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The problem of the transfer of energy from one end of a resonant cavity in the TM_{010} mode to the other is of great interest in the theory of linear accelerators, especially in regard to the coupling of oscillators and oscillator control problems for long accelerators. Since the general problem of the transient behavior of waveguides is extremely complex[1], a somewhat simpler approach was made. What was primarily desired was a time scale for the transfer of large quantities of energy. The assumption made was that the cavity was being excited at one end with a field such that H_ϕ had the radial dependence for the TM_{01} mode of the waveguide, with a time frequency ω.

The TM_{01} mode propagates in the guide with a propagation constant which depends on the frequency. Thus the waveguide represents a dispersive medium for the propagation of electromagnetic waves. The problem then resolves itself to finding how a signal travels in the waveguide. By a signal one means a wave train with at least a beginning. Such a wave train has a spectrum of frequencies in the Fourier sense and each of these frequencies travels with different velocities on the waveguide. The re-synthesis of these Fourier components at each distance and time will show how the signal propagates.

The mathematical analysis used in this report follows the treatment of anomalous dispersion by Sommerfeld[2] and Brillouin[3].

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[1]M. Cerrillo, Transient Phenomena in Waveguids, Massachusetts Inst. of Technology Technical Report No. 33, 1948.

[2]A. Sommerfeld, Ann. de Physik 44, 177 (1914).

[3]L. Brillouin, Ann. de Physik 44, 203, (1914).

Propagation of TM_{01} Mode in Cylindrical Wave Guide with Losses.

In order to get a finite velocity for energy transfer at the resonant frequency it is necessary to include some energy loss terms in Maxwell's equations. If this is done by assuming a loss term independent of frequency and having the correct value at the resonant frequency one can show that the propagation constant for such a guide is given by:

$$\gamma(\omega) = \frac{1}{c}\sqrt{\omega^2 - \omega_c^2 + 2i\omega\alpha}$$

where

$$\alpha = \frac{\omega_R}{Q} \qquad \omega_R = \text{resonant frequency}$$

ω_c = resonant frequency without losses.

$\omega_R = \omega_c$ if $\alpha = 0$.

If we consider only the TM_{01} mode of the guide and an infinitely long guide one can formally express $H_\phi(z,t)$ in terms of $H_\phi(0,t)$ by a Fourier transform.

$$H(z,t) = \int_{-\infty}^{\infty} A(\omega)\, e^{i\left(\omega t - \sqrt{\omega^2 - \omega_c^2 + 2i\alpha\omega}\,\frac{z}{c}\right)}\, d\omega$$

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(0,t)\, e^{i\omega t}\, dt$$

The simplest signal to feed in at $z = 0$ is:

$$t < 0 \quad H(0,t) = 0$$

$$t > 0 \quad H(0,t) = e^{i\omega t}$$

This type of a signal cannot be represented by a Fourier integral but can be represented by a Laplace transform. Making the usual substitution $s = i\omega$, one has:

$$H(z,t) = \int_{-\gamma-i\infty}^{\gamma+i\infty} \frac{1}{2\pi i}\,\frac{e^{\left(st - \sqrt{s^2 + \omega_c^2 + 2\alpha s}\,\frac{z}{c}\right)}}{s - i\omega}\, ds$$

In order to evaluate the integral we will have to integrate around a contour in the complex s plane. This integrand has a pole at $s = i\omega$ and branch points at the roots of the square root.

$$s = -\alpha \pm i\sqrt{\omega_c^2 - \alpha^2}$$

$$\omega_R = \sqrt{\omega_c^2 - \alpha^2}$$

$$s = -\alpha \pm i\omega_R$$

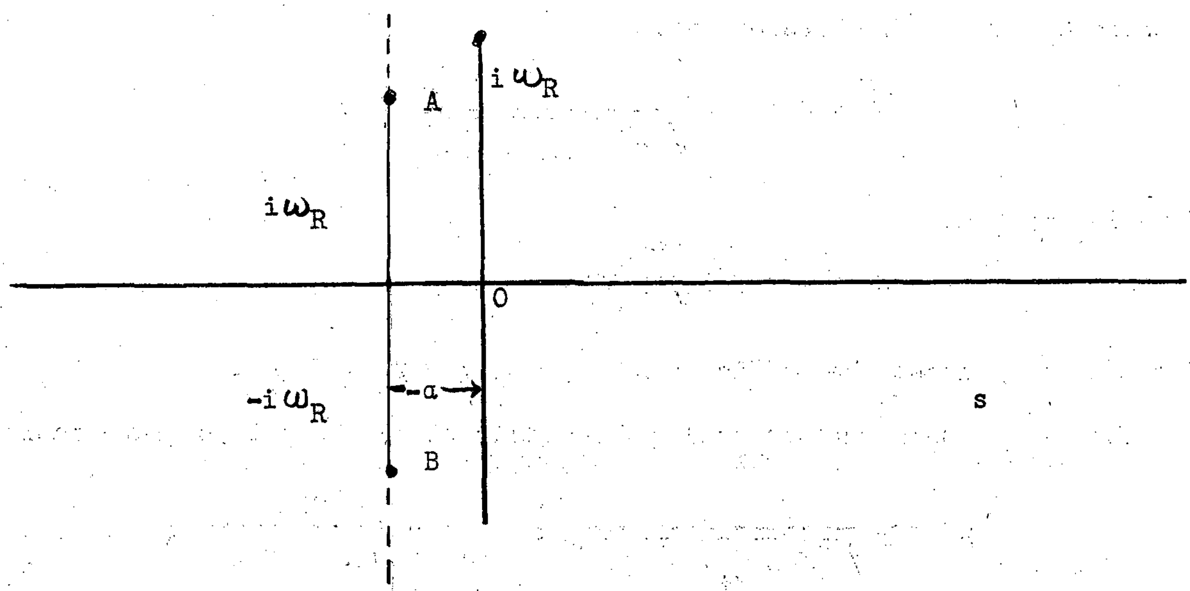

Fig. 1

These are shown in Fig. 1.

In order to make the function single valued in the plane a cut must be made between the two branch points. The contour must not cross the cut.

For large s the exponential becomes:

$$e^{s(t - \frac{z}{c})}$$ hence if $t < \frac{z}{c}$

one can form the contour to the right of all singularities and hence get 0 for the integral.

For $t > \frac{z}{c}$ the contour must be closed in the left half plane; this however includes both the pole and branch points. While the contribution from the pole is easy to evaluate, the contribution from the integral around the two branch points is not easily evaluated. Hence it is necessary to resort to a saddle point approximation for the integral.

The phase of the exponential is given by:

$$\phi = st - \sqrt{s^2 + \omega_c^2 + 2as} \quad \frac{z}{c}$$

the saddle points are therefore at:

$$\frac{d\phi}{ds} = 0 = t - \frac{s + a}{\sqrt{s^2 + \omega_c^2 + 2as}} \; \frac{z}{c}$$

If we let $\frac{ct}{z} = u$,

$$S_s = - a \pm i \frac{u\omega_R}{\sqrt{u^2 - 1}}$$

If we expand ϕ about the saddle point we have $\phi = \phi_o + \phi'' \xi^2$.

The second term must be real and negative along the path of integration.

$$\phi'' = - \frac{1}{\sqrt{s^2 + \omega_c^2 + 2as}} \; \frac{z}{c} + \frac{(s + a)^2}{(s^2 + \omega_c^2 + 2as)} \; \frac{z}{c}$$

at $s = - a \pm i \frac{u\omega_R}{\sqrt{u^2-1}}$

$$\phi'' = i \left(\frac{\sqrt{u^2 - 1}}{R}\right)\left(u^2 + 1\right) \frac{z}{c}$$

hence $\xi^2 = i\eta^2$

$$\xi = \frac{1 \pm i}{\sqrt{2}} \eta$$

hence we must integrate along a 45° line through the saddle point, until the integrand becomes very small, and then close the contour in such a way that there is negligible contribution along the remainder of the path. As long as we can keep our contour far away from the pole the integral can be approximated by taking the constant value of the phase at the saddle point and integrating over along the correct contour.

This gives a solution for $\frac{ct}{z} << \sqrt{Q}$

$$H(z,t) = \frac{1}{2\pi i} \frac{e^{S_s - \sqrt{S_s^2 + \omega_c^2 + 2\alpha S_s}\ \frac{z}{c}}}{S_s - i\omega_o} \; e^{-i\frac{\pi}{4}} \int_{-\infty}^{\infty} e^{-\frac{z\sqrt{u^2-1}}{c\omega_R}(u^2+1)\eta^2} \, d\eta$$

$$H(z,t) = \frac{1}{2\pi i} \frac{e^{-\alpha t} e^{i\left(\frac{u\omega_R}{\sqrt{u^2-1}} t - \frac{\omega_R}{\sqrt{\omega^2-1}} \frac{z}{c}\right)}}{i\left(\frac{u\omega_R}{\sqrt{u^2-1}} - \omega_o\right) - \alpha} \; e^{-i\frac{\pi}{4}} \sqrt{\frac{\pi\omega_R c}{\sqrt{u^2-1}(u^2+1)\, z}}$$

The case of interest is $\omega_o = \omega_R$ i.e., the input signal is at the resonant frequency. As u increases (t increasing at a given z) the saddle point moves from ∞ down the line $-\alpha + iy$ and approaches the branch point. However, at a time u which is such that the 45° line from the saddle point goes through the pole at $i\omega_R$, there will be a sudden increase in the field at z; mathematically this means we will get a direct contribution to the integral from the pole in our contour integration.

Physically this implies that we are finally receiving signals with a frequency which is within the band pass of the resonator. This time can be interpreted as the time it takes the bulk of the signal to arrive at z. The value of u for this to happen is given by:

$$\frac{u\omega_R}{\sqrt{u^2-1}} = \omega_R + \alpha$$

$$u = \sqrt{\frac{\omega_R}{2\alpha}}$$

if we ignore α compared to and $\frac{\alpha^2}{\omega_R}$ compared to $\frac{\alpha}{\omega_R}$

$$\alpha = \frac{\omega_R}{2Q}$$

hence $u = \sqrt{Q}$

$$\frac{z}{t} = v_s = \frac{c}{\sqrt{Q}}$$

For later times one has the first part of the solution decreasing as e^{-at} and now the pole must be integrated around a separate loop. The main contribution from this time on comes from the integral around the pole which gives the residual at the pole.

This can be evaluated very simply and gives a steady state solution of the form:

$$H(z,t) = e^{i\left(\omega_R t - \frac{\omega_R}{\sqrt{2Q}}\frac{z}{c}\right)} e^{-\frac{\omega_R}{\sqrt{2Q}}\frac{z}{c}}$$

which is a wave moving with a phase velocity:

$$v_P = \frac{c}{\sqrt{2Q}}$$

Since the transient solution we have found is good for only small values of $t > \frac{z}{c}$, no numerical evaluation of the transient part of the solution has been made. The important numbers one has obtained from this analysis are the velocity of the signal

$$\frac{v_s}{c} \sim \frac{1}{\sqrt{Q}}$$

and the steady state velocity,

$$\frac{v_P}{c} = \frac{1}{\sqrt{2Q}} .$$