Title
Quadratic ideals and Rogers–Ramanujan recursions

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Abstract. We give an explicit recursive description of the Hilbert series and Gröbner bases for the family of quadratic ideals defining the jet schemes of a double point. We relate these recursions to the Rogers-Ramanujan identity and prove a conjecture of the second author, Oblomkov and Rasmussen.

1. Introduction

In this paper, we study a family of quadratic ideals defining the jet schemes for the double point $D = \text{Spec} \ k[x]/x^2$. Here $k$ is a field of characteristic zero. Recall that the $(n-1)$-jet scheme of $X$ is defined as the space of formal maps $\text{Spec} \ k[t]/t^n \to X$. In the case of the double point, such a formal map is defined by a polynomial $x(t) = x_0 + x_1 t + \cdots + x_{n-1} t^{n-1}$, such that $x(t)^2 \equiv 0 \mod t^n$. By expanding this equation, we get a system of equations

$$f_1 = x_0^2, f_2 = 2x_0x_1, \ldots, f_n = \sum_{i=0}^{n-1} x_ix_{n-1-i}.$$  

We denote the defining ideal of Jet$^{n-1}D \subseteq \mathbb{A}^n$ by

$$I_n := \langle f_1, \ldots, f_n \rangle \subseteq R_n := k[x_0, \ldots, x_{n-1}].$$

The ring $R_n$ is $\mathbb{Z}_{\geq 0}$-graded by assigning the grading $(i, 1)$ to $x_i$. It is then clear that the ideal $I_n$ is bihomogeneous. Let

$$H_n(q, t) = \sum_{i,j \geq 0} \dim_k(R_n/I_n)_{i,j}q^it^j \in \mathbb{Z}[\![q, t]\!].$$

denote the bigraded Hilbert series for $R_n/I_n$. Our first main result is the following.

**Theorem 1.1.** The series $H_n(q, t)$ satisfies the recursion relation

$$H_n(q, t) = H_{n-2}(q, qt) + tH_{n-3}(q, q^2t)$$

with initial conditions

$$H_0(q, t) = 1, \ H_1(q, t) = 1 + t, \ H_2(q, t) = \frac{1}{1 - qt} + t.$$  

Using this recursion relation, we obtain explicit combinatorial formulas for $H_n(q, t)$:

**Theorem 1.2.** The Hilbert series $H_n(q, t)$ is given by the following explicit formula:

$$H_n(q, t) = \sum_{p=0}^{\infty} \frac{\binom{h(n,p)+1}{p}_q \cdot q^{p(p-1)t^p}}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-1}t)},$$

where $h(n, p) = \left\lfloor \frac{n-p}{2} \right\rfloor$.  


In the limit $n \to \infty$, we reprove the theorem of Bruschek, Mourtada and Schepers \cite{[1]}, which relates the Hilbert series of the arc space for the double point to the Rogers-Ramanujan identity. In fact, we refine their result by considering an additional grading, see equation (7.1). Similar results for $n = \infty$ were obtained by Feigin-Stoyanovsky \cite{[9]} \cite{[10]}. Lepowsky et al. \cite{[5]} \cite{[6]}, and the second author, Oblomkov and Rasmussen in \cite{[8]}. Although our approach to the computation of the Hilbert series is inspired by \cite{[4]}, it is quite different. The key result in \cite{[4]} shows that for $n = \infty$ the polynomials $f_k$ form a Gröbner basis of the ideal $I_\infty$. As we will see below, the Gröbner basis of the ideal $I_n$ for finite $n$ is larger and has a very subtle recursive structure. We completely describe such a basis in Theorems 1.2 and 1.6. In particular, we prove the following.

**Theorem 1.3.** Let $k > 2$. Then the reduced Gröbner basis for $I_n$ contains $\binom{n+2}{k-2}$ polynomials of degree $k$.

Our proof of Theorem 1.3 does not use Gröbner bases at all. First, by an explicit inductive argument in Theorem 2.2 we give a complete description of the first syzygy module for $f_i$. Then, we define a “shift operator” $S : R_n \to R_{n+1}$, which sends $x_i$ to $x_{i+1}$, and identify $I_n \cap x_0 R_n$ and $I_n / (I_n \cap x_0 R_n)$ with the images of $I_{n-3}$ and $I_{n-2}$ under appropriate powers of $S$. This implies the recursion relation in Theorem 1.4.

We also observe a recursive structure in the minimal free resolution of $R_n / I_n$. In particular, we prove the following:

**Theorem 1.4.** Let $b(i,n)$ denote the rank of the $i$-th term in the minimal free resolution for $R_n / I_n$, in other words the $i$-th Betti number. Then

$$b(i,n) = b(i, n-1) + b(i-1, n-3) + b(i-2, n-3).$$

As a consequence, we can compute the projective dimension of $R_n / I_n$.

**Corollary 1.5.** The projective dimension of $R_n / I_n$ equals $\lceil \frac{2n}{3} \rceil$.

**Remark 1.6.** It is easy to see that the reduced scheme $(\text{Jet}^{n-1}D)_{\text{red}}$ is a linear subspace given by the equations $x_0 = \ldots = x_i^{n+1} = 0$ and has dimension

$$\dim \text{Jet}^{n-1}D = n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil.$$

A more careful analysis of the gradings in Theorem 1.4 implies another formula for the series $H_n(q,t)$ which was first conjectured in \cite{[8]}.

**Theorem 1.7.** The Hilbert series of $R_n / I_n$ has the following form:

$$H_n(q,t) = \prod_{i=0}^{n-1} \frac{1}{(1 - qt)} \sum_{p=0}^{\infty} (-1)^p \prod_{k=0}^{p-1} \left(1 - q^{k+1}t\right) \times$$

$$q^{\frac{5p^2+2p}{2}} t_{2p} \binom{n-2p+1}{p} q^{\frac{5p^2+5p}{2}} t_{2p+2} \binom{n-2p-1}{p}.$$
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2. Ideals and syzygies

2.1. Ideals. Let \( R_n = k[x_0, \ldots, x_{n-1}] \) and \( f_k = \sum_{i=0}^{k-1} x_ix_{k-1-i} \). Define \( I_n \subseteq R_n \) to be the ideal generated by \( f_1, \ldots, f_n \). Let \( F_n \) be the free \( R_n \)-module with the basis \( e_1, \ldots, e_n \).

Consider the map \( \phi_n : F_n \to R_n \) given by the equation

\[
\phi_n(\alpha_1, \ldots, \alpha_n) = f_1\alpha_1 + \ldots + f_n\alpha_n.
\]

The \( R_n \)-module \( \ker(\phi_n) \) is called the first syzygy module of \( I_n \).

Lemma 2.1. One has

\[
\sum_{i=0}^{n} (n - 3i)x_if_{n+1-i} = 0.
\]

Proof. Indeed,

\[
\sum_{i=0}^{n} (n - 3i)x_if_{n+1-i} = \sum_{i+k+l=n} (n - 3i)x_ix_l.
\]

The coefficient at each monomial \( x_ix_l \) equals

\[
(n - 3i) + (n - 3k) + (n - 3l) = 3n - 3(i + k + l) = 3n - 3n = 0.
\]

For \( 0 < k < n \), define

\[
\mu_k := (-2kx_k, (-2k+3)x_{k-1}, \ldots, kx_0, 0, \ldots, 0) \in F_n.
\]

By (2.1), we have \( \phi_n(\mu_k) = 0 \). Denote also \( \nu_{ij} = f_ie_j - f_je_i \) (for \( i \neq j \)). It is clear that \( \phi_n(\nu_{ij}) = 0 \). The main result of this section is the following.

Theorem 2.2. The first syzygy module \( \ker(\phi_n) \) is generated by \( \mu_k \) and \( \nu_{i,j} \) over \( R_n \).

We prove Theorem 2.2 in Section 2.4.

2.2. The shift operator. We define a ring homomorphism \( S : R_n \to R_{n+1} \) by the equation \( S(x_i) = x_{i+1} \). Note that \( S \) is injective and we can uniquely write any polynomial in \( R_n \) in the form

\[
f = x_0f' + S(f''), \ f' \in R_n, f'' \in R_{n-1}.
\]

The following equation is clear from the definition and will be very useful below:

\[
f_n = 2x_0x_{n-1} + S(f_{n-2}).
\]

By abuse of notation, denote also \( S : F_n \to F_{n+2} \) the map which is given by

\[
S(\alpha_1, \ldots, \alpha_n) = (0, 0, S(\alpha_1), \ldots, S(\alpha_n)).
\]

Lemma 2.3. Let \( \alpha \in F_n \). Then \( \phi_{n+2}(S(\alpha)) \) is divisible by \( x_0 \) if and only if \( \phi_n(\alpha) = 0 \).
Proof. By \( (2.2) \) we have
\[
\phi_{n+2}(S(\alpha)) = \sum_{i=1}^{n} S(\alpha_i)f_{i+2} \equiv S\left( \sum_{i=1}^{n} \alpha_if_i \right) \mod x_0.
\]
Therefore \( \phi_{n+2}(S(\alpha)) \) is divisible by \( x_0 \) if and only if \( S(\sum \alpha_if_i) \) is divisible by \( x_0 \). But since no shift contains \( x_0 \), this happens if and only if
\[
S\left( \sum \alpha_if_i \right) = 0 \iff \sum \alpha_if_i = \phi_n(\alpha) = 0.
\]

Since \( \phi_n(\mu_k) = \phi_n(\nu_{ij}) = 0 \), by Lemma \( [2,3] \) the images of \( S(\mu_k) \) and \( S(\nu_{ij}) \) under \( \phi_{n+2} \) are divisible by \( x_0 \). The following lemma describes these images explicitly.

**Lemma 2.4.** One has \( \phi_{n+2}(S(\mu_k)) = (2k + 6)x_{k+3}f_1 + (2k + 3)x_{k+2}f_2 - (k + 3)x_0f_{k+4} \), \( \phi_{n+2}(S(\nu_{ij})) = 2x_0x_{j+1}f_{i+2} - 2x_0x_{i+1}f_{j+2} \).

**Proof.** By definition,
\[
S(\mu_k) = (0, 0, -2kx_{k+1}, (-2k + 3)x_k, \ldots, kx_1, 0, \ldots, 0) = \\
\mu_{k+3} + (2k + 6)x_{k+3}e_1 + (2k + 3)x_{k+2}e_2 - (k + 3)x_0e_{k+4},
\]
so
\[
\phi_{n+2}(S(\mu_k)) = (2k + 6)x_{k+3}f_1 + (2k + 3)x_{k+2}f_2 - (k + 3)x_0f_{k+4}.
\]
Also, \( S(\nu_{ij}) = S(f_i)e_{j+2} - S(f_j)e_{i+2} \), so
\[
\phi_{n+2}(S(\nu_{ij})) = S(f_i)f_{j+2} - S(f_j)f_{i+2} = (f_{i+2} - 2x_0x_{i+1})f_{j+2} - (f_{j+2} - 2x_0x_{j+1})f_{i+2} = \\
2x_0x_{j+1}f_{i+2} - 2x_0x_{i+1}f_{j+2}.
\]

**Corollary 2.5.** One has
\[
\phi_{n+2}(S(\mu_k)) = (2k + 3)x_{k+2}f_2 - (k + 3)x_0S(f_{k+2}) = kx_{k+2}f_2 - (k + 3)x_0S^2(f_k).
\]

**Proof.**
\[
\phi_{n+2}(S(\mu_k)) = (2k + 6)x_{k+3}f_1 + (2k + 3)x_{k+2}f_2 - (k + 3)x_0f_{k+4} = \\
(2k + 6)x_{k+3}f_1 + (2k + 3)x_{k+2}f_2 - (k + 3)(2x_0^2x_{k+3} + 2x_0x_{k+1}x_{k+2} + x_0S^2(f_k)) = \\
(2k + 3)x_{k+2}f_2 - (k + 3)x_0S(f_{k+2}) = kx_{k+2}f_2 - (k + 3)x_0S^2(f_k).
\]

**Example 2.6.** \( \mu_1 = (-2x_1, x_0) \), so \( S(\mu_1) = (0, 0, -2x_2, x_1) \), and
\[
\phi_4(S(\mu_1)) = -2x_2(2x_0x_2 + x_1^2) + x_1(2x_0x_3 + 2x_1x_2) = \\
2x_3x_0x_1 - 4x_0x_2^2 = x_3f_2 - 4x_0S^2(x_0^2).
\]

**Lemma 2.7.** The polynomial \( x_1S(f_{n-2}) \) can be expressed via \( f_1, \ldots, f_{n-1} \) modulo \( x_0 \).

**Proof.** We have \( (n - 3)x_0f_{n-2} + (n - 6)x_1f_{n-3} + \ldots + 2(n - 3)x_{n-2}f_0 = 0 \), so
\[
(n - 3)x_1S(f_{n-2}) + (n - 6)x_2S(f_{n-3}) + \ldots + 2(n - 3)x_{n-1}S(f_0) = 0.
\]
It remains to notice that \( S(f_i) \equiv f_{i+2} \mod x_0 \).

**Lemma 2.8.** Assume that \( \text{Ker}(\phi_{n-2}) \) is generated by \( \mu_k \) and \( \nu_{ij} \) and suppose that \( \phi_n(\alpha) \) is divisible by \( x_0 \). Then \( \alpha_n = Ax_0 + Bx_1 + \sum_{i=3}^{n-1} \gamma_if_i \) for some \( A, B \) and \( \gamma_i \).
Proof. As above, we can write $\alpha_i = x_0\alpha'_i + S(\alpha''_{i-2})$ for $i \geq 3$. Since $f_1$ and $f_2$ are divisible by $x_0$, we get

$$\phi_n(S(\alpha'')) = \sum_{i=3}^{n} S(\alpha''_{i-2})f_i \equiv \sum_{i=1}^{n} \alpha_if_i \equiv 0 \mod x_0.$$  

By Lemma 2.3 we get $\phi_{n-2}(\alpha'') = 0$. By the assumption, we can write

$$\alpha'' = \sum_{k<n-2} \beta_k\mu_k + \sum_{i<j\leq n-2} \gamma_{i,j}\nu_{ij}.$$  

Therefore

$$\alpha''_{n-2} = \beta_{n-1}x_0 + \sum_{j\leq n-3} \gamma_{j,n-2}f_j,$$  

and

$$\alpha_n = x_0\alpha'_1 + S(\alpha''_{n-2}) = x_0\alpha'_1 + S(\beta_{n-1})x_1 + \sum_{j\leq n-3} S(\gamma_{j,n-2})(f_{j+2} - 2x_0x_{j+1}).$$  

\square

2.3. Examples. Before proving Theorem 2.2 we would like to present the proof for $n \leq 4$.

Example 2.9. For $n = 2$ we have $f_1 = x_0^2$ and $f_2 = 2x_0x_1$, so the module of syzygies is clearly generated by $(-2x_1, x_0) = \mu_1$.

Example 2.10. Let $n = 3$, suppose that $\alpha_1f_1 + \alpha_2f_2 + \alpha_3f_3 = 0$. We can write $\alpha_3 = \alpha'_3x_0 + \alpha''_3$, where $\alpha''_3$ does not contain $x_0$. Since $f_1$ and $f_2$ are divisible by $x_0$ and $f_3 = 2x_0x_2 + x_1^2$, we get $x_1^2\alpha''_3 = 0$, so $\alpha''_3 = 0$. Now $\alpha = \frac{1}{2}\alpha'_3\mu_2 + \gamma$, where $\gamma$ is a syzygy between $f_i$ with $\gamma_3 = 0$. By the previous example, $\gamma$ is a multiple of $\mu_1$, so the module of syzygies is actually generated by $\mu_1$ and $\mu_2$.

Example 2.11. Let $n = 4$, suppose that $\alpha$ is a syzygy. We can write $\alpha_3 = \alpha'_3x_0 + \alpha''_3$ and $\alpha_4 = \alpha'_4x_0 + \alpha''_4$ where $\alpha''_i$ do not contain $x_0$. Similarly to the previous case, we obtain

$$\alpha''_3x_0^2 + \alpha''_4 \cdot 2x_1x_2 = 0.$$

This means that there exists some $\beta$ such that $\alpha''_3 = -2x_2\beta$ and $\alpha''_4 = x_1\beta$. Now

$$\alpha_1x_0^2 + 2\cdot 2x_0x_1 + (\alpha'_3x_0 - 2x_2\beta)(2x_0x_2 + x_1^2) + (\alpha'_4x_0 + x_1\beta)(2x_0x_3 + 2x_1x_2) = 0.$$  

The terms without $x_0$ cancel, and the linear terms in $x_0$ are the following:

$$x_0(2\alpha_2x_1 + \alpha'_3x_1^2 - 4x_2^2\beta + 2\alpha'_4x_1x_2 + 2\beta x_1x_3) = 0.$$  

Note that all terms but $-4x_2^2\beta$ are divisible by $x_1$, so $\beta$ is divisible by $x_1$, $\beta = mx_1$. Then

$$\alpha_4 = \alpha'_4x_0 + mx_1^2 = (\alpha'_4 - 2x_2m)x_0 + mf_3.$$  

By subtracting $m\nu_{3,4} + \frac{1}{3}(\alpha'_4 - 2x_2m)\mu_3$ from $\alpha$, we obtain a syzygy between $f_1, f_2, f_3$ and reduce to the previous case.
2.4. Syzygies. In this section, we prove Theorem 2.2 by induction on \( n \). The base cases were covered in Section 2.3. Suppose that \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \Ker(\phi_n) \), i.e., is a linear relation between \( f_1, \ldots, f_n \). As above, write \( \alpha_i = \alpha'_i x_0 + S(\alpha''_i) \) for \( i \geq 3 \). Without loss of generality, we can assume that \( \alpha'_i \) do not contain \( x_0 \) (otherwise we can subtract a multiple of \( \nu_{1,i} \)). Since

\[
\alpha'' = \sum_{i=3}^{n-1} \beta_{i+1} \mu_{i-2} + \sum_{3 \leq j < k \leq n, j \neq k} \beta_{j,k} v_{j-2,k-2}.
\]

Because

\[
S(\nu_{j-2,k-2}) = -S(f_{k-2}) e_j + S(f_{j-2}) e_k = \nu_{j,k} + 2x_0 x_k e_j - 2x_0 x_j e_k,
\]

without loss of generality we can assume \( \alpha'' = S(\sum_{i=3}^{n-1} \beta_{i+1} \mu_{i-2}) \). By Corollary 2.5 we get

\[
\phi_n(S(\mu_{i-2})) = -(i + 1)x_0 S(f_i) + (2i - 1)x_{i-1}f_2,
\]

hence

\[
\phi_n(\alpha) = \alpha_1 f_1 + (\alpha_2 + \sum_{i=3}^{n-1} (2i - 1) S(\beta_{i+1}) x_{i-1}) f_2 + \sum_{i=3}^{n} x_0 \alpha'_i f_i - \sum_{i=3}^{n-1} (i + 1) S(\beta_{i+1}) x_0 S(f_i) = 0.
\]

By collecting the terms linear in \( x_0 \), we get

\[
(\alpha_2 + \sum_{i=3}^{n-1} (2i - 1) S(\beta_{i+1}) x_{i-1}) 2x_1 + \sum_{i=3}^{n} \alpha'_i S(f_{i-2}) - \sum_{i=3}^{n-1} (i + 1) S(\beta_{i+1}) S(f_i) = 0,
\]

so

\[
\sum_{i=3}^{n} \alpha'_i S(f_{i-2}) - \sum_{i=3}^{n-1} (i + 1) S(\beta_{i+1}) S(f_i)
\]

is divisible by \( x_1 \), and

\[
\sum_{i=3}^{n} \alpha''_{i-2} f_{i-2} - \sum_{i=3}^{n-1} (i + 1) \beta_{i+1} f_i
\]

is divisible by \( x_0 \), where \( \alpha'_i = S(\alpha''_i) \). By Lemma 2.8 this implies

\[
\beta_n = Bx_0 + Cx_1 + \sum_{i=3}^{n-2} \gamma_i f_i
\]

for some constants \( B, C \). Now we can rewrite

\[
\alpha_n = \alpha'_n x_0 + S(\beta_n x_0) = \alpha'_n x_0 + Bx_1^2 + C x_1 x_2 + \sum_{i=3}^{n-3} \gamma_i x_1 (f_{i+2} - 2x_0 x_{n-1}) + \gamma_{n-2} x_1 S(f_{n-2}).
\]

Observe that \( x_1^2 = f_3 - 2x_0 x_2, x_1 x_2 = \frac{1}{2}(f_3 - 2x_0 x_3) \) and by Lemma 2.7 \( x_1 S(f_{n-2}) \) can be expressed via \( f_1, \ldots, f_{n-1} \) modulo \( x_0 \). In other words,

\[
\alpha_n = \delta x_0 + \sum_{i=3}^{n-1} \delta_i f_i
\]

for some coefficients \( \delta_i \). Then \( \alpha - \frac{1}{n-1} \delta \mu_{n-1} - \sum_{i=3}^{n-1} \delta_i \nu_{i,j} \) is a syzygy between \( f_1, \ldots, f_{n-1} \), so by the induction assumption it can be expressed as an \( R_{n-1} \)-linear combination of the \( \mu_i \) and \( \nu_{i,j} \).
Remark 2.12. The above proof shows that the syzygies $\nu_{1,k}$ and $\nu_{2,k}$ are not necessary, and can be expressed as linear combinations of other syzygies. Indeed, since the coefficients at $e_k$ are divisible by $x_0$, one can subtract an appropriate multiple of $\mu_{k-1}^{-1}$ and get a syzygy involving $e_1, \ldots, e_{k-1}$ only.

3. Hilbert Series

In this section, we prove Theorem 3.5 by studying the relation between the ideals $I_n$ and $x_0R_n$.

Lemma 3.1. One has

$$R_n/(x_0R_n + I_n) \simeq S(R_{n-2}/I_{n-2})[x_{n-1}]$$

as $R_n$-modules, the module structure on the right coming from $S : R_{n-1} \to R_n$.

Proof. We have $x_0R_n + I_n = \langle x_0, f_1, \ldots, f_n \rangle = \langle x_0, S(f_1), \ldots, S(f_{n-2}) \rangle$, so

$$R_n/(x_0R_n + I_n) = R_n/\langle x_0, S(f_1), \ldots, S(f_{n-2}) \rangle = S(R_{n-2}/I_{n-2})[x_{n-1}] .$$

Lemma 3.2. The subspace $x_0S^2(I_{n-3})[x_{n-1}]$ does not intersect the ideal $\langle f_1, f_2 \rangle$ in $R_n$. Furthermore, $x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle$ is an ideal in $R_n$ which is contained in $I_n \cap x_0R_n$.

Proof. Given a nonzero polynomial $g \in I_{n-3}$, the iterated shift $S^2(g)$ does not contain $x_0$ or $x_1$, so that $x_0S^2(g)$ is not contained in $\langle f_1, f_2 \rangle$. Furthermore, $I_{n-3}$ is stable under multiplication by $x_0, \ldots, x_{n-4}$, so $S^2(I_{n-3})$ is stable under multiplication by $x_2, \ldots, x_{n-2}$, and $x_0S^2(I_{n-3})[x_{n-1}]$ is stable under multiplication by $x_2, \ldots, x_{n-1}$. Multiplication by $x_0$ or $x_1$ sends the latter subspace to $\langle f_1, f_2 \rangle$, so $x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle$ is an ideal in $R_n$.

Finally, to prove that this ideal is contained in $I_n$, it is sufficient to prove that $x_0S^2(f_k) \in I_n$ for $k \leq n - 3$. On the other hand, by Corollary 2.5

$$x_0S^2(f_k) = \frac{1}{k+3} \phi_n(S(\mu_k)) \mod \langle f_1, f_2 \rangle .$$

Lemma 3.3. One has

$$I_n \cap x_0R_n = x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle .$$

Proof. By Lemma 3.2, the right hand side is a submodule of the left hand side, so it remains to prove the reverse inclusion. We have

$$f_i = 2x_0x_{i-1} + S(f_{i-2}) = 2x_0x_{i-1} + 2x_1x_{i-2} + S^2(f_{i-4}) .$$

Suppose that $\sum_{i=1}^n \alpha_if_i \in I_n \cap x_0R_n$. Then by Lemma 2.3

$$\alpha_n = Ax_0 + Bx_1 + \sum_j \gamma_jf_j = A'x_0 + B'x_1 + \sum_j \gamma_jS^2(f_{j-4}) .$$

Now by (2.1) and Corollary 2.5 $x_0f_n$ and $x_1f_n$ can be expressed as $R_n$-linear combinations of $f_1, \ldots, f_{n-1}$ and elements of $x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle$, so $\sum_{i=1}^n \alpha_if_i$ can be expressed as such a combination as well. Induction on $n$ finishes the proof.

Corollary 3.4. One has

$$x_0R_n/(I_n \cap x_0R_n) = x_0S^2(R_{n-3}/I_{n-3})[x_{n-1}] .$$
Proof. We have
\[ x_0R_n/(f_1, f_2) = x_0R_n/(x_0^2, x_0x_1) = x_0k[x_2, \ldots, x_{n-1}] = x_0S^2(R_{n-3})[x_{n-1}] \]
Therefore
\[ x_0R_n/(I_n \cap x_0R_n) = x_0R_n/(x_0S^2(I_{n-3})[x_{n-1}] + \langle f_1, f_2 \rangle) = x_0S^2(R_{n-3}/I_{n-3})[x_{n-1}] \]
\[ \square \]

**Theorem 3.5.** Let \( H_n(q, t) \) denote the bigraded Hilbert series of the quotient \( R_n/I_n \). Then one has the following recursion relation
\[ H_n(q, t) = \frac{H_{n-2}(q, qt) + tH_{n-3}(q, q^2t)}{1 - q^{n-1}t} \]
with initial conditions
\[ H_0(q, t) = 1, \quad H_1(q, t) = 1 + t, \quad H_2(q, t) = \frac{1}{1 - qt} + t. \]

**Remark 3.6.** This recursion is similar, but not identical to the various recursions considered by Andrews \([1, 2, 3]\) in his proofs of the Rogers-Ramanujan identity. It is also similar to the recursions recently considered by Paramonov \([12]\) in a different context.

**Proof.** We have an exact sequence
\[ 0 \to x_0R_n/(x_0R_n \cap I_n) \to R_n/I_n \to R_n/(x_0R_n + I_n) \to 0. \]
By Lemma \(3.1\) the Hilbert series of \( R_n/(x_0R_n + I_n) \) equals \( \frac{H_{n-2}(q, qt)}{1 - q^{n-1}t} \), and by Corollary \(3.4\) the Hilbert series of \( x_0R_n/(x_0R_n \cap I_n) \) equals \( \frac{tH_{n-3}(q, q^2t)}{1 - q^{n-1}t} \).
\[ \square \]

4. **Gröbner bases**

We will now compute Gröbner bases for the ideals \( I_n \). Recall that a Gröbner basis for an ideal \( I \) is a subset \( G = \{g_1, \ldots, g_s\} \subset I \) such that, for a chosen monomial ordering \(<\),
\[ \langle \text{LT}_<(g_1), \ldots, \text{LT}_<(g_s) \rangle = \text{LT}_<(I), \]
where \( \text{LT}_< \) denotes leading term.

Let us order the monomials in \( R_n \) in grevlex order, that is
\[ x^\alpha < x^\beta \]
if \(|\alpha| < |\beta|\) or \(|\alpha| = |\beta|\) and the rightmost entry of \( \alpha - \beta \) is negative.

**Remark 4.1.** In fact, any order refining the reverse lexicographic order will work, but for definiteness and its popularity in computer algebra systems we shall fix grevlex order throughout.

**Theorem 4.2.** Let \( G_1 = \{f_1\} \subseteq R_1, G_2 = \{f_1, f_2\} \subset R_2 \)
and recursively define the sets \( G_n, n \geq 3 \) as follows:
\[ G_n = x_0S^2(G_{n-3}) \cup \{f_1, f_2\} \cup \tilde{S}(G_{n-2}), \]
where \( \tilde{S} \) is a modified shift operator as explained below. Then \( G_n \) is a Gröbner basis for \( I_n \).
Remark 4.3. The notation requires explanation. Note that any $G_m$ is naturally a subset of $R_n$, $n \geq m$ so we can and will identify $G_m$ inside a larger polynomial ring without explicit mention. Furthermore, we denote by $x_0S^2(G_{n-3})$ the image of $G_{n-3}$ under $S^2 : R_{n-2} \to R_n$ multiplied by $x_0$. The “operator” $\tilde{S}$ is defined on elements $p \in I_{n-2}$ as follows: write $p = \sum_{i=1}^n \varphi_i f_i$, and let

$$\tilde{S}(p) = \sum_{i=1}^n S(\varphi_i)f_{i+2}.$$  

Note that by (2.22), we have $\tilde{S}(p) = S(p) + \sum_{i=1}^n x_0 x_{i+2} S(\varphi_i) \in I_{n+2}$. In particular, if $p \neq 0$ and $p$ is homogeneous then $\text{LT}_<(\tilde{S}(p)) = S(\text{LT}_<(p))$. Therefore the construction of $\tilde{S}(p)$ requires a choice if $\varphi_i$, but the leading term of the result does not depend on this choice.

Proof. We will proceed by induction. The base cases $n = 1, 2$ are clear because the ideals are monomial. Consider now the ideal $\text{LT}_<(I_n)$ generated by all the leading terms of elements of $I_n$. It is clear by Lemma 3.1 and the fact that $S$ respects the reverse lexicographic order that if $g \in I_n$ is not divisible by $x_0$, its leading term is the image of a leading term in $I_{n-2}$ under $S$. Since we assumed $G_{n-2}$ to be a Gröbner basis, we must have $\text{LT}_<(g)$ divisible by some monomial in $S(\text{LT}_<(G_{n-2}))$.

Similarly, if $g$ is divisible by $x_0$, we know by Lemma 3.2 and order preservation that its leading term is the image under $x_0S^2$ of a leading term in $I_{n-3}$ or divisible by $f_1, f_2$. By the induction assumption $\text{LT}_<(g)$ is then divisible by an element of $x_0S^2(\text{LT}_<(G_{n-3})) \cup \{f_1, f_2\}$. In particular, $\text{LT}_<(I_n) \subseteq (\text{LT}_<(G_n))$. But the reverse inclusion is clear, so we have

$$\text{LT}_<(I_n) = (\text{LT}_<(G_n))$$

as desired, and $G_n$ is a Gröbner basis for $I_n$. \hfill \Box

Example 4.4. We have

- $G_3 = \{f_1, f_2, f_3\}$
- $G_4 = \{f_1, f_2, f_3, f_4, x_0x_2^2\}$
- $G_5 = \{f_1, f_2, f_3, f_4, f_5, x_0x_2x_3\}$
- $G_6 = \{f_1, \ldots, f_6, x_0x_2^2 + 2x_0x_2x_4, 2x_1x_3^2 + 3x_0x_3x_4 - x_0x_2x_5\}$

Note that the last polynomial in $G_6$ can be identified with $\tilde{S}(x_0x_2^2) \in \tilde{S}(G_4)$. Indeed, $4x_0x_2^2 = 2x_2(2x_0x_2 + x_1^2) - x_1(2x_0x_3 + 2x_1x_2) + x_3(2x_0x_1) = 2x_2f_3 - x_1f_4 + x_3f_2$.

so

$$\tilde{S}(4x_0x_2^2) = 2x_3f_5 - x_2f_6 + x_4f_4 =$$

$$2x_3(2x_0x_4 + 2x_1x_3 + x_2^2) - x_2(2x_0x_5 + 2x_1x_4 + 2x_2x_3) + x_4(2x_0x_3 + 2x_1x_2) =$$

$$4x_1x_3^2 + 6x_0x_3x_4 - 2x_0x_2x_5.$$

Remark 4.5. The Gröbner basis constructed in Theorem 1.2 is far from being reduced. The following theorem describes the reduced basis implicitly.

Since all $G_n$ contain $\{f_1, \ldots, f_n\}$ and none of their leading terms divides one another, we can throw away other polynomials in $G_n$ in a controlled manner to obtain a minimal Gröbner basis. That is to say, if the leading terms of $G_n \setminus \{g\}$ still generate the leading ideal we are in business. Therefore after appropriate reduction [7 Proposition 6 on p. 92] we get a reduced Gröbner basis with the same leading terms.
Let us call a monomial $\prod x_i^{a_i}$ admissible if $a_i + a_{i+1} \leq 1$ for all $i$, that is, it is not divisible by $x_i^2$ or by $x_ix_{i+1}$.

**Theorem 4.6.** Fix $k > 2$. The leading terms of (t-)degree $k$ in a reduced Gröbner basis for $I_n$ have the form $m(x)LT_<(f_{n+k-2})$ where $m(x)$ is an admissible monomial of degree $k-2$ in variables $x_0, \ldots, x_{\left\lfloor \frac{n-4}{2} \right\rfloor}$. The number of degree $k$ polynomials in the reduced Gröbner basis equals $\left( \frac{n-1}{k-2} \right)$.

**Remark 4.7.** It is easy to see that there are no linear polynomials in the Gröbner basis (or in the ideal $I_n$), and $f_1, \ldots, f_n$ are the only quadratic polynomials in the reduced Gröbner basis.

**Proof.** We prove the statement by induction in $n$. Suppose that it is true for $G_{n-2}$ and $G_{n-3}$. By Theorem 4.2 the leading monomials in the degree $k$ part of $G_n$ consist of shifted degree $k$ monomials in $G_{n-2}$, and twice shifted degree $(k-1)$ monomials in $G_{n-3}$, multiplied by $x_0$.

Consider first the case $k = 3$. We will prove that the leading terms in the reduced Gröbner basis have the form $x_j LT_<(f_{n+1})$ for $j \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Indeed, in the first case we get $S(x_j LT_<(f_{n+1})) = x_j+1 LT_<(f_{n+1})$. In the second case we have to consider the polynomials $x_0S^2(f_i)$ for all $i \leq n-3$. Observe that for $i \leq n-4$ we get $LT_<(x_0S^2(f_i)) = x_0 LT_<(f_{i+4})$ and hence divisible by the leading term of $f_{i+4}$ and can be eliminated. For $i = n-3$ we get $LT_<(x_0S^2(f_{n-3})) = x_0 LT_<(f_{n+1})$.

Assume now that $k > 3$. In the first case we get

$$S(m(x)LT_<(f_{(n-2)+k-2})) = S(m(x))LT_<(f_{n+k-2}).$$

If $m(x)$ is an admissible monomial in $x_j$, $0 \leq j \leq \left\lfloor \frac{n-2+k-7}{2} \right\rfloor$ then $S(m(x))$ is an admissible monomial in $x_j$, $1 \leq j \leq \left\lfloor \frac{n-3+k-7}{2} \right\rfloor + 1 = \left\lfloor \frac{n+k-7}{2} \right\rfloor$.

In the second case we get

$$x_0S^2(m(x))LT_<(f_{(n-3)+(k-1)-2}) = x_0S^2(m(x))LT_<(f_{n+k-2}).$$

Now $S^2(m(x))$ is an admissible monomial in $x_j$, $2 \leq j \leq \left\lfloor \frac{n-3+(k-1)-7}{2} \right\rfloor + 2 = \left\lfloor \frac{n+k-7}{2} \right\rfloor$, so $x_0S^2(m(x))$ is also an admissible in a correct set of variables. In fact, all such monomials not divisible by $x_0$ appear from the first case, and the ones divisible by $x_0$ appear from the second case.

It is easy to see that none of these leading monomials are divisible by each other. Therefore after appropriate reduction we get a reduced Gröbner basis with the same leading terms.

Finally, we can count monomials of given degree $k$. The number of admissible monomials of degree $l$ in $s$ variables equals $\binom{s-l+1}{l}$, so the number of polynomials in $G_n$ of degree $k$ equals

$$\left( 1 + \left\lfloor \frac{n+k-7}{2} \right\rfloor - (k-2) + 1 \right) = \binom{n+k-6}{k-2}. \tag*{□}$$

**Example 4.8.** Let $n = 12$. The reduced Gröbner basis for $I_{12}$ contains quadratic polynomials $f_1, \ldots, f_{12}$. It also contains 5 cubic polynomials with leading terms

$$x_0x_6^2, x_1x_6^2, x_2x_6^2, x_3x_6^2, x_4x_6^2,$$

6 quartic polynomials with leading terms

$$x_0x_2x_6x_7, x_2x_3x_6x_7, x_0x_4x_6x_7, x_1x_3x_6x_7, x_1x_4x_6x_7, x_2x_4x_6x_7$$
and 4 quintic polynomials with leading terms
\[ x_0x_2x_4x_7^2, x_0x_2x_5x_7^2, x_0x_3x_5x_7^2, x_1x_3x_5x_7^2. \]
Observe that \( \text{LT}_< (f_{13}) = x_0^2 \), \( \text{LT}_< (f_{14}) = x_6x_7 \) and \( \text{LT}_< (f_{15}) = x_7^2 \).

5. Minimal resolution

In this section we describe the bigraded minimal free resolutions of \( I_n \) and \( R_n/I_n \). We write them as follows:

\[
0 \leftarrow I_n \leftarrow F(1, n) \leftarrow F(2, n) \leftarrow F(3, n) \cdots
\]

and

\[
0 \leftarrow R_n/I_n \leftarrow R_n = F(0, n) \leftarrow F(1, n) \leftarrow F(2, n) \leftarrow F(3, n) \cdots
\]

**Theorem 5.1.** Let \( F(i, n) \) be the \( i \)-th term in the minimal free resolution for \( I_n \). Then there is an injection \( F(i, n - 1) \hookrightarrow F(i, n) \), and

\[
F(i, n)/F(i, n - 1) \cong S(F(i - 1, n - 3)) \oplus x_0S(F(i - 2, n - 3))
\]

as \( R_n \)-modules, and the shift of a free \( R_n \)-module is as in \((2, 3)\). Note that the gradings in the right hand side are shifted by the bidegree of \( f_n \) (which equals \( q^{n-1}t^2 \)).

**Proof.** Observe that the ideal generated by \( f_1, \ldots, f_{n-1} \) in \( R_n \) is isomorphic to \( I_{n-1}[x_{n-1}] \), so its minimal resolution over \( R_n \) is identical to the one for \( I_{n-1} \) over \( R_{n-1} \) tensored over \( R_n \). Moreover, since \( I_n = \langle f_1, \ldots, f_n \rangle \), the minimal free \( R_n \)-resolution of \( I_{n-1}[x_{n-1}] \) is naturally a subcomplex of the minimal free resolution for \( I_n \). In other words, \( F(i, n - 1) \cong R_n \otimes_{R_{n-1}} R_n \) can be identified with a subspace in \( F(i, n) \), which we will by abuse of notation also denote \( F(i, n - 1) \). We have a short exact sequence

\[
0 \to F(i, n - 1) \to F(i, n) \to F(i, n)/F(i, n - 1) \to 0.
\]

From the long exact sequence in cohomology, it is easy to see that \( F(i, n)/F(i, n - 1) \) is acyclic in positive degrees. Now \( I_n = \langle f_1, \ldots, f_n \rangle \), so \( F(1, n)/F(1, n - 1) \cong R_n \) is generated by a single vector corresponding to \( f_n \). Furthermore, by Theorem 2.2 \( F(2, n) \) has generators corresponding to \( \mu_1, \ldots, \mu_{n-1} \) and \( \nu_{i,j} \) for \( 3 \leq i < j \leq n \), so \( F(2, n)/F(2, n - 1) \cong R_n^{n-2} \) is spanned by the basis elements corresponding to \( \mu_{n-1} \) and \( \nu_{i,n} \) for \( 3 \leq i \leq n - 1 \). The differential \( d : F(2, n) \to F(1, n) \) descends to \( d : F(2, n)/F(2, n - 1) \to F(1, n)/F(1, n - 1) \), sending \( \mu_{n-1} \) to \( x_0f_n \) and \( \nu_{i,n} \) to \( f_i \cdot f_n \).

Therefore, the quotient complex with terms \( F(i, n)/F(i, n - 1) \) is isomorphic to the minimal resolution of \( R_n/\langle x_0, f_3, \ldots, f_{n-1} \rangle = R_n/\langle x_0, S(f_1), \ldots, S(f_{n-3}) \rangle \). The latter is nothing but the (shifted) minimal resolution for \( I_{n-3} \) tensored with the two-term complex \( R_n \overset{x_0}{\longrightarrow} R_n \).

**Corollary 5.2.** Let \( b(i, n) \) denote the rank of \( F(i, n) \). Then

\[
(5.1) \quad b(i, n) = b(i, n - 1) + b(i - 1, n - 3) + b(i - 2, n - 3).
\]

**Corollary 5.3.** Let \( H_n(q, t) \) denote the Hilbert series for \( R_n/I_n \), and let \( \tilde{H}_n(q, t) = H_n(q, t) \prod_{i=0}^{n-1} (1 - q^it) \). Then \( \tilde{H}_n(q, t) \) satisfies the following recursion relation:

\[
(5.2) \quad \tilde{H}_n(q, t) = \tilde{H}_{n-1}(q, t) - q^{n-1}t^2(1 - t^2)\tilde{H}_{n-3}(q, qt).
\]

**Corollary 5.4.** The projective dimension of \( I_n \) equals \( \left\lceil \frac{2n}{3} \right\rceil - 1 \). The projective dimension of \( R_n/I_n \) equals \( \left\lceil \frac{2n}{3} \right\rceil \).
Proof. By definition, the projective dimension $\text{pd}(I_n)$ is equal to the length of the minimal free (or projective) resolution. By (5.1) we have $\text{pd}(I_n) = \text{pd}(I_{n-3}) + 2$. The minimal free resolutions for $I_1$, $I_2$ and $I_3$ are easy to compute:

$$I_1 \leftarrow \frac{(f_1)}{R_1}$$

$$I_2 \leftarrow \frac{(f_1 \ f_2)}{R_2} \leftarrow \frac{(-2x_1) \ x_0}{R_2}$$

$$I_3 \leftarrow \frac{(f_1 \ f_2 \ f_3)}{R_3} \leftarrow \frac{(-2x_0 \ -4x_2 \ x_1 \ -x_1 \ 0 \ 2x_0)}{R_3}$$

The minimal resolution of $R_n/I_n$ is one step longer than the one for $I_n$. \qed

6. COMBINATORIAL IDENTITIES

We define

$$\binom{a}{b}_q = \frac{(1 - q) \cdots (1 - q^a)}{(1 - q) \cdots (1 - q^b) \cdot (1 - q) \cdots (1 - q^{a-b})}.$$ If $a < b$, we set $\binom{a}{b}_q = 0$. The following lemma is well known.

Lemma 6.1. The following identities holds:

$$\binom{a}{b}_q + q^{b+1} \binom{a + 1}{b + 1}_q = \binom{a + 1}{b + 1}_q = q^{a-b} \binom{a}{b}_q + \binom{a}{b + 1}_q.$$

Proof. One has

$$\binom{a}{b + 1}_q = \frac{(1 - q^{a-b})}{(1 - q^{b+1})} \binom{a}{b}_q,$$

hence

$$\binom{a}{b}_q + q^{b+1} \binom{a + 1}{b + 1}_q = \binom{a}{b}_q \left(1 + q^{b+1} \frac{1 - q^{a-b}}{(1 - q^{b+1})} \right) = \binom{a}{b}_q \frac{1 - q^{a+1}}{(1 - q^{b+1})} = \binom{a + 1}{b + 1}_q.$$

\qed

Theorem 6.2. The Hilbert series $H_n(q, t)$ is given by the following explicit formula:

$$H_n(q, t) = \sum_{p=0}^{\infty} \frac{\binom{h(n,p)+1}{p}_q \cdot q^{p(p-1)t}p}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-1})},$$

where $h(n, p) = \left\lfloor \frac{n-2p}{2} \right\rfloor$.

Proof. By Theorem 3.3 it is sufficient to prove that the right hand side of (6.1) satisfies the recursion relation (3.1). Let us denote the $p$-th term in (6.1) by $H_{n,p}(q, t)$ so that $H_n(q, t) = \sum_p H_{n,p}(q, t)$. We have $h(n - 2, p) = h(n - 3, p - 1) = h(n, p) - 1$, so

$$H_{n-2,p}(q, t) = \frac{\binom{h(n,p)}{p}_q \cdot q^{p(p-1)t}p \cdot q^p}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-2t})},$$

$$H_{n-3,p-1}(q, t^2) = \frac{\binom{h(n,p)}{p-1}_q \cdot q^{(p-1)(p-2)t}p \cdot q^{2p^2}}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-2t})}.$$
therefore

\begin{equation}
H_{n-2,p}(q,t) + tH_{n-3,p-1}(q,q^2t) = \frac{q^{p(p-1)p}}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-2t})} \left[ q^{p} \left( \begin{array}{c} h(n,p) \\ p \end{array} \right)_q + \left( \begin{array}{c} h(n,p) \\ p-1 \end{array} \right)_q \right] = \frac{q^{p(p-1)p}}{(1 - q^{n-h(n,p)t}) \cdots (1 - q^{n-2t})} \left( h(n,p) + 1 \right) \left( \begin{array}{c} p \\ n \end{array} \right)_q = (1 - q^{n-1t})H_{n,p}(q,t).
\end{equation}

This proves (5.1), and the initial conditions are easy to check. \qed

The free resolution of $I_n$ gives another formula for the Hilbert series of $R_n/I_n$.

**Proposition 6.3.** Let $b(i,n)$, as above, denote the rank of $i$-th module in the free resolution of $R_n/I_n$. Then

\begin{equation}
b(i,n) = \sum_p \left[ \left( \begin{array}{c} n-2p+1 \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p \end{array} \right) + \left( \begin{array}{c} n-2p-1 \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p-1 \end{array} \right) \right]
\end{equation}

**Remark 6.4.** The terms in the first sum are nonzero if $p \leq (n+1)/3$ and $i/2 \leq p \leq i$. The terms in the second sum are nonzero if $p \leq (n-1)/3$ and $(i-1)/2 \leq p \leq (i-1)$.

**Proof.** Let

$$A(n,p,i) = \left( \begin{array}{c} n-2p+1 \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p \end{array} \right),
B(n,p,i) = \left( \begin{array}{c} n-2p-1 \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p-1 \end{array} \right).$$

Then

$$A(n-1,p,i) + A(n-3,p-1,i-1) + A(n-3,p-1,i-2) =
\left( \begin{array}{c} n-2p \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p \end{array} \right) + \left( \begin{array}{c} n-2p \\ p-1 \end{array} \right)_i \left( \begin{array}{c} p-1 \\ i-p \end{array} \right) + \left( \begin{array}{c} n-2p \\ p-1 \end{array} \right)_i \left( \begin{array}{c} p-1 \\ i-p-1 \end{array} \right) =
\left( \begin{array}{c} n-2p \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p \end{array} \right) + \left( \begin{array}{c} n-2p \\ p-1 \end{array} \right)_i \left( \begin{array}{c} p \\ i-p \end{array} \right) = \left( \begin{array}{c} n-2p+1 \\ p \end{array} \right)_i \left( \begin{array}{c} p \\ i-p \end{array} \right) = A(n,p,i).$$

Similarly, $B(n-1,p,i) + B(n-3,p-1,i-1) + B(n-3,p-1,i-2) = B(n,p,i)$, so the right hand side satisfies the recursion relation (5.1). It remains to check the base cases:

$$f(0,n) = 1 = \left( \begin{array}{c} n-1 \\ 0 \end{array} \right),
\quad
f(1,n) = n = \left( \begin{array}{c} n-1 \\ 1 \end{array} \right) + \left( \begin{array}{c} n-3 \\ 0 \end{array} \right),
\quad
f(2,n) = (n-1) + \left( \begin{array}{c} n-2 \\ 2 \end{array} \right) = \left( \begin{array}{c} n-1 \\ 1 \end{array} \right) + \left( \begin{array}{c} n-3 \\ 1 \end{array} \right) + \left( \begin{array}{c} n-3 \\ 2 \end{array} \right).$$

By Corollary 5.1 $b(i,n) = 0$ for $i > 2$ and $n \leq 3$. \qed

We have the following $(q,t)$-analogue of Proposition 6.3.
Proposition 6.5. Let \( \hat{b}(i, n) \) denote the bigraded Hilbert polynomial for the generating set in \( F(i, n) \). Then

\[
\begin{align*}
\hat{b}(i, n) &= \sum_{p>0} q^{\frac{5p^2 - 3p + (i-p)(i-p-1)}{2}} t^{2p+(i-p)} (n - 2p + 1) \binom{n - 2p + 1}{p} q^i \binom{i-p}{q} \\
&+ q^{\frac{5p^2 + 5p + (i-p)(i-p-1)}{2}} t^{2p+2+(i-p)} (n - 2p - 1) \binom{n - 2p - 1}{p} q^{i-p-1} \binom{i-p}{q}
\end{align*}
\]

Proof. The proof is completely analogous to the proof of Proposition 6.3, but we include it here for completeness. By Theorem 5.1 we have a recursion relation

\[
\hat{b}(i, n) = \hat{b}(i, n - 1) + q^{n-1} t^2 \hat{b}(i-1, n-3)(q, qt) + q^{n-1} t^3 \hat{b}(i-2, n-3)(q, qt).
\]

We need to prove that the right hand side of (6.3) satisfies (6.4). Let

\[
\hat{A}(n, p, i) = q^{\frac{5p^2 - 3p + (i-p)(i-p-1)}{2}} t^{2p+(i-p)} (n - 2p + 1) \binom{n - 2p + 1}{p} q^i \binom{i-p}{q}.
\]

Then

\[
\begin{align*}
\hat{A}(n-3, p-1, i-1)(q, qt) &= q^{\frac{5p^2 - 9p + 4(i-p)(i-p-1)}{2}} t^{2p-2+(i-p)} (n - 2p) \binom{n - 2p}{p-1} q^{i-1} \binom{i-p}{q}, \\
\hat{A}(n-3, p-1, i-2)(q, qt) &= q^{\frac{5p^2 - 9p + 4(i-p)(i-p-1)}{2}} t^{2p-2+(i-p-1)} (n - 2p) \binom{n - 2p}{p-1} q^{i-2} \binom{i-p-1}{q},
\end{align*}
\]

so

\[
\hat{A}(n-3, p-1, i-1)(q, qt) + t \hat{A}(n-3, p-1, i-2)(q, qt) = q^{\frac{5p^2 - 9p + 4(i-p)(i-p-1)}{2}} t^{2p-2+(i-p)} (n - 2p) \binom{n - 2p}{p-1} q^i \binom{i-p}{q}.
\]

Now

\[
\begin{align*}
\hat{A}(n-1, p, i) + q^{n-1} t^2 \hat{A}(n-3, p-1, i-1)(q, qt) + q^{n-1} t^3 \hat{A}(n-3, p-1, i-2)(q, qt) =
\end{align*}
\]

\[
\begin{align*}
q^{\frac{5p^2 - 3p + (i-p)(i-p-1)}{2}} t^{2p+(i-p)} &\left[ (n - 2p) \binom{n - 2p}{p} q^i \binom{i-p}{q} + q^{n-3p+1} \binom{n - 2p}{p-1} q^i \binom{i-p}{q} \right] \\
= q^{\frac{5p^2 - 3p + (i-p)(i-p-1)}{2}} t^{2p+(i-p)} (n - 2p + 1) \binom{n - 2p + 1}{p} q^i \binom{i-p}{q} = \hat{A}(n, p, i).
\end{align*}
\]

A similar recursion holds for \( \hat{B}(n, p, i) \). It remains to check the initial conditions:

\[
\begin{align*}
\hat{b}(0, n) &= 1, \\
\hat{b}(1, n) &= (t^2 + qt^2 + \ldots + q^{n-1}t^2) = q^{\frac{n-1}{2}} t^2 \binom{n-1}{1}, \\
\hat{b}(2, n) &= qt^3 \binom{n - 2}{2} = qt^3 \binom{n-1}{1} + q^5 t^4 \binom{n-3}{1} + q^7 t^4 \binom{n-3}{2}.
\end{align*}
\]

The following result was conjectured by the second author, Oblomkov and Rasmussen in \( \square \) Conjecture 4.1].
Theorem 6.6. The Hilbert series of $R_n/I_n$ has the following form:

\[
H_n(q,t) = \frac{1}{\prod_{i=0}^{n-1} (1-q^it)} \sum_{p=0}^{n-1} (-1)^p \prod_{k=0}^{p-1} (1-q^kt) \times \left( q^{\frac{5p^2-3p}{2}}i^p \left( \frac{n-2p+1}{p} \right)_q - q^{\frac{5p^2+5p}{2}}i^{p+2} \left( \frac{n-2p-1}{p} \right)_q \right).
\]

Proof. It is clear that $H_n(q,t) = \frac{1}{\prod_{i=0}^{n-1} (1-q^it)} \sum_{i=0}^{\infty} (-1)^i \hat{b}(i,n)$. The latter can be computed by (6.3), and it remains to use the identity

\[
\prod_{k=0}^{p-1} (1-q^kt) = \sum_{j=0}^{p} (-1)^j q^{j(j-1)/2} t^j \binom{p}{j}.
\]

\[\square\]

7. LIMIT AT $n \to \infty$

In the limit $n \to \infty$ both formulas for the Hilbert series simplify. Indeed, for fixed $p$ we have

\[
\lim_{n \to \infty} \binom{n}{p}_q = \frac{1}{(1-q)\cdots(1-q^p)},
\]

so we can take the limit of all the above results.

Proposition 7.1. The limit of the Hilbert series $H_n(q,t)$ has the following form:

\[
H_\infty(q,t) = \sum_{p=0}^{\infty} \frac{q^{p(p-1)} t^p}{(1-q)(1-q^2)\cdots(1-q^p)}.
\]

Proposition 7.2. The limit of the bigraded rank of the $i$-th syzygy module $F(i,n)$ equals

\[
\hat{b}(i, \infty) = \sum_{p>0} q^{\frac{5p^2-3p+i(p-1)}{2}} t^{p+(i-p)} \binom{p}{i-p} q^{\frac{5p^2+5p+i(p-1)}{2}} t^{2p+2+(i-p)} \binom{p}{i-p-1}.
\]

Proposition 7.3. The limit of the Hilbert series $H_n(q,t)$ has the following form:

\[
H_n(q,t) = \frac{1}{\prod_{i=0}^{\infty} (1-q^it)} \sum_{p=0}^{\infty} (-1)^p \prod_{k=0}^{p-1} \frac{1-q^kt}{1-q^{k+1}} \left( q^{\frac{5p^2-3p}{2}} t^{2p} - q^{\frac{5p^2+5p}{2}} t^{2p+2} \right).
\]

The equality between the right hand sides of (7.3) and (7.1) was proved in [10, Theorem 3.3.2(b)]. At $t=1$ and $t=q$ one recovers more familiar Rogers-Ramanujan identities.

The following proposition concerning Gröbner bases in the limit was proved first in [4, but we give an alternative proof here. In fact, [4] use a slightly different basis of Bell polynomials. Yet another proof can be obtained by taking the limit in Theorem 4.6.

Proposition 7.4. For $n \to \infty$ the polynomials $f_i$ form a Gröbner basis for the ideal $I_\infty$. 

Before embarking on the proof, we record the following lemmas concerning Gröbner bases here for the convenience of the reader.

**Lemma 7.5** ([7] Proposition 8 on p. 106). Given \((g_1, \ldots, g_s) \in F_s\), the S-pairs

\[
S_{ij} := \frac{\text{lcm}(\text{LT}_<(g_i), \text{LT}_<(g_j))}{\text{LT}_<(g_i)} e_i - \frac{\text{lcm}(\text{LT}_<(g_i), \text{LT}_<(g_j))}{\text{LT}_<(g_j)} e_j
\]

form a homogeneous basis for the syzygies on \(\{\text{LT}_<(g_1), \ldots, \text{LT}_<(g_s)\}\).

**Lemma 7.6** ([7] Proposition 9 on p. 107). Let \(I = \langle g_1, \ldots, g_s \rangle\). Then \(G = \{g_1, \ldots, g_s\}\) is a Gröbner basis for \(I\) if and only if every element of a homogeneous basis for the syzygies on \(\text{LT}_<(G)\) reduces to zero modulo \(G\).

**Lemma 7.7** ([7] Proposition 4 on p. 103). \(G = \{g_1, \ldots, g_s\} \subseteq R_n\), and suppose \(g_i, g_j \in G\) have relatively prime leading monomials. Then the S-polynomial

\[
S(g_i, g_j) := \phi_n(S_{ij}) = \frac{\text{lcm}(\text{LT}_<(g_i), \text{LT}_<(g_j))}{\text{LT}_<(g_i)} g_i - \frac{\text{lcm}(\text{LT}_<(g_i), \text{LT}_<(g_j))}{\text{LT}_<(g_j)} g_j
\]

reduces to zero modulo \(G\).

**Proof of Proposition 7.4.** Consider \(S(f_i, f_j)\). By Lemma 7.7 \(\gcd(\text{LT}_<(f_i), \text{LT}_<(f_j)) = 1\) implies that \(S(f_i, f_j)\) reduces to zero modulo \(\{f_k\}_{k=1}^\infty\). Write \(i = 2q + r\), where \(r = 0, 1\). Then \(\text{LT}_<(f_i) = x^2_q\) if \(i\) is even and \(\text{LT}_<(f_i) = 2x_q x_{q+1}\) if \(i\) is odd. So the only case we need to consider is \(j = i + 1\). In this case, we have

\[
\text{lcm}(\text{LT}_<(f_i), \text{LT}_<(f_{i+1})) = \begin{cases} 2x^2_q x_{q+1}, & \text{i even} \\ 2x_q x^2_{q+1}, & \text{i odd} \end{cases}
\]

Additionally

\[
S(f_i, f_{i+1}) = \begin{cases} 2x_{q+1} f_i - x_q f_{i+1}, & \text{i even} \\ x_q f_i - 2x_{q+1} f_{i+1}, & \text{i odd} \end{cases}
\]

But from (2.1) it follows that these S-pairs appear in the relations \(\phi_n(\mu_{n-1}) = 0\) for \(n \gg 0\). Since \(n = \infty\), we always have these relations in \(I_\infty\). Additionally, moving the S-pair to the right-hand side we reduce \(S(f_i, f_{i+1}) \equiv 0\) modulo \(\{f_k\}_{k=1}^\infty\). In particular, Lemma 7.6 implies that \(\{f_k\}_{k=1}^\infty\) is a Gröbner basis for \(I_\infty\). \(\square\)

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