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RAPID SPECTROMETER ANALYSIS

Arthur B. Wicklund

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ABSTRACT

We have discovered a general solution of Maxwell's equations which describes the field of a spectrometer magnet with precision and accuracy. For example, starting with 70 000 field measurements made on an $18 \times 36 \times 8$-in. bending magnet (AEC 67 522) we emerged with a distillate of 200 parameters which describes all components of the field everywhere at all currents with errors less than 4×10^{-4} of the central field values (typically 1 or 2 gauss out of 15 000); measured line integrals are reproduced within one part in 10^4.

To summarize the solution: we describe the field by cuts and poles near the real axis defined by the longitudinal path through the field. Particle orbits have a corresponding singularity structure. Inhomogeneities in the field are introduced as perturbations on the residue functions which depend on the orbit characteristics; by means of the residue theorem, one can take these inhomogeneities into account by simply adding up the perturbed residue functions. The essential mathematical simplification which we have achieved is this; instead of expanding field inhomogeneities as polynomials in the coordinates and integrating these along a path, we expand them in a series of increasingly higher-order poles and then, because of the residue theorem, we need retain only the first term in this series.

Finally we note that nature is happy with this solution; the accuracy of any momentum analysis done with the spectrometer is limited only by uncertainties in the original field measurements. With modifications the ideas presented here can be extended to larger, more complicated fields. Analysis capabilities are buttressed by the fact that with a typical spark chamber set-up, one can fit orbits, compute χ^2's and measure momenta at the rate of 16 000 tracks per minute on the CDC 6600, using the techniques outlined below.

I. GENERAL REMARKS

A dipole magnet, approximated by two arbitrary Helmholz coils, has a return field at large distances which behaves like R^{-3}. Our solution does not have this feature; the field never changes sign and falls off as R^{-4}. Measurements show that the return field is undetectable in our magnet, which differs from a simple Helmholz pair by having iron and by having upswept coils.

Direct calculation shows that, for a Helmholz pair with upswept coils, B_z is

$$B_z(0,y,0) \simeq \left[\frac{y-y_0}{[g_1{}^2+(y\pm y_0)^2]^{1/2}} + \frac{g_1}{g_2}\left[\frac{y-y_0}{[g_2{}^2+(y\pm y_0)^2]^{1/2}}\right]\right]_{\text{Symmetrized in } y},$$

where y_0 is the longitudinal dimension, $g_1 \approx$ transverse dimension, and $g_2 \approx [g_1{}^2 + (\text{gap}/2)^2]^{1/2}$. In general, for arbitrary x, z the solution has the form

$$\sum_{i=1}^{2} \frac{y\pm y_0}{[(y\pm y_0)^2+g_i{}^2]^{1/2}} \left[\sum_{n=0}^{\infty} P_n(x,z)/[(y\pm y_0)^2+g_i{}^2]^n\right]_{\text{Symmetrized in } y}, \qquad \text{(I-a)}$$

where the P_n are increasingly higher-order polynomials in x, z.

The first term is a branch-cut singularity in the complex plane at $y = \pm y_0 \pm i g_i$; the remaining terms are multiple poles, which provide a natural cutoff in the expansion in x, z, since for n sufficiently large, the pole term acts like a delta function affecting only the regions $y \simeq \pm y_0$. The solution which we obtain has:

(1) $\ell n(y-z_1)$ singularity instead of $(y-z_1)^{-1/2}$

(2) A real pole $B_3/(y-z_3)$

(3) An imaginary pole $iB_2/(y-z_2)$.

Each term is expanded in a series like the one above, e.g.:

$$\text{Real}\left\{\ln(y-z_1)\quad\times\left[\sum_0^\infty P_n^{\,1}(x,z)/(y-z_1)^n\right]\right.$$
$$+\quad i/y-z_2\quad\times\left[\sum_0^\infty P_n^{\,2}(x,z)/(y-z_2)^n\right]$$
$$\left.+\quad 1/(y-z_3)\times\left[\sum_0^\infty P_n^{\,3}(x,z)/(y-z_3)^n\right]\right\}. \qquad \text{(I-b)}$$

Because there are no terms in $(y^2+G^2)^{1/2} \to R$ at infinity, the asymptotic dependence on R can only be $1/R^{2n}$. The necessity for three singularities instead of the two required above stems from the asymptotic requirement that $B_z \sim 1/R^4$ (not $1/R^2$), as will be shown presently.

The branch cut gives rise to a step-like function (the phase of $(y-z_1)$ which turns on when we enter the magnet $(y<y_0)$ and turns off when we leave it.

The poles correct the shape of the gradient at y_0 and fix up the behavior at infinity.

Any resemblance between (I-a) and (I-b) must be between the branch-cut terms [We could add poles to (I-a) in order to smear out the singularities at y_0 and change the shape of the gradient].

In (I-a) we have

$$\frac{-(y-y_0)}{[(y-y_0)^2+g^2]^{1/2}}+\frac{(y+y_0)}{[(y+y_0)^2+g^2]^{1/2}}\ .$$

If we denote $Y\pm(y_0\pm ig) = Re^{i\phi\pm}$, then these terms are just $B_z = \cos\phi_+ - \cos\phi_-$ (see Fig. 1a). This is just a localized function which approaches a square well as g goes to zero.

Solution (1-b) accomplishes the same effect by the phase alone rather than the cosine of the phase. Again $y-(y_0+ig) = Re^{i\phi-}$; $y-(-y_0+ig) = Re^{i\phi+}$. Then $B_z = \phi_+ - \phi_-$ (Fig. 1b).

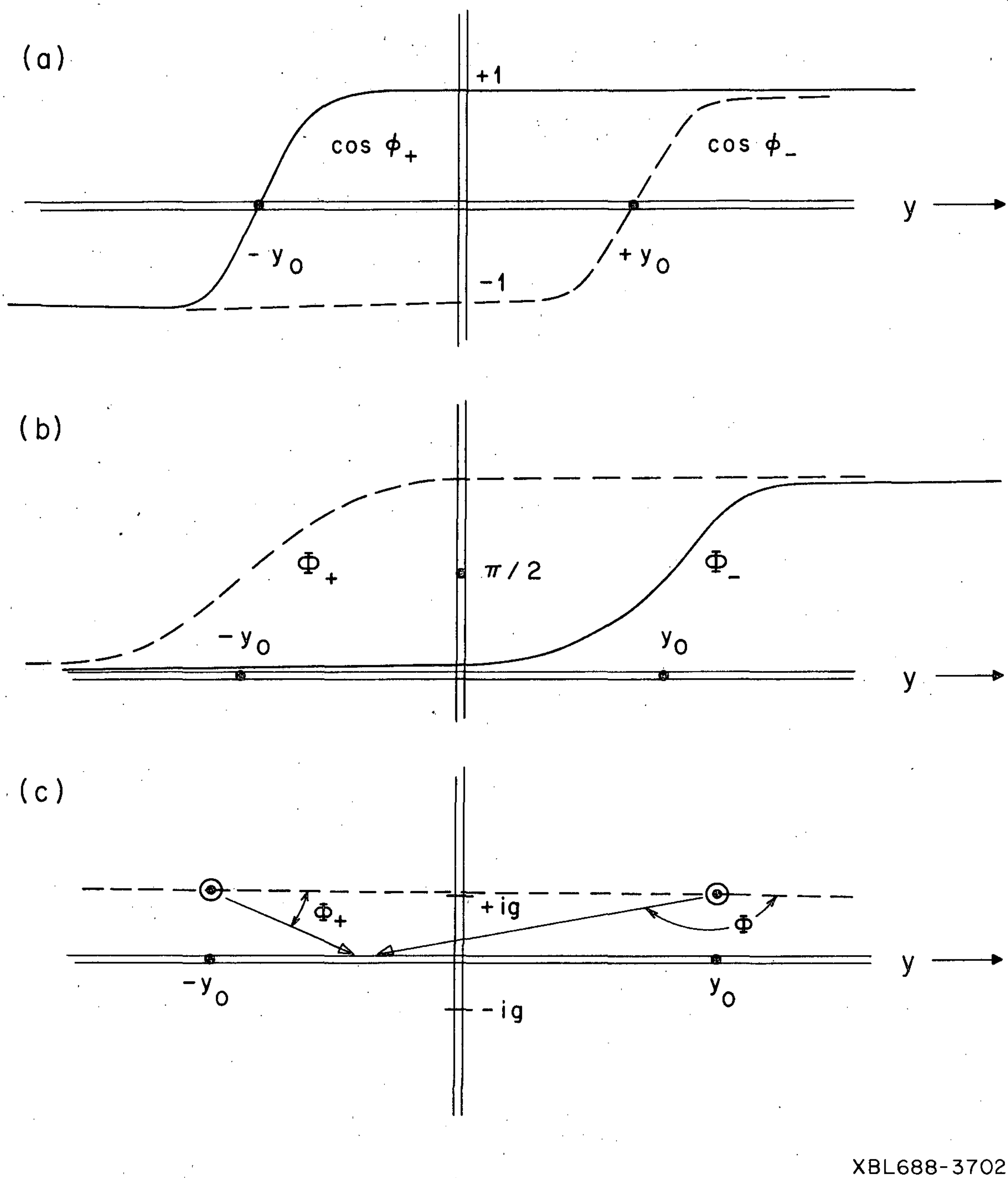

Fig. I-1. Plots of the field B_z as a function of (a) the cosine of the phase and (b) the phase alone. B_z is given by the difference of the top and bottom curves in each case. (c) shows the meaning of the phase angles.

The $\cos\phi_+ - \cos\phi_-$ function has a sharper edge than $\phi_+ - \phi_-$. However, both functions have a gradient at $y = \pm y_0$ which is almost symmetric about y_0 in the limit $g/y_0 \ll 1$, and to introduce any appreciable asymmetry in this gradient, extra singularities located at $y_i \neq y_0$ are needed, but they must be added in such a way as to satisfy the asymptotic requirement that $B_z \to 1/R^n$ where $N \geqslant 3$. This limit is dictated by the fact that, for a track in the midplane,

$$P \frac{d^2x}{dy^2} \simeq B_z.$$

$\int_{-\infty}^{0} x(y)$ must be finite in the particular case that $dx/dy(-\infty) = 0$. For $B_z \propto 1/R^2$, we have $\int_{-\infty}^{0} x(y) \sim \ln(-\infty)$, which violates our requirement.

II. THE SOLUTION

No matter how intractable the field, we can describe $B_z(x, y, 0)$ and deduce the behavior for $Z \neq 0$ from Maxwell's equations. For convenience, we do Taylor series in x and z, assuming that for real particles traveling through the field, x and z will be bounded so that no insolubly divergent integrals can result.

Let $B_z(x, y, 0) = \psi(x, y) = \psi[y, \vec{\alpha}(x)]$, where the $\vec{\alpha}(x)$ are parameters giving the positions and residues of the singularities in the complex y plane.

Then if $Bi = d/dxi\ \Phi(x, y, z)$, we get

$$\Phi(x, y, z) = z\psi(x, y) + z^3\psi^2(x, y) + z^5\psi^4(x, y) \cdots .$$

Applying $\nabla^2 \Phi = 0$ we deduce

$$\Phi(x, y, z) = z\psi - z^3/6\,[dx^2 + dy^2]\psi + z^5/120\,[dx^2 + dy^2]^2\psi + \cdots .$$

Similarly we expand

$$\psi(x, y) \simeq \psi(y, 0) + d/d\alpha i\ \psi(y, 0)\ d\alpha i/dx$$
$$\alpha i = \alpha i^0 + \alpha i^2 x^2 + \alpha i^4 x^4\ \alpha i^6 x^6 \cdots . \qquad \text{(II-c)}$$

Note that the general structure of the solution, if ψ or $d\psi/dy$ is proportional to $[(y - y_0)^2 + g^2]^{-1}$ from poles at $y_0 \pm ig$, is simply

$$\Phi(x,y,z) = z \sum_{n=0}^{\infty} T^{(2n+2m)}_{(x,z)} \left(\frac{d}{dy}\right)^{2n} \left[\frac{1}{(y-y_0)^2+g^2}\right]. \tag{II-d}$$

Here T^{2n} is a polynomial of order $(2n + 2m)$ in x, y, where m is the degree of the polynomial at $z = 0$. Roughly, this says that higher-order inhomogeneities become increasingly localized at the pole positions (e.g., at the physical edge of the magnet, where the gradient in y is largest.)

Specifically, our solution for $\psi\{y[\alpha(x)]\}$ is

$$\begin{aligned}\psi(y,\alpha(0)) &= B_1\left[\tan^{-1}\left(\frac{y+y_1}{g_1}\right) - \tan^{-1}\left(\frac{y-y_1}{g_1}\right)\right] \\ &+ B_2\left\{\frac{g_2}{[g_2{}^2+ *y+y_2)^2]} + \frac{g_2}{[g_2{}^2+(y-y_2)2]}\right\} \\ &+ B_3\left\{\frac{(y+y_3)}{[g_3{}^2+(y+y_3)^2]} - \frac{(y-y_3)}{[g_3{}^2+(y-y_3)^2]}\right\}.\end{aligned}$$

Equivalently we have

$$\begin{aligned}\psi(y,\alpha(0)) &= \frac{B_1}{2i}\left[\ell n(y-\zeta_1{}^*) - \ell n(y-\zeta_1) + \ell n(y+\zeta_1{}^*) - \ell n(y+\zeta_1)\right], \\ &+ \frac{B_2}{2i}\left[\frac{1}{y-\zeta_2} + \frac{1}{y-\zeta_2{}^*} - \frac{1}{y+\zeta_2} - \frac{1}{y+\zeta_2{}^*}\right] \\ &+ \frac{B_3}{2}\left[\frac{1}{y+\zeta_3{}^*} + \frac{1}{y+\zeta_3} - \frac{1}{y-\zeta_3} - \frac{1}{y-\zeta_3{}^*}\right],\end{aligned}$$

where $\zeta_i = y_i + ig_i$. (II-e)

We incorporate the x and z dependence as described above, fitting each parameter to a power series in x and applying Maxwell's equations. For any given pole we get an expansion

$$\psi(\text{pole}) = \sum_{n=0}^{\infty} T^n(x,z) \frac{d^n}{d\zeta_i^{\,n}} \left(\frac{1}{y-\zeta_i}\right). \qquad \text{(II-f)}$$

Denoting $B_i(x) = B_i(x) - B_i^0$ and $W_i(x) = \zeta_i(x) - \zeta_i(0)$, where B_i^0 are the constants for $x = z = 0$ used in (I-e) above, we get:

$$T_i^{\,0} = B_i^{\,0} + \left[B_i(x) - z^2/6 \; B_i^{(2)}(x) + z^4/120 \; B_i^{(4)}(x) \cdots \right]$$

$$T_i^{\,2} = \left[-z^2/6 \; B_i(x) + 2z^4/120 \; B_i^{(2)}(x) \right]$$

$$T_i^{\,4} = \left[z^4/120 \; B_i(x) \right]$$

$$T_i^{\,1} = B_i^{\,0} \times \left[W_i(x) - z^2/6 \; W_i^{(2)}(x) + z^4/120 \; W_i^{(4)}(x) \cdots \right]$$

$$T_i^{\,3} = B_i^{\,0} \times \left[-z^2/6 \; W_i(x) + 2z^4/120 \; W_i^{(2)}(x) \cdots \right]$$

$$T_i^{\,5} = B_i^{\,0} \times \left[z^4/120 \; W_i(x) \right]. \qquad \text{(II-g)}$$

Note that in expansion (II-f) the $\ell n(y-\zeta_1)$ terms have

$$\left(\frac{1}{y-\zeta_1}\right) \to -\ell n \left(y-\zeta_1\right)$$

and

$$d^n_{\zeta_1} \left(\frac{1}{y-\zeta_1}\right) \to d^{n-1}_{\zeta_1} \left(\frac{1}{y-\zeta_1}\right).$$

Expansion (II-f) gives B_x, B_z from the polynomial (II-g) easily; B_y is just

$$B_y = -z \sum_{n=0}^{\infty} T^n(x,z) \frac{d^{n+1}}{d\zeta_i^{\,n+1}} \left(\frac{1}{y-\zeta_1}\right).$$

The asymptotic behavior is given by

$$B_z \underset{(y\to\infty)}{\to} \left[B_1 y_1 g_1 + B_2 g_2 - B_3 y_3 \right] / y^2 + K/y^4.$$

Since B_z must fall off faster than Y^{-2}, we get an important constraint on B_3:

$$B_3 = \left[B_1 y_1 g_1 + B_2 g_2 \right] / y_3, \qquad \text{(II-h)}$$

which must hold for all x. Then correct asymptotic behavior in y is assured for all x, z.

If the field is not symmetric in y, then the pole positions and residues can be varied at each physical end. Asymmetries in x and z are created by introducing a linear x term in $\alpha_i(x)$ and corresponding terms in z^2 and z^4 in $\phi\,(x, y, z)$.

If either dimension x or z is expected to be large for real trajectories, so that a polynomial is undesirable, we can fit $\alpha_i(x)$ or $\alpha_i(z)$ to more tractable functions (e.g., the same functions as were used for the y dependence--examine the complex x-plane for example) and do the Taylor series in only one variable.

In what follows we denote the B_1, B_2, and B_3 terms in (II-f) by ψ_1, ψ_2, $\psi_3\,(x, y, x)$. Figures II-1 and II-2 show the accuracy of the fit to Eqs. (II-e) and (II-f). Figure II-5 shows the current dependence of the parameters B_1, B_2 and B_3.

III. TRAJECTORIES

The equations of motion for a particle in a magnetic field can be expressed as

$$P\frac{d}{dy}\left(\frac{dx}{ds}\right) = Bz(x, y, z) - \frac{dz}{dy}\;By(x, y, z)$$

and

$$P\frac{d}{dy}\left(\frac{dz}{ds}\right) = \frac{dx}{dy}\;By(x, y, z) - Bx(x, y, x), \qquad \text{(III-a)}$$

where $ds = (dx^2 + dy^2 + dz^2)^{1/2}$, and p is measured in arbitrary units.

Conventionally (e.g., in most bubble chamber calculations) starting with the equations of motion,

$$P\;\frac{d}{ds}\left(\frac{dx_i}{ds}\right) = \epsilon ijk\;\frac{dx_j}{ds}\;Bk(x, y, z),$$

one derives

$$Pd\,\psi/ds = Bz - \frac{\sin\lambda}{\cos\lambda}\,(\cos\psi\;By + \sin\psi Bx)$$
$$Pd\,\lambda/ds = \sin\psi\;By - \cos\psi\;Bx$$

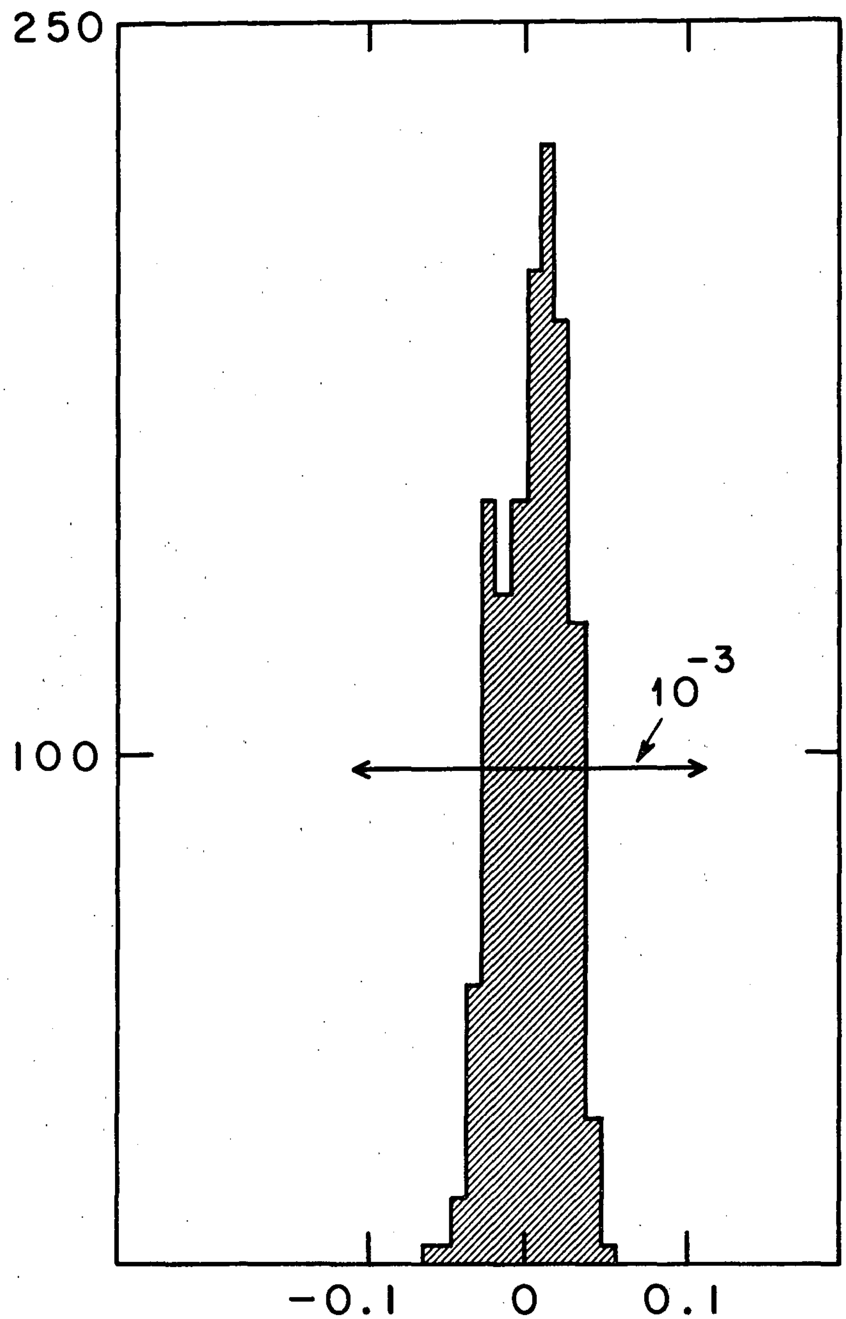

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Fig. II-1. Histogram of the differences [B_z fitted to Eq. (II-e) minus measured B_z] for all currents with $x = 0$, $z = 0$. These differences are expressed in units of $B_z(0,0,0) \times 10^{-4}$.

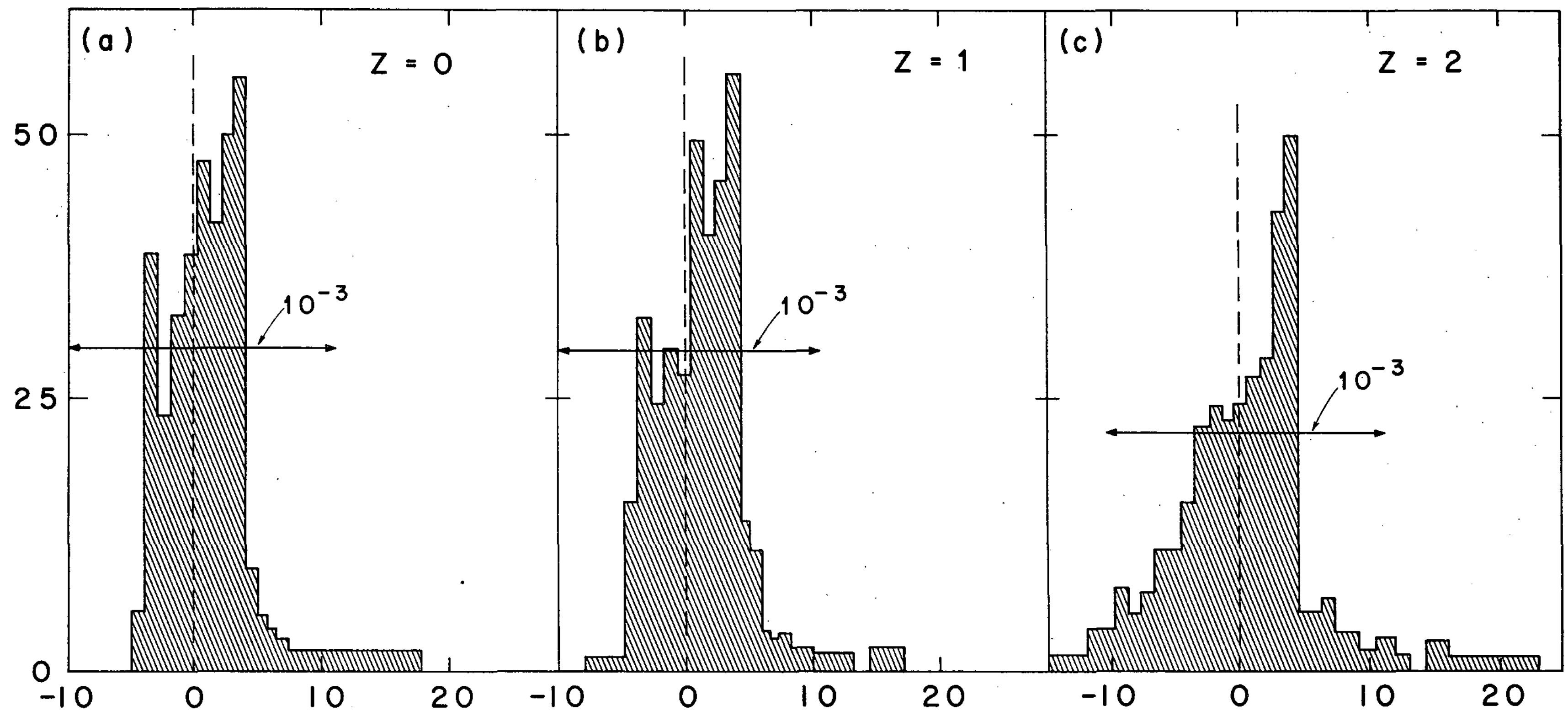

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Fig. II-2. Histogram of the differences [B_z fitted to Eq. II-e minus measured B_z] in units of $B_z(0,0,0) \times 10^{-4}$. These data are for all $|x| \leq 7$ inches at 1100 A. (a), (b), and (c) show $z = 0, 1,$ and 2 in. respectively. The quality of the fit is as good at all other currents.

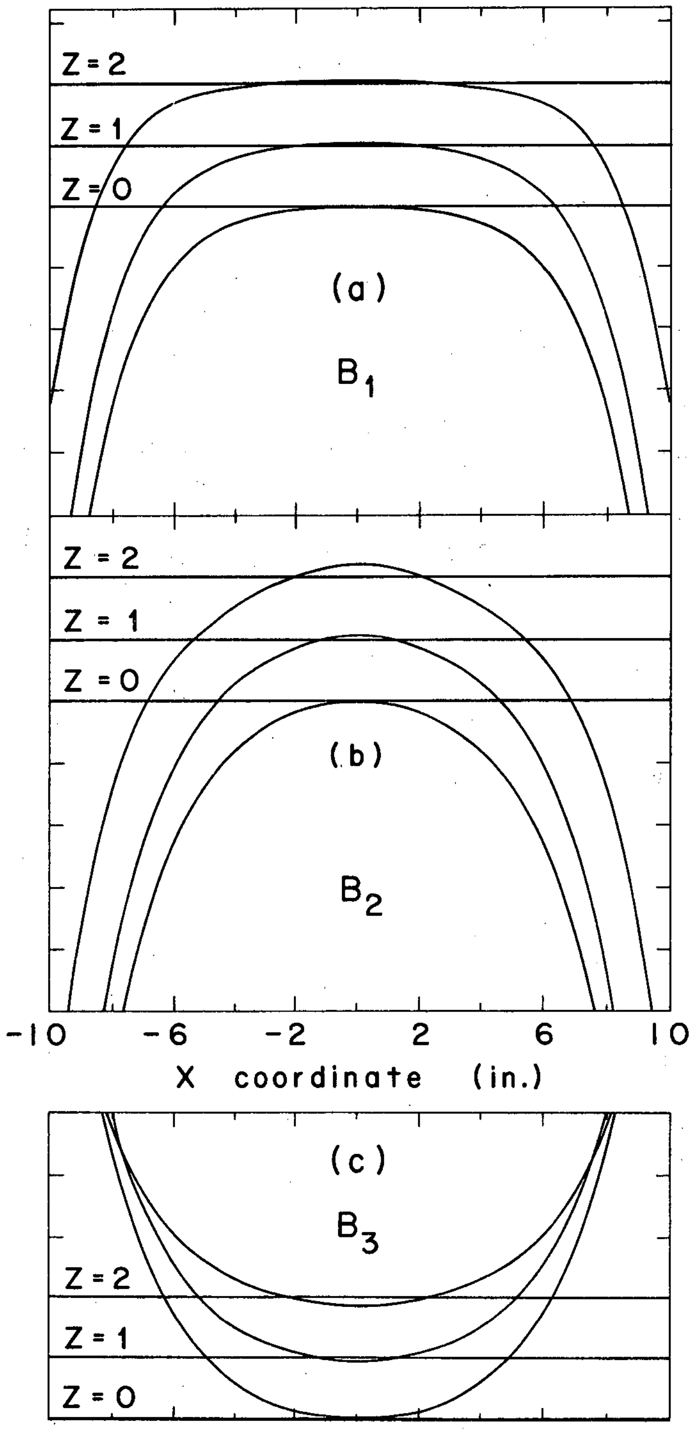

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Fig. II-3. Plots of the quantity $100\times(T_i^0(x, z) - B_i)/B_i$ versus x for z = 0, 1, and 2 in. (a), (b), and (c) are for i = 1, 2, and 3 respectively. For clarity the origins are displaced upwards 5 units for z = 1 and 10 units for z = 2; this also establishes the ordinate scale. The other parameters y_i and g_i have similar spatial variations. The data are taken at 900 A.

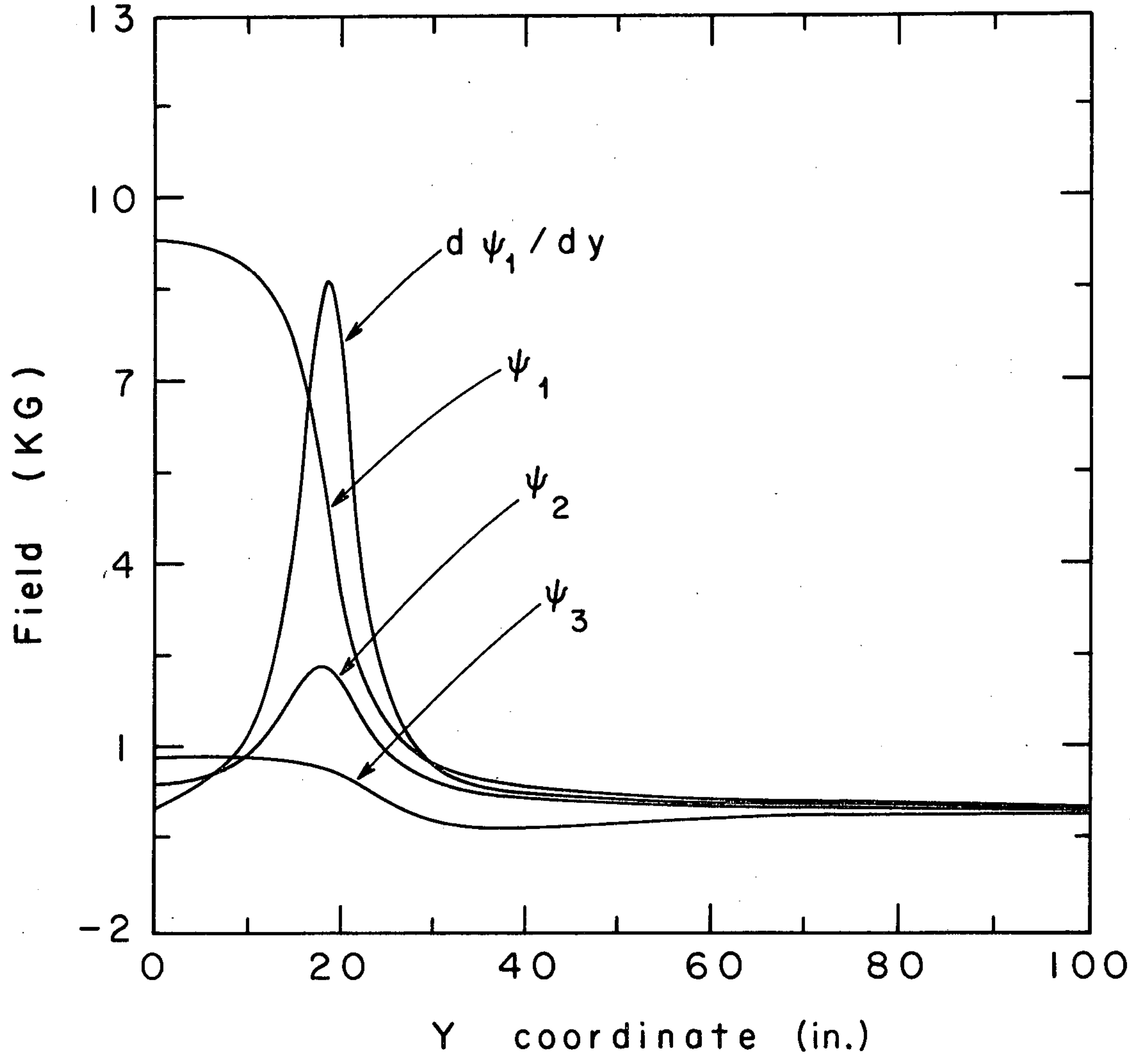

Fig. II-4. Field in kilogauss versus y axis showing the contributions of each term ψ_1, ψ_2, and ψ_3 and also the derivative $d\psi_1/dy$. The data are taken at 900 A.

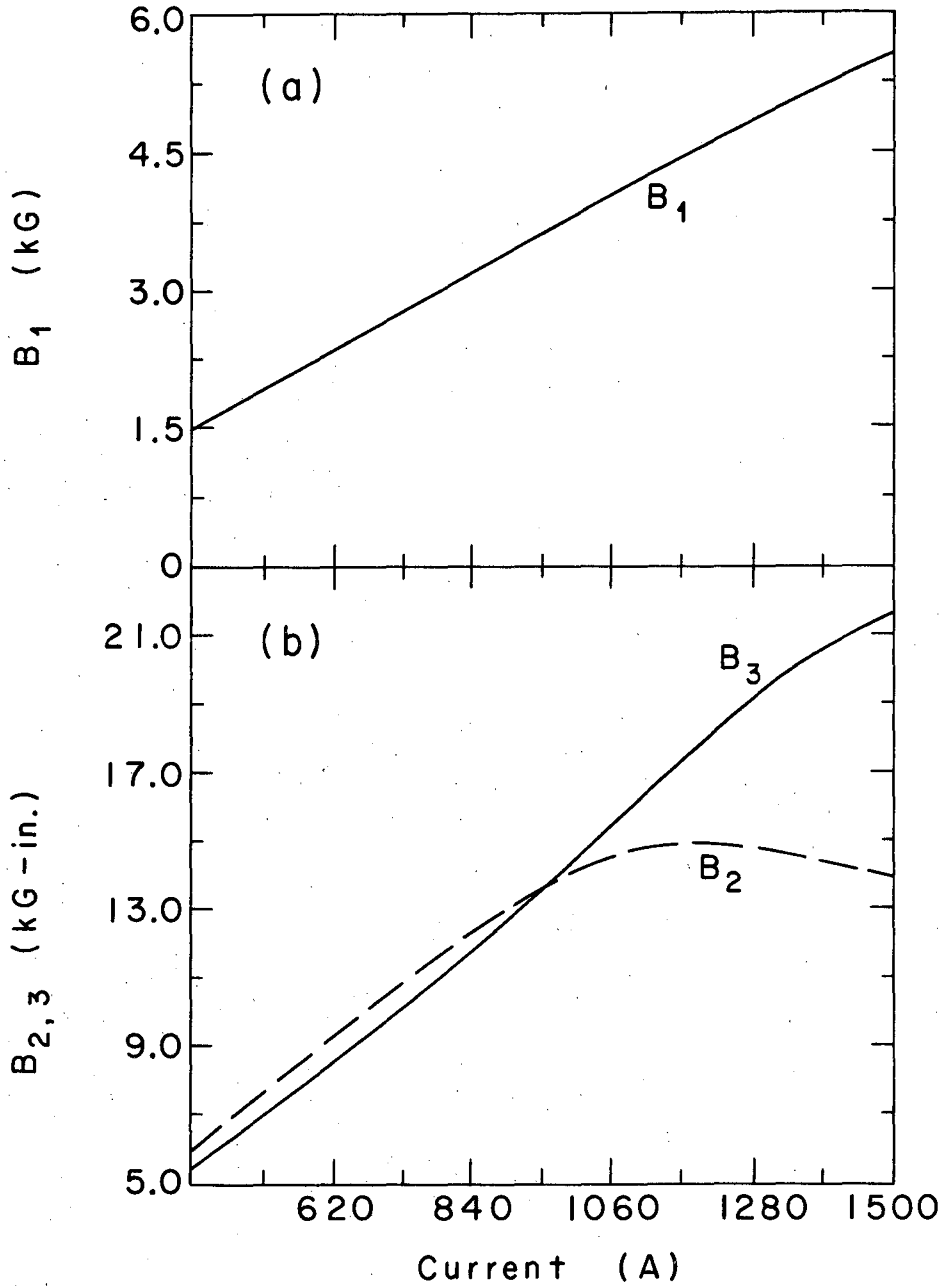

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Fig. II-5. Plot of the parameters B_1(a), B_2(b), and B_3(c) expressed respectively in kilogauss, kilogauss-inches, and kilogauss-inches. The abscissa gives the current in amperes. The remaining parameters y_i and g_i vary less than 15% over the full range of currents.

where

$$dx/ds = \sin\psi \cos\lambda$$
$$dy/ds = \cos\psi \cos\lambda$$
$$dz/ds = \sin\lambda.$$

The advantage of the formulation in (III-a) is that we do not need to know the path length precisely, although this could in principle be calculated from our analytic expression for the field. Also, from the point of view of a spark-chamber experiment, dx/ds, dy/ds, and dz/ds are the directly measurable quantities, and p is determined by

$$P = \left[\int_{-\infty}^{\infty} B_z(\vec{x})\, dy + \int_{-\infty}^{\infty} \frac{dz}{dy} By(\vec{x})\, dy\right] / [_{-\infty}^{\infty} dx/ds. \qquad \text{(III-b)}$$

To obtain P accurately, we need to know

$$\int_{-\infty}^{\infty} B_z(0, y, 0)\, dy + \int_{-\infty}^{\infty} \left[B_z(x, y, z) + \lambda\, By(x, y, z) - B_z(0, y, 0)\right] dy.$$

The latter term is typically less than 1% of the first term (more nearly 0.2 to 0.4%).

The sequence of calculations required is:

1. Obtain $P = \int_{-\infty}^{\infty} B_z(0, y, 0)\, dy / [_{-\infty}^{\infty} dx/ds.$

Now $1/P \simeq (1 - \delta P/P_0)/P_0$; $(P = P_0 + \delta P)$, where $\delta P/P_0$ is less than 0.01, and we want to get δP.

2. For any track the same calculation gives us x(y), z(y) to an accuracy of at least 0.2 in. inside the magnet, and from this we can generate the correction integral due to inhomogeneity to an accuracy of .04% of the leading term.

It is relevant to point out at this stage that we do not know $\int_{-\infty}^{\infty} B_z(0, y, 0)dy$ to better than 0.2% because of uncertainties in the current, but this only results in an overall shift of the spectrum in P (a constant correction $\delta P/P_0$ rather than a random error which affects resolution). The inhomogeneity correction can be calculated with confidence; it is the principle factor in determining the resolution.

3. We use the correction integral to correct the original trajectories so that at large y, $x(y) \simeq Ay + B$ is known to an accuracy $0.05\% \times y$.

We solve

$$\frac{d^2x}{dy^2} \simeq \frac{d}{dy}\left(\frac{dx}{ds}\right) \simeq \sum_i \Psi_i\,(y, 0, \alpha_i^{\,0})/P.$$

The left side is too large by ~0.8%, since $\cos\psi \simeq 1 - \psi^2/2 \simeq 0.992$ for a 15 deg bend. The right side is too large by the correction for inhomogeneity or about 0.5% to -0.5% (for large z). The correction for the B_y term is only .02% for $dz/dy < 0.02$.

Thus the major errors tend to cancel, and for given X_0, dX/dy_0, P_0 we can calculate a trajectory good to ~ 1%, which is perfectly adequate to calculate the correction integral due to x, z dependence.

$$x(y) = X_0 + \xi_0(y-y_0) + \left[\sum_{i=1}^{3} M_i(y)\right]/P_0$$

$$dx/dy(y) = \xi_0 + \left(\sum_1^3 dM_i/dy\right)/P_0. \qquad \text{(III-c)}$$

Finally, $d^2x/dy^2(y) = (\sum_1^3 d^2M_i/dy^2)/P_0 = \sum_i \Psi_i\,(y, 0, \alpha^0)/P_0$. The M_i's are specified by $\lim_{y\to-\infty} M_i(y) = 0$, $\lim_{y\to-\infty} dM_i/dy = 0$, and $d^2/dy^2\, M_i(y) = \psi_i(y, 0, \alpha_i^{\,0})$.
The solution is

$$M_1(y) = -B_1/2\left\{yg_1\,\ell n\,(R_1^+/R_1^-)^2 + \alpha_1^+[(y+y_1)^2 - g_1^{\,2}] + \alpha_1^-[(y-y_1)^2] - 6y_1g_1\right\}$$

$$M_2(y) = -B_2\left\{\alpha_2^+(y+y_2) - \alpha_2^-(y-y_2) - 2g_2\right\}$$

$$M_3(y) = B_3\left\{y/2\,\ell n\,(R_3^+/R_3^-)^2 - g_3(\alpha_3^- + \alpha_3^+) - 2y_3\right\}$$

$$M_0(y) = -B_1y_1g_1/2\,\ell n(R_1^+R_1^-/R_3^+R_3^-)^2 - B_2g_2/2\,\ell n\,(R_2^+R_2^-/R_2^+R_2^-)^2, \qquad \text{(III-d)}$$

where:

$$R_i^{\pm 2} = (y\pm y_i)^2 + g_i^{\,2},\quad d/dy\, R_i^{2\pm} = 2(y\pm y_i),$$

$$\alpha_i^\pm = \mp \tan^{-1}(g/(y_i\pm y)),$$

$$d/dy\,\alpha_i^\pm = \mp\, g_i/[g_i^{\,2} + (y\pm y_i)^2].$$

We see that R_i and α_i together constitute the magnitude and phase of the distances from the four poles in Φ_i; we have adopted the convention that the phase of $(y - z_i)$ is (-) the phase of $(y - z_i^*)$, with the result:

$$\alpha_i^{\pm}(y) \underset{y \to -\infty}{\rightarrow} 0 \ ; \ \alpha_i^{\pm}(y) \underset{y \to \infty}{\rightarrow} \mp \pi.$$

Note that $\Phi_1 = -B_1 \times (\alpha_1^- + \alpha_1^+)$.

In Eq. (III-d) the term $M_0(y)$ is a combination of terms from M_1, M_2, and M_3, each of which diverges separately as $\ln y_0$ when y_0 goes to $-\infty$. Because of the asymptotic condition (II-h), the combination does not diverge. Finally, we have:

$$\begin{aligned} dM_1/dy &= -B_1 \left\{ g_1/2 \ \ln (R_1^+/R_1^-)^2 + (y+y_1)\alpha_1^+ (y-y_1)\alpha_1^- \right\} \\ dM_2/dy &= B_2 \left\{ \alpha_2^- - \alpha_2^+ \right\} \\ dM_3/dy &= B_3/2 \ \ln (R_3^+/R_3^-)^2. \end{aligned} \qquad \text{(III-e)}$$

We have included in each M_i appropriate contributions from dM_0/dy; this is possible because dM_i/dy approaches y^{-1} as y goes to infinity, and so the choice of initial coordinate is irrelevant to any dM_i/dy.

Asymptotically, it is true that

$$\begin{aligned} \lim_{y \to \infty} \ & dM_1/dy = 2\pi B_1 y_1 \\ & dM_2/dy = 2\pi B_2 \\ & dM_3/dy = 0 \end{aligned}$$

$$\lim_{y \to \infty} \int_{-y}^{y} dx/ds\,(y) = 2\pi (B_1 y_1 + B_2)/P.$$

Also, as required: the limits on $M_i(y)$ are

$$\lim_{y \to -\infty} M_i(y) = 0; \qquad \lim_{y \to \infty} M_i(y) = y(\lim_{y \to \infty} dM_i/dy) \qquad \text{(III-f)}$$

We will derive (III-f) directly in Section IV by calculus of residues. See Fig. III-1.

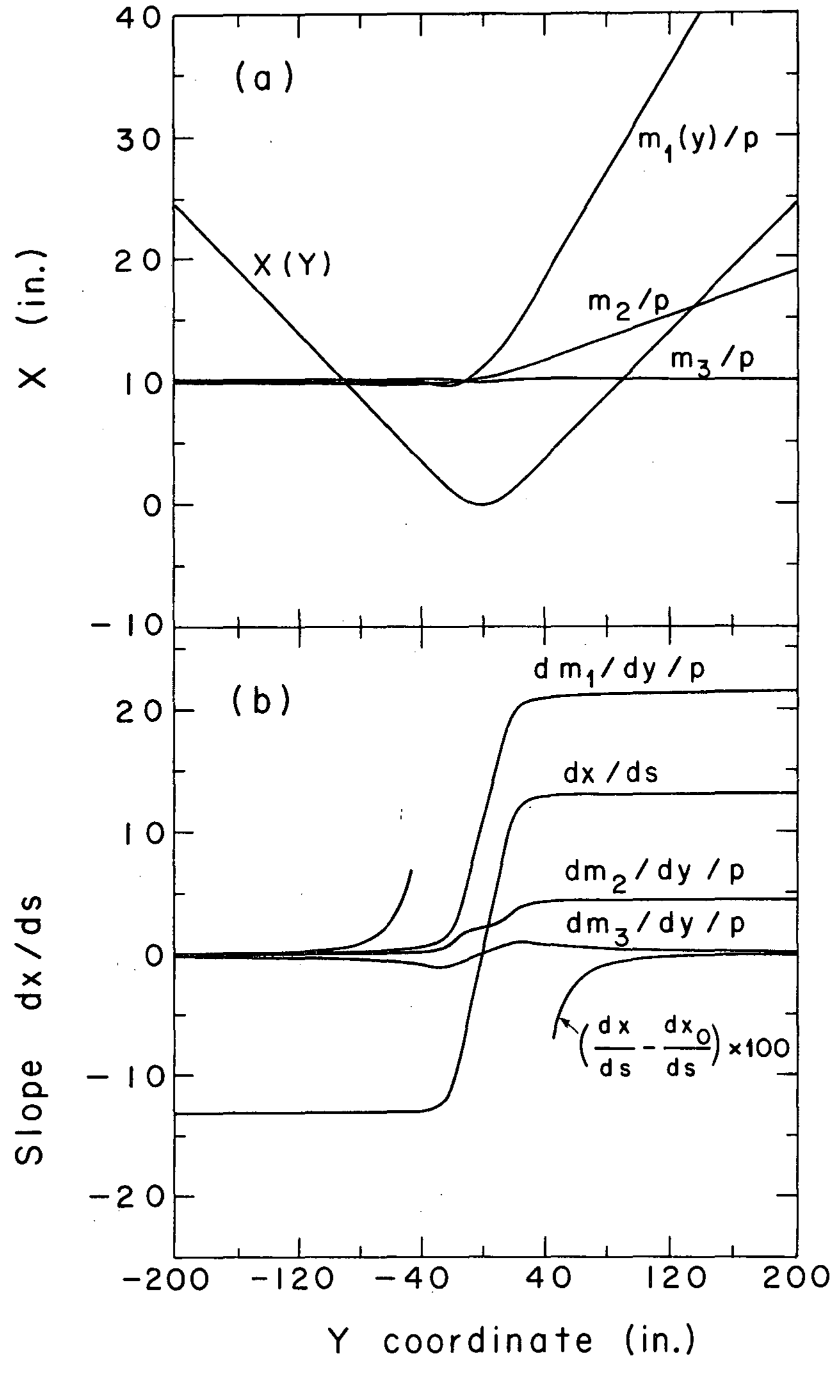

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Fig. III-1. (a) Plot of the trajectory x(y) and the contributions of the functions M_i/P. (b) Plot of the slope dx/ds and the contributions of the functions $(dM_i/dy)/P$. Initial conditions are chosen to make x(y) symmetric in y.

There are two corrections to the M functions which we now take into account:

(1) the correction due to $ds/dy \neq 1$

(2) the correction due to $B(x, y, z) - B(0, y, 0) \neq 0$.

We treat these as small perturbations and start with (1), assuming no x-z inhomogeneity.

Then (III-c) above must be revised to read:

$$X(y) = X_0 + \xi_0(S - S_0) + \left[\sum_1^3 [M_i(y) + \delta M_i(y)]/P\right]$$

$$\frac{dx}{ds} = \xi_0 + (dy/ds)\left[\sum_1^3 \frac{d}{dy}(M_i + \delta M_i)/P\right]$$

$$\frac{d^2x}{dyds} = d^2 y M_i/P, \qquad \text{(III-g)}$$

where δM_i is a correction to the M function.

Since $dy/ds = 1-(dx/ds)^2/2$ to sufficient accuracy, and since the second of Eqs. (III-c) is exact, we require

$$\left[1-(dx/ds)^2\right]\left[\sum d/dy(M_i + \delta M_i)\right] = \sum d/dyM_i. \qquad \text{(III-h)}$$

To this order of accuracy [no terms in $(dx/ds)^4$], we find

$$\sum \delta M_i(y) = (dx/ds)^2/2 \cdot \sum M_i(y) - \xi_0 \int_{-\infty}^{y} \frac{MM''}{P} - \int_{-\infty}^{y} \frac{MM'M''\,dy}{P^2},$$

where

$$(dx/ds)^2 = \left(\xi_0 + \frac{dM/dy}{P}\right)^2$$

and

$$M(y) = \sum_{i=1}^{3} M_i(y). \qquad \text{(III-i)}$$

Proof:

Equation (III-h) implies

$$-(dx/ds)^2/2\ dM/dy + d/dy\ \delta M = 0.$$

Equation (III-i) gives

$$d/dy\delta M = (dx/ds)^2/2\ dM/dy + M\,dx/ds\ d/dy(dx/ds)$$
$$- \xi_0\ MM''/P - MM'M''/P^2,$$

or

$$\left\{ M dx/ds \ d/dy(dx/ds) = M(\xi_0 + M'/P)\, M''/P \right\}.$$

So the result is established. Thus we can write:

$$X(y) = X_0 + \xi_0(S - S_0) + M(y)/P[(1 + dx/ds^2)/2] + \delta M/P$$

$$\delta M(y) = -\int_{-\infty}^{y} dy(\xi_0 MM''/P + MM'/P^2)$$

$$dx/ds(y) = \xi_0 + M'/P$$

$$d^2x/dyds = M''/P = B_z(0, y, 0).$$

Note that the integral for $\delta M''$ strongly convergent ($M'' \to y^{-4}$, $M \to y$, $M' \to$ constant as $y \to \infty$.

In addition, it is necessary to calculate $s(y) = s_0 + \int_{-\infty}^{y} ds/dy\, dy$.

By analogy with the technique used above to get the path integral of M', write

$$s(y) = y\left[(1 + \frac{dx^2}{ds})/2\right] + \delta s(y).$$

Then

$$\frac{d}{dy}\delta s(y) = \frac{ds}{dy} - \left[(1 + \frac{dx^2}{ds})/2\right] + y\,\frac{dx}{ds}\,\frac{d^2x}{dsdy}$$

$$\to \delta s(y) = -\int_0^y y(\xi_0 + M'/P)\, M''\, dy.$$

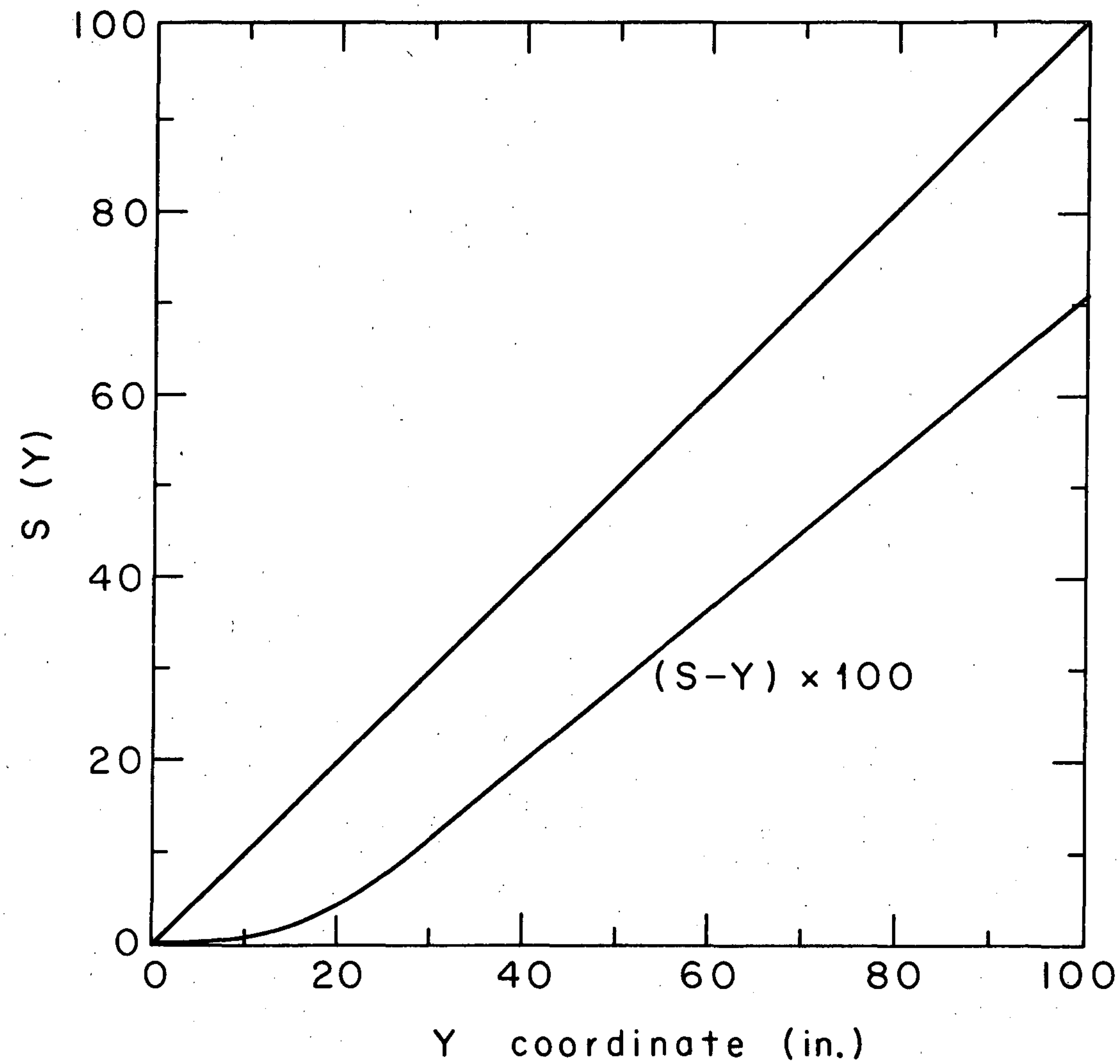
See Fig. III-4. Combining $\xi_0 \delta s(y) + \delta M(y)$ in (III-g) above, we have

$$X(y) = X_0 + \xi_0 y + \int P^{-1}\left\{M(1 + \frac{dx^2}{ds})/2 + \Delta M(y)\right\}$$

$$\Delta M(y) = -\int_{-\infty}^{\infty} (\xi_0 y + M/P))(\xi_0 + M'/P)M''/P\, dy$$

$$= \int_{-\infty}^{\infty} (x - x_0)\,\frac{dx}{dy}\,\frac{d^2x}{dy^2}\, dy.$$

We have done the integrals over $[-\infty, \infty]$ to get the total correction due to curvature to the quantity $X(y\to\infty) - X(y\to -\infty)$; for large y, the integrand is proportional to y^{-3}.



XBL688-3705

Fig. III-4. Plot of s(y) versus y for a symmetric track having an initial slope of 7.5 deg, together with (s-y) × 100. s(y) is obtained from the computation outlined in the text.

The correction is zero for a symmetric track, since the integrand is then an odd function of y.

In general, we can write:

$$\xi_0 = \xi_s + \delta\xi$$

$$\xi_s = -\frac{1}{2P_0}\int_{-\infty}^{\infty} B_z\, dy,$$

where $\delta\xi$ denotes the deviation of ξ_0 from ξ_0 for a symmetric track of momentum P_0. The result is:

$$\Delta M(y) = \delta\xi/{P_0}^2 \int_{-\infty}^{\infty} M'' \frac{d}{dy}(yM)\, dy$$

$$\simeq -1/2\ (\xi_0+\xi_f)/P_0 \times \int_{-\infty}^{\infty} M'' \frac{d}{dy}(yM)\, dy. \qquad \text{(III-j)}$$

The correction is typically 10^{-3} in. and can be easily computed for a given event, since the integral is only a single constant common to all tracks.

The second correction to the M functions is due to the correction for inhomogeneity in B_z. Here we only have to consider the asymptotic effect of the correction, and insert it in such a way that

$$M \to M(y) + \delta M_h(y),$$

where δM_h is due to field inhomogeneity in x and z. Then, denoting,

$$\delta I = \int_{-\infty}^{\infty} B_z(x, y, z) - \lambda\, B_y(x, y, z) - B(0, y, 0)\, dy,$$

we require:

$$\lim_{y\to\infty} \frac{d}{ds}\, \delta M_h(y) = \delta I$$

$$\lim_{y\to-\infty} d/ds\, \delta M_h = 0.$$

Correspondingly:

$$\lim_{y\to\infty} \delta M_h(y) = \delta I \cdot (S - S_0)$$

$$\lim_{y\to-\infty} \delta M_h(y) = 0.$$

We will show that the field inhomogeneities act like residues of poles at $\pm y_i + ig_i y_3$. Therefore we can turn on inhomogeneities at $\pm y_i$ of magnitude $\pi T_i\,[x, z(y_i)]$ by means of the functions

$$\phi(y_i, y) = \frac{1}{\pi}\int^{y}\int_{-\infty}^{-y} \frac{g_i^{2}\,dy'dy''}{[g_i^{2} + (y-y_i)^{2}} \simeq (y-y_i)\cdot\left[\left(\frac{\pi}{2}\right) + \tan^{-1}\left(\frac{y-y_i}{g_i}\right)\right] + g_i.$$

These functions are the same as $\mp\,(y - y_i)\,\alpha_i^{\pm}(y)$ defined above. Then:

$$\delta M_h(y) \simeq \sum_{i=1}^{3}\left[\Phi(-y_i, y)\;T_i(-y_i) + \Phi(y_i, y)\;T_i(y_i)\right],$$

where, as we explain in IV

$$\delta I = \pi\sum_{1}^{3} T_i(-y_i) + T_i(y_i)$$

$$\lim y\to\infty \;\; \delta M_h = y\,\delta I \text{ as required.}$$

There are other ways of inserting the inhomogeneities; for example:

$$\delta M_h = M(y)\cdot\delta I/I.$$

For truly asymptotic regions $(y - y_i/g_i \gg 1$ for all i) it does not matter what form we choose, to the accuracy in x(y) desired. See Fig. III-2.

To summarize:

$$X(y) = X_0\xi_0(s - s_0) + \frac{M(y)}{P}\left\{1 + \left(\frac{dx}{ds}\right)^{2}/2\right\} + \frac{\Delta M}{P}\,(\xi_{-\infty}, \xi_{\infty})$$
$$+ \frac{1}{P}\sum_{1}^{3}\Phi(-y_i, y)\;T_i(-y_i) + \Phi_i(y_i, y)\;T_i(y_i). \qquad \text{(III-k)}$$

Where $T_i(y_i)$ are given in (V-d).

Z TRAJECTORIES

To get the inhomogeneity integral, we have to know z(y) to about 0.25 in. in the region y = ±20 in., as well as x(y). To do this we solve

$$P\frac{d^{2}z}{ds^{2}} = \frac{dx}{ds}\;B_y - B_x \simeq P\;\frac{d^{2}z}{dy^{2}}.$$

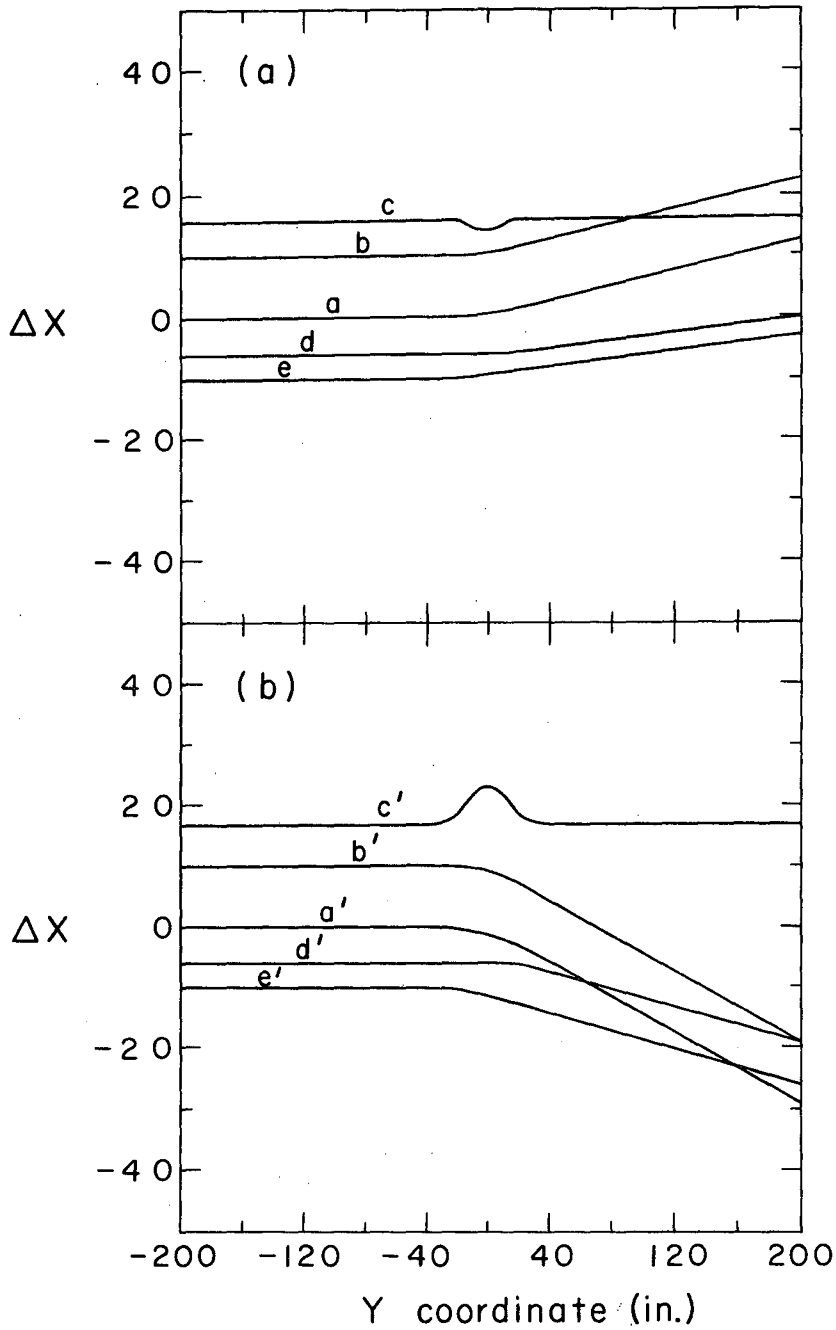


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Fig. III-2. Trajectory correction caused by inhomogeneities in units of 10^{-2} in. In (a) the trajectory is defined by $x_0 = -2$, $z = +2.5$; in (b) $x_0 = +2$, $z = 2.5$. The momentum and initial conditions are chosen to make the track symmetric with a 15-deg bend. The curves represent respectively:

$a, a' = \Sigma T_i(\pm y_i)\phi(\pm y_i)$
$b, b' = M(y) \times \delta I/I_0$
$c = (a-b) \times 1000;\quad c' = (a'-b') \times 10^3$
$d, d' = \Sigma T_i(+y_i)\phi(+y_i)$
$e, e' = \Sigma T_i(-y_i)\phi(-y_i)$.

Then

$$z(y) \simeq z_0 + \lambda_0(s - s_0) + M_z(y)/P,$$

where

$$\lim_{y\to\infty} M_z(y) = 0 \qquad \lim_{y\to\infty} M_z'(y) = 0$$

and

$$\frac{d^2}{dy^2} M_z(y) \simeq \frac{dx}{ds}(y)\, B_y - B_x.$$

We do not have to solve this analytically, although it is possible to do so with reasonable accuracy.

Since B_x and B_y are small perturbations on the orbit, we treat them exactly like the inhomogeneities which affect the orbit in the xy plane. Thus, to an adequate approximation, $z = z_0 + \lambda_0(s - s_0)$ for $y < 0$ and even to $y < +20$ in. B_y acts to converge rays from a distant source, while B_x diverges these rays (for trajectories which enter at $X_0 \simeq 0$ and $dx/ds\,(-\infty) < 0$), the net effect being a slight convergence of rays for large z_0.

Typically, for a ray with $z_0 \simeq 2.5$, $\lambda_0 \simeq 0.02$, we get $\lambda(+\infty) \simeq 0.016$, with symmetric 15-deg bending trajectories. Thus we write

$$\int_{-\infty}^{\infty} \frac{dx}{ds}\, B_y\, dy = T_y(z_0, \lambda_0, x_0, \xi_0)$$

and

$$\int_{-\infty}^{\infty} B_x\, dy = T_x(z_0, \lambda_0, x_0, \xi_0).$$

Both B_x and B_y are nearly linear in z. The B_y integral is trivial, as will be shown in general, using the method of residues:

$$T_y \simeq \pi B_1 \left\{ \frac{dx}{ds}(-y_1)\,[z_0 - \lambda_0 y_1] - \frac{dx}{ds}(y_1)[z_0 + \lambda_0 y_1] \right\}. \qquad \text{(III-1)}$$

T_x can be computed along with the inhomogeneity terms in B_z (see Section V below), provided z_0 and λ_0 are known. Unfortunately T_x must be calculated first to give a fit to determine z_0, λ_0. Because of the strong linearity in z, it turns out that

$$T_x \simeq -1.1\, z_0(x_0 + 2.6)\,[2\pi(B_1 y_1 + B_2]/10^{-3}]. \qquad \text{(III-m)}$$

We then parameterize $z(y)$ as

$$z(y) \simeq z_0 + \lambda_0(s-s_0) + \frac{1}{P}\left\{\Phi(-y_1, y)[T_y(-y_1) + T_x/2] + \Phi(y_1, y)[T_y(y_1) + T_x/2]\right\}, \qquad \text{(III-n)}$$

where

$$T_y(\pm y_1) = \mp \pi B_1 \frac{dx}{ds}(\pm y_1)(z_0 \pm \lambda_0 y_1).$$

Thus (III-n) gives correctly the asymptotic dip angles dz/dy and reproduces $z(y)$ at large distances. To facilitate track analysis, it is useful to note that $z(y)$ can be written

$$z(y) = B(y)\, z_0 + C(y)\, \lambda_0.$$

If $B(y_i)$, $C(y_i)$ are measured at a set of fixed points (y_i is the fixed position of a spark chamber), then z_0 and λ_0 can be fitted by least-squares analysis, and $z(y)$ can then be determined at any other fixed set of points (for example, at the locations of the poles in the field). See Fig. III-3.

From an initial fit to $x(y)$ and $dx/ds(y)$ we can determine $\mathbf{P}_0$, X_0, $dx/ds\,(\pm y_1)$ (y_1 is the pole in Φ_1 with residue B_1). Then we have

$$B(y_i) \simeq 1 + \mathbf{P}_0^{-1}\left\{\Phi(-y_1, y_i)[B_1\xi(-y_1) + T_x/2] + \Phi(y_1, y_i)[-B_1\xi(y_1) + T_x/2]\right\}$$

and

$$C(y_i) \simeq y_i - \mathbf{P}_0^{-1}\left[\Phi(-y_1, y_i)\, B_1 y_1\, \xi(-y_1) + \Phi(y_1, y_i)\, B_1 y_1\, \xi(y_1)\right].$$

$z(y)$ is then easily analyzed, since we can express B and C as

$$B(y_m) = 1 + \alpha_1^{\,m}\xi_-/\mathbf{P}_0 + \alpha_2^{\,m}\xi_+/\mathbf{P}_0 + \alpha_3^{\,m} x_0/\mathbf{P}_0 + \alpha_4^{\,m}/\mathbf{P}_0$$

and

$$C(y_m) = y_m + B_1^{\,m}\xi_-/\mathbf{P}_0 + B_2^{\,m}\xi_+/\mathbf{P}_0.$$

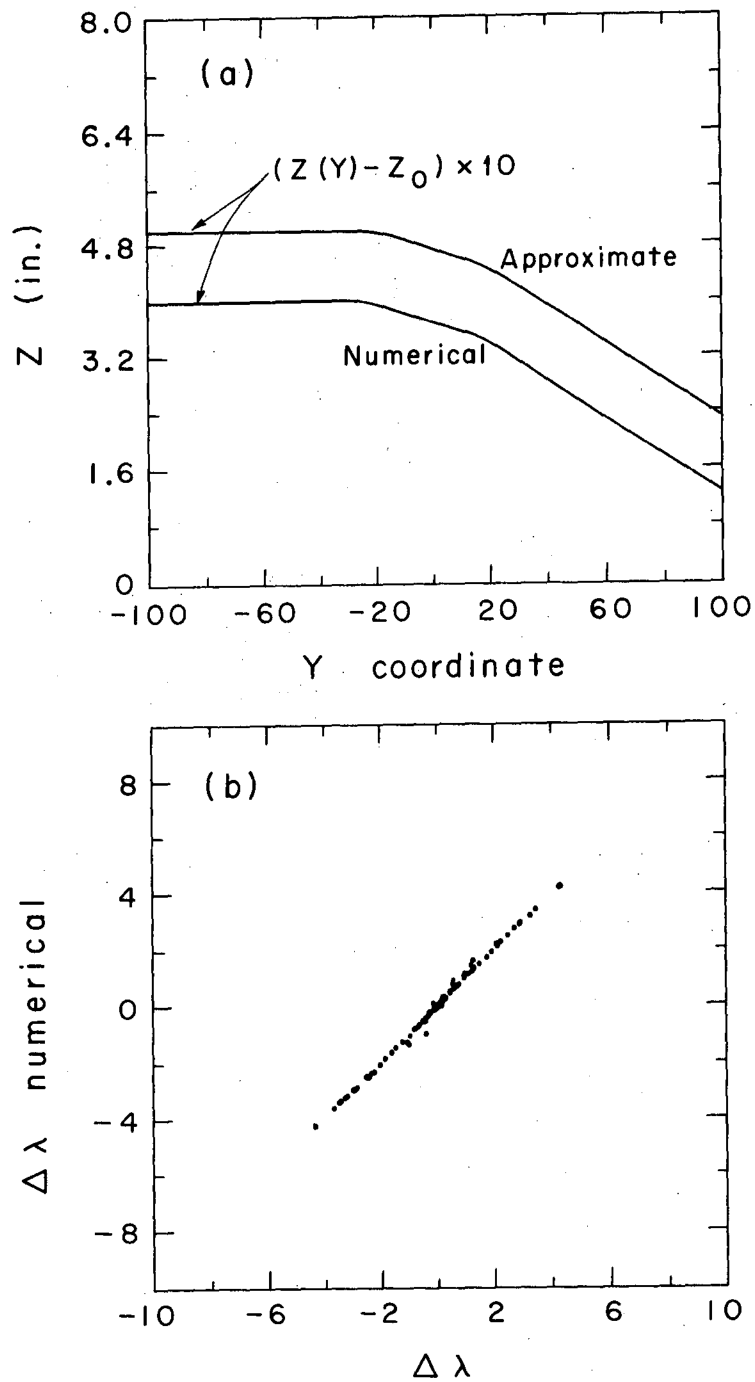

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Fig. III-3. (a) Variation of z(y) -

Fig. III-3. (a) Variation of $z(y) - z_0$ for a symmetric track with $z_0 = 2.5$, $\lambda_0 = 0$, $x_0 = 0$. The "approximate curve" is given by Eq. (III-n); the "numerical curve" is obtained from numerical integration of the differential equation. The ordinate is in tenths of an inch and the curves have been arbitrarily separated for clarity. (b) Net change in the dip angle in milliradians as computed by numerical integration of the differential equations (ordinate) versus the approximate integral given by (III-1) plus (III-m). The tracks were generated by Monte Carlo simulation of random initial conditions.

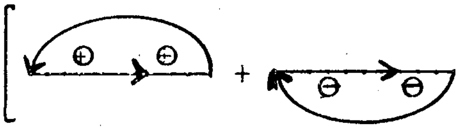
The $\alpha_i^{\ m}$, $B_i^{\ m}$ are constants associated with the spark chamber locations y_m.

IV. CALCULATION OF STRAIGHT-LINE INTEGRALS

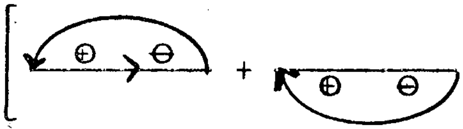
We want $\int_{-\infty}^{\infty} \Psi_i(x, y, z)\, dy$, with X, Z fixed. Referring to (II-f), the only term that contributes to the integral from a given pole in Ψ_i is

$$(1/y-\zeta_i)\,[B(x) - Z^2/2\ B^2(x) + Z^4/120\ B^{(4)}(x)].$$

We can close the contour either in the upper or lower half plane. For convenience, later we will do both and take half the sum, i. e.,

$$\int \psi_2\, dy = \frac{1}{2}\left[\ \ \right.$$  $$\left.\ \ \right]$$

and

$$\int \psi_3\, dy = \frac{1}{2}\left[\ \ \right.$$  $$\left.\ \ \right].$$

Trivially, we have

$$\int_{-\infty}^{\infty} \psi_2\, dy = 2\pi\left[B_2(x) - Z^2/2\ B_2^{(2)}(x) + Z^4/120\ B_2^{(4)}\right]$$

$$\int_{-\infty}^{\infty} \psi_3\, dy = 0\,.$$

To do ψ_1 terms, we use integration by parts and get

$$\int \frac{B_1}{2i}\left[-\ell n\,(y-\zeta_1)\right] dy = -\frac{1}{2i}\left[B_1\,\Big|_{-\infty}^{+\infty}\ \ell n(y-\zeta_1) - \int_{-\infty}^{\infty} dy\,\frac{\int^{y} B dy}{y-\zeta_1}\right].$$

The first term on the RHS does not contribute, because the term due to the pole at ζ_1 cancels with the term from the pole at $-\zeta_1^{\ *}$, and we are left with

$$\int_{-\infty}^{\infty} \psi_1\, dy = \pi\int_{-y_1}^{y_1} B_1(y)\, dy = 2\pi\ y_1\ B_1(x, z).$$

These results agree precisely with the values of dMi/dy obtained in III by directly integrating the trajectories.

Empirically, referring to the Apollo magnet we find that

$$\sum_{y=-53}^{y=+53} B(y_i)\,\Delta y_i = \left[1-0.003\right]\left[2\pi(B_1 y_1 + B_2)\right]\Big|_{x=z=0} .$$

This empirical relation holds for all currents, so the correction factor 0.003 is just the integral of the fringe. Measured deviations in the line integrals from the central line integrals are reproduced within 1/10000 (central value) by the X-Z-dependent B_i's.

V. CALCULATIONS FOR ARBITRARY TRAJECTORIES

When x(y), z(y) are not constant, two major difficulties arise which have to be accounted for satisfactorily.

1. Higher-order poles $\partial_{\zeta i}{}^{n}(1/y-\zeta i)$ need to be included, since the residue functions are now not independent of y. If we write (II-f) as

$$\psi_i(\zeta) = \sum_{n=0} \partial_\zeta{}^n (1/y-\zeta)\, T^n \left[x(y),\ z(y)\right] ,$$

then

$$\int \psi_i(\zeta) = \int dy\, \frac{1}{(y-\zeta)} \sum_n \left\{\frac{1}{n!}\ \partial_\zeta{}^n\, T^n\, [x(\zeta),\ z(\zeta)]\right\} . \qquad \text{(V-a)}$$

Note that the terms T^0, T^2, and T^4 in (II-g) are associated with Bi(x), while T_1, T_3, and T_5 are associated with $\zeta(x)$. They have the property that, if

$$\partial_\zeta{}^n\, T^n \quad \text{has a term} \quad z^{2m}\partial_x{}^{2m}\, B(x), \text{ then}$$

$$\partial_\zeta{}^{n+2}\, T^{n+2} \quad \text{has a term} \quad z^{2m}\partial_\zeta{}^{2m-2}\, B(x). \text{ Now}$$

$\partial_\zeta B \simeq \partial_x B \cdot dx/dy$, so the higher-order corrections give rise to $T^{n+2}/T^n \approx (dx/dy)^2$, which is typically 0.01, while the initial correction terms themselves, T^0 and T^1, are onlh 1 to 2% to begin with. Thus we consider only T^0 and T^1 in a reasonable analysis.

2. The residue functions are analytic only in a limited domain of the x-y plane and blow up for large x and z; asymptotically x is proportional to y(dx/dy), and so all the integrals, in principle, diverge. The source of this difficulty is that, for large x we need to join the polynomial in x and z with an analytic function that falls off with large x, z as X^{-n}, Z^{-n}, for $n>2$.

The residue functions must have distant cuts and singularities in order to fall off at large x, z. These cannot be determined. Furthermore, we do not know how to make the analytic continuation of $B[x(y)]$ into the complex y plane where the poles are located. In principle this is a serious difficulty, since the trajectory x(y) itself has singularities at the poles of Ψ_i in the y plane. It turns out from an inspection of the residue functions that these are all singularities of the type $(y-\zeta)\,\ell n\,(y-\zeta)$ or $(y-\zeta)^2\,\ell n\,(y-\zeta)$, which vanish at the poles ζ and produce no mathematical catastrophes.

To avoid the divergence difficulty, we impose a cutoff y_c in the integral $\int_{-\infty}^{\infty}[\psi(x,y,z) - \psi(0,y,0)]\,dy$, and for $y>y_c$ we assume $\psi(x,y,z)\sim K(R_0/R)^4$ as discussed in section IV. Since the field is accurately determined only to $x \simeq 7$ in., we place a cutoff at $y \simeq 40$ in. We also test that this choice does not have any systematic effect on the answer by varying y_c for a random bundle of rays. (See Fig. V-1). The parameters start to change rapidly beyond $x \simeq 6.5$ in., and the role of the cutoff is to obtain a reasonable answer for $x > 6.5$, where $y > y_c$.

The correction integral can be written

$$\int_{-\infty}^{\infty} B_z(x,y,z) - B_z(0,y,0)\,dy = \sum_i \int_{-y_c}^{y_c} \delta\psi_i[x(y), z(y)]\,dy - \left[\int_{y_c}^{\infty} + \int_{-\infty}^{-y_c} K\left(\frac{y_0}{y}\right)^4 \frac{\xi^2(y)}{2}\right], \qquad \text{(V-b)}$$

where $\xi(y) = d\sqrt{R^2-y^2}/dy$, and $Ky_0^4 = 2[B_1 g_1^3 y_1 + B_2 g_2(g_2^2+y_2^2) + B_3 y_3(g_3^2+y_2^2)]$. (V-b)

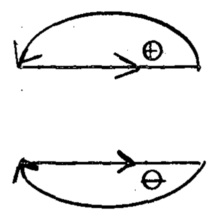
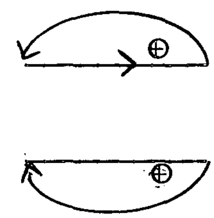
The last term is just

$$-\frac{2}{3}\,\frac{Ky_0^4}{y_c^3}\,(\xi^2_{-\infty} + \xi^2_{-\infty}),$$

which is expressed in terms of the asymptotic slopes and is typically $3\times 10^{-4}\times I_0$.

The integral of $\delta\psi_i$ is the x-dependent part of each pole contribution, namely T^0 and T^1 above with no constant terms in the residues.

If we use calculus of residues and ignore the effect of the cutoffs, we find two types of integrals:

(a)  (b) 

(a) $\frac{1}{2i}\int \frac{B_a}{y-3} - \frac{B_a^*}{y-3^*}$ (b) $\frac{1}{2}\int \frac{B_b}{y-3} + \frac{B_b^*}{y-3^*}$

We assume the residues $\beta(x, z)$ are real analytic in the Y plane.

Assume

$$\beta[x(y),\ z(y)] \simeq \beta(y_0) + \frac{\partial\beta}{\partial y_0}(y-y_0) + \frac{\partial^2\beta}{\partial y_0^2}\left(\frac{y-y_0}{2}\right)^2 .$$

This assumption is reasonable if β is analytic in Y near the real axis. At $y - y_0 = ig$, we see that $\mathrm{Im}\beta \simeq (\partial\beta/\partial y)g$. The series is expected to converge rapidly because

$$\frac{\partial\beta}{\partial y} = \frac{\partial\beta}{\partial x}\,\xi + \frac{\partial\beta}{\partial z}\,\lambda. \qquad (\xi = dx/ds)$$

Thus at a pole $(y - y_0 = ig)$,

$$\mathrm{Re}\ \beta \simeq \beta_0 + \xi^2\,\beta_0''/2 + \cdots$$
$$\mathrm{Im}\,\beta \simeq g\xi_0\,(\beta_0' + g^2\,\xi^2/6\,\beta_0''' + \cdots).$$

We ignore derivatives in z which depend on λ, since $\lambda/\xi \simeq 0.1$; the Taylor series is, in any case, perfectly well determined for any finite polynomial in x, z. Higher-order terms in ξ are negligible, like higher-order poles.

For the integrals we get:

$$\text{(a)}\quad I = +\pi\ \mathrm{Re}\ \beta(\zeta_0)$$
$$\text{(b)}\quad I = -\pi\ \mathrm{Im}\beta\ (\zeta_0)\ . \qquad \text{(V-c)}$$

Note that the infinity semicircles contribute but cancel.

In the Taylor series we ignored $d/dy(dx/ds)$ terms; it can be shown that these terms are reduced by a factor of the order x/y_0 from the terms we have employed.

Having expanded all residues in a Taylor series up to order $(y \mp y_i)^2$ about the poles at $\pm y_i$, we note that for certain parameters, the first or even the second derivatives are needed to give nonvanishing contributions. Thus y_2 and g_3 need $\partial\beta/\partial y_i$ because they have double poles at $\pm y_i$; B_3 needs $\partial\beta/\partial y_3$ because the B_0 term does not contribute to $\mathrm{Im}(\beta)$. Parameters g_2 and y_3 need $\partial^2\beta/\partial y^2$ because they have a double pole with imaginary residue. Terms like B_1, y_1, B_2 with single poles and real residues need only the $\beta_0(y_i)$ term.

To take into account the cutoff, we have to subtract contributions to each integral from $y > y_c$, which turns out to be equivalent to replacing π in the residue theorem by:

(1) $\pi \to 2\tan^{-1} U_c$ for zeroth order terms

(2) $\pi \to 2(\tan^{-1} U_c)\, U_c$ for first derivatives

(3) $\pi \to 2\left\{\tan^{-1} U_c - [U_c(U_c^2+2)]/[2(U_c^2+1)]\right\}$ for second derivatives,

where $U_c = [(y_c - y_i)/g_i]$.

These factors represent respectively:

(1) $$\int_{-U_c}^{U_c} \frac{1}{1+U^2}\, dU$$

(2) $$\int_{-U_c}^{U_c} \frac{U^2}{1+U^2}\, dU$$

(3) $$\int_{-U_c}^{U_c} \left[\frac{1}{1+U^2} + U\frac{\partial}{\partial U}\left(\frac{1}{1+U^2}\right)\right] dU.$$

We can regard the mathematics in several ways:

1. We are integrating the field over a finite range—where the power series in x and z and the first order expansion in the parameters themselves (e.g. I-c) is justified.

2. We can replace the (exact but limited) polynomials in x by an approximate Taylor series in y. By limiting the expansion to order y^2 we can make the integrals converge, and hence use the residue theorem.

Thus, from the asymptotic condition (I-h), we have

$$B_z \sim \left[(K_1 + K_2 y + K_3 y^2)/y^2\right] + Ky^2/y^4.$$

The first term cancels for large y, provided the Taylor expansion is made about the same point y_0 for all poles.

Even though the integrals converge, they are still not quite right, since we want the asymptotic behavior to be $1/R^4$. Thus we must subtract the contribution beyond a certain cutoff y_c, and replace this part of the integral by $\int 1/R^4$.

All procedures are essentially equivalent, in that some cutoff in y is needed to produce satisfactory asymptotic behavior. By treating each term individually and using the functions of U_c to replace π as described above, we reproduce exactly the integral of each set of poles out to the cutoff. On the other hand, if we just integrate the Taylor expansion (using $\pi \times \Sigma$ residues) we get a reasonably close answer for the total integral, but the individual pole terms come out wildly wrong; the individual terms have relatively different contributions from the asymptotic region, which affect the individual integrals but which tend to cancel when all poles are added together.

We now summarize the terms needed in a table.

Write:

$$T_i^B(x, z) = \delta B_i^0(x) - (z^2/6)\, B_i^{(2)}(x) + (z^4/120)\, B_i^{(4)}(x)$$
$$T_i^y(x, z) = B_i^0\, [\delta y_i(x) - (z^2/6)\, y_i^{(2)}(x) + (z^4/120)\, y_i^{(4)}(x)]$$
$$T_i^g(x, z) = B_i^0\, [\delta g_i(x) - (z^2/6)\, g_i^{(2)}(x) + (z^4/120)\, g_i^{(4)}(x)] \quad .$$

These define polynomials in x and z which can be Taylor-series-expanded in y.

Also let $F_1 = 2 \tan^{-1} U_c/\pi$

$F_2 = 2\,(\tan^{-1} U_c - U_c)/\pi$ and

$F_3 = 2\,\{\tan^{-1} U_c - U_c/2\ [U_c^2 + 2/U_c^2 + 1]\}\,.\,/\pi$

Then we get the following "residues" with $\pi \to F_i$ in the residue theorem:

Parameter	Residue ($y = -y_i$)	Residue ($y = - y_i$)
B_2	$T_2^{\,B}(-y_i)\, F_1$	$T_2^{\,B}(+y_i)\, F_1$
B_3	$-g_3 T_3^{\,B'}(-y_3) F_2$	$+g_3\, T_c^{\,B'}(+y_3)\, F_2$
y_1	$T_1^{\,y}(-y_1)\, F_1$	$T_1^{\,y}(+y_1)\, F_1$
y_2	$-T_2^{\,y'}(-y_2)\, F_1$	$+T_2^{\,y'}(+y_2)\, F_1$
y_3	$+g_3\, T_3^{\,g''}(-y_3)\, F_3$	$+g_3\, T_3^{\,g''}(y_3)\, F_3$
g_1	$+g_1\, T_1^{\,g'}(-y_1)\, F_2$	$-g_1\, T_1^{\,g'}(+y_1)\, F_2$
g_2	$-g_2\, T_2^{\,g''}(-y_2)\, F_3$	$-g_2\, T_2^{\,g''}(y_2)\, F_3$
g_3	$-T_3^{\,g'}(-y_3)\, F_1$	$+T_3^{\,g'}(y_3)\, F_1$

Prime denotes $\partial/\partial y$. (V-d)

The B_1 term deserves extra effort, since it requires integration by parts as explained above.

$$\int_{-y_c}^{y_c} \delta B_1(y) \left\{ \tan^{-1} U_+ - \tan^{-1} U_- \right\} dy =$$

$$\left[I(y) \left\{ \tan^{-1}[(y+y_1)/g] - \tan^{-1}[y - y_1/g] \right\} \right]_{-y_c}^{y_c}$$

$$- \int_{-y_c}^{y_c} I(y) \left[\frac{1}{1+U_+^2} \quad \frac{1}{1+U_-^2} \right],$$

where

$$\delta B_1(y) = T_1^{\,B}(x, z), \quad \partial/\partial y\, I(y) = \delta B_1(y), \quad U\pm = (y \pm y_1)/g_1 .$$

As above, we neglect the pole at $-y_1$ in integrating over $+y$ and vice versa, and arrive at two contributions:

$$I(y_c)\ \delta_c(U_c) + \int_{-U_c}^{U_c} I(y)\ dU/\sqrt{1+U^2},$$

where

$$\delta_c(U_c) = \left\{\tan^{-1}[(y_c + y_1)/g_1] - \tan^{-1}[(y_c - y_1)/g_1]\right\},$$

$$I(y) = \int_0^y \delta B_1(y)\ dy,$$

$$I(-y) = \int_0^y \delta B_1(-y)\ dy,$$

and we get two residue functions, one for each set of poles $(-y_1, \pm y_1)$. To do the integral of δB_1, we expand $x(y)$ and $\delta B_1[x(y)]$ in a Taylor series. Also, to improve accuracy slightly, we take

$$I(y) \simeq I(y_1) + (y - y_1)\ I'(y_1) + \left(\frac{y - y_1}{2}\right) I''(y_1).$$

Then we get for the B_1 term

$$\left[I(y_c)\ \delta_c(U_c) + I(y_1)\ F_1 + g_1^{\,2}\ I''(y_1)\ F_2/2\right] \tag{V-e}$$

where $I''(y_1) = \delta B_1{}'(y_1)$. To get $I(y_1)$ we write

$$\delta B_1(y) = \delta B_1(0) + \delta B_1{}'\ y + (\delta B_1{}''/2)\ y^2,$$

where

$$\delta B_1{}' = \xi_0\ \partial(\delta B)/\partial x$$

$$\delta B_1{}'' = \xi_0^2\ \partial^2/\partial x^2\ \delta B + \frac{d\xi_0}{dy}\ \partial(\delta B)/\partial x$$

$$\xi_0 = d/dy\ x(0)$$

$$\frac{d\xi_0}{dy} = B_z(x_0, 0, z_0)/P_0 .$$

Denoting $\partial B_1(0) = B_0$, $\partial/\partial x\ \delta B_1 = B_1$, $\partial^2/\partial x^2\ \delta B_1(0) = B_2$, we have

$$\delta B_1(y) = B_0 + yB_1\xi_0 + y^2\left[B_1\xi_0{}^1 + \xi_0{}^2 B_2/2\right] + y^3\left[\xi_0\frac{d\xi_0}{dy} B_2\right] + y^4\left[\left(\frac{d\xi_0}{dy}\right) B_2/2\right],$$

and
$$I(y) \simeq B_0 y + (B_1\xi_0)y^2/2 + \left[B_1\xi_0{}' + \xi_0{}^2 B_2/2\right]y^2/3$$
$$(y \leq y_1) \qquad + \left[\xi_0\xi_0{}^1 B_2\right]y^4/4 + \left[\xi_0{}^{12} B_2/2\right]y^5/5. \qquad \text{(V-f)}$$

This approximation, checked against numerical computation, is perfect.

For $I(y_c)$, a good approximation is

$$I(y_c) = I(y_1) + \left(\int_{x_1}^{x_c} \delta B(x)\,dx\right)/\xi_\infty,$$

where $x_c = x(y_c)$, $x_1 = x(y)$, and $\xi_\infty = \lim dx/dy(\infty)$.

To summarize, we have shown that the contribution from inhomogeneities in all ψ_i terms can be written as

$$\pi\sum_{i=1}^{3}\left[T_i(-y_1) + T_i(y_i)\right]F_i(y_c) + \left[\right]_{y'=-y_c}^{y'=y_c} -\frac{2}{3}y' B_z(y')\left(\frac{dx}{ds}\right)^2(y') + \delta c(y')I(y'),$$

where the T_i's are obtained from (V-b), or for the B_1 term, from (V-e) together with (V-f).

From the trajectory functions (no x, z dependence) we can calculate $x(y_i)$ and $z(y_i)$ to good accuracy from

$$x(y_i) \simeq x_0 + \xi_0 y_i + M(y_i)/\mathbf{P}_0$$
$$z(y_i) \simeq z_0 + \lambda_0 y_i\,.$$

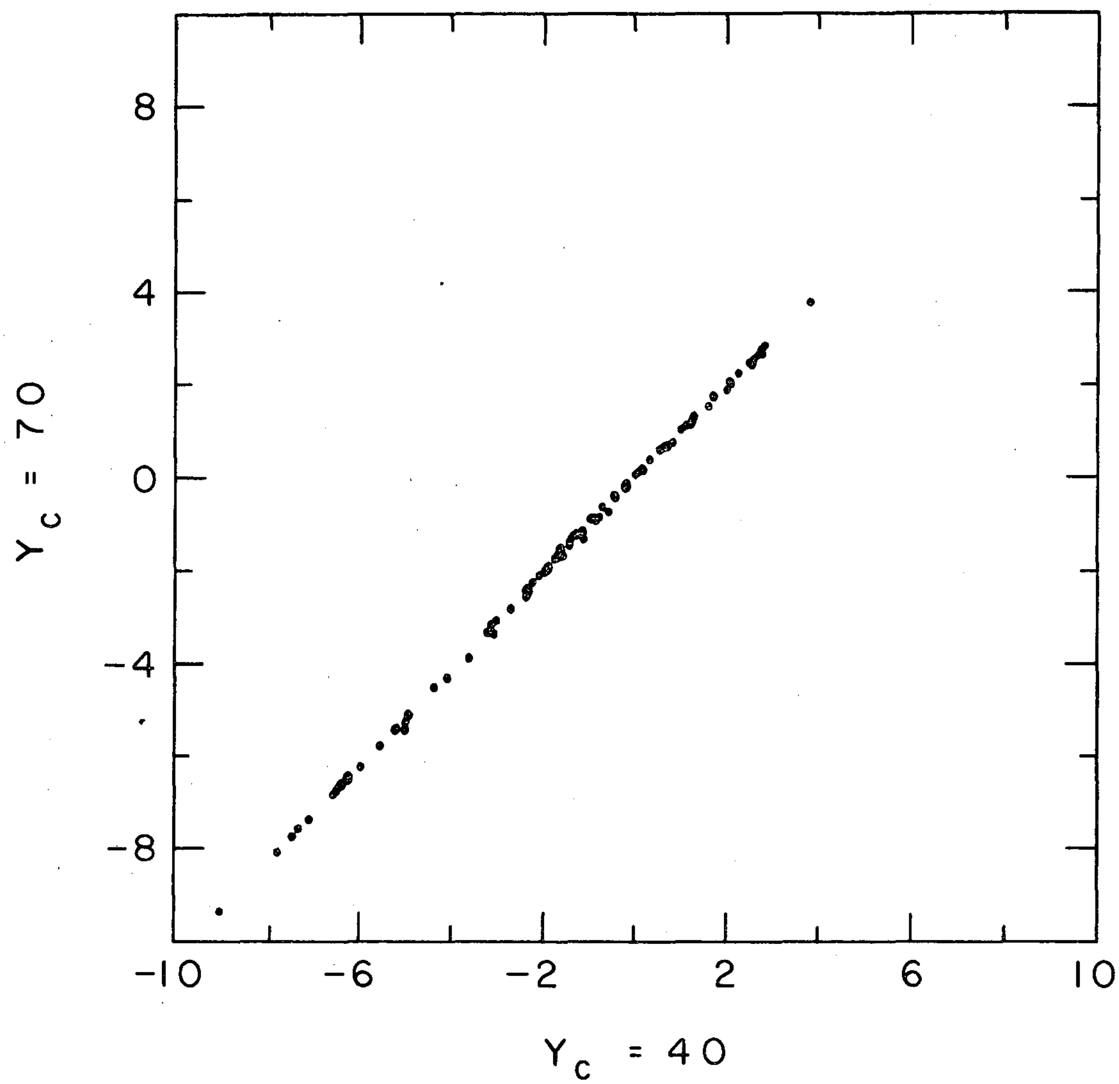
To the accuracy needed (x, z to 0.25-in.), ξ_0, λ_0, z_0, and $\mathbf{P}_0^{-1}$ can be inferred from the input data, so $x(y_i)$ is a linear function of input data [$M(y_i)$ are constants, calculated only once in the experiment].

For an iterative procedure (though this should not be necessary)(III-k) with $\mathbf{P}_0 \to \mathbf{P}_0'$ can be used for a slightly better approximation to $x(y)$. Also, $\int B_x$ can be written as a sum of residues and inserted in (III-n) for a better approximation to the z trajectory.

For nonnegligible dip angles (> 0.03) all residue functions must be replaced by:

$$T'(y) = \frac{dx}{dy}\,\partial T/\partial x \rightarrow \frac{dx}{dy}\frac{\partial T}{\partial x} + \frac{dz}{dy}\frac{\partial T}{\partial z}\,.$$

Figures V-1 through V-3 show the results (in 0.1% of the central line integral) of residues plus cutoff versus numerical integration in determining these corrections due to inhomogeneity. Evidently the calculation is reliable to 0.05%.



XBL688-3704

Fig. V-1. Plot of the numerical integral of the inhomogeneities in 0.1% of the central line integral for random tracks; the ordinate has $y_c = 70$ in.; the abscissa has $y_c = 40$ in. This shows that the choice of a cutoff does not affect the actual value of the integral, and so the only problem is to find a good approximation to the numerical integral by the residue technique.

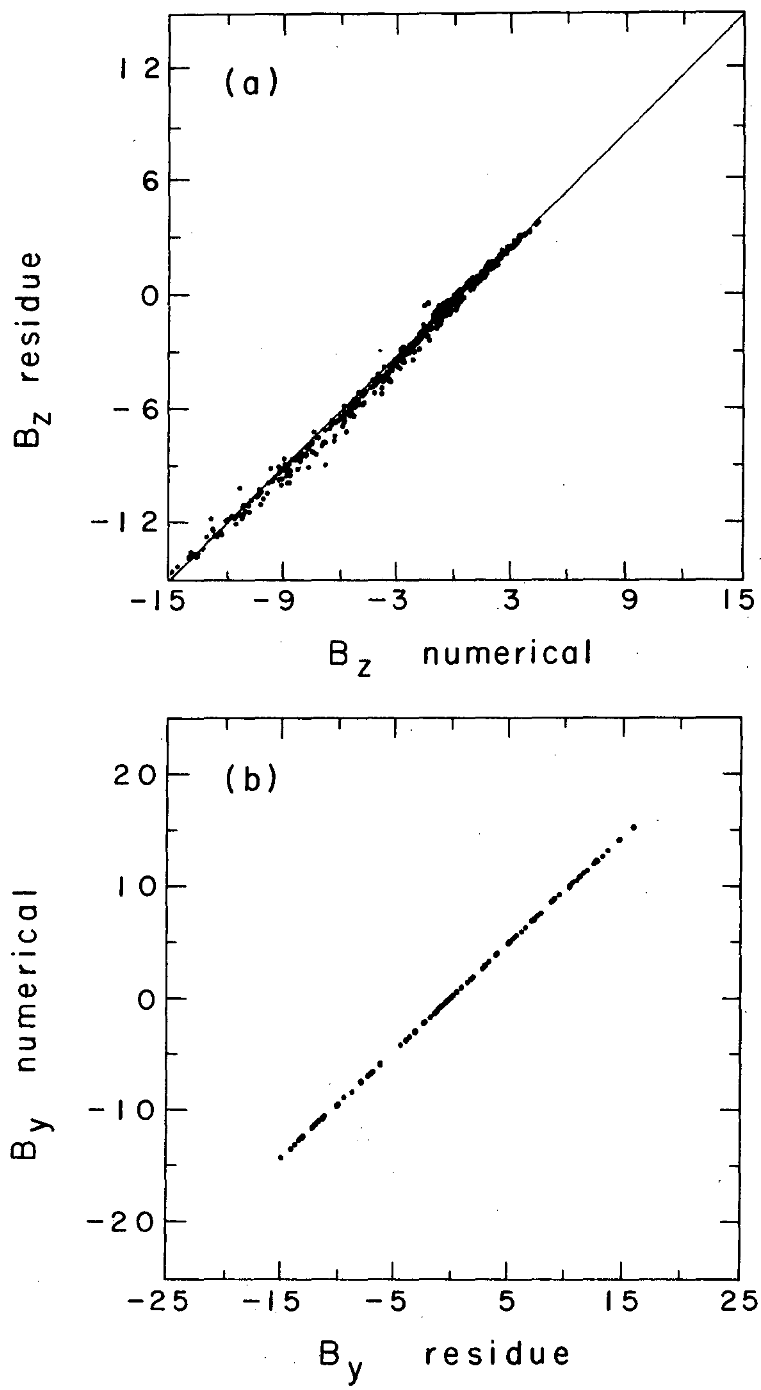


Fig. V-2. (a) Integral of B_z inhomogeneity by the method of residues (ordinate) versus numerical integration (abscissa). (b) Numerical integral $\int_{-\infty}^{\infty}$ (dx/ds) B_y(y)dy(ordinate) versus integral by residues (abscissa) in units of 0.1% of the central line integral.

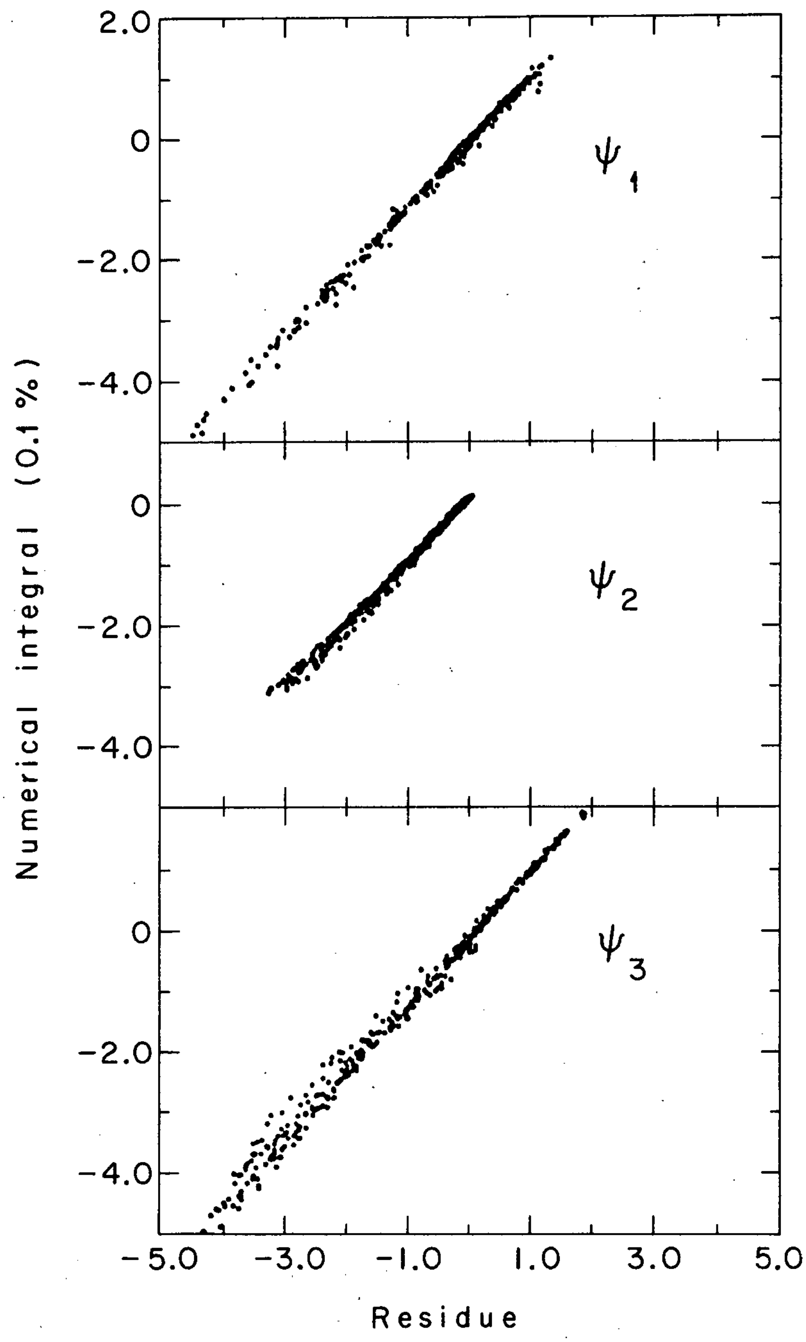


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Fig. V-3. Plots of residue (abscissa) versus numerical (ordinate) integration of inhomogeneities due to ψ_1, ψ_2, and ψ_3. Y_c is chosen to be 40 in. and units are 0.1% of central line integral.