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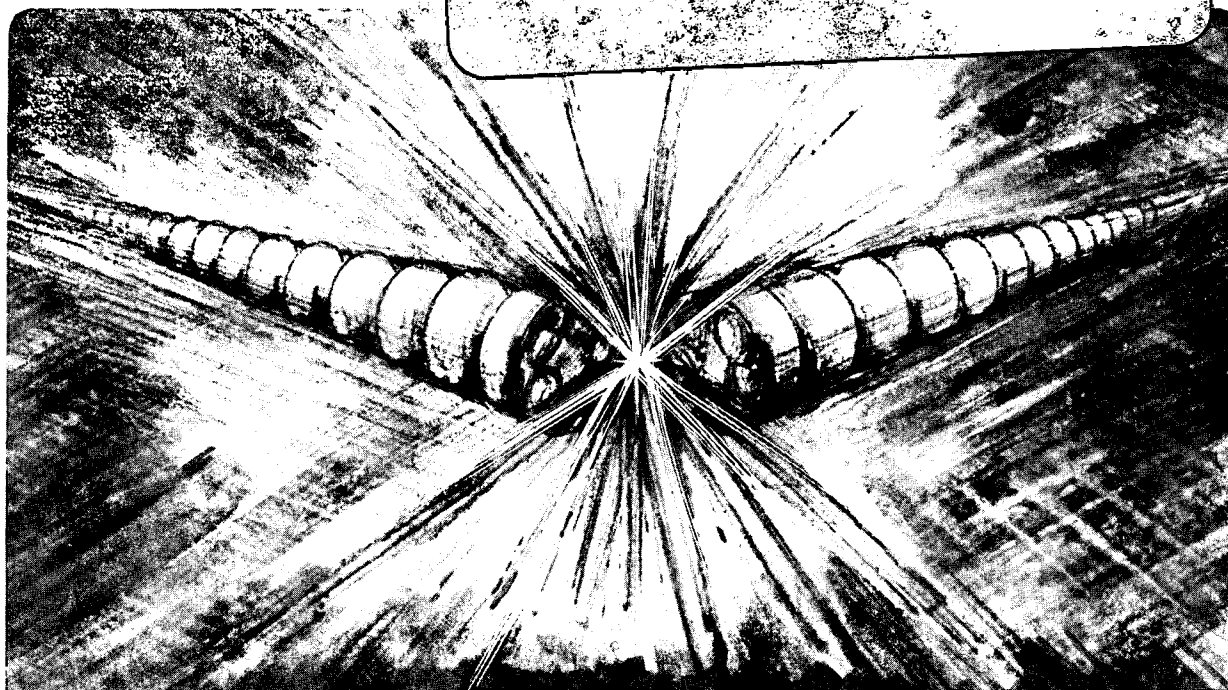
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E.P. Lee

June 1985

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MODEL EQUATIONS FOR HIGH CURRENT TRANSPORT*

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MODEL EQUATIONS FOR HIGH CURRENT TRANSPORT*

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Abstract The use of distribution functions to model transverse beam dynamics is discussed. Emphasis is placed on envelope equations, moments, the Vlasov equation, and the Kapchinski-Vladimirskij distribution.

INTRODUCTION

A beam transported by a linear focal system is commonly treated using the well-known approach of the transport matrix. In situations involving the transport of high currents the matrix approach is of limited value because the evaluation of interparticle forces depends on details of the time-dependent distribution function. Some of the analytical tools used in this situation are described here, along with their limitations in practice. The dominant emerging tool for the study of high current beams is particle-in-cell simulation. This trend results from the availability of high speed computers and simulation techniques developed in plasma physics, as well as the limitations of analytical methods described here.

LINEAR FORCES

We start the discussion by considering a single particle subject to a linear force⁽¹⁾

$$\frac{d^2x}{ds^2} = -K(s)x \quad (1)$$

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There are two independent solutions of Eq. (1), from which the 2x2 transfer matrix $M(s)$ can be formed, which generates the general solution (x, x') resulting from initial conditions (x_0, x'_0) :

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} . \quad (2)$$

This may be viewed as an incompressible mapping of phase space, with an ellipse of conserved phase area $\pi\epsilon$ (emittance) rotated and elongated by the action of M .

Even though there is as yet no mention of distribution functions or even a beam described by Eq. (1), there is an envelope formulation for the phase ellipse dynamics given by the Twiss or Courant-Snyder parameters $(\alpha, \beta, \gamma)^{(1)}$. A particle coordinate within the ellipse is given by

$$x(s) = \sqrt{\epsilon\beta(s)} \cos(\psi(s) + \theta) \quad (3)$$

with

$$\psi(s) = \int_{s_0}^s \frac{ds}{\beta(s')} . \quad (4)$$

The quantity $a = \sqrt{\epsilon\beta}$ is the maximum x contained in the ellipse, and it satisfies the envelope equation

$$\frac{d^2 a}{ds^2} = -Ka + \frac{\epsilon^2}{a^3} . \quad (5)$$

The parameter $\alpha = -\beta'/2$ is related to the rate of (spatial) contraction of the envelope, and $\gamma = (1 + \alpha^2)/\beta$ is used to obtain $x'_{\max} = \sqrt{\epsilon\gamma}$.

Several features of the transfer matrix formulation are stressed here:

1. (α, β, γ) can be advanced in time using only the transfer matrix constructed from $K(s)$. The particle distribution is not needed.
2. The formulation is essentially a single particle one.
3. Emittance is a phase space area.

Despite these limitations this approach is an adequate basis for much design work and also motivates the multiparticle, finite current Kapchinski-Vladimirskij (K-V) model.

THE K-V DISTRIBUTION

A particle distribution in (4-d) phase space satisfies the Vlasov equation:

$$0 = \frac{df}{ds} = \frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + x'' \frac{\partial f}{\partial x'} + y'' \frac{\partial f}{\partial y'} . \quad (6)$$

Therefore f can be constructed from particle constants of the motion. In general (time varying, non-linear force) no constants can be found. However for particles satisfying the linear equation (1), the Courant-Snyder invariant⁽¹⁾ applies:

$$\Gamma_x = \gamma x^2 + 2\alpha x x' + \beta x'^2 \quad (7)$$

is conserved. The analogous quantity Γ_y exists for the (y, y') space. The K-V distribution⁽²⁾ is

$$f = \frac{N}{\pi^2 \epsilon_x \epsilon_y} \delta \left(\frac{\Gamma_x}{\epsilon_x} + \frac{\Gamma_y}{\epsilon_y} - 1 \right) , \quad (8)$$

where N is the number of beam particles per unit length, and the beam's spatial profile is an upright ellipse of uniform density $n = N/\pi ab$, where a and b are the x and y radii.

The K-V distribution is valid with finite line charge density (λ) because the single particle equations are linear; we have

$$\frac{d^2 x}{ds^2} = -K(s) x + \frac{2Qx}{a(a+b)} . \quad (9)$$

where

$$Q = \frac{2e\lambda}{\beta^2 \gamma^3 M c^2} \frac{1}{4\pi \epsilon_0} \quad (10)$$

is a dimensionless measure of line charge or current called perveance (β and γ are relativistic factors), and $K(s)$ is any externally applied force (focal elements). The envelope equation for a is

$$a'' = -Ka + \frac{\epsilon^2 x^2}{3a} + \frac{2Q}{a+b} . \quad (11)$$

Significant features of the K-V distribution with space charge include:

1. All the given relations involving the Twiss parameters are valid.
2. The transfer matrix (M) depends on the envelope radii $a(s)$ and $b(s)$. Therefore the envelope equation must be solved in advance of a determination of any particle orbits, and there is a loss of general predictive power -- each set of initial conditions requires a new solution of the envelope equation.
3. The envelope equation is intrinsically non-linear when space charge is included and therefore requires numerical solution in most cases, even with simple focal elements. This is not the case in the absence of space charge despite the apparent non-linearity of Eq. (5).

MOMENT EQUATIONS

The distribution of a real beam is not K-V, but is determined by source conditions, scrape-off, etc. However the K-V distribution is the only one known for the case of discontinuous focussing, so it is frequently used as an approximate solution. Fortunately, in the absence of an analytical distribution or even detailed experimental information about a beam an "equivalent" K-V distribution can be defined provided the rms values of radii and emittance are known. The important fact is that the envelope equation is unchanged, and in the absence of finite charge effects emittance is conserved.

We return to Eq. (1) and take moments over an arbitrary (centered) profile, e.g.

$$\tilde{a}^2 \equiv \overline{x^2} = \frac{1}{N} \sum_1^N x_i^2, \quad (12)$$

$$\tilde{\epsilon}^2 \equiv \overline{x^2} \left[\overline{x'^2} - (\tilde{a}')^2 \right]. \quad (13)$$

Then the virial (x) and velocity (x') moments of Eq. (1) are

$$\frac{d^2}{ds^2} \overline{\frac{x^2}{2}} - \overline{x'^2} = -K \overline{x^2}, \quad (14)$$

$$\frac{d}{ds} \overline{x'^2} = -K \frac{d}{ds} \overline{x^2}. \quad (15)$$

Some algebraic manipulations yield

$$\frac{d^2 \tilde{a}}{ds^2} = -K \tilde{a} + \frac{\tilde{\epsilon}^2}{\tilde{a}^3}, \quad (16)$$

$$\frac{d \tilde{\epsilon}}{ds} = 0. \quad (17)$$

The equivalent K-V beam has $a = 2\tilde{a}$, $\epsilon = 4\tilde{\epsilon}$. We may also define rms Twiss parameters:

$$\tilde{\beta} = \tilde{a}^2 / \tilde{\epsilon} = \overline{x^2} / \tilde{\epsilon}, \quad (18)$$

$$\tilde{\alpha} = -\tilde{\beta}' / 2 = -\overline{x'^2} / 2\tilde{\epsilon}, \quad (19)$$

$$\tilde{\gamma} = (1 + \tilde{\alpha}^2) / \tilde{\beta} = \overline{x'^2} / \tilde{\epsilon}. \quad (20)$$

When finite charge is considered the rms formulation is a useful guide, but is no longer exact except in special cases (such as K-V). If the beam and focal system are axisymmetric, then the envelope equation is

$$\frac{d^2 \tilde{a}}{ds^2} = -K \tilde{a} + \frac{\tilde{\epsilon}^2}{\tilde{a}^3} + \frac{Q}{4\tilde{a}}. \quad (21)$$

This is exact; however the emittance ($\tilde{\epsilon}$) varies with s . Analysis of the original moment equations has recently provided some guide to the

variations of $\tilde{\epsilon}$, but there is no developed theory.^(3,4,5) In general we expect some damping of envelope oscillations resulting from an initial mismatch of the distribution to the focal system -- because of the spread of oscillation frequencies in a rounded charge distribution. This suggests that $\tilde{\epsilon}$ should oscillate 90° out of phase from \tilde{a} to produce a damping effect in Eq. (21), and a phenomenological model of $\tilde{\epsilon}$ has been based on this idea.⁽⁶⁾ In general one must resort to particle-in-cell simulations to treat this situation.

THE VLASOV EQUATION

A plasma physicist might wonder why the Vlasov equation [Eq. (6)] does not play a large role in the study of transverse dynamics (it is very useful for longitudinal dynamics). The reasons of course are the absence of analytical equilibria other than K-V and the fact that particle orbits are not localized in the transverse plane by any adiabatic invariants (unlike most plasmas). The Vlasov equation has played an important conceptual role and is the starting point for some calculations, e.g. fluid and envelope models.

The K-V distribution is also unrepresentative of physical beams because it is often unstable in a periodic focal system.⁽⁷⁾ Growth of small amplitude perturbations is predicted for the K-V distribution over a wide range of lattice tunes (phase advance per lattice period) and degree of space charge loading (tune depression). Experiment has shown that laboratory beams are more robust, with a growing envelope mode observed for a lattice tune above 90° and stability observed otherwise.⁽⁸⁾

CONTINUOUS AXISYMMETRIC SYSTEM

The Vlasov equation is of considerable use in the special case of axisymmetry with focussing field independent of s . Physically this is achieved with a long solenoid, or as an approximation to alternate gradient focussing with low lattice tune ($\ll 60^\circ$). Other physical systems which meet these conditions include a magnetically pinched beam propagating in a plasma and a spherical cluster of stars.

A single particle in such a system satisfies:

$$\frac{d^2 \underline{r}}{ds^2} = -k^2 \underline{r} - \nabla \phi \quad (22)$$

where k is constant and ϕ is determined from the Poisson's equation. In equilibrium $\phi = \phi(r)$ only, and

$$E = \frac{|\underline{r}'|^2}{2} + \frac{k^2 r^2}{2} + \phi \quad (23)$$

is a constant of the motion. Isotropic distributions are constructed from E :

$$f(\underline{r}, \underline{r}') = F(E) . \quad (24)$$

The conserved angular momentum $L = r^2 \theta'$ can be used to construct non-isotropic distributions if desired. An assumed form for $F(E)$ leads to a detailed calculation of beam properties. For example the waterbag distribution

$$F(E) \propto H(E_0 - E) , \quad (25)$$

where H is the Heaviside function, yields a rounded profile for number density⁽⁹⁾

$$n(r) \propto \left[1 - \frac{I_0(\alpha r)}{I_0(\alpha a)} \right] . \quad (26)$$

Here a is edge radius and α is a constant depending on the degree of tune depression.

An important result for distributions of the form (repulsive self forces) (24) is the sufficient condition of stability⁽¹⁰⁾

$$F'(E) \leq 0 . \quad (27)$$

The K-V distribution (known to be unstable) fails this condition while the waterbag is seen to be marginally stable.

A non-K-V distribution satisfying Eq. (27), but injected with a slightly mismatched radius is expected to undergo stable oscillations which eventually damp as the beam approaches an equilibrium state. This simple property is predicted on the basis of general considerations, but can only be examined by simulation.

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