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Aspects of Symmetries in Quantum Field Theories

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Aspects of Symmetries in Quantum Field Theories

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by

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The Dissertation of Zhengdi Sun is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

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DEDICATION

To my wife, my parents, and my grandparents.

EPIGRAPH

Every science, once it is treated not as an instrument for gaining dominion and power, but as part of the adventure of knowledge of our species through the ages, may be nothing but that harmony, more or less rich, more or less grand depending on the times, which unfolds over generations and centuries through the delicate counterpoint of each of its themes as they appear one by one, as if summoned forth from the void to join up and intermingle with each other.

Alexander Grothendieck, translated by *Roy Lisker*

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ABSTRACT OF THE DISSERTATION

Aspects of Symmetries in Quantum Field Theories

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We study the properties and applications of generalized symmetries in the quantum field theories. We explore how to use the spacetime symmetry and the internal symmetry to refine the Cardy formula in 2-dimensional conformal field theories. We use the technique of group theoretical fusion categories to study the physical implications of triality defects in 2d CFT. We also explore when the non-invertible symmetries are non-intrinsically non-invertible, that is, when they can be constructed from topological manipulations of invertible symmetries.

Chapter 1

Introduction

The central theme in this thesis is the *symmetry* in quantum field theories. Roughly speaking, symmetry is an operation S of a given system, under which the the system is unchanged. The symmetry would then imply relations between observable physical quantities; therefore it is an important guideline for the study of the physical system. In this thesis, we will discuss symmetry in the context of quantum field theory (QFT). There, the additional structure leads to a more refined structure of the symmetry, and several important generalizations of the notion of symmetry have be made in this context recently. This thesis aims to expand the understanding of how ordinary symmetries refine physical observables, as well as explore the structure of the generalized global symmetries and its physical implication.

Given a physical system, let's consider the set G of all operations that leave the system invariant. It is expected that G has the mathematical structure of a group. This means that given two such operations, we can combine the two to a single operation. This corresponds to the multiplication in the group. Furthermore, a trivial operation to the system will certainly keep the system unchanged, hence belong to G and correspond to the identity element in the group. Finally, given any operation $g \in G$, it is expected that we could perform some other operation to undo the operation g. This corresponds to the each element in a group has an inverse under multiplication.

In the context of quantum field theory (QFT), the additional structure such as locality or spacetime symmetry leads to a more refined structure of the symmetry. Here, the symmetry group G is realized as extended operators in the QFT. Assuming G acts faithfully, then this means for every $g \in G$, there is a unitary operator $U(g)$ supported on an equal time slice realizing the symmetry transformation on the Hilbert space. If G is continuous, then the Noether's theorem further allows us to associate a local conserved current $j^\mu(x)$ satisfying $\partial_\mu j^\mu(x) = 0$ to a given continuous symmetry, and the unitary operator can be constructed as $e^{i\alpha \oint_{\Sigma_{d-1}} *j}$. But we could alternatively considering the symmetry operator $U(g, \Sigma_{d-1})$ supported on other codim-1 manifold Σ_{d-1}; and instead of global fusion where we put two symmetry operators $U(g_1, \Sigma_{d-1})$ and $U(g_2, \Sigma_{d-1})$ on top of each other, we can consider the local fusion where we only put part of the symmetry operators on top of each other. If we have multiple symmetry operators, then in general there are different ways of doing local fusions; and their difference captures the additional structure known as the 't Hooft anomaly of the symmetry G. When G is continuous, the 't Hooft anomaly is easier to extract from the correlation functions of $j^\mu(x)$'s. The 't Hooft anomaly is an important observable of the theory and can be used to constrain the possible IR phases of a given QFT.

The modern description and generalization of the ordinary symmetry starts from the two important features of these symmetry operators. First, for any $g \in G$, there is a codim-1 surface operator $U(g, \Sigma_{d-1})$ supported on the codim-1 manifold Σ_{d-1}, and their fusion rules are governed by the group multiplication law of G. Second, the symmetry operator $U(g, \Sigma_{d-1})$ is topological, which means it is invariant under the local deformation of its support Σ_{d-1}. For continuous symmetry, this is guaranteed by the Stoke's theorem together with the property that $\partial_\mu j^\mu(x) = 0$.

Several generalizations have been made by relaxing some of the properties mentioned above. For instance, we could generalize the support of the invertible topological operator to be codim-$(p+1)$ surfaces for $p > 0$. This leads to the notion of p-form invertible symmetries and the ordinary symmetry then corresponds to 0-form symmetry. For a p-form symmetry $A^{(p)}$ where $p > 0$ and $A^{(p)}$ is an Abelian group, there is a topological surface operator $U^{(p)}(a, \Sigma_{d-p-1})$ for every group element $a \in A^{(p)}$ supported on a closed codim-$(p+1)$ surface Σ_{d-p-1} in the spacetime. And the fusion rule of $U^{(p)}(a, \Sigma_{d-p-1})$ is governed by the group multiplication law of $A^{(p)}$. In the case where $A^{(p)}$ is continuous, there is a conserved $(p+1)$-form current $(j^{(p)})^{\mu_1 \cdots \mu_{p+1}}$ satisfying $\partial_{\mu_1}(j^{(p)})^{\mu_1 \cdots \mu_{p+1}}(x) = 0$ and the topological surface operator is constructed as $e^{i\alpha \oint_{\Sigma_{d-p-1}} *j^{(p)}}$ for some codim-$(p+1)$ surface Σ_{d-p-1}. The ordinary 0-form symmetries can interact non-trivially with the higher form symmetries, and this leads to the structure of higher group symmetries.

Another direction of the generalization is to study the topological surface operator which does not admit an inverse under the fusion. The study of codim-1 non-invertible topological operators in 2d conformal field theory (CFT) has a long history, and the most famous example is the Kramers-Wannier duality line N in the Ising CFT. Together with two lines $\mathbb{1}$ and η in an ordinary $\mathbb{Z}_2$ symmetry, their fusion rules form the Ising fusion algebra:

$$\eta N = N\eta = N, \quad N^2 = \mathbb{1} + \eta. \tag{1.0.1}$$

These line defects can fuse locally, and the F-symbols characterize the difference between two distinct ways of locally fusing three line defects. It is convenient to package the fusion algebra together with the F-symbols as a structure known as the fusion category. Because of this, the non-invertible symmetries are also known as categorical symmetries. Notice that this framework is useful even for ordinary invertible 0-form symmetries. Symmetries with the same finite group G but with different 't Hooft anomalies will form different fusion categories. Therefore, an important question in the study of the categorical symmetries in QFTs to understand how to physically characterize these additional data other than fusion rules in QFTs and derive their physical implications.

There are two types of non-invertible symmetries. Sometimes, non-invertible symmetries can be engineered from invertible symmetries by some topological manipulations. Such non-invertible symmetries are called non-intrinsically non-invertible; otherwise they are called intrinsically non-invertible.

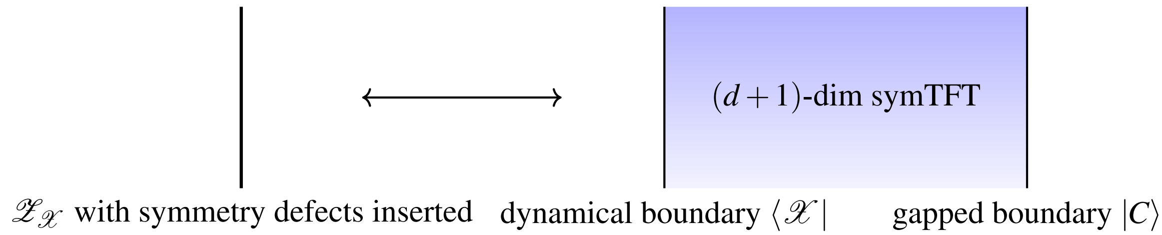


Figure 1.1. The idea of the symmetry TFT. The bulk of the slab is the $(d+1)$-dim symmetry TFT, and the right boundary is a topological boundary $|C\rangle$ which characterizes the topological defects in $\mathscr{C}$, while the left boundary is a non-topological boundary depending only on the theory $\mathscr{X}$ denoted as $|\mathscr{X}\rangle$.

An important tool to study the categorical symmetries is the notion of symmetry TFT. Consider a d-dimensional QFT $\mathscr{X}$ with categorical symmetries $\mathscr{C}$. The corresponding symmetry TFT is a $(d+1)$-dimensional topological field theory which allows us to expand the partition functions of $\mathscr{X}$ with topological defects inserted to a $(d+1)$-dimensional slab. The bulk of the slab is the $(d+1)$-dim symmetry TFT, and the right boundary is a topological boundary which characterizes the insertion of the topological defects in $\mathscr{C}$ while the left boundary is a non-topological boundary depending only on the theory $\mathscr{X}$, as depicted in the Figure 1.1. We will see examples of symmetry TFT and its application in determining whether a non-invertible symmetry is non-intrinsically non-invertible later in the thesis.

This thesis is organized as follows. We first study that, in 2-dim CFT, how the global symmetries can be used to refine the well-known Cardy formula, an asymptotic formula for the density of states derived from modular invariance. We then move on to study the properties of generalized global symmetries. We will first explore properties and physical implications of the triality defects in 2-dim CFT. Then we will study when the non-invertible symmetries are non-intrinsically non-invertible.

In Chapter 2, we prove a 2 dimensional Tauberian theorem in context of 2 dimensional conformal field theory. The asymptotic density of states with conformal weight $(h,\bar{h}) \to (\infty,\infty)$ for any arbitrary spin is derived using the theorem. We further rigorously show that the error term is controlled by the twist parameter and insensitive to spin. The sensitivity of the leading piece towards spin is discussed. We identify a universal piece in microcanonical entropy when the averaging window is large. An asymptotic spectral gap on $(h,\bar{h})$ plane, hence the asymptotic twist gap is derived. We prove an universal inequality stating that in a compact unitary 2D CFT without any conserved current $Ag \leq \frac{\pi(c-1)r^2}{24}$ is satisfied, where g is the twist gap over vacuum and A is the minimal "areal gap", generalizing the minimal gap in dimension to $(h',\bar{h}')$ plane and $r = \frac{4\sqrt{3}}{\pi} \simeq 2.21$. We investigate density of states in the regime where spin is parametrically larger than twist with both going to infinity. Moreover, the large central charge regime is studied. We also probe finite twist, large spin behavior of density of states.

In Chapter 3, we derive Cardy-like formulas for the growth of operators in different sectors of unitary 2 dimensional CFT in the presence of topological defect lines by putting an upper and lower bound on the number of states with scaling dimension in the interval $[\Delta-\delta,\Delta+\delta]$ for large Δ at fixed δ. Consequently we prove that given any unitary modular invariant 2D CFT symmetric under finite global symmetry G (acting faithfully), all the irreducible representations of G appear in the spectra of the untwisted sector; the growth of states is Cardy like and proportional to the "square" of the dimension of the irrep. In the Schwarzian limit, the result matches onto that of JT gravity with a bulk gauge theory. If the symmetry is non-anomalous, the result applies to any sector twisted by a group element. For $c>1$, the statements are true for Virasoro primaries. Furthermore, the results are applicable to large c CFTs. We also extend our results for the continuous $U(1)$ group.

In Chapter 4, we consider the triality fusion category discovered in the $c=1$ Kosterlitz-Thouless theory [181]. We analyze this fusion category using the tools from the group theoretical fusion category and compute the simple lines, fusion rules and F-symbols. We then studied the physical implication of this fusion category including deriving the spin selection rule, computing the asymptotic density of states of irreducible representations of the fusion category symmetries, and analyzing its anomaly and constraints under the renormalization group flow. There is another set of F-symbols for the fusion categories with the same fusion rule known in the literature [179]. We find these two solutions are different as they lead to different spin selection rules. This gives a complete list of the fusion categories with the same fusion rule by the classification result in [117].

Finally, in Chapter 5, we start with noticing that a quantum field theory with a finite abelian symmetry G admits a non-invertible duality defect if it is invariant under gauging G. For certain G, duality defects admit an alternative construction where one starts with invertible symmetries with certain 't Hooft anomaly, and gauging a non-anomalous subgroup. This special type of duality defects are termed group theoretical. In this work, we determine when duality defects are group theoretical, among $G = \mathbb{Z}_N^{(0)}$ and $\mathbb{Z}_N^{(1)}$ in 2d and 4d quantum field theories, respectively. We show that a duality defect is group theoretical if and only if its Symmetry TFT is a Dijkgraaf-Witten theory, which further translates to a stability condition of the topological boundary conditions of the G gauge theory. By solving the stability condition, we find that a $\mathbb{Z}_N^{(0)}$ duality defect in 2d is group theoretical if and only if N is a perfect square, and under certain assumptions a $\mathbb{Z}_N^{(1)}$ duality defect in 4d is group theoretical if and only if $N = L^2M$ where -1 is a quadratic residue of M. For these subset of N, we construct explicit topological manipulations that map the non-invertible duality defects to invertible ones. We also comment on the connection between our results and the recent discussion of obstruction to duality-preserving gapped phases.

Chapter 2

Tauberian-Cardy formula with spin

2.1 Summary & Discussion

The Cardy formula [42] for the asymptotic density of states has recently been rigorously derived with an estimate for the error term in [158, 92]. A natural question is to ask whether one can generalize the formalism so as to make it sensitive to the spin or equivalently to the conformal weights $h,\bar{h}$ separately. This necessitates working out a 2 dimensional Tauberian theorem, which we achieve here. The motivations for investigating Cardy formula on $(h',\bar{h}')$ plane are several. First of all, the notion of infinity on a 2d plane is richer than $\Delta \to \infty$ limit. We will see that the finer details of the Cardy formula actually depend on how infinity is approached unless one makes extra assumption about the spectrum. Furthermore, there have been interesting developments in the direction of lightcone bootstrap in recent times [134, 135, 55, 146, 28], our analysis puts some of these results on rigorous footing. Another amazing feature is the ability to investigate the "areal" notion of spectral gap. If we probe the $(h',\bar{h}')$ plane with circular areas of radius R, centered at $(h,\bar{h})$, then we find the optimal value of R which guarantees that the area contains at least one state. Again unless we put in extra assumption, the value of R depends on how infinity is approached and thus showing a richer asymptotic behavior. If we one assumes existence of twist gap, it turns out that the twist gap is complementary to asymptotic spectral gap in some sense, which we will make precise in due course.

The naive Cardy like analysis provides us with an expression for the asymptotic density of states where h and $\bar{h}$ are of the same order. One can re-express this as a function of dimension Δ and spin J with $\Delta \simeq J$. Now a natural question is to ask whether the result is valid when Δ and J is not of the same order. For example, in the large charge expansion literature [107, 106, 154, 65, 64, 18, 161, 81, 132, 133], the regime where $J \simeq \Delta^{1/n}$ with $n > 1$ is being probed. It turns out that only a part of the answer coming from the naive Cardy like analysis is meaningful while the rest of it is comparable to the error term. We emphasize that the analysis is only possible because now we have a rigorous estimate of the error term due to the Tauberian theorem that we prove in this paper.

With our rigorous treatment, it is possible to address issues regarding whether we can trust the naive Cardy formula when h and $\bar{h}$ are not of the same order. It turns out that the answer to this question is intimately connected with the existence of twist gap. We show that we can trust the naive Cardy formula for all the operators when $\max(h,\bar{h}) = \min(h,\bar{h})^{\Upsilon}$ with $1 \leq \Upsilon < 2$. It is also shown that with the assumption of twist gap, the validity of Cardy formula for primaries for $c > 1$ CFTs does not require any restriction on Υ.

The another motivation for taking up a rigorous study of Cardy formula is to be able to probe the large central charge (c) sector, to be specific, to derive the density of states when $h/c, \bar{h}/c$ are finite but c is very large. This part is in the spirit of result derived in [100]. A nice feature that reveals itself through the rigorous treatment is a curious connection between validity of Cardy regime and the twist gap above the vacuum. These features are important in the context of holography.

The plan of the paper is to quote the main results here in the beginning and discuss its consequences in terms of CFT data, such that the current section can be thought of as mostly self contained. The next section §2.2 gives some intuitive understanding of the technical stuff that follows. From §2.3 onwards, we plunge into technical proofs with a healthy relaxing intermission in §2.6, where we numerically verify our results. For readers going for a really quick ride, we have highlighted the main equations and results in what follows.

2.1.1 Integrated density of states

We prove a 2 dimensional Tauberian theorem in context of 2 dimensional conformal field theory. The asymptotic density of states with conformal weight $(h,\bar{h}) \to (\infty,\infty)$ is derived using the theorem. We find that the error term is controlled by the twist parameter. We note that as $(h,\bar{h}) \to (\infty,\infty)$, the twist also goes to ∞. We remark that the regime of validity depends on whether we put in the assumption of having a twist gap.

Definition: by finite twist gap, we mean there exists a number $\tau_* > 0$ such that there is no operator with twist $\tau \in (0,\tau_*)$ and there are finite number of zero twist operators[1] *with dimension less than $c/12$.*

We make two remarks: a) the fact that there are finite number of zero twist operators with dimension less than $c/12$ is always true since there are finite number of operators with dimension less than $c/12$ for finite central charge, b) Not having any operator with twist $\tau \in (0,\tau_*)$ disallows having 0 as twist accumulation point.

[1] Usually, by finite twist gap, it is assumed that there is no zero twist primaries except the Identity. Here we are using it in a slightly different manner, so one needs to be careful about using bounds on twist gap, such as the one appearing in [56].

Main theorems on integrated density of states

No assumption on twist gap:

We show that for finite central charge c, the number of states with conformal weights less than or equal to some specified large conformal weight $h, \bar{h}$ is given by:

$$\boxed{\begin{aligned} F(h,\bar{h}) &\equiv \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \rho(h',\bar{h}') \\ &\underset{\substack{h/\bar{h}=O(1) \\ h,\bar{h}\to\infty}}{=} \frac{1}{4\pi^2}\left(\frac{36}{c^2 h\bar{h}}\right)^{1/4} \exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\left[1+O\left(\tau^{-1/4}\right)\right]. \end{aligned}} \tag{2.1.1}$$

where τ is the twist of the state with $h, \bar{h}$ and given by $\tau = 2\min\{h, \bar{h}\}$. Here we have assumed that $h/\bar{h} = O(1)$ number[2]. As a result one could have written the error term as $O\left(h^{-1/4}\right)$ or $O\left(\bar{h}^{-1/4}\right)$.

Assuming a twist gap:

It turns out that if we assume a finite twist gap, we can trust eq. (2.1.1) even when h and $\bar{h}$ are not of the same order but $h = \bar{h}^{\upsilon}$ with $1/2 < \upsilon < 2$. In such a scenario, the error term becomes $O(\tau^{\frac{\Upsilon}{4}-1/2})$, where $\Upsilon = \max(\upsilon, 1/\upsilon)$. The Υ characterizes how h and $\bar{h}$ are of different order asymptotically in a symmetrized fashion, for example, if we approach the infinity along the curve $h = \bar{h}^{1.1}$ or $\bar{h} = h^{1.1}$, we have $\Upsilon = 1.1$. Thus our error estimation is symmetric if we reflect the line of approach to infinity about $h = \bar{h}$ line.

[2]If we say $f = O(1)$, we mean $|f| < M$ for a fixed positive number M.

We have for $1 \leq \Upsilon < 2$,

$$\boxed{\begin{aligned} F(h,\bar{h}) &\equiv \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \rho(h',\bar{h}') \\ &\underset{\substack{h,\bar{h}\to\infty \\ \frac{1}{2}<\frac{\log h}{\log \bar{h}}<2}}{=} \frac{1}{4\pi^2}\left(\frac{36}{c^2 h\bar{h}}\right)^{1/4} \exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\left[1+O\left(\tau^{\frac{\Upsilon}{4}-1/2}\right)\right], \quad \Upsilon < 2. \end{aligned}} \tag{2.1.2}$$

The eq. (2.1.1) and eq. (2.1.2) are two of the central results obtained in this paper. If h and $\bar{h}$ are not of the same order, we basically probe the large spin sector of density of states[3], to be precise, the regime where spin is parametrically larger than the twist but both goes to infinity.

The basic structure of both the eq. (2.1.1) and eq. (2.1.2) is that they have leading exponential piece multiplied with a subleading polynomial suppression. The error term is then further suppressed by a polynomial piece. Now if $\Upsilon \geq 2$, one can see the error term in (2.1.2) is not really suppressed, hence is not in fact an error term. Thus we can not trust the polynomially suppressed terms. In this regime, we are able to show that

$$\boxed{F(h,\bar{h}) \underset{h,\bar{h}\to\infty}{=} \exp\left[2\pi\sqrt{\frac{ch}{6}}+2\pi\sqrt{\frac{c\bar{h}}{6}}\right] O\left(\tau^{-3/4}\right), \quad \Upsilon \geq 2.} \tag{2.1.3}$$

We further remark that for CFTs where the partition function nicely factorizes into holomorphic and antiholomorphic pieces, the leading result directly follows from the analogous result for large $\Delta = h + \bar{h}$, proven in [158], nonetheless the error term in analogues of eq. (2.1.1) and eq. (2.1.2) goes like $O(h^{-1/2})$, hence, in such a case, we have more control over the approximation.

[3] A cautionary remark is that here in this paper unless otherwise mentioned, the twist is **NOT** kept finite while taking this limit. This can be contrasted to the scenario in the usual large spin expansion [6], where one keeps the twist finite.

Corollaries of the theorems [Eq. (2.1.1) **and Eq.** (2.1.2)**] on integrated density of states**

Below we will digress a bit and touch upon some of the interesting results that can be extracted from the above before coming back to summazing our main results in the next subsection §2.1.2.

Rich structure of asymptotic approach:

The integrated density of states show distinct leading behavior depending on how the asymptotic infinity is approached. In [158], it has been shown that as $\Delta \to \infty$, we have

$$\begin{aligned} F^{\mathrm{MZ}}(\Delta) &\equiv \int_0^{\Delta} \mathrm{d}\Delta' \rho(\Delta') \\ &\underset{\Delta\to\infty}{=} \frac{1}{2\pi}\left(\frac{3}{c\Delta}\right)^{1/4} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right]\left[1+O\left(\Delta^{-1/2}\right)\right]. \end{aligned} \tag{2.1.4}$$

We remark that in the asymptotic limit, both $F^{\mathrm{MZ}}(\Delta\to\infty)$ and $F(h\to\infty,\bar{h}\to\infty)$ count the total number of operators. But these functions approach infinity in a different manner (see the figure 2.1). To be concrete, let us choose $h=\bar{h}=\Delta/2$, thus we have

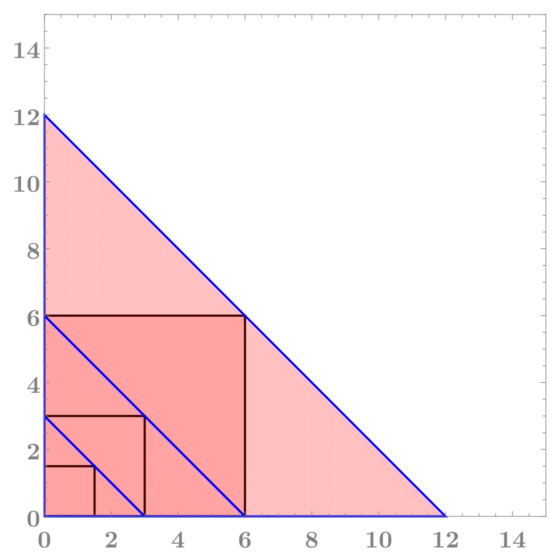


Figure 2.1. Approaching to infinity on $(h',\bar{h}')$ plane: The number of operators bounded by the blue lines is counted by $F^{\mathrm{MZ}}(\Delta)$ originally calculated in [158]. The number of operators bounded by the black lines is counded by $F(\Delta/2,\Delta/2)$ calculated in this paper.

$$F(\Delta/2,\Delta/2) \underset{\Delta\to\infty}{=} \frac{1}{2\pi^2}\left(\frac{3}{c\Delta}\right)^{1/2} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right]\left[1+O\left(\Delta^{-1/4}\right)\right]. \tag{2.1.5}$$

So we can see that $\lim_{\Delta\to\infty} F(\Delta/2,\Delta/2)$ is power law suppressed compared to $\lim_{\Delta\to\infty} F^{\mathrm{MZ}}(\Delta)$ i.e.

$$\lim_{\Delta\to\infty}\left(\frac{F(\Delta/2,\Delta/2)}{F^{\mathrm{MZ}}(\Delta)}\right) = O(\Delta^{-1/4}).$$

We see that the square $\mathbb{S}_{\text{II}}$ of size $\Delta/2$ with one vertex at origin and another one at $(\Delta/2,\Delta/2)$ is always contained within the rightangled triangular region $\mathbb{T}$, created by h' axis, $\bar{h}'$ axis and $h'+\bar{h}' = \Delta$ line. This is consistent with the observation that leading behavior of $F(\Delta/2,\Delta/2)$ is suppressed compared to $F^{\mathrm{MZ}}(\Delta)$. In fact, one can similarly study the distribution of the operators in rectangular (or square) areas such that the rectangle is contained within $\mathbb{T}$, and one vertex is on the line $h'+\bar{h}' = \Delta$ (see the figure 2.2). This study reveals that the among such areas, the

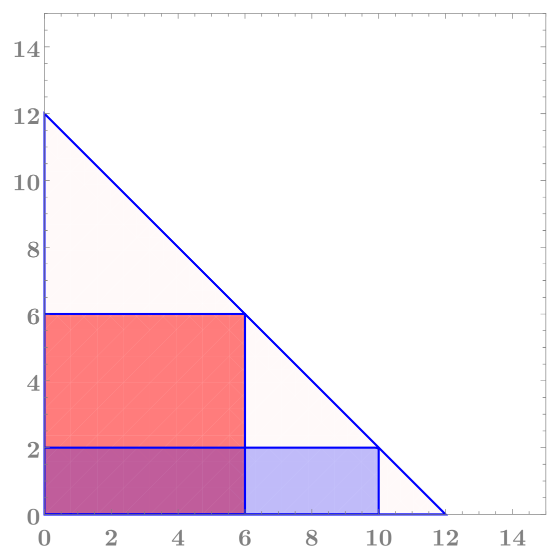


Figure 2.2. $(h',\bar{h}')$ plane : asymptotically, the rectangular region (blue shaded) contains exponentially less number of operators compared to square region (red shaded). They are contained within the rightangled triangle, created by h' axis, $\bar{h}'$ axis and $h'+\bar{h}' = \Delta$ line. Here $\Delta = 12$.

square $\mathbb{S}_{\text{II}}$ contains the most number of operators while any other rectangular region contains fewer number of operators, in fact the number is exponentially suppressed compared to that of the square $\mathbb{S}_{\text{II}}$.

Spin sensitivity of the asymptotics:

One can make a detailed analysis of spin sensitivity of the above result, which we expound on §2.5.2.

Windowed entropy with respect to h and $\bar{h}$:

An immediate consequence of the eq. (2.1.1) is the expression for "windowed" entropy $S_{\delta,\bar{\delta}}$. The windowed entropy is defined as logarithm of number of states within a rectangular window of side length 2δ and $2\bar{\delta}$, centered at $(h,\bar{h})$. This is analogous to entropy defined as in microcanonical ensemble by proper "binning", where the bin size is dictated by $\delta,\bar{\delta}$. As we take $h\to\infty,\bar{h}\to\infty$, we can keep the bin size $\delta,\bar{\delta}$ order one or let them scale like h^{α} and $\bar{h}^{\alpha}$ respectively. We find that

$$\begin{aligned} S_{\delta,\bar{\delta}} &\equiv \log\left(\int_{h-\delta}^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}' \rho(h',\bar{h}')\right) \\ &\underset{h,\bar{h}\to\infty}{=} 2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)+\frac{1}{4}\log\left[\frac{c^2\delta^4\bar{\delta}^4}{36h^3\bar{h}^3}\right]+s(\delta,\bar{\delta},h,\bar{h}), \end{aligned} \tag{2.1.6}$$

where for $3/8<\alpha\leq 1/2$, we have, :

$$\begin{cases} \delta\simeq h^{\alpha} \\ \bar{\delta}\simeq \bar{h}^{\alpha} \end{cases} \quad s(\delta,\bar{\delta},h,\bar{h}) = \log\left(\frac{\sinh\left(\pi\sqrt{\frac{c}{6}}\frac{\delta}{\sqrt{h}}\right)}{\pi\sqrt{\frac{c}{6}}\frac{\delta}{\sqrt{h}}}\right)+\log\left(\frac{\sinh\left(\pi\sqrt{\frac{c}{6}}\frac{\bar{\delta}}{\sqrt{\bar{h}}}\right)}{\pi\sqrt{\frac{c}{6}}\frac{\bar{\delta}}{\sqrt{\bar{h}}}}\right)+O(\tau^{3/4-2\alpha}), \tag{2.1.7}$$

$$\delta,\bar{\delta}\simeq O(1) \qquad s_-(\delta,\bar{\delta})\leq s(\delta,\bar{\delta},h,\bar{h})\leq s_+(\delta,\bar{\delta}) \tag{2.1.8}$$

where the functions $s_{\pm}(\delta,\bar{\delta})$ are determined in the section §2.3, in particular, we have $s_{\pm}\equiv\exp(c_{\pm})$, and $c_{\pm}$ is given by (2.3.25). We remark that when the bin size is large, there is a universal correction to Cardy formula given by the sinhyperbolic functions. This is analogous to what is found in [158] from the analysis sensitive to dimension only.

Windowed entropy with respect to $\Delta+J$:

One can define a microcanonical entropy with respect to $\Delta+J = 2\max\{h,\bar{h}\}$ (name this parameter κ) as

$$S_{\delta}^{\kappa} \equiv \log\left[F(\kappa/2+\delta,\kappa/2+\delta) - F(\kappa/2-\delta,\kappa/2-\delta)\right], \tag{2.1.9}$$

The asymptotic behavior of S_{δ}^{κ} is given by

$$S_{\delta}^{\kappa} = 4\pi\sqrt{\frac{c\kappa}{12}} + \log\left(\frac{2\delta}{\pi\kappa}\right) + s(\delta,\kappa), \tag{2.1.10}$$

where for large enough bin size ($\delta \simeq \kappa^{\alpha}$) we have

$$\delta \simeq \kappa^{\alpha} : s(\delta,\tau) = \log\left(\frac{\sinh\left(2\pi\sqrt{\frac{c}{3}}\frac{\delta}{\sqrt{\kappa}}\right)}{2\pi\sqrt{\frac{c}{3}}\frac{\delta}{\sqrt{\kappa}}}\right) + O\left(\kappa^{1/4-\alpha}\right),\ 1/4 < \alpha \leq 1/2. \tag{2.1.11}$$

2.1.2 $c > 1$ CFTs-results specific for primaries

One can make the results in the previous subsection specific to Virasoro primaries only, in fact do better. This boils down essentially repeating the argument presented in §2.3,§2.4 and §2.5 with minor modification. The idea of extending the argument from §2.3,§2.4 and §2.5 to this case is similar in spirit and practice to how [158] obtained the specific results for primary using methods suitable to study all the operators. The details can be found in §2.3, specifically eq. (2.3.37) onwards. Without much ado, here is the result: for finite central charge c, we find the integrated density of states specific for primaries behave like (from now on, we will be using the superscript "Vir" to denote the result specific for primaries):

$$\boxed{\begin{aligned} F^{\mathrm{Vir}}(h,\bar{h}) &\equiv \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}'\, \rho^{\mathrm{Vir}}(h',\bar{h}') \\ &\underset{h,\bar{h}\to\infty}{=} \frac{1}{\pi^2}\left(\frac{3}{c-1}\right)\exp\left[2\pi\left(\sqrt{\frac{(c-1)h}{6}}+\sqrt{\frac{(c-1)\bar{h}}{6}}\right)\right]\left[1+O\left(\tau^{-1/4}\right)\right]. \end{aligned}} \tag{2.1.12}$$

The "windowed" entropy (we have considered bin of size 2δ by $2\bar{\delta}$ just like what we did for the analysis of all the operators) for Virasoro primaries is given by

$$\begin{aligned} S^{\mathrm{Vir}}_{\delta,\bar{\delta}} &\equiv \log\left(\int_{h-\delta}^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}' \rho^{\mathrm{Vir}}(h',\bar{h}')\right) \\ &\underset{h,\bar{h}\to\infty}{=} 2\pi\left(\sqrt{\frac{(c-1)h}{6}}+\sqrt{\frac{(c-1)\bar{h}}{6}}\right)-\frac{1}{2}\log\left[\frac{h\bar{h}}{4\delta^2\bar{\delta}^2}\right]+s^{\mathrm{Vir}}(\delta,\bar{\delta},h,\bar{h}), \end{aligned} \tag{2.1.13}$$

where for $1/8<\alpha\leq 1/2$, we have :

$$\begin{aligned} &\begin{cases} \delta \simeq h^{\alpha} \\ \bar{\delta} \simeq \bar{h}^{\alpha} \end{cases} : s^{\mathrm{Vir}}(\delta,\bar{\delta},h,\bar{h}) \\ &\qquad = \log\left(\frac{\sinh\left(\pi\sqrt{\frac{c-1}{6}}\frac{\delta}{\sqrt{h}}\right)}{\pi\sqrt{\frac{c-1}{6}}\frac{\delta}{\sqrt{h}}}\right)+\log\left(\frac{\sinh\left(\pi\sqrt{\frac{c-1}{6}}\frac{\bar{\delta}}{\sqrt{\bar{h}}}\right)}{\pi\sqrt{\frac{c-1}{6}}\frac{\bar{\delta}}{\sqrt{\bar{h}}}}\right)+O(\tau^{1/4-2\alpha}), \\ &\delta,\bar{\delta}\simeq O(1) : s_-(\delta,\bar{\delta}) \leq s^{\mathrm{Vir}}(\delta,\bar{\delta},h,\bar{h}) \leq s_+(\delta,\bar{\delta}) \end{aligned} \tag{2.1.14}$$

where the functions $s_{\pm}(\delta,\bar{\delta})$ are the same functions that appear in the analysis for all the operators.

Large spin, large twist sector for primaries:

If we assume a finite twist gap (as defined in $\{2.1.1\}$), the result given in eq. (2.1.12) is true irrespective of whether h and $\bar{h}$ are of the order one or not. Thus unlike the case for all the operators, here we can trust the polynomially suppressed correction for all values of υ, where $h = \bar{h}^{\upsilon}$.

2.1.3 Large spin, finite twist sector

The large spin, finite twist sector is not entirely asymptotic regime since the quantity knows about low lying spectrum in one of the weights. It turns out we can only put an upper bound in this case. There is an $O(1)$ error in the estimation. While for the upper bound this does not cause any trouble, for the lower bound, it makes thing tricky. In particular, the lower bound on the density of states, appropriately integrated, contains an exponential piece as expected from extended Cardy formula [134, 135, 146, 28] but it comes with a multiplicative order one number, which can become negative unless proven otherwise.

Analysis for all the operators:

In what follows, we will keep h finite and let $\bar{h} \to \infty$, the windowed entropy $S^{\text{ft}}_{\delta,\bar{\delta}}$ is found to be bounded above by

$$\boxed{S^{\text{ft}}_{\delta,\bar{\delta}} \leq \mathbb{S}^{\text{ft}}_{h,\delta,\bar{\delta}} \leq 2\pi\sqrt{\frac{c\bar{h}}{6}} - \frac{1}{4}\log\left(\frac{\bar{h}^3}{16\bar{\delta}^4}\right) + M\,,} \tag{2.1.15}$$

where M is an order one number. Here $S^{\text{ft}}_{\delta,\bar{\delta}}$ and $\mathbb{S}^{\text{ft}}_{h,\delta,\bar{\delta}}$ are defined as

$$\exp\left[S^{\text{ft}}_{\delta,\bar{\delta}}\right] \equiv \int_{h-\delta}^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}'\, \rho(h',\bar{h}')\,, \quad \exp\left[\mathbb{S}^{\text{ft}}_{h,\delta,\bar{\delta}}\right] \equiv \int_{0}^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}'\, \rho(h',\bar{h}')\,.$$

The number M is given (or estimated) by

$$M = 2\pi\left(h+\delta-\frac{c}{24}\right) + \log\left[c_{+}\sum_{\tilde{h}}\chi_{\tilde{h}}(e^{-2\pi})\right]\,. \tag{2.1.16}$$

where $\chi_{\tilde{h}}$ is the character for the conserved current with weight $(\tilde{h},0)$, $\tilde{h}\geq 0$ (including the Identity) and $c_\pm$ is an order one $h,\bar{h}$ independent number, defined in §2.7. M is a finite number as the absolute value of the sum over $\tilde{h}$ is bounded above by the partition function evaluated at $\beta=\bar{\beta}=2\pi$, which is a finite number.

Analysis for primaries with/without conserved currents:

The above result can also be made specific to primaries:

$$\boxed{S^{\text{Vir,ft}}_{\delta,\bar{\delta}} \leq \mathbb{S}^{\text{Vir,ft}}_{h,\delta,\bar{\delta}} \leq 2\pi\sqrt{\frac{(c-1)\bar{h}}{6}} - \frac{1}{2}\log\left(\frac{\bar{h}}{4\bar{\delta}^2}\right) + M^{\text{Vir}}\,.} \tag{2.1.17}$$

Here $S^{\text{Vir,ft}}_{\delta,\bar{\delta}}$ and $\mathbb{S}^{\text{Vir,ft}}_{h,\delta,\bar{\delta}}$ are defined as

$$\begin{aligned}
\exp\left[\mathbb{S}^{\text{Vir,ft}}_{h,\delta,\bar{\delta}}\right] &\equiv \int_0^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}'\ \rho^{\text{Vir}}(h',\bar{h}')\,,\\
\exp\left[S^{\text{Vir,ft}}_{\delta,\bar{\delta}}\right] &\equiv \int_{h-\delta}^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}'\ \rho^{\text{Vir}}(h',\bar{h}')\,.
\end{aligned}$$

and M^{Vir} is an order one number, given by

$$M^{\text{Vir}} = 2\pi\left(h+\delta-\frac{c-1}{24}\right) + \log\left[c_+\sum_{\tilde{h}} e^{-2\pi\left(\tilde{h}-\frac{c-1}{24}\right)}\right], \tag{2.1.18}$$

where the zero twist primaries have weight $(\tilde{h},0)$, $\tilde{h}\geq 0$ (including the Identity) and c_+ is an order one $h,\bar{h}$ independent number, defined in §2.7. M^{Vir} is a finite number since the sum inside the log is convergent. This happens because the absolute value of the sum is bounded by the partition function evaluated at $\beta=\bar{\beta}=2\pi$, which is a finite number.

Analysis for primaries for CFT with no nontrivial conserved current:

If we assume that there is no nontrivial conserved current i.e the only zero twist primary is the Identity and there is a finite twist gap (the finite twist gap as defined in $\{2.1.1\}$, combined with the absence of nontrivial conserved current implies the usual twist gap condition used in the literature, for example in [56]). we show that

$$\boxed{\begin{aligned} S^{\text{Vir,ft}}_{\delta,\bar{\delta}} \leq \mathbb{S}^{\text{Vir,ft}}_{h,\delta,\bar{\delta}} \leq 2\pi\sqrt{\frac{(c-1)\bar{h}}{6}} - \frac{1}{2}\log\left(\frac{\bar{h}}{4\bar{\delta}^2}\right) + \log\left(\sqrt{h+\delta-\frac{c-1}{24}}\right) \\ + \frac{\pi^2}{6}(c-1)\left(h+\delta+\frac{c-1}{24}\right) + \log\left(1-e^{-4\pi^2\left(h+\delta-\frac{c-1}{24}\right)}\right) + M', \end{aligned}} \tag{2.1.19}$$

where M' is an order one h independent number. If we assume that $(h+\delta-\frac{c-1}{24})$ is a very small number compared to $\frac{1}{c-1}$, this matches with the leading result appeared in the lightcone bootstrap program [134, 135, 55, 146, 28] i.e.

$$\boxed{S^{\text{Vir,ft}}_{\delta,\bar{\delta}} \leq \mathbb{S}^{\text{Vir,ft}}_{h,\delta,\bar{\delta}} \leq 2\pi\sqrt{\frac{(c-1)\bar{h}}{6}} - \frac{1}{2}\log\left(\frac{\bar{h}}{4\bar{\delta}^2}\right) + \frac{3}{2}\log\left(h+\delta-\frac{c-1}{24}\right) + \tilde{M}',} \tag{2.1.20}$$

where $\tilde{M}' = M + \log(4\pi^2)$. We remark that the limit is very subtle here. There are several scales. The scale set by $\bar{h}$ is the largest one and we are seeking an asymptotic behavior in $\bar{h}$. Then there are two fixed parameters h and c. We are probing the regime where $(h+\delta-\frac{c-1}{24})$ is a very small number compared to $\frac{1}{c-1}$. The details of the calculation can be found at the end of §2.7.

2.1.4 Asymptotic spectral gap

The idea about deriving an upper bound on spectral gap comes from binning the states. If we make the bin size very small, we can not prove a positive lower bound on the number of states in that bin, because the bin might not have any state at all. As we increase the bin size, the chances are more that we find such positive lower bound. If we find a positive lower bound for a specific bin size centered at some large $h,\bar{h}$; that would immediately imply existence of an upper bound on the asymptotic spectral gap.

Probing spectral gap via Circle of order one area

With/without twist gap:

Here we do not put any assumption on twist gap.

Let us consider a square $\mathbb{S}$ of side $\frac{4\sqrt{3}\gamma}{\pi}+\varepsilon_g$ centered at $(h,\bar{h})$ on $(h',\bar{h}')$ plane. Here ε_g can be any arbitrarily small positive number. In the limit $h\to\infty,\bar{h}\to\infty$ we have

$$\int_{\mathbb{S}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho^{\mathrm{Vir}}(h',\bar{h}')>0\,, \tag{2.1.21}$$

where the asymptotic region is reached along a curve for which $\max(h,\bar{h})\simeq\frac{\gamma^4\tau}{2}$. Thus the spectral gap along this curve is bounded above by a circle of radius $\frac{\gamma r}{\sqrt{2}}$ and the best possible value of r that we find is $r=\frac{4\sqrt{3}}{\pi}$, this being the circle circumscribing the square.

An immediate corollary is that the asymptotic twist gap is upper bounded by $\frac{8\sqrt{3}\gamma}{\pi}\simeq 4.42\gamma$. For $c>1$, the argument can be made specific for primaries, hence the asymptotic gap. This in some sense complements the bound on primary twist gap over the vacuum[4] [56, 28].

We suspect that either by suitable choice of function or by the better estimate of heavy sector of the partition function, r and/or length of a side of the bounding square can be made to 1. If this can be done, then the bound becomes optimal for $\gamma=1$, (assuming we always consider circular/square region) since tensoring chiral Monster CFT with antichiral Monster CFT saturates the bound. One can see the saturation by circumscribing a square of unit length by a circle of radius $\frac{1}{\sqrt{2}}$ on $(h',\bar{h}')$ plane [See the fig. 2.3]. Nonetheless, the optimality along a curve for $\gamma\neq 1$ is not guaranteed. An immediate corollary of finding such a circle is that the asymptotic twist

[4] In [56], it is mentioned that the argument is due to Tom Hartman.

gap is upper bounded by $2\sqrt{2}$ along $h = \bar{h}$ curve. The same bound holds for asymptotic primary twist gap. We remark that in terms of twist, the above gap might not be optimal, since if we tensor chiral Monster CFT with anti-chiral monster CFT, the asymptotic twist gap is 2. If one can find a bounding square of side length given by 1, that would reproduce the optimal twist gap 2.

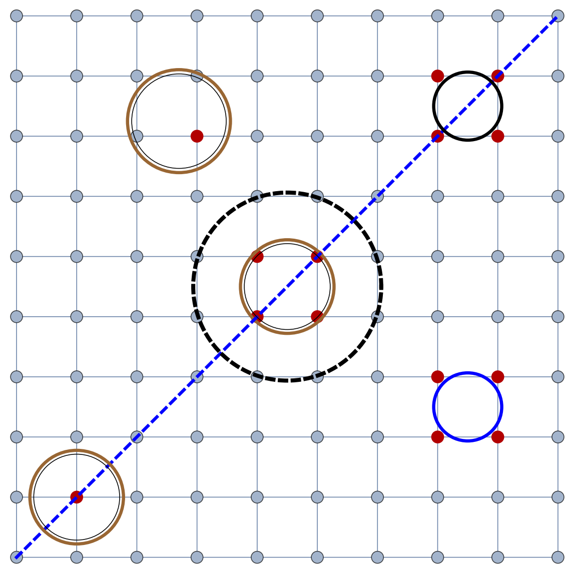


Figure 2.3. Operator spectrum of chiral Monster CFT tensored with its antichiral avatar on $(h', \bar{h}')$ plane: each vertex in the lattice represents the presence of operators. Any circle centered at $(h, \bar{h})$ and of radius $\frac{1}{\sqrt{2}} + \varepsilon_g$ with $\varepsilon_g > 0$ would contain at least an operator.

The above result and the conjectures can not be applied to a scenario, where infinity is approached along a curve where h is of widely different order compared to $\bar{h}$, in particular, say, if we want to approach the infinity along the curve $h^{\upsilon} = \bar{h}$ with $\upsilon \neq 1$. To circumnavigate this issue, we assume existence of twist gap g. We remind the readers that by finite twist gap, we mean there exists a number $\tau_* > 0$ such that there is no operator with twist $\tau \in (0, \tau_*)$ and there are finite number of zero twist operators with dimension less than $c/12$, and $g \geq \tau_*$. Moreover, assuming existence of g helps us to get rid of dependence on γ.

CFT with twist gap g:

Now we assume that the CFT has a twist gap as defined in $\{2.1.1\}$.

For a CFT with twist gap g (as defined in $\{2.1.1\}$) and central charge $c > 1$, one can have a bounding circle $\mathbb{C}$ specific to primaries having a radius $\frac{\sigma r}{\sqrt{2}} + \varepsilon_g$ with $\varepsilon_g > 0$, where

$$\sigma = \max\left(1, \sqrt{\frac{c-1}{12g}}\right), \quad r = \frac{4\sqrt{3}}{\pi}. \tag{2.1.22}$$

irrespective of how infinity is approached, such that the bounding circle contains at least one operator.

Thus for such a scenario there exists h_* and $\bar{h}_*$, two order one numbers such that

$$\int_{\mathbb{C}} \mathrm{d}h' \, \mathrm{d}\bar{h}' \, \rho^{\mathrm{Vir}}(h', \bar{h}') > 0 \,, \; \forall \, h > h_*, \bar{h} > \bar{h}_* \,. \tag{2.1.23}$$

Again this is obtained by circumscribing the appropriate bounding square [See the fig. 2.4]. The superscript "Vir" on ρ^{Vir} denotes that it is density of primary operators as opposed to all the operators. In a compact unitary 2D CFT without any conserved current, one can use the upper bound of twist gap due to Hartman, appearing in [56], to deduce

$$\sigma = \max\left(1, \sqrt{\frac{c-1}{12g}}\right) = \sqrt{\frac{c-1}{12g}}. \tag{2.1.24}$$

Now we will explain the sense in which the minimal gap is complementary to twist gap:

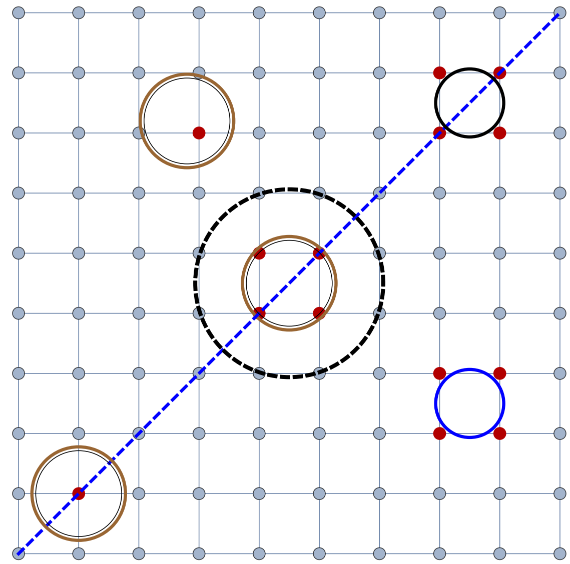


Figure 2.4. Assuming twist gap g: operator spectrum of chiral Monster CFT tensored with its antichiral avatar on $(h', \bar{h}')$ plane: each vertex in the lattice represents the presence of operators. Any circle centered at $(h, \bar{h})$ and of radius $\frac{\kappa}{\sqrt{2}} + \varepsilon_g$ with $\varepsilon_g > 0, \kappa \geq 1$ would contain at least an operator.

Suppose we consider a 2D compact unitary CFT with twist gap g such that it does not have any zero twist primaries (conserved currents) except the Identity: if asymptotically there exists a circle of minimal area[a] A on $(h', \bar{h}')$ plane such that it does not contain any operator, we immediately deduce the following inequality

$$Ag \leq \frac{\pi(c-1)r^2}{24}\,. \tag{2.1.25}$$

[a]One can imagine that operators having conformal weight $(h, \bar{h})$ are denoted by the point $(h, \bar{h})$ on $(h', \bar{h}')$ plane. We name this set to be $\mathbb{S}$. Now consider the set $\mathbb{S}_d \equiv \{d(a,b) : a, b \in \mathbb{S}\}$, where $d(a,b)$ is the Euclidean distance on the plane between the points a and b. Existence of a circle with minimal area means $\mathrm{Inf}\, \mathbb{S}_d > 0$. Asymptotically minimal area means that we look at the plane for $h' > h_*$ & $\bar{h}' > \bar{h}_*$ and construct the set $\mathbb{S}_d$ and consider its infimum.

This can be thought of as an upper bound on twist gap if the minimal areal gap A is known (note that minimal areal gap obtained from the full spectra has to be less than or equal to the asymptotic minimal gap). If one can make $r = 1$ and show that $\frac{\pi}{2} < A = \frac{k\pi}{2}$ then it is possible to lower the upper bound on twist gap from $\frac{c-1}{12}$ to $\frac{c-1}{12k}$ with $k > 1$. This might be of importance for proving the proposed upper bound $\frac{c-1}{16}$ in [28], if it is true. To rephrase, if one can show that the minimal area $A=\frac{2\pi}{3}$ (thus the diameter of the circle would be $2\sqrt{\frac{2}{3}}$), it would imply the proposed bound. Similarly, any lower bound on twist gap translates to upper bound on minimal areal gap A.

The similar analysis can be done for all the operators assuming a twist gap. The only difference is that σ would be given by $\sigma=\max\left(1,\sqrt{\frac{c}{12g}}\right)$. We elucidate on these bounds in the §2.3.

Probing spectral gap via Strips

Instead of squares , we can think of covering the $(h',\bar{h}')$ plane via strips of finite width and ask what is the minimum width of the strip that guarantees existence of at least one state (one can make the analysis sensitive to primaries such that the state is Virasoro primary) in the strip. We can consider three kind of strips (see figure 2.5) :

$$H_1(h)\equiv\{(h',\bar{h}'):h'\in[h-\delta_1,h+\delta_1],\bar{h}'\geq 0\}\,, \tag{2.1.26}$$

$$H_2(\bar{h})\equiv\{(h',\bar{h}'):\bar{h}'\in[\bar{h}-\delta_2,\bar{h}+\delta_2]\,,h'\geq 0\}, \tag{2.1.27}$$

$$H_3(\Delta)\equiv\{(h',\bar{h}'):h'+\bar{h}'\in[\Delta-\delta_3,\Delta+\delta_3],h',\bar{h}'\geq 0\}\,. \tag{2.1.28}$$

It is shown in [92] that if $\delta_3>\frac{1}{2}$, we have

$$\int_{H_3(\Delta\to\infty)}\mathrm{d}h'\,\mathrm{d}\bar{h}'\,\rho^{\mathrm{Vir}}(h',\bar{h}')>0 \tag{2.1.29}$$

Thus the asymptotic spectral gap is bounded above by 1.

In this work we show that

$$\boxed{\int_{H_1(h\to\infty)}\mathrm{d}h'\,\mathrm{d}\bar{h}'\,\rho^{\mathrm{Vir}}(h',\bar{h}')>0\text{ for }\delta_1>\frac{1}{\sqrt{2}}} \tag{2.1.30}$$

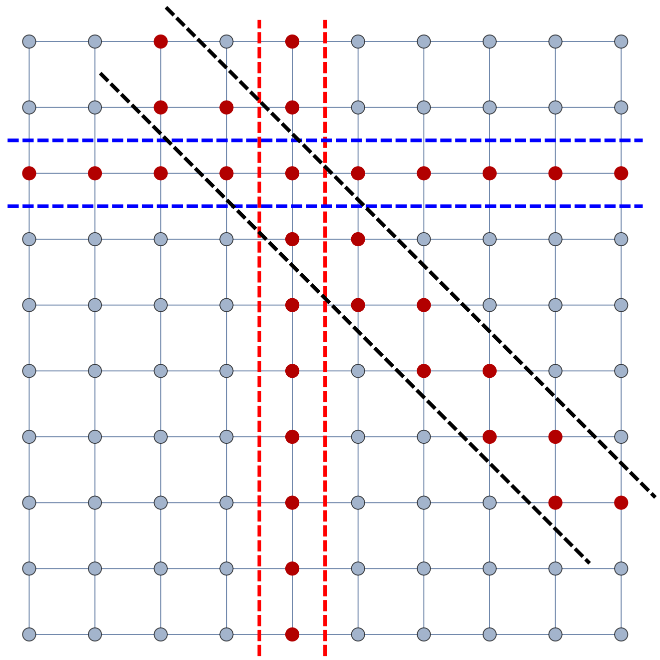


Figure 2.5. We consider three kind of strips: the red vertical one is $H_2(\bar{h})$, the blue horizontal one is $H_1(h)$ and the black one is $H_3(\Delta)$. In each cases, we see that there is a minimum width of the strip such that the strips contain at least one operator.

The same result holds true for the H_2 strip with h replaced by $\bar{h}$, where H_1, H_2 are defined in (2.1.26). This comes from putting a positive lower bound on the right hand side of the following inequality:

$$\int_{H_1(h\to\infty)} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho^{\mathrm{Vir}}(h',\bar{h}') \geq \int_{H_1(h\to\infty)} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho^{\mathrm{Vir}}(h',\bar{h}')e^{-\bar{\beta}\bar{h}} \tag{2.1.31}$$

We achieve this as a corollary of the lemma proven in §2.4 (a similar lemma can be proven for primaries only and then we use the above inequality) . This shows that

> Asymptotically on the $(h',\bar{h}')$ plane, if the width of the horizontal or vertical strip is bigger than $\sqrt{2}$, it contains at least one Virasoro primary. This might not be optimal since the gap we find by tensoring chiral Monster CFT with its anti-chiral avatar is 1.

2.1.5 Analysis at large central charge

We consider the $c \to \infty$ limit and parametrize the conformal weights in following way:

$$h = c\left(\frac{1}{24}+\varepsilon\right), \bar{h} = c\left(\frac{1}{24}+\bar{\varepsilon}\right); \quad \varepsilon, \bar{\varepsilon} \text{ fixed}. \tag{2.1.32}$$

Let us define $\varepsilon_* = \min(\varepsilon, \bar{\varepsilon})$ and $\varepsilon^* = \max(\varepsilon, \bar{\varepsilon})$.

With/without twist gap:

We show that[5]

For $\min\left\{(h/c-1/24),(\bar{h}/c-1/24)\right\} = \varepsilon_* > \frac{1}{6}$, the microcanonical entropy for order one window $\delta, \bar{\delta} \simeq O(1)$ is given by

$$\boxed{S_{\delta,\bar{\delta}} \underset{c\to\infty}{\simeq} 2\pi\left(\sqrt{\frac{c}{6}\left(h-\frac{c}{24}\right)}+\sqrt{\frac{c}{6}\left(\bar{h}-\frac{c}{24}\right)}\right)-\log(c)+O(1),} \tag{2.1.33}$$

while for $\delta, \bar{\delta} \simeq c^{\alpha}$ with $0 < \alpha \leq 1$, we have

$$\boxed{S_{\delta} \underset{c\to\infty}{\simeq} 2\pi\left(\sqrt{\frac{c}{6}\left(h+\delta-\frac{c}{24}\right)}+\sqrt{\frac{c}{6}\left(\bar{h}+\bar{\delta}-\frac{c}{24}\right)}\right)-\log(c)+O(1).} \tag{2.1.34}$$

Assuming finite twist gap:

If we assume a finite twist gap g (as defined in $\{2.1.1\}$), then

[5] It might be possible to extend the region of validity beyond this, in particular, following [100], one might expect it to be valid for $\varepsilon\bar{\varepsilon} > \frac{1}{24^2}$!

The regime of validity of the Cardy result as in (2.1.33) can be extended to

$$\boxed{\varepsilon_* > \frac{1}{6}\left[\max\left\{\frac{1}{2},\left(1-\frac{6g}{c}\right)^2\right\}\right], \quad \varepsilon_* \equiv \min\left\{(h/c-1/24),(\bar{h}/c-1/24)\right\}.} \tag{2.1.35}$$

Relevant recent work:

There has been recent surge in analyzing the asymptotics of CFT data on a rigorous footing. The results have been obtained [168, 171] using techniques borrowed from a part of mathematics literature, which goes by the name of Tauberian theorems. The appendix C of [69] emphasizes the importance of Ingham theorem [114] in analyzing Cardy's result [42] for the asymptotic density of states in 2D CFT. Subsequently, the complex Tauberian theorems, as appeared originally in [174] is utilized in the work of [157]. A complete rigorous treatment of Cardy formula appeared in the work [158], where they figured out the density of states in $\Delta \to \infty$ limit with a rigorous optimal estimate of the error term. The improvement of the result along with a proof of the conjecture made in [158] has been put forward in [92]. An rigorous analysis of the asymptotics of three point coefficients [130] appeared recently [165] where the main challenge was to circumnavigate the negativity issue for the analysis of three point coefficients.

2.2 Set up

We consider a 2D CFT with spectrum of operators having conformal weights $(h',\bar{h}')$. We assign different real temperatures $\beta,\bar{\beta}$ to the left-moving and the right-moving sectors respectively. The partition function $Z(\beta,\bar{\beta})$ is given by:

$$Z(\beta,\bar{\beta}) = \sum_{h',\bar{h}'} e^{-\beta(h'-c/24)-\bar{\beta}(\bar{h}'-c/24)}\,. \tag{2.2.1}$$

The modular invariance of the partition function yields:

$$Z(\beta,\bar{\beta}) = Z(\beta',\bar{\beta}'),\ \ \beta' = \frac{4\pi^2}{\beta},\ \ \bar{\beta}' = \frac{4\pi^2}{\bar{\beta}}\,. \tag{2.2.2}$$

We further define the following measure

$$\mathrm{d}F(h',\bar{h}') = \sum d(h_i,\bar{h}_i)\delta(h'-h_i)\delta(\bar{h}'-\bar{h}_i)\,, \tag{2.2.3}$$

where $d(h_i,\bar{h}_i)$ is the degeneracy of the state with conformal weight $(h_i,\bar{h}_i)$. Our goal is to estimate the integral of the measure $\mathrm{d}F$ over different regions.

2.2.1 A semi technical glimpse of the subtleties

One of the key step in deriving the Cardy formula is the intuitive understanding that at high temperature, the partition function is dominated by the heavy states, thus doing an inverse Laplace transform of the high temperature behavior of the partition function should produce the asymptotic density of states. Schematically,

$$Z(\beta \to 0) = \underbrace{\text{Leading Term}}_{\text{Produces asymptotic density of states}} + \text{Error}\,, \tag{2.2.4}$$

$$\text{Inverse Laplace}\,[\text{Error}] \overset{???}{=} \text{Error in asymptotic density of states}\,. \tag{2.2.5}$$

The underlying assumption while doing the above is that the inverse Laplace transform of the error term is bounded as well, thus producing an error compared to the leading behavior of the asymptotic density of states. The Tauberian formalism justifies this step by carefully estimating the error terms. The way it works is following: one bounds the number of states within an order an window centered at some heavy Δ, from above and below by some convolution ($\circledast$) of partition function at high temperature (β) and bandlimited function $\phi_\pm$, schematically this looks like

$$[Z(\beta+it)\circledast\phi_-(t)] \leq \#\text{states in window} \leq [Z(\beta+it)\circledast\phi_+(t)]\,. \tag{2.2.6}$$

Intuitively, at this stage, we know that heavy states contribute to this partition function at high temperature. Now we implement modular transformation, the partition function at high temperature becomes partition function at low temperature. Schematically we have

$$\left[Z\left(\frac{4\pi^2}{\beta+it}\right)\circledast\phi_-(t)\right] \leq \#\text{states in window} \leq \left[Z\left(\frac{4\pi^2}{\beta+it}\right)\circledast\phi_+(t)\right]\,. \tag{2.2.7}$$

At low temperature, low lying states contribute the most i.e $Z\left(\frac{4\pi^2}{\beta}\to\infty\right)$ is dominated by low lying states. So, following [100], we separate the low lying states from heavy states; the low lying states constitute the "light" part, while the "heavy" part is complement of that. The "light" part contains finite number of operators at finite central charge. So we do the inverse Laplace transform of this "Light" part to get the leading answer ρ_*. Thus ρ_* reproduces the $Z\left(\frac{4\pi^2}{\beta}\to\infty\right)$ behavior upon doing Laplace transformation. We are still left with the "heavy" part contribution of $Z\left(\frac{4\pi^2}{\beta+it}\right)$. This part can be shown to produce a subleading correction to the leading piece of asymptotic density of states, thus justifying (2.2.5) using the bound proven in [100]. This requires relating the "heavy" part at temperature β to the "light" part at temperature β'. The upshot of this discussion is that we have a full control of the error term and its inverse Laplace transformation. In practice, we only consider the Identity operator among all the operators in the "light" region. So one might worry about the error coming from that but since there are finite number of operators in the "Light" region, one can do inverse Laplace transformation term by term and show that each of them is exponentially suppressed and hence the finite sum of them. We emphasize the "finiteness" of finite sum is really very important for this and this is precisely why we need to treat heavy part separately[6]. In fact, we remind the readers that in the large central charge limit, we have infinite number of operators even in the "light" sector, hence we

[6]One could have imagined doing inverse Laplace transformation term by term including the operators from "heavy" sector, even if each of them is suppressed, there's no guarantee that the infinite sum is suppressed.

need an extra assumption of *sparseness*, as done in [100].

The immediate generalization of this technique used in the [158] has obstacles because of the cross terms present in the analysis, for example, say contribution from the states where h' is large but $\bar{h}'$ is not that large. The most obvious way to generalize the argument is to use the generalized HKS [100] cut: dividing the $(h',\bar{h}')$ plane into two regions, where the "light" region (call it $\mathbb{L}$) contains all the operators with one of h' or $\bar{h}'$ being less than $c/24$ while the heavy region is the complement of them. It is possible to make a similar statement about this "heavy" region, relating it to the "light" part using HKS like argument. Nonetheless, one then stumbles upon the issue of defining ρ_*, which is supposed to reproduce the leading contribution to the partition function at high temperature ($\beta \to 0, \beta' \to \infty$), to be precise, the light part of the partition function at temperature β'. Now the issue is that there are infinite number of operators in this region $\mathbb{L}$. Unlike the case in [158], we just can not take the Identity operator to prove that this produces the leading behavior and say the rest are suppressed. In particular, the previous argument of term by term exponential suppression fails because there are infinite of them. The take home message is that it is not a priori clear whether just considering the vacuum to calculate ρ_* is good enough, because infinite number of other operators might conspire to spoil the "leading contribution", even if each one of them is exponentially suppressed. We reemphasize that [158] did not face this problem, since in their case, the light region was $\Delta' < c/12$ and the region has finite number of operators and everything is under control, so in principle their ρ_* was defined having contribution from all those states with $\Delta' < c/12$ and in practice derivable from the vacuum. To circumnavigate this problem, in §2.3, we use the original HKS cut i.e. we define the "light" region to be the one where $h' + \bar{h}' < c/12$ and the "heavy" region is the complement of the light region. See fig. 2.6. The immediate cost for doing this is that we can make comment only when h and $\bar{h}$ is of the same order asymptotically. The details depend on how infinity is approached. But this is expected because intuitively the infinity can be reached several ways on a plane. This issue persists for the analysis sensitive to primary as well. It turns out that one can

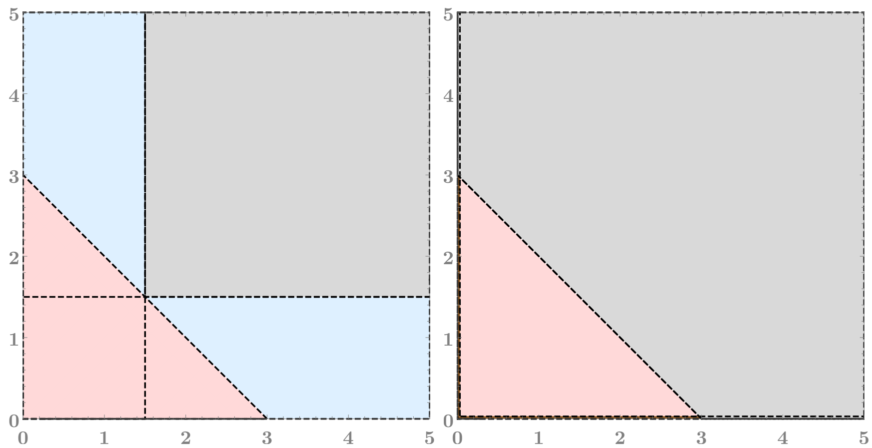


Figure 2.6. The $(h', \bar{h}')$ plane: the light red shaded region contributes to Z_L and the complement of this contributes to Z_H. The naive HKS cut makes the union of red and blue shaded region to be the "Light" region which brings in the problem of having infinite number of operators in the light region.

bypass this restriction of h and $\bar{h}$ being of the same order, if one assumes a twist gap (as defined in $\{2.1.1\}$ we require finite number of conserved currents with dimension less than $c/12$ to be precise). In that scenario, the results in §2.3 holds true even if h and $\bar{h}$ are not of the same order asymptotically. The estimation of the "heavy" region becomes really involved in this case and carried out in details in §2.3. One has to further separate out the "heavy" zero twist operators and estimate their contribution separately (see the right fig. 2.6).

We further point out that estimating the integrated density of states require us to prove another lemma, which is special to 2 D Tauberian theorem. We achieve this in §2.4. The lemma is then fed into the main proof in §2.5. The result for the integrated density of states also requires an estimate of number of states within an order one area on $(h', \bar{h}')$ plane, which is achieved in §2.3. In §2.6, we verify our results using 2D Ising model. The large spin, finite twist is discussed in §2.7. The large central charge regime is discussed in §2.8. We conclude with a list of open of problems along with a brief discussion.

2.3 $O(1)$ rectangular window

In this section, we study the number of states lying on an order one rectangular/square area of $(h',\bar{h}')$ plane, centered at some large $(h,\bar{h})$ and having sides of length 2δ and $2\bar{\delta}$. We are interested in $h\to\infty,\bar{h}\to\infty$ limit with $\delta,\bar{\delta}$ being fixed order one numbers. We do it in two ways, in the first subsection, we do it generically without any assumption on twist gap while in the next subsection, we assume existence of a twist gap. The assumption of the twist gap (as defined in $\{2.1.1\}$) facilitates studying the regime where h is not of the same order as $\bar{h}$.

2.3.1 Generic analysis: with/without Twist gap

Following [158], let us choose functions $\Phi_\pm(h,\bar{h})$ such that

$$\Phi_-(h',\bar{h}') \leq \Theta_{h,\bar{h},\delta,\bar{\delta}}(h',\bar{h}') \leq \Phi_+(h',\bar{h}'), \tag{2.3.1}$$

where $\Theta_{h,\bar{h},\delta,\delta'}(h',\bar{h}')$ is the indicator function of the rectangle and defined as

$$\Theta_{h,\bar{h},\delta,\delta'}(h',\bar{h}') = \theta_{[h-\delta,h+\delta]}(h')\theta_{[\bar{h}-\bar{\delta},\bar{h}+\bar{\delta}]}(\bar{h}'), \tag{2.3.2}$$

In principle, one can choose the energy window for the microcanonical ensemble to be of a different shape, for example, a circle. But for now, we consider it to be a rectangle. Now, from eq. (2.3.1) we obtain

$$e^{\beta(h-\delta)+\bar{\beta}(\bar{h}-\bar{\delta})}e^{-\beta h'-\bar{\beta}\bar{h}'}\Phi_-(h',\bar{h}') \leq \Theta_{h,\bar{h},\delta,\bar{\delta}}(h',\bar{h}') \leq e^{\beta(h+\delta)+\bar{\beta}(\bar{h}+\bar{\delta})}e^{-\beta h'-\bar{\beta}\bar{h}'}\Phi_+(h',\bar{h}'). \tag{2.3.3}$$

Multiplying the above by the density of states $\rho(h',\bar{h}')$ and integrating yields the following inequality:

$$
\begin{aligned}
&e^{\beta(h-\delta)+\bar{\beta}(\bar{h}-\bar{\delta})}\int \mathrm{d}F(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\Phi_-(h',\bar{h}')\\
&\leq \int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}F(h,\bar{h}') \leq \\
&e^{\beta(h+\delta)+\bar{\beta}(\bar{h}+\bar{\delta})}\int \mathrm{d}F(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\Phi_+(h',\bar{h}').
\end{aligned}
\tag{2.3.4}
$$

At this point, we use the Fourier transform $\hat{\Phi}_{\pm}(t,\bar{t})$, defined as

$$
\Phi_{\pm}(h',\bar{h}') \equiv \int_{-\infty}^{\infty}\mathrm{d}t\,\int_{-\infty}^{\infty}\mathrm{d}\bar{t}\;\hat{\Phi}_{\pm}(t,\bar{t})e^{-ih't-i\bar{h}'\bar{t}}.
\tag{2.3.5}
$$

This facilitates us to rewrite the inequality in (2.3.4) as

$$
\begin{aligned}
&\int \mathrm{d}h'\,\mathrm{d}\bar{h}'\mathscr{L}_\rho(\beta+it,\bar{\beta}+i\bar{t})\hat{\Phi}_-(t,\bar{t})\\
&\leq \int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}F(h,\bar{h}') \leq \\
&\int \mathrm{d}h'\,\mathrm{d}\bar{h}'\mathscr{L}_\rho(\beta+it,\bar{\beta}+i\bar{t})\hat{\Phi}_+(t,\bar{t}),
\end{aligned}
\tag{2.3.6}
$$

where we have

$$
\mathscr{L}_\rho(\beta,\bar{\beta}) \equiv \int_0^{\infty}\mathrm{d}h\,\int_0^{\infty}\mathrm{d}\bar{h}\;\rho(h,\bar{h})e^{-\beta h-\bar{\beta}\bar{h}}.
\tag{2.3.7}
$$

Next we need to split $Z = Z_L + Z_H$. In [100], for the mixed temperature analysis, the light sector is chosen to be $\{(h,\bar{h})|h<\frac{c}{24} \text{ or } \bar{h}<\frac{c}{24}\}$ and the heavy sector is the complement of that. Here, we choose a different light sector $\{(h,\bar{h})|h+\bar{h}<\frac{c}{12}\}$ which has finite size (for large central charge, this is not true and one needs to have an extra sparseness condition on the low lying spectra [100, 158]). So at least, in principle, we can choose $\rho_*(h,\bar{h})$ such that ρ_* reproduce the contributions from all operators in the light sector. In practice, we take ρ_* such that it reproduces only the vacuum state contribution, that is,

$$\rho_*(h,\bar{h}) = \left[\pi\sqrt{\frac{c}{6}}\frac{I_1\left(2\pi\sqrt{\frac{c}{6}(h-\frac{c}{24})}\right)}{\sqrt{h-\frac{c}{24}}}\theta\left(h-\frac{c}{24}\right)+\delta\left(h-\frac{c}{24}\right)\right]\times(h\to\bar{h}), \tag{2.3.8}$$

which has the asymptotic behavior

$$\rho_*(h,\bar{h}) \sim \sqrt{\frac{c}{96}}\left(\frac{1}{h^3\bar{h}^3}\right)^{1/4} e^{2\pi\sqrt{\frac{ch}{6}}+2\pi\sqrt{\frac{c\bar{h}}{6}}}\left[1+O\left(h^{-1/2}\right)+O\left(\bar{h}^{-1/2}\right)\right]. \tag{2.3.9}$$

Then we can write $e^{-(\beta+it+\bar{\beta}+i\bar{t})c/24}Z_L\left(\frac{4\pi^2}{\beta+it},\frac{4\pi^2}{\bar{\beta}+i\bar{t}}\right)=\mathscr{L}_{\rho_*,L}(\beta+it,\bar{\beta}+i\bar{t})$. So we get

$$\begin{aligned}
&e^{\beta(h-\delta)+\bar{\beta}(\bar{h}-\bar{\delta})}\Bigg(\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{d}\bar{t}\ \hat{\Phi}_-(t,\bar{t})\mathscr{L}_{\rho_*,L}(\beta+it,\bar{\beta}+i\bar{t})\\
&\qquad\qquad -\left|\int_{-\infty}^{\infty}\mathrm{d}t\mathrm{d}\bar{t}\ e^{-(\beta+it+\bar{\beta}+i\bar{t})c/24}\hat{\Phi}_-(t,\bar{t})Z_H\left(\frac{4\pi^2}{\beta+it},\frac{4\pi^2}{\bar{\beta}+i\bar{t}}\right)\right|\Bigg)\\
&\leq \int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}}\mathrm{d}F(h,\bar{h}')\leq \\
&e^{\beta(h+\delta)+\bar{\beta}(\bar{h}+\bar{\delta})}\Bigg(\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{d}\bar{t}\ \hat{\Phi}_+(t,\bar{t})\mathscr{L}_{\rho_*,L}(\beta+it,\bar{\beta}+i\bar{t})\\
&\qquad\qquad +\left|\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{d}\bar{t}\ e^{-(\beta+it+\bar{\beta}+i\bar{t})c/24}\hat{\Phi}_+(t,\bar{t})Z_H\left(\frac{4\pi^2}{\beta+it},\frac{4\pi^2}{\bar{\beta}+i\bar{t}}\right)\right|\Bigg),
\end{aligned} \tag{2.3.10}$$

Now we can estimate the contribution from the heavy sector using the HKS bound. We choose $\Phi_\pm$ such that $\hat{\Phi}_\pm(t,\bar{t})$ have finite support $[-\Lambda_\pm,\Lambda_\pm]$ for t and $\bar{t}$. A possible choice can be made via modifying the functions appearing in [158, 92] a little bit. To be concrete, let us make the following choices:

$$\Phi_+(h',\bar{h}') = \frac{f_+(h-h')f_+(\bar{h}-\bar{h}')}{f_+(\delta)f_+(\bar{\delta})}, \tag{2.3.11}$$

$$\Phi_-(h',\bar{h}') = f_-(h-h')f_-(\bar{h}-\bar{h}')\left(1-\left(\frac{h-h'}{\delta}\right)^2-\left(\frac{\bar{h}-\bar{h}'}{\bar{\delta}}\right)^2\right), \tag{2.3.12}$$

where we have

$$f_{\pm}(x) = \left[\operatorname{sinc}\left(\frac{\Lambda_{\pm}x}{4}\right)\right]^4. \tag{2.3.13}$$

We remark that for Φ_-, the locus $1-(a/\delta)^2-(b/\bar{\delta})^2=0$ is inside the rectangle region, hence it is a valid choice conforming to the inequality (2.3.1). Here $a=h-h'$ and $b=\bar{h}-\bar{h}'$. At this point, our aim is to show that

$$I_{\pm} = e^{\beta(h\pm\delta)+\bar{\beta}(\bar{h}\pm\bar{\delta})}\left|\int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{d}\bar{t}\ e^{-(\beta+it+\bar{\beta}+i\bar{t})c/24}\hat{\Phi}_{\pm}(t,\bar{t})Z_H\left(\frac{4\pi^2}{\beta+it},\frac{4\pi^2}{\bar{\beta}+i\bar{t}}\right)\right| \tag{2.3.14}$$

is sub-leading. We will make use of the most basic HKS bound [100] for Δ in a clever way. We remark that by requiring the saddle of the light sector is located at $(h,\bar{h})$, we find $\beta=\pi\sqrt{\frac{c}{6h}}<<1, \bar{\beta}=\pi\sqrt{\frac{c}{6\bar{h}}}<<1$, so some terms such as $\beta\delta$ can be dropped from the bounds as it goes to 0 for large $h,\bar{h}$. Then using the fact that $\Phi_{\pm}$ is a bandlimited function, we have

$$I_{\pm} \leq e^{\beta h+\bar{\beta}\bar{h}}\int \mathrm{d}t\ \mathrm{d}\bar{t}\ Z_H\left(\frac{4\pi^2\beta}{\beta^2+t^2},\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\bar{t}^2}\right)\left|\hat{\Phi}_{\pm}(t,\bar{t})\right|. \tag{2.3.15}$$

Now Z_H has contribution from states where either h' or $\bar{h}'$ is greater than $c/24$. Since the contributing states have $h'+\bar{h}'>c/12$, both can not be less than or equal to $\frac{c}{24}$. We illustrate the case for $h'>c/24$.

$$\begin{aligned}
Z_H &\ni \exp\left[-\frac{4\pi^2\beta}{\beta^2+t^2}\left(h'-\frac{c}{24}\right)-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\bar{t}^2}\left(\bar{h}'-\frac{c}{24}\right)\right] \\
&\leq e^{\frac{\pi^2 c}{6\bar{\beta}}}\exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda_{\pm}^2}\left(h'-\frac{c}{24}\right)-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_{\pm}^2}\bar{h}'\right] \\
&\leq e^{\frac{\pi^2 c}{6\bar{\beta}}}e^{-\frac{\pi^2\beta_* c}{6(\beta_*^2+\Lambda_{\pm}^2)}}\exp\left[-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_{\pm}^2}\left(h'+\bar{h}'-\frac{c}{12}\right)\right],
\end{aligned} \tag{2.3.16}$$

where β_* is defined as $\beta_* = \min\left(\beta, \bar{\beta}\right)$, hence we have (for small enough β and $\bar{\beta}$)

$$\min\left(\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}, \frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\right) = \frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\,, \tag{2.3.17}$$

Thus we have

$$I_\pm \le e^{\beta h+\bar{\beta}\bar{h}}\left(e^{\frac{\pi^2 c}{6\beta}} + e^{\frac{\pi^2 c}{6\bar{\beta}}}\right) e^{-\frac{\pi^2\beta_* c}{6(\beta_*^2+\Lambda_\pm^2)}} Z_{H,\Delta}\left(\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\right)\,, \tag{2.3.18}$$

where the last $Z_{H,\Delta} = \sum_{\Delta > c/12} e^{-\beta(\Delta - c/12)}$ is the heavy contribution from the original HKS bound for Δ. We will be showing that the above term is subleading. There are two pieces, one with $e^{\frac{\pi^2 c}{6\beta}}$ and another with $e^{\frac{\pi^2 c}{6\bar{\beta}}}$. Let us illustrate the subleading nature of the term with $e^{\frac{\pi^2 c}{6\bar{\beta}}}$. The other term can be treated similarly.

In $\beta, \bar{\beta} \to 0$ limit, we have

$$\begin{aligned}
e^{\beta h+\bar{\beta}\bar{h}} e^{\frac{\pi^2 c}{6\bar{\beta}}} e^{-\frac{\pi^2\beta_* c}{6(\beta_*^2+\Lambda_\pm^2)}} Z_{H,\Delta}\left(\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\right) &\le e^{\beta h+\bar{\beta}\bar{h}} e^{\frac{\pi^2 c}{6\bar{\beta}}} Z_{H,\Delta}\left(\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\right) \\
&\sim e^{\pi\sqrt{\frac{ch}{6}}+2\pi\sqrt{\frac{c\bar{h}}{6}}+2\pi\sqrt{\frac{ch^*}{6}}\left(\frac{\Lambda_\pm}{2\pi}\right)^2} \\
&\le \begin{cases} h \ge \bar{h}: \; e^{2\pi\sqrt{\frac{ch}{6}}+2\pi\sqrt{\frac{c\bar{h}}{6}}+2\pi\sqrt{\frac{ch}{6}}\left(\left(\frac{\Lambda_\pm}{2\pi}\right)^2-\frac{1}{2}\right)}. \\ h < \bar{h}: \; e^{\pi\sqrt{\frac{ch}{6}}+2\pi\sqrt{\frac{c\bar{h}}{6}}+2\pi\sqrt{\frac{c\bar{h}}{6}}\left(\frac{\Lambda_\pm}{2\pi}\right)^2}. \end{cases}
\end{aligned} \tag{2.3.19}$$

where $h^* = \max(h, \bar{h})$ and $\tau = 2\,\min(h, \bar{h})$. To make the contribution from the heavy sector sub-leading, we need

$$\Lambda_\pm < \min\left(\frac{2\pi}{\sqrt{2}}, \frac{2\pi}{\sqrt{2}\gamma}\right)\,, \tag{2.3.20}$$

where $\frac{\tau}{2}\gamma^4 \simeq h^*$ (clearly, $\gamma > 1$). This can be achieved by choosing

$$\Lambda_{\pm} < \frac{\sqrt{2}\pi}{\gamma}\,. \tag{2.3.21}$$

Then at large $h, \bar{h}$, we have the following bound, starting from (2.3.10) (the second term i.e. the term with the absolute value in (2.3.10) has already been shown to be subleading and we rewrite the first term as an integral over $h', \bar{h}'$ below):

$$\begin{aligned}
&e^{\beta(h-\delta)+\bar{\beta}(\bar{h}-\bar{\delta})}\int \mathrm{d}h'\ \mathrm{d}\bar{h}'\rho_*(h',\bar{h}')\Phi_-(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'} \\
&\leq \int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}F(h,\bar{h}') \leq \\
&e^{\beta(h+\delta)+\bar{\beta}(\bar{h}+\bar{\delta})}\int \mathrm{d}h'\ \mathrm{d}\bar{h}'\rho_*(h',\bar{h}')\Phi_+(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\,.
\end{aligned} \tag{2.3.22}$$

We can evaluate this integral by saddle point approximation,

$$c_-\rho_*(h,\bar{h}) \leq \frac{1}{4\delta\bar{\delta}}\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}F(h,\bar{h}') \leq c_+\rho_*(h,\bar{h})\,, \tag{2.3.23}$$

where $c_\pm$ is defined as

$$c_\pm = \frac{1}{4}\int\ \mathrm{d}x\ \mathrm{d}y\ \Phi_\pm(h+\delta x, \bar{h}+\bar{\delta}y)\,. \tag{2.3.24}$$

With the previous choice of $\Phi_\pm$, we have

$$\begin{aligned}
c_+ &= \frac{16\pi^2}{9}\frac{1}{\delta\bar{\delta}\Lambda_+^2\,\mathrm{sinc}^4(\delta\Lambda_+/4)\mathrm{sinc}^4(\bar{\delta}\Lambda_+/4)}, \\
c_- &= \frac{16\pi^2}{9}\frac{\delta^2\bar{\delta}^2\Lambda_-^2 - 12\delta^2 - 12\bar{\delta}^2}{\delta^3\bar{\delta}^3\Lambda_-^4}.
\end{aligned} \tag{2.3.25}$$

where we must optimize over $0 < \Lambda_\pm < \frac{\sqrt{2}\pi}{\gamma}$ to get the tightest bound while keeping $\delta, \bar{\delta}$ arbitrary.

The condition for the lower bound to be positive is

$$\frac{1}{\delta^2}+\frac{1}{\bar{\delta}^2}<\frac{\Lambda_-^2}{12}\,. \tag{2.3.26}$$

The allowed region increases as we increase Λ_-. And the minimum area such that there has to be at least one operator is given by $4\delta\bar{\delta}=\frac{48\gamma^2}{\pi^2}=4.86\gamma^2$ at $\delta=\bar{\delta}=\frac{2\sqrt{3}\gamma}{\pi}$. This analysis can be made sensitive to primaries only, thus gives an asymptotic gap between primaries. We suspect that the above does not give the tightest bound for spectral gap (this intuition is coming from the similar analysis done for the spectral gap in Δ, appearing in [158, 92])!

Next we can keep $\delta,\bar{\delta}$ arbitrary and optimize over $0<\Lambda_\pm<\frac{\sqrt{2}\pi}{\gamma}$ to get the tightest bound.

(a) For lower bound:

$$\begin{aligned}
&\frac{1}{4\delta\bar{\delta}}\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}}\mathrm{d}F(h,\bar{h}')\geq\frac{\pi^2}{27}\frac{\delta\bar{\delta}}{\delta^2+\bar{\delta}^2}\rho_*(h,\bar{h}),\;\;\frac{\sqrt{\delta^2+\bar{\delta}^2}}{\delta\bar{\delta}}<\frac{\pi}{2\sqrt{3}\gamma},\\
&\frac{1}{4\delta\bar{\delta}}\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}}\mathrm{d}F(h,\bar{h}')\geq\frac{8\pi^2\gamma^2\delta^2\bar{\delta}^2-48\gamma^4(\delta^2+\bar{\delta}^2)}{9\pi^2\delta^3\bar{\delta}^3}\rho_*(h,\bar{h}),\;\;\frac{\sqrt{\delta^2+\bar{\delta}^2}}{\delta\bar{\delta}}\geq\frac{\pi}{2\sqrt{3}\gamma}.
\end{aligned} \tag{2.3.27}$$

(b) For upper bound:

$$\begin{aligned}
&\frac{1}{4\delta\bar{\delta}}\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}}\mathrm{d}F(h,\bar{h}')\leq\frac{16\pi^2}{9\delta\bar{\delta}(\Lambda_+^*)^2}\frac{\rho_*(h,\bar{h})}{\mathrm{sinc}\big(\frac{\delta\Lambda_+^*}{4}\big)^4\mathrm{sinc}\big(\frac{\bar{\delta}\Lambda_+^*}{4}\big)^4},\;\;\Lambda_+^*\leq\frac{\sqrt{2}\pi}{\gamma},\\
&\frac{1}{4\delta\bar{\delta}}\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}}\mathrm{d}F(h,\bar{h}')\leq\frac{8\gamma^2}{9\delta\bar{\delta}}\frac{\rho_*(h,\bar{h})}{\mathrm{sinc}\big(\frac{\delta\sqrt{2}\pi}{4\gamma}\big)^4\mathrm{sinc}\big(\frac{\bar{\delta}\sqrt{2}\pi}{4\gamma}\big)^4},\;\;\Lambda_+^*>\frac{\sqrt{2}\pi}{\gamma}.
\end{aligned} \tag{2.3.28}$$

where $\Lambda_+^*(\delta,\bar{\delta})$ is the non-zero least positive solution of

$$\delta\Lambda_+\cot\left(\frac{\delta\Lambda_+}{4}\right)+\bar{\delta}\Lambda_+\cot\left(\frac{\bar{\delta}\Lambda_+}{4}\right)=6. \tag{2.3.29}$$

2.3.2 Analysis of $O(1)$ window assuming a twist gap

We have seen that the result proven in the previous subsection holds true when h and $\bar{h}$ are of the same order asymptotically. When we make the analysis sensitive to primary, this feature persists. Nonetheless, we can circumnavigate this issue by assuming an existence of twist gap (as defined in $\{2.1.1\}$). One can also do this for the analysis sensitive to all the operators. The only catch is that one has to separately treat the zero twist operators with dimension greater than $c/12$. We revisit the analysis of suppression of heavy region in the light of the above discussion. At first, we take up the analysis for all the operators.

We go back to the eq. (2.3.16) and redo the first part of the analysis. We separate out Z_H into two pieces, one with zero twist heavy operators, we name it $Z_H^{(0)}$, while the other one contains all the heavy operators with non-zero twist, we name it $Z_H^{(\tau)}$. We start with analyzing $Z_H^{(\tau)}$, assuming a twist gap. We have two scenarios.

Scenario I: If $g \leq \frac{c}{12}$, we have

$$
\begin{aligned}
Z_H^{(\tau)} \ni & \exp\left[-\frac{4\pi^2\beta}{\beta^2+t^2}\left(h'-\frac{c}{24}\right)-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\bar{t}^2}\left(\bar{h}'-\frac{c}{24}\right)\right] \\
& \leq e^{\frac{\pi^2 c\left(1-\frac{12g}{c}\right)}{6\bar{\beta}}+\frac{\pi^2 c\left(1-\frac{12g}{c}\right)}{6\beta}} \exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\left(h'-\frac{g}{2}\right)-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\left(\bar{h}'-\frac{g}{2}\right)\right] \\
& \leq e^{\frac{\pi^2 c\left(1-\frac{12g}{c}\right)}{6\bar{\beta}}+\frac{\pi^2 c\left(1-\frac{12g}{c}\right)}{6\beta}} \exp\left[-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\left(h'+\bar{h}'-g\right)\right] \\
& \leq e^{\frac{\pi^2 c\left(1-\frac{12g}{c}\right)\left(\frac{1}{\beta}+\frac{1}{\bar{\beta}}\right)}{6}} \exp\left[-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\left(h'+\bar{h}'-\frac{c}{12}\right)\right].
\end{aligned}
\tag{2.3.30}
$$

Here going from the first inequality to the second one, we used $2h' > g, 2\bar{h}' > g$; going from the second one to the third (last) one, we used $g \leq c/12$. The rest of the analysis goes in a similar manner, as done subsequently after (2.3.16) and we deduce:

$$\Lambda_{\pm} < \frac{\sqrt{2}\pi}{\sigma}\,, \quad \sigma^2 \equiv \left(\frac{c}{12g}\right)\,. \tag{2.3.31}$$

Scenario II: If $g \geq \frac{c}{12}$, we have $h' > c/24$ and $\bar{h}' > c/24$, leading to

$$\begin{aligned} Z_H^{(\tau)} \ni \exp\left[-\frac{4\pi^2\beta}{\beta^2+t^2}\left(h'-\frac{c}{24}\right)-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\bar{t}^2}\left(\bar{h}'-\frac{c}{24}\right)\right] \\ \leq \exp\left[-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_{\pm}^2}\left(h'+\bar{h}'-\frac{c}{12}\right)\right]\,. \end{aligned} \tag{2.3.32}$$

The rest of the analysis goes in a similar manner and we deduce:

$$\Lambda_{\pm} < \sqrt{2}\pi\,. \tag{2.3.33}$$

Now we come back to analyzing the zero twist heavy sector. For this sector, $h^* = \max(h,\bar{h}) =$

$\Delta > c/12 > c/24$, thus we have

$$
\begin{aligned}
&Z_H^{(0)} \\
&= \sum_{h'} \exp\left[-\frac{4\pi^2\beta}{\beta^2+t^2}\left(h'-\frac{c}{24}\right)+\frac{\pi^2\bar{\beta}c}{6(\bar{\beta}^2+\bar{t}^2)}\right]+\sum_{\bar{h}'}\exp\left[\frac{\pi^2\beta c}{6(\beta^2+t^2)}-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\bar{t}^2}\left(\bar{h}'-\frac{c}{24}\right)\right] \\
&\leq \sum_{h'} \exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda^2}\left(\Delta_{h',0}-\frac{c}{24}\right)+\frac{\pi^2 c}{6\bar{\beta}}\right]+\sum_{\bar{h}'}\exp\left[\frac{\pi^2 c}{6\beta}-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda^2}\left(\Delta_{0,\bar{h}'}-\frac{c}{24}\right)\right] \\
&\leq e^{\frac{\pi^2 c}{6\bar{\beta}}}\sum_{h'} \exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda^2}\left(\Delta_{h',0}-\frac{c}{24}\right)\right]+e^{\frac{\pi^2 c}{6\beta}}\sum_{\bar{h}'}\exp\left[-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\left(\Delta_{0,\bar{h}'}-\frac{c}{24}\right)\right] \\
&\leq e^{\frac{\pi^2 c}{6\bar{\beta}}}\sum_{h'} \exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\left(\Delta_{h',0}-\frac{c}{12}\right)\right]+e^{\frac{\pi^2 c}{6\beta}}\sum_{\bar{h}'}\exp\left[-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\left(\Delta_{0,\bar{h}'}-\frac{c}{12}\right)\right] \\
&\leq e^{\frac{\pi^2 c}{6\bar{\beta}}}\sum_{h'} \exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\left(\Delta_{h',0}-\frac{c}{12}\right)\right]+e^{\frac{\pi^2 c}{6\beta}}\sum_{\bar{h}'}\exp\left[-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\left(\Delta_{0,\bar{h}'}-\frac{c}{12}\right)\right] \\
&\leq e^{\frac{\pi^2 c}{6\bar{\beta}}}\sum_{\Delta} \exp\left[-\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\left(\Delta-\frac{c}{12}\right)\right]+\sum_{\Delta}e^{\frac{\pi^2 c}{6\beta}}\exp\left[-\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\left(\Delta-\frac{c}{12}\right)\right] \\
&\leq e^{\pi\sqrt{\frac{c\bar{h}}{6}}}Z_{H,\Delta}\left(\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\right)+e^{\pi\sqrt{\frac{ch}{6}}}Z_{H,\Delta}\left(\frac{4\pi^2\bar{\beta}}{\bar{\beta}^2+\Lambda_\pm^2}\right).
\end{aligned}
\tag{2.3.34}
$$

The subscript on Δ in the second line denotes the actual conformal weights of the operator and in the penultimate line, we have extended the sum to all the heavy operators. Now one can see the zero twist heavy sector is suppressed as long as we choose $\Lambda_\pm < \sqrt{2}\pi$. Thus, combining everything, we have

$$
\Lambda_\pm < \begin{cases} \min\left(\sqrt{2}\pi, \frac{\sqrt{2}\pi}{\zeta}\right) = \frac{\sqrt{2}\pi}{\zeta}, & g \leq c/12 \\ \sqrt{2}\pi, & g \geq c/12 \end{cases}
\tag{2.3.35}
$$

where $\zeta^2 = \frac{c}{12g}$. We can combine the above to write for all g,

$$
\Lambda_\pm < \min\left(\sqrt{2}\pi, \frac{\sqrt{2}\pi}{\zeta}\right).
\tag{2.3.36}
$$

The above has immediate implication in terms of asymptotic gap for all the operators, in particular, the gap does not depend on the term γ anymore. Nonetheless, as we already know existence of descendants asymptotically, the asymptotic gap for all the operators is not so illuminating, so we will not illustrate upon this. Rather we come back to this when we make our analysis sensitive to primaries and in that scenario, the result about asymptotic gap is indeed illuminating.

Analysis for primaries: twist gap complementary to asymptotic spectral gap:

The analysis for primaries proceeds in a similar manner. We will be estimating the following object

$$\exp\left[S^{\text{Vir}}_{\delta,\bar{\delta}}\right] = \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \int_{h-\delta}^{h+\delta} \mathrm{d}h'\, \mathrm{d}\bar{h}' \rho^{\text{Vir}}(h',\bar{h}')\,, \tag{2.3.37}$$

where $\rho^{\text{Vir}}(h',\bar{h}')$ is the density of primaries. Instead of the partition function we consider the following object (for $c > 1$, the expansion of this object is universal)

$$\begin{aligned}
&Z_{\text{primary}}(\beta,\bar{\beta}) \\
&\equiv \eta(\beta)\eta(\bar{\beta})Z(\beta) = e^{\beta\frac{c-1}{24}+\bar{\beta}\frac{c-1}{24}}\left[\left(1-e^{-\beta}\right)\left(1-e^{-\bar{\beta}}\right) + \sum_{h'\neq 0,\bar{h}'\neq 0} d_{h',\bar{h}'} e^{-\beta h'-\bar{\beta}\bar{h}'}\right].
\end{aligned} \tag{2.3.38}$$

Under modular transformation, we have

$$Z_{\text{primary}}(\beta,\bar{\beta}) = \sqrt{\frac{2\pi}{\beta}}\sqrt{\frac{2\pi}{\bar{\beta}}} Z_{\text{primary}}\left(\frac{4\pi^2}{\beta},\frac{4\pi^2}{\bar{\beta}}\right). \tag{2.3.39}$$

Then we define the crossing $\rho^{\text{Vir}}_{*}(h',\bar{h}') = \rho^{\text{Vir}}_{*}(h')\rho^{\text{Vir}}_{*}(\bar{h}')$ to reproduce the high temperature behavior of $Z_{\text{primary}}(\beta,\bar{\beta})$ i.e we have

$$\begin{aligned}
&\int_0^\infty \mathrm{d}h'\, e^{-\beta\left(h'-\frac{c-1}{24}\right)}\rho^{\text{Vir}}_{*}(h') = \sqrt{\frac{2\pi}{\beta}}\left(\exp\left[\frac{\pi^2(c-1)}{6\beta}\right] - \exp\left[\frac{\pi^2(c-25)}{6\beta}\right]\right), \\
&\int_0^\infty \mathrm{d}\bar{h}'\, e^{-\bar{\beta}\left(\bar{h}'-\frac{c-1}{24}\right)}\rho^{\text{Vir}}_{*}(\bar{h}') = \sqrt{\frac{2\pi}{\bar{\beta}}}\left(\exp\left[\frac{\pi^2(c-1)}{6\beta}\right] - \exp\left[\frac{\pi^2(c-25)}{6\bar{\beta}}\right]\right).
\end{aligned} \tag{2.3.40}$$

Explicitly, ρ_*^{Vir} would be given by the following function:

$$\rho_*^{\text{Vir}}(h') = \begin{cases} 0 \text{ if } h' < \frac{c-1}{24} \\ \frac{\sqrt{2}}{\sqrt{h-\frac{c-1}{24}}} \left[\cosh\left(4\pi\sqrt{\frac{(c-1)}{24}\left(h-\frac{c-1}{24}\right)}\right) - \cosh\left(4\pi\sqrt{\frac{(c-25)}{24}\left(h-\frac{c-1}{24}\right)}\right)\right] . \end{cases} \tag{2.3.41}$$

The analysis pertaining to the estimation of the heavy part presented before for the analysis of all the operators can be used as a stepping stone for a similar analysis for primaries for $c > 1$ CFTs. We again use bandlimited functions and we deduce that the support $\Lambda_\pm$ has to satisfy[7]:

$$\Lambda_\pm < \min\left(\frac{\sqrt{2}\pi}{\zeta_p}, \sqrt{2}\pi\right), \quad \zeta_p^2 \equiv \left(\frac{c-1}{12g}\right). \tag{2.3.42}$$

The leading answer comes out to be

$$\begin{aligned} &\frac{1}{2}\frac{c_-}{\sqrt{h-\frac{c-1}{24}}\sqrt{\bar{h}-\frac{c-1}{24}}} \exp\left[2\pi\left(\sqrt{\frac{(c-1)h}{6}}+\sqrt{\frac{(c-1)\bar{h}}{6}}\right)\right] \\ &\leq \frac{1}{4\delta\bar{\delta}} \exp\left[S^{\text{Vir}}_{\delta,\bar{\delta}}\right] \leq \\ &\frac{1}{2}\frac{c_+}{\sqrt{h-\frac{c-1}{24}}\sqrt{\bar{h}-\frac{c-1}{24}}} \exp\left[2\pi\left(\sqrt{\frac{(c-1)h}{6}}+\sqrt{\frac{(c-1)\bar{h}}{6}}\right)\right], \end{aligned} \tag{2.3.43}$$

where $c_\pm$ is defines as in the Eq. (2.3.24).

Asymptotic gap:

Now we come back to our discussion of asymptotic gap of primaries. We use the function given in Eq. (2.3.12) but now with constraint as given in Eq. (2.3.42). Thus the asymptotic binding square will have length $\frac{4\sqrt{3}\sigma}{\pi}$ and the binding circle would have radius $\frac{r\sigma}{\sqrt{2}} + \varepsilon_g$ with

[7] Should we not assume twist gap, we would have $\Lambda_\pm < \frac{\sqrt{2}\pi}{\gamma}$, just like the analysis for all the operators without assuming twist gap.

$\varepsilon_g > 0$, and σ, r are given by

$$\sigma = \max\left(1, \frac{c-1}{12g}\right), \quad r = \frac{4\sqrt{3}}{\pi}\,. \tag{2.3.44}$$

If we consider tensoring the chiral Monster CFT with its antichiral avatar, we find $g = 4$ and our result predicts that the asymptotic spectral gap involves a circle of radius $\frac{2\sqrt{6}}{\pi}$ irrespective of how infinity is approached. This is above the suspected optimal value 1 (see fig. 2.4). In a unitary compact CFT without conserved currents, there is a bound on twist gap[56]:

$$g \leq \frac{c-1}{12}\,. \tag{2.3.45}$$

In that scenario, we have

$$\Lambda_{\pm} < \min\left(\frac{\sqrt{2}\pi}{\zeta_p}, \sqrt{2}\pi\right) = \frac{\sqrt{2}\pi}{\zeta_p}, \quad \zeta_p^2 \equiv \left(\frac{c-1}{12g}\right)\,. \tag{2.3.46}$$

As a result, we deduce the universal inequality satisfied by the "areal" spectral gap A and twist gap g:

$$Ag \leq \frac{\pi(c-1)r^2}{12}\,, \tag{2.3.47}$$

where we have shown $r = \frac{4\sqrt{3}}{\pi} \simeq 2.21 > 1$ and we suspect that it can be made to 1.

2.4 Lemma: density of states on strip of order one width

In this section, we prove a lemma which is going to play a pivotal role in the next section, where we are going to prove an asymptotic result for the integrated density of states i.e number of states upto a large $(h, \bar{h})$ threshold. This also helps us to derive the asymptotic spectral gap via strip like regions as defined in $\{2.1.4\}$. We start by defining the following functions

$$\begin{aligned} Q(h,\bar{\beta}) &\equiv \int_{h-\delta}^{h+\delta} \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\, \rho(h',\bar{h}') e^{-\bar{\beta}\bar{h}'}\,, \\ P(\bar{h},\beta) &\equiv \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}' \int_0^\infty \mathrm{d}h'\, \rho(h',\bar{h}') e^{-\beta h'}\,. \end{aligned} \tag{2.4.1}$$

The aim of this section is to prove the following lemma:

$$e^{\bar{\beta}\bar{h}} Q(h,\bar{\beta}) \underset{\bar{\beta}=\pi\sqrt{\frac{c}{6\bar{h}}}}{=} O\left(h^{-3/4}\exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\right)\,, \tag{2.4.2}$$

$$e^{\beta h} P(\bar{h},\beta) \underset{\beta=\pi\sqrt{\frac{c}{6h}}}{=} O\left(\bar{h}^{-3/4}\exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\right)\,. \tag{2.4.3}$$

Let us focus on the quantity Q, the argument for P follows in a similar manner. In order to estimate Q, we write down the master inequality:

$$\begin{aligned} & e^{\beta(h-\delta)} \int_0^\infty dh' \int_0^\infty \mathrm{d}\bar{h}'\, \rho(h',\bar{h}')\phi_-(h') e^{-\beta h'-\bar{\beta}\bar{h}'} \\ & \leq Q(h,\bar{\beta}) \\ & \leq e^{\beta(h+\delta)} \int_0^\infty dh' \int_0^\infty \mathrm{d}\bar{h}'\, \rho(h',\bar{h}')\phi_+(h') e^{-\beta h'-\bar{\beta}\bar{h}'}\,, \end{aligned} \tag{2.4.4}$$

where we have used

$$\phi_-(h') \leq \Theta\left(h' \in [h-\delta, h+\delta]\right) \leq \phi_+(h')\,. \tag{2.4.5}$$

Next we note that

$$\begin{aligned} & \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\, \rho(h',\bar{h}')\phi_\pm(h') e^{-\beta h'-\bar{\beta}\bar{h}'} \\ & = e^{-\beta c/24-\bar{\beta}c/24} \int_{-\infty}^\infty \mathrm{d}t\; e^{-\imath t c/24} Z(\beta+\imath t,\bar{\beta})\hat{\phi}_\pm(t) \\ & = e^{-\beta c/24-\bar{\beta}c/24} \int_{-\infty}^\infty \mathrm{d}t\; e^{-\imath t c/24} Z\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar{\beta}}\right)\hat{\phi}_\pm(t)\,. \end{aligned} \tag{2.4.6}$$

Now we separate the contribution to $Z\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)$ into two pieces Z_L (contribution from the "light" sector) and Z_H (contribution from the "heavy" sector). The inequality in (2.4.4) can be written as

$$
\begin{aligned}
& e^{\beta(h-c/24-\delta)-\bar\beta c/24}\int_{-\infty}^{\infty}\mathrm{d}t\; e^{-\imath t c/24} Z_L\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)\hat\phi_-(t) \\
& - e^{\beta(h-c/24-\delta)-\bar\beta c/24}\left|\int_{-\infty}^{\infty}\mathrm{d}t\; e^{-\imath t c/24} Z_H\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)\hat\phi_-(t)\right| \\
& \leq Q(h,\bar\beta) \qquad (2.4.7)\\
& \leq e^{\beta(h-c/24+\delta)-\bar\beta c/24}\int_{-\infty}^{\infty}\mathrm{d}t\; e^{-\imath t c/24} Z_L\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)\hat\phi_+(t) \\
& + e^{\beta(h-c/24+\delta)-\bar\beta c/24}\left|\int_{-\infty}^{\infty}\mathrm{d}t\; e^{-\imath t c/24} Z_H\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)\hat\phi_+(t)\right|
\end{aligned}
$$

For notational simplicity, let us name the terms

$$
I^1_\pm = e^{\beta(h-c/24\pm\delta)-\bar\beta c/24}\int_{-\infty}^{\infty}\mathrm{d}t\; e^{-\imath t c/24} Z_L\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)\hat\phi_\pm(t)\,, \qquad (2.4.8)
$$

$$
I^2_\pm = e^{\beta(h-c/24\pm\delta)-\bar\beta c/24}\left|\int_{-\infty}^{\infty}\mathrm{d}t\; e^{-\imath t c/24} Z_H\left(\frac{4\pi^2}{\beta+\imath t},\frac{4\pi^2}{\bar\beta}\right)\hat\phi_\pm(t)\right|. \qquad (2.4.9)
$$

The idea is to show that $I^2_\pm$ is subleading with respect to $I^1_\pm$. The argument closely follows the argument presented in §2.3. Let us concentrate on $I^2_\pm$ first. We have

$$
I^2_\pm \leq e^{\beta(h-c/24\pm\delta)-\bar\beta c/24}\int_{-\infty}^{\infty}\mathrm{d}t\; Z_H\left(\frac{4\pi^2\beta}{\beta^2+t^2},\frac{4\pi^2}{\bar\beta}\right)\left|\hat\phi_\pm(t)\right|. \qquad (2.4.10)
$$

We notice that (for the heavy sector, $h'+\bar h' > c/12$, thus one of them has to be greater than $c/24$)

$$Z_H\left(\frac{4\pi^2\beta}{\beta^2+t^2},\frac{4\pi^2}{\bar\beta}\right)\ni e^{-\frac{4\pi^2\beta}{\beta^2+t^2}\left(h'-\frac{c}{24}\right)-\frac{4\pi^2}{\bar\beta}\left(\bar h'-\frac{c}{24}\right)}$$

$$\le\begin{cases} e^{\frac{\pi^2 c}{6\beta}}e^{-\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}h'-\frac{4\pi^2\bar\beta}{\bar\beta^2+\Lambda_\pm^2}\left(\bar h'-\frac{c}{24}\right)}, & \bar h'>c/24\\ e^{\frac{\pi^2 c}{6\bar\beta}}e^{-\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\left(h'-\frac{c}{24}\right)-\frac{4\pi^2\bar\beta}{\bar\beta^2+\Lambda_\pm^2}\bar h'}, & h'>c/24\end{cases}$$

$$\le\begin{cases} e^{\frac{\pi^2 c}{6\beta}}e^{-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}h'-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\left(\bar h'-\frac{c}{24}\right)}, & \bar h'>c/24\\ e^{\frac{\pi^2 c}{6\bar\beta}}e^{-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}h'-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\left(\bar h'-\frac{c}{24}\right)}, & h'>c/24\end{cases} \tag{2.4.11}$$

$$\le\begin{cases} e^{\frac{\pi^2 c}{6\beta}}e^{\frac{-\pi^2 c\beta_*}{6(\beta_*^2+\Lambda_\pm^2)}}e^{-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\left(h'+\bar h'-\frac{c}{12}\right)}, & \bar h'>c/24\\ e^{\frac{\pi^2 c}{6\bar\beta}}e^{-\frac{\pi^2 c\beta_*}{6(\beta_*^2+\Lambda_\pm^2)}}e^{-\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\left(h'+\bar h'-\frac{c}{12}\right)}. & h'>c/24\end{cases}$$

Thus we have

$$\begin{aligned} I_\pm^2 &\le e^{\beta(h+c/24\pm\delta)-\bar\beta c/24}\int_{-\infty}^{\infty}\mathrm{d}t\ Z_H\left(\frac{4\pi^2\beta}{\beta^2+t^2},\frac{4\pi^2}{\bar\beta}\right)\left|\hat\phi_\pm(t)\right| \\ &\underset{\bar\beta,\beta<2\pi}{\le} e^{\beta(h-c/24\pm\delta)-\bar\beta c/24}\left(e^{\frac{\pi^2 c}{6\bar\beta}}+e^{\frac{\pi^2 c}{6\beta}}\right)e^{-\frac{\pi^2\beta_* c}{6(\beta_*^2+\Lambda_\pm^2)}}Z_{H,\Delta}\left(\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\right)\int_{-\infty}^{\infty}\mathrm{d}t\ \left|\hat\phi_\pm(t)\right|. \end{aligned} \tag{2.4.12}$$

The above analysis is analogous to the one presented in §2.3. Now we choose

$$\beta=\pi\sqrt{\frac{c}{6h}},\bar\beta=\pi\sqrt{\frac{c}{6\bar h}}, \tag{2.4.13}$$

and use the HKS bound [100] to estimate $Z_{H,\Delta}\left(\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda_\pm^2}\right)$. This leads to the following inequality:

$$I_\pm^2\le e^{\pi\sqrt{\frac{ch}{6}}}\left(e^{\pi\sqrt{\frac{c\bar h}{6}}}+e^{\pi\sqrt{\frac{ch}{6}}}\right)\int_{-\infty}^{\infty}\mathrm{d}t\ \left|\hat\phi_\pm(t)\right|\begin{cases} e^{2\pi\sqrt{\frac{ch}{6}}\frac{\Lambda_\pm^2}{4\pi^2}} & h\ge\bar h\\ e^{2\pi\sqrt{\frac{c\bar h}{6}}\frac{\Lambda_\pm^2}{4\pi^2}} & h<\bar h\end{cases}, \tag{2.4.14}$$

To estimate $I^1_\pm$, we consider $\rho_*(h,\bar{h})$, the crossing kernel, defined as

$$\exp\left[\frac{\pi^2 c}{6\beta}+\frac{\pi^2 c}{6\bar{\beta}}\right]=\int_0^\infty dh\,\int_0^\infty d\bar{h}\;\rho_*(h,\bar{h})e^{-\beta(h-c/24)-\bar{\beta}(\bar{h}-c/24)}\,. \tag{2.4.15}$$

Here $\rho_*(h,\bar{h})=\rho_*(h)\rho_*(\bar{h})$, and $\rho_*(h),\rho_*(\bar{h})$ are given by

$$\begin{aligned}\rho_*(x)&=\pi\sqrt{\frac{c}{6}}\frac{I_1\left(2\pi\sqrt{\frac{c}{3}\left(x-\frac{c}{24}\right)}\right)}{\sqrt{x-\frac{c}{24}}}\theta\left(x-\frac{c}{24}\right)+\delta\left(x-\frac{c}{24}\right)\,,\\ &\underset{x\to\infty}{=}\left(\frac{c}{96x^3}\right)^{\frac{1}{4}}\exp\left[2\pi\sqrt{\frac{cx}{6}}\right]\,.\end{aligned} \tag{2.4.16}$$

Hence, we have

$$\begin{aligned}I^1_\pm&=e^{\pi\sqrt{\frac{ch}{6}}}\int_0^\infty \mathrm{d}h'\,\int_0^\infty \mathrm{d}\bar{h}'\;\rho_*(h',\bar{h}')e^{-\sqrt{\frac{c}{6h}}h'-\sqrt{\frac{c}{6\bar{h}}}\bar{h}'}\phi_\pm(h')\\ &=2\delta c_\pm e^{\pi\sqrt{\frac{c\bar{h}}{6}}}\rho_*(h)\,,\end{aligned} \tag{2.4.17}$$

where we have used separability of ρ_* in the variable h and $\bar{h}$ and have defined

$$c_\pm=\frac{1}{2}\int_{-\infty}^\infty dx\;\phi_\pm(h+\delta x)\,. \tag{2.4.18}$$

Comparing the inequalities (2.4.17) and (2.4.14), we see that by choosing $\gamma\Lambda_\pm<\sqrt{2}\pi$, with $\max(h,\bar{h})=\gamma^4\min(h,\bar{h})$; one can make $I^2_\pm$ subleading, consequently in the $h,\bar{h}\to\infty$ limit we have

$$2\delta c_-\left(\frac{c}{96\bar{h}^3}\right)^{-\frac{1}{4}}\rho_*(h,\bar{h})\le e^{\pi\sqrt{\frac{c\bar{h}}{6}}}Q(h)\le 2\delta c_+\left(\frac{c}{96\bar{h}^3}\right)^{-\frac{1}{4}}\rho_*(h,\bar{h})\,. \tag{2.4.19}$$

By symmetry we obtain

$$2\delta c'_-\left(\frac{c}{96h^3}\right)^{-\frac{1}{4}}\rho_*(h,\bar{h})\le e^{\pi\sqrt{\frac{ch}{6}}}P(\bar{h})\le 2\delta c'_+\left(\frac{c}{96h^3}\right)^{-\frac{1}{4}}\rho_*(h,\bar{h})\,. \tag{2.4.20}$$

The best possible value of $c_\pm$ ($c'_\pm$) can be obtained from [92]. For the verification purpose, here we choose the function given in [158] for estimating Q

$$\phi_+(h') = \left(\frac{\sin\left(\frac{\Lambda_+\delta}{4}\right)}{\frac{\Lambda_+\delta}{4}}\right)^{-4}\left(\frac{\sin\left(\frac{\Lambda_+(h-h')}{4}\right)}{\frac{\Lambda_+(h-h')}{4}}\right)^{4}, \tag{2.4.21}$$

$$\phi_-(h') = \left(1-\left(\frac{h-h'}{\delta}\right)^2\right)\left(\frac{\sin\left(\frac{\Lambda_-(h-h')}{4}\right)}{\frac{\Lambda_-(h-h')}{4}}\right)^{4}. \tag{2.4.22}$$

The above function yields almost the same bound as found in [158], except for the fact that we have to take care of the constraint $\Lambda < \frac{\sqrt{2}\pi}{\gamma}$. In particular we find the following bounding function $s_\pm(\delta) = \log c_\pm$:

$$\begin{gathered}
c_+ = \begin{cases} \frac{\pi}{3}\left(\frac{\pi\delta}{2\sqrt{2}\gamma}\right)^3\left(\sin\left(\frac{\pi\delta}{2\sqrt{2}\gamma}\right)\right)^{-4} & , \quad \delta < \frac{\gamma a_*}{\sqrt{2}\pi} \\ 2.02 & , \quad \delta \geq \frac{\gamma a_*}{\sqrt{2}\pi} \end{cases} \\
c_- = \begin{cases} \frac{2\sqrt{2}\gamma}{3\pi\delta^3}\left(\delta^2 - \frac{6\gamma^2}{\pi^2}\right) & , \quad \delta < \frac{6\gamma}{\sqrt{2}\pi} \\ 0.46 & , \quad \delta \geq \frac{6\gamma}{\sqrt{2}\pi} \end{cases}
\end{gathered} \tag{2.4.23}$$

where $a_* = 3.38$ [158]. We verify the eq. (2.4.19) in the §2.6 using the above values of $c_\pm$ (in particular, see the fig. 2.11). Similar verification can be done for the eq. (2.4.20) as well.

The analysis with the assumption of twist gap g proceeds as in the end of §2.3. We do not repeat the analysis here. We just state the result. In that scenario, one obtains

$$\Lambda_\pm < \min\left(\frac{\sqrt{2}\pi}{\zeta}, \sqrt{2}\pi\right), \quad \zeta^2 \equiv \left(\frac{c}{12g}\right). \tag{2.4.24}$$

If we make it specific for primaries, $c \mapsto c-1$ and we have

$$\begin{aligned} \Lambda_{\pm} &< \min\left(\frac{\sqrt{2}\pi}{\zeta_p}, \sqrt{2}\pi\right) \\ &= \frac{\sqrt{2}\pi}{\zeta_p}, \quad \zeta_p^2 \equiv \left(\frac{c-1}{12g}\right) \geq 1\,. \end{aligned} \tag{2.4.25}$$

where the second equality follows only if there is no conserved currents because of the bound on twist gap.

We can use this lemma to prove the result about asymptotic spectral gap in terms of strips, as mentioned in $\{2.1.4\}$. For this purpose, we use the magic function introduced in [92]. We can let $\gamma = 1$ in the above analysis. We have to keep in mind that now the support of the Fourier transform of ϕ_- satisfies $\Lambda < \frac{\sqrt{2}\pi}{\gamma}$, thus the minimal value of δ comes out to be $\frac{1}{\sqrt{2}}$ in stead of $1/2$ as in [92].

2.5 The integrated density of states

2.5.1 The main 2D Tauberian theorem

We prove in this section

$$\begin{aligned} F(h,\bar{h}) &\equiv \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \rho(h',\bar{h}') \\ &\underset{h,\bar{h}\to\infty}{=} \frac{1}{4\pi^2}\left(\frac{36}{c^2 h\bar{h}}\right)^{1/4} \exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\left[1+O\left(\tau^{\frac{\Upsilon}{4}-1/2}\right)\right]. \end{aligned}$$

where τ is the twist of the state with $h,\bar{h}$ and given by $\tau = 2\min\{h,\bar{h}\}$ and $h = \bar{h}^{\upsilon}$ with $1/2 < \upsilon < 2$ and $\Upsilon = \max(\upsilon, 1/\upsilon)$. When $\Upsilon = 1$, this reduces to the eq (2.1.1). In order to prove this we define

$$\delta\rho(h,\bar{h}) = \rho(h,\bar{h}) - \rho_*(h,\bar{h}), \tag{2.5.1}$$

$$\delta\mathscr{L}(\beta,\bar{\beta}) = \mathscr{L}_\rho(\beta,\bar{\beta}) - \mathscr{L}_{\rho_*}(\beta,\bar{\beta}). \tag{2.5.2}$$

Since the leading term is already produced by $\rho_*(h,\bar{h})$, our job is to show that

$$\begin{aligned}&\int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}') \\ &= \frac{1}{4\pi^2}\left(\frac{36c^2}{h\bar{h}}\right)^{1/4} \exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right] O\left(\tau^{\Upsilon/4-1/2}\right).\end{aligned} \tag{2.5.3}$$

In particular, we will be showing that

$$\begin{aligned}&\int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}') \\ &= \frac{\sqrt{6c}}{4\pi^2} \exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right] \left[O\left(h^{-3/4}\right)+O\left(\bar{h}^{-3/4}\right)\right].\end{aligned} \tag{2.5.4}$$

Thus if $\upsilon \in (1/2,2)$, the error term is suppressed by maximum of $\frac{h^{1/4}}{\sqrt{\bar{h}}}$ and $\frac{\bar{h}^{1/4}}{\sqrt{h}}$, arriving at (2.5.3).

In order to prove the above, we proceed as in [158] and introduce the following kernel:

$$G(\nu) = \frac{1}{2\pi\imath}\int_{\beta-\imath\Lambda}^{\beta+\imath\Lambda} \frac{dz}{z}\frac{\Lambda^2+(z-\beta)^2}{\Lambda^2+\beta^2} e^{-\nu z}, \quad \nu = h'-h, \tag{2.5.5}$$

$$G(\bar{\nu}) = \frac{1}{2\pi\imath}\int_{\bar{\beta}-\imath\Lambda}^{\bar{\beta}+\imath\Lambda} \frac{d\bar{z}}{\bar{z}}\frac{\Lambda^2+(\bar{z}-\bar{\beta})^2}{\Lambda^2+\bar{\beta}^2} e^{-\bar{\nu}\bar{z}}, \quad \bar{\nu} = \bar{h}'-\bar{h}. \tag{2.5.6}$$

Here we have done slight abuse of notation. It is implicitly assumed that the function $G(\bar{\nu})$

depends on $\bar{\beta}$ instead of β. Now it can be shown that [158]:

$$G(\nu)G(\bar{\nu}) = [\theta(-\nu)+G_+(\nu)\theta(\nu)+G_-(\nu)\theta(-\nu)]\,[\theta(-\bar{\nu})+G_+(\bar{\nu})\theta(\bar{\nu})+G_-(\bar{\nu})\theta(-\bar{\nu})]\,, \tag{2.5.7}$$

where $G_\pm$ is defined exactly like in [158] for both the variable ν and $\bar{\nu}$. At this point, we use the kernel given in (2.5.7) and integrate it against $\delta\rho(\Delta)$. This yields us the following equation

$$\int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}') = \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}') G(\nu)G(\bar{\nu})$$

$$+\int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}')\,[-\theta(-\bar{\nu})\theta(\nu)G_+(\nu)-\theta(-\bar{\nu})\theta(-\nu)G_-(\nu)-(\nu\to\bar{\nu})] \tag{2.5.8}$$

$$+\int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}')\,[-\theta(-\bar{\nu})G_-(\bar{\nu})\theta(\nu)G_+(\nu)-\theta(-\bar{\nu})G_-(\bar{\nu})\theta(-\nu)G_-(\nu)] \tag{2.5.9}$$

$$+\int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}')\,[-\theta(\bar{\nu})G_+(\bar{\nu})\theta(\nu)G_+(\nu)-\theta(\bar{\nu})G_+(\bar{\nu})\theta(-\nu)G_-(\nu)]\,. \tag{2.5.10}$$

Most of the terms can be estimated using techniques from [158]. The new players in the game are the cross terms, for example the term:

$$Z = \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \delta\rho(h',\bar{h}')\,[\theta(-\bar{\nu})\theta(\nu)G_+(\nu)+\theta(-\bar{\nu})\theta(-\nu)G_-(\nu)+(\nu\to\bar{\nu})]\,. \tag{2.5.11}$$

Below we will illustrate how to handle these terms. We remark that this is what requires us to prove the lemma in the previous section. For concreteness, consider the following term $\theta(-\bar{\nu})\theta(\nu)G_+(\nu)$ and analyze it carefully. The analysis for the other terms in Z goes exactly in the same manner. In what follows, we will be using the inequalities for $\beta > 0$:

$$\theta(-\bar{\nu}) \le e^{-\bar{\beta}\bar{\nu}}\,, \tag{2.5.12}$$

$$|G_\pm(\nu)| \le 2e^{-\beta\nu}\min\left(1,(h-h')^{-2}\right)\,, \tag{2.5.13}$$

$$|G_\pm(\bar{\nu})| \le 2e^{-\bar{\beta}\bar{\nu}}\min\left(1,(\bar{h}-\bar{h}')^{-2}\right)\,. \tag{2.5.14}$$

The inequalities (2.5.13) and (2.5.14) have been derived in the appendix of [158].

Consider the term

$$Z1 = \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\, \theta(-\bar{\nu})\Theta(\nu)G(\nu)\delta\rho(h',\bar{h}')\,. \tag{2.5.15}$$

Clearly, we have

$$|Z1| \leq 2e^{\beta h+\bar{\beta}\bar{h}} \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\, e^{-\beta h'-\bar{\beta}\bar{h}'} \left[\rho(h',\bar{h}')+\rho_*(h',\bar{h}')\right] \min\left(1,(h-h')^{-2}\right)\,. \tag{2.5.16}$$

For the term with $\rho_*(h',\bar{h}')$, the estimation procedure mimics the one presented in the section 5 of [158]. In particular, we have

$$\begin{aligned} &e^{\beta h+\bar{\beta}\bar{h}} \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\, e^{-\beta h'-\bar{\beta}\bar{h}'} \rho_*(h',\bar{h}')\min\left(1,(h-h')^{-2}\right) \\ &= O\left(h^{-3/4} e^{2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)}\right)\,. \end{aligned} \tag{2.5.17}$$

The h' integral is done via saddle point method, note it is important to have the factor $\min\left(1,(h-h')^{-2}\right)$ for the validity of saddle point approximation. This is why, the $\bar{h}'$ integral can not be done using saddle, hence does not produce any polynomial suppression in $\bar{h}$. Now consider the term

$$2e^{\beta h+\bar{\beta}\bar{h}} \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\, e^{-\beta h'-\bar{\beta}\bar{h}'} \rho(h',\bar{h}')\min\left(1,(h-h')^{2}\right)\,,$$

and we divide it into three pieces

$$a_1 = 2e^{\beta h+\bar{\beta}\bar{h}} \int_0^{h-h^{3/8}} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\bar{h}' \, e^{-\beta h' - \bar{\beta}\bar{h}'} \rho(h',\bar{h}') \min\left(1,(h-h')^{-2}\right) , \tag{2.5.18}$$

$$a_2 = 2e^{\beta h+\bar{\beta}\bar{h}} \int_{h-h^{3/8}}^{h+h^{3/8}} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\bar{h}' \, e^{-\beta h' - \bar{\beta}\bar{h}'} \rho(h',\bar{h}') \min\left(1,(h-h')^{-2}\right) , \tag{2.5.19}$$

$$a_3 = 2e^{\beta h+\bar{\beta}\bar{h}} \int_{h+h^{3/8}}^{\infty} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\bar{h}' \, e^{-\beta h' - \bar{\beta}\bar{h}'} \rho(h',\bar{h}') \min\left(1,(h-h')^{-2}\right) . \tag{2.5.20}$$

The estimate for a_1 and a_3 again proceeds like in [158] and we obtain

$$a_1 = O\left(h^{-3/4} e^{2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)} \right) , \tag{2.5.21}$$

$$a_3 = O\left(h^{-3/4} e^{2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)} \right) . \tag{2.5.22}$$

The estimation of the term a_2 would require the lemma from the previous section §2.4. We subdivide a_2 into three different parts:

$$a_{21} = 2e^{\beta h+\bar{\beta}\bar{h}} \int_{h-h^{3/8}}^{h-1} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\bar{h}' \, e^{-\beta h' - \bar{\beta}\bar{h}'} \rho(h',\bar{h}') \min\left(1,(h-h')^{-2}\right) , \tag{2.5.23}$$

$$a_{22} = 2e^{\beta h+\bar{\beta}\bar{h}} \int_{h-1}^{h+1} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\bar{h}' \, e^{-\beta h' - \bar{\beta}\bar{h}'} \rho(h',\bar{h}') \min\left(1,(h-h')^{-2}\right) , \tag{2.5.24}$$

$$a_{23} = 2e^{\beta h+\bar{\beta}\bar{h}} \int_{h+1}^{h+h^{3/8}} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\bar{h}' \, e^{-\beta h' - \bar{\beta}\bar{h}'} \rho(h',\bar{h}') \min\left(1,(h-h')^{-2}\right) . \tag{2.5.25}$$

We have already estimated a_{22} in the previous section §2.4. This is basically order one window. Since, the estimation of a_{21} and a_{23} proceeds in similar manner, we would demonstrate the estimation of the term a_{21}.

$$\begin{aligned}
a_{21} &= 2e^{\beta h+\bar{\beta}\bar{h}}\int_{h-h^{3/8}}^{h-1}\mathrm{d}h'\int_0^\infty \mathrm{d}\bar{h}'\, e^{-\beta h'-\bar{\beta}\bar{h}'}\rho(h',\bar{h}')\min\left(1,(h-h')^{-2}\right)\\
&= 2e^{\bar{\beta}\bar{h}}\sum_{k=2}^{h^{3/8}}\int_{h-k}^{h-k+1}\mathrm{d}h'\int_0^\infty \mathrm{d}\bar{h}'\, e^{\beta(h-h')-\bar{\beta}\bar{h}'}\rho(h',\bar{h}')(h-h')^{-2}\\
&\le 2e^{\bar{\beta}\bar{h}}\sum_{k=2}^{h^{3/8}}\frac{e^{\beta k}}{(k-1)^2}\int_{h-k}^{h-k+1}\mathrm{d}h'\int_0^\infty \mathrm{d}\bar{h}'\, e^{-\bar{\beta}\bar{h}'}\rho(h',\bar{h}')\\
&= O\left(h^{-3/4}e^{2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)}\sum_{k=2}^{h^{3/8}}\frac{e^{\beta k}}{(k-1)^2}\right)\\
&= O\left(h^{-3/4}e^{2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)}\right),
\end{aligned}\tag{2.5.26}$$

where going from the third line to the fourth line requires use of lemma proven in the previous section §2.4.

The estimation of the terms in (2.5.9) and (2.5.10) requires us to divide the $(h,\bar{h})$ plane into 9 regions (see figure 2.7):

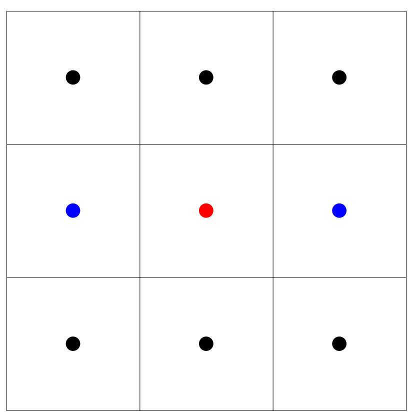


Figure 2.7. The estimation of the terms in (2.5.9) and (2.5.10) requires us to divide the $(h,\bar{h})$ plane into 9 regions. In the figure, the horizontal lines are $h' = 0, h-h^{3/8}, h+h^{3/8}, \infty$ while the vertical lines are $\bar{h}' = 0, \bar{h}-\bar{h}^{3/8}, \bar{h}+\bar{h}^{3/8}, \infty$. The different colors denote the different methods of treating them.

$$
\begin{aligned}
R_1 &= \left\{(h',\bar{h}') :\ h' \in [0, h-h^{3/8}], \bar{h}' \in [0, \bar{h}-\bar{h}^{3/8}]\right\}, \\
R_2 &= \left\{(h',\bar{h}') :\ h' \in [0, h-h^{3/8}], \bar{h}' \in [\bar{h}-\bar{h}^{3/8}, \bar{h}+\bar{h}^{3/8}]\right\}, \\
R_3 &= \left\{(h',\bar{h}') :\ h' \in [0, h-h^{3/8}], \bar{h}' \in [\bar{h}+\bar{h}^{3/8}, \infty]\right\}, \\
R_4 &= \left\{(h',\bar{h}') :\ h' \in [h-h^{3/8}, h+h^{3/8}], \bar{h}' \in [0, \bar{h}-\bar{h}^{3/8}]\right\}, \\
R_5 &= \left\{(h',\bar{h}') :\ h' \in [h-h^{3/8}, h+h^{3/8}], \bar{h}' \in [\bar{h}-\bar{h}^{3/8}, \bar{h}+\bar{h}^{3/8}]\right\}, \\
R_6 &= \left\{(h',\bar{h}') :\ h' \in [h-h^{3/8}, h+h^{3/8}], \bar{h}' \in [\bar{h}+\bar{h}^{3/8}, \infty]\right\}, \\
R_7 &= \left\{(h',\bar{h}') :\ h' \in [h+h^{3/8}, \infty], \bar{h}' \in [0, \bar{h}-\bar{h}^{3/8}]\right\}, \\
R_8 &= \left\{(h',\bar{h}') :\ h' \in [h+h^{3/8}, \infty], \bar{h}' \in [\bar{h}-\bar{h}^{3/8}, \bar{h}+\bar{h}^{3/8}]\right\}, \\
R_9 &= \left\{(h',\bar{h}') :\ h' \in [h+h^{3/8}, \infty], \bar{h}' \in [\bar{h}+\bar{h}^{3/8}, \infty]\right\}.
\end{aligned}
\tag{2.5.27}
$$

Basically, we have to estimate $\sum_i S_i$ where S_i is given by

$$
\begin{aligned}
S_i &= \int_{R_i} \mathrm{d}h'\ \mathrm{d}\bar{h}' |\delta\rho(h',\bar{h}')| \Big[|G_+(\nu)G_+(\bar{\nu})| + |G_-(\bar{\nu})G_+(\nu)| + (+ \leftrightarrow -) \Big] \\
&\leq 4(S_i^{(1)} + S_i^{(2)}),
\end{aligned}
\tag{2.5.28}
$$

where we have used $|\delta\rho| \leq \rho + \rho_*$, $|G_\pm(\nu) \leq 2e^{-\beta\nu} \min\big(1, (h-h')^{-2}\big)$, and $|G_\pm(\bar{\nu})| \leq 2e^{-\bar{\beta}\bar{\nu}} \min\big(1, (\bar{h}-\bar{h}')^{-2}\big)$. The appearance of 4 is due to the fact that there are 4 terms in the integrand defining S_i and each is suppressed in a same manner. Here $S_i^{(1)}$ and $S_i^{(2)}$ are defined as

$$
\begin{aligned}
S_i^{(1)} &= 4\int_{R_i} \mathrm{d}h'\ \mathrm{d}\bar{h}' \rho(h',\bar{h}') e^{-\beta\nu-\bar{\beta}\bar{\nu}} \min\big(1, (h-h')^{-2}\big) \min\big(1, (\bar{h}-\bar{h}')^{-2}\big), \\
S_i^{(2)} &= 4\int_{R_i} \mathrm{d}h'\ \mathrm{d}\bar{h}' \rho_*(h',\bar{h}') e^{-\beta\nu-\bar{\beta}\bar{\nu}} \min\big(1, (h-h')^{-2}\big) \min\big(1, (\bar{h}-\bar{h}')^{-2}\big).
\end{aligned}
\tag{2.5.29}
$$

Now for R_i with $i \neq 4,5,6$ we have

$$\begin{aligned}
\frac{1}{4}S^{(1)}_{i\neq 4,5,6} &= e^{\beta h+\bar{\beta}\bar{h}}\int_{R_{i\neq 4,5,6}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\min\left(1,(h-h')^{-2}\right)\min\left(1,(\bar{h}-\bar{h}')^{-2}\right)\\
&\leq e^{\beta h+\bar{\beta}\bar{h}}\int_{R_{i\neq 4,5,6}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\min\left(1,(\bar{h}-\bar{h}')^{-2}\right)\\
&= O\left(\bar{h}^{-3/4}\exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\right)\,.
\end{aligned} \tag{2.5.30}$$

For R_4 and R_6 we observe the following

$$\begin{aligned}
\frac{1}{4}S^{(1)}_{i=4,6} =& e^{\beta h+\bar{\beta}\bar{h}}\int_{R_{i=4,6}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\min\left(1,(h-h')^{-2}\right)\min\left(1,(\bar{h}-\bar{h}')^{-2}\right)\\
&\leq e^{\beta h+\bar{\beta}\bar{h}}\int_{R_{i=4,6}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\min\left(1,(h-h')^{-2}\right)\\
&\leq e^{\beta h+\bar{\beta}\bar{h}}\bar{h}^{-3/4}\int_{R_{i=4,6}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\\
&= O\left(h^{-3/4}\exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\right)\,.
\end{aligned} \tag{2.5.31}$$

For analyzing the region R_5, we are required to subdivide it into 9 regions again where each of the region is Cartesian product of order one interval in h' and $\bar{h}'$. Now one can use the lemma proven in the section §2.3 to show that

$$\begin{aligned}
\frac{1}{4}S^{(1)}_{5} =& e^{\beta h+\bar{\beta}\bar{h}}\int_{R_5} \mathrm{d}h'\ \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\min\left(1,(h-h')^{-2}\right)\min\left(1,(\bar{h}-\bar{h}')^{-2}\right)\\
&= O\left(h^{-3/4}\bar{h}^{-3/4}\exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\right)\,.
\end{aligned} \tag{2.5.32}$$

The estimation for $S^{(2)} = \sum_i S_i^{(2)}$ can be done by saddle point method:

$$\frac{1}{4}S^{(2)} = e^{\beta h+\bar{\beta}\bar{h}}\int_0^\infty\int_0^\infty \mathrm{d}h'\,\mathrm{d}\bar{h}'\,\rho_*(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\min\left(1,\frac{1}{(h-h')^2}\right)\min\left(1,\frac{1}{(\bar{h}-\bar{h}')^2}\right) \tag{2.5.33}$$

$$= O\left(h^{-3/4}\bar{h}^{-3/4}\exp\left[2\pi\left(\sqrt{\frac{ch}{6}}+\sqrt{\frac{c\bar{h}}{6}}\right)\right]\right). \tag{2.5.34}$$

We are yet to estimate the following term

$$m \equiv \int_0^\infty \mathrm{d}h' \int_0^\infty \mathrm{d}\bar{h}'\delta\rho(h',\bar{h}')G(\nu)G(\bar{\nu}). \tag{2.5.35}$$

which appears in the expression for $\int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}'\delta\rho(h',\bar{h}')$. Using the definition of $G(\nu)$ and $G(\bar{\nu})$, we arrive at

$$m = -\frac{1}{4\pi^2}\int_{\beta-\imath\Lambda}^{\beta+\imath\Lambda}\int_{\bar{\beta}-\imath\Lambda}^{\bar{\beta}+\imath\Lambda}\frac{dzd\bar{z}}{z\bar{z}}\frac{\Lambda^2+(z-\beta)^2}{\Lambda^2+\beta^2}\frac{\Lambda^2+(\bar{z}-\bar{\beta})^2}{\Lambda^2+\bar{\beta}^2}e^{zh+\bar{z}\bar{h}}\delta\mathscr{L}(z,\bar{z}). \tag{2.5.36}$$

Thus we have

$$|m| \leq \frac{1}{4\pi^2}\int_{-\Lambda}^{\Lambda}\int_{-\Lambda}^{\Lambda}\frac{\mathrm{d}t d\bar{t}}{|\beta+\imath t||\bar{\beta}+\imath\bar{t}|}\left(\frac{\Lambda^2-t^2}{\Lambda^2+\beta^2}\right)\left(\frac{\Lambda^2-\bar{t}^2}{\Lambda^2+\bar{\beta}^2}\right)e^{\beta h+\bar{\beta}\bar{h}}|\delta\mathscr{L}(z,\bar{z})|. \tag{2.5.37}$$

Now we use the inequality

$$|\delta\mathscr{L}(z,\bar{z})| \leq e^{-(Re[z]+Re[\bar{z}])c/24}Z_H\left(\frac{4\pi^2 Re[z]}{|z|^2},\frac{4\pi^2 Re[\bar{z}]}{|\bar{z}|^2}\right), \tag{2.5.38}$$

and subsequently the method utilized in §2.3 to put a bound by $Z_{H,\Delta}$:

$$|m| \leq \frac{4\Lambda^6}{9\beta\bar{\beta}\pi^2}\frac{e^{\beta(h-c/24)}e^{\bar{\beta}(\bar{h}-c/24)}e^{\frac{\pi^2 c}{6y}}}{(\Lambda^2+\beta^2)\left(\Lambda^2+\bar{\beta}^2\right)}Z_{H,\Delta}\left(\frac{4\pi^2\beta_*}{\beta_*^2+\Lambda^2}\right), \tag{2.5.39}$$

where $y = \beta$ or $\bar{\beta}$. Essentially this is exactly the same argument as in §2.3. Then by choosing $\Lambda < \sqrt{2}\pi$ (when $h \simeq \bar{h}$, otherwise we need to choose $\gamma\Lambda < \sqrt{2}\pi$, see the discussion in §2.3) and using the HKS [100] argument, one can show that the above term is exponentially suppressed compared to the leading answer coming from $\rho_*(h,\bar{h})$. When h is not of the order of $\bar{h}$, we need to assume existence of twist gap (as defined in $\{2.1.1\}$) and proceed like we did in §2.3. This concludes our analysis and hence the proof of the main theorem.

2.5.2 Sensitivity of Asymptotics towards spin J

The asymptotic formula given in eq. (2.1.1) and derived above can be rewritten in terms of dimension $\Delta = h + \bar{h}$ and spin $J = |h - \bar{h}|$:

$$
\begin{aligned}
&F(h \to \infty, \bar{h} \to \infty) = \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \rho(h', \bar{h}') \\
&= \frac{1}{4\pi^2} \left(\frac{36}{c^2 \left(\Delta^2 - J^2\right)} \right)^{1/4} \exp\left[2\pi \left(\sqrt{\frac{c(\Delta + J)}{12}} + \sqrt{\frac{c(\Delta - J)}{12}} \right) \right] \left[1 + O\left(\Delta^{-1/4} \right) \right],
\end{aligned}
\tag{2.5.40}
$$

which is true when $1 < \frac{\Delta}{J} = O(1)$. It turns out that even when Δ and J is not of the same order, we can do order by order correction to this integrated density of states by spin.

First of all, when J is of order one, we should just ignore J dependence of the eq. (2.5.40). In fact, $J = \Delta^{1/n}$ with $n > 4/3$, one can ignore J dependence. Thus in this regime, we have

$$
\begin{aligned}
&F(\Delta \to \infty, J = \Delta^{1/n} \to \infty) = \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}' \; \rho(h', \bar{h}'), \quad 4/3 < n \leq \infty \\
&= \frac{1}{4\pi^2} \left(\frac{6}{c\Delta} \right)^{1/2} e^{2\pi \sqrt{\frac{c\Delta}{3}}} \left[1 + O\left(\Delta^{-1/4} \right) \right].
\end{aligned}
\tag{2.5.41}
$$

When $8/7 \leq n \leq 4/3$, the J dependence is meaningful only within the exponential i.e.

we have

$$
\begin{aligned}
F(\Delta\to\infty, J=\Delta^{1/n}\to\infty) &= \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')\,, \quad 8/7\le n\le 4/3 \\
&= \frac{1}{4\pi^2}\left(\frac{6}{c\Delta}\right)^{1/2} e^{2\pi\sqrt{\frac{c\Delta}{3}}\left(1-\frac{J^2}{8\Delta^2}\right)}\left[1+O\left(\Delta^{-1/4}\right)\right].
\end{aligned}
\tag{2.5.42}
$$

When $16/15 < n < 8/7$ (n can not less than one because of unitarity bound), we have

$$
\begin{aligned}
F(\Delta\to\infty, J=\Delta^{1/n}\to\infty) &= \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')\,, \quad 1< n< 8/7 \\
&= \frac{1}{4\pi^2}\left(\frac{6}{c\Delta}\right)^{1/2}\left(1+\frac{J^2}{4\Delta^2}\right)\exp\left[2\pi\left(\sqrt{\frac{c(\Delta+J)}{12}}+\sqrt{\frac{c(\Delta-J)}{12}}\right)\right]\left[1+O\left(\Delta^{-1/4}\right)\right].
\end{aligned}
\tag{2.5.43}
$$

We remark that not all the term in the exponential are meaningful, we have to do an J/Δ expansion and the only meaningful terms are of the form $\exp\left(\Delta^{-\ell}\right)$ with $\ell<1/4$ (since $\exp\left(\Delta^{-\ell}\right)\simeq 1+\Delta^{-\ell}$ becomes comparable to error term for $\ell<1/4$). For example, when $n=\frac{15}{14}$, it is meaningful to keep the following terms only:

$$
e^{2\pi\left(\sqrt{\frac{c(\Delta+J)}{12}}+\sqrt{\frac{c(\Delta-J)}{12}}\right)} = e^{2\pi\sqrt{\frac{c\Delta}{3}}\left(1-\frac{J^2}{8\Delta^2}-\frac{5J^4}{128\Delta^4}-\frac{21J^6}{1024\Delta^6}-\frac{429J^8}{32768\Delta^8}-\frac{2431J^{10}}{262144\Delta^{10}}\right)}\left[1+O\left(\Delta^{-3/10}\right)\right].
$$

One can generalize the equation (2.5.43) for $J=\Delta^{1/n}$ with $\frac{2^{m+3}}{2^{m+3}-1}<n<\frac{2^{m+2}}{2^{m+2}-1}\le 8/7$ where $m\ge 1$ is a fixed integer. We define $f(m)=\frac{2^m}{2^m-1}$ and we have :

$$
\begin{aligned}
F(\Delta\to\infty, J=\Delta^{1/n}\to\infty) &= \int_0^h \mathrm{d}h' \int_0^{\bar{h}} \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')\,, \quad f(m+3)<n<f(m+2) \\
&= \frac{1}{4\pi^2}\left(\frac{6}{c\Delta}\right)^{1/2}\left(\sum_{k=0}^{m} a_k\left(\frac{J^2}{\Delta^2}\right)^k\right)\exp\left[2\pi\left(\sqrt{\frac{c(\Delta+J)}{12}}+\sqrt{\frac{c(\Delta-J)}{12}}\right)\right]\left[1+O\left(\Delta^{-1/4}\right)\right].
\end{aligned}
\tag{2.5.44}
$$

where a_k's are defined as

$$
\left(1-\frac{J^2}{\Delta^2}\right)^{1/4} = \sum_{k=0}^{\infty} a_k\left(\frac{J^2}{\Delta^2}\right)^k,\ \Delta>J\,. \tag{2.5.45}
$$

2.6 Verification: 2D Ising model

We verify the bounds on $O(1)$ correction to entropy associated with order one window centered at some large h and $\bar{h}$, proven in §2.3 as shown in the figure 2.8, 2.9 and 2.10. In §2.3, we have obtained two kinds of bounds, one without assuming any twist gap while the other one assumes existence of a twist gap. For 2D Ising model, it turns out that the bound coming from assuming a twist gap (indeed 2D Ising model has a twist gap) is stronger than the one without using the information about twist gap. Thus in the figures below, we verify the bounds that uses the information about twist gap. In fact, as we have mentioned in §2.3, the use of twist gap actually enables us to probe regions where h and $\bar{h}$ are not of the same order. The figure 2.10 elucidates such a scenario. We verify the lemma proven in §2.4 as shown in the figure 2.11 and

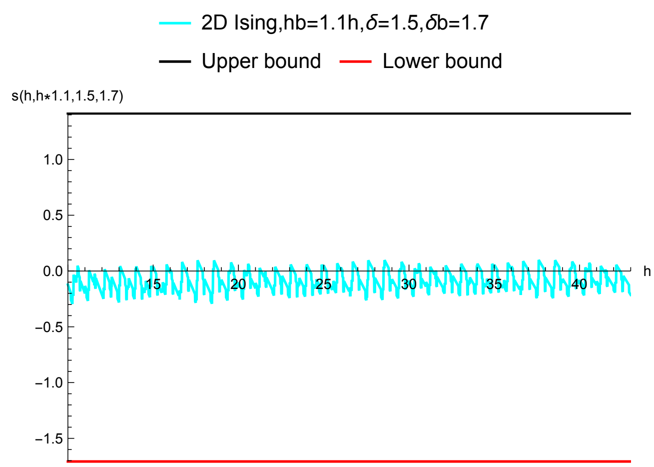


Figure 2.8. Verifying the bounds on $O(1)$ correction to entropy associated with order one window centered at some large h and $\bar{h}$ of the same order. The curvy cyan line is the difference between the actual number of states lying within the window and leading answer coming from the Cardy formula.

figure 2.12 using 2 dimensional Ising CFT. The partition function for 2 dimensional Ising CFT is given by

$$Z_{\text{Ising}}(\beta,\bar{\beta}) = \frac{1}{2}\left(\sqrt{\frac{\theta_2(\beta)}{\eta(\beta)}}\sqrt{\frac{\theta_2(\bar{\beta})}{\eta(\bar{\beta})}} + \sqrt{\frac{\theta_3(\beta)}{\eta(\beta)}}\sqrt{\frac{\theta_3(\bar{\beta})}{\eta(\bar{\beta})}} + \sqrt{\frac{\theta_4(\beta)}{\eta(\beta)}}\sqrt{\frac{\theta_4(\bar{\beta})}{\eta(\bar{\beta})}}\right). \tag{2.6.1}$$

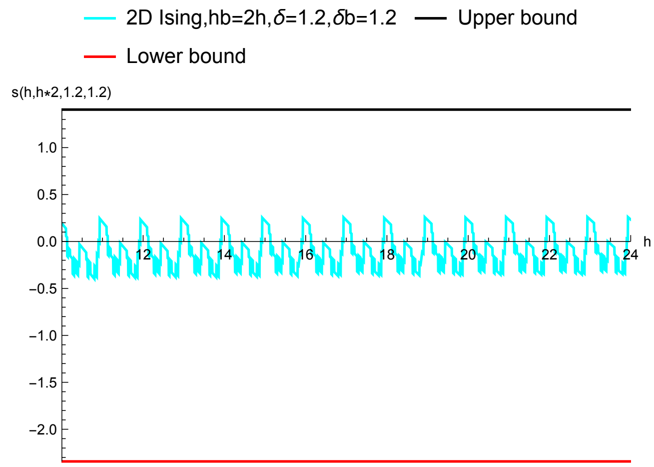


Figure 2.9. Verifying the bounds on $O(1)$ correction to entropy associated with order one window centered at some large h and $\bar{h}$ of the same order. The curvy cyan line is the difference between the actual number of states lying within the window and leading answer coming from the Cardy formula.

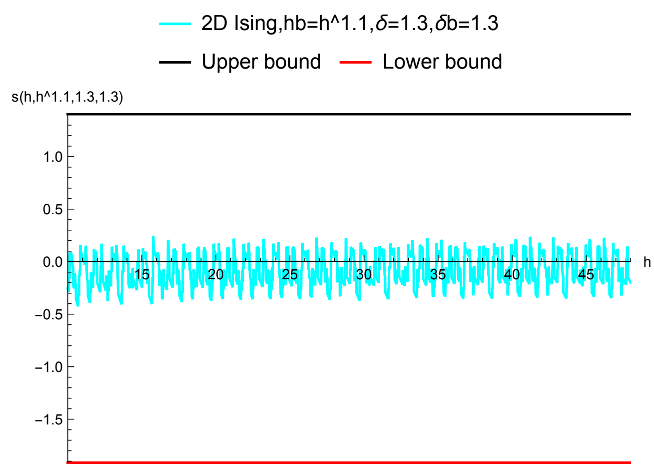


Figure 2.10. Verifying the bounds on $O(1)$ correction to entropy associated with order one window centered at some large h and $\bar{h}$ of different order. The curvy cyan line is the difference between the actual number of states lying within the window and leading answer coming from the Cardy formula.

We verify the main theorem proven in §2.5 in fig. 2.13 and 2.14. We plot the total number of states upto some $(h,\bar{h})$ as a function of h. We have considered two different cases where the asymptote is approached along different curves. We compare it against the asymptotic formula we derive in §2.5.

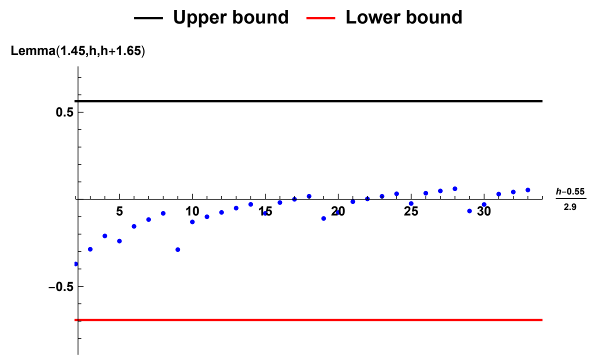


Figure 2.11. The blue dots denotes the values of Lemma$(\delta,h,\bar{h})$, here we have $\delta=1.45$ and $\bar{h}=h+1.65$, for different values of h. The blue dots are bounded by an order one i.e. $h,\bar{h}$ independent number, denoted by the red and the black curve. The values of $c_{\pm}$ found in §2.4 are used as the bounds.

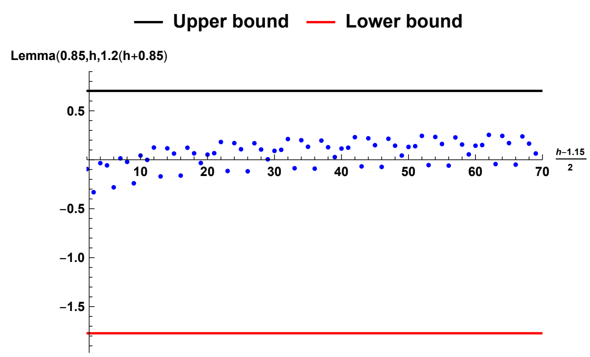


Figure 2.12. The blue dots denotes the values of Lemma$(\delta,h,\bar{h})$, where $\delta=1.7$ and $\bar{h}=1.2(h+0.85)$, for different values of h. The blue dots are bounded by an order one i.e. $h,\bar{h}$ independent number, denoted by the red and the black curve. The $c_{\pm}$ found in the §2.4 are used as the bounds.

2.7 Finite twist-Large spin

In this section, our aim is to estimate the finite twist and large spin sector of the density of states. Without loss of generality, we will assume h is finite and $\bar{h}\to\infty$. We will probe the following quantity $U_h(\bar{h})$ in the limit $\bar{h}\to\infty$:

$$U_h(\bar{h})=\int_0^h \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}'\ \rho(h',\bar{h}')\,. \tag{2.7.1}$$

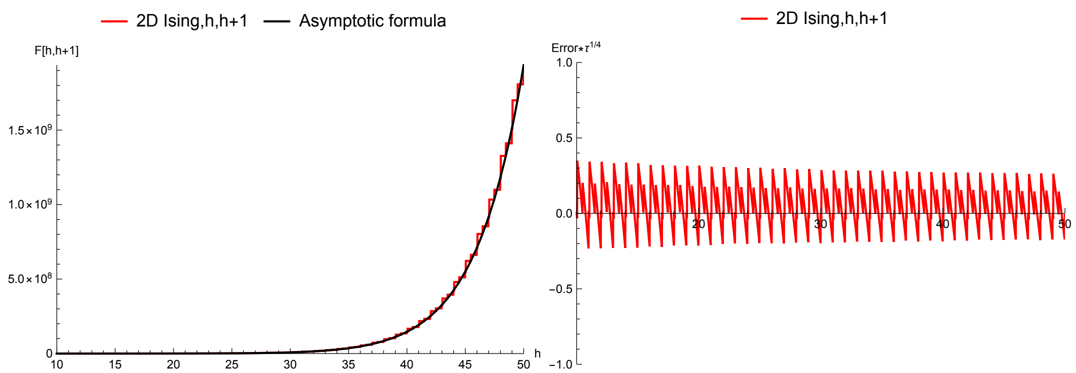


Figure 2.13. On the left, the black curve is the asymptotic Cardy formula while the red curve is the actual number of operators till $(h,\bar{h})$in the Ising model. Here $h = \bar{h} - 1$. The picture on the right hand side plot the relative error times $h^{1/4}$ which is bounded above and below by an order one number.

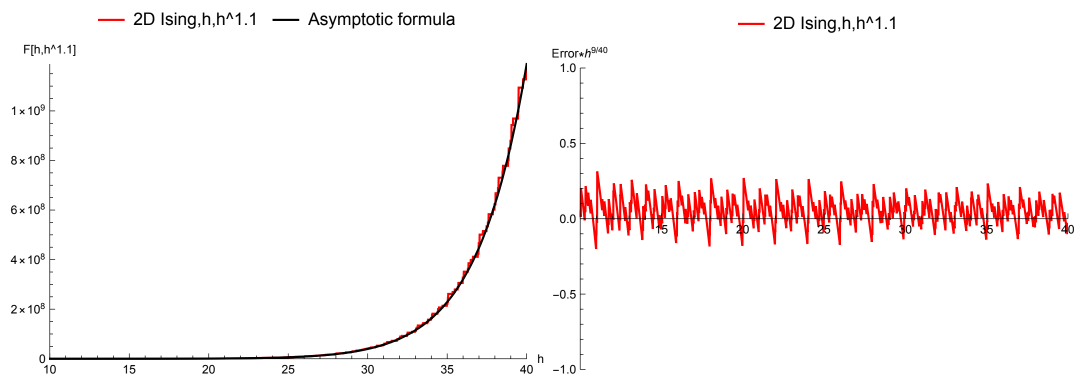


Figure 2.14. On the left, the black curve is the asymptotic Cardy formula while the red curve is the actual number of operators till $(h,\bar{h})$ in the Ising model. Here $h^{1.1} = \bar{h}$. The picture on the right hand side plot the relative error times $h^{9/40}$, which is bounded above and below by an order one number.

This is because we have

$$\int_{h-\delta}^{h+\delta} \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}\bar{h}' \, \rho(h',\bar{h}') \leq U_{h+\delta}(\bar{h}) \,. \tag{2.7.2}$$

Intuitively, it is clear that one can not make both β and $\bar{\beta}$ to approach zero while looking at the partition function, as this would probe the regime $h,\bar{h} \to \infty$. This suggests that we should let $\bar{\beta} \to 0$ and keep β fixed.

We can prove an upper bound on density of states in this sector. Let us write down the following inequality:

$$U_h(\bar{h}) \leq e^{\beta(h-c/24)} \int_0^\infty \, \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \, \mathrm{d}\bar{h}' \, \rho(h',\bar{h}') e^{-\beta(h'-c/24)}, \tag{2.7.3}$$

from which we can write

$$U_h(\bar{h}) \leq e^{\beta\left(h-\frac{c}{24}\right)+\bar{\beta}\bar{h}+\bar{\beta}\bar{\delta}} \int_0^\infty \, \mathrm{d}h' \int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \, \mathrm{d}\bar{h}' \, \rho(h',\bar{h}')\phi_+(\bar{h}') e^{-\beta\left(h'-\frac{c}{24}\right)-\bar{\beta}\bar{h}'}. \tag{2.7.4}$$

At this point we choose

$$\beta = 2\pi, \quad \bar{\beta} = \pi\sqrt{\frac{c}{6\bar{h}}}, \; \bar{h} \to \infty. \tag{2.7.5}$$

Again we separate the partition function into two pieces; the light sector, where the contribution comes from two kinds of states: a) all the states with conformal weight $(h',0)$ with $h' \geq 0$ and b) the states such that $h' + \bar{h}' < \frac{c}{12}$, and the heavy sector, which is defined to be the complement of this. We remark the heavy sector does not contain any operator with conformal weight $(h',0)$. In the usual asymptotic analysis as done previously, the operators with $(h',0)$ such that $h' > \frac{c}{12}$ is put into the heavy sector, but here we can not do that since β is finite, there is no separation between light or heavy in the sense of HKS [100] i.e. the operators with $(h',0)$ contribute on equal footing as the operator $(0,0)$. The upshot of the above discussion is that one can define a kernel ρ_*^{ft} ("ft" stands for finite twist) such that

$$\int_0^\infty \mathrm{d}h' \, \int_0^\infty \mathrm{d}\bar{h}' \, \rho_*^{\mathrm{ft}}(h',\bar{h}') e^{-\beta\left(h'-\frac{c}{24}\right)-\bar{\beta}\bar{h}'} = e^{\frac{\pi^2 c}{6\bar{\beta}}} \sum_{h_i} e^{-\beta\left(h_i-\frac{c}{24}\right)}, \tag{2.7.6}$$

where the sum on the right hand side is over all the states with weight $(h_i,0)$ and we have used

$\beta = \frac{4\pi^2}{\beta} = 2\pi$. As a result $\rho_*^{\rm ft}$ is given by

$$\rho_*^{\rm ft}(h',\bar{h}') = \rho_*(\bar{h}')\sum_{h_i}\delta\left(h'-h_i\right). \tag{2.7.7}$$

Once we have defined ρ_*, we follow the usual method and our next aim is to show that the heavy part has a subleading contribution. The estimation of the heavy part is done depending on whether both h' and $\bar{h}'$ is greater than $c/24$ or one of them is less than $c/24$. In the later case, we have to assume existence of twist gap. Last but not the least, we have to estimate the contribution from the states with $(0,\bar{h}')$ with $\bar{h}' > c/12$. Following the methods in §2.3, we can show that they are indeed subleading. The leading answer is then given by

$$U_h(\bar{h}) \leq e^{2\pi\left(h-\frac{c}{24}\right)+\pi\sqrt{\frac{c\bar{h}}{6}}}\int \mathrm{d}h' \int \mathrm{d}\bar{h}'\rho_*^{\rm ft}(h',\bar{h}')e^{-2\pi\left(h'-\frac{c}{24}\right)-\pi\sqrt{\frac{c}{6\bar{h}}}\bar{h}'}\Phi_+(\bar{h}'). \tag{2.7.8}$$

The $\bar{h}'$ integral can be done using saddle point approximation and we obtain

$$\begin{aligned}
\frac{1}{2\bar{\delta}}U_h(\bar{h}) &\leq e^{-\frac{\pi c}{12}}e^{2\pi h}c_+\rho_*(\bar{h}')\sum_{h_i}e^{-2\pi\left(h_i-\frac{c}{24}\right)},\\
\frac{1}{2\bar{\delta}}U_h(\bar{h}) &\leq e^{-\frac{\pi c}{12}}e^{2\pi h}c_+\rho_*(\bar{h}')\chi_0(e^{-2\pi}).
\end{aligned} \tag{2.7.9}$$

where $\chi_0(q)$ is the vacuum character and $q = e^{-\beta}$ where we have assumed that there is no nontrivial conserved current. Here c_+ is defined as

$$c_+ = \int_{-\infty}^{\infty} dx\, \Phi_+(\bar{h}'+\bar{\delta}x). \tag{2.7.10}$$

One can extend this argument to a scenario where we have nontrivial conserved currents, then we would have a sum over characters for all the conserved currents

$$\frac{1}{2\bar{\delta}}U_h(\bar{h}) \leq c_+\rho_*(\bar{h}')e^{2\pi\left(h-\frac{c}{24}\right)}\sum_{\tilde{h}}\chi_{\tilde{h}}(e^{-2\pi}). \tag{2.7.11}$$

This sum over $\tilde{h}$ is convergent as the absolute value of the sum is bounded above by partition function evaluated at $\beta = \bar{\beta} = 2\pi$. Similar result applies to $h, \bar{h}$ getting swapped. Thus for finite h and $\bar{h} \to \infty$ limit we have

$$S_{\delta,\bar{\delta}} \leq \mathbb{S}_{h,\delta,\bar{\delta}} \leq 2\pi\sqrt{\frac{c\bar{h}}{6}} - \frac{3}{4}\log\left(\bar{h}\right) + 2\pi\left(h + \delta - \frac{c}{24}\right) + \log\left[2c_{+}\bar{\delta}\sum_{\tilde{h}}\chi_{\tilde{h}}(e^{-2\pi})\right], \quad (2.7.12)$$

where $\chi_{\tilde{h}}$ is the character for the conserved current with weight $(\tilde{h}, 0)$.

The similar result specific for the primaries with finite h and $\bar{h} \to \infty$ limit, would read:

$$S^{\text{Vir}}_{\delta,\bar{\delta}} \leq \mathbb{S}^{\text{Vir}}_{h,\delta,\bar{\delta}} \leq 2\pi\sqrt{\frac{(c-1)\bar{h}}{6}} + 2\pi\left(h + \delta - \frac{c-1}{24}\right) + \log\left[c_{+}\frac{2\bar{\delta}}{\sqrt{\bar{h}}}\sum_{\tilde{h}} e^{-2\pi\left(\tilde{h} - \frac{c-1}{24}\right)}\right], \quad (2.7.13)$$

where the zero twist primaries have weight $(\tilde{h}, 0)$. Here again, the sum over $\tilde{h}$ is convergent as the absolute value of the sum is bounded above by partition function evaluated at $\beta = \bar{\beta} = 2\pi$.

CFT without nontrivial zero twist primary:

In case the CFT does not have any conserved current, we do not need to worry about $(h', 0)$ operators anymore and we can do much better as it is possible to choose $\beta \neq 2\pi$ and still define $\rho^{\text{ft-Vir}}_{*}(h', \bar{h}') = \rho^{\text{ft-Vir}}_{*}(h')\rho^{\text{ft-Vir}}_{*}(\bar{h}')$ as a solution to the following equality:

$$\int_{0}^{\infty} \mathrm{d}\bar{h}'\ \rho^{\text{ft-Vir}}_{*}(\bar{h}') e^{-\bar{\beta}\left(\bar{h}' - \frac{c-1}{24}\right)} = \sqrt{\frac{2\pi}{\bar{\beta}}}\left(1 - e^{-\frac{4\pi^2}{\bar{\beta}}}\right). \quad (2.7.14)$$

This parallels the analysis for Virasoro primary in §2.3 (see (2.3.41)).

In the present case, using the kernel $\rho_*^{\text{ft-Vir}}(h',\bar{h}')$ the leading answer turns out to be

$$\frac{1}{2\bar{\delta}}U_h^{\mathrm{Vir}}(\bar{h}) \leq c_+\rho_*(\bar{h})e^{\beta\left(h-\frac{c-1}{24}\right)}e^{\frac{\pi^2(c-1)}{6\beta}}\sqrt{\frac{2\pi}{\beta}}\left(1-e^{-\frac{4\pi^2}{\beta}}\right). \tag{2.7.15}$$

Now by appropriately choosing β we can recover the "square-root" edge present in the analysis in [146]. The square root edge in h dependence of the density of states should produce a factor of $(h-(c-1)/24)^{3/2}$. In particular, we choose $\beta = \frac{1}{(h-\frac{c-1}{24})}$ and let $h-\frac{c}{24}$ to be very small[8]

$$\begin{aligned}
\frac{1}{2\bar{\delta}}U_h^{\mathrm{Vir}}(\bar{h}) &\leq c_+e\rho_*(\bar{h})e^{\frac{\pi^2(c-1)\left(h-\frac{c-1}{24}\right)}{6}}\sqrt{2\pi\left(h-\frac{c-1}{24}\right)}\left(1-e^{-4\pi^2\left(h-\frac{c-1}{24}\right)}\right) \\
&\underset{h-c/24<<\frac{1}{c-1}}{\simeq} c_+\left(4\sqrt{2}\pi^{5/2}e\right)\rho_*(\bar{h})\left(h-\frac{c-1}{24}\right)^{3/2},
\end{aligned} \tag{2.7.16}$$

The above result is consistent with the leading result as reported in [134, 135, 146, 28].

2.8 Holographic CFTs

Holographic CFTs are the ones characterized by a sparse low lying spectrum and large central charge. The sparseness condition is first derived in [100], then rederived in [158], where it emerges naturally out of the Tauberian formalism. In the context of asymptotic behavior of OPE coefficients in large c CFTs, a stronger sparseness condition appears as elucidated in [165, 153]. In this section we will be exploring such CFTs with large central charge and a low lying sparse spectra. In particular, we derive an expression for the density of states in the limit $h,\bar{h}\sim c\to\infty$. Following [158], we parameterize $h,\bar{h}$ as

$$h = c\left(\varepsilon+\frac{1}{24}\right),\ \bar{h} = c\left(\bar{\varepsilon}+\frac{1}{24}\right),\ c\to\infty, \varepsilon-\text{fixed}, \bar{\varepsilon}-\text{fixed}. \tag{2.8.1}$$

[8] This is analogous to the condition written down in [146] as $0<\bar{h}-\frac{c-1}{24}<<1/c$, there $\bar{h}$ is finite and h is let to infinity.

In this limit the asymptotic of the vacuum crossing kernel is given by

$$\rho_*(h,\bar{h}) = \frac{\sqrt{6}}{24}c^{-1}\left(\frac{1}{\varepsilon\bar{\varepsilon}}\right)^{\frac{3}{4}} e^{2\pi c\left(\sqrt{\frac{\varepsilon}{6}}+\sqrt{\frac{\bar{\varepsilon}}{6}}\right)}\theta(\varepsilon)\theta(\bar{\varepsilon})+\cdots. \tag{2.8.2}$$

As done in §2.3 we separate out the contribution to the partition function into two pieces: the light Z_L, and the heavy Z_H.Here we choose

$$\beta = \frac{\pi}{\sqrt{6\varepsilon}},\ \bar{\beta} = \frac{\pi}{\sqrt{6\bar{\varepsilon}}}. \tag{2.8.3}$$

We are required to show that Z_H term is sub-leading. Z_H term gets contribution whenever h' or $\bar{h}'$ is greater than $c/24$ and $\Delta' > c/12$.

It can be shown that Z_H contribution is sub-leading (the method is exactly similar as in §2.3, one has to be careful about $e^{-\beta c/24-\bar{\beta}c/24}$ factor, since c is not finite in the analysis.). The result of the analysis is summarized below:

$$\Lambda_{\pm}^2 < \left(\frac{\sqrt{2}\pi}{\gamma}\right)^2\left(1-\frac{\gamma^2}{12\varepsilon^*}\right), \tag{2.8.4}$$

where $\gamma^4 = \frac{\varepsilon^*}{\varepsilon_*} \geq 1, \varepsilon^* = \max(\varepsilon,\bar{\varepsilon}), \varepsilon_* = \min(\varepsilon,\bar{\varepsilon})$. The above requires that $1-\frac{\gamma^2}{12\varepsilon^*} > 0$, that is, $\varepsilon^*\varepsilon_* = \varepsilon\bar{\varepsilon} > \frac{1}{12^2}$.

In fact, it turns out that we will be requiring much more stronger condition on $\varepsilon,\bar{\varepsilon}$:

$$\varepsilon_* > \frac{1}{6},\ \varepsilon^* > \max\left(\frac{\gamma^4}{6},\frac{\gamma^2}{12}\right) = \frac{\gamma^4}{6} > \frac{1}{6}, \tag{2.8.5}$$

$$\tau > \frac{5c}{12}. \tag{2.8.6}$$

This condition justifies the assumption that the first term is dominated by the vacuum. We compare this with the result in [100], where the Cardy formula is reported to be applicable for $\varepsilon_*\varepsilon^* = \varepsilon\bar{\varepsilon} > \frac{1}{24^2}$. This implies that there is further scope to improve our result and reach the

HKS threshold rigorously. If we assume existence of a twist gap g, we can push the regime of validity to following:

$$\varepsilon_* > \frac{1}{6}\left[\max\left\{\frac{1}{2},\left(1-\frac{6g}{c}\right)^2\right\}\right]. \tag{2.8.7}$$

We will come back to the derivation of the bound on $\varepsilon,\bar{\varepsilon}$ at the end of this section. For now, with this assumption of vacuum dominance, we find :

$$e^{-2\pi c\left(\sqrt{\frac{\varepsilon}{6}}\delta+\sqrt{\frac{\bar{\varepsilon}}{6}}\bar{\delta}\right)}\rho_*(\varepsilon,\bar{\varepsilon})c_- \leq \frac{1}{4\delta\bar{\delta}}\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}F(h,\bar{h}') \leq e^{2\pi c\left(\sqrt{\frac{\varepsilon}{6}}\delta+\sqrt{\frac{\bar{\varepsilon}}{6}}\bar{\delta}\right)}\rho_*(\varepsilon,\bar{\varepsilon})\tilde{c}_+. \tag{2.8.8}$$

As a consequence, we find that for fixed $\delta,\bar{\delta} > \delta_{gap}$,

$$S_{h,\bar{h}}(\delta,\bar{\delta}) = 2\pi\sqrt{\frac{c}{6}\left(h-\frac{c}{24}\right)}+2\pi\sqrt{\frac{c}{6}\left(\bar{h}-\frac{c}{24}\right)}-\log c+O(1),\ c\to\infty. \tag{2.8.9}$$

We can extend the above result to the case where $\delta,\bar{\delta}\sim c^{\alpha}$ where $0<\alpha<1$ by splitting the integral domain into squares of unit area:

$$\int_{h-\delta}^{h+\delta}\int_{\bar{h}-\bar{\delta}}^{\bar{h}+\bar{\delta}} \mathrm{d}h'\ \mathrm{d}\bar{h}'\rho(h',\bar{h}') = \sum_{m=1}^{2\delta}\sum_{n=1}^{2\bar{\delta}}\int_{h-\delta+m-1}^{h-\delta+m}\int_{\bar{h}-\bar{\delta}+n-1}^{\bar{h}-\bar{\delta}+n} \mathrm{d}h'\ \mathrm{d}\bar{h}'\rho(h',\bar{h}'), \tag{2.8.10}$$

and then, using the previous bound, we find

$$S_{h,\bar{h}}(\delta,\bar{\delta}) = 2\pi\sqrt{\frac{c}{6}\left(h+\delta-\frac{c}{24}\right)}+2\pi\sqrt{\frac{c}{6}\left(\bar{h}+\bar{\delta}-\frac{c}{24}\right)}-\log c+O(1),\ c\to\infty. \tag{2.8.11}$$

Now we show that the vacuum contribution dominants the contribution from the light sector. This is straightforward for the sector where $h_L < c/24$ and $\bar{h}_L < c/24$. For this region, we consider the sum

$$A=e^{\beta h+\bar{\beta}\bar{h}}\sum_{h_L,\bar{h}_L\leq c/24}\int \mathrm{d}h'\,\mathrm{d}\bar{h}'\rho_{h_L,\bar{h}_L}(h',\bar{h}')e^{-\beta h'-\bar{\beta}\bar{h}'}\Phi_+(h',\bar{h}')\,, \tag{2.8.12}$$

where the crossing kernel of the operator with conformal dimension $(h_L,\bar{h}_L)$ is given by

$$\begin{aligned}
&\rho_{h_L}(h)=2\pi\sqrt{\frac{\frac{c}{24}-h_L}{h-\frac{c}{24}}}I_1\left(4\pi\sqrt{(\frac{c}{24}-h_L)(h-\frac{c}{24})}\right)\theta\left(h-\frac{c}{24}\right)+\delta\left(h-\frac{c}{24}\right),\\
&\rho_{h_L,\bar{h}_L}(h,\bar{h})=\rho_{h_L}(h)\rho_{\bar{h}_L}(\bar{h}),
\end{aligned} \tag{2.8.13}$$

and the above reproduces the contribution of this operator in the dual channel i.e. at high temperature:

$$\int dhd\bar{h}\;\rho_{h_L,\bar{h}_L}(h,\bar{h})\;e^{-\beta h-\bar{\beta}\bar{h}}=e^{-\frac{4\pi^2}{\beta}(h_L-c/24)-\frac{4\pi^2}{\beta}(\bar{h}_L-c/24)}. \tag{2.8.14}$$

The asymptotic of the function $\rho_{h_L}(h)$ is given by

$$\rho_{h_L}(\varepsilon)\sim\frac{1}{2}\frac{1}{6^{1/4}}c^{-1/2}\varepsilon^{-3/4}\left(1-\frac{24h_L}{c}\right)^{1/4}e^{2\pi c\sqrt{\frac{\varepsilon}{6}\left(1-\frac{24h_L}{c}\right)}}. \tag{2.8.15}$$

Evaluating the each integral by saddle point approximation, we find

$$A=O\left(c^{-1}e^{2\pi c\sqrt{\frac{\varepsilon}{6}}+2\pi c\sqrt{\frac{\bar{\varepsilon}}{6}}}\sum_{h_L+\bar{h}_L\leq\Delta_H}e^{-4\pi\sqrt{6\varepsilon}h_L-4\pi\sqrt{6\bar{\varepsilon}}\bar{h}_L}\right). \tag{2.8.16}$$

Subsequently, using the following sparseness condition

$$\sum_{h_L+\bar{h}_L\leq\Delta_H}e^{-\beta h_L-\bar{\beta}\bar{h}_L}=O(1),\;\;\beta,\bar{\beta}>2\pi,\;\;c\to\infty, \tag{2.8.17}$$

we find that

$$A\sim O(\rho_*(h,\bar{h})). \tag{2.8.18}$$

where the condition $\beta,\bar{\beta}>2\pi$ translates to $\varepsilon,\bar{\varepsilon}>1/24$. Below we will see that we need to have a more stronger condition i.e $\varepsilon,\bar{\varepsilon}>1/6$.

We are left to investigate the region where either of the h_L or $\bar{h}_L$ is greater than $c/24$. This is basically given by the brown region in fig. 2.15. Without loss of generality, let us look at

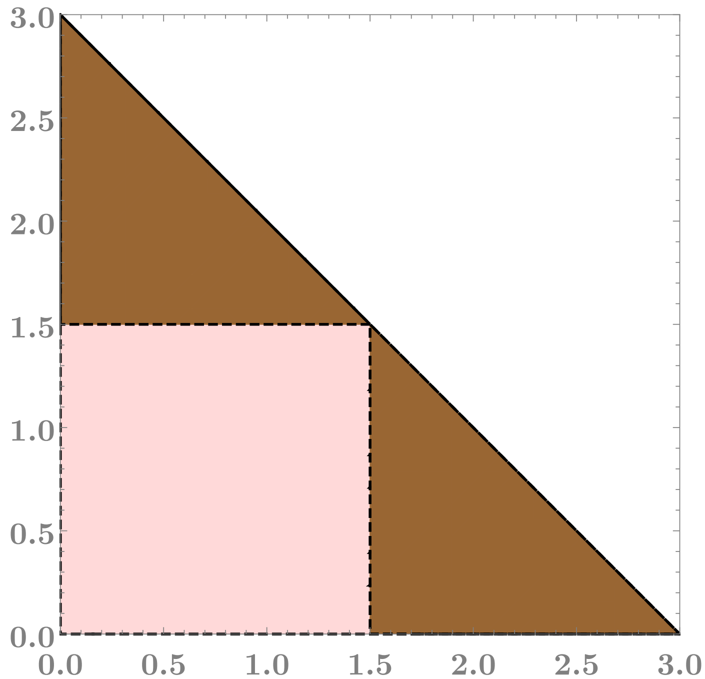


Figure 2.15. The light region is the union of the region colored with pink and the region colored with brown. The brown regions requires a more careful treatment.

the region where $h_L > c/24$. Now the crossing would be given by

$$\begin{aligned}
&\rho_{h_L}(h) = 2\pi\sqrt{-\left(\frac{h_L-\frac{c}{24}}{h-\frac{c}{24}}\right)}I_1\left(4\pi\sqrt{-\left(h_L-\frac{c}{24}\right)\left(h-\frac{c}{24}\right)}\right)\theta\left(h-\frac{c}{24}\right)+\delta\left(h-\frac{c}{24}\right), \\
&\rho_{h_L,\bar{h}_L}(h,\bar{h}) = \rho_{h_L}(h)\rho_{\bar{h}_L}(\bar{h}),
\end{aligned} \tag{2.8.19}$$

We are going to estimate the following:

$$\begin{aligned}
B &= e^{\beta h+\bar{\beta}\bar{h}}\int \mathrm{d}h'\,\mathrm{d}\bar{h}'\,\rho_{h_L,\bar{h}_L}e^{-\beta h'-\bar{\beta}\bar{h}'}\Phi_+(h',\bar{h}') \\
&= e^{\beta(h-c/24)+\bar{\beta}(\bar{h}-c/24)}\int \mathrm{d}h'\,\mathrm{d}\bar{h}'\,\rho_{h_L,\bar{h}_L}e^{-\beta(h'-c/24)-\bar{\beta}(\bar{h}'-c/24)}\Phi_+(h',\bar{h}')\,.
\end{aligned} \tag{2.8.20}$$

The integral over $\bar{h}'$ proceeds in usual manner since $\bar{h}_L < c/24$. The integral over $\bar{h}'$ proceeds in usual manner since $\bar{h}_L < c/24$. The integral over h' requires bit of care. Let us focus on

$$
\begin{aligned}
&\left|e^{\beta(h-c/24)}\int \mathrm{d}h'\rho_{h_L}(h')e^{-\beta(h'-c/24)}\Phi_+(h',\bar{h}')\right| \\
&=\left|e^{\beta(h-c/24)}\int_{c/24}^{\infty} \mathrm{d}h'\rho_{h_L}(h')e^{-\beta(h'-c/24)}\Phi_+(h',\bar{h}')\right| \\
&=e^{\beta(h-c/24)}\int_{0}^{\infty} dE'\left|\rho_{h_L}(h')e^{-\beta E'}\Phi_+(h',\bar{h}')\right| \\
&\leq Me^{\pi c\sqrt{\frac{\varepsilon}{6}}}\int dE'\frac{1}{\sqrt{E'}}e^{-\beta E'} \\
&\leq M\sqrt{\left(h_L-\frac{c}{24}\right)}\exp\left[\pi c\sqrt{\frac{\varepsilon}{6}}\right] \\
&\underset{c\to\infty}{\leq} M\sqrt{\frac{1}{c}}\exp\left[2\pi c\sqrt{\frac{\varepsilon}{6}}\right]\exp\left[-2\pi\sqrt{6\varepsilon}h_L\right]\,,
\end{aligned}
\tag{2.8.21}
$$

where in the second line we have used the Θ function present in the expression for ρ_{h_L}. In the penultimate line, we have used $h_L < c/12$. The strict inequality is important to make sure replacing $\sqrt{\left(h_L-\frac{c}{24}\right)}$ with $\frac{1}{\sqrt{c}}$ does not spoil the inequality because of the presence of exponential term. Now for the rest of the argument, we will require that

$$
\min\left(4\varepsilon,\bar{\varepsilon}\right)>\frac{1}{6}\ \&\ \min\left(\varepsilon,4\bar{\varepsilon}\right)>\frac{1}{6}\Rightarrow \varepsilon,\bar{\varepsilon}>\frac{1}{6}\,. \tag{2.8.22}
$$

The above leads to (2.8.5). Now we sum over all the light states in the brown region to obtain

$$
\begin{aligned}
B&=O\left(\frac{e^{2\pi c\sqrt{\frac{\varepsilon}{6}}+2\pi c\sqrt{\frac{\bar{\varepsilon}}{6}}}}{c}\left[\sum_{h_L>\bar{h}_L}e^{-2\pi\sqrt{6\varepsilon}h_L-4\pi\sqrt{6\bar{\varepsilon}}\bar{h}_L}+\sum_{h_L\leq\bar{h}_L}e^{-4\pi\sqrt{6\varepsilon}h_L-2\pi\sqrt{6\bar{\varepsilon}}\bar{h}_L}\right]\right) \\
&=O\left(\rho_*(h,\bar{h})\right)\,.
\end{aligned}
\tag{2.8.23}
$$

Twistgap:

If we assume a finite twist gap g, the (2.8.21) can be revisited in the light of the twist gap:

$$\begin{aligned}
&\left|e^{\beta(h-c/24)}\int \mathrm{d}h'\rho_{h_L}(h')e^{-\beta(h'-c/24)}\Phi_+(h',\bar{h}')\right| \\
&=\left|e^{\beta(h-c/24)}\int_{c/24}^{\infty} \mathrm{d}h'\rho_{h_L}(h')e^{-\beta(h'-c/24)}\Phi_+(h',\bar{h}')\right| \\
&=e^{\beta(h-c/24)}\int_0^{\infty} dE'\left|\rho_{h_L}(h')e^{-\beta E'}\Phi_+(h',\bar{h}')\right| \\
&\leq Me^{\pi c\sqrt{\frac{\varepsilon}{6}}}\int dE'\frac{1}{\sqrt{E'}}e^{-\beta E'} \\
&\leq M\sqrt{\left(h_L-\frac{c}{24}\right)}\exp\left[\pi c\sqrt{\frac{\varepsilon}{6}}\right] \\
&\underset{c\to\infty}{\leq} M\sqrt{\frac{1}{c}}\exp\left[2\pi c\sqrt{\frac{\varepsilon}{6}}\right]\exp\left[-2\left(1-\frac{6g}{c}\right)^{-1}\pi\sqrt{6\varepsilon}h_L\right],
\end{aligned} \tag{2.8.24}$$

where now in the penultimate step we use $h_L \leq c/12 - g/2$. And this leads to the modified validity regime as given in (2.8.7).

2.9 Open problems

We end with a list of open problems which would be nice to figure out:

1. One can hope to use these techniques to investigate the asymptotic OPE coefficients [130, 69, 78, 55, **?**, 40, 70, 38, 108, 172, 153] and make it spin sensitive. A richer structure in such scenarios is expected as well.

2. For the large spin, finite twist, we have derived an upper bound on the windowed entropy. It would be nice to put a lower bound as well and match up to the results of [134, 135, 55, 146, 28]. On a conservative note, we remark that even if one can prove a lower bound, it seems hard to distinguish the order one error coming from considering a bin of width 2δ in spin and the dependence of the extended Cardy formula on finite twist. At present, all our attempts to prove a lower bound provided us with a Cardy like growth multiplied by a

negative number, which is trivially true, hence we omitted the details of the trivial lower bound in the text. Furthermore, it would be nice to find out the asymptotic formula for the integrated version (integrated from spin 0 upto some large spin) of the density of states.

3. It would be nice to have an expression for integrated density of states upto some particular Δ within a specified range of spin. To be specific, we want to have an estimate of the following quantity:

$$\int_0^{\Delta} \mathrm{d}\Delta' \int_{J_1}^{J_2} \mathrm{d}J' \, \rho(\Delta',J') , \quad \text{where } J_1, J_2 \in \left[-\Delta', \Delta'\right] . \tag{2.9.1}$$

4. It would be nice to improve on the value of r and possibly prove that $r = 1$ (the parameter appearing in the asymptotic "areal" spectral gap) either by some suitable choice of magic functions or by better estimate of the heavy sector of the partition function. The naive generalization from [92] would not suffice. So one needs to be more creative. And this might shed light on the twist gap and provide a way to expound on the proposed gap in [28].

Our work should be thought of a part of modular bootstrap program[105, 84, 56, 49, 7, 4, 3, 40, 130, 69, 70, 38, 131, 15, 37, 5]. On a more general ground, it would be interesting to see whether Tauberian theorems and/or Modular bootstrap program can say anything about the chaotic, irrational CFTs. An approach borrowing ideas from Tauberian techniques and that of extremal functionals appearing in [147, 149, 148, 101, **?**, **?**, **?**] might be useful in this regard. Furthermore, for holographic CFTs, we can only achieve a reduced regime of validity of Cardy formula compared to what is reported in [100]. It might be possible to improve our result. Albeit, we remark that if the twist gap is greater than $c/12$, it is possible to achieve the regime of validity of Cardy formula as predicted in [100]. We hope to come back to these problems in future.

Chapter 2, in full, is a reprint of the material as it appears in Sridip Pal, Zhengdi Sun, JHEP **01**, 135 (2020). The dissertation author was one of the primary investigator and author of

this paper.

Chapter 3

High Energy Modular Bootstrap, Global Symmetries and Defects

3.1 Summary & Discussion

Symmetry plays a key role in studying Quantum Field theories (QFT). To study a QFT admitting a symmetry, we consider irreducible representations (irreps) of the group and declare that the quantum fields transform as irreps. A very natural and fundamental question is to ask whether there is any consistency condition telling us existence or absence of particular kind of irreps. These conditions can come about due to mathematical consistency and/or due to physical requirements like unitarity. Some of the famous examples in this genre are Coleman Mandula theorem [54], which roughly implies the impossibility of mixing space-time (Poincare) symmetry with internal symmetry unless one has supersymmetry; the unitarity bounds in $(3+1)$-D CFT by Mack [145], Weinberg-Witten theorem [186], which shows that impossiblity of having massless particles with higher spin in a theory with Lorentz covariant energy momentum tensor/conserved current.

In this work, we consider unitary modular invariant 2D conformal field theory. The consistency condition that we are going to leverage is modular transformation properties of Torus partition function with/without possible insertion of some operators. A standard result along this line is the existence of infinite number of Virasoro primaries for $c>1$ CFTs [42, 158]. Recently, it has been established that every integer spin has to appear in the bosonic CFT [156] by projecting the grand canonical partition function of 2D CFT onto a particular spin and studying the high temperature behavior of this fixed spin partition function. In a similar spirit, to study the different sectors of a 2D CFT with global symmetries (more generically with insertion of topological defect lines), we project the canonical partition function (with/without the insertion of topological defect lines) onto relevant sectors and study the high temperature behavior to extract the growth of operators within each sector. Further use of modular crossing equations can be found in [130, 40, 100, 70, 108, 172, 38, 28, 5, 57, 37, 158, 92, 165, 167, 156] and some aspects has been made symmetry sensitive in [69, 79, 17, 140].

One of the motivations for undertaking such investigation stems from a related question in holography. In the context of AdS-CFT, it is widely believed that all the irreps of internal gauge group appears in the gravity side a.k.a "completeness hypothesis"; on the CFT side, the gauge symmetry becomes a global symmetry and hence it implies the existence of all the irreps of the global symmetry modulo some fine prints[170, 19, 97, 99, 98]. To understand it better, consider the case of $U(1)$. If we know that an operator with minimal charge exists, we can create black holes of arbitrary charge by collapsing such minimal charged objects in arbitrary number. On the CFT side, this amounts to taking OPE and generating operators of arbitrary charge. One of the main challenges is to show that the such minimal charged object exists, i.e. $U(1)$ acts faithfully. Here we will not be saying anything about faithfulness. Rather given the faithfulness condition on the CFT side, we will pose the following question of whether one can generate operators of arbitrary charge with arbitrarily high dimension in the way mentioned above. On the gravity side this amounts to having black holes with arbitrary charge. By the OPE argument, one can generate primaries of arbitrary charge and one needs to consider heavy descendants to answer positively to the above question. Hence, a more refined and nontrivial question is to ask whether we can say anything about heavy primaries with arbitrary charge and if possible, whether we can estimate the growth of each irreps. It turns out that in a 2D CFT, one can investigate this leveraging the modular invariance.

The recent study of partition function of 2-D JT gravity [173] with bulk gauge field

[124, 112] motivates us as well. It is well appreciated that the genus zero contribution to the partition function can be obtained by looking at dual quantum mechanical system, which is known to be the Schwarzian limit of a 2-D CFT[152, 95]. Now considering a bulk gauge field amounts to having a CFT with a global symmetry and then taking the Schwarzian limit. One curious feature present in the calculation of [124] is the square of dimension of irrep in the expression for density of states corresponding to the genus zero partition function of JT gravity with bulk gauge field. Here we take up a CFT calculation to precisely reproduce this curious factor.

Given a continuous global symmetry, we can turn on fugacity corresponding to the conserved current and consider the grand canonical partition function. This idea can be generalized to discrete symmetries by thinking of inserting topological defect lines (TDL) while doing the path integral over the relevant manifold to define the grand canonical partition function. In fact, one can allow non invertible TDLs (which does not correspond any global symmetry in conventional sense, nonetheless meaningful object, see section 1 of [141]) and define grand canonical partition functions. In this work, the relevant manifold is square torus, i.e. we consider 2D CFT on a spatial circle of length 2π, at inverse temperature β. If the topological defect line is inserted along the spatial circle, it is exactly the grand canonical partition function. If the topological line is inserted along the temporal circle, it creates a defect in the spatial manifold, thereby defines a "defect" Hilbert space of operators. The partition function constructed out of operators in the defect Hilbert space is related to the grand canonical partition function by a S modular transformation. Roughly speaking, a S modular transformation exchanges the spatial and temporal circle, thereby changes the role of TDLs. Given this set up, we ask following questions:

- Can we estimate the growth of operators in the defect Hilbert space ? The spectrum of operators in the defect Hilbert space is not same as the original Hilbert space. On the other hand, one might think that introducing a defect only modifies the theory globally by modifying the boundary conditions of the field, thus one should not expect any change in asymptotic growth of operators compared to the original Hilbert space. We will confirm this intuition in part by doing a rigorous calculation in this work. Even though, the spectrum changes, the averaged behavior remains same (apart from a possible multiplicative factor, which we explain below in the results) even in the presence of defects.

- Given a 2D CFT with a global symmetry (finite group), do all the irreps of the global symmetry group appear in the spectra of local operators? The answer turns out to be yes.
- If the symmetry group is non-anomalous, it is possible to group the operators appearing in the defect Hilbert space into irreps of the group and we ask whether all the irreps of the global symmetry group appear in the defect spectra. Here also the answer turns out to be yes.

The basic strategy that we follow to answer these questions is to consider a partition function of the sector of the CFT which we want to study and then to look at its high temperature behavior. The relevant sector specific partition function can be obtained by using appropriate projection operators onto the partition function in appropriate channel. The precise way of doing this is explained in details in the paper. Below we summarize our results and discuss the implications.

Results:

1. We consider a CFT on a torus with the topological defect line (TDL) being inserted along the temporal direction. We estimate the growth of operators in the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ as $\Delta \to \infty$:

$$\text{growth of operators in } \mathscr{H}_{\mathscr{L}} \simeq N_0 \rho_0(\Delta) \tag{3.1.1}$$

where

$$\rho_0(\Delta) = \left(\frac{c}{48\Delta^3}\right)^{\frac{1}{4}} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right]. \tag{3.1.2}$$

A rigorous statement is made in theorem [1] and in the eq. (3.3.2). The TDL can either correspond to a Global symmetry or correspond to a non-invertible defect such as duality defect. Here N_0 is "quantum dimension", obtained from the action of TDL on the $\Delta = 0$ state. For TDL corresponding to global symmetry the vacuum remains invariant and $N_0 = 1$, for TDL corresponding to duality defects, N_0 may not be 1.

2. We consider a CFT with a finite global symmetry group (acting faithfully). We find that every irreducible representation has to appear in the spectrum of operators in the untwisted sector and they have a Cardy like growth as $\Delta \to \infty$. In particular, we have

$$\begin{aligned} &\text{growth of occurence of particular irrep } \alpha \simeq d_\alpha |G|^{-1} \rho_0(\Delta)\,, \\ &\text{growth of states in an irrep } \alpha \simeq d_\alpha^2 |G|^{-1} \rho_0(\Delta)\,. \end{aligned} \tag{3.1.3}$$

where $\rho_0(\Delta)$ is defined in eq. (3.1.2). Here $|G|$ is the order of the group and d_α is the dimension of the representation of irrep α. We remark that if we sum over all the irreps, we get back the usual Cardy like growth for all the operators, i.e. $\rho_0(\Delta)$. A more rigourous statement is made in theorem [2] and in eq. (3.4.10). If the symmetry is non-anomalous, the result is true for any particular twisted sector. The rigorous statement can be found in theorem [3]. To illustrate, in the example of $\mathbb{Z}_2$, α can be even or odd, $d_\alpha = 1$ and $|G| = 2$.

★ A unified version of the above two results is presented in theorem [4] and in eq. (3.5.6).

★ *Schwarzian sector-JT gravity*: 2-D CFT is known to have a schwarzian sector [95], which is relevant for the study of JT gravity. The partition function corresponding to the disk topology [173] corresponds to the identity character in some particular limit, as explained in [152, 95]. Having a global symmetry on the CFT side induces a bulk gauge field on the gravity side. In the set up [124], the bulk gauge theory is taken to be topological BF theory. The corresponding partition function has been calculated in [124, 112], the density of states has been shown to have a $d_\alpha^2 |G|^{-1}$ factor multiplied with the seed gravity answer without the gauge field. Our result precisely reproduces this factor, since we can readily take the Schwarzian limit of our answer following [152, 95].

★ The factor d_α^2 as opposed to d_α might be surprising because we expect the extra d_α-fold degeneracy due to the symmetry. Intuitively, this comes about due to smearing[1]. The growth formula is valid only after smearing over an order one window, and it turns out that the order one window has d_α number of α irreps. This might hint at emergence of some approximate symmetry. This is exactly similar to the scenario in [124] where they speculate about emergence of extra approximate symmetry. We discuss this after (3.4.6) as one of the remarks and we explicitly look at 3-state Potts model ($c = 4/5$) to back up our claim.

★ The rigorous bounds in theorem [1] and theorem [2] have order one error. Without any further input, that's the best order of error that one can achieve. To optimize over the order one error, we need to use Selberg-Beurling extremizers as elucidated in [156] .

3. All of the above estimates can be made in the limit $c \to \infty$ and $\Delta = c\left(\frac{1}{12} + \varepsilon\right)$ for $\varepsilon > \frac{1}{12}$ following [100, 158]. We use $\Delta - \frac{c}{12}$ instead of Δ everywhere in the above formulas.

4. All of the above estimates can be made sensitive to Virasoro primaries for $c > 1$ following [158]. Instead of $\rho_0(\Delta)$ we will have

$$\rho_0^{\text{Vir}}(\Delta) = \left(\frac{c-1}{3}\Delta\right)^{\frac{1}{4}} \exp\left[2\pi\sqrt{\frac{(c-1)\Delta}{3}}\right].$$

5. We find the analogous result for continuous group $U(1)$ (acting faithfully). Under a technical assumption, we show that every charged state has to appear in the spectrum and they do have a Cardy like growth at large Δ given by $\sqrt{\frac{c}{3k}}\frac{1}{\Delta}\exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right]$. The rigorous statement can be found in eq. (3.4.24). Again one can generalize this to Virasoro primaries for $c > 1$ in one hand and on the other hand to the large central charge regime.

[1] We thank Raghu Mahajan for discussion along this line.

• **Application and future avenues:**

We have already mentioned one application of our result upon taking the Schwarzian limit and making contact with the results of JT gravity with a bulk gauge field. Here we list out few more applications. For example, we can consider the following table 3.1. A similar one appears in [109]. Same is explored in the context of $\mathbb{Z}_2$ symmetry of Monster CFT in [141]. We consider a theory A with a non-anomalous $\mathbb{Z}_2$. The untwisted sectors can be divided into two pieces: even and odd, named as P and Q. This is obtained when the TDL corresponding to the $\mathbb{Z}_2$ symmetry is extended along the spatial direction. The twisted sector is obtained by keeping the TDL along the time direction, thereby creating a defect. Since, $\mathbb{Z}_2$ is non-anomalous, one can have even and odd states in the twisted sector as well, we call them R and S respectively. Gauging this $\mathbb{Z}_2$ symmetry lands us onto the theory D. Both the theory A and D can be fermionized to theory F and $\tilde{F}$. The effect of this amounts to permuting and relabelling the different sectors P,Q,R,S. Using our result, we can estimate the growth of operators for each of the sector P,Q,R,S. All of them have a Cardy like growth given by $\frac{1}{2}\rho_0(\Delta)$ (corresponding to $d_\alpha = 1$ and $|G| = 2$) for large Δ.

A	untwisted	twisted
even	P	R
odd	Q	S

D	untwisted	twisted
even	P	Q
odd	R	S

F	NS	R
bosonic	P	R
fermionic	S	Q

$\tilde{F}$	NS	R
bosonic	P	Q
fermionic	S	R

Figure 3.1. The theory A and D are related by orbifolding by $\mathbb{Z}_2$. The theory A and F are related by Bosonization-Fermionization and so are D and $\tilde{F}$.

One can think of further applications of these ideas generalizing the results appearing in [130, 70, 108, 172, 38]. Moreover, one can also make all of the above results spin-sensitive following [156]. It would also be interesting to explore other aspects of modular bootstrap for example bounding the dimension of lowest nontrivial Virasoro primary, constructing the extremal functionals [105, 56, 49, 16, 148, 149, 4, 101, 15] in presence of TDLs.

As a technique, we generalize the application of Tauberian formalism in context of CFT beyond S modular invariant partition functions. In particular, the method can be applied to vector valued modular functions as elucidated in § 3.5. One immediate application would be generalizing the results of [70] for $LL'H$-squared for two different operators using the Tauberian technique. Note that the positivity is guaranteed in one of the channels while in other channel, it is not there. This scenario is reminiscent of the partition function of the defect Hilbert space, where positivity is guaranteed but in the S transformed channel, positivity is not guaranteed.

- **Organization**

The paper is organized in following manner. The § 3.2 reviews the idea of TDLs as generalization of global symmetry. A nice and brief exposition can also be found in the introduction of [141]. In § 3.3, we study the defect Hilbert space. In § 3.4 we study the growth of operators within an irrep. The § 3.4.1 expounds on a simple example of $\mathbb{Z}_2$ symmetry, which we generalize and make rigorous in § 3.4.2. The similar question relevant to $U(1)$ symmetry is analyzed in § 3.4.3. The § 3.5 encapsulates the gist of applying the Tauberian technique to the vector valued modular functions. In § A.1, we provide some numerics on known models to cross-check our results. In § A.2, we review the derivation of spin selection rule for anomalous global symmetry.

3.2 Lightning review of Topological defect line

Given any continuous global symmetry, one can define Noether's current j_μ and the charge Q is given by $Q = \int \mathrm{d}^{d-1}x\, j^0$, an integral of j_μ over a codimension one surface, here the surface is given by $x^0 = \text{constant}$. In general, one can define an operator, supported on any codim-1 surface Σ and given by $\exp(\imath\theta \int_\Sigma \star j)$. The statement that the charge conservation, $\mathrm{d}\star j = 0$ boils down to the statement that the operator is invariant under continuous small deformation Σ.

We also note that here the charge Q is a scalar, we name it 0 form symmetry. Now instead of codimension 1 surface, one can in general consider topological surface operator of codim-$(q+1)$ and define q-form global symmetries [90]. For a 0 form symmetry, when the surface Σ is chosen to be the full spatial slice, this operator is exactly the symmetry operator acting on the Hilbert space; while if one of the direction of Σ is the time direction, then this operator creates a codim-1 defect in the space–any local operator undergoes a twisting when crossing the defect. For this reason, topological surface operators are sometimes called the topological defects.

In 2-dim, ordinary 0-form symmetries correspond to topological defects lines (TDL). A natural question to ask if whether converse is true. The answer is generically no for the following reason. The fusion of the TDLs associated with global symmetries must respect the group multiplication. Therefore, for any TDL corresponding to an group, there must exist an inverse TDL; in fact, the inverse line can be obtained by simply reversing the orientation of the line. However, there do exist the so-called non-invertible line operators which don't have an inverse, (e.g. the duality line N in the Ising CFT or Monster CFT [140, 141]).

As in the general dimension space-time, we can place the TDL $\mathscr{L}$ along the time direction on $\mathbb{R}_t \times S_1$, which amounts to imposing the twisted boundary condition on S_1. The resulting Hilbert space is called the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ whose states can be labelled by the usual weights $(h,\bar{h})$. This is possible because the energy momentum tensor commutes with TDL. Via state-operator correspondence, a state in $\mathscr{H}$ corresponds to an operator, sitting at the end of the $\mathscr{L}$. A particular important question for our analysis is whether there's a state with conformal weight $(0,0)$ in the defect Hilbert space. As in [140], if we require that the global symmetry acts faithfully on the Hilbert space of local operators, that is, the only line operator that commutes with every local operators is the identity line, then the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ contains no weight-$(0,0)$ state. Otherwise, the existence of such state would allow line $\mathscr{L}$ to connect to the identity line via the corresponding operator, thus it would commute with every local operator, violating our requirement (see fig. 3.2). As we will see, this makes sure the leading result in our analysis is universal in the sense that it only depends on the central charge c and the symmetry group G. We also remark here if the symmetry is anomalous (if one can not define action of the symmetry in the defect Hilbert space consistently), then the ground state in the defect Hilbert space has $\Delta > 0$. This follows from the spin selection rule [140]. We review it in the appendix §A.2.

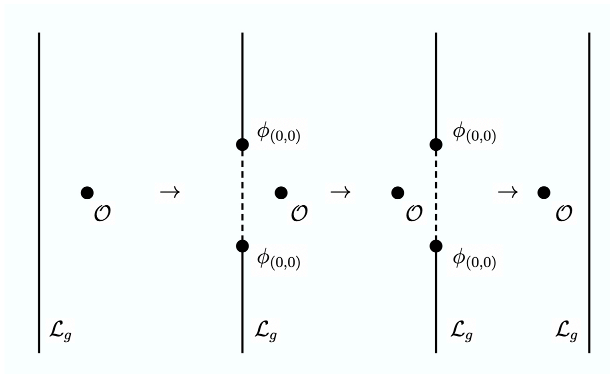


Figure 3.2. $\mathscr{L}_g$ denotes the g symmetry TDL; dashed line denotes the trivial line; $\mathscr{O}$ is arbitrary local operator; and $\phi_{(0,0)}$ is the operator correspond to the weight $(0,0)$ state in $\mathscr{H}_{\mathscr{L}_g}$. The existence of $\phi_{(0,0)}$ allows us to open the TDL to show that the $\mathscr{L}_g$ commutes with every local operator $\mathscr{O}$.

On the other hand, if we place the TDL along the spatial direction, then it acts as a group element on the Hilbert space of local operators. Instead of $\mathbb{R}_t \times S_1$, one can consider $S_1 \times S_1$ and generalize the above story. Since the modular transformation exchanges two cycles of $S_1 \times S_1$, the configuration of TDL along the spatial circle must be related to the configuration of TDL along the temporal circle. This brings us to the key property of the partition function of defect Hilbert space, that is, it is related to the partition function with the insertion of the corresponding charge operator (see fig. 3.3) along the spatial cycle. To be concrete, we define

$$\begin{aligned} Z_{\mathscr{L}}(\beta, g) &:= \mathrm{Tr}_{\mathscr{H}_{\mathscr{L}_g}}(q^{L_0 - c/24}\overline{q}^{\overline{L} - c/24}), \\ Z^{\mathscr{L}}(\beta, g) &:= \mathrm{Tr}_{\mathscr{H}}(\hat{g}\; q^{L_0 - c/24}\overline{q}^{\overline{L} - c/24}), \end{aligned} \tag{3.2.1}$$

and modular transformation tells us that

$$Z_{\mathscr{L}}(\beta,g)=Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta},g\right). \tag{3.2.2}$$

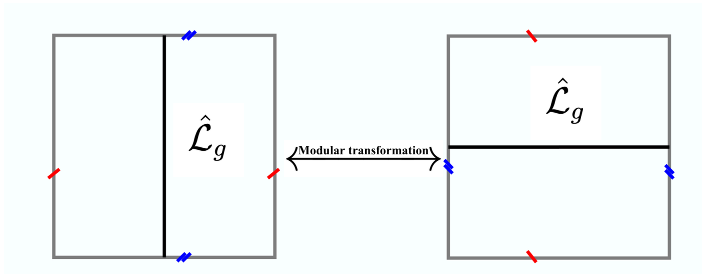


Figure 3.3. The partition function of the defect Hilbert space (the left figure, which we will denote $Z_{\mathscr{L}}(\beta,g)$) is related to the partition function with the insertion of the corresponding charge operator (the right figure, which we will denote $Z^{\mathscr{L}}(\beta,g)$).

We end this section by making a crucial remark that the low temperature expansion coefficient of $Z_{\mathscr{L}}(\beta,g)$ is positive, hence falls under the purview of Tauberian formalism whereas in the dual channel, positivity is not guaranteed. One needs to keep this in mind while expecting whether a Cardy like statement is true or not. For example, whereas we can hope to prove the asymptotic growth of low temperature expansion coefficient of $Z_{\mathscr{L}}(\beta,g)$, the same is not true for $Z^{\mathscr{L}}(\beta,g)$ without any further assumption because the positivity is not guaranteed in this channel.

3.3 Charting Defect Hilbert Space $\mathscr{H}_{\mathscr{L}}$ associated with TDL $\mathscr{L}$

In the usual Cardy formula, the asymptotic growth of operators is controlled by the low temperature limit of the partition function in the dual (S transformed) channel. As explained in the previous section § 3.2, the dual channel corresponding to the partition function of a defect Hilbert space ($Z_{\mathscr{L}}(\beta,g)$) is the partition function evaluated on the original Hilbert space with an insertion of group element g, which we call $Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta},g\right)$. The leading behavior (low temperature) in the later channel is controlled by the vacuum operator. Thus one can expect a

Cardy like growth for operators in the original channel i.e in the defect Hilbert space.

$$Z_{\mathscr{L}}(\beta \to 0, g) = Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta} \to 0, g\right) \simeq e^{\frac{\pi^2 c}{3\beta}} \tag{3.3.1}$$

and hence we expect the growth of the operators in the defect Hilbert space is given by inverse Laplace of $e^{\frac{\pi^2 c}{3\beta}}$, which is $\rho_0(\Delta)$. In what follows, we will be making this idea rigorous using Tauberian techniques.

3.3.1 Cardy Formula for Defect Hilbert Space

Theorem 1. *Given a TDL $\mathscr{L}$, the asymptotic behavior ($\Delta \to \infty$) of the growth of states in an order one window of width 2δ, centered at Δ in the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ is given by*

$$c_- N_0 \rho_0(\Delta) \leq \frac{1}{2\delta} \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \; \rho_{\mathscr{H}_{\mathscr{L}}}(\Delta') \leq c_+ N_0 \rho_0(\Delta) \tag{3.3.2}$$

where $N_0 = 1$ if the TDL is associated with a global symmetry, i.e. invertible one, otherwise it is taken to be a positive number as defined below and $\rho_0(\Delta)$ is defined in eq. (3.1.2). *Here $c_\pm$ order one positive numbers. These numbers can be determined using the extremal functionals appearing in [156]. In particular, we have $c_\pm = 1 \pm 1/2\delta$. The above statement is true under the following technical assumptions:*

- *The action of $\mathscr{L}$ on the states are uniformly bounded, i.e. $|\langle \Delta|\mathscr{L}|\Delta\rangle| \leq N$ for all Δ in the physical spectra. For example, if we consider $\mathbb{Z}_n$ then, $|\langle \Delta|\mathscr{L}|\Delta\rangle| \leq 1$, since the matrix element is always a phase. In fact, this is true for any TDL associated with a finite group. For non invertible TDLs, i.e. the ones which are not associated with global symmetry, we take this as an assumption, which is true for a wide class of non invertible TDLs.*

- *The vacuum is invariant under any topological defect line associated with global symmetry. Thus we have*

$$\mathscr{L}|0\rangle = |0\rangle, \tag{3.3.3}$$

- *The action of a non-invertible topological defect line $\mathscr{L}$ (such as duality defects, not associated with any global symmetry) on the vacuum state is given by:*

$$\mathscr{L}|0\rangle = N_0|0\rangle, \;\; N_0 > 0. \tag{3.3.4}$$

For example, in Ising model, we have duality defect line $\widehat{N}$ and $\widehat{N}|0\rangle = \sqrt{2}|0\rangle$.

The basic structure of the proof is similar to the one appeared in [158, 167, 156], though the deatils are different as we will see. This comment applies to theorems proven in subsequent sections as well. We start by considering two functions $\phi_\pm(\Delta)$ whose Fourier transformation has finite support $[-\Lambda,\Lambda]$ and they majorise and minorise the characteristic function for the interval $[\Delta-\delta,\Delta+\delta]$:

$$\phi_-(\Delta') \leq \theta_{[\Delta-\delta,\Delta+\delta]}(\Delta') \leq \phi_+(\Delta')\,. \tag{3.3.5}$$

From the above it follows that

$$e^{\beta(\Delta-\delta)}e^{-\beta\Delta'}\phi_-(\Delta') \leq \theta_{[\Delta-\delta,\Delta+\delta]}(\Delta') \leq e^{\beta(\Delta+\delta)}e^{-\beta\Delta'}\phi_+(\Delta')\,. \tag{3.3.6}$$

Multiplying both sides by the density of states of the twisted Hilbert space $\rho_{\mathscr{H}_{\mathscr{L}}}$ and integrating from 0 to ∞, we find

$$e^{\beta(\Delta-\delta)}\int_0^\infty dF(\Delta')e^{-\beta\Delta'}\phi_-(\Delta') \leq \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') \leq e^{\beta(\Delta+\delta)}\int_0^\infty dF(\Delta')e^{-\beta\Delta'}\phi_+(\Delta')\,, \tag{3.3.7}$$

where $dF(\Delta') = \rho_{\mathscr{H}_{\mathscr{L}}}(\Delta')d\Delta'$. We emphasize β,δ are free parameters. We consider the Fourier transformation of $\phi_\pm(\Delta) = \int_{-\infty}^{\infty}\mathrm{d}t\;\hat{\phi}_\pm(t)^{-\imath\Delta t}$, such that in Fourier domain the above inequality becomes

$$\begin{aligned}
&e^{\beta(\Delta-\delta)}\int_{-\infty}^{\infty} dt\,\hat{\phi}_-(t) Z_{\mathscr{L}}(\beta+\imath t) e^{-(\beta+\imath t)c/12} \\
&\leq \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') \leq \\
&e^{\beta(\Delta+\delta)}\int_{-\infty}^{\infty} dt\,\hat{\phi}_+(t) Z_{\mathscr{L}}(\beta+\imath t) e^{-(\beta+\imath t)c/12}.
\end{aligned} \tag{3.3.8}$$

The modular property implies

$$Z_{\mathscr{L}}(\beta+\imath t) = Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right). \tag{3.3.9}$$

Thus in the dual channel we have an expression in terms of the original Hilbert space. We split this original Hilbert space $\mathscr{H}$ into light part and heavy part:

$$Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) = Z_L^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) + Z_H^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right). \tag{3.3.10}$$

Notice that the contribution from the light sector $Z_L^{\mathscr{L}}$ is not necessary real if it contains operators arbitrarily charged under global symmetry group G. For example, if we consider the $\mathbb{Z}_3$ symmetry, then the TDL $\mathscr{L}$ can act on a state such the state picks up a phase of $e^{2\pi\imath/3}$. One can circumnavigate this by assuming charge conjugation invariance.

$$\left(\int_{-\infty}^{\infty} dt\,\hat{\phi}_\pm(t) Z_{\mathscr{L}}(\beta+\imath t) e^{-(\beta+\imath t)c/12}\right) = \left(\int_{-\infty}^{\infty} dt\,\hat{\phi}_\pm(t) Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) e^{-(\beta+\imath t)c/12}\right) \in \mathbb{R} \tag{3.3.11}$$

Then we can split it as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}t\,\hat{\phi}_\pm(t) Z^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) e^{-(\beta+\imath t)c/12} \\
&= \int_{-\infty}^{\infty} \mathrm{d}t\,\hat{\phi}_\pm(t) Z_L^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) e^{-(\beta+\imath t)c/12} + \int_{-\infty}^{\infty} dt\,\hat{\phi}_\pm(t) Z_H^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) e^{-(\beta+\imath t)c/12}.
\end{aligned} \tag{3.3.12}$$

At first, we consider the light sector where $\Delta \le c/12$ choose a $\rho_0^{\mathscr{L}}(\Delta)$ such that

$$\int_0^\infty \mathrm{d}\Delta\, \rho_0^{\mathscr{L}}(\Delta) e^{-\beta(\Delta - c/12)} = Z_L^{\mathscr{L}}\left(\frac{4\pi^2}{\beta}\right). \tag{3.3.13}$$

As a result, the contribution from the light sector can be written as

$$e^{\beta(\Delta\pm\delta)} \int_{-\infty}^{\infty} \mathrm{d}t\ \hat{\phi}_\pm(t) Z_L^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) e^{-(\beta+\imath t)c/12} = e^{\beta(\Delta\pm\delta)} \int_0^\infty \mathrm{d}\Delta'\ \rho_0^{\mathscr{L}}(\Delta')\phi_\pm(\Delta')e^{-\beta\Delta'}. \tag{3.3.14}$$

Notice that, in general, the light sector contains all the states with $\Delta \le c/12$. $\rho_0^{\mathscr{L}}(\Delta)$ contains more than just contribution from the vacuum state $N_0\rho_0(\Delta)$ where $\rho_0(\Delta)$ is the crossing kernel of the vacuum state. The extra light operators would give exponentially suppressed corrections and are not universal (model dependent). Since there are finite number of operators below $c/12$, so that sum of the contribution coming from each of the extra light operators is still suppressed. In the following, we shall only consider the vacuum contribution.

Next, we treat contribution from the heavy sector and show they are suppressed in magnitude, hence can be dropped from both the lower and the upper bound.

$$e^{\beta(\Delta\pm\delta)} \left| \int_{-\infty}^{\infty} dt\, \hat{\phi}_\pm(t) Z_H^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right) e^{-(\beta+\imath t)c/12} \right| \le e^{\beta(\Delta - c/12 \pm\delta)} \int_{-\infty}^{\infty} dt \left| Z_H^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right)\right| |\hat{\phi}_\pm(t)| \tag{3.3.15}$$

Now we do the following estimation

$$\begin{aligned}
\left| Z_H^{\mathscr{L}}\left(\frac{4\pi^2}{\beta+\imath t}\right)\right| &= \left| \sum_{\Delta > c/12} N_\Delta \exp\left[-\frac{4\pi^2}{\beta+\imath t}\left(\Delta - \frac{c}{12}\right)\right]\right| \\
&\le N \sum_{\Delta > c/12} \exp\left[-\frac{4\pi^2\beta}{\beta^2+t^2}\left(\Delta - \frac{c}{12}\right)\right] \\
&= N Z_H\left[\frac{4\pi^2\beta}{\beta^2+t^2}\right] \le N Z_H\left[\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\right] \text{ for } t^2 \le \Lambda_\pm^2
\end{aligned} \tag{3.3.16}$$

where N_Δ denote the action of $\mathscr{L}$ on a state with conformal dimension Δ and N denote the upper bound of all N_Δ's. We use this bound in (3.3.15) and the fact that $\hat{\phi}_\pm(t)$ has finite support $[-\Lambda_\pm,\Lambda_\pm]$ to have the following inequality

$$
\begin{aligned}
&e^{\beta(\Delta-\delta)}\left[\int_0^\infty d\Delta'\,\rho_0^{\mathscr{L}}(\Delta')e^{-\beta\Delta'}\phi_-(\Delta')-Ne^{-\beta c/12}Z_H\left(\frac{4\pi^2\beta}{\beta^2+\Lambda_-^2}\right)\int_{-\Lambda_-}^{\Lambda_-}dt\,|\hat{\phi}_-(t)|\right]\\
&\leq\int_{\Delta-\delta}^{\Delta+\delta}dF(\Delta')\leq\\
&e^{\beta(\Delta+\delta)}\left[\int_0^\infty d\Delta'\,\rho_0^{\mathscr{L}}(\Delta')e^{-\beta\Delta'}\phi_+(\Delta')+Ne^{-\beta c/12}Z_H\left(\frac{4\pi^2\beta}{\beta^2+\Lambda_+^2}\right)\int_{-\Lambda_+}^{\Lambda_+}dt\,|\hat{\phi}_+(t)|\right].
\end{aligned}
\tag{3.3.17}
$$

The bounds get greatly simplified once we consider the large Δ region. Indeed, as in [158] using HKS bound, one can show

$$
e^{\beta\Delta}Z_H\left(\frac{4\pi^2\beta}{\beta^2+\Lambda_\pm^2}\right)\sim e^{\beta\Delta}e^{\frac{\pi^2c}{3\beta}\left(\frac{\Lambda_\pm}{2\pi}\right)^2}\sim\rho_0^{\mathscr{L}}(\Delta)^{1+\frac{1}{2}\left(\left(\frac{\Lambda_\pm^2}{2\pi}\right)-1\right)},
\tag{3.3.18}
$$

where we choose $\beta=\pi\sqrt{\frac{c}{3\Delta}}<<1$. Therefore the contribution from Z_H is sub-leading once we choose $\Lambda_\pm<2\pi$. Then the upper bound at large Δ (the lower bound is similar $\phi_+\to\phi_-$) simplifies to

$$
\int_{\Delta-\delta}^{\Delta+\delta}\mathrm{d}\Delta'\,\rho_{\mathscr{H}_{\mathscr{L}}}(\Delta')\leq e^{\beta\Delta}\int_0^\infty d\Delta'\,N_0\rho_0(\Delta')\phi_+(\Delta')e^{-\beta\Delta'}.
\tag{3.3.19}
$$

Upon doing integrals by the saddle point approximation, we have in the $\Delta\to\infty$ limit

$$
N_0c_-\rho_0(\Delta)\leq\frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta}dF(\Delta')\leq N_0c_+\rho_0(\Delta),\quad\text{where } c_\pm=\frac{\pi}{\delta}\hat{\phi}_\pm(0).
\tag{3.3.20}
$$

This concludes the proof of the theorem. For $c>1$ CFTs, the analysis can be made sensitive to primaries only. We end this subsection with two remarks.

- As in [92, 156], we can derive a spectral gap for the defect Hilbert $\mathscr{H}_{\mathscr{L}}$. The upper bound on the gap is found to be 1. This is the optimal gap as one can consider the Monster CFT with insertion of Identity line; now the defect Hilbert space is same as the original Hilbert space, as a result the gap is exactly 1. For further discussion related to optimality, we refer the readers to [156].

- Following [158], we can also derive a global approximation of the number of states $F_{\mathscr{L}}(\Delta)$ in the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ valid for large Δ:

$$F_{\mathscr{L}}(\Delta) \equiv \int_0^{\Delta} \mathrm{d}\Delta' \rho_{\mathscr{L}}(\Delta') = \frac{N_0}{2\pi}\left(\frac{3}{c\Delta}\right)^{1/4} e^{2\pi\sqrt{\frac{c}{3}\Delta}}\left[1+O(\Delta^{-1/2})\right],\ \Delta \to \infty. \tag{3.3.21}$$

3.4 Charting Hilbert Space $\mathscr{H}^{\mathscr{L}}$ associated with invertible TDL $\mathscr{L}$

In this section we consider invertible TDLs associated with a global symmetry G. In particular, we will be focussing on the case where the symmetry group is finite. The primary goal is to focus on the untwisted sector (we are imposing periodic boundary condition along the spatial circle) estimate the growth of operators which transforms under a particular irreducible representation of the group G. Later on we will generalize our result to a given twisted sector, where another TDL is inserted along the temporal direction if the symmetry is non-anomalous.

3.4.1 Warm up: $G = \mathbb{Z}_2$

The symmetry group $\mathbb{Z}_2$ has two elements: identity e and the element p, which squares to Identity. We set up the following notation for any group element $g \in G$:

$$Z^{\mathscr{L}}(\beta, g) = \mathrm{Tr}\left(g e^{-\beta\left(L_0+\bar{L}_0-\frac{c}{12}\right)}\right). \tag{3.4.1}$$

Thus for $g=e$ we have the usual partition function while for $g=p$ we have

$$Z^{\mathscr{L}}(\beta, p) = Z_{\mathrm{even}}(\beta) - Z_{\mathrm{odd}}(\beta), \tag{3.4.2}$$

where Z_{even} (Z_{odd}) is the partition function for all the even (odd) operators. Clearly, $Z_{\mathrm{even}}(\beta)+$

$Z_{\mathrm{odd}}(\beta) = Z^{\mathscr{L}}(\beta, e)$. In the usual Cardy formula, we want to have an estimate of partition function at high temperature. Similarly, we want to have an expression for $Z_{\mathrm{even}}(\beta)$ and $Z_{\mathrm{odd}}(\beta)$ in the $\beta \to 0$ limit.

We have

$$\begin{aligned} Z_{\mathrm{even}}(\beta) &= \frac{1}{2}\Big(Z^{\mathscr{L}}(\beta,e) + Z^{\mathscr{L}}(\beta,p)\Big) = \frac{\dim(even)}{|G|}\sum_g \chi^*_{\mathrm{even}}(g) Z^{\mathscr{L}}(\beta,g)\,, \\ Z_{\mathrm{odd}}(\beta) &= \frac{1}{2}\Big(Z^{\mathscr{L}}(\beta,e) - Z^{\mathscr{L}}(\beta,p)\Big) = \frac{\dim(odd)}{|G|}\sum_g \chi^*_{\mathrm{odd}}(g) Z^{\mathscr{L}}(\beta,g)\,. \end{aligned} \tag{3.4.3}$$

We remark that $\frac{1}{|G|}\sum_g \chi^*_\alpha(g) Z^{\mathscr{L}}(\beta,g)$ calculates the number (weighted by $e^{-\beta(\Delta - c/12)}$, where Δ is the conformal weight) of times the irrep α is appearing, and the number of states is obtained by multiplying the dimension of irrep to the quantity. We briefly review the representation theory of finite group in § A.3. For any Abelian group, the dimension of irrep is 1 always, so it is simpler in that scenario. The reason we wrote it in terms of characters χ is that they immediately generalize to any finite group. For $\mathbb{Z}_2$ the trivial representation is the one where $\chi_{\mathrm{even}}(g) = 1$ for all $g \in G = \mathbb{Z}_2$. The nontrivial irrep is the one where we have $\chi_{\mathrm{odd}}(e) = 1$ and $\chi_{\mathrm{odd}}(p) = -1$. For $G = \mathbb{Z}_2$, we have $|G| = 2$, the order of the finite group.

Before delving into the rigorous Tauberian formalism, let us gain some intuition by doing usual Cardy like analysis. For brevity, let us write $Z_+ \equiv Z_{\mathrm{even}}$, $Z_- \equiv Z_{\mathrm{odd}}$ and similarly $\chi_{\mathrm{even}} \equiv \chi_+$, $\chi_{\mathrm{odd}} \equiv \chi_-$; $\dim(even) \equiv d_+$, $\dim(odd) \equiv d_-$. Now in the dual channel, we have

$$\begin{aligned} Z_\pm(\beta \to 0) &= \frac{d_\pm}{|G|}\left[\chi^*_\pm(e) Z_{\mathscr{L}}(\beta' \to \infty, e) + \chi^*_\pm(p) Z_{\mathscr{L}}(\beta' \to \infty, p)\right], \quad \beta' = \frac{4\pi^2}{\beta} \\ &= \frac{1}{2}\left[Z_{\mathscr{L}}(\beta' \to \infty, e) \pm Z_{\mathscr{L}}(\beta' \to \infty, p)\right], \quad \beta' = \frac{4\pi^2}{\beta}\,. \end{aligned} \tag{3.4.4}$$

Here $Z_{\mathscr{L}}(\beta', e)$ is the usual partition function evaluated at the dual temperature β'. The quantity $Z_{\mathscr{L}}(\beta', p)$ is obtained by doing modular transformation on $Z^{\mathscr{L}}(\beta, p)$. Now $Z^{\mathscr{L}}(\beta, p)$ is not modular invariant, because it has an insertion of TDL along spatial direction. Under S modular transformation, cycles of the torus get exchanged, thus we have a torus configuration where the TDL is along the time direction. We can interpret this as having a defect in the spatial circle. Thus $Z_{\mathscr{L}}(\beta', p)$ is the partition function for the defect Hilbert space.

If the ground state in the defect Hilbert space (corresponding to $g \neq e$) has $\Delta > 0$, we have

$$Z_{\pm}(\beta \to 0) = \frac{d_{\pm}}{|G|} \chi_{\pm}^{*}(e) Z_{\mathscr{L}}(\beta' \to \infty, e) = \frac{d_{\pm}^{2}}{|G|} \exp\left[\frac{\pi^2 c}{3\beta}\right] = \frac{1}{2} \exp\left[\frac{\pi^2 c}{3\beta}\right]. \tag{3.4.5}$$

Let us pause for a moment and discuss when we can ensure that $\Delta > 0$ in the defect Hilbert space. According to [140], if the $\mathbb{Z}_2$ is anomalous, then the spin is constrained to be of the form $\frac{1}{4} + \mathbb{Z}/2$, thus excludes the possibility of having $\Delta = 0$ state. Similar argument is true for anomalous $\mathbb{Z}_n$ for any n. Since any finite group has a subgroup $\mathbb{Z}_m$ for some $m \in \mathbb{Z}_+$, if the subgroup is anomalous, the argument applies and we can not have $\Delta = 0$ state in the defect Hilbert space corresponding to that subgroup. If $\mathbb{Z}_2$ is non-anomalous, then we can not apply this argument. Nonetheless, we can gauge the $\mathbb{Z}_2$ group in such scenario to obtain the orbifold theory. We note that $\Delta = 0$ states is an even state, so it will be in the even sector of the defect Hilbert space if it is there in defect Hilbert space to begin with. The orbifolded theory has even operators from the usual Hilbert space (untwisted sector) and the even operators from the defect Hilbert space (twisted sector). Now if we assume the uniqueness of the $\Delta = 0$ state in the orbifolded theory, the defect Hilbert space can not have any $\Delta = 0$ state.The another way to phrase the statement is to demand that the action of symmetry group is faithful, thus the only TDL which commutes with all the operators is the Identity line as explained in the § 3.2. In what follows, we will assume this as a generic condition that in the defect Hilbert space $\Delta > 0$. We mention that the assumption is true for the Ising model.

From (3.4.5), we immediately derive the growth of operators in even and odd sector:

$$\rho_{\pm}(\Delta) \underset{\Delta\to\infty}{\simeq} \frac{1}{2}\left(\frac{c}{48\Delta^3}\right)^{\frac{1}{4}} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right], \tag{3.4.6}$$

where $\rho_{\pm}$ stands for density of states for even and odd operators respectively.

We make some remarks below:

★ **Smearing turns d_α into d_α^2:** Presence of symmetry predicts an extra-fold degeneracy of d_α where d_α is the dimension of α irrep. Thus it is somewhat surprising to find d_α^2 in the expression for density of states. But as we will show in the next subsection, the expression for the density of states is true only after smearing over an order one window. This smearing[2] allows for an effective extra-fold degeneracy of d_α^2. This becomes particularly clear if we examine the 3-state Potts model ($c = 4/5$), which has S_3 symmetry (See [46] for a quick and nice exposition of this theory with an emphasis on TDLs). S_3 is a generated by two elements: one element generates the $\mathbb{Z}_3$ symmetry, while the other element acts as $\mathbb{Z}_3$ charge conjugation. There are two doublet of primaries in this CFT sitting in the nontrivial 2 dimensional S_3 representation. Each of the doublet contains a primary of $\mathbb{Z}_3$ charge ω and a primary of $\mathbb{Z}_3$ charge $\omega^* = \omega^2$. One doublet has dimension $2/15$ while the other one has dimension $4/3$. All the descendants of these primaries sit in the same representation. If we consider a window of width $2\delta \gtrapprox 1$, it contains descendants of both the doublets. Thus it gives a factor of $2^2 = 4$. Should we able to resolve the actual density of states, we would have found degeneracy of 2 as predicted by the actual symmetry. Furthermore, note that for S_3, we have $|G| = 6$, thus we have a growth of $4/6\rho_0$ for the doublet irrep. From the perspective of $\mathbb{Z}_3$, we are counting all the operators with charge ω and ω^2, thus we should have a growth of $(1+1)/3\rho_0$, lo and behold $4/6 = 2/3$. Roughly speaking, the irrep α has to appear d_α times in a window of width $2\delta \to 1^+$, this might hint at some approximate

[2]We thank Raghu Mahajan for discussion along this line.

symmetry which emerges only because we smear. The scenario is very much similar to the one present in the calculation of disk partition function of JT gravity and bulk gauge field theory [124].

★ It might seem very tempting to discuss the growth of $\rho_+ - \rho_-$. Naively, asymptotic growth of $\rho_+ - \rho_-$ is controlled by inverse Laplace transformation of $Z^{\mathscr{L}}(\beta \to 0, p) = Z_{\mathscr{L}}(\beta' \to \infty, p)$ [140]. Nonetheless this argument does not pass the rigorous treatment of Tauberian since the positivity of $\rho_+ - \rho_-$ is not guaranteed, in fact it can in principle widely oscillate. Nonetheless, it is also possible to prove the following as a corollary of the theorem proven in the next section.

$$\left| \int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta' \, \left[\rho_+(\Delta') - \rho_-(\Delta')\right] \right| \leq \left(\frac{c}{48\Delta^3}\right)^{\frac{1}{4}} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right] \tag{3.4.7}$$

where we have used the extremal functions appearing in [156] to fix the order one number.

★ For $c > 1$, the analysis can be made sensitive to Virasoro primaries only. In the following section, we will be generalizing the idea to arbitrary finite group G using the notion of character as well as we will make our analysis rigorous using Tauberian formalism [158, 92, 167].

3.4.2 Arbitrary finite group G ala Tauberian

Untwisted sector

The partition function for the operators transforming under particular irreducible representation α is given by

$$Z_\alpha^{\mathscr{L}}(\beta) = \frac{d_\alpha}{|G|} \sum_{g \in G} Z^{\mathscr{L}}(\beta, g) \chi_\alpha^*(g) \equiv \int_0^\infty \mathrm{d}\Delta' \, \rho_\alpha(\Delta') \tag{3.4.8}$$

where d_α is the dimension of the irrep α. Under S modular transformation we have

$$Z_\alpha^{\mathscr{L}}(\beta) \underset{S}{\to} Z_{\mathscr{L}\alpha}(\beta') = \frac{d_\alpha}{|G|} \sum_{g \in G} Z_{\mathscr{L}}(\beta', g) \chi_\alpha^*(g) \tag{3.4.9}$$

where $\beta' = \frac{4\pi^2}{\beta}$. Our objective is to establish the following theorem:

Theorem 2. *We consider untwisted sector of a CFT admitting a global symmetry under a finite group G. The states transforming under the irreducible representation α, has an asymptotic growth, which is given by:*

$$\frac{c_- d_\alpha^2}{|G|}\rho_0(\Delta) \leq \frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} d\Delta'\ \rho_\alpha(\Delta') \leq \frac{c_+ d_\alpha^2}{|G|}\rho_0(\Delta) \tag{3.4.10}$$

Here $\rho_0(\Delta)$ is defined in (3.1.2) *and $c_\pm$ are order one positive numbers. These numbers can be determined using the extremal functionals appearing in [156]. In particular, we have $c_\pm = 1 \pm 1/2\delta$. The above statement is true under the assumption that $\mathscr{H}_{\mathscr{L}}(g)$ does not contain $\Delta = 0$ state for g not equal to the identity (e) element. This ensures that the sum defining $Z_{\mathscr{L}\alpha}(\beta')$ in eq.* (3.4.9) *is dominated by the $\Delta = 0$ state coming from the original Hilbert space, i.e.from $Z_{\mathscr{L}}(\beta', e)$.*

The proof of the theorem closely resembles the one in the previous section. The leading answer comes from inverse Laplace transformation of $\frac{d_\alpha}{|G|}Z_{\mathscr{L}}(\beta',e)\chi_\alpha^*(e) = \frac{d_\alpha^2}{|G|}Z(\beta')$. The only non-trivial part is to show the suppression of the heavy part of $Z_{\mathscr{L}\alpha}(4\pi^2/(\beta+\imath t))$. Now we have two ingredients, the character and the heavy part of the defect partition function. Like before, the absolute value of the heavy part is dominated by $|t| = \Lambda$. Then we can use the following chain of inequality

$$Z_{\mathscr{L}H}\left(\frac{4\pi^2\beta}{\beta^2+\Lambda^2}, g\right) \leq Z_{\mathscr{L}}\left(\frac{4\pi^2\beta}{\beta^2+\Lambda^2}, g\right) = Z^{\mathscr{L}}\left(\frac{\beta^2+\Lambda^2}{\beta}, g\right) \leq NZ\left(\frac{\beta^2+\Lambda^2}{\beta}\right) \underset{\beta\to 0}{\simeq} Ne^{\frac{\Lambda^2}{\beta}}.$$

On the other hand, the character can be bound using

$$|\chi_\alpha^*(g)|^2 \leq \left(\sum_g |\chi_\alpha^*(g)|^2\right) = |G| \quad \Rightarrow \quad \frac{1}{G}|\chi_\alpha^*(g)| \leq \frac{1}{|G|^{1/2}}. \tag{3.4.11}$$

Using the above two, we estimate the heavy part integrand for $|t| \leq \Lambda$

$$\left|\frac{d_\alpha}{|G|}\sum_{g\in G} Z_{\mathscr{L}H}(\beta',g)\chi^*_\alpha(g)\right| \leq \frac{d_\alpha}{|G|^{1/2}}\sum_{g} Z_{\mathscr{L}H}\left(\frac{4\pi^2\beta}{\beta^2+\Lambda^2},g\right) \underset{\beta\to 0}{\leq} N|G|^{1/2} d_\alpha \exp\left[\frac{\Lambda^2}{\beta}\right] \tag{3.4.12}$$

Thus we have

$$\Bigg|\text{heavy part}\Bigg| \leq N|G|^{1/2} d_\alpha \exp\left[\frac{\Lambda^2}{\beta}\right]\int_{-\Lambda}^{\Lambda} \mathrm{d}t\ |\hat{\phi}_\pm(t)| \tag{3.4.13}$$

Again we use the bandlimited functions $\phi_\pm$ and choose the support of $\hat{\phi}_\pm$ to be $[-\Lambda,\Lambda]$ with $\Lambda = 2\pi$. One can choose $\Lambda < 2\pi$. In fact by careful treatment, it is possible to choose $\Lambda = 2\pi$ and the extremal functions appearing in [156] to deduce the value of order one numbers $c_\pm$ appearing in the theorem.

Twisted sector

One can consider the twisted sector by introducing the TDL corresponding to the global symmetry G along temporal direction. This is what we called defect Hilbert space. Now if the symmetry is non-anomalous G, we can insert another TDL along the spatial direction and unambiguously resolve the crossing of two TDLs. Within a twisted sector (twisted by a given element $g \in G$) one can estimate the growth of operators transforming under particular irrep of G.

Here we use slightly different notations because now we have to deal insertion of two TDLs. By $Z_{\mathscr{L}}(\beta, g_0, g)$ we mean the partition function evaluated with TDL corresponding to g_0 inserted along temporal direction and TDL corresponding to g inserted along spatial direction. We also put in g_0 as argument of density of states to remind ourselves that we are dealing with the twisted sector. Thus the partition function $Z^{(\alpha)}_{\mathscr{L}}(\beta, g_0)$ for the operators in the α irrep in the twisted sector is given by

$$Z_{\mathscr{L}}^{(\alpha)}(\beta,g_0) \equiv \frac{d_\alpha}{|G|}\sum_{g\in G} Z_{\mathscr{L}}(\beta,g_0,g)\chi_\alpha^*(g) \equiv \int_0^\infty \mathrm{d}\Delta'\, \rho_\alpha(g_0,\Delta') \tag{3.4.14}$$

where d_α is the dimension of the irrep α. Under S modular transformation[3] we have

$$Z_{\mathscr{L}}^{(\alpha)}(\beta,g_0) \underset{S}{\to} \frac{d_\alpha}{|G|}\sum_{g\in G} Z_{\mathscr{L}}(\beta',g,g_0^{-1})\chi_\alpha^*(g)\,,\ \text{where } \beta' = \frac{4\pi^2}{\beta}\,. \tag{3.4.15}$$

The final result is again given by eq. (3.4.10). In particular we have

Theorem 3. *We consider twisted sector (twisted by the $g_0 \in G$) of a CFT admitting a symmetry (non-anomalous) under a group G. The* (3.4.10) *holds true for the growth of operators in this sector. The assumptions are same as the one in theorem [2].*

3.4.3 TDL associated with continuous symmetry $U(1)$

The idea presented above for the finite group can be generalized to continuous group as well. The tricky part is to determine the behavior of $Z^{\mathscr{L}_g}(\beta\to 0,\alpha) = Z_{\mathscr{L}_g}(\beta'\to\infty,\alpha)$. Earlier knowing that for $g\neq e$, the defect Hilbert space has states with $\Delta > 0$ only sufficed because we have a sum over group elements. But here we have an integral over the group manifold and as $g\to e$, the ground state of the defect Hilbert space goes to 0. Thus we need to know the behavior of the ground state of the defect Hilbert space as $g\to e$, to say something concrete.

In what follows, we can consider the compact $U(1)$ group, which is generated by $J \equiv J_0 - \bar{J}_0$, coming from the Kac-Moody algebra. For a nice discussion related to compact vs non compact we refer the readers to [29]. The partition function is given by

$$Z^{\mathscr{L}_g}(\beta,\nu,\bar{\nu}) = \sum_{n,J} e^{-\beta\left(\Delta_n - \frac{c}{12}\right)} e^{2\pi\imath\nu J} \tag{3.4.16}$$

[3] When g and g_0 are both non-identity elements, under S modular transformation, the relative orientation of the TDLs corresponding to them changes. Hence in the dual channel we have g_0^{-1} inserted along the spatial direction.

where $g = e^{2\pi\iota\nu} \in G = U(1)$ and $\nu \in (-1/2, 1/2]$. Usually we think of this as partition function for Grand Canonical ensemble. Alternatively, we can think of this as a partition function evaluated on Torus with insertion of TDLs corresponding to $U(1)$.

We wish to estimate asymptotic growth of states with a definite J. We write down a partition function for a fixed $J \equiv Q$:

$$Z_Q^{\mathscr{L}_g}(\beta) = \int_{-1/2}^{1/2} \mathrm{d}\nu \; e^{-2\pi\iota\nu Q} Z^{\mathscr{L}_g}(\tau) = \sum_n d_{n,Q} e^{-\beta\left(\Delta_n - \frac{c}{12}\right)} \tag{3.4.17}$$

We pause here to comment about the integral range of ν, i.e. $\nu \in (-1/2, 1/2]$. This implies that we are considering "single" cover of $U(1)$ and all the charges are integer. We further assume that this action is faithful. Thus we exclude scenarios like where all the charges are even. Instead of "single" cover, we can also consider $N \in \mathbb{Z}_+$ cover , so that possible charges are of the form $\frac{q}{N}$ with $N-1 > |q| \in \mathbb{N}$; in that scenario the ν integral would have been from $-N/2$ to $N/2$ with a multiplicative factor of $\frac{1}{N}$ for correct normalization. In this way of thinking, the scenario, where all the charges are even can be treated as effectively making $N = \frac{1}{2}$. In what follows, we will be considering $N = 1$ case. Without loss of generality, we also assume the spectra is invariant under $Q \to -Q$ as they correspond to taking $\nu \to -\nu$. As an example, readers might keep in mind compact boson with level $k = 1$ and radius $R = 2$, where the charge under J_0 is $\frac{e}{R} + \frac{mR}{2}$ and the charge under $\bar{J}_0$ is $\frac{e}{R} - \frac{mR}{2}$ with $e, m \in \mathbb{Z}$.

Modular transformation of $Z^{\mathscr{L}_g}(\beta)$ gives us the partition function of the defect Hilbert space and we have

$$\begin{aligned} Z_{\mathscr{L}_g Q}\left(\frac{4\pi^2}{\beta}\right) &= \int_{-1/2}^{1/2} \mathrm{d}\nu \; e^{-2\pi\iota\nu Q} Z_{\mathscr{L}_g}\left(\frac{4\pi^2}{\beta}\right) \\ &= \sum_{n,q,\bar{q}} d_{n,q,\bar{q}} \int_{-1/2}^{1/2} \mathrm{d}\nu \; e^{-2\pi\iota\nu Q} e^{-\frac{4\pi^2}{\beta}\left(\Delta_n - \frac{c}{12} + k\nu^2 - \nu(q+\bar{q})\right)}, \end{aligned} \tag{3.4.18}$$

where k is a parameter coming from the Kac-Moody algebra, q,$\bar{q}$ are the charge under J_0 and $\bar{J}_0$.

We want to evaluate this integral in the $\beta \to 0$ limit.

$$Z_{\mathscr{L}_g Q}\left(\frac{4\pi^2}{\beta}\right) \simeq \sqrt{\frac{\beta}{4\pi k}} \sum_{\substack{\widehat{\Delta}_n \\ |q+\bar{q}|\leq' k}} d_{\widehat{\Delta}_n,q,\bar{q}}\, e^{-\frac{4\pi^2}{\beta}\left(\widehat{\Delta}_n - \frac{c}{12}\right) - \beta\frac{Q^2}{4k} - \imath\pi\frac{Q(q+\bar{q})}{k}} \tag{3.4.19}$$

where $\widehat{\Delta}_n = \left(\Delta_n - \frac{(q+\bar{q})^2}{4k}\right)$. The prime over $\leq$ indicates whenever $q+\bar{q}$ becomes $\pm k$, there is a factor of $\frac{1}{2}$ associated with the ν integral. The crucial point is to observe that the states in the defect Hilbert space has dimension $\widehat{\Delta} + (\nu - \frac{q+\bar{q}}{2k})^2$. Thus in the $\beta \to 0$ limit, the ν integral can contribute only if $(\nu - \frac{q+\bar{q}}{2k}) = 0$ for some $\nu \in (-1/2, 1/2]$. Thus the sum over $q, \bar{q}$ is restricted. Thus the leading piece is given by $\widehat{\Delta} = 0$ states. Of course $\Delta = q = \bar{q} = 0$ would contribute. We observe that the unitarity bound tells us that

$$\Delta \geq \frac{q^2 + \bar{q}^2}{2k} \geq \frac{(q+\bar{q})^2}{4k}, \tag{3.4.20}$$

where the saturation of the second inequality can happen only if $q = \bar{q}$. Thus the states that would contribute to the leading order is given by $\widehat{\Delta} = 0, q = \bar{q}, |q| \leq' \frac{k}{4}$. Hence we have

$$Z^{\mathscr{L}_g}_{Q,\bar{Q}}(\beta \to 0) \simeq N_0 \sqrt{\frac{\beta}{4\pi k}} e^{\frac{\pi^2 c}{3\beta}} \tag{3.4.21}$$

where $N_0 = \sum_{q=\bar{q}, |q|\leq' k/2} w(q) e^{-\frac{2\pi\imath q Q}{k}}$ and $w(q) = 1$ if $|q| < k/2$ and $w(q) = \frac{1}{2}$ if $|q| = \frac{k}{2}$. Since we have assumed $q \to -q$ symmetry, N_0 is a real number. $N_0 = 1$ if only such state is the vacuum. In what follows, we will assume that this is the case[4] and $N_0 = 1$. Strictly speaking, in the Tauberian analysis, we would require the above argument to hold for complex $\beta + \imath t$.

The next step is to split up the Hilbert space into the light and the heavy sector. Now we divide the Hilbert space using the quantity $\widehat{\Delta} \equiv \Delta - \frac{q^2+\bar{q}^2}{4k}$. The light sector is defined as

[4] We note that if the action of $U(1)$ is not faithful, for example, if all the charges are even, then for odd charges, the asymptotic expression should give 0, as a result N_0 should have been equal to 0, in those cases the phases in the sum defining N_0 play a key role.

$\widehat{\Delta}+(\nu-\frac{q}{2k})^2 \leq c/12$ while the heavy sector is the one with $\widehat{\Delta}+(\nu-\frac{q}{2k})^2 > c/12$. Thereafter, we restrict our attention to the heavy sector and show it is suppressed. Recall the quantity related to the heavy sector that appears in the Tauberian analysis is following:

$$I \equiv e^{\beta(\Delta\pm\delta-c/12)}\left|\int_{-\Lambda}^{\Lambda}\mathrm{d}t\ e^{-\imath tc/12}\hat{\phi}(t)\ \sum_{n,q,\bar{q}} d_{n,q,\bar{q}}\int_{-1/2}^{1/2}\mathrm{d}\nu\ e^{-2\pi\imath\nu Q}e^{-\frac{4\pi^2}{\beta+\imath t}\left(\Delta_n-\frac{c}{12}+k\nu^2+\nu(q+\bar{q})\right)}\right|.$$

Now we pull in the absolute value inside the integral and notice the exponential is maximized for $|t|=\Lambda$. Thus we have

$$\begin{aligned}
I &\leq e^{\beta(\Delta\pm\delta-c/12)}\int_{-1/2}^{1/2}\mathrm{d}\nu \sum_{\substack{n,q,\bar{q}\\ \text{heavy}}} d_{n,q,\bar{q}}\ e^{-\frac{4\pi^2\beta}{\beta^2+\Lambda^2}\left(\Delta_n-\frac{c}{12}+k\nu^2+\nu(q+\bar{q})\right)}\int_{-\Lambda}^{\Lambda}\mathrm{d}t\ |\hat{\phi}(t)| \\
&\leq e^{\beta(\Delta\pm\delta-c/12)}\int_{-1/2}^{1/2}\mathrm{d}\nu\ Z_{\mathscr{L}_g}\left(\frac{4\pi^2\beta}{\beta^2+\Lambda^2}\right)\int_{-\Lambda}^{\Lambda}\mathrm{d}t\ |\hat{\phi}(t)| \\
&= e^{\beta(\Delta\pm\delta-c/12)}\int_{-1/2}^{1/2}\mathrm{d}\nu\ Z^{\mathscr{L}_g}\left(\frac{\beta^2+\Lambda^2}{\beta},\nu\right)\int_{-\Lambda}^{\Lambda}\mathrm{d}t\ |\hat{\phi}(t)| \\
&\leq e^{\beta(\Delta\pm\delta-c/12)}\int_{-1/2}^{1/2}\mathrm{d}\nu\ Z^{\mathscr{L}_g}\left(\frac{\beta^2+\Lambda^2}{\beta},\nu=0\right)\int_{-\Lambda}^{\Lambda}\mathrm{d}t\ |\hat{\phi}(t)| \\
&\underset{\beta\to 0}{\simeq} e^{\beta(\Delta\pm\delta-c/12)}e^{\frac{\pi^2c}{3\beta}\frac{\Lambda^2}{4\pi^2}}\int_{-\Lambda}^{\Lambda}\mathrm{d}t\ |\hat{\phi}(t)| \underset{\beta=\pi\sqrt{\frac{c}{3\Delta}}}{\simeq}\exp\left[\frac{\Lambda^2}{4\pi^2}2\pi\sqrt{\frac{c\Delta}{3}}\right]
\end{aligned}\tag{3.4.22}$$

We will see that the suppression requires $\Lambda < 2\pi$. The light sector produces the leading Cardy like behavior for density of states $\rho_{Q,\bar{Q}}(\Delta')$ of operators with fixed order one charge $Q,\bar{Q}$ and large conformal dimension Δ. This can be obtained by doing the following integral and realizing that the integral is dominated by $t=0$ in the $\beta\to 0$ limit:

$$\begin{aligned}
&e^{\beta(\Delta\pm\delta-c/12)}\int_{-\Lambda}^{\Lambda}e^{-\imath tc/12}\sqrt{\frac{\beta+\imath t}{4\pi k}}\exp\left[\frac{\pi^2 c}{3(\beta+\imath t)}\right]\hat{\phi}_{\pm}(t) \\
&= e^{\beta(\Delta\pm\delta-c/12)}\hat{\phi}_{\pm}(0)\exp\left[\frac{\pi^2c}{3\beta}\right]\sqrt{\frac{\beta}{4\pi k}}\left(\frac{3}{\pi c}\right)^{1/2}\beta^{3/2} \\
&= 2\pi\hat{\phi}(0)\sqrt{\frac{1}{k}}\left(\frac{1}{4\Delta}\sqrt{\frac{c}{3}}\exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right]\right)
\end{aligned}\tag{3.4.23}$$

where the factor $(\frac{3}{\pi c})^{1/2}\beta^{3/2}$ comes from the integrating over the fluctuation around the saddle at $t=0$. Thus we have following estimate:

$$c_{-}\sqrt{\frac{c}{48k\Delta^2}}\exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right] \leq \frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta'\ \rho_Q(\Delta') \leq c_{+}\sqrt{\frac{c}{48k\Delta^2}}\exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right]. \tag{3.4.24}$$

Here $c_{\pm} = \frac{2\pi}{2\delta}\hat{\phi}_{\pm}(0)$ is order one positive number.

We conclude this section with two final remarks that one can generalize the analysis for Virasoro primaries for CFT with $c>1$ and one can generalize this to large central charge.

3.5 Tauberian for Vector-valued modular function

The results for the finite group can nicely be encapsulated in terms of something known as vector valued modular function. The vector-valued modular function $\mathbf{Z}$ obeys the following transformation law under S modular transformation:

$$\mathbf{Z}\left(\frac{4\pi^2}{\beta+\imath t}\right) = \mathbf{F}\cdot\mathbf{Z}(\beta+\imath t) \tag{3.5.1}$$

where $\mathbf{Z}$ is a column vector consisting of bunch of functions and $\mathbf{F}$ is a constant (β independent) matrix. The condition $S^2=\mathbb{I}$ boils down to $\mathbf{F}^2=\mathbf{I}$.

- In the example of CFT with $\mathbb{Z}_2$ symmetry we can consider

$$\mathbf{Z} = (Z_{+}, Z_{-}, Z_{p})^T,$$

where $Z_{\pm}$ are the partition functions for even and odd operator and Z_p is the defect Hilbert space with insertion of non-identity $\mathbb{Z}_2$ TDL. The matrix $\mathbf{F}$, in this case, is given by

$$\mathbf{F} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \tag{3.5.2}$$

and $\mathbf{F}^2$ is indeed identity.

• For a generic compact group G, the vector $\mathbf{Z}$ will have $2k-1$ entry , where k is the number of conjugacy classes of the group G. The k entries correspond to k different irreps (recall the number of conjugacy class is equal to number of irreps) and $k-1$ entries correspond to the partition function for the defect Hilbert space with insertion of non-identity element. It suffices to consider one representative element from each conjugacy class as the partition function with insertion of TDL along spatial direction is sensitive to conjugacy class only. For $\mathbb{Z}_n$, we have n different conjugacy classes, i.e. n irreps.

To estimate the growth of operators in each of the sectors, we define a vector valued density of states $\vec{\rho}(\Delta)$. The upper bound (the lower bound is similar) on the integral of $\vec{\rho}(\Delta)$ is given by a matrix inequality

$$\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta'\, \vec{\rho}(\Delta') \leq e^{\beta(\Delta+\delta)} \int_{-\Lambda}^{\Lambda} \mathrm{d}t\, \mathbf{F}\cdot \left[\mathbf{Z}\left(\frac{4\pi^2}{\beta+\imath t}\right)\right] e^{-\imath tc/12}\phi_+(t) \tag{3.5.3}$$

Thus we need to estimate the integrals of the form

$$e^{\beta(\Delta+\delta)} \int_{-\Lambda}^{\Lambda} \mathrm{d}t\, \mathbf{F}\cdot \left[\mathbf{Z}\left(\frac{4\pi^2}{\beta+\imath t}\right)\right] e^{-\imath tc/12}\phi_+(t)$$

in the $\beta \to 0$ limit.

At this point, we separate out the light contribution and the heavy contribution in the usual way. If we further assume that $\Delta = 0$ state appears in one and only one of the sectors, without loss of generality we can keep it as first entry. Then the light sector $\mathbf{Z}^{(L)}$ will give

$$\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta' \, \vec{\rho}(\Delta') \leq e^{\beta(\Delta+\delta)} \int_{-\Lambda}^{\Lambda} \mathrm{d}t \; \mathbf{F} \cdot \left[\mathbf{Z}^{(L)} \left(\frac{4\pi^2}{\beta + \imath t} \right) \right] e^{-\imath t c/12} \phi_+(t) = 2\pi \hat{\phi}_+(0) \rho_0(\Delta) \vec{\mathbf{F}}_1 \,,$$

where $\vec{\mathbf{F}}_1$ is the first column of the $\mathbf{F}$ matrix. This determines the parameter N_0 (or d_α^2) appearing previously.

We still need to show that the heavy sector $\mathbf{Z}^{(H)}$ gives a suppressed contribution in magnitude. In order to achieve that we will use that $|\mathbf{F}_{ij}| < K_i$. This is true for all the calculations done previously and generically true because $\mathbf{F}$ is finite matrix and $\mathbf{F}^2 = \mathbf{I}$. A more mathematical way to saying this is that

$$||\mathbf{F}||_\infty = \mathrm{Max}_i \left\{ \sum_j |\mathbf{F}_{ij}| \right\} \text{ is finite} \,.$$

We note that for $|t| \leq \Lambda$,

$$\left| \left(\mathbf{F} \cdot \left[\mathbf{Z}^{(H)} \left(\frac{4\pi^2}{\beta + \imath t} \right) \right] \right)_i \right| \leq \sum_j |\mathbf{F}_{ij}| \left| \mathbf{Z}_j^{(H)} \left(\frac{4\pi^2}{\beta + \imath t} \right) \right| \leq ||\mathbf{F}||_\infty \sum_j \mathbf{Z}_j^{(H)} \left(\frac{4\pi^2 \beta}{\beta^2 + \Lambda^2} \right) . \quad (3.5.4)$$

To estimate the sum appearing in the rightmost, we observe that

$$\sum_j \mathbf{Z}_j^{(H)} \left(\frac{4\pi^2 \beta}{\beta^2 + \Lambda^2} \right) \leq \sum_j \mathbf{Z}_j \left(\frac{4\pi^2 \beta}{\beta^2 + \Lambda^2} \right) = \sum_j \mathbf{F} \cdot \left[\mathbf{Z} \left(\frac{\Lambda^2 + \beta^2}{\beta} \right) \right] . \qquad (3.5.5)$$

Again we use the fact that $\Delta = 0$ appears in one and only one sector to have

$$\sum_j \mathbf{F} \cdot \left[\mathbf{Z} \left(\frac{\Lambda^2 + \beta^2}{\beta} \right) \right] \simeq \mathbf{Z}_1 \left(\frac{\Lambda^2 + \beta^2}{\beta} \right) \sum_j \mathbf{F}_{j1} \simeq e^{\frac{\Lambda^2 c}{12\beta}} \,.$$

Choosing $\Lambda < 2\pi$ suppress the heavy part. Thus we arrive at our general theorem.

Theorem 4. *We consider vector valued modular function as defined in* (3.5.1). *Each entry in the* *column vector* $\mathbf{Z}$ *denotes different sector of the CFT. The growth of operators in each of these sectors obey the following inequality:*

$$c_{-}\rho_0(\Delta)\mathbf{F}_{j1} \leq \frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} d\Delta'\, \rho_j(\Delta') \leq c_{+}\rho_0(\Delta)\mathbf{F}_{j1}\,, \tag{3.5.6}$$

where $c_\pm$ *are order one numbers. These numbers can be determined using the extremal functionals appearing in [156]. In particular, we have* $c_\pm = 1 \pm 1/2\delta$.

One can further apply similar technique to any rational CFT where characters are indeed vector-valued modular functions, this would facilitate estimation of growth of descendants for each primary (primary of the full chiral algebra). For $c < 1$, one can apply this to Minimal models and estimate the growth of descendants of each Virasoro primary.

Chapter 3, in full, is a reprint of the marterial as it appears in Sridip Pal, Zhengdi Sun, JHEP **08**, 064 (2020). The dissertation author was one of the primary investigator and author of this paper.

Chapter 4

On Triality Defects in 2d CFT

4.1 Introduction

Global symmetry is an important guideline for constructing and analyzing quantum field theories. In modern language, the global symmetries, whether continuous or discrete, can be represented as invertible topological operators supported on a codimensions-1 (codim-1) surface. Any correlation functions with the topological surface operator insertion are invariant under the small deformation of the codim-1 surface where the topological operator is supported. As the result, the topological operator commute with the stress-energy tensor. For a continuous symmetry given by a conserved current $j_\mu(x)$, the corresponding topological surface operator is constructed as $e^{i\alpha \oint_{\Sigma_{d-1}} d^{d-1}x n^\mu(x) j_\mu(x)}$ and the invariance under the small deformation then follows from the conservation equation $\partial_\mu j^\mu(x) = 0$.

Several generalizations have been made from this point of view, which leads to the concept of generalized global symmetries [90]. For instance, the support of the invertible topological operator can be generalized to codim-p surfaces for $p > 1$, and the corresponding symmetries are called $(p-1)$-form symmetries [90, 125]. The standard 0-form symmetries can interact non-trivially with the 1-form symmetries, and this leads to the structure of 2-group symmetries [60, 61, 62]. Another direction of generalization is to study the topological operators which are not invertible. For instance, the non-invertible 0-form symmetries in 2-dimensional field theory can be described by the fusion categories, and the topological operators correspond to the objects in the fusion categories. Therefore, these non-invertible symmetries are also called categorical symmetries. There is a lot of progress in the study of non-invertible symmetries in dimension $d \geq 3$ recently [14, 102, 52, 63, 25, 68, 67, 123]. Of course, the categorical 0-form symmetries can interact non-trivially with the categorical higher form symmetries, this leads to the concept of higher categorical symmetries [30].

For a more detailed review and a more complete list of references on the development of the generalized global symmetries and their applications from the high energy physics perspective and the condensed matter physics perspective, we refer the readers to the two reviews [62, 150].

There are two classes of non-invertible symmetries [123, 122]. Consider a field theory $\mathscr{T}$ with non-invertible symmetries $\mathscr{C}$, if there exists some topological manipulation ϕ such as finite gauging on the theory $\mathscr{T}$ such that the theory $\phi(\mathscr{T})$ contains only invertible symmetries $\phi(\mathscr{C})$ and the constraints on the RG flows of the theory $\mathscr{T}$ from the non-invertible symmetries $\mathscr{C}$ can be completely determined from the constraints on the RG flows from the invertible symmetries $\phi(\mathscr{C})$ of the theory $\phi(\mathscr{T})$, we refer the non-invertible symmetries $\mathscr{C}$ as non-intrinsic [118]. Otherwise, the non-invertible symmetries are called intrinsic. A simple example of non-intrinsic invertible symmetries is the $\mathrm{Rep}(S^3)$ symmetries in the 2-dimensional field theory, acquired from the topological manipulation of gauging the S^3 global symmetries. (For example, see [36].)

The study of codim-1 non-invertible topological operators (also known as topological defect lines, or TDLs) in 2-dimensional CFT has a long history [41, 163, 162, 169, 85, 86, 87, 88, 82], focusing on their connection to boundary CFT, twisted partition function on various 2-manifold, orbifolds and symmetry TFT. Recent studies not only focus on searching for interesting non-invertible TDLs [39, 180, 181, 45, 111, 44], but also extending the applications of the TDLs, including their lattice construction [2, 1], constraints for 2d modular bootstrap [141, 142, 166, 58, 119, 136], constraints on the RG flow [180, 46, 126, 127], gauging non-invertible symmetries [36, 20] etc.

Perhaps the most well-studied non-invertible TDL is the duality line N under the $\mathbb{Z}_2$ gauging. The duality line N and two lines $\mathbb{1}, \eta$ in the $\mathbb{Z}_2$ symmetries form the Ising fusion category with the fusion rule

$$\eta N = N\eta = N, \quad N^2 = I + \eta. \tag{4.1.1}$$

The Ising fusion category has been extensively studied in the literature [39, 91, 141, 2, 1, 163, 162, 180, 181]. The existence of the duality line N implies the conformal field theory is invariant under the $\mathbb{Z}_2$ gauging. One possible generalization is of course to consider the duality line N under A-gauging where A is a finite Abelian group. The corresponding fusion category is known as the Tambara-Yamagami Fusion category $\mathscr{T}(A,\chi,\varepsilon)$ [177, 176] where χ is a symmetric bicharacter of the group A and $\varepsilon = \pm 1$ is the Frobenius-Schur indicator. The $\mathscr{T}(A,\chi,\varepsilon)$ with the same A but different χ and ε satisfies the same fusion rules, yet they are different fusion categories distinguished by the F-symbols (also known as the crossing kernels $\mathscr{K}$ in [46]), which measures the difference between two different ways of resolving the crossing of two TDLs. The F-symbols reduce to the familiar group anomaly measured by $H^3(G,U(1))$ when considering only invertible TDLs. For instance, there are two types of Ising fusion categories with different FS indicators $\varepsilon = \pm 1$ and they can also be distinguished from the so-called spin selection rules [46]. The studies of the Tambara-Yamagami fusion category symmetries in the physics literature include [39, 141, 46, 180, 181].

Another generalization of the Ising fusion category is the triality fusion category and recently is studied in [180, 181]. The simple TDLs contain symmetry operators which generate $\mathbb{Z}_2\times\mathbb{Z}_2$ global symmetries, as well as a triality line $\mathscr{L}_Q$ and its orientation reversal $\mathscr{L}_{\overline{Q}}$, satisfying the fusion rule [1] generalizing (4.1.1)

$$\mathscr{L}_Q\times\mathscr{L}_{\overline{Q}} = \sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} g, \quad \mathscr{L}_Q\times\mathscr{L}_Q = 2\mathscr{L}_{\overline{Q}}, \quad g\times\mathscr{L}_Q = \mathscr{L}_Q\times g = \mathscr{L}_Q, \quad g\in\mathbb{Z}_2\times\mathbb{Z}_2. \tag{4.1.2}$$

[1] In general, the invertible symmetries in a triality fusion category does not have to be $\mathbb{Z}_2\times\mathbb{Z}_2$. For simplicity, however, the notion of the triality fusion category would specifically mean the case where the invertible symmetries are $\mathbb{Z}_2\times\mathbb{Z}_2$.

The triality fusion category is obtained by gauging the $\mathbb{Z}_2$ global symmetry in A_4 global symmetry of the $SU(2)_1$ theory. In general, consider a 2-dimensional theory $\mathscr{T}$ with the global symmetry G and anomaly $\omega \in H^3(G,U(1))$. One can gauge a non-anomalous subgroup $H \subset G$ and additional data which is the discrete torsion $\psi \in H^2(H,U(1))$ needs to be specified in this gauging process. The fusion category that describes the categorical symmetries of the gauged theory $\mathscr{T}/H$ is called the group theoretical fusion category, and it is denoted by $\mathscr{C}(G,\omega,H,\psi)$. In this language, the triality fusion category in the KT theory is $\mathscr{C}(A_4,1,\mathbb{Z}_2,1)$.

In this paper, we study this triality fusion category using the tools from the group theoretical fusion category. We compute the spectrum of simple TDLs, their fusion rules, and the F-symbols by using the description of the group theoretical fusion category in terms of bimodules [164].

We then study the physical implication of the triality fusion category. We derive the spin selection rules from the F-symbols we acquire. We also derive the Cardy formula for densities of states using the result in [166]. Since these triality fusion categories are group-theoretical, its constraint on the RG flow can be determined by the group G and the anomaly ω. In general, this means that finite non-intrinsic non-invertible 0-form symmetries are completely characterized by group theoretical fusion categories in a 2-dimensional bosonic theory. We then consider the $c = 1$ KT theory as an example, and compute its twisted partition function explicitly and show the result indeed agree with our general analysis.

It is a natural question to ask if there are more allowed F-symbols with the same fusion relations. Just like the TY-categories characterizing the duality can have distinct FS indicators with $\varepsilon = \pm 1$, the possible FS indicators of the triality fusion category are given by $\alpha = e^{2\pi ik/3} \in \mathbb{Z}_3$ with $k = 0,1,2$. One can see this from the fact that there are $\mathbb{Z}_3$ phase rotations of the F-symbols which preserve the pentagon equations and are not gauge transformations. Physically, this means one can construct triality fusion category with different FS indicator α' from a known with α by taking the theory $\mathscr{T}$ with the triality defect $\mathscr{L}_Q$, staking a decoupled theory $\mathscr{T}'$ with an anomalous $\mathbb{Z}_3$ global symmetry generated by η, and defining a new triality line $\mathscr{L}'_Q \equiv \mathscr{L}_Q \eta$. From the group theoretical fusion category point of view, to acquire triality fusion categories with the FS indicator $\alpha = e^{2\pi i k/3}$ means we are gauging the non-anomalous $\mathbb{Z}_2$ symmetries in A_4 but now with non-trivial 't Hooft anomaly for A_4. The anomaly for A_4 is classified by $H^3(A_4, U(1)) \simeq \mathbb{Z}_6$ and let's denote its generator as ω_0. The triality fusion categories with different FS indicators are $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2, 1)$ for $k = 0, 1, 2$.

However, just taking into account the FS indicators does not enumerate all the possible triality fusion categories. Indeed, another set of F-symbols is computed in the condensed matter literature [179] and it is natural to ask if these F-symbols lead to the same fusion categories as ours. We show that the two sets of fusion categories are different and can be distinguished using spin selection rules. Then this enumerates all the possible inequivalent F-symbols for the triality fusion categories according to the classification result in [117].

4.1.1 Outline of the Paper

The goal of the paper is to provide a description of group theoretical triality fusion categories $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2, 1)$ in terms of the bimodules of the $\mathbb{Z}_2$ group algebra, and use it to compute the fusion rule and F-symbols. The simplicity of the group theoretical fusion category is that the objects naturally have a $\mathbb{C}$-vector spaces structure, therefore every data we need can be described using linear algebras.[2] We also pointed out there are triality fusion categories that are not group theoretical, therefore are intrinsic. We show that whether the triality fusion category is intrinsic or non-intrinsic can be determined from the spins of the defect Hilbert space of the triality line.

In section 4.2, we briefly review the TDLs in CFT. In section 4.2.1, we introduce the basic notions relate to the TDLs in 2d CFT. In section 4.2.2, we review the modular bootstrap program, and describe how to relate the twisted partition function computes the action of TDL

[2] Notice that there is no $\mathbb{C}$-vector spaces structure on objects in a generic fusion category $\mathscr{C}$. For finite 0-form symmetries in a bosonic theory, non-intrinsic non-invertible symmetries form group theoretical fusion categories. This means non-intrinsic non-invertible symmetries naturally have $\mathbb{C}$-vector space structure while the intrinsic non-invertible symmetries do not. It would be interesting to check if there's any physical understanding or implication of this difference.

$\mathscr{L}$ on Hilbert space $\mathscr{H}$ and to the twisted partition function computes the states of the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$. Then, in section 4.2.3, we briefly review the result in [166] on the asymptotic density of states in various Hilbert spaces, which will be useful for us later.

In section 4.3, we describe how to understand the triality fusion categories discovered in [180, 181] as group theoretical fusion category $\mathscr{C}(A_4, 1, \mathbb{Z}_2, 1)$. In section 4.3.1, we briefly review the triality fusion category, which is acquired from gauging $\mathbb{Z}_2$ subgroup in A_4. Then, we introduce the notion of group theoretical fusion category in section 4.3.2 and show the triality fusion category in the KT theory can be described as $\mathscr{C}(A_4, 1, \mathbb{Z}_2, 1)$. We then begin to describe the data of this triality fusion category from the $\mathscr{C}(A_4, 1, \mathbb{Z}_2, 1)$ using the language of bimodules. In the rest of this section, we give an explicit calculation of the spectrum of simple TDLs, their fusion rules, and the F-symbols for $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2, 1)$ using bimodules.

In section 4.4, we use the above result to derive physical consequences. In section 4.4.1, we compute the spin selection rules following the techniques in [46]. Since $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2, 1)$ is acquired from gauging the $\mathbb{Z}_2$ subgroup in a theory with global symmetry A_4, we can match the irreducible representations (irreps) of the fusion ring of $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2, 1)$ to different sectors in $\mathscr{T}$. This allows us to derive the Cardy-like formulas for different irreps of $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2, 1)$ in the section 4.4.2. Finally in the section 4.4.3, we show that the anomaly of the group theoretical fusion categories $\mathscr{C}(G, \omega, H, \psi)$ is equivalent to the group anomaly ω of G, in the sense that the symmetric gapped phases of $\mathscr{C}(G, \omega, H, \psi)$ are equivalent to the symmetric gapped phases of $\mathrm{Vec}_G^{\omega} \equiv \mathscr{C}(G, \omega, \mathbb{Z}_1, 1)$. This means that the finite non-intrinsic non-invertible symmetries are completely characterized by group theoretical fusion categories for 2-dimensional bosonic field theory.

In section 4.5, we first review the Kosterlitz-Thouless (KT) theory and its triality fusion categories discovered in [180, 181]. We then compute the twisted partition function of the triality defect and show the spins in the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$ indeed match the spin selection rules in the section 4.5.2. Since the KT theory has $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset (U(1)^{\tilde{\theta}} \times U(1)^{\tilde{\phi}}) \rtimes \mathbb{Z}_2$ global symmetries, it is interesting to check if one can construct new triality line via $\mathscr{L}_Q \eta$, where η

generates the one of the $\mathbb{Z}_3$ symmetry. We show explicitly this is not possible by checking that the line $\mathscr{L}_Q\eta$ does not satisfy the fusion rule in the section 4.5.3.

In section 4.6, we consider other triality fusion categories. We first review the classification of the triality fusion categories in [117] in the section 4.6.1. We then list the F-symbols for the intrinsic triality fusion categories computed in [179] in the section 4.6.2. In the section 4.6.3, We then compute the spin selection for the triality line $\mathscr{L}_Q$ for the intrinsic triality fusion categories and show that the triality fusion categories can be distinguished by the spins of the states in the $\mathscr{H}_{\mathscr{L}_Q}$. We conclude the paper with a comment on when the spin selection rules should be saturated.

4.1.2 Future Directions

We outline some of the future directions to explore.

Exploring generic group theoretical fusion categories

In this paper, we mainly focus on describing the group theoretical fusion categories $\mathscr{C}(A_4, 1, \mathbb{Z}_2^\sigma, 1)$. Since our approach can be easily generalized to other group theoretical fusion categories, it would be interesting to systematically explore the group theoretical fusion categories using Mathematica, for example, computing the simple TDLs, their fusion rules, and the F-symbols, as well as the physical implications such as the spin selection rules, classification of $\mathscr{C}$-symmetric gapped phases, and solutions of general modular bootstrap equations, etc. Group theoretical fusion categories provide a systematic way of constructing N-ality fusion categories in 2-dimension. Some of these will be done in the upcoming work by the authors [144].

Intrinsic triality defects

In this paper, we only study the spin selection rules of the intrinsic triality fusion categories, which is sufficient to distinguish them from the non-intrinsic ones. Unlike the group theoretical fusion category symmetries, it's harder to find examples of CFT with non-intrinsic triality symmetries. A possible candidate is the bosonization of the theory of 8 Majorana fermions discussed in [183]. Furthermore, it would be interesting to understand the anomaly and in general, the possible symmetric gapped phases, which are described to module categories $\mathscr{M}$ over the fusion category $\mathscr{C}$. This can be done using the techniques developed in [151]. More generally, a class of intrinsic N-ality fusion categories is constructed in [117] and it would be interesting to answer the above questions for these as well.

Note added: By the time we are about to post this paper, a nice paper [139] generalizing the result in [166] appears and their general result can be used to produce our results in the section 4.4.2. Also, the three papers [138, 35, 21] appear on the arXiv the same day as the authors post the first version of the draft on arXiv, which generalizes the idea of gauging non-normal or non-Abelian subgroup will lead to non-invertible symmetries to higher form and higher group symmetries in higher dimensions.[3]

4.2 Review on TDLs and their applications in 2d CFT

In this section, we give a brief review of TDLs in 2d CFT to fix the convention, and readers who are interested in a more detailed review should look at [46, 36]. We also review results on modular bootstraps and Cardy-like formulas which will be useful later.

4.2.1 TDLs in 2d CFT

In 2-dimensional conformal field theory, the symmetry defects of the ordinary symmetry are line operators and they are the very first examples of TDLs. For symmetry G of a CFT, the corresponding TDLs are denoted as $\mathscr{L}_g, g \in G$. The juxtaposition of two such TDLs satisfies the group algebra, $\mathscr{L}_g \times \mathscr{L}_h = \mathscr{L}_{gh}$. The TDL corresponds to the identity $\mathbb{1}$ in the group G is the identity line which we will also denote as $\mathbb{1} \equiv \mathscr{L}_{\mathbb{1}}$. Since every group element has its inverse,

[3]For understanding the non-invertible symmetries from holographic point of view, see recent papers [10, 93, 104, 8].

these TDLs also have their inverse in the sense that there exists another line such that they fuse into a single identity line $\mathbb{1}$, i.e. they are invertible TDLs.

However, TDLs are ubiquitous in the CFT, and the way they fuse is beyond the group algebra, more generally satisfying the fusion algebra. These TDLs then generate the non-invertible symmetries or the fusion category symmetries, since the TDLs may not have their inverse. The Kramer-Wannier duality line in the Ising CFT and most of the Verlinde lines in rational CFT are examples of the non-invertible TDLs.

The TDL $\mathscr{L}$ can be deformed locally without changing the correlation functions $\langle \mathscr{L} \cdots \rangle$ where $\cdots$ denotes any other operator insertions. The topological nature of these TDLs imply that they commute with stress energy tensor, therefore can be algebraically expressed as,

$$[L_n, \mathscr{L}_a] = [\bar{L}_n, \mathscr{L}_a] = 0, \tag{4.2.1}$$

where $L_n, \bar{L}_n$ are the generators of the Virasoro algebra. When the TDL $\mathscr{L}_a$ acts on a state $|\phi\rangle$ in Hilbert space $\mathscr{H}$, the resulting state $\mathscr{L}_a |\phi\rangle$ is still in $\mathscr{H}$.

When two TDLs are brought close to each other, their juxtaposition satisfies the fusion rule,

$$\mathscr{L}_a \times \mathscr{L}_b = \bigoplus_c N_{ab}^c \mathscr{L}_c \tag{4.2.2}$$

where N_{ab}^c is the fusion multiplicity which can only be non-negative integers. More specifically, the TDLs can join at the point-like junction, which is equipped with a Hilbert space. For example, the fusion of TDLs $\mathscr{L}_1, \mathscr{L}_2$ into $\mathscr{L}_3$ corresponds to a vector space with dimension depending on the fusion multiplicity, we dub the vector space as fusion space and denote by $V_{\mathscr{L}_1, \mathscr{L}_2}^{\mathscr{L}_3}$. Correspondingly, the TDL $\mathscr{L}_3$ can split into $\mathscr{L}_1, \mathscr{L}_2$, whose vector space is split space and denoted by $V_{\mathscr{L}_3}^{\mathscr{L}_1, \mathscr{L}_2}$.

More complicated fusion/split process can be decomposed into the fusion/split space with 3 TDLs. However, the decomposition is not unique but under the isomorphism, $\bigoplus_{\mathscr{L}_5} V_{\mathscr{L}_5}^{\mathscr{L}_1, \mathscr{L}_2} \otimes$

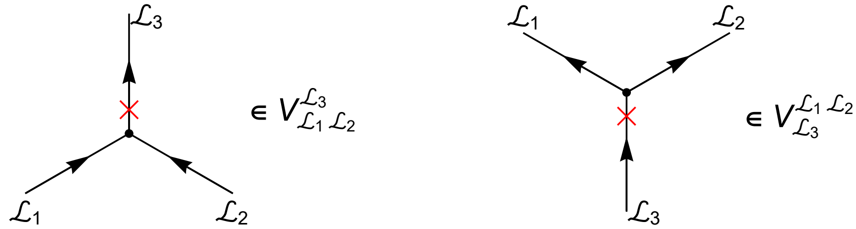


Figure 4.1. The fusion vertex of TDLs $\mathscr{L}_1, \mathscr{L}_2$ fuse into $\mathscr{L}_3$ and split vertex of TDLs $\mathscr{L}_3$ splits into $\mathscr{L}_1, \mathscr{L}_2$. The red cross marks the last leg, which determines the ordering of each vertex.

$V^{\mathscr{L}_5,\mathscr{L}_3}_{\mathscr{L}_4} \cong \bigoplus_{\mathscr{L}_6} V^{\mathscr{L}_2,\mathscr{L}_3}_{\mathscr{L}_6} \otimes V^{\mathscr{L}_1,\mathscr{L}_6}_{\mathscr{L}_4}$. This is called F-move and can be written diagrammatically as shown in Figure 4.2. Notice that here we use the notion of F-symbol, which is same as the crossing kernels $\mathscr{K}$ in [46] up to the flipping of some of the orientation of the TDLs.

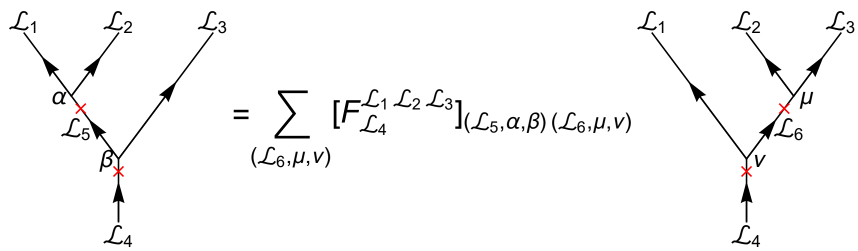


Figure 4.2. The fusion of three TDLs has two different ways, but they are related by the F-move and characterized by the F-symbol defined in this diagram.

The F-symbols are constrained by the self-consistent conditions when applying F-moves to the split process with 5 TDLs. Two sequences of F-moves start with the same state and end with the same state should be equivalent. These consistent conditions diagrammatically shown in Figure 4.3 yield the pentagon equations on the F-symbols,

$$\begin{aligned}
&\sum_{\delta} \left[F^{fcd}_{e}\right]_{(g,\beta,\gamma)(l,\nu,\delta)} \left[F^{abl}_{e}\right]_{(f,\alpha,\delta)(k,\mu,\lambda)} \\
&= \sum_{h,\sigma,\psi,\rho} \left[F^{abc}_{g}\right]_{(f,\alpha,\beta)(h,\psi,\sigma)} \left[F^{ahd}_{e}\right]_{(g,\sigma,\gamma)(k,\rho,\lambda)} \left[F^{bcd}_{k}\right]_{(h,\psi,\rho)(l,\nu,\mu)}.
\end{aligned} \tag{4.2.3}$$

For example, the invertible symmetry G with anomaly ω is described by the category of G-graded vector spaces, denoted by $\mathrm{Vec}^{\omega}_{G}$. The simple objects are $1-$dimensional $\mathbb{C}$-vector space labeled by $g \in G$, and physically they correspond to the TDLs which generates the g-action. The

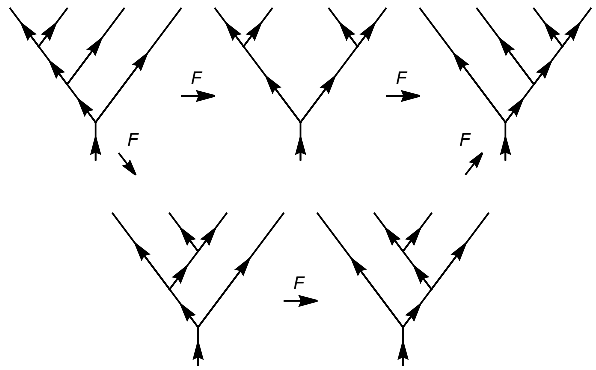


Figure 4.3. The upper 2 F-moves and the lower 3 F-moves yield the same diagram, which gives the pentagon equation in (4.2.3).

fusion rule is $\mathscr{L}_g \times \mathscr{L}_h = \mathscr{L}_{gh}$. The F-symbols of this category are $U(1)$ phase factors (rather than generic complex numbers in order to preserve the normalization), $\omega(g,h,k) \equiv F^{\mathscr{L}_g,\mathscr{L}_h,\mathscr{L}_k}_{\mathscr{L}_{ghk}}$ for $\mathscr{L}_{1,2,3} = \mathscr{L}_{g,h,k}$ as in Figure 4.2. These $U(1)$ phase factors satisfy the pentagon equations,

$$\omega(g,h,kl)\omega(gh,k,l) = \omega(h,k,l)\omega(g,hk,l)\omega(g,h,k), \tag{4.2.4}$$

which is the cocycle condition for the 3-cocycle $\omega : G \times G \times G \to U(1)$.

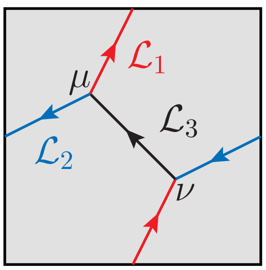
Notice that one can consider shifting the basis in $V^{g,h}_{gh}$ by a phase $\beta(g,h)$, which is an element in $C^2(G,U(1))$. This basis change does not change the physics and should be understood as a gauge transformation of the F-symbols. The F-symbol changes as

$$\omega(g,h,k) \to \omega(g,h,k)\frac{\beta(g,hk)\beta(h,k)}{\beta(g,h)\beta(gh,k)}, \tag{4.2.5}$$

and is interpreted as changing ω by an exact element in the set of 3-coboundaries $Z^3(G,U(1)) \subset C^3(G,U(1))$. Therefore, inequivalent Vec^{ω}_G's are labelled by a finite group G and its anomaly $\omega \in H^3(G,U(1)) = C^3(G,U(1))/Z^3(G,U(1))$.

4.2.2 Modular bootstrap with TDLs

By utilizing the modular transformation properties of the partition functions or correlation functions of a CFT on a torus with complex structure τ, one can extract many useful information of the CFT [43, 40, 167, 166, 141, 142, 58, 119, 136, 159, 156]. Here, we are interested in studying the modular properties of the CFT partition functions with networks of TDLs inserted on the torus, which is called the twisted partition function on the torus. We will adopt the convention from [181] on the twisted partition function over torus, as in Figure 4.4. To reveal the physical



$$:= Z^{\mathcal{L}_3,\mu,\nu}_{\mathcal{L}_1\mathcal{L}_2}(\tau,\overline{\tau}) = \mathrm{tr}_{\mathcal{H}_{\mathcal{L}_1}}(\widehat{\mathcal{L}}_2)_{\mathcal{L}_3,\mu,\nu}q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}$$

Figure 4.4. Convention on the twisted partition function $Z^{\mathscr{L}_3}_{\mathscr{L}_1\mathscr{L}_2}(\tau,\overline{\tau})$.

meaning of $Z^{\mathscr{L}_3,\mu,\nu}_{\mathscr{L}_1,\mathscr{L}_2}(\tau,\overline{\tau})$, we first consider the simple case where $\mathscr{L}_2 = \mathbb{1}$ and $\mathscr{L}_3 = \mathscr{L}_1 = \mathscr{L}$. For convenience, we will sometimes abbreviate this partition function as $Z_{\mathscr{L}}(\tau,\overline{\tau})$. Because the topological defect $\mathscr{L}$ is inserted along the time direction, it should be interpreted as a defect along the spatial direction, leading to a different Hilbert space on the spatial circle S^1, and we will denote this Hilbert space as the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$.

$$Z_{\mathscr{L}}(\tau,\overline{\tau}) = Z^{\mathscr{L},1,1}_{\mathscr{L},\mathbb{1}}(\tau,\overline{\tau}) = \mathrm{Tr}_{\mathscr{H}_{\mathscr{L}}}q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}. \tag{4.2.6}$$

For instance, if $\mathscr{L}$ is a $\mathbb{Z}_2$ symmetry defect η, then the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ is the $\mathbb{Z}_2$ twisted Hilbert space acquired by imposing the twisted boundary condition.

As an another simple example, let's consider instead $\mathscr{L}_1 = I$ and $\mathscr{L}_2 = \mathscr{L}_3 = \mathscr{L}$. Similarly, we will sometimes abbreviate this partition function as $Z^{\mathscr{L}}(\tau,\overline{\tau})$. In this configuration, the

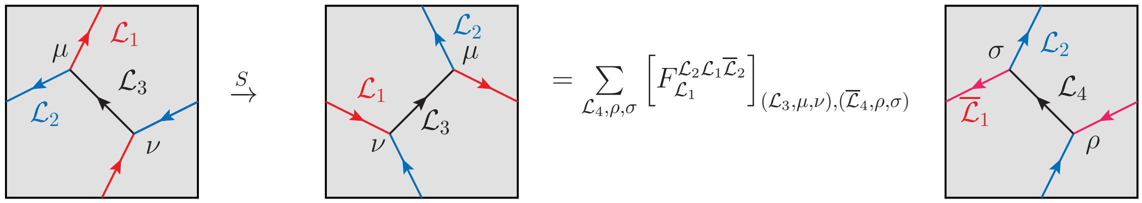


Figure 4.5. Generalized modular bootstrap equations from S modular transformation of the twisted partition function $Z^{\mathscr{L}_3,\mu,\nu}_{\mathscr{L}_1,\mathscr{L}_2}(\tau,\overline{\tau})$.

TDL $\mathscr{L}$ is inserted on an equal time slice, therefore should be interpreted as a quantum operator $\widehat{\mathscr{L}}$ on the Hilbert space $\mathscr{H}$:

$$Z^{\mathscr{L}}(\tau,\overline{\tau}) \equiv Z^{\mathscr{L},1,1}_{\mathbb{1},\mathscr{L}}(\tau,\overline{\tau}) = \mathrm{Tr}_{\mathscr{H}}\,\widehat{\mathscr{L}} q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}. \tag{4.2.7}$$

More generally, if we consider the twisted partition function $Z^{\mathscr{L}_3,\mu,\nu}_{\mathscr{L}_1,\mathscr{L}_2}(\tau,\overline{\tau})$, we should interpret $\mathscr{L}_2$ as a quantum operator $(\widehat{\mathscr{L}_2})_{\mathscr{L}_3,\mu,\nu}$ acting on the defect Hilbert space $\mathscr{H}_{\mathscr{L}_1}$, where the subscript $(\mathscr{L}_3,\mu,\nu)$ indicates that for different intermediate lines $\mathscr{L}_3$ and different vertex labels μ,ν when the fusion multiplicities are greater than 1, we will have a different operator in general.

$$Z^{\mathscr{L}_3,\mu,\nu}_{\mathscr{L}_1,\mathscr{L}_2}(\tau,\overline{\tau}) = \mathrm{Tr}_{\mathscr{H}_{\mathscr{L}_1}}(\widehat{\mathscr{L}_2})_{\mathscr{L}_3,\mu,\nu} q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}. \tag{4.2.8}$$

Under the modular transformation, the twisted partition function $Z^{\mathscr{L}_3,\mu,\nu}_{\mathscr{L}_1,\mathscr{L}_2}(\tau,\overline{\tau})$ transforms as the Figure 4.5,

$$Z^{\mathscr{L}_3,\mu,\nu}_{\mathscr{L}_1,\mathscr{L}_2}\left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right) = \sum_{\mathscr{L}_4,\rho,\sigma}\left[F^{\mathscr{L}_2\mathscr{L}_1\overline{\mathscr{L}}_2}_{\mathscr{L}_1}\right]_{(\mathscr{L}_3,\mu,\nu),(\overline{\mathscr{L}}_4,\rho,\sigma)} Z^{\mathscr{L}_4,\sigma,\rho}_{\mathscr{L}_2,\overline{\mathscr{L}}_1}(\tau,\overline{\tau}). \tag{4.2.9}$$

As a simple example, we may consider taking $\mathscr{L}_1$ to be the identity line. Then the F-symbol is always trivial in this case (for example, see [46]) and $\mu,\nu = 1$ as the fusion is always one dimensional. Then we have

$$Z_{\mathscr{L}}\left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right)\equiv Z_{\mathbb{1},\mathscr{L}}^{\mathscr{L},1,1}\left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right)=Z_{\mathscr{L},\mathbb{1}}^{\mathscr{L}}(\tau,\overline{\tau})\equiv Z^{\mathscr{L}}(\tau,\overline{\tau}). \tag{4.2.10}$$

As one can see, the S modular transformation relates the partition functions that count states the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ to the partition functions that compute the action of the TDL $\mathscr{L}$ on the Hilbert space $\mathscr{H}$. This allows us to derive interesting Cardy-like formula which we will now review in the next subsection.

4.2.3 Asymptotic density of states

As an application of the modular bootstrap reviewed above, one can derive the asymptotic density of states of a CFT [43, 159]. Intuitively, the modular bootstrap equation,

$$Z^{\mathscr{L}}\left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right)=Z_{\mathscr{L}}(\tau,\overline{\tau}), \tag{4.2.11}$$

relates the high-temperature limit of the partition function $Z^{\mathscr{L}}$ which computes the action of the TDL $\mathscr{L}$ on $\mathscr{H}$ to the low-temperature limit of the partition function $Z_{\mathscr{L}}$ which computes the spectrum of the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ and vice versa. Since in the low-temperature limit, the partition function is always dominated by the ground state, (4.2.11) essentially allows us to determine the partition function $Z^{\mathscr{L}}$ and $Z_{\mathscr{L}}$ in the high-temperature limit. In the high-temperature limit, states with different energy would contribute equally to the partition sum, knowing $Z_{\mathscr{L}}$ in such limit would allow us to derive an approximation of the density of states for the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$. This idea is made into a rigorous mathematical statement in [166] using the techniques in [159], and we will only mention the result below.

Let $F_{\mathscr{L}}(\Delta)$ denote the total number of states with scaling dimension $\Delta' < \Delta$ in the defect Hilbert space. By asymptotic density of states, we mean a continuous function $\rho_{0,\mathscr{L}}(\Delta)$ which approximates the actual density of states in the sense that

$$F_{\mathscr{L}}(\Delta) = \int_0^{\Delta} d\Delta' \rho_{0,\mathscr{L}}(\Delta') + O(\Delta^{-1/2}), \quad \Delta \to \infty. \tag{4.2.12}$$

Let

$$\rho_0(\Delta) = \left(\frac{c}{48\Delta^3}\right)^{1/4} \exp\left[2\pi\sqrt{\frac{c\Delta}{3}}\right], \tag{4.2.13}$$

it is shown in [166] that the asymptotic density of states $\rho_{0,\mathscr{L}}$ of the defect Hilbert space $\mathscr{H}_{\mathscr{L}}$ is simply given by

$$\rho_{0,\mathscr{L}}(\Delta) = \langle 0|\mathscr{L}|0\rangle \rho_0(\Delta), \tag{4.2.14}$$

where $|0\rangle$ is the ground state and $\langle 0|\mathscr{L}|0\rangle$ is the quantum dimension of the TDL $\mathscr{L}$.

Furthermore, let's consider the theory with finite global symmetry G. Then the states in the Hilbert space will organize into irreducible representations of G. For simplicity, we assume G acts faithfully. Recall that a particular type of irreducible representation α can be counted by $\frac{1}{|G|}\sum_{g\in G}\chi_\alpha(g)^* \mathrm{Tr} g$ in a reducible representation of G where $\chi_\alpha(g)$ is the character function of the irrep α and $|G|$ is the order of the group, then the partition function counts the number of irrep α in the Hilbert space is given by

$$\frac{1}{|G|}\sum_{g\in G}\chi_\alpha(g)^* Z^g(\tau,\overline{\tau}), \tag{4.2.15}$$

whose high-temperature limit is known since each Z^g's high-temperature limit is known. Moreover, the assumption that G acts faithfully implies that the ground state in the defect Hilbert space has $h+\overline{h}>0$, therefore (4.2.15) is dominated by $Z^{\mathbb{1}}(\tau,\overline{\tau}) = Z(\tau,\overline{\tau})$ in the high temperature limit. This allows us to derive the following result. Every irreducible representation has to appear in the Hilbert space $\mathscr{H}$ and they also have a Cardy-like growth. Specifically, we can consider the asymptotic growth $\rho_{0,\alpha}(\Delta)$ of the occurrence of a particular irrep α of G as a function of the scaling dimension Δ takes the form,

$$\rho_{0,\alpha}(\Delta) = \frac{d_\alpha}{|G|}\rho_0(\Delta). \tag{4.2.16}$$

This result will be useful for us later.

4.3 Non-intrinsic triality fusion category as group theoretical fusion category

As pointed out in [181], the origin of the triality fusion rule in the KT theory arises from gauging $\mathbb{Z}_2$ subgroup of $A_4 \subset SO(4)$ in the $SU(2)_1$ theory. Generically, gauging a subgroup H which is not normal in G or not Abelian leads to non-invertible symmetries [160]. In this section, we first review the construction of triality acquired in [180, 181]. We then discuss how to understand the group theoretical triality fusion category using the mathematical tools of bimodule categories in [164]. Specifically, we describe how to understand and compute simple TDLs, fusion rules and F-symbols.

4.3.1 A brief review of the triality category discovered in [180, 181]

The origin of the triality fusion category discovered in [180, 181] is a result of gauging $\mathbb{Z}_2$ subgroup in a theory with A_4 global symmetry. Notice that A_4 is an order 12 group and can be presented as

$$A_4 = \langle \sigma, \eta, q | q^3 = \sigma^2 = \eta^2, q\sigma q^{-1} = \sigma\eta = \eta\sigma, q\eta q^{-1} = \sigma \rangle. \tag{4.3.1}$$

This means we can think of A_4 containing a $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\eta$ subgroup, and the conjugation by the $\mathbb{Z}_3^q$ generator q will permute the three $\mathbb{Z}_2$ generators in $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\eta$. After gauging the $\mathbb{Z}_2^\sigma$ symmetry, the $\mathbb{Z}_2^\eta$ subgroup will survive since it commutes with the $\mathbb{Z}_2^\sigma$ subgroup, together with the quantum $\mathbb{Z}_2$ which we will denote as $\mathbb{Z}_2^{\hat{\sigma}}$, they form the $\mathbb{Z}_2^{\hat{\sigma}} \times \mathbb{Z}_2^\eta$ invertible symmetries in the gauged theory. The symmetry operator q does not commute with σ, therefore is not gauge invariant, and will not appear as a genuine topological line operator in the gauged theory. However, the linear combination

$$q+\sigma q\sigma \tag{4.3.2}$$

does commute with σ and will survive as a genuine topological line operator in the gauged theory, which we will denote as $\mathscr{L}_Q$. However, this operator is not invertible, as it has quantum dimension 2. Similarly, its orientation reversal $\mathscr{L}_{\overline{Q}}$ relates to the gauge-invariant linear combination $q^{-1}+\sigma q^{-1}\sigma$. It is pointed out in [180, 181] that $\mathscr{L}_Q,\mathscr{L}_{\overline{Q}}$ together with the $\mathbb{Z}_2^{\hat{\sigma}}\times\mathbb{Z}_2^{\eta}$ forms the triality fusion categories with the fusion rules,

$$\begin{aligned}
&g\times\mathscr{L}_Q=\mathscr{L}_Q\times g=\mathscr{L}_Q, \quad g\times\mathscr{L}_{\overline{Q}}=\mathscr{L}_{\overline{Q}}\times g=\mathscr{L}_{\overline{Q}},\\
&\mathscr{L}_Q\times\mathscr{L}_Q=2\mathscr{L}_{\overline{Q}}, \quad \mathscr{L}_{\overline{Q}}\times\mathscr{L}_{\overline{Q}}=2\mathscr{L}_Q, \quad \mathscr{L}_{\overline{Q}}\times\mathscr{L}_Q=\mathscr{L}_Q\times\mathscr{L}_{\overline{Q}}=\sum_{g\in\mathbb{Z}_2^{\hat{\sigma}}\times\mathbb{Z}_2^{\eta}} g.
\end{aligned} \tag{4.3.3}$$

The existence of the triality fusion category implies the theory is invariant under the $\mathbb{Z}_2^{\hat{\sigma}}\times\mathbb{Z}_2^{\eta}$ gauging, but $\mathbb{Z}_2^{\hat{\sigma}}\times\mathbb{Z}_2^{\eta}$ charge assignments will be permuted. This can be understood as the following. Gauging the $\mathbb{Z}_2^{\hat{\sigma}}$ quantum symmetry gives back the original theory with A_4 symmetry, then gauging $\mathbb{Z}_2^{\eta}$ symmetry will give the same theory acquired from gauging the $\mathbb{Z}_2^{\sigma}$ symmetry, since $\mathbb{Z}_2^{\eta}$ and $\mathbb{Z}_2^{\sigma}$ are related by the conjugation of q. However, this will change the $\mathbb{Z}_2\times\mathbb{Z}_2$ charge assignment in the gauged theory since we are gauging different but equivalent $\mathbb{Z}_2$'s.

4.3.2 Group theoretical fusion category

Let's consider the fusion category Vec_G^{ω}, where G is a finite group and $\omega\in H^3(G,U(1))$. Physically, this fusion category describes the global symmetry G with anomaly ω for 2-dimensional field theory. The simple elements are labeled by group element $g\in G$ and the fusion rule is simply the product of the group elements. The F-symbols are given by

$$F_l^{ghk}=\omega(g,h,k). \tag{4.3.4}$$

If we consider a finite subgroup H in G, then the anomaly-free condition for H is that ω as a function from G^3 to $U(1)$ restricting to H^3 is trivial, that is,

$$\omega(h_1,h_2,h_3)=1, \quad \forall h_i \in H. \tag{4.3.5}$$

When H is anomaly free, we can consider gauging the subgroup H and we have to choose a discrete torsion $\psi \in H^2(G,U(1))$. The resulting fusion category which describes the global symmetry in the gauged theory is denoted as $\mathscr{C}(G,\omega,H,\psi)$ and this class of fusion category is called the group theoretical fusion category. In particular, $\mathrm{Vec}_G^\omega = \mathscr{C}(G,\omega,\mathbb{Z}_1,1)$.

Generically, when H is not a normal subgroup of G or is not Abelian, then the resulting group theoretical fusion category is not of the form Vec_G^ω, meaning it contains non-invertible simple lines. For example, gauging the $\mathbb{Z}_2$ symmetry of S^3 with $\omega=1$ leads to the fusion category $\mathrm{Rep}(S^3)$ [36] and gauging the $\mathbb{Z}_2$ symmetry of A_4 with $\omega=1$ leads to the triality fusion category [180, 181].

Below, we will describe how to understand and compute the simple TDLs, fusion rules, and F-symbols of the triality fusion category from the data of the group theoretical fusion category. It would be helpful to point out that although the F-symbols for the group theoretical fusion category can be computed by the more general method given in [20] by considering the tube algebra $\mathbf{Tub}_{\mathscr{C}}(\mathscr{M})$, it is more convenient to compute the F-symbols using our approach. For instance, the calculation of the F-symbol for $\mathscr{C}(A_4,1,\mathbb{Z}_2,1)$ using [20] requires to construct an explicit Artin-Wedderburn isomorphism from the tube algebra $\mathbf{Tub}_{\mathrm{Vec}_{A_4}}(\mathscr{M})$ to a direct sum of several matrix algebras, but since the tube algebra $\mathbf{Tub}_{\mathrm{Vec}_{A_4}}(\mathscr{M})$ has a very large dimension 432 in this case, it's computationally hard to explicitly construct the Artin-Wedderburn isomorphism.

4.3.3 Simple lines

As pointed out in [164], the simple TDLs in the gauged theory are described by indecomposable $A-A$ bimodules in Vec_G^ω, where A is the group algebra of H twisted by some 2-cocycle of $\psi \in H^2(H,U(1))$, corresponding to a choice of discrete torsion. In the case of $SU(2)_1$ theory, we take $G = A_4$ and G is anomaly free so that $\omega = 1 \in H^3(G,U(1))$. $H = \mathbb{Z}_2 \subset G$ and there's no non-trivial discrete torsion for $\mathbb{Z}_2$, so we can choose ψ to be trivial as well. Hence, A is simply the group algebra of $\mathbb{Z}_2$, namely a 2-dimensional vector space over $\mathbb{C}$ with a basis $\{1,\sigma\}$ equipped with multiplication $1^2 = \sigma^2 = 1$ and $\sigma 1 = 1\sigma = \sigma$ which is the group multiplication of $\mathbb{Z}_2$.

In the special case of group-theoretical fusion category, by $A-A$ bimodule M, we mean a $\mathbb{C}$-vector space M together a left multiplication $A \times M \to M$ denoted as am for $a \in A, m \in M$ and a right multiplication $M \times A \to M$ denoted as ma for $a \in A, m \in M$ such that $\forall a_i \in A$ and $m \in M$:

$$a_1(a_2m) = (a_1a_2)m \equiv a_1a_2m, \quad (ma_1)a_2 = m(a_1a_2) \equiv ma_1a_2, \quad (a_1m)a_2 = a_1(ma_2) \equiv a_1ma_2. \tag{4.3.6}$$

The indecomposable $A-A$ bimodule M is of the following form. Consider a double coset HgH of H in G, and let H^g denote the little group of HgH:

$$H^g = \{h \in H | \exists h' \in H\, s.t.\, hgh' = g\} \subset H. \tag{4.3.7}$$

As one can check from the above definition, the little group $H^{g'}$ does not depend on the choice of representative $g' \in HgH$ up to isomorphism.

Each indecomposable $A-A$ bimodule is labelled by a double coset HgH and an irreducible representation ρ (potentially twisted by some 2-cocycle of the little group) of the little group of an arbitrary element in the double coset HgH. More specifically, given (HgH,ρ), an indecomposable $A-A$ bimodule M^{ρ}_{HgH} is a vector space over $\mathbb{C}$ such that

$$M^{\rho}_{HgH} = \bigoplus_{g' \in HgH} M^{\rho}_{g'}. \tag{4.3.8}$$

To describe the action of A on M^{ρ}_{HgH}, we only need to describe the action of the basis $\{h\}_{h\in H}$

on M^ρ_{HgH}. And multiplication of h on the left induces an isomorphism from $M^\rho_{g'}$ to $M^\rho_{hg'}$ while the multiplication of h on the right induces an isomorphism from $M^\rho_{g'}$ to $M^\rho_{g'h}$. This implies $M^\rho_{g'}$'s are isomorphic to each other. Consider M^ρ_g and we find for $h \in H^g$, multiplying h on the left and multiplying the corresponding h' on the right leads to an isomorphism between M^ρ_g and itself (since $hgh' = g$), this is why H^g is called the little group. M^ρ_g is the vector space of the irreducible representation ρ of H^g. Notice that in the case where the group algebra A is twisted by non-trivial 2-cocycle ψ, the composition of two left(right) multiplications on M^ρ_g can composite non-trivially. But in the case of the gauging $\mathbb{Z}_2$ symmetry in A_4 with no anomaly, such data is trivial. The fusion of the TDLs in the gauged theory is described by the tensor product of $A-A$ bimodule over the algebra A.

We now list the indecomposable bimodule in $\mathscr{C}(A_4, 1, \mathbb{Z}_2^\sigma, 1)$ and compute its fusion rule, and show indeed we reproduce the fusion rule of the triality fusion category. There are 4 double cosets of $H = \mathbb{Z}_2 = \{(), (12)(34)\} \equiv \{1, \sigma\}$ in A_4:[4]

$$\begin{aligned} &I = \{(), (12)(34)\} \equiv \{1, \sigma\}, \quad J = \{(13)(24), (14)(23)\} \equiv \{\eta, \eta\sigma\}, \\ &Q = \{(143), (124), (132), (234)\}, \quad \overline{Q} = \{(134), (142), (123), (243)\}. \end{aligned} \tag{4.3.9}$$

The little group for the first two double cosets is H itself while for the last two double cosets are trivial. Hence, for the first two double cosets I, J, we need to label the irreducible representation $\pm$ of $H = \mathbb{Z}_2$ as well. Hence, we find the following 6 indecomposable $A-A$ bimodules,

$$\begin{aligned} &M_I^\pm = M_1^\pm \oplus M_\sigma^\pm, \quad M_J^\pm = M_\eta^\pm \oplus M_{\sigma\eta}^\pm, \\ &M_Q = M_{(143)} \oplus M_{(124)} \oplus M_{(132)} \oplus M_{(234)}, \\ &M_{\overline{Q}} = M_{(134)} \oplus M_{(142)} \oplus M_{(123)} \oplus M_{(243)}. \end{aligned} \tag{4.3.10}$$

Notice that each M^ρ_g appears in the direct sum is a 1-dimensional vector space, so we will choose

[4]The q used in (4.3.1) is the cycle (143).

a basis vector m_g^ρ for each M_g^ρ. We choose the action of $h \in A$ on m_g^ρ's as,

$$1m_g^\rho = m_g^\rho 1 = m_g^\rho, \quad \sigma m_g^\pm = \pm m_{\sigma g}^\pm, \quad m_g^\pm \sigma = m_{g\sigma}^\pm. \tag{4.3.11}$$

As one can check, for M_I^- and M_J^-, the multiplication of σ on left and right simultaneously does generate non-trivial action of $\mathbb{Z}_2$ on M_g^- for $g \in I, J$. The bimodules $M_I^\pm$ correspond to the quantum $\mathbb{Z}_2$ symmetry in the gauged theory, while M_J^+ generates the unbroken $\mathbb{Z}_2$ symmetry in A_4. Together they form the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in the gauged theory. M_Q and $M_{\overline{Q}}$ are identified with the triality line $\mathscr{L}_Q$ and $\mathscr{L}_{\overline{Q}}$.

To conclude, we point it out there is a natural dual module $\mathrm{Hom}_A(M_{HgH}^\rho, A)$ for each bimodule M_{HgH}^ρ, which is isomorphic to $M_{Hg^{-1}H}^{\rho^\dagger}$.[5] Physically this corresponds to taking the orientation reversal of the TDL.

4.3.4 Fusion rules

The fusion of the TDLs in the gauged theory corresponds to the tensor product of bimodules over the algebra A. Generically, let M and N be $A-A$ bimodules, then

$$M \otimes_A N = M \otimes N / \sim \tag{4.3.12}$$

where the first tensor product is the tensor product of vector spaces and $\sim$ is the equivalence relation

$$(ma) \otimes n \sim m \otimes (an), \quad a \in A, m \in M, n \in N. \tag{4.3.13}$$

The $M \otimes_A N$ is naturally an $A-A$ bimodule and we can then decompose it as a direct sum over indecomposable $A-A$ bimodules.

Using this, one can compute the tensor product of bimodules explicitly and acquire the

[5]Generically, for an $A-B$ bimodule M, one can either consider the dual being $\mathrm{Hom}_A(M,A)$ or $\mathrm{Hom}_B(M,B)$, which are both $B-A$ bimodules. Here, since we are considering $A-A$ bimodule, we can consider either choice and the results should be isomorphic.

fusion rule. There is an additional rule one needs to know is that the grading of $M_g^\rho \otimes_A M_{g'}^{\rho'}$ is gg'.

We will not list all the calculations but provide several examples to help the reader understand the procedure.

To start, let's consider the fusion between M_I^+ with a generic M_{HgH}^ρ. The tensor product $M_I^+ \otimes M_{HgH}^\rho$ has a basis $\{m_1^+ \otimes m_{g'}^\rho, m_\sigma^+ \otimes m_{g'}^\rho\}_{g' \in HgH}$. After the identification with the equivalence relation, we denote the equivalence class as $m_1^+ \otimes_A m_{g'}^\rho = \rho(\sigma) m_\sigma^+ \otimes_A m_{\sigma g'}^\rho$. The resulting bimodule $M_I^+ \otimes_A M_g^\rho$ is isomorphic to M_g^ρ itself where $m_1^+ \otimes_A m_g^\rho \simeq m_g^\rho$. We can check the left action of σ on $m_1^+ \otimes_A m_g^\rho$ gives the correct sign for $\rho = -$:

$$\sigma(m_1^+ \otimes_A m_g^\rho) = (\sigma m_1^+) \otimes_A m_g^\rho = (m_1^+ \sigma) \otimes_A m_g^\rho = m_1^+ \otimes_A (\sigma m_g^\rho) = \rho(\sigma) m_1^+ \otimes_A m_{\sigma g}^\rho, \quad (4.3.14)$$

where $\rho(g)$ will produce the desired sign when $\rho = -$. Hence, we find the fusion rule $M_I^+ \otimes_A M_{HgH}^\rho = M_{HgH}^\rho$.

As another example, let's consider the fusion between M_Q and M_Q. There are 16 basis vectors in $M_Q \otimes M_Q$ and after modding out the equivalence relation we are left with 8 basis vectors. The grading suggests there are two copies of $M_{\overline{Q}}$ and we take them to be

$$\begin{aligned}
&m_{(143)} \otimes_A m_{(143)} \simeq m_{(134),1}, \quad m_{(143)} \otimes_A m_{(124)} \simeq m_{(123),1}, \\
&m_{(132)} \otimes_A m_{(143)} \simeq m_{(142),1}, \quad m_{(132)} \otimes_A m_{(124)} \simeq m_{(243),1}, \\
&m_{(134)} \otimes_A m_{(234)} \simeq m_{(142),2}, \quad m_{(143)} \otimes_A m_{(132)} \simeq m_{(243),2}, \\
&m_{(132)} \otimes_A m_{(132)} \simeq m_{(123),2}, \quad m_{(132)} \otimes_A m_{(234)} \simeq m_{(134),2},
\end{aligned} \quad (4.3.15)$$

where the subscript $m_{g,i}, i = 1,2$ indicates which copies of $M_{\overline{Q}}$.

As the final example, let's consider the fusion between M_Q and $M_{\overline{Q}}$. There are 16 basis vectors in $M_Q \otimes M_{\overline{Q}}$ and after modding out the equivalence relation there are only 8 left. After rewrite them as different linear combinations, we find they generate $M_I^+ \oplus M_I^- \oplus M_J^+ \oplus M_J^-$

where

$$\begin{aligned}
\frac{m_{(132)}\otimes_A m_{(123)}+m_{(143)}\otimes_A m_{(134)}}{\sqrt{2}} &\simeq m_1^+, & \frac{m_{(132)}\otimes_A m_{(134)}+m_{(143)}\otimes_A m_{(123)}}{\sqrt{2}} &\simeq m_\sigma^+,\\
\frac{m_{(132)}\otimes_A m_{(123)}-m_{(143)}\otimes_A m_{(134)}}{\sqrt{2}} &\simeq m_1^-, & \frac{m_{(132)}\otimes_A m_{(134)}-m_{(143)}\otimes_A m_{(123)}}{\sqrt{2}} &\simeq m_\sigma^-,\\
\frac{m_{(132)}\otimes_A m_{(243)}+m_{(143)}\otimes_A m_{(142)}}{\sqrt{2}} &\simeq m_\eta^+, & \frac{m_{(132)}\otimes_A m_{(142)}+m_{(143)}\otimes_A m_{(243)}}{\sqrt{2}} &\simeq m_{\sigma\eta}^+,\\
\frac{m_{(132)}\otimes_A m_{(243)}-m_{(143)}\otimes_A m_{(142)}}{\sqrt{2}} &\simeq m_\eta^-, & \frac{m_{(132)}\otimes_A m_{(142)}-m_{(143)}\otimes_A m_{(243)}}{\sqrt{2}} &\simeq m_{\sigma\eta}^-.
\end{aligned} \tag{4.3.16}$$

One can check the above identification is consistent with the action of σ. For instance,

$$\sigma m_1^- \simeq \sigma \frac{m_{(132)}\otimes_A m_{(123)}-m_{(143)}\otimes_A m_{(134)}}{\sqrt{2}} = \frac{m_{(143)}\otimes_A m_{(123)}-m_{(132)}\otimes_A m_{(134)}}{\sqrt{2}} \simeq -m_\sigma^-. \tag{4.3.17}$$

Table 4.1. Result of tensor product of $A-A$ bimodules.

$\otimes_A$	M_I^+	M_I^-	M_J^+	M_J^-	M_Q	$M_{\overline{Q}}$
M_I^+	M_I^+	M_I^-	M_J^+	M_J^-	M_Q	$M_{\overline{Q}}$
M_I^-	M_I^-	M_I^+	M_J^-	M_J^+	M_Q	$M_{\overline{Q}}$
M_J^+	M_J^+	M_J^-	M_I^+	M_I^-	M_Q	$M_{\overline{Q}}$
M_J^-	M_J^-	M_J^+	M_I^-	M_I^+	M_Q	$M_{\overline{Q}}$
M_Q	M_Q	M_Q	M_Q	M_Q	$M_{\overline{Q}}\oplus M_{\overline{Q}}$	$M_I^+\oplus M_I^-\oplus M_J^+\oplus N_J^-$
$M_{\overline{Q}}$	$M_{\overline{Q}}$	$M_{\overline{Q}}$	$M_{\overline{Q}}$	$M_{\overline{Q}}$	$M_I^+\oplus M_I^-\oplus M_J^+\oplus N_J^-$	$M_Q\oplus M_Q$

The rest of the tensor products can be computed as above and we list the result in Tab. 4.1. With the identifications between bimodules and TDLs given previously, we reproduce the fusion rule of the triality fusion category.

4.3.5 F-symbols

The advantage of the description using bimodules is that we can easily compute the F-symbols for the resulting fusion category. The local fusion of two TDLs labeled by bimodules M,N into the TDL labeled by bimodule L can be identified as the vector space of bimodule homomorphism from $M \otimes_A N$ to L:

$$\mathrm{Hom}_{A-A}(M \otimes_A N, L). \tag{4.3.18}$$

Here, a $A-A$ bimodule homomorphism $\phi \in \mathrm{Hom}_{A-A}(M,N)$ from $A-A$ bimodule M to $A-A$ bimodule N is a $\mathbb{C}$-linear map from M to N such that

$$a\phi(m)a' = \phi(ama'), \quad \forall a, a' \in A, m \in M. \tag{4.3.19}$$

We can then consider choose a basis for each vector space $\mathrm{Hom}_{A-A}(M \otimes_A N, L)$ and use the Greek letter $\mu, \nu, \cdots = 1, 2, \cdots$ to label the basis vectors.

For instance, let's consider the fusion $M_1^+ \otimes_A M_g^\rho$ studied previously. Since $M_1^+ \otimes_A M_g^\rho \simeq M_g^\rho$, there is only one fusion channel and $\mathrm{Hom}_{A-A}(M \otimes_A N, L)$ is a 1-dim vector space. The isomorphism

$$m_1^+ \otimes_A m_g^\rho \simeq m_g^\rho \tag{4.3.20}$$

is a basis vector for $\mathrm{Hom}_{A-A}(M \otimes_A N, L)$.

As an another example, let's consider $M_Q \otimes_A M_Q \simeq M_{\overline{Q}} \oplus M_{\overline{Q}}$. In this case, $M_{\overline{Q}}$ appears twice in the direct sum, therefore $\mathrm{Hom}_{A-A}(M_Q \otimes_A M_Q, M_{\overline{Q}})$ would be 2-dimensional. The choice of identifications in (4.3.15) leads a basis in $\mathrm{Hom}_{A-A}(M_Q \otimes_A M_Q, M_{\overline{Q}})$, which are given by two $A-A$ bimodule homomorphism ϕ_1, ϕ_2 from $M_Q \otimes_A M_Q \to M_{\overline{Q}}$:

$$\phi_1 : \begin{pmatrix} m_{(143)} \otimes_A m_{(143)} \\ m_{(143)} \otimes_A m_{(124)} \\ m_{(132)} \otimes_A m_{(124)} \\ m_{(132)} \otimes_A m_{(143)} \end{pmatrix} \mapsto \begin{pmatrix} m_{(134)} \\ m_{(123)} \\ m_{(243)} \\ m_{(142)} \end{pmatrix}, \tag{4.3.21}$$

and,

$$\phi_2 : \begin{pmatrix} m_{(143)} \otimes_A m_{(234)} \\ m_{(143)} \otimes_A m_{(132)} \\ m_{(132)} \otimes_A m_{(132)} \\ m_{(132)} \otimes_A m_{(234)} \end{pmatrix} \mapsto \begin{pmatrix} m_{(142)} \\ m_{(243)} \\ m_{(123)} \\ m_{(134)} \end{pmatrix}. \tag{4.3.22}$$

Finally, let's consider the fusion $M_Q \otimes_A M_{\overline{Q}} \simeq M_I^+ \oplus M_I^- \oplus M_J^+ \oplus M_J^-$. Since there are 4 modules that appear in the direct sum and each only appears once, there are 4 Hom spaces and each has dimension 1. For instance, $\mathrm{Hom}_{A-A}(M_Q \times_A M_{\overline{Q}}, M_I^-)$ is the space of projection maps from $M_Q \otimes_A M_{\overline{Q}}$ to M_I^-. Our identification in (4.3.16) also determines a basis vector (which is a projection map) in $\mathrm{Hom}_{A-A}(M_Q \times_A M_{\overline{Q}}, M_I^-)$, given by the following $A-A$ bimodule homomorphism,

$$\begin{pmatrix} m_{(132)} \otimes_A m_{(123)} \simeq m_{(234)} \otimes_A m_{(243)} \\ m_{(143)} \otimes_A m_{(134)} \simeq m_{(124)} \otimes_A m_{(142)} \\ m_{(143)} \otimes_A m_{(123)} \simeq m_{(124)} \otimes_A m_{(243)} \\ m_{(132)} \otimes_A m_{(134)} \simeq m_{(234)} \otimes_A m_{(142)} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}} m_1^- \\ -\frac{1}{\sqrt{2}} m_1^- \\ -\frac{1}{\sqrt{2}} m_\sigma^- \\ \frac{1}{\sqrt{2}} m_\sigma^- \end{pmatrix} \tag{4.3.23}$$

where unlisted elements are mapped to 0.

Let's consider the F-symbol. The local fusions of the diagrams on both sides of the diagram give $A-A$ bimodule homomorphisms from $M \otimes_A N \otimes_A L$ to G in $\mathrm{Hom}_{A-A}(M \otimes_A N \otimes_A L, G)$, and the F-symbol can be interpreted as the matrix elements of the linear transformation between two sets of elements in $\mathrm{Hom}_{A-A}(M \otimes_A N \otimes_A L, G)$.

With a choice of basis for each $\mathrm{Hom}_A(M \otimes_A N, P)$, we can acquire F-symbols which are the matrix elements of the above transformation. Practically, let's consider the fixed bimodule M, N, L, G and choose basis vectors $\phi_{M \otimes_A N \to P, \mu} \in \mathrm{Hom}_A(M \otimes_A N, L)$ for every junction space $\mathrm{Hom}_A(M \otimes_A N, L)$, then the diagram on the left hand side leads to an element $\phi_{P \otimes_A L \to G, \nu} \circ$

$\phi_{M\otimes_A N\to P,\mu} \in \mathrm{Hom}_{A-A}(M\otimes_A N\otimes_A L, G)$:

$$\phi_{P\otimes_A L\to G,\nu}\circ\phi_{M\otimes_A N\to P,\mu}: m\otimes_A n\otimes_A l \mapsto \phi_{P\otimes_A L\to G,\nu}(\phi_{M\otimes_A N,\mu}(m\otimes_A n)\otimes_A l). \tag{4.3.24}$$

Similarly, $\phi_{M\otimes_A Q\to G,\mu}\circ\phi_{N\otimes_A L\to Q,\nu} \in \mathrm{Hom}_{A-A}(M\otimes_A N\otimes_A L, G)$ is defined as

$$\phi_{M\otimes_A Q\to G,\mu}\circ\phi_{N\otimes_A L\to Q,\nu}: m\otimes_A n\otimes_A l \mapsto \phi_{M\otimes_A Q\to G,\mu}(m\otimes_A \phi_{N\otimes_A L\to Q,\nu}(n\otimes_A l)). \tag{4.3.25}$$

The F-symbols are just $\mathbb{C}$-numbers such that the following equations of $A-A$ bimodule homomorphisms from $M\otimes_A N\otimes_A L$ to G hold

$$\phi_{P\otimes_A L\to G,\nu}\circ\phi_{M\otimes_A N\to P,\mu} = \sum_{Q,\alpha,\beta}\left[F^{MNL}_G\right]_{(P,\mu,\nu),(Q,\alpha,\beta)}\phi_{M\otimes_A Q\to G,\beta}\circ\phi_{N\otimes_A L\to Q,\alpha}. \tag{4.3.26}$$

To solve the above equation, we only need to evaluate the A-homomorphism on basis vectors of $M\otimes_A N\otimes_A L\to G$, which will produce a set of linear equations and can be solved quite easily.

As an example, let's consider taking $M = N = Q = M_Q, L = M_1^-, P = G = M_{\overline{Q}}$. This would compute the F-symbol $\left[F^{\mathscr{L}_Q\mathscr{L}_Q\hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_Q,1,\beta)}$, that is, we want to solve,

$$\begin{aligned}&\phi_{M_{\overline{Q}}\otimes_A M_1^-\to M_{\overline{Q}},1}\circ\phi_{M_Q\otimes_A M_Q\to M_{\overline{Q}},\mu}\\ =&\sum_{\beta=1,2}\left[F^{\mathscr{L}_Q\mathscr{L}_Q\hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_Q,1,\beta)}\phi_{M_Q\otimes_A M_Q\to M_{\overline{Q}},\beta}\circ\phi_{M_Q\otimes_A M_1^-\to M_Q,1}.\end{aligned} \tag{4.3.27}$$

We consider evaluate the above equation on $m_{(143)}\otimes_A m_{(143)}\otimes_A m_1^-$ and $m_{(132)}\otimes_A m_{(132)}\otimes_A m_1^-$.

As an example, we do this for $m_{(143)} \otimes_A m_{(143)} \otimes_A m_1^-$. On the left-hand side, we have,

$$\begin{aligned}
&\phi_{M_{\overline{Q}} \otimes_A M_1^- \to M_{\overline{Q}},1} \circ \phi_{M_Q \otimes_A M_Q \to M_{\overline{Q}},\mu}(m_{(143)} \otimes_A m_{(143)} \otimes_A m_1^-) \\
=&\phi_{M_{\overline{Q}} \otimes_A M_1^- \to M_{\overline{Q}},1}(\phi_{M_Q \otimes_A M_Q \to M_{\overline{Q}},\mu}(m_{(143)} \otimes_A m_{(143)}) \otimes_A m_1^-) \\
=&\phi_{M_{\overline{Q}} \otimes_A M_1^- \to M_{\overline{Q}},1}(\delta_{\mu,1} m_{(134)} \otimes_A m_1^-) \\
=&\delta_{\mu,1} m_{(134)},
\end{aligned} \tag{4.3.28}$$

and on the right-hand side, we have,

$$\begin{aligned}
&\phi_{M_Q \otimes_A M_Q \to M_{\overline{Q}},\beta} \circ \phi_{M_Q \otimes_A M_1^- \to M_Q,1}(m_{(143)} \otimes_A m_{(143)} \otimes_A m_1^-) \\
=&\phi_{M_Q \otimes_A M_Q \to M_{\overline{Q}},\beta}(m_{(143)} \otimes_A \phi_{M_Q \otimes_A M_1^- \to M_Q,1}(m_{(143)} \otimes_A m_1^-)) \\
=&\phi_{M_Q \otimes_A M_Q \to M_{\overline{Q}},\beta}(m_{(143)} \otimes_A m_{(143)}) \\
=&\delta_{\beta,1} m_{(134)}.
\end{aligned} \tag{4.3.29}$$

Hence, we get the following 2 linear equations,

$$1 = \left[F^{\mathscr{L}_Q \mathscr{L}_Q \hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},1,1),(\mathscr{L}_Q,1,1)}, \quad 0 = \left[F^{\mathscr{L}_Q \mathscr{L}_Q \hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},2,1),(\mathscr{L}_Q,1,1)}. \tag{4.3.30}$$

Similarly, we get another two equations when evaluating on $m_{(132)} \otimes_A m_{(132)} \otimes_A m_1^-$:

$$0 = \left[F^{\mathscr{L}_Q \mathscr{L}_Q \hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},1,1),(\mathscr{L}_Q,1,2)}, \quad -1 = \left[F^{\mathscr{L}_Q \mathscr{L}_Q \hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},2,1),(\mathscr{L}_Q,1,2)}. \tag{4.3.31}$$

Solving the four equations, we found,

$$\left[F^{\mathscr{L}_Q \mathscr{L}_Q \hat{\sigma}}_{\mathscr{L}_{\overline{Q}}}\right]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_Q,1,\beta)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3. \tag{4.3.32}$$

One can solve the rest of the F-symbols in a similar way.

Just as the Ising fusion category containing duality line N is determined up to the FS indicator $\varepsilon \in H^2(\mathbb{Z}_2, U(1))$, the fusion category containing triality defect also has a FS indicator $\alpha \in H^3(\mathbb{Z}_3, U(1))$. From the F-symbol point of view, given a solution of F-symbols for the pentagon equations of the triality fusion categories, one can generate a new set of solutions [179] by

$$\begin{aligned}
&F^{\mathscr{L}_Q \mathscr{L}_{\overline{Q}} \mathscr{L}_Q}_{\mathscr{L}_Q} \to e^{2\pi m\mathrm{i}/3} F^{\mathscr{L}_Q \mathscr{L}_{\overline{Q}} \mathscr{L}_Q}_{\mathscr{L}_Q}, \quad F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_Q \mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}} \to e^{-2\pi m\mathrm{i}/3} F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_Q \mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}}, \\
&F^{\mathscr{L}_Q \mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}} \to e^{-2\pi m\mathrm{i}/3} F^{\mathscr{L}_Q \mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}}, \quad F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_Q \mathscr{L}_Q}_{\mathscr{L}_Q} \to e^{2\pi m\mathrm{i}/3} F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_Q \mathscr{L}_Q}_{\mathscr{L}_Q}, \\
&F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}} \mathscr{L}_Q}_{\mathscr{L}_{\overline{Q}}} \to e^{2\pi m\mathrm{i}/3} F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}} \mathscr{L}_Q}_{\mathscr{L}_{\overline{Q}}}, \quad F_g^{\mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}}} \to e^{-2\pi m\mathrm{i}/3} F_g^{\mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}} \mathscr{L}_{\overline{Q}}}.
\end{aligned} \tag{4.3.33}$$

Alternatively, given a triality fusion category, we can construct the triality fusion category with different FS indicator by stacking the theory with another theory with anomalous $\mathbb{Z}_3$ symmetry $\tilde{\eta}$ and identify the new triality line as $\widetilde{\mathscr{L}_Q} = \mathscr{L}_Q \tilde{\eta}$. If we gauge the quantum $\mathbb{Z}_2$ symmetry, then we would recover the theory with $\widetilde{A_4}$ symmetry but now the $\widetilde{A_4}$ symmetry has an anomaly due to the anomaly of $\mathbb{Z}_3$. This implies the triality fusion category with different FS indicators can be realized by $\mathbb{Z}_2$ gauging of the A_4 global symmetry with different anomalies. Indeed, $H^3(A_4, U(1)) = \mathbb{Z}_6$ and let ω_0 denote the generator of $H^3(A_4, U(1))$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of A_4 is anomaly free only when the anomaly of A_4 is ω_0^{2k} for $k = 0, 1, 2$ (where we use the multiplicative notation for $\mathbb{Z}_6$), and gauging one of the $\mathbb{Z}_2$ subgroup leads to the triality fusion category with FS indicator $\alpha = e^{2\pi k\mathrm{i}/3}$. [6]

[6]Notice that $k = 0$ (i.e. the anomaly of A_4 is trivial) always leads to the trivial FS indicator $\alpha = 1$. We choose the generator ω_0 such that $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2^{\sigma}, 1)$ has FS indicator $\alpha = e^{2\pi \mathrm{i}k/3}$.

Let $\alpha = e^{2\pi \mathrm{i}/3} \in U(1)$, and we list the F-symbols below.

$$
\begin{aligned}
&F_{\mathscr{L}_Q}^{\mathscr{L}_Q g,h} = F_h^{\mathscr{L}_{\overline{Q}}\mathscr{L}_Q g} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad F_{\mathscr{L}_Q}^{g\mathscr{L}_Q h} = F_{\mathscr{L}_{\overline{Q}}}^{g\mathscr{L}_{\overline{Q}} h} = F_h^{\mathscr{L}_{\overline{Q}} g \mathscr{L}_Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \\
&F_{\mathscr{L}_Q}^{g,h,\mathscr{L}_Q} = F_{\mathscr{L}_{\overline{Q}}}^{\mathscr{L}_{\overline{Q}},g,h} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad F_h^{\mathscr{L}_Q\mathscr{L}_{\overline{Q}} g} = F_h^{g\mathscr{L}_Q\mathscr{L}_{\overline{Q}}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \\
&F_{\mathscr{L}_{\overline{Q}}}^{g,h,\mathscr{L}_{\overline{Q}}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, F_h^{g\mathscr{L}_{\overline{Q}}\mathscr{L}_Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, F_h^{\mathscr{L}_Q g \mathscr{L}_{\overline{Q}}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \\
&\left[F_{\mathscr{L}_Q}^{\mathscr{L}_Q\mathscr{L}_{\overline{Q}}\mathscr{L}_Q}\right]_{g,h} = \frac{\alpha}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \quad \left[F_{\mathscr{L}_{\overline{Q}}}^{\mathscr{L}_{\overline{Q}}\mathscr{L}_Q\mathscr{L}_{\overline{Q}}}\right]_{g,h} = \frac{\alpha^{-1}}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.
\end{aligned}
\tag{4.3.34}
$$

The rest of the F-symbols are listed in Table 4.2.

Table 4.2. The F-symbols of triality fusion categories $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2^{\sigma}, 1)$ for fusion multiplicity 2 where $\alpha = e^{2\pi k\mathrm{i}/3}$. The σ^i denotes the Pauli i matrix and σ^0 is the 2×2 identity matrix.

g	1	$\hat{\sigma}$	η	$\eta\hat{\sigma}$
$[F_{\mathscr{L}_{\overline{Q}}}^{\mathscr{L}_Q\mathscr{L}_Q g}]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_Q,1,\nu)}$	σ^0	σ^3	σ^1	$\mathrm{i}\sigma^2$
$[F_{\mathscr{L}_{\overline{Q}}}^{\mathscr{L}_Q g \mathscr{L}_Q}]_{(\mathscr{L}_Q,1,\mu),(\mathscr{L}_Q,1,\nu)}$	σ^0	σ^3	σ^1	$-\mathrm{i}\sigma^2$
$[F_{\mathscr{L}_{\overline{Q}}}^{g\mathscr{L}_Q\mathscr{L}_Q}]_{(\mathscr{L}_Q,1,\mu),(\mathscr{L}_{\overline{Q}},\nu,1)}$	σ^0	σ^3	σ^1	$-\sigma^2$
$[F_{\mathscr{L}_Q}^{\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}} g}]_{(\mathscr{L}_Q,\mu,1),(\mathscr{L}_{\overline{Q}},1,\nu)}$	σ^0	σ^3	σ^1	$\mathrm{i}\sigma^2$
$[F_{\mathscr{L}_Q}^{\mathscr{L}_{\overline{Q}} g \mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_{\overline{Q}},1,\mu),(\mathscr{L}_{\overline{Q}},1,\nu)}$	σ^0	σ^3	σ^1	$\mathrm{i}\sigma^2$
$[F_{\mathscr{L}_Q}^{g\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_{\overline{Q}},1,\mu),(\mathscr{L}_Q,\nu,1)}$	σ^0	σ^3	σ^1	$-\mathrm{i}\sigma^2$
$[F_g^{\mathscr{L}_Q\mathscr{L}_Q\mathscr{L}_Q}]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_{\overline{Q}},\nu,1)}$	σ^0	$-\sigma^3$	σ^1	$\mathrm{i}\sigma^2$
$[F_g^{\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_Q,\mu,1),(\mathscr{L}_Q,\nu,1)}$	$\alpha^{-1}\sigma^0$	$-\alpha^{-1}\sigma^3$	$\alpha^{-1}\sigma^1$	$\mathrm{i}\alpha^{-1}\sigma^2$
$[F_{\mathscr{L}_Q}^{\mathscr{L}_Q\mathscr{L}_Q\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_Q,\mu,\nu),(g,1,1)}$	$\frac{1}{\sqrt{2}}\sigma^0$	$\frac{-1}{\sqrt{2}}\sigma^3$	$\frac{1}{\sqrt{2}}\sigma^1$	$\frac{-\mathrm{i}}{\sqrt{2}}\sigma^2$
$[F_{\mathscr{L}_{\overline{Q}}}^{\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}\mathscr{L}_Q}]_{(\mathscr{L}_Q,\mu,\nu),(g,1,1)}$	$\frac{\alpha}{\sqrt{2}}\sigma^0$	$\frac{\alpha}{\sqrt{2}}\sigma^3$	$\frac{\alpha}{\sqrt{2}}\sigma^1$	$\frac{-\mathrm{i}\alpha}{\sqrt{2}}\sigma^2$
$[F_{\mathscr{L}_Q}^{\mathscr{L}_{\overline{Q}}\mathscr{L}_Q\mathscr{L}_Q}]_{(g,1,1),(\mathscr{L}_{\overline{Q}},\mu,\nu)}$	$\frac{\alpha}{\sqrt{2}}\sigma^0$	$\frac{\alpha}{\sqrt{2}}\sigma^3$	$\frac{\alpha}{\sqrt{2}}\sigma^1$	$\frac{\mathrm{i}\alpha}{\sqrt{2}}\sigma^2$
$[F_{\mathscr{L}_{\overline{Q}}}^{\mathscr{L}_Q\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}}]_{(g,1,1),(\mathscr{L}_Q,\mu,\nu)}$	$\frac{\alpha^{-1}}{\sqrt{2}}\sigma^0$	$\frac{-\alpha^{-1}}{\sqrt{2}}\sigma^3$	$\frac{\alpha^{-1}}{\sqrt{2}}\sigma^1$	$\frac{-\mathrm{i}\alpha^{-1}}{\sqrt{2}}\sigma^2$

4.4 Physical implication of the group theoretical triality fusion categories

In this section, we derive the physical implication of the group theoretical triality fusion categories $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2^\sigma, 1)$. We first derive the spin selection rules for the triality defect. Then we show how to match the states in the Hilbert space which transforms in different irreducible representations of the fusion category symmetry $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2^\sigma, 1)$ of the theory $\mathscr{T}/\mathbb{Z}_2^\sigma$ with the states in the Hilbert space $\mathscr{H}$ or the defect Hilbert space $\mathscr{H}_\sigma$ of the theory $\mathscr{T}$. This allows us to derive the asymptotic density of states in $\mathscr{H}$ transforms in different irreducible representations of $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2^\sigma, 1)$ by applying the result in [166]. Finally, we show that the constraint on the RG flow from the fusion category symmetry $\mathscr{C}(A_4, \omega_0^{2k}, \mathbb{Z}_2^\sigma, 1)$ is equivalent to the constraint on the RG flow of $\mathrm{Vec}_{A_4}^{\omega_0^{2k}}$.

4.4.1 Spin selection rules

We now derive the spin selection rules for using the F-symbols computed in section 4.3.5.

To do this, we first consider the action of $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ on the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$. Following in the convention in (4.2.8) and suppressing the $1,1$ indices for the fusion channel (since the multiplicity is just 1 in this case), we denote the operator as $\hat{g}_{\mathscr{L}_Q}$. This is depicted in Figure 4.6.

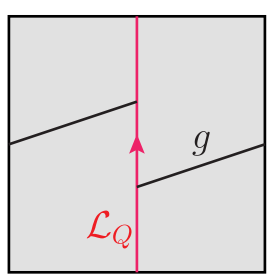


Figure 4.6. Symmetry operator $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$, which we denote as $\hat{g}_{\mathscr{L}_Q}$.

Notice that the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathscr{H}_{\mathscr{L}_Q}$ can be twisted by 2-cocycle $\gamma(g,h) \in Z^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ such that

$$\hat{h}_{\mathscr{L}_Q} \cdot \hat{g}_{\mathscr{L}_Q} = \gamma(h,g) \widehat{hg}_{\mathscr{L}_Q}. \tag{4.4.1}$$

We can compute γ from the F-symbols via following configuration shown in the Figure 4.7.

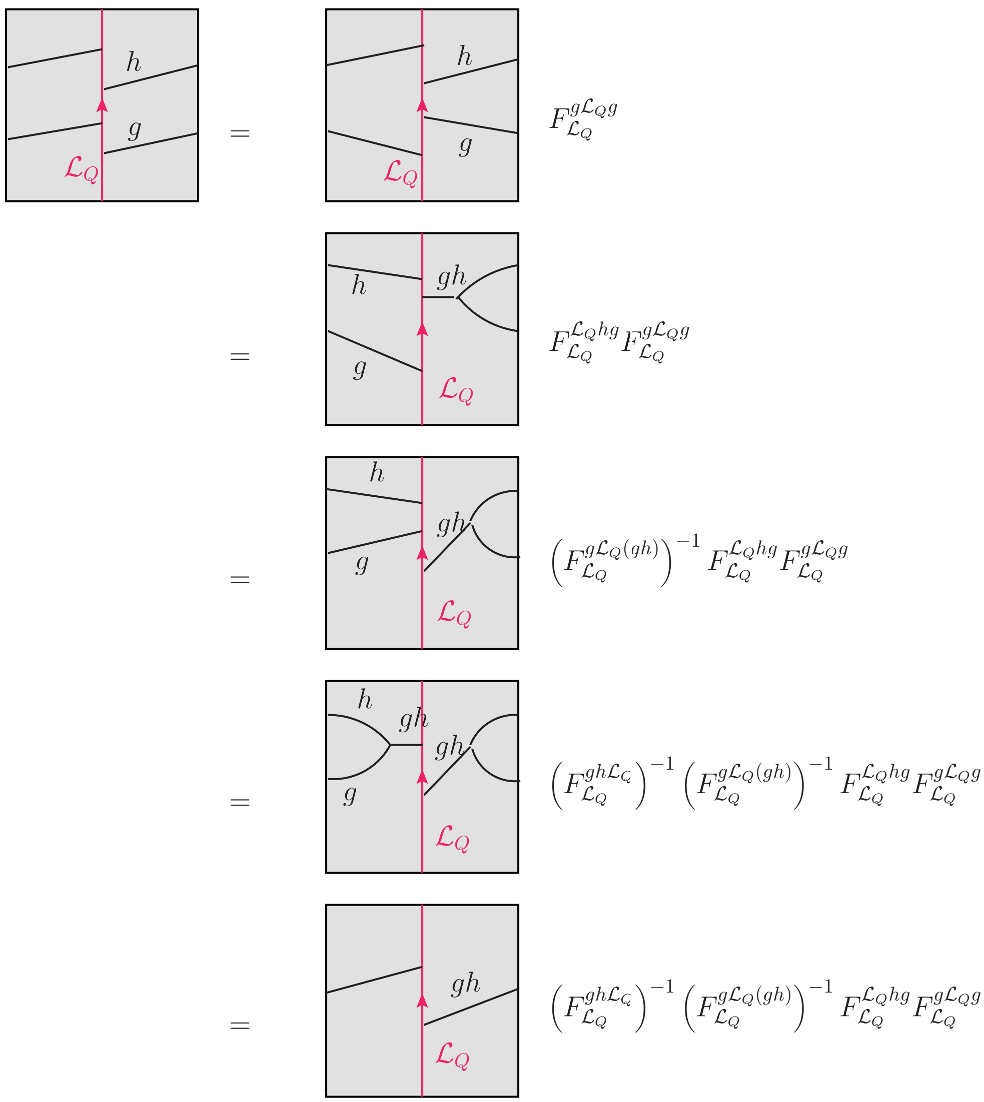


Figure 4.7. The calculation of the product of $\hat{g}_{\mathscr{L}_Q}$ and $\hat{h}_{\mathscr{L}_Q}$ using F-moves.

Under a sequence of F-move, we relate the phase $\gamma(g,h)$ to products of F-symbols as following

$$\gamma(h,g) = \left(F^{gh\mathscr{L}_Q}_{\mathscr{L}_Q}\right)^{-1} \left(F^{g\mathscr{L}_Q(gh)}_{\mathscr{L}_Q}\right)^{-1} F^{\mathscr{L}_Q hg}_{\mathscr{L}_Q} F^{g\mathscr{L}_Q g}_{\mathscr{L}_Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{4.4.2}$$

We list the eigenvalues of allowed irreducible representations as the following

$$(1,1,1,-1), \quad (1,1,-1,1), \quad (1,-1,1,1), \quad (1,-1,-1,-1). \tag{4.4.3}$$

Notice that the 2-cocycle γ above is cohomologically trivial in the group cohomology, which is consistent with the fact that we have 4 1-dimensional irreducible representations.[7] Yet we will see its importance when deriving the spin selection rule for the intrinsic triality defects in later sections.

Next, to derive the spin selection rules, we consider twisted partition function $Z_{\mathscr{L}_Q}(\tau)$ and apply T modular transformation three times, see Figure 4.8. This amounts to inserting $e^{6\pi is}$ in the trace over the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$ where s is the spin of the state.

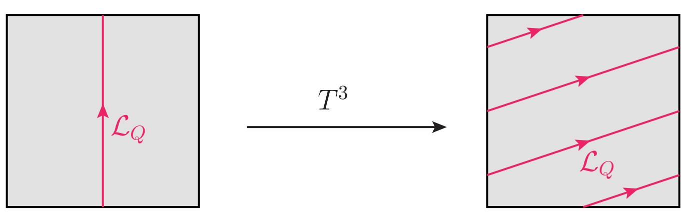


Figure 4.8. Applying T^3 to the twisted partition function $Z_{\mathscr{L}_Q}(\tau)$. This is equivalent to insert $e^{6\pi is}$ in the trace over $\mathscr{H}_{\mathscr{L}_Q}$.

[7] $\mathbb{Z}_2 \times \mathbb{Z}_2$ only has a single 2-dimensional irreducible representation when twisted by the cohomologically non-trivial 2-cocycle.

We then apply a sequence of F-moves to relate $Z_{\mathscr{L}_Q}(\tau+3)$ to the action of the symmetry $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ on the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$, as shown in Figure 4.9. We find the following relation:

$$
\begin{aligned}
Z_{\mathscr{L}_Q}(\tau+3) &= \sum_{\substack{\mu,\nu=1,2,\\ g\in\mathbb{Z}_2\times\mathbb{Z}_2}} \left[F^{\mathscr{L}_{\overline{Q}}\mathscr{L}_Q\mathscr{L}_Q}_{\mathscr{L}_Q}\right]_{(\mathbb{1},0,0)(\mathscr{L}_Q,\mu,\nu)} \left[F^{\mathscr{L}_Q\mathscr{L}_Q\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_Q}\right]_{(\mathscr{L}_{\overline{Q}},\mu,\nu)(g,0,0)} Z^{\mathscr{L}_Q}_{\mathscr{L}_Q g}(\tau) \\
&= \alpha Z^{\mathscr{L}_Q}_{\mathscr{L}_Q \mathbb{1}}(\tau).
\end{aligned}
\tag{4.4.4}
$$

This implies the spin s of the states in defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$ satisfies the following relation:

$$
e^{6\pi \mathrm{i} s} = \alpha, \tag{4.4.5}
$$

which implies,

$$
e^{2\pi \mathrm{i} s} = \begin{cases} e^{\frac{2\pi \mathrm{i} k}{3}}, \quad k=0,1,2, \quad \text{when} \quad \alpha = 1 \\ e^{\frac{2\pi \mathrm{i} k}{3}+\frac{2\pi \mathrm{i}}{9}}, \quad k=0,1,2, \quad \text{when} \quad \alpha = e^{2\pi \mathrm{i}/3}, \\ e^{\frac{2\pi \mathrm{i} k}{3}-\frac{2\pi \mathrm{i}}{9}}, \quad k=0,1,2, \quad \text{when} \quad \alpha = e^{-2\pi \mathrm{i}/3}. \end{cases} \tag{4.4.6}
$$

Now, we provide an alternative derivation of the same spin selection rule by constructing the twisted partition function from the ungauged theory. Consider a CFT $\mathscr{T}$ with A_4 global symmetry with the anomaly parameterized by $\omega_0^{2k} \in H^3(A_4, U(1)) \simeq \mathbb{Z}_6$ where $k=0,1,2$. The $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\eta$ is free of anomaly but the $\mathbb{Z}_3$ subgroup generated by q has 't Hooft anomaly. As pointed out before, by gauging a $\mathbb{Z}_2$ symmetry, we get the non-intrinsic triality fusion category with FS indicator $\alpha = e^{2\pi \mathrm{i} k/3}$. By the fusion rule of the triality defect $\mathscr{L}_Q$,

$$
\mathscr{L}_Q \times \mathscr{L}_Q \times \mathscr{L}_Q = \mathscr{L}_{\overline{Q}} \times \mathscr{L}_{\overline{Q}} \times \mathscr{L}_{\overline{Q}} = 2 \sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} g. \tag{4.4.7}
$$

In the gauged theory $\mathscr{T}/\mathbb{Z}_2$, the twisted sector is odd under the quantum $\mathbb{Z}_2$-symmetry, hence the entire twisted sector is annihilated by the triality defects $\mathscr{L}_Q$ or $\mathscr{L}_{\overline{Q}}$. Then, we can construct the twisted partition function $Z^{\mathscr{L}_Q}_{1,\mathscr{L}_{\overline{Q}}}(\tau)$ of the gauged theory from the twisted partition function of the original theory as follows,

$$
\begin{aligned}
&(Z_{\mathscr{T}/\mathbb{Z}_2})^{\mathscr{L}_Q}_{1,\mathscr{L}_{\overline{Q}}}(\tau)\\
=&\mathrm{Tr}_{\mathscr{H}_{\mathscr{T}/\mathbb{Z}_2}}(\mathscr{L}_Q q^{L_0-1/24}\overline{q}^{\overline{L}_0-1/24})\\
=&\mathrm{Tr}_{\mathscr{H}_{\mathscr{T}/\mathbb{Z}_2,\text{untwisted}}}(\mathscr{L}_Q q^{L_0-1/24}\overline{q}^{\overline{L}_0-1/24})\\
=&\mathrm{Tr}_{\mathscr{H}_{\mathscr{T}}}((q+\sigma q\sigma)\frac{1+\sigma}{2}q^{L_0-1/24}\overline{q}^{\overline{L}_0-1/24})\\
=&\frac{(Z_{\mathscr{T}})^{q}(\tau)+(Z_{\mathscr{T}})^{\sigma q}(\tau)+(Z_{\mathscr{T}})^{q\sigma}(\tau)+(Z_{\mathscr{T}})^{\sigma q\sigma}(\tau)}{2}\\
=&(Z_{\mathscr{T}})^{q}(\tau)+(Z_{\mathscr{T}})^{\sigma q}(\tau),
\end{aligned}
\tag{4.4.8}
$$

where we used the above twisted partition function only depending on the conjugacy class. Applying the S-modular transformation on both sides, we find

$$
(Z_{\mathscr{T}/\mathbb{Z}_2})^{\mathscr{L}_Q}_{\mathscr{L}_Q,1}(\tau)=(Z_{\mathscr{T}})_{q}(\tau)+(Z_{\mathscr{T}})_{\sigma q}(\tau). \tag{4.4.9}
$$

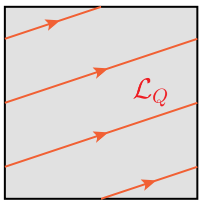
Then, the spin selection rules of the triality defect $\mathscr{L}_Q$ are the same as the symmetry defect q and σq which generates $\mathbb{Z}_3$ symmetries in A_4. The spin selection rules of $\mathbb{Z}_3$-symmetry defect has been derived in [46], which takes the form

$$
s\in\frac{1}{3}\mathbb{Z}+\frac{k}{9}. \tag{4.4.10}
$$

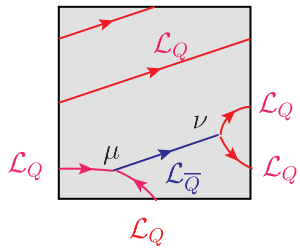
We then find agreement between (4.4.6) and (4.4.10).

4.4.2 Asymptotic density of states

In this subsection, we derive several asymptotic density of states for different sectors in the Hilbert space $\mathscr{H}$ for a theory with the group theoretical triality category symmetries.

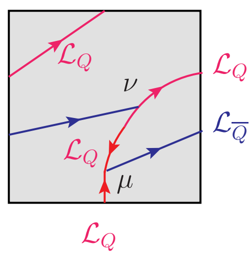


$$= \sum_{\mu,\nu=1,2}$$



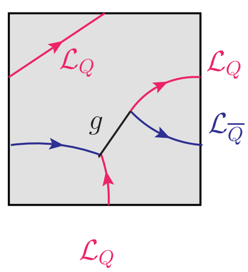
$$\left[F^{\mathcal{L}_{\overline{Q}}\mathcal{L}_Q\mathcal{L}_Q}_{\mathcal{L}_Q}\right]_{(\mathbb{1},1,1)(\mathcal{L}_{\overline{Q}},\mu,\nu)}$$

$$= \sum_{\mu,\nu=1,2}$$



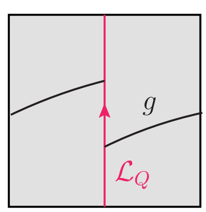
$$\left[F^{\mathcal{L}_{\overline{Q}}\mathcal{L}_Q\mathcal{L}_Q}_{\mathcal{L}_Q}\right]_{(\mathbb{1},1,1)(\mathcal{L}_{\overline{Q}},\mu,\nu)}$$

$$= \sum_{\substack{\mu,\nu=1,2,\\ g\in\mathbb{Z}_2\times\mathbb{Z}_2}}$$



$$\left[F^{\mathcal{L}_{\overline{Q}}\mathcal{L}_Q\mathcal{L}_Q}_{\mathcal{L}_Q}\right]_{(\mathbb{1},1,1)(\mathcal{L}_{\overline{Q}},\mu,\nu)}\left[F^{\mathcal{L}_Q\mathcal{L}_Q\mathcal{L}_{\overline{Q}}}_{\mathcal{L}_Q}\right]_{(\mathcal{L}_{\overline{Q}},\mu,\nu)(g,1,1)}$$

$$= \sum_{\substack{\mu,\nu=1,2,\\ g\in\mathbb{Z}_2\times\mathbb{Z}_2}}$$



$$\left[F^{\mathcal{L}_{\overline{Q}}\mathcal{L}_Q\mathcal{L}_Q}_{\mathcal{L}_Q}\right]_{(\mathbb{1},1,1)(\mathcal{L}_{\overline{Q}},\mu,\nu)}\left[F^{\mathcal{L}_Q\mathcal{L}_Q\mathcal{L}_{\overline{Q}}}_{\mathcal{L}_Q}\right]_{(\mathcal{L}_{\overline{Q}},\mu,\nu)(g,1,1)}$$

Figure 4.9. Here, we start with the configuration which computes the $e^{6\pi is}$. Under sequence of F-moves, we relate it to the action of $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ on the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$.

When acting on the Hilbert space $\mathscr{H}$, the 6 simple lines in the triality fusion category $I, \hat{\sigma}, \eta, \hat{\sigma}\eta, \mathscr{L}_Q, \mathscr{L}_{\overline{Q}}$ correspond to 6 operators whose product satisfies the fusion rule (4.6.1), from which we learn they all commute with each other, hence can be simultaneously diagonalized. We enumerate eigenvalues for all 6 possible 1-dim irreducible representations as follows,

$$\begin{aligned} &(1,1,1,1,2,2), \quad (1,1,1,1,2\omega,2\omega^2), \quad (1,1,1,1,2\omega^2,2\omega), \\ &(1,1,-1,-1,0,0), \quad (1,-1,1,-1,0,0), \quad (1,-1,1,-1,0,0). \end{aligned} \tag{4.4.11}$$

We now derive a formula of the asymptotic density of states for all the 6 irreps by relating the above 6 irreps to representations of finite group symmetries in the ungauged theory and utilizing the result in [166], which we reviewed in section 4.2.3. Notice that since the asymptotic density of states for different irreps of $\mathscr{H}$ only depends on the fusion ring structure of the fusion category, we can derive the asymptotic density of states from the simplest case when the fusion category is acquired from gauging $\mathbb{Z}_2^\sigma$ subgroup of A_4 with the trivial anomaly. Then, by relating the above 6 irreps to the representation of the A_4 in the ungauged theory, we can use the result in [166] to get the result. Let's first consider the twisted sector in the gauged theory $\mathscr{T}/\mathbb{Z}_2^\sigma$, which is odd under the quantum symmetry $\hat{\sigma}$. In the ungauged theory $\mathscr{T}$, these states correspond to the $\mathbb{Z}_2^\sigma$-even states in the defect Hilbert space $\mathscr{H}_{\mathscr{T},\sigma}$. Since there's no mixed 't Hooft anomaly between $\mathbb{Z}_2^\sigma$ and $\mathbb{Z}_2^\eta$, there are well-defined $\mathbb{Z}_2^\eta$ charges for states in the defect Hilbert space $\mathscr{H}_{\mathscr{T},\sigma}$, which are the $\mathbb{Z}_2^\eta$ charges for states in the gauged theory.

Next, let's consider the untwisted sector from the $\mathbb{Z}_2^\sigma$ gauging. This sector is given by the $\mathbb{Z}_2^\sigma$ even states in the ungauged theory with the global A_4 symmetry. Here, we can use the fact that the triality line acts on the untwisted sector as $q+\sigma q\sigma$. We will relate the eigenvalues of $q+\sigma q\sigma$ of the states in the $\mathbb{Z}_2^\sigma$-invariant space in each irreducible representation of A_4. This group has 4 irreducible representation, labelled by the dimension $\mathbf{1}, \mathbf{1}_A, \mathbf{1}_B, \mathbf{3}$, where $\mathbf{1}$ is the trivial irrep and $\mathbf{1}_A$ and $\mathbf{1}_B$ are two irreps where σ, η acts as trivially and q has eigenvalues ω and ω^2 respectively, which means the states in these two irreps are invariant under the $\mathbb{Z}_2^\sigma$ and have eigenvalues 2ω and $2\omega^2$ under $\mathscr{L}_Q \simeq q + \sigma q \sigma$ respectively. For the three-dimensional representation **3**, the representation matrices are given by

$$U_{\mathbf{3}}(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U_{\mathbf{3}}(\eta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U_{\mathbf{3}}(q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.4.12}$$

As one can check, $\mathbb{Z}_2^\sigma$-invariant space is 1-dimensional, and the operator $\mathscr{L}_Q \simeq q + \sigma q \sigma$ annihilates this state.

We summarize the results in the Tab. 4.3.

Table 4.3. Relating the states in different irreps of the group-theoretical triality fusion category $\mathscr{C}(A_4, 1, \mathbb{Z}_2^\sigma, 1)$ in $\mathscr{T}/\mathbb{Z}_2^\sigma$ to states in the ungauged theory $\mathscr{T}$.

Irrep in $\mathscr{H}_{\mathscr{T}/\mathbb{Z}_2^\sigma}$ in the gauged theory $\mathscr{T}/\mathbb{Z}_2$	Corresponding states in the theory $\mathscr{T}$
$(1,1,1,1,2,2)$	the irrep **1** of A_4 in $\mathscr{H}_{\mathscr{T}}$
$(1,1,1,1,2\omega,2\omega)$	the irrep $\mathbf{1}_A$ of A_4 in $\mathscr{H}_{\mathscr{T}}$
$(1,1,1,1,2\omega^2,2\omega^2)$	the irrep $\mathbf{1}_B$ of A_4 in $\mathscr{H}_{\mathscr{T}}$
$(1,1,-1,-1,0,0)$	the $\mathbb{Z}_2^\sigma$-even state in the **3** of A_4 in $\mathscr{H}_{\mathscr{T}}$
$(1,-1,1,-1,0,0)$	$\mathbb{Z}_2^\eta$-even, $\mathbb{Z}_2^\sigma$-even states in $\mathscr{H}_{\mathscr{T},\sigma}$
$(1,-1,-1,1,0,0)$	$\mathbb{Z}_2^\eta$-odd, $\mathbb{Z}_2^\sigma$-even states in $\mathscr{H}_{\mathscr{T},\sigma}$

The last 3 irreps of $\mathscr{C}(A_4, 1, \mathbb{Z}_2^\sigma, 1)$ corresponds to the 3 irreps of $\mathbb{Z}_2^{\hat{\sigma}} \times \mathbb{Z}_2^\eta$ in the gauged theory $\mathscr{T}/\mathbb{Z}_2^\sigma$, therefore, we can directly apply the result in [166] in the gauged theory and find the asymptotic density of states to be

$$\rho_{0,(1,1,-1,-1,0,0)}(\Delta) = \rho_{0,(1,-1,1,-1,0,0)}(\Delta) = \rho_{0,(1,-1,-1,1,0,0)}(\Delta) = \frac{1}{4}\rho_0(\Delta), \tag{4.4.13}$$

where $\rho_0(\Delta)$ is defined in (4.2.13). To determine the asymptotic density of states in irreps $(1,1,1,1,2\omega^k,2\omega^{-k})$ we can simply use the relation in Table 4.3 and apply the result in [166].

We then find

$$\rho_{0,(1,1,1,1,2\omega^k,2\omega^{-k})}(\Delta) = \frac{1}{12}\rho_0(\Delta), \quad k = 0,1,2. \tag{4.4.14}$$

4.4.3 Constraints on RG flow

To study the constraints on the RG flow, we want to determine if the fusion category symmetry $\mathscr{C}$ is anomalous, in the sense that if it obstructs a $\mathscr{C}$-symmetric trivially gapped phase. As pointed out in [180], module categories $\mathscr{M}$ of $\mathscr{C}$ are in bijection with $\mathscr{C}$ symmetric gapped phases such that the ground-states are in bijection with the simple objects in $\mathscr{M}$. Therefore, to check whether a fusion category symmetry $\mathscr{C}$ has a trivially gapped phase is to check if it has a module category $\mathscr{M}$ with a single simple object. Equivalently, one can check if $\mathscr{C}$ admits a fiber functor.

In general, this is not an easy problem. For the case of the group-theoretic fusion category, this is relatively easy because the module categories over $\mathscr{C}$ and over the dual module $\mathscr{C}^*_{\mathscr{M}}$ are in the canonical bijection as pointed out in [164]. Since the group theoretical fusion category $\mathscr{C}(G,\omega,H,\psi)$ is the dual module of $\mathscr{C}(G,\omega,\mathbb{Z}_1,1) \simeq \mathrm{Vec}_G^{\omega}$, the anomaly of $\mathscr{C}(G,\omega,H,\psi)$ is equivalent to the anomaly of Vec_G^{ω}.

Physically, this can be seen as follows. Let's consider a relevant operator $O(x)$ in a CFT $\mathscr{T}/H$ which preserves the fusion category symmetries $\mathscr{C}(G,\omega,H,\psi)$. We then consider gauging the quantum symmetry $\mathrm{Rep}(H)$ to get back the theory $\mathscr{T}$ with global symmetry Vec_G^{ω}. The relevant operator $O(x)$ remains a local operator in the CFT $\mathscr{T}$. This operator will trigger the RG flow in $\mathscr{T}$. If the theory $\mathscr{T}/H$ flows to the trivially gapped phase after perturbing by the operator $O(x)$, then this means the theory $\mathscr{T}$ would also flow to the trivially gapped phase after perturbing by the operator $O(x)$. However, we would run into contradiction if the 3-cocycle ω characterizes the G-anomaly is not trivial. Hence, we conclude the theory with the group theoretical fusion category symmetries $\mathscr{C}(G,\omega,H,\psi)$ can not flow to a trivially gapped phase when the anomaly $\omega \neq 1$.

4.5 Example: $c = 1$ Compact boson at Kosterlitz-Thouless point

In this section, we consider the example of $c = 1$ compact boson at the Kosterlitz-Thouless (KT) point and compute the twisted partition functions of triality defect in the $c = 1$ compact boson at the KT point. We match the spin selection rule and also show one can not construct a new triality fusion category by combining the triality defect $\mathscr{L}_Q$ with another generator η of the $\mathbb{Z}_3$ symmetry in the KT theory.

4.5.1 A lightning review of $c = 1$ compact boson and the triality defect

We first briefly review the $c = 1$ compact boson following the convention in [181]. The theory is described by a scalar field X with period $2\pi R$,

$$X \simeq X + 2\pi R. \tag{4.5.1}$$

It is convenient to define 2π-periodic field θ and 2π-periodic conjugate momentum ϕ and introduce the left and right moving fields $X_{L,R}$,

$$\theta = R^{-1}(X_L + X_R), \quad \phi = R(X_L - X_R)/2. \tag{4.5.2}$$

The global symmetry at a generic radius R is

$$G_{bos} = (U(1)^\theta \times U(1)^\phi) \rtimes \mathbb{Z}_2^C, \tag{4.5.3}$$

where $U(1)^\theta$ and $U(1)^\phi$ are the shifting symmetry of θ and ϕ respectively, and the charge conjugation C flips the sign of θ and ϕ simultaneously.

At a generic radius R, the primary local operators in this theory contain vertex operators,

$$V_{n,w} = e^{i\left(\frac{n}{R}+\frac{wR}{2}\right)X_L} e^{i\left(\frac{n}{R}-\frac{wR}{2}\right)X_R} = e^{in\theta} e^{iw\phi}, \quad n,w \in \mathbb{Z} \tag{4.5.4}$$

with the scaling dimension,

$$(h,\overline{h}) = \left(\frac{1}{2}\left(\frac{n}{R}+\frac{wR}{2}\right)^2, \frac{1}{2}\left(\frac{n}{R}-\frac{wR}{2}\right)^2\right), \tag{4.5.5}$$

together with the normal ordered Schur symmetric polynomials in the $U(1)$ currents $j_1 = \partial X_L$ and $\overline{j}_1 = \overline{\partial} X_R$ and their derivatives, denoted as,

$$j_{n^2}\overline{j}_{m^2} \quad (h,\overline{h}) = (n^2, m^2). \tag{4.5.6}$$

The spectrum can also be seen from the partition function,

$$Z(R) = \frac{1}{|\eta(\tau)|^2} \sum_{n,m\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{n}{R}+\frac{wR}{2}\right)^2} \overline{q}^{\frac{1}{2}\left(\frac{n}{R}-\frac{wR}{2}\right)}. \tag{4.5.7}$$

For $c = 1$ CFT, there are null states in the descendent states when the Virasoro primary state has scaling dimension $h = \frac{n^2}{4}$ for $n \in \mathbb{Z}$. For a generic h, there is no null states in the descendent states of a Virasoro primary state and the Virasoro character is given by,

$$\chi_h(\tau) = \frac{q^h}{\eta(\tau)}. \tag{4.5.8}$$

For the primary state with $h = \frac{n^2}{4}$ with $n \in \mathbb{Z}$, because of the null states, its character takes the form,

$$\chi_{h=\left(\frac{n}{2}\right)^2}(\tau) = \frac{q^{\left(\frac{n}{2}\right)^2} - q^{\left(\frac{n}{2}+1\right)^2}}{\eta(\tau)}. \tag{4.5.9}$$

At a generic point of the moduli space, terms with $n \neq 0$ or $m \neq 0$ correspond to characters with primaries $V_{n,m}$'s containing no null states. However, the term $\frac{1}{\eta(\tau)\overline{\eta}(\overline{\tau})}$ with $n = m = 0$ cannot be a character of Virasoro primary (the identity operator) due to the appearance of null states, but should correspond to the sum of characters of primary states and can be seen via the following rewriting:

$$\frac{1}{\eta(\tau)\overline{\eta}(\overline{\tau})}=\frac{1}{\eta(\tau)\overline{\eta}(\overline{\tau})}\sum_{n,m=0}^{\infty}(q^{n^2}-q^{(n+1)^2})(\overline{q}^{m^2}-\overline{q}^{(m+1)^2}), \tag{4.5.10}$$

where each term in the sum is a character for the primary operator with scaling dimension $(h,\overline{h})=(n^2,m^2)$, corresponding to the primary operator $j_{n^2}\overline{j}_{m^2}$ mentioned above.

It is worth mentioning at the special radius $R=\sqrt{2}$, the theory becomes $SU(2)_1$, and the global symmetry is enhanced to $SO(4)=\frac{SU(2)_L\times SU(2)_R}{\mathbb{Z}_2}$. We can represent this $SO(4)$ in its vector representation, where the basis is given by 4 operators $(\sin\theta,\cos\theta,\sin\phi,\cos\phi)$ [181]. The charge conjugation is represented as,

$$C=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}. \tag{4.5.11}$$

Similarly, the $U(1)_\theta$ and $U(1)_\phi$ can be represented as,

$$R_\theta(\alpha)=\begin{pmatrix}\cos\alpha&-\sin\alpha&0&0\\\sin\alpha&\cos\alpha&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad R_\phi(\alpha)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\alpha&-\sin\alpha\\0&0&\sin\alpha&\cos\alpha\end{pmatrix}. \tag{4.5.12}$$

The spectrum of primary operators of the $SU(2)_1$ theory can be derived by decomposing the partition function in terms of characters (4.5.9) of irreducible representations of the Virasoro algebra [74, 89], and the details are presented in the Appendix B.1. The $SU(2)_1$ Hilbert space decomposes as,

$$\mathscr{H}_{SU(2)_1} = \bigoplus_{\substack{j,\bar{j}\in\frac{1}{2}\mathbb{Z}_{\geq 0},\\ j+\bar{j}\in\mathbb{Z}}} V_j \otimes V_{\bar{j}} \otimes H^{Vir}_{j^2} \otimes H^{\overline{Vir}}_{\bar{j}^2}, \tag{4.5.13}$$

where by V_j ($\overline{V}_{\bar{j}}$) we denote the spin-j(spin-$\bar{j}$) representation of $SU(2)_L$ ($SU(2)_R$) and by $H^{Vir}_{j^2}$ we denote the Virasoro representation with $h = j^2$. Notice that here j and $\bar{j}$ label the irrep of $SU(2)_L$ and $SU(2)_R$ symmetry rather than the affine $SU(2)_L$ or $SU(2)_R$, therefore the affine cut-off of j or $\bar{j}$ is not at presence. It is clear from this decomposition that how the $\frac{SU(2)_L\times SU(2)_R}{\mathbb{Z}_2}$ acts on the $\mathscr{H}_{SU(2)_1}$.

The $\mathbb{Z}_2^C$ symmetry is free of anomaly, so one could consider gauging it. The resulting theories are a class of theories also parameterized by the radius R of the compact boson, and we call the resulting theories the orbifold branch. The spectrum of Virasoro primaries on the $c = 1$ orbifold branch consists of two sectors, the untwisted sector which contains $\mathbb{Z}_2^C$ invariant operators of the corresponding compact boson theory, and the twisted sector which contains the $\mathbb{Z}_2^C$ invariant non-local operators ending on the C defect line in the compact boson theory.

The $\mathbb{Z}_2^C$-invariant twisted sector is constructed by acting on the two ground states $\left|\frac{1}{16},\frac{1}{16}\right\rangle_i$ $i = 1,2$ with even powers of the operators α_{-n} and $\overline{\alpha}_{-n'}$ (where now $n,n' \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$) appearing in the mode expansion of the compact boson ϕ with twisted boundary condition [96],

$$\mathscr{H}_{\text{orbifold, twisted}} = \left\{ \alpha_{-n_1}\cdots\alpha_{-n_l}\overline{\alpha}_{-n_{l+1}}\cdots\overline{\alpha}_{-n_{2k}} \left|\frac{1}{16},\frac{1}{16}\right\rangle_j : n_i \in \frac{1}{2}+\mathbb{Z}_{\geq 0} \right\}. \tag{4.5.14}$$

The two ground states $\left|\frac{1}{16},\frac{1}{16}\right\rangle_i$ are denoted as σ_i where $i = 1,2$.[8] The first two excited states are primary states given by $\alpha_{-\frac{1}{2}}\overline{\alpha}_{-\frac{1}{2}}\left|\frac{1}{16},\frac{1}{16}\right\rangle_i$ which both have scaling dimensions $(\frac{9}{16},\frac{9}{16})$ and we will denote the two as τ_i where $i = 1,2$.

[8]We abuse the notation slightly here. These σ_i should be distinguished from σ appear in the previous section, which denotes an element of the A_4 group. The readers should be able to distinguish the two based on context.

The untwisted sector contains,

$$V^+_{n,w} = \frac{V_{n,w} + V_{-n,-w}}{\sqrt{2}} \tag{4.5.15}$$

which are invariant under the $\mathbb{Z}_2^C$, as well as the $\mathbb{Z}_2^C$ invariant normal-ordered Schur polynomials which are given by,

$$j_{n^2} j_{m^2}, \quad \text{with} \quad m - n \in 2\mathbb{Z}. \tag{4.5.16}$$

At a generic point of the orbifold branch, there is a $D_8 = \langle s, r | s^2 = r^4 = (rs)^2 = 1 \rangle$ global symmetry, acting on the untwisted sector as,

$$r : (\theta, \phi) \to (\theta + \pi, \phi + \pi), \quad s : (\theta, \phi) \to (\theta, \phi + \pi). \tag{4.5.17}$$

For the operators in the twisted sector, D_8 acts as follows. The generator s exchanges two ground states σ_i while r acts as

$$r : (\sigma_1, \sigma_2) \mapsto (i\sigma_1, -i\sigma_2) \tag{4.5.18}$$

and

$$r : \alpha_{-n} \mapsto -\alpha_{-n}, \quad r : \overline{\alpha}_{-m} \mapsto -\overline{\alpha}_{-m}, \quad n, m \in \mathbb{Z}_{\geq 0} + \frac{1}{2}. \tag{4.5.19}$$

This implies $r : (\tau_1, \tau_2) \mapsto (i\tau_1, -i\tau_2)$. There are two important D_4 subgroups of D_8:

$$D_4^A \equiv \langle r^2, s \rangle, \quad D_4^B \equiv \langle r^2, sr \rangle. \tag{4.5.20}$$

An important result we will use later to determine the action of the triality line $\mathscr{L}_Q$ on the twisted sector $\mathscr{H}_{KT,\text{twisted}}$ is that the action of r^2 acts as -1 on the entire twisted sector, which can be seen from (4.5.14)(4.5.18)(4.5.19). Furthermore, since r^2 acts trivially on the entire untwisted sector and acts as -1 on the entire twisted sector, we identify as the generator $\hat{C}$ of the quantum $\mathbb{Z}_2$ symmetry from the $\mathbb{Z}_2^C$ gauging.

4.5.2 Spectrum of triality defect and twisted partition functions on torus

The KT theory can be acquired by gauging the $\mathbb{Z}_2^C$ symmetry of the $SU(2)_1$ theory, which locates at the intersection point between the circle branch and the orbifold branch. And in [181], the triality defect $\mathscr{L}_Q$ has been identified with the element $Q \in \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$ global symmetries of the $SU(2)_1$ theory. In the representation of $SO(4)$ we used above, Q can be represented as

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.5.21}$$

However, the symmetry operator Q does not commute with C, therefore, in the gauged theory, the reminiscent of the symmetry operator Q is given by the triality line $\mathscr{L}_Q$, related to Q as

$$\mathscr{L}_Q = Q + CQC, \tag{4.5.22}$$

with the fusion rule [181]

$$\mathscr{L}_Q \times \mathscr{L}_Q = 2\mathscr{L}_{\overline{Q}}, \quad \mathscr{L}_Q \times \mathscr{L}_{\overline{Q}} = \sum_{g \in D_{4B}} g. \tag{4.5.23}$$

As one can see, the charge conjugation C corresponds to $\sigma \in A_4$ and Q corresponds to $q \in A_4$ discussed previously and as one can check using the matrix representation above the minimal subgroup of $SO(4)$ containing C and Q is indeed A_4.

From the above fusion rule and the irreducible representations of $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ given by

$$(1,1,1,1), \quad (1,1,-1,-1), \quad (1,-1,1,-1) \quad (1,-1,-1,1), \tag{4.5.24}$$

we find the action of $\mathscr{L}_Q$ on a state $|\psi\rangle$ in the KT theory are non-trivial only if $|\psi\rangle$ transforms in the trivial representation of D_4.

Now, we determine the action of the triality line $\mathscr{L}_Q$ on the states in the KT theory. For the states in twisted sector, all twisted sector states transform non-trivially under $\mathbb{Z}_2 = \langle r^2 \rangle$, thus non-trivially under D_{4B} as well. Hence, by the fusion rule (4.5.23), the triality operator $\mathscr{L}_Q$ must annihilate all the states in the twisted sector.

The action of $\mathscr{L}_Q$ on the untwisted sector can be determined by (4.5.22). We simply need to construct the representation matrices of C and Q for $(j, \bar{j})$ irreducible representation of $SO(4)$ global symmetry in the $SU(2)_1$ theory. And the action of $\mathscr{L}_Q$ on the untwisted sector is simply given by $Q + CQC$ restricted on the $\mathbb{Z}_2^C$-invariant sector of each $(j, \bar{j})$ irreducible representation of $SO(4)$.

Knowing the action of $\mathscr{L}_Q$ on the KT theory Hilbert space $\mathscr{H}_{KT}$ allows us to compute the twisted partition function $(Z_{KT})^{\mathscr{L}_Q}$. Since $\mathscr{L}_Q$ annihilates the twisted sector, the twisted partition function $(Z_{KT})^{\mathscr{L}_Q} = \mathrm{Tr}_{\mathscr{H}_{KT}}(\mathscr{L}_Q q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24})$ can be reduced to the untwisted sector and expressed as the following sum of the twisted partition function of the $SU(2)_1$ theory,

$$\begin{aligned}
(Z_{KT})^{\mathscr{L}_Q} &= \mathrm{Tr}_{\mathscr{H}_{KT}}(\mathscr{L}_Q q^{L_0-1/24} \bar{q}^{\bar{L}-1/24}) \\
&= \mathrm{Tr}_{\mathscr{H}_{KT,\mathrm{untwisted}}}(\mathscr{L}_Q q^{L_0-1/24} \bar{q}^{\bar{L}-1/24}) \\
&= \mathrm{Tr}_{\mathscr{H}_{SU(2)_1}}((Q+CQC)\frac{1+C}{2} q^{L_0-1/24} \bar{q}^{\bar{L}_0-1/24}) \\
&= \frac{(Z_{SU(2)_1})^Q + (Z_{SU(2)})^{CQ} + (Z_{SU(2)_1})^{QC} + (Z_{SU(2)_1})^{CQC})}{2}.
\end{aligned} \tag{4.5.25}$$

To evaluate the twisted partition function in $SU(2)_1$ theories, we must first rewrite the partition function in terms of irreps of $\frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$ global symmetries, as in [74, 89]. As shown in the Appendix B.1, this is given by,

$$Z_{SU(2)_1}(\tau, \bar{\tau}) = \sum_{\substack{j, \bar{j} \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \\ j + \bar{j} \in \mathbb{Z}}} (2j+1)(2\bar{j}+1) \chi_{j^2}(\tau) \bar{\chi}_{\bar{j}^2}(\bar{\tau}). \tag{4.5.26}$$

Since TDL commutes with the stress energy tensor $T(z)$ and $\overline{T}(\bar{z})$, we only need to study its action on the $V_j \otimes \overline{V}_{\bar{j}}$. For this purpose, we can represent the group element $Q, C \in SO(4) = \frac{SU(2)_L \times SU(2)_R}{2}$ as the tensor product of representations of $SU(2)_L$ and $SU(2)_R$, that is,

$$Q = Q_L \otimes Q_R, \quad C = C_L \otimes C_R, \tag{4.5.27}$$

where Q_L, C_L are matrices of a spin-j representation of $SU(2)_L$ and Q_R, C_R are matrices of a spin-$\bar{j}$ representation of $SU(2)_R$. This allows us to compute trace easily since the generic form of the character of $SU(2)$ is well-known. Following the calculation in Appendix B.2, we find,

$$\begin{aligned}
&Tr_{V_j \otimes \overline{V}_{\bar{j}}} Q = Tr_{V_j \otimes \overline{V}_{\bar{j}}} (CQC) = (Tr_{V_j} Q_L)(Tr_{\overline{V}_{\bar{j}}} Q_R) = \frac{\sin((2j+1)\pi/3)}{\sin(\pi/3)} \frac{\sin((2\bar{j}+1)\pi/3)}{\sin(\pi/3)}, \\
&Tr_{V_j \otimes \overline{V}_{\bar{j}}} CQ = Tr_{V_j \otimes \overline{V}_{\bar{j}}} (QC) = (Tr_{V_j} Q_L C_L)(Tr_{\overline{V}_{\bar{j}}} Q_R C_R) \\
=&\frac{\sin((2j+1)2\pi/3)}{\sin(2\pi/3)} \frac{\sin((2\bar{j}+1)2\pi/3)}{\sin(2\pi/3)} \\
=&\frac{\sin((2j+1)\pi/3)}{\sin(\pi/3)} \frac{\sin((2\bar{j}+1)\pi/3)}{\sin(\pi/3)}, \text{ for } (j,\bar{j}) \in (\mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}) \cup \left((\frac{1}{2} + \mathbb{Z}_{\geq 0}) \oplus (\frac{1}{2} + \mathbb{Z}_{\geq 0}) \right).
\end{aligned} \tag{4.5.28}$$

The twisted partition function is therefore given by,

$$(Z_{KT})^{\mathscr{L}_Q} = \frac{8}{3|\eta(\tau)|^2} \sum_{\substack{j,\bar{j} \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \\ j+\bar{j} \in \mathbb{Z}}} \sin(\frac{\pi(2j+1)}{3}) \sin(\frac{\pi(2\bar{j}+1)}{3}) (q^{j^2} - q^{(j+1)^2})(\bar{q}^{\bar{j}^2} - \bar{q}^{(\bar{j}+1)^2}). \tag{4.5.29}$$

We can then rewrite the partition function over the familiar sum over the Narain lattice,

$$(Z_{KT})^{\mathscr{L}_Q}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{n,w \in \mathbb{Z}} \left(\cos(\frac{2\pi n}{3}) + \cos(\frac{2\pi w}{3}) \right) q^{(\frac{n+w}{2})^2} \bar{q}^{(\frac{n-w}{2})^2}. \tag{4.5.30}$$

Using the S-modular transformation, we find,

$$(Z_{KT})_{\mathscr{L}_Q}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{n,w \in \mathbb{Z}} q^{\frac{(n+w+\frac{1}{3})^2}{4}} \bar{q}^{\frac{(n-w+\frac{1}{3})^2}{4}} + q^{\frac{(n+w-\frac{1}{3})^2}{4}} \bar{q}^{\frac{(n-w+\frac{1}{3})^2}{4}}. \tag{4.5.31}$$

This twisted partition function computes the states in the defect Hilbert space $\mathscr{H}_{KT,\mathscr{L}_Q}$ and is consistent as it has integer coefficients in the q and $\overline{q}$ expansion.

Then applying T-transformation, we find

$$\begin{aligned}
&(Z_{KT})^{I}_{\mathscr{L}_Q,\mathscr{L}_{\overline{Q}}}(\tau)\\
=&(Z_{KT})_{\mathscr{L}_Q}(\tau+1)\\
=&\frac{1}{|\eta(\tau)|^2}\sum_{n,w\in\mathbb{Z}} e^{\frac{2\pi i w}{3}} q^{\frac{(n+w+\frac{1}{3})^2}{4}}\overline{q}^{\frac{(n-w+\frac{1}{3})^2}{4}} + e^{\frac{-2\pi i n}{3}} q^{\frac{(n+w-\frac{1}{3})^2}{4}}\overline{q}^{\frac{(n-w+\frac{1}{3})^2}{4}}.
\end{aligned} \tag{4.5.32}$$

This twisted partition function computes the spin of the twisted Hilbert space $\mathscr{H}_{KT,\mathscr{L}_Q}$, which is given by the phase in front of the q of $\overline{q}$ expansion. And the result is consistent with the spin selection rule derived in (4.4.6) for the case where the FS indicator $\alpha = 1$.

Next, we move to compute the twisted partition function of $\mathscr{L}_{\overline{Q}}$. By the fusion rule $\mathscr{L}_{\overline{Q}} = \mathscr{L}_Q \times \mathscr{L}_Q$, $\mathscr{L}_{\overline{Q}}$ annihilates the twisted sector in the KT theory and therefore to compute $(Z_{KT})^{\mathscr{L}_{\overline{Q}}}$, we only need to focus on the untwisted sector. There are two ways to represent the actions of $\mathscr{L}_{\overline{Q}}$ on the untwisted sector. The first is to consider the action of

$$\mathscr{L}_{\overline{Q}} = Q^2 + CQ^2C \tag{4.5.33}$$

on the C-invariant subspace of $H_{SU(2)_1}$. Alternatively, we may consider using the fusion rule and compute the action of $\mathscr{L}_Q^2$ on the C-invariant subspace of $H_{SU(2)_1}$. As a consistency check, one can show the two approaches agree with each other.

The twisted partition function is given,

$$(Z_{KT})^{\mathscr{L}_{\overline{Q}}} = \frac{2}{|\eta(\tau)|^2}\sum_{\substack{j,\overline{j}\in\frac{1}{2}\mathbb{Z}_{\geq 0},\\ j+\overline{j}\in\mathbb{Z}}} \frac{\sin(\frac{(2j+1)2\pi}{3})}{\sin\frac{2\pi}{3}}\frac{\sin(\frac{(2\overline{j}+1)2\pi}{3})}{\sin\frac{2\pi}{3}}(q^{j^2}-q^{(j+1)^2})(\overline{q}^{\overline{j}^2}-\overline{q}^{(\overline{j}+1)^2}), \tag{4.5.34}$$

and can be written as a sum over the Narain lattice where $j = \frac{n+w}{2}, \overline{j} = \frac{n-w}{2}$,

$$(Z_{KT})^{\overline{\mathscr{L}_Q}}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{n,w\in\mathbb{Z}} \left(\cos(\frac{2\pi n}{3}) + \cos(\frac{2\pi w}{3}) \right) q^{(\frac{n+w}{2})^2} \overline{q}^{(\frac{n-w}{2})^2} \tag{4.5.35}$$

taken the same form as $(Z_{KT})^{\mathscr{L}_Q}(\tau,\overline{\tau})$.

4.5.3 Constructing more triality lines from the known ones

Now we explore the possibility of constructing more triality line $\mathscr{L}'_Q$ from the known one via combining the known triality line $\mathscr{L}_Q$ with the global symmetry G_{bos} at the KT point.

The most apparent strategy is to take the generator η of some $\mathbb{Z}_3 \subset G_{bos} \equiv (U(1)^{\tilde{\theta}} \times U(1)^{\tilde{\phi}}) \rtimes \mathbb{Z}_2^{\tilde{C}}$, and attempt to construct the line operator,

$$\mathscr{L}'_Q = \mathscr{L}_Q \eta, \quad \mathscr{L}'_Q = \eta \mathscr{L}_Q, \tag{4.5.36}$$

which has been considered in [181, 180, 46] to construct the duality line N with different FS indicator. However, for this to preserve the fusion rule in general, $\mathscr{L}_Q \times \mathscr{L}_Q \times \mathscr{L}_Q = 2\sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} g$, η has to commute with $\mathscr{L}_Q$. Indeed, the construction used in [181, 180, 46] is to tensor product one theory with duality line N and another theory with anomalous $\mathbb{Z}_2$ global symmetry η, and consider the operator $N\eta$, where the duality line N in one theory apparently commutes with the operator η in another theory. As we will see, however, the candidate $\mathbb{Z}_3$ subgroups are the $\mathbb{Z}_3$ subgroups of $U(1) \times U(1)$, which does not commute with $\mathscr{L}_Q$, therefore fusion $\mathscr{L}_Q$ with generators of $\mathbb{Z}_3$ will not lead to new triality lines.

To see this is the case, we consider the action of $\mathscr{L}_Q\eta$ or $\eta\mathscr{L}_Q$ on the untwisted sector $\mathscr{H}_{KT,\text{untwisted}}$ and check whether $(\mathscr{L}_Q\eta)^3$ or $(\eta\mathscr{L}_Q)$ only has eigenvalues $0, 8$ or not.

For that, we need to understand the origin of the $U(1)^{\widetilde{\theta}} \times U(1)^{\widetilde{\phi}}$ in the KT theory from the $SU(2)_1$ theory. Under the $\mathbb{Z}_2^C$-gauging, the subgroup of the $SO(4)$ global symmetry commute with the $\mathbb{Z}_2^C$ would survive the gauging and remain as the global symmetry of the resulting KT theory. Since the charge conjugation C in the adjoint representation of $SO(4)$ is given by (4.5.11), the commutant of $\mathbb{Z}_2^C$ therefore contains

$$R_1(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & 0 & \sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\beta & 0 & \cos\beta \end{pmatrix}. \tag{4.5.37}$$

Notice that C is identified as the π-rotation $R_1(\pi)$. Hence, gauging $\mathbb{Z}_2^C$ would half the radius of $R_1(\alpha)$ which we identify as $U(1)_{\tilde{\theta}}$ (that is, $R_1(\pi)$ acts trivially on every state in the KT theory) and double the radius of $R_2(\alpha)$ which we identify as $U(1)_{\tilde{\phi}}$ (that is $R_2(2\pi)$ acts non-trivially on the twisted sector in the KT theory).

To check the fusion rule, we only need to consider the action of $\eta\mathscr{L}_Q$ or $\mathscr{L}_Q\eta$ on the untwisted sector, as $\eta\mathscr{L}_Q$ or $\mathscr{L}_Q\eta$ automatically annihilates the twisted sector therefore satisfies the fusion rule when acting on the twisted sector. The $\mathbb{Z}_3^{\tilde{\theta}} \subset U(1)^{\tilde{\theta}}$ is generated by either $R_1(\pi/3)$ or $R_1(2\pi/3)$ while the $\mathbb{Z}_3^{\tilde{\phi}} \subset U(1)^{\tilde{\phi}}$ is generated by either $R_2(8\pi/3) = R_2(2\pi/3)$ or $R_2(4\pi/3)$ when acting on the untwisted sector. For convenience, we take the generator $\eta_{\tilde{\theta}}$ of $\mathbb{Z}_3^{\tilde{\theta}}$ to be $R_1(2\pi/3)$ and the generator of $\eta_{\tilde{\phi}}$ of $\mathbb{Z}_3^{\tilde{\phi}}$ to be $R_2(2\pi/3)$ as well.

We can check explicitly that on the $(j,\bar{j}) = (3/2,3/2)$ irrep of $SO(4)$ that $\mathscr{L}_Q$ does not commute with η and their product $\mathscr{L}_Q\eta$ does not lead to triality line. Following the convention in Appendix B.2, we construct the matrix of $Q+CQC$ as well as η and diagonalize it using C eigenstates as a basis. For $(j,\bar{j}) = (3/2,3/2)$ irrep, the dimension of C invariant states is 5 and project $Q+CQC$ to this subspace we find,

$$Q+CQC=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & -\frac{3}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{3}{2} & 0 \\ 0 & -\frac{3}{2} & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{3}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.5.38}$$

and the generator $\eta_{\tilde{\theta}}$ of $\mathbb{Z}_3^{\tilde{\theta}} \subset U(1)^{\tilde{\theta}}$ and the generator $\eta_{\tilde{\phi}}$ of $\mathbb{Z}_3^{\tilde{\phi}} \subset U(1)^{\tilde{\phi}}$ are

$$\begin{aligned}
\eta_{\tilde{\theta}} &= \begin{pmatrix}
\frac{7}{16} & \frac{3}{8} & \frac{3\sqrt{3}}{16} & 0 & -\frac{3}{8} & -\frac{9}{16} & 0 & -\frac{3\sqrt{3}}{16} \\
-\frac{3}{8} & \frac{1}{16} & \frac{\sqrt{3}}{4} & \frac{3\sqrt{3}}{16} & \frac{9}{16} & -\frac{3}{8} & -\frac{3\sqrt{3}}{16} & 0 \\
\frac{3\sqrt{3}}{16} & -\frac{\sqrt{3}}{4} & \frac{1}{16} & \frac{3}{8} & 0 & \frac{3\sqrt{3}}{16} & -\frac{3}{8} & -\frac{9}{16} \\
0 & \frac{3\sqrt{3}}{16} & -\frac{3}{8} & \frac{7}{16} & \frac{3\sqrt{3}}{16} & 0 & \frac{9}{16} & -\frac{3}{8} \\
\frac{3}{8} & \frac{9}{16} & 0 & \frac{3\sqrt{3}}{16} & \frac{1}{16} & \frac{3}{8} & -\frac{3\sqrt{3}}{16} & \frac{\sqrt{3}}{4} \\
-\frac{9}{16} & \frac{3}{8} & \frac{3\sqrt{3}}{16} & 0 & -\frac{3}{8} & \frac{7}{16} & 0 & -\frac{3\sqrt{3}}{16} \\
0 & -\frac{3\sqrt{3}}{16} & \frac{3}{8} & \frac{9}{16} & -\frac{3\sqrt{3}}{16} & 0 & \frac{7}{16} & \frac{3}{8} \\
-\frac{3\sqrt{3}}{16} & 0 & -\frac{9}{16} & \frac{3}{8} & -\frac{\sqrt{3}}{4} & -\frac{3\sqrt{3}}{16} & -\frac{3}{8} & \frac{1}{16}
\end{pmatrix}, \\
\eta_{\tilde{\phi}} &= \begin{pmatrix}
-\frac{13}{32} & \frac{15}{32} & -\frac{3\sqrt{3}}{32} & \frac{3\sqrt{3}}{32} & \frac{15}{32} & -\frac{9}{32} & \frac{9\sqrt{3}}{32} & \frac{3\sqrt{3}}{32} \\
-\frac{15}{32} & -\frac{7}{32} & \frac{7\sqrt{3}}{32} & -\frac{3\sqrt{3}}{32} & \frac{9}{32} & -\frac{3}{32} & -\frac{9\sqrt{3}}{32} & \frac{9\sqrt{3}}{32} \\
-\frac{3\sqrt{3}}{32} & -\frac{7\sqrt{3}}{32} & -\frac{7}{32} & \frac{15}{32} & \frac{9\sqrt{3}}{32} & \frac{9\sqrt{3}}{32} & -\frac{3}{32} & -\frac{9}{32} \\
-\frac{3\sqrt{3}}{32} & -\frac{3\sqrt{3}}{32} & -\frac{15}{32} & -\frac{13}{32} & -\frac{3\sqrt{3}}{32} & \frac{9\sqrt{3}}{32} & \frac{9}{32} & \frac{15}{32} \\
-\frac{15}{32} & \frac{9}{32} & -\frac{9\sqrt{3}}{32} & -\frac{3\sqrt{3}}{32} & -\frac{7}{32} & -\frac{3}{32} & -\frac{9\sqrt{3}}{32} & -\frac{7\sqrt{3}}{32} \\
-\frac{9}{32} & \frac{3}{32} & \frac{9\sqrt{3}}{32} & -\frac{9\sqrt{3}}{32} & \frac{3}{32} & \frac{11}{32} & \frac{5\sqrt{3}}{32} & -\frac{9\sqrt{3}}{32} \\
-\frac{9\sqrt{3}}{32} & -\frac{9\sqrt{3}}{32} & \frac{3}{32} & \frac{9}{32} & -\frac{9\sqrt{3}}{32} & -\frac{5\sqrt{3}}{32} & \frac{11}{32} & -\frac{3}{32} \\
\frac{3\sqrt{3}}{32} & -\frac{9\sqrt{3}}{32} & -\frac{9}{32} & -\frac{15}{32} & \frac{7\sqrt{3}}{32} & -\frac{9\sqrt{3}}{32} & \frac{3}{32} & -\frac{7}{32}
\end{pmatrix},
\end{aligned} \tag{4.5.39}$$

and the possible $\eta = \eta_\alpha^i \eta_\beta^j$ where $(i,j) \neq (0,0)$. As one can check explicitly, the product $\mathscr{L}_Q \eta$ or $\eta \mathscr{L}_Q$ does not lead to new duality line, as $(\mathscr{L}_Q \eta)^3$ or $(\eta \mathscr{L}_Q)^3$ does not have eigenvalues which are either 0 or 8. Hence, we conclude we can't build new triality out of the known one $\mathscr{L}_Q$ from this procedure.

Since the generator η of $\mathbb{Z}_3$ and, in fact, elements of $U(1) \times U(1)$ in general, does not commute with $\mathscr{L}_Q$, we can consider another possible construction, namely to conjugate $\mathscr{L}_Q$ by an element $h \in U(1) \times U(1)$,

$$\mathscr{L}_Q' = h^{-1} \mathscr{L}_Q h \tag{4.5.40}$$

with the fusion rule,

$$\mathscr{L}'_Q \times \mathscr{L}'_Q \times \mathscr{L}'_Q = 2 \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} h^{-1} g h, \tag{4.5.41}$$

and $\mathscr{L}'_{\overline{Q}} = h^{-1} \mathscr{L}_{\overline{Q}} h$. Notice that we do get a "new" triality category under this procedure, since $(h^{-1}gh)^2 = 1$. However, this "new" triality defect should be Morita equivalent to the old one.

4.6 More Triality Fusion Categories

One might wonder if there exist more fusion categories besides the ones described previously satisfying the same fusion rule. Indeed, there are another set of F-symbols that have been computed in the condensed matter literature [179]. It is natural to ask if their F-symbols give the same fusion categories as ours and if there are more inequivalent F-symbols. We will answer these questions in this section.

4.6.1 The classification of triality fusion category

We first review the result in [117] which classifies the fusion category whose simple objects containing $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathscr{L}_Q$ and $\mathscr{L}_{\overline{Q}}$. They satisfy the following fusion relations,

$$\begin{aligned}
&g \times \mathscr{L}_Q = \mathscr{L}_Q \times g = \mathscr{L}_Q, \quad g \times \mathscr{L}_{\overline{Q}} = \mathscr{L}_{\overline{Q}} \times g = \mathscr{L}_{\overline{Q}}, \\
&\mathscr{L}_Q \times \mathscr{L}_Q = 2\mathscr{L}_{\overline{Q}}, \quad \mathscr{L}_{\overline{Q}} \times \mathscr{L}_{\overline{Q}} = 2\mathscr{L}_Q, \\
&\mathscr{L}_Q \times \mathscr{L}_{\overline{Q}} = \mathscr{L}_{\overline{Q}} \times \mathscr{L}_Q = \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} g.
\end{aligned} \tag{4.6.1}$$

Theorem 1.4 in [117] then implies there are 6 inequivalent fusion categories with the simple objects satisfying the above fusion relations. Since the Frobenius-Schur indicator is given by an element in $H^3(\mathbb{Z}_3, U(1)) = \mathbb{Z}_3$, the 6 inequivalent fusion categories organize into 2 classes each containing 3 related by choosing different FS indicator.

Theorem 1.1 in [117] further describes the two classes of fusion categories. The first one is the group theoretic fusion category with different FS indicator α (which are the ones we studied in section 4.3) while the second class is constructed explicitly in [117] in terms of the classification data (but the F-symbols are not explicitly given). We will argue the F-symbols computed in [179] correspond to the second class as they lead to different fusion categories from ours. Since the second class is not group theoretical fusion category, these triality fusion categories are intrinsic non-invertible in the sense of [123].

4.6.2 F-symbols of intrinsic triality fusion category

We follow [179] to list the F-symbols of the triality category. Recall that the triality line satisfies,

$$\mathscr{L}_Q \times \mathscr{L}_Q = 2\mathscr{L}_{\overline{Q}}, \quad \mathscr{L}_Q \times \mathscr{L}_{\overline{Q}} = \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} g. \tag{4.6.2}$$

and invertible symmetry lines g satisfy the fusion rule of $\mathsf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}$. We represent invertible symmetry line g by $\mathbb{Z}_2$-valued vectors $\{(0,0),(1,0),(0,1),(1,1)\}$ (which corresponds to $(\mathbb{1},\hat{\sigma},\eta,\hat{\sigma}\eta)$ in previous notation). In this representation, the triality symmetry is also a $\mathbb{Z}_2$-valued matrix,

$$\Lambda_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.6.3}$$

The R-symbols between different invertible symmetry lines are,

$$R^{g,h} = (-1)^{g^\intercal \sigma^1 \Lambda_3^2 h}, \quad g,h \in \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{4.6.4}$$

where σ^i are the Pauli i matrices. The braiding phase is $\mathscr{D}\mathscr{S}_{g,h} = R^{g,h}R^{h,g} = (-1)^{g^\intercal \sigma^1 h}$. The F-symbols can be understood as a representation of the double cover of A_4. We choose the 2-d representations as,

$$\mathscr{A}_{(0,0)} = \sigma^0,\ \mathscr{A}_{(1,0)} = \mathrm{i}\sigma^1,\ \mathscr{A}_{(0,1)} = -\mathrm{i}\sigma^2,\ \mathscr{A}_{(1,1)} = \mathrm{i}\sigma^3. \tag{4.6.5}$$

The F-symbols consist of a free parameter $\alpha = e^{2\pi k i/3}, k = 0,1,2$ which is the Frobenius-Schur indicator. The list of F-symbols is given by,

$$F^{ghk}_{g+h+k} = 1, \quad g,h,k \in \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{4.6.6}$$

$$F^{\mathscr{L}_Q gh}_{\mathscr{L}_Q} = R^{g,(\Lambda_3 h)}, \quad F^{gh\mathscr{L}_Q}_{\mathscr{L}_Q} = R^{h,(\Lambda_3^2 g)}, \quad F^{g\mathscr{L}_Q h}_{\mathscr{L}_Q} = \mathscr{D}\mathscr{S}_{(\Lambda_3 g),h}, \tag{4.6.7}$$

$$F^{\mathscr{L}_Q \mathscr{L}_{\overline{Q}} g}_{h} = R^{g,\Lambda_3^2 h}, \quad F^{g\mathscr{L}_Q \mathscr{L}_{\overline{Q}}}_{h} = R^{g\times h,\Lambda_3^2 g}, \quad F^{\mathscr{L}_Q g \mathscr{L}_{\overline{Q}}}_{h} = \mathscr{D}\mathscr{S}_{g,\Lambda_3 h} R^{g,\Lambda_3 g}. \tag{4.6.8}$$

When exchanging $\mathscr{L}_Q \leftrightarrow \mathscr{L}_{\overline{Q}}$, the F-symbols are obtained by replacing $\Lambda_3 \leftrightarrow \Lambda_3^2$,

$$F^{\mathscr{L}_{\overline{Q}} gh}_{\mathscr{L}_{\overline{Q}}} = R^{g,(\Lambda_3^2 h)}, \quad F^{gh\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}} = R^{h,(\Lambda_3 g)}, \quad F^{g\mathscr{L}_{\overline{Q}} h}_{\mathscr{L}_{\overline{Q}}} = \mathscr{D}\mathscr{S}_{(\Lambda_3^2 g),h}, \tag{4.6.9}$$

$$F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_Q g}_{h} = R^{g,\Lambda_3 h}, \quad F^{g\mathscr{L}_{\overline{Q}} \mathscr{L}_Q}_{h} = R^{g\times h,\Lambda_3 g}, \quad F^{\mathscr{L}_{\overline{Q}} g \mathscr{L}_Q}_{h} = \mathscr{D}\mathscr{S}_{g,\Lambda_3^2 h} R^{g,\Lambda_3^2 g}, \tag{4.6.10}$$

and

$$[F^{\mathscr{L}_Q \mathscr{L}_{\overline{Q}} \mathscr{L}_Q}_{\mathscr{L}_Q}]_{g,h} = -\frac{\alpha}{2}\mathscr{D}\mathscr{S}_{\Lambda_3 g,h} R^{h,\Lambda_3 h}, \quad [F^{\mathscr{L}_{\overline{Q}} \mathscr{L}_Q \mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}}]_{g,h} = -\frac{\alpha^{-1}}{2}\mathscr{D}\mathscr{S}_{\Lambda_3^2 g,h} R^{h,\Lambda_3^2 h}. \tag{4.6.11}$$

Other F-symbols are listed in Tab. 4.4.

4.6.3 Spin selection rules

We now derive the spin selection rules from the above F-symbols. Repeating the same calculation in section 4.4.1, we find,

$$\gamma(h,g) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{4.6.12}$$

Table 4.4. The rest of F-symbols computed in [179] where $\sigma_{\text{sym}} = \exp(\frac{\pi}{3}\frac{\mathscr{A}_{sym}}{\sqrt{3}})$ and $\mathscr{A}_{sym} = \sum_{g=\{(1,0),(0,1),(1,1)\}} \mathscr{A}_g$. $\alpha = e^{\mathrm{i}2\pi k/3}$ is the FS indicator.

g	$\mathbb{1}$	$\hat{\sigma}$	η	$\hat{\sigma}\eta$
$[F^{\mathscr{L}_Q\mathscr{L}_Q g}_{\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_Q,1,\nu)}$	σ^0	$-\mathrm{i}\sigma^1$	$\mathrm{i}\sigma^2$	$-\mathrm{i}\sigma^3$
$[F^{\mathscr{L}_Q g\mathscr{L}_Q}_{\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_Q,1,\mu),(\mathscr{L}_Q,1,\nu)}$	σ^0	$\mathrm{i}\sigma^2$	$-\mathrm{i}\sigma^3$	$-\mathrm{i}\sigma^1$
$[F^{g\mathscr{L}_Q\mathscr{L}_Q}_{\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_Q,1,\mu),(\mathscr{L}_{\overline{Q}},\nu,1)}$	σ^0	$\mathrm{i}\sigma^3$	$\mathrm{i}\sigma^1$	$-\mathrm{i}\sigma^2$
$[F^{\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}} g}_{\mathscr{L}_Q}]_{(\mathscr{L}_Q,\mu,1),(\mathscr{L}_{\overline{Q}},1,\nu)}$	σ^0	$-\mathrm{i}\sigma^1$	$\mathrm{i}\sigma^2$	$-\mathrm{i}\sigma^3$
$[F^{\mathscr{L}_{\overline{Q}} g\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_Q}]_{(\mathscr{L}_{\overline{Q}},1,\mu),(\mathscr{L}_{\overline{Q}},1,\nu)}$	σ^0	$\mathrm{i}\sigma^3$	$\mathrm{i}\sigma^1$	$-\mathrm{i}\sigma^2$
$[F^{g\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_Q}]_{(\mathscr{L}_{\overline{Q}},1,\mu),(\mathscr{L}_Q,\nu,1)}$	σ^0	$-\mathrm{i}\sigma^2$	$\mathrm{i}\sigma^3$	$\mathrm{i}\sigma^1$
$[F_g^{\mathscr{L}_Q\mathscr{L}_Q\mathscr{L}_Q}]_{(\mathscr{L}_{\overline{Q}},\mu,1),(\mathscr{L}_{\overline{Q}},\nu,1)}$	$\alpha^{-1}\sigma_{\text{sym}}$	$\mathrm{i}\alpha^{-1}\sigma^1\sigma_{\text{sym}}$	$-\mathrm{i}\alpha^{-1}\sigma^2\sigma_{\text{sym}}$	$\mathrm{i}\alpha^{-1}\sigma^3\sigma_{\text{sym}}$
$[F_g^{\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_Q,\mu,1),(\mathscr{L}_Q,\nu,1)}$	$\alpha\sigma_{\text{sym}}^{-1}$	$\mathrm{i}\alpha\sigma^1\sigma_{\text{sym}}^{-1}$	$-\mathrm{i}\alpha\sigma^2\sigma_{\text{sym}}^{-1}$	$\mathrm{i}\alpha\sigma^3\sigma_{\text{sym}}^{-1}$
$[F^{\mathscr{L}_Q\mathscr{L}_Q\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_Q}]_{(\mathscr{L}_Q,\mu,\nu),(g,1,1)}$	$\frac{1}{\sqrt{2}}\mathrm{i}\sigma_{\text{sym}}\sigma^1$	$\frac{1}{\sqrt{2}}\sigma_{\text{sym}}\sigma^0$	$-\frac{1}{\sqrt{2}}\mathrm{i}\sigma_{\text{sym}}\sigma^3$	$-\frac{1}{\sqrt{2}}\mathrm{i}\sigma_{\text{sym}}\sigma^2$
$[F^{\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}\mathscr{L}_Q}_{\mathscr{L}_{\overline{Q}}}]_{(\mathscr{L}_Q,\mu,\nu),(g,1,1)}$	$\frac{\alpha}{\sqrt{2}}\mathrm{i}\sigma_{\text{sym}}^{-1}\sigma^3$	$-\frac{\alpha}{\sqrt{2}}\mathrm{i}\sigma_{\text{sym}}^{-1}\sigma^2$	$-\frac{\alpha}{\sqrt{2}}\mathrm{i}\sigma_{\text{sym}}^{-1}\sigma^1$	$-\frac{\alpha}{\sqrt{2}}\sigma_{\text{sym}}^{-1}\sigma^0$
$[F^{\mathscr{L}_{\overline{Q}}\mathscr{L}_Q\mathscr{L}_Q}_{\mathscr{L}_Q}]_{(g,1,1),(\mathscr{L}_{\overline{Q}},\mu,\nu)}$	$\frac{\mathrm{i}\alpha^{-1}\sigma^2}{\sqrt{2}}$	$\frac{-\alpha^{-1}\sigma^0}{\sqrt{2}}$	$\frac{-\mathrm{i}\alpha^{-1}\sigma^1}{\sqrt{2}}$	$\frac{\mathrm{i}\alpha^{-1}\sigma^3}{\sqrt{2}}$
$[F^{\mathscr{L}_Q\mathscr{L}_{\overline{Q}}\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_{\overline{Q}}}]_{(g,1,1),(\mathscr{L}_Q,\mu,\nu)}$	$\frac{\mathrm{i}\sigma^2}{\sqrt{2}}$	$\frac{-\mathrm{i}\sigma^1}{\sqrt{2}}$	$\frac{\mathrm{i}\sigma^3}{\sqrt{2}}$	$\frac{-\sigma^0}{\sqrt{2}}$

with the same four 1-dimensional irreducible representations as in the non-intrinsic or group-theoretical case,

$$(1,1,1,-1), \quad (1,1,-1,1), \quad (1,-1,1,1), \quad (1,-1,-1,-1). \tag{4.6.13}$$

Then, consider the same calculation in Figure 4.9 and plug in the F-symbols for the intrinsic triality defects, we find,

$$\begin{aligned} Z_{\mathscr{L}_Q}(\tau+3) &= \sum_{\substack{\mu,\nu=1,2,\\ g\in\mathbb{Z}_2\times\mathbb{Z}_2}} \left[F^{\mathscr{L}_{\overline{Q}}\mathscr{L}_Q\mathscr{L}_Q}_{\mathscr{L}_Q}\right]_{(\mathbb{1},1,1)(\mathscr{L}_Q,\mu,\nu)} \left[F^{\mathscr{L}_Q\mathscr{L}_Q\mathscr{L}_{\overline{Q}}}_{\mathscr{L}_Q}\right]_{(\mathscr{L}_{\overline{Q}},\mu,\nu)(g,1,1)} Z^{\mathscr{L}_Q}_{\mathscr{L}_Q g}(\tau) \\ &= \frac{\alpha}{2}\sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} Z^{\mathscr{L}_Q}_{\mathscr{L}_Q g}(\tau). \end{aligned} \tag{4.6.14}$$

where $\alpha = e^{2\pi \mathrm{i}k/3}, k = 0,1,2$ is the FS indicator. This implies,

$$e^{6\pi \mathrm{i}s} = \frac{\alpha}{2}\sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} \widehat{g}_{\mathscr{L}_Q}. \tag{4.6.15}$$

Using the eigenvalues of the irreducible representations of $\mathbb{Z}_2\times\mathbb{Z}_2$ with phase $\gamma(g,h)$ given by (4.6.13), we have,

$$e^{6\pi \mathrm{i}s} = \pm\alpha. \tag{4.6.16}$$

Notice that 2-cocycle $\gamma(g,h)$ ensures the eigenvalues lead to consistent result in (4.6.14), as the result on the right-hand side has to be a phase, which is not true for the eigenvalues of the usual irreducible representations of $\mathbb{Z}_2\times\mathbb{Z}_2$. We find the allowed spin is given by,

$$e^{2\pi \mathrm{i}s} = \begin{cases} e^{k\pi\mathrm{i}/3}, \quad k=0,1,2,3,4,5, \quad \alpha = 1, \\ e^{\frac{2\pi\mathrm{i}}{9}+\frac{k\pi\mathrm{i}}{3}}, \quad k=0,1,2,3,4,5, \quad \alpha = e^{2\pi\mathrm{i}/3}, \\ e^{-\frac{2\pi\mathrm{i}}{9}+\frac{k\pi\mathrm{i}}{3}}, \quad k=0,1,2,3,4,5, \quad \alpha = e^{4\pi\mathrm{i}/3}. \end{cases} \tag{4.6.17}$$

Determine the triality fusion category from spin selection rules

We now argue that one can determine the triality fusion category from the spins of the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$. It is clear that we can determine the FS indicator from the spins that appear in $\mathscr{H}_{\mathscr{L}_Q}$.

We now argue that we can distinguish between intrinsic and non-intrinsic triality fusion categories from the spins. For example, considering the case when the FS indicator $\alpha = 1$, then the allowed spin s for the non-intrinsic triality fusion categories satisfies $s \in \frac{1}{3}\mathbb{Z}$ while the allowed spin s for the intrinsic triality fusion categories satisfies $s \in \left(\frac{1}{3}\mathbb{Z}\right) \cup \left(\frac{1}{3}\mathbb{Z} + \frac{1}{2}\right)$. To distinguish the two cases, we only need to show the additional spin where $s \in \frac{1}{3}\mathbb{Z} + \frac{1}{2}$ must appear.

To see this, we can show all the allowed irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in (4.6.13) has to appear using the technique in [166]. Specifically, consider the following partition function

$$\frac{1}{4} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_\alpha(g) Z^{\mathscr{L}_Q}_{\mathscr{L}_Q, g}(\tau = \mathsf{i}\beta) \tag{4.6.18}$$

where χ_α is the character associated with the irreducible representation α, which can be seen from (4.6.13). For each choice of χ_α, we keep only the contribution from the particular irreducible representation α. We only need to show this is non-zero for any α.

For this, we consider applying the S-modular transformation and get,

$$\frac{1}{4} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_\alpha(g) Z^{\mathscr{L}_Q}_{\mathscr{L}_Q, g}(\tau = \mathsf{i}\beta) = \frac{1}{4} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_\alpha(g) \operatorname{Tr}_{\mathscr{H}_g} \left(\widehat{\mathscr{L}_Q}\right)_{g,+} e^{-\frac{4\pi^2}{\beta}\left(H - \frac{c}{12}\right)}. \tag{4.6.19}$$

Considering the high-temperature limit $\beta \to 0$, we can see in the dual channel on the right-hand side, the partition sum is dominated by the ground state in each defect Hilbert space $\mathscr{H}_g$. As long as the $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts faithfully in the theory, the ground state in $\mathscr{H}_g$ for $g \neq 1$ has positive energy, hence the R.H.S. is dominated by the vacuum state in $\mathscr{H}_1$, which implies

$$\frac{1}{4} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_\alpha(g) Z^{\mathscr{L}_Q}_{\mathscr{L}_Q, g}(\tau = \mathsf{i}\beta) \xrightarrow{\beta \to 0} \frac{1}{2} e^{\frac{3\pi^2 c}{\beta}} > 0, \tag{4.6.20}$$

where in the last step we used $\chi_\alpha(1) = 1$ for every α as in (4.6.13) and $\mathscr{L}_{\overline{Q}}$ acts on vacuum state as its quantum dimension 2. Since in the high-temperature limit the partition sum over a particular fixed irreducible representation is positive, we know each irreducible representation must appear. This then implies the spin s such that $s \in \frac{1}{3}\mathbb{Z} + \frac{1}{2}$ has to appear since it comes from the states with irreducible representation $(1,-1,-1,-1)$.

Hence, we can distinguish the different triality fusion categories by the spins that appeared in the defect Hilbert space $\mathscr{H}_{\mathscr{L}_Q}$.

To conclude this subsection, we briefly comment on when the spin selection rule should be saturated.[9] For illustration, let's consider the three-state Potts model. This RCFT contains two $\mathbb{Z}_3$ self-duality lines N, N' [46] with the fusion rules:

$$N^2 = (N')^2 = I + \eta + \overline{\eta}, \tag{4.6.21}$$

where η generates the $\mathbb{Z}_3$ global symmetries. The spins of the defect Hilbert space satisfy,

$$\mathscr{H}_N : s \in \frac{1}{2}\mathbb{Z} + \left\{\frac{1}{8}, -\frac{1}{24}\right\}, \quad \mathscr{H}_{N'} : s \in \frac{\mathbb{Z}}{2} + \left\{-\frac{1}{8}, \frac{1}{24}\right\}. \tag{4.6.22}$$

The spin selection rule is derived in [46] from the relation,

$$e^{4\pi i s}\langle\psi'|\psi\rangle = \frac{1}{\sqrt{3}}\langle\psi'|1 + \hat{\eta}_- + \hat{\overline{\eta}}_-|\psi\rangle = \frac{1+\omega^a+\omega^b}{\sqrt{3}} \tag{4.6.23}$$

by requiring that $\frac{1+\omega^a+\omega^b}{\sqrt{3}}$ where $a, b = 0, 1, 2$ and $\omega = e^{2\pi i/3}$ is a phase which takes the form,

$$s \in \frac{1}{2}\mathbb{Z} \pm \left\{\frac{1}{24}, \frac{1}{8}\right\}. \tag{4.6.24}$$

At first glance, the spin selection rule is not saturated. This is because the spin selection rule is not derived from the eigenvalues of the (projective) representation. For instance, let's consider

[9] The authors thank Yifan Wang for mentioning this example which leads to this discussion.

the irrep of $\mathbb{Z}_2 \times \mathbb{Z}_2$ which leads to $e^{4\pi \mathrm{i} s} = e^{-\frac{2\pi \mathrm{i}}{12}}$ in the defect Hilbert space $\mathscr{H}_{\mathcal{N}}$, say $(1, \omega^2, 1)$. From this, we can derive the 2-cocycle $\gamma(g,h)$,

$$\gamma(g,h) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \tag{4.6.25}$$

The allowed irreducible representations twisted by this 2-cocycle are given by,

$$(1, \omega^2, 1), \quad (1, 1, \omega^2), \quad (1, \omega, \omega), \tag{4.6.26}$$

where the first two lead to spin such that $e^{4\pi \mathrm{i} s} = e^{-\frac{2\pi \mathrm{i}}{12}}$ and the last one leads to the spin such that $e^{4\pi \mathrm{i} s} = e^{\frac{2\pi \mathrm{i}}{4}}$. By the same argument, each irreducible representation has to appear and this spin selection rule must be saturated, which indeed is the case. Similarly, if we consider the defect Hilbert space of $\mathscr{H}_{\mathcal{N}'}$, the 2-cocycle $\gamma'(g,h)$ is now given by

$$\gamma'(g,h) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \tag{4.6.27}$$

and the allowed irreducible representations are given by

$$(1, \omega, 1), \quad (1, 1, \omega), \quad (1, \omega^2, \omega^2), \tag{4.6.28}$$

where the first two leads to spin such that $e^{4\pi \mathrm{i} s} = e^{2\pi \mathrm{i}/12}$ and the last one leads to spin such that $e^{4\pi \mathrm{i} s} = e^{-2\pi \mathrm{i}/4}$. Similarly, this spin selection rule is also saturated. One might wonder why there could be two different 2-cocycles arising from the same $\mathbb{Z}_3$-duality category. This is because when deriving the crossing kernels, even after fixing the FS-indicator $\varepsilon = 1$, one needs to choose the ω to be either $e^{2\pi \mathrm{i}/3}$ or $e^{-2\pi \mathrm{i}/3}$. Two different choices relate to each other by relabeling η

as $\overline{\eta}$. Such relabelling will change $\gamma(g,h)$ to $\gamma'(g,h)$ as well. In the case of two duality lines N and N', both form a $\mathbb{Z}_3$-duality category with the same $\mathbb{Z}_3$ symmetry, hence there's no way we can relabel the η in one $\mathbb{Z}_3$ fusion category without doing the same relabeling for the other. Therefore, the choices of $\omega = e^{\pm 2\pi \mathrm{i} s/3}$ matter here.

Chapter 4, in full, is a reprint of the material as it appears in Da-Chuan Lu, Zhengdi Sun, JHEP **02**, 173 (2023). The dissertation author was the primary investigator and author of this paper.

Chapter 5

When are Duality Defects Group-Theoretical?

5.1 Introduction and summary

5.1.1 The problem

Duality defects:

A d-dimensional quantum field theory (QFT) $\mathscr{X}$ with a finite, non-anomalous, p-form abelian global symmetry $G^{(p)}$ has a *non-invertible duality symmetry* when $\mathscr{X}$ is invariant under gauging $G^{(p)}$ [50, 51],

$$\mathscr{X} = \mathscr{X}/G^{(p)}. \tag{5.1.1}$$

This in particular requires $p = \frac{d}{2} - 1$, so that the gauged theory $\mathscr{X}/G^{(p)}$ also has the same $G^{(p)}$ global symmetry.[1] Each finite symmetry is associated with a topological defect [90], and in the present case the *duality defect*. The duality defect can be constructed by gauging $G^{(p)}$ on half-space and imposing the Dirichlet boundary condition for $G^{(p)}$ at the defect locus. It has

[1] For simplicity, we only consider the symmetry of a single form-degree. One can also consider symmetries with multiple form-degrees, say, a p-form symmetry and a q-form symmetry. The theory can also be self-dual under gauging both symmetries if $d-2 = p+q$. See e.g. [121, 72] for such examples. More generally, a finite invertible symmetry in d dimensional quantum field theory is described by a higher-group, and gauging it leads to the higher representation category of a higher group. Self-duality under gauging the higher group requires its higher representation category coincides with itself. See [30, 35, 83, 34, 22, 32, 33, 24, 23, 59, 31] for the recent developments of higher categorical theory description of global symmetries.

been shown that the duality defect satisfies *non-invertible* fusion rule [50, 51]. By construction, duality defect implements gauging $G^{(p)}$. Note that there are additional data specifying the duality defects, such as the (symmetric and non-degenerate) bicharacter and the Frobenius-Schur indicator. Different QFT $\mathscr{X}$ satisfying (5.1.1) may yield duality defects with different choices of such additional data. Throughout this work, for simplicity, we will denote the duality defect associated with gauging $G^{(p)}$ as $G^{(p)}$ *duality defect*, to emphasize the underlying invertible symmetry, and leave the dependence of additional data implicit.

A generic $G^{(p)}$ symmetric QFT does not satisfy (5.1.1). However, there are some well-known examples satisfying (5.1.1), including compact scalars in 2d with $\mathbb{Z}_N^{(0)}$ symmetry [115, 50], Maxwell theories in 4d with $\mathbb{Z}_N^{(1)}$ symmetry [50], $\mathscr{N}=4$ super Yang-Mills theories in 4d with $\mathbb{Z}_N^{(1)}$ symmetry [121], etc. For these theories, showing (5.1.1) often requires a T-duality in 2d or S-duality in 4d,[2] and these dualities are usually found only in highly fine-tuned theories. Thus it is highly non-trivial and interesting to find deformations preserving the duality symmetry, i.e. the relation (5.1.1). See [66] for recent discussions on duality-preserving deformation of the $\mathscr{N}=4$ super Yang-Mills theory.

An alternative construction:

In [121], an alternative construction of theories with duality symmetry was proposed for certain $G^{(p)}$. See also [30, 31] for further generalizations. The idea is to start with a theory $\mathscr{Y}$ with invertible symmetries only, and a mixed anomaly. Gauging a non-anomalous subgroup of $\mathscr{Y}$ yields another theory $\mathscr{X}$, and the mixed anomaly in $\mathscr{Y}$ enforces the existence of duality symmetry in $\mathscr{X}$.

Let's illustrate the idea by an example. Take a 4d QFT $\mathscr{Y}$ with global symmetry $G^{(1)}\times H^{(0)}=\mathbb{Z}_2^{(1)}\times\mathbb{Z}_4^{(0)}$. We also assume a mixed anomaly characterized by the 5d anomaly

[2]One should distinguish the T or S-dualities from the duality transformation associated with gauging $G^{(p)}$.

theory[3]

$$\pi \int_{X_5} A^{(1)} \cup \frac{\mathscr{P}(B^{(2)})}{2}. \tag{5.1.2}$$

The anomaly means that the partition function of $\mathscr{Y}$ obeys

$$\mathscr{Z}_{\mathscr{Y}}[B^{(2)}] = \mathscr{Z}_{\mathscr{Y}}[B^{(2)}] e^{i\pi \int_{X_4} \mathscr{P}(B^{(2)})/2}. \tag{5.1.3}$$

In particular, when $e^{i\pi \int_{X_4} \mathscr{P}(B^{(2)})/2}$ evaluates to be a non-trivial phase, the partition function $\mathscr{Z}_{\mathscr{Y}}[B^{(2)}]$ vanishes (for example due to the presence of zero modes). Next, we construct the theory $\mathscr{X}$ by gauging $G^{(1)} = \mathbb{Z}_2^{(1)}$ of $\mathscr{Y}$ and then stacking an $\mathbb{Z}_2^{(1)}$ SPT,[4]

$$\mathscr{Z}_{\mathscr{X}}[B^{(2)}] = \sum_{b^{(2)} \in H^2(X_4, \mathbb{Z}_2)} \mathscr{Z}_{\mathscr{Y}}[b^{(2)}] e^{i\pi \int_{X_4} b^{(2)} B^{(2)} + i\pi \int_{X_4} \mathscr{P}(B^{(2)})/2}. \tag{5.1.4}$$

Combining with (5.1.3), it is straightforward to check that $\mathscr{X} = \mathscr{X}/\mathbb{Z}_2^{(1)}$, i.e.

$$\mathscr{Z}_{\mathscr{X}}[B^{(2)}] = \sum_{b^{(2)} \in H^2(X_4, \mathbb{Z}_2)} \mathscr{Z}_{\mathscr{X}}[b^{(2)}] e^{i\pi \int_{X_4} b^{(2)} B^{(2)}}. \tag{5.1.5}$$

The associated duality defect can be obtained using half-gauging. Hence we have found an alternative, yet systematic, way to construct theories with duality symmetry associated with gauging $\mathbb{Z}_2^{(1)}$, as well as the $\mathbb{Z}_2^{(1)}$ duality defect.

It is useful to know that the duality symmetry associated with gauging $\mathbb{Z}_2^{(1)}$ admits the above alternative construction. Note that this construction does not require any detailed dynamical information of $\mathscr{Y}$ (such as whether $\mathscr{Y}$ is a CFT or a free field theory). Any $\mathscr{Y}$, as long as it has the requested symmetry and anomaly can be fed into the construction. In particular,

[3]Only $\mathbb{Z}_2^{(0)}$ normal subgroup of $\mathbb{Z}_4^{(0)}$ is anomalous. But rigorously speaking, (5.1.2) is well-defined only when $A^{(1)}$ belongs to $H^1(X_5, \mathbb{Z}_4)$ (rather than $H^1(X_5, \mathbb{Z}_2)$) due to the $1/2$ factor.

[4]We suppress the overall normalization throughout the paper. Please refer to [50, 122] for systematic discussions of the overall normalizations.

because only the invertible symmetries of the theory $\mathscr{Y}$ play a role in the construction, it is easy to turn on perturbations leaving the symmetry and anomaly unchanged. After gauging $\mathbb{Z}_2^{(1)}$, such symmetric perturbation in $\mathscr{Y}$ becomes a duality-symmetry preserving perturbation in $\mathscr{X}$. Hence it is easy to turn on duality-preserving deformations, and allows one to study the consequence of duality symmetry along the RG flow. Moreover, this alternative construction allows one to uncover new duality defects in gauge theories. For instance, this construction can be used to show the presence of duality defects in a large class of gauge theories, including non-invertible time reversal symmetries in the 4d Yang-Mills theories [121, 30, 53], non-invertible axial symmetries in 4d QED and QCD [52, 63, 184], etc.

It turns out that for an arbitrary $G^{(p)}$, such an alternative construction may or may not exist. It is therefore useful to ask for which $G^{(p)}$ such an alternative construction exists. When it exists, the non-invertible duality defect in QFT $\mathscr{X}$ can be mapped to an invertible defect in QFT $\mathscr{Y}$ under a *topological manipulation* ξ, which includes gauging a non-anomalous subgroup and stacking an SPT etc, and such duality defect was named *non-intrinsically non-invertible* [123]. Conversely, a duality defect which does not admit the alternative construction will be called intrinsically non-invertible.

On the other hand, for 2d QFTs with a generic finite Abelian symmetry $G^{(0)}$, the condition of when the alternative construction exists has been classified in mathematical literatures [94]. The duality defect was named *group theoretical* if the alternative construction exists. In this work, we will follow the math notation and determine for which $G^{(p)}$ the duality defect is group theoretical.

As the answer in 2d is known, we will first review the results in [94], and the goal is to present the discussion there using a more physical language, and pave the way for generalization to higher dimensions. For concreteness, we focus on the $\mathbb{Z}_N^{(0)}$ symmetry, although generalization to more complicated Abelian symmetries is possible.

We then generalize the 2d discussion and proceed to determine when duality defects associated with gauging $\mathbb{Z}_N^{(1)}$ in 4d QFTs are group theoretical. A partial list of group theoretical duality defects have been identified in [51, 26]. Our results not only reproduce the known ones in [51], but also uncover new ones. As a systematic theory of higher category is less well-established than those in lower dimensions, we are unable to fully generalize the proofs in [94] to higher dimensions. Hence the completeness of our list of group theoretical duality defects only holds under certain assumptions which we will specify in Sec. 5.2.

5.1.2 Main results

The key idea to find the group theoretical duality defect is to use the *Symmetry Topological Field Theory* (SymTFT) [122, 11, 83, 128, 116, 91, 39, 33, 139, 129].[5] A d-dimensional QFT with a global symmetry described by a (higher) fusion category $\mathscr{C}$ is equivalent to a $(d+1)$-dimensional "sandwich" where in the bulk is a SymTFT describing the Drinfeld center of $\mathscr{C}$, the left d-dimensional boundary condition is a canonical/Dirichlet boundary condition where the defects labeled by $\mathscr{C}$ are supported, and the right boundary condition is a non-topological boundary condition encoding all the dynamical information of the QFT. Since the bulk is topological, one can shrink the sandwich by colliding the left and right boundaries to recover the d-dimensional QFT.

One of the advantages of the sandwich construction is that it automatically encodes the additional data of the duality defect mentioned in Sec. 5.1.1, i.e. the bicharacters and the Frobenius-Schur indicator, while they are not explicit from (5.1.1).

The sandwich construction also enjoys two interesting properties [122, 11, 83]. First, when the symmetry of the QFT is invertible, the SymTFT is a gauged anomaly theory, i.e. a Dijkgraaf-Witten theory, whose partition function is

$$\mathscr{Z} = \sum_{g\in G} e^{i\omega(g)} \tag{5.1.6}$$

where G is the finite (higher) gauge group of the Dijkgraaf-Witten theory, g is a (higher form)

[5] See also [113, 33, 24, 143, 120, 189, 155, 185, 47, 12, 10, 9] for applications of the SymTFT to the dynamics of QFTs, lattice models and string/M-theories.

gauge field valued in G,[6] and $\omega(g)$ is the twist term which specifies the G-anomaly.[7] Second, because all the symmetry defects (or background fields when the symmetry is invertible) are supported on the left topological boundary, topological manipulations (including gauging and stacking an invertible phase) do not change the SymTFT. Combining these two properties, we conclude that *a duality defect is group theoretical if and only if its SymTFT is a Dijkgraaf-Witten theory*.

The SymTFT of a duality defect associated with gauging $G^{(p)}$ admits a convenient construction [122, 189]. One starts with the SymTFT of $G^{(p)}$ symmetry (without a duality defect), which is simply a $G^{(p)}$ $(p+1)$-form gauge theory in $2p+3$ dimensions. Such a theory admits an electro-magnetic (EM) exchange symmetry whose order depends the parity of p. The SymTFT of the duality defect (including $G^{(p)}$ symmetry) is obtained by gauging the EM exchange symmetry of the $G^{(p)}$ $(p+1)$-form gauge theory. As we will review in Sec. 5.2, such SymTFT is a Dijkgraaf-Witten theory amounts to the existence of an EM stable topological boundary condition of the $G^{(p)}$ $(p+1)$-form gauge theory. This latter condition will be explicitly checked in the following sections.

In this paper, we focus on the duality defects associated with gauging $G^{(p)} = \mathbb{Z}_N^{(0)}$ in 2d, and $G^{(p)} = \mathbb{Z}_N^{(1)}$ in 4d. We determine for which N together with the choice of bicharacters and the Frobenius-Schur indicator, the duality defect is group theoretical by examining when the EM stable topological boundary condition of $\mathbb{Z}_N^{(0)}$ (or $\mathbb{Z}_N^{(1)}$) gauge theory exists in 3d (or 5d). We find the following results:

$\mathbb{Z}_N^{(0)}$ duality defects in 2d:

The $\mathbb{Z}_N^{(0)}$ duality defect is group theoretical if and only if N is a perfect square.

[6] For simplicity we use $g \in G$ to represent a g gauge field taking value in G, although a more appropriate way is to write $g \in H^*(X_{d+1}, G)$. We hope this simplified notation does not lead to confusion.

[7] There are more restricted definition of the Dijkgraaf-Witten in the literature, where G is an ordinary group. For more general G, e.g. G is a 2-group, the corresponding 4d TFT is called Yetter TFT [187]. See [113] for a review of various TFTs in higher dimensions. Here we denote the G gauge theory with any finite (higher) group G as Dijkgraaf-Witten.

$\mathbb{Z}_N^{(1)}$ duality defects in 4d:

On spin manifolds, the $\mathbb{Z}_N^{(1)}$ duality defect is group theoretical if and only if $N = L^2 M$ where -1 is a quadratic residue of M.

The above conclusion holds for arbitrary choices of additional data, including the bicharacters and the Frobenius-Schur indicators.

The 2d results were already proven in a mathematics reference [94]. In the main text, for both 2d and 4d cases, we will show the if direction by explicitly demonstrating the SymTFT to be a Dijkgraaf-Witten, and also working out the explicit topological manipulation. In particular, the special case of $L = 1$ in 4d was known in [51], and our result shows that there are new cases for $L > 1$. However, unlike the 2d case where the only if direction is proven,[8] for the 4d case the only if direction remains a conjecture.

We also note that the question of whether a duality defect is group theoretical has also been extensively discussed in the context of string/M-theories. See [103, 137, 12]. In particular, in [12] whether the $\mathbb{Z}_N^{(1)}$ duality defect is group theoretical was phrased in terms of Hanany-Witten transition between strings and 7-branes in the holographic IIB setup, and the authors found the same sequence of N for $N \leq 29$.[9]

Note that we assumed the 4d spacetime to be a spin manifold. On non-spin manifolds, the criteria for odd N remains the same. However, for even N, the situation is more complicated, and we will comment on them in the main text.

The organization of the paper is as follows. In Sec. 5.2, we discuss the general strategy to determine when a duality defect is group theoretical. This section is largely based on [94], presented in a way that applies to higher dimensions as well. Sec. 5.3 and Sec. 5.4 are in parallel,

[8]In [122], the authors used the SymTFT to give a physical derivation of the only if direction in 2d. The observation there is that the SymTFT of the duality defect has line operators of quantum dimension $\sqrt{N}$. Note that all line operators in any bosonic Dijkgraaf-Witten theory are of integer quantum dimension [110, 75, 71], the SymTFT can be Dijkgraaf-Witten only if N is a perfect square.

[9]After our preprint appeared on arXiv, we were informed by Fabio Apruzzi that [12] contains overlapping results with the present work.

which discuss the group theoretical duality defects in 2d and 4d respectively. In both sections, we first identify the group theoretical N's using the stable topological boundary condition following the strategy of Sec. 5.2. Then for each N, we explicitly demonstrate that the SymTFT is a Dijkgraaf-Witten theory, and also propose the explicit topological manipulation which maps the duality defect to the invertible defect. In Sec. 5.5, we comment on the relation between group theoretical duality defects and obstructions to duality preserving gapped phases.

5.2 Criteria of group-theoretical duality defects

In this section, we study in general when the duality defects are group theoretical. A rigorous mathematical discussion of this problem in 2d was already available in [94]. The goal of this section is to translate the discussion to a more physicist-friendly language. We also try to present the discussion in a way applicable to 4d.

5.2.1 Symmetry TFT of duality defects

As reviewed in the introduction, the SymTFT is a useful tool to identify whether a fusion category is group theoretical. We thus first review the properties of the SymTFT of duality defects, focusing on $\mathbb{Z}_N^{(d/2-1)}$ symmetry in d dimensions, for $d = 2,4$. We will follow the discussion in [122].

Consider a d dimensional QFT $\mathscr{X}$ with a non-anomalous $\mathbb{Z}_N^{(d/2-1)}$ symmetry. Let's denote its partition function as $\mathscr{Z}_{\mathscr{X}}[B^{(d/2)}]$. Any such theory can be expanded into a $d+1$ dimensional slab, as shown in Fig. 5.1. In the bulk of the slab, there is a $d+1$ dimensional $\mathbb{Z}_N$ $d/2$-form gauge theory, whose Lagrangian is

$$\mathscr{L} = \frac{2\pi}{N}\widehat{b}^{(d/2)}\delta b^{(d/2)}. \tag{5.2.1}$$

This is the SymTFT of the $\mathbb{Z}_N^{(d/2-1)}$ symmetry. On the left boundary, there is a Dirichlet topological boundary condition/state obtained by setting the electric field $b^{(d/2)}$ to background

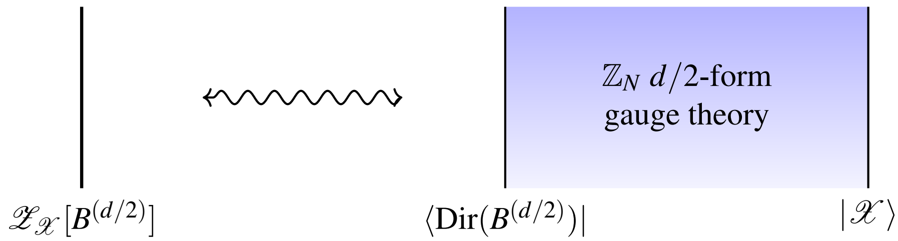


Figure 5.1. A d dimensional QFT $\mathscr{X}$ with a non-anomalous $\mathbb{Z}_N^{(d/2-1)}$ symmetry can be expanded into a $d+1$ dimensional slab, where the bulk of the slab is the SymTFT of the $\mathbb{Z}_N^{(d/2-1)}$ symmetry—$\mathbb{Z}_N$ $d/2$-form gauge theory.

value,

$$\langle \mathrm{Dir}(B^{(d/2)})| = \sum_{b^{(d/2)} \in H^{d/2}(X_d, \mathbb{Z}_N)} \delta(b^{(d/2)} - B^{(d/2)}) \langle b^{(d/2)}| \,. \tag{5.2.2}$$

On the right boundary, there is a dynamical boundary encoding all the information of the QFT $\mathscr{X}$, where the boundary state is

$$|\mathscr{X}\rangle = \sum_{b^{(d/2)} \in H^{d/2}(X_d, \mathbb{Z}_N)} \mathscr{Z}_{\mathscr{X}}[b^{(d/2)}]\, |b^{(d/2)}\rangle \,. \tag{5.2.3}$$

Shrinking the slab amounts to colliding the boundary states (5.2.1) and (5.2.2), which reproduces the partition function $\mathscr{Z}_{\mathscr{X}}[B^{(d/2)}]$.

We further require the QFT $\mathscr{X}$ to be invariant under gauging $\mathbb{Z}_N^{(d/2-1)}$, i.e. $\mathscr{X} = \mathscr{X}/\mathbb{Z}_N^{(d/2-1)}$, so that the symmetry of $\mathscr{X}$ contains not only $\mathbb{Z}_N^{(d/2-1)}$ but also self-duality. To obtain the SymTFT of the full symmetry, we start with the $\mathbb{Z}_N$ $d/2$-form gauge theory in $d+1$ dimensions and gauge the electro-magnetic (EM) exchange symmetry. This symmetry basically exchanges $b^{(d/2)}$ and $\widehat{b}^{(d/2)}$, to be more precise,

$$b^{(d/2)} \to u\widehat{b}^{(d/2)}, \qquad \widehat{b}^{(d/2)} \to (-1)^{d/2+1} v b^{(d/2)}, \tag{5.2.4}$$

where $uv = 1 \mod N$. By Chinese Remainder Theorem, both u and v are coprime with N.

The minus sign means that the EM exchange symmetry is $\mathbb{Z}_2^{\text{em}}$ when $d = 2$ mod 4, and is $\mathbb{Z}_4^{\text{em}}$ when $d = 0$ mod 4.[10] Below for simplicity the EM symmetry will be denoted as $\mathbb{Z}_\chi^{\text{em}}$, with $\chi = 4/2^{[d/2]_2}$. Note that when gauging $\mathbb{Z}_\chi^{\text{em}}$, there is a freedom of choosing the discrete theta term labeled by $\varepsilon \in H^{d+1}(\mathbb{Z}_\chi^{\text{em}}, U(1))$. When $d = 2$, different choices of u (and hence v) correspond to bicharacters of the Tambara-Yamagami fusion category $\text{TY}(\mathbb{Z}_N^{(0)}, u, \varepsilon)$, while different choices of the discrete theta term ε correspond to the Frobenius-Schur indicator [178, 189, 180]. For simplicity, we will adopt the same notation in $d = 4$ as well. In short,

$$\text{SymTFT of duality symmetry} \;=\; \mathbb{Z}_N \; d/2\text{-form gauge theory}/(\mathbb{Z}_\chi^{\text{em}})_{u,\varepsilon}. \tag{5.2.5}$$

5.2.2 Criteria for group-theoretical duality defects

We proceed to use the SymTFT to determine when the $\mathbb{Z}_N^{(d/2-1)}$ duality defect is group theoretical. The schematic idea is shown in Fig. 5.2. We first recall, as discussed in the introduction, that a duality defect is group theoretical if and only if its SymTFT is a Dijkgraaf-Witten theory. Using (5.2.5), the problem boils down to showing that the $\mathbb{Z}_N$ $d/2$-form gauge theory $/(\mathbb{Z}_\chi^{\text{em}})_{u,\varepsilon}$ is a Dijkgraaf-Witten theory. This completes the first two arrows on the top of Fig. 5.2.

Below, we argue for the remaining arrows. Sec. 5.2.2 establishes the two arrows in the middle of Fig. 5.2. Sec. 5.2.2 and Sec. 5.2.2 discusses the upward and dashed downward arrow at the bottom, respectively.

Dijkgraaf-Witten = existence of Lagrangian subcategory

Let's start with the observation that a SymTFT (5.2.5) is Dijkgraaf-Witten if and only if there exist a set of topological operators S_α's such that

1. S_α's form a (higher) representation category of some symmetry group G, and for simplicity we schematically denote them as $\text{Rep}(G)$. [11]

[10] When $N = 2$ and $d = 0$ mod 4, the EM exchange symmetry is still $\mathbb{Z}_2^{(0)}$ because $b^{(d/2)} = -b^{(d/2)}$ mod 2.

[11] The higher representation category, e.g. $2\text{Rep}(G)$, has been discussed recently in the context of generalized

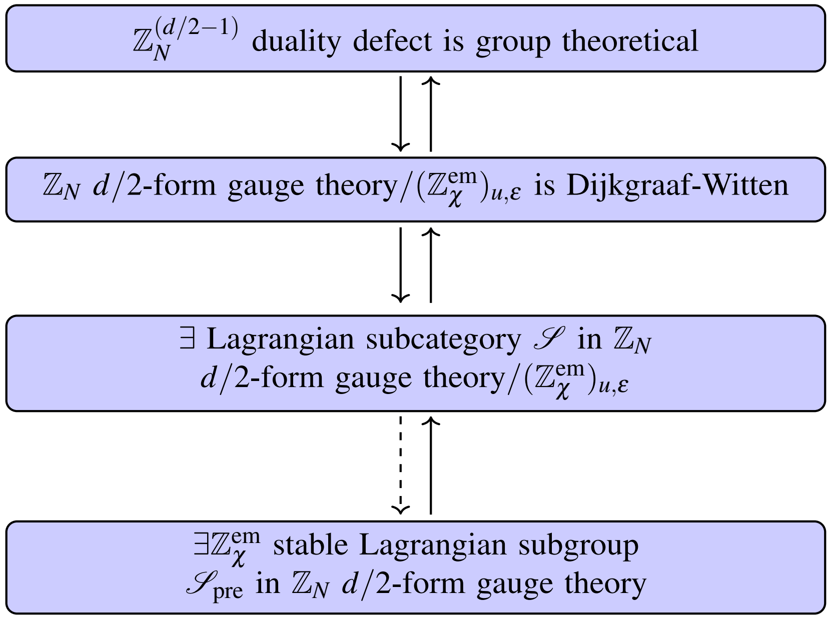


Figure 5.2. Idea of determining when a $\mathbb{Z}_N^{(d/2-1)}$ duality defect is group theoretical. We will argue (at the physics level of rigor) for the solid arrows, while for the dashed arrow, our argument is based on an assumption that is only proved in $d = 2$.

2. S_α's are gaugable and gauging them leads to an invertible theory. [12]

Let's denote the subcategory whose objects are S_α as the *Lagrangian subcategory* $\mathscr{S}$. One direction of this claim is obvious: in the Dijkgraaf-Witten theory with gauge group G and cocycle ω, obviously the set of Wilson operators form $\text{Rep}(G)$. Hence the first property is satisfied. Furthermore, $\text{Rep}(G)$ is gaugable because it is the quantum symmetry from gauging G of the G symmetric invertible theory (i.e. G-SPT) ω. This further means that gauging $\text{Rep}(G)$ leads to an invertible theory. This shows the second property, and completes the only if direction, i.e. the $\downarrow$ in the middle of Fig. 5.2.

For the if direction, suppose a SymTFT has a set of gaugable topological operators

symmetries and generalized charges in [21, 35, 23, 32, 22, 31, 33]. In this section, we will schematically denote both the ordinary and higher representation categories as $\text{Rep}(G)$.

[12] Gauging $\mathscr{S}$ amounts to first form a algebra object $\mathscr{A}$ from S_α's, and insert a mesh of $\mathscr{A}$ in the path integral. For 3d TFT, an algebra such that after gauging it leads to an invertible theory is known as Lagrangian algebra, which is shown to be a gaugable algebra, whose quantum dimension is the total quantum dimension of the TFT. In higher dimension, the theory of Lagrangian algebra is less well established, and we will simply define the Lagrangian algebra to be gaugable (i.e. can consistently insert a mesh of it) and has the property that gauging it would lead to an invertible theory. See [191] for a recent discussion of the Lagrangian algebra of 4d TFTs. Mathematically rigorous discussions on gauging a fusion 2-category can be found in [73, 72].

labelled by $\mathrm{Rep}(G)$ and gauging them leads to an invertible theory whose partition function is a phase $e^{i\omega}$, then the quantum symmetry is G and we can gauge G to recover the original SymTFT. This ensures the original SymTFT is Dijkgraaf-Witten. This completes the if direction, i.e. the $\uparrow$ in the middle of Fig. 5.2.

The above definition of the "Lagrangian subcategory" was motivated by the results of 3d TFT. In the context of 3d TFT, the set of anyons satisfying the above conditions form a fusion subcategory known as the Lagrangian subcategory L [94, 76]. It satisfies the following two conditions

1. L is of the form $\mathrm{Rep}(G)$ equipped with the standard symmetric braiding for some finite group G. The subcategory of this form is called *Tannakian*.

2. $L = L'$ where L' is the centralizer defined as the fusion subcategory (of the entire braided modular tensor category of the 3d TFT) which contains all the anyon a having trivial braiding with every anyon in L.

Notice that the first condition ensures the 1-form symmetries generated by $\mathrm{Rep}(G)$ is gaugable, and to gauge it we can consider condensing the algebra A_G which is the regular representation in $\mathrm{Rep}(G)$. The second condition implies every line operator outside L is charged non-trivially under the 1-form symmetry generated by the lines in L, hence condensing L would project out every line operator and end up with an invertible theory. In other words, the algebra A_G is Lagrangian. We reformulate the second condition such that the statement works for higher dimension as well. For this reason, we will also name the higher dimensional generalization of L, i.e. $\mathscr{S}$, as the Lagrangian subcategory.

Given a Lagrangian algebra A_G as the regular representation in $\mathrm{Rep}(G)$, we can consider half-gauging it to engineer a gapped boundary for the SymTFT. In the half space with invertible theory, there's apparently a quantum G symmetry whose symmetry defects can terminate on the gapped boundary and the intersections are the G-symmetry defects on the gapped boundary.

Up arrow at the bottom of Fig. 5.2

We proceed to argue for the up arrow at the bottom of Fig. 5.2, i.e.

$$\begin{array}{c} \boxed{\begin{array}{c} \exists \text{ Lagrangian subcategory } \mathscr{S} \text{ in } \mathbb{Z}_N \\ d/2\text{-form gauge theory}/(\mathbb{Z}_\chi^{\text{em}})_{u,\varepsilon} \end{array}} \\ \big\uparrow \\ \boxed{\begin{array}{c} \exists \mathbb{Z}_\chi^{\text{em}} \text{ stable Lagrangian subgroup} \\ \mathscr{S}_{\text{pre}} \text{ in } \mathbb{Z}_N \; d/2\text{-form gauge theory} \end{array}} \end{array} \tag{5.2.6}$$

Here we denote the Lagrangian subgroup of the $\mathbb{Z}_N$ $d/2$-form gauge theory (without gauging $\mathbb{Z}_\chi^{\text{em}}$) as $\mathscr{S}_{\text{pre}}$, where the subscript pre is to distinguish it from the Lagrangian subcategory $\mathscr{S}$ of the $\mathbb{Z}_\chi^{\text{em}}$ gauged theory (5.2.5). Since $\mathbb{Z}_N$ $d/2$-form gauge theory is an abelian TQFT, its boundary condition is specified by the Lagrangian (higher) subgroup [48]. The operators in the Lagrangian subgroup automatically form a $d/2$ representation category, i.e. $\mathscr{S}_{\text{pre}} = \text{Rep}(G)$, for certain finite group G. Thus the Lagrangian subgroup is automatically a Lagrangian subcategory. For this reason, we will use the Lagrangian subgroup to denote the Lagrangian subcategory of an Abelian TQFT throughout this paper.

A Lagrangian subgroup $\mathscr{S}_{\text{pre}}$ is $\mathbb{Z}_\chi^{\text{em}}$ stable means that $\mathbb{Z}_\chi^{\text{em}}$ is a group endomorphism of $\mathscr{S}_{\text{pre}}$, i.e. under $\mathbb{Z}_\chi^{\text{em}}$, $\mathscr{S}_{\text{pre}}$ is mapped to $\mathscr{S}_{\text{pre}}$ itself, while each object in $\mathscr{S}_{\text{pre}}$ might transform non-trivially. Suppose $\mathscr{S}_{\text{pre}}$ is $\mathbb{Z}_\chi^{\text{em}}$ stable, we can recover a $\mathbb{Z}_\chi^\rho$ automorphism acting the group G itself from this. In the $\mathbb{Z}_\chi^{\text{em}}$ gauged theory, we can then construct a fusion subcategory of the form $\text{Rep}(G \rtimes_\rho \mathbb{Z}_\chi)$, generated by $\mathbb{Z}_\chi$ orbit of $\text{Rep}(G)$ and the quantum $\mathbb{Z}_\chi$ symmetry defect K. Indeed, an irreducible representation in $G \rtimes_\rho \mathbb{Z}_\chi$ is either constructed as a direct sum of two irreducible representations of G related by $\mathbb{Z}_\chi$, or a tensor product between a $\mathbb{Z}_\chi$-invariant irreducible representation of G and a representation of the $\mathbb{Z}_\chi^\rho$.

It remains to show that $\mathscr{S}' = \text{Rep}(G \rtimes_\rho \mathbb{Z}_\chi)$ is gaugable and that gauging it leads to an invertible theory. This is the case because gauging the entire $\mathrm{Rep}(G \rtimes_\rho \mathbb{Z}_\chi)$ can be achieved by sequential gauging where we first gauge $\mathbb{Z}_\chi^{(d-2)}$ generated by the quantum $\mathbb{Z}_\chi$ defect K to recover $\mathrm{Rep}(G)$ and then gauge the anomaly free $\mathrm{Rep}(G)$ which will leads to the trivial theory. Hence $\mathrm{Rep}(G \rtimes_\rho \mathbb{Z}_\chi)$ is the Lagrangian subcategory of the $\mathbb{Z}_\chi^{\mathrm{em}}$ gauged theory. This completes the proof.

We comment that the $\mathbb{Z}_\chi^{\mathrm{em}}$ stable Lagrangian subgroup is equivalent to $\mathbb{Z}_\chi^{\mathrm{em}}$ stable boundary condition or state of the $\mathbb{Z}_N$ $d/2$-form gauge theory. Given a Lagrangian subgroup consisting of the operators S_α, where $\alpha \in \mathscr{I}$ and $\mathscr{I}$ is the index set labeling the simple object in $\mathscr{S}_{\mathrm{pre}}$, the topological boundary state $|\psi_{\mathscr{I}}\rangle$ is determined via $S_\alpha |\psi\rangle = |\psi\rangle$ for all $\alpha \in \mathscr{I}$. Note that the boundary state is unique, and can be constructed by gauging the Lagrangian subgroup on half space. Denote the symmetry operator of $\mathbb{Z}_\chi^{\mathrm{em}}$ as U, stable Lagrangian subcategory means $U S_\alpha U^{-1} = S_\beta$ where $\beta \in \mathscr{I}$. It follows that

$$S_\beta U |\psi\rangle = U S_\alpha U^{-1} U |\psi\rangle = U S_\alpha |\psi\rangle = U |\psi\rangle \tag{5.2.7}$$

meaning that $U|\psi\rangle$ is stabilized by the same Lagrangian subgroup. By uniqueness, we have $U|\psi\rangle = |\psi\rangle$, which means the boundary state $|\psi\rangle$ is also $\mathbb{Z}_\chi^{\mathrm{em}}$ stable.

In Sec. 5.3 and Sec. 5.4, we will enumerate all the Lagrangian subgroups of the $\mathbb{Z}_N$ $d/2$-form gauge theory and classify when they are $\mathbb{Z}_\chi^{\mathrm{em}}$ stable. When there exist stable Lagrangian subgroups, we will explicitly show that gauging the $\mathbb{Z}_\chi^{\mathrm{em}}$ symmetry leads to Dijkgraaf-Witten theory, and also find explicit topological manipulations that map the duality defect to an invertible defect. Such an explicit construction gives a practical proof of the up arrows, from the bottom to the top, in Fig. 5.2.

Dashed down arrow at the bottom of Fig. 5.2

We proceed to show the dashed down arrow at the bottom of Fig. 5.2, i.e.

$$\begin{tikzcd} \boxed{\begin{array}{c} \exists \text{ Lagrangian subcategory } \mathscr{S} \text{ in } \mathbb{Z}_N \\ d/2\text{-form gauge theory}/(\mathbb{Z}_\chi^{\text{em}})_{u,\varepsilon} \end{array}} \arrow[d, dashed] \\ \boxed{\begin{array}{c} \exists \mathbb{Z}_\chi^{\text{em}} \text{ stable Lagrangian subgroup} \\ \mathscr{S}_{\text{pre}} \text{ in } \mathbb{Z}_N \ d/2\text{-form gauge theory} \end{array}} \end{tikzcd} \tag{5.2.8}$$

We use the dashed arrow to emphasize that it is proven only in $d = 2$ [94], while for higher d, we are only able to generalize the results in [94] under an assumption which we specify below.

Suppose the $\mathbb{Z}_N$ $d/2$-form gauge theory $/(\mathbb{Z}_\chi^{\text{em}})_{u,v,\varepsilon}$ contains a Lagrangian subcategory $\mathscr{S} = \text{Rep}(G)$. Since the theory comes from gauging $\mathbb{Z}_\chi^{\text{em}}$, there must be an $\mathbb{Z}_\chi^{\text{em}}$ bosonic topological line operator K corresponding to the quantum $\mathbb{Z}_\chi$ $(d-1)$-form symmetry of the gauged theory. Clearly, the subcategory generated by K is of the form $\text{Rep}(\mathbb{Z}_\chi^K)$. Below we will make a key assumption:

Key assumption:

$\text{Rep}(\mathbb{Z}_\chi^K)$ is a subcategory of $\mathscr{S} = \text{Rep}(G)$.

Since K obeys the $\mathbb{Z}_\chi$ fusion rule, it must be labelled by an order χ representation of G. Equivalently, we can describe K as a surjective homomorphism $G \to \mathbb{Z}_\chi$, and denote the kernel of this homomorphism as H then we have the following short exact sequence:

$$0 \to H \to G \to \mathbb{Z}_\chi \to 0. \tag{5.2.9}$$

After gauging $\mathbb{Z}_\chi^K$ by condensing the K-line, the Lagrangian subcategory $\mathrm{Rep}(G)$ then becomes $\mathrm{Rep}(H)$. To see $\mathrm{Rep}(H)$ is Lagrangian, notice that gauging H is the second step in the sequential gauging, which corresponds to gauge $\mathrm{Rep}(G)$ in the original theory. Therefore, gauging $\mathrm{Rep}(H)$ must lead to an invertible theory and $\mathrm{Rep}(H)$ is Lagrangian. It is also clear that $\mathrm{Rep}(H)$ must be stable under the 0-form $\mathbb{Z}_\chi^{\mathrm{em}}$ symmetry from condensing K. Hence, up to the assumption that the Lagrangian subcategory $\mathrm{Rep}(G)$ contains $\mathrm{Rep}(\mathbb{Z}_\chi^K)$ as a subcategory, we argued that (5.2.8) holds.

When $d = 2$, however, there is a theorem guarantees that the **key assumption** is true: we can always find a $\mathrm{Rep}(G)$ such that it contains the K-line[77, 80]. Given a Tannakian subcategory $\mathrm{Rep}(H)$,[13] in general it may be contained as a subcategory in another Tannakian subcategory $\mathrm{Rep}(H')$. A Tannakian subcategory which is not properly[14] contained in another Tannakian subcategory is called maximal [77, 80]. Notice that a Lagrangian subcategory is automatically a maximal Tannakian subcategory. Since a Tannakian subcategory describes gaugable 1-form symmetries, one can consider condense those anyons to get a smaller TFT. In [77, 80], it was shown that condensing maximal Tannakian subcategory would lead to equivalent TFT which doesn't depend on the choice of the maximal Tannakian subcategory.[15] Using this, we can start with the Tannakian subcategory $\mathrm{Rep}(\mathbb{Z}_2^K)$ and find the maximal Tannakian subcategory $\mathrm{Rep}(G)$ containing it. Then, by the above theorem, condensing $\mathrm{Rep}(G)$ would lead to trivial theory and therefore $\mathrm{Rep}(G)$ is Lagrangian. This completes the proof for the necessary condition for $d = 2$. For higher dimensions, we do not know if the analog of such theorem exists, so (5.2.8) is only a conjecture.

[13]Note that H here is a generic group and should not be confused with the H in (5.2.9).

[14]Given two categories $\mathscr{C}$ and $\mathscr{D}$, $\mathscr{D}$ is properly contained in $\mathscr{C}$ if $\mathscr{D}$ is a subcategory of $\mathscr{C}$ but $\mathscr{D}$ is not the same as $\mathscr{C}$.

[15]The TFT acquired from the gauging is called the core of the original TFT, which seems to give a measure on the intrinsically non-invertibleness of a symmetry TFT.

5.3 Group theoretical duality defects in 2d

The SymTFT of $\mathbb{Z}_N^{(0)}$ duality defects is a 3d $\mathbb{Z}_N$ gauge theory with $\mathbb{Z}_2^{\text{em}}$ symmetry gauged. It has been shown in [94], which we revisited in Sec. 5.2, that the duality defect is group theoretical if and only if the 3d $\mathbb{Z}_N$ gauge theory admits $\mathbb{Z}_2^{\text{em}}$ stable topological boundary condition. In this section, we first review the topological boundary condition of the 3d $\mathbb{Z}_N$ gauge theories, and determine when a $\mathbb{Z}_2^{\text{em}}$ stable topological boundary condition is allowed. We then explicitly show that for those allowed N the SymTFT is a Dijkgraaf-Witten theory, and find explicit topological manipulations under which the duality defects are mapped to invertible defects.

5.3.1 Lagrangian subgroups of 3d $\mathbb{Z}_N$ gauge theory

The action of the $\mathbb{Z}_N$ gauge theory is

$$\mathscr{L} = \frac{2\pi}{N}\widehat{a}\delta a \tag{5.3.1}$$

where $\widehat{a}, a$ are both $\mathbb{Z}_N$ cochains. It has a $\mathbb{Z}_2^{\text{em}}$ exchange symmetry

$$a \to u\widehat{a}, \qquad \widehat{a} \to va, \tag{5.3.2}$$

with $uv = 1 \bmod N$. The topological lines are

$$L_{(e,m)}(\gamma) = e^{\frac{2\pi i e}{N}\oint_\gamma a} e^{\frac{2\pi i m}{N}\oint_\gamma \widehat{a}}. \tag{5.3.3}$$

The topological boundary conditions of an Abelian TQFT are classified by the Lagrangian subgroups. The Lagrangian subgroup $\mathscr{A}$ consists of N topological line operators $L_{(e,m)}$ with the following conditions,

1. $L_{(e,m)}$ has trivial topological spin, i.e. $e^{i2\pi em/N} = 1$.

2. Any two lines $L_{(e,m)}$ and $L_{(e',m')}$ in the Lagrangian subgroup $\mathscr{A}$ have trivial mutual braiding, i.e. $e^{2\pi i(em'+e'm)/N} = 1$.

3. Any other line operator that does not belong to $\mathscr{A}$ braids non-trivially with at least one line in $\mathscr{A}$.

We remark that the first condition automatically implies the second condition, since the mutual braiding phase $B_{(e,m),(e',m')}$ between two lines $L_{(e,m)}$ and $L_{(e',m')}$ is determined by their self spins $\theta_{(e,m)}$ as $B_{(e,m),(e',m')} = \theta_{(e+e',m+m')}/(\theta_{(e,m)}\theta_{(e',m')})$. However, in order to contrast with the analogue condition in higher dimensions, we still present the second property explicitly. The third property is guaranteed by the fact that there are N lines in $\mathscr{A}$, which we will verify at the end of this subsection.

To enumerate all possible Lagrangian subgroups, we first assume that a particular line $L_{(e,m)}$ belongs to $\mathscr{A}$, hence $L_{(ke,km)}$ also belongs to $\mathscr{A}$ due to group structure, for any $k \in \mathbb{Z}$.

The trivial topological spin means e and m can not be simultaneously coprime with N. This means that there must exist a $k < N$ such that $L_{(ke,km)}$ is a purely electric line $L_{(p',0)}$ or purely magnetic line $L_{(0,q')}$. So any Lagrangian subgroup contains at least one non-trivial purely electric or magnetic line. Without loss of generality, we assume that a electric line $L_{(p',0)}$ belongs to $\mathscr{A}$, where $1 < p' < N$. Under multiplication, $L_{(p,0)}$ also belongs to $\mathscr{A}$, where $p = \gcd(p',N)$. Note that $L_{(p,0)}$ is the purely electric line with the *smallest* electric charge. For convenience, we also denote $q = N/p$. Summarizing the above, we have shown that $\mathscr{A}$ contains q purely electric lines among N lines in total

$$L_{(kp,0)}, \qquad k = 0,1,...,q-1. \tag{5.3.4}$$

Suppose $L_{(e,m)}$ also belongs to $\mathscr{A}$ (which is independent of the $L_{(e,m)}$ in the previous paragraph), with $m \neq 0 \bmod N$. Trivial mutual braiding with purely electric lines requires $pm = 0$

mod N, or equivalently

$$m = sq \tag{5.3.5}$$

for *certain* $s \in \mathbb{Z}$. Denoting $t = \gcd(s, p)$, it follows that under multiplication, $L_{(e',tq)}$ also belongs to $\mathscr{A}$, for certain e'. Trivial topological spin of $L_{(e',tq)}$ requires

$$te' = 0 \bmod p. \tag{5.3.6}$$

Note that tq is the minimal magnetic charge mod N in the orbit $L_{(ke,ksq)}$ for different $k \in \mathbb{Z}$. Supplementing $L_{(e',tq)}$ to the set (5.3.4), we find the following lines in the Lagrangian subgroup $\mathscr{A}$,

$$L_{(kp+k'e',k'tq)}, \qquad k = 0,1,...,q-1, \qquad k' = 0,1,...,p/t-1. \tag{5.3.7}$$

Thus we find $qp/t = N/t$ lines in the Lagrangian subgroup. Since the Lagrangian subgroup contains N lines, unless $t = 1$, the above set does not contain enough lines to form a Lagrangian subgroup. However, since the charge lattice is two dimensional, any missing line should be generated by the two generators $L_{(p,0)}$ and $L_{(e',tq)}$ (note that they are not linearly dependently when $t > 1$), which is a contradiction. This implies that $t = 1$, and by (5.3.6), $e' = 0 \bmod p$.[16] In conclusion, the Lagrangian subgroup is completely specified by p where $p|N$, which is generated

[16]Another way to see $t = 1$ is as follows. We prove by contradiction. Assume $t > 1$, and we would like to find an operator that is not generated by $L_{(p,0)}$ and $L_{(e',tq)}$. By (5.3.6), $e' = (p/t)x$ for $x \in \mathbb{Z}$. When $x = 0 \bmod t$, $e' = 0 \bmod p$, and one can compose $L_{(e',tq)}$ with $L_{(p,0)}$ to get $L_{(0,tq)}$. Clearly, the generator $L_{(0,q)}$ can not be generated by $L_{(p,0)}$ and $L_{(0,tq)}$. When $x \neq 0 \bmod t$, we can assume $0 < x < t$, i.e. $0 < e' < p$ without loss of generality. We can then consider $L_{(0,tq)}$, which has the trivial topological spin and the trivial mutual braiding with both $L_{(p,0)}$ and $L_{(e',tq)}$ thanks to (5.3.6). Because $0 < e' < p$, $L_{(0,tq)}$ is not generated by $L_{(p,0)}$ and $L_{(e',tq)}$. In both cases, we find at least one operator that can be added into the Lagrangian subgroup, showing that $L_{(p,0)}$ and $L_{(e',tq)}$ do not generate the full Lagrangian subgroup. When $t = 1$ however, (5.3.6) shows $e' = 0 \bmod p$, and clearly $L_{(p,0)}$ and $L_{(e',q)}$ generate the entire Lagrangian subgroup.

by $L_{(p,0)}$ and $L_{(0,N/p)}$, i.e.

$$\mathscr{A}_p = \{L_{(xp,yN/p)} | x \in \mathbb{Z}_{N/p}, y \in \mathbb{Z}_p\}. \tag{5.3.8}$$

Finally, we show the third property of the Lagrangian subgroup, that for any line $L_{(e,m)}$ not within $\mathscr{A}_p$, it must braid non-trivially with at least one element in $\mathscr{A}_p$. We prove by contradiction. Suppose there exists $L_{(e,m)} \notin \mathscr{A}_p$ which braids trivially with every element in $\mathscr{A}_p$. The assumption implies $eyN/p + mxp = 0 \bmod N$ for any x, y. We first take $y = 0$, then $mxp = 0 \bmod N$ for any x implies $m = 0 \bmod N/p$. Similarly, by taking $x = 0$, then $eyN/p = 0 \bmod N$ for any y implies $e = 0 \bmod p$. Thus $L_{(e,m)} \in \mathscr{A}_p$, which contradicts with the assumption. This completes the proof.

5.3.2 $\mathbb{Z}_2^{\mathrm{em}}$ stable Lagrangian subgroup

We further classify which topological boundary condition is $\mathbb{Z}_2^{\mathrm{em}}$ stable. The $\mathbb{Z}_2^{\mathrm{em}}$ symmetry (5.3.2) maps the charges (e,m) to (vm, ue). The generators of the Lagrangian subgroup $\mathscr{A}_p$ are mapped to

$$L_{(p,0)} \to L_{(0,up)}, \qquad L_{(0,N/p)} \to L_{(vN/p,0)}. \tag{5.3.9}$$

$\mathbb{Z}_2^{\mathrm{em}}$ stability implies the above $\mathbb{Z}_2^{\mathrm{em}}$ of the generators also belong to $\mathscr{A}_p$, i.e.

$$L_{(0,up)} = L_{(xp,yN/p)}, \qquad L_{(vN/p,0)} = L_{(zp,wN/p)}. \tag{5.3.10}$$

The above implies $x \in (N/p)\mathbb{Z}$, $w \in p\mathbb{Z}$, and

$$up = yN/p \bmod N, \qquad vN/p = zp \bmod N. \tag{5.3.11}$$

Combining the two conditions, we find $zy = 1 \bmod p$ as well as $zy = 1 \bmod N/p$. This in particular implies y and z are coprime with both p and N/p. On the other hand, the first condition in (5.3.11) implies $p = (vy + p\alpha)N/p$ for certain integer α. However, because y is coprime with p and v is coprime with N (hence p), $(vy + p\alpha)$ is also coprime with p. So this condition can be satisfied only when $(N/p)|p$. Similarly, the second equality in (5.3.11) implies $p|(N/p)$. Combining the two conditions, we find

$$p = N/p \tag{5.3.12}$$

which means $N = p^2$ is a perfect square. By the results in Sec. 5.2, we conclude that the $\mathbb{Z}_N^{(0)}$ duality defect is group theoretical if and only if N is a perfect square, for any choice of u, v (i.e. the bicharacter). Since the choice of the Frobenius-Schur indicator ε does not enter the above discussion, the group-theoretical condition is also independent of ε as well. We emphasize that this result has been already proven in [94], and hope that our discussion is more accessible to physicists.

5.3.3 SymTFT as a Dijkgraaf-Witten theory

In Sec. 5.3.2, we showed that a $\mathbb{Z}_N$ duality defect is group theoretical if and only if N is a perfect square, for any choice of u, v, ε. In this section, we would like to explicitly show that the SymTFT for a perfect square N is indeed a Dijkgraaf-Witten theory.

Since $N = p^2$, the $\mathbb{Z}_N$ cochains $a, \widehat{a}$ can be rewritten as

$$a = p\widehat{b} + c', \qquad \widehat{a} = p\widehat{c} + b' \tag{5.3.13}$$

where $\widehat{c}$, $\widehat{b}$ are $\mathbb{Z}_p$ cochains, and b', c' are $\mathbb{Z}_N$ cochains. In terms of these new variables, (5.3.1) is rewritten as

$$\mathscr{L}=\frac{2\pi}{N}(p\widehat{c}+b')\delta(p\widehat{b}+c')=\frac{2\pi}{p}b\delta\widehat{b}+\frac{2\pi}{p}\widehat{c}\delta c+\frac{2\pi}{N}b'\delta c'. \tag{5.3.14}$$

In the second equality, we dropped $2\pi\widehat{b}\delta\widehat{c}$ since it belongs to $2\pi\mathbb{Z}$. On the right hand side, the first two terms are the standard BF couplings where $b=b'$ mod p and $c=c'$ mod p, while the last term is the DW twist term. Thus the $\mathbb{Z}_N$ gauge theory without a DW term is equivalent to $\mathbb{Z}_p\times\mathbb{Z}_p$ gauge theory with a DW term.

The advantage of recasting (5.3.1) into (5.3.14) is that the electric-magnetic exchange symmetry exchanging a and $\widehat{a}$ becomes a symmetry that only exchanges among the electric fields b,c, and among the magnetic fields $\widehat{b},\widehat{c}$, separately. But the electric and magnetic fields do not mix under $\mathbb{Z}_2^{\text{em}}$. Concretely, the (5.3.2) acts on the $\mathbb{Z}_p$ cochains via

$$b\to vc,\qquad c\to ub,\qquad \widehat{b}\to u\widehat{c},\qquad \widehat{c}\to v\widehat{b}. \tag{5.3.15}$$

Thus gauging (5.3.2) of (5.3.1) is equivalent to gauging (5.3.15) of (5.3.14). From the latter, it is almost by definition that the gauged theory is Dijkgraaf-Witten, where the definition is reviewed in Sec. 5.1.2.

We remark that the new variables introduced in (5.3.13) is motivated from the Lagrangian subgroup derived in Sec. 5.3.2. The Lagrangian subgroup is generated by $e^{\frac{2\pi i}{N}p\oint a}=e^{\frac{2\pi i}{p}\oint c}$ and $e^{\frac{2\pi i}{N}p\oint\widehat{a}}=e^{\frac{2\pi i}{p}\oint b}$, and we require that these operators are closed under $\mathbb{Z}_2^{\text{em}}$ transformation. This is obvious from (5.3.15).

Let's derive the Lagrangian from gauging (5.3.15) of (5.3.14). The first step is to couple to the $\mathbb{Z}_2^{\text{em}}$ background field x, and sum over all the flat configuration of x, i.e. gauge $\mathbb{Z}_2^{\text{em}}$. Coupling to the $\mathbb{Z}_2^{\text{em}}$ background field amounts to changing the ordinary differential operator δ to the twisted differential operator δ_x, and ordinary cup product $\cup$ to twisted cup product $\cup_x$. See [27, App. A] for a review of twisted cochains, differentials and cup products. In components, the

$\mathbb{Z}_2^{\rm em}$ gauged Lagrangian on a 3-simplex $(ijkl)$ is

$$\mathscr{L}_{ijkl}^{\rm gauged}=\frac{2\pi}{p}\widehat{\mathbf{b}}_{ij}^T K^{x_{ij}}\left(K^{x_{jk}}\mathbf{b}_{kl}-\mathbf{b}_{jl}+\mathbf{b}_{jk}\right)+\frac{\pi}{N}\mathbf{b}'^T_{ij}\sigma^x K^{x_{ij}}\left(K^{x_{jk}}\mathbf{b}'_{kl}-\mathbf{b}'_{jl}+\mathbf{b}'_{jk}\right)+\pi\varepsilon x_{ij}x_{jk}x_{kl} \tag{5.3.16}$$

where

$$K=\begin{pmatrix}0 & v\\ u & 0\end{pmatrix},\qquad \mathbf{b}=\begin{pmatrix}b\\ c\end{pmatrix},\qquad \mathbf{b}'=\begin{pmatrix}b'\\ c'\end{pmatrix} \tag{5.3.17}$$

and x is the dynamical, flat, $\mathbb{Z}_2^{\rm em}$ gauge field. The last term is the twist one can add upon gauging $\mathbb{Z}_2$, whose coefficient ε is related to the FS indicator $(-1)^{\varepsilon}$. The action is invariant under the gauge transformations

$$\mathbf{b}_{ij}\to K^{-\gamma_i}(\mathbf{b}_{ij}+K^{x_{ij}}\beta_j-\beta_i),\quad \widehat{\mathbf{b}}_{ij}\to (K^T)^{-\gamma_i}(\widehat{\mathbf{b}}_{ij}+K^{x_{ij}}\widehat{\beta}_j-\widehat{\beta}_i),\quad x_{ij}\to x_{ij}+\gamma_j-\gamma_i. \tag{5.3.18}$$

Summing over $\widehat{\mathbf{b}}$ constrains $\mathbf{b}$ to be a twisted cocycle (i.e. it is twisted-flat), in components, $(\delta_x\mathbf{b})_{jkl}=K^{x_{jk}}\mathbf{b}_{kl}-\mathbf{b}_{jl}+\mathbf{b}_{jk}=0$ mod p. Thus the partition function is independent of how the $\mathbb{Z}_p$ fields $\mathbf{b}$ are lifted to $\mathbb{Z}_N$ fields $\mathbf{b}'$. The full partition function is

$$\mathscr{Z}_{\rm gauged}=\sum_{(\mathbf{b},x)}\prod_{ijkl}\exp\left(\frac{\pi i}{N}\mathbf{b}_{ij}^T\sigma^x K^{x_{ij}}\left(K^{x_{jk}}\mathbf{b}_{kl}-\mathbf{b}_{jl}+\mathbf{b}_{jk}\right)+\pi i\varepsilon x_{ij}x_{jk}x_{kl}\right). \tag{5.3.19}$$

The pair $(\mathbf{b},x)$ with the above flatness condition is the gauge field for the gauge group $(\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_2$, with the $\mathbb{Z}_2$ exchanging the two $\mathbb{Z}_p$'s.[17] The partition function is clearly of the form of DW, for any choice of u,v,ε.

[17]The special case of $N=4,p=2$ was discussed in a lattice model in [190].

5.3.4 Explicit topological manipulation

We finally proceed to find an explicit topological manipulation that maps the duality defect to an invertible defect.

Suppose a 2d QFT $\mathscr{X}$ is self-dual. This means

$$\mathscr{Z}_{\mathscr{X}}[A] = \sum_{a\in\mathbb{Z}_N} \mathscr{Z}_{\mathscr{X}}[a] e^{\frac{2\pi i \nu}{N}\int_{M_2} aA}. \tag{5.3.20}$$

The summation $a \in \mathbb{Z}_N$ means summing over 1-cochain a valued in $\mathbb{Z}_N$, with flatness condition $\delta a = 0 \bmod N$, modulo gauge transformations. The parameter ν (which is coprime with N) paramaterizes different ways of gauging $\mathbb{Z}_N$. By gauging $\mathbb{Z}_N$ on half of the spacetime, we get a duality defect $\mathscr{N}$.

To motivate the topological manipulation, we first note that the QFT $\mathscr{X}$ corresponds the SymTFT being $\mathbb{Z}_N$ gauge theory and Dirichlet boundary condition of a. Under the decomposition (5.3.13), the Dirichlet boundary condition of a translates to the Dirichlet boundary condition of c and Dirichlet boundary condition of $\widehat{b}$. Given a Dijkgraaf-Witten theory as the SymTFT, the invertible symmetry corresponds to the Dirichlet boundary condition of electric fields i.e. b, c only, so the topological manipulation should transform the above topological boundary condition to the Dirichlet boundary condition of both b and c. Concretely, in terms of the boundary states defined in Sec. 5.2.1,

$$\langle A| \to \sum_{\widehat{b}\in\mathbb{Z}_p} \langle p\widehat{b} + C| e^{\frac{2\pi i}{N}\int_{X_2} \widehat{b}B} \tag{5.3.21}$$

where $\langle A|$ is the Dirichlet boundary state with $\mathbb{Z}_N = \mathbb{Z}_{p^2}$ background field A, the second factor on the RHS is the boundary term coming from the integration by parts of the first term in (5.3.14), so that it has the standard BF coupling $\frac{2\pi}{p}\widehat{b}\delta b$. We also set the electric fields b, c in the SymTFT (5.3.14) to background fields B, C respectively. Taking the inner product between the new topological boundary state (5.3.21) and the dynamical boundary state (5.2.3), we get the new partition function

$$\mathscr{Z}_{\widetilde{\mathscr{X}}}[B,C] = \sum_{\widehat{b}\in\mathbb{Z}_p, \delta\widehat{b}=-\beta C} \mathscr{Z}_{\mathscr{X}}[p\widehat{b}+C] e^{\frac{2\pi i}{p}\int_{M_2}\widehat{b}B}. \tag{5.3.22}$$

The summation is over all $\widehat{b}$ with the constraint $\delta\widehat{b} = -\beta C$, modulo gauge transformations.[18] The constraint comes from the flatness condition of A before topological manipulation, which descends to the flatness condition of $p\widehat{b}+C$. The constraint also enforces that the exponent $\frac{2\pi i}{p}\int_{M_2}\widehat{b}B$ is not gauge invariant, and one way to make it gauge invariant is to introduce a 3d bulk[175],

$$\frac{2\pi i}{p}\int_{M_3} B\beta C. \tag{5.3.23}$$

This shows that after gauging, there is a mixed anomaly between two $\mathbb{Z}_p$ symmetries.

We claim that (5.3.22) defines the desired topological manipulation mapping the self duality under gauging $\mathbb{Z}_N$ to an invertible symmetry. To see this, we perform a self-duality transformation on the right hand side of (5.3.22). Concretely, we substitute (5.3.20) to the right hand side of (5.3.22), and get

$$\mathscr{Z}_{\widetilde{\mathscr{X}}}[B,C] = \sum_{\substack{a\in\mathbb{Z}_N, \widehat{b}\in\mathbb{Z}_p, \\ \delta\widehat{b}=-\beta C}} \mathscr{Z}_{\mathscr{X}}[a] e^{\frac{2\pi i}{N}\int_{M_2} p\widehat{b}B + va(p\widehat{b}+C)}. \tag{5.3.24}$$

Summing over $\widehat{b}$ enforces $a = uB \bmod p$, which is equivalent to $a = pc + uB \bmod N$. The flatness condition of a enforces $\delta c = -u\beta B$. Substituting this solution to the partition function, the exponent becomes $\frac{2\pi i}{N}\int_{M_2} v(pc+uB)C = \frac{2\pi i}{p}\int_{M_2} vcC + \frac{2\pi i}{N}BC$. The last term is a counter term, which can be absorbed to the bulk anomaly SPT action (5.3.23) by exchanging $B \leftrightarrow C$, i.e.

[18] β is the Bockstein homomorphism $\beta : H^1(M_2,\mathbb{Z}_p) \to H^2(M_2,\mathbb{Z}_p)$.

$\frac{2\pi i}{N}\int_{M_2} BC + \frac{2\pi i}{p}\int_{M_3} B\beta C = \frac{2\pi i}{p}\int_{M_3} C\beta B$. Combining the above, (5.3.24) finally becomes

$$\mathscr{Z}_{\widetilde{\mathscr{X}}}[B,C] = \sum_{c\in\mathbb{Z}_p, \delta c=-u\beta B} \mathscr{Z}_{\mathscr{X}}[pc+uB]e^{\frac{2\pi i}{p}\int_{M_2} vcC} = \mathscr{Z}_{\widetilde{\mathscr{X}}}[vC,uB]. \tag{5.3.25}$$

This shows that the self duality symmetry in $\mathscr{X}$ is mapped to an invertible $\mathbb{Z}_2$ symmetry that simply maps the background fields as

$$B \to vC, \qquad C \to uB \tag{5.3.26}$$

which is indeed consistent with the transformation (5.3.15) in the SymTFT.

5.4 Group theoretical duality defects in 4d

We generalize the discussion of duality defects in 2d to duality defects in 4d. The discussion is overall in parallel with Sec. 5.3, but there are additional subtleties which we highlight.

5.4.1 Lagrangian subgroups of 5d $\mathbb{Z}_N$ 2-form gauge theory

The action of the $\mathbb{Z}_N$ 2-form gauge theory is

$$\mathscr{L} = \frac{2\pi}{N}\widehat{b}\delta b, \tag{5.4.1}$$

where $b,\widehat{b}$ are both $\mathbb{Z}_N$ 2-cochains. It has a $\mathbb{Z}_4^{\text{em}}$ exchange symmetry

$$b \to u\widehat{b}, \qquad \widehat{b} \to -vb, \tag{5.4.2}$$

where $uv = 1 \bmod N$. Comparing with (5.3.2), the additional minus sign comes from the change of form degree of the gauge fields, which consequently makes the exchange symmetry to be order four, rather than order two.[19] The topological surfaces are

$$S_{(e,m)}(\sigma) = e^{\frac{2\pi i e}{N}\oint_\sigma b} e^{\frac{2\pi i m}{N}\oint_\sigma \widehat{b}}. \tag{5.4.3}$$

One significance is that the topological surface operator can be modified by a local counter term $e^{\frac{2\pi i k}{2N}\mathscr{P}(\sigma)}$ for even N, and $e^{\frac{2\pi i}{N}\frac{N+1}{2}\langle\sigma,\sigma\rangle}$ for odd N, where we $\mathscr{P}(\sigma)$ is the Pontryagin square of $\sigma \in H_2(X,\mathbb{Z}_N)$. In the absence of such counter term, the surface operators are not closed under fusion, $S_{(e,m)}(\sigma)S_{(e',m')}(\sigma) = e^{\frac{2\pi i}{N}me'\langle\sigma,\sigma\rangle}S_{(e+e',m+m')}(\sigma)$.

A full classification of topological boundary conditions of a generic 5d TQFT is still under development. However, for 5d $\mathbb{Z}_N$ 2-form gauge theory, its topological boundary conditions are expected to be classified by the Lagrangian subgroup of its surface operators, with additional data specifying the topological refinements [48]. We will take this as an assumption throughout the rest of the paper. In particular, the surface operators within the Lagrangian subgroup are closed under fusion.[20] The Lagrangian subgroup $\mathscr{A}$ consists of N topological surface operators $S_{(e,m)}$ with the following conditions,

1. $S_{(e,m)}$ has trivial topological spin. This is automatically satisfied for any (e,m) [48].
2. Any two surfaces $S_{(e,m)}$ and $S_{(e',m')}$ in the Lagrangian subgroup $\mathscr{A}$ have trivial mutual braiding, i.e. $e^{\frac{2\pi i}{N}(em'-me')} = 1$.
3. Any other surface operator that does not belong to $\mathscr{A}$ braids non-trivially with at least one line in $\mathscr{A}$.

[19] When $N=2$, due to $b=-b$ mod 2, $\mathbb{Z}_4^{\text{em}}$ reduces to $\mathbb{Z}_2^{\text{em}}$.

[20] One way to understand the closedness under fusion is as follows. Assuming two operators $\widetilde{S}_{(e,m)}$ and $\widetilde{S}_{(e',m')}$ are within the Lagrangian subalgebra, where $\widetilde{S}_{(e,m)}$ is related to $S_{(e,m)}$ by stacking a counter term specified in the previous paragraph. This means that the associated boundary state $|\mathscr{B}\rangle$ is stabilized by both of them. $\widetilde{S}_{(e,m)}|\mathscr{B}\rangle = |\mathscr{B}\rangle$, $\widetilde{S}_{(e',m')}|\mathscr{B}\rangle = |\mathscr{B}\rangle$. This means that their product $\widetilde{S}_{(e,m)}\widetilde{S}_{(e',m')}$ also stabilizes the boundary state $|\mathscr{B}\rangle$, hence belongs to the Lagrangian subalgebra. Now, let $(e+e',m+m') = (0,0)$ mod N, hence $\widetilde{S}_{(e,m)}\widetilde{S}_{(e',m')}$ is at most a phase specified by the counter term. If the counter term is non-trivial, the boundary state must vanish. Thus we should carefully choose the counter term such that $\widetilde{S}_{(e,m)}\widetilde{S}_{(e',m')} = 1$ whenever $(e+e',m+m') = (0,0)$ mod N. This is ensured if we demand the operators $\widetilde{S}_{(e,m)}$ are closed under fusion for arbitrary (e,m), i.e. $\widetilde{S}_{(e,m)}\widetilde{S}_{(e',m')} = \widetilde{S}_{(e+e',m+m')}$.

Since the trivial self-braiding condition is automatically satisfied, the Lagrangian subgroup in 5d is less constrained than that in 3d, hence the structure of $\mathscr{A}$ is richer. For example, we will see that the Lagrangian subgroups can be generated by an arbitrary single surface $\widetilde{S}_{(e,m)}$ as long as $\gcd(e,m,N)=1$, while in the 3d $\mathbb{Z}_N$ gauge theory we should further require it to be a self-boson $em=0 \bmod N$ which is much more constraining.

Like in 3d $\mathbb{Z}_N$ gauge theory, the third condition is guaranteed when there are N topological surface operators in $\mathscr{A}$. We will verify this at the end of this subsection.

Below, we classify the topological boundary condition of $\mathbb{Z}_N$ 2-form gauge theory by classifying its Lagrangian subgroups. We first derive the classification at the level of charges in Sec. 5.4.1, by assuming that the surface operators within it are closed fusion, i.e.

$$\begin{aligned}
\widetilde{S}_{(e,m)}(\sigma)\widetilde{S}_{(e,m)}(\sigma') &= \widetilde{S}_{(e,m)}(\sigma+\sigma'), &(5.4.4)\\
\widetilde{S}_{(e,m)}(\sigma)\widetilde{S}_{(e',m')}(\sigma) &= \widetilde{S}_{(e+e',m+m')}(\sigma), &(5.4.5)
\end{aligned}$$

where $\widetilde{S}_{(e,m)}=S_{(e,m)}e^{iK_{e,m}\langle\sigma,\sigma\rangle}$ with $K_{e,m}$ determined by the closeness under fusion. We then discuss the phase factors in Sec. 5.4.1, where we will find for each charge (e,m) there can be distinct topological refinements, as emphasized in [48].

Classifying Lagrangian subgroup: charges

We first classify the charges of the surface operators in the Lagrangian subgroup, following the discussion in Sec. 5.3.1. In App. C.1 we provide an alternative derivation of the Lagrangian subgroups.

Suppose there are pure electric surface operators in the Lagrangian subgroup, and the one with *minimal* electric charge is $\widetilde{S}_{(p,0)}$, where $p|N$ and $1\le p\le N$.[21] It is clear that there are additional operators $\widetilde{S}_{(\ell',sN/p)}$ having trivial mutual braiding with $\widetilde{S}_{(p,0)}$, hence can be supplemented into the Lagrangian subgroup. Denoting $t=\gcd(s,p)$, it follows that $\widetilde{S}_{(\ell,tN/p)}$ also

[21] When $p=N$, there is no pure electric operator.

belongs to the Lagrangian subgroup, where $\ell = x\ell'$ mod p and $sx = t$ mod p.

We proceed to show that $t = 1$ for $\widetilde{S}_{(p,0)}$ and $\widetilde{S}_{(\ell,tN/p)}$ to generate the entire Lagrangian subgroup. To see this, we denote $\gcd(t,\ell) = \tilde{\ell}$. It follows that $\widetilde{S}_{(\ell/\tilde{\ell},tN/p\tilde{\ell})}$ also has trivial braiding with the above two generators, and should belong to the Lagrangian subgroup, meaning that it can be expended by the generators. This is possible only when $\tilde{\ell} = 1$, meaning t and ℓ are coprime. Then we fuse p/t copies of the second generator to get $\widetilde{S}_{(\ell p/t,0)}$, hence should be generated by the first generator, i.e. $\ell p/t = px$ for an integer x. This means $\ell = xt$, in other words, $\gcd(\ell,t) = t = 1$, as desired. When $t = 1$, indeed $\widetilde{S}_{(p,0)}$ and $\widetilde{S}_{(\ell,N/p)}$ generate $(N/p)\times p = N$ surfaces, which is the size of the Lagrangian subgroup. In summary, Lagrangian subgroups are specified by (ℓ, p), generated by $\widetilde{S}_{(p,0)}$ and $\widetilde{S}_{(\ell,N/p)}$, i.e.

$$\mathscr{A}_{(\ell,p)} = \{\widetilde{S}_{(xp+y\ell,yN/p)} | x \in \mathbb{Z}_{N/p}, y \in \mathbb{Z}_p\}. \tag{5.4.6}$$

We finally verify the third property of the Lagrangian subgroup, that for any surface $S_{(e,m)}$ not within $\mathscr{A}_{(\ell,p)}$, it must braid non-trivially with at least one element in $\mathscr{A}_{(\ell,p)}$. We again prove by contradiction. Suppose there exists $S_{(e,m)} \notin \mathscr{A}_{(\ell,p)}$ which braids trivially with every element in $\mathscr{A}_{(\ell,p)}$. The assumption means

$$ey\frac{N}{p} - m(xp+y\ell) = 0 \bmod N, \qquad \forall x \in \mathbb{Z}_{N/p}, y \in \mathbb{Z}_p. \tag{5.4.7}$$

Setting $y = 0$, we get $mxp = 0$ mod N for any $x \in \mathbb{Z}_{N/p}$, which means $m = 0$ mod N/p. Let's denote $m = \widehat{m}N/p$. We can alternatively set $x = 0$, then $(eN/p - m\ell)y = 0$ mod N for any y, meaning $e - \widehat{m}\ell = 0$ mod p. Let's then denote $e = p\widehat{e} + \ell\widehat{m}$. So

$$(e,m) = (p\widehat{e} + \ell\widehat{m}, \widehat{m}N/p) = (p,0)\widehat{e} + (\ell, N/p)\widehat{m} \in \mathscr{A}_{(\ell,p)}. \tag{5.4.8}$$

This contradicts with the assumption that (e,m) is not within $\mathscr{A}_{(\ell,p)}$. Hence the third property in the Lagrangian subgroup is verified.

Classifying Lagrangian subgroup: including phases

In Sec. 5.4.1, we have determined the charges in the Lagrangian subgroups. In this section, we explicitly construct the surface operators, in particular specify the counter terms within $\widetilde{S}_{(e,m)}$. The most general ansatz is

$$\widetilde{S}_{(e,m)}(\sigma) = S_{(e,0)}(\sigma) S_{(0,m)}(\sigma) e^{iK_{e,m}\langle\sigma,\sigma\rangle}. \tag{5.4.9}$$

Closedness under fusion requires (5.4.4) and (5.4.5). In the above, (e,m) belongs to $k(p,q)$ for one generator case, and $(px+\ell y, yN/p)$ for two generators case. We discuss these two cases separately.

We substitute $(e,m) = (xp+y\ell, yN/p)$ with $x = 0,1,...,N/p-1$ and $y = 0,1,...,p-1$. Similarly, $(e',m') = (x'p+y'\ell, y'N/p)$. However, $(e+e',m+m') = ((x+x')p+(y+y')\ell,(y+y')N/p)$ should be more precisely written as

$$(e',m') = [x+x'+(y+y'-[y+y']_p)\frac{\ell}{p}]_{N/p}(p,0)+[y+y']_p(\ell,N/p), \tag{5.4.10}$$

where by $[x]_p$ is the mod p function taking value in $0,\cdots,p-1$. Then (5.4.4) yields

$$K_{xp+yLr,yN/p} = \begin{cases} \frac{2\pi}{N}\frac{N+1}{2}(xp+y\ell)yN/p \bmod 2\pi, & N \text{ odd} \\ \frac{2\pi}{2N}(xp+y\ell)yN/p+\pi J_{x,y} \bmod 2\pi, & N \text{ even} \end{cases}. \tag{5.4.11}$$

Substituting (5.4.11) into (5.4.5), we find that for odd N (5.4.11) is automatically satisfied, while for even N, $J_{x,y}$ should satisfy

$$\begin{aligned} &\frac{\ell}{p}((y+y')^2-[y+y']_p^2)+xy+x'y'-[x+x'+(y+y'-[y+y']_p)\ell/p]_{N/p}[y+y']_p \\ &= J_{[x+x'+(y+y'-[y+y']_p)\ell/p]_{N/p},[y+y']_p} - J_{x,y} - J_{x',y'} \bmod 2. \end{aligned} \tag{5.4.12}$$

It is not straightforward to explicitly solve for $J_{x,y}$ analytically. However, by numerically solving (5.4.12) for $N \leq 100$, we confirmed that the solution exist for all even N's: there are four solutions when $p, N/p, \ell$ are all even, and two solutions otherwise. In particular, from the numerical results, we find that when $p, N/p, \ell$ are all even, the four solutions are of the form $J_{x,y} = xy + c_1 x + c_2 y \bmod 2$, where $c_1, c_2 = 0, 1$ parameterize the four solutions. Another situation where we are able to solve $J_{x,y}$ is when the Lagrangian subgroup is generated by only one surface operator, which we will discuss in App. C.2.

The situation significantly simplifies when we restrict the 4d boundary of the 5d SymTFT to be spin manifolds. In this case, the $\pi J_{x,y}$ term in (5.4.11) can be ignored due to $\langle \sigma, \sigma \rangle = 0$ mod 2.

5.4.2 $\mathbb{Z}_4^{\text{em}}$ stable Lagrangian subgroup

We proceed to examine when the Lagrangian subalgebra is stable under $\mathbb{Z}_4^{\text{em}}$. Recall that under $\mathbb{Z}_4^{\text{em}}$, the gauge field transforms as (5.4.2). So $S_{(e,0)}(\sigma) \to S_{(0,ue)}(\sigma)$, and $S_{(0,m)}(\sigma) \to S_{(-vm,0)}(\sigma)$.

When the 4d spacetime X_4 is a spin manifold, the counter terms for even and odd N are given by (5.4.11) with $J_{x,y} = 0$. Under $\mathbb{Z}_4^{\text{em}}$, the generators $\widetilde{S}_{(p,0)}$ and $\widetilde{S}_{(\ell,N/p)}$ are mapped to $\widetilde{S}_{(0,up)}$ and $\widetilde{S}_{(-vN/p,u\ell)}$ respectively for both even and odd N. Stability under $\mathbb{Z}_4^{\text{em}}$ implies $(0, up) = (xp + y\ell, yN/p) \bmod N$, and $(-vN/p, u\ell) = (zp + w\ell, wN/p) \bmod N$. In components, we have

$$
\begin{aligned}
xp + y\ell &= 0 \bmod N, && (5.4.13)\\
yN/p &= up \bmod N, && (5.4.14)\\
zp + w\ell &= -vN/p \bmod N, && (5.4.15)\\
wN/p &= u\ell \bmod N. && (5.4.16)
\end{aligned}
$$

Once the $\mathbb{Z}_4^{\text{em}}$ images of the generators are within the Lagrangian subgroup, it is not hard to show that the $\mathbb{Z}_4^{\text{em}}$ images of all other surface operators in the Lagrangian subgroup are all within the

Lagrangian subgroup as well.

Given N, u, v, can we find p, ℓ, x, y, z, w such that (5.4.13)∼(5.4.16) are satisfied? Below, we provide an equivalent but simpler-looking criteria of N for which the solutions exists.[22] First, multiplying p on both sides of (5.4.14) yields $p^2 = 0 \bmod N$ (note that u, v are coprime with N). This means

$$p^2 = NM \tag{5.4.17}$$

for some integer M. As a consequence, we have $p = (N/p)M$, which means

$$L := \gcd(p, N/p) = N/p, \tag{5.4.18}$$

hence we write $p = ML$. Substituting this into (5.4.17), we have $(ML)^2 = NM$, which gives rise to

$$N = L^2 M. \tag{5.4.19}$$

So far, we only used (5.4.14). To see the condition of M, we multiply ℓ on both sides of (5.4.16),

$$\ell^2 = vw\ell N/p = v(-vN/p - zp)N/p = -(vN/p)^2 = -v^2 L^2 \bmod N \tag{5.4.20}$$

where in the second equality we used (5.4.15). Multiplying u^2 and dividing L^2 on both sides, we find

$$(u\ell/L)^2 = -1 \bmod M. \tag{5.4.21}$$

In other words, $\ell = Lrv$ with $r^2 = -1 \bmod M$. Finally, we note that (5.4.13) does not impose

[22]We are grateful to Justin Kaidi for discussing the following proof.

Table 5.1. N's that admit $\mathbb{Z}_4^{\text{em}}$ stable Lagrangian subgroups, i.e. $N = L^2M$ labeled by colors. The red and blue ones are the N's with $L = 1$ and $L > 1$ respectively. They correspond to the Lagrangian subgroups that are generated by one and two generators.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99

further constraints and can be solved by $x = -r$ and $y = uM$.

So far, we have shown that (5.4.13)$\sim$(5.4.16) imply $N = L^2M$ where -1 is a quadratic residue of M. To show the converse is also true, we simply take

$$(N, x, y, z, w, p, \ell) = (L^2M, -r, uM, -v\lambda, r, LM, Lrv) \tag{5.4.22}$$

where r satisfies $r^2 + 1 = \lambda M$, and it is straightforward to check that (5.4.13)$\sim$(5.4.16) are satisfied. In summary, we have the criteria:

Stability criteria:

When the 4d spacetime is a spin manifold, a $\mathbb{Z}_4^{\text{em}}$ stable Lagrangian subgroup exists if and only if $N = L^2M$ and for every choice of bicharacter labeled by u, v with $uv = 1 \bmod N$, where -1 is a quadratic residue of M. We enumerate such N's in Tab. 5.1.

Let's make several comments.

1. Combining the above criteria with the general discussion in Sec. 5.2, we claim that the duality defect associated with gauging $\mathbb{Z}_N^{(1)}$ is group theoretical if and only if $N = L^2M$ where -1 is a quadratic residue of M. The if direction is proven in Sec. 5.2, and will be further supported by explicitly showing the SymTFT is a Dijkgraaf-Witten theory and also explicitly constructing the topological manipulation for every such N. The condition is also independent of the choice of bicharacters u, v and the higher dimensional generalization of the Frobenius-Schur indicator ε, which will be further discussed in Sec. 5.4.3. However

the only if direction is less solid, and remains a conjecture.

2. On non-spin manifolds, the criteria for odd N remains the same. For even N, we don't have the full classification of stable Lagrangian subgroups, but let's comment on two special cases. The first case is when the Lagrangian subgroup is generated by one surface operator, then we are able to show that the stable Lagrangian subgroup does not exist. We present the details in App. C.2. The second case is when $N = L^2$ is a perfect square so that $(p, N/p, \ell) = (L, L, 0)$ are all even for even L. On top of the stability condition on charges (5.4.13)∼(5.4.16) whose solution is $(N, x, y, z, w, p, \ell) = (L^2, 0, u, -v, 0, L, 0)$ for even L, there are additional stability conditions on the topological refinements $J_{x,y} = xy + c_1 x + c_2 y$ mod 2 enforcing $c_1 = c_2$ mod 2. In Sec. 5.4.4 we will comment on the explicit topological manipulation on non-spin manifold corresponding to this special case.

3. When $L = 1$, there is only one generator $\widetilde{S}_{(r,1)}$, with $(vr)^2 = -1 \bmod N$. This is precisely the condition found in [53, App. C], as well as in [26] for prime N's, for the $\mathbb{Z}_N^{(1)}$ duality defect to be group theoretical on spin manifolds.

4. In [12], the problem of determining group theoretical-ness was phrased in terms of Hanany-Witten transition between strings and 7-branes in the holographic IIB setup. In particular, the main result, summarized in Tab. 5 of [12], coincides with Tab. 5.1 of the current work for $N \leq 29$.

5. It is intriguing to note that the same series $N = L^2 M$ (with -1 being a quadratic residue of M) was found in a completely different context in [13], where the authors showed that a (spin) TQFT with $\mathbb{Z}_N^{(1)}$ symmetry N satisfies exactly the same condition. In Sec. 5.5, we explain such a coincidence.

5.4.3 SymTFT as a Dijkgraaf-Witten theory

In this subsection, we show explicitly that the SymTFT for the entire $\mathbb{Z}_N$ duality symmetry is a Dijkgraaf-Witten theory when the $\mathbb{Z}_4^{\mathrm{em}}$ stable Lagrangian subgroup exists. From Sec. 5.3.3, we learned that for the SymTFT obtained from gauging the electric-magnetic exchange symmetry of the $\mathbb{Z}_N$ gauge theory to be a Dijkgraaf-Witten theory, we should rewrite the original $\mathbb{Z}_N$ gauge theory in terms of a set of new gauge fields so that the em exchange symmetry acts on the electric and magnetic gauge fields separately, and does not exchange them. We will find below how this can be achieved for 5d $\mathbb{Z}_N$ 2-form gauge theory.

Odd N:

From Sec. 5.4.2, we found that the $\mathbb{Z}_4^{\text{em}}$ stable Lagrangian subgroup is generated by surface operators $\widetilde{S}_{(LM,0)}$ and $\widetilde{S}_{(Lrv,L)}$ with the constraints that $r^2+1=\lambda M$ for some integer λ. Inspired by the discussion in Sec. 5.3.3, we decompose the $\mathbb{Z}_N$ 2-form gauge fields into new ones as follows

$$(\widehat{b},b)=(LM,0)\widehat{a}+(Lrv,L)\widehat{c}-(\lambda,0)c-(rv,1)a \tag{5.4.23}$$

where $(\widehat{a},\widehat{c},a,c)$ are $(\mathbb{Z}_L,\mathbb{Z}_{LM},\mathbb{Z}_N,\mathbb{Z}_N)$ cochains respectively. We again label the magnetic fields by hatted letters, while the electric fields by unhatted letters. Under $\mathbb{Z}_4^{\text{em}}$, (5.4.2) implies

$$\begin{aligned}(\widehat{b},b)\to(-vb,u\widehat{b})&=(0,uLM)\widehat{a}+(-vL,Lr)\widehat{c}-(0,u\lambda)c-(-v,r)a\\&=[-(LM,0)r+(Lrv,L)Mu]\widehat{a}+[-(LM,0)\lambda v+(Lrv,L)r]\widehat{c}\\&\quad-[-(\lambda,0)r+(vr,1)\lambda u]c-[-(\lambda,0)Mv+(vr,1)r]a\end{aligned} \tag{5.4.24}$$

where in the second line we used (5.4.13)$\sim$(5.4.16). Comparing (5.4.23) and (5.4.24), we find the transformation of new gauge fields as[23]

$$
\begin{aligned}
&\widehat{a} \to -r\widehat{a} - \lambda v\widehat{c},\\
&\widehat{c} \to uM\widehat{a} + r\widehat{c},\\
&a \to ra + u\lambda c,\\
&c \to -Mva - rc,
\end{aligned}
\tag{5.4.25}
$$

where the electric and magnetic fields do not mix under $\mathbb{Z}_4^{\text{em}}$. Indeed, the generators of the Lagrangian subgroup $e^{\frac{2\pi i p}{N}\oint b} = e^{-\frac{2\pi i}{L}\oint a}$ and $e^{\frac{2\pi i}{N}\oint Lrvb + L\widehat{b}} = e^{-\frac{2\pi i\lambda}{LM}\oint c}$ are all written in terms of the electric fields, hence their generating set is stable under $\mathbb{Z}_4^{\text{em}}$. The Lagrangian of the $\mathbb{Z}_N$ 2-form gauge theory can be rewritten in terms of the new gauge fields as

$$
\begin{aligned}
&\frac{2\pi}{N}\widehat{b}\delta b\\
&= -\frac{2\pi}{L}\widehat{a}\delta a + \frac{2\pi\lambda}{LM}\widehat{c}\delta c + \frac{2\pi\lambda}{L^2M}a\delta c + \frac{2\pi}{L^2M}\delta\Big(-\lambda L\widehat{c}c + \frac{N+1}{2}L^2rv\widehat{c}\widehat{c} - Lrva\widehat{c} + \frac{N+1}{2}rvaa\Big)\\
&= -\frac{2\pi}{L}\widehat{a}\delta a + \frac{2\pi\lambda}{LM}\widehat{c}\delta c + \frac{2\pi\lambda}{L^2M}a\delta c + \mathscr{L}_{\text{bdy}}.
\end{aligned}
\tag{5.4.26}
$$

In the second line, the first two terms are the BF terms that couple electric fields (a,c) to the magnetic fields $(\widehat{a},\widehat{c})$, while the third term is the mixed coupling (Dijkgraaf-Witten twist term) that only depends on the electric fields a,c. The last term $\mathscr{L}_{\text{bdy}}$ is the boundary term which does not play any role in this subsection, while it will be crucial in Sec. 5.4.4. Because the $\mathbb{Z}_4^{\text{em}}$ only exchanges among the electric fields and the magnetic fields separately, we conclude that after gauging $\mathbb{Z}_4^{\text{em}}$, the theory is still a Dijkgraaf-Witten. The explicit form the Dijkgraaf-Witten can be analogously constructed as in Sec. 5.3.3. See also [122]. Concretely, after gauging $\mathbb{Z}_4^{\text{em}}$, we introduce the $\mathbb{Z}_4^{\text{em}}$ cocycle x as a dynamical gauge field. Denote the two component 2-form

[23] The coefficient of a and c in (5.4.24) is designed to ensure that a,c transform in a conjugate way as $\widehat{a},\widehat{c}$ and also $(\widehat{a},a)$ and $(\widehat{c},c)$ appear as conjugate fields in the Lagrangian.

gauge field

$$\mathbf{a} = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad \widehat{\mathbf{a}} = \begin{pmatrix} \widehat{a} \\ \widehat{c} \end{pmatrix}. \tag{5.4.27}$$

The gauged $\mathbb{Z}_N$ 2-form gauge theory becomes

$$\begin{aligned} \mathscr{L}^{\text{gauged}}_{ijklmn} =& \frac{2\pi}{LM}\widehat{\mathbf{a}}^T_{ijk} W K^{x_{ik}}(K^{x_{kl}}\mathbf{a}_{lmn} - \mathbf{a}_{kmn} + \mathbf{a}_{kln} - \mathbf{a}_{klm}) \\ &+ \frac{\pi\lambda}{N}\mathbf{a}^T_{ijk} V K^{x_{ik}}(K^{x_{kl}}\mathbf{a}_{lmn} - \mathbf{a}_{kmn} + \mathbf{a}_{kln} - \mathbf{a}_{klm}) + \frac{\pi\varepsilon}{2}x_{ij}(\beta x)_{jkl}(\beta x)_{lmn} \end{aligned} \tag{5.4.28}$$

where

$$K = \begin{pmatrix} r & u\lambda \\ -Mv & -r \end{pmatrix}, \quad \widehat{K} = \begin{pmatrix} -r & -\lambda v \\ uM & r \end{pmatrix}, \quad W = \begin{pmatrix} -M & 0 \\ 0 & \lambda \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.4.29}$$

and $(\beta x)_{jkl} := \frac{1}{4}(\delta x)_{jkl} = \frac{1}{4}(x_{kl} - x_{jl} + x_{jk})$ is always an integer valued since x is a $\mathbb{Z}_4$ cocycle. The gauged theory is invariant under gauge transformations

$$\begin{aligned} \mathbf{a}_{ijk} &\to K^{-\gamma_i}(\mathbf{a}_{ijk} + K^{x_{ij}}\alpha_{jk} - \alpha_{ik} + \alpha_{ij}), \\ \widehat{\mathbf{a}}_{ijk} &\to \widehat{K}^{-\gamma_i}(\widehat{\mathbf{a}}_{ijk} + \widehat{K}^{x_{ij}}\widehat{\alpha}_{jk} - \widehat{\alpha}_{ik} + \widehat{\alpha}_{ij}), \\ x_{ij} &\to x_{ij} + \gamma_j - \gamma_i, \end{aligned} \tag{5.4.30}$$

thanks to the identities $\widehat{K}^T W K = W$ and $K^T V K = V$. Summing over $\widehat{\mathbf{a}}$ constrains $\mathbf{a}$ to be a twisted cocycle, i.e. $\mathbf{a}$ is twisted-flat. In components, $W(K^{x_{kl}}\mathbf{a}_{lmn} - \mathbf{a}_{kmn} + \mathbf{a}_{kln} - \mathbf{a}_{klm}) = 0$ mod LM. This means that $\mathbf{a}, x$ form a non-trivial 2-group. The full partition function is

$$\mathscr{Z}_{\text{gauged}} = \sum_{\mathbf{a},x}\prod_{ijklmn}\exp\left(\frac{i\pi\lambda}{N}\mathbf{a}^T_{ijk} V K^{x_{ik}}(K^{x_{kl}}\mathbf{a}_{lmn} - \mathbf{a}_{kmn} + \mathbf{a}_{kln} - \mathbf{a}_{klm}) + \frac{i\pi\varepsilon}{2}x_{ij}(\beta x)_{jkl}(\beta x)_{lmn}\right) \tag{5.4.31}$$

where we only sum over the gauge fields $\mathbf{a}, x$ with the twisted-flatness condition. The partition function is obviously of the form of Dijkgraaf-Witten (see Eq. (5.1.6)).

Even N:

When the 4d spacetime is a spin manifold, the discussion is almost the same as above, such as (5.4.23), (5.4.24) and (5.4.25). The only modification is to replace the factor $\frac{N+1}{2}$ in the Lagrangian (5.4.26) by $\frac{1}{2}$, and the self pairing e.g. aa by the Pontryagin square of a, i.e. $\mathscr{P}(a)$. When the 4d spacetime is a non-spin manifold, in the special case where the Lagrangian subgroup has only a single generator, we find in App. C.2 that there is no stable Lagrangian subgroup, hence the SymTFT obtained by gauging $\mathbb{Z}_4^{\text{em}}$ can not be a Dijkgraaf-Witten. When there are two generators, we do not solve the stability condition in this paper.

In summary, for all the cases where the stable Lagrangian subgroup exist, we showed that the SymTFT for the duality symmetry is a Dijkgraaf-Witten theory.

5.4.4 Explicit topological manipulations

We finally proceed to find explicit topological manipulations that map the duality defect to an invertible defect.

Suppose a 4d QFT with $\mathbb{Z}_N^{(1)}$ one-form symmetry is self-dual under gauging $\mathbb{Z}_N^{(1)}$. This means

$$\mathscr{Z}_{\mathscr{X}}[B] = \sum_{b \in \mathbb{Z}_N} \mathscr{Z}_{\mathscr{X}}[b] e^{-\frac{2\pi i}{N} \int_{M_4} vbB}. \tag{5.4.32}$$

The summation $b \in \mathbb{Z}_N$ means summing over 2-cochain b valued in $\mathbb{Z}_N$ with flatness condition $\delta b = 0 \bmod N$, modulo gauge transformations. The idea of identifying the topological manipulation which maps the duality defect to an invertible defect is the same as in Sec. 5.3.4. The self-dual theory $\mathscr{X}$ corresponds to Dirichlet boundary condition of b. The desired topological manipulation should map such boundary condition to a new topological boundary condition where all the new electric fields have Dirichlet boundary conditions. In the meantime, the discrete theta terms should be introduced coming from the boundary terms of the SymTFT $\mathscr{L}_{\text{bdy}}$ in (5.4.26).

Odd N:

In this case, N has the form $N = L^2 M$ where -1 is a quadratic residue of M. The topological manipulation should map the Dirichlet boundary condition of b to the Dirichlet boundary conditions of a, c defined via (5.4.23). Denoting the background field for the former as B, and those for the latter as A and C, we have

$$\langle B| \to \sum_{\widehat{c}\in\mathbb{Z}_{LM}} \langle L\widehat{c} - A|\, e^{-\frac{2\pi i}{LM}\int_{M_4}(\lambda C + rvA)\widehat{c} + \frac{2\pi i}{N}\frac{N+1}{2}\int_{M_4}(L^2 rv\widehat{c}\widehat{c} + rvAA)}. \tag{5.4.33}$$

The phase factor on the RHS is the $\mathscr{L}_{\text{bdy}}$ in (5.4.26). This implies that the new partition function obtained by topological manipulation is

$$\mathscr{Z}_{\widetilde{\mathscr{X}}}[A,C] = \sum_{\widehat{c}\in\mathbb{Z}_{LM}} \mathscr{Z}_{\mathscr{X}}[L\widehat{c} - A] e^{-\frac{2\pi i}{LM}\int_{M_4}(\lambda C + rvA)\widehat{c} + \frac{2\pi i}{N}\frac{N+1}{2}\int_{M_4}(L^2 rv\widehat{c}\widehat{c} + rvAA)}. \tag{5.4.34}$$

Notice that we start with taking the background gauge field A, C to be $\mathbb{Z}_N$ gauge field for convenience. However, the action (5.4.34) is invariant up to terms containing only background fields under the following transformation:

$$A \to A + LX, \qquad C \to C + LMY, \qquad \widehat{c} \to \widehat{c} + X, \tag{5.4.35}$$

where $X, Y \in \mathbb{Z}_N$. Eq. (5.4.35) shows that A is actually a $\mathbb{Z}_L$ background field and C is a $\mathbb{Z}_{LM}$ background field.

We proceed to check what does the duality symmetry of $\mathscr{X}$ is mapped to in $\widetilde{\mathscr{X}}$. Substituting (5.4.32) into (5.4.34), we find

$$\mathscr{Z}_{\widetilde{\mathscr{X}}}[A,C] = \sum_{\widehat{c}\in\mathbb{Z}_{LM},b\in\mathbb{Z}_N} \mathscr{Z}_{\mathscr{X}}[b] e^{-\frac{2\pi i}{N}\int_{M_4} vb(L\widehat{c}-A) - \frac{2\pi i}{LM}\int_{M_4}(\lambda C+rvA)\widehat{c} + \frac{2\pi i}{N}\frac{N+1}{2}\int_{M_4}(L^2 rv\widehat{c}\widehat{c}+rvAA)}. \tag{5.4.36}$$

We first sum over $\widehat{c}$, where the relevant terms are $\sum_{\widehat{c}\in\mathbb{Z}_{LM}} e^{-\frac{2\pi i}{LM}\int_{M_4}(vb+\lambda C+rvA)\widehat{c}+\frac{2\pi i}{M}\frac{N+1}{2}rv\int_{M_4}\widehat{c}\widehat{c}}$. This sum is non-vanishing only when $vb+\lambda C+rvA$ can be divided by L. [24] We thus define $b = Lb' - u(\lambda C + rvA)$. Substituting this into (5.4.36) and summing over $\widehat{c}$, we get

$$\begin{aligned}\mathscr{Z}_{\widetilde{\mathscr{X}}}[A,C] &= \sum_{\widehat{c},b'\in\mathbb{Z}_{LM}} \mathscr{Z}_{\mathscr{X}}[Lb'-u(\lambda C+rvA)] e^{-\frac{2\pi i}{N}\int_{M_4}(-vLb'+\lambda C+rvA)A+\frac{2\pi i}{N}\frac{N+1}{2}\int_{M_4}rvAA+L^2 rvb'b'} \\ &= \mathscr{Z}_{\widetilde{\mathscr{X}}}[u\lambda C+rA, -vMA-rC] e^{-\frac{2\pi i}{N}\frac{N+1}{2}\lambda\int_{M_4}(2\lambda MAC+ru\lambda CC+rvMAA)}.\end{aligned} \tag{5.4.37}$$

Therefore we have shown that the duality symmetry of $\mathscr{X}$ becomes an invertible symmetry mapping on the background fields of the new theory $\widetilde{\mathscr{X}}$ as

$$A \to u\lambda C + rA \bmod L, \qquad C \to -vMA - rC \bmod ML. \tag{5.4.38}$$

This agrees with (5.4.25).

Even N:

When the 4d spacetime is a spin manifold, the discussion is almost identical to the odd N case. The only modifications are to replace $\frac{N+1}{2}$ by $\frac{1}{2}$, and self pairing, e.g. BB, by $\mathscr{P}(B)$ in (5.4.33), (5.4.34), (5.4.36), (5.4.37). The duality symmetry (5.4.32) is mapped to an invertible symmetry (5.4.38).

When the 4d spacetime is a non-spin manifold, we haven't classified all the solutions. However, as mentioned in Sec. 5.4.2, when $N = L^2$ for even L, there is a stable Lagrangian sub-

[24] This can be seen by replacing $\widehat{c}$ with $\widehat{c}+s$ where s is an integer valued cochain, and demand the sum does not change as $\widehat{c}$ is a dummy variable.

group. Inspired by a similar discussion in Sec. 5.3.4, we claim that the topological manipulation is simply gauging the $\mathbb{Z}_L^{(1)}$ subgroup, i.e. $\widetilde{\mathscr{X}} = \mathscr{X}/\mathbb{Z}_L^{(1)}$. Concretely,

$$\mathscr{Z}_{\widetilde{\mathscr{X}}}[A,C] = \sum_{a\in\mathbb{Z}_L} \mathscr{Z}_{\mathscr{X}}[L\widehat{c}-A]e^{-\frac{2\pi i}{L}\int_{X_4}\widehat{c}C} \tag{5.4.39}$$

where A,C are both $\mathbb{Z}_L$ 2-form gauge fields. Substituting (5.4.32) into (5.4.39), we find the background fields A,C are mapped as

$$A \to uC \bmod L, \qquad C \to -vA \bmod L\,. \tag{5.4.40}$$

Hence when the 4d spacetime is a non-spin manifold, we have at least confirmed a spacial case where $N = L^2$ such that the $\mathbb{Z}_N^{(1)}$ duality defect is group theoretical.

In summary, for all the cases where the stable Lagrangian subgroup exist (on spin manifolds), we have constructed explicit topological manipulation which maps the duality symmetry to an invertible symmetry.

5.5 Connection with obstruction to duality-preserving gapped phases

We have observed in the above that for the $\mathbb{Z}_N^{(d/2-1)}$ duality defect to be group theoretical, some number-theoretic condition should be satisfied, and exactly the same condition has appeared in other contexts, including whether there exists an SPT or TQFT invariant under gauging $\mathbb{Z}_N^{(d/2-1)}$[50, 51, 13, 189]. In this section, we comment on the relation between them.

5.5.1 Duality defects, gauging, and SPT

Given a finite abelian group $\mathbb{Z}_N^{(d/2-1)}$ and an integer u coprime with N which specifies how to gauge $\mathbb{Z}_N^{(d/2-1)}$, one may look for $\mathbb{Z}_N^{(d/2-1)}$ SPTs in d dimensions satisfying the one of the following closely related properties:

(a) The $\mathbb{Z}_N^{(d/2-1)}$ SPT is equipped with a $\mathbb{Z}_N^{(d/2-1)}$ duality defect with an *unspecified* Frobenius-Schur indicator (labeled by ε). Note that the bicharacter is specified by u. Equivalently, the $\mathbb{Z}_N^{(d/2-1)}$ SPT is invariant under gauging $\mathbb{Z}_N^{(d/2-1)}$;

(b) The $\mathbb{Z}_N^{(d/2-1)}$ SPT is equipped with a $\mathbb{Z}_N^{(d/2-1)}$ duality defect with a *given* Frobenius-Schur indicator (labeled by ε).

A $\mathbb{Z}_N^{(d/2-1)}$-SPT satisfying property (b) automatically satisfies property (a) since $\mathbb{Z}_N^{(d/2-1)}$ duality defect implements gauging $\mathbb{Z}_N^{(d/2-1)}$. However, the converse is not true: it is possible that for a given coprime pair (N,u), the duality defect with certain ε may not be realized by any $\mathbb{Z}_N^{(d/2-1)}$ SPT, or equivalently, the duality defect may be anomalous for some choice of ε. See [180] for the general criteria of the anomaly in 2d.

In [50, 51, 13, 122, 180], the authors discussed the G-SPTs with property (a). It was found that

1. in 2d and among all N's, the only $\mathbb{Z}_N^{(0)}$ SPT is a trivial SPT with partition function $\mathscr{Z}=1$. Furthermore, the trivial SPT is not invariant under gauging $\mathbb{Z}_N^{(0)}$ for any coprime pair (N,u);

2. in 4d and among all N's, an $\mathbb{Z}_N^{(1)}$ SPT is classified by

$$\begin{gathered} \frac{2\pi k}{N}\frac{N+1}{2}\int_{X_4} B^{(2)}B^{(2)}, \quad k\in\mathbb{Z}_N, \quad X_4= \text{ spin or non-spin}, \\ \frac{2\pi k}{2N}\int_{X_4}\mathscr{P}(B^{(2)}), \quad \begin{cases} k\in\mathbb{Z}_{2N}, & X_4= \text{ non-spin}, \\ k\in\mathbb{Z}_N, & X_4= \text{ spin}, \end{cases} \end{gathered} \tag{5.5.1}$$

and there exists an $\mathbb{Z}_N^{(1)}$ SPT invariant under gauging $\mathbb{Z}_N^{(1)}$ for a given u if and only if there is an integer r solving the following equation[25]

$$r^2 = -1 \bmod \begin{cases} N, & \text{odd } N \text{ on spin and non-spin } X_4, \\ N, & \text{even } N \text{ on spin } X_4, \\ 2N, & \text{even } N \text{ on non-spin } X_4. \end{cases} \tag{5.5.2}$$

Solving (b) is harder. In 2d, for general G, a classification of G SPTs equipped with a G duality defect was achieved in [180], see also [189]. In 3d, some related recent works on the anomaly of fusion 2-categories can be found in [73, 72].

5.5.2 Duality defects, gauging, TQFT, and relation with group theoretical duality defects

Given an abelian group $\mathbb{Z}_N^{(1)}$ and an integer u coprime with N which specifies how to gauge $\mathbb{Z}_N^{(1)}$, one may look for $\mathbb{Z}_N^{(1)}$ symmetric 4d TQFTs[26] satisfying the one of the following closely related properties:

(a') The $\mathbb{Z}_N^{(1)}$ symmetric TQFT (with one ground state on S^3 spatial manifold) is equipped with a $\mathbb{Z}_N^{(1)}$ duality defect with an *unspecified* Frobenius-Schur indicator (labeled by ε). Note that the bicharacter is specified by u. Equivalently, the $\mathbb{Z}_N^{(1)}$ symmetric TQFT is invariant under gauging $\mathbb{Z}_N^{(1)}$;

(b') The $\mathbb{Z}_N^{(1)}$ symmetric TQFT (with one ground state on S^3 spatial manifold) is equipped with a $\mathbb{Z}_N^{(1)}$ duality defect with a *given* Frobenius-Schur indicator (labeled by ε).

The requirement of one ground state on S^3 spatial manifold ensures that the $\mathbb{Z}_N^{(1)}$ duality symmetry is not spontaneously broken, however, the $\mathbb{Z}_N^{(1)}$ symmetry is allowed to be spontaneously broken since we don't require one ground state on other spatial manifolds such as T^3. Similar to the

[25] In [50] only the case $u = 1$ was discussed. But it is straightforward to check that the same condition holds for any u coprime with N.

[26] In this subsection, we constrain our discussion to duality defects in 4d. The reason is that in 2d, the TQFTs with one ground state on S^1 spatial manifold is always an SPT, hence the discussion reduces to Sec. 5.5.1.

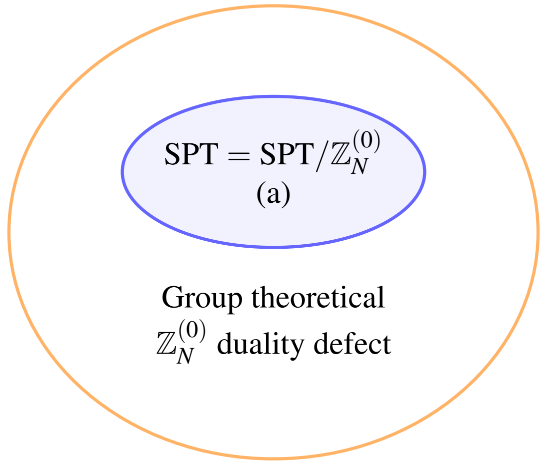


Figure 5.3. For 2d theory: space of N where group theoretical $\mathbb{Z}_N^{(0)}$ duality defects exist (bounded by orange circle and contains N being a perfect square), and where there exists a $\mathbb{Z}_N^{(0)}$-SPT invariant under gauging $\mathbb{Z}_N^{(0)}$ (bounded by blue circle and contains only $N = 1$).

discussion in Sec. 5.5.1, a $\mathbb{Z}_N^{(1)}$ symmetric TQFT satisfying property (b') automatically satisfies property (a'). However, the converse is not true.

In [13], a classification for (a'), when the 4d manifold is spin, has been achieved. It was found that a 4d (spin) TQFT is invariant under gauging $\mathbb{Z}_N^{(1)}$ if and only if N has the form

$$N = L^2 M, \qquad L \in \mathbb{Z}, \text{ and } \exists \text{ an integer } r \text{ solving } r^2 = -1 \bmod M. \tag{5.5.3}$$

This is precisely the condition for the $\mathbb{Z}_N^{(1)}$ duality defects to be group theoretical.

To understand why there is such a coincidence, we again use the SymTFT. It turns out that the SymTFT is a natural set up in discussing 4d TQFTs with the property (a') or (b'). Any 4d TQFT with a non-anomalous $\mathbb{Z}_N^{(1)}$ global symmetry can be expanded into a 5d slab where in the 5d bulk is a $\mathbb{Z}_N$ 2-form gauge theory (5.2.1), the left boundary is the Dirichlet boundary condition (5.2.2), and the right boundary is another topological boundary condition (5.2.3) specified by the 4d TQFT. Since the SymTFT does not contain genuinue line or point operator, it is clear that after shrinking the 5d slab to get a genuine 4d theory, there isn't any non-trivial topological local operator, neither directly coming from point operator in the bulk nor from compactifying

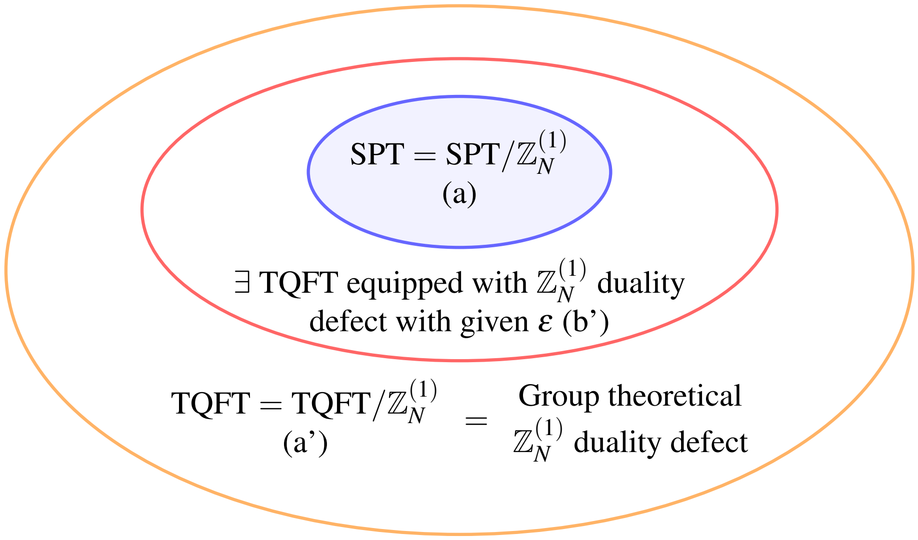


Figure 5.4. The Venn diagram captures the result for 4d theory: the orange circle bounds N satisfying (a'), the red circle bounds the N satisfying (b') while the blue circle bounds the N satisfying (a).

line operators along the shrinked direction. Hence the only topological local operator is the trivial identity, and the unique ground state on S^3 is guaranteed. Furthermore, it is also known [122, 83, 33, 189, 8] that gauging $\mathbb{Z}_N^{(1)}$ of the 4d TQFT can be equivalently achieved by fusing a $\mathbb{Z}_4^{\text{em}}$ symmetry defect (constructed as a condensation defect in [122, 8]) of the 5d $\mathbb{Z}_N^{(1)}$ gauge theory to either of the topological boundary. Requiring (a'), i.e. the TQFT to be invariant under gauging $\mathbb{Z}_N^{(1)}$, amounts to requiring the right topological boundary state (5.2.3) to be invariant under fusing with the $\mathbb{Z}_4^{\text{em}}$ symmetry defect. This is precisely the stability condition in Sec. 5.4.2, from where we derived the same condition as (5.5.3).

Solving (b') is again harder. To achieve a full classification of TQFTs equipped with a duality defect, we need to consider the SymTFT of the duality symmetry, i.e. $\mathbb{Z}_N^{(1)}$ 2-form gauge theory with $\mathbb{Z}_4^{\text{em}}$ gauged, as in (5.2.5). Note that the resulting SymTFT depends on both the choice of bicharacters (see the discussion below (5.2.5)), as well as the choice of a discrete theta term of the $\mathbb{Z}_4^{\text{em}}$ gauge field x. The choice of discrete theta term can be understood as a higher dimensional generalization of the Frobenius Schur indicator as discussed in Sec. 5.4.3. There is a topological line operator $K = e^{\frac{i\pi}{2}\oint x}$, which by construction topologically terminates on the left topological boundary, just as the electric surface operator $e^{\frac{2\pi i}{N}\oint b}$ does. To ensure that the duality symmetry is not spontaneously broken (or equivalently there is one ground state on S^3), we should demand that the K-line can not topologically terminate on the right topological boundary. This imposes additional constraints for (b'), which will be left for future study.

We summarize the main results of $\mathbb{Z}_N^{(0)}$ duality defects in 2d in Fig. 5.3, and those of $\mathbb{Z}_N^{(1)}$ duality defects in 4d in Fig. 5.4. [27]

Chapter 5, in full, is currently being prepared for submission for publication of the material. Zhengdi Sun, Yunqin Zheng, arXiv:2307.14428 [hep-th]. The dissertation author was one of the primary investigator and author of this material.

[27] We thank Philip Boyle Smith for the discussions on the Venn diagrams.

Appendix A

High Energy Modular Bootstrap, Global Symmetries and Defects

A.1 Verification

A.1.1 Ising CFT with $\mathbb{Z}_2$ symmetry

The Ising model has three primaries: $\mathbb{I}, \varepsilon, \sigma$. Under the $\mathbb{Z}_2$ symmetry, $\mathbb{I}$ (with $h = \bar{h} = 0$) and ε (with $h = \bar{h} = \frac{1}{2}$) are even while the σ (with $h = \bar{h} = \frac{1}{16}$) is odd. The central charge is given by $c = \frac{1}{2}$. Let us denote the characters corresponding to $\mathbb{I}, \varepsilon, \sigma$ as $\chi_0, \chi_{1/2}, \chi_{1/16}$ respectively. A nice exposition of the Ising model in context of TDLs can be found in [140]. Here we briefly recapitulate the necessary ingredients for verifying our formulas (3.4.10) and (3.3.2) against the Ising model.

The partition function of the Ising model with TDL (name it η) corresponding to $\mathbb{Z}_2$ group element (the identity element e and the non-identity element p) inserted along the spatial direction is given by

$$\begin{aligned} Z^{\eta}(\beta, e) &= Z(\beta) = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2, \\ Z^{\eta}(\beta, p) &= |\chi_0|^2 + |\chi_{1/2}|^2 - |\chi_{1/16}|^2. \end{aligned} \tag{A.1.1}$$

- Irreps: The growth of the even and the odd operators (denoted by $\rho_{\pm}$ respectively) are controlled by $\frac{1}{2}(Z^{\eta}(\beta,e) \pm Z^{\eta}(\beta,p))$ in the $\beta \to 0$ limit. From eq. (3.4.10), we have

$$s_-(\delta) \leq \log\left[\frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta' \rho_{\pm}(\Delta')\right] - 2\pi\sqrt{\frac{\Delta}{6}} - \frac{1}{4}\log\left(\frac{1}{96\Delta^3}\right) + \log(2) \leq s_+(\delta) \qquad \text{(A.1.2)}$$

where $s_{\pm} = \log(c_{\pm})$. We use the value of $c_{\pm}$ presented in [92]. We verify the above inequality in fig. A.1.

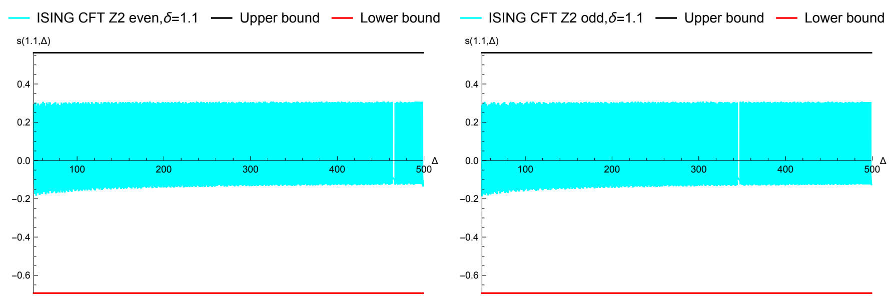


Figure A.1. The estimate of the number of even and odd operators (under $\mathbb{Z}_2$) in Ising CFT. We plot the logarithm of the ratio of actual number of operators in the interval of size $2\delta = 2.2$ and the leading prediction from Tauberian-Cardy analysis.

• Defect Hilbert space: The partition function corresponding to the defect Hilbert space is given by the S modular transformation of $Z^{\eta}(\beta,p)$:

$$Z_{\eta}(\beta) = \chi_0\bar{\chi}_{1/2} + \chi_{1/2}\bar{\chi}_0 + |\chi_{1/16}|^2 . \qquad \text{(A.1.3)}$$

The Virasoro primaries have weights $(0,1/2)$, $(1/2,0)$ and $(1/16,1/16)$. We note that there is no $\Delta = 0$ state in the defect Hilbert space. The states with $\Delta = 1/2$ corresponds to Fermions. We can verify following estimate of the growth of number of operators in the defect Hilbert space (defect corresponding to $\mathbb{Z}_2$, here the TDL is extended along the time direction) of the Ising CFT:

$$s_-(\delta) \leq \log\left[\frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta' \rho_{\mathscr{H}_\eta}(\Delta')\right] - 2\pi\sqrt{\frac{\Delta}{6}} - \frac{1}{4}\log\left(\frac{1}{96\Delta^3}\right) \leq s_+(\delta) \qquad \text{(A.1.4)}$$

where $s_{\pm} = \log(c_{\pm})$. The above follows from eq. (3.3.2). Again we use the value of $c_{\pm}$ presented in [92] and verify the inequality in fig. A.2.

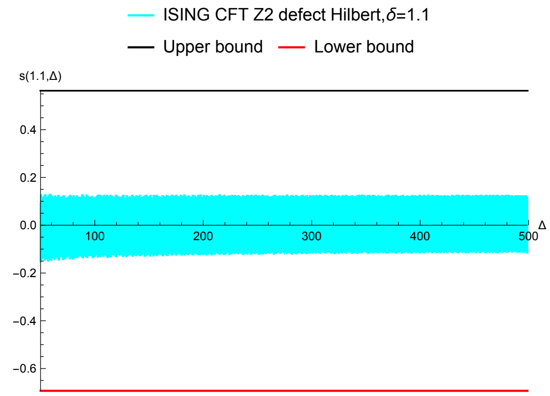


Figure A.2. The estimate of the number of operators in the defect Hilbert space corresponding to $\mathbb{Z}_2$ in Ising CFT. We plot the logarithm of the ratio of actual number of operators in the interval of size $2\delta = 2.2$ and the leading prediction from Tauberian-Cardy analysis.

The Ising model also has a duality defect line $\widehat{N}$. This is non invertible TDL. The fusion rule is given by $\widehat{N} \times \widehat{N} = \mathbb{I} + \eta$, thus the action of $\widehat{N}$ is given by

$$\widehat{N}|\text{even}\rangle = \sqrt{2}|\text{even}\rangle\,, \quad \widehat{N}|\text{odd}\rangle = 0\rangle\,.$$

The growth of the operators in the defect Hilbert space corresponding to the duality line can be estimated via eq. (3.3.2):

$$s_-(\delta) \leq \log\left[\frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta' \rho_{\mathscr{H}_{\widehat{N}}}(\Delta')\right] - 2\pi\sqrt{\frac{\Delta}{6}} - \frac{1}{4}\log\left(\frac{1}{96\Delta^3}\right) - \frac{1}{2}\log(2) \leq s_+(\delta)\,, \quad \text{(A.1.5)}$$

which we verify in the fig. A.3.

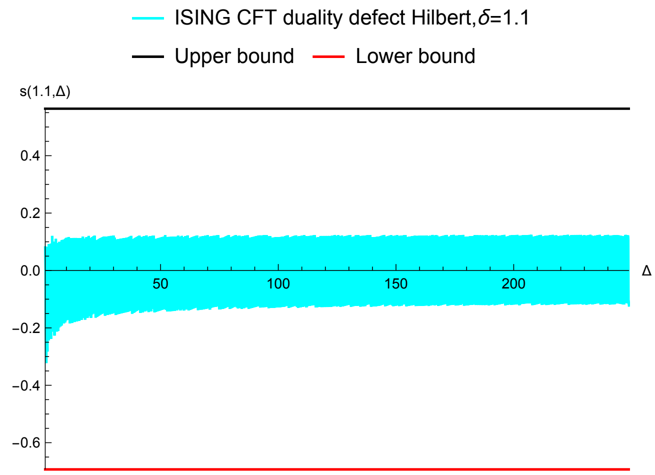


Figure A.3. The estimate of the number of operators in the defect Hilbert space of the duality defect line $\widehat{N}$ in the Ising CFT. We plot the logarithm of the ratio of actual number of operators in the interval of size $2\delta = 2.2$ and the leading prediction from Tauberian-Cardy analysis.

A.1.2 Compact Boson at $R = \frac{1}{2}$ with $U(1)$ symmetry

For compact Boson at radius $R = \frac{1}{2}$, the $U(1)$ generated by $J_0 - \bar{J}_0$ acts faithfully. The partition function for the charge Q is given by

$$Z^Q(q) = q^{\frac{m^2}{4} - \frac{1}{12}} \left[\frac{\theta_3(q)}{\eta^2}\right] = q^{\frac{m^2}{4} - \frac{1}{12}} \left(1 + 4q + 9q^2 + 20q^3 + O(q^4)\right) . \tag{A.1.6}$$

For compact boson $k = 1$, thus the growth of operators with charge Q is given by

$$s_-(\delta) \leq \log\left[\frac{1}{2\delta}\int_{\Delta-\delta}^{\Delta+\delta} \mathrm{d}\Delta' \rho_{\mathscr{H}_\eta}(\Delta')\right] - 2\pi\sqrt{\frac{\Delta}{3}} - \log\left(\frac{1}{4\Delta}\right) + \frac{1}{2}\log(3k) \leq s_+(\delta), \tag{A.1.7}$$

which follows from eq. (3.4.24). This is verified in fig. A.4.

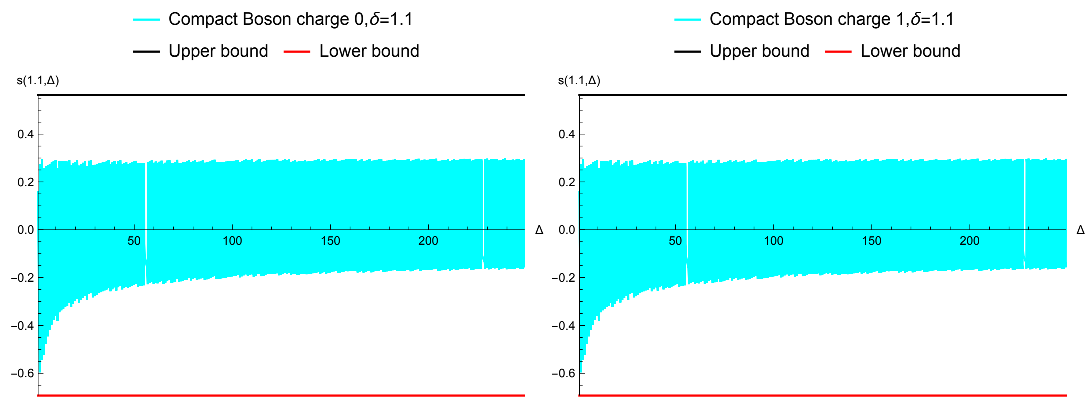


Figure A.4. The estimate of the number of operators with charge $Q = 0, 1$ of the $U(1)$ symmetry in compact boson at $R = \frac{1}{2}$. We plot the logarithm of the ratio of actual number of operators in the interval of size $2\delta = 2.2$ and the leading prediction from Tauberian-Cardy analysis.

A.2 Spin selection rule for anomalous symmetry

The defect Hilbert space is defined by having a TDL along the time like direction. Now if we want to define the action of the symmetry in the defect Hilbert space, we need to introduce another TDL along the spatial direction. Since, the two TDLs cross each other, we need to resolve the crossing. And this is how the global symmetry can turn out to have 't Hooft anomaly, which is related to the ambiguity in locally resolving the crossing configuration of two TDL (see fig. A.5). Two different ways of resolution leads to defining two operators $\mathscr{L}_{\pm}$ acting on the states in the defect Hilbert space. Relationship between these two different ways of resolving ambiguity leads to the "crossing relations", which naturally generalize to the any TDLs (not only the one corresponding to the global symmetry). We will see that such 't Hooft anomaly of global symmetry will impose spin selection rules on the defect Hilbert space (see fig. A.7).

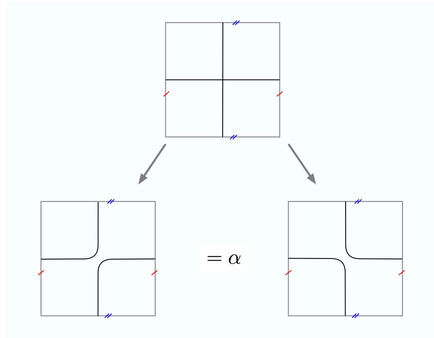


Figure A.5. Here, we consider the $\mathbb{Z}_2$ symmetry line on a torus. There are two ways to resolve the crossing configuration on the top, which are related by $\alpha = 1$ for non-anomalous $\mathbb{Z}_2$ and $\alpha = -1$ is for the anomalous $\mathbb{Z}_2$. We note the left configuration as $\hat{\mathscr{L}}_+$ and the right configuration as $\hat{\mathscr{L}}_-$.

We will focus on the group $\mathbb{Z}_2$ for rest of the appendix. Following [140], to derive a spin selection rule, we first determine the action of $\hat{\mathscr{L}}_\pm$ on the defect Hilbert space and then consider a specific configuration which relates the action of $\hat{\mathscr{L}}_\pm$ to the spin of the state. We consider the fig. A.6 to derive $\alpha^2 = 1$.

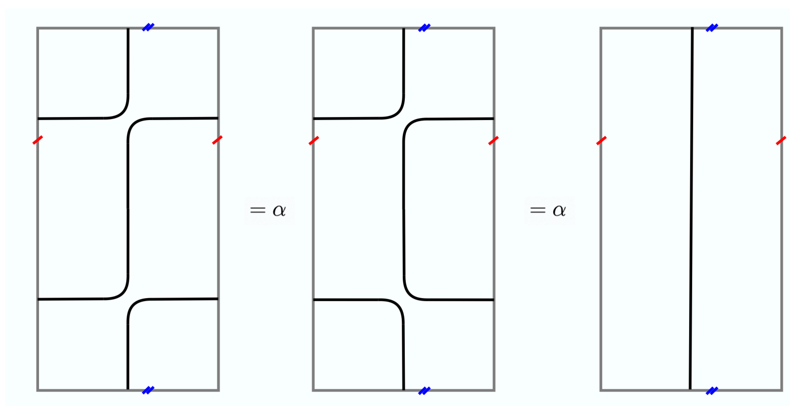


Figure A.6. Here we consider two $\hat{\mathscr{L}}_+$ on the left figure and show it is equal to α acting on states in the defect Hilbert space.

On the other hand, we have

$$(\hat{\mathscr{L}}_+)^2|h,\bar{h}\rangle = \alpha|h,\bar{h}\rangle \Rightarrow \hat{\mathscr{L}}_+|h,\bar{h}\rangle \quad = \pm\sqrt{\alpha}|h,\bar{h}\rangle\,. \tag{A.2.1}$$

For the next step, we consider mapping $\hat{\mathscr{L}}_+|h,\bar{h}\rangle$ from $\mathbb{R}_t \times S^1$ to $\mathbb{R}^2$ and unwind the $\mathscr{L}^+$

to deduce

$$\hat{\mathscr{L}}_{\pm}|h,\overline{h}\rangle = e^{\pm 2\pi \imath s}|h,\overline{h}\rangle. \tag{A.2.2}$$

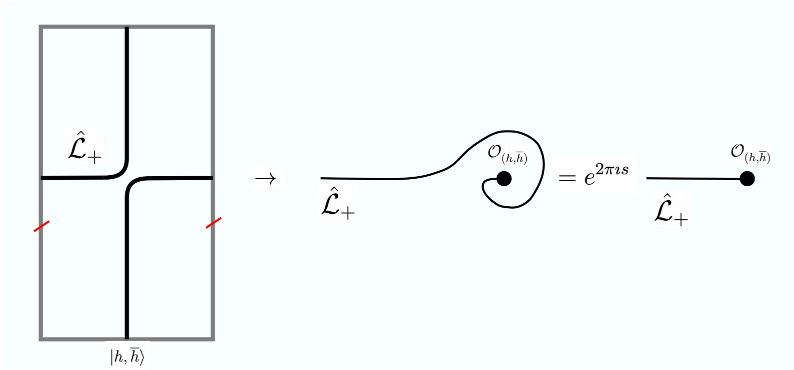


Figure A.7. Here, we consider the action of $\hat{\mathscr{L}}_+$ acting on the state $|h,\overline{h}\rangle$ on $\mathbb{R}_t \times S^1$ and maps to $\mathbb{R}^2$ via the operator-state correspondence map. Then unwinding $\hat{\mathscr{L}}_+$ shows $\hat{\mathscr{L}}_+|h,\overline{h}\rangle = e^{2\pi \imath s}|h,\overline{h}\rangle$.

Combining the previous results, we find:

- in the non-anomalous case where $\alpha = 1$, we have

$$s \in \begin{cases} \mathbb{Z} & \text{if } \hat{\mathscr{L}}_+ \text{ acts as } +1, \\ \frac{1}{2} + \mathbb{Z} & \text{if } \hat{\mathscr{L}}_+ \text{ acts as } -1; \end{cases} \tag{A.2.3}$$

- in the anomalous case where $\alpha = -1$, we have

$$s \in \begin{cases} +\frac{1}{4} + \mathbb{Z} & \text{if } \hat{\mathscr{L}}_+ \text{ acts as } +i, \\ -\frac{1}{4} + \mathbb{Z} & \text{if } \hat{\mathscr{L}}_+ \text{ acts as } -i. \end{cases} \tag{A.2.4}$$

Analogously, one can generalize the above result to $\mathbb{Z}_n$ [46]. Thus the spin selection rule automatically rules out the existence of $\Delta = 0$ states in the defect Hilbert space if the symmetry is anomalous. For completeness, we remark here that if the symmetry is non-anomalous, we can rule out the existence of $\Delta = 0$ state by requiring that that symmetry group acts faithfully on the Hilbert space.

A.3 Review of Representation Theory for Finite Group

We review some basic notions and results in the representation theory for finite group. For a more detailed exposition including proofs and jokes, see II.1 and II.2 of [188].

Given a finite group G and a unitary (reducible or irreducible) representation r of G given by matrices $D^{(r)}(g)$, we define the character $\chi^{(r)}(g)$ to be

$$\chi^{(r)}(g) \equiv \mathrm{tr} D^{(r)}(g). \tag{A.3.1}$$

The Great Orthogonality Theorem together with one of its corollary states that, given two irreducible representation r, s,

$$\sum_g D^{(r)\dagger}(g)^i{}_j D^{(s)}(g)^k{}_l = \frac{|G|}{d_r} \delta^{rs} \delta^i{}_l \delta^k{}_j \tag{A.3.2}$$

where $|G|$ is the order of the group, d_r is the dimension of the irrep r, and $\delta^{rs} = 1$ if two irreps are the same and $\delta^{rs} = 0$ otherwise. For a proof of this result, see II.2 of [188].

From the above result, one can derive the so-called character orthogonality. By taking trace, we find

$$\sum_g (\chi^{(r)}(g))^* \chi^{(s)}(g) = |G| \delta^{rs}. \tag{A.3.3}$$

We can use the character orthogonality to count how many times a given irrep r appears in a reducible representation. First, notice that if a reducible representation R can be decompose into a direct sum of irreps r_i, then

$$\chi^{(R)}(g) = \sum_{i=1}^{N} \chi^{(r_i)}(g). \tag{A.3.4}$$

Applying the character orthogonality, we find

$$\frac{1}{|G|}\sum_{g}(\chi^{(r)}(g))^{*}\chi^{(R)}(g) = \text{No of times irrep "r" appears}\,. \tag{A.3.5}$$

In context of conformal field theory, the finite symmetry group G commutes with the Virasoro algebra, thus the states with the same scaling dimension Δ form a reducible representation of G. Therefore,

$$\begin{aligned}\frac{1}{|G|}\sum_{g}\chi^{\alpha}(g)^{*}Z^{\mathscr{L}}(\beta,g) &= \frac{1}{|G|}\sum_{g}\chi^{\alpha}(g)^{*}\mathrm{Tr}\left(\hat{g}q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right)\\ &= \frac{1}{|G|}\sum_{g}\sum_{\Delta}\chi^{\alpha}(g)^{*}(\mathrm{Tr}_{H_\Delta}\hat{g})e^{-\beta(\Delta-c/12)}\\ &= \sum_{\Delta}N_{\alpha,\Delta}e^{-\beta(\Delta-c/12)}\end{aligned} \tag{A.3.6}$$

where $N_{\alpha,\Delta}$ is the number of irrep α with scaling dimension Δ. We used this basic fact in the statements below (3.4.3).

Appendix B

On Triality Defects in 2d CFT

B.1 Details on the compact boson partition function

We show that the compact boson partition function, which is derived using (4.5.13), can be rewritten as the familiar sum over lattice. We start with breaking $Z_{SU(2)_1}$ into the partition sum over irreps with integer spins $j, \overline{j}$ denoted as $Z_{SU(2)_1,I}$ and the partition sum over irreps with half-integer spins $j, \overline{j}$ denoted as $Z_{SU(2)_1,II}$.

$$
\begin{aligned}
&|\eta(\tau)|^2 Z_{SU(2)_1,I} \\
&= \sum_{\substack{j=0,1,\cdots \\ \overline{j}=0,1,\cdots}} (2j+1)(2\overline{j}+1)(q^{j^2} - q^{(j+1)^2})(\overline{q}^{\overline{j}^2} - \overline{q}^{(\overline{j}+1)^2}) \\
&= \sum_{\substack{j=0,1,\cdots \\ \overline{j}=0,1,\cdots}} \left[(2j+1)(2\overline{j}+1) q^{j^2} \overline{q}^{\overline{j}^2}\right] - \left(\sum_{\substack{j=0,1,\cdots, \\ \overline{j}=0,1,\cdots}} \left[(2j+1)(2\overline{j}-1) q^{j^2} \overline{q}^{\overline{j}^2}\right] - \sum_{j=0,1,\cdots} (2j+1)(-1) q^{j^2} \right) \\
&\quad - \left(\sum_{\substack{j=0,1,\cdots \\ \overline{j}=0,1,\cdots}} \left[(2j-1)(2\overline{j}+1) q^{j^2} \overline{q}^{\overline{j}^2}\right] - \sum_{\overline{j}=0,1,\cdots} (2\overline{j}+1)(-1) \overline{q}^{\overline{j}^2} \right) \\
&\quad + \left(\sum_{\substack{j=0,1,\cdots, \\ \overline{j}=0,1,\cdots}} \left[(2j-1)(2\overline{j}-1) q^{j^2} \overline{q}^{\overline{j}^2}\right] - \sum_{j=0,1,\cdots} \left[(2j-1)(-1) q^{j^2}\right] - \sum_{\overline{j}=0,1,\cdots} \left[(2\overline{j}-1)(-1) \overline{q}^{\overline{j}^2}\right] + 1 \right) \\
&= \left(\sum_{\substack{j=0,1,\cdots, \\ \overline{j}=0,1,\cdots}} 4 q^{j^2} \overline{q}^{\overline{j}^2} \right) - \left(\sum_{j=0,1,\cdots} 2 q^{j^2} \right) - \left(\sum_{\overline{j}=0,1,\cdots} 2 \overline{q}^{\overline{j}^2} \right) + 1
\end{aligned}
\tag{B.1.1}
$$

and similarly,

$$|\eta(\tau)|^2 Z_{SU(2)_1,II} = \sum_{\substack{j=\frac{1}{2},\frac{3}{2},\cdots, \\ \bar{j}=\frac{1}{2},\frac{3}{2},\cdots}} 4q^{j^2}\bar{q}^{\bar{j}^2}. \tag{B.1.2}$$

It is then straightforward to see from the above that summing over two parts lead to the familiar sum over $2d$ Narain lattice,

$$|\eta(\tau)|^2 (Z_{SU(2)_1,I} + Z_{SU(2)_1,II}) = \sum_{n,w\in\mathbb{Z}} q^{(\frac{n+w}{2})^2}\bar{q}^{(\frac{n-w}{2})^2}. \tag{B.1.3}$$

The twisted partition functions (4.5.29) and (4.5.34) can be rewritten as a sum over Narain lattice (4.5.30) and (4.5.35) using the same method.

B.2 Group theory convention

We normalize the generators T^i ($i = 1,2,3$) of $su(2)$ Lie algebra such that,

$$[T^i, T^j] = \varepsilon^{ijk} T^k. \tag{B.2.1}$$

We match the generators of the vector representation of $SO(4)$ with $SU(2)_L \times SU(2)_R$ as,

$$T_L^1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad T_L^2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad T_L^3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \tag{B.2.2}$$

and,

$$T_R^1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad T_R^2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad T_R^3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}. \tag{B.2.3}$$

The symmetry operator $Q \in SO(4)$ can be written as,

$$Q = R^{\theta}\left(\frac{\pi}{2}\right) R^{\phi'}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.2.4}$$

where,

$$R^{\theta}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^{\phi'}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.2.5}$$

Notice that we can write,

$$R^{\theta}\left(\frac{\pi}{2}\right) = \exp\left(\frac{\pi}{2}(T_L^3 + T_R^3)\right), \quad R^{\phi'}\left(\frac{\pi}{2}\right) = \exp\left(\frac{\pi}{2}(T_L^2 - T_R^2)\right). \tag{B.2.6}$$

This allows us to break Q into tensor product of $Q_L \in SU(2)_L$ and $Q_R \in SU(2)_R$ where,

$$
\begin{aligned}
Q_L &= \exp(\frac{\pi}{2}t_L^3)\exp(\frac{\pi}{2}t_L^2) = \begin{pmatrix} \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}} & -\frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}} \\ \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}} & \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}} \end{pmatrix}, \\
Q_R &= \exp(\frac{\pi}{2}t_R^3)\exp(-\frac{\pi}{2}t_R^2) = \begin{pmatrix} \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}} & \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}} \\ -\frac{e^{\frac{i\pi}{4}}}{\sqrt{2}} & \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}} \end{pmatrix},
\end{aligned}
\tag{B.2.7}
$$

where in the adjoint representation of $SU(2)$, $t^i = -\frac{i}{2}\sigma^i$ satisfying $[t^i, t^j] = \varepsilon^{ijk}t^k$.

The charge conjugation C of $c = 1$ compact boson is chosen to be $R^{\phi'}(\pi)$ which can also be written as a tensor product $C = C_L \otimes C_R$ where,

$$
C_L = -C_R = -i\sigma^2. \tag{B.2.8}
$$

Notice that since Q_L, Q_R, C_L, C_R are all $SU(2)$ elements, their trace over a spin$-j$ representation is nothing but the character $\chi_j(\phi)$ of $SU(2)$ for some ϕ, given by,

$$
\chi_j(\phi) = \frac{\sin((2j+1)\phi)}{\sin\phi}. \tag{B.2.9}
$$

And the ϕ can be solved by matching the $j = \frac{1}{2}$ result known from the above representation. We find,

$$
\begin{aligned}
Tr_{V_j}(Q_L) &= \frac{\sin((2j+1)\pi/3)}{\sin(\pi/3)}, \quad Tr_{V_j}(Q_L C_L) = \frac{\sin((2j+1)2\pi/3)}{\sin(2\pi/3)}, \\
Tr_{\overline{V}_{\bar{j}}}(Q_R) &= \frac{\sin((2\bar{j}+1)\pi/3)}{\sin(\pi/3)}, \quad Tr_{\overline{V}_{\bar{j}}}(Q_R C_R) = \frac{\sin((2\bar{j}+1)2\pi/3)}{\sin(2\pi/3)}.
\end{aligned}
\tag{B.2.10}
$$

Next, we list the group theory result for $\overline{Q} = Q^2$. The decomposition is given by,

$$
\overline{Q}_L = \begin{pmatrix} -\frac{1}{2}-\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} \\ \frac{1}{2}+\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} \end{pmatrix}, \quad \overline{Q}_R = \begin{pmatrix} -\frac{1}{2}-\frac{i}{2} & \frac{1}{2}-\frac{i}{2} \\ -\frac{1}{2}-\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} \end{pmatrix}. \tag{B.2.11}
$$

We then have,

$$Tr_{V_j}(\overline{Q}_L) = Tr_{V_j}(\overline{Q}_L C_L) = \frac{\sin((2j+1)2\pi/3)}{\sin(2\pi/3)},$$
$$Tr_{\overline{V}_{\overline{j}}}(\overline{Q}_R) = Tr_{\overline{V}_{\overline{j}}}(\overline{Q}_R C_R) = \frac{\sin((2\overline{j}+1)2\pi/3)}{\sin(2\pi/3)}. \tag{B.2.12}$$

We will also need to construct the representation matrices of the group $SO(4)$ on the $(j,\overline{j})$ irrep as well, which can be done via the tensor product of the representation matrices of the group $SU(2)_L$ and $SU(2)_R$. The spin-j representation matrices of the $SU(2)$ can be acquired by exponentiate the generators $S^i_{(j)}$ where j labels the spin and $i = 1, \cdots, 3$. The $S^i_{(j)}$ is constructed from,

$$[S^+_{(j)}]_{mn} = \delta_{m+1,n}\sqrt{j(j+1)-(j+1-m)(j-m)}, \tag{B.2.13}$$

and,

$$S^1_{(j)} = \frac{S^+_{(j)} + (S^+_{(j)})^\dagger}{2i}, \quad S^2_{(j)} = -\frac{S^+_{(j)} - (S^+_{(j)})^\dagger}{2}, \quad S^3_{(j)} = [S^1_{(j)}, S^2_{(j)}]. \tag{B.2.14}$$

As one can check,

$$[S^k_{(j)}, S^l_{(j)}] = \varepsilon^{klm} S^m_{(j)}. \tag{B.2.15}$$

B.3 Detail on the basis in $\mathrm{Hom}_A(M \otimes_A N, L)$

In this section, we list our choices of basis in $\mathrm{Hom}_A(M \otimes_A N, L)$. Notice that if $\phi \in \mathrm{Hom}_A(M \otimes_A N, L)$, then,

$$a_1 \phi(m \otimes_A n) a_2 = \phi((a_1 m) \otimes_A (n a_2)), \quad \forall a_i \in A, m \in M, n \in N. \tag{B.3.1}$$

Let's consider the example $H_A(M^-_\eta \otimes_A M_{\overline{Q}}, M_{\overline{Q}})$. If we choose,

$$\phi_{M^-_\eta \otimes_A M_{\overline{Q}} \to M_{\overline{Q}}}(m^-_\eta \otimes_A m_{(134)}) = m_{(243)}, \tag{B.3.2}$$

then acting σ on left or right or both side on $\phi_{M_\eta^- \otimes_A M_{\overline{Q}} \to M_{\overline{Q}}}(m_\eta^- \otimes_A m_{(134)})$ allows us to determine,

$$\phi_{M_\eta^- \otimes_A M_{\overline{Q}} \to M_{\overline{Q}}} : \begin{pmatrix} m_\eta^- \otimes_A m_{(134)} \\ m_\eta^- \otimes_A m_{(123)} \\ m_\eta^- \otimes_A m_{(142)} \\ m_\eta^- \otimes_A m_{(243)} \end{pmatrix} \mapsto \begin{pmatrix} m_{(243)} \\ m_{(142)} \\ -m_{(123)} \\ -m_{(134)} \end{pmatrix}. \tag{B.3.3}$$

Hence, we will only list the action of $\phi \in \mathrm{Hom}_A(M \otimes_A N, L)$ on a single element which allows one to determine its action on the rest of the elements as follows,

$$\begin{aligned}
&\phi_{M_I^+ \otimes_A M_{HgH}^\rho \to M_{HgH}^\rho} : m_1^+ \otimes_A m_g^\rho \mapsto m_g^\rho, \quad \phi_{M_{HgH}^\rho \otimes_A M_1^+ \to M_{HgH}^\rho} : m_g^\rho \otimes_A m_1^+ \mapsto m_g^\rho, \\
&\phi_{M_I^- \otimes_A M_J^+ \to M_J^-} : m_1^- \otimes_A m_\mu^+ \mapsto m_\mu^-, \quad \phi_{M_I^- \otimes_A M_J^- \to M_J^+} : m_1^- \otimes_A m_\mu^- \mapsto m_\mu^+, \\
&\phi_{M_J^+ \otimes_A M_J^+ \to M_I^+} : m_\eta^+ \otimes_A m_\eta^+ \mapsto m_1^+, \quad \phi_{M_J^- \otimes_A M_J^+ \to M_I^-} : m_\eta^- \otimes_A m_\eta^+ \mapsto m_1^-, \\
&\phi_{M_J^+ \otimes_A M_J^- \to M_I^-} : m_\eta^+ \otimes_A m_\eta^- \mapsto m_1^-, \quad \phi_{M_J^- \otimes_A M_J^- \to M_I^-} : m_\eta^- \otimes_A m_\eta^- \mapsto m_1^+, \\
&\phi_{M_Q \otimes_A M_I^- \to M_Q} : m_{(143)} \otimes_A m_1^- \mapsto m_{(143)}, \quad \phi_{M_I^- \otimes_A M_Q \to M_Q} : m_1^- \otimes_A m_{(143)} \mapsto m_{(143)}, \\
&\phi_{M_Q \otimes_A M_J^+ \to M_Q} : m_{(143)} \otimes_A m_\eta^+ \mapsto m_{(234)}, \quad \phi_{M_J^+ \otimes_A M_Q \to M_Q} : m_\eta^+ \otimes_A m_{(143)} \mapsto m_{(124)}, \\
&\phi_{M_Q \otimes_A M_J^- \to M_Q} : m_{(143)} \otimes_A m_\eta^- \mapsto m_{(234)}, \quad \phi_{M_J^- \otimes_A M_Q \to M_Q} : m_\eta^- \otimes_A m_{(143)} \mapsto m_{(124)}, \\
&\phi_{M_{\overline{Q}} \otimes_A M_I^- \to M_{\overline{Q}}} : m_{(134)} \otimes_A m_1^- \mapsto m_{(134)}, \quad \phi_{M_I^- \otimes_A M_{\overline{Q}} \to M_{\overline{Q}}} : m_1^- \otimes_A m_{(134)} \mapsto m_{(134)}, \\
&\phi_{M_{\overline{Q}} \otimes_A M_J^+ \to M_{\overline{Q}}} : m_{(134)} \otimes_A m_\eta^+ \mapsto m_{(142)}, \quad \phi_{M_J^+ \otimes_A M_{\overline{Q}} \to M_{\overline{Q}}} : m_\eta^+ \otimes_A m_{(134)} \mapsto m_{(243)}, \\
&\phi_{M_{\overline{Q}} \otimes_A M_J^- \to M_{\overline{Q}}} : m_{(134)} \otimes_A m_\eta^- \mapsto m_{(142)}, \quad \phi_{M_J^- \otimes_A M_{\overline{Q}} \to M_{\overline{Q}}} : m_\eta^- \otimes_A m_{(134)} \mapsto m_{(243)},
\end{aligned} \tag{B.3.4}$$

and,

$$
\begin{aligned}
&\phi_{M_Q\otimes_A M_Q\to M_{\overline{Q}},1}: m_{(143)}\otimes_A m_{(143)}\mapsto m_{(134)}, \quad \phi_{M_Q\otimes_A M_Q\to M_{\overline{Q}},2}: m_{(143)}\otimes_A m_{(234)}\mapsto m_{(142)},\\
&\phi_{M_{\overline{Q}}\otimes_A M_{\overline{Q}}\to M_Q,1}: m_{(134)}\otimes_A m_{(134)}\mapsto m_{(143)}, \quad \phi_{M_{\overline{Q}}\otimes_A M_{\overline{Q}}\to M_Q,2}: m_{(134)}\otimes_A m_{(142)}\mapsto m_{(234)},\\
&\phi_{M_Q\otimes_A M_{\overline{Q}}\to M_I^+}: m_{(132)}\otimes_A m_{(123)}\mapsto \frac{1}{\sqrt{2}}m_1^+, \quad \phi_{M_Q\otimes_A M_{\overline{Q}}\to M_I^-}: m_{(132)}\otimes_A m_{(123)}\mapsto \frac{1}{\sqrt{2}}m_1^-,\\
&\phi_{M_Q\otimes_A M_{\overline{Q}}\to M_J^+}: m_{(132)}\otimes_A m_{(243)}\mapsto \frac{1}{\sqrt{2}}m_\eta^+, \quad \phi_{M_Q\otimes_A M_{\overline{Q}}\to M_J^-}: m_{(132)}\otimes_A m_{(243)}\mapsto \frac{1}{\sqrt{2}}m_\eta^-,\\
&\phi_{M_{\overline{Q}}\otimes_A M_Q\to M_I^+}: m_{(134)}\otimes_A m_{(143)}\mapsto \frac{1}{\sqrt{2}}m_1^+, \quad \phi_{M_{\overline{Q}}\otimes_A M_Q\to M_I^-}: m_{(134)}\otimes_A m_{(143)}\mapsto \frac{1}{\sqrt{2}}m_1^-,\\
&\phi_{M_{\overline{Q}}\otimes_A M_Q\to M_J^+}: m_{(142)}\otimes m_{(132)}\mapsto \frac{1}{\sqrt{2}}m_\eta^+, \quad \phi_{M_{\overline{Q}}\otimes_A M_Q\to M_J^-}: m_{(142)}\otimes m_{(132)}\mapsto \frac{1}{\sqrt{2}}m_\eta^-.
\end{aligned}
\tag{B.3.5}
$$

The $\phi(m\otimes_A n)$'s which can not be determined using (B.3.1) are set to zero.

Appendix C

When are Duality Defects Group-Theoretical?

C.1 Lagrangian subgroup for $\mathbb{Z}_N$ 2-form gauge theory in 5d

In this appendix, we provide an alternative derivation of the Lagrangian subgroups for $\mathbb{Z}_N$ 2-form gauge theory in 5d. Recall the Lagrangian subgroup consists of maximal number of pairs (e,m) such that any two pairs (e,m) and (e',m') satisfy $e^{-\frac{2\pi i}{N}(em'-me')\langle\sigma,\sigma'\rangle} = 1$. The maximal condition ensures that if we condense all the surface operators in the Lagrangian subgroup, every surface operator outside the Lagrangian subgroup braids non-trivially with at least one surface operator in the Lagrangian subgroup, and hence is projected out.

To proceed, we quote a theorem from [182, Sec. 4.2], which claims that the Lagrangian subgroups are classified by a subgroup Q of $H = \mathbb{Z}_N$ and a symmetric bilinear form Ψ on Q, that is, $\Psi : Q\times Q \to U(1)$ where

$$\Psi(h_1,h_2) = \Psi(h_2,h_1), \quad \Psi(h_1h_2,h_3) = \Psi(h_1,h_3)\Psi(h_2,h_3). \tag{C.1.1}$$

In this case, $Q = \mathbb{Z}_p$ for some integer p dividing N. To see all symmetric bilinear forms Ψ on $\mathbb{Z}_p$, let η denote the generator of $\mathbb{Z}_p$, and from (C.1.1) we find

$$\Psi(\eta^m,\eta^n) = \Psi(\eta,\eta)^{mn}. \tag{C.1.2}$$

Hence, Ψ is completely determined by $\Psi(\eta,\eta)$. First note that $\Psi(\eta,1)=\Psi(1,\eta)=1$.[1] Taking $m=p$ and $n=1$ in the above equation, we find $\Psi(\eta,\eta)=e^{\frac{2\pi i\ell}{p}}$ where $\ell=0,1,\cdots,p-1$. Hence, we will use the $\Psi_{p,\ell}$ to denote the symmetric bilinear form such that

$$\Psi_{p,\ell}(\eta,\eta)=e^{\frac{2\pi i}{p}\ell}. \tag{C.1.3}$$

The Ref. [182, Sec. 4.2] also specifies how the elements in the Lagrangian subgroup can be constructed from $(Q,\Psi_{p,\ell})$. Denote an arbitrary operator with electric and magnetic charge (x,y) as $\alpha^x\beta^y$, with $y\in\mathbb{Z}_N$ and $x\in\mathrm{Hom}(\mathbb{Z}_N,U(1))\simeq\mathbb{Z}_N$. We also define the standard pairing $\alpha(\beta)=e^{\frac{2\pi i}{N}}$. The key statement is that $\alpha^x\beta^y$ belongs to the Lagrangian subgroup specified by $(Q,\Psi_{p,\ell})$ if

- the magnetic charge takes value in Q, i.e. $y=\frac{N}{p}y'$ with $y'\in Q$;
- the electric charge is constrained by the pairing relation: $\alpha^z(\beta^y)=\Psi_{p,\ell}(b^y,b^z)$ for any $z=\frac{N}{p}z'$ with $z'\in Q=\mathbb{Z}_p$.

By definition, $\alpha^z(\beta^y)=e^{\frac{2\pi i}{N}\frac{N}{p}z'x}$. Using (C.1.3), the above condition gives

$$z'(x-\ell y')=0 \bmod p, \qquad \forall z'\in\mathbb{Z}_p. \tag{C.1.4}$$

This enforces $x=\ell y'+px'$ with $x'\in\mathbb{Z}_{N/p}$. Thus the charges are

$$(x,y)=(\ell y'+px',y'N/p)=x'(p,0)+y'(\ell,N/p), \qquad x'\in\mathbb{Z}_{N/p}, \quad y'\in\mathbb{Z}_p. \tag{C.1.5}$$

In other words, the charges in the Lagrangian subgroup are generated by

$$(\ell,N/p)\,, \quad (p,0)\,. \tag{C.1.6}$$

[1] This follows from the second condition in (C.1.1), where $h_1=h_2=1$ and $h_3=\eta$.

Notice that the two generators could be linearly dependent in general, but they nevertheless generated the full Lagrangian algebra.

C.2 Lagrangian subgroup with one generator

In this appendix, we focus on the case where the Lagrangian subgroup of 5d $\mathbb{Z}_N$ 2-form gauge theory is generated by a single surface operator $\widetilde{S}_{(p,q)}$, with the coprime condition $\gcd(p,q,N)=1$. This special case has been explored in [48].

$$\mathscr{A}_{(p,q)} = \{\widetilde{S}_{(kp,kq)} | k \in \mathbb{Z}_N, \gcd(p,q,N)=1\}. \tag{C.2.1}$$

Because of the coprime condition, it generates N distinct operators $\widetilde{S}_{(kp,kq)}$ with $k=0,1,...,N-1$. The trivial mutual braiding condition is clearly satisfied. Different pairs (p,q) may generate the same Lagrangian subgroup, for instance when $N=5$, $(p,q)=(1,1)$ and $(p',q')=(3,3)$ generate the same Lagrangian subgroup $\{\widetilde{S}_{(k,k)} | k \in \mathbb{Z}_5\}$. Such redundancy will not be a problem for our purposes.

To determine the counter term within $\widetilde{S}_{(kp,kq)}$, we substitute $(e,m)=k(p,q)$ in (5.4.4), and find

$$K_{kp,kq} = \begin{cases} \frac{2\pi}{N}\frac{N+1}{2}k^2 pq \bmod 2\pi, & \text{odd } N \\ \frac{2\pi}{2N}k^2 pq + \pi J_k \bmod 2\pi, & \text{even } N \end{cases}. \tag{C.2.2}$$

Substituting (C.2.2) into (5.4.5), we find that for odd N (5.4.5) is automatically satisfied, while for even N, J_k should satisfy

$$J_k = kJ \bmod 2 \tag{C.2.3}$$

where $J=0$ or 1. These two solutions are precisely the topological refinement discussed extensively in [48].

For even N, it is useful to extend the range of charges from $\mathbb{Z}_N$ to $\mathbb{Z}_{2N}$, so that shifting the topological refinement $J \to J+1$ can be replaced by shifting the electric or magnetic charge by N. For simplicity, take $k=1$. Since $\gcd(p,q,N)=1$, p,q can not be both even. Suppose p is odd. Then $J \to J+1$ can be achieved by shifting $q \to q+N$.

In summary, when there is a single generator, the Lagrangian subgroup of 5d $\mathbb{Z}_N$ 2-form gauge theory is generated by the surface operator

$$\widetilde{S}_{(p,q)}(\sigma) = \begin{cases} S_{(p,0)}(\sigma)S_{(0,q)}(\sigma)e^{\frac{2\pi i}{N}\frac{N+1}{2}pq\langle\sigma,\sigma\rangle}, & \text{odd } N \\ S_{(p,0)}(\sigma)S_{(0,q)}(\sigma)e^{\frac{2\pi i}{2N}pq\mathscr{P}(\sigma)+i\pi J\langle\sigma,\sigma\rangle}, & \text{even } N \end{cases} \tag{C.2.4}$$

where $J=0,1$ specifies the topological refinement in [48] on 4d non-spin spacetime manifold. When the 4d spacetime manifold is spin, the J dependence is trivialized.

$\mathbb{Z}_4^{\mathbf{em}}$ stable Lagrangian subgroup for odd N:

Under $\mathbb{Z}_4^{\text{em}}$, it is mapped to $\widetilde{S}_{(-vq,up)}(\sigma)$, which should also belong to the Lagrangian subalgebra, if stable. So stable Lagrangian subalgebra implies that there exists k, such that

$$k(p,q) = (-vq,up) \bmod N. \tag{C.2.5}$$

Given (C.2.5), the $\mathbb{Z}_4^{\text{em}}$ image of any other element $a(p,q)$ is also within the Lagrangian subalgebra, $a(-vq,up) = ka(p,q) \bmod N$. So the Lagrangian subalgebra is $\mathbb{Z}_4^{\text{em}}$ stable if and only if (C.2.5) is satisfied.

For which N,u,v do there exist k,p,q such that (C.2.5) holds? We first note that (C.2.5) implies

$$(k^2+1)p = 0 \bmod N, \qquad (k^2+1)q = 0 \bmod N. \tag{C.2.6}$$

Further combining with $\gcd(p,q,N)=1$, we have $xp+yq=1 \bmod N$. Multiplying x and y to the above two equations in (C.2.6), we find $k^2 = -1 \bmod N$. Conversely, given k satisfying $k^2 = -1 \bmod N$, we simply take $(p,q) = (vk,u)$ such that $\gcd(p,q,N) = 1$ and (C.2.5) is satisfied. Thus we have shown that (C.2.5) holds if and only if

$$k^2 + 1 = 0 \bmod N \tag{C.2.7}$$

holds, for any u, v. This corresponds to the special case $L = 1$ in Sec. 5.4.2. As commented there, (C.2.7) is precisely the condition where the $\mathbb{Z}_N^{(1)}$ duality defect in 4d with odd N can be mapped to an invertible defect discussed in [51, App.C], and also in [26] for prime N. Such odd N's belong to the red series listed in Tab. 5.1.

$\mathbb{Z}_4^{\mathrm{em}}$ stable Lagrangian subgroup for even N:

As pointed out in Sec. 5.4.1, the generator also depends on the choice of topological refinement $J = 0, 1$, and different choices can be packaged by extending the range of electric and magnetic charges from $\mathbb{Z}_N$ to $\mathbb{Z}_{2N}$, hence we take $p, q \in \mathbb{Z}_{2N}$ below for convenience. Under $\mathbb{Z}_4^{\mathrm{em}}$, the generator $\widetilde{S}_{(p,q)}(\sigma)$ is mapped to

$$\widetilde{S}_{(p,q)}(\sigma) \to \widetilde{S}_{(-vq,up)}(\sigma), \tag{C.2.8}$$

and the Lagrangian subgroup is $\mathbb{Z}_4^{\mathrm{em}}$ stable if and only if there exists k such that

$$(-vq, up) = k(p,q) \bmod 2N. \tag{C.2.9}$$

By applying the same argument as for (C.2.7), we find that (C.2.9) is again equivalent to

$$k^2 = -1 \bmod 2N. \tag{C.2.10}$$

There is no solution to this for any even N. However, if we restrict to 4d spin manifolds, different topological refinements are trivialized and the electric and magnetic charges obey

$p \sim p+N$ and $q \sim q+N$, i.e. the charges are back to $\mathbb{Z}_N$ valued. Thus (C.2.9) reduces to $(-vq, up) = k(p,q) \bmod N$, which is equivalent to $k^2 = -1 \bmod N$. This again reproduces the special case $L = 1$ in Sec. 5.4.2 as well as the results in [51, App. C].

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