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TURBULENCE AND VORTEX STRETCHING ON A LATTICE¹

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Alexandre Joel Chorin

ABSTRACT

A lattice model of the inertial range in turbulence theory is presented; it has features in common both with the qualitative theory and with a vortex method solution of Euler's equations. It conserves volume, circulation, energy, connectivity, and a vorticity/energy relation, and proceeds through random vortex stretching and a sequence of scaling transformations. It yields a vorticity dimension, an inertial range exponent and inertial range statistics.

Introduction

The inertial range in turbulence theory is the range of scales small enough to sustain universal statistics but not so small that viscous effects are important in their dynamics (see e.g. [2],[9],[30]). An understanding of the inertial range is important both for the theoretical understanding of turbulence and for the design of practical modeling methods. The inertial range has been the object of an elegant qualitative theory, summarized below, that predicts a universal form for the energy spectrum but provides little information for practical use and makes no use of the equations of motion. This range has also been studied numerically; the calculations have yielded much useful information but are too expensive for routine use and have not been convincingly reconciled with the qualitative theory.

The purpose of the present paper is to provide a middle ground between the qualitative theory and the numerical solution of Euler's equations, and to clarify their relation to each other. In addition, the statistics of the inertial range will be generated in a form that can be put to practical use. A lattice vortex model will be presented that conserves energy, volume, connectivity and circulation and, when appropriate, an energy/vorticity relation; it models plausibly the random evolution of a system of vortex lines in inviscid flow. The main ingredients in the model are a random vortex stretching procedure that resembles Metropolis sampling [29] and a sequence of scaling transformations.

The model reproduces the salient qualitative features of earlier numerical calculations and it elucidates the role of vortex stretching and folding and the origin of intermittency; in homogeneous flow it yields a Hausdorff dimension for the vorticity set $D \sim 2.3$, and an inertial range exponent close to the Kolmogorov value $\gamma \sim 5/3$. It also provides an explanation for the difficulty

in estimating γ from a straightforward numerical solution of the equations of motion.

The present paper can be viewed as a direct sequel to [11] and [12]; the calculations in [12] suggested some of the main features of the model, and the scaling transformations were introduced in [11]. The paper is organized as follows: in the next two sections the qualitative theory and numerical work with vortex methods are briefly reviewed. The model is then presented in three steps: the random stretching is introduced, the scaling transformations are explained and reveal the need for an intermittency correction, and then that correction is implemented. The model is then applied to homogeneous flow, where an energy/vorticity relation fixes the value of D and provides an estimate for γ . Conclusions are drawn in a final section.

The practical uses of the model will be explored elsewhere; they stem mainly from the fact that the model provides not only the spectrum of the inertial range but also a full random model of that range that can be sampled and used as input in a larger calculation.

The Qualitative Theory of the Inertial Range

We begin with a quick review of Kolmogorov's theory of the inertial range, in the version suggested by Kraichnan ([17],[20],[21]). Consider a collection of eddies whose characteristic length scales take the discrete values $\ell_n = \ell_0 2^{-n}$, with ℓ_0 a given length and $n = 1, 2, \dots$. The eddies of characteristic length ℓ_n have a characteristic energy E_n (the notion of energy connected with eddies of a given length scale can be given a plausible definition with the help of a Fourier transform, see e.g. [2],[17]), and they have a characteristic velocity u_n , with $E_n \sim u_n^2$. The characteristic time (or "turnover time") of an eddy of size ℓ_n is $t_n \sim \ell_n/u_n$.

Assume that in a time $\sim t_n$ the eddies of size ℓ_n yield their energy to the eddies of size ℓ_{n+1} ; the eddies of size ℓ_n are now the n -th "generation" of eddies. If the rate at which energy moves across scales is constant, then

$$\frac{E_n}{t_n} \sim \varepsilon \equiv \text{constant rate of energy transfer} \quad , \quad (1)$$

(note for later use that $E_n \sim \text{constant} \times t_n$); thus

$$\frac{u_n^3}{\ell_n} \sim \varepsilon \quad , \quad u_n^2 \sim \ell_n^{2/3} \quad ,$$

and a Fourier transform yields

$$E(k) \sim k^{-\gamma} \quad , \quad \gamma = 5/3 \quad , \quad (2)$$

where k is a wave number and $E(k)$ is the energy spectrum (for definitions see, e.g. [9],[17],[21],[30]). This is the Kolmogorov law and $\gamma = 5/3$ is the Kolmogorov exponent; this exponent can also be derived by dimensional considerations (see e.g. [2],[9],[30]).

It was noted by Landau [23] that the Kolmogorov argument ignores intermittency, i.e., the uneven distribution of energy and of energy dissipation in physical space. The argument was therefore modified by a sequence of authors (for historical review, see [17],[30]). In the formulation of [17], one assumes that the "active eddies" (whatever they are) of the n -th generation occupy only a fraction β of the volume occupied by the eddies of the previous generation, i.e., the "active eddies" are not space-filling. Assume that $E_n \sim \beta u_n^2$, and write $\beta = 2^D/2^3$; D is a similarity dimension similar to Hausdorff dimension (see [27],[28]). If $\ell_n/\ell_0 = 2^{-n}$, $t_n = \ell_n/u_n$, a quick calculation yields

$$E_n \sim \beta^n u_n^2 \sim \ell_n^{(2/3 + (3-D)/3)} \quad ,$$

and a Fourier transform yields

$$E(k) \sim k^{-\gamma}, \quad \gamma = \frac{5}{3} + \frac{1}{3} (3-D). \quad (3)$$

We thus find an intermittency correction to the Kolmogorov exponent. The argument that leads to (3) is, however, not without flaws. All the quantities under discussion ("active eddies", "characteristic times") are only loosely defined. It is far from obvious why one should keep $\ell_n = \ell_0 2^{-n}$ even when the eddies are occupying volumes much smaller than $(\ell_0 2^{-n})^3$. If D were an integer one could imagine that each active eddy occupies a solid cube, with some of the subcubes of the previous generation's cube remaining empty; if D is not an integer ℓ_n should depend on the shape of the active eddies in some way that is not readily analyzable and not unique. There is no reasonable physical mechanism that would allow the energy to be effectively confined to a volume fraction. The lack of uniqueness of the intermittency correction was discussed to some extent in [27] and, in the context of a different model, in [22]. We shall see below that there is no turbulence without intermittency and thus an intermittency "correction" to γ makes little sense and, furthermore, equation (3) is incompatible with the physics of vortex stretching. However, the idea that "activity" concentrates in a fractal set has received support from several sources, including experiment, and the value $D \sim 2.5$ was suggested in [11],[12],[17],[27]. The possibility that Kolmogorov's exponent is correct even if the original argument is not was raised in [22].

Note that the argument of this section makes no use of the equations of motion. What we shall retain from this qualitative theory is mostly the idea that energy cascades across a sequence of eddies of shrinking size, and that "activity" (to be defined more precisely) concentrates in a shrinking fraction of the total volume.

The Euler Equations and Vortex Stretching

The equations of motion that describe the flow in the inertial range are Euler's equations (see e.g. [8],[11],[12]); these equations can be written in the form

$$\partial_t \underline{\xi} + (\underline{u} \cdot \underline{\nabla}) \underline{\xi} - (\underline{\xi} \cdot \underline{\nabla}) \underline{u} = 0 \quad , \quad (4a)$$

$$\underline{\xi} = \text{curl } \underline{u} \quad , \quad \text{div } \underline{u} = 0 \quad , \quad (4b,c)$$

where $\underline{u}$ is the velocity, $\underline{\xi}$ is the vorticity, t is the time, and $\underline{\nabla}$ is the differentiation vector. (For an introduction to these equations, see [14]; for a recent review of their theory, see [26]). The kinetic energy T of a fluid whose flow satisfies equations (4) is given by

$$T = \frac{1}{2} \int |\underline{u}|^2 d\mathbf{x} = \frac{1}{8\pi} \int d\mathbf{x} \int d\mathbf{x}' \frac{\underline{\xi}(\mathbf{x}) \underline{\xi}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad , \quad (5)$$

([3], p.520) where $\mathbf{x}$ is the position vector. Furthermore, $\frac{dT}{dt} < 0$. The mean squared vorticity Z is defined as

$$Z = \int d\mathbf{x} |\underline{\xi}|^2 \quad .$$

Equations (4) describe a process in which $\underline{\xi}$ is transported by $\underline{u}$, and $\underline{u}$ can be computed if $\underline{\xi} = \underline{\xi}(\mathbf{x})$ is known. It is generally believed that Z increases sharply in time, and may even become infinite in finite time ([11],[12],[33]). Z increases through the stretching of vortex lines; indeed, when vortex lines stretch, conservation of volume and conservation of circulation produce an increase in Z . As vortex lines stretch, lateral distances between neighboring vortex lines shrink and create velocity variations over ever decreasing scales. We wish to identify this process of vortex stretching with the energy cascade described in the preceding section, and to identify "active eddies" with regions containing stretched vorticity.

The qualitative theory of the preceding section now becomes attached to a specific set of equations in three-dimensional space.

The numerical calculations in [11],[12],[33] show that vortex lines fold as they stretch. This folding can be understood in two ways: (a) Vorticity produces velocity. Since the vorticity is increasing but the energy is not, the vorticity must arrange itself in such a way that cancellations occur between the velocity fields produced by neighboring patches of vorticity. (b) As vortex lines stretch the vorticity concentrates in a decreasing fraction of the available volume; that fraction must converge to a set that can carry a non-zero vorticity without giving rise to an infinite energy [12]. In an incompressible two-dimensional flow the vorticity-carrying set must have positive capacity, and this occurs when it has positive Hausdorff dimension [18]. In three-dimensional space the vector nature of vorticity invalidates a direct comparison with electrostatics, but it is still highly plausible that the vorticity set must have a positive capacity, which in three dimensions means a Hausdorff dimension larger than 1. The process of vortex folding is the process by which a non-trivial set of positive capacity can be formed. (a) and (b) are equivalent ways of saying that vortex lines that stretch must also fold tightly if the energy cannot increase. This remark connects the similarity dimension D that appeared in the previous section with properties of the equations of motion (4).

Consider what is likely to happen qualitatively if the vorticity concentrates on a set that is too small (e.g. on a set whose Hausdorff dimension D is less than 1). Suppose a vortex line stretches and folds over and over into a bundle that is smooth in one direction (Figure 1; Siggia [33] has challenged the reasonableness of the last assumption). Consider further the intersection of the line with a given surface. If $D < 1$, the intersection,

if not empty, will consist of a sparse set that can be thought of as consisting of a countable collection of points. If the distances between these points are finite the energy will be infinite (by the two-dimensional result); if infinite energy is not allowed then the points must merge and the resulting energy will be zero.

The calculations presented in [11],[12],[33] exhibit a tangle of vortex lines that is very unstable; indeed, columnar vortices and pairs of counterrotating vortex lines are well known to be unstable [15],[24]. After a certain interaction time, the detailed dynamics of the computation are affected more by small numerical errors than by physical effects, while certain global properties of the solution, such as conservation of energy and circulation or presence of folding, are reasonably well approximated and are independent of numerical effects. These observations suggest that, since one has to be content with a description of turbulent flow that is less than perfect in all its details, one could just as well assume a priori that the description will be less than perfect and make it economical in terms of computer time. Furthermore, since the imperfect description in [11],[12] is already quite informative, one can hope that imperfect descriptions would be useful. For an explanation of vortex methods, see e.g., [1],[4],[10],[19],[25].

In the next few sections we shall exhibit a crude description of turbulence and vortex stretching that we believe to be appropriate and that does make substantial use of the equations of motion.

Vortex Stretching on a Lattice

In the present section we begin the presentation of our simplified lattice model of vortex stretching and the inertial range. Consider a three-dimensional cubic lattice with vertices $(i\delta, j\delta, k\delta)$, i, j, k integers, $1 \leq i, j, k \leq m$,

$m\delta = 1$, where δ is the lattice spacing. Assume that at time $t=0$ a "vortex line" is set on this lattice, and that the whole configuration is then continued periodically with period 1 in all directions. The vortex line (we shall henceforth omit the quotation marks) is made up of oriented segments of finite thickness whose center lines coincide with edges of the lattice; the vortex line does not intersect itself, i.e., each vertex of the lattice separates either zero or two segments. Write $I = (i,j,k)$ for short instead of $(i\delta, j\delta, k\delta)$; denote by $h_I = h_{i,j,k}$ a horizontal active segment whose left-most end coincides with I ("active" means "belonging to the vortex line", h stands for "horizontal", "horizontal" means "parallel to the $x=i\delta$ axis", "left-most" means "corresponding to the smallest value of x "); similarly, denote by v a segment that is vertical (parallel to the $z=k\delta$ axis) and by w_I a segment that is transverse (parallel to the $y=j\delta$ axis); see Figure 2. Denote also by h_I, v_I, w_I a variable attached to these segments and taking on the value +1 if the segments are oriented in the direction of increasing x, y or z and taking on the value -1 otherwise.

We endow this line with an energy T , constructed so as to mimic equation (5):

$$T = T_E + T_S, \quad (6)$$

$$T_E = \sum_I \sum_{I' \neq I} h_I h_{I'} \frac{1}{|I-I'|} + \sum_I \sum_{I' \neq I} v_I v_{I'} \frac{1}{|I-I'|} + \sum_I \sum_{I' \neq I} w_I w_{I'} \frac{1}{|I-I'|},$$

$$T_S = \sum_I S(h_I) + \sum_I S(v_I) + \sum_I S(w_I).$$

The sums in T_E are meant to mimic the integral in (5); the constant $(8\pi)^{-1}$ is immaterial and is therefore omitted, and the subscript E stands for "exchange". In a periodic system each segment should interact with every

other segment as well as with all periodic images of every other segment; to save labor we keep only the largest one of these interactions (a similar device was used in [11],[12]); thus we set

$$I = (i,j,k), \quad I' = (i',j',k') ,$$

$$\phi(q) = \begin{cases} q^\delta & \text{if } 1/2 < q^\delta \leq 1/2 , \\ q^\delta - 1 & \text{if } q^\delta > 1/2 , \\ q^\delta + 1 & \text{if } q^\delta \leq 1/2 , \end{cases}$$

and

$$|I-I'| = (\phi^2(i-i') + \phi^2(j-j') + \phi^2(k-k'))^{1/2} ;$$

the sums in T_E are over all active segments.

The domain of integration in formula (5) does however include values of $\underline{x}$ and $\underline{x}'$ belonging to the same segment, giving rise to a "self-energy" term for the segment. This term cannot be omitted; a vortical flow without a self-energy cannot have an infinite self-induction when it stretches and it avoids by construction the more interesting pathological features of fluid turbulence. Assume that at time $t=0$ a self-energy term has been computed for each segment; if the segment is horizontal denote this self-energy by $S(u_I)$, and similarly for $S(v_I)$, $S(w_I)$. The sum of these terms over all active segments makes up T_S . T_S is always positive if the vorticity in any one segment points in a fixed direction, an assumption we shall henceforth make.

We now undertake to stretch the vortex line by a sequence of elementary stretchings, which are not assumed to have a physical meaning individually; it is the end result of the sequence that will be compared with the breakdown of an "eddy". Pick an active segment at random, e.g. by picking the segment whose origin is given by $i = [\eta_1 m + 1]$, $j = [\eta_2 m + 1]$, $k = [\eta_3 m + 1]$,

where η_1, η_2, η_3 are random numbers picked independently from the uniform distribution on (0,1) and $[]$ denotes an integer part. If there is no such segment, try again. Consider the possible stretchings of the segment into a U-shaped configuration of three segments (Figure 3). There are four such configurations for each segment (e.g., for a transverse segment, one could stretch up, down, to the right, or to the left). Consider one of the possibilities at random. If the proposed stretching configuration leads to a self-intersecting vortex line, reject it and pick another segment at random. Self-intersection is unacceptable because it leads to an infinite energy, see equation (5), and our goal is energy conservation. If the proposed stretching avoids self-intersection, it will be accepted only if a certain energy constraint is satisfied (somewhat similar constructions are widely used in statistical mechanics, see e.g. [5],[29],[31]); we shall now explain the constraint.

The change in energy due to the proposed stretching consists of several parts: T_E is changed because (a) all interaction terms involving the old segment must be deleted; (b) interaction terms involving the three new segments must be added, and (c) two of the new segments interact, and the interaction is always negative (Figure 3). T_S is modified because the self-energy changes when vortices stretch.

To evaluate the change in T_S , consider a vertical segment, of length δ , cross section σ , and vorticity $\underline{\xi} = (0,0,\xi)$. Suppose the segment is stretched by a factor L ; its cross-section is reduced by a factor L and its diameter is reduced by a factor $\sqrt{L}$, assuming as we shall that the segment was and remains a circular cylinder. By conservation of circulation ξ is multiplied in an appropriate average by L . Let the vorticity in the stretched segment be $\tilde{\underline{\xi}} = (0,0,\tilde{\xi})$, and assume that the vorticity is self-similar, i.e. $\tilde{\xi}(x/\sqrt{L}, y/\sqrt{L}, z)/L = \xi(x,y,z)$, a condition satisfied in particular if ξ and $\tilde{\xi}$

are constants over the cross-section of the segment. Let $\tilde{\sigma}$ be the new cross-section. The self-energy of the stretched segment is

$$\tilde{S} = \int_{\tilde{\sigma}} dx dy \int_0^{L\delta} dz \int_{\tilde{\sigma}} dx' dy' \int_0^{L\delta} dz' \frac{\tilde{\xi}(\underline{x})\tilde{\xi}(\underline{x}')}{|\underline{x}-\underline{x}'|} \quad (7)$$

The upper limit of integration $L\delta$ in the z and z' integration shows that $\tilde{S}$ depends quadratically on the length of the segment, a fact that will be of some importance below. Here however we are considering one piece of a stretched vortex whose length is equal to the length of the original segment.

Make the change of variables $\underline{x} = \underline{x}\sqrt{L}$, $\tilde{z} = \tilde{z}/L$; suppose $\delta = \infty$ (an infinitely long segment). It is easy to see that under the change of variables the expression for $\tilde{S}$ will satisfy

$$\tilde{S} = \sqrt{L} S, \quad (8)$$

where S is the self-energy of the unstretched vortex. If the radius of the segment is much smaller than its length, the expression (8) will still hold approximately. We shall assume expression (8) holds, with $L=3$. Each stretched segment thus acquires a self-energy $\sqrt{3}$ times larger than the original self-energy, and since there are three segments replacing one old segment, the change in T_S is $(3\sqrt{3} - 1)$ times the self-energy of the original unstretched segment.

Suppose we stretch a vortex line that is initially vertical (as in Figure 2), with $S(w_I)$ originally small. T_E will initially decrease: the vertical sum in equation (6), i.e., the sum involving $w_I w_{I'}/|I-I'|$, will decrease because the distances $|I-I'|$ to one of the segments will increase, and the horizontal interaction that is added is negative; a small initial $S(w_I)$ will not interfere too much. If one keeps on performing stretchings and accepts all stretchings that do not lead to self-intersection, the $S(w_I)$ will ultimately increase enough to take over; $T = T_E + T_S$ will start to increase. A typical graph of

T as a function of the number of stretchings will look like Figure 5, computed with $m = 8$ and $S(w_T) = 1$ initially.

When T returns to its initial value we have a new vortex line: thinner, longer, and more convoluted, but having the same volume, circulation, connectivity and energy as the initial line. We shall take this line to be the new stretched line, and we shall assume that the transition from the initial to the final line mimics the physical process of vortex stretching. The final line is not unique. The criterion for accepting stretchings that do not lead to self-intersection is the following: if the energy remains below the original energy, the stretching is accepted. Since the energy changes are discrete, the energy in general does not return exactly to its initial value and the criterion must be applied approximately. One can decide to accept an energy that is just below or just above the right value; in the next section we shall explain how to ensure that the cumulative effect of such decisions does not become significant.

The construction just given avoids the problem of following accurately in time the increasingly important effect of small vortex deformations on the local vortex velocity; the lattice cuts off the smallest scales. The question of whether a real vortex line whose motion satisfies Euler's equations can move from a position approximating the initial lattice vortex to a position approximating the final lattice vortex remains open. In view of the dominant role of local effects the answer is likely to be positive. The problem of estimating how much physical time elapses during the stretching will be discussed below.

In [33],[34], Siggia has argued that one has to account carefully for the curvature of a vortex line in order to see a certain mechanism of possible blow-up of solutions of Euler's equations. Our model will not satisfy his

criteria. The special role of vortices of circular cross-section, like the ones we are using, was explained in [7],[8], where it was argued that a union of such vortices is an approximately stationary solution of Euler's equations, and a real flow should hover in its neighborhood.

In Figure 5 we reproduce the final configuration of a vortex line on a 4^3 lattice, produced from an initially vertical vortex with $S(w_I) = 1$. It is instructive to contrast this figure with the pictures of folded vortices in [12].

Scaling of the Lattice

Once the calculation of the preceding section has come to a halt, no further stretching with conservation of energy is possible on the given lattice. However, if the lattice were refined (e.g. by halving δ , doubling m and thus multiplying the number of vertices by 8), then stretching would become possible again into the newly created smaller loops; indeed, if δ is smaller, a stretching as in Figure 1, which produces a negative interaction proportional to $1/\delta$, will be able to make up for the increase in energy. It is of course impractical to continuously refine the lattice spacing, since the amount of computational labor would soon become intolerable - this is the standard conundrum of numerical turbulence theory.

What we shall do is simultaneously halve δ and focus attention on an eighth of the computational domain, along the lines of [11]. We pick out an eighth of the lattice, making sure it contains active segments, and halve δ , so that the number of vertices remains the same, and then, to simplify the bookkeeping, scale up the eighth so that the new vertices coincide with the vertices we had before. The bookkeeping is simpler if m is even, a condition we shall enforce from now on.

For the calculation to proceed, the newly scaled up lattice requires

boundary conditions; we provide them by assuming that the lattice is continued periodically in all directions. Clearly this assumption changes the energy of the lattice, since whatever balance had been achieved between T_E and T_S may be undone by the periodic continuation. This problem can be slightly attenuated if one requires that the non-empty eighth that is being scaled up contain as many active segments as possible, and further if one centers the active segments in such a way that as few of them as possible lie directly on the boundary. However, since the interactions in T_E are long-range, these precautions will not help much.

Fortunately, most of the time we shall be concerned with variables that depend on the stretching process but not on initial or boundary conditions. Assume that the scaling is done repeatedly, and let ψ be a variable that that depends on the state of the system. Let $\tilde{\psi}_n^0$ be the value of ψ at the beginning of the n -th stretching, and $\tilde{\psi}_n'$ be the value of ψ at the end of n -th stretching, as computed by the procedure we are describing. $\tilde{\psi}_n^0 \neq \tilde{\psi}_{n-1}'$ because there is a scaling and a periodic continuation inbetween. We shall set

$$\psi_n^0 = C_n \tilde{\psi}_n^0, \quad \psi_n' = C_n \tilde{\psi}_n', \quad (9a)$$

$$C_n = \frac{n}{1-1} (\tilde{\psi}_j^0 / \tilde{\psi}_{j-1}') , \quad (9b)$$

i.e., we multiply out the effect of the scalings and periodic continuations on ψ . We shall do this for all the functions computed below except where stated otherwise.

In each scaling we wish to preserve the balance between T_E and T_S . Each segment has doubled its length, and there are four interactions in T_E where there had been one before the scaling. The distance between segments has been doubled and thus T_E has been doubled; T_S should also double,

and thus the balance between T_E and T_S will be conserved if each segment inherits the self-energy that its parent had before the scaling. One conclusion from this fact that in the scaling process as presented here, vortex stretching will eventually come to a halt: the self-energy of the segment will increase in each round of stretching while T_E is clearly bounded from below; soon no stretching will be possible without an increase in energy. In language of the next section, indefinite vortex stretching is impossible if the active eddies are space-filling, i.e., without intermittency.

One flaw the of the scaling procedures just described is that however thin the segments were to begin with, eventually the scaled lattices will have scales smaller than the initial thickness of the segments, and some provision should be made to smear segments over some of their neighbors; we shall not make any such provision, since such smearing will be removed by the effect of equation (9). Note however that we are defining scale by external volume and not by vortex cross-section as in [11],[12]. The fact that these scales are generally different suggests that vortex calculations in three dimensions with Euler's equations require care. Note that the scaling does not preserve connectivity at the edge of the lattice.

It was noted in the preceding section that after a stretching the energy does not return to its original value exactly; it may be a little smaller or a little larger than the initial energy. If stretchings are carried out many times on the scaled lattices, the cumulative effect of such decisions may become significant. An obvious remedy is to make the decision adaptively: consider at the end of each scaling the energy at the end of the previous scaling T_{n-1} (adjusted as in equations (9)) divided by the energy T_1 at the beginning of the first stretching. If $T_{n-1}/T_1 < 1$ allow the stretching to go slightly above T_n , if $T_{n-1}/T_1 > 1$ remain slightly below T_n .

Scaling with Intermittency

We have seen that vortex stretching comes to a halt if the scaling is carried out as described in the preceding section. In order for the vorticity to stretch indefinitely, one has to allow it to congregate into tighter bundles than are possible on a regular lattice; indeed, if the segments are allowed to congregate T_E becomes larger and can compensate for the increase in T_S . It is the presence of T_E that allows stretching, as we have seen above; this is consistent with results from vortex theory in the self-induction approximation [6],[32], in which T_E is absent and vortex arc length is preserved [6].

One can cram the vorticity into tighter bundles by assuming that each time an eighth of the lattice is extracted for further consideration it is scaled up into a volume smaller than the initial volume by a factor $\beta = 2^D/2^3$, where $D < 3$ is, as above, a similarity dimension. Equivalently, one can squeeze the eighth into a fraction β of its volume before scaling it up. The calculations in [12] lend support to the idea that such squeezing does correspond to a physical phenomenon. The squeezing does not violate conservation of volume since the segments are surrounded by irrotational fluid that can be evicted to make room for vorticity.

A decision has to be made regarding the shape of the β fraction of the volume into which the segments are squeezed. In the remainder of the paper we shall make the arbitrary assumption that the shape is cubic and all length scales are decreased by a factor $d = \sqrt[3]{\beta} = 2^{(D-3)/3}$. This assumption seems to destroy the connectivity of the vortex line; however, one could view it as the result of keeping one preferred direction parallel to the local vortex direction fixed and squeezing the other two, and then averaging over all

vortex directions. When this squeezing is done, each term in both T_E and T_S is multiplied by d^2 (because all interactions are quadratic functions of the length of the segments); each term in T_S is further multiplied by $\sqrt{d}$ (by conservation of circulation, see equation (8)) and each term in T_E is multiplied by d^{-1} (see equation (6)); thus, after the n -th scaling,

$$T_{n+1} = (T_S)_{n+1} + (T_E)_{n+1} = d^{5/2}(T_S)_n + d(T_E)_n. \quad (10)$$

When $d < 1$, T_S is reduced in comparison with T_E and the stretching has a chance to proceed. The smaller cubes of side d are to be identified with the "active eddies" of the qualitative theory; they are vorticity rather than velocity eddies. After p individual stretchings and n scalings of the lattice, the self-energy of a segment is proportional to $(\sqrt{3}) d^{(5/2)}$; equations (10) set up a competition between the effects of stretching and scaling on T_S . Note that T_{n+1} can be smaller than T_n ; the remaining energy is left behind in the unscaled portion of the lattice.

As explained above, if one tries to squeeze the vorticity into a volume that is too small while not allowing the energy to grow, the energy should collapse to zero; it was conjectured that this would happen if $D < 1$. Equations (10) provide a mechanism for the collapse: for D too small, there may not be enough segments available in the lattice to stretch the vortex line sufficiently so that the self-energy term can overcome the T_E term. In Figure 6 we display T_n as a function of the number of scalings n for $D=0.7$ and $m=6$ (i.e., a lattice with 6^3 vertices). T_n is adjusted as in equation (9) and the fluctuation in T_n in the early stages are due to the imperfect returns to initial energy in the several stretchings. T_n collapses to zero when the lattice becomes full before T_n returns to the initial energy. We set $T_n = 0$ whenever the calculation yields $T_n < 0$. In Figure 7 we display T_n as a

function of D after 12 scalings for $m=6$. The transition occurs between $D=0.8$ and $D=0.9$. A calculation with $m=4$ yields a similar picture with transition between $D=0.9$ and $D=1.0$. A calculation with $m=2$ is hard to interpret because the energy change in each stretching is a substantial fraction of the total energy and it is hard to enforce energy conservation. These results are strongly supportive of the conjecture that a set with $D < 1$ cannot support the vorticity while keeping the energy finite and positive.

The results with $m=8$ are harder to interpret. The picture is still qualitatively similar but the transition occurs between $D=.5$ and $D=.6$. One would have hoped that a larger m would alleviate the conceptual problems that result from the periodic continuation at each scaling, but since the interactions in equations (6) are long-range, this is a vain hope. As m increases the problem of enforcing a return to the initial energy after stretching is clearly alleviated. However, each individual stretching involves loops of size $1/m$; energy is transferred from scales comparable to the lattice size to scales m times smaller. If m is large this violates the assumption of energy transfer between scales whose ratio is moderate. Thus $m=4$ and $m=6$ are likely to be the best choices for m , since m must be even; we reject the results obtained with $m=8$ and conclude that $D > 1$.

Dimension and Spectrum for Homogeneous Turbulence

We have considered the behavior of the vortex line on the scaled lattices as a function of D but have not established the values of D . Clearly the value of D depends on the boundary condition imposed on the lattice. For example, if the lattice is viewed as imbedded in a boundary layer, the imposed shear will stretch those vortex lines that have a component parallel to the shear, and values of D that do not allow enough stretching will be excluded.

This conclusion is compatible with the idea that vortex stretching is responsible for the energy cascade. Stretching on energy-containing scales has an effect on scales comparable to a vortex diameter, which may well lie in the inertial range; that range may be less universal in structure than is assumed in the Kolmogorov theory.

Consider however a lattice imbedded in homogeneous turbulence; there is no mean shear. Let ξ be the vorticity and consider Z_n , the lattice analogue of the mean squared vorticity $\int |\xi|^2 dx$. If S is the cross-section of a segment and ξ is constant over S , then $S\xi = \text{constant}$ by conservation of circulation, and the contribution of a segment to Z_n is proportional to its length and to $\xi^2 S = S^{-1}$. After p stretchings and n scalings, $S^{-1} = 3^p d^{2n}$, and thus

$$Z_n \sim \sum_{\text{segments}} 3^p d^{2n} \quad (11)$$

If $T_n \sim \ell_n^\alpha$ in homogeneous turbulence, then $Z_n \sim \ell_n^{(\alpha-2)}$ (see e.g., [2],[9],[11]); this relation usually appears in its Fourier-transformed form $Z(k) \sim k^2 E(k)$, where $Z(k)$ is the vorticity spectrum and $E(k)$ is the energy spectrum. It is a rigorous consequence of the equations of motion). α depends on the distribution of characteristic times (see equation (1) or the discussion below) and does not concern us here; the relation between Z_n and T_n can be written as

$$\frac{Z_{n+1}}{Z_n} = 4 \frac{T_{n+1}}{T_n} \quad (12)$$

The passage from the n -th to the $(n+1)$ -st lattice consists of a stretching and a scaling. Consider the stretching first. The energy is conserved and if the effect of the scaling is neglected we should have $q_n = (Z_{n+1}/Z_n) = 4$.

In Table I we display values of T_n , Z_n , q_n , and the averaged quantities

$$Q_n = \frac{1}{n-1} \sum_{j=2}^n q_j .$$

Since q_n is random it contains a statistical error; Q_n is introduced to reduce variance, according to the usual practice. Since the initial conditions consist of a vertical vortex the first stretching is special and is omitted from Q_n . The oscillations in T_n are due as usual to an imperfect return to the initial conditions. The table displays both the substantial statistical scatter in the results and the fact that a reliable trend does emerge.

In Figure 8 we display Q_n as a function of D with $m=4$ and $m=6$. The results seem to fall roughly on a straight line; the dotted line is a rough sketch of where that line would be. The conclusion is that equation (12) is satisfied when $D \sim 2.3$, if the effect of scaling is neglected.

Consider now the effect of scaling. Z_n is multiplied by d^2 in this scaling (see equation (11)). The dependence of T_n on d is unclear since T_E, T_S are multiplied by different powers of d and T_E can be negative. However, for D not far from 3, the self-energy is likely to be the dominant part of T_n and then we should have approximately $T_{n+1} \sim d^{5/2} T_n$, and equation (12) is satisfied if $Q_{n+1}/Q_n = 4\sqrt{d}$. For $D \sim 2.3$, $\sqrt{d} = 2^{-0.7/6} = 0.922$ and as can be seen from Figure 7 the value of D is affected by less than the statistical scatter and the conclusion $D \sim 2.3$ stands.

We can now indulge in some qualitative speculation. Consider the spectrum in the inertial range. Consider a large volume containing many vortex structures in the process of decaying. The mean energy E_n in the n -th scale is proportional to $T_n t_n$, where t_n is the time the energy T_n spends in the n -th lattice; the constant of proportionality has dimension $[\text{time}]^{-1}$ and depends on the total life-span of the vortical structure.

Table I: Vortex stretching, $D=2.3$, $m=6$.

n	E_n	Z_n	$q_n = Z_{n+1}/Z_n$	$Q_n = \frac{1}{n-1} \sum_{j=2}^n q_j$
1	1.00	1.80×10^1	—	—
2	.99	1.59×10^4	3.73	3.73
3	1.11	5.94×10^4	2.95	3.34
4	1.10	1.75×10^5	4.74	3.81
5	1.10	8.33×10^5	5.16	4.14
6	1.09	4.30×10^6	3.64	4.05
7	1.07	1.57×10^7	4.47	4.11
8	1.07	7.01×10^7	4.25	4.13
9	1.06	2.98×10^8	3.25	4.02
10	1.06	9.70×10^8	3.52	3.97
11	1.06	3.42×10^9	4.03	3.98
12	1.06	1.38×10^{10}	—	—

Assume t_n is proportional to the turnover time of the n -th lattice, $t_n = \ell_n/u_n$. We are not imposing the requirement that all the energy move from one lattice size to the next; indeed, equations (10) allow $T_{n+1} < T_n$ if scaling is included. If we find $E(k) = k^{-\gamma}$, $\gamma \neq 5/3$, we shall have to allow energy to cascade to larger as well as smaller scales and make appropriate corrections in whatever law we find. We can set $\ell_n = \ell_0 2^{-n}$ even in the presence of intermittency, since velocity is space filling even when vorticity is not; similarly $T_n \sim u_n^2$ and thus $t_n \sim 2^{-n}/\sqrt{T_n}$. If $T_n \sim T_0(d^{5/2})^n$, we find

$$\begin{aligned} E_n &\sim T_n t_n \sim (d^{5/2})^n \frac{1}{(d^{5/2})^{n/2}} \ell_n \\ &\sim \ell_n^{1 - (5/12)(3-D)} \quad \text{since } d = 2^{(D-3)/3}; \end{aligned}$$

A Fourier transform yields

$$E(k) \sim k^{-\gamma}, \quad \gamma = 2 - (5/12)(3-D). \quad (13)$$

Equations (13) and (3) agree if $D \sim 2.55$. Equation (13) has meaning only for the one true value of D ; all other values of D correspond to hypothetical flows of no real significance. Note however that in equations (13) and (3) $d\gamma/dD$ have opposite signs, and the one in (13) is the right one from the point of view of vortex dynamics: if D decreases the vorticity becomes more singular and γ should decrease.

If $D \sim 2.3$, we find from (13) $\gamma = 1 + 17/24$, which, in view of all the approximations made, is indistinguishable from Kolmogorov's value $\gamma = 5/3$. Note that D can be evaluated in a calculation that starts with a single vortex line, but the evaluation of γ requires us to consider an ensemble of such lines and would demand a much larger calculation if it were to be done numerically; this remark agrees with the observations made in [12].

Conclusions

We have constructed a model of the inertial range of turbulence that is intermediate between the qualitative theory and a full numerical solution of Euler's equations. The model captures the salient features of the numerical solution, provides a qualitative picture of the origin of turbulence and of its relation to vortex stretching, and suggests values of the Kolmogorov exponent γ and of the vorticity dimension D . The model can and will be generalized to other situations where self-similar turbulence exists, e.g. in the turbulent boundary layer.

As was pointed out in the introduction, the most interesting aspect of the model is that it provides not only a spectrum but also a spatial distribution of vorticity in the inertial scales, a distribution that can be sampled and used in the modeling of complex turbulent flows, in the spirit of the random vortex method [10] and of large eddy simulation [16].

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Note: The program used above is available from the author.

LIST OF FIGURE CAPTIONS

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- Fig. 8. Vorticity ratio as a function of dimension D .

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Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

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