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Hodge Structures with Hodge Numbers $(n, 0, \ldots, 0, n)$ and their Geometric Realizations

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Laure Bonahon Flapan

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Abstract of the Dissertation

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Professor Burt Totaro, Chair

The focus of this thesis is $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$, which are $\mathbb{Q}$-vector spaces V equipped with a decomposition into n-dimensional complex subspaces $V \otimes_{\mathbb{Q}} \mathbb{C} = V^{w,0} \oplus V^{0,w}$ such that the two subspaces $V^{w,0}$ and $V^{0,w}$ are conjugate to each other. In this first part of the thesis, we investigate the possible Hodge groups of simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. In particular, we generalize work of Moonen-Zarhin, Ribet, and Tankeev to completely determine the possible Hodge groups of such Hodge structures when n is equal to 1, 4, or a prime p. In addition, we determine, under certain conditions on the endomorphism algebra, the possible Hodge groups when $n = 2p$, for p an odd prime. A consequence of these results is that, for all powers of a simple complex $2p$-dimensional abelian variety whose endomorphism algebra is of a particular specified type, both the Hodge and General Hodge Conjectures hold. In the second part of the thesis, we investigate the geometry of a particular class of examples of smooth projective varieties whose rational cohomology realizes a $\mathbb{Q}$-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$.

The dissertation of Laure Bonahon Flapan is approved.

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2017

À EF et FB

qui m'ont montré ce que je voulais

et à M

qui m'a montré que je pouvais

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CHAPTER 1

Introduction

If X is a smooth complex projective variety, then, for any $w \geq 1$ there is a decomposition of the rational cohomology of X given by

$$H^w(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{p+q=w} H^{p,q},$$

such that the induced action of complex conjugation yields $\overline{H^{p,q}} = H^{q,p}$. Moreover, any codimension-p subvariety of X corresponds to a rational cohomology class which lies in the $H^{p,p}$ subspace of $H^{2p}(X, \mathbb{C}) \cong H^{2p}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ under the above decomposition. The long-standing Hodge conjecture predicts that all of the *Hodge classes*, meaning the elements of $H^{2p}(X, \mathbb{Q})$ which lie in $H^{p,p}$, come from $\mathbb{Q}$-linear combinations of subvarieties of X.

Although few cases of the Hodge Conjecture have been proven, one class of varieties for which it is hoped that the Hodge Conjecture may be more tractable is the class of abelian varieties; the reason being that, since abelian varieties are topologically tori, an abelian variety X has an isomorphism on its cohomology ring given by $H^\bullet(X, \mathbb{Q}) \cong \bigwedge^\bullet H^1(X, \mathbb{Q})$, meaning that all of the rational cohomology of the abelian variety X, and even of any power X^k, is determined by $H^1(X, \mathbb{Q})$.

Putting the above in a more abstract framework, a $\mathbb{Q}$-*Hodge structure* of weight $w \geq 1$ is a $\mathbb{Q}$-vector space V together with a decomposition into linear subspaces

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=w} V^{p,q}, \tag{1.1}$$

such that $\overline{V^{p,q}} = V^{q,p}$. If the weight w is even, the *Hodge classes* of the $\mathbb{Q}$-Hodge structure V are the elements $v \in V$ which lie in $V^{w/2,w/2}$ under the above decomposition.

If V is a $\mathbb{Q}$-Hodge structure, then for any $m, n \geq 1$, we get an induced $\mathbb{Q}$-Hodge structure on

$$T^{m,n} := V^{\otimes m} \otimes (V^*)^{\otimes n}.$$

The *Hodge group* $Hg(V)$ of V is then the connected reductive algebraic $\mathbb{Q}$-subgroup of $SL(V)$ whose invariants in the algebra $\bigoplus_{m,n} T^{m,n}$ are exactly the Hodge classes.

Thus, very roughly speaking, the established strategy for verifying the Hodge Conjecture for certain classes of abelian varieties is to show that the Hodge group of the $\mathbb{Q}$-Hodge structure $V = H^1(X, \mathbb{Q})$ is large, and thus the cohomology ring $H^\bullet(X, \mathbb{Q})$ has few Hodge classes, for which it is easy to verify the Hodge Conjecture. This was, for instance, the strategy used by Ribet [Rib83] and Tankeev [Tan82] to prove the Hodge Conjecture for all powers of a simple complex abelian variety of prime dimension.

As mentioned, the $\mathbb{Q}$-Hodge structure considered in the case of abelian varieties is $V = H^1(X, \mathbb{Q})$, which has Hodge decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$, where $V^{1,0}$ and $V^{0,1}$ each have dimension $n = \dim X$.

The topic of this thesis is the slightly more general situtation of $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$, meaning $\mathbb{Q}$-Hodge structures V with Hodge decomposition satisfying

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V^{w,0} \bigoplus V^{0,w},$$

where $V^{w,0}$ and $V^{0,w}$ have dimension n.

Aside from the case of complex abelian varieties, such $\mathbb{Q}$-Hodge structures arise naturally in geometry as, for instance, the degree 3 rational cohomology of a a rigid Calabi-Yau threefold. Moreover, we can produce a $\mathbb{Q}$-Hodge structure with Hodge numbers $(n, 0, n)$ geometrically as follows. If X is a smooth complex projective surface with maximal Picard number, namely the Picard number of X is equal to $h^{1,1}(X)$, then $H^2(X, \mathbb{Q})$ modulo the subspace of Hodge classes is a $\mathbb{Q}$-Hodge structure with Hodge numbers $(p_g(X), 0, p_g(X))$. For larger weights w, Arapura shows in [Ara16] how to construct a $\mathbb{Q}$-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ as a direct summand of the rational cohomology of a power

E^N of a CM elliptic curve and Bergeron-Millson-Moeglin show that $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ arise arithmetically as summands in the rational cohomology of Shimura varieties associated with a standard unitary group [BMM16, Corollary 6.2].

We thus say that a $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ *comes from geometry* if it is a summand of the rational cohomology of a smooth complex projective variety defined by a correspondence. Griffiths tranversality implies that a variation of Hodge structures of weight at least 2 with no two adjacent non-zero Hodge numbers is locally constant [Voi02, Theorem 10.2]. In particular, this applies to $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ and implies that only countably many $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ and weight at least 2 can come from geometry. Although there are concrete examples of such Hodge structures that come from geometry, such as those mentioned above, little is known about the subset of the period domain of all $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ consisting of those that come from geometry.

The first half of this thesis is concerned with $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ in general, meaning not necessarily those that come from geometry. In particular, we investigate what sort of Hodge groups such $\mathbb{Q}$-Hodge structures can have, generalizing many of the known results about Hodge groups of complex abelian varieties to the case of $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. In doing so, using new results of Totaro [Tot15, Theorem 4.1] and Green-Griffiths-Kerr [GGK12], one may in fact recover some new results about Hodge groups of complex abelian varieties in the context of the Hodge conjecture.

The second half of this thesis is concerned with a specific class of varieties constructed by Schreieder whose rational cohomology realizes a $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. In particular, we investigate the geometry of these varieties and show that they are equipped with a fibration over $\mathbb{P}^1$. Moreover, in the case when the varieties are of dimension 2, we show that they fall into a special class of surfaces investigated by Shioda [Shi72], called elliptic modular surfaces.

CHAPTER 2

Hodge Groups of Hodge Structures with Hodge Numbers $(n, 0, \dots, 0, n)$

2.1 Introduction

Recall that a $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \dots, 0, n)$ is a $\mathbb{Q}$-vector space V together with a decomposition into n-dimensional complex subspaces

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V^{w,0} \oplus V^{0,w}$$

such that the two subspaces $V^{w,0}$ and $V^{0,w}$ are conjugate to each other. Recently, Totaro [Tot15, Theorem 4.1] classified all of the possible endomorphism algebras of simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \dots, 0, n)$. These possible endomorphism algebras fall into four broad types, under a classification by Albert of division algebras with positive involution [Alb39], which are referred to as Type I, Type II, Type III, and Type IV.

Complex abelian varieties of dimension n are equivalent, up to isogeny, to polarizable $\mathbb{Q}$-Hodge structures of weight 1 with Hodge numbers (n, n). Thus, Totaro's result generalizes a result of Shimura's [Shi63, Theorem 5], which classifies all the possible endomorphism algebras of a complex abelian variety of fixed dimension.

If V is a $\mathbb{Q}$-Hodge structure of even weight w with decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=w} V^{p,q},$$

then the *Hodge classes* of V are the elements of V that lie in the subspace $V^{w/2,w/2}$ in this decomposition. These Hodge classes are the subject of the long-standing Hodge Conjecture.

The Hodge Conjecture states that if X is a smooth projective variety, then, for any $p \geq 1$, all of the Hodge classes of $H^{2p}(X, \mathbb{Q})$ are $\mathbb{Q}$-linear combinations of rational cohomology classes of algebraic subvarieties of codimension p in X.

For a rational sub-Hodge structure W of a $\mathbb{Q}$-Hodge structure V, meaning a subspace $W \subset V$ such that the summands $W^{p,q} := V^{p,q} \cap W \otimes_{\mathbb{Q}} \mathbb{C}$ endow W with a $\mathbb{Q}$-Hodge structure, the *level* of W is $l(W) = \max\{p - q \mid W^{p,q} \neq 0\}$. The General Hodge Conjecture states that for a rational sub-Hodge structure $W \subset H^w(X, \mathbb{Q})$ such that $l(W) = w - 2p$, there exists a Zariski-closed subset Z of codimension p in X such that W is contained in $\ker(H^w(X, \mathbb{Q}) \to H^w(X - Z, \mathbb{Q}))$.

The *Hodge group* of a $\mathbb{Q}$-Hodge structure V is a connected algebraic $\mathbb{Q}$-subgroup of $SL(V)$ whose invariants in the tensor algebra generated by V and its dual V^* are exactly the Hodge classes. Thus Hodge groups are objects of interest towards a better understanding of both the Hodge and General Hodge Conjectures.

In this chapter, we characterize the possible Hodge groups of simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ by applying Totaro's results about endomorphism algebras as well as by extending techniques of Moonen-Zarhin [MZ99], Ribet [Rib83], and Tankeev [Tan82] for determining Hodge groups and combining these with more recent work of Green-Griffiths-Kerr [GGK12] about domains of polarizable $\mathbb{Q}$-Hodge structures with specified Hodge group. In particular, Proposition 2.12.1, Theorem 2.12.2, and Theorem 2.12.3 determine the possible Hodge groups when n is equal to 1, a prime p, or 4, respectively. Moreover, for $n = 2p$, where p is an odd prime, Theorem 2.12.7 determines the possible Hodge groups when the endomorphism algebra is of Types I, II, or III as well as when the endomorphism algebra is of Type IV and has a particular type of action by an imaginary quadratic field.

The results for $n = 1$, p, and 4 generalize known results about the possible Hodge groups of simple complex abelian varieties, while the results for $n = 2p$ are new.

In particular, Ribet and Tankeev showed in [Rib83] and [Tan82], respectively, that the

Hodge group of a simple complex abelian variety X of prime dimension is always equal to the Lefschetz group of X, meaning, roughly speaking, that the Hodge group of X is always as large as possible. We show that, in fact, the above holds for all simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(p, 0, \ldots, 0, p)$. Moreover, for simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(2p, 0, \ldots, 0, 2p)$, we establish conditions under which the above holds and conditions under which it does not.

Moonen and Zarhin addressed the case of simple complex abelian fourfolds in [MZ99]. They showed that the Hodge group of a simple complex abelian fourfold X is equal to the Lefschetz group of X except in the cases when the endomorphism algebra of X is $\mathbb{Q}$ or a CM field of degree 2 or 8 over $\mathbb{Q}$. In all of these exceptional cases, an additional group is also possible.

We show that for a simple polarizable $\mathbb{Q}$-Hodge structure V with Hodge numbers $(4, 0 \ldots, 0, 4)$, an additional group other than the Lefschetz group can arise as the Hodge group in all the same exceptional cases that Moonen-Zarhin determined. When the endomorphism algebra of V is a CM field, the additional possible group is analogous to the one found by Moonen and Zarhin. In the case when the endomorphism algebra of V is equal to $\mathbb{Q}$, Moonen and Zarhin used a construction of Mumford's [Mum69, Section 4] to show that the additional group $SL(2) \times SO(4)$ arises. We show that the analogous group for an even-weight simple polarizable Hodge structure with Hodge numbers $(4, 0, \ldots, 0, 4)$ does not arise, however that a different non-Lefschetz group, namely the group $SO(7)$ acting by the spin representation does arise.

The *Hodge ring* of a smooth projective variety X is defined by

$$\mathcal{B}^\bullet(X) = \bigoplus_{l \geq 0} \left(H^{2l}(X, \mathbb{Q}) \cap H^{l,l} \right).$$

Most of the proven cases of the Hodge Conjecture and General Hodge Conjecture, particularly for abelian varieties, are obtained by proving that the *divisor ring* $\mathcal{D}^\bullet(X)$ of X, meaning the $\mathbb{Q}$-subalgebra of $\mathcal{B}^\bullet(X)$ generated by divisor classes, is equal to $\mathcal{B}^\bullet(X)$. Theorem 2.12.7 about simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(2p, 0, \ldots, 0, 2p)$ allows one to use this approach in Corollary 2.13.5 to prove the Hodge and General Hodge Conjectures for all powers of a simple $2p$-dimensional abelian variety whose endomorphism algebra is of Type I or II in Albert's classification.

Mumford gave the first construction of a variety X such that $\mathcal{B}^\bullet(X) \neq \mathcal{D}^\bullet(X)$ (see [Poh68]). Weil later observed that the exceptional Hodge classes in Mumford's example, meaning the Hodge classes that did not come from divisor classes, were still, in some sense, described by the endomorphisms of the rational cohomology of X [Wei79] (see Section 2.13). The exceptional Hodge classes satisfying the property Weil described came to be known an *Weil classes*. We show in Corollary 2.13.4 that for any simple abelian variety X of dimension $2p$ satisfying the hypotheses of Theorem 2.12.7, the Hodge ring $\mathcal{B}^\bullet(X^k)$ of any power X^k is generated by divisors and Weil classes.

The organization of this chapter is as follows. In Section 2.2, we introduce the necessary background on $\mathbb{Q}$-Hodge structures and their endomorphism algebras. Sections 2.3-2.8 focus on preliminary properties of Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. Sections 2.9-2.11 then focus on results about the Hodge groups of these Hodge structures depending on the endomorphism algebra type in Albert's classification. The main results of the paper occur in Section 2.12, where the possible Hodge groups for $n = 1$, 4, p, and $2p$ are characterized. Section 2.13 then gives consequences of the results in Section 2.12 in the context of the Hodge Conjecture and General Hodge Conjecture for simple abelian varieties of dimension $2p$.

2.2 Background on Hodge Structures

2.2.1 Hodge Structures and Polarizations

A $\mathbb{Q}$-*Hodge structure* V is a finite dimensional $\mathbb{Q}$-vector space together with a decomposition into linear subspaces

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q},$$

such that $\overline{V^{p,q}} = V^{q,p}$ and such that the *weight grading* $p+q$ is defined over $\mathbb{Q}$. Unless stated otherwise, the term "Hodge structure" in this paper will always refer to a $\mathbb{Q}$-Hodge structure. A $\mathbb{Q}$-Hodge structure V is of *(pure) weight* w if $V^{p,q} = 0$ whenever $p+q$ is not equal to w. For example, any smooth complex projective variety X has a Hodge structure of (pure) weight w on $H^w(X, \mathbb{Q})$.

Alternatively, a $\mathbb{Q}$-Hodge structure can be defined as a finite dimensional $\mathbb{Q}$-vector space V together with a homomorphism of $\mathbb{R}$-algebraic groups

$$h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to GL(V \otimes_{\mathbb{Q}} \mathbb{R}), \tag{2.1}$$

where $R_{\mathbb{C}/\mathbb{R}}$ denotes the Weil restriction functor from $\mathbb{C}$ to $\mathbb{R}$. Here $h(z)$ acts on $V^{p,q}$ as multiplication by $z^{-p}\overline{z}^{-q}$.

A $\mathbb{Q}$-Hodge structure V has *Hodge numbers* $(a_0, a_1, \ldots, a_w)$ if V has weight $w \geq 0$ and

$$\dim_{\mathbb{C}} V^{i,w-i} = \begin{cases} a_i & \text{for } 0 \leq i \leq w \\ 0 & \text{otherwise.} \end{cases}$$

A *polarization* of a $\mathbb{Q}$-Hodge structure V of weight w is a bilinear form $\langle , \rangle : V \times V \to \mathbb{Q}$ that is alternating if w is odd, symmetric if w is even, and whose extension to $V \otimes_{\mathbb{Q}} \mathbb{C}$ satisfies:

1. $\langle V^{p,q}, V^{p',q'} \rangle = 0$ if $p' \neq w-p$
2. $i^{p-q}(-1)^{\frac{w(w-1)}{2}}\langle x, \overline{x} \rangle > 0$ for all nonzero $x \in V^{p,q}$.

Note, for instance, that for X a smooth complex projective variety, a choice of ample line bundle on X will determine a polarization on the Hodge structure $H^w(X, \mathbb{Q})$. The category of polarizable $\mathbb{Q}$-Hodge structures is a semisimple abelian category [Moo99, Theorem 1.16]. All Hodge structures considered in this paper will be polarizable.

2.2.2 The Endomorphism Algebra

Let V be a simple $\mathbb{Q}$-Hodge structure with polarization $\langle,\rangle : V \times V \to \mathbb{Q}$. Then the endomorphism algebra $L = \mathrm{End}_{\mathbb{Q}-\mathrm{HS}}(V)$ of V as a $\mathbb{Q}$-Hodge structure is a division algebra over $\mathbb{Q}$ with an involution $a \to \overline{a}$, given by $\langle ax, y\rangle = \langle x, \overline{a}y\rangle$ for all $x, y \in V$. This involution $^-$ is called the *Rosati involution*.

The Rosati involution is a *positive involution*, meaning that, if $\Sigma(L)$ is the set of embeddings of the endomorphism algebra L into $\mathbb{C}$, then for every element $\sigma \in \Sigma(L)$, the reduced trace $\sigma(\mathrm{tr}_{\mathbb{Q}}^{L}(x\overline{x}))$ is positive as an element of $\mathbb{R}$ for all nonzero $x \in V$ [Moo99, Remark 1.20]. It follows that if L is a field, then L is either a totally real or a CM field, where a *CM field* means a totally imaginary quadratic extension of a totally real number field. Namely, if L is a field, then the Rosati involution on L just corresponds to complex conjugation.

Letting F_0 be the center of L and F the subfield of F_0 fixed by the Rosati involution, Albert's classification of division algebras over a number field that have positive involution [Alb39, Chapter X, §11] yields that L is one of the following four types:

1. Type I: $L = F$ is totally real
2. Type II: L is a totally indefinite quaternion algebra over the totally real field F
3. Type III: L is a totally definite quaternion algebra over the totally real field F
4. Type IV: L is a central simple algebra over the CM field F_0.

2.2.3 The Mumford-Tate Group and Hodge Group

The *Mumford-Tate group* $MT(V)$ of a polarizable $\mathbb{Q}$-Hodge structure V is the $\mathbb{Q}$-Zariski closure of the homomorphism $h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to GL(V \otimes_{\mathbb{Q}} \mathbb{R})$ which defines the Hodge structure on V. Note that the Mumford-Tate group is thus a connected group. Define the cocharacter

$$\mu : \mathbb{G}_m \to R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$$

to be the unique cocharacter such that $z \circ \mu$ is the identity in $\mathrm{End}(\mathbb{G}_{\mathrm{m}})$ and $\overline{z} \circ \mu$ is trivial. Then the Mumford-Tate group of V may be alternatively described as the smallest $\mathbb{Q}$-algebraic group contained in $GL(V)$ such that

$$h \circ \mu : \mathbb{G}_{m,\mathbb{C}} \to GL(V \otimes_{\mathbb{Q}} \mathbb{C})$$

factors through $MT(V)_{\mathbb{C}}$.

Instead of working with the Mumford-Tate group, we will generally work with a slightly smaller connected group, called the *Hodge group* $Hg(V)$ of V. The Hodge group is the $\mathbb{Q}$-Zariski closure of the restriction of the homomorphism h to the circle group

$$U_1 = \ker(\mathrm{Norm} : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbb{G}_m).$$

If the Hodge structure V is of weight 0, then $Hg(V)$ and $MT(V)$ coincide. If V is of nonzero weight, then $MT(V)$ contains $\mathbb{G}_m$, and in fact, is equal to the almost direct product $\mathbb{G}_m \cdot Hg(V)$ in $GL(V)$.

The category of polarizable $\mathbb{Q}$-Hodge structures is a semi-simple Tannakian category, which, in particular, implies that $MT(V)$ and $Hg(V)$ are reductive $\mathbb{Q}$-groups [Moo99, Theorem 1.16].

Remark 2.2.1. The key property of the Hodge group $Hg(V)$ of a polarizable $\mathbb{Q}$-Hodge structure V is that $Hg(V)$ is the subgroup of $SL(V)$ preserving the Hodge classes in all the $\mathbb{Q}$-Hodge structures T formed as finite direct sums of spaces of the form $T^{k,l} := V^{\otimes k} \otimes (V^*)^{\otimes l}$, where V^* denotes the dual of the vector space V [GGK12, I.B.1].

Remark 2.2.2. If V is a polarizable $\mathbb{Q}$-Hodge structure with Hodge group $\{1\}$, then $h(z)$ acts as the identity on nonzero $V^{p,q}$ in the decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$. However, by definition $h(z)$ acts on $V^{p,q}$ as multiplication by $z^{-p}\overline{z}^{-q}$. Hence, if $Hg(V) = \{1\}$, then $V^{p,q}$ is zero for all p, q such that $p \neq q$.

Remark 2.2.3. If the endomorphism algebra L of a polarizable $\mathbb{Q}$-Hodge structure V has no simple factors of Type IV, then the Hodge group of V is semisimple [Moo99, Proposition 1.24].

2.2.4 The Lefschetz Group

Since elements of the endomorphism algebra $L = \mathrm{End}_{\mathbb{Q}-\mathrm{HS}}(V)$ of a polarizable $\mathbb{Q}$-Hodge structure V preserve the Hodge decomposition, elements of L may be viewed as Hodge classes of the $\mathbb{Q}$-Hodge structure $\mathrm{End}_{\mathbb{Q}}(V) \cong V \otimes V^*$. Remark 2.2.1 yields that these Hodge classes are the elements of $\mathrm{End}_{\mathbb{Q}}(V)$ which are invariant under the action of the Hodge group $Hg(V)$. Namely we have

$$L = [\mathrm{End}_{\mathbb{Q}}(V)]^{Hg(V)}.$$

In particular, the Hodge group of V is contained in the connected component of the centralizer of L in the $\mathbb{Q}$-group $SL(V)$.

Let the *Lefschetz group* $Lef(V)$ of V be the connected component of the centralizer of L in:

$$\begin{cases} Sp(V) & \text{if } V \text{ is of odd weight} \\ SO(V) & \text{if } V \text{ is of even weight.} \end{cases}$$

It should be noted that the definition of the Lefschetz group used here corresponds with the definition used by Murty in [Mur84, Section 2], which is the connected component of the identity in the definition used by Murty in [Mur00, Section 3.6.2].

Remark 2.2.4. If $\langle , \rangle$ is a polarization on V, then $Hg(V)$ preserves $\langle , \rangle$, which yields the inclusions

$$Hg(V) \subseteq Lef(V).$$

2.3 Hodge Structures with Hodge Numbers $(n, 0, \ldots, 0, n)$

We now specify our discussion to the main subject of this paper, namely polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. The first crucial observation to make about these Hodge structures is the following.

Remark 2.3.1. There is an equivalence of categories between the category of $\mathbb{Q}$-Hodge structures of weight w and Hodge numbers $(n, 0 \ldots, 0, n)$, and the category of $\mathbb{Q}$-Hodge structures of weight 1 and Hodge numbers (n, n). This equivalence is given simply by identifying $V^{w,0} \subset V \otimes_{\mathbb{Q}} \mathbb{C}$ with $V^{1,0}$. When w is odd, this equivalence preserves polarizability. However, this is not the case when w is even.

However, there is also an equivalence of categories between the category of complex abelian varieties up to isogeny and the category of polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers (n, n) given by identifying a complex abelian variety with its weight-1 rational cohomology. Thus when analyzing properties of polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \dots, 0, n)$ in the context, for instance, of their endomorphism algebras, Hodge groups, or Lefschetz groups, we may use existing machinery about complex abelian varieties to deal with the odd-weight case. For the even weight case, however, new techniques are needed.

In [Shi63], Shimura classifies all of the possible endomorphism algebras of simple polarized complex abelian varieties, which in light of the above equivalence, yields a classification of of the possible endomorphism algebras of odd-weight simple polarizable Hodge structures with Hodge numbers $(n, 0, \dots, 0, n)$. Totaro completes this classification in [Tot15] to include all even-weight such Hodge structures as well. For reference, we include this classification below.

2.3.1 Endomorphism Algebra Classification

Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \dots, 0, n)$ and L its endomorphism algebra. Let F_0 be the center of L, so that F_0 is a totally real or CM field (see Section 2.2.2), and let F be the maximal totally real subfield of F_0. Writing $g = [F : \mathbb{Q}]$, $2n = m[L : \mathbb{Q}]$, $q^2 = [L : F_0]$, let $B \cong M_m(L^{\text{op}})$ be the centralizer of L in $\text{End}_{\mathbb{Q}}(V)$.

If L is of Type IV in Albert's classification, so that F_0 is a CM field, let $\Sigma(F_0) = \{\sigma_1, \dots, \sigma_g, \overline{\sigma}_1, \dots, \overline{\sigma}_g\}$ be the set of embeddings of F_0 into $\mathbb{C}$. Then $L \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to $2g$ copies of $M_q(\mathbb{C})$, one for each embedding $\sigma_i \in \Sigma(F_0)$. This decomposition of $L \otimes_{\mathbb{Q}} \mathbb{C}$

yields a decomposition of $V^{w,0} \subset V \otimes_{\mathbb{Q}} \mathbb{C}$ into summands $V^{w,0}(\sigma_i)$ on which F_0 acts via the embedding $\sigma_i \in \Sigma(F_0)$. Letting $n_{\sigma_i} = \dim_{\mathbb{C}} V^{w,0}(\sigma_i)$, we then have $n_{\sigma_i} + n_{\overline{\sigma}_i} = mq$ for each $i = 1, \ldots, g$.

Theorem 2.3.2. *(Totaro [Tot15, Theorem 4.1]) In the above notation, if V is a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$, then $[L : \mathbb{Q}]$ divides $2n$ and $[F : \mathbb{Q}]$ divides n.*

Conversely, every division algebra with positive involution satisfying these two bounds is the endomorphism algebra of some simple polarizable $\mathbb{Q}$-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ except for the following 5 odd-weight and 7 even-weight exceptional cases.

Odd-weight exceptional cases:

1. *Type III and $m = 1$*
2. *Type III, $m = 2$, $\operatorname{disc}(B, ^{-}) = 1$ in $F^*/(F^*)^2$*
3. *Type IV and $\sum_{i=1}^{g} n_{\sigma_i} n_{\overline{\sigma}_i} = 0$, unless $m = q = 1$*
4. *Type IV, $m = 2$, $q = 1$, and $n_{\sigma_i} = n_{\overline{\sigma}_i} = 1$ for all $i = 1, \ldots, g$*
5. *Type IV, $m = 1$, $q = 2$, and $n_{\sigma_i} = n_{\overline{\sigma}_i} = 1$ for all $i = 1, \ldots, g$*

Even-weight exceptional cases

1. *Type II and $m = 1$*
2. *Type II, $m = 2$, $\operatorname{disc}(B, ^{-}) = 1$ in $F^*/(F^*)^2$*
3. *Type IV and $\sum_{i=1}^{g} n_{\sigma_i} n_{\overline{\sigma}_i} = 0$, unless $m = q = 1$*
4. *Type IV, $m = 2$, $q = 1$, and $n_{\sigma_i} = n_{\overline{\sigma}_i} = 1$ for all $i = 1, \ldots, g$*
5. *Type IV, $m = 1$, $q = 2$, and $n_{\sigma_i} = n_{\overline{\sigma}_i} = 1$ for all $i = 1, \ldots, g$*
6. *Type I and $m = 2$*
7. *Type I, $m = 4$, and $(V, \langle , \rangle)$ has discriminant 1 in $F^*/(F^*)^2$*

2.4 The Lefschetz Group of a $\mathbb{Q}$-Hodge Structure with Hodge Numbers $(n, 0, \ldots, 0, n)$

Let A be a central simple algebra over a field K_0 with involution $^-$. Let K be the subfield of K_0 fixed by the involution $^-$. Define

$$\mathrm{Sym}(A, ^-) = \{f \in A \mid \overline{f} = f\}$$

$$\mathrm{Alt}(A, ^-) = \{f \in A \mid \overline{f} = -f\}.$$

Then, following the Book of Involutions [KMRT98] and writing $[A : K_0] = q^2$, we say that the involution $^-$ on A is *orthogonal* if $K_0 = K$ and $\dim_K \mathrm{Sym}(A, ^-) = \frac{q(q+1)}{2}$. We say the involution is *symplectic* if $K_0 = K$ and $\dim_K \mathrm{Sym}(A, ^-) = \frac{q(q-1)}{2}$. Finally, we say the involution is *unitary* if $K_0 \neq K$. There are no other possibilities for the involution $^-$ [KMRT98, Proposition I.2.6].

Letting A^* denote the group of invertible elements of A, define the group of *isometries* of A by

$$\mathrm{Iso}(A, ^-) = \{g \in A^* \mid \overline{g} = g^{-1}\}.$$

Then write

$$\mathrm{Iso}(A, ^-) = \begin{cases} O(A, ^-) & \text{if } ^- \text{ is orthogonal} \\ Sp(A, ^-) & \text{if } ^- \text{ is symplectic} \\ U(A, ^-) & \text{if } ^- \text{ is unitary.} \end{cases}$$

If the involution $^-$ is orthogonal, then, as an algebraic group, the kernel $O^+(A, ^-)$ of the reduced norm map $O(A, ^-) \to \{\pm 1\}$ is a K-form of $SO(q)$, meaning the two groups are isomorphic over an algebraic closure of K. If the involution $^-$ is symplectic, then $Sp(A, ^-)$ is a K-form of $Sp(q)$ (in this case q must be even). If the involution $^-$ is unitary, then $U(A, ^-)$ is a K-form of $GL(q)$ and $SU(A, ^-) := \ker(\mathrm{Norm}_{A/K} \colon U(A, ^-) \to \mathbb{G}_{m,K})$ is a K-form of $SL(q)$.

Let V be a simple polarizable $\mathbb{Q}$-Hodge structure V with Hodge numbers $(n, 0, \ldots, 0, n)$ and let L be its endomorphism algebra. Using the notation introduced above as well as the notation of Section 2.3.1, we now list in Table 2.1 Lefschetz group of V depending on the type that L has in Albert's classification. Note that in the table, the notation ${}_FV$ denotes V considered as an F-vector space.

Table 2.1: The Lefschetz Group of V Depending on $L = \mathrm{End}_{\mathbb{Q}-HS}(V)$

L	Lef(V)	
	Odd Weight	Even Weight
Type I	$R_{F/\mathbb{Q}}Sp({}_FV)$	$R_{F/\mathbb{Q}}SO({}_FV)$
Type II	$R_{F/\mathbb{Q}}Sp(B,^{-})$	$R_{F/\mathbb{Q}}O^{+}(B,^{-})$
Type III	$R_{F/\mathbb{Q}}O^{+}(B,^{-})$	$R_{F/\mathbb{Q}}Sp(B,^{-})$
Type IV	$R_{F/\mathbb{Q}}U(B,^{-})$	$R_{F/\mathbb{Q}}U(B,^{-})$

2.5 Irreducible Representations of the Hodge Group

Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ and endomorphism algebra L. Let $H = Hg(V)$ be the Hodge group of V.

If L is of Types I, II, or III in Albert's classification, then $F_0 = F$ and so, letting $\Sigma(F)$ denote the set of embeddings of F into $\mathbb{C}$, the action of $L \otimes_{\mathbb{Q}} \mathbb{C}$ on $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$ induces a decomposition as an $H_{\mathbb{C}}$-representation

$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(F)} qW_{\sigma}. \tag{2.2}$$

If L is of Type IV, then F_0 is a CM field with maximal totally real subfield F and $V_{\mathbb{C}}$ decomposes as an $H_{\mathbb{C}}$-representation as

$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(F)} \left(qW_{\sigma} \oplus qW_{\sigma}^{*}\right). \tag{2.3}$$

In both cases, each of the summands W_{σ} for $\sigma \in \Sigma(F)$ has complex dimension mq.

Lemma 2.5.1. *Each representation W_σ of the group $H_\mathbb{C}$ is irreducible.*

Proof. By construction each representation W_σ is nonzero and hence $[W_\sigma \otimes W_\sigma^*]^{H_\mathbb{C}}$ is nonzero. Namely in the decomposition of $[V_\mathbb{C} \otimes V_\mathbb{C}^*]^{H_\mathbb{C}}$ coming from (2.2) respectively (2.3), the term

$$\left[\bigoplus_{\sigma\in\Sigma(F)} q^2(W_\sigma \otimes W_\sigma^*)\right]^{H_\mathbb{C}} \quad \text{respectively} \quad \left[\bigoplus_{\sigma\in\Sigma(F)} 2q^2(W_\sigma \otimes W_\sigma^*)\right]^{H_\mathbb{C}},$$

has dimension at least $q^2[F_0 : \mathbb{Q}] = [L : \mathbb{Q}]$. But since L is the endomorphism algebra of V, we know $[V_\mathbb{C} \otimes V_\mathbb{C}^*]^{H_\mathbb{C}}$ has dimension exactly equal to $[L : \mathbb{Q}]$. Therefore, each term $[W_\sigma \otimes W_\sigma^*]^{H_\mathbb{C}}$ has dimension equal to 1, meaning that the representation of $H_\mathbb{C}$ on W_σ is irreducible. □

The group $H_\mathbb{C}$, up to permutation of factors, has a canonical decomposition as an almost direct product of its center $Z(H)_\mathbb{C}$ and its simple factors H_i given by

$$H_\mathbb{C} = Z(H)_\mathbb{C} \cdot H_1 \cdots H_s.$$

Passing to Lie algebras and writing $\mathfrak{c} = \mathrm{Lie}(Z(H))_\mathbb{C}$ and $\mathfrak{g}_i = \mathrm{Lie}(H_i)$ yields

$$\mathrm{Lie}(H)_\mathbb{C} = \mathfrak{c} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s.$$

Hence if $W_\sigma \subset V_\mathbb{C}$ is one of the irreducible $H_\mathbb{C}$-submodules introduced above, we have a decomposition of W_σ as a representation of $\mathrm{Lie}(H)_\mathbb{C}$ given by

$$W_\sigma = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s, \tag{2.4}$$

where χ is a character of $\mathfrak{c}$ and the ρ_i are irreducible representations of the simple factors $\mathfrak{g}_i$. Note that in this notation the ρ_i are allowed to be trivial.

In order to study this irreducible representation W_σ, we first recall some facts and terminology about representations of semisimple Lie algebras.

2.5.1 Representations of Semisimple Lie Algebras

Suppose $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field K and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let R be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let $B = \{\alpha_1, \dots, \alpha_l\}$ be a set of simple roots of R with corresponding set of coroots $B^\vee = \{\alpha^\vee \mid \alpha \in B\}$. If w_0 is the longest element of the Weyl group of R with respect to the basis B, let $\lambda \to \lambda' := -w_0(\lambda)$ denote the opposition involution on $\mathfrak{h}^*$. If λ is a dominant weight, we can write

$$\lambda = \sum_{\alpha \in B} c_\alpha \cdot \alpha,$$

where all the c_α are in $\mathbb{Q}_{\geq 0}$. By Lemma 3.3 in [Moo99], we have that $c_\alpha + c_{\alpha'}$ lies in $\mathbb{Z}_{\geq 0}$ for all $\alpha \in B$. Then define

$$\text{length}(\lambda) = \min_{\alpha \in B} c_\alpha + c_{\alpha'},$$

If R is an irreducible root system, we say that a dominant weight λ is *minuscule* if $\langle \lambda, \alpha^\vee \rangle$ lies in $\{-1, 0, 1\}$ for all $\alpha \in R$ [Bou75, Chapter VIII, §7.3]. Then length(λ) is equal to 1 if and only if λ is a minuscule weight and R is of classical type, meaning of type A_l, B_l, C_l, or D_l [Moo99, Example 3.6].

We make extensive use of Table 2.2, reproduced from [Moo99]. The table describes for a given root system with minuscule weight λ, the corresponding representation $V(\lambda)$ (omitting the weight 0). The table gives the dimension and autoduality of the representation $V(\lambda)$. Note that in the table, the symbol $-$ denotes a symplectic representation, the symbol $+$ denotes an orthogonal representation, and 0 denotes a non-self-dual representation.

2.5.2 The Representation W_σ

We now return to our consideration of the mq-dimensional irreducible representation $W_\sigma = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s$ of the complexified Hodge group $H_{\mathbb{C}}$.

Lemma 2.5.2. *Each nontrivial ρ_i for $1 \leq i \leq s$ in the above notation satisfies*

1. *The highest weight of ρ_i is minuscule and $\mathfrak{g}_i$ is of classical type*

Table 2.2: Minuscule weights in irreducible root systems

Root system	Minuscule weight	Representation	Dimension	Autoduality
A_l	$\overline{\omega}_j (1 \leq j \leq l)$	$\wedge^j$(Standard)	$\binom{l+1}{j}$	$(-1)^j$ if $l = 2j - 1$ 0 otherwise
B_l	$\overline{\omega}_l$	Spin	2^l	$+$ if $l \equiv 0, 3 \mod 4$ $-$ if $l \equiv 1, 2 \mod 4$
C_l	$\overline{\omega}_1$	Standard	$2l$	$-$
D_l	$\overline{\omega}_1$ $\overline{\omega}_{l-1},\ \overline{\omega}_l$	Standard Spin^-, resp. Spin^+	$2l$ 2^{l-1}	$+$ $+$ if $l \equiv 0 \mod 4$ $-$ if $l \equiv 2 \mod 4$ 0 if $l \equiv 1 \mod 2$
E_6	$\overline{\omega}_1$ $\overline{\omega}_6$		27 27	0 0
E_7	$\overline{\omega}_7$		56	-1

2. *If ρ_i is self dual, then ρ_i is even-dimensional*

3. *If ρ_i is symplectic with $\dim_{\mathbb{C}}(\rho_i) \equiv 2 \pmod 4$, then $\mathfrak{g}_i$ is of type C_l, where $l \geq 1$ is odd, and the representation is the standard representation of $\mathfrak{sp}_{2l}$*

4. *If ρ_i is orthogonal with $\dim_{\mathbb{C}}(\rho_i) \equiv 2 \pmod 4$, then $\mathfrak{g}_i$ is of type either:*

   (a) *D_l, where $l \geq 1$ is odd, and the representation is the standard representation of $\mathfrak{so}_{2l}$*

   (b) *A_{2^k-1}, where $k \geq 3$, and the representation is $\bigwedge^{2^{k-1}}(\mathrm{St})$, where St denotes the standard representation of $\mathfrak{sl}_{2^k}$*

Proof. Moonen proves in [Moo99, Theorem 3.11] that for any polarizable $\mathbb{Q}$-Hodge structure V of weight w, the length of the highest weight of ρ_i is bounded above by the integer N, where $N+1$ is the number of integers p such that $V^{p,w-p} \neq 0$ in the Hodge decomposition of $V_{\mathbb{C}}$. A $\mathbb{Q}$-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ thus has $N = 1$. So the highest weight λ_i of ρ_i must have length 1, meaning that λ_i is minuscule and $\mathfrak{g}_i$ is of classical type [Moo99, Section 3.6]. Table 2.2 then yields the rest of the result, making use, for Item 4b, of the combinatorial fact that, for any integer $z \geq 0$, the binomial coefficient $\binom{2z}{z}$ is congruent to 2 mod 4 if and only if z is a power of 2. □

Lemma 2.5.3. *Suppose that the Hodge group $H = Hg(V)$ is semisimple and that for each of irreducible representations W_σ for $\sigma \in \Sigma(F)$, there is only one nontrivial ρ_i. If none of these representations ρ_i are of type D_4, then we have the equality*

$$\mathrm{Lie}(H)_{\mathbb{C}} = \prod_{\sigma \in \Sigma(F)} (\mathfrak{g}_i)_\sigma.$$

Proof. For any polarizable $\mathbb{Q}$-Hodge structure U and any positive integer $m \geq 1$, the Hodge group of the direct sum of m copies of U is isomorphic to the Hodge group of U acting diagonally on mU [Moo99, Remark 1.8]. Thus for each of the representations W_σ, we have $Hg(qW_\sigma) = Hg(W_\sigma)$. Hence the decompositions of (2.2) and (2.3) yield the inclusion

$$\mathrm{Lie}(H)_{\mathbb{C}} \subseteq \prod_{\sigma \in \Sigma(F)} (\mathfrak{g}_i)_\sigma, \tag{2.5}$$

where, in addition, we know that $\text{Lie}(H)_{\mathbb{C}}$ surjects onto each of the $(\mathfrak{g}_i)_\sigma$ factors.

If this inclusion is strict, then there must exist an isomorphism between two factors $(\mathfrak{g}_i)_{\sigma_1}$ and $(\mathfrak{g}_i)_{\sigma_2}$ whose graph gives the representation $(\rho_i)_{\sigma_1} \oplus (\rho_i)_{\sigma_2}$. However, since $(\mathfrak{g}_i)_{\sigma_1}$ is not of type D_4, its outer automorphism group is either trivial or is $\mathbb{Z}/2\mathbb{Z}$ [FH04, Chapter 20]. Hence, an isomorphism between $(\mathfrak{g}_i)_{\sigma_1}$ and $(\mathfrak{g}_i)_{\sigma_2}$ is induced by conjugation by an isomorphism between the underlying representations W_{σ_1} and W_{σ_2}. But this contradicts the fact that the representations W_σ were defined according to the decomposition $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\sigma \in \Sigma(F)} M_q(\mathbb{C})$. Hence no such isomorphism exists and so the inclusion in (2.5) must be an equality. □

2.6 $SL(2)$-factors and the Hodge Group

Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ and endomorphism algebra L. Recall the definition of the Mumford-Tate group $M = MT(V)$ of V as the smallest $\mathbb{Q}$-algebraic group contained in $GL(V)$ such that the homomorphism

$$\gamma := h \circ \mu \colon \mathbb{G}_{m,\mathbb{C}} \to GL(V)_{\mathbb{C}}$$

factors through $M_{\mathbb{C}}$. Defining the representation ϕ to be the tautological representation

$$\phi \colon M \to GL(V),$$

the weights in $V_{\mathbb{C}}$ of the composition $\phi \circ \gamma$ are exactly the cocharacters $z \mapsto z^{-p}$, for p an integer such that $V^{p,w-p} \neq 0$, meaning that the only two possible weights are $z \mapsto z^{-w}$ or $z \mapsto 1$.

Observe that if the Hodge group $H = Hg(V)$ has decomposition $H_{\mathbb{C}} = Z(H)_{\mathbb{C}} \cdot H_1 \cdots H_s$, then the Mumford-Tate group has decomposition

$$M_{\mathbb{C}} = Z(M)_{\mathbb{C}} \cdot H_1 \cdots H_s.$$

Passing to Lie algebras and writing $\mathfrak{c} = \text{Lie}(Z(H))_{\mathbb{C}}$, $\mathfrak{c}' = \text{Lie}(Z(M))_{\mathbb{C}}$, and $\mathfrak{g}_i = \text{Lie}(H_i)$ yields

$$\text{Lie}(H)_{\mathbb{C}} = \mathfrak{c} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s$$

$$\mathrm{Lie}(M)_{\mathbb{C}} = \mathfrak{c}' \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s.$$

Now, for $\sigma \in \Sigma(F)$, let W_σ be an irreducible representation of $M_{\mathbb{C}}$ defined as in Section 2.5 with decomposition as a representation of $\mathrm{Lie}(M)_{\mathbb{C}}$ given by

$$W = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s, \tag{2.6}$$

where χ is a character of $\mathfrak{c}'$ and the ρ_i are irreducible representations of the simple factors $\mathfrak{g}_i$.

Write the weights in W_σ of the homomorphism $\gamma\colon \mathbb{G}_{m,\mathbb{C}} \to M_{\mathbb{C}}$ as $z \mapsto (z^{-l_0}, z^{-l_1}, \ldots, z^{-l_s})$ according to the decomposition $W_\sigma = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s$.

Lemma 2.6.1. *Suppose that $\mathfrak{g}_i \cong \mathfrak{sl}_2$ for some $1 \leq i \leq s$. Denote the two possible values of l_i by α and β and denote the possible values of $\left(\sum_{j=0}^{s} l_j\right) - l_i$ by $\lambda_1, \ldots, \lambda_r$ for $r = \frac{\dim_{\mathbb{C}} W_\sigma}{2}$. Then either $\alpha = \beta = 0$ or $\lambda_1 = \cdots = \lambda_r = 0$.*

Proof. The only two weights of $\phi \circ \gamma$ in W_σ are the cocharacters $z \mapsto z^{-w}$ and $z \mapsto 1$. Hence for any weight of $\gamma\colon \mathbb{G}_{m,\mathbb{C}} \to M_{\mathbb{C}}$ written as $z \mapsto (z^{-c}, z^{-l_1}, \ldots, z^{-l_s})$, we must have the sum $\sum_{j=1}^{s} l_j$ either equal to $w - c$ or equal to 0. In other words, half of the elements in the set

$$\{\alpha + \lambda_k \mid 1 \leq k \leq r\} \cup \{\beta + \lambda_k \mid 1 \leq k \leq r\}$$

are equal to 0 and half are equal to $w - c$.

If $\alpha + \lambda_k = 0$ and $\beta + \lambda_k = w - c$ for all $1 \leq k \leq r$, then $\lambda_k = -\alpha = w - c - \beta$ for all $1 \leq k \leq r$. But then the representation $\boxtimes_{k \neq i} \rho_k$ is just multiplication by z^{α}. But $\prod_{k \neq i} \mathfrak{g}_k$ is contained in $\mathfrak{sl}_r$ and hence $\sum_{k=1}^{r} \lambda_k = r(-\alpha) = 0$. Thus $\lambda_k = 0$ for all $1 \leq k \leq r$ and we are done.

If it is not the case that $\alpha + \lambda_k = 0$ and $\beta + \lambda_k = w - c$ for all $1 \leq k \leq r$, then there exists $t \in \{1, \ldots, r\}$ such that without loss of generality $\lambda_1 = \cdots = \lambda_t = -\alpha$ and $\lambda_{t+1} = \cdots = \lambda_r = w - c - \alpha$. Adding β yields $\beta - \alpha = 0$. Then since ρ_i acts as multiplication by $z^{-\alpha}$ and $\mathfrak{g}_{\mathfrak{i}} = \mathfrak{sl}_2$, we have 2α=0, hence $\alpha = \beta = 0$. $\square$

Corollary 2.6.2. *There is no simple polarizable $\mathbb{Q}$-Hodge structure V of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ such that*

$$Hg(V)_{\mathbb{C}} = SL(2, \mathbb{C}) \times G,$$

where G is a nontrivial semisimple $\mathbb{C}$-group having no simple factors isomorphic to $SL(2, \mathbb{C})$ and $Hg(V)_{\mathbb{C}}$ acts irreducibly on $V_{\mathbb{C}}$ by the product of the standard representation of $SL(2, \mathbb{C})$ with a representation of G.

Proof. If such a Hodge structure V existed, then by Lemma 2.6.1, the homomorphism γ would actually factor through either $Z(M)_{\mathbb{C}} \cdot SL(2, \mathbb{C})$ or through $Z(M)_{\mathbb{C}} \cdot G$. But then, since G has no simple factors isomorphic to $SL(2, \mathbb{C})$, the Hodge group would be a $\mathbb{Q}$-form of $SL(2)$ or a $\mathbb{Q}$-form of G, which is a contradiction. □

2.7 Lower Bound on the Rank of the Hodge Group

The following lemma restates a result proved by Orr [Orr15, Theorem 1.1] for abelian varieties, which generalized an earlier result by Ribet [Rib81] for abelian varieties of CM-type.

Lemma 2.7.1. *Let V be a polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ whose endomorphism algebra L is commutative. Then the rank, as a $\mathbb{Q}$-algebraic group, of the Hodge group $Hg(V)$ satisfies*

$$\mathrm{Rank}(Hg(V)) \geq \log_2(2n).$$

Proof. As in previous sections, for the Mumford-Tate group $M = MT(V)$, we have decompositions $\mathrm{Lie}(M)_{\mathbb{C}} = \mathfrak{c} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s$ and $W_\sigma = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s$, where W_σ is an irreducible $M_{\mathbb{C}}$-module coming from the decomposition of $L \otimes_{\mathbb{Q}} \mathbb{C}$ according to the embeddings $\sigma \in \Sigma(F)$. By Lemma 2.5.2, the nontrivial ρ_i have highest weights which are minuscule and so their weight spaces are all one-dimensional.

Let T be a maximal torus of M and let r be its rank. Consider the restricted representation on W_σ given by $(\chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s)\mid_T$. Since the characters of T in a minuscule representation have multiplicity 1, we know

$$\dim W_\sigma = (\text{ number of characters of } (\chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s)\mid_T).$$

Moreover, since L is commutative, none of the representations W_σ indexed by the embeddings $\sigma \in \Sigma(F)$ can be isomorphic to each other and so, since non-isomorphic minuscule representations have disjoint characters, these W_σ have disjoint characters. Letting $\phi\colon M \to GL(V)$ be the tautological representation, the sum $V_{\mathbb{C}}$ of the representations W_σ satisfies $\dim V_{\mathbb{C}} = 2n$, so we have

$$2n = (\text{ number of characters of } \phi|_T). \tag{2.7}$$

Let S be the set of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-conjugates of the homomorphism γ and let S' be the set of cocharacters of the torus $T_{\mathbb{C}}$ that are $M(\mathbb{C})$-conjugate to an element of S. Note that the images of the $M(\mathbb{C})$-conjugates of elements of S generate $M_{\mathbb{C}}$. Moreover, because every cocharacter of M is $M(\mathbb{C})$-conjugate to a cocharacter of $T_{\mathbb{C}}$, the images of the $M(\mathbb{C})$-conjugates of elements of S' still generate $M_{\mathbb{C}}$.

Consider the action of the Weyl group of M on $X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q} = \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})\otimes_{\mathbb{Z}}\mathbb{Q}$. Since S' is closed under this action and $M_{\mathbb{C}}$ is generated by the images of the $M(\mathbb{C})$-conjugates of elements of S', it must be the case that S' spans $X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ as a $\mathbb{Q}$-vector space. Thus, let Δ be a basis for $X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ contained in S'. So, $|\Delta|$ is equal to the rank r of T.

Any weight $\lambda \in \mathrm{Hom}(T_{\mathbb{C}}, \mathbb{G}_m)$ of the representation ϕ is then determined by the integers $\langle \lambda, \delta\rangle$ for $\delta \in \Delta$. But any δ in Δ is also in S' and thus is some $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$- and $M(\mathbb{C})$-conjugate of γ.

Recall that for any $z \in \mathbb{C}^*$, the map $\phi\circ\gamma$ acts as multiplication by z^{-w} on $V^{w,0}$ and as the identity on $V^{0,w}$. Hence, if $\delta \in \Delta$, the integer $\langle\lambda,\delta\rangle$ can only be 0 or $-w$. Namely,

$$(\text{ number of characters of } \phi|_T) \leq 2^r.$$

However, this bound may be reduced by noting that M contains the homotheties. Namely, there is a unique cocharacter $\nu : \mathbb{G}_m \to M_{\mathbb{C}}$ such that for $z \in \mathbb{C}^*$, the map $\phi\circ\nu$ acts by multiplication by z^{-w}. So, in fact, ν may be viewed as an element of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and, moreover, $\langle \lambda, \nu \rangle$ is equal to $-w$ for any character λ of $\phi \mid_T$. We may thus choose a new subset Δ' of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and S' such that $\nu \cup \Delta'$ forms a basis of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Repeating the above arguments for Δ' then yields

$$\text{(number of characters of } \phi|_T) \leq 2^{r-1}. \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$\log_2(2n) \leq r - 1,$$

where r is the rank of M as an algebraic group over $\mathbb{Q}$. However we know M is the almost direct product inside $GL(V)$ of $\mathbb{G}_m$ and $Hg(V)$, so $Hg(V)$ has rank $r - 1$. □

2.8 Hodge Representations and Mumford-Tate domains

Following [GGK12, Section IV.A], we introduce the notions of Hodge representations and Mumford-Tate domains, which will prove useful in later sections.

Definition 2.8.1. Suppose V is a $\mathbb{Q}$-vector space and $\langle , \rangle \colon V \otimes V \to \mathbb{Q}$ is a bilinear form on V such that $\langle u, v \rangle = (-1)^w \langle v, u \rangle$ for some integer $w \geq 1$. A *Hodge representation* (H, ρ, ϕ) is the data of a representation defined over $\mathbb{Q}$

$$\rho \colon H \to \mathrm{Aut}(V, \langle , \rangle)$$

of a connected $\mathbb{Q}$-algebraic group H and a non-constant homomorphism

$$\phi \colon U_1 \to H_{\mathbb{R}}$$

such that the data $(V, \langle , \rangle, h := \rho \circ \phi)$ is a polarized $\mathbb{Q}$-Hodge structure of weight w.

Let (H, ρ, ϕ) be a Hodge representation attached to the $\mathbb{Q}$-vector space V with bilinear form $\langle , \rangle$ and let $h = \rho \circ \phi$ be its associated polarized $\mathbb{Q}$-Hodge structure of weight w. Let D be the period domain of all $\mathbb{Q}$-Hodge structures with the same Hodge numbers as V.

Consider the Mumford-Tate domain D_{H_h} given by the $H(\mathbb{R})$-orbit of the Hodge structure h inside of the period domain D. Let $D^0_{H_h}$ be a connected component of this Mumford-Tate domain D_{H_h}.

Note that $D^0_{H_h}$ is a closed analytic space and thus we may speak of a *very general* $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ to mean a $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ occurring outside of countably many closed analytic subspaces not equal to $D^0_{H_h}$.

Lemma 2.8.2. *A very general $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ has Hodge group a connected normal $\mathbb{Q}$-subgroup of H.*

Proof. The following proof is heavily inspired by the very similar proofs of Totaro's [Tot15, Page 4110] and Green-Griffiths-Kerr [GGK12, Proposition VI.A.5]. From [GGK12, Proposition IV.A.2], every $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ has Hodge group contained in the group H. Moreover, the Hodge group G of a very general $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ is determined by the data $(V, \langle, \rangle, \rho, D^0_{H_h})$. Using that $D^0_{H_h}$ is connected, the action of the group $H(\mathbb{Q})$ preserves this data and hence normalizes the algebraic group G. Since H is a connected group over the perfect field $\mathbb{Q}$, the group $H(\mathbb{Q})$ is Zariski dense in H [Bor12, Corollary 18.3]. Thus, in fact, the group H normalizes the group G. Since $G \subset H$, we have that G is a connected normal $\mathbb{Q}$-subgroup of H. □

Corollary 2.8.3. *Let (H, ρ, ϕ) be a Hodge representation of a polarized $\mathbb{Q}$-Hodge structure $(V, \langle, \rangle, h := \rho \circ \phi)$ such that there exists $p \neq q$ such that $V^{p,q} \neq 0$ in the Hodge decomposition of $V_{\mathbb{C}}$. If H is a $\mathbb{Q}$-simple group, then a very general $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ has Hodge group equal to H.*

Proof. By Lemma 2.8.2, since H is $\mathbb{Q}$-simple, the Hodge group G of a very general $\mathbb{Q}$-Hodge structure in $D^0_{H_h}$ is either 1 or all of H. But since there exists $p \neq q$ such that $V^{p,q}$ is nonzero, by Remark 2.2.2, the group G must be nontrivial. □

2.9 Hodge Groups for Type I Endomorphism Algebras

We now begin our characterization of Hodge groups of simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$, making use of the notation for Hodge groups and Lefschetz groups established in Section 2.4.

Proposition 2.9.1. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ and endomorphism algebra L a totally real number field such that $l = \frac{n}{[L:\mathbb{Q}]}$ is odd. Then,*

$$Hg(V) = \begin{cases} R_{L/\mathbb{Q}} Sp({}_L V) & \text{if } w \text{ is odd} \\ R_{L/\mathbb{Q}} SO({}_L V) \text{ or } R_{L/\mathbb{Q}} SU(2^k) \ \left(\text{for } k \geq 3 \text{ and } 2l = \binom{2^k}{2^{k-1}}\right) & \text{if } w \text{ is even.} \end{cases}$$

Moreover, if w is even, then $l \geq 3$ and both possible groups occur.

Proof. When V is of odd weight, the result follows using the equivalence of Remark 2.3.1 together with a result of Ribet's [Rib83, Theorem 1], which proves the result for simple complex abelian varieties. Thus we may assume that the weight w of V is even.

The case when w is even and $l = 1$ is exceptional case (6) in Totaro's classification of the of the possible endomorphism algebras of $\mathbb{Q}$-Hodge structures of the specified type (see Section 2.3.1), and thus, since l is assumed to be odd, we know $l \geq 3$.

Let $H = Hg(V)$ be the Hodge group of V, which by Remark 2.2.3 is semisimple. Remark 2.2.4 and Table 2.1 about the Lefschetz group of V imply

$$H \subseteq R_{L/\mathbb{Q}} SO({}_L V).$$

Now, in the notation of Section 2.5, consider an irreducible representation $W_\sigma = \rho_1 \boxtimes \cdots \boxtimes \rho_s$ of $\text{Lie}(H)_\mathbb{C}$ induced by the decomposition $L \otimes_\mathbb{Q} \mathbb{C} = \prod_{\sigma \in \Sigma(L)} \mathbb{C}$.

Note that since the representation W_σ is orthogonal, each of the nontrivial ρ_i is a self-dual representation, meaning either symplectic or orthogonal, and the number of i such that ρ_i is symplectic must be even. Since W_σ has dimension $2l$ with l odd, using Part (2) of Lemma

2.5.2, none of the dimensions of the representations ρ_i can be divisible by 4 and there can be only one i such that ρ_i is nontrivial. Hence this nontrivial ρ_i must be orthogonal. Applying Part (4) of Lemma 2.5.2 yields that this nontrivial ρ_i is either of type D_l, acting on W_σ by the standard representation of $\mathfrak{so}_{2l}$ or, in the case that $2l = \binom{2^k}{2^k-1}$ for some $k \geq 3$, of type A_{2^k-1}, acting on W_σ by the (2^k-1)-th exterior product of the standard representation of $\mathfrak{sl}_{2k}$.

Applying Lemma 2.5.3 then yields

$$\mathrm{Lie}(H)_{\mathbb{C}} = \prod_{\sigma \in \Sigma(L)} \mathfrak{g}_\sigma,$$

where $\mathfrak{g}_\sigma$ is either equal to $\mathfrak{so}_{2l}$ or to $\mathfrak{sl}_{2^k}$ for $2l = \binom{2^k}{2^k-1}$ for some $k \geq 3$. But since the group H must be defined over $\mathbb{Q}$, we must have either $\mathfrak{g}_\sigma = \mathfrak{so}_{2l}$ for all $\sigma \in \Sigma(L)$ or $\mathfrak{g}_\sigma = \mathfrak{sl}_{2^k}$ for all $\sigma \in \Sigma(L)$.

Hence either $H = R_{L/\mathbb{Q}}SO({}_LV)$, which is the generic case [GGK12, Corollary II.A.6], or $H = R_{L/\mathbb{Q}}SU(2^k)$ acting via the representation $\rho\colon R_{L/\mathbb{Q}}SU(2^k) \to SO(V)$ given by the product over all embeddings $\sigma \in \Sigma(L)$ of the $2^k - 1$-th exterior product of the standard representation of $SU(2^k)$. We must show that this second case really can occur.

Let $r = [L : \mathbb{Q}]$ and consider the homomorphism

$$\phi\colon \mathbb{U}_1 \to SU(2^k, \mathbb{R})^r$$

given by

$$z \in \mathbb{C}^* \mapsto \left(\begin{array}{c|c} z^{-\frac{w}{2^k-1}} \cdot \mathrm{Id}_{2^k-1} & 0 \\ \hline 0 & z^{\frac{(2^k-1)w}{2^k-1}} \end{array}\right)^r.$$

Let h be the composition $h = \rho \circ \phi$. Then observe that h defines a weight w polarized Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ on V. Namely, the data $(R_{L/\mathbb{Q}}SU(2^k), \rho, \phi)$ defines a Hodge representation with Hodge numbers $(n, 0, \ldots, 0, n)$.

Since $R_{L/\mathbb{Q}}SU(2^k)$ is $\mathbb{Q}$-simple, by Corollary 2.8.3 a very general $\mathbb{Q}$-Hodge structure in a connected component of the Mumford-Tate domain $D_{R_{L/\mathbb{Q}}SU(2^k)_h}$ will have Hodge group equal to $R_{L/\mathbb{Q}}SU(2^k)$.

□

Proposition 2.9.2. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ and endomorphism algebra L a totally real number field such that $\frac{n}{[L:\mathbb{Q}]} = 2$. Then*

$$Hg(V) = \begin{cases} R_{L/\mathbb{Q}} Sp({}_L V) & \text{if } w \text{ is odd} \\ R_{L/\mathbb{Q}} SO({}_L V) & \text{if } w \text{ is even.} \end{cases}$$

Proof. Let $H = Hg(V)$ be the Hodge group of V, which is semisimple by Remark 2.2.3. So then, using Remark 2.2.4 and Table 2.1, when w is even (respectively when w is odd), we have

$$H \subseteq R_{L/\mathbb{Q}} SO({}_L V) \quad (\text{respectively } H \subseteq R_{L/\mathbb{Q}} Sp({}_L V)).$$

As in the proof Proposition 2.9.1 and using the notation of Section 2.5, let $W_\sigma = \rho_1 \boxtimes \cdots \boxtimes \rho_s$ be an irreducible representation of $\mathrm{Lie}(H)_\mathbb{C}$ induced by the decomposition $L \otimes_\mathbb{Q} \mathbb{C} = \prod_{\sigma \in \Sigma(L)} \mathbb{C}$. These W_σ are 4-dimensional and are orthogonal (respectively symplectic), which implies that each nontrivial ρ_i is self-dual and the number of i such that ρ_i is symplectic must be even (respectively odd). Hence Lemma 2.5.2 yields that the representation W_σ is of type $A_1 \times A_1$ acting by the product of the standard representations of $\mathfrak{sl}_2$ (respectively of type C_2 acting by the standard representation of $\mathfrak{sp}_4$).

In the latter case, namely when w is odd and hence the representation W_σ is of type C_2, Lemma 2.5.3 yields $\mathrm{Lie}(H)_\mathbb{C} = \prod_{\sigma \in \Sigma(L)} \mathfrak{sp}_4$ and so $H = R_{L/\mathbb{Q}} Sp({}_L V)$.

Now consider the case when w is even and hence the representation W_σ is of type $A_1 \times A_1$. If the Hodge structure V is such that the Lefschetz group $R_{L/\mathbb{Q}} SO({}_L V)$ is simple, then either H is all of $R_{L/\mathbb{Q}} SO({}_L V)$ or, as in the proof of Lemma 2.5.3, there is a $\mathrm{Lie}(H)_\mathbb{C}$-module isomorphism α between factors $\mathfrak{so}_4$ and $\mathfrak{so}_4$. Such an isomorphism α may be viewed as a matrix

$$\begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}, \tag{2.9}$$

where each $\psi_{ij} \in \mathrm{Aut}(\mathfrak{sl}_2)$. Since automorphisms of $\mathfrak{sl}_2$ are all inner, as in the proof of Lemma 2.5.3, the nonzero ψ_{ij} induce isomorphisms of the standard representations of their corresponding $\mathfrak{sl}_2$ factors. Namely, if α is an isomorphism between the copy of $\mathfrak{so}_4$ acting on $W_1 = U_{11} \otimes U_{12}$ and the copy of $\mathfrak{so}_4$ acting on $W_2 = U_{21} \otimes U_{22}$, where each U_{ij} is the standard representation of $\mathfrak{sl}_2$, then each nonzero ψ_{ij} induces a $\mathrm{Lie}(H)_{\mathbb{C}}$-module isomorphism between U_{1j} and U_{2i}. Since the matrix in (2.9) is invertible, we get $U_{11} \cong U_{21}, U_{12} \cong U_{22}$ or $U_{11} \cong U_{22}, U_{12} \cong U_{21}$, which in either case yields $W_1 \cong W_2$, contradicting the assumption that the endomorphism algebra L is a field. Hence, if $R_{L/\mathbb{Q}}SO({}_LV)$ is simple, then the Hodge group H is all of $R_{L/\mathbb{Q}}SO({}_LV)$.

Now suppose that the Lefschetz group $R_{L/\mathbb{Q}}SO({}_LV)$ is not simple. This occurs when ${}_LV$ has discriminant 1 in $L^*/(L^*)^2$ and thus $SO({}_LV)$ is the product of two subgroups $SL(1, L')$ and $SL(1, L'^{\mathrm{op}})$, where L' is a quaternion algebra over L [KMRT98, Corollary 15.12]. So then we know that the Hodge group H as a $\mathbb{Q}$-group satisfies

$$H \subset SL(1, L') \cdot SL(1, L'^{\mathrm{op}}) \tag{2.10}$$

and that H surjects onto each of these two factors. Hence H is either the entire product or H is the graph of an isomorphism between the two simple factors.

The Mumford-Tate domain D of Hodge structures with Hodge numbers $(n, 0 \ldots, 0, n)$ and endomorphism algebra contained in L is isomorphic to $(\mathbb{CP}^1 \sqcup \mathbb{CP}^1)^g$, where $g = [L : \mathbb{Q}]$. Consider the homomorphism $\gamma \colon \mathbb{G}_{m,\mathbb{C}} \to MT(V)_{\mathbb{C}}$ defining the Mumford-Tate group of V. Then by Lemma 2.6.1, the weights in each irreducible representation $W_\sigma = U_{\sigma,1} \otimes U_{\sigma,2}$ of γ must either be trivial on $U_{\sigma,1}$ or on $U_{\sigma,2}$.

Observe that if the Hodge group H is the graph of an isomorphism between the simple factors $SL(1, L')$ and $SL(1, L'^{\mathrm{op}})$ in (2.10), then either the weights of γ on $U_{\sigma,1}$ are trivial for all $\sigma \in \Sigma(L)$ or the weights of γ on $U_{\sigma,2}$ are trivial for all $\sigma \in \Sigma(L)$. Namely the Hodge structure V lies exactly on the two connected components of the Mumford-Tate domain D whose generic elements are non-simple, as proved by Totaro in [Tot15, Theorem 4.1]. Namely, if H is the graph of an isomorphism between the two factors in (2.10), then the

Hodge structure V is not simple, which is a contradiction. Hence the Hodge group H is all of $SL(1, L') \cdot SL(1, L'^{\text{op}}) \cong R_{L/\mathbb{Q}}SO({}_LV)$. □

Proposition 2.9.3. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$, where n is twice an odd number, such that the endomorphism algebra L of V is equal to $\mathbb{Q}$. Then*

$$Hg(V) = \begin{cases} Sp(V) & \textit{if } w \textit{ is odd} \\ SO(V) \textit{ or } R_{L/\mathbb{Q}}SU(2^k) \ \left(\textit{for } k \geq 3 \textit{ and } 2n = \binom{2^k}{2^{k-1}}\right) & \textit{if } w \textit{ is even.} \end{cases}$$

In the case when w is even, both possible groups occur.

Proof. Let $H = Hg(V)$ be the Hodge group of V. Then, using Remark 2.2.4 and Table 2.1, when w is even (respectively when w is odd), we have

$$H \subseteq SO(V) \quad \text{(respectively } H \subseteq Sp(V)).$$

Since the endomorphism algebra L is equal to $\mathbb{Q}$, the representation V of H is irreducible. Applying Lemma 2.5.2, the possibilities for H acting on V are:

1. $SO(2n)$ acting by the standard representation
2. $SO(2^k)$, with $2n = \binom{2^k}{2^{k-1}}$ for $k \geq 3$, acting by $\bigwedge^{2^{k-1}}$(Standard)
3. $SU(2) \times SO(n)$ acting by the tensor product of the two standard representations

(respectively,

1. $Sp(2n)$ acting by the standard representation
2. $SL(2) \times SO(n)$ acting by the tensor product of the two standard representations
3. $SL(2) \times SL(2^k)$, with $n = \binom{2^k}{2^{k-1}}$ for $k \geq 3$, acting by the standard representation of $SL(2)$ tensor $\bigwedge^{2^{k-1}}$(Standard).

However, using Corollary 2.6.2 we may eliminate $SL(2) \times SO(n)$ (respectively $SL(2) \times SO(n)$ and $SL(2) \times SL(2^k)$) as possibilities. Hence $H_{\mathbb{C}}$ must be $SO(V) \cong SO(2n)$ or $SO(2^k)$ acting by $\bigwedge^{2^{k-1}}$(Standard) (respectively $Sp(V) \cong Sp(2n)$).

As in the proof of Proposition 2.9.1, the case $H = SO(V)$ is the generic case and thus is always possible [GGK12, Corollary II.A.6]. Thus, it remains to show that when w is even and $2n = \binom{2^k}{2^{k-1}}$ for $k \geq 3$, the Hodge group $SU(2^k)$ acting by the representation

$$\rho\colon R_{L/\mathbb{Q}} SU(2^k, L) \to SO(V),$$

given by the $2^k - 1$-th exterior product of the standard representation of $SU(2^k, L)$, is also possible. However, as in the proof of Proposition 2.9.1, the homomorphism

$$\phi\colon \mathbb{U}_1 \to SU(2^k, \mathbb{R})$$

given by

$$z \in \mathbb{C}^* \mapsto \left(\begin{array}{c|c} z^{-\frac{w}{2^{k-1}}} \cdot \mathrm{Id}_{2^k-1} & 0 \\ \hline 0 & z^{\frac{(2^k-1)w}{2^{k-1}}} \end{array} \right)$$

defines a Hodge representation $(SU(2^k), \rho, \phi)$ with Hodge numbers $(n, 0, \ldots, 0, n)$. Since the $\mathbb{Q}$-group $SU(2^k)$ is simple, Corollary 2.8.3 finishes the proof. □

Proposition 2.9.4. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(4, 0, \ldots, 0, 4)$ and endomorphism algebra L equal to $\mathbb{Q}$. Then*

$$Hg(V) = \begin{cases} Sp(V) \text{ or } SL(2) \times SO(4) \text{ acting by product of standard representations} & \text{if } w \text{ is odd} \\ SO(V) \text{ or } SO(7) \text{ acting by spin representation} & \text{if } w \text{ is even.} \end{cases}$$

Moreover, all of the above groups occur.

Proof. When V is of odd weight, the result follows using the equivalence of Remark 2.3.1 together with the analogous statement for abelian fourfolds proved by Moonen and Zarhin [MZ99, 4.1]. Thus we may assume that V is of even weight.

In this case, using Table 2.1, the Lefschetz group of V is $SO(V) \cong SO(8)$. Since the endomorphism algebra L is equal to $\mathbb{Q}$, the representation V of H is irreducible. Applying Lemma 2.5.2, the possibilities for H acting on V are:

1. $SO(8)$ acting by the standard representation,

2. $SO(7)$ acting by the spin representation

3. $SL(2) \times Sp(4)$ acting by the tensor product of the two standard representations.

However, using Corollary 2.6.2 we may eliminate $SL(2) \times Sp(4)$ as a possibility. As in the preceding proofs, the case $H = SO(V) \cong SO(8)$ is the generic case and thus is always possible [GGK12, Corollary II.A.6]. Thus, it remains to show that the Hodge group $SO(7)$ acting by the spin representation is also possible.

To construct the spin representation ρ, choose a pair (W, W^*) of maximal isotropic subspaces of $\mathbb{Q}^7$ equipped with a symmetric bilinear form $(,)$ such that $W \cap W^* = 0$. If a_1, a_2, a_3 is a basis for W, then there is a unique basis $\alpha_1, \alpha_2, \alpha_3$ of W^* such that for all i and j $(a_i, \alpha_j) = \delta_{ij}$. Now consider the element $\psi_{x \wedge y}$ of the Lie algebra $\mathfrak{so}_7$ given by

$$\psi_{x \wedge y}(v) = 2((y, v)x - (x, v)y)).$$

We can identify $\bigwedge^2 \mathbb{Q}^7$ with the Lie algebra $\mathfrak{so}_7$ via the map

$$x \wedge y \mapsto \psi_{x \wedge y}.$$

Now let $\phi \colon U_1 \to SO(7, \mathbb{R})$ be induced by the map on Lie algebras sending

$$1 \mapsto 2w(\alpha_1 \wedge a_1).$$

Then the data $(SO(7), \rho, \phi)$ defines a Hodge representation with Hodge numbers $(4, 0, \ldots, 0, 4)$. Since $SO(7)$ is $\mathbb{Q}$-simple, Corollary 2.8.3 finishes the proof.

□

2.10 Hodge Groups for Type II/III Endomorphism Algebras

Proposition 2.10.1. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ and endomorphism algebra L of Type II or III such that*

$m = \frac{2n}{[L:\mathbb{Q}]}$ *is odd. Let* $B \cong M_m(L^{\mathrm{op}})$ *be the centralizer of* L *in* $\mathrm{End}_{\mathbb{Q}}(V)$ *and let* F *be the center of* L*. Then*

1. *If* L *is of Type II, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}Sp(B,^-) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}}O^+(B,^-) \text{ or } R_{F/\mathbb{Q}}SU(2^k) \ \left(\text{with } 2m = \binom{2^k}{2^{k-1}} \text{ for } k \geq 3\right) & \text{if } w \text{ is even.} \end{cases}$$

2. *If* L *is of Type III, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}O^+(B,^-) \text{ or } R_{F/\mathbb{Q}}SU(2^k) \ \left(\text{with } 2m = \binom{2^k}{2^{k-1}} \text{ for } k \geq 3\right) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}}Sp(B,^-) & \text{if } w \text{ is even.} \end{cases}$$

Additionally, when L *is of Type II and* w *is even or when* L *is of Type III and* w *is odd, then* $m \geq 3$ *and both possible groups occur.*

Proof. The restriction on m when L is of Type II and w is even or when L is of Type III and w is odd follows from Totaro's classification (see Section 2.3.1).

Let $H = Hg(V)$ be the Hodge group of V, which by Remark 2.2.3 is semisimple. Remark 2.2.4 and Table 2.1, when L is of Type II and w is even or when L is of Type III and w is odd (respectively when L is of Type II and w is odd or when L is of Type III and w is even) yield

$$H \subseteq R_{F/\mathbb{Q}}O^+(B,^-) \quad (\text{respectively } R_{F/\mathbb{Q}}Sp(B,^-).)$$

Here $O^+(B,^-)$ is an F-form of $SO(2m)$ and $Sp(B,^-)$ is an F-form of $Sp(2m)$.

Thus, in the notation of Section 2.5, the decomposition $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod M_2(\mathbb{C})$ indexed by the set of embeddings $\Sigma(F)$ induces a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(F)} 2W_{\sigma},$$

where each W_{σ} is a $2m$-dimensional irreducible orthogonal (respectively symplectic) $H_{\mathbb{C}}$-representation, where m is odd. The rest of the proof then proceeds identically to the proof of Proposition 2.9.1.

□

Proposition 2.10.2. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \dots, 0, n)$ and endomorphism algebra L of V of Type II or III such that $[L : \mathbb{Q}] = n$. Let $B \cong M_2(L^{\mathrm{op}})$ be the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$ and let F be the center of L. Then*

1. *If L is of Type II, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}} Sp(B, ^-) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}} O^+(B, ^-) & \text{if } w \text{ is even.} \end{cases}$$

2. *If L is of Type III, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}} O^+(B, ^-) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}} Sp(B, ^-) & \text{if } w \text{ is even.} \end{cases}$$

Proof. By identical arguments as those in Proposition 2.10.1, the Hodge group H is semisimple and the decomposition $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod M_2(\mathbb{C})$ indexed by the set of embeddings $\Sigma(F)$ induces a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(F)} 2W_\sigma,$$

where each W_σ is a 4-dimensional irreducible orthogonal (respectively symplectic) $H_{\mathbb{C}}$-representation. The rest of the proof then proceeds identically to the proof of Proposition 2.9.2. □

Proposition 2.10.3. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \dots, 0, n)$, where n is four times an odd number, such that the endomorphism algebra L of V is a quaternion algebra over $\mathbb{Q}$. Let $B \cong M_{n/2}(L^{\mathrm{op}})$ be the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$. Then:*

1. *If L is of Type II, then:*

$$Hg(V) = \begin{cases} Sp(B, ^-) & \text{if } w \text{ is odd} \\ O^+(B, ^-) \text{ or } SU(2^k) \ \left(\text{for } k \geq 3 \text{ and } n = \binom{2^k}{2^{k-1}} \right) & \text{if } w \text{ is even.} \end{cases}$$

2. If L is of Type III, then:

$$Hg(V) = \begin{cases} O^{+}(B,^{-}) \text{ or } SU(2^k) \ \left(\text{for } k \geq 3 \text{ and } n = \binom{2^k}{2^{k-1}} \right) & \text{if } w \text{ is odd} \\ Sp(B,^{-}) & \text{if } w \text{ is even.} \end{cases}$$

Proof. Again, we proceed as in the first half of the proof of Proposition 2.10.1, using Lemma 2.5.1 to get an irreducible n-dimensional orthogonal (respectively symplectic) representation W of the semisimple group $H_{\mathbb{C}}$ such that $V_{\mathbb{C}} = W \oplus W$. Since $n = 2k$, where k is, by hypothesis, twice an odd number, the rest of the proof then proceeds identically to the proof of Proposition 2.9.3. □

2.11 Hodge Groups for Type IV Endomorphism Algebras

Let us briefly recall the notation introduced in Section 2.3.1 about the Type IV endomorphism algebra case. Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ whose endomorphism algebra L is of Type IV in Albert's classification. Then the center F_0 of L is a CM field (see Section 2.2.2) with maximal totally real subfield denoted by F.

Let us write $g = [F : \mathbb{Q}]$, $2n = m[L : \mathbb{Q}]$, and $q^2 = [L : F_0]$. Now if $\sigma(F_0) = \{\sigma_1, \ldots, \sigma_g, \overline{\sigma}_1, \ldots, \overline{\sigma}_g\}$ is the set of embeddings of F_0 into $\mathbb{C}$, then $L \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to $2g$ copies of $M_q(\mathbb{C})$, one for each embedding $\sigma \in \Sigma(F_0)$, and this decomposition of $L \otimes_{\mathbb{Q}} \mathbb{C}$ yields a decomposition of $V^{w,0} \subset V_{\mathbb{C}}$ into summands $V^{w,0}(\sigma)$ on which F_0 acts via the embedding $\sigma \in \Sigma(F_0)$. Letting n_σ denote the complex dimension of $V^{w,0}(\sigma)$, we then have that $n_{\sigma_i} + n_{\overline{\sigma}_i} = mq$ for $i = 1, \ldots, g$.

Proposition 2.11.1. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$ and endomorphism algebra L an imaginary quadratic field. Letting $\Sigma(L) = \{\sigma, \overline{\sigma}\}$ be the set of embeddings of L into $\mathbb{C}$ and $B \cong M_n(L^{\mathrm{op}})$ be the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$, if n_σ and $n_{\overline{\sigma}}$, as defined above, are coprime, then $Hg(V) = U(B,^{-})$.*

Proof. The above result and its proof are analogous to those given in [Rib83, Theorem 3] for simple complex abelian varieties. Let $H = Hg(V)$ be the Hodge group of V. From Remark 2.2.4 and Table 2.1, we know $H \subseteq U(B, ^-)$, so we just need to show this containment is an equality.

As introduced in Section 2.5 in Equation (2.3), the action of $L \otimes_{\mathbb{Q}} \mathbb{C}$ induces a decomposition

$$V_{\mathbb{C}} = W_\sigma \oplus W_\sigma^*. \tag{2.11}$$

Let $M = MT(V)$ be the Mumford-Tate group of V and consider the map

$$\overline{\rho} : M_{\mathbb{C}} \to \mathrm{GL}(W_\sigma)$$

induced by the action of $M_{\mathbb{C}}$ on W_σ. Let N be the image of $\overline{\rho}$.

Note firstly of all that N is a reductive, connected subgroup of $\mathrm{GL}(W_\sigma)$. Secondly, because $L = \mathrm{End}_M(V)$, where $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, and because the action of M is compatible with the decomposition in (2.11), we have must have $\mathrm{End}_N W_\sigma = \mathbb{C}$. Thirdly, note that for $z \in \mathbb{C}^*$ the composition

$$\overline{\rho} \circ h \circ \mu : \mathbb{G}_m \to \mathrm{GL}(W_\sigma)$$

acts as multiplication by z^{-w} on $V^{w,0}(\sigma)$ and as the identity on $V^{0,w}(\sigma)$. Namely, N contains the group of automorphisms of W_σ that are a homothety on $V^{w,0}(\sigma)$ and the identity on $V^{0,w}(\sigma)$. Fourthly, the dimensions $n_\sigma = \dim V^{w,0}(\sigma)$ and $n_{\overline{\sigma}} = \dim V^{0,w}(\sigma)$ are coprime.

These four observations are exactly the situation of a result of Serre [Ser67, Proposition 5] which establishes that, under these circumstances, we have $N = \mathrm{GL}(W_\sigma)$. In particular, the fact that $\overline{\rho}$ surjects onto $\mathrm{GL}(W_\sigma)$ implies that the commutator subgroup of $M_{\mathbb{C}}$ surjects onto $\mathrm{SL}(W_\sigma)$.

The Lefschetz group $U(B, ^-)$ of V is a $\mathbb{Q}$-form of $\mathrm{GL}(n)$. Hence, by dimension arguments, in order to show that the Hodge group H is equal to $U(B, ^-)$, it is enough to show that the center of M has dimension at least 2. To do this, it is enough to produce a two-dimensional torus which is a quotient of M.

The inclusion $M \to \mathrm{GL}({}_LV)$, recalling that ${}_LV$ denotes V considered as an L-vector space, yields a determinant map

$$\delta : M \to R_{L/\mathbb{Q}}\mathbb{G}_m.$$

Now, the character group of $R_{L/\mathbb{Q}}\mathbb{G}_m$ is just the free abelian group on σ and $\overline{\sigma}$. Thus

$$(R_{L/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m.$$

The Mumford-Tate group M contains the torus of homotheties of $\mathrm{GL}(V)$. In the image over $\mathbb{C}$ of the map δ, this torus of homotheties corresponds to the diagonal $\mathbb{G}_m$ in the expression $\mathbb{G}_m \times \mathbb{G}_m$.

The composition

$$\delta \circ \mu : \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$$

is given by

$$z \in \mathbb{C}^* \mapsto (z^{n_\sigma}, z^{n_{\overline{\sigma}}}).$$

But since n_σ and $n_{\overline{\sigma}}$ are coprime, they are not equal and so the image of $\delta \circ \mu$ is a subtorus of $\mathbb{G}_m \times \mathbb{G}_m$ which is not the diagonal. Hence δ is surjective. So, the Mumford-Tate group M indeed has a quotient which is a two-dimensional torus and thus we have $H = U(B, ^-)$. $\square$

2.11.1 E-Hodge Structures

Let E be a number field. Define an *E-Hodge structure* to be a $\mathbb{Q}$-Hodge structure V together with a homomorphism of $\mathbb{Q}$-algebras $E \to \mathrm{End}_{\mathbb{Q}-HS}(V)$.

Suppose E is, in fact, a totally real or CM field. Then, writing $a \to \overline{a}$ for the involution on E given by complex conjugation (which is the identity involution if E is totally real), we may define a *polarized E-Hodge structure* to be a polarized $\mathbb{Q}$-Hodge structure together with a homomorphism $E \to \mathrm{End}_{\mathbb{Q}-HS}(V)$ of $\mathbb{Q}$-algebras with involution. Namely, the form $\langle , \rangle : V \times V \to \mathbb{Q}$ satisfies $\langle ax, y \rangle = \langle x, \overline{a}y \rangle$ for all $a \in E$ and all $x, y \in V$. In fact, if V is an E-Hodge structure whose underlying $\mathbb{Q}$-Hodge structure is polarizable, then V is polarizable as an E-Hodge structure [Tot15, Lemma 2.1]. There does not seem to be a good definition of a polarized E-Hodge structure for E a number field which is not totally real or a CM field.

If V is an E-Hodge structure of weight w, then each $V^{p,q}$ in the decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$ is an $E \otimes_{\mathbb{Q}} \mathbb{C}$-module and so splits as a direct sum

$$V^{p,q} = \bigoplus_{\sigma \in \Sigma(E)} V^{p,q}(\sigma),$$

where $V^{p,q}(\sigma)$ is the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ where E acts via σ.

We say that V has *Hodge numbers* $(a_0, \ldots, a_w)$ as an E-Hodge structure if, for each embedding $\sigma \in \Sigma(E)$, the summand $V^{j,w-j}(\sigma)$ has complex dimension a_j for all j. Note that if V is an E-Hodge structure with Hodge numbers $(a_0, \ldots, a_w)$ and $[E : \mathbb{Q}] = r$, then V is a $\mathbb{Q}$-Hodge structure with Hodge numbers $(ra_0, \ldots, ra_w)$.

Lemma 2.11.2. *Let V be a polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$. Suppose there exists a CM field E embedding into the endomorphism algebra L of V. Writing $[E : \mathbb{Q}] = r$ and $2n = lr$, let J be the maximal totally real subfield of E and let $C \cong M_l(E^{\mathrm{op}})$ be the centralizer of E in $\mathrm{End}_{\mathbb{Q}}(V)$. Then we have the inclusion*

$$Hg(V) \subseteq R_{J/\mathbb{Q}} SU(C, ^-)$$

if and only if V is an E-Hodge structure with Hodge numbers $(\frac{n}{r}, 0, \ldots, 0, \frac{n}{r})$.

Proof. We know $V \otimes_{\mathbb{Q}} \mathbb{C}$ has rank l as a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module, so consider the exterior product $\bigwedge_E^l V$. An element α in $Hg(V)$ acts on $\bigwedge_E^l V$ as multiplication by $\mathrm{Norm}_E(\alpha)$. Since E is contained in L, we already know that $Hg(V)$ is contained in $R_{J/\mathbb{Q}} U(C, ^-)$. Hence we have $Hg(V) \subseteq R_{J/\mathbb{Q}} SU(C, ^-)$ if and only if $\bigwedge_E^l V$ is invariant under the action of $Hg(V)$, meaning if and only if $\bigwedge_E^l V$ is purely of type $(\frac{l}{2}, \frac{l}{2})$ as a sub-Hodge structure of $\bigwedge_{\mathbb{Q}}^l V$.

For each $\sigma \in \Sigma(E)$, let $e_\sigma = \dim_{\mathbb{C}} V^{w,0}(\sigma)$ in the decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(E)} \left(V^{w,0}(\sigma) \oplus V^{0,w}(\sigma) \right).$$

Then we have,

$$\begin{aligned}\left(\bigwedge_E^l V\right)\otimes_{\mathbb{Q}}\mathbb{C} &= \bigwedge_{E\otimes_{\mathbb{Q}}\mathbb{C}}^{l}(V\otimes_{\mathbb{Q}}\mathbb{C})\\ &= \bigoplus_{\sigma\in\Sigma(E)}\bigwedge^{l}\left(V^{w,0}(\sigma)\oplus V^{0,w}(\sigma)\right)\\ &= \bigoplus_{\sigma\in\Sigma(E)}\bigoplus_{e_\sigma}\left(\bigwedge^{e_\sigma}V^{w,0}(\sigma)\otimes\bigwedge^{l-e_\sigma}V^{0,w}(\sigma)\right).\end{aligned}$$

Thus $\bigwedge_E^l V$ is purely of type $(\frac{l}{2},\frac{l}{2})$ if and only if $e_\sigma=\frac{l}{2}=l-e_\sigma$ for all embeddings $\sigma\in\Sigma(E)$ of E into $\mathbb{C}$. But the property of having $e_\sigma=\frac{l}{2}=l-e_\sigma$ for all embeddings $\sigma\in\Sigma(E)$ is exactly what it means for V to be an E-Hodge structure with Hodge numbers $(\frac{n}{r},0,\dots,0,\frac{n}{r})$, so this finishes the proof.

□

Proposition 2.11.3. *Let V be a simple polarizable E-Hodge structure with Hodge numbers $(p,0,\dots,0,p)$, where p is prime and E is a CM field of degree r over $\mathbb{Q}$. Suppose the endomorphism algebra of V as a $\mathbb{Q}$-Hodge structure is E. Let $C\cong M_{2p}(E^{\mathrm{op}})$ be the centralizer of E in $\mathrm{End}_{\mathbb{Q}}(V)$ and let J be the maximal totally real subfield of E. Then $Hg(V)=R_{J/\mathbb{Q}}SU(C,^{-})$.*

Proof. Let $H=Hg(V)$ be the Hodge group of V. Since V is an E-Hodge structure, by Lemma 2.11.2 we know $H\subseteq R_{J/\mathbb{Q}}SU(C,^{-})$, where $SU(C,^{-})$ is a J-form of $SL(2p)$. We just need to prove equality.

Since $E=[\mathrm{End}_{\mathbb{Q}}(V)]^H$, the center of H is contained in the center of E, which in this case is all of E. Hence the center of H is contained in $R_{J/\mathbb{Q}}SU(C,^{-})\cap E$. So elements of the center of H must be $2n$-th roots of unity in E, of which there are only finitely many. Thus, the center of H is finite, which, since H is also reductive, implies that H is semisimple.

In the notation of Section 2.5, let

$$W_\sigma=\rho_1\boxtimes\cdots\boxtimes\rho_s$$

be an irreducible representation of $\mathrm{Lie}(H)_{\mathbb{C}}$ in the decomposition

$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(J)} (W_\sigma \oplus W_\sigma^*).$$

Namely the representation W_σ is $2p$-dimensional and not self-dual. So, we may assume without loss of generality than none of the ρ_j are self-dual either. Lemma 2.5.2 yields that the highest weight of each of the ρ_j is minuscule and each $\mathfrak{g}_j$ is of classical type. Hence consulting Table 2.2 yields the following possibilities for the Lie algebra $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s$ acting on W_i:

1. The Lie algebra $\mathfrak{sl}_{2p}$ acting by the standard representation
2. When $p = 5$, the Lie algebra $\mathfrak{sl}_5$ acting by $\bigwedge^2$(Standard)
3. When $p \neq 2$, the Lie algebra $\mathfrak{sl}_2 \times \mathfrak{sl}_p$ acting by the product of the standard representations.

Now let $M_{\mathbb{C}}$ be the complexified Mumford-Tate group of V and consider Case (2). Write the weights in W_i of the homomorphism $\gamma\colon \mathbb{G}_{m,\mathbb{C}} \to M_{\mathbb{C}}$ as $z \mapsto (z^{-c}, z^{-l_1})$, according to the decomposition $W = \chi \boxtimes \rho_1$, where χ is a character of $\mathfrak{c}$ and ρ_1 is the representation of $\mathfrak{sl}_5$ acting by $\bigwedge^2$(Standard).

Let $\lambda_1, \ldots, \lambda_5$ be the 5 possible values of l_1. Then we know, half of the elements in the set $S := \{\lambda_i + \lambda_j \mid 1 \leq i < j \leq 5\}$ are equal to 0 and half are equal to $w - c$. Using a pigeon-hole argument, there exists i and $j, k \neq i$ such that $\lambda_i = -\lambda_j = -\lambda_k$. But since either $\lambda_j = -\lambda_k$ or $\lambda_j = w - c - \lambda_k$, we must have either $\lambda_i = \lambda_j = \lambda_k = 0$ or $\lambda_i = -\frac{w-c}{2}$ and $\lambda_j = \lambda_k = \frac{w-c}{2}$. In either case though, there is no way to choose the two remaining values of l_1 so as to satisfy the requirement on the set S. So the Case (2) in the above list not possible.

Similarly, for Case (3) on the list, write the weights in W_i of the homomorphism $\gamma\colon \mathbb{G}_{m,\mathbb{C}} \to M_{\mathbb{C}}$ as $z \mapsto (z^{-c}, z^{-l_1}, z^{-l_2})$ according to the decomposition $W = \chi \boxtimes \rho_1 \boxtimes \rho_2$, where ρ_1 is the standard representation of $\mathfrak{sl}_2$ and ρ_2 is the standard representation of $\mathfrak{sl}_p$. Denote the two possible values of l_1 by α and β and denote the possible values of l_2 by $\lambda_1, \ldots, \lambda_p$. Then by Lemma 2.6.1, either $\alpha = \beta = 0$ or $\lambda_1 = \cdots = \lambda_p = 0$. But if $\alpha = \beta = 0$, then half of the λ_k for must be equal to $w - c$ and half of the λ_k must be equal to 0. Since p is odd, this is impossible. Therefore, we must have $\lambda_1 = \cdots = \lambda_p = 0$ and hence $\alpha = 0$ and $\beta = w - c$. But then the homomorphism γ factors through $Z(M)_{\mathbb{C}} \cdot SL(2, \mathbb{C})^{r/2}$. So then $H_{\mathbb{C}} \subset SL(2, \mathbb{C})^{r/2}$. But since $\mathfrak{sl}_2 \times \mathfrak{sl}_p$ acts on the representation W by the product of the standard representations, the group $H_{\mathbb{C}}$ must surject onto $SL(p, \mathbb{C})$, which contradicts the statement that $H_{\mathbb{C}}$ is contained in $SL(2, \mathbb{C})^{r/2}$. Hence, Case (3) is not possible.

Hence the representation W_σ must be the standard representation of $\mathfrak{sl}_{2p}$. Applying Lemma 2.5.3, then gives $H_{\mathbb{C}} = \prod_{\sigma \in \Sigma(J)} SL(2p)$, which completes the proof.

□

Proposition 2.11.4. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ which is an E-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ for some CM field E. Writing $[E : \mathbb{Q}] = r$, suppose that the endomorphism algebra L of V is a CM field such that $[L : \mathbb{Q}] = 2r$. Moreover suppose that $n_\sigma = \dim V^{w,0}(\sigma)$ and $n_{\overline{\sigma}} = \dim V^{w,0}(\overline{\sigma})$ are coprime for all embeddings $\sigma \in \Sigma(L)$. Then, letting $B \cong M_n(L^{\mathrm{op}})$ denote the centralizer of L in* $\mathrm{End}_{\mathbb{Q}}(V)$ *and F the maximal totally real subfield of L, we have*

$$Hg(V) = R_{F/\mathbb{Q}} SU(B, ^{-}).$$

Proof. Let $M = MT(V)$ be the Mumford-Tate group of V and, as in the proof of Proposition 2.11.1, consider the map

$$\overline{\rho} : M_{\mathbb{C}} \to \mathrm{GL}(W_\sigma)$$

induced by the action of $M_{\mathbb{C}}$ on an n-dimensional summand W_σ in the decomposition $V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(F)} W_\sigma \oplus W_\sigma^*$. Since $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\sigma \in \Sigma(F)} (\mathbb{C} \oplus \mathbb{C})$, if N is the image of $\overline{\rho}$, we have $\mathrm{End}_N(W_\sigma) = \mathbb{C}$. So by an argument identical to that given in the proof of Proposition 2.11, the map $\overline{\rho}$ is surjective. Hence the commutator subgroup of $M_{\mathbb{C}}$ surjects onto $\mathrm{SL}(W_\sigma)$.

Now, since the Hodge group $Hg(V)$ is contained in the Lefschetz group we know

$$R_{F/\mathbb{Q}}SU(B,^-) \subseteq H \subseteq R_{F/\mathbb{Q}}U(B,^-). \tag{2.12}$$

Note that $U(B,^-)$ is an F-form of $GL(n)$ and $SU(B,^-)$ is an F-form of $SL(n)$.

But since V is an E-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$, by Lemma 2.11.2 we also know

$$H \subseteq R_{J/\mathbb{Q}}SU(C,^-),$$

where J is the subfield of E fixed by complex conjugation and $C \cong M_{2n}(E^{\mathrm{op}})$ is the centralizer of E in $\mathrm{End}_{\mathbb{Q}}(V)$. Note that the group $SU(C,^-)$ is a J-form of $SL(2n)$.

We know $L = [\mathrm{End}_{\mathbb{Q}}(V)]^H$, hence the center of H is contained in L. Hence the center of H is contained in $R_{F/\mathbb{Q}}U(B,^-) \cap L$. So if λ_1 and λ_2 are elements of the center of H, both must have norm 1. Moreover, because H is contained in $R_{J/\mathbb{Q}}SU(C,^-)$, considering an embedding of $R_{F/\mathbb{Q}}U(B,^-) \cap L$ into $R_{J/\mathbb{Q}}SU(C,^-)$, we must have $\lambda_1\lambda_2$ an n-th root of unity in L. There are only finitely many such λ_1 and λ_2 in L, so the center of H is finite, which, since H is reductive, implies H is semisimple. The inclusions in (2.12) thus yield $H = R_{F/\mathbb{Q}}SU(B,^-)$. □

Proposition 2.11.5. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ which is an E-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ for some CM field E. Writing $[E : \mathbb{Q}] = r$, suppose that the endomorphism algebra L of V is a CM field such that $[L : \mathbb{Q}] = nr$. Then, letting $B \cong M_2(L^{\mathrm{op}})$ be the centralizer of L in* $\mathrm{End}_{\mathbb{Q}}(V)$ *and F the maximal totally real subfield of L, we have*

$$Hg(V) = R_{F/\mathbb{Q}}SU(B,^-).$$

Proof. Let $H = Hg(V)$. The Lefschetz group of V is $R_{F/\mathbb{Q}}U(B,^-)$, so we have:

$$H \subseteq R_{F/\mathbb{Q}}U(B,^-), \tag{2.13}$$

where $U(B,^-)$ is an F-form of $GL(2)$.

Since V is an E-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$. Thus by Lemma 2.11.2 we have:

$$H \subseteq R_{J/\mathbb{Q}} SU(C, ^-),$$

where J is the maximal totally real subfield of E and $C \cong M_{2n}(E^{\mathrm{op}})$ is the centralizer of E in $\mathrm{End}_{\mathbb{Q}}(V)$. Here, the group $SU(C, ^-)$ is a J-form of $SL(2n)$.

By the same argument as in the proof of Proposition 2.11.4, the center of H is contained in $R_{F/\mathbb{Q}} U(B, ^-) \cap L$ and so all its elements have norm 1 and since H is contained in $R_{J/\mathbb{Q}} SU(C, ^-)$ any product of n such elements is equal to ± 1. Since there are only finitely many elements of L with this property, the center of H is finite and so, since H is reductive, we have that H is semisimple. The inclusion of (2.13) then yields

$$H \subseteq R_{F/\mathbb{Q}} SU(B, ^-),$$

where $SU(B, ^-)$ is an F-form of $SL(2)$. Then in the decomposition $V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma(F)} W_\sigma \oplus W_\sigma^*$ coming from the action of $L \otimes_{\mathbb{Q}} \mathbb{C}$, the 2-dimensional irreducible representation W_σ of $\mathrm{Lie}(H)_{\mathbb{C}}$ is just the standard representation of $\mathfrak{sl}_2$. By Lemma 2.5.3, we get $\mathrm{Lie}(H)_{\mathbb{C}} = \prod_{\sigma \in \Sigma(F)} \mathfrak{sl}_2$, which finishes the proof. □

Consider the torus given by

$$U_L = \ker(\mathrm{Norm}_L \colon R_{L/\mathbb{Q}} \mathbb{G}_m \to R_{F/\mathbb{Q}} \mathbb{G}_m).$$

Letting J denote the maximal totally real subfield of E, define the torus $SU_{L/E}$ to be the subtorus of U_L given by

$$SU_{L/E} = \ker(\mathrm{Norm}_{L/E} \colon U_L \to R_{J/\mathbb{Q}} \mathbb{G}_m).$$

Observe that $SU_{L/E}$ has codimension $[J : \mathbb{Q}]$ in U_L.

Proposition 2.11.6. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ which is an E-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$ for some CM field E. Writing $[E : \mathbb{Q}] = r$, suppose that the endomorphism algebra L of V is a CM field and $[L : \mathbb{Q}] = 2nr$. Then we have the inclusion $Hg(V) \subseteq SU_{L/E}$.*

Proof. Since $\dim_L V = 1$, the Lefschetz group of V is just the torus U_L and thus we know $Hg(V) \subseteq U_L$. Now, as in previous proofs, by Lemma 2.11.2, we know $Hg(V) \subseteq R_{J/\mathbb{Q}}SU(C,^-)$. Recall that $SU(C,^-)$ is defined to be the kernel of the norm map $\mathrm{Norm}_{C/E}$ acting on $U(C,^-)$, and so, in particular, given the inclusion $Hg(V) \subseteq U_L$, all the elements of $Hg(V)$ lie in the kernel of the norm map $\mathrm{Norm}_{L/E}$, which implies $Hg(V) \subseteq SU_{L/E}$. □

Proposition 2.11.7. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ which is an E-Hodge structure with Hodge numbers $(p, 0, \ldots, 0, p)$, where p is an odd prime, for some imaginary quadratic field E. Suppose that the endomorphism algebra L of V satisfies:*

1. *L is of Type IV*
2. *$L/\mathbb{Q}$ is a Galois extension*
3. *$[L : \mathbb{Q}] = 4p$*

Then L is a CM field and $Hg(V) = SU_{L/E}$.

Proof. Note, first of all, that using Albert's classification, since L is of Type IV, we know the center F_0 of L is a CM field and hence $[F_0 : \mathbb{Q}]$ is even. Writing $q^2 = [L : F_0]$, since $[L : \mathbb{Q}] = 4p$ with p odd, we have that $q = 1$ and so L is a CM field. Thus, Proposition 2.11.6 implies $Hg(V) \subseteq SU_{L/E}$.

Since E is an imaginary quadratic field, we know this torus $SU_{L/E}$ has codimension 1 in U_L, which has rank $[F : \mathbb{Q}] = 2p$. Hence the rank of $SU_{L/E}$ is $2p - 1$.

Following [Dod87] and [Kub65], for $\Sigma(L) = \{\sigma_1, \ldots, \sigma_{4p}\}$ the set of embeddings of L into $\mathbb{C}$, a *CM type* $\Theta \subset \Sigma(L)$ is defined by the criterion $\Theta \cup \overline{\Theta} = \Sigma(L)$ and the Hodge structure V corresponds to a unique CM type Θ on L [GGK12, Section V.C].

Now let $\tilde{L}$ be the Galois closure of L and consider $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$. For every $g \in \mathrm{Gal}(\tilde{L}/\mathbb{Q})$ and for every $\sigma_i \in \Theta$, let $\sigma_i^g \colon L \to \mathbb{C}$ be the element of $\Sigma(L)$ defined by $x \mapsto g \cdot \sigma_i(x)$. We then have a CM-type on L given by

$$\Theta^g = \{\sigma_1^g, \ldots, \sigma_{4p}^g\}.$$

The *Kubota rank* of Θ, denoted $\mathrm{Rank}(\Theta)$, is the rank over $\mathbb{Z}$ of the submodule of $\mathbb{Z}[\Sigma(L)]$ spanned by the set $\{\Theta^g \mid g \in \mathrm{Gal}(\tilde{L}/\mathbb{Q})\}$.

We then have [GGK12, Proposition V.D.5]

$$\mathrm{Rank}(\Theta) = \dim_{\mathbb{Q}} Hg(V). \tag{2.14}$$

However, Tankeev proves in [Tan01, Corollary 3.15], that for a simple CM type Θ on a CM field L of degree $2p$ over $\mathbb{Q}$, where p is an odd prime, we have:

$$\mathrm{Rank}(\Theta) \geq 2p - 1.$$

Hence it follows from (2.14) that we have $\dim_{\mathbb{Q}} Hg(V) \geq 2p - 1$ and so the result is proved. □

We now combine the above results to obtain the following theorem about simple polarizable $\mathbb{Q}$-Hodge structures that have endomorphism algebra of Type IV and that are also E-Hodge structures with Hodge numbers $(p, 0, \ldots, 0, p)$.

Theorem 2.11.8. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ which is an E-Hodge structure with Hodge numbers $(p, 0, \ldots, 0, p)$, where p is a prime and E is an imaginary quadratic field. If the endomorphism algebra $L = \mathrm{End}_{\mathbb{Q}-HS}(V)$ is of Type IV, then L is a CM field. Writing $4p = m[L:\mathbb{Q}]$, letting $B \cong M_m(L^{\mathrm{op}})$ be the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$, and letting F be the maximal totally real subfield of L, we have*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}} SU(B, ^{-}) & \text{if } [L:\mathbb{Q}] \neq 4p \\ SU_{L/E} & \text{if } [L:\mathbb{Q}] = 4p \text{ and either } p = 2 \text{ or } L/\mathbb{Q} \text{ is a Galois extension.} \end{cases}$$

Proof. Observe that since $\dim_E V = 2p$, we must have, using the embedding $E \hookrightarrow L$, that $[L : E]$ is equal to one of 1, 2, p, or $2p$. The case when $[L : E] = 1$ is taken care of by Proposition 2.11.3.

In the case when $[L : E] = 2$, we know $[L : \mathbb{Q}] = 4$. Since L is assumed to be of Type IV, Albert's classification (see Section 2.2.2) yields that L is a CM field. So $m = p$ and

$q = 1$, and thus we have $n_\sigma + n_{\overline{\sigma}} = p$ for all embeddings $\sigma \in \Sigma(L)$. But since p is prime, the numbers n_σ and $n_{\overline{\sigma}}$ are coprime and so the result follows from Proposition 2.11.4.

When $[L : E] = p$, we know $[L : \mathbb{Q}] = 2p$, and so, because L is of Type IV, Albert's classification yields that L is a CM field. Hence the result follows from Proposition 2.11.5.

Lastly, consider the case when $[L : E] = 2p$, namely when $[L : \mathbb{Q}] = 4p$. If p is an odd prime, then the result follows by Proposition 2.11.7. So suppose we have $p = 2$. Then Albert's classification yields that either L is a CM field of degree 8 over $\mathbb{Q}$ or L is a division algebra of degree 4 over an imaginary quadratic field F_0. In the latter case, we would have $n_\sigma + n_{\overline{\sigma}} = 2$, where σ and $\overline{\sigma}$ are the 2 embeddings of F_0 into $\mathbb{C}$. Namely, we have either $n_\sigma = n_{\overline{\sigma}} = 1$ or $n_\sigma = 2$ and $n_{\overline{\sigma}} = 0$, which correspond to exceptional cases (3) and (4) in Totaro's classification (see Theorem 2.3.2). Hence the endomorphism algebra L cannot be a division algebra and therefore must be a CM field of degree 8 over $\mathbb{Q}$. Moreover, by Lemma 2.11.2, because V is an E-Hodge structure, we know $Hg(V)$ is contained in $SU_{L/E}$, which has dimension 3. However by Lemma 2.7.1, the group $Hg(V)$ has dimension at least 3. So $Hg(V)$ is equal to $SU_{L/E}$, which finishes the proof. □

2.12 Main Results

Proposition 2.12.1. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ and Hodge numbers $(1, 0, \ldots, 0, 1)$. Then $Hg(V) = Lef(V)$.*

Proof. Albert's classification yields that the endomorphism algebra L of V is either $\mathbb{Q}$ or an imaginary quadratic field. Writing $2 = m[L : \mathbb{Q}]$, this corresponds to L being of Type I with $m = 2$ or L being of Type IV with $m = 1$. When the weight w is odd, the first case corresponds to exceptional case (6) in Totaro's classification. Thus, if w is odd, then the endomorphism algebra L of V is either $\mathbb{Q}$ or an imaginary quadratic field, while if w is even, then L must be an imaginary quadratic field.

In the case when L is $\mathbb{Q}$, then by Table 2.1, we have $Hg(V) \subset SL(V)$, where $SL(V)$ has rank 1. However, by Remark 2.2.2, the Hodge group $Hg(V)$ is nontrivial and by Remark 2.2.3 the group $Hg(V)$ is semisimple. Hence $Hg(V)$ is equal to $SL(2)$.

In the case when L is an imaginary quadratic field, Table 2.1 yields $Hg(V) \subset U_L$. Since U_L has dimension 1 and $Hg(V)$ must be nontrivial, we get $Hg(V) = U_L$. □

2.12.1 Hodge Numbers $(p, 0, \ldots, 0, p)$

Theorem 2.12.2. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(p, 0, \ldots, 0, p)$, where p is prime. Then $Hg(V) = Lef(V)$.*

Proof. When V is of odd weight, the result follows using the equivalence of Remark 2.3.1 together with results of Ribet [Rib83, Theorems 1,2], from whose proofs we extensively borrow, and Tankeev [Tan82], which prove the result for simple complex abelian varieties of prime dimension. Thus we may assume that V is of even weight.

Assume first that p is odd. Then Albert's classification yields that the endomorphism algebra L of V is either a totally real field or a CM field. Moreover, Totaro's exceptional case (6) allows us to eliminate the possibility that L is a totally real field of degree p over $\mathbb{Q}$. Thus we are left with the following possibilities for L:

1. $\mathbb{Q}$ (Type I)

2. An imaginary quadratic field (Type IV)

3. A CM-field of degree $2p$ over $\mathbb{Q}$ (Type IV).

First consider Case (1). By Table 2.1, the Lefschetz group $Lef(V)$ of V is $SO(V)$. Letting $H = Hg(V)$ denote the Hodge group of V, Proposition 2.9.1 yields that H is either equal to $Lef(V)$ or is equal to $SU(2^k)$, where $2p = \binom{2^k}{2^{k-1}}$ for some $k \geq 3$. Thus, to prove the result we must show that this second option cannot occur. To do this, we use the following simple combinatorial argument to show that we cannot have $2p = \binom{2^k}{2^{k-1}}$ for any $k \geq 3$. We know

$$\binom{2^k}{2^{k-1}} = 2^{2^{k-1}} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2^k - 1)}{(2^{k-1})!}.$$

Moreover, by De Polignac's formula for the prime factorization of $n!$, the 2-adic order of the term $(2^{k-1})!$ is $2^{k-1}-1$. So, after cancellation, we have

$$\binom{2^k}{2^{k-1}} = 2 \cdot \frac{\text{(product of all odd numbers less than } 2^k)}{\text{(product of some odd numbers all less than } 2^{k-1})}. \tag{2.15}$$

Using known bounds on the prime-counting function π [RS62, Corollary 1] yields $\pi(2^k) - \pi(2^{k-1}) \geq 2$ for $k \geq 3$. Namely, the numerator in (2.15) always contains at least two terms not cancelled by the denominator. Hence we cannot have $2p = \binom{2^k}{2^{k-1}}$ for any $k \geq 3$. This finishes Case (1).

Now consider Case (2). In this case, the Lefschetz group $Lef(V)$ of V is $U(B, ^-)$. As in previous proofs, since $m = p$ and $q = 1$ in this case, we have $n_\sigma + n_{\overline{\sigma}} = p$ in the decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$ induced by $L \otimes_{\mathbb{Q}} \mathbb{C}$. Hence n_σ and $n_{\overline{\sigma}}$ are coprime and thus the result follows from Proposition 2.11.1.

Finally, consider Case (3). In this case, the Lefschetz group of V is $U_L = \ker(\mathrm{Norm}_L \colon R_{L/\mathbb{Q}}\mathbb{G}_m \to R_{F/\mathbb{Q}}\mathbb{G}_m)$, so we know $H \subset U_L$. Since $[F : \mathbb{Q}] = p$, the torus U_L has dimension p over $\mathbb{Q}$, and so to show $H = U_L$, we just need to show that $\dim_{\mathbb{Q}}(H) = p$.

Consider the character groups $X^*(H)$, $X^*(U_L)$, and $X^*(R_{L/\mathbb{Q}}\mathbb{G}_m)$. Since both H and U_L are contained in $R_{L/\mathbb{Q}}\mathbb{G}_m$, the groups $X^*(H)$ and $X^*(U_L)$ are quotients of $X^*(R_{L/\mathbb{Q}}\mathbb{G}_m)$. Here $X^*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$ is the free abelian group on the embeddings $\sigma \in \Sigma(L)$. The group $X^*(R_{L/\mathbb{Q}}\mathbb{G}_m)$ has a natural left action by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The group $X^*(U_L)$ is the quotient of $X^*(R_{L/\mathbb{Q}}\mathbb{G}_m)$ by the relation $\sigma + \overline{\sigma} = 0$ for all embeddings $\sigma \in \Sigma(L)$. The group $X^*(H)$ is a quotient of $X^*(U_L)$ with the property that the images of the embeddings σ in $X^*(H)$ are all distinct. Indeed, since $L = \mathrm{End}_H(V)$ is commutative, the H-module V cannot have multiplicities greater than 1 in its decomposition into simple modules over H, each corresponding to an embedding $\sigma \in \Sigma(L)$.

Now, consider the homomorphism

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(X^*(H))$$

giving the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Aut}(X^*(H))$. Because the elements of $X^*(H)$ are the embeddings $\sigma \in \Sigma(L)$, each one occurring once, we have

$$\mathrm{Ker}(\rho) = \mathrm{Gal}(\overline{\mathbb{Q}}/\tilde{L}),$$

where $\tilde{L}$ denotes the Galois closure of L. Hence the order of $\mathrm{Im}(\rho)$ is $[\tilde{L} : \mathbb{Q}]$. But $[\tilde{L} : \mathbb{Q}]$ is divisible by the prime number p, since $[L : \mathbb{Q}] = 2p$. Hence we may choose some $g \in \mathrm{Im}(\rho)$ of order p. Since g is in $\mathrm{Aut}(X^*(H))$, the action of g on the $\mathbb{Q}$-vector space

$$Y = X^*(H) \otimes \mathbb{Q}$$

makes Y into a module over $\mathbb{Q}[x]/(x^p - 1)$. We may write $\mathbb{Q}[x]/(x^p - 1) = \mathbb{Q}(\mu_p) \times \mathbb{Q}$, where μ_p is a p-th root of unity. Thus write

$$Y = Y_1 \oplus Y_2,$$

where Y_1 is a $\mathbb{Q}(\mu_p)$-vector space and Y_2 is a $\mathbb{Q}$-vector space. Because g has order p and thus does not have order 1, the element x does not act as the identity on Y_1. So Y_1 is nonzero. Then the dimension of Y_1 over $\mathbb{Q}$ is a multiple of $p - 1$, which yields

$$\dim_{\mathbb{Q}}(H) \geq p - 1.$$

To show that $\dim_{\mathbb{Q}}(H) = p$, it just remains to show that Y_2 is nonzero. Namely we need to show that Y contains a nonzero element fixed under the action of g. Choosing some embedding $\sigma_0 : L \to \mathbb{C}$, it is clear that the element

$$\chi = \sigma_0 + g\sigma_0 + \cdots + g^{p-1}\sigma_0 \tag{2.16}$$

in $X^*(H)$ is fixed by g. So we just need to show that χ is nonzero.

Let M be the Mumford-Tate group of V and consider the cocharacter groups $X_*(H)$, $X_*(M)$, and $X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$. As in Proposition 2.11.7, let $\Theta \subset \Sigma(L)$ be the CM type corresponding to the Hodge structure V [GGK12, Section V.C]. We may view Θ as an element of $X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$ by identifying it with $\sigma_1 + \cdots + \sigma_p$ in $X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$.

Since M is the smallest $\mathbb{Q}$-algebraic group such that the cocharacter $\gamma\colon \mathbb{G}_m \to GL(V_{\mathbb{C}})$ factors through $M_{\mathbb{C}}$, the element Θ lies in $X_*(M)$ and, in fact, the $\mathbb{Z}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$ generated by Θ is contained in $X_*(M)$. But $X_*(H)$ consists of the elements of $X_*(M)$ which have degree 0 in $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$. Thus, in particular, $\eta = \Theta - \overline{\Theta}$ is an element of $X_*(H)$. Since each embedding $\sigma \in \Sigma(L)$ has coefficient $\pm w$ in η, we have:

$$\langle \eta, \sigma \rangle = \pm w \text{ for all } \sigma \in \Sigma(L),$$

where $\langle\ ,\ \rangle : X_*(H) \times X^*(H) \to \mathbb{Z}$ denotes the natural bilinear pairing. Then, from the description of χ in (2.16), the integer $\langle \eta, \chi \rangle$ is the sum of p terms each of which is $\pm w$. Since p is odd, this means $\langle \eta, \chi \rangle$ is nonzero. Hence χ must be nonzero. So indeed Y contains a nonzero element fixed under the action of g and hence we have $\dim_{\mathbb{Q}} H = p$. This finishes Case (3) and so the statement of the theorem holds whenever p is odd.

So now assume $p = 2$. As before, Totaro's exceptional case (6) eliminates the possibility that L is a totally real quadratic field. Additionally, Totaro's exceptional case (1) eliminates the possibility that L is a totally indefinite quaternion algebra over $\mathbb{C}$. Now consider the case when L is of Type IV. Since $[L : \mathbb{Q}] = 4$, Albert's classification yields that L is a CM field of degree 2 or 4 over $\mathbb{Q}$. In the first case, namely when L is an imaginary quadratic field, we have $m = 2$ and $q = 1$ and so $n_\sigma + n_{\overline{\sigma}} = 2$ for the two embeddings $\sigma, \overline{\sigma} \in \Sigma(L)$. Hence, either $n_\sigma = 0$ and $n_{\overline{\sigma}} = 2$ or $n_\sigma = n_{\overline{\sigma}} = 1$. These correspond to exceptional cases (3) and (4) in Totaro's classification, so L cannot be an imaginary quadratic field. We are thus left with the following possibilities for L:

1. $\mathbb{Q}$ (Type I)

2. A totally definite quaternion algebra over $\mathbb{Q}$ (Type III)

3. A CM-field of degree 4 (Type IV).

Consider Case (1) first. In this case, the Lefschetz group of V is $SO(V)$, where $SO(V)$ has rank 2. Observe that by Lemma 2.7.1 the rank of H as an algebraic group over $\mathbb{Q}$ must be greater than or equal to 2. Moreover, by Remark 2.2.3, the group H is semisimple. Since $SO(V)$ is a $\mathbb{Q}$-form of $SO(4)$, it contains no semisimple proper subgroups of rank at least 2. So indeed H is equal to the Lefschetz group $SO(V)$.

For Case (2), the Lefschetz group is equal to $Sp(L,^-)$, which is a $\mathbb{Q}$-form of SL_2 and hence has rank 1. Thus we must have $H = Lef(V)$.

For Case (3), the Lefschetz group is U_L which is a torus of dimension 2. Applying Lemma 2.7.1 yields that H has dimension at least 2, so we must have $H = U_L$. Thus, indeed, when $p = 2$ the Hodge group of V is always equal to the Lefschetz group of V, which finishes the proof. □

2.12.2 Hodge Numbers $(4, 0, \ldots, 0, 4)$

Theorem 2.12.3. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(4, 0, \ldots, 0, 4)$ with endomorphism algebra L. Then the Hodge group $Hg(V)$ of V is described by Table 2.3. In particular, we have $Hg(V) = Lef(V)$ except in the following cases:*

1. *If $L = \mathbb{Q}$ and w is odd, then we can also have $Hg(V) = SL(2) \times SO(4)$, acting on V by the product of the standard representations*

2. *If $L = \mathbb{Q}$ and w is even, then we can also have $Hg(V) = SO(7)$, acting on V by the spin representation.*

3. *If L is an imaginary quadratic field such that V is an L-Hodge structure with Hodge numbers $(2, 0, \ldots, 0, 2)$, then $Hg(V) = R_{F/\mathbb{Q}} SU(B,^-)$, where $B \cong M_4(L^{\mathrm{op}})$ is the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$*

4. *If L is a CM field of degree 8 containing an imaginary quadratic field E such that V is an E-Hodge structure with Hodge numbers $(2, 0, \ldots, 0, 2)$, then $Hg(V) = SU_{L/E}$.*

Proof. If the endomorphism algebra L is of Type I, then L is either $\mathbb{Q}$, a totally real quadratic field, or a totally real field of degree 4 over $\mathbb{Q}$. However, the case when w is even and L is a totally real field of degree 4 corresponds to Totaro's exceptional case (6) and thus cannot occur.

Table 2.3: Hodge groups for $\mathbb{Q}$-Hodge structures with Hodge numbers $(4, 0, \ldots, 0, 4)$

L	$[L : \mathbb{Q}]$	Possible Hodge Groups		Equal to Lefschetz Group?
		Odd Weight	Even Weight	
Type I	1	$Sp(8)$	$SO(8)$	Yes
		$SL(2) \times SO(4)$	-	No
		-	$SO(7)$	No
	2	$R_{F/\mathbb{Q}}Sp({}_FV)$	$R_{F/\mathbb{Q}}SO({}_FV)$	Yes
	4	$R_{F/\mathbb{Q}}Sp({}_FV)$	-	Yes
Type II	4	$Sp(B,^-)$	$O^+(B,^-)$	Yes
	8	$R_{F/\mathbb{Q}}Sp(L,^-)$	-	Yes
Type III	4	$O^+(B,^-)$	$Sp(B,^-)$	Yes
	8	-	$R_{F/\mathbb{Q}}Sp(L,^-)$	Yes
Type IV	2	$U(B,^-)$	$U(B,^-)$	Yes
		$SU(B,^-)$	$SU(B,^-)$	No
	4	$R_{F/\mathbb{Q}}U(B,^-)$	$R_{F/\mathbb{Q}}U(B,^-)$	Yes
	8	U_L	U_L	Yes
		$SU_{L/E}$	$SU_{L/E}$	No

Similarly, if L is of Type II or III, then L is either a quaternion algebra over $\mathbb{Q}$ or a quaternion algebra over a real quadratic field. However, Totaro's exceptional cases (1) eliminate the possibility of L being a Type II (respectively Type III) quaternion algebra over a totally real quadratic field if w is even (respectively odd).

Lastly, writing $q^2 = [L : F_0]$ for the degree of L over its center F_0, if L is of Type IV and $q = 2$, then Albert's classification implies $m = 1$. Namely, we have $[L : \mathbb{Q}] = 8$ and L is a central simple algebra over the imaginary quadratic field F_0. But, as argued in the last paragraph of the proof of Theorem 2.11.8, such an L is not possible. Namely, if L is of Type IV, then L is a CM field.

We thus have the following list of possibilities for L:

1. $\mathbb{Q}$ (Type I)

2. A totally real quadratic field (Type I)

3. A totally real field of degree 4 (if w odd) (Type I)

4. A quaternion algebra over $\mathbb{Q}$ (Type II/ Type III)

5. A quaternion algebra over F with $[F:\mathbb{Q}] = 2$ (Type II if w odd/ Type III if w even)

6. An imaginary quadratic field (Type IV)

7. A CM field of degree 4 (Type IV)

8. A CM field of degree 8 (Type IV).

Using Table 2.1, when L is of Type I, then the Lefschetz group of V is $R_{F/\mathbb{Q}}Sp({}_FV)$ when w is odd and $R_{F/\mathbb{Q}}SO({}_FV)$ when w is even. Case (1) is then taken care of by Proposition 2.9.4, Case (2) is taken care of by Proposition 2.9.2, and Case (3) is taken care of by Proposition 2.9.1. From these, we conclude that when L is of Type I, the Hodge group of V is always equal to the Lefschetz group of V, except when $L = \mathbb{Q}$, in which case, the two additional groups $SL(2)\times SO(4)$ in the odd weight case, acting by the product of the standard representations, and $SO(7)$ in the even-weight case, acting by the spin representation, are also possible.

Similarly, Case (4) is taken care of by Proposition 2.10.2 as well as by Proposition 2.10.3 and Case (5) is taken care of by Proposition 2.10.1. From these, we conclude that when L is of Type II or III, then the Hodge group of V is always equal to the Lefschetz group of V. This leaves only the cases when L is of Type IV to consider. The result of the theorem then follows from the following propositions below: Proposition 2.12.4, Proposition 2.12.5, and Proposition 2.12.6, which address Cases (6), (7), and (8) respectively.

Proposition 2.12.4. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ and Hodge numbers $(4, 0, \ldots, 0, 4)$ such that the endomorphism algebra L of V is an imaginary quadratic field. Let $B \cong M_4(L^{\mathrm{op}})$ be the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$, let $\Sigma(L) = \{\sigma, \overline{\sigma}\}$ be* *the set of embeddings of L into $\mathbb{C}$, and let $n_\sigma = \dim V^{w,0}(\sigma)$ and $n_{\overline{\sigma}} = \dim V^{w,0}(\overline{\sigma})$. Then, either $\{n_\sigma, n_{\overline{\sigma}}\} = \{1,3\}$ or $\{n_\sigma, n_{\overline{\sigma}}\} = \{2,2\}$ and we have:*

$$Hg(V) = \begin{cases} U(B,^-) & \text{if } \{n_\sigma, n_{\overline{\sigma}}\} = \{1,3\} \\ SU(B,^-) & \text{if } \{n_\sigma, n_{\overline{\sigma}}\} = \{2,2\}. \end{cases}$$

Proof. Since L is an imaginary quadratic field, we know $m = 4$ and $q = 1$, so $n_\sigma + n_{\overline{\sigma}} = 4$. Totaro's exceptional case (3) implies that we cannot have $\{n_\sigma, n_{\overline{\sigma}}\} = \{0,4\}$. Therefore either we have $\{n_\sigma, n_{\overline{\sigma}}\} = \{1,3\}$ or we have $\{n_\sigma, n_{\overline{\sigma}}\} = \{2,2\}$. The case when $\{n_\sigma, n_{\overline{\sigma}}\} = \{1,3\}$ is taken care of by Proposition 2.11.1. In the case when $\{n_\sigma, n_{\overline{\sigma}}\} = \{2,2\}$, then V is an L-Hodge structure with Hodge numbers $(2,0,\ldots,0,2)$. Hence the result follows from Proposition 2.11.3. □

Proposition 2.12.5. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ and Hodge numbers $(4,0,\ldots,0,4)$ such that the endomorphism algebra L of V is a CM field of degree 4. Let F be maximal totally real subfield of L and let $B \cong M_2(L^{\mathrm{op}})$ be the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$. Then $Hg(V) = R_{F/\mathbb{Q}}U(B,^-)$.*

Proof. The following proof borrows from Moonen and Zarhin's proof [MZ99, 7.5] of the analogous result for simple abelian fourfolds.

Consider the set of embeddings $\Sigma(L) = \{\sigma_1, \overline{\sigma}_1, \sigma_2, \overline{\sigma}_2\}$ of the CM field L into $\mathbb{C}$. Then, since $[L:\mathbb{Q}] = 4$, we know $m = 2$ and $q = 1$, hence for each $i \in \{1,2\}$, we have $n_{\sigma_i} + n_{\overline{\sigma}_i} = 2$. However, the case $n_{\sigma_1}n_{\overline{\sigma}_1} + n_{\sigma_2}n_{\overline{\sigma}_2} = 0$ corresponds to exceptional case (3) in Totaro's classification and the case $n_{\sigma_1} = n_{\sigma_2} = 1$ corresponds to exceptional case (4). So neither of these can occur. Thus, without loss of generality, we have

$$(n_{\sigma_1}, n_{\overline{\sigma}_1}) = (2,0) \text{ and } (n_{\sigma_2}, n_{\overline{\sigma}_2}) = (1,1). \tag{2.17}$$

Let Z denote the center of H. Then since H is contained in the Lefschetz group $R_{F/\mathbb{Q}}U(B,^-)$, we must have Z contained in the 2-dimensional torus $U_L := \ker(\mathrm{Norm}_L \colon R_{L/\mathbb{Q}}\mathbb{G}_m \to R_{F/\mathbb{Q}}\mathbb{G}_m)$.

Suppose Z is trivial. This implies $H \subseteq R_{F/\mathbb{Q}}SU(B,^-)$. However the centralizer of $R_{F/\mathbb{Q}}SU(B,^-)$ in $\mathrm{End}_{\mathbb{Q}}(V)$ is a quaternion algebra over F, whereas the centralizer of H in $\mathrm{End}_{\mathbb{Q}}(V)$ is L. Hence Z must be nontrivial.

Suppose Z has dimension 1. Then by Lemma 7.3 in [MZ99] there exists an imaginary quadratic subfield E of L such that $Z = SU_{L/E}$. Now let $C \cong M_4(E^{\mathrm{op}})$ be the centralizer of E in $\mathrm{End}_{\mathbb{Q}}(V)$. Then having the center Z of H equal to $SU_{L/E}$ implies $H \subseteq SU(C,^-)$. Let ν and $\overline{\nu}$ be the two embeddings of E into $\mathbb{C}$. Then letting $n_\nu = \dim V^{w,0}(\nu)$, we have $n_\nu + n_{\overline{\nu}} = 4$. By Lemma 2.11.2, we must have $n_\nu = 2 = n_{\overline{\nu}}$. However, since E is contained in L, the equalities in (2.17) imply $\{n_\nu, n_{\overline{\nu}}\} = \{1,3\}$, which is a contradiction. Hence Z cannot have dimension 1 and so we have shown that Z is 2-dimensional, meaning $Z = U_L$.

Now let ρ_1 and ρ_2 be the two embeddings of F into $\mathbb{C}$. Without loss of generality, we may assume that the two embeddings $\sigma_1, \overline{\sigma}_1 \in \Sigma(L)$ both extend ρ_1 and that the two embeddings $\sigma_2, \overline{\sigma}_2 \in \Sigma(L)$ both extend ρ_2.

The action of $F \otimes_{\mathbb{Q}} \mathbb{C}$ on $V \otimes_{\mathbb{Q}} \mathbb{C}$ yields a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = X_{\rho_1} \oplus X_{\rho_2},$$

where X_{ρ_1} and X_{ρ_2} are 4-dimensional $\mathbb{C}$-vector spaces.

For a fixed polarization $\langle , \rangle$ of V let $\psi : V \times V \to F$ be the bilinear form such that $\langle , \rangle = \mathrm{Tr}_{\mathbb{Q}}^F \circ \psi$ and let ψ_{ρ_1} and ψ_{ρ_2} be the restrictions of ψ to to X_{ρ_1} and X_{ρ_2} respectively. For any $v, w \in V$ and any $f \in L$ we have $\psi(fv, w) = \psi(v, \overline{f}w)$. Hence, since the field F is fixed under the Rosati involution on L, the vector spaces X_{ρ_1} and X_{ρ_2} are orthogonal with respect to ψ. So ψ_{ρ_1} and ψ_{ρ_2} are nondegenerate alternating bilinear forms such that

$$H_{\mathbb{C}} \subseteq U(X_{\rho_1}, \psi_{\rho_1}) \oplus U(X_{\rho_2}, \psi_{\rho_2}). \tag{2.18}$$

Hence, if $H_{\mathbb{C}}^{ss}$ denotes the semisimple part of $H_{\mathbb{C}}$ we have

$$H_{\mathbb{C}}^{ss} \subseteq SU(X_{\rho_1}, \psi_{\rho_1}) \oplus SU(X_{\rho_2}, \psi_{\rho_2}). \tag{2.19}$$

For $i \in \{1, 2\}$ write

$$X_{\rho_i} = V(\sigma_i) \oplus V(\overline{\sigma_i}), \tag{2.20}$$

where $V(\sigma_i)$ and $V(\overline{\sigma_i})$ are 2-dimensional irreducible $H_{\mathbb{C}}$-modules.

Using the decompositions (2.18) and (2.20) in combination, we may write the center $Z_{\mathbb{C}}$ of $H_{\mathbb{C}}$ as

$$Z_{\mathbb{C}} = \{(z_1 \cdot \mathrm{Id}, -z_1 \cdot \mathrm{Id}, z_2 \cdot \mathrm{Id}, -z_2 \cdot \mathrm{Id}) \mid z_1, z_2 \in \mathbb{C}\} \subset U(X_{\rho_1}, \psi_{\rho_1}) \oplus U(X_{\rho_2}, \psi_{\rho_2}).$$

Because the $V(\sigma_i)$ and $V(\overline{\sigma_i})$ are 2-dimensional irreducible $H_{\mathbb{C}}$-modules, the above description of $Z_{\mathbb{C}}$ yields that the projection of $H_{\mathbb{C}}^{ss}$ onto each factor $SU(X_{\rho_i}, \psi_{\rho_i})$ must be nonzero. But both of the $SU(X_{\rho_i}, \psi_{\rho_i})$ factors are simple. Thus, $H_{\mathbb{C}}^{ss}$ surjects onto each factor $SU(X_{\rho_i}, \psi_{\rho_i})$ in the inclusion in (2.19).

The argument in the proof of Lemma 2.5.3 then yields that the inclusion in (2.18) is in fact an equality. Since we already showed that the center of $H_{\mathbb{C}}$ is all of U_L, we have thus shown that H is equal to the Lefschetz group $R_{F/\mathbb{Q}}U(B, ^{-})$. □

The last remaining case to deal with in the proof of Theorem 2.12.3 is Case (8), namely the case when the endomorphism algebra is a CM field of degree 8.

Proposition 2.12.6. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ and Hodge numbers $(4, 0, \ldots, 0, 4)$ with endomorphism algebra L a CM field of degree 8. Letting F be the maximal totally real subfield of L, we have*

$$Hg(V) = \begin{cases} SU_{L/E} & \text{if } L \text{ contains an imaginary quadratic field } E \text{ such that } V \text{ is an} \\ & \quad E\text{-Hodge structure with Hodge numbers } (2, 0, \ldots, 0, 2) \\ U_L & \text{otherwise.} \end{cases}$$

Proof. Let $H = Hg(V)$ be the Hodge group of V. By Table 2.1, the Lefschetz group of V is the 4-dimensional torus U_L. By Lemma 2.7.1, the rank of H is greater than or equal to $\log_2(8) = 3$. So either H is a 3-dimensional subtorus of U_L or H is all of U_L.

Suppose L contains an imaginary quadratic field E such that V is an E-Hodge structure with Hodge numbers $(2, 0, \ldots, 0, 2)$. Then by Proposition 2.11.6, the group H is contained in the 3-dimensional torus $SU_{L/E}$. Hence in this case $H = SU_{L/E}$.

Conversely, if H is 3-dimensional, then by Lemma 7.3 in [MZ99], there exists an imaginary quadratic field E in L such that $H = SU_{L/E}$. Moreover, by Lemma 2.11.2 this means V must be an E-Hodge structure with Hodge numbers $(2, 0, \ldots, 0, 2)$.

Hence, if L contains no such field E, then H must be 4-dimensional and hence H must be equal to the Lefschetz group U_L. □

Proposition 2.12.4, Proposition 2.12.5, and Proposition 2.12.6 thus indeed verify the statement of Theorem 2.12.3 in Cases (6), (7), and (8) respectively. Since we have previously confirmed the statement of Theorem 2.12.3 in Cases (1)-(5), this completes the proof of Theorem 2.12.3.

□

2.12.3 Hodge Numbers $(2p, 0, \ldots, 0, 2p)$

Theorem 2.12.7. *Let V be a simple polarizable $\mathbb{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(2p, 0, \ldots, 0, 2p)$, where p is an odd prime. If the endomorphism algebra L of V is of Type I, II, or III, then*

$$Hg(V) = Lef(V).$$

However, if L is of Type IV and L contains an imaginary quadratic field E such that V is an E-Hodge structure with Hodge numbers $(p, 0, \ldots, 0, p)$, then

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}} SU(B, ^{-}) & \text{if } [L : \mathbb{Q}] \neq 4p \\ SU_{L/E} & \text{if } [L : \mathbb{Q}] = 4p \text{ and } L/\mathbb{Q} \text{ is a Galois extension.} \end{cases}$$

Proof. If L is of Type I, II, or III, we use Albert's and Totaro's classifications to obtain a list of possibilities for L. As in previous proofs, the case when w is even and L is a totally real field of degree $2p$ corresponds to Totaro's exceptional case (6) and thus cannot occur. Additionally, Totaro's exceptional cases (1) eliminate the possibility of L being a Type II (respectively Type III) quaternion algebra over a totally real field of degree p over $\mathbb{Q}$ if w is even (respectively odd).

If L is of Type IV and satisfies the assumptions of the theorem, then by Theorem 2.11.8, we know that L must be a CM field.

We thus have the following list of possibilities for L:

1. $\mathbb{Q}$ (Type I)
2. A real quadratic field (Type I)
3. A totally real field of degree p (Type I)
4. A totally real field of degree $2p$ (if w odd) (Type I)
5. A quaternion algebra over $\mathbb{Q}$ (Type II/ Type III)
6. A quaternion algebra over F, where $[F:\mathbb{Q}] = p$ (Type II if w odd/ Type III if w even)
7. A CM field of degree 2 (Type IV)
8. A CM field of degree 4 (Type IV)
9. A CM field of degree $2p$ (Type IV)
10. A CM field of degree $4p$ (Type IV).

First consider the cases when L is of Type I. Then by Table 2.1, the Lefschetz group of V is $R_{F/\mathbb{Q}}Sp({}_FV)$, when w is odd, and $R_{F/\mathbb{Q}}SO({}_FV)$, when w is even. So consider Case (1). By Proposition 2.9.3, the Hodge group $H = Hg(V)$ is either equal to $Lef(V)$ or, when w is even, we may also have $R_{L/\mathbb{Q}}SU(2^k)$, where $4p = \binom{2^k}{2^{k-1}}$ for some $k \geq 3$. However, by a combinatorial argument similar to the one used in the proof of Theorem 2.12.2, this latter case is impossible.

In Case (2), Proposition 2.9.1 yields that H is either equal to $Lef(V)$ or, when w is even, we may also have $R_{L/\mathbb{Q}}SU(2^k)$, where $2p = \binom{2^k}{2^{k-1}}$ for some $k \geq 3$. However, as verified in the proof of Theorem 2.12.2, this latter case is impossible. Case (3) is taken care of by Proposition 2.9.2. Case (4) is taken care of by Proposition 2.9.1 since in this case $\frac{n}{[L:\mathbb{Q}]} = 1$ and thus in both even and odd weights the only possibility for H is $Lef(V)$. This finishes the cases for L of Type I.

For L of Type II or Type III, referring to Table 2.1, Case (5) is taken care of by Proposition 2.10.1, again using that $2p$ cannot be of the form $\binom{2^k}{2^{k-1}}$ for $k \geq 3$. Case (6) is also taken care of by Proposition 2.10.1.

When L is of Type IV, Table 2.1 yields that the Lefschetz group is $R_{F/\mathbb{Q}}U(B,^{-})$. Namely, under the hypotheses of the theorem, the predicted Hodge group in the cases when L is of Type IV is strictly smaller than the Lefschetz group. The statement of the theorem for these Type IV endomorphism algebra cases, meaning Cases (7) through (10), follows from Theorem 2.11.8. □

2.13 Applications to the Hodge Conjecture for Abelian Varieties

In Section 2.12, we determined the possible Hodge groups of simple polarizable $\mathbb{Q}$-Hodge structures with Hodge numbers $(n, 0, \dots, 0, n)$ when n was equal to 1, a prime p, 4 and $2p$. The results for $n = 1$, p, and 4 are generalizations of previous results in [Rib83], [Tan82], and [MZ99] about the possible Hodge groups of simple n-dimensional abelian varieties. However, the results in Section 2.12 about the Hodge groups when n is equal $2p$ are new.

Since, by Remark 2.3.1, there is a polarization-preserving equivalence of categories between the category of $\mathbb{Q}$-Hodge structures of odd weight and Hodge numbers $(n, 0 \dots, 0, n)$, and the category of complex abelian varieties of dimension n, it is natural to ask about the implications of Theorem 2.12.7 for complex abelian varieties. In particular, it is natural to wonder about the implications in terms of both the Hodge Conjecture and the General Hodge conjecture for these simple complex abelian varieties of dimension $2p$.

In order to simplify notation, in the case of an abelian variety A, we will denote by $Hg(A)$ and $Lef(A)$ the Hodge and Lefschetz groups respectively of the $\mathbb{Q}$-Hodge structure

$V = H^1(A, \mathbb{Q})$.

If A has dimension n, we also introduce the notation $W(A)$ to denote the set of CM fields E such that V is an E-Hodge structure with Hodge numbers $(\frac{n}{[E:\mathbb{Q}]}, 0, \ldots, 0, \frac{n}{[E:\mathbb{Q}]})$.

Corollary 2.13.1. *Let A be a simple complex abelian variety of dimension $2p$, where p is an odd prime. Suppose the endomorphism algebra L of the $\mathbb{Q}$-Hodge structure $V = H^1(A, \mathbb{Q})$ satisfies either*

1. *L is of Type I, II, or III*
2. *L is of Type IV, there exists an imaginary quadratic field $E \in W(A)$, and $[L : \mathbb{Q}] \neq 4p$.*

Then, if the Hodge conjecture is true for all powers of A, then the General Hodge Conjecture is true for all powers of A.

Proof. By Theorem 2.12.7, the Hodge group $Hg(A)$ is semisimple and is equal to the semisimple part $Lef(A)^{ss}$ of the Lefschetz group of A. Moreover, when L is of Type III, then by Shimura's classification of the possible endomorphism algebras of a simple abelian variety [Shi63, Theorem 5], we must have L a quaternion algebra over $\mathbb{Q}$. Namely $\frac{4p}{[L:\mathbb{Q}]}$ is equal to p, where p is odd. So A satisfies the following two conditions:

1. $Hg(A) = Lef(A)^{ss}$
2. If L is of Type III, then $\frac{2 \dim A}{[L:\mathbb{Q}]}$ is odd.

These conditions are exactly the hypotheses of a result of Abdulali [Abd97, Theorem 5.1], which then shows that under the above circumstances, the Hodge Conjecture for all powers of A implies the General Hodge Conjecture for all powers of A. □

For any $E \in W(A)$, let J be the maximal totally real subfield of E and let C be the the centralizer of E in $\mathrm{End}_{\mathbb{Q}}(V)$. The Hodge structure V may be viewed as an E-vector space, say of dimension l. Thus define

$$W_E = \bigwedge_E^l V.$$

Since V is an E-Hodge structure, a result of Moonen and Zarhin [MZ98, Section 6] shows that W_E consists entirely of Hodge classes. The elements of W_E are called *Weil classes*.

We now introduce the group $M(A)$, originally defined by Murty in [Mur00, Section 3.6.4].

Definition 2.13.2. For a simple complex abelian variety A, let the *Murty group* $M(A)$ be given by

$$M(A) = Lef(A) \bigcap \left(\bigcap_{E \in W(A)} R_{J/\mathbb{Q}} SU(C,^{-}) \right).$$

Thus for any simple complex abelian variety A, we have:

$$Hg(A) \subseteq M(A) \subseteq Lef(A). \tag{2.21}$$

In [Mur00], Murty proves the following property about the Murty group:

Proposition 2.13.3. *[Mur00, Proposition 3.8] For a complex abelian variety A, the Hodge ring*

$$\mathcal{B}^{\bullet}(A^k) = \bigoplus_{l \geq 0} \left(H^{2l}(A^k, \mathbb{Q}) \cap H^{l,l} \right)$$

is generated by divisors and Weil classes for all $k \geq 1$ if and only if $Hg(A) = M(A)$.

Corollary 2.13.4. *Let A be a simple complex abelian variety of dimension $2p$, where p is an odd prime. Suppose the endomorphism algebra L of the $\mathbb{Q}$-Hodge structure $V = H^1(A, \mathbb{Q})$ satisfies either*

1. *L is of Type I, II, or III*
2. *L is of Type IV, there exists an imaginary quadratic field $E \in W(A)$, and if $[L : \mathbb{Q}] = 4p$, then $L/\mathbb{Q}$ is Galois.*

Then for every $k \geq 1$, the Hodge ring $\mathcal{B}^{\bullet}(A^k) = \bigoplus_{l \geq 0} \left(H^{2l}(A^k, \mathbb{Q}) \cap H^{l,l} \right)$ is generated by divisors and Weil classes.

Proof. When L is of Type I, II, or III in Albert's classification, then by 2.13.3 we have $Lef(A) = M(A)$. Since A corresponds to a simple polarizable $\mathbb{Q}$-Hodge structure V with

Hodge numbers $(2p, 2p)$, when L is of Type I, II, or III, then by Theorem 2.12.7, we have $Hg(A) = Lef(A)$. Hence, in Case (1) in the statement of the corollary, the Hodge group of A is indeed equal to the Murty group of A.

In the situation of Case (2), by Theorem 2.12.7 we have

$$Hg(A) = \begin{cases} R_{F/\mathbb{Q}}SU(B,^-) & \text{if } [L:\mathbb{Q}] \neq 4p \\ SU_{L/E} & \text{if } [L:\mathbb{Q}] = 4p, \end{cases}$$

where B is the centralizer of L in $\mathrm{End}_{\mathbb{Q}}(V)$. Moreover, in this case, the Lefschetz group of A is:

$$Lef(A) = \begin{cases} R_{F/\mathbb{Q}}U(B,^-) & \text{if } [L:\mathbb{Q}] \neq 4p \\ U_{L/E} & \text{if } [L:\mathbb{Q}] = 4p, \end{cases}$$

However, observe that this means for some $E \in W(A)$ with totally real subfield J and centralizer C in $\mathrm{End}_{\mathbb{Q}}(V)$, we have:

$$Hg(A) = Lef(A) \cap R_{J/\mathbb{Q}}SU(C,^-).$$

Namely, $M(A) \subseteq Hg(A)$ and so by (2.21) above, we have $M(A) = Hg(A)$. Then Murty's result, Proposition 2.13.3, implies that for every $k \geq 1$ the Hodge ring $\mathcal{B}^\bullet(A^k)$ is generated by divisors and Weil classes. □

Corollary 2.13.5. *Let A be a simple abelian variety of dimension $2p$, where p is an odd prime. Suppose the endomorphism algebra L of the corresponding Hodge structure $V = H^1(X, \mathbb{Q})$ is of Type I or II in Albert's classification. Then both the Hodge and General Hodge Conjectures are satisfied for every power of A.*

Proof. In [MZ98, Section 13], Moonen and Zarhin show that both the Type III and Type IV cases in the statement of Corollary 2.13.4 will yield exceptional Hodge classes in W_E, but that this will not occur in the Type I and Type II cases. Namely if L is of Type I or Type II, then all of the Weil classes in W_E are actually just divisor classes. Hence the Hodge ring $\mathcal{B}^\bullet(A^k)$ is generated by divisors and so the Hodge Conjecture is satisfied for every power of

A. However, by Corollary 2.13.1, since the Hodge Conjecture holds for all powers of A, the General Hodge Conjecture holds for all powers of A. $\square$

CHAPTER 3

Geometry of Schreieder's Varieties with Hodge Numbers $(g, 0, \ldots, 0, g)$

3.1 Introduction

In this chapter, we turn our attention to one particular class of examples, constructed by Schreieder in [Sch15], of smooth projective varieties whose rational cohomology realizes a $\mathbb{Q}$-Hodge structure with Hodge numbers $(n, 0, \ldots, 0, n)$. Fixing an integer $c \geq 1$ and a dimension $d \geq 1$, these smooth projective varieties X_c are constructed as smooth models of a quotient C_g^d/G, where C_g is a hyperelliptic curve of genus $g = \frac{3^c-1}{2}$ and where $G \cong (\mathbb{Z}/3^c\mathbb{Z})^{d-1}$ is a finite group of automorphism acting on the product C_g^d. Recalling the notation for the Hodge numbers $h^{p,q} = \dim H^{p,q}$ in the decomposition $H^k(X_c, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}$, Schreieder proves that these smooth projective varieties X_c have the following property:

Proposition 3.1.1. *[Sch15, Theorem 17] For any $c \geq 1$, the Hodge numbers of the d-dimensional variety X_c satisfy $h^{d,0} = h^{0,d} = g$ and $h^{p,q} = 0$ for all other $p \neq q$.*

Moreover, Schreieder proves that all Hodge classes of X_c are algebraic [Sch15, Section 8.2]. Hence, in addition to their noteworthy Hodge numbers, these varieties X_c give an infinite collection of varieties for which the group $H^d(X_c, \mathbb{Z})_{\text{alg}}$ of algebraic classes in $H^d(X_c, \mathbb{Z})$ has rank $h^{d/2,d/2}$, which is the maximum possible. In particular, in dimension $d = 2$ these varieties X_c give examples of surfaces with arbitrarily large geometric genus having maximal Picard number. They thus add to recent examples of Picard maximal surfaces given by Beauville in [Bea14] and Arapura-Solapurkar in [Ara16]. In higher dimensions, Shioda [Shi79, Section

3] and Beauville [Bea14, Propsition 11] showed that $2d$-dimensional Fermat hypersurfaces F of degree 3 and 4 have their group of algebraic classes $H^{2d}(F,\mathbb{Z})_{\mathrm{alg}}$ of maximal rank equal to $h^{d/2,d/2}$ and Beauville [Bea14, Proposition 12] constructed examples of cubic fourfolds which have this property as well, however aside from these, very few other nontrivial examples (for instance, with nonzero Hodge number $h^{d,0}$) were previously known. All of this motivates the study of this particular class of examples.

In the context of this thesis, the particular features of the Hodge numbers and Hodge classes of these varieties X_c allows us to prove:

Theorem 3.2.2. *For the d-dimensional variety X_c, the subspace $V_c \subset H^d(X_c,\mathbb{Q})$ of transcendental classes is a $\mathbb{Q}$-Hodge structure of CM type with Hodge numbers $(g,0,\ldots,0,g)$. It decomposes into a sum $V_c = \bigoplus_{i=0}^{c-1} W_i$, where each W_i is a simple polarizable $\mathbb{Q}$-Hodge structure of CM type with Hodge numbers $(3^i,0,\ldots,0,3^i)$ and endomorphism algebra $\mathbb{Q}(\zeta^{3^{c-(i+1)}})$.*

Note that if the dimension d of the variety X_c is odd, then we have $V_c = H^d(X_c,\mathbb{Q})$, meaning that the degree d rational cohomology of X_c is itself a $\mathbb{Q}$-Hodge structure with Hodge numbers $(g,0,\ldots,0,g)$.

The fact that these varieties X_c thus give an explicit geometric realization of a $\mathbb{Q}$-Hodge structure with Hodge numbers $(g,0,\ldots,0,g)$ as the rational cohomology of a smooth projective variety motivates our investigation of the geometry of these varieties X_c. In this chapter, we study the Iitaka fibration of the varieties X_c and prove the following:

Theorem 3.5.1. *For every $c \geq 2$, the Iitaka fibration of X_c has image $\mathbb{P}^1$ and yields a fibration $f\colon X_c \to \mathbb{P}^1$ with exactly 3^c+2 singular fibers, where the reducible fibers occur above 0 and ∞ and the irreducible fibers occur above the 3^c-th roots of $(-1)^d$.*

In the case, when the variety X_c has dimension 2, we may give a more detailed geometric analysis of the Iitaka fibration in order to conclude that for a particular subgroup Γ_c of index $6\cdot 3^c$ in $SL(2,\mathbb{Z})$:

Theorem 3.6.8. *For $c \geq 2$, the surface X_c is the elliptic modular surface attached to Γ_c.*

The organization of this chapter is as follows. In Section 3.2, we discuss initial cohomological properties of the varieties X_c that may be obtained directly from their construction. In Section 3.3, we present the details of the construction of these varieties X_c. In Section 3.4, we calculate the Kodaira dimension of the varieties X_c, showing first in Section 3.4.2 that X_c has Kodaira dimension 1 when X_c has dimension 2 and then proving inductively in Section 3.4.3.1 that the varieties X_c have Kodaira dimension 1 in any dimension d. In Section 3.5, we investigate the Iitaka fibration of X_c in arbitrary dimension d. Then in Section 3.6, we investigate the Iitaka fibration of X_c in greater detail in the case $d = 2$. We analyze the induced elliptic fibration $f\colon X_c \to \mathbb{P}^1$, showing in Section 3.6.3 that X_c is an extremal ellitpic surface with Mordell-Weil group $\mathbb{Z}/4\mathbb{Z}$. In Section 3.6.4 we analyze the j-invariant of the fibration and use this to show in Section 3.6.5 that the surface X_c is an elliptic modular surface.

3.2 Initial Properties

Before going into the details of the construction of the varieties X_c, we begin by presenting some of the initial cohomological properties of these varieties, so as to motivate further investigation of their geometry and place them within the broader context of this thesis.

Proposition 3.2.1. *[Sch15, Section 8.2] The d-dimensional Schreieder variety X_c comes equipped with an automorphism ϕ of order 3^c such that the pair (X_c, ϕ) satisfies:*

1. *The Hodge numbers $h^{p,q}$ of X_c are given by $h^{d,0} = h^{0,d} = g$ and $h^{p,q} = 0$ for all other $p \neq q$.*
2. *The action of ϕ on $H^{d,0}(X_c)$ has eigenvalues $\zeta, \zeta^2, \dots, \zeta^g$, where ζ is a primitive 3^c-th root of unity.*
3. *For all $p \geq 0$, the group $H^{p,p}(X_c)$ is generated by algebraic classes which are fixed by the action of ϕ.*

Since $H^{p,p}(X_c)$ is generated by algebraic classes, the varieties X_c perforce satisfy the

Hodge Conjecture. Moreover, we have the stronger statement that for any $0 \le p \le d$, the subgroup of algebraic classes $H^{2p}(X_c, \mathbb{Z})_{\text{alg}}$ in $H^{2p}(X_c, \mathbb{Z})$ has maximal rank $h^{p,p}$. These rather special properties of the varieties X_c allow us to prove the following:

Theorem 3.2.2. *For the d-dimensional variety X_c, the subspace $V_c \subset H^d(X_c, \mathbb{Q})$ of transcendental classes is a $\mathbb{Q}$-Hodge structure of CM type with Hodge numbers $(g, 0, \dots, 0, g)$. It decomposes into a sum $V_c = \bigoplus_{i=0}^{c-1} W_i$, where each W_i is a simple polarizable $\mathbb{Q}$-Hodge structure of CM type with Hodge numbers $(3^i, 0, \dots, 0, 3^i)$ and endomorphism algebra $\mathbb{Q}(\zeta^{3^{c-(i+1)}})$.*

Proof. From Proposition 3.2.1, if d is even, we have a decomposition $H^d(X_c, \mathbb{C}) = H^{d,0} \oplus H^{d/2,d/2} \oplus H^{0,d}$, where the subspaces $H^{d,0}$ and $H^{0,d}$ have dimension g and where the subspace $H^{d/2,d/2}$ is generated entirely by algebraic classes. Thus since $H^{d,0} \oplus H^{0,d}$ is the orthogonal complement of $H^{d/2,d/2}$ in $H^d(X_c, \mathbb{C})$ under the intersection pairing, which is a bilinear form defined over $\mathbb{Q}$, we know that the space $H^{d,0} \oplus H^{0,d}$ is defined over $\mathbb{Q}$. Namely, the subspace $V_c \subset H^d(X_c, \mathbb{Q})$ of transcendental classes has the property that $V_c \otimes_{\mathbb{Q}} \mathbb{C} = H^{d,0} \oplus H^{0,d}$ and thus V_c is a $\mathbb{Q}$-Hodge structure with Hodge numbers $(g, 0, \dots, 0, g)$. If d is odd, then we just have $H^d(X_c, \mathbb{Q}) = H^{d,0} \oplus H^{0,d}$ and so $V_c = H^d(X_c, \mathbb{Q})$.

It thus remains to show that the Hodge structure V_c is of CM type with endomorphism algebra having the stated property.

By Proposition 3.2.1, the variety X_c comes equipped with an automorphism ϕ of order 3^c. We now consider the induced action ϕ^* of ϕ on the Hodge structure V_c. From (2) of Proposition 3.2.1, this induced action ϕ^* of ϕ on $H^{d,0}$ has eigenvalues $\zeta, \zeta^2, \dots, \zeta^g$. Since ϕ has order 3^c, we know that ϕ^* satisfies the equation $x^{3^c} - 1 = 0$. Now, we may factor the polynomial $x^{3^c} - 1$ into a product of the k-th cyclotomic polynomial $\Phi_k(x)$ for k dividing 3^c as follows :

$$x^{3^c} - 1 = \sum_{k|3^c} \Phi_k(x) = \Phi_1(x)\Phi_3(x) \cdots \Phi_{3^c}(x).$$

Observe that ζ^{3^j} is a root of $\Phi_{3^{c-j}}$ for every $0 \le j \le c-1$ and that moreover every such ζ^{3^j} appears in the list $\zeta, \zeta^2, \dots, \zeta^g$. Therefore, the lowest degree polynomial $f(x)$ of which ϕ^* is a root is the polynomial:

$$f(x) = \prod_{j=0}^{c-1} \Phi_{3^{c-j}}(x) = \frac{x^{3^c-1}}{x-1}$$

Consider the endomorphism algebra $L = \mathrm{End}_{\mathbb{Q}-HS}(V_c)$ of V_c as a Hodge structure. Since ϕ^* gives an automorphism of the Hodge structure V_c we know:

$$\mathbb{Q}[x]/(f) \subset L. \tag{3.1}$$

Now observe that the degree of $f(x)$ is $3^c - 1$, which is equal to $2g$. So we have

$$2g = [\mathbb{Q}[x]/(f) : \mathbb{Q}] \leq [L : \mathbb{Q}] \leq 2g.$$

Hence we must have the equality $L = \mathbb{Q}[x]/(f)$. Then using the factorization of the polynomial $f(x)$ and the semisimplicity of the category of $\mathbb{Q}$-Hodge structures, we have that the $\mathbb{Q}$-Hodge structure V_c decomposes into a sum$V_c = \bigoplus_{i=0}^{c-1} W_i$, where each W_i is a simple polarizable $\mathbb{Q}$-Hodge structure with Hodge numbers $(3^i, 0, \ldots, 0, 3^i)$ and endomorphism algebra $\mathbb{Q}(\zeta^{3^{c-(i+1)}})$. Since each Hodge structure W_i has endomorphism algebra with an embedded CM field of degree equal to the dimension of W_i, it follows that each W_i is of CM type and consequently so is V_c [GGK12, Proposition V.3]. □

3.3 Construction of X_c

Now that we have provided some motivation for the study of the varieties X_c, we present the details of their construction as presented in [Sch15, Section 8].

For a fixed $c \geq 1$ consider the hyperelliptic curve C_g of genus $g = \frac{3^c-1}{2}$ given by a smooth projective model of the affine curve

$$\{y^2 = x^{2g+1} + 1\}$$

The curve C_g is obtained from this affine curve by adding a point at ∞, which is covered by an affine piece

$$\{v^2 = u^{2g+2} + u\},$$

where $x = u^{-1}$ and $y = v \cdot u^{-g-1}$. Note that to unify the discussion, here a hyperelliptic curve of genus 1 means an elliptic curve.

Consider the automorphism ψ on C_g of order 3^c given by, for ζ a primitive 3^c-th root of unity,

$$(x, y) \mapsto (\zeta \cdot x, y)$$
$$(u, v) \mapsto (\zeta^{-1} \cdot u, \zeta^g v)$$

The d-dimensional variety X_c is constructed inductively, so let us denote by $X_c(d)$ the d-dimensional such variety. Then, each $X_c(d)$ comes equipped with a distinguished automorphism, which we denote ϕ_d. In the case $d = 1$, the variety $X_c(1)$ is just the curve C_g and its distinguished automorphism ϕ_1 is the automorphism ψ.

The idea for the inductive construction is that if two such varieties $(X_c(d_1), \phi_{d_1})$ and $(X_c(d_2), \phi_{d_2})$ have been constructed, the new variety $(X_c(d_1+d_2), \phi_{d_1+d_2})$ can be constructed as follows:

Consider the subgroup of $\mathrm{Aut}(X_c(d_1) \times X_c(d_2))$ given by

$$H := \langle \phi_{d_1}^{-1} \times \mathrm{id}, \mathrm{id} \times \phi_{d_2} \rangle.$$

For each $i = 1, \ldots, c$, consider the element of order 3^i inside of H given by

$$\eta_i := (\phi_{d_1}^{-1} \times \phi_{d_2})^{3^{c-i}}.$$

Thus η_i generates a cyclic subgroup $H_i := \langle \eta_i \rangle \subset H$, which gives a filtration

$$0 = H_0 \subset H_1 \subset \cdots \subset H_c = \langle \psi^{-1} \times \phi_d \rangle$$

such that each quotient H_i/H_{i-1} is cyclic of order 3. Now, let

$$Y_0 = X_k(d_1) \times X_k(d_2),$$
$$Y_0' = \text{Blow up of } Y_0 \text{ along } \mathrm{Fix}_{Y_0}(\eta_1),$$
$$Y_0'' = \text{Blow up of } Y_0' \text{ along } \mathrm{Fix}_{Y_0'}(\eta_1).$$

Observe that since the action of the group H restricts to an action on $\mathrm{Fix}_{Y_0}(\eta_1)$, the action of H on Y_0 lifts to an action on Y_0' and then similarly to an action on Y_0''. Here, by abuse of notation, we let $\langle \eta_1 \rangle$ denote both the subgroups of $\mathrm{Aut}(Y_0')$ and $\mathrm{Aut}(Y_0'')$ generated by the action of $\eta_1 \in H$.

Define for $i \in \{1, \ldots, k\}$:

$$\begin{aligned} Y_i &= Y_{i-1}''/\langle \eta_i \rangle, \\ Y_i' &= \text{Blow up of } Y_i \text{ along } \mathrm{Fix}_{Y_i}(\eta_{i+1}), \\ Y_i'' &= \text{Blow up of } Y_i' \text{ along } \mathrm{Fix}_{Y_i'}(\eta_{i+1}). \end{aligned}$$

Namely we have the following diagram:

$$\begin{array}{ccccccccc} & Y_0'' & & Y_1'' & & \cdots & & Y_{k-1}'' & \\ \swarrow & & \searrow\;\swarrow & & \searrow\;\swarrow & & \searrow\;\swarrow & & \searrow \\ Y_0 & & Y_1 & & Y_2 & & Y_{k-1} & & Y_k \end{array}$$

Each arrow to the left in the above diagram corresponds to a sequence of two blow-up maps and each arrow to the right corresponds to a $3:1$ cover.

Schreieder proves [Sch15, Proposition 19] that each Y_{i+1} is a smooth model of $Y_i/\langle \eta_i \rangle$ and thus, in particular, the variety $X_c(d_1 + d_2) := Y_c$ is a smooth model of $Y_0/\langle \phi_{d_1}^{-1} \times \phi_{d_2} \rangle$. The distinguished automorphism $\phi_{d_1+d_2}$ on $X_c(d_1 + d_2)$ is then defined to be the one induced by the automorphism $\mathrm{id} \times \phi_{d_2}$ on Y_0.

Thus, the inductive construction of $X_c(d)$ is as follows. We know we can construct the variety $X_c(1)$ with distinguished automorphism ϕ_1, since, as discussed, this is just the curve C_g with automorphism ψ. Assume we can construct the variety $X_c(d)$ with distinguished automorphism ϕ_d, then the above construction allows us to construct the variety $X_c(d + 1)$ with distinguished automophism ϕ_{d+1}.

We will by abuse of notation assume that these varieties X_c are minimal. As constructed above, they are not, but since subvarieties correspond to Hodge classes in cohomology, after sufficient contractions they can be taken to be minimal, without affecting the desired form of the Hodge diamond.

3.4 The Kodaira Dimension of X_c

For a smooth algebraic variety V and any $m > 0$, the *m-th plurigenus* of V is given by$P_m = h^0(V, K_V^{\otimes m})$. The *Kodaira dimension* κ of V is $-\infty$ if $P_m = 0$ for all $m > 0$ and otherwise it is the minimum κ such that P_m/m^κ is bounded. If V has dimension d, then the Kodaira dimension of V is either $-\infty$ or an integer $0 \leq \kappa \leq d$.

In order to compute the Kodaira dimension of the variety X_c, we thus wish to consider the plurigenera $P_m = h^0(X_c, K_{X_c}^{\otimes m})$ for $m > 0$. Since X_c is constructed from C_g^d by a sequence of blow-ups and quotients, it is necessary to establish what happens to global sections of powers of the canonical bundle of C_g^d under these blow-ups and quotients. More generally, we establish how the vanishing of forms on X_c in any dimension changes under blow-ups and quotients.

3.4.1 Forms Under Quotients

Recalling the notation from the construction of X_c in Section 3.3, consider the $3:1$ cover maps $f_i : Y_i'' \to Y_{i+1}$, where Y_i'' and Y_{i+1} have dimension d. The Riemann-Hurwitz formula gives:

$$K_{Y_i''} = f_i^* \left(K_{Y_{i+1}} + \sum_{D \in \mathrm{Div}(Y_{i+1})} \frac{a-1}{a} D \right) \tag{3.2}$$

where a is the order of the group fixing the divisors in Y_i'' mapping to D under f_i. However, by construction, the group G_{i+1}/G_i acting on Y_i'' is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Namely for every irreducible divisor $D \in \mathrm{Div}(Y_{i+1})$, either $a = 1$ or $a = 3$. Moreover, the irreducible D for which $a = 3$ are exactly the images of the irreducible components of the exceptional divisors E_i'' obtained from the blow-up map $Y_i'' \to Y_i'$, where it should be noted that it may happen that $E_i'' \cong E_i'$. Let $E_{i,1}'' \ldots, E_{i,k_i}''$ be the irreducible components of E_i''. Observe that since η_{i+1} fixes each of the $E_{i,j}''$, each component $E_{i,j}''$ descends to an irreducible divisor on Y_{i+1}.

Equation (3.2) then yields:

$$K_{Y_i''} = f_i^* \left(K_{Y_{i+1}}\right) + \sum_{j=1}^{k_i} 2E_{i,j}''.$$

This gives:

$$K_{Y_i''}^{\otimes m} - \sum_{j=1}^{k_i} 2mE_{i,j}'' = f_i^* \left(K_{Y_{i+1}}^{\otimes m}\right). \tag{3.3}$$

Now for an algebraic variety V with a coordinate patch $(z_1, \ldots, z_d)$ having the standard action of $\mathbb{G}_m^d$ on $\mathbb{C}^d$, we say that a form ω is *toric* on the patch $(z_1, \ldots, z_d)$ if the divisor of zeros of ω is invariant under the action of $\mathbb{G}_m^d$.

Definition 3.4.1. A toric form ω on a coordinate patch $(z_1, \ldots, z_d)$ on an algebraic variety V has *vanishing sequence* $(\beta_1, \ldots, \beta_d)$ on the point $(z_1, \ldots, z_d) = (0, \ldots, 0)$ if ω vanishes to order β_i along the hypersurface $z_i = 0$.

Now consider a G_{i+1}-invariant form σ on Y_i''. Suppose Y_i'' has local coordinates $(z_1, \ldots, z_d)$ around some $E_{i,j}''$ such that without loss of generality $E_{i,j}''$ is given by $z_1 = 0$ and $E_{i,j}''$ is fixed by the action of G_{i+1}.

Consider the point $R = (0, \ldots, 0)$ on $E_{i,j}''$. Suppose the vanishing sequence of σ on R is

$$(\alpha_1, \ldots, \alpha_d).$$

Then, using Equation (3.3), the vanishing sequence of the pushforward of σ to Y_{i+1} has vanishing sequence on the image of R in Y_{i+1} given by:

$$(\frac{1}{3}(\alpha_1 - 2m), \alpha_2, \ldots, \alpha_d). \tag{3.4}$$

3.4.1.1 Forms Under Blow-Ups

Let V be a d-dimensional variety with a global section σ of $K_V^{\otimes m}$. Consider a codimension-k subvariety Z of V, where $k \geq 2$, given locally by $z_1 = \cdots = z_k = 0$. Suppose we have a form σ on V with vanishing sequence $(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)$ on Z. Namely, σ is given locally by:

$$f(z_1, \ldots, z_d)(dz_1 \cdots dz_d)^{\otimes m},$$

and the polynomial f has vanishing sequence $(\alpha_1, \dots, \alpha_k, 0, \dots, 0)$ on $z_1 = \cdots = z_k = 0$. Blowing up V at Z introduces new coordinates $(z'_1, \dots, z'_k)$, with $z_i z'_j = z_j z'_i$ for all i, j. Then, on the coordinate patch of the blown-up variety V' given by $z'_i \neq 0$, we have coordinates:

$$(\tilde{z}_1, \dots, \tilde{z}_{i-1}, z_i, \tilde{z}_{i+1}, \dots, \tilde{z}_k, z_{k+1}, \dots, z_d),$$

where $\tilde{z}_j = \frac{z'_j}{z'_i}$ and thus $z_j = z_i \tilde{z}_j$. Thus, locally around the exceptional divisor E, the form σ pulls back to the form:

$$z_i^{m(k-1)} f(z_i\tilde{z}_1, \dots, z_i\tilde{z}_{i-1}, z_i, z_i\tilde{z}_{i+1}, \dots, z_i\tilde{z}_k, z_{k+1}, \dots, z_d)(d\tilde{z}_1 \cdots d\tilde{z}_{i-1} dz_i d\tilde{z}_{i+1} \cdots d\tilde{z}_k dz_{k+1} \cdots dz_d)^{\otimes m}.$$

In the new coordinates $(\tilde{z}_1, \dots, \tilde{z}_{i-1}, z_i, \tilde{z}_{i+1}, \dots, \tilde{z}_k, z_{k+1}, \dots, z_d)$, consider the point $R = (0, \dots, 0)$. Then the vanishing sequence on R of the pullback of σ to V' is given by:

$$(\alpha_1, \dots, \alpha_{i-1}, \sum_{l=1}^{k} \alpha_l + m(k-1), \alpha_{i+1}, \dots, \alpha_d). \tag{3.5}$$

3.4.2 The 2-Dimensional Case

We begin by computing the Kodaira dimension of X_c in the case when X_c is a surface. Consider the product $C_g \times C_g$ and the automorphism of $C_g \times C_g$ given by $\psi_g^{-1} \times \psi_g$. The surface X_c is then the minimal resolution of the quotient $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$.

Observe that the fixed set of the action of ψ_g on C_g consists of 3 points. In the coordinate patch given by $\{y^2 = x^{2g+1} + 1\}$ we have two fixed points:

$$P_1 : (x, y) = (0, 1) \text{ and } P_2 : (x, y) = (0, -1).$$

In the coordinate patch given by $\{v^2 = u^{2g+2} + u\}$, we have a third fixed point:

$$Q : (u, v) = (0, 0).$$

Hence, the fixed set of $\psi_g^{-1} \times \psi_g$ on $C_g \times C_g$ is 9 points consisting of pairs of points taken from the set $\{P_1, P_2, Q\}$. The surface X_c is then obtained from $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$ by resolving these 9 singular points.

Now using the Implicit Function Theorem one may verify that the coordinate x is a local coordinate in the coordinate patch on C_g given by $\{y^2 = x^{2g+1} + 1\}$ and the coordinate v is a local coordinate in the coordinate patch on C_g given by $\{v^2 = u^{2g+2} + u\}$. Hence ψ_g acts with weight 1 around P_1 and P_2 and acts with weight g around Q.

Thus the $\mathbb{Z}/3^c\mathbb{Z}$-action of $\psi_g^{-1} \times \psi_g$ on $C_g \times C_g$ has weights $(-1, 1)$ around fixed points of the form (P_i, P_j), weights $(-1, g)$ around fixed points of the form (P_i, Q), weights $(-g, 1)$ around fixed points of the form (Q, P_i), and weights $(-g, g)$ around the fixed point (Q, Q).

We will call the 5 fixed points on which $\psi_g^{-1} \times \psi_g$ has weights $(-1, 1)$ or $(-g, g)$ the *Type I fixed points* of $C_g \times C_g$ and the 4 fixed points on which $\psi_g^{-1} \times \psi_g$ has weights $(-1, g)$ or $(-g, 1)$ the *Type II fixed points* of $C_g \times C_g$.

Proposition 3.4.2. *For $c \geq 2$, the surface X_c has Kodaira dimension* 1*.*

Proof. Let σ be a global section of $K_{Y_0}^{\otimes m}$ with vanishing sequence (α_1, α_2) on the η_c-fixed point (Q, P_1) and with vanishing sequence (β_1, β_2) on the η_c-fixed point (P_1, Q).

Now, as discussed in Section 3.3, the automorphism η_c acts with weights $(-g, 1)$ around (Q, P_1). Thus η_c acts locally on the preimage E_0' of (Q, P_1) in $\mathrm{Fix}_{Y_0'}(\eta_c)$ with weights

$$(-g, g + 1) \text{ and } (g, 1).$$

Hence η_c acts locally on the preimage E_0'' of (Q, P_1) in $\mathrm{Fix}_{Y_0''}(\eta_c)$with weights

$$(-g, 0), (0, g + 1), (g, g + 2), \text{ and } (-(g + 2), 1).$$

In particular, there is a coordinate patch W_0'' on E_0'' on which η_c acts with weights $(-g, 0)$. In the coordinate patch W_0'', consider the point $R_0 = (0, 0)$.

By Equation (3.5) in Section 3.4.1.1, the image of σ on Y_0'' has vanishing sequence on R_0 given by:

$$(\alpha_1 + 2\alpha_2 + 2m, \alpha_2).$$

Let W_1 denote the image of W_0'' under the map $f_0 : Y_0'' \to Y_1$ and let Z_1 denote the image of E_0'' under f_0. Then, by Equation (3.4) in Section 3.4.1, the image of σ on Y_1 has vanishing sequence on $f_0(R_0)$ given by:

$$\left(\frac{1}{3}(\alpha_1 + 2\alpha_2), \alpha_2\right).$$

Now observe that η_c acts with weights $(-g, 0)$ on W_1, meaning the fixed locus of the action of η_c on W_1 is all of Z_1. But Z_1 has codimension 1 in Y_1. Hence, locally on W_1, the blow up maps $Y_1'' \to Y_1' \to Y_1$ are isomorphisms. Namely, we have $W_1 \cong W_1' \cong W_1''$ and $Z_1 \cong Z_1' \cong Z_1''$, where η_c acts on W_1'' with weights $(-g, 0)$ and this action has fixed locus Z_1''. Then, if W_2 is the image of W_1'' under the map $f_1 : Y_1'' \to Y_2$ and Z_2 is the image of Z_1'' under f_0, then η_c acts on W_2 with weights $(-g, 0)$ and this action has fixed locus Z_2.

Inductively, define W_i to be the image of W_{i-1}'' under the map $f_{i-1} : Y_{i-1}'' \to Y_i$ and let Z_i be the image of Z_{i-1}'' under f_{i-1}. Then by induction, for all $1 \leq i \leq c-1$, we have that η_c acts on W_i with weights $(-g, 0)$ and this action has fixed locus Z_i. So we always have $W_i \cong W_i' \cong W_i''$ and $Z_i \cong Z_i' \cong Z_i''$.

Tracing the point R_0 through this sequence of blow-ups and quotients, consider the image of R_0 in Y_c. Inductively using Equations (3.4) and (3.5), the image of σ on Y_c has vanishing sequence on the image of R_0 given by:

$$\left(\frac{1}{3^c}(\alpha_1 + 2\alpha_2 - 3m(3^{c-1} - 1)), \alpha_2\right).$$

So, from Equation (3.21), for σ to correspond to a global section of $K_{X_c}^{\otimes m}$, we need:

$$\alpha_1 + 2\alpha_2 \geq 3m(3^{c-1} - 1) \quad \text{and} \quad \alpha_2 \geq 0. \tag{3.6}$$

Symmetrically, since η_c acts with weights $(-1, g)$ around (P_1, Q), there is a coordinate patch $\tilde{W}_0''$ on the preimage of (P_1, Q) in $\mathrm{Fix}_{Y_0''}(\eta_c)$ on which η_c acts with weights $(0, g)$. In this coordinate patch $\tilde{W}_0''$, consider the point $\tilde{R}_0 = (0, 0)$.

Using Equation (3.5), the image of σ on Y_0'' has vanishing sequence on $\tilde{R}_0$ given by:

$$(\beta_1, 2\beta_1 + \beta_2 + 2m).$$

Thus, by Equation (3.4), the image of σ on Y_1 has vanishing sequence on $f_0(\tilde{R}_0)$ given by:

$$\left(\beta_1, \frac{1}{3}(2\beta_1 + \beta_2)\right)$$

in the coordinate patch $\tilde{W}_1 := f_0(\tilde{W}_0'')$.

Once again, considering the image of the point $\tilde{R}_0$ in Y_c, we have inductively, using Equations (3.4) and (3.5), that the image of σ on Y_c has vanishing sequence on the image of $\tilde{R}_0$ given by:

$$\left(\beta_1, \frac{1}{3^c}(2\beta_1 + \beta_2 - 3m(3^{c-1} - 1))\right).$$

Hence for σ to correspond to a global section of $K_{X_c}^{\otimes m}$, we need:

$$\beta_1 \geq 0 \quad \text{and} \quad 2\beta_1 + \beta_2 \geq 3m(3^{c-1} - 1). \tag{3.7}$$

We now make use of the following theorem of Köck and Tait:

Theorem 3.4.3. *[KT15, Theorem 5.1] Let C be a hyperelliptic curve of genus $g \geq 2$ of the form $y^2 = f(x)$ for some f and let $\omega \in K_C^{\otimes m}$ be given by $\omega = \frac{dx^{\otimes m}}{y^m}$. Then an explicit basis for $H^0(C, K_C^{\otimes m})$ is given by the following:*

$$\begin{cases} \omega, x\omega, \ldots, x^{g-1}\omega & \text{if } m = 1 \\ \omega, x\omega, x^2\omega & \text{if } m = 2 \text{ and } g = 2 \\ \omega, x\omega, \ldots, x^{m(g-1)}\omega; y\omega, xy\omega, \ldots, x^{(m-1)(g-1)-2}y\omega & \text{otherwise} \end{cases} \cdot$$

Observe that for the purposes of computing Kodaira dimension, we may assume $m > 1$. Thus we consider global sections of $K_{C_g}^{\otimes m}$ of the form $x^a\omega$, where $0 \leq a \leq m(g-1)$ or of the form $x^a y\omega$, where $0 \leq a \leq (m-1)(g-1) - 2$.

Begin by considering the affine patch of C_g given by $\{y^2 = x^{2g+1} + 1\}$. We know the variable x is a local coordinate for C_g at the points P_1 and P_2. Since we have

$$\omega = \frac{dx^{\otimes m}}{y^m},$$

the form ω has order of vanishing equal to 0 at P_1 and P_2. Hence both forms $x^a\omega$ and $x^a y\omega$ have order of vanishing a at fixed points P_1 and P_2.

In (u, v)-coordinates, the form ω is given by:

$$\frac{(-1)^d u^{m(g-1)} du^{\otimes m}}{v^m}$$

Recall that v is a local coordinate near the point Q. The equation $v^2 = u^{2g+2} + u$ yields $2v \cdot dv = ((2g+2)u^{2g+1} + 1) \cdot du$. Namely, du and v vanish to the same order. Moreover, u has order of vanishing 2 with respect to v, hence the order of vanishing of ω at the point Q is $2m(g-1)$, which is equal to $3m(3^{c-1} - 1)$.

Hence a form $x^a\omega = u^{-a}\omega$ has order of vanishing at Q given by:

$$3m(3^{c-1}) - 2a$$

and a form $x^a y\omega = u^{-(a+g+1)}v\omega$ has order of vanishing at Q given by:

$$2m(g-1) - 2(a+g+1) + 1 = 3m(3^{c-1}) - 2a - 3^c.$$

Now, from Theorem 3.4.3, without loss of generality global sections of $K_{Y_0}^{\otimes m}$ are of three possible forms:

1. $x_1^{a_1}\omega_1 \times x_2^{a_2}\omega_2$
2. $x_1^{a_1}\omega_1 \times x_2^{a_2}y_2\omega_2$
3. $x_1^{a_1}y_1\omega_1 \times x_2^{a_2}y_2\omega_2$.

Consider first the case of a global section σ of $K_{Y_0}^{\otimes m}$ of the form $x_1^{a_1}\omega_1 \times x_2^{a_2}\omega_2$. Then in the above notation we have:

$$(\alpha_1, \alpha_2) = (3m(3^{c-1} - 1) - 2a_1, a_2) \tag{3.8}$$

$$(\beta_1, \beta_2) = (a_1, 3m(3^{c-1} - 1) - 2a_2) \tag{3.9}$$

Then Equations (3.6) and (3.7) yield, after simplification, that we must have:

$$a_1 = a_2.$$

In the case of a global section σ of $K_{Y_0}^{\otimes m}$ of the form $x_1^{a_1}\omega_1 \times x_2^{a_2}y_2\omega_2$, we have:

$$(\alpha_1, \alpha_2) = (3m(3^{c-1} - 1) - 2a_1, a_2) \tag{3.10}$$

$$(\beta_1, \beta_2) = (a_1, 3m(3^{c-1} - 1) - 2a_2 - 3^c) \tag{3.11}$$

Then Equations (3.6) and (3.7) yield, after simplification, that we must have:

$$2a_2 \geq 2a_2 + 3^c,$$

which is impossible. So no such σ can exist.

Finally, in the case of a global section σ of $K_{Y_0}^{\otimes m}$ of the form $x_1^{a_1} y_1 \omega_1 \times x_2^{a_2} y_2 \omega_2$, we have:

$$(\alpha_1, \alpha_2) = (3m(3^{c-1} - 1) - 2a_1 - 3^c, a_2) \tag{3.12}$$

$$(\beta_1, \beta_2) = (a_1, 3m(3^{c-1} - 1) - 2a_2 - 3^c) \tag{3.13}$$

Then Equations (3.6) and (3.7) yield, after simplification, that we must have:

$$2a_2 \geq 2a_2 + 2 \cdot 3^c,$$

which is impossible. So no such σ can exist.

Therefore, any global section of $K_{X_c}^{\otimes m}$ must necessarily come from a G_c-invariant global section of $K_{Y_0}^{\otimes m}$ of the form:

$$x_1^a \omega_1 \times x_2^a \omega_2$$

for $0 \leq a \leq m(g-1)$.

Now observe that ψ_g sends the form ω to $\zeta^m \omega$. Hence the form $x_1^a \omega_1 \times x_2^a \omega_2$ is G_c-invariant if and only if $(-m-a) + (a+m) \equiv 0 \mod 3^c$, which is always the case.

Therefore, any global section of $K_{X_c}^{\otimes m}$ corresponds exactly to a form $x_1^a \omega_1 \times x_2^a \omega_2$ with $0 \leq a \leq m(g-1)$. This is a linear condition on m, therefore the Kodaira dimension of X_c is at most equal to 1.

However, by construction, we have $h^0(X_c, K_{X_c}) = h^{2,0} = g$. In particular, this means $h^0(X_c, K_{X_c})$ is greater than 1, so the Kodaira dimension of X_c is at least equal to 1. Hence, the Kodaira dimension of X_c is exactly equal to 1.

□

3.4.3 Kodaira Dimension of X_c in Arbitrary Dimension

Recall the notation and construction established in Section 3.3. In particular, recall that the fixed set of the action of ψ on C_g consists of 3 points. In the coordinate patch given by $\{y^2 = x^{2g+1} + 1\}$ we have two fixed points:

$$P_1 : (x, y) = (0, 1) \text{ and } P_2 : (x, y) = (0, -1).$$

In the coordinate patch given by $\{v^2 = u^{2g+2} + u\}$, we have a third fixed point:

$$Q : (u, v) = (0, 0).$$

Lemma 3.4.4. *For $c \geq 2$, consider the d-dimensional variety X_c. There exists a coordinate patch V_1 on $\mathrm{Fix}_{X_c}(\phi)$ on which ϕ acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, \ldots, 0, 1)$ and there exists a coordinate patch V_g on $\mathrm{Fix}_{X_c}(\phi)$ on which ϕ acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, \ldots, 0, g)$.*

Moreover, for $1 \leq j \leq d$ let σ_j be a global section of $K_{C_g}^{\otimes m}$ vanishing to order α_j on P_1 and P_2 and vanishing to order β_j on Q. Then V_1 and V_g may be chosen so that the form on X_c induced by $\sigma_1 \times \cdots \times \sigma_n$ has vanishing sequence $(\gamma_1, \ldots, \gamma_{d-1}, \alpha_d)$ on V_1 for some $\gamma_1, \ldots, \gamma_{d-1}$ and vanishing sequence $(\lambda_1, \ldots, \lambda_{d-1}, \beta_d)$ on V_g.

Proof. We proceed by induction on d. In order to keep track of the dimension, let us denote the d-dimensional variety associated to $c \geq 2$ by $X_c(d)$ and denote by ϕ_d its distinguished automorphism of order 3^c.

Recall that using the Implicit Function Theorem one may verify that the coordinate x is a local coordinate in the coordinate patch on C_g given by $\{y^2 = x^{2g+1}+1\}$ and the coordinate v is a local coordinate in the coordinate patch on C_g given by $\{v^2 = u^{2g+2} + u\}$. Hence ψ acts with $\mathbb{Z}/3^c\mathbb{Z}$-weight 1 around P_1 and P_2 and acts with $\mathbb{Z}/3^c\mathbb{Z}$-weight g around Q. This verifies the $d = 1$ case.

Assume the result holds for $d-1$. Then there is a coordinate patch V_1 on $\mathrm{Fix}_{X_c(d-1)}(\phi_{d-1})$ on which ϕ_{d-1} acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, \ldots, 0, 1)$ and there exists a coordinate patch V_g on $\mathrm{Fix}_{X_c(d-1)}(\phi_{d-1})$ on which ϕ_{d-1} acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, \ldots, 0, g)$. Moreover, we may choose V_1 and V_g so that the form on X_{d-1} induced by $\sigma_2 \times \cdots \sigma_d$ has vanishing sequence $(\gamma_1, \ldots, \gamma_{d-1}, \alpha_d)$ on V_1 for some $\gamma_1, \ldots, \gamma_{d-1}$ and vanishing sequence $(\lambda_1, \ldots, \lambda_{d-1}, \beta_d)$ on V_g.

Now let

$$Y_0 := C_g \times X_c(d-1).$$

Consider the action of $\mathrm{id} \times \phi_{d-1}$ on Y_0. Then V_1 corresponds to a coordinate patch V_1' on $\mathrm{Fix}_{Y_0}(\mathrm{id}\times\phi_{d-1})$ on which $\mathrm{id}\times\phi_{d-1}$ acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, 0, \ldots, 0, 1)$ and V_g corresponds to a coordinate patch V_g' on which $\mathrm{id}\times\phi_{d-1}$ acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, 0, \ldots, 0, g)$. Moreover, the form on Y_0 induced by $\sigma_1 \times \sigma_2 \times \cdots \sigma_d$ has vanishing sequence on V_1' given by:

$$(0, \gamma_1, \ldots, \gamma_{d-1}, \alpha_d)$$

and has vanishing sequence on V_g' given by:

$$(0, \lambda_1, \ldots, \lambda_{d-1}, \beta_d).$$

Now consider the action on Y_0 of

$$\eta_c := \psi^{-1} \times \phi_{d-1}.$$

On $V_1' \cap \mathrm{Fix}_{Y_0}(\eta_1)$ the $\mathbb{Z}/3^c\mathbb{Z}$-action of η_c has weights $(-1, 0 \ldots, 0, 1)$ and $(-g, 0, \ldots, 0, 1)$ and on $V_g' \cap \mathrm{Fix}_{Y_0}(\eta_1)$ the $\mathbb{Z}/3\mathbb{Z}$-action of η_c has weights $(-1, 0 \ldots, 0, g)$ and $(-g, 0 \ldots, 0, g)$.

Denote by $\tilde{V}_1$ the coordinate patch on $V_1' \cap \mathrm{Fix}_{Y_0}(\eta_1)$ on which the $\mathbb{Z}/3^c\mathbb{Z}$-action of η_c has weights $(-g, 0, \ldots, 0, 1)$ and denote by $\tilde{V}_g$ the coordinate patch on $V_g' \cap \mathrm{Fix}_{Y_0}(\eta_1)$ on which the $\mathbb{Z}/3^c\mathbb{Z}$-action of η_c has weights $(-1, 0, \ldots, 0, g)$. Then the form on Y_0 induced by $\sigma_1 \times \sigma_2 \times \cdots \sigma_d$ has vanishing sequence on $\tilde{V}_1$ given by:

$$(0, \gamma_1, \ldots, \gamma_{d-1}, \alpha_d)$$

and vanishing sequence on $\tilde{V}_g$ given by:

$$(0, \lambda_1, \ldots, \lambda_{d-1}, \beta_d).$$

The variety $X_c(d)$ is obtained from $Y_0 := C_g \times X_c(d-1)$ by performing a sequence of blow-ups along the fixed set of some η_i-action and $\mathbb{Z}/3\mathbb{Z}$-quotients. Consider the variety Y_0', obtained by blowing up Y_0 at $\mathrm{Fix}_{Y_0}(\eta_1)$. Locally in $\tilde{V}_1$ and $\tilde{V}_g$, the blow up introduces new coordinates $\tilde{z_1}, \tilde{z_n}$. In the $\tilde{z_1} \neq 0$ patches, the $\mathbb{Z}/3^c\mathbb{Z}$-action of η_c has weights $(-g, 0, \ldots, 0, g+1)$ and $(-1, 0, \ldots, 0, g+1)$ respectively. The $\mathbb{Z}/3^c\mathbb{Z}$-action of $\mathrm{id}\times\phi_{n-1}$ has weights $(0, 0, \ldots, 0, 1)$ and $(0, 0, \ldots, 0, g)$ respectively. Moreover, using Section 3.4.1.1 the form on Y_0' induced by $\sigma_1 \times \sigma_2 \times \cdots \sigma_d$ has vanishing sequence on the strict transform of $\tilde{V}_1$ given by:

$$(\alpha_d + m, \gamma_1, \ldots, \gamma_{d-1}, \alpha_d)$$

and has vanishing sequence on the strict transform of $\tilde{V}_g$ given by:

$$(\beta_d + m, \lambda_1, \ldots, \lambda_{d-1}, \beta_d).$$

Inductively, considering at each blow-up step the coordinate patch given by $\tilde{z}_1 \neq 0$, the corresponding coordinate patches in $Y_c = X_d$ will have $\mathbb{Z}/3^c\mathbb{Z}$-action of $\mathrm{id} \times \phi_{d-1}$ having weights $(0, 0, \ldots, 0, 1)$ and $(0, 0, \ldots, 0, g)$ respectively. Note that since since both 1 and g are congruent to 1 mod 3, these weights will be unaffected by the $\mathbb{Z}/3\mathbb{Z}$-quotients. Additionally, the form induced by $\sigma_1 \times \sigma_2 \times \cdots \sigma_d$ has vanishing sequences on them given by:

$$(\delta_1\alpha_d + \delta_2 m, \gamma_1, \ldots, \gamma_{d-1}, \alpha_d)$$

and

$$(\epsilon_1\beta_d + \epsilon_2 m, \lambda_1, \ldots, \lambda_{d-1}, \beta_d)$$

for some constants $\delta_1, \delta_2, \epsilon_1, \epsilon_2$. However, by construction ϕ_d is just the image of $\mathrm{id} \times \phi_{d-1}$ on X_d, so this finishes the proof. □

3.4.3.1 Kodaira Dimension Computation for X_c

Theorem 3.4.5. *For $c \geq 2$, the d-dimensional Schreieder variety X_c has Kodaira dimension 1.*

Proof. We proceed inductively on d. As in the proof of Lemma 3.4.4, let $X_c(d)$ denote the d-dimensional Schreieder variety associated to $c \geq 2$ and let ϕ_d denote its distinguished automorphism of order 3^c.

Using the proof of Theorem 3.4.2, we know that global sections of $K_{X_{c,2}}^{\otimes m}$ correspond to global sections of $K_{C_g \times C_g}^{\otimes m}$ of the form

$$x_1^a \omega_1 \times x_2^a \omega_2$$

where $0 \leq a \leq m(g-1)$. Thus, for the induction hypothesis, assume that global sections of $K_{X_c(d-1)}^{\otimes m}$ correspond to global sections of $K_{C_g^{d-1}}^{\otimes m}$ of the form

$$x_1^a \omega_1 \times \cdots \times x_{d-1}^a \omega_{d-1}$$

where $0 \leq a \leq m(g-1)$. Then we may proceed much in the same way as in the proof of Theorem 3.4.2.

Let $Y_0 = C_g \times X_c(d-1)$. Let σ be a global section of $K_{Y_0}^{\otimes m}$. Then we may write

$$\sigma = \sigma_1 \times \tau,$$

where σ_1 is a global section of $K_{C_g}^{\otimes m}$ and τ is a global section of $K_{X_c(d-1)}^{\otimes m}$.

Inductively, the form τ corresponds to a global section of $K_{C_g^{d-1}}^{\otimes m}$ of the form

$$x_2^a \omega_2 \times \cdots \times x_d^a \omega_d,$$

where $0 \leq a \leq m(g-1)$.

Suppose that σ_1 vanishes to order β_1 on the ψ-fixed points P_1 and P_2 of C_g and suppose that σ_1 vanishes to order α_1 on the ψ-fixed point Q of C_g.

Moreover, for $2 \leq j \leq d$ suppose the form $x_j^a \omega_j$ vanishes to order α_j on P_1 and P_2 and vanishes to order β_j on Q. By Lemma 3.4.4 there exists a coordinate patch V_1 on $\mathrm{Fix}_{X_{d-1}}(\phi_{d-1})$ on which ϕ_{d-1} acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, \ldots, 0, 1)$ and there exists a coordinate patch V_g on $\mathrm{Fix}_{X_{n-1}}(\phi_{n-1})$ on which ϕ_{n-1} acts with $\mathbb{Z}/3^c\mathbb{Z}$-weights $(0, \ldots, 0, g)$. Additionally, the patches V_1 and V_g may be chosen so that τ has vanishing sequence on the point $R_1 = (0, \ldots, 0)$ in V_1 given by:

$$(\gamma_1, \ldots, \gamma_{d-2}, \alpha_{d-1})$$

for some $\gamma_1, \dots, \gamma_{d-2}$ and vanishing sequence on the point $R_g = (0, \dots, 0)$ in V_g given by:

$$(\lambda_1, \dots, \lambda_{d-2}, \beta_{d-1}).$$

Consider the action of

$$\eta_1 := (\psi^{-1} \times \phi_{d-1})^{3^{c-1}}$$

on Y_0. We have:

$$\mathrm{Fix}_{Y_0}(\eta_1) = \mathrm{Fix}_{C_g}(\psi^{-1}) \times \mathrm{Fix}_{X_{d-1}}(\phi_{d-1}).$$

Hence, the action of

$$\eta_c = \psi^{-1} \times \phi_{d-1}$$

has $\mathbb{Z}/3^c\mathbb{Z}$-weights

$$(-g, 0, \dots, 0, 1)$$

around $Q \times V_1$ and has $\mathbb{Z}/3^c\mathbb{Z}$-weights

$$(-1, 0, \dots, 0, g)$$

around $P_1 \times V_g$. Moreover, the form σ has vanishing sequence

$$(\alpha_1, \gamma_1, \dots, \gamma_{d-2}, \alpha_{d-1})$$

on $Q \times R_1$ and has vanishing sequence

$$(\beta_1, \lambda_1, \dots, \lambda_{d-2}, \beta_{d-1})$$

on $P_1 \times R_g$.

Since η_c acts with weights $(-g, 0, \dots, 0, 1)$ around $Q \times V_1$, we have that η_c acts with weights

$$(-g, 0, \dots, 0, g+1) \text{ and } (g, 0, \dots, 0, 1)$$

locally on the preimages of $Q \times V_1$ in $\mathrm{Fix}_{Y_0'}(\eta_c)$. Hence η_c acts with weights

$$(-g, 0, \dots, 0, 0), (0, 0, \dots, 0, g+1), (g, 0, \dots, 0, g+2), \text{ and } (-(g+2), 0, \dots, 0, 1)$$

on the preimages of $Q \times V_1$ in $\mathrm{Fix}_{Y_0''}(\eta_c)$.

In particular, there is a coordinate patch W_0'' on the preimage E_0'' of $Q \times V_1$ in $\mathrm{Fix}_{Y_0''}(\eta_c)$ on which η_c acts with weights $(-g, 0, \ldots, 0, 0)$. Consider the point $S_0 = (0, \ldots, 0)$ in W_0''.

By Equation (3.5) in Section 3.4.1.1, the image of σ on Y_0'' has vanishing sequence on S_0 given by:

$$(\alpha_1 + 2\alpha_{d-1} + 2m, \gamma_1, \ldots, \gamma_{d-2}, \alpha_{d-1}).$$

Let W_1 denote the image of W_0'' under the map $f_0 : Y_0'' \to Y_1$ and let Z_1 denote the image of E_0'' under f_0. Then, by Equation (3.4)in Section 3.4.1, the image of σ on Y_1 has vanishing sequence on $f_0(S_0)$ given by:

$$(\frac{1}{3}(\alpha_1 + 2\alpha_{d-1}), \gamma_1, \ldots, \gamma_{d-2}, \alpha_{d-1}).$$

Now observe that η_c acts with weights $(-g, 0, \ldots, 0, 0)$ on W_1, meaning the fixed locus of the action of η_c on W_1 is all of Z_1. But Z_1 has codimension 1 in Y_1. Hence, locally on W_1, the blow up maps $Y_1'' \to Y_1' \to Y_1$ are isomorphisms. Namely, we have $W_1 \cong W_1' \cong W_1''$ and $Z_1 \cong Z_1' \cong Z_1''$, where η_c acts on W_1'' with weights $(-g, 0, \ldots, 0, 0)$ and this action has fixed locus Z_1''. Then, if W_2 is the image of W_1'' under the map $f_1 : Y_1'' \to Y_2$ and Z_2 is the image of Z_1'' under f_0, then η_c acts on W_2 with weights $(-g, 0, \ldots, 0, 0)$ and this action has fixed locus Z_2.

Inductively, define W_i to be the image of W_{i-1}'' under the map $f_{i-1} : Y_{i-1}'' \to Y_i$ and let Z_i be the image of Z_{i-1}'' under $f_{i=1}$. Then by induction, for all $1 \leq i \leq c-1$, we have that η_c acts on W_i with weights $(-g, 0, \ldots, 0, 0)$ and this action has fixed locus Z_i. So we always have $W_i \cong W_i' \cong W_i''$ and $Z_i \cong Z_i' \cong Z_i''$.

Tracing the point S_0 through this sequence of blowups and quotients, consider the image of S_0 in Y_c. Inductively using Equations (3.4) and (3.5), the image of σ on Y_c has vanishing sequence on the image of S_0 given by:

$$(\frac{1}{3^c}(\alpha_1 + 2\alpha_{d-1} - 3m(3^{c-1} - 1)), \gamma_1, \ldots, \gamma_{d-2}, \alpha_{d-1}).$$

So for σ to correspond to a global section of $K_X^{\otimes m}$, we need:

$$\alpha_1 + 2\alpha_{d-1} \geq 3m(3^{c-1} - 1), \gamma_1, \dots, \gamma_{d-2} \geq 0, \text{ and } \alpha_{d-2} \geq 0. \tag{3.14}$$

Symmetrically, since η_c acts with weights $(-1, 0, \dots, 0, g)$ around $P_1 \times V_g$, there is a coordinate patch $\tilde{W}_0''$ on the preimage of $P_1 \times V_g$ in $\mathrm{Fix}_{Y_0''}(\eta_c)$ on which η_c acts with weights $(0, 0, \dots, 0, g)$. In this coordinate patch $\tilde{W}_0''$, consider the point $\tilde{S}_0 = (0, \dots, 0)$.

Using Equation (3.5), the image of σ on Y_0'' has vanishing sequence on $\tilde{S}_0$ given by:

$$(\beta_1, \lambda_1, \dots, \lambda_{d-2}, 2\beta_1 + \beta_{d-1} + 2m).$$

Thus, by Equation (3.4), the image of σ on Y_1 has vanishing sequence on $f_0(\tilde{S}_0)$ given by:

$$(\beta_1, \lambda_1, \dots, \lambda_{d-2}, \frac{1}{3}(2\beta_1 + \beta_{d-1}))$$

in the coordinate patch $\tilde{W}_1 := f_0(\tilde{W}_0'')$.

Once again, considering the image of the point $\tilde{S}_0$ in Y_c, we have inductively, using Equations (3.4) and (3.5), that the image of σ on Y_c has vanishing sequence on the image of $\tilde{S}_0$ given by:

$$(\beta_1, \lambda_1, \dots, \lambda_{d-2}, \frac{1}{3^c}(2\beta_1 + \beta_{d-1} - 3d(3^{c-1} - 1))).$$

Hence for σ to correspond to a global section of $K_X^{\otimes m}$, we need:

$$\beta_1 \geq 0, \lambda_1, \dots, \lambda_{d-2} \geq, \text{ and } 2\beta_1 + \beta_{d-1} \geq 3d(3^{c-1} - 1). \tag{3.15}$$

By Theorem 3.4.3, the form σ_1 is of the form $x_1^b\omega_1$, where $0 \leq b \leq m(g-1)$ or of the form $x_1^b y_1 \omega_1$, where $0 \leq b \leq (m-1)(g-1) - 2$. Moreover, by the argument used in the proof of Theorem 3.4.2, both forms $x_1^b\omega_1$ and $x_1^b y_1\omega_1$ have order of vanishing b at fixed points P_1 and P_2. Forms $x_1^b\omega_1$ have order of vanishing $3m(3^{c-1}) - 2b$ at Q and forms $x_1^b y_1 \omega_1$ have order of vanishing $3m(3^{c-1}) - 2b - 3^c$ at Q.

Now without loss of generality σ is of two possible forms:

1. $x_1^b\omega_1 \times \tau$

2. $x_1^b y_1 \omega_1 \times \tau$

Consider first the case of σ of the form $x_1^{a_1}\omega_1 \times \tau$. Then in the above notation we have:

$$(\alpha_1, \gamma_1, \ldots, \gamma_{d-2}, \alpha_{d-1}) = (3m(3^{c-1} - 1) - 2b, \gamma_1, \ldots, \gamma_{d-2}, a) \quad (3.16)$$

$$(\beta_1, \lambda_1, \ldots, \lambda_{d-2}, \beta_{d-1}) = (b, \lambda_1, \ldots, \lambda_{d-2}, 3m(3^{c-1} - 1) - 2a) \quad (3.17)$$

Then Equations (3.14) and (3.15) yield, after simplification, that we must have:

$$a = b.$$

In the case of σ of the form $x_1^{a_1} y_1 \omega_1 \times \tau$, we have:

$$(\alpha_1, \gamma_1, \ldots, \gamma_{d-2}, \alpha_{d-1}) = (3m(3^{c-1} - 1) - 2b - 3^c, \gamma_1, \ldots, \gamma_{d-2}, a) \quad (3.18)$$

$$(\beta_1, \lambda_1, \ldots, \lambda_{d-2}, \beta_{d-1}) = (b, \lambda_1, \ldots, \lambda_{d-2}, 3m(3^{c-1} - 1) - 2a) \quad (3.19)$$

Then Equations (3.14) and (3.15) yield, after simplification, that we must have:

$$2a_2 \geq 2a_2 + 3^c,$$

which is impossible. So no such σ can exist.

Therefore, any global section of $K_{X_c(d)}^{\otimes m}$ must necessarily come from a G_c-invariant global section σ of $K_{Y_0}^{\otimes m}$ of the form:

$$x_1^a \omega_1 \times \tau$$

for $0 \leq a \leq m(g-1)$. Hence, global sections of $K_{X_c(d)}^{\otimes m}$ correspond to J-invariant global sections of $K_{C_g^d}^{\otimes m}$ of the form:

$$x_1^a \omega_1 \times \cdots x_d^a \omega_d,$$

for $0 \leq a \leq m(g-1)$. In particular, since the number of forms $x_1^a \omega_1 \times \cdots x_d^a \omega_d$ is linear in m, the Kodaira dimension of $X_c(d)$ is at most 1.

However, by construction, we have $h^0(X_c(d), K_{X_c(d)}) = g$, where we know $g \geq 4$. In particular, this means $h^0(X_c(d), K_{X_c(d)})$ is greater than 1, so the Kodaira dimension of $X_c(d)$ is at least equal to 1. Hence, the Kodaira dimension of $X_c(d)$ is exactly equal to 1.

□

3.5 The Iitaka Fibration of X_c

By Theorem 3.4.5, the surface X_c has Kodaira dimension 1. Let $\mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$ denote the space of hyperplanes in $H^0(X_c, K_{X_c}^{\otimes m})$. Then the image of the Iitaka fibration of X_c

$$f\colon X_c \to \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m})),$$

given by sending a point x to its evaluation on a basis of global sections of $K_{X_c}^{\otimes m}$ for m sufficiently divisible (see [Laz04, Theorem 2.1.33]), is a curve. Moreover, since X_c has $h^{1,0} = h^{0,1} = 0$, this curve must have genus 0. Moreover, the smooth fibers of the morphism f have Kodaira dimension 0 [Laz04, Theorem 2.1.33]).

Theorem 3.5.1. *For $c \geq 2$, the morphism $f\colon X_c \to \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$ has the following properties*

1. *The image of f is isomorphic to $\mathbb{P}^1$.*
2. *The fibration $f\colon X_c \to \mathbb{P}^1$ has exactly $3^c + 2$ singular fibers.*
3. *The reducible fibers occur above 0 and ∞ in $\mathbb{P}^1$.*
4. *The irreducible fibers occur above the 3^c roots of the polynomial $t^{3^c} - (-1)^d$ and each has a single singular point.*

Proof. For m sufficiently divisible we have the following diagram:

$$\begin{array}{ccc} X_c & \xrightarrow{f} & \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m})) \\ \uparrow & & \uparrow \\ C_g^d & \longrightarrow & \mathbb{P}(H^0(C_g^d, K_{C_g^d}^{\otimes m})) \end{array} \tag{3.20}$$

The horizontal maps are the Iitaka fibrations associated to $K_{X_c}^{\otimes m}$ and $K_{C_g^d}^{\otimes m}$ and the rational vertical map on the left is the sequence of blow-ups, blow-downs, and quotients needed to obtain X_c from C_g^d.

To understand the fibration $f : X_c \to \mathbb{P}^1 \subset \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$, we thus first would like to understand the composition

$$\alpha : C_g^d \dashrightarrow \mathbb{P}(H^0(X, K_X^{\otimes m})).$$

Recall from Theorem 3.4.3 that global sections of $K_{C_g}^{\otimes m}$ are of the form

$$x^a\omega \text{ for } 0 \leq a \leq m(g-1).$$

Observe that the only points of C_g on which the form $x^a\omega$ can vanish are the points

$$P_1 : (x, y) = (0, 1) \text{ and } P_2 : (x, y) = (0, -1)$$

in the coordinate patch on C_g given by $\{y^2 = x^{2g+1} + 1\}$ and the point

$$Q : (u, v) = (0, 0)$$

in the coordinate patch given by $\{v^2 = u^{2g+2} + u\}$. In fact, we have:

$$x^a\omega(P_1) = \begin{cases} dx^m & \text{if } a = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.21}$$

$$x^a\omega(P_2) = \begin{cases} (-1)^m dx^m & \text{if } a = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.22}$$

$$x^a\omega(Q) = \begin{cases} (-1)^m & \text{if } a = m(g-1) \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

From the proof of Theorem 3.4.5, global sections of $K_{X_c}^{\otimes m}$ correspond to global sections of $K_{C_g \times C_g}^{\otimes m}$ of the form:

$$s_a = x_1^a\omega_1 \times \cdots x_d^a\omega_d,$$

for $0 \leq a \leq m(g-1)$. Thus we may view the map $\alpha : C_g^d \dashrightarrow \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$ as the rational map sending:

$$(z_1, \ldots, z_d) \mapsto [s_0(z_1, \ldots, z_d) : \cdots : s_{m(g-1)}(z_1, \ldots, z_d)].$$

Say that $\mathbb{P}(H^0(\tilde{X}_c, K_{\tilde{X}_c}^{\otimes m}))$ has coordinates $[w_0 : \cdots : w_{m(g-1)}]$. Then on the affine patch of $\mathbb{P}(H^0(\tilde{X}_c, K_{\tilde{X}_c}^{\otimes m}))$ given by $w_0 \neq 0$, the image of α is of the form

$$(t, t^2, \ldots, t^{m(g-1)}),$$

where $t = x_1(z_1) \cdots x_d(z_d)$.

The images on the other affine patches of $\mathbb{P}(H^0(\tilde{X}_c, K_{\tilde{X}_c}^{\otimes m}))$ take similar forms. Hence the image of $\alpha : C_g^d \dashrightarrow \mathbb{P}(H^0(\tilde{X}_c, K_{\tilde{X}_c}^{\otimes m}))$ is the rational curve $\mathbb{P}^1$. This establishes statement (1) in the statement of the theorem.

Observe that α sends any curve in C_g^d of the form $C_g \times P_{i_1} \times \cdots P_{i_{n-1}}$ or any permutation of these coordinates to the point $[1 : 0 : \cdots : 0]$ in $\mathbb{P}(H^0(\tilde{X}_c, K_{\tilde{X}_c}^{\otimes m}))$, which corresponds to the point $[1 : 0]$ in $\mathbb{P}^1$. Moreover, the action of the group G which acts on C_g^d does not identify these various curves. Namely, the images of all of these curves are curves in the fiber of $[1 : 0]$ under the fibration f. In particular, the fiber of $[1 : 0]$ under the fibration f is singular.

Similarly, the map α sends all d curves of the form $C_g \times Q \times \cdots \times Q$ or any permutation of these coordinates to the point $[0 : \cdots : 0 : 1]$ in $\mathbb{P}(H^0(\tilde{X}_c, K_{\tilde{X}_c}^{\otimes m}))$, which corresponds to the point $[0 : 1]$ in $\mathbb{P}^1$. Moreover, the action of the group G does not identify these curves. Namely, the images of all of these curves are curves in the fiber of $[0 : 1]$ under the fibration f. In particular, the fiber of $[0 : 1]$ under the fibration f is singular.

So we have identified two reducible singular fibers of f: one occurring at the point $[1 : 0]$ in $\mathbb{P}^1$ and one occurring at the point $[0 : 1]$ in $\mathbb{P}^1$, which proves statement (3) of the theorem. Away from these two points, the image of α is given by points $(t, t^2, \ldots, t^{d(g-1)})$, where $t = x_1(z_1) \cdots x_d(z_d)$ is not equal to zero.

The fibers of α away from $[1 : 0]$ and $[0 : 1]$ are then the G-invariant hypersurfaces F_t in C_g^d defined by:

$$(y_1^2 = x_1^{2g+1} + 1,\ y_2^2 = x_2^{2g+1} + 1,\ \ldots,\ y_d^2 = x_d^{2g+1} + 1,\ x_1 \cdots x_d = t),$$

where we are assuming $t \neq 0$. Such hypersurfaces F_t have Jacobian:

$$\begin{pmatrix} (2g+1)x_1^{2g} & 2y_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (2g+1)x_2^{2g} & 2y_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & (2g+1)x_d^{2g} & 2y_d \\ x_2\cdots x_d & 0 & x_1x_3\cdots x_d & 0 & \cdots & x_1\cdots x_{d-1} & 0 \end{pmatrix}$$

Hence the fiber F_t is singular exactly when $y_1 = \cdots = y_d = 0$. When this is the case, then for each $i = 1, \ldots, d$ we have that x_i satisfies the equation $x_i^{2g+1} + 1 = 0$, namely x_i of the form $\xi^{2\gamma_i+1}$, where ξ is a primitive $2 \cdot 3^c$-th root of unity and $0 \leq \gamma_i \leq 3^c - 1$. Hence we have

$$t = \xi^{\left(2\left(\sum_{i=1}^d \gamma_i\right)+d\right)}$$

and so

$$t^{3^c} = \xi^{d \cdot 3^c} = (-1)^d.$$

In other words, it $t \in \mathbb{C}*$ is such that $t^{3^c} = (-1)^d$, then the fiber F_t is singular and has singularities at the points of the form

$$((x_1, y_1), \ldots, (x_d, y_d)) = ((\xi^{2\gamma_1+1}, y_1), \ldots, (\xi^{2\gamma_d+1}, y_d)).$$

Now since ζ is an even power of ξ, the action of the group G permutes the points on F_t. Namely, the image of F_t in X_c is a hypersurface with a single singular point. Thus there are 3^c irreducible singular fibers of f, each having a single singular point. Since we have already shown there are 2 reducible singular fibers, this finishes the proof. □

3.6 Geometry of X_c in the Dimension 2 Case

By Theorem 3.5.1, when X_c is a surface, the Iitaka fibration $f : X_c \to \mathbb{P}^1$ endows X_c with an elliptic fibration. In this section, we study this elliptic fibration in greater detail, exploiting many of the known tools for studying elliptic surfaces. One may find many of the basic properties of elliptic surfaces in the surveys [Mir89] and [SS10].

In order to understand the geometry of the elliptic surface $f : X_c \to \mathbb{P}^1$, it will be necessary to understand the resolutions of the singular points in $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$. To do this, we will make use of established facts about surface cyclic quotient singularities and Hirzebruch-Jung resolutions. A brief survey of these can be found in [Kol07, Section 2.4] and a more detailed explanation can be found in [Rei12].

3.6.1 Surface Cyclic Quotient Singularities and X_c

Consider the action of the cyclic group $\mathbb{Z}/r\mathbb{Z}$ on $\mathbb{C}^2$ given by

$$(z_1, z_2) \mapsto (\epsilon z_1, \epsilon^a z_2),$$

for some a coprime to r, where ϵ is a primitive r-th root of unity. We denote this action by:

$$\frac{1}{r}(1, a).$$

For coprime integers r and a, the *Hirzebruch-Jung continued fraction* of $\frac{r}{a}$ is the expansion:

$$\frac{r}{a} = b_0 - \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \frac{1}{\cdots}}}}.$$

The *Hirzebruch-Jung expansion* of $\frac{r}{a}$ is then the sequence

$$[b_0, b_1, b_2, b_3, \ldots b_s].$$

Then the minimal resolution of the singularity $\mathbb{C}^2/\frac{1}{r}(1, a)$ consists of a chain of $s+1$ exceptional curves $E_0, E_1, \ldots, E_s$ with nonzero intersection numbers $E_i.E_i = -b_i$ and $E_i.E_{i+1} = 1$ [Kol07, Proposition 2.32].

Recall from Section 3.3 that $C_g \times C_g$ has 9 fixed points under the action of $\psi_g^{-1} \times \psi_g$: five Type I fixed points of the form (P_i, P_j) or (Q, Q) and four Type II fixed points of the form (P_i, Q) or (Q, P_i). The automorphism $\psi_g^{-1} \times \psi_g$ locally acts on the Type I fixed points with weights $(-1, 1)$ on points (P_i, P_j) and with weights $(-g, g)$ on the point (Q, Q). It locally acts on the Type II fixed points with weights $(-1, g)$ on points (P_i, Q) and with weights $(-g, 1)$ on points (Q, P_i).

Hence, the Type I singular points of $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$, meaning those points corresponding to Type I fixed points of $C_g \times C_g$ under the action of $\psi_g^{-1} \times \psi_g$, have Hirzebruch-Jung expansion:

$$\underbrace{[2, \ldots, 2]}_{(3^c-1)-\text{times}} .$$

Namely the Type I singular points of $C_g \times C_g/\langle \psi_g^{-1} \times \psi_g \rangle$ are DuVal singularities of type A_{3^c-1}. In other words, the minimal resolution of each Type I singular point consists of a chain of $3^c - 1$ rational curves, each with self-intersection -2.

The Type II singular points of $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$ have Hirzebruch-Jung expansion:

$$[2, g + 1].$$

Hence the minimal resolution of each Type II singular point consists of a chain of two rational curves, one with self-intersection -2 and one with self-intersection $-(g + 1)$.

3.6.2 Singular Fibers of $f : X_c \to \mathbb{P}^1$

We now study in detail the geometry of the elliptic fibration $f : X_c \to \mathbb{P}^1$. In particular, although we know from Theorem 3.5.1, where the singular fibers of this fibration occur, we would like to study the specifics of these fibers. We extensively use Kodaira's classification, in [Kod63] and [Kod60], of the possible singular fibers of an elliptic surface. For a survey of the possible fiber types, see [Mir89, I.4] and [SS10, Section 4].

As we will see, the two kinds of singular fibers that appear here are the singular fibers of type I_b for $b > 0$ and singular fibers of type I_b^* for $b \geq 0$. Singular fibers of type I_b consist of b smooth rational curves meeting in a cycle, namely meeting with dual graph the affine Dynkin diagram $\tilde{A}_b$. Singular fibers of type I_b^* consist of $b+5$ smooth rational curves meeting with dual graph the affine Dynkin diagram $\tilde{D}_{b+4}$.

Theorem 3.6.1. *For $c \geq 2$, the elliptic surface $f : X_c \to \mathbb{P}^1$ has $3^c + 2$ singular fibers: one of type $I_{4 \cdot 3^c}$ located at 0, one of type $I_{3^c}^*$ located at ∞, and the remaining 3^c of type I_1 and located at the points ζ^i, for ζ a primitive 3^c-th root of unity.*

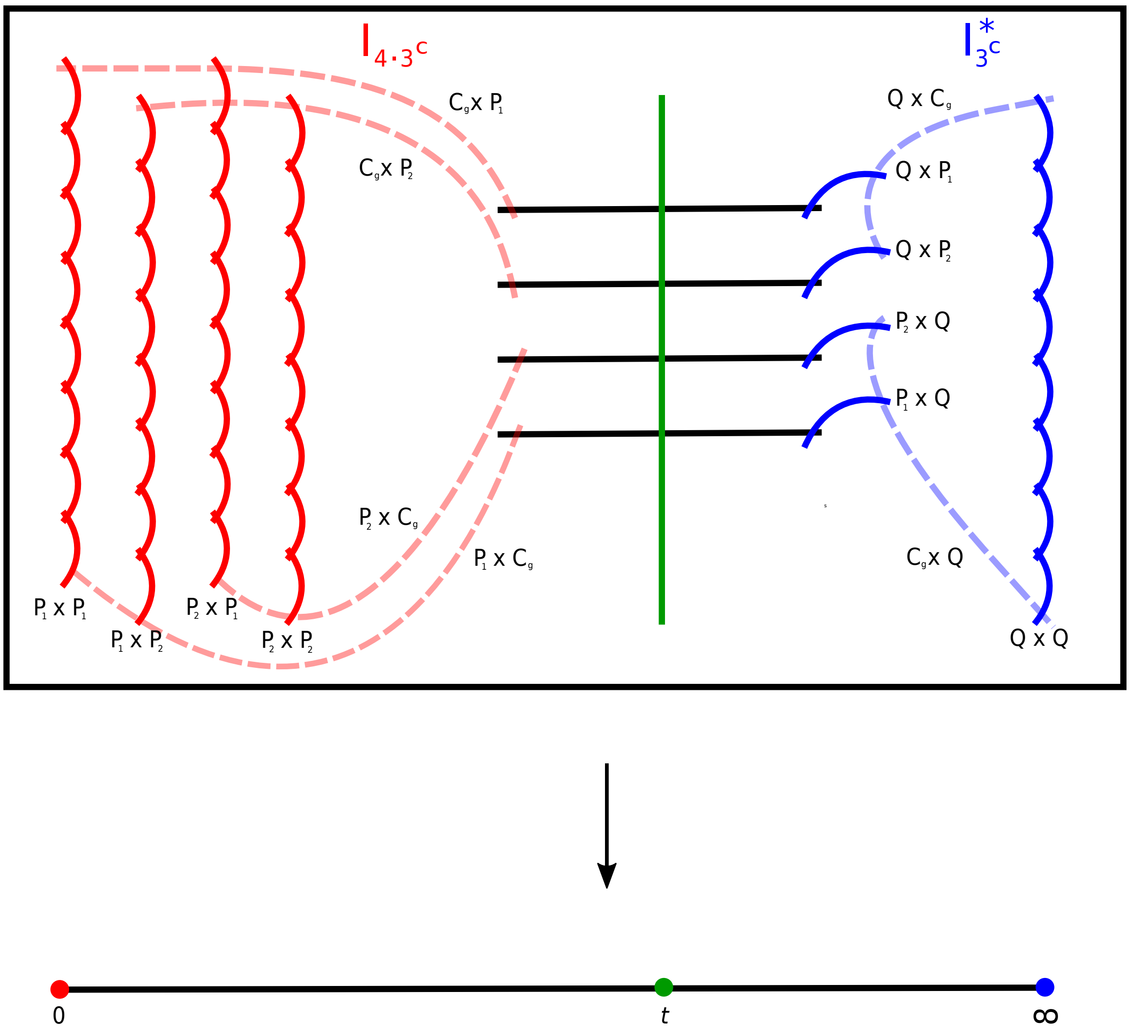

Figure 3.1: The elliptic surface $f : X_c \to \mathbb{P}^1$

Proof. Recall from the proof of Theorem 3.5.1 that for m sufficiently divisible, we have the following diagram

$$\begin{array}{ccc} X_c & \xrightarrow{f} & \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m})) \\ \uparrow & & \uparrow \\ C_g \times C_g & \longrightarrow & \mathbb{P}(H^0(C_g \times C_g, K_{C_g \times C_g}^{\otimes m})) \end{array} \tag{3.24}$$

and we consider the composition

$$\alpha : C_g \times C_g \dashrightarrow \mathbb{P}(H^0(X, K_X^{\otimes m}))$$

given by

$$(z_1, z_2) \mapsto [s_0(z_1, z_2) : \cdots : s_{m(g-1)}(z_1, z_2)],$$

where $s_a = x_1^a \omega_1 \times x_2^a \omega_2$. Then using the vanishing of forms established in Equations (3.21), (3.22), (3.23), we have

$$\begin{aligned} \alpha^{-1}([1:0:\cdots:0]) =& (P_1 \times (C_g - Q)) \cup (P_2 \times (C_g - Q)) \cup ((C_g - Q) \times P_1) \cup ((C_g - Q) \times P_2) \\ \alpha^{-1}([0:\cdots:0:1]) =& (Q \times Q) \cup (Q \times (C_g - P_1 - P_2)) \cup ((C_g - P_1 - P_2) \times Q). \end{aligned} \tag{3.25}$$

Consider the fixed points in $C_g \times C_g$ of the form (P_i, P_j). By (3.25), we know that the image under α of these points is the point $[1 : 0 : \cdots : 0]$ in $\mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$.

As established in Section 3.6.1, the points (P_i, P_j) are DuVal singularities of type A_{3^c-1} under the action of $\psi_g^{-1} \times \psi_g$. Thus the resolution of each such point consists of a chain of 3^{c-1} rational curves of self-intersection -2. By the diagram (3.24), for each such point (P_i, P_j), all of these 3^{c-1} rational curves in X_c must get mapped to the point $[1 : 0 : \cdots : 0]$ by the fibration $f : X_c \to \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$.

Moreover, by (3.24) in conjunction with (3.25), since f is a morphism we must have that the strict transforms in X_c of the curves

$$P_1 \times C_g,\ P_2 \times C_g,\ C_g \times P_1, \text{ and } C_g \times P_2$$

also get sent to $[1 : 0 : \cdots : 0]$. Note that the strict transform of $C_g \times P_j$ will intersect the chain of rational curves resolving the singularity $P_i \times P_j$ at one end of the chain and the strict transform of $P_i \times C_g$ will intersect the chain at the other end of the chain.

Now, consider the fixed point (Q, Q) in $C_g \times C_g$. It gets sent to the point $[0 : \cdots : 0 : 1]$ by α. As established in Section 3.6.1, the point (Q, Q) is also a DuVal singularity of type A_{3^c-1} under the action of $\psi_g^{-1} \times \psi_g$. Thus the resolution of (Q, Q) consists of a chain of 3^{c-1} rational curves of self-intersection -2. So, all of these 3^{c-1} rational curves in X get mapped to the point $[0 : \cdots : 0 : 1]$ by the fibration $f : X_c \to \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$.

Moreover, again by (3.24) in conjunction with (3.25), since f is a morphism we must have that the strict transforms in X of the curves

$$Q \times C_g \text{ and } C_g \times Q$$

get sent to $[0 : \cdots : 0 : 1]$ as well. Again, the strict transform of $Q \times C_g$ will intersect the chain of rational curves resolving the singularity (Q, Q) at one end and the strict transform of $C_g \times Q$ will intersect the chain at the other end.

So f sends the strict transforms in X_c of the curves

$$P_1 \times C_g,\ P_2 \times C_g,\ C_g \times P_1,\ \text{and } C_g \times P_2 \tag{3.26}$$

to $[1 : 0 : \cdots : 0]$ and the strict transforms of the curves

$$Q \times C_g \text{ and } C_g \times Q \tag{3.27}$$

to $[0 : \cdots : 0 : 1]$. Each of the four Type II fixed points in $C_g \times C_g$, namely those of the form either $Q \times P_i$ or $P_i \times Q$, has two of the curves in (3.26) and (3.27) passing through it. Observe that these four Type II fixed points are exactly the points on which the rational map $\alpha : C_g \times C_g \dashrightarrow \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$ is not defined.

Now, as established in Section 3.6.1, these Type II singular points have minimal resolution consisting of a chain of two rational curves, one with self-intersection -2 and one with self-intersection $-(g+1)$. Observe that for a Type II point of the form $Q \times P_i$, the strict transform in X_c of the curve $Q \times C_g$ will intersect this chain of rational curves at the end of the (-2)-curve and the strict transform of the curve $C_g \times P_i$ will intersect the chain of rational curves at the end of the $-(g+1)$-curve.

By the adjunction formula, the irreducible curves on X_c contained in a fiber of $f : X_c \to \mathbb{P}^1 \subset \mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$ are exactly those curves $F \subset X_c$ with $K_{X_c}.F = 0$. So, of the two curves in the resolution of a Type II singular point, the curve with self-intersection -2 gets mapped to a point by f and the curve with self-intersection $-(g+1)$ gets mapped to all of $\mathbb{P}^1$ by f.

Since the (-2)-curve in the resolution of each Type II singular point intersects either the curve $Q \times C_g$ or the curve $C_g \times Q$, both of which get sent to the point $[0 : \cdots : 0 : 1]$ by α, we have that f sends these (-2)-curves in the resolutions of the Type II singular points to $[0 : \cdots : 0 : 1]$ as well.

So we have accounted for all of the curves in the resolutions of the singular points of $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$. In summary, the fiber in X_c of the point $[1 : 0 : \cdots : 0]$ in $\mathbb{P}(H^0(X_c, K_{X_c}^{\otimes m}))$ is a cycle consisting of the four sets of $3^c - 1$ rational curves coming from the resolutions of the points $P_i \times P_j$ together with the four curves in (3.26). Namely, the fiber under f of $[1 : 0 : \cdots : 0]$ consists of

$$4(3^c - 1) + 4 = 4 \cdot 3^c$$

rational curves. This is a fiber of type $I_{4 \cdot 3^c}$ in Kodaira's classification of the singular fibers of an elliptic surface.

Similarly, the fiber in X_c of the point $[0 : \cdots : 0 : 1]$ consists of the $3^c - 1$ rational curves resolving the singularity $Q \times Q$ together with the curves in (3.27) and the four (-2)-curves, each from a resolution of a Type II point. Namely, the fiber under f of $[0 : \cdots : 0 : 1]$ consists of a chain of $3^c + 1$ rational curves, where each curve on the ends of the chain has two additional curves coming off it. This is a fiber of type $I_{3^c}^*$.

Now, by Theorem 3.5.1, we know that away from the points 0 and ∞ in $\mathbb{P}^1$, the fibration $f \colon X_c \to \mathbb{P}^1$ has singular fibers given by curves with a single singularity above the points ζ^i for $0 \leq i \leq 2g$. By Kodaira's classification of the singular fibers of an elliptic surface, the fiber $\bar{F}_t$ must, in fact, be a rational nodal curve, namely a fiber of type I_1.

Therefore, in summary, we have identified a singular fiber of type $I_{4 \cdot 3^c}$ at the point 0 in $\mathbb{P}^1$, we have identified a singular fiber of type $I_{3^c}^*$ at the point ∞, and we have identified 3^c singular fibers of type I_1, located at the points ζ^i.

We may confirm that these are indeed all of the singular fibers of f using the following description of the topological Euler number of an elliptic surface. From [CD89, Proposition 5.16] we have that for a complex elliptic surface $f : S \to C$ with fiber F_v at $v \in C$ having m_v components, we have

$$\chi_{\mathrm{top}}(S) = \sum_{v \in C} e(F_v), \tag{3.28}$$

where

$$e(F_v) = \begin{cases} 0 & \text{if } F_v \text{ is smooth} \\ m_v & \text{if } F_v \text{ is of type } I_n \\ m_v + 1 & \text{otherwise.} \end{cases}$$

For any elliptic surface, Noether's formula implies that the topological Euler number is 12 times the geometric Euler number. Since the surface X_c has irregularity $q = 0$ and geometric genus $p_g = g$, we thus have

$$\chi_{\text{top}}(X_c) = 12(g+1) = 6 \cdot 3^c + 6.$$

Considering the fibration $f : X_c \to \mathbb{P}^1$, the $4 \cdot 3^c$ components from the singular fiber of type $I_{4 \cdot 3^c}$, the $3^c + 5$ components from the singular fiber of type $I^*_{3^c}$, and the 3^c components from the 3^c singular fibers of type I_1 account for exactly $6 \cdot 3^c + 6$ on the right hand side of Equation (3.28). Thus, we have verified that we indeed have identified all the singular fibers of the elliptic surface X_c. □

3.6.3 The Mordell-Weil group of X_c

The Mordell-Weil group of an elliptic surface $f : S \to C$ is the group of K-rational points on the generic fiber of f, where $K = \mathbb{C}(C)$. Such an elliptic surface S is called *extremal* if it has maximal Picard rank $\rho(S)$, meaning $\rho(S) = h^{1,1}(S)$, and if it has Mordell-Weil rank $r = 0$.

Corollary 3.6.2. *The surface $f : X_c \to \mathbb{P}^1$ is an extremal elliptic surface.*

Proof. Following the notation in [SS10], for the fibration $f : X_c \to \mathbb{P}^1$ and for any $v \in \mathbb{P}^1$, let F_v denote the fiber $f^{-1}(v)$. Let m_v denote the number of components of F_v and let R denote the points of $\mathbb{P}^1$ underneath reducible fibers. Namely, we have

$$R = \{v \in \mathbb{P}^1 \mid F_v \text{ is reducible}\}.$$

The Shioda-Tate formula [SS10, Corollary 6.13] gives the Picard number $\rho(X_c)$ in terms of both the reducible singular fibers of $f : X_c \to \mathbb{P}^1$ and the rank r of the Mordell-Weil group:

$$\rho(X_c) = 2 + \sum_{v \in R} (m_v - 1) + r. \tag{3.29}$$

From Theorem 3.6.1, the elliptic fibration $f : X_c \to \mathbb{P}^1$ has two reducible singular fibers: one of type $I_{4 \cdot 3^c}$ at 0 and one of type $I^*_{3^c}$ at ∞. These have $4 \cdot 3^c$ and $3^c + 5$ components respectively. Hence we have:

$$\begin{aligned}\sum_{v \in R} (m_v - 1) &= (4 \cdot 3^c - 1) + (3^c + 4) \\ &= 5 \cdot 3^c + 3.\end{aligned}$$

So then Equation (3.29) becomes:

$$\rho(X_c) = 5 \cdot 3^c + 5 + r.$$

However we know the Picard number $\rho(X_c)$ of X_c satisfies $\rho(X_c) \leq h^{1,1}(X_c)$.

As discussed in the proof of Theorem 3.6.1, we have $\chi_{\text{top}}(X_c) = 12(g+1)$. Since $h^{1,0}(X_c) = h^{0,1}(X_c) = 0$ and $h^{2,0}(X_c) = h^{0,2}(X_c) = g$, it follows that $h^{1,1}(X_c) = 10(g + 1) = 5 \cdot 3^c + 5$. Therefore $r = 0$ and $\rho(X_c) = h^{1,1}(X_c)$. □

Since the Mordell-Weil group of the elliptic fibration $f : X_c \to \mathbb{P}^1$ is the group of $\mathbb{C}(\mathbb{P}^1)$-rational points on the generic fiber of f, elements of the Mordell-Weil group determine sections $\mathbb{P}^1 \to X_c$ and so may be viewed as curves in X_c. In order to determine the Mordell-Weil group of $f : X_c \to \mathbb{P}^1$, we thus would like to understand the sections of f as curves in X_c.

Recall that the action of $\psi_g^{-1} \times \psi_g$ on $C_g \times C_g$ has 9 fixed points: 5 of Type I, meaning of the form (P_i, P_j) or (Q, Q), and 4 of Type II, meaning of the form (P_i, Q) or (Q, P_i). Let us denote these four Type II points by:

$$\delta_1 = Q \times P_1,\ \delta_2 = Q \times P_2,\ \delta_3 = P_1 \times Q,\ \delta_4 = P_2 \times Q.$$

We have seen in the proof of Theorem 3.6.1 that the images in X_c of the points δ_i consist of two rational curves: one with self-intersection -2 and one with self-intersection $-(g+1)$. Let us denote the $-(g+1)$-curves corresponding to δ_1, δ_2, δ_3, and δ_4, respectively, by:

$$\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3,\ \text{and } \mathcal{C}_4.$$

Then we have the following:

Proposition 3.6.3. *For $c \geq 2$, the elliptic fibration $f : X_c \to \mathbb{P}^1$ has exactly four sections. Viewed as curves in X_c, these sections correspond exactly to the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$.*

Proof. By Corollary 3.6.2, the Mordell-Weil group of $f : X_c \to \mathbb{P}^1$ has rank 0. Namely, the fibration $f : X \to \mathbb{P}^1$ has only finitely many sections.

Now suppose we have a section $s : \mathbb{P}^1 \to X_c$ of f. Let us view s as a curve $\overline{s}$ in X_c. So $\overline{s}$ is a rational curve sitting inside of X_c and for every $t \in \mathbb{P}^1$, the curve $\overline{s}$ meets $f^{-1}(t)$ at exactly one point.

If $\overline{s}$ is not one of the rational curves introduced in resolving the singular points of $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$, then $\overline{s}$ meets the set of rational curves resolving such a singular point in at most one point. Namely, the curve $\overline{s}$ is isomorphic to a curve $\tilde{s}$ in $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$. Moreover, the curve $\overline{s}$ intersects the fiber $F_0 = f^{-1}(0)$ exactly once and intersects the fiber $F_\infty = f^{-1}(\infty)$ exactly once. Hence the preimage $\hat{s}$ of $\tilde{s}$ in $C_g \times C_g$ has at most two points fixed by the action of $\psi_g^{-1} \times \psi_g$. Since $\tilde{s}$ has genus 0, the Riemann-Hurwitz formula yields:

$$2g(\hat{s}) - 2 = 3^c(-2) + (3^c - 1)(l) \quad \text{for} \quad 0 \leq l \leq 2.$$

Simplifying using the bounds on l yields:

$$g(\hat{s}) \leq 1.$$

Namely, we have a morphism $\hat{s} \to C_g \times C_g$, where $\hat{s}$ has genus less than or equal to 1. However this is impossible since any such morphism can be composed with the projection map to get a morphism $\hat{s} \to C_g$, which cannot exist since C_g has genus $g \geq 4$. Hence we have shown that any section $s : \mathbb{P}^1 \to X_c$ must correspond to one of the rational curves introduced in resolving the singular points of $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$.

Moreover, using the canonical bundle formula for an elliptic surface, any section $s : \mathbb{P}^1 \to X_c$ considered as a curve in X_c must have self-intersection given by the negative of the geometric Euler characteristic $-\chi(X_c) = -(g+1)$ [SS10, Corollary 6.9].

However, we have established in the proof of Theorem 3.6.1 that all of the curves in the resolution of the singular points of $(C_g \times C_g)/\langle \psi_g^{-1} \times \psi_g \rangle$ have self-intersection -2 except for the four curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ coming from the resolution of the Type II fixed points δ_1, δ_2, δ_3, and δ_4. It thus remains only to verify that these four $-(g+1)$-curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ indeed give sections of $f : X_c \to \mathbb{P}^1$.

Consider the point $Q \times P_i$ in $C_g \times C_g$. As discussed in the proof of Theorem 3.4.2, the coordinate v is a local coordinate for C_g near the point Q and the coordinate x is a local coordinate for C_g near the points P_i. So then $\psi_g^{-1} \times \psi_g$ locally acts on the coordinate patch (v_1, x_2) around $Q \times P_i$ with weights $(-g, 1)$. After blowing up the point $Q \times P_i$, the automorphism $\psi_g^{-1} \times \psi_g$ locally acts with weights:

$$(-g, g+1) \text{ and } (g, 1)$$

on the exceptional curve.

Explicitly, if the coordinates of the exceptional curve are $[w_1 : w_2]$, then $\psi_g^{-1} \times \psi_g$ acts on the affine patch having coordinates (v_1, w_2) with weights $(-g, g+1)$ and acts on the affine patch having coordinates (w_1, x_2) with weights $(g, 1)$.

There are two fixed points of this action, corresponding to the points at 0 and at ∞ on the exceptional curve. Blowing up these two points yields local action by $\psi_g^{-1} \times \psi_g$ having weights:

$$(-g, 0), \ (0, g+1), \ (g, g+2), \text{ and } (-(g+2), 1).$$

Thus observe that the exceptional curve E obtained from blowing up the point $[w_1 : w_2] = [1 : 0]$ is point-wise fixed by the action of $\psi_g^{-1} \times \psi_g$. Namely, the curve E is unaffected by all subsequent blow-ups and quotients involved in producing Y_c from $C_g \times C_g$, aside from the fact that each $\mathbb{Z}/3\mathbb{Z}$-quotient multiplies the self-intersection number of E by 3. Hence the image of E in Y_c has self-intersection $-3^c = -(2g+1)$.

One may verify that obtaining the minimal surface X_c from Y_c requires g rounds of contracting (-1)-curves in the chain of curves resolving $Q \times P_i$. At each stage, a (-1)-curve lies adjacent to E in the chain. This is how in the minimal resolution of $Q \times P_i$, the image of E has self-intersection $-(g+1)$.

Now, as discussed in the proof of Theorem 3.6.1, away from the points at 0 and at ∞ in $\mathbb{P}^1$, a fiber $\bar{F}_t$ of the fibration $f : X_c \to \mathbb{P}^1$ may be viewed as the image in X_c of the curve $F_t \subset C_g \times C_g$ given by:

$$(y_1^2 = x_1^{2g+1} + 1,\ y_2^2 = x_2^{2g+1} + 1,\ x_1 x_2 = t).$$

As we approach the point $Q \times P_i$, we would like to switch coordinates and view this curve F_t as the curve given by:

$$(v_1^2 = u_1^{2g+2} + u_1,\ y_2^2 = x_2^{2g+1} + 1,\ u_1^{-1} x_2 = t).$$

Since we have local coordinates (v_1, x_2) near the point $Q \times P_i$, very close to $Q \times P_i$ we may think of F_t as being given simply by:

$$\gamma(v_1)^{-1} x_2 = t,$$

where $\gamma(v_1)$ is a continuous function in v_1. Namely, very close to $Q \times P_i$ the curve F_t has coordinates given by:

$$(v_1, x_2) = (v_1, t\gamma(v_1)).$$

It follows that the slope of F_t as we approach the point $Q \times P_i$ given by $(v_1, x_2) = (0, 0)$ is:

$$\lim_{v_1 \to 0} \frac{t\gamma(v_1)}{v_1} = 0.$$

Hence the strict transform F_t' of F_t in the blow-up at $Q \times P_i$ intersects the exceptional curve at the point $[w_1 : w_2] = [1 : 0]$. Near this point, we have local coordinates (v_1, w_2), where $x_2 = v_1 w_2$. So very close to the point $[w_1 : w_2] = [1 : 0]$, also given by $(v_1, w_2) = (0, 0)$, the strict transform F'_t is given by $v_1 w_2 = t\gamma(v_1)$.

But we know that the point $[w_1 : w_2] = [1 : 0]$ gets blown up and its exceptional curve is E. Hence the strict transform F''_t of F'_t after this second blow-up must intersect E. Very close to the point $[w_1 : w_2] = [1 : 0]$, the strict transform F'_t has coordinates given by:

$$(v_1, w_2) = \left(v_1, \frac{t\gamma(v_1)}{v_1}\right).$$

Hence the slope of F'_t as we approach $[w_1 : w_2] = [1 : 0]$ is:

$$\lim_{v_1 \to 0} \frac{t\gamma(v_1)}{v_1^2} = t.$$

Namely, the curve F''_t intersects E at the point with coordinate t on E. Hence the image $\bar{F}_t$ of F''_t in X_c intersects the image of E at the point with coordinate t.

So we have shown that a fiber $\bar{F}_t$ of the fibration $f : X_c \to \mathbb{P}^1$ intersects the $-(g+1)$-curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in each of the resolutions of the points $\delta_1 = Q \times P_1$ and $\delta_2 = Q \times P_2$ at the point on $\mathcal{C}_i$-curve with coordinate t. Symmetrically, the same also holds for the intersection of $\bar{F}_t$ with the $-(g+1)$-curves $\mathcal{C}_3$ and $\mathcal{C}_4$ in each of the resolutions of the points $\delta_3 = P_1 \times Q$ and $\delta_4 = P_2 \times Q$. Hence for each $i = 1, \ldots, 4$, the morphism $s_i : \mathbb{P}^1 \to X_c$ given by sending the point t to the point on $\mathcal{C}_i$ with coordinate t defines a section of the fibration $f : X_c \to \mathbb{P}^1$. So we indeed have four sections of f given by the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$. □

Corollary 3.6.4. *For $c \geq 2$, the Mordell-Weil group of $f : X_c \to \mathbb{P}^1$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.*

Proof. From Proposition 3.6.3, the Mordell-Weil group of X_c has order 4, namely it must be either the cyclic group $\mathbb{Z}/4\mathbb{Z}$ or the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Viewed as curves in X_c, the elements of the Mordell-Weil group correspond to the $-(g+1)$-curves

$$\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3,\ \text{and}\ \mathcal{C}_4$$

in the resolutions of the four Type II singular points

$$\delta_1 = Q \times P_1,\ \delta_2 = Q \times P_2,\ \delta_3 = P_1 \times Q,\ \delta_4 = P_2 \times Q$$

in $C_g \times C_g / \langle \psi^{-1} \times \psi \rangle$.

Recall from the proof of Theorem 3.6.1 that each resolution of a Type II singular point δ_i resulted in two curves: the $-(g+1)$-curve $\mathcal{C}_i$ and a (-2)-curve, which we denote by T_i. These curves T_i form the outermost four components of the fiber of f of type I_{3^c}, which is located above ∞.

Now let

$$S_1,\ S_2,\ S_3,\ \text{and } S_4$$

denote the strict transforms in X_c of the curves (respectively):

$$C_g \times P_1,\ C_g \times P_2,\ P_1 \times C_g,\ \text{and } P_2 \times C_g.$$

Then, from the proof of Theorem 3.6.1, the curves S_i all lie in the fiber of f of type $I_{4 \cdot 3^c}$, which is located above 0. Each curve $\mathcal{C}_i$ intersects the curve S_i.

Now an elliptic surface $f : S \to C$ has a group structure on all of its fibers, not just on its smooth fibers. The group elements in a singular fiber F_v are in correspondence with the components of multiplicity 1 in F_v (see [Mir89, VII.2.6]).

In particular, singular fibers of type I_b have all their components of multiplicity 1 and have group structure $G(I_b) \cong \mathbb{Z}/b\mathbb{Z}$. Singular fibers of type I_b^* have 4 components of multiplicity 1, corresponding to the four extremal components, and have group structure $G(I_b^*) \cong (\mathbb{Z}/2\mathbb{Z})^2$ if b is even and $G(I_b^*) \cong \mathbb{Z}/4\mathbb{Z}$ if b is odd [SS10, Lemma 7.3].

For any such elliptic surface $f : S \to C$, let R denote the set of reducible fibers of f. Then if $MW(S)$ is the Mordell-Weil group of $f : S \to C$, we have, see [SS10, Corollary 7.5], an injective homomorphism

$$\gamma : MW(S)_{tors} \to \sum_{v \in R} G(F_v),$$

given by sending a section to the respective fiber components that it meets.

Namely, in the case of the elliptic surface $f : X_c \to \mathbb{P}^1$, we have an injection

$$\gamma : MW(X_c) \to \mathbb{Z}/(4 \cdot 3^c)\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},$$

such that

$$\gamma(\mathcal{C}_i) = (S_i, T_i).$$

Hence, we indeed have $MW(X_c) \cong \mathbb{Z}/4\mathbb{Z}$. □

3.6.4 The j-Invariant of $f : X_c \to \mathbb{P}^1$

Following Kodaira [Kod63], for an elliptic surface $f : S \to C$ without multiple fibers, let $j : C \dashrightarrow \mathbb{P}^1$ be the rational map such that for every $P \in C$ such that $f^{-1}(P)$ is nonsingular, we have $j(P)$ is the j-invariant of the elliptic curve $f^{-1}(P)$. This rational map j can in fact be extended to all of C. The morphism $j : C \to \mathbb{P}^1$ is called the *j-invariant* of the elliptic surface $f : S \to C$. If $P \in C$ is such that $f^{-1}(P)$ is singular, then we have the following (reproduced from [Klo04]):

Fiber Type over P	$j(P)$
I_0^*	$\neq \infty$
I_b, I_b* $(b > 0)$	∞
II, IV, IV^*, II^*	0
III, III^*	1728

Lemma 3.6.5. *For $c \geq 2$, the j-invariant $j : \mathbb{P}^1 \to \mathbb{P}^1$ of $f : X_c \to \mathbb{P}^1$ is non-constant.*

Proof. From Theorem 3.6.1, all of the singular fibers of $f : X_c \to \mathbb{P}^1$ are of type I_b or I_b^* with $b > 0$. Hence the j-invariant of $f : X_c \to \mathbb{P}^1$ satisfies $j(P) = \infty$ for all $P \in \mathbb{P}^1$ such that $f^{-1}(P)$ is singular. However, since generically for $P \in \mathbb{P}^1$ the j-invariant $j(P)$ is the j-invariant of the elliptic curve $f^{-1}(P)$, generically j cannot be ∞. Thus j is non-constant. □

Proposition 3.6.6. *For $c \geq 2$, the j-invariant $j : \mathbb{P}^1 \to \mathbb{P}^1$ of $f : X_c \to \mathbb{P}^1$ has degree $6 \cdot 3^c$ and is ramified at the points 0, 1728, and ∞. There are $2 \cdot 3^c$ branch points above 0, all of ramification index 3. There are $3 \cdot 3^c$ branch points above 1728, all of ramification index 2. Finally, there are 2 branch points above ∞, one with ramification index $4 \cdot 3^c$ corresponding to the point $0 \in \mathbb{P}^1$ with singular fiber under f of type $I_{4 \cdot 3^c}$, and one with ramification index 3^c corresponding to the point $\infty \in \mathbb{P}^1$ with singular fiber of type I_{3^c}.*

Proof. This follows directly from results of Mangala Nori in [Nor85]. Since $f : X_c \to \mathbb{P}^1$ is extremal with non-constant j-invariant, it follows from [Nor85, Theorem 3.1] that $j : \mathbb{P}^1 \to \mathbb{P}^1$ is ramified only over the points 0, 1728, and ∞ and that we have:

$$\deg(j) = \sum_{I_b} b + \sum_{I_b^*} b,$$

where the two sums occur over all the singular fibers of f of type I_b and of type I_b^* respectively.

From Theorem 3.6.1, the surface X_c has one fiber of type $I_{4 \cdot 3^c}$, a total of 3^c fibers of type I_1, and one fiber of type $I_{3^c}^*$. We thus have:

$$\deg(j) = 6 \cdot 3^c.$$

So indeed the j-invariant of $f : X_c \to \mathbb{P}^1$ has degree $6 \cdot 3^c$. Now let

$$\mathcal{R}_0 = \{v \in \mathbb{P}^1 \mid j(v) = 0\}$$

$$\mathcal{R}_{1728} = \{v \in \mathbb{P}^1 \mid j(v) = 1728\}.$$

Then if e_v denotes the ramification index of a point $v \in \mathbb{P}^1$, let

$$R_0 = \sum_{v \in \mathcal{R}_0} (e_v - 1)$$

$$R_{1728} = \sum_{v \in \mathcal{R}_{1728}} (e_v - 1)$$

Then [Nor85, Lemma 3.2] in conjunction with Theorem 3.6.1 implies we have the following 3 equations:

$$R_0 + R_{1728} = \frac{7 \cdot \deg(j)}{6} \tag{3.30}$$

$$R_0 - \frac{2 \cdot \deg(j)}{3} \geq 0 \tag{3.31}$$

$$R_{1728} - \frac{\deg(j)}{2} \geq 0 \tag{3.32}$$

Observe that

$$\frac{2 \cdot \deg(j)}{3} + \frac{\deg(j)}{2} = \frac{7 \cdot \deg(j)}{6}.$$

Therefore we must have equality in Equations (3.31) and (3.32). Namely, we have

$$R_0 = \frac{2 \cdot \deg(j)}{3} = 4 \cdot 3^c$$

$$R_{1728} = \frac{\deg(j)}{2} = 3 \cdot 3^c.$$

Moreover, because equality holds in (3.31), Nori's proof in [Nor85, Lemma 3.2] also implies that

$$\deg(j) = 3|\mathcal{R}_0|.$$

Hence we have:

$$|\mathcal{R}_0| = 2 \cdot 3^c. \tag{3.33}$$

Now from [Nor85, Theorem 3.1], for any $v \in \mathcal{R}_0$, we must have $e_v \leq 3$. Hence using (3.33),

$$R_0 \leq 4 \cdot 3^c.$$

But we have already shown that in fact equality holds, therefore we have $e_v = 3$ for all $v \in \mathcal{R}_0$.

Since X_c is extremal and $f : X_c \to \mathbb{P}^1$ has no singular fibers of type III^*, Nori's results [Nor85, Theorem 3.1] also imply that $e_v = 2$ for all $v \in \mathcal{R}_{1728}$.

Finally, we know j has a pole of order b_i at points $v_i \in \mathbb{P}^1$ where the fiber over v_i is of type I_{b_i} or of type $I_{b_i}^*$. Hence the result follows from Theorem 3.6.1. □

3.6.5 The Surface X_c is Elliptic Modular

We begin by giving a brief introduction to elliptic modular surfaces as defined by Shioda [Shi72].

3.6.5.1 Preliminaries on Elliptic Modular Surfaces

Following Nori [Nor85], for an elliptic surface $f : S \to C$ with j-invariant $j : C \to \mathbb{P}^1$, let us define

$$C' = C \backslash j^{-1}\{0, 1728, \infty\}.$$

In particular, for every $v \in C'$, the fiber $F_v = f^{-1}(v)$ is smooth.

The sheaf $G = R^1 f_* \mathbb{Z}$ on C is the *homological invariant* of the elliptic surface S. The restriction of G to C' is then a locally constant sheaf of rank two $\mathbb{Z}$-modules.

Consider the monodromy homomorphism $\rho : \pi_1(C') \to SL(2, \mathbb{Z})$ associated to $f : S \to C$. Observe that ρ both determines and is determined by the sheaf G.

Conversely, let $j : C \to \mathbb{P}^1$ be a holomorphic map from an algebraic curve C to $\mathbb{P}^1$ and let $C' = C \backslash j^{-1}\{0, 1728, \infty\}$. Let $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane in $\mathbb{C}$ and consider the elliptic modular function $J : \mathcal{H} \to \mathbb{P}^1 \backslash \{0, 1728, \infty\}$. Finally let U' be the universal cover of C'. Then there exists a holomorphic map $w : U' \to \mathcal{H}$ such that the following diagram commutes:

$$\begin{array}{ccc} U' & \xrightarrow{\quad w \quad} & \mathcal{H} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle J} \\ C' & \xrightarrow{\ j\ } & \mathbb{P}^1 \backslash \{0, 1728, \infty\}. \end{array} \tag{3.34}$$

This map w thus induces a homomorphism $\overline{\rho} : \pi_1(C') \to PSL(2, \mathbb{Z})$.

Now suppose we have a homomorphism $\rho : \pi_1(C') \to SL(2, \mathbb{Z})$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(C') & \xrightarrow{\quad \rho \quad} & SL(2, \mathbb{Z}) \\ & {\scriptstyle \overline{\rho}} \searrow \quad \swarrow & \\ & PSL(2, \mathbb{Z}) & \end{array} .$$

Then it is possible to construct a unique elliptic surface $f : S \to C$ having j-invariant given by the holomorphic map $j : C \to \mathbb{P}^1$ and having homological invariant given by the sheaf G associated to the homomorphism ρ [Kod63, Section 8].

So now consider any finite-index subgroup Γ of the modular group $SL(2, \mathbb{Z})$ not containing $-\mathrm{Id}$. Then Γ acts on the upper half plane $\mathcal{H}$ and the quotient $\Gamma \backslash \mathcal{H}$, together with a finite number of cusps, forms an algebraic curve C_Γ. For any other such subgroup Γ', if $\Gamma \subset \Gamma'$, then the canonical map $\Gamma \backslash \mathcal{H} \to \Gamma' \backslash \mathcal{H}$ extends to a holomorphic map $C_\Gamma \to C_{\Gamma'}$. In particular, taking $\Gamma' = SL(2, \mathbb{Z})$ and identifying $C_{\Gamma'}$ with $\mathbb{P}^1$ via the elliptic modular function J, we get a holomorphic map

$$j_\Gamma : C_\Gamma \to \mathbb{P}^1.$$

Hence, as discussed, there exists a $w : U' \to \mathcal{H}$ fitting into a diagram (3.34) which induces a representation $\overline{\rho} : \pi_1(C') \to \overline{\Gamma} \subset PSL(2,\mathbb{Z})$, where $\overline{\Gamma}$ is the image of Γ in $PSL(2,\mathbb{Z})$. Because Γ contains no element of order 2, this homomorphism $\overline{\rho}$ lifts to a homomorphism $\rho : \pi_1(C') \to SL(2,\mathbb{Z})$, which then gives rise to a sheaf G_Γ on C_Γ.

Definition 3.6.7. [Shi72] For any finite index subgroup Γ of $SL(2,\mathbb{Z})$ not containing $-\mathrm{Id}$, the associated elliptic surface $f : S_\Gamma \to C_\Gamma$ having j-invariant j_Γ and homological invariant G_Γ is called the *elliptic modular surface* attached to Γ.

3.6.5.2 The case of X_c

We now return to considering the elliptic surface $f : X_c \to \mathbb{P}^1$. Recall that we have studied the j-invariant of $f : X_c \to \mathbb{P}^1$ in detail in Proposition 3.6.6.

Let us define the following elements $A_0, A_1 \ldots, A_{3^c}, A_\infty$ of $SL(2,\mathbb{Z})$ as elements of the following conjugacy classes:

$$A_0 \in \left[\begin{pmatrix} 1 & 4\cdot 3^c \\ 0 & 1 \end{pmatrix}\right] \qquad A_1, \ldots, A_{3^c} \in \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] \qquad A_\infty \in \left[\begin{pmatrix} -1 & -3^c \\ 0 & -1 \end{pmatrix}\right]$$

Then consider the subgroup Γ_c of index $6\cdot 3^c$ in $SL(2,\mathbb{Z})$ with the following presentation:

$$\Gamma_c := \langle A_0, A_1, \ldots, A_{3^c}, A_\infty \mid A_0 A_1 \cdots A_{3^c} A_\infty = \mathrm{Id}\rangle.$$

We remark that Γ_c is not a congruence subgroup as it does not appear on the list in [CP03] of the genus 0 congruence subgroups of $SL(2,\mathbb{Z})$ (see [CLY04] for more details on such subgroups).

Theorem 3.6.8. *For $c \geq 2$, the surface X_c is the elliptic modular surface attached to Γ_c.*

Proof. A result of Mangala Nori shows that an extremal elliptic surface $f : S \to C$ with a section and with non-constant j-invariant is an elliptic modular surface as long as $f : S \to C$

has no singular fibers of type II^* or III^* in Kodaira's classification [Nor85, Theorem 3.5]. Therefore, since the surface $f : X_c \to \mathbb{P}^1$ is extremal (by Corollary 3.6.2), has a section (by Proposition 3.6.3), has non-constant j-invariant (by Lemma 3.6.5), and only has fibers of type I_b and I_b^* (by Theorem 3.6.1), we know X_c is indeed an elliptic modular surface.

So let Γ be the finite-index subgroup of $SL(2, \mathbb{Z})$ attached to X_c. By Proposition 3.6.6, the degree of the j-invariant of $f : X_c \to \mathbb{P}^1$ is $6 \cdot 3^c$. Hence the group Γ has index $6 \cdot 3^c$ in $SL(2, \mathbb{Z})$.

Now consider the j-invariant $j : \mathbb{P}^1 \to \mathbb{P}^1$ of X_c, which we have investigated in Proposition 3.6.6. Let $C' = \mathbb{P}^1 \backslash \{0, 1728, \infty\}$.

Because X_c is elliptic modular, its j-invariant induces a homomorphism

$$\rho : \pi_1(C') \to \Gamma \subset SL(2, \mathbb{Z}).$$

Let us write the set

$$j^{-1}\{0, 1728, \infty\} = \{v_1, \ldots, v_s\}.$$

By Proposition 3.6.5, we know $s = 5 \cdot 3^c + 2$. For each point v_i let α_i be the loop element in $\pi_1(C')$ going around v_i. Then $\pi_1(C')$ is the free group on these generators $\alpha_1, \ldots, \alpha_s$ subject to the relation (taken in cyclic order) $\alpha_1 \cdots \alpha_s = 1$ [BT03, Lemma 2.1].

Now, by Proposition 3.6.5 all of the points v_i such that $j(v_i) = 0$ have ramification index 3. Hence by [Nor85, Proposition 1.4], for the corresponding α_i, we have $\rho(\alpha_i) = \pm\text{Id}$. However since X_c is elliptic modular, the subgroup Γ cannot contain $-\text{Id}$. Hence, for all i such that $j(v_i) = 0$, we must have $\rho(\alpha_i) = \text{Id}$.

Similarly, by Proposition 3.6.5 all of the points v_i such that $j(v_i) = 1728$ have ramification index 2. But then by [Nor85, Proposition 1.4], for all such i, we have $\rho(\alpha_i) = \pm\text{Id}$ and thus, in fact, $\rho(\alpha_i) \in \text{Id}$.

Therefore the only points $v_i \in j^{-1}\{0, 1728, \infty\}$ that contribute non-identity elements to Γ are the points sent to ∞ by j. These are exactly the points of $\mathbb{P}^1$ underneath the singular fibers of $f : X_c \to \mathbb{P}^1$. From [Shi72, Proposition 4.2], if a point v_i has singular fiber of type

I_b with $b > 0$, then we have:

$$\rho(\alpha_i) \in \left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right].$$

If a point v_i has singular fiber of type I_b^* with $b > 0$, then we have:

$$\rho(\alpha_i) \in \left[\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} \right].$$

Therefore, using Theorem 3.6.1, in the case of $f : X_c \to \mathbb{P}^1$, the point 0 contributes a generator A_0 of Γ in the conjugacy class of

$$\begin{pmatrix} 1 & 4 \cdot 3^c \\ 0 & 1 \end{pmatrix}$$

in $SL(2, \mathbb{Z})$. Each point ζ^i, for ζ a 3^c-th root of unity, contributes a generator A_{i+1} in the conjugacy class of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Finally, the point ∞ contributes a generator A_∞ in the conjugacy class of

$$\begin{pmatrix} -1 & -3^c \\ 0 & -1 \end{pmatrix}.$$

Then Γ is the free group on these generators $A_0, A_1, \ldots, A_{3^c}, A_\infty$ subject to the relation

$$A_0 A_1 \cdots A_{3^c} A_\infty = \mathrm{Id}.$$

Hence we indeed have that Γ is the group Γ_c defined above. □

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