchange is necessarily *general* equilibrium; at least three commodities are needed. We hope to demonstrate Tobin's suggestion that monetary exchange leads to a more efficient equilibrium allocation than barter. Those are the examples of Section 14.

The Tobin and Golub notion of "self-justifying" designation of the monetary instrument is formalized in the treatment below .

Our view is to emphasize price theory as explaining the designation of 'money,' where the relevant price is the price of liquidity, essentially a bid/ask spread on the medium of exchange. In this we agree with Menger (1892),

> A commodity should be given up by its owner in exchange for another more useful to him. But ... exchange ... for little metal disks apparently useless as such, or for documents representing the latter, is ... mysterious.
>
> Goods [are] ...more or less saleable [*absatzfahig* marketable], according to the greater or less facility with which they can be disposed of ... at current purchasing prices, or with less or more diminution.
>
> The theory of money necessarily presupposes a theory of the saleableness [*Absatzfahigkeit*, marketability] of goods.
>
> when any one has brought goods not highly saleable to market, the idea uppermost in his mind is to exchange them, not only for such as he happens to be in need of, but ... for other goods ... more saleable than his own .... By ... a mediate exchange, he gains the prospect of accomplishing his purpose more surely and economically than if he had confined himself to direct exchange .... Men have been led ... without convention, without legal compulsion,... to exchange ... their wares ... for other goods ... more saleable ... which ... have ... become generally acceptable media of exchange.
>
> legislation...is neither the only, nor the primary mode in which money has taken its origin...Money has not been generated by law. In its origin it is a social, and not a state institution.

A common shortcut in modeling a monetary economy is to put money in the utility function. That is assuming the conclusion, that money is useful and valued. Wallace (1980) comments "fiat money is never wanted for its own sake; it is not legitimate to take fiat money to be an argument of anyone's utility function or of any engineering production function." Tobin (1980) concurs that money stocks do not belong in a utility function. "Clearly enough, the value of paper money does not derive from the beauty of the engravings; the practice of putting money stocks in utility functions is reprehensible..."

Properties (i) - (iv) of Section 1 can be derived as results based on a notion of large scale economies generating natural monopoly, Starr (2003). But that treatment, with its monopolistic structure does not allow for a competitive general equilibrium. Hahn (1971) recognized this issue but did not resolve it,

> But I shall argue here that at least the pure transaction theory of money seems to require in an essential way that the set of marketing activities exhibits a certain kind of increasing returns. I have not been able to formulate this to my full satisfaction. But if the argument is correct, difficulties present themselves in the study of equilibrium.

Hahn's concern is that competitive general equilibrium is most directly modeled assuming convexity of technology —- no increasing returns at the firm level— but that he notes that there is a scale economy at the level of the economy as a whole. This essay formalizes Hahn's concept as a thick market externality. The notion is that there is an external effect; transactions are expedited by increased liquidity from concentration of economic activity in a common medium of exchange. The thick market externality is consistent with maintaining convexity at the firm level. The scale economy is external to a firm's transaction technology. Then a price-taking general competitive equilibrium can sustain the observations.

This essay will examine these issues in the following way. Each transaction incurs transaction cost, and must fulfil a *quid pro quo* budget constraint. Media of exchange , carriers of value in exchanges, allow equal values to be exchanged at each trade. Other commodities could act as carriers of value between exchanges; 'money' (commodity money or fiat money) does so at low transaction cost. The low transaction cost comes not from intrinsic properties of the monetary instrument, but from the high volume of monetary trade, a thick markets externality. Of course this is a near-circular argument, but economists are familiar with this class of issues as a fixed point. Government's role is to give value to its fiat currency by accepting it in payment of taxes and by using it in government expenditures.

Rey (2001) develops a model of transaction cost focusing on trading volume. High-volume markets have lower marginal transaction costs than low-volume markets. Rey (2001) denotes this as a 'thick markets externality.' Rey (2001) focuses on foreign exchange markets. There is an external economy; increasing trading activity in a (currency pairwise) market for some agents reduces marginal costs for others. We adapt this notion to commodity pairwise exchanges. That is the scale economy that Hahn (1971) noted, but there is a useful technicality here. Because the scale economy is external to each individual firm it is consistent with convex technology at the firm level. That facilitates an Arrow-Debreu general equilibrium approach.

3 Double Coincidence of Wants

In Tobin's remarks above, he notes that monetary exchange is used because double coincidence of wants rarely happens. He is agreeing with Jevons (1875). Kiyotaki & Wright (1993) concur, modeling the resolution of traders meeting with a double coincidence of wants as fulfilling their complementary demands by a barter exchange. The implication is that when double coincidence of wants occurs, it should result in direct barter. Double coincidence is unusual, but there are many instances where double coincidence occurs in actual economies. They do not result in a barter trade.

University of California faculty often send their children to college at a UC campus. The faculty member's employer is also the seller of the educational service they acquire. Faculty and the University could barter lecturing and instruction time for the students' tuition. That's not how it happens. The faculty pay their children's tuition fees in money, not in kind.

Ford Motor Corp. employees could barter labor for a Ford car. That's not how it works. Auto-workers pay for their cars in money, not in kind.

When Albertson's supermarket employees buy food at the supermarket, they pay for their groceries in money, not by barter.

Common opportunities for barter in an otherwise monetary economy do not take place as barter transactions. This observation suggests that the focus on the absence of double coincidence of wants - as distinct from transaction costs - as an explanation for the monetization of trade may miss a significant part of the underlying causal mechanism. Barter trades with double coincidence necessarily take place in thin markets. The examples of Section 14 suggest that the thick market of monetary trade reduces transaction costs. Thin markets may lead to high transaction costs, a wide bid/ask spread. So the transactors who execute potential barter transactions in money, not by barter in kind, are being fully rational, despite the apparent convenience of barter in the double coincidence setting. The thin markets of double coincidence generate high transaction costs, resulting in the apparent anomaly of monetary trade even in the instance of reciprocally satisfactory demands.

4 Fiat Money

Over several centuries, there is a recurrent argument that government-issued fiat money, though apparently worthless paper, commands a positive price because it is backed by an undertaking by the issuing government to accept the paper for something valuable: fulfillment of the tax obligation, (Desan 2014).

Smith (1776) notes,

> A prince, who should enact that a certain proportion of his taxes should be paid in a paper money of a certain kind might thereby give a certain value to this paper money, even though the term of its final discharge and redemption should depend altogether upon the will of the prince." (Volume I, Book II, Chap. II).

Knapp (1905) writes

> We keep most closely to the facts if we take as our test, that the money is accepted in payments made to the State's offices....... On this basis it is not the issue, but the acceptation, as we call it, which is decisive. State acceptation delimits the monetary system. By the expression 'State-acceptation' is to be understood only the acceptance at State pay offices where the State is the recipient.

Lerner (1947) concurs

> The modern state can make anything it chooses generally acceptable as money and thus establish its value quite apart from any connection, even of the most formal kind, with gold or with backing of any kind. It is true that a simple declaration that such and such is money will not do, even if backed by the most convincing constitutional evidence of the state's absolute sovereignty. But if the state is willing to accept the proposed money in payment of taxes and other obligations to itself the trick is done. Everyone who has obligations to the state will be willing to accept the pieces of paper with which he can settle the obligations, and all other people will be willing to accept these pieces of paper because they know that the taxpayers, etc., will accept them in turn.

5 Trading Posts

The trading post model consists of N commodities traded pairwise at $\frac{1}{2}N(N-1)$ trading posts, each with a *quid pro quo* budget constraint. Acquistions are repaid by delivery of equal value. There are distinct bid and ask prices; the bid/ask spread reflects transaction costs. Walras (1874) forms the picture this way:

> we shall imagine that the place which serves as a market for the exchange of all the commodities... for one another is divided into as many sectors as there are pairs of commodities exchanged. We should then have $\frac{m(m-1)}{2}$ special markets each identified by a signboard indicating the names of the two commodities exchanged there as well as their ... rates of exchange...

The determination of which trading posts are active in equilibrium is endogenous and characterizes the monetary or barter character of trade. In an equilibrium where most of the $\frac{1}{2}N(N-1)$ trading posts are active, the equilibrium is barter. Conversely, when a household acquires a commodity at one trading post and disposes of it at another, the commodity is acting as commodity money. The equilibrium is monetary with a unique money if only N trading posts out of $\frac{1}{2}N(N-1)$ are active, those trading all goods against 'money,' Starr (2003, 2012). (Starr 2003, 2012).

6 Commodities, Prices, Bid/Ask Spread

Let $N =$ number of elementary commodities, $N \geq 3$, each of which may trade against any other, generating

$N(N-1)$ dimensions of activity at bid prices and another $N(N-1)$ at ask prices. There are $\frac{1}{2}N(N-1)$ trading posts with two goods traded at each. Further, commodities enter both in bid and ask price transactions (wholesale or retail transactions). In addition, commodities act as inputs to transaction costs.

Price vectors are q, π each $\in \mathbb{R}_{+}^{N(N-1)}$. Eventually, homogeneity of degree zero of supply and demand correspondences will be demonstrated, so that attention can be confined to the unit simplex in $\mathbb{R}_{+}^{2N(N-1)}$ for purposes of demonstrating existence of general equilibrium. For demonstrating examples it is more convenient to allow a more general specification. q represents the prevailing bid prices, $q+\pi$ is the vector of prevailing ask prices, π is the vector of the premium above bid prices of ask prices.

A typical co-ordinate entry will be denoted $x(k,l), 1 \leq k, l \leq N, k \neq l$, representing commodity k at the trading post of k for l. This is the same trading post as for l for k. $x^S(k,l)$ represents good k traded for l at bid price. $x^B(k,l)$ represents good k traded for l at ask price. The bid price of (k,l) is $q(k,l)$; the ask price is $q(k,l)+\pi(k,l)$. $\pi(k,l)$ is the ask premium or retail premium. Purchases are positive co-ordinates, sales are negative. Much of this notation follows Foley (1970).

Let:

$$\begin{aligned} B(p) \equiv \{ x \in R^{2N(N-1)} | \; & \forall k, l \in \{1 \leq k, l \leq N, k \neq l\} \\ & q(k,l)x^S(k,l) + (q(k,l) + \pi(k,l))x^B(k,l) \\ & + q(l,k)x^S(l,k) + (q(l,k) + \pi(l,k))x^B(l,k) \leq 0 \} \end{aligned}$$

$B(p)$ is the trading budget presenting the collection of transactions priced for *quid pro quo* at each trading post. Budgets are balanced at each trading post separately. Purchases enter with positive signs, sales with negative signs. Then the budget constraint is that sales must be (weakly) more valuable than purchases. The budget constraint in this model distinguishes it from the Arrow-Debreu model. Each agent in this setting faces $\frac{1}{2}N(N-1)$ budget constraints, one at each trading post.

H is the finite population of households. For each $i \in H$, let $x^i \in R^{2N(N-1)}$ be i's trade vector. Household i has an endowment $r^i \in R^{2N(N-1)}$. The dimension of r^i is set at $2N(N-1)$ for notational consistency. This usage is discussed more completely below. Household i has utility function u^i . Households sell at bid prices, q, and buy at ask prices, $q+\pi$.

7 General Equilibrium in a Trading Post Model

We consider here a trading post model of $N \geq 3$ commodities and transaction costs. That treatment generates $\frac{1}{2}N(N-1)$ separate budget constraints with distinct bid and ask prices. General equilibrium, market-clearing prices and transactions at each trading post, exists under conventional continuity and convexity conditions. Commodities acquired by an agent at one trading post and disbursed at another constitute commodity money.

In Arrow & Debreu (1954) each agent's budget is a single equation: the value of sales minus the value of purchases is a firm's profit; the value of endowment plus the value of dividends is the limit of a household's purchases. That is a powerful simplification. The alternative here is to recognize a separate budget at each of many distinct transactions. That treatment generates a need for a carrier of value between transactions. Hence a model of commodity money.

In Arrow & Hahn (1971), section 6.2, extends the Arrow-Debreu analysis to the case of external effects, the generalization we make use of in the present paper.

The trading post model consists of N commodities traded pairwise at $\frac{1}{2}N(N-1)$ trading posts, each with a *quid pro quo* budget constraint. Acquistions are repaid by delivery of equal value. There are distinct bid and ask prices; the bid/ask spread reflects transaction costs.

Theorem 1 demonstrates existence of general equilibrium with familiar continuity and convexity assumptions, adapted to this much larger price and action space.

7.1 Money

Commodity moneys occur endogenously in the market equilibrium reflecting the constraints of *quid pro quo* embodied in $B(p)$.

In a competitive general equilibrium, let $x^{iS}(k,\ell) < 0$, $x^{iB}(k,m) > 0$, for some i, k, ℓ, m. That is, household i both buys and sells good k in exchange for two different goods. Then k is a medium of exchange, a commodity money. If k is a government-issued instrument without backing, then k is fiat money.

Recall that $x^{iS}(k,l) \leq 0$ is i's sales of k for l. $x^{iB}(k,l) \geq 0$ is i's purchases of k for l. Then $\sum_{l\neq k} -x^{iS}(k,l) \ + \sum_{l\neq k} x^{iB}(k,l) \geq 0$ is i's gross trade in k. $\sum_{l\neq k} x^{iS}(k,l) \ + \sum_{l\neq k} x^{iB}(k,l)$ is i's net trade in k. Then $\sum_{l\neq k} -x^{iS}(k,l) \ + \sum_{l\neq k} x^{iB}(k,l) - |[\sum_{l\neq k} x^{iS}(k,l) \ + \sum_{l\neq k} x^{iB}(k,l)]|$ is the volume of transactions in good k money, flow of good k as medium of exchange, gross transactions minus net transactions.

The price system will designate one or several commodity moneys typically as the low transaction cost instrument(s).

8 Firms

There is a finite population of firms $j \in F$. Firms deal in the trading process, buying and selling, incurring transaction costs in commodities. There is no production in the model. Firm j formulates a transaction plan $(y^{jS}, y^{jB}, w^j) \in R^{3N(N-1)}$. Positive co-ordinates of y^{jB}, y^{jS} indicate purchases. Negative co-ordinates indicate sales. Negative co-ordinates in w indicate inputs to the trading technology, transaction costs. y^{jS} is the vector of transactions, purchases and sales, the firm makes at bid (wholesale) prices. y^{jB} is the vector of purchases and sales subject to the premium buying (retail) price. Note that in contrast to the households, for the firm, both y^{jS} and y^{jB} may have both positive and negative co-ordinates. The firm's ability or inability to deal in positive or negative actions at bid or ask prices is formalized in its technology Y^j. The budget constraint on firm transactions is for each two commodities $k, \ell, = 1, 2, \ldots, N$,

$$\begin{aligned} &q(k,\ell) \cdot y^{jS}(k,\ell) + (q(k,\ell) + \pi(k,\ell)) \cdot y^{jB}(k,\ell) \\ &+ (q(\ell,k) + \pi(\ell,k)) \cdot y^{jB}(\ell,k) + q(\ell,k) \cdot y^{jS}(\ell,k) \leq 0 \quad (\mathcal{B}') \end{aligned}$$

Equivalently:

$$\begin{aligned} (y^{jS}, y^{jB}) \in B(p) \equiv \{x \in R^{2N(N-1)} | &q(k,l)x^S(k,l) + (q(k,l) + \pi(k,l))x^B(k,l) \\ &+ q(l,k)x^S(l,k) + (q(l,k) + \pi(l,k))x^B(l,k) \leq 0, \\ &\text{for all } 1 \leq k, l \leq N, k \neq l\} \end{aligned}$$

A suitable maximand for the firm needs to be defined. It is simplest to ask the firm to maximize profit, $(q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j)$. This applies though the firm cannot easily distribute the accounting value of profit to shareholders.

The technically possible mix (y^{jS}, y^{jB}, w^j) of purchases, inputs, and sales of firm j is contained in the closed convex set $Y^j \subseteq R^{3N(N-1)}$. Firm j's supply decision then is:

$$\begin{aligned} S^{j\dagger}(p) \equiv \{(y^S, y^B, w) | &(y^S, y^B, w) = \arg\max \ (q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j) \\ &\text{subject to } \ (y^S, y^B, w) \in (Y^j \cap [B(p) \times \mathbb{R}^{N(N-1)}])\} \end{aligned}$$

Firm j's marketed supply behavior then is:

$$\begin{aligned} S^j(p) \equiv \{(y^S, y^B) | &(y^S, y^B, w) = \arg\max \ (q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j) \\ &\text{subject to } \ (y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}])\} \end{aligned}$$

There is a finite set of households, H. For each $i \in H$, let $x^i \in \mathbb{R}^{2N(N-1)}$ represent i's transaction offers. Define $\hat{x} = (x^1, x^2, ..., x^{\#H}) \in R^{\#H 2N(N-1)}$. Household i owns a proportion Θ^{ij} of firm j (Foley notation), $1 \geq \Theta^{ij} \geq 0$, $\sum_{i \in H} \Theta^{ij} = 1$. The distribution of firms' earnings to households cannot easily be summarized as a money dividend, so it will enter as a commodity dividend distribution from firms to households, $\Theta^{ij}(y^{jS}, y^{jB} + w^j)$.

9 External Effects

The concept of a thick market externality, Rey (2001), leads to concentration on a medium of exchange. The economic notion is that thick markets, very active trading posts, will be low transaction cost posts. The next step for formal modeling is to generalize the trading post model to include external effects. The concept is to allow trading offers of households and other firms to affect the transactions technology of a trading firm.

The treatment of external effects here follows Arrow & Hahn (1971). The issue is to provide a theoretical foundation for allowing a thick markets externality. The trading activity of household i is represented as $x^i \in R^{2N(N-1)}$. The complex of activities by all households $i \in H$, is represented as $\hat{x} \in R^{(\#H)2N(N-1)}$. The trading activity of firm j is represented as $y^j \in R^{2N(N-1)}$. The complex of trading activity by all firms $j \in F$, is represented as $\hat{y} \in R^{(\#F)2N(N-1)}$. Then the external effect is represented as

$Y^j \equiv \varphi^j(\hat{x}, \hat{y})$, a point-to-set mapping.

$\varphi^j : R^{(\#H+\#F)2N(N-1)} \to R^{3N(N-1)}$; $\varphi^j(\hat{x}, \hat{y}) \subseteq R^{3N(N-1)}$

There is a no-self-referential-effect assumption. We posit the following assumptions E.I to E.V to formalize firm technology and external effects here, very much in the fashion of Arrow & Hahn (1971).

(E.I) Let $\hat{y} \in R^{(\#F)2N(N-1)}$ and $\hat{y}' \in R^{(\#F)2N(N-1)}$ so that $\hat{y}$ and $\hat{y}'$ differ only in the jth co-ordinates. Then $\varphi^j(\hat{x}, \hat{y}) = \varphi^j(\hat{x}, \hat{y}')$. (E.I) says that trading offer volumes of households and other firms may affect the trading technology of firm j but that its own trading activities do not affect its trading technology.

(E.II) For all $(\hat{x}, \hat{y}) \in R^{(\#H+\#F)2N(N-1)}$, $Y^j \equiv \varphi^j(\hat{x}, \hat{y})$ fulfills

(P.I) Y^j is convex for all$j$.

(P.II) $0 \in Y^j$ for each j.

(P.III) Y^j is a closed convex cone for all j.

(P.IV) No Free Lunch (No Free Transaction) Let $(y^S, y^B, w) \in Y^j$ then

(i)If $(y^S, y^B, w) \neq 0$ then $w \neq 0, w \leq 0$ co-ordinatewise.

(ii) for each $k = 1, 2, \ldots, N, \sum_\ell y^S(k,\ell) + \sum_\ell y^B(k,\ell) + \sum_\ell w(k,\ell) \geq 0$.

.

(E.III) For all $(\hat{x}, \hat{y}) \in R^{(\#H+\#F)2N(N-1)}$, $\varphi^j(\hat{x}, \hat{y})$ is continuous, upper and lower hemicontinuous.

Let $Y^{j**} \equiv \{(y^B, y^S, w) \in \varphi^j(\hat{x}, \hat{y}) | (\hat{x}, \hat{y}) \in R^{(\#H+\#F)2N(N-1)}\}$. That is, Y^{j**} is the array of all conceivable technically possible trading plans for firm j considering the full range of conceivable external effects interacting to create technically possible trading plans. Let Y^{j*} be the closed convex hull of Y^{j**}. Then introduce assumption (E.IV)

(E.IV) Let $(y^B, y^S, w) \in Y^{j*}$, $(y^B, y^S, w) \neq 0$. Then $w \neq 0, w \leq 0$. The inequality applies co-ordinatewise.

(E.V) For each $j \in F$ let $(y^{jB}, y^{jS}, w^j) \in Y^{j*}$ so that $\sum_{j\in F}(y^{jB}, y^{jS}, w^j) = 0$ where 0 is the zero vector. Then for each $j \in F$, $(y^{jB}, y^{jS}, w^j) = 0$.

Assumption (E.IV) extends the No Free Lunch (No Free Transaction) assumption to a much grander space: the closed convex hull of all conceivable technically possible transactions for firm j. And (E.V) says that irreversibility holds in Y^j and in the limit as we take the closed convex hull of Y^{j**}.

We now define the supply correspondence $S^j(p, \hat{x}, \hat{y})$ in terms of prices and external effects:

$$S^j(p, \hat{x}, \hat{y}) \equiv \{(y^S, y^B)|(y^S, y^B, w) = \arg\max(q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j)$$
$$\text{subject to } (y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}]), Y^j = \varphi^j(\hat{x}, \hat{y})\}$$

This definition of S^j differs from the previous definition by taking account of the external effect. Households' and other firms' trading activities enter into firm j's technology, $Y^j = \varphi^j(\hat{x}, \hat{y})$. Thus the supply behavior of firm j subject to the external effects of $(\hat{x}, \hat{y})$ is optimizing subject to choosing a plan consistent with budget balance and the technology $\varphi^j(\hat{x}, \hat{y})$.

10 Attainable Transactions

The aggregate trading technology is $Y \equiv \sum\limits_{j\in F} Y^j$, where $Y^j = \varphi^j(\hat{x}, \hat{y})$. The economy's initial resource vector is

$r = \sum\limits_{i\in H} r^i \in R_+^{2N(N-1)}$. Then $(y^S, y^B, w) \in Y$ is said to be attainable if for each $k = 1, 2, \ldots, N$, we have:

$$\sum_{\ell} [y^S(k, \ell) + y^B(k, \ell)] \leq \sum_{\ell} r(k, \ell), \text{and}$$
$$\sum_{\ell} [w(k, \ell)] \geq -\sum_{\ell} r(k, \ell)$$

$(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ is said to be attainable in $Y^{j\prime}$ if there is $(y^{jS}, y^{jB}, w^j) \in Y^j$ for all $j \in F, j \neq j\prime$, so that $(y^{j\prime S}, y^{j\prime B}, w^{j\prime B}) + \sum\limits_{j\in F, j\neq j\prime} (y^{jS}, y^{jB}, w^j)$ is attainable.

$(y^{jS}, y^{jB}) \in S^j(p)$ is said to be attainable if there is $(y^{jS}, y^{jB}, w^j) \in Y^j$ so that (y^{jS}, y^{jB}, w^j) is attainable in Y^j.

Lemma 1. *Assume (E.I) through (E.V). Let:*

$$\Xi^j \equiv \{(y^{jB}, y^{jS}, w^j) \in Y^{j*}| \text{ For each } k = 1, 2, \ldots, N$$
$$\sum_{\ell} [w^j(k, \ell)] \geq -\sum_{\ell} r(k, \ell)\}$$

Then Ξ^j is compact.

Lemma 1 (above) demonstrates that the attainable set of transactions for the economy and for any single firm is bounded. This property is developed more completely below for the external effect model.

Proof Section 17.

Corollary Assume E.II (that is, P.I through P.IV) . Let $(\hat{x}, \hat{y})$ be (arbitrarily) specified, not necessarily attainable. Let $Y^j = \varphi^j(\hat{x}, \hat{y})$. Then the set of attainable elements $(y^S, y^B, w) \in Y$ is bounded. And for each $j\prime \in F$, the set of $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ is bounded.

Proof: Section 17

Choose $C > 0$, so that $C > |(y^{jB}, y^{jS}, w^j)|$ for all $(y^{jB}, y^{jS}, w^j) \in \Xi^j$ for all $j \in F$. Note the strict inequality. Let $\Psi \equiv \{x \in R^{N(N-1)} | |x| \leq C\}$. Thus, $\Psi \subseteq R^{N(N-1)}$ is a closed ball in $N(N-1)$ space of radius C centered at the origin.

Then $C > |(y^S, y^B, w)|$ (note the strict inequality) for all attainable $(y^S, y^B, w) \in Y$, and so that $C > |(y^{j\prime S}, y^{j\prime B}, w^{j\prime})|$ (note the strict inequality) for all $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ for all $j\prime \in F$. That is, there is a constant C so that all of the attainable points in Y and in any Y^j are strictly contained in a ball of radius C centered at the origin.

Let $\Psi^2 = \Psi \times \Psi, \quad \Psi^3 = \Psi^2 \times \Psi$. Let $\bar{\Psi}$ be a closed ball of radius $C(\#H + \#F)$ in $N(N-1)$ space. That is, $\bar{\Psi}$ is a closed ball of radius sufficiently large to encompass all #H-fold plus #F-fold sums taken from Ψ . Let $\bar{\Psi}^2$ be a closed ball of radius $C(\#H + \#F) \in 2N(N-1)$ space.

Let $Y^j \equiv \varphi^j(\hat{x}, \hat{y})$. Let $p = (q, \pi)$ be the vector of prices.

Define $\begin{array}{l} \tilde{S}^{j\dagger}(p, \hat{x}, \hat{y}) \equiv \{(y^S, y^B, w) | (y^S, y^B) = \arg\max\ (q, q+\pi) \cdot (y^{jS}, y^{jB} + w^j) \\ \text{subject to}\ (y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}] \cap \Psi^3), Y^j = \varphi^j(\hat{x}, \hat{y})\} \end{array}$

$\tilde{S}^{j\dagger}(p, \hat{x}, \hat{y})$ represents a profit maximizing decision for firm j based on prevailing external effects, $Y^j = \varphi^j(\hat{x}, \hat{y})$, and a restriction to a bounded set Ψ^3 that strictly includes all attainable points. The restriction to the bounded set Ψ^3 is not based on technology or prices; we expect to demonstrate that in a price-guided equilibrium, the restriction is not a binding constraint and can be deleted.

Lemma 2. *Let $p \in \mathbb{R}_+^{2N(N-1)}$. $B(p)$, $S^j(p, \hat{x}, \hat{y})$, and $S^{j\dagger}(p, \hat{x}, \hat{y})$ are homogeneous of degree zero in p.*

Proof: Section 17.

As a consequence of Lemma 2, it is sufficient to consider $p \in \Delta$ where Δ is the unit simplex in $\mathbb{R}_+^{2N(N-1)}$. Firm j's provisionally bounded supply decision then is:

$$\tilde{S}^j(p,\hat{x},\hat{y}) \equiv \{(y^S,y^B)|(y^S,y^B,w) = \arg\max(q,q+\pi)\cdot(y^{jS},y^{jB}+w^j)$$
$$\text{subject to } (y^S,y^B,w) \in (Y^j \cap [B(p)\times R^{N(N-1)}] \cap \Psi^3)\}$$

Lemma 3. *Assume E.I through E.V. Then* $\tilde{S}^{j\dagger}(p,\hat{x},\hat{y})$ *and* $\tilde{S}^j(p,\hat{x},\hat{y})$ *are nonempty, convex-valued, and upper hemicontinuous in* p *throughout* $p\in\Delta$. *Let* $(y^{jS},y^{jB}) \in \tilde{S}^j(p,\hat{x},\hat{y})$ *be attainable. Then* $(y^{jS},y^{jB}) \in S^j(p,\hat{x},\hat{y})$. *Let* $(y^{jS},y^{jB},w^j) \in \tilde{S}^{j\dagger}(p,\hat{x},\hat{y})$ *be attainable. Then* $(y^{jS},y^{jB},w^j) \in S^{j\dagger}(p,\hat{x},\hat{y})$.

Proof: Section 17.

Lemma 3 says that a firm's price-guided supply behavior, when it is attainable, does not treat the restriction to the bounded set Ψ^3 as a binding constraint. Under the prevailing convexity conditions, then the bounding restriction is redundant and can be deleted.

11 Households

There is a finite set of households H with typical element $i\in H$. Household i's possible consumption set is $W^i \subseteq R^{2N(N-1)}$. We can specify W^i more precisely. Define

$$W^i \equiv \{x' \in R^{2N(N-1)}| \sum_{l=1,k\neq l}^{N}(x'^B(k,l)+x'^S(k,l)) \geq 0, \text{ for each } k=1,2,\dots,N\}.$$

(The notation x' is intended to avoid confusion with household transaction offers.) Household i has an endowment $r^i \in R_+^{2N(N-1)}$. It is sufficient to characterize household preferences by a well-behaved continuous concave utility function u^i on $W^i \subseteq R^{2N(N-1)}$.

The following conditions on household trading and preferences are familiar in the general equilibrium theory, with adaptation to the current setting. They are intended to parallel their counterparts in Starr (2011). (C.I), (C.II), (C.III) are fulfilled by the definition of W^i above.

(C.I) W^i is closed and nonempty.
(C.II) $W^i \subseteq R^{2N(N-1)}$ is bounded below and unbounded above.
(C.III) W^i is convex.
(C.IV) (nonsatiation) Let $x\in W^i$. Then there is $x''\in W^i$ so that $u^i(x'') > u^i(x)$.
(C.V) (continuity) $u^i : W^i \to R$. u^i is well defined, and continuous.

(C.VI) (convexity of preferences) u^i is quasi-concave. That is, let there be $x, x' \in W^i$ so that $u^i(x') \geq u^i(x), 0 \leq \alpha \leq 1$. Then $u^i(\alpha x + (1-\alpha)x') \geq u^i(x)$.

(C.VII) (strict positivity of income and endowment) $r^i \in W^i$. r^i is strictly positive co-ordinatewise, $r^i >> 0$, where 0 is the zero vector in $R^{2N(N-1)}$.

Household i has a share Θ^{ij} of firm j . Firm j makes a distribution to shareholders $[(y^{jS}, y^{jB} + w^j)] \in R^{2N(N-1)}$ of which i receives $\Theta^{ij}[(y^{jS}, y^{jB} + w^j)]$ leading to a total of dividend distributions $\sum_{j\in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]$. i makes trades $x^i \in R^{2N(N-1)}$. $x^i = (x^{iS}, x^{iB})$. $x^{iB} \geq 0, x^{iB} \in R^{N(N-1)}$, is the vector of i's purchases. $x^{iS} \leq 0, x^{iS} \in R^{N(N-1)}$ is the vector of i's sales. The household sells at bid prices, and buys at ask prices. (Informally, it buys retail and sells wholesale.)

The budget constraint on household transactions is $x^i \in B(p)$. Let $\hat{y} = (y^{1S}, y^{1B}, w^1, \ldots, y^{jS}, y^{jB}, w^j, \ldots.) \in Y^1 \times Y^2 \times \ldots \times Y^{\#F}$. The household opportunity set is defined as:

$$A^i(p, \hat{y}) \equiv B(p) + \{r^i\} + \sum_{j\in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]$$

Demand behavior is given by:

$$\begin{aligned} D^i(p, \hat{y}) &= \{x \in B(p) | x = \arg\max u^i(x + r^i + \sum_j \Theta^{ij}(y^{jS}, y^{jB} + w^j)) \\ &\qquad \text{subject to } (x + r^i + \sum_j \Theta^{ij}(y^{jS}, y^{jB} + w^j)) \in W^i\} \\ &= \Big\{\{\arg\max u^i(x) \text{ for } x \in [A^i(p, \hat{y}) \cap W^i]\} \\ &\qquad - \{r^i + \sum_{j\in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]\}\Big\} \end{aligned}$$

The provisionally bounded household opportunity set is defined as

$$\tilde{A}^i(p, \hat{y}) \equiv \{x \in B(p) | \big[x + r^i + \sum_{j\in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]\big] \cap \Psi^2\}$$

Provisionally bounded household demand behavior is described as:

$$\begin{aligned} \tilde{D}^i(p, \hat{y}) &= \{x \in B(p) | x = \arg\max u^i(x + r^i + \sum_j \Theta^{ij}(y^{jS}, y^{jB} + w^j)) \\ &\qquad \text{subject to } [x + r^i + \sum_j \Theta^{ij}(y^{jS}, y^{jB} + w^j)] \in \Psi^2\} \\ &= \Big\{\{\arg\max u^i(x) \text{ for } x \in [\tilde{A}^i(p, \hat{y}) \cap W^i]\} \\ &\qquad - \{r^i + \sum_{j\in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]\}\Big\}. \end{aligned}$$

Let $\Psi^{3\#F}$ be the #F-fold Cartesian product of Ψ^3 .

Lemma 4. *Assume E.I through E.V. Let $\hat{y} \in \Psi^{3\#F}$.*
(i) Then $\tilde{D}^i(p, \hat{y})$ is nonempty and homogeneous of degree zero in p. $\tilde{A}^i(p, \hat{y})$ is continuous (upper and lower hemicontinuous) throughout $\Delta \times \Psi^{3\#F}$ and convex-valued. $\tilde{D}^i(p, \hat{y})$ is upper hemicontinuous throughout Δ and convex-valued.
(ii) Let $x^i \in \tilde{D}^i(p, \hat{y})$ be attainable. Then $x^i \in D^i(p, \hat{y})$.

Proof: Section 17.

The reader may find it awkward to have endowments be elements of the space $R_+^{2N(N-1)}$ rather than elements of the more natural space of commodities R^N. This distinction though is ultimately one of notational convenience only, since any given vector of commodity endowments in R^N corresponds to a equivalence class of vectors in the trading post endowmnents in $R_+^{2N(N-1)}$ via the map $\omega : R_+^{2N(N-1)} \rightarrow R^N$:

$$\omega : r_i \rightarrow \sum_{l=1, k \neq l}^{N} r_i^B(k, l) + r_i^S(k, l)$$

Let $\rho_i \in R^N$ be a vector of commodity endowmnets. The pre-image of a given vector of commodity endowments ρ_i under ω gives the collection arrangements of the commodity endowments across trading posts retail and wholesale market. Because of the definition of the household possibility set we show that for two vectors that coincide on the commodity endowments, they lead to identical possibility sets for the household.

Proposition . *[Endownment Invariance] For $\rho_i \in R^N$, and any $r_i, r_i' \in \omega^{-1}(\rho_i) \iff W_i - r_i = W_i - r_i'$*

Proof. Let $x_i' \in W_i - r_i$, then x_i satisfies $\sum_{l=1,k\neq l}^{N} x' S_i(k,l) + x' B_i(k,l) \geq \sum_{l=1,k\neq l}^{N} r_i^B(k,l) + r_i^S(k,l) = \rho_i(k)$ for all $k = 1, \cdots, N$. But then since r_i' is also in $w^{-1}(\rho_i)$ it must be the case that $\sum_{l=1,k\neq l}^{N} x' S_i(k,l) + x' B_i(k,l) \geq \sum_{l=1,k\neq l}^{N} r_i'^B(k,l) + r_i'^S(k,l)$. But then, by the definition of W_i, $x_i' \in W_i - r_i$. The corverse follows the same argument. $\square$

For this reason one can take a trading post endowment to be a representer of the equivalence class of arrangements corresponding to the same commodity endowment given by $\omega^{-1}\omega(r_i)$.

12 General Equilibrium

A market equilibrium is a vector of prices (q^*, π^*), a vector $c^{i*} \in W^i$, and $x^{i*} \in R^{2N(N-1)}$ for each household, $i \in H$, and a vector $(y^{jS*}, y^{jB*}, w^{j*}) \in Y^j$ for each firm$j \in F$, such that

(i) $c^{i*} = r^i + x^{i*} + \sum_{j \in F} \Theta^{ij}[(y^{*jS}, y^{*jB} + w^{*j})]$ is maximal with respect to u^i in W^i subject to $x^i \in B(p^*)$ at $p^* = (q^*, \pi^*)$,

(ii) (y^{jS*}, y^{jB*}, w^*) maximizes $[(q, q+\pi) \cdot (y^{jS}, y^{jB} + w^j)]$ subject to $(y^{jS}, y^{jB}) \in B(p^*)$, and $(y^{jS}, y^{jB}, w^j) \in Y^j$.

(iii) $\sum_i (x^{iS*}, x^{iB*}) + \sum_j (y^{jS*}, y^{jB*}) \leq 0$ co-ordinatewise

(iv) $q^* \geq 0, \pi^* \geq 0$ (the inequalities hold co-ordinatewise).

Note that in (iii) supplies enter with negative signs and demands with positive signs co-ordinatewise.

Let $\hat{x} \in \Psi^{2\#H}; \hat{y} \in \Psi^{2\#F}$.
Excess demand is defined as $Z(\hat{x}, \hat{y}) \equiv \sum_i (x^{iS}, x^{iB}) + \sum_j (y^{jS}, y^{jB})$.

Lemma 5. *(Walras's Law) Let $p = (q, \pi) \in \Delta$, and let $\hat{x}, \hat{y}$ be arbitrarily specified. Let $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, \hat{y})$and let $(y^{jS}, y^{jB}) \in \tilde{S}^j(p, \hat{x}, \hat{y})$.*
Then $q \cdot [\sum_i x^{iS} + \sum_j y^{jS}] + (q+\pi) \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.
Equivalently, $q \cdot [\sum_i x^{iS} + \sum_j y^{jS} + \sum_i x^{iB} + \sum_j y^{jB}] + \pi \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.

Proof: Section 17.

No Externality Case The 'No Externality Case' is defined in the following fashion. For each $j \in F, Y^j = \varphi^j(\hat{x}, \hat{y}) =$ constant. In this case prevailing $(\hat{x}, \hat{y})$ has no effect on the underlying transaction technology. In this case we can without loss of generality omit the functional notation. Thus we abbreviate the expressions $S^j(p, \hat{x}, \hat{y})$ and so forth as $S^j(p)$ omitting the notation $\hat{x}, \hat{y}$. In this case E. II (that is P.I through P.IV), and (C.I) through (C.VII) are sufficient to assure existence of a market clearing equilibrium price vector $p = (q, \pi)$. (C.VII) is used to avoid discontinuities at the boundary.

Theorem 1. *For the No Externality Case, Assume(E.I) through (E.IV) and (C.I) through (C.VII). Then the economy has a competitive equilibrium.*

Proof. Let $\prod_j, \prod_i$ indicate multiple Cartesian product.

Let $\hat{x} \in \prod_{i\in H} \tilde{D}^i(p,\hat{y}) \subseteq \Psi^{2\#H}$; $\hat{x} \equiv (x^{1S}, x^{1B}; x^{2S}, x^{2B}; ...; x^{\#HS}, x^{\#HB})$

$\hat{y} \in \prod_{j\in F} \tilde{S}^{j\dagger}(p) \subseteq \Psi^{3\#F}$; $\hat{y} \equiv (y^{1S}, y^{1B}, w^1; y^{2S}, y^{2B}, w^2; ...; y^{\#FS}, y^{\#FB}, w^{\#F})$

Excess demand is defined as: $Z(\hat{x},\hat{y}) = \sum_{i\in H}(x^{iS}, x^{iB}) + \sum_{j\in F}(y^{jS}, y^{jB})$; $z = (z^S, z^B)$

Let $\Gamma(z) \equiv \{(q', \pi') \in \Delta | (q', \pi') = \arg\max_{(q,\pi)\in\Delta}(q,\pi)\cdot(z^S + z^B, z^B)\}$ be the price adjustment correspondence.

Let $T(p,\hat{x},\hat{y},z) \equiv \Gamma(z) \times \prod_{i\in H} \tilde{D}^i(p,\hat{y}) \times \prod_{j\in F} \tilde{S}^{j\dagger}(p) \times Z(\hat{x},\hat{y})$.

That is, T is a set-valued mapping
$T : \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2 \to \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$.

Note that $\Gamma, \tilde{D}^i, \tilde{S}^{j\dagger}$, and $Z(\hat{x},\hat{y})$ are each well defined, upper hemicontinuous, and convex-valued throughout $\Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$. Then the proof will apply the Kakutani Fixed Point Theorem, to generate a fixed point, $(p^\circ, \hat{x}^\circ, \hat{y}^\circ, z^\circ)$. To begin the proof, we start with the following un-numbered lemma.

Lemma A. *The correspondence $T(p,\hat{x},\hat{y},z)$ is non-empty, upper hemicontinuous and convex valued.*

Proof of Lemma A: $\tilde{D}^i(p,\hat{y})$ is non-empty, convex valued and upper-hemicontinuous by Lemma 4. Similarly, each $\tilde{S}^j$ is non-empty, convex valued and upper-hemicontinuous by Lemma 3. $Z(\hat{x},\hat{y})$ is an additive function between finite dimensional spaces, hence continuous. Trivially it is also non-empty and convex valued since it is the finite sum of correspondences that satisfy these properties. Finally Γ is nonempty, since $(q,\pi)\cdot(z^S + z^B, z^B)$ is a continuous function on a compact set Δ, hence, by the Weierstrass theorem, it achieves a maximum. Because $(q,\pi)\cdot(z^S + z^B, z^B)$ is linear in the z's, the set of maximizers of that function is a convex set. Hence Γ is convex valued.

Now we show that Γ is upper-hemicontinuous. Think of z as the parameter affecting the continuous function $f(p,z) := (q,\pi)\cdot(z^S + z^B, z^B)$, the maximand. Notice that the constraint set Δ can be viewed as a constant, compact correspondence of z. Hence, Δ, as a correspondence, is trivially continuous. Then, by the maximum theorem, the set of maximizers with respect to $p = (q,\pi)$ of $f(z,p)$, is an upper-hemicontinous correspondence of the parameter z. Since we defined $\Gamma(z)$ as such a set of maximizers, the desired result follows.

Notably, the finite cartesian product of correspondences preserves upper-hemicontinuity, convex valued-ness and non-emptiness. This completes the proof of Lemma A.

The proof of the theorem continues. Δ is clearly a compact, convex, non-empty set. The above lemma satisfies the assumptions of the Kakutani fixed point theorem, hence T has a fixed point, i.e. $(p^o, x^o, y^o, z^o) \in T(p^o, x^o, y^o, z^o)$.

We now show that (p^o, x^o, y^o, z^o) is a market clearing equilibrium.

By the Weak Walras's Law, Lemma 5, $(q^\circ, \pi^\circ) \cdot (z^{\circ S} + z^{\circ B}, z^{\circ B}) \leq 0$.

Notice that $(q^\circ, \pi^\circ) \geq 0$ and (q°, π°) is $\text{argmax}_{(q,\pi)\in\Delta}[q \cdot (z^{oS} + z^{oB}) + \pi \cdot z^{oB}]$ so $z^{^o} \leq 0$. If the inequality were not to hold, the maximand could be increased by increasing the price of the positive component of z. So it must be the case that $z^o \leq 0$, co-ordinatewise.

We have $x^{i\circ} \in \tilde{D}^i(p^\circ, \hat{y}^\circ)$, $y^{j\circ} \in \tilde{S}^j(p^\circ)$. Note the tilde $\tilde{}$ notation. We now seek to demonstrate that, for each $i \in H$ and each $j \in F$, $x^{i\circ} \in D^i(p^\circ, \hat{y}^\circ)$, $y^{j\circ} \in S^j(p^\circ)$. Recall $z^o = \sum\limits_{i\in H} x^{io} + \sum\limits_{j\in F} y^{jo}$ where $x^{io} \in \tilde{D}^i(p^o, \hat{y}^o)$ and $y^{jo} \in \tilde{S}^j(p^o)$.
But $\sum\limits_{i\in H} x^{io} + \sum\limits_{j\in F} y^{jo} \leq 0$, so $x^{io}, i \in H$ is attainable, so $|x^{io}| < C$. But $|x^{io}| < C$ and $x^{io} \in \tilde{D}^i(p^{^o}, \hat{y}^o)$ implies that the constraint to length C is not binding, so by Lemma 4, $x^{io} \in D^i(p^o, \hat{y}^o)$. Similarly, by Lemma 3, $y^{jo} \in S^j(p^o, \hat{x}^o)$. Hence markets clear and the households and firms are optimizing subject to budget and technology constraints. The length constraint is not binding. The price and allocation is a general equilibrium.

$\square$

13 General Equilibrium with External Effects

Lemma 6. *Assume (E.I) through (E.V). Then $\tilde{S}^{j\dagger}(p, \hat{x}, \hat{y})$ is non-empty, convex-valued, and upper hemicontinuous in $(p, \hat{x}, \hat{y})$.*

Proof Section 17.

Definition: Let $(p, \hat{x}, \hat{y}) \in R^{2N(N-1)} \times R^{\#H2N(N-1)} \times R^{\#F2N(N-1)}$. A competitive equilibrium is a vector of prices $p = (q^*, \pi^*)$, a vector $c^{i*} \in W^i$, and $x^{i*} \in R^{2N(N-1)}$ for each household, $i \in H$, and a vector $(y^{jS*}, y^{jB*}, w^{j*}) \in Y^j = \varphi^j(\hat{x}, \hat{y})$ for each firm $j \in F$, such that:

(i) $c^{i*} = r^i + x^{i*} + \sum\limits_{j\in F} \Theta^{ij}[(y^{*jS}, y^{*jB} + w^{*j})]$ is maximal with respect to u^i in W^i subject to $x^i \in B(p^*)$ at $p^* = (q^*, \pi^*)$,

(ii) (y^{jS*}, y^{jB*}, w^*) maximizes $[(q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j)]$ subject to $(y^{jS}, y^{jB}) \in B(p^*)$, and $(y^{jS}, y^{jB}, w^j) \in Y^j = \varphi^j(\hat{x}, \hat{y})$

(iii) $\sum\limits_i (x^{iS*}, x^{iB*}) + \sum\limits_j (y^{jS*}, y^{jB*}) \leq 0$ co-ordinatewise

(iv) $q^* \geq 0, \pi^* \geq 0$ (the inequalities hold co-ordinatewise).

Note that, in *(iii)*, supplies enter with negative signs and demands with positive signs co-ordinatewise.

Theorem 2. *Assume (C.I) through (C.VII), (E.I) through (E.V). Then the economy has a competitive equilibrium.*

Plan of the proof: The plan for the proof is to characterize price and quantity adjustments as an upper hemicontinuous convex-valued mapping from compact convex price and output spaces into themselves. The impact of external effects, under (E.I) through (E.V) is upper hemicontinuous and convex-valued by Lemma 6. A fixed point will exist by the Kakutani Fixed Point Theorem. The difficulty is that this argument needs the opporunity sets for firms and households to be bounded, in order for demand and supply correspondences to be assured of nonemptiness. But the price system need not restrict agents to bounded sets, so it is necessary to show that boundedness constraints are not binding and that the same (fixed point) actions will be undertaken, even in the absence of a boundedness constraint.

Proof of Theorem 2

The following usages are introduced in the proof of Theorem 1. Let

$\hat{x} \in \prod_{i\in H} \tilde{D}^i(p,\hat{y}) \subseteq \Psi^{2\#H}$; $\hat{x} \equiv (x'^{1S}, x'^{1B}; x'^{2S}, x'^{2B}; ...; x'^{\#HS}, x'^{\#HB})$;

$\hat{y} \in \prod_{j\in F} \tilde{S}^{j\dagger}(p,\hat{x},\hat{y}) \subseteq \Psi^{3\#F}$; $\hat{y} \equiv (y'^{1S}, y'^{1B}, w'^{1}; y'^{2S}, y'^{2B}, w'^{2}; \ldots; y'^{\#FS}, y'^{\#FB}, w'^{\#F})$;

$Z(\hat{x},\hat{y}) = \sum_{i\in H} (x'^{iS}, x'^{iB}) + \sum_{j\in F} (y'^{jS}, y'^{jB})$; $z = (z^S, z^B)$;

$\Gamma(z) \equiv \{\arg\max_{(q,\pi)\in\Delta} (q,\pi)\cdot(z^S + z^B, z^B)\}$ is the price adjustment function. The grand adjustment correspondence is T, encompassing price adjustment, supplies, demands, and external effects.

Let $T(p,\hat{x},\hat{y},z) \equiv \Gamma(z) \times \prod_{i\in H} \tilde{D}^i(p,\hat{y}) \times \prod_{j\in F} \tilde{S}^j(p,\hat{x},\hat{y}) \times Z(\hat{x},\hat{y})$

$T : \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2 \to \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$

We will argue that $\Gamma, \tilde{D}^i, \tilde{S}^j, \varphi^j$, and$\tilde{Z}(\hat{x},\hat{y})$ are each well defined, upper hemicontinuous, and convex- valued throughout $\Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$. Then the proof will apply the Kakutani Fixed Point Theorem, to generate a fixed point, $(p^\circ, \hat{x}^\circ, \hat{y}^\circ, z^\circ)$. The proof will argue that the fixed point is a market-clearing equilibrium. Application of the Walras's Law, Lemma 5, as in Theorem 1 implies that the length constraints in Ψ^2 and Ψ^3 are not binding. Hence markets clear and the households and firms are optimizing subject to budget and technology (but not length) constraints. The price and allocation is a general equilibrium accounting for the external effects.

First we show that the correspondence defined by $T : \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2 \to \Delta \times \Psi^{2\#H} \times \Psi^{3\#F} \times \bar{\Psi}^2$ is non-empty, convex valued and upper hemicontinuous.

T is clearly nonempty and convex since it is the finite cartesian product of nonempty and convex-valued correspondences. To see this notice that, for any point $(p,\hat{x},\hat{y},z)$, we have $T(p,\hat{x},\hat{y},z) \equiv \Gamma(z) \times \prod_{i\in H} \tilde{D}^i(p,\hat{y}) \times \prod_{j\in F} \tilde{S}^{j\dagger}(p,\hat{x},\hat{y}) \times Z(\hat{x},\hat{y})$ and $\Gamma(z)$ is nonempty since it's the maximizer set of a convex function on a compact set. For

the same reason, $\Gamma(z)$ is also convex. Now by Lemma 4, for each $i \in H$ we have $\tilde{D}^i(p,\hat{y})$ nonempty, convex-valued. Hence the finite cartesian product $\prod_{i\in H} \tilde{D}^i(p,\hat{y})$ is also nonempty and convex-valued. By Lemma 6, for each $j \in F$ we have $\tilde{S}^{j\dagger}(p,\hat{x},\hat{y})$ is nonempty and convex valued. So again $\prod_{j\in F} \tilde{S}^{j\dagger}(p,\hat{x},\hat{y})$ is nonempty and convex-valued. Finally $Z(\hat{x},\hat{y})$ is a function so it is trivially nonempty and convex valued. Hence T is nonempty and convex valued as a corrrespondence.
Now we show upper-hemicontinuity. $\Gamma(z) \equiv \{\arg\max_{(q,\pi)\in\Delta}(q,\pi)\cdot(z^S+z^B,z^B)\}$ is the set of maximizers $p=(q,\pi)\in\Delta$ of the continuous function $(q,\pi)\cdot(z^S+z^B,z^B)$. Then notice how Δ is a constant correspondence of the parameter z. But then by the maximum theorem $\Gamma(z) \equiv \{\arg\max_{(q,\pi)\in\Delta}(q,\pi)\cdot(z^S+z^B,z^B)\}$ is upper-hemicontinuous as a function of the parameter z, which is the desired result. Upper hemicontinuity of the finite cartesian product $\prod_{i\in H} \tilde{D}^i(p,\hat{y})$ and $\prod_{j\in F} \tilde{S}^{j\dagger}(p,\hat{x},\hat{y})$ follow directly from Lemma 4 and 6, respectively. Finally $Z(\hat{x},\hat{y})$ is a continuous function and hence an upper-hemicontinuous correspondence. Hence T is again upper-hemicontinuous.

By the Kakutani fixed point theorem we have that T has a fixed point so there is $(p^\circ,\hat{x}^\circ,\hat{y}^\circ,z^\circ)$ such that $(p^\circ,\hat{x}^\circ,\hat{y}^\circ,z^\circ) \in T(p^\circ,\hat{x}^\circ,\hat{y}^\circ,z^\circ)$. We now show that $(p^\circ,\hat{x}^\circ,\hat{y}^\circ,z^\circ)$ constitutes an equilibrium according to the definition above. The restriction of $\tilde{D}^i(p^\circ,\hat{y}^\circ)$ to Ψ^2, and of $\tilde{S}^{j\dagger}(p^\circ,\hat{x}^\circ,\hat{y}^\circ)$ to Ψ^3 assures their nonemptiness. The essential point is to show that the restriction is not a binding constraint. Then the restrictions to Ψ^2 and Ψ^3 can be eliminated, applying Lemmas 3 and 4. Then $\tilde{D}^i(p^\circ,\hat{y}^\circ) = D^i(p^\circ,\hat{y}^\circ)$ and $\tilde{S}^{j\dagger}(p^\circ,\hat{x}^\circ,\hat{y}^\circ) = S^{j\dagger}(p^\circ,\hat{x}^\circ,\hat{y}^\circ)$.

Consider $(p^\circ,\hat{x}^\circ,\hat{y}^\circ,z^\circ)$. We know from Lemma 5 that $(q^\circ,\pi^\circ)\cdot(z^{\circ S}+z^{\circ B},z^{\circ B}) \le 0$. Now each co-ordinate of z^o is ≤ 0. If not then $(q,\pi)\cdot(z^{\circ S}+z^{\circ B},z^{\circ B})$ could be increased by increasing the the prices on the positive coordinate of z^o. But then $(q^o,\pi^o) \notin \Gamma(z^o) = \{\arg\max_{(q,\pi)}(q,\pi)\cdot(z^S+z^B,z^B)\}$ so $(p^\circ,\hat{x}^\circ,\hat{y}^\circ,z^\circ)$ would not be a fixed point, a contradiction. Hence $z^o \le 0$ coordinatewise. Now $\sum_{i\in H}(x^{\circ iS},x^{\circ iB}) + \sum_{j\in F}(y^{\circ jS},y^{\circ jB}) \le 0$. Now to finish the proof we want to claim that the $x^{\circ i}$ is attainable and that $y^{\circ j} \in \Xi^j$. We have:

$$\sum_{i\in H}(x^{\circ iS},x^{\circ iB}) + \sum_{j\in F}(y^{\circ jS},y^{\circ jB}) \le 0$$

Clearly we must have:

$$\sum_{i\in H} x^{\circ iS} + x^{\circ iB} + \sum_{j\in F} y^{\circ jS} + y^{\circ jB} \le 0$$

Restating the characterization of attainability:

The aggregate trading technology is $Y \equiv \sum\limits_{j\in F} Y^j$. The economy's initial resource vector is
$r = \sum\limits_{i\in H} r^i \in R_+^{2N(N-1)}$. Then $(y^S, y^B, w) \in Y$ is said to be attainable if for each $k = 1, 2, \ldots, N$, we have:

$$\sum_{\ell} [w(k,\ell)] \geq -\sum_{\ell} r(k,\ell)$$

To demonstrate attainability it is sufficient to prove the inequality at $(p^\circ, \hat{x}^\circ, \hat{y}^\circ)$.

Now suppose $\hat{y}^\circ$ is not attainable, then it must be the case that the inequality above does not hold. Thus :

$$\sum_{l}\sum_{j\in F} w^{\circ j}(k,l) + \sum_{l}\sum_{i\in H} r^i(k,l) < 0$$

But then we may write:

$$\sum_{j\in F}\sum_{l\neq k}(y^{\circ jS}(k,l)+y^{\circ jB}(k,l))+\sum_{l\neq k}\sum_{i\in H}(x^{\circ iS}(k,l)+x^{\circ iB}(k,l))+\sum_{j\in F}\sum_{l\neq k} w^{\circ j}(k,l)+\sum_{i\in H}\sum_{l\neq k} r^i(k,l) < 0$$

since we are adding a strictly negative term to the sum. On the other hand, observe that since each household i must choose $x^i \in W^i$ we have:

$$\sum_{l\neq k}(x^{i\circ S}(k,l)+x^{i\circ B}(k,l))+\sum_{j\in F}\Theta^{ij}\sum_{l\neq k}(\sum_{j\in F} y^{\circ jS}(k,l)+y^{\circ jB}(k,l)+w^{\circ j}(k,l))+\sum_{l\neq k} r^i(k,l) \geq 0$$

Summing across all households $i \in H$ we obtain:

$$\begin{aligned}
&\sum_{i\in H}\sum_{l\neq k} x^{i\circ S}(k,l) + x^{i\circ B}(k,l) + \sum_{i\in H}\sum_{j\in F}\Theta^{ij}(y^{\circ jS}(k,l) + y^{\circ jB}(k,l) + w^{\circ j}(k,l)) + \sum_{i\in H}\sum_{l\neq k} r^i(k,l)\\
&= \sum_{i\in H}\sum_{l\neq k}(x^{i\circ S}(k,l) + x^{i\circ B}(k,l)) + \sum_{j\in F}\sum_{l\neq k}(y^{\circ jS}(k,l) + y^{\circ jB}(k,l) + w^{\circ j}(k,l)) + \sum_{i\in H}\sum_{l\neq k} r^i(k,l)\\
&\geq 0
\end{aligned}$$

But this is a contradiction. Hence it must be the case that $\sum_{j\in F} w^{j\circ} \geq -\sum_{i\in H} r^i$ hence $(y^{\circ S}, y^{\circ B}, w^\circ)$ is attainable and for each $j \in F$ we have $(y^{\circ jS}, y^{\circ jB}, w^{\circ j}) \in \Xi^j$. But then by Lemma 4 we know that $(y^{\circ jS}, y^{\circ jB}, w^{\circ j}) \in \tilde{S}^{\dagger j}(p) \implies (y^{\circ jS}, y^{\circ jB}, w^{\circ j}) \in S^{\dagger j}(p)$.

Moreover $x^{\circ i}$ is attainable and hence by Lemma 4 we have: $x^{\circ i} \in \tilde{D}^i(p^\circ, \hat{y}^\circ) \implies x^{\circ i} \in D^i(p^\circ, \hat{y}^\circ)$. Hence $(p^\circ, \hat{x}^\circ, \hat{y}^\circ, z^\circ)$ is an equilibrium for the unbounded economy.

QED

14 Examples of Monetary Equilibrium and and Barter Equilibrium

For ease of notation in this section, $q, \pi \in R_+^{N(N-1)}$. There is no restriction to the unit simplex. There is no loss of generality.

14.1 Transaction Costs with Thick Market Externality

Define the (wholesale) sales offers for commodity l and l' at trading post $\{l, l'\}$ as:

$Q(l, l') \equiv |\sum_{h\in H} x^{hS}(l, l')|$ and $Q(l', l) \equiv |\sum_{h\in H} x^{hS}(l', l)|$.

Recall that the superscript S denotes a seller at price q and that superscript B denotes a buyer at price $q + \pi$. Typically, for households $h \in H$, $x^{hB}(l, l') \geq 0$, and $x^{hS}(l, l') \leq 0$.

Transaction costs are modeled in the following way. The firms active in the trading post $\{l, l'\}$ treat parametrically the volume of offers at the post. The trading offer volumes help to determine the marginal cost of firms at the post, assumed to be average cost. Marginal costs are assumed to have the following character. Let marginal transaction cost be $\alpha + f(Q)$, assumed equal to average cost.

Q will be specified as the gross transaction quantity, including quantities expended in transaction cost. That is, if ten units are to be acquired and the transaction markup is 40% then the gross transaction quantity is 14 and $Q = 14$. So in the expressions above, the relevant quantities Q are $Q(l, l') \equiv |\sum_{h\in H} x^{hS}(l, l')|$ and $Q(l', l) \equiv |\sum_{h\in H} x^{hS}(l', l)|$. In a general equilibrium markets will clear, so as an equilibrium condition

$$Q(l, l') = [\sum_{h\in H} x^{hB}(l, l')] \times [q(l, l') + \pi(l, l')].$$

As an equilibrium condition, for each commodity, total consumption plus total expenditure in transaction costs needs to be less or equal to total endowment. Solving for that resource clearing allocation is pursued in subsubsection 14.7.4.

$f(Q)$ is the variable portion of marginal transaction cost. Specification of f is to be arranged. Ideally $f(0) > 0; f'(Q) \leq 0$ for $Q > 0$; $f(Q) \to 0$ as $Q \to +\infty$. Typical specifications would be $f(Q) = \min[E > 0, \frac{1}{Q}]$ for $Q > 0$ or $\min[1, \frac{\sqrt{Q}}{Q}]$ where $Q > 0$.

Specification Primo is specificied as follows.

$WW^j(l, l') = \left[\alpha + \beta \times \left[\max(\frac{1}{Q(l,l')}, \frac{1}{Q(l',l)})\right]\right] = WW^j(l', l)$, where $\alpha, \beta > 0$.

when this expression is well defined, or let $WW^j(l, l') = E >> 0$ when not.

Then actual costs incurred by firm j will be $[WW^j(l, l') + WW^j(l', l)] \times (|y^{jB}(l, l')| + |y^{jS}(l, l')|)$. In a market equilibrium these costs will be allocated between buyers and sellers as trading markups $\pi(l, l')$ and $\pi(l', l)$.

The cost structure of Specification Primo reflects a thick markets externality. High offer volume generates low marginal and average transaction cost. Transaction costs for each firm j are linear in the transactions executed at the trading post but inversely proportional to the total volume of household offers at the trading post (l, l'). Marginal cost pricing of the transaction cost markup gives

$$\pi(l,l') + \pi(l',l) = WW^j(l,l') + WW^j(l',l) = 2\left[\alpha + \beta \times \left[\max(\frac{1}{Q(l,l')}, \frac{1}{Q(l',l)})\right]\right].$$

14.2 The $\kappa - N$ Model Economy

Let's consider one example now. Denote the following specification as the $\kappa - N$ economy. Let the population be $H \equiv \{1, 2, \ldots, \kappa N\}$. Let N be even, $N \geq 6$. There are κ replicas of each household totaling κN households. Let the notation $\ell \oplus k$ denote $(\ell + k)(modN)$. Each $i \in H$ is endowed with 1 unit of commodity $i(modN)$ and desires consumption of commodities $i \oplus 1 \equiv i + 1(modN)$ through $i \oplus ((N/2) - 1)$. Think of the commodities arrayed over a clock-face.

We'll consider two alternative specifications of demand.

No Coincidence Demand is a Leontieff specification where each household desires the $(N/2) - 1$ goods clockwise from his own. That is, think of each household i, endowed with good $i(modN)$ and desiring goods numbered $i \oplus 1$ through $i \oplus ((N/2) - 1)$ .The commodities are divisible. This specification represents a complete absence of double coincidence of wants. There are κ households endowed with good $i(modN)$ each desiring goods $i \oplus 1$ through $i \oplus ((N/2) - 1)$.

$$u^i = \min\{x_n | n = i \oplus 1, \ldots, i \oplus (\frac{N}{2} - 1),\}$$

The alternative, **Double Coincidence Demand** is a Leontieff specification where each household desires the $N - 1$ goods other than his endowment.

$$u^i = \min\{x_n | n = i \oplus 1, i \oplus 2, \ldots, i \oplus N, n \neq i(modN)\}$$

It is perhaps not surprising that money arises in the first specification with no coincidence of wants. Importantly, we show that the externality is so strong that money also arises naturally in the second specification where there is indeed a double conincidence of wants.

14.3 Commodity Money Equilibrium

In the following example, any commodity can become 'money' by the process of being treated as the common medium of exchange. Using Specification Primo and No Coincidence Demand, the resulting high trading volume of the 'money' commodity implies low transaction cost. The low transaction cost implies high trading volume. The designation as 'money' is "self-justifying" as in Tobin & Golub (1997). Individual trading households, are guided to use the 'money' by the price system.

The designated commodity medium of exchange is good m chosen without distinction among the N commodities. For each of the N commodities, its liquidity is priced in its bid/ask spread π. m becomes the most liquid good by its use as money. Using Specification Primo, the high trading volume from that use enforces and confirms its liquidity.

There are $N-1$ active trading posts, where m is traded against each of $N-1$ other goods. At each of those active trading posts, typically trading $(m, i(modN))$, there are $\kappa((N/2)-1)$ households offering good m seeking to purchase good $i(modN)$ and κ households selling one unit of $i(modN)$ for m. Total trading volume at each active trading post is κ.

What is going on here? Everyone , i, (with the exception of $i(modN) = m$) trades his endowed i for m. Upon acquiring m, household i spends the m acquiring $i \oplus 1$ through $i \oplus (N/2-1)$. The price system guides the population to this pattern of trade.

Household i considers trading directly for $i \oplus 1$ through $i \oplus (N/2-1)$ at posts $\{i(modN), i \oplus 1\}$ through $\{i(modN), i \oplus (N/2-1)\}$ but is dissuaded by the wide bid/ask spread. Why is the spread wide? Because the trading volume is low. Trading through m is considerably more rewarding, since the bid/ask spreads, $\pi(i,m), \pi(i \oplus 1, m)$ through $\pi(i \oplus (N/2-1), m)$ are narrow. Why are they narrow? Because the trading volumes are large. Why are the volumes large? Because the spreads are narrow.

Let $m \in \{1, 2, \ldots, N\}$.
$q(i,j) = 1$, all $i, j \in \{1, 2, \ldots, N\}$.
$\pi(i,j) = E$ all $i, j \neq m$, $i, j \in \{1, 2, \ldots, N\}$.
$\pi(i,m) = 2\left[\alpha + \beta \times \left[\frac{1}{\kappa}\right]\right]$,
$\pi(m,i) = 0$
$x^{iS}(i,m) = -1$
$x^{iB}(m,i) = 1(\frac{1}{1+\pi(m,i)})$
$x^{iS}(m, i \oplus n) = \frac{1}{(N/2)-1}, n = 1, 2, ..., (N/2)-1$
$x^{iB}(i \oplus n, m) = \frac{1}{(N/2)-1}(\frac{1}{1+\pi(i \oplus n, m)}), n = 1, 2, ..., (N/2)-1$

Households that are initially endowed with commodity m, the common medium of exchange, do not trade twice, they just distribute m over the trading posts of the commodities they want. Thus, for the case $i(modN) = m$ we have
$x^{iS}(m, m \oplus n) = -\frac{1}{(N/2)-1}, n = 1, 2, ..., (N/2)-1$
$x^{iB}(m \oplus n, m) = \frac{1}{(N/2)-1}(\frac{1}{1+\pi(m, m \oplus n)}), n = 1, 2, ..., (N/2)-1$. In this equilibrium, there is an endogenous notion of seigniorage. In these symmetric equilibria seigniorage can be defined as the difference in consumption between agent $i(modN) = m$ (who incurs only one transaction cost in each trade) and the other agents (who incur transaction costs twice).

For κ sufficiently large, $\pi(i,m) = 2\big[\alpha + \beta \times \big[\frac{1}{\kappa}\big]\big]$ converges to 2α.

How should we interpret the example? There are many pairwise low-volume trading posts, typically $\{k, l\}$ with a high $\pi(k,l) = E$. There is a small number of high volume pairwise trading posts $\{i, m\}$, with the relatively low markup $2\big[\alpha + \beta \times \big[\frac{1}{\kappa}\big]\big]$. Where does the low markup come from? From the high volume of household trading

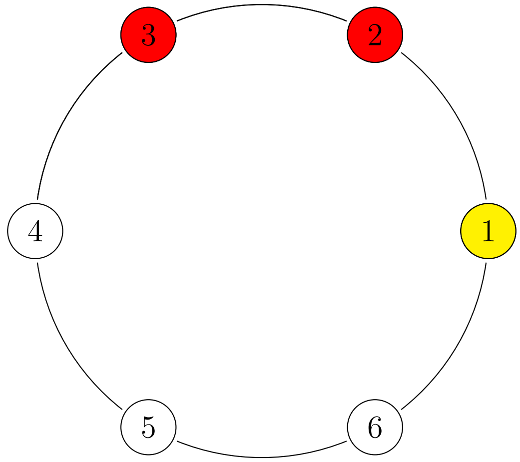

Figure 1: Example of commodities for household $i = 1$. Household 1 is endowed with commodity 1 (in yellow) but desires commodities 2 and 3.

offers. There is an external effect, reducing the transaction costs at trading post $\{i, m\}$. The external effect shows up in the specification of $W^j(i, m)$. It formalizes the notion that a high volume trading post, reflecting a thick markets externality, is a low transaction cost trading post.

Bottom line: Commodity m is the universal common medium of exchange because it is priced with a narrow bid/ask spread, $\pi(i, m)$, a low transaction cost. Why does it have a low transaction cost? Because it has a high trading volume. Why does it have a high trading volume? Because it has a low transaction cost. This represents "circularity in acceptability," Tobin (1980). The price system itself designates 'money' and guides transactors to trade using 'money'.

14.4 Barter Equilibrium

The $\kappa - N$ economy with No Coincidence Demand can have a barter equilibrium. There are $\frac{1}{2}N(N-1)$ trading posts. Most posts are active. There are $\frac{1}{2}N$ inactive posts, those trading diametrically opposed goods, $\{n(modN), (n + \frac{1}{2}N)(modN)\}$. In this setting there is no double coincidence of wants. κ buyers come to each active post (ℓ, k) each seeking $\frac{1}{((N/2)-1)}$ of ℓ. Trading volume there is $\kappa\frac{1}{((N/2)-1)}$.

$x^{iS}(i, i \oplus n) = -\frac{1}{((N/2)-1)}, n = 1, 2, \ldots, (N/2) - 1$

$x^{iB}(i \oplus n, i) = [\frac{1}{((N/2)-1)}]\frac{1}{1+\pi(i \oplus n, i)}, n = 1, 2, \ldots, (N/2) - 1$

$q(i, i \oplus n) = 1$

$\pi(i, i \oplus n) = 0$, reflecting no demand for $i(modN)$ by those endowed with $i \oplus n$. Households so endowed will trade their endowment, but will not willingly pay a premium for i. The market clears with the i's incurring all the transaction cost in $\pi(i \oplus n, i)$.

$\pi(i \oplus n, i) = 2\left[\alpha + \beta \times (\frac{1}{\kappa[\frac{1}{(N/2)-1}]})\right] = 2\left[\alpha + \beta \times (\frac{(N/2)-1}{\kappa})\right]$

The barter market equilibrium here leads to thin markets. Thin markets lead to high transaction costs, displayed in the high value of $\pi(i \oplus n, i)$ compared to the monetary equilibrium of the previous subsection. There is a resultant welfare loss. At the post $\{i, i \oplus n\}$ household i is the buyer, desiring $i \oplus n$. Household $i \oplus n$ has no interest in selling, having no use for i, accepting it only as equally saleable as his own $i \oplus n$. Hence all of the transaction cost is incurred by i accounting for the factor 2 in the expression $\pi(i \oplus n, i)$.

14.5 Double Coincidence of Wants

For purposes of this section and the next, the endowments of $\kappa - N$ remain; we restate preferences to allow a double coincidence of wants. It will be an exchange economy displaying a full double coincidence of wants. Then Jevons (1875), Kiyotaki & Wright (1993), and Tobin (1980) suggest that money and monetary trade need not arise. There is a non-monetary equilibrium of direct trade and there is alternatively a monetary equilibrium. The latter is Pareto superior.

For each $i \in H$, assume **Double Coincidence Demand** , a Leontieff specification where each household desires the $N - 1$ goods other than his endowment.

$$u^i = \min\{x_n | n = i \oplus 1, i \oplus 2, \ldots, i \oplus N, n \neq i(modN)\}$$

That is, think of each household i, endowed with good $i(modN)$ and desiring to exchange $i(modN)$ for all other goods.

In barter trade the array of trades looks as follows. Each good i is traded for each of the $N - 1$ other goods. There are $\frac{1}{2}N(N-1)$ pairwise trading posts where the goods can be directly traded for one another. Trading volumes are necessarily thin, since the quantity $\frac{\kappa}{(N-1)}$ of each good is traded against the quantity $\frac{\kappa}{(N-1)}$ of each other good.

14.6 Barter Equilibrium in the Double Coincidence Economy; Pricing Transaction Cost

We will consider a barter equilibrium and a monetary equilibrium in the double coincidence of wants economy. Pricing in the barter equilibrium starts with $q(i, n) = 1$ for all n.

Assuming Specification Primo, this gives transaction cost pricing
$\pi(l, l') = WW^j(l, l') = \left[\alpha + \beta \times \left[\max(\frac{1}{Q(l,l')}, \frac{1}{Q(l',l)})\right]\right]$

But in the direct trade trading posts $Q(l, l') = \frac{\kappa}{(N-1)} = Q(l', l)$.

Then the trading markup is (if allocated equally to both sides of a transaction)
$\pi(i, n) = WW^j(l, l') = \left[\alpha + \beta \times \left[\max([1/\frac{\kappa}{(N-1)}], [1/\frac{\kappa}{(N-1)}])\right]\right] = \left[\alpha + \beta \times \left[\frac{(N-1)}{\kappa}\right]\right]$

Taking account of transaction costs on both sides of the transaction at a typical trading post gives overall marginal cost of a transaction as
$WW^j(l, l') + WW^j(l', l) = 2 \times \left[\alpha + \beta \times \left[\max([1/\frac{\kappa}{(N-1)}], [1/\frac{\kappa}{(N-1)}])\right]\right] = 2 \times \left[\alpha + \beta \times \left[\frac{(N-1)}{\kappa}\right]\right]$

14.6.1 Monetary Equilibrium in the Double Coincidence of Wants Economy

Alternatively, let commodity m be the common medium of exchange. Instead of $\frac{1}{2}N(N-1)$ active pairwise trading posts, there are $(N-1)$. The remaining trading posts are priced, but inactive. Assume Specification Primo.

Allocate transaction costs to the markup of purchases of m for sales of $i(modN)$, rather than to sales of m, reflecting an advantage to households endowed with the common medium of exchange, $i(modN) = m$. At a typical active trading post $\{i(modN), m\}$, there are $\kappa(N-1)$ households that each arrive to purchase $\frac{1}{(N-1)}$ of $i(modN) = n$ (net of transaction cost), and κ households $i(modN) = n$ arrive to sell 1 unit of $i(modN)$ in exchange for 1 of m (net of transaction cost). $Q(n, m) = (N-1) \times [\kappa/(N-1)]$, $Q(m, n) = \kappa$. Then we have

$Q(l, m) = \kappa = Q(m, l), l = 1, 2, ..., N, l \neq m$

$q(n, m) = 1$

$\pi(m,n) = 0$

$\pi(n,m) = WW^j(i,m) + WW^j(m,i) = 2\big[\alpha + \beta \times \big[\max(\frac{1}{\kappa}, \frac{1}{\kappa})\big]\big] = 2\big[\alpha + \beta \times \big[\frac{1}{\kappa}\big]\big]$

Note that trading volume in the monetary equilibrium is twice the volume in the barter equilibrium. So the relevant cost comparison is

For the barter equilibrium: $= 2 \times \left[\alpha + \beta \times \left[\frac{(N-1)}{\kappa}\right]\right]$

For the monetary equilibrium: $2 \times 2\big[\alpha + \beta \times \big[\frac{1}{\kappa}\big]\big]$

The barter equilibrium has the benefit of full double coincidence of wants. But the monetary equilibrium benefits from the thick markets externality. Thus when the ratio of N to κ is sufficiently large, the monetary equilibrium has lower transaction costs. Then the monetary equilibrium is Pareto superior.

To suggest a numerical example, let $N = 100, \alpha = 0.1, \beta = 0.010, \kappa = 1$. Then in the barter economy with full double coincidence of wants, transaction costs are approximately five times those in the monetary economy. The monetary equilibrium is Pareto superior. Note that no specific qualities are assumed of the commodity money; it merely benefits from high trading volume. A full calculation of the manifold of parameters leading to cost determination is beyond the scope of this article.

This cost structure reflects a thick markets externality. High — and matching — offer volume generates low marginal and average transaction cost. Transaction costs for each firm j are linear in the transactions executed at the trading post but inversely proportional to the matched household offers at the trading post (l, l').

14.7 Government Taxation: Fiat Money and Barter Equilibria in the κ - N Economy

Here, in subsection 14.7, we'll consider the $\kappa - N$ economy with taxation, assuming specification primo. The first treatment deals with a fiat money version where the money is suitable for payment of taxes. Then, we consider the same economy without money; taxes are paid in kind and trade is barter. The principal conclusion is that the monetary economy has significantly lower transaction costs because its markets are thicker, taking advantage of the the thick markets externality.

Subsubsection 14.7.1 presents a discussion of the theory of fiat money.

Subsubsection 14.7.2 specifies a fiat money equilibrium model with taxation. Taxes and purchases there are paid in fiat money. Transaction costs are specified as marginal costs per unit volume equal to average cost, assuming specification primo. Transaction costs are modeled as marginal costs dependent on transaction volume, allowing for a thick-market externality.

Subsubsection 14.7.3 presents the same specification in a barter economy. Taxes and purchases there are paid by households directly from commodity endowments .

Subsubsection 14.7.4 describes the equilibrium trading volume in the barter economy as the endogenous solution of market clearing, accounting for resources expended in transaction cost.

Subsection 14.7.5 makes a welfare comparison between the monetary economy and the barter economy.

14.7.1 Fiat Money

Fiat money is a puzzle in two dimensions: It is inherently worthless so why is it valuable? Why is it (and its close substitutes) the universal unique common medium of exchange? The answer to the first question is taxation payable in fiat money. The answer to the second comes from the thick market externality in transaction costs that makes money a natural monopoly. Government's large scale creates the market thickness that secures the monopoly to government's fiat instrument.

One of the observations this article begins with is that money is almost universally uniquely government-issued fiat money (and instruments denominated and convertible thereto) trading at a positive price though it produces no output or utility. There are two issues here: why the positive price, why the universal usage. Positive price comes from acceptability in payment in taxes (a notion that goes back to Smith (1776)). In the treatment developed below, universal usage comes from a thick markets externality, Rey (2001). Government's large scale creates the market thickness, leading the economy to a corner solution where government money is the unique actively traded medium of exchange.

In order to study fiat money in the model, we introduce a government, denoted G, with the unique power to issue fiat money. Fiat money is intrinsically worthless; it enters no one's utility function. But government is capable of declaring it acceptable in payment of taxes, and fulfillment of tax obligations does enter utility functions.

Taxation and fiat money's value in payment of taxes explains the positive equilibrium price of fiat money. Acceptability in payment of taxes at a fixed rate creates a market-based value. Government's large scale interacts with thick market externality in transaction costs to explain fiat money's uniqueness as the medium of exchange. Everyone needs money to pay his/her taxes, hence actively trading his/her endowment for money. Government buys most goods, paying in money. High trading volume with thick market externality creates low transaction costs in the commodity pairwise trading posts of money for goods. Thick money-goods trading posts contrast with thin barter trading posts between goods. Thick markets have narrow bid/ask spreads; thin markets have wide spreads. The bid/ask spread is the price of liquidity. Narrow spreads induce high trading volumes.

14.7.2 Fiat Money Equilibrium Model

Good $N+2$ represents fiat money. Government, G, sells $N+1$ (tax receipts) for $N+2$ at a fixed ratio of one-for-one. The trading post $\{N+1, N+2\}$ where tax receipts are traded for $N+2$ operates, for simplicity, with zero transaction cost. Acceptability in payment of taxes ensures $N+2$'s positive value. If, in addition, trading posts for other goods n for $N+2$ have sufficiently low transaction cost, then $N+2$ becomes the common medium of exchange.

This is a single period flow model, so all transactions are modeled to take place simultaneously. There is no role for stocks of money or inventories of goods. Each household i has endowment of one unit of commodity $i(modN)$. The household sells its endowment for $N+2$, fiat money. The household spends this income buying its desired goods, $i+1(modN)$ through $(i+(\frac{N}{2}-1))(modN))$ and in paying taxes for $N+1$, tax receipts. $i+1(modN)$ through $(i+(\frac{N}{2}-1))(modN))$ are denoted $i \oplus 1, ..., i \oplus (\frac{N}{2}-1)$. Goods markets, $n = 1, 2, ..., N$, and money market, $N+2$, clear.

In the κ-N economy, all households i are endowed with one unit of good $i(modN)$. They desire goods $i \oplus 1, ..., i \oplus (\frac{N}{2} - 1)$. And they each desire to pay their tax bill.

There is a large finite set of households, H, numbered $1, 2, \ldots, \kappa N$, totalling an integer multiple κ of N elements. Each household $i \in H$ is has an endowment of one unit of good $i(modN)$ and desires goods $i \oplus 1, \ldots, i \oplus (\frac{N}{2} - 1)$, plus payment of taxes. Thus, this is a large finite economy with complete absence of double coincidence of wants.

Let marginal transaction cost be $\alpha + [\beta \cdot f(Q)]$, assumed equal to average cost. We will use Specification Primo. Consistent with Specification Primo, $\beta \cdot f(Q)$ is the variable portion of marginal transaction cost.

Q is specified as the gross transaction quantity, including transaction cost. That is, if ten units are to be acquired and the transaction markup is 40% then the gross transaction quantity is 14 and $Q = 14$.

Household i has a tax bill of τ units of good $N + 1$. τ is assumed to be sufficiently small that the household, after paying taxes and transaction costs on sale of endowment for $N + 2$, has a positive amount of $i(modN)$ left for trade.

For goods $k, N + 2$, $k = 1, 2, \ldots, N$, transacting between fiat money and real goods, the transaction costs reflect the transaction technology above. For goods

$k, l \neq N + 2$, direct barter trading posts will be thin, and reflect the thin market transaction cost above.

The typical household utility function shows a desire to pay taxes and is Leontieff style in goods $n = i \oplus 1, \ldots, i \oplus ((\frac{N}{2}) - 1)$. Household i's utility function is

$u^i(x) = 2(\min(-\tau + x_{N+1}, 0)) + \min\{x_n | n = i \oplus 1, \ldots, i \oplus (\frac{N}{2} - 1)\}$.

Transaction costs and retail markup between $N + 1$ and $N + 2$ are assumed nil,

$\pi(N + 2, N + 1) = 0 = \pi(N + 1, N + 2) = 0$.

$N + 1$ designates the tax payment receipt typically purchased by a household for money. $N + 2$ designates fiat money. For $n = 1, 2, \ldots, N$,

$x^{GB}(n, N + 2) = \tau\kappa$, government buys n for fiat money in the amount of κ times the household tax bill.

$x^{GS}(N + 2, n) = -\tau\kappa$, government delivers (sells) $N + 2$ in exchange for good n in the amount of κ times the household tax bill. .

$x^{GB}(N+2, N+1) = \tau N\kappa$, government acquires $N+2$, fiat money, in the amount of population times the household tax bill, in exchange for tax receipts.

$x^{GS}(N+1, N+2) = -\tau N\kappa$, government sells $N+1$, tax receipts, in the amount of population times the household tax bill.

For household $i = 1, 2, ..., N, ..., \kappa N$.

$x^{iS}(i(modN), N+2) = -1$, i sells all endowment for $N+2$, fiat money.

$x^{iB}(N+2, i(modN)) = 1 \times \frac{1}{1+\pi(N+2,i(modN))}$, i buys fiat money from sale of endowment, discounted by transaction markup.

$x^{iB}(N+1, N+2) = \tau$, i buys tax receipts, in the amount of tax bill, for fiat money.

$x^{iS}(N+2, N+1) = -\tau$, i delivers fiat money to pay for tax receipts.

Net of disbursement for payment of taxes, household i's remaining money balance is $\left[\frac{1}{1+\pi(N+2,i(modN))} - \tau\right]$. This expression is the money value of endowment net of transaction cost, less the value of tax payment. To avoid boundary solutions we will assume this expression is positive.

For indices, n, $n = (i+1)(modN)$, to $(i + (\frac{N}{2} - 1))(modN)$,

$x^{iB}(n, N+2) = \frac{1}{(\frac{N}{2}-1)}\left[1-\tau[1+\pi(N+2, i(modN))]\right]\frac{1}{1+\pi(n,N+2)}$; household i takes his disposable money and allocates it to acquire equal amounts of goods $i \oplus 1$ through $i \oplus (\frac{N}{2} - 1)$, net of their transaction costs.

$x^{iS}(N+2, n) = -\frac{1}{(\frac{N}{2}-1)}\left[1 - \tau[1 + \pi(N+2, i(modN))]\right]$; household i pays for his purchases.

$q(i, j) = 1$ all $i, j = 1, 2, ..., N, N+1, N+2, i \neq j$; bid price for all goods is unity.

$\pi(i, j) = \alpha + \beta \cdot f(0), i \neq N+1, N+2 \neq j$; retail markup on barter trading posts reflects nil volume.

$\pi(N+2, N+1) = 0$; assumed transaction cost of zero from money $N+2$ to tax receipt $N+1$, so the retail markup there is zero. .

$Q(i(modN), N+2) = \kappa$. Household i sells all of his endowmwent for money, $N+2$. There are κ households of type $i(modN)$.

$\pi(i(modN), N+2) = 2\left[\alpha+\beta\cdot f(Q(i(modN), N+2))\right] = 2\left[\alpha+\beta\cdot f(\kappa)\right]$. Ask price markup consistent with formula and trading volume κ for each $i(modN) = 1, 2, \ldots, N$.

For each i, n so that $n, i(modN) = 1, 2, \ldots, N$

$Q(i(modN), N+2) = \kappa$. All transactions go through $N+2$.

$Q(i(modN), n) = 0$. There are no direct barter trades.

Total transaction cost expended $= N \times \kappa \times 2\Bigg[\alpha + \beta \cdot (f(\kappa))\Bigg]$. There are N active trading posts, each with trading volume κ.

14.7.3 Barter Equilibrium Model

Here's the story of the barter equilibrium. We maintain the same tax structure, but there is no money instrument. Each household i sends τ of his endowment to G accepting a transaction cost of $\pi(i(modN), N+1)$ in exchange for τ of $N+1$, a tax receipt. The remainder $[1 - [\tau(1 + \pi(i(modN), N+1)]]$ is disbursed acquiring equal portions of $n = [i+1](modN)$ to $n = [i + (\frac{N}{2} - 1)](modN)$. Now to formalize.

For each $i = 1, 2, ..., \kappa N$.

$x^{GB}(i(modN), N+1) = \tau\kappa$; each household i pays a tax of τ of $i(modN)$ to G. There are κ such households.

$x^{GS}(N+1, n(modN)) = -\tau\kappa$; G returns to each i a tax receipt $N+1$ in the amount τ.

For each $i = 1, 2, ..., \kappa N$.

$x^{iS}(i(modN), N+1) = -\tau[1 + \pi(i(modN), N+1)]$; household i delivers τ of endowment to G for

$x^{iB}(N+1, i(modN)) = \tau$

For each $n = i \oplus 1, ..., i \oplus (\frac{N}{2} - 1)$.

$x^{iB}(n, i(modN)) = [1 - \tau[1 + \pi(i(modN), N+1)]]\frac{1}{(\frac{N}{2}-1))} \times \frac{1}{1+\pi(n,i(modN))}$. Household i takes the remaining $i(modN)$ net of his tax payments and their transaction cost, and spends it at trading posts $i \oplus 1$ through $i \oplus (\frac{N}{2} - 1)$.

$x^{iS}(i(modN), n) = -[1 - \tau[1 + \pi(i(modN), N+1)]]\frac{1}{(\frac{N}{2}-1)}$

$q(i, n) = q(n, i) = 1$

$\pi(i(modN), N+1) = 2\Bigg[\alpha + \beta \cdot f(Q(i(modN), N+1))\Bigg]$

$Q(i(modN), N+1)) = \kappa\tau$

$\pi(i(modN), N+1) = 2\Big[\alpha + \beta \cdot (f(\kappa\tau))\Big]$

$\pi(i(modN), n) = 2\Big[\alpha + \beta \cdot f(Q(i(modN), n))\Big]$

$Q(i(modN), n) = \kappa \frac{1}{(\frac{N}{2}-1)}\Big[1 - \tau[1 + \pi(i(modN), N+1)]\Big] \times \frac{1}{1+\pi(i(modN),n)}$.

$\pi(n, i(modN)) = \Big[\alpha + \beta \cdot f(Q^{\circ}(n, i(modN)))\Big]$ where $Q^{\circ}(n, i(modN))$ is defined in subsubsection 7.7.4 below.

Total transaction costs are composed of two parts,

(i) tax-paying transaction costs at $\{n, N+1\}, n = 1, 2, \ldots, N$. These occur at N active trading posts.

$N \times Q(i(modN), N+1)) \times \pi(i(modN), N+1) = N \times \kappa\tau \times 2\Big[\alpha + \beta \cdot (f(\kappa\tau)\Big]$

(ii) barter exchange transaction costs at $\{i(modN), j\}, i(modN) \neq j; i = 1, 2, \ldots, \kappa N;$

$j = i \oplus 1, i \oplus 2, \ldots, i \oplus (\frac{N}{2} - 1)$. These costs occur at one side of $N(\frac{N}{2} - 1)$ active trading posts.

$N(\frac{N}{2} - 1) \times 2 \times Q(n, i(modN)) \times \pi(n, i(modN)) =$

$N(\frac{N}{2}-1) \times 2 \times \kappa \frac{1}{(\frac{N}{2}-1)}\Big[1 - \tau[1 + \pi(i(modN), N+1)]\Big] \times \Big[\alpha + \beta \cdot f(Q(n, i(modN)))\Big]$

14.7.4 Finding trading post volume Q° in barter equilibrium

For each household i, $i(modN) = 1, 2, \ldots, N$, there are κ households endowed with $i(modN)$ trading for $i \oplus 1, i \oplus 2, ..., i \oplus (\frac{N}{2} - 1)$. They are the only households trading at the the trading posts $\{i(modN), i \oplus 1\}, \{i(modN), i \oplus 2\}, ..., \{i(modN), i \oplus (\frac{N}{2} - 1)\}$. After paying taxes, each has $\big[1 - \tau[1 + \pi(i(modN), N+1)]$ left of its endowment of $i(modN)$. Household i divides the remaining endowment in $(\frac{N}{2} - 1)$ equal portions to purchase the desired goods gross of transaction costs. All of the transaction costs are assessed on the purchasers $i(modN)$, so acquisition cost is $\big[1 + 2[\alpha + \beta \cdot f(Q)]\big]$ per unit on each purchase .

Solving for the volume of trade at a typical barter trading post. Q° is the solution to the following expression.

$\kappa \frac{1}{\frac{N}{2}-1}\big[1 - \tau[1 + \pi(i(modN), N+1)]\big] = Q\big[1 + \pi(i(modN), n)\big]$
$= Q\big[1 + 2[\alpha + \beta \cdot f(Q)]\big]$

14.7.5 Welfare Comparison, Monetary vs. Barter Economy

This section includes a numerical example of the welfare comparison between monetary and barter equilibria for the economy described in section 14.7. After fixing the structure of transaction costs by fixing $\alpha = 0.01, \beta = 0.05$ and $\tau = 0.2$ and the total number of commodities to $N = 100$, the key features that inform the welfare comparison are taste for variety and size of the economy, which we describe below. The "size of the economy" is captured by the parameter κ. It controls the number of replicas of each houshold type and we let it range from a minimum of 1 to a maximum of 50. The parameter N_D controls the number of commodities that a household desires, out of 100 availble ones. It is labeled "taste for variety" capturing the fact that the households Leontieff preferences can be concentrated on a few commodities (low taste for variety) or many commodities (high taste for variety). In section 14.7, the "taste for variety" parameter amounts to half of all commodities available so $N_D = 50$. In this section we study its impact on equilibrium allocations by letting it range from a minimum of 2 to a maximum of 50. Each possible choice of these two parameters results in a different welfare for the associated barter and monetary equilibria. The graph below shows the difference between the total household welfare under monetary and barter equilibrium as level sets for each possible pair of the "size of the economy" and "taste for variety" parameters. Each solid line represents the level set associated to a particular difference between monetary and barter equilibrium welfare. For example, the blue line labelled "0" represents the configurations of "size of the economy" and "taste for variety" that result in a 0 welfare gap between the economy's monetary and barter equilibria. Put differently, along the solid line, monetary exchange does not deliver any welfare gains relative to barter. Any level set to the north-west of the "0" line, is associated to a positive welfare gap and lighter colors. For all such configurations, monetary equilibria are welfare-superior to barter equilibria. Conversely, any line to the south-east of the "0" line corresponds to a negative welfare gap and darker colors. For all such configurations, barter equilibria are welfare-superior to monetary equilibria. Why is that the case? The welfare gains from concentrating on a unique medium of exchage depend on how large are the thick market externalities, which itself depends on the "taste for variety" and "size of the economy" parameters.

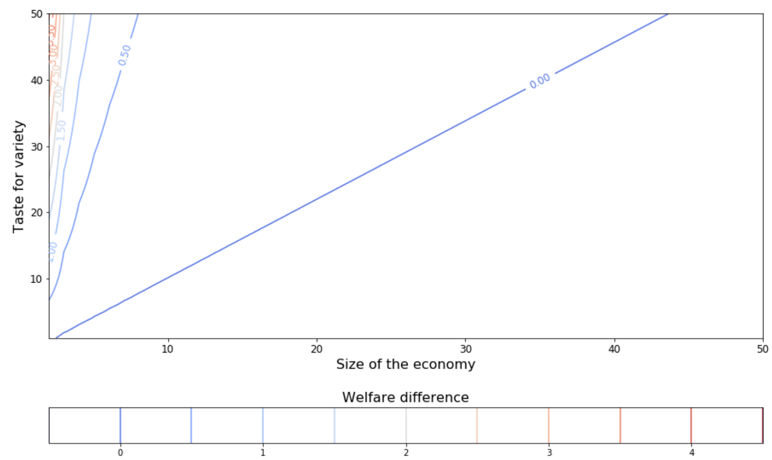

Figure 2: The figure shows two parameters of interest: a taste for variety parameter capturing the number of different commodities that households want (ranging from 2 to 50) and a size of the economy parameter, capturing how many copies of the household of each type are in the economy. The figure shows, in shaded colors, the level sets of the welfare gain (i.e. the value of the welfare of the monetary equilibrium minus the value of the welfare of the barter equilibrium) for each value of the parameters. Here the transaction cost parameters are set to $\alpha = 0.5$ and $\beta = 0.1$. We can see that for values of the parameters to the south-east of the 0 level set, the barter equilibrium is welfare superior. This happens when the taste for variety parameter is small relative to the size of the economy. If the economy is relatively homogeneous in tastes and each taste class is large, a few *quid pro quo* exchanges can already benefit from the scale economy and the concentration of transactions on a single medium of exchange is less attractive. Conversely, when the taste classes are small, the scale economy within each taste class is not strong enough and a single medum of exchange is more attractive.

One can see that if the "taste for variety" is sufficiently high relative to the size of the economy, the monetary equilibrium leads to higher welfare than barter. At a high level, this reconciles a general idea about the adoption of money (and therefore the prevalence of a monetary equilibrium). In economies where the preferences are varied enough (households desire many commodities), concentrating trade on a single medium of exchage is better than many disparate trades. If the preferences are concentrated enough relative to the size of the economy, direct barter is better because there are only a few commodities to acquire. While this numerical example tackles the special case of symmetric endowments and Leontief preferences (which induce symmetric equilibria) it is still instructive to tease out the relationship between deep parameters like "taste for variety" and "size of the economy" and the prevalence of monetary equilibria over barter equilibria.

15 Alternative Approaches

15.1 Money Demand and Store of Value

This essay presents a single period model, so it does not treat directly the stock of money held over time. Particularly germane in this context is the Baumol (1952)-Tobin (1956) model. Treating that model in an Arrow-Debreu general equilibrium context is beyond the scope of this paper, but the outlines of an interpretation can be developed. Note that the literature includes some notable work on the topic, Hahn (1971), Heller (1974), Heller & Starr (1976), Kurz (1974), Starrett (1973). The long-standing Arrow-Debreu model is interpreted over time to have complete futures and contingent commodity markets. The clear next step is to treat that model with sufficiently rich transaction costs to require a distinct store of value, the Arrow futures markets of Arrow (1964).

15.2 Matching Models

How does commodity money in the trading post model compare to commodity money in the trade search literature, (Kiyotaki & Wright 1989, 1993)? In the trading post model, the commodity money can arise in equilibrium because the firm's transaction cost structure can support a bid-ask spread that is different for each commodity pair. There is a similarity to the transaction costs in Kiyotaki & Wright (1989). In that model, trade costs endogenously arise as a function of both agents' trading strategies and storage costs. The storage cost is a deep parameter of the model which determines the type of commodity money equilibria that can arise.

In the case where there is a thick market externality, Rey (2001) and below section 14, in the trading post model, there is another construction here paralleling the matching model. In the trading post model, coordination around a commodity money would be mediated by the effect of trade volumes in the firm's technology and depends on market forces. Kiyotaki & Wright (1989) uses a belief system about other agents' adoption of commodity money as a self-verifying coordination device, designating *speculative* equilibria. Hence, in both models, the nature of commodity money equilibria can be constructed to be self-enforcing, obtaining the same qualitative economic result.

Finally, in both models, commodity money equilibria may support more than one commodity money at a time, albeit for households' with different sets of preferences.

An additional framework is the one presented by Howitt (2005) which combines elements of cash-in-advance and search theoretic models. Our treatment shares the existence of a monetary equilibrium under very general assumptions on preferences. In Howitt (2005) monetary and commodity *quid-pro-quo* transactions may coexist. Conversely in the trading post model, full monetary equilibrium may arise even when there is double coincidence of wants, analogous to Howitt (2005) robust monetary equilibrium, which is not always guaranteed to exist. We believe this latter feature reflects the almost exclusive monetary nature of real world transactions.

An early account of money in a strategic setting is also given in Shubik (1973). The author notes that "money enters into trade in a way that distinguishes it strategically from other commodities". The present paper incorporates this lesson in that commodity money is an equilibrium notion that arises from households' trading strategies as opposed to households' preferences.

15.3 Existence of General Equilibrium with Commodity Money

In the No Externalities Case, Starr (2008) presents a general equilibrium in a trading post model, paralleling Theorem 1 here. The distinctive result there is to demonstrate the existence and transaction function of commodity money, endogenously as the result of elementary properties of the economy and its equilibrium. There is endogenous determination of commodity money(s). At the trading post of commodities k and ℓ, $\{k, \ell\}$,

$\frac{q(k,\ell)}{q(\ell,k)+\pi(\ell,k)}$ in this paper is equivalent there to the bid price of k, $b_k^{\{k,\ell\}}$, in exchange for ℓ at its ask price , $a_\ell^{\{k,\ell\}}$. Hence $\frac{q(k,\ell)}{q(\ell,k)+\pi(\ell,k)} \approx \frac{b_k^{\{k,\ell\}}}{a_\ell^{\{k,\ell\}}}$.

16 Conclusion

Menger (1892) gives us a price theory of money. Liquidity is characterized by a narrow bid-ask spread. Goods are traded for more liquid goods that evolve into 'generally acceptable media of exchange.' This paper formalizes that argument mathematically. The concept of transaction cost is embodied in the bid-ask spread denoted here π, (Foley 1970). General acceptability is formalized here as a thick market externality, (Rey 2001). Actively traded goods have low transaction cost; low transaction cost encourages active trade as media of exchange.

This paper treats three distinctive modeling approaches. For each household, budgets are enforced separately at each of many transactions. At each transaction there is a bid-ask spread. External effects enter into technology following (Arrow & Hahn 1971). Here the technology is transaction technology determining resources used up in the exchange process.

Tobin (1980) argues that money is a public good, analogous to a common language. The approach here focuses on the related concept of external economy. In the case of language, each speaker of English in the USA enhances the communication possibilities for other English-speakers. In the case of money, each user of US dollars enhances the trading possibilities of other dollar users. That is the thick markets externality.

The present model derives the need for an instrument to act as a carrier of value (fiat or commodity money) between trades. It does so — in contrast to the Arrow-Debreu general equilibrium —- by allowing households and firms to deal with many separate budget constraints at a succession of transactions. The treatment explicitly accounts for external effects in a general equilibrium of the model with those multiple budget constraints. The external effect accounts for the public good quality of a common money.

17 Proofs of Lemmas

17.1 Lemma 1

Lemma 1. *Assume (E.I) through (E.V). Let:*

$$\Xi^j \equiv \{(y^{jB}, y^{jS}, w^j) \in Y^{j*} | \text{ For each } k = 1, 2, \ldots, N$$

$$\sum_{\ell} [w(k, \ell)] \geq -\sum_{\ell} r(k, \ell)\}$$

Then Ξ^j is compact.

Proof. It's immediately clear that Ξ^j is closed since Y^{j*} is closed and Ξ^j is defined by a set of linear inequalities. It remains to show that Ξ^j is bounded. We show this by contradiction: suppose Ξ^j is not bounded for some $j \in F$. Then, there exists a sequence $\{y^{\nu j'}\}$ such that:

(i) $\lim_{\nu\to\infty} |y^{\nu j}| = +\infty$

(ii) $y^{\nu j} \in \Xi^j$ for all ν sufficiently large.

Now, since $y^{\nu j} \in \Xi^j$ this implies that $y^{\nu j} \in Y^{*j}$ and that the following inequality is satified:

$$\sum_\ell [w^{\nu j}(k,\ell)] \geq -\sum_\ell r(k,\ell)$$

Observe that $0 \in \Xi^j$. Now set $\mu_\nu := |y^{\nu j}|$. By convexity of Ξ^j we may construct the vector $\tilde{y}^{\nu j} := \frac{1}{\mu_\nu} \cdot y^{\nu j} + (1 - \frac{1}{\mu_\nu}) \cdot 0$. But then, by convexity $\tilde{y}^{\nu j} \in \Xi^j$. But then, for the fact $y^{\nu j} \in \Xi^j$ automatically $\tilde{y}^{\nu j}$ must satisfy:

$$\sum_\ell [\tilde{w}^{\nu j}(k,\ell)] \geq \frac{1}{\mu_\nu}\left(-\sum_\ell r(k,\ell)\right)$$

Moreover $\tilde{y}^{\nu j}$ is clearly bounded so it has a converging subsequence so $\tilde{y}^{\nu j} \to y^{jo}$. In particular, by construction $\lim_{\nu\to\infty} |\tilde{y}^{\nu j}| = 1$. But the inequality above requires, since $\lim_{\nu\to\infty} \frac{1}{\mu_\nu} \sum_\ell r(k,\ell) \to 0$ so that $\sum_l \tilde{y}^{jS\circ}(k,l) + \sum_l \tilde{y}^{jB\circ}(k,l) \leq 0$.

Additionally note that the second inequality implies $\lim_{\nu\to\infty} \tilde{w}^{\nu j}(k,\ell) \geq 0$ which, jointly with the requirement that $\tilde{w}^{jo}(k,\ell) \leq 0$ implies $\tilde{w}^{jo}(k,\ell) = 0$. To see that this second requirement is true observe that by E.II we have, for any $(y^{jS}, y^{jB}, w^j) \in$

$\phi(\hat{x}, \hat{y})$ we have $\tilde{w}^{jo}(k, \ell) \leq 0$. Then, let $(y^{jS}, y^{jB}, w^j) \in Y^{**j} = \bigcup_{\hat{x},\hat{y}} \varphi(\hat{x}, \hat{y})$. Then $(y^{jS}, y^{jB}, w^j) \in \varphi(\hat{x}, \hat{y})$ for some $(\hat{x}, \hat{y})$ so $w^j \leq 0$ by E.II. We now show that this must be true for vectors in Y^{*j}, the closed convex hull of Y^{**j}. Take $(y^{jS}, y^{jB}, w^j) \in Y^{*j}$. By definition, it can be written as $(y^{jS}, y^{jB}, w^j) = \lim_{m \to \infty}(y^{mjS}, y^{mjB}, w^{mj})$ where each vector $(y^{mjS}, y^{mjB}, w^{mj}) \in co(Y^{**j})$. But then we have:

$$
\begin{aligned}
(y^{jS}, y^{jB}, w^j) &= \lim_{m \to \infty}(y^{mjS}, y^{mjB}, w^{mj}) \\
&= \lim_{m \to \infty} \alpha_m(y^{'mjS}, y^{'mjB}, w^{'mj}) + (1 - \alpha_m)(y^{''mjS}, y^{''mjB}, w^{''mj})
\end{aligned}
$$

for $(y^{'mjS}, y^{'mjB}, w^{'mj}), (y^{''mjS}, y^{''mjB}, w^{''mj}) \in Y^{**j}$. Then ceratinly:

$$
\begin{aligned}
w^j &= \lim_{m \to \infty} \alpha_m w^{'mj} + (1 - \alpha_m)w^{''mj} \\
&\leq \lim_{m \to \infty} \alpha_m 0 + (1 - \alpha_m)0 \\
&\leq 0
\end{aligned}
$$

Then for any $(y^{jS}, y^{jB}, w^j) \in Y^{*j}$ we have $w^j \leq 0$ coordinate-wise.

So, as discussed above $\tilde{w}^{jo}(k, \ell) = 0$. Observe that the transaction cost (sub)-vector $\tilde{w}^{jo}(k, \ell) = 0$ is the transaction cost associated to the production vector $(\tilde{y}^{joB}, \tilde{y}^{joB})$. But then, since $(\tilde{y}^{joB}, \tilde{y}^{joB}, \tilde{w}^{jo}) \in Y^{*j}$ by convexity we must conclude that $\tilde{w}^{jo} = 0$. On the other hand $|\tilde{y}^{jo}| = 1$, so certainly $\tilde{y}^{jo} \neq 0$. Yet $\tilde{w}^{jo} = 0$ which contraddicts E.IV. But this is impossible. Hence Ξ^j is bounded and closed. By the Heine-Borel theorem, since Ξ^j closed and bounded subset of a finite dimensional space, it's compact. □

Remark: Here convexity of the $Y^j = \varphi^j(\hat{x}, \hat{y}) \subset \mathbb{R}^{3N(N-1)}$ disciplines the shape that the w and the (y^B, y^s) can jointly take. In particular requiring the no free lunch condition for the set Y^{*j}, together with the convexity of Y^{*j} has very strong implications on the shape of the transaction costs. For example, it rules out parametric dependeces of the "quadratic type" like the one discussed below. Setting $v(k,l)(\hat{x}, \hat{y}) = |\sum_{h \in H} x^{hS}(k,l)| + |\sum_{h \in H} x^{hB}(k,l)|$ to be the volume of demands on a given trading post pair.

$$w^j(l,k)(\hat{x}, \hat{y}) = \frac{1}{|v(k,l)|^2}(|y^{jS}(k,l)| + |y^{jB}(k,l)|)$$

Then consider the 3-dimensional subspace of $\mathbb{R}^{3N(N-1)}$ given by the (k,l) coordinates of y^j.
This is simply $(y^{jS}(k,l), y^{jB}(k,l), w^j(k,l))$. Then Y^{**j} must contain points arbitrarily close to the plane $(a, b, 0)$ for arbitray $a \in \mathbb{R}$ and $b \in \mathbb{R}$ because for a large enough $\hat{x}$ $v(l,k)$ can be made arbitraily large and hence $w^j(l,k)$ can be made arbitraily small. But then, because Y^{*j} is the closed convex hull of Y^{**j}, it must contain the line $(0, a, b)$. But there is a cost free, non-zero transaction plan available in Y^{*j}. This violates E.IV.

Remark: The treatment of Lemma 1 is more general but the above example of a firm that takes households and other firms transaction plans parametrically is still very instructive. Consider the following formulation of firms transaction costs which generalizes the equation above:

$$w^j(l,k)(\hat{x}, \hat{y}) := \kappa(\hat{x}, \hat{y})(|y^{jS}(k,l)| + |y^{jB}(k,l)|)$$

For example, in the special case we treated above $\kappa(\hat{x}, \hat{y}) = \frac{1}{|v(k,l)|^2}$.

With this formulation, E.IV imples a very concrete condition: the coefficient $\kappa(\hat{x}, \hat{y})$ must be uniformly (in $\hat{x}$ and $\hat{y}$) bounded below by some $\kappa^* = \inf_{\hat{x}\hat{y}} \kappa(\hat{x}, \hat{y}) > 0$.

Corollary Assume E.I through E.IV. Then the set of attainable elements $(y^S, y^B, w) \in$

Y is bounded. And for each $j\prime \in F$, the set of $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ is bounded.

We recall the specification of attainable set. A vector of exchanges (y^S, y^B, w) is said to be attainable if it satisfies:

$$\begin{aligned}
&\sum_{j\in F}\sum_{l} y^{jS}(k,l) + \sum_{j\in F}\sum_{l} y^{jB}(k,l) \leq \sum_{i\in H}\sum_{l} r^i(k,l) \\
&-\sum_{j\in F}\sum_{l} w^j(k,l) \leq \sum_{i\in H}\sum_{l} r^i(k,l)
\end{aligned}$$

The first condition says that the net purchase of resources cannot exceed the endowment of the economy as a whole. The second inequality says that total disbursement in transaction costs cannot exceed the total endowment of the economy.

Now a vector of exchanges $y^{j'} := (y^{j'S}, y^{j'B}, w^{j'})$is attainable in $Y^{j'}$ if, for each $j \neq j'$ there is $y^j := (y^{jS}, y^{jB}, w^j) \in Y^j$ such that $y^{j'} + \sum_{j\neq j'} y^j$ is attainable.

Recall E.II (that is, P.I through P.IV).

Proof: The proof proceeds by contradiction. We will denote $y^{\nu j'} := (y^{S\nu j'}, y^{B\nu j'}, w^{\nu j'})$ First suppose there exist indeed one $j' \in F$ such that its attainable set is unbounded. Then, there must be a sequence $\{y^{\nu j}\} \subset Y^j$, for each $j \in F$ such that the following properties hold:

i) $|(y^{S\nu j'}, y^{B\nu j'}, w^{\nu j'})| \rightarrow +\infty$

ii) $y^{\nu j} \in Y^j \ \forall j \in F$

iii) $y^{\nu} := \sum_{j\in F} y^{\nu j}$ are all attainable.

set $\mu_v := \max_{j\in F} |y^{\nu j}|$. Because for at least one firm j', the set of attainable y is unbounded we immediately have $\mu_v \rightarrow +\infty$ as $\nu \rightarrow +\infty$.

By P.II, $0 \in Y^j$. By P.I, convexity of the Y^j, for ν sufficiently large, we can write $\tilde{y}^{vj} = \frac{1}{\mu_\nu} y^{\nu j} + (1 - \frac{1}{\mu_\nu})0 \in Y^j$. The attainability condition requires:

$\sum_{j\in F}\sum_l [\tilde{y}^{Svj}(k,l) + \tilde{y}^{Bvj}(k,l)] \leq \frac{1}{\mu_\nu}\sum_{i\in H}\sum_l r^i(k,l)$

$\sum_{j\in F}\sum_l \tilde{w}^{vj}(k,l) \geq -\frac{1}{\mu_\nu}\sum_{i\in H}\sum_l r^i(k,l)$

Observe that by definition the sequences $\tilde{y}^{\nu j}$ are bounded since $|\tilde{y}^{\nu j}| \leq 1$. Consider the vector $(\tilde{y}^{\nu 1}, \tilde{y}^{\nu 2}, \cdots \tilde{y}^{\nu \# F})$. Because this sequence is bounded (immediate to show that it is bounded by $|F|$) there exist a converging sub-sequence that converges to a limit $(\tilde{y}^{o1}, \tilde{y}^{o2}, \cdots, \tilde{y}^{o\#F})$ where $\tilde{y}^{oj} \in Y^j$, by P.III (Y^j closed). The right hand side of the inequalities above imply, in the limit: $-\frac{1}{\mu_\nu} \sum_{i \in H} \sum_l r^i(k,l) \to 0$. We have:

$\sum_{j \in F} \sum_l \tilde{y}^{ojS}(k,l) + \sum_{j \in F} \sum_l \tilde{y}^{ojB}(k,l) \leq 0$

$\sum_{j \in F} \sum_l \tilde{w}^{oj}(k,l) \geq 0$

Hence they imply:

$\sum_{j \in F} \sum_l \tilde{y}^{ojS}(k,l) + \sum_{j \in F} \sum_l \tilde{y}^{ojB}(k,l) + \sum_{j \in F} \sum_l \tilde{w}^{oj}(k,l) \leq 0$ since $\sum_l \tilde{w}^{oj}(k,l)$ is always non-positive. Moreover, from the nonpositivity of $\tilde{w}^{oj}(k,l)$ it must be that

$\sum_l \sum_{j \in F} \tilde{w}^{oj}(k,l) \leq 0$. Together with $\sum_{j \in F} \sum_l \tilde{w}^{oj}(k,l) \geq 0$ it must be the case that:

$\sum_l \sum_{j \in F} \tilde{w}^{oj}(k,l) = 0$

This term, for each k, is the sum of $(N-1) \cdot \#F$ terms. But then, since each $\tilde{w}^{oj}(k,l)$ is non positive it must be the case that for each (l,k), for each $j \in F$, $\tilde{w}^{oj}(k,l) = 0$. Hence, for each $j \in F$ we have: $\tilde{w}^{oj} = 0$. Notice now that, since Y^j is closed for every j, $(\tilde{y}^{ojS}, \tilde{y}^{ojB}, \tilde{w}^{oj}) \in Y^j$. But then, by condition P.IV (i), $\tilde{w}^{oj} = 0$ implies $(\tilde{y}^{ojS}, \tilde{y}^{ojB}, \tilde{w}^{oj}) = 0$.

But then for each $j \in F$, $|\tilde{y}^{vj}| \to |\tilde{y}^{oj}| = 0 \neq 1$ which shows the desired contradiction. That is, there is no such firm j' for which the attainable set is unbounded. Then the set of attainable transactions as a subset of Y^j is bounded for every $j \in F$.

Remark: Incidentally, since the attainable set for every firm j is bounded and there are a finite number F of firms, the attainable set for the whole economy is also bounded.

17.2 Lemma 2

Lemma 2. *Let $p \in \mathbb{R}_+^{2N(N-1)}$. $B(p)$, $S^j(p)$, and $S^{j\dagger}(p)$ are homogeneous of degree zero in p.*

Proof: By inspection.

17.3 Lemma 3

Lemma 3. *Assume E.I through E.IV. Then* $\tilde{S}^{j\dagger}(p)$ *and* $\tilde{S}^j(p)$ *are nonempty, convex-valued, and upper hemicontinuous throughout* $p \in \Delta$. *Let* $(y^{jS}, y^{jB}) \in \tilde{S}^j(p)$ *be attainable. Then* $(y^{jS}, y^{jB}) \in S^j(p)$. *Let* $(y^{jS}, y^{jB}, w^j) \in \tilde{S}^{j\dagger}(p)$ *be attainable. Then* $(y^{jS}, y^{jB}, w^j) \in S^{j\dagger}(p)$.

Recall E.I, E.II, E.III, E.IV.

$$\begin{aligned}\tilde{S}^j(p,\pi) :=& \{(y^B, y^S) | (y^B, y^S, w) = \arg\max\ (q, q+\pi)\cdot(y^{jS}, y^{jB} + w^j) \\ & \text{subject to } (y^B, y^S, w) \in Y^j \cap B(q,\pi) \times R^{N(N-1)} \cap \Psi^3\}\end{aligned}$$

Proof. The proof will use the maximum theorem. We first check the hypotheses. Define $C(q,\pi) := \{(y^B, y^S, w) \in Y^j \cap B(q,\pi) \times R^{N(N-1)}\} \cap \Psi^3$. Observe that $C(q,\pi)$ is a continuous correspondence in (q,π) because $B(q,\pi)$ is a continuous correspondence. Moreover, for any $(q,\pi), C(q,\pi)$ is closed and bounded. Closed-ness follows from definition as the intersection of closed sets. Bounded-ness follows from Ψ^3 being bounded. By the Heine-Borel theorem $C(q,\pi)$ is compact. $C(q,\pi)$ is compact valued. Moreover recall that $(q, q+\pi)\cdot(y^{jS}, y^{jB} + w^j)$ is a continuous function of p. Then, by the maximum theorem, the correspondence

$C^*(q,\pi) := \{\arg\max\ (q, q+\pi)\cdot(y^{jS}, y^{jB} + w^j) | (y^B, y^S, w) \in C(q,\pi)\}$ is nonempty, compact valued and upper hemicontinous. It remains to show that $C^*(q,\pi)$ is convex valued. Recall that $(q, q+\pi)\cdot(y^{jS}, y^{jB} + w^j)$ is linear in (y^{jS}, y^{jB}, w^j), implying that the upper level set is convex. Fix (q,π). Let c^* be the maximum for $(q,q+\pi)\cdot(y^{jS},y^{jB}+w^j)$ on $C(q,\pi)$. The upper level set $\{(y^B,y^S,w)|(q,q+\pi)\cdot(y^{jS},y^{jB}+w^j)\geq c,\ (y^B,y^S,w)\in C(q,\pi)\}=C^*(q,\pi)$ is convex. Because the choice of (q,π) was arbitrary we have convex valued-ness of the $C^*(\cdot)$ correspondence. Recognize that $C^*(q,\pi)=\tilde{S}^{j\dagger}(q,\pi)$ which finishes the proof of the first part.

$$\tilde{S}^j(p)\equiv\{(y^S,y^B)|(y^S,y^B,w)=\arg\max(q,q+\pi)\cdot(y^{jS},y^{jB}+w^j)$$

$$\text{subject to }(y^S,y^B,w)\in(Y^j\cap[B(p)\times R^{N(N-1)}]\cap\Psi^3)\}.$$

Let $(y^{\circ S},y^{\circ B},w^\circ)\in\tilde{S}^{j\dagger}(p)$. Suppose contrary to the lemma there is $(y^S,y^B,w)\in Y^j$, $(y^S,y^B,w)\in S^{j\dagger}(p)$ so that $(q,q+\pi)\cdot(y^S,y^B+w)>(q,q+\pi)\cdot(y^{\circ jS},y^{\circ jB}+w^{\circ j})$. Then for $1>\alpha>0,\alpha\neq 0,1$, sufficiently large, there is $\alpha(y^{\circ S},y^{\circ B},w^\circ)+(1-\alpha)(y^S,y^B,w)\in Y^j\cap\Psi^3$ by convexity of Y^j.

But then $(q,q+\pi)\cdot[\alpha(y^{\circ S},y^{\circ B}+w^\circ)+(1-\alpha)(y^S,y^B+w)]>(q,q+\pi)\cdot(y^{\circ jS},y^{\circ jB}+w^{\circ j})$; a contradiction.

$\square$

17.4 Lemma 4

Lemma 4. *Let* $\hat{y}\in\Psi^{3\#F}$.

(i) Then $\tilde{D}^i(p,\hat{y})$ *is nonempty and homogeneous of degree zero in p.* $\tilde{A}^i(p,\hat{y})$ *is continuous (upper and lower hemicontinuous) throughout* $\Delta\times\Psi^{3\#F}$ *and convex-valued.*

$\tilde{D}^i(p, \hat{y})$ is upper hemicontinuous throughout Δ and convex-valued.

(ii) Let $x^i \in \tilde{D}^i(p, \hat{y})$ be attainable. Then $x^i \in D^i(p, \hat{y})$.

Remark: Part (ii) says that if the length constraint in $\tilde{D}^i(p, \hat{y})$ is not a binding constraint then it can be deleted, and x^i is optimizing subject to distribution and budget only, without the requirement that household i limit the size of its plans.

The proof of Lemma 4 takes place in several steps. In order to demonstrate upper and lower hemicontinuity of $\tilde{A}^i(p, \hat{y})$ we'll characterize it as the intersection of many more elementary sets, $\Lambda^i_{l,k}(p, \hat{y})$, each of which is shown to be upper and lower hemicontinuous. We introduce a lemma of Green and Heller (1981) to demonstrate that the intersection — and hence $\tilde{A}^i(p, \hat{y})$ — is continuous. $\tilde{D}^i(p, \hat{y})$ is the set of maximizers of u^i in a translate of $\tilde{A}^i(p, \hat{y})$. Then the theorem of the maximum leads to upper hemicontinuity of $\tilde{D}^i(p, \hat{y})$. A conventional argument focusing on convexity leads to equivalence to $D^i(p, \hat{y})$.

Proof. We recall the definition of the quantities of interest:

$$\tilde{A}^i(p, \hat{y}) := B(p) + \{\textstyle\sum_{j \in F} \Theta^{ij} \left(\hat{y}^{jS}, \hat{y}^{jB} + \hat{w}^j\right) + r^i\} \cap \Psi^2$$

$$\tilde{D}^i(p, \hat{y}) = \left\{\{\arg\max u^i(x) | x \in [\tilde{A}^i(p, \hat{y}) \cap W^i]\} - \{r^i + \sum_{j \in F} \Theta^{ij}[(y^{jS}, y^{jB} + w^j)]\}\right\}.$$

where

$$B(p) := \{x \in R^{2N(N-1)} | q(k,l)x^S(k,l) + [q(k,l) + \pi(k,l)]x^B(k,l)$$
$$+ q(l,k)x^S(l,k) + [q(l,k) + \pi(l,k)]x^B(l,k) \leq 0, \text{for } 1 \leq k \neq l \leq N\}$$

Let $\Psi^{3\#F}$ be the #F-fold Cartesian product of Ψ^3 .

First we will characterize properties of the $\tilde{A}^i(p,\hat{y})$ correspondence. Non-emptiness follows from compactness of Ψ^2, $0 \in B(p)$, and $r^i \in \mathbb{R}_+^{2N(N-1)}$, $r^i >> 0$. Further, $\sum_j \sum_{l \neq k} \Theta^{ij} \left(\hat{y}^{jS}(k,l), \hat{y}^{jB}(k,l) + \hat{w}^j(k,l)\right)$ is non-negative, by P.IV(ii). Homogeneity of degree 0 follows immediately from homogeneity of degree 0 of $B(p)$ and the definition of $\tilde{A}^i(p,\hat{y})$. Incidentally $\tilde{A}^i(p,\hat{y})$ is compact because it's the (translated) intersection of a compact set and a closed set, so the intersection is closed and bounded and, by Heine-Borel theorem, compact. $B(p)$ is convex by construction. Moreover, $\tilde{A}^i(p,\hat{y})$ is defined as the intersection of convex sets, hence it's convex.

The core of the proof is to show continuity of $\tilde{A}^i(p,\hat{y})$. We will show upper hemicontinuity and lower hemicontinuity separately with the use of an auxiliary lemma. Consider $x \in \tilde{A}^i(p^o,\hat{y}^o)$. Then $x - (\sum_{j \in F} \Theta^{ij} \cdot (\hat{y}^{joS}, \hat{y}^{joB} + w^{jo}) + r^i) \in B(p^o)$. We introduce a useful construction below.

For any pair (l,k) define the correspondence:

$$\Lambda^i_{l,k}(p,\hat{y}) := \Bigg\{ x \in \mathbb{R}^{2N(N-1)} \text{ such that:}$$

$$x + \left(\sum_{j \in F} \Theta^{ij} \cdot [\hat{y}^{jS}, \hat{y}^{jB} + \hat{w}^j] + r^i \right) \in W^i \text{ and}$$

$$q(k,l) \cdot x^S(k,l) + [q(k,l) + \pi(k,l)] \cdot x^B(k,l)$$

$$+ q(l,k) \cdot x^S(l,k) + [q(l,k) + \pi(l,k)] \cdot x^B(l,k) \leq 0 \Bigg\}$$

Note that $\Lambda^i_{l,k}(p,\hat{y}) = \Lambda^i_{k,l}(p,\hat{y})$.

Remark: For a given $(p,\hat{y}), \Lambda^i_{l,k}(p,\hat{y})$ is the space of all trade vectors available to i that respect the budget constraint at trading post (l,k) at the prevailing prices p and economic activity $\hat{y}$. The relevant information for prices and transactions only concerns the (l,k) and (k,l) components (for both bid and ask, wholesale and retail) of the price and transaction vectors because these are the only two commodities traded at the particular trading post. There are $\frac{N(N-1)}{2}$ such trading posts.

We know that, for a trade commodity vector to be feasible it needs to respect all budget constraints at all trading posts simultaneously. As such we can write:

$$\tilde{A}^i(p,\hat{y}) = \bigcap_{l,k} \Lambda^i_{l,k}(p,\hat{y}) \cap \Psi^2$$

Recall that no provisional bound is imposed on each of the $\Lambda_{l,k}$, that is, each of the x in the $\Lambda_{l,k}$ are not required to lie in Ψ^2. Conversely, elements of the $\tilde{A}^i(p,\hat{y})$ are in the provisionally bounded set.

The reason of this construction lies in the following lemma:

Lemma (Green and Heller, 1981). *Let X and Y be subsets of Euclidean space*[1]*. Let $\Gamma_1 : X \rightrightarrows Y$. $\Gamma_2 : X \rightrightarrows Y$. $\Gamma_3 : X \rightrightarrows Y$. $\Gamma_4 : X \rightrightarrows Y$ be correspondences.*

(i) If Γ_1, Γ_2 are two upper hemicontinous closed-valued correspondences such that $\Gamma_1(x^o) \cap \Gamma_2(x^o) \neq \emptyset$ then $\Gamma(x^o) := \Gamma_1(x^o) \cap \Gamma_2(x^o)$ is upper hemicontinuous.

[1]We note that Y is simply a generic subset of the Euclidean space, it bears no relation with the Y or Y^j in the treatment of the paper.

(ii) If Γ_3, Γ_4 are two lower hemicontinous convex-valued correspondences such that int $(\Gamma_3(x^o)) \cap$ int $(\Gamma_4(x^o)) \neq \emptyset$ then $\Gamma(x^o) := \Gamma_3(x^o) \cap \Gamma_4(x^o)$ is lower hemicontinuous.

A plan of the proof of Lemma 4 is :

- 1) Fix the pair (l, k). Show that the correspondence $\Lambda^i_{l,k}(p, \hat{y})$ is upper and lower hemicontinuous in $(p, \hat{y})$.
- 2) Verify that the conditions for Green & Heller (1981) hold.
- 3) Conclude by induction that $\tilde{A}^i(p, \hat{y})$ is upper and lower hemicontinuous.

Fix a pair (l, k). Consider the correspondence $\Lambda^i_{l,k}(p, \hat{y})$ defined above.

1a) **Upper Hemicontinuity of** $\Lambda^i_{l,k}(p, \hat{y})$:

Consider a sequence $(p^\nu, \hat{y}^\nu) \in \Delta \times Y$ such that $(p^\nu, \hat{y}^\nu) \to (p^o, \hat{y}^o)$ and $x^\nu \in \tilde{A}^i(p^\nu, \hat{y}^\nu)$ as well as $x^\nu \to x^o$. We want to show $x^o \in \tilde{A}^i(p^o, \hat{y}^o)$. Suppose not. Because $\tilde{A}^i(p^o, \hat{y}^o)$ is closed we can take an open set U around x^o which guarantees that $U \cap \tilde{A}^i(p^o, \hat{y}^o) = \emptyset$. But then, we must have:

$$q^o(k,l) \cdot x^{oS}(k,l) + [q^o(k,l) + \pi^o(k,l)] \cdot x^{oB}(k,l)$$
$$+ q^o(l,k) \cdot x^{oS}(l,k) + [q^o(l,k) + \pi^o(l,k)] \cdot x^{oB}(l,k) > \epsilon$$

with $\epsilon > 0$, because x^o must be unfeasible at $(p^o, \hat{y}^o)$. But then, since each of the $x^\nu \in \tilde{A}^i(p^\nu, \hat{y}^\nu)$ we have, by the definition of the budget set:

$$q^{\nu}(k,l)\cdot x^{\nu S}(k,l)+[q^{\nu}(k,l)+\pi^{\nu}(k,l)]\cdot x^{\nu B}(k,l)$$

$$+\,q^{\nu}(l,k)\cdot x^{\nu S}(l,k)+[q^{\nu}(l,k)+\pi^{\nu}(l,k)]\cdot x^{\nu B}(l,k)\leq 0$$

for all ν. But then, by the order limit theorem, we must have $\lim_{\nu\to\infty} LHS_{\nu} \leq 0$. Thus, a contradiction. We conclude that $x^{o} \in \Lambda^{i}_{l,k}(p,\hat{y})$ so $\Lambda^{i}_{l,k}(p,\hat{y})$ is upper hemi-continuous.

Remark: There is a shorter proof that leaves out many of the details. Consider the graph of the correspondence $\Lambda^{i}_{l,k}$ given by $G_{k,l} := \{(p,\hat{y},\Lambda^{i}_{l,k}(p,\hat{y})\} \in \Delta\times\prod_{j} Y^{j}\times W^{i}$. Then the budget constraint at trading post (k,l) can be expressed as a function $f_{k,l} : \Delta\times\prod_{j} Y^{j}\times W^{i}\to\mathbb{R}$ where the explicit formula for f is given in the equation above. Moreover, $G_{k,l} = f^{-1}_{k,l}((-\infty,0])$ when the budget constraint is respected. But $f_{k,l}$ is clearly a continuous function (it only involves sums and multiplications) and since $(-\infty,0]$ is closed in $\mathbb{R}$, it follows that $G_{k,l}$ is also closed in $\Delta\times\prod_{j} Y^{j}\times\Psi^{3}$ since it's the pre-image of a closed set under a continuous function. By the Closed Graph Theorem $\Lambda^{i}_{l,k}(p,\hat{y})$ is an upper-continuous correspondence which is the desired result.

1b) **Lower Hemicontinuity of $\Lambda^{i}_{l,k}(p,\hat{y})$:**

Let $(p^{\nu},\hat{y}^{\nu})\to(p^{\circ},\hat{y}^{\circ})$, $x^{\circ}\in\Lambda^{i}_{l,k}(p^{\circ},\hat{y}^{\circ})$. We seek $x^{\nu}\in\Lambda^{i}_{l,k}(p^{\nu},\hat{y}^{\nu})$ so that $x^{\nu}\to x^{\circ}$.

Recall that $x^{\circ S}(k,l)\leq 0$, $x^{\circ B}(k,l)\geq 0$.

Here's how we'll construct x^{ν} . For $k',l'\neq k,l$ the choice of $x^{\nu}(k',l'), x^{\nu}(l',k')$

is unrestricted by the specification of $\Lambda^i_{l,k}(p^\nu, \hat{y}^\nu)$ so we can set $x^\nu(k', l'), x^\nu(l', k') = x^\circ(k', l'), x^\circ(l', k')$.

For k, l here's the logic of the construction. Recall that the superscript S denotes i's delivery at bid prices to the trading post. For $x^{\nu S}(k, l)$, how much (k, l) can household i arrange to deliver to the trading post? It is limited by its availability of k from all sources — endowment and dividends. If that's less than $x^{\circ S}(k, l)$ then i should deliver what it can. For ν sufficiently large, the availability will approach $x^{\circ S}(k, l)$. If the availability is greater, then it is sufficient to deliver $x^{\circ S}(k, l)$. Similarly for $x^{\nu S}(l, k)$.

The superscript B denotes i's acquisitions at ask prices from the trading post. For $x^{\nu B}(k, l)$ how much (k, l) can i afford to acquire at $(p^\nu, \hat{y}^\nu)$? That depends mainly on the values just established, $x^{\nu S}(k, l)$, $x^{\nu S}(l, k)$, that i delivers to (k, l), evaluated at p^ν. How does that compare to the value of $x^{\circ B}(k, l)$ and $x^{\circ B}(l, k)$ at p^ν? That's the fraction in the specification for $x^{\nu B}(k, l)$ below. When the budget in the numerator is bigger than the budget in the denominator then i can afford $x^{\circ B}(k, l)$ at p^ν. And for ν sufficiently large, i will converge on fully affordable. When i has an even more ample budget, it is sufficient just to plan on $x^{\nu B}(k, l) = x^{\circ B}(k, l)$. The calculation becomes trickier when budgets are zero, as may occur if prices are zero's. Then $x^{\circ B}(k, l)$ is fully affordable and is set equal to $x^{\nu B}(k, l)$. $x^{\nu S}(k, l), x^{\nu S}(l, k), x^{\nu B}(k, l), x^{\nu B}(l, k)$ are described below.

There are two settings to keep in mind to determine $x^{\nu B}(k, l)$ and $x^{\nu B}(l, k)$, depending on the value at p^ν of $x^{\circ B}(k,l)$ and $x^{\circ B}(l,k)$.

If $[q^\circ(k,l)+\pi^\circ(k,l)]\cdot x^{\circ B}(k,l)+[q^\circ(l,k)+\pi^\circ(l,k)]\cdot x^{\circ B}(l,k)>0$, then for ν large, $[q^\nu(k,l)+\pi^\nu(k,l)]\cdot x^{\nu B}(k,l)+[q^\nu(l,k)+\pi^\nu(l,k)]\cdot x^{\nu B}(l,k)>0$ and the fraction in the specification of $x^{\nu B}(k,l)$ and $x^{\nu B}(l,k)$ below is well defined.

On the contrary, when $[q^\circ(k,l)+\pi^\circ(k,l)]\cdot x^{\circ B}(k,l)+[q^\circ(l,k)+\pi^\circ(l,k)]\cdot x^{\circ B}(l,k)=0$, then the fraction in the description of $x^{\nu B}(k,l)$ below may not be well defined. If so the alternative there applies.

Proposed values of $x^{\nu S}(k,l), x^{\nu S}(l,k)$ follow, below. Recall that typically $x^{\circ S}(k,l)\leq 0$, so the "max" notation below means choosing the smaller absolute value. Under (C.VII) the summations in square brackets (that is [ ]) will typically be positive, and negative when denoted with a minus sign (−).

Let $x^{\nu S}(k,l)$ be defined as the maximum between $x^{\circ S}(k,l)$ and

$$-\sum_{m=1,m\neq k}^{N}\left[r^{iS}(k,m)+r^{iB}(k,m)+\sum_{j\in F}\Theta^{ij}[\hat{y}^{\nu jS}(k,m)+\hat{y}^{\nu jB}(k,m)+\hat{w}^{\nu j}(k,m)]\right]$$

Simlarly let $x^{\nu S}(l,k)$ be defined as the maximum between $x^{\circ S}(l,k)$ and

$$-\sum_{m=1,m\neq l}^{N}\left[r^{iS}(l,m)+r^{iB}(l,m)+\sum_{j\in F}\Theta^{ij}[\hat{y}^{\nu jS}(l,m)+\hat{y}^{\nu jB}(l,m)+\hat{w}^{\nu j}(l,m)]\right]$$

Let $x^{\nu B}(k,l)$ be defined as the minimum between $x^{\circ B}(k,l)$ and

$$\frac{|q^\nu(k,l)x^{\nu S}(k,l)+q^\nu(l,k)x^{\nu S}(l,k)|}{[q^\nu(k,l)+\pi^\nu(k,l)]x^{\circ B}(k,l)+[q^\nu(l,k)+\pi^\nu(l,k)]x^{\circ B}(l,k)}x^{\circ B}(k,l)$$

when the fraction is well defined.

Similarly, let $x^{\nu B}(l,k)$ be defined as the minimum between $x^{\circ B}(l,k)$ and

$\frac{|q^\nu(k,l)x^{\nu S}(k,l)+q^\nu(l,k)x^{\nu S}(l,k)|}{[q^\nu(k,l)+\pi^\nu(k,l)]x^{\circ B}(k,l)+[q^\nu(l,k)+\pi^\nu(l,k)]x^{\circ B}(l,k)} x^{\circ B}(l,k)$

when the fraction is well defined.

Then $x^{\nu B}(k,l)$ is the amount of (k,l) to be purchased. It is the smaller of $x^{\circ B}(k,l)$ and the affordable fraction of $x^{\circ B}(k,l)$ based on sales of $x^{\nu S}(k,l)$ and $x^{\nu S}(l,k)$ (each determined in the specification above). The affordable proportion is described in the fraction. The numerator is the budget at bid prices based on the sales. The denominator is the expenditure, evaluated at p^ν of the purchases on the (k,l) market $x^{\circ B}(k,l)$ and $x^{\circ B}(l,k)$. Not all of these terms need be nonzero, but that is endogenous. For $(k',l') \neq (k,l)$, let $x^{\nu S}(k',l') = x^{\circ S}(k',l'), x^{\nu B}(k',l') = x^{\circ B}(k',l')$; $x^{\nu S}(l',k') = x^{\circ S}(l',k'), x^{\nu B}(l',k') = x^{\circ B}(l',k')$. Then $x^\nu \in \Lambda^i_{k,l}(p^\nu, \hat{y}^\nu)$ and $x^\nu \to x^\circ$. Hence x^ν is the required sequence.

2) **Verifying Conditions for Green and Heller (1981) Lemma:** We now verify the conditions for the Green and Heller (1981) lemma. First, it follows immediately from the definition by a linear inequality that $\Lambda^i_{l,k}(p,\hat{y})$ is closed (it's the inverse image of a closed set $(-\infty, 0]$ under a continuous function, hence closed). Moreover, since Ψ^2 is bounded, so is $\Lambda^i_{l,k}(p,\hat{y})$. By the Heine-Borel theorem, $\Lambda^i_{l,k}(p,\hat{y})$ is compact. Now, clearly, $0 \in \Lambda^i_{l,k}(p,\hat{y})$ since the zero transaction vector, 0 , satisfies all the inequalities for any $(p,\hat{y})$. Hence $\Lambda^i_{l,k}(p,\hat{y})$ is not empty. Moreover, $0 \in \Lambda^i_{l,k}(p,\hat{y})$ regardless of the choice of the pair (l,k). Convexity is also immediate for $\Lambda^i_{l,k}(p,\hat{y})$ is the intersection of a set defined by a linear inequality, (which is con-

vex) and the convex set Ψ^2. Now, recall $r^i >> 0$. We need to show that for any given price and dividend distribution, the interior of the correspondence $\Lambda_{l,k}(\hat{p}, \hat{y})$ is not empty. To show this, it is sufficient to show that there is a collection $c_{l,k}$ of 5 transaction vector points that are in $\Lambda_{l,k}(\hat{p}, \hat{y})$ and are in general position. Recall that the construction of $\Lambda_{l,k}(\cdot)$ does not constrain any of the non-(k, l) coordinates of x, other than requiring them to be in W^i. Hence we can always augment a set of 5 points (as we obtain below) to a set of $2N(N-1)+1$ points $\{c_{l,k}, d\}$ by choosing a collection $\{d\}$ to be (the negative of) the standard basis vectors $-\{e_{l'}\}$ for all the non-(k, l) wholesale coordinates and the standard basis vector $\{e_{l'}\}$ for all the non-(k, l) retail coordinates of $\mathbb{R}^{2N(N-1)}$. Such vectors clearly live in W^i. We show below how to pick the 5 points in the $c_{k,l}$ collection. We emphasize the coordinates for the k, l, all other coordinates, indicated by the dots, are taken to be 0. Without loss of generality we may assume that none of the prices are 0, if they are then take local transaction vectors $x_1 = (\cdots, -100, 0, 0, 0, \cdots), x_2 = (\cdots, 0, 100, 0, 0, \cdots), x_3 = (\cdots, 0, 0, -100, 0, \cdots), x_4 = (\cdots, 0, 0, 0, 100, \cdots)$ respectively, whenever x_i's price is 0. In general, for nonzero prices take the 0 transaction vector, together with:

$$x_1 = \begin{pmatrix} \vdots \\ -\frac{r(k,l)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix};$$

$$x_2 = \begin{pmatrix} \vdots \\ -\frac{r(k,l)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ \frac{[r(k,l)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')]\cdot q(k,l)}{2\cdot[q(k,l)+\pi(k,l)]} \\ 0 \\ 0 \\ \vdots \end{pmatrix};$$

$$x_3 = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ -\frac{r(l,k)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ 0 \\ \vdots \end{pmatrix};$$

$$x_4 = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ -\frac{r(l,k)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')}{2} \\ \frac{[r(l,k)+\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l')]\cdot q(l,k)}{2\cdot[q(l,k)+\pi(l,k)]} \\ \vdots \end{pmatrix}$$

where the $x_{k,l}$ coordinates are listed above and all other coordinates are 0. which are guaranteed to be in general position since $r(k,l), r(l,k) > 0$ and the quantity $\sum_{j\in F}\sum_{l'\neq k}\Theta^{ij}\hat{y}^j(k,l') \geq 0$. Moreover, they are clearly in W^i, hence they satisfy the desired conditions. We take $c_{k,l} = \{x_0, x_1, x_2, x_3, x_4\}$. The collection of points $\{c_{l,k}, d\}$ preserves general position. Because $\Lambda^i_{l,k}(p,\hat{y})$ is convex, their convex hull, $\mathrm{co}(\{c_{l,k}, d\})$ will be contained in $\Lambda_{l,k}(\hat{p},\hat{y})$. By construction $\mathrm{co}(\{c_{l,k}, d\})$ is a $2N(N+1)$ dimensional polytope and has non-empty interior, showing the desired requirement.

Hence, there exists an open neighborhood $\mathcal{V}_{k,l}$ (in the subset topology) containing 0. Similarly, for a general correspondence $\Lambda^i_{l',k'}(p,\hat{y})$ there exists a neighborhood $\mathcal{V}_{k',l'}$ containing 0. But then, the neighborhood of 0 given by $\mathcal{V}_{k,l} \cap \mathcal{V}_{k',l'} \neq \emptyset$ and by the definition of neighborhood there exist a non-empty open set $\mathcal{V} \subset \text{int}(\mathcal{V}_{k,l}) \cap \text{int}(\mathcal{V}_{k',l'}) \subset \text{int}(\Lambda^i_{l,k}) \cap \text{int}(\Lambda^i_{l',k'})$ which satisfies the condition in the lemma above. Note that Ψ^2 is a constant correspondence which is closed and trivially upper and lower hemicontinous.

3) Now, by the Green & Heller (1981) lemma and an induction argument, we show that $\tilde{A}^i(p,\hat{y}) = \bigcap_{l,k} \Lambda^i_{l,k}(p,\hat{y}) \cap \Psi^2$ is the finite intersection of closed valued, convex-valued upper and lower hemicontinuous correspondences, hence it's upper and lower hemicontinuous.

We have shown that $\tilde{A}^i(p,\hat{y})$ is a continuous correspondence. Then by continuity of $u^i(\cdot)$ and the theorem of the maximum we have: $C(p,\hat{y}) = \{\arg\max u^i(x) | x \in \tilde{A}^i(p,\hat{y})\}$ is continuous. In this case $C(p,\hat{y})$ is exactly the provisionally bounded demand correspondence $\tilde{D}^i(p,\hat{y})$. This finishes the proof of part (i).

Now a short proof of part (ii) of the lemma. Because $x \in \tilde{D}^i(p,\hat{y})$ and x is attainable, we must have $|x| < C$. We claim $x \in D^i(p,\hat{y})$. Suppose not; then there is $x' \in W^i \cap A(p,\hat{y})$ such that $u^i(x') > u^i(x)$. Then $u^i(\alpha x + (1-\alpha)x') > u^i(x)$ for any $\alpha \in (0,1)$. But choosing $0 < \alpha < 1$ large enough must imply that $x'' := \alpha x + (1-\alpha)x' \in \tilde{A}^i(p,\hat{y})$. But then, $u^i(x'') > u^i(x)$ and both x and $x'' \in \Psi$. So

$x \notin \tilde{D}(p, \hat{y})$, a contradiction. □

17.5 Lemma 5

Lemma 5. *Let $p = (q, \pi) \in \Delta$. Let $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, \hat{y})$ and let $(y^{jS}, y^{jB}) \in \tilde{S}^j(p)$. Then $q \cdot [\sum_i x^{iS} + \sum_j y^{jS}] + (q + \pi) \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.*
Equivalently, $q \cdot [\sum_i x^{iS} + \sum_j y^{jS} + \sum_i x^{iB} + \sum_j y^{jB}] + \pi \cdot [\sum_i x^{iB} + \sum_j y^{jB}] \leq 0$.

Proof. Because $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, y)$, it follows from the definition of the demand correspondence that, for each individual household we must have:

$$q \cdot x^{iS} + (\pi + q) \cdot x^{iB} \leq 0$$

Similarly, for each firm j we have $(y^{jS}, y^{jB}) \in \tilde{S}(p)$ so it must be the case that, for each j:

$$q \cdot y^{jS} + (\pi + q) \cdot y^{jB} \leq 0$$

Consider the value of aggregate excess demand:

$$\begin{aligned} & q \cdot \left(\sum_i x^{iS} + \sum_j y^{jS} \right) + (\pi + q) \cdot \left(\sum_i x^{iB} + \sum_j y^{jB} \right) \\ &= \left(q \cdot \sum_i x^{iS} + (\pi + q) \cdot \sum_i x^{iB} \right) + \left(q \cdot \sum_j y^{jS} + (\pi + q) \cdot \sum_j y^{jB} \right) \end{aligned}$$

Since, each of $(x^{iS}, x^{iB}) \in \tilde{D}^i(p, \hat{y})$ we have

$$q \cdot \sum_i x^{iS} + (\pi + q) \cdot \sum_i x^{iB} = \sum_i \left(q \cdot x^{iS} + (\pi + q) \cdot x^{iB} \right) \leq 0$$

since if the first inequality holds for each household i, all the more it holds for the sum. Hence, the first term of the right-hand-side is not greater than 0. Similarly, since the second inequality holds for every j, all the more:

$$q \cdot \sum_j y^{jS} + (\pi + q) \cdot \sum_j y^{jB} = \sum_j \left(q \cdot y^{jS} + (\pi + q) \cdot y^{jB} \right) \leq 0$$

so the second term of right-hand-side is also no greater than 0. Finally, combining these two one gets:

$$q \cdot \left(\sum_i x^{iS} + \sum_j y^{jS} \right) + (\pi + q) \cdot \left(\sum_i x^{iB} + \sum_j y^{jB} \right) \leq 0$$

which is the desired result.

$\square$

Moving on to Lemma 6, we restate some technical details.

Choose $C > 0$, so that $C > |(y^{jB}, y^{jS}, w^j)|$ for all $(y^{jB}, y^{jS}, w^j) \in \Xi^j$ for all j. Let $\Psi^3 := \{(y^{jB}, y^{jS}, w^j) \in \mathbb{R}^{3N(N-1)}$ such that $|(y^{jB}, y^{jS}, w^j)| \leq C\}$.

Define $\begin{array}{l} \tilde{S}^{j\dagger}(p, \hat{x}, \hat{y}) \equiv \{(y^S, y^B, w) | (y^S, y^B) = \arg\max\ (q, q + \pi) \cdot (y^{jS}, y^{jB} + w^j) \\ \text{subject to}\,(y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}] \cap \Psi^3), Y^j = \varphi^j(\hat{x}, \hat{y})\} \end{array}$

Lemma 6. *Assume Assume (E.I) through (E.V). Then $\tilde{S}^{j\dagger}(p, \hat{x}, \hat{y})$ is convex-valued and upper hemicontinuous in $(p, \hat{x}, \hat{y})$.*

Proof. Denote the space $C(\hat{x}, \hat{y}, p) := \{(y^S, y^B, w) \in (Y^j \cap [B(p) \times R^{N(N-1)}] \cap \Psi^3)\}$. We first will show that $C(\hat{x}, \hat{y}, p)$ is a continuous correspondence. Notably $C(\hat{x}, \hat{y}, p)$ is compact: it is closed because it is the intersection of closed sets and it's bounded because Ψ^3 is bounded. First observe that by assumption, $Y^j(\hat{x}, \hat{y})$ is a continous correspondence in $(\hat{x}, \hat{y})$. Hence we can define a correspondence $Y^{\dagger j} : (\hat{x}, \hat{y}, p) \rightrightarrows Y^j(\hat{x}, \hat{y})$. Note that for any p we have $Y^{\dagger j}(\hat{x}, \hat{y}, p) = Y^j(\hat{x}, \hat{y})$. Then $Y^{j\dagger}$ is a continous correspondence [2].

Applying a similar argument we can define $B^\dagger(p)$ as a continuous correspondence in $(\hat{x}, \hat{y}, p)$ since it's constant in $(\hat{x}, \hat{y})$ and it's continous in p. By the Green and Heller lemma the intersection of continuous correspondences is continous so $C(\hat{x}, \hat{y}, p)$ is a continuous correspondence.

Now observe that the the function $f(\hat{x}, \hat{y}, p, y^S, y^B, w) = (y^S, y^B + w) \cdot (q, q + \pi)$ is continuous in all of its arguments. But then, by the maximum theorem, the set $C^*(\hat{x}, \hat{y}, p) := \{(y^S, y^B, w) \arg\max_{C(\hat{x},\hat{y},p)} f(\hat{x}, \hat{y}, p, y^S, y^B, w)\}$ is nonempty, compact valued and upper-hemicontinous. Additionally, because f is clearly quasiconvex, the set of maximizers of f on the convex set $C(\hat{x}, \hat{y}, p)$ is convex. Hence $C^*(\hat{x}, \hat{y}, p)$ is convex valued. But clearly $\hat{S}^j(\hat{x}, \hat{y}, p)$ is just the (y^S, y^B) coordinates of the $C^*(\hat{x}, \hat{y}, p)$. Hence $\hat{S}^j(\hat{x}, \hat{y}, p)$ is nonempty, compact-valued, upper-hemicontinous and convex-

[2] To see this take an arbitrary sequence $(\hat{x}_n, \hat{y}_n, p_n) \to (\hat{x}, \hat{y}, p)$ and suppose we take $y_n^j \in Y^{\dagger j}(\hat{x}_n, \hat{y}_n, p_n)$ such that $y_n^j \to y$. Then by construction $y_n^j \in Y^{\dagger j}(\hat{x}, \hat{y}, p_n) = Y^j(\hat{x}, \hat{y})$ and hence $y \in Y^j(\hat{x}, \hat{y}) = Y^{\dagger j}(\hat{x}, \hat{y}, p)$ by hemicontinuity of Y^j so $Y^{\dagger j}$ is also upper-hemicontinuous. Similarly take a sequence $(\hat{x}_n, \hat{y}_n, p_n) \to (\hat{x}, \hat{y}, p)$. Then for all $y \in Y^{\dagger j}(\hat{x}, \hat{y}, p) = Y^j(\hat{x}, \hat{y})$, by lower hemicontinuity of Y^j there exist a subsequence $(\hat{x}_{n_k}, \hat{y}_{n_k})$ and a subsequence y_{n_k} such that $y_{n_k} \in Y^j(\hat{x}_{n_k}, \hat{y}_{n_k})$ and $y_{n_k} \to y$. Now choose any subsequence p_{n_k} of p_n. Then the sequence $(\hat{x}_{n_k}, \hat{y}_{n_k}, p_{n_k})$ satisfies $y_{n_k} \in Y^{\dagger j}(\hat{x}_{n_k}, \hat{y}_{n_k}, p_n) = Y^j(\hat{x}_{n_k}, \hat{y}_{n_k})$ and $y_{n_k} \to y$. So $Y^{\dagger j}$ is also lower hemicontinuous.

valued whcih is the desired result. $\square$

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