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REVIEW ARTICLE

The Enactive Roots of STEM: Rethinking Educational Design in Mathematics

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Abstract New and radically reformative thinking about the enactive and embodied basis of cognition holds out the promise of moving forward age-old debates about whether we learn and how we learn. The radical enactive, embodied view of cognition (REC) poses a direct, and unmitigated, challenge to the trademark assumptions of traditional cognitivist theories of mind—those that characterize cognition as always and everywhere grounded in the manipulation of contentful representations of some kind. REC has had some success in understanding how sports skills and expertise are acquired. But, REC approaches appear to encounter a natural obstacle when it comes to understanding skill acquisition in knowledge-rich, conceptually based domains like the hard sciences and mathematics. This paper offers a proof of concept that REC's reach can be usefully extended into the domain of science, technology, engineering, and mathematics (STEM) learning, especially when it comes to understanding the deep roots of such learning. In making this case, this paper has five main parts. The section "Ancient Intellectualism and the REC Challenge" briefly introduces REC and situates it with respect to rival views about the cognitive basis of learning. The "Learning REConceived: from Sports to STEM?" section outlines the substantive contribution REC makes to understanding skill acquisition in the domain of sports and identifies reasons for doubting that it will be possible to apply the same approach to knowledge-rich STEM domains. The "Mathematics as Embodied Practice" section gives the general layout for how to understand mathematics as an embodied practice. The section "The Importance of Attentional Anchors" introduces the concept "attentional anchor" and establishes why attentional anchors are important to educational design in STEM domains like mathematics. Finally, drawing on some exciting new empirical studies, the section "Seeing Attentional Anchors" demonstrates how REC

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can contribute to understanding the roots of STEM learning and inform its learning design, focusing on the case of mathematics.

Keywords Enactivism · Ecological dynamics · Attentional anchor · Mathematics

New and radically reformative thinking about the enactive and embodied basis of cognition holds out the promise of moving forward age-old debates about whether we learn and how we learn. Embodied and enactive approaches to cognition emphasize the role of embodied and situated activity and how, in human cases at least, such activity is scaffolded by shared practices as crucial to a full understanding of cognitive abilities (Gallagher 2005; Varela et al. 1991). The most radical of enactive and embodied theories of cognition poses a direct, and unmitigated, challenge to the trademark assumptions of traditional cognitivist theories of mind—those that characterize cognition in strongly intellectualist terms, seeing it as always and everywhere grounded in the manipulation of contentful representations of some kind (Hutto and Myin 2013).

Radically enactive and embodied approaches to cognition (REC, for short) have had some success in understanding how sports skills and expertise are acquired. Sport psychologists have developed methods for situated skill training that focuses on how individuals interact with and adjust to environments as opposed to focusing on how their subpersonal components internally represent such features. The design requires putting learners into scenarios that closely resemble performance environments and giving them opportunities to attune to key features of such situations. These representative learning designs assume, in line with REC, that learners are dynamical systems that refine their responses to possibilities for action afforded by their environments (Davids et al. 2008; Renshaw et al. 2010). Where REC and such approaches really come together is in understanding training development as a matter of selectively modifying specific bodily, environmental, and task constraints and not, initially through any kind of explicit instruction. Adjusting such constraints would involve, for example, changing the size of the playing field, adjusting distances between players, fatiguing players as opposed to, *pace* Dreyfus and Dreyfus (1986), inculcating skills by, in the first stage, breaking them in analytic steps and sequences by means of instruction (Hutto and Sánchez-García 2014).

While it might be allowed that REC-inspired approaches to learning design could work in the acquisition of embodied skills and expertise, in domains such as sports, many will suppose that they encounter a natural obstacle when it comes to understanding skill acquisition in knowledge-rich, conceptually based domains like the hard sciences and mathematics.

Scepticism about REC's ability to inform educational design in areas such as mathematics goes hand in hand with adherence to strong nativist views which maintain that "concepts for at least some specific natural numbers are innate and that these innate concepts are a crucial factor in the explanation of why the human mind is suited for mathematics" (Laurence and Margolis 2007, p. 139). Recent interest in strong nativism of this brand has been revived by new findings about infant's basic number sense which some take as evidence that "preverbal, non-symbolic numerical capacities exhibited by human infants in the first year of life serve as a conceptual basis for learning to count and acquiring symbolic mathematical knowledge" (Starra et al. 2013, p. 18116). A direct conflict with REC arises if it is assumed that these capacities entail the existence of "a domain-specific nonverbal numerical representation" (Starra et al. 2013, p. 18116). Thinking that infants have a conceptually and contentfully grounded number sense has practical implications. It sponsors a particular vision of how mathematics is best learned and taught. For it assumes a "correspondence between external expressions (such as speech, written mathematical forms, and gestures) and some internal mechanism generating these expressions" (Roth 2015, p. 1). Unsurprisingly, for those who adopt this view, it is natural to think that mathematics education is best done through explicit instruction and training, through carefully staged sequences of learning (counting, followed by addition and subtraction, then multiplication and division, and so on). Such an approach assumes that children can build upon the assumed implicit conceptual knowledge that is already in place and serves them as a solid foundation.

The aim of this paper is to offer a proof of concept that REC's reach can be usefully extended into the domain of STEM—viz., science, technology, engineering, and mathematics—education, especially when it comes to understanding the deep roots of such learning. In making this case, this paper has five main parts. The section "Ancient Intellectualism and the REC Challenge" briefly introduces REC and situates it with respect to rival views about the cognitive basis of learning. The "Learning REConceived: from Sports to STEM?" section outlines the substantive contribution REC makes to understanding skill acquisition in the domain of sports and identifies reasons for doubting that it will be possible to apply the same approach to knowledge-rich STEM domains. The "Mathematics as Embodied Practice" section gives the general layout for how to understand mathematics as an embodied practice. The section "The Importance of Attentional Anchors" introduces the concept "attentional anchor" and establishes why attentional anchors are important to educational design in STEM domains like mathematics. Finally, drawing on some exciting new empirical studies, the section "Seeing Attentional Anchors" demonstrates how REC can contribute to understanding the roots of STEM learning and inform its learning design, focusing on the case of mathematics.

Ancient Intellectualism and the REC Challenge

The most extreme versions of cognitivist intellectualism assume that concept learning is, strictly speaking, impossible. For example, in contemporary guise Fodor (1975), p. 61, revived an ancient argument about the impossibility of genuine concept learning. It defends the view that we must postulate a set of pre-existing representations with precisely the same expressive power as any conceptual system or language to be "learned". This is necessary in order to explain how we form the hypotheses needed to fuel the learning processes in such cases. To learn a concept (or a local linguistic label for one), one must already be able to think about what is to be learnt as such. This, apparently, requires having contentful mental representations in order to couch one's initial hypotheses.

This argument only works if we are prepared to accept the key premise that new learners must always form content-based hypotheses about specific concepts to be learnt, which are then put to the test.[1] Since formation of such hypotheses must precede the learning of public language labels, such contentful hypotheses cannot be based in public language sentences; hence, contentful mental representations are required. Those who take this argument seriously think the very possibility of any rival proposals about what lies at the roots of concept learning can be ruled out in advance.

Nevertheless, even some staunch cognitivists are unconvinced and skeptical of this sort of "mad-dog" conceptual nativism (Prinz 2002, p. 235): They seek to motivate their cognitivist

[1] For a compelling critique of the need to invoke this Fodorian hypothesis-formation assumption, see Shea's (2011) commentary on Carey (2011).

intellectualism in other, softer ways. For example, theorists might adhere to a different Fodorian presupposition: that "Tracking requires a way to represent the trackee" (Fodor 2003, p. 20).

One formulation of this idea takes it that to track some X necessarily requires having some concept of what is being tracked. Accordingly, tracking involves representing-as.[2] Yet the demand that representations are needed for tracking can be softened further: One can respect Fodor's proposal even if it is assumed that no concepts are in place or play in the first stages of learning. All Fodor's view demands is that representational contents of some sort must be involved at the first stages of any learning process; nothing forbids that such contents might be non-conceptual. But, it is widely held that even if concepts are not in play, some representational content must be involved in basic stages of learning: How else could learners possibly think about the sorts of things on which they are trying to get a conceptual grip?

All versions of cognitivism hold, in one way or another, that learning always involves, as a matter of necessity, the manipulation of contentful mental representations. Put otherwise, all cognitivist theories of learning are thus committed to a thesis we dub Content Involving Cognition, or CIC, for short (Hutto and Myin 2013).[3]

REC, by contrast, promotes the view that cognition, in its fundamental form, is a matter of interactive dynamics that operates without the aid of contents or concepts. As such, it does not regard cognition as essentially a matter of acquiring and processing informational content or forming representations about the world. Rather, cognition is at root fundamentally world involving: Organisms interact with specific features of the world long before they have a capacity to conceptually categorize such features or form thoughts that may be true or false about them (Thompson 2007; Chemero 2009; Hutto and Myin 2013; Hutto et al. 2014).[4]

REC asks a simple question, which has significant implications: Is it possible to understand the basis of cognitive activity in fundamentally non-contentful terms? In *Radicalizing Enactivism*, Hutto and Myin (2013) articulate and defend the view that the best way to understand the dynamics of basic cognitive activity is in non-contentful terms, foregrounding the view that mind (or cognitive activity) is the result of an individual's situated embodied interactions with environmental affordances, i.e., opportunities for action ground in reciprocal agent–environment dynamics. REC thus strongly denies that a content-involving view of cognition (or, CIC) is the best explanation of, and framework for understanding, what lies at the roots of cognition.

Learning REConceived: from Sports to STEM?

The insight that cognitive activity is inherently interactive and engaged has influenced thinking about how skills are acquired in the domain of sports and how best to design learning

[2] Borrowing from Fodor, again, the assumption is that representing as necessarily involves concepts: "To represent (e.g., mentally) Mr. James *as* a cat is to represent him falling under the concept CAT" (Fodor 2007, p. 105). This line of thought motivates and (apparently) justifies believing in atomistic conceptual primitives. Carruthers (2011) articulates this working assumption well and makes his commitment to it indelibly clear: "many mental states are realized discretely in the brain and possess causally relevant component structure … they possess a discrete existence and are structured out of component concepts" (Carruthers 2011, p. xiv; see the preface of Fodor and Pylyshyn 2015 for similar view and an expanded list of related working assumptions).

[3] Here "content" is understood designating representational content—where, canonically, the notion of representation content assumes the existence of some kind of correctness condition such that the world is taken ("said," "represented," or "claimed") to be in a certain way that it might not be in.

[4] See (Hutto et al. 2014) for an elaborated account of the notion extensive, including how the idea of an extensive mind differs from that of an extended mind.

environments for such skill acquisition (Davids 2012; Davids et al. 2013; Renshaw et al. 2010). As noted above, REC adds value to such accounts by enabling them to break with the last residues of CIC thinking—by providing an entirely content-free way of understanding the cognitive basis of ecological dynamics and constraints-led approaches to learning (van Dijk et al. 2015).

Why think learning mathematics requires a fundamentally different sort of cognition than mastering embodied skills? Is there really something substantially different between learning mathematics and, say, learning to play tennis that requires that explanations of the former must involve contentful mental representations? Again, as discussed above, *prima facie*, if any domain is beyond REC's reach, surely, mathematics is: It involves manipulating and grasping concepts and rules, making the kind of learning required seem fundamentally different from that involved in acquiring embodied skills. In general, sporting activities are inherently about the expert manipulation of material items—such as the body and equipment—and often this is done in concert with fellow athletes. Mathematics, in contrast, appears to require building on already implicit forms of "heady" basic conceptual knowledge, as such engaged manipulation of material objects is not what needs to be primarily manipulated when learning mathematics. In this light, it is not unreasonable to suppose that the abstract, disembodied, and theoretical nature of mathematics make it such that its mastery demands and is rooted in mental activity involving contentful mental representations. Mastering mathematics, so a strong intuition suggests, is rooted in getting a firmer grip on already known, basic abstract concepts and rules.

In the rest of the paper, our focus will be on promoting a different take on learning and teaching mathematics—a cardinal area of STEM education. Our goal is to demonstrate that it is possible to understand this problem space without presupposing underlying contentful mental representations governing or even constraining learning mathematics. In particular, we will show that REC's notion of "attentional anchor" can be usefully applied in educational research both in sports science and mathematics. Our aim is to demonstrate REC's potential to drive educational psychology research in productive new directions. In what follows, we will explain what attentional anchors are in theory and how they are used in practice in research on mathematics educational design. Before doing so, however, it is important to say a few words about how sports and mathematics are both embodied practices, and more fundamentally similar than they might seem at first glance.

Mathematics as Embodied Practice

Human cognitive abilities evolved via our species' adaptation to its ecological niche, whose features afford dynamical interaction. Because of this, one would expect traditional mathematical images—those two-dimensional symbolic depictions of idealized objects—to be difficult to handle with our developed abilities. And yet this is not the case, in large part because mathematically fluent adults attribute to these visual stimuli similar qualities as are characteristic of tangible objects occupying space, viz., intact entities that can be lifted and moved about (Wittmann et al. 2013). Indeed, the most fundamental structures and mechanisms of our shared cognitive architecture are such that our primary intuition is to perceive mathematical symbols not as weightless indices of abstract objects but as bona fide voluminous and tangible features of the environment (Núñez et al. 1999). Micro-ethnographical developmental studies suggest that goal-oriented physical interaction with objects is the psychological basis of an individual's ability to recognize and reproduce symbolic tokens of these objects (Bruner 1960; Piaget et al. 1960; Roth and Thom 2009).

Later on in elementary school, semiotic–cultural analyses of classroom episodes involve mathematical symbols as students' objectifications of their pre-symbolic multimodal notions (Radford 2013). For example, a child might entertain the sense of two groups of discrete elements coming together to form a single larger group, and she might gesture at this transformation, and yet only later accept the arithmetic operation of addition, along with its sign, "+", as expressing this hitherto implicit, pre-reflective sense. What this indicates, alongside studies conducted at the level of high school and beyond, is that it is only as tangible objects that individuals are able to handle and master mathematical inscriptions, even operations on symbolic notations (Davis and Hersh 1981; Goldstone et al. 2009; Marghetis and Núñez 2013).

Mathematics learners must be sensitive to the normative practices characteristic of mathematics. For example, students are to construe particular orientations, patterns, or quantitative relations of diagrammatic elements in a manner compatible with the discipline's conventions (Rotman 2000). As such, one objective of education is to enculturate students into understanding static images as offering opportunities for action (Barab et al. 2007). To an extent, mathematics instruction is just the process of guiding students toward developing normative visualizations of flat images, such as diagrams and symbols (Sinclair and Gol Tabaghi 2010). To these ends, mathematics educators use a range of pedagogical methodologies in their attempts to steer students toward sensitivity to the discipline's canonical visual images such as making sense of a graph line (Abrahamson 2012a; Bartolini Bussi and Mariotti 2008; Sfard 2002; Stevens and Hall 1998).

Having outlined some arguments for a material phenomenology of mathematical displays, we underscore that canonical mathematical displays are expressly unlike basketballs, for example, so that embodied interaction with these displays requires elaborate mental construction (de Freitas and Sinclair 2012). In a sense, visualizing displays in the covertly embodied disciplines, such as mathematics, is often a degenerate form of engaging imagery in the overtly embodied disciplines, such as sports, because the interaction cannot be physically enacted in these semiotic domains. For example, compare the rungs of an upright ladder to the horizontal lines of a Cartesian grid. In both cases, an eye that scans this display vertically will traverse a sequence of lines, and yet only the ladder rungs are given to embodied interaction. At the same time, it is these very constraints on interaction posed by symbolical objects on flat media that have afforded the development of semiotic systems and discursive forms critical to the evolution of the mathematical discipline (Brown et al. 2009; Kirsh 2010; Olson 1994; Schmandt-Besserat 1992; Smith and Gasser 2005; Uttal and O'Doherty 2008).

This is why the study of mathematical practice as embodied is both fascinating and important—it stands to explicate the historical, developmental, and pedagogical possibility of ascribing meaning to mathematical signs. And, this is where dialogue across the overtly and covertly embodied disciplines may enable mutual insights on learning processes. For example, scholars of mathematics education stand to learn from scholars of sports education how instructors' imagistic information promotes learning. Specifically, we are interested in how imagery serves instructors in steering students to visualize a less familiar situation similar to a more familiar situation. In so doing, we acknowledge a body of earlier research on metaphorical visualization of mathematical displays (Presmeg 2006; Sfard 1994)—research without which we could not be asking our current questions—yet we seek to revisit the phenomenon via ecological dynamics.

From our perspective, the term "visualization" may be doing a disservice to the field of mathematics education research, because the term threatens to undergird a conception of mathematics as passive and mono-modal, whereas it is a proactive multimodal sensorimotor interaction (Abrahamson et al. 2014; Nemirovsky 2003; Roth 2010). Once we ascribe to mathematical objects the same kind of qualities as concrete objects (see above), then the parallels between mathematics and other practices become more evident. Coming to terms with the complementary feature of overt and covert embodied practices is ultimately, we submit, more constructive for all stakeholders in educational activities (Trninic and Abrahamson 2012). In particular, instructional designers sensitive this is parallel stand to create learning environments for mathematical practice much better suited for enactive and embodied minds (Abrahamson 2009; Abrahamson et al. 2011, 2012; Chahine 2013; Fischer et al. 2011; Gerofsky 2011). Collaborations between researchers of sports and mathematics education stand to promote an exciting turn in educational research toward embodied design principles (Abrahamson and Lindgren 2014). If human learning in mathematics and sports is more similar than meets the eye, it follows that these disciplines might share a set of unified principles of instructional methodology. At least, as von Glasersfeld (1983) observed, mathematics methodologists have much to learn from their sports colleagues.

As section two made clear, it is now believed that athletes develop more effective embodied skills through engaged activity rather than by absorbing and following dictates (Chow et al. 2007). The implications for explicit instructional teaching are nothing short of monumental. Instead of telling a student what to do, a teacher ought rather to set an environment, assign a task, and then productively steer the student's engagement of the task by responsively implementing formative constraints into the environment. This fundamental idea, which Chow and others refer to as non-linear or constraints-led pedagogy, turns on the following hypothesis: You cannot directly teach anyone anything; at best, you can create activities that foster opportunities for a person to construct some targeted knowledge for themselves. In this light, mathematics and sports share much more than is initially evident.

The Importance of Attentional Anchors

Having established some gross systemic similarities between pedagogical practices in the overt (sports) and covert (mathematics) disciplines, we are now in a position to understand attentional anchors and why they are important to educational design in STEM domains such as mathematics.

An attentional anchor is the focus of an actor's interaction with the environment that is brought forth as the agent's skill set grows through engaging in a task (Ingold 2000). It is a real or imagined object, area, or other aspect of a situation that facilitates coordination (Hutto and Sánchez-García 2014). Attentional anchors productively hone and channel attention during perception–action couplings, thus functioning as enabling constraints on action. Attentional anchors reduce operational complexity, rendering ergonomic and feasible an otherwise overwhelming task. The agent acting on the attentional anchor experiences it as a "steering wheel" overlaid upon the perceptual field—the attentional anchor is a focus of both operating on the environment and responding to the environment. Specifically, responding to attentional anchors can bring forth potential affordances by foregrounding relevant task-oriented aspects in the environment.

Attentional anchors are the centerpieces of a non-linear pedagogical methodology. Numerous studies in skill acquisition of sport techniques suggest the comparative advantage of directing learners' focus away from internal kinesiological components toward external environmental structures (Wulf and Su 2007; Zarghami et al. 2012). For example, when swimmers practice the arm stroke in crawl style, they perform better by focusing on “pushing the water back” than on “pulling your hands back” (Stoate and Wulf 2011). Further support for attentional anchors comes from dynamical-systems-theory models of motor action (Thelen and Smith 1994), particularly those inspired by ecological psychology (Gibson 1977). Newell and Ranganathan (2010) as well as Kelso and collaborators (Kostrubiec et al. 2012) characterize the emergence of situated motor-action skill as the development of a higher-order invariant that emerges for the agent through her environmentally coupled interaction to reduce performance complexity and enhance control. For example, a rodent preparing to leap across a rivulet will perform looming actions, lurching its body back and forth and appraising shifts in perspective, as a perceptual means of anticipating and calibrating the necessary effort for fording the gap. Attentional anchors, then, serve individuals in better engaging the environment or, as Merleau–Ponty calls this iterative optimization process, maximizing one’s grip on the world (Dreyfus and Dreyfus 1999).

Attentional anchors highlight REC in action. They eschew mentalist preoccupation with contentful representations as the would-be basic apparatus of learning precisely because of their focus on embodied and situated activity. Equipped with attentional anchors, we need not trouble ourselves about the learning paradox Fodor raises. There is no paradox. What we call learning is developed responsiveness to attentional anchors in the “field of promoted action” (Reed and Bril 1996), thereby enabling perception to guide action. In honing their ways of responding to attentional anchors, learners develop more effective worldly engagements. Teachers who recognize attentional anchors as the progenitors of knowledge might create opportunities for students to reinvent these same “steering wheels” by productively constraining their engagement in pedagogical tasks. What we call learning is no more yet no less than a gradual or abrupt systemic shift (Kelso and Engstrøm 2006). Still, attentional anchors would be bona fide explanans for REC assertions only if (somehow) we could measure them independently, for otherwise they would be cast away indefinitely to the ignominy of mere theory. Enter technology.

Seeing Attentional Anchors

Using eye-tracking methodology, it is now possible to target the creation and unfolding of attentional anchors. For example, it is possible to see a child construct a mathematical concept as the reflective articulation of a new situated and embodied activity. The relevant experimental work has been made possible by a combination of two recent techno-methodological developments adopted by the Embodied Design Research Laboratory (EDRL) and their collaborators—natural-user interfaces (NUIs) and multimodal learning analytics (MMLA). Together, this equipment is creating opportunities in educational research that permits experimentation on attentional anchors. In particular, tracking students’ eye gaze as they engage in problem-solving activities is creating promising opportunities for scholars of mathematics education to investigate empirically relations between perception, action, and learning. This is supported by findings from a set of semi-structured task-based clinical interviews with young students operating a multitouchscreen tablet Mathematical Imagery Trainer for Proportion (MIT-P). In addition to videotaping the participants’ multimodal utterance in dialogue with the interviewer, it has been possible to non-invasively log their manipulation of virtual elements as well as their visual pathway across the interface. In what follows, we unpack the pedagogical rationale, early implementations, and current empirical findings relevant to REC.

Empirical Context: Design-Based Research of the Mathematical Imagery Trainer

The *Kinemathics* project (Abrahamson et al. 2011, 2012; Abrahamson and Trninic 2015; Howison et al. 2011) took on the educational design problem of students' enduring challenges with learning the mathematical concept of proportional relations. When students look at 6:10= 9:*x*, for example, they are liable to make sense of these symbols through an "additive lens" instead of a "multiplicative lens" (Behr et al. 1993; Karplus et al. 1983; Van Dooren et al. 2010). For example, they might attend only to the differences among the numbers: seeing a difference of 4 between 6 and 10 (or seeing a difference of 3 between the 6 and 9), they would infer that the other pair has the same difference, so that the unknown number is 13 (whereas it should be 15).

We assumed that students have scarce sense of what proportional equivalence is, feels, or looks like. We began by choreographing a bimanual motor-action scheme that enacts proportional equivalence, and then we envisioned, designed, and engineered conditions in which students could learn to move in a new way that emulates this scheme. Our two-step activity plan was for students to (1) develop the target motor-action schemes as dynamical solutions to situated problems bearing no mathematical symbolism and (2) describe these schemes mathematically, using semiotic means we interpolate into the action problem space.

Our design solution was the Mathematical Imagery Trainer for Proportion (MIT-P). We seat a student at a desk in front of a large, red-colored screen and ask the student to "make the screen green" (Fig. 1). The screen will be green only if the cursors' heights along the screen relate by the correct ratio (e.g., 1:2). Participants are asked first to make the screen green and then to maintain a green screen while they move their hands.

The activity advances along a sequence of stages, each launched when the instructor introduces a new display overlay (see Fig. 2) immediately after the student has satisfied a protocol criterion. For example, consider a student who is working with the cursors against a blank background (Fig. 2b). Once she articulates a strategy for moving her hands while keeping the screen green, the activity facilitator introduces the grid (see Fig. 2c). In a culminating stage, not discussed here, the students switch to controlling the cursors indirectly via inserting numbers into a ratio table.

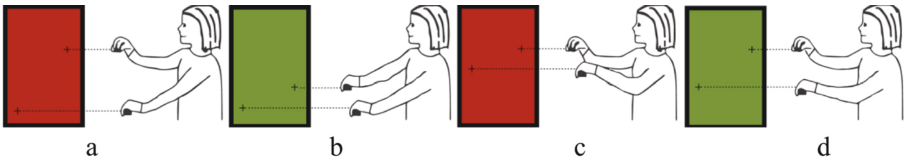

Fig. 1 The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the favorable sensory feedback (a green background) is activated only when the right hand is two times as high along the monitor as the left hand. This figure sketches out our grade 4–6 study participants' paradigmatic interaction sequence toward discovering an effective operatory scheme: **a** While exploring, the student first positions the hands incorrectly (*red feedback*), **b** stumbles upon a correct position (*green*), **c** raises hands maintaining a fixed interval between them (*red*), and **d** corrects position (*green*). Compare (**b**) and (**d**) to note the different vertical intervals between the virtual objects (Color figure online)

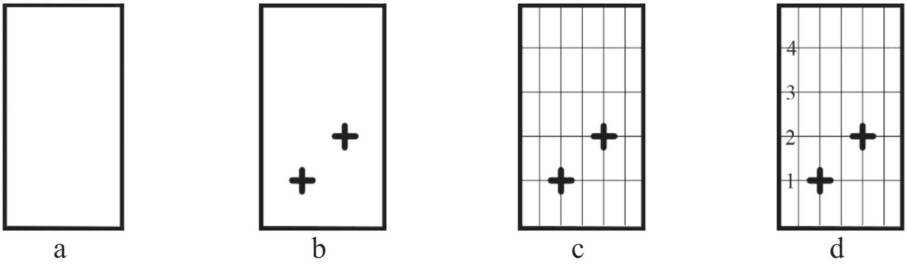

Fig. 2 MIT-P display schematics, beginning with **a** a blank screen, and then featuring the virtual objects (symbolic artifacts) that the facilitator incrementally overlays onto the display: **b** cursors, **c** a grid, and **d** numerals along the *y*-axis of the grid. For the purposes of this figure, the schematics are not drawn to scale. Also, the actual device enables the tutor to flexibly calibrate the grid, numerals, and target ratio in between trials

Figure 3 illustrates an early moment in the MIT-P activity, where a child has found a pair of locations on the screen that make it green and is considering her next move. Note the absence of any mathematical instruments or symbols on the screen.

We implemented the MIT-P design in the form of a tutorial task-based clinical interview with 22 grade 4–6 students, who participated either individually or in pairs, and these sessions were audio–video-recorded for subsequent analysis (Howison et al. 2011). Our primary methodological approach is for the collective of researchers to engage in collaborative ethnographic micro-analysis of selected brief episodes from the entire data corpus (Siegler 2006), where we focus on the study participants' range of physical actions and multimodal utterance around the available media (Ferrara 2014). The analytic process is iterative and in dialogue with the learning-sciences literature, leading to the progressive identification, labeling, and refinement of emergent categories (Strauss and Corbin 1990). In the course of this analytic work, new constructs might emerge that constitute ontological innovations extending beyond the study context (diSessa and Cobb 2004). In particular, the construct of attentional

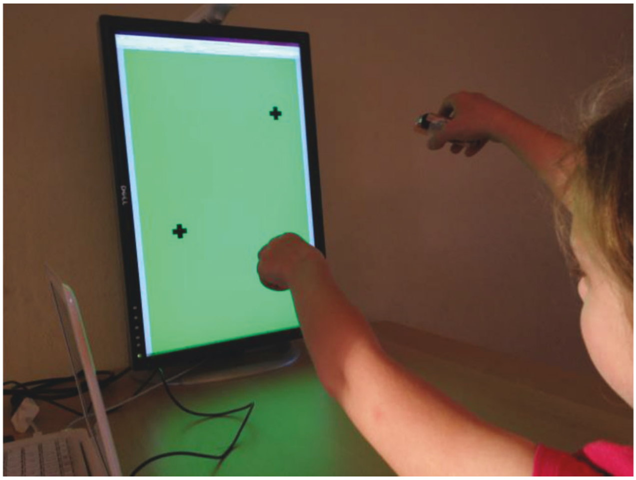

Fig. 3 Neomi is working with the Mathematical Imagery Trainer for Proportion. The unknown ratio has been set by the tutor at 1:2, and so the screen will be green only if the cursors' respective heights above the screen base relate at a 1:2 ratio. Neomi is holding the cursors at appropriate heights above the screen base, and so the screen is green

anchor informed our analysis and served as a capstone to our budding ecological-dynamics/REC interpretation of the empirical data. In what follows, we will demonstrate the role an attentional anchor played in mediating a shift in students' grasp of mathematical notions.

The Emergence of an Attentional Anchor Mediating System Dynamics

Students typically begin the task by lifting the controls and, in an attempt to make the screen green, waving them up and down in a variety of patterns. Eventually, the students discover that the positions of their hands need to be related one to the other in specific ways. In particular, as one student commented, the hands "have to be a certain distance" from each other. At first, they attempt to keep this distance fixed. But, as they further explore lower and higher screen regions, they figure out that "the higher you go, the bigger the distance." Students thus discover, articulate, and empirically validate a systemic action principle governing a phenomenon under inquiry: a proposed correlation between two qualitative properties of a new object—the height and size of a linear interval subtended between their hands.

We have been intrigued by students' initial discovery of the interval between their hands as a means of controlling the screen color as well as by their subsequent shift from keeping this interval fixed as they elevate their hands along the screen to varying the interval size in proportion with its elevation. Crafted spontaneously, the interval articulates into being, foregrounded as a new auxiliary stimulus wedged between agent and object. The interval coalesces as a means of both visualizing latent correlations and enacting them to achieve environmental effects.

From the ecological-dynamics or REC view, the interval served the students as a spontaneous self-constraint—an attentional anchor that promoted their performance. The videography documents the birth of this "steering wheel" (Kostrubiec et al. 2012; Newell and Ranganathan 2010) as evident in the child's actions and multimodal utterance. In fact, comparison among students who achieved this attentional anchor suggests a collective convergence of multiple idiosyncratic solutions.[5]

When people attempt to perform an unfamiliar task in an unfamiliar setting, they begin by exploring the space in undirected ways. As they engage, their actions may affect features of the environment: They might move, remove, or transform an object or set of objects, whether they do so deliberately or inadvertently. These exploratory actions can give rise to new apparent structures that in turn create new affordances for interaction, and so on (Kirsh and Maglio 1994; Loader 2012; Schwartz and Martin 2006). Such systemic engagements are ultimately constrained by agent, task, and environment, even as they create opportunities to detect and tune toward emergent patterns that further establish and regulate stable interaction routines (Aguilera et al. 2013; Kelso 1995; Newell and Ranganathan 2010; Thelen and Smith 2006).

Features of the environment affected by the agent's explorative interactions might be external to its body, such as properties of material or virtual objects in a technological system, as well as properties of embodied resources, such as changes to the location of the agent's own hands relative either to each other or to features of the environment. Moreover, changes in the relative location of two or more features of the environment can be experienced as geometrical

[5] Other publications explain the critical role that the interval plays in fostering mathematical learning through the MIT-P activity, and in particular its mediating function in students' micro-actions of adopting the mathematical frames of reference (i.e., the grid and numerals; see (Abrahamson and Trninic 2015); (Abrahamson et al. 2011)) and linking among competing visualizations of the bimanual "green" enactment (Abrahamson et al. 2014).

change, that is, as change in the magnitude of invisible lines between two points or the overall shape configured by three or more points. These geometrical structures are emergent gestalts—constellations of selected features that both highlight embedded structures in dynamical perceptual displays and thus create opportunity to maintain these structures for effective interaction (Liao and Masters 2001). In the MIT-P project, one such gestalt of interest is the spatial interval subtended by two hands (or two cursors). The interval—literally empty space with nothing but air in it—becomes foregrounded as a thing. It is thus a compelling case of what Bamberger (2011) calls an entity "carved out … where none was to be seen."

As our study participants moved their hands in an attempt to keep the screen green, they were attracted to the interval between their hands as a thing that is a new and present entity that can be manipulated and monitored. Initially, the participants' repertory of operatory schemes constrained them to manipulate the interval only as a thing that should remain of constant size. Eventually, through further exploration, they noticed that this space between their hands was changing, and should change, as their hands were moving. First, they said, "It gets bigger as you go up." Next, they assumed agency in deliberately manipulating the size of this empty space correlative to the elevation of this space, because doing so appeared to enhance their control of the technological object. Still referring to the interval, they then said, "It has to be…," "It has to go…," or "I need to make it bigger as I go up." We thus submit that students become able to directly manipulate the interval effectively (making the screen green) only after they have been proficient in enacting this new scheme. Indeed, research in kinesiology demonstrates that agents will often adapt their physical interaction with the environment still before noticing this change consciously (Kelso 1984), albeit explicit awareness may then "kick in" to enhance the performance (Boutin et al. 2014; Ginsburg 2010). As such, enactive engagement is at the vanguard of explorative reasoning.

In sum, the interval between the hands, a span of negative space, becomes entified as positive space. The interval comes into being, foregrounded as an auxiliary stimulus wedged between agent and object. The interval emerges through explorative interaction as a thing—a handle, a lever, or a utensil crafted ad hoc as a means of anchoring focused engagement with the environment.

Yet why do study participants first latch onto the interval between their hands as an object that can be manipulated, even before they have articulated an effective global manipulation strategy for accomplishing their task objective? How does this attentional anchor emerge ex nihilo into the dynamics of an agent–environment task-oriented interacting system? How precisely does the interval come to serve action?

The interval emerges because doing so favorably collapses two motor-action schemes into a single scheme—from moving two hands to manipulating one thing. This simpler scheme is thus oriented on an external focusing medium, the interval. Indeed, as they manipulated the interval, our study participants never spoke about what each individual hand should do but rather about the activity, handling, and causality of the interval. From the ecological-dynamics perspective, we say that the interval served the students as an attentional anchor that promoted their performance.

The MIT-P design, first published in 2010 (Reinholz et al. 2010), has been evaluated through further research. In a controlled experiment conducted with 128 students, participants who directly or vicariously engaged in activities with the MIT-P outperformed a control group on conceptual items (Petrick and Martin 2011). Several tablet variations on this design are now available (Abrahamson 2012c; Rick 2012). Our final section will highlight eye-tracking research on the MIT-P that utilized a new version of the application.

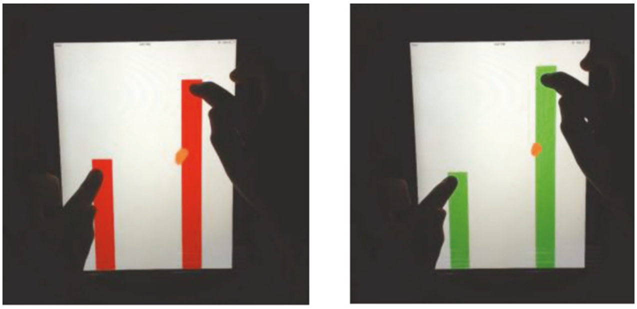

Fig. 4 Two temporally consecutive frames of gaze locations before and after finding the next green (the *orange dot* indicates the fixation of the eye gaze). The gaze lies not on any shape contours but on an unmarked location halfway up the right-hand bar, and the hands then adjust—lowering the left hand, raising the right hand—to effect the 1:2 ratio (Color figure online)

Seeing Attentional Anchors: Empirical Data from an Eye-Tracking Study

A collaboration between Abrahamson and a research team at the University of Utrecht (Bakker and van der Schaaf, PI's) has led to a research project that is attempting both to expand MIT-P design toward Cartesian concepts (per the vision outlined in (Lee et al. 2013)) and to deepen an understanding of the micro-evolution of proportional enskilment. In particular, the project seeks to combine videography of student actions and multimodal student–tutor discourse with streaming touchscreen logging and eye-tracking so as to develop a more detailed and comprehensive model for the spontaneous emergence of new perceptual-motor coordinations leading to mathematical insight. We use non-invasive eye-tracking technology to monitor the child's visual pathways on the screen, and our qualitative and computational analyses attend to patterns in the students' search for effective bimanual manipulation strategies. More broadly, educational researchers are increasingly capitalizing on methodological opportunities presented by embodied interaction to perform multimodal learning analytics on students' learning process (Schneider et al. 2015).

The REC notion of an attentional anchor was readily infused into the analysis process—it guided our hypotheses as to what we might find in the data and in particular framed our

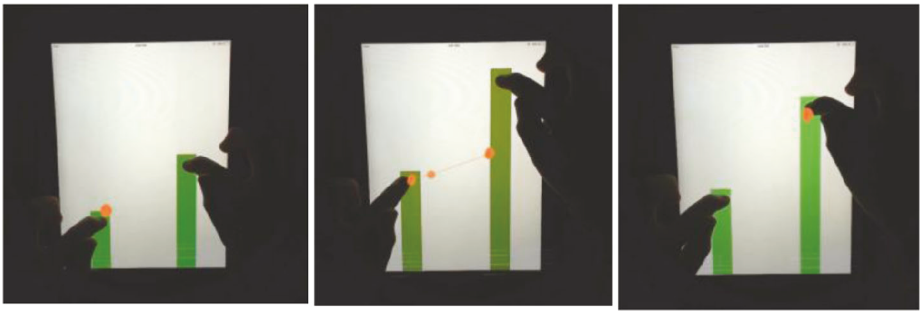

Fig. 5 Three consecutive frames of a time-sequenced triangular attentional anchor—top-left, middle right, top right—for keeping the bars green while moving the hands. The middle frame shows where the eye gaze (on the right bar) anticipates the next height of the left bar

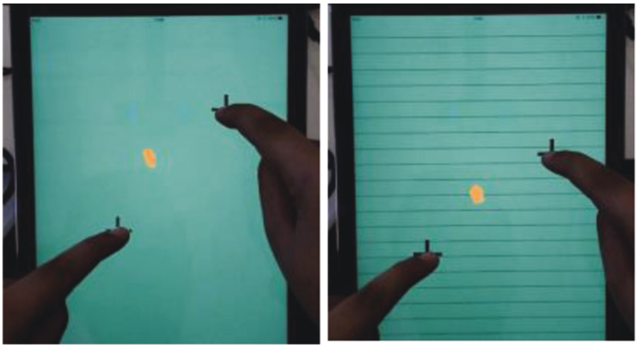

Fig. 6 Spontaneous evolution of an attentional anchor. The "hovering" eye gaze attends to a spot on the screen that contains no information in-and-of-itself but only with respect to the dynamical motor-action coordination. A grid then offers a frame of reference for bringing forth the attentional anchor into mathematical consciousness

coordination of the multiple data streams. As we now elaborate, empirical findings from a study with 80 participants (9–12 years old) support our conjecture that attentional anchors constitute the missing link between action and concept (Abrahamson and Sánchez–García 2014; Duijzer 2015; Shayan et al. 2015, submitted).

Students started the task with purely random movements of the two fingers and constant gaze shifts alternating between the two fingertips. Yet then most participants apparently formed an attentional anchor that helped them succeed in making the bars/screen green and keeping them green while moving their fingers. In particular, the eye-tracking/action-logging data reveal spontaneous discovery of new attentional foci on the tablet. These visual foci, we argue, facilitated students' enactment of a new motor-action coordination by which to keep the interface elements green while moving the fingers across the screen.

Figures 4, 5, and 6 offer snapshot examples of our empirical data from several of the various activity sequences enacted in this design. We focus on images marking the first emergence, for different participants under different conditions, of new gaze patterns accompanying new hand–eye coordination. Typically, these moments were followed by utterances marking conscious discovery (Aha!). For example, students gazed at the relevant position (e.g., halfway up the taller bar) well before they expressed that one of the bars should be half as tall as the other. We could never have detected this coordination-to-consciousness lag without eye-tracking technology.

It is thus that natural user interfaces (NUIs), such as touchscreens, are generating empirical data that can be tracked via multimodal learning analytics (MMLA) to offer new insights, inspired by REC onto enduring research problems.[6] We think—in line with REC—that it is time to step out of the head and call attention back to action as the genesis of acculturation into disciplinary skill.

[6] At the Embodied Design Research Laboratory (EDRL), we are particularly excited by the opportunities created by these new intellectual perspectives and research designs for revisiting and corroborating the construct of reflective abstraction (Piaget 1968). Piaget implicated sensorimotor coordination as a critical achievement in the development of a new conceptual schema. Attentional anchors enable us to underscore Piaget's revolutionary adoption of structuralism (Piaget 1970) as the systemic alternative to cognitivism.

Conclusion

In this paper, we have considered a proof of concept for REC that, if correct, shows REC has real empirical and practical implications. We have done so not only in the context of sports and the acquisition of sport skills in which REC already is showing explanatory promise but also in the domain of mastery of mathematics (a core feature of the STEM educational program). If our results are correct, they add some support to the view that the roots of learning about abstract mathematical concepts such as proportional equivalence do not require appealing to contentful representations but are based on content-free enactive explorations that unfold within learning contexts.

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