Title
Dualities and Curved Space Partition Functions of Supersymmetric Theories

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Publication Date
2015

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Dualities and Curved Space Partition Functions of Supersymmetric Theories

A dissertation submitted in partial satisfaction of the
requirements for the degree of Doctor of Philosophy

in

Physics

by

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Committee in charge:

Professor Kenneth A. Intriligator, Chair
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Professor Frank Wuerthwein

2015

The Dissertation of Prarit Agarwal is approved and is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2015

DEDICATION

In recognition of all the sacrifices made by parents and my brother in order for me to be able to focus on my research without being burdened by the thought of my responsibilities towards them.

EPIGRAPH

O my dear Susy, where art thou?
Thy game of hide and seek has gone on for too long now.
I have searched for thee,
and have scoured the landscape all around,
but it continues to baffle me,
that thou art nowhere to be found.
Without thee by my side,
the cure for this affliction continues to hide.
This malady that besets me - a humble scalar,
has sent my world into a state of squalor.
The top and the bottom, the up and down,
run in loops making my head go round and round.
They have charm but they are strange,
they give me chills every time they exchange.
Unbeknownst to thee, their forces are strong,
they have vexed me for very long.
O ye Susy, I beseech thee,
reveal thyself lest this ailment will impeach me.

- *Higgs Boson*

(Prarit Agarwal)

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ACKNOWLEDGEMENTS

About a little more than a decade ago, I decided to embark on my journey towards becoming a physicist. I was a starry eyed kid who knew very little about physics but was nonetheless fascinated by the adventures that lay ahead. And today when I look back, I realize that if I have come this far, it is only because I was immensely lucky to have had friends and family who stood there by my side without fail whenever and in whatever manner I needed them. I would therefore like to take this moment to thank them for their unwavering support and confidence in my pursuits.

I'd like to start by thanking my advisor Ken Intriligator who was kind enough to take me under his wing when I came to UCSD. Over the past six years or so, he has overseen my metamorphosis from a naive graduate student into a physicist. His advice and direction has been crucial to my research and whatever little field theory I know is due to what I learnt from him either in the form of direct lectures or through highly illuminating discussions I had with him. His comments and direct input to my research has enriched my appreciation for physics in general and supersymmetry in particular. And the fact that he has always done this with so much humility is all the more inspiring.

Next I'd like to thank Antonio Amariti. He has been a collaborator, a friend and a mentor to me and I can't even begin to imagine what my stint at UCSD would have been like without him around. On countless occasions I have barged into his office puzzled about inane details regarding papers that I was trying to understand and every time he would drop all his work to help me out. Outside of work he taught me through his jovial attitude that there is so much more to life than just physics. His goofiness would make me forget all my worries and through this he taught me to live in the moment and enjoy the journey.

I was very fortunate to have had Jaewon Song as my collaborator and friend during the last few years of my time here at UCSD. Through my association with him, I was able to explore new vistas in physics. Discussion with him always led to a better and clearer understanding of various ideas. His encouraging and friendly advice would always buoy my hopes and fill me with a renewed energy to delve into the depths of my work.

I also wish to thank my committee members Adriano Garsia, Ben Grinstein, Ken Intriligator, Justin Roberts, Frank Wuerthwein for putting their time and effort into my defense. My advancement committee also included Mark Gross as one of the members and I thank him for that.

I have also benefited from my discussions with Ben Grinstein, Aneesh Manohar, John McGreevy and Julius Kutti. I am sure I will miss all the Friday beers and Thanksgiving dinners I have had due to the courtesy of Ben. I will specially like to thank Ben for continuing to lecture us on field theory well past the completion of 3rd quarter in 2011. This is one of the many occasions when I was able to witness Ben's dedication towards teaching. Through this he has set the example of an ideal teacher for me, something I hope I am able to emulate as and when the opportunity presents itself.

My stay at UCSD could not have been as comfortable and hassle free if not for all the help and love I got from Michael Anderson, Hilari Ford, Catherine McConney, Antoinette Moore and Sharmila Poddar.

I thank my teachers in school and college for making me who I am. Mehta sir, Ashok sir, Dahiya sir and Bro. Dominic Jacob have had a profound influence on me. Even today, when in doubt, I am reminded of their advice and teachings. At IIT Bombay, I had the opportunity to interact with and learn from Prof.P. Ramadevi, Prof.Tapobrata Sarkar, Prof.P. P. Singh and Prof.Urjit Yajnik. Prof.Ramadevi was extremely kind to me during my time there. Every time I needed help I would just go to her without any fear of judgment or ridicule. For this I cannot thank her enough. I would also like to take this moment to thank Nitin Pawar for teaching us the electronics labs at college. Without him those labs could not have been so informative and yet so much fun.

Special thanks to my collaborators Antonio Amariti, Ibrahima Bah, Ken Intriligator, Alberto Mariotti, Kazunobu Maruyoshi, P. Ramadevi, Tapobrata Sarkar, Massimo Siani and Jaewon Song for patiently putting up with me during our collaborations. I would also like to thank the graduate students and post-docs at UCSD for bearing with me while I asked stupid questions and bounced crazy ideas off of them: Rodrigo Alonso, Jorge Camalich, Diptarka Das, Jean-Francois Fortin, Selim Gomez-Avila, Chris Murphy, Emily Nardoni, David Pirskhalava, Andreas Stergiou, David Stone, Ze'ev Surujon, Patipan Uttayarat and Wouter Waalewijn.

Prerit Jaiswal and Kirit Makwana deserve a special mention for helping me learn physics and collaborating with me on various projects during college. I continue to feel enriched through my physics and non-physics discussions with you guys.

Jayadev Acharya, Michael Erickstad, David Olson, Son Pham and Saurabh Prasad made me feel welcomed when I first came to UCSD. I have always treasured their friendship and will continue to do so. I'd like to offer a special thanks to David Olson's family for having me over for my first christmas celebrations in the US. I would also like to thank Nitin Tiwari and Mini Tiwari for treating me like their family.

My time in the US was made particularly memorable due to the frequent phone calls I have had with Paridhi Athe, Maigha Garg, Nikhil Ghare, Harshad Ghodke, Divya Gupta, Prerit Jaiswal, Madhura Joglekar, Omesh Johar, Kirit Makwana, Kajal Miyan and Vaishnavi Rajgopal. You guys were always a phone call away and became my family in a foreign land. I would specially like to thank Harshad Ghodke, Prerit Jaiswal and Kirit Makwana for all the trips that we have taken together.

My interaction with my non-physics friends at UCSD has brought me immense joy. My housemates at Costa Verde: Sudeep Kamath, Ranjeet Kumar, Sivasankar Radhakrishnan and Ashish Tawari and my current housemates Bharatan Balaji, Manish Gupta and Rishi Kapoor have had to accomadate my various idiosyncarcies and they did this without complain. I have also had many joyous evening that I will cherish forever due to the company of many friends including but not limited to Abhay Bhopat, Ashish Cherukuri, Mansa Lanka Cherukuri, Neeraja Dashaputre, Viraj Deshpande, Arpit Gupta, Siddharth Joshi, Siddhesh Kamat, Siddhartha Nath, Amol Patwardhan, Shweta Purawat, Abhay Ramchandra, Alok Singh, Jenny Stergiou, Priyal Thakur.

My days at IIT Bombay were blessed with such friends as Jitendra Dwivedi, Sunil Gupta, Abhishek Kumar, Shiv Nandan Patel and Sanved Raut. I have formed treasurable memories with you. Your love and support during both happy and my most troubled times has made you irreplaceable in my life.

I spent a summer at IIT Kanpur where I met Ajay Singh. He and his wife Swati Ajay Singh continue to be a treasured friends.

At school, I formed unbreakable bonds of friendship with Anant Bhatia, Maigha Garg, Aman Goel, Mranal Gupta, Vanya Jaitly, Omesh Johar, Gunjan Juyal, Kajal Miyan, Saumya Oli, Ashraf Rahman, Ankur Sexana and Subharth Saha. I'd like to specially thank Saumya Oli for her unwavering confidence and support when we were preparing for JEE. Thanks also to Maigha Garg, Kajal Miyan and Omesh Johar for all the support that you gave me in some particularly troubled times.

The winters of 2008 are particularly memorable for the time spent with Malini Raju, Pragna Sarkar and Ankur Sexana. The evenings that we spent by the lake side are one of the best times I have ever had in my life.

I would also like to thank Teenu Didi, Sarika Agarwal and Mukesh Prasad for all the love that they have showered upon me.

In the end I wish to thank the most important co-passengers of my journey so far: my parents and my brother. My dad, Bhagwan Swaroop Agarwal toiled hard to give us a happy life. He was my first role model. Every moment of my life I have aspired to live through the hardships of life with the same calm and humble demeanor that he has. He taught me how to work hard, stay focused and never to lose sight of my goals. My mother Sunita Agarwal has been no less of an inspiration. With her around, I have always felt safe. Even to this day it matters more when it is her who tells me to be patient and keep plowing through the times when life makes it a little more difficult to bear. My brother Ankit Agarwal was my first and best friend in life. He is one person I have always depended upon. Since childhood I have confided in him my deepest fears and he has never failed to be there by my side whenever I needed him. None of this would have been possible had he not always selflessly taken over what should have been my responsibilities towards the family. First during my preparations for JEE and then while I completed my PhD. here in the US, he would never fail to tell me "bhai, you focus on your studies. I am here to provide whatever help mom and dad need."

Finally I'd like to thank the citizens of both United States and India whose monetary contributions towards the development of science made all this possible.

Technical Acknowledgments: Chapter 2 and Appendix A is a reprint of the material as it appears in "BPS states and their reductions ", Prarit Agarwal, Antonio Amariti, Alberto Mariotti, Massimo Siani, JHEP 1308 (2013) 011. Chapter 3 and Appendix B is a reprint of the material as it appears in "A Zig-Zag Index ", Prarit Agarwal, Antonio Amariti, Alberto Mariotti, arXiv:1304.6733. Chapter 4 and Appendix C is a reprint of the material as it appears in "Refined Checks and Exact Dualities in Three Dimensions " , Prarit Agarwal, Antonio Amariti, Massimo Siani , JHEP 1210 (2012) 178. Chapter 5 and Appendix D is a reprint of the material as it appears in "New $N = 1$ Dualities from $M5$-branes and Outer-automorphism Twists ", Prarit Agarwal, Jaewon Song, JHEP 1403 (2014) 133. Chapter 6 and Appendix E is a reprint of the material as it appears in "Quiver tails and $N = 1$ SCFTs from $M5$-branes ", Prarit Agarwal, Ibrahima Bah, Kazunobu Maruyoshi, Jaewon Song, JHEP 1503 (2015) 049. Chapter 7 is a reprint of the material as it appears in "Infinitely many $N = 1$ dualities from $m + 1 - m = 1$",

Prarit Agarwal, Kenneth Intriligator, Jaewon Song, arXiv:1505.00255.

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7. Prarit Agarwal, Kenneth Intriligator , Jaewon Song, "Infinitely many $N = 1$ dualities from $m + 1 - m = 1$", arXiv:1505.00255

ABSTRACT OF THE DISSERTATION

Dualities and Curved Space Partition Functions of Supersymmetric Theories

by

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Doctor of Philosophy in Physics

University of California, San Diego, 2015

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In this dissertation we discuss some conjectured dualities in supersymmetric field theories and provide non-trivial checks for these conjectures. A quick review of supersymmetry and related topics is provided in chapter 1. In chapter 2, we develop a method to identify the so called BPS states in the Hilbert space of a supersymmetric field theory (that preserves at least two real supercharges) on a generic curved space. As an application we obtain the superconformal index (SCI) of 4d theories. The large N SCI of quiver gauge theories has been previously noticed to factorize over the set of extremal BPS mesonic operators. In chapter 3, we reformulate this factorization in

terms of the zigzag paths in the dimer model associated to the quiver and extend the factorization theorem of the index to include theories obtained from D-branes probing orbifold singularities.

In chapter 4, we consider the dualities in two classes of 3 dimensional theories. The first class consist of dualities of certain necklace type Chern-Simons (CS) quiver gauge theories. A non trivial check of these dualities is provided by matching their squashed sphere partition functions. The second class consists of theories whose duals are described by a collection of free fields. In such cases, due to mixing between the superconformal R-symmetry and accidental symmetries, the matching of electric and magnetic partition functions is not straightforward. We provide a prescription to rectify this mismatch.

In chapter 5, we consider some the $\mathcal{N} = 1$ $4d$ theories with orthogonal and symplectic gauge groups, arising from $\mathcal{N} = 1$ preserving reduction of $6d$ theories on a Riemann surface. This construction allows us to dual descriptions of $4d$ theories. Some of the dual frames have no known Lagrangian description. We check the dualities by computing the anomaly coefficients and the superconformal indices. We also give a prescription to write the index of the theory obtained by reduction of $6d$ theories on a three punctured sphere with Z_2 and Z_3 twist lines and verify that it exhibits the conjectured symmetry enhancement from $G_2 \times USp(6)$ to E_7. In chapter 6, we continue our study of $4d$ theories obtained from reduction of $6d$ theories. We introduce a new type of object that we call the 'Fan' and show how to construct new $\mathcal{N} = 1$ superconformal theories using the Fan. In chapter 7, we demonstrate the existence of an infinite number of theories that are either dual to or exhibit a cascade of RG flows down to the $SU(N)$ SQCD with four flavors and a quartic superpotential.

Chapter 1

Introduction

In this chapter, I attempt to provide context for the work contained in the following chapters and background material for the lay-reader. Those who already know basic particle physics and supersymmetry should skip to section 1.3. Specialists should skip this introduction altogether.

1.1 A Flavor of Particle Physics for Non-Physicists

Towards the end of the 19*th* century it was realized that the chemical properties of all matter can be easily described if all kinds of matter is assumed to be built from different possible combinations of a little more than 100 types of atoms. This can be viewed as our first concrete step towards a possible unified framework to understand the laws governing nature. However, the fact that there are so many different atoms with different chemical properties immediately begged the question "what causes different types of atoms to have different chemical properties?" Further investigations, revealed that each atoms can itself be viewed as electrons revolving around a central nucleus much like planets revolve around the sun. The only difference being that planets are held in their orbits due to the gravitational pull of the sun while the electrons are held in their orbits due to the electrostatic attraction that they feel from the nucleus [1]. The different chemical properties of the atoms are due to the difference in the number of electrons that are present in different atoms. This also implied that there should be about 100 different nuclei, one for each type of atoms. Further inquires into the nature of these nuclei revealed that the difference between the various possible nuclei can be easily explained if they are thought of as a collection of just two different kinds of particles - protons and neutrons. The various types of nuclei were different from each other because of the difference in the number of protons and neutrons that were used to make up the nuclei.

At this point the following picture of nature seems to emerge: All objects around us are constituted from electrons, protons and neutrons. The electrons and the protons exert electrostatic forces on each other. At the same time there is the gravitational force which barely affects the interaction between the electron and the proton but becomes one of the most dominant forces when considering the interaction between the heavenly bodies. It will also serve well to remind ourselves that the electrostatic forces are a little different from gravity. This can be seen from the fact that while the gravitational attraction between any two objects is always attractive, the electrostatic forces are such that they cause electrons and protons to attract each other but two electrons in the vicinity of each other will feel a mutual repulsive force. The latter is also true for the force felt by two protons in the vicinity of each other! An astute reader will immediately realize that the fact that protons repel each other should make it difficult to hold two or more protons in close proximity of each other. How is it then that they stay bunched up inside the nucleus? Why doesn't the nucleus explode into its constituents? This seems to suggest that along with gravitation and electrostatics there should be another force that

[1] The electrostatic force is the same force that causes electricity to flow around in electric appliances and at times our hair to stand up when we go down a slide in the park. It is also the reason why at times we get shocked when we touch the doorknob.

helps bind the protons and neutrons into a nucleus. The electrostatic force between two particles grows rapidly as the particles are brought close to each other. Inside a nucleus the protons are extremely close to each other and therefore their mutual electrostatic repulsion must be enormous. It therefore follows that whatever new force it is that keeps the protons bound inside the nucleus, it must be extremely strong in order to overcome the electrostatic repulsion. This force was therefore called the strong force.

Around this same time another curious phenomenon was discovered. It was found that the neutron could itself decay into a proton, an electron and a new kind of particle called the neutrino. This discovery paved the way to the uncovering of a fourth kind of force, much weaker than the electromagnetic [2] or the strong force. This was dubbed the weak force.

Experiments performed during the middle to late $20th$ century have revealed that the protons and neutrons are not elementary either but are made up of smaller building blocks called the quarks which are held together into the proton/neutron by the strong force. As of now we know that there are 6 different varieties of quarks (depending upon how you count) and 3 different kinds of neutrinos. There is also the electron and 2 other particles that are close cousins of the electron. These are what we will call the elementary building blocks of nature. The different combinatorial ways of putting these particles together then gives rise to almost everything [3] that we see around us. There are also the four forces - the strong, the weak, the electromagnetic and gravity- that mediate interactions between these particles. The strong and the weak force do not operate at distances much larger than the size of an atomic nucleus and therefore we do not experience them directly in our everyday lives in the same manner as we experience the electromagnetic and gravitational phenomenon. A few years ago the particle accelerator

[2] Around mid $19th$ it was realized that the electric and the magnetic forces are related to each other and are therefore grouped into a single force called the electromagnetic force.

[3] I say "almost everything" because there is still the issue of dark matter.

in Europe revealed the existence of a new particle called the Higgs boson.

This is where our experimental understanding currently stands. These elementary particles and the forces that they exert on each other are the gears that go into the beautiful clockwork that nature is. But, we will also need a theoretical framework to help us explain how exactly are these gears supposed to be put together. This is where "quantum field theory" (QFT) comes in. Quantum field theory is a mathematical framework which allows us to incorporate the rules of special relativity and quantum mechanics when explaining what any given elementary particle will do in a given situation. According to QFT we should think of an elementary particle as a fluctuation in a field (much like ripples in water). Each particle then has a field associated with it. In this framework, even the forces are the outcome of fields called the gauge bosons. According to QFT, the mechanism through which the quarks etc. exert forces on each other is an act of exchanging gauge bosons. QFT thus lays down the basic mathematical machinery that we can use to predict the behavior of these particles. However, much like any framework, QFT should not be expected to predict the number of distinct elementary particles that nature can have. That is an experimental input we feed into QFT to come up with a theoretical model that properly describes nature. The model that does this for us is popularly called "The Standard Model" (SM) in the physics community.

The story so far, gives us the impression that in order to be able to use SM and answer various questions, physicists might have to trudge through some hard to solve mathematical equations. So why are physicists even able to solve these equations? The answer lies in the fact that most of the times when physicists are able to solve these equations, it is because they have certain symmetry principles guiding them. Consider the following problem for example: let us put some amount of positive charge on the surface of a spherical ball. Let the charges be free to move around on this surface. Now we know that "like charges" repel each other while "unlike charges" attract. Given that all the charges involved in this situation are positively charged, they will therefore repel each other and try to move away from each other. In fact, they will continue moving around on the surface of the ball and redistribute themselves until they no longer feel any net force from their neighbors (i.e. the neighbors can be paired up in manner such that repulsion from one neighbor counteracts the repulsion from the other). We now ask what is the configuration of charges that achieves this? We can try to answer this question using some involved math formulas but the truth is that it is very simple to guess what the answer will be. Notice that the surface of the ball is very symmetric - no one point on the ball's surface is any different from any other point. This implies that there is no preferred point on the surface of the ball where the density of charge should be particularly large or small. It therefore follows that the charges will be evenly distributed on the surface of the ball. This is in-fact the correct answer. Notice that we did not have to solve any complicated equations in order to arrive at this conclusion. The argument that led us to our conclusion was the fact that all the points on the surface of the ball are similar to each other. Another way of saying this is - imagine you look away for a little bit. While you were looking away I am free to rotate. When you look at the ball again, will you be able to tell if I rotated the ball or not? The ball is therefore said to be invariant under rotations. In fact our whole universe is largely invariant under rotations. Another simple example of symmetry is the translation symmetry. This symmetry reflects the invariance of natural laws under the shift of our position from one place to another. We don't expect our electrical appliances to start behaving in a weird manner if we move our residence to a different city. Similarly we do not expect our experiments to give us different results based on which place on earth the experiment is performed. The example of charges on the surface of a ball was just one of the many examples where symmetries helped us solve a problem. But we can go further than this. Not only can we solve our mathematical equations using symmetries of the problem, but we can use them to uncover the physical laws themselves. By that I mean that we can use symmetries to guess the math that describes the laws of nature. Let me explain - since we know that universe is invariant under certain symmetries (e.g. the rotational and translational symmetries), it therefore must be that the natural laws and the equations describing these laws must also be invariant under these symmetries. This is a very strong constraint which usually leaves very few possibilities to choose from and often sheds light on how different phenomenon might be related to each other.

The rotational and translational symmetries that we talked about a little bit ago are examples of what are called space-time symmetries. The idea being that they arise due to operations on position and orientation of various objects. But it turns out that our natural laws obey certain internal symmetries too. These internal symmetries result from permutation of the elementary particles. Very loosely speaking some of the elementary particles behave very similar to each other[4]. Thus if we know that two given particles are related by an internal symmetry, then we can guess the behavior of the second particle by using our knowledge about the behavior of the first particle. Again, we can use this symmetry the other way around i.e. if through some experiment or by means of some clever deductions we are able to conclude that nature should exhibit a certain internal symmetry, then it follows that there must exist a very precise set of elementary particles whose permutations realize this symmetry. In this scenario it may happen that all but one of the particles belonging to this set have been already discovered experimentally, we then know that the missing particle must also exist in nature. It's just that we have not yet discovered it. We thus have a prediction! This is precisely what happened with the discovery of the so called "charm quark" and the "top quark". The existence of these quarks was predicted by theoretical physicists much before the experimentalists found

[4]Recall that we mentioned that there along with the electron, there are two other elementary particles that are close cousins of the electrons. This is exactly what we meant when we claimed that they are "cousins". A similar statement can also be made about the three neutrinos.

them. For those who know the story of Mendeleev's periodic table, this way of predicting new, hitherto undiscovered elementary particles is not too different from the manner in which Mendeleev used his periodic table to predict the existence of missing elements.

There is an interesting classification of particles that I have not mentioned yet. According to this classification all particles can be grouped into two mutually exclusive sets. The first of these sets is called fermions and the second is bosons. Without elaborating upon what properties should a particle have in order for it be classified as a boson or a fermion, I'll just point out that the electrons, the quarks and the neutrinos are fermions while the gauge fields and the Higgs boson are bosons. Given this classification, there is an interesting observation that physicists have made. It turns out that the equations that describe fermions can be mapped to the equations that describe bosons through some not so difficult mathematical gymnastics. This might look like a mere coincidence but if we take it more seriously then we arrive at the conclusion that there is some symmetry between the behavior of fermions and bosons. We will call this symmetry "supersymmetry" (SUSY).

Whether nature is supersymmetric or not is still a mystery. One of the requirements of supersymmetry is that all particles in nature should occur in pairs of a boson and a fermion. A straightforward counting of the particles that we have found so far, shows that this is not the case. It seems like nature, as we currently understand it, is not supersymmetric. However, we can postulate that nature is supersymmetric and predict that the missing particles exist but remain to be discovered. This is certainly an interesting idea and many physicists believe that this is the case. But even if this were not true, it is not futile to consider what can we learn about nature if it were to be supersymmetric. One of the reasons physicists like to study supersymmetric QFTs (SUSY QFTs) is that we don't have a complete understanding of how the various quantum fields behave. We know that small fluctuations in these fields manifest themselves as the particles that we have already talked about but this is not the end of the story. This can be considered analogous to the fact that ripples are not the only thing that happens in water. At times we can have vortices and tsunamis too. These vortices and tsunamis are an example of a more general phenomenon that physicists call “solitons. Other common examples of solitons include hurricanes and twistors. The great red spot on the surface of Jupiter is another example of a soliton. It turns out that just like water, even quantum fields can have solitons. These are complex objects whose behavior is not so straightforward to analyze. Nonetheless, we know that they exist and their presence has important implications for physics. Solitons are but one of the many possible things that can happen in QFT. In general QFT exhibits a very rich phenomena, not all of which has yet been tamed through mathematics. It turns out that imposing supersymmetry on QFT puts quite stringent constraints on how the various fields should behave and therefore makes QFT more amenable to mathematical treatment. We can therefore first analyze what happens in a supersymmetric QFT and then ask ourselves which phenomena can still be present in a QFT without supersymmetry. Studying supersymmetric QFTs can therefore be very rewarding in terms of giving us general insights into quantum field theory itself.

One such phenomena that SUSY QFTs have helped us understand better is that of electromagnetic duality. To understand what we mean by electromagnetic duality we'll have to remind ourselves what a magnetic monopole is. Though this was not mentioned in our discussion so far but we hope that the reader is well acquainted with the fact that any magnet has two poles which are customarily called the north pole and the south pole of the magnet [5]. Much like the case of electric charges, in this case like poles repel and unlike poles attract i.e. if we try to bring two magnets close to each other with their

[5] This nomenclature arises from the fact that a freely suspended magnet will always orient itself such that the north pole of the magnet points in the direction of the north pole of the earth.

north poles facing each other , the magnets will repel each other. The same story repeats if we bring them close to each other with their south poles facing each other. However, if we bring the magnets close to each other such that the south pole of one is facing the north pole of the other, then the magnets attract each other. Now in practice any attempts to break a magnet in order to separate its north and the south pole will fail. We only end up getting two smaller magnets each with its own pair of north and the south pole. However, we shall not let this impede ourselves from imagining that the north and the south poles of a magnet can be separated from each other. We then get a pair of magnetic monopoles one of which is the north pole while the other is the south pole. It turns out that the manner in which magnetic monopoles give rise to magnetic fields is similar to the manner in which an electric charge gives rise to electrostatic fields. It is also probably not unknown to the reader that if we pass current through a wire, there is a magnetic field that gets created around the wire [6]. Since current is nothing but an electric charge moving through the wire, we find that moving electric charges produce magnetic fields. Similarly the turbines in the hydroelectric damns produce electricity by using the power of water to move magnets around an axis. When this is done we get a current flowing around in the coils of wire placed around the magnet. We thus see that moving magnets result in electrostatic fields which then push charges around the coils of wire in the turbine. Notice that there seems to be some similarity in the behavior of electric and magnetic charges - they both obey the colloquial rule of “likes repel and unlikes attract”; They both produce their respective fields in similar fashion; Moving charges/monopoles produce magnetic/electric fields respectively. This symmetry between electric and magnetic phenomena is called electromagnetic duality. This is a simplified version of a more complex phenomena that is conjectured to occur in QFTs. It

[6]Try placing a compass close to a wire connected to the two ends of a battery. You’ll find that the magnetic needle no longer points towards the earth’s north pole.

is believed that a generalized version of electromagnetic duality in QFTs should involve exchanging elementary particles with solitons. As has been mentioned before, solitons are complex objects that are difficult to treat mathematically. It is therefore not an easy task to test our conjecture. However, through studying SUSY QFTs, physicists have come to the conclusion that such a phenomenon does occur.

I'd like to conclude this section with the hope that I have been able to convey a flavor of what particle physics is about and that the reader is convinced of the important role that symmetries play in helping us learn about nature. In what follows, I will elaborate upon the various ideas that were discussed here and try to fill in many gaps that were left behind in order not to bog down a non-expert reader with technical details.

1.2 Supersymmetry and What Is It Good For?

In the previous section we mentioned that there are two kinds of symmetries that are realized in nature. The first of these are the space time symmetries. These result from invariance of the physical laws under rotations and translations in space-time. Together they form the Poincaré group. The second kind of symmetries are the internal symmetries. These act in non-trivial ways on local fields in a QFT but do not have any effect on their position and orientation in space-time . Due to Emmy Noether, we now know that the presence of these symmetries give rise to conserved quantities which are constants of motion. For example, invariance of physical laws under translations in space imply that momentum is a constant of motion. Invariance under rotations of space-time leads to conservation of angular momentum and spin. Similarly invariance of physics under local variations of the phase of complex fields in a QFT has electric charge as its constant of motion.

It is an interesting exercise to ask ourselves if it is possible to have QFTs which exhbit a non-trivial combination of the symmetries of the Poincaré group and the internal symmetries. Coleman and Madula investigated this question in [62]. They concluded that this cannot be the case and the symmetry group of all QFTs must be nothing more than the direct product of the group of internal symmetries and the Poincaré group. However, in order to come to this conclusion they had made some very mild and general assumptions. There are two ways to relax their assumptions. The first of these loopholes leads to QFTs with scale and conformal invariance while the other loophole gives us supersymmetry (SUSY).

In general, the conserved charges associated with internal symmetries do not transform under the elements of Poincaré group. This is a direct consequence of the fact that internal symmetries and Poincaré symmetries do not usually mix. Since SUSY is an exception to this rule, it therefore should not be a surprise that the conserved charges associated with SUSY have non-trivial spinorial properties. It then follows that the action of supersymmetry on a field, also transmutes its spin quantum number. More precisely SUSY establishes a symmetry between fields whose spin quantum numbers differ by 1/2. Now, the famous spin-statistics theorem of Pauli tells that in $4d$ fields with half-integral spin obey Fermi statistics and are called fermions while those with integers as their spin quantum number obey Bose statistics and are called bosons. We therefore see that by changing the spin of the field that it acts on, supersymmetry relates a boson to a fermion and vice-versa.

Supersymmetric theories can be interesting from both phenomenological and formal points of view. Phenomenologically, one of the most promising aspects of SUSY is that it provides a very elegant solution to the naturalness problem to the Higgs mass. Let us start by a quick review of this problem. The Higgs boson which was recently discovered at CERN has a mass of about 126 GeV. Now the thing is, we expect the mass of the Higgs boson to get quantum mechanical corrections due to its interaction with the various fermionic and bosonic fields in the Standard Model (SM). These correction will in general push the mass of the Higgs to be infinite unless there are significant cancellations between the corrections from the fermionic and bosonic sectors of SM. This cancellation implies that the interaction of the Higgs boson with the various fields should be very finely tuned to match each other. Such a fine-tuning being non-generic appears to be unnatural. However if nature is also supersymmetric, then for every bosonic field there is a fermionic partner and vice-versa. Invariance under SUSY then gives a natural answer to the fine-tuning problem mentioned above. Supersymmetric models also usually contain a natural candidate for the dark matter.

From a formal point of view , let us start by pointing out that our understanding of QFTs in general is quite limited. Even after many towering successes such as quantum electrodynamics, the standard model of electroweak theory, BCS theory of superconductivity etc., there still are a lot of open questions. Thus, we still lack a complete understanding of how quarks stay confined in hadrons such that it is impossible to isolate a bare quark in particle accelerators. Similarly the behavior of non-Fermi liquid metals and quantum critical points are open problems in the condensed matter community.

Most of these problems stem from the fact that we can apply perturbative techniques to a given QFT when its coupling constants are small but as the quantum mechanical corrections to these couplings become large we are rapidly driven into the non-perturbative regime where perturbative methods are no longer applicable. Probing the physics of this regime requires inventing new mathematical tools. Clearly, having some degree of intuition about the non-perturbative phenomena itself, will go a long way in helping us develop the mathematical machinery we seek. One way to make progress in this direction comes from studying SUSY QFTs. The idea is to probe the non-perturbative regimes of SUSY QFTs and then ask ourselves: which phenomena are independent of the constraints imposed by supersymmetry? The reason we expect this strategy to work is because, in supersymmetric set-ups, the Bose-Fermi symmetry adds more structure to the allowed interactions between the elementary constituents of the system, thereby making its physics more amenable to mathematical treatment.

1.3 The Superconformal Index

As an example of the afore mentioned mathematical tractability, we'd like to point out that recent development of localization techniques in SUSY theories has made it possible to analytically compute certain SUSY preserving quantities. These have helped us put various conjectures about QFTs on a firmer ground and at the same time have lead to a wealth of new insights which have enabled us to probe non-perturbative quantum effects in physics. In principle localization can be used to calculate the partition function of any given supersymmetric theory on a generic background manifold that preserves at least one complex supercharge. Partition functions on different manifolds capture different information about the flat space theory. For example, the Witten index [191], which is the partition function of 4d SUSY theories on $\mathrm{T}^3 \times \mathrm{R}$, counts the number of supersymmetric vacua of the corresponding flat space theory. Similarly for a superconformal theory, we can calculate its partition function on $\mathrm{S}^3 \times \mathrm{S}^1$. This is dubbed the superconformal index (SCI) [142, 168]. It is a topological quantity, independent of the coupling constants and counts the number of protected BPS operators of the theory. In [175], it was shown that the supersymmetric index on $\mathrm{S}^3 \times \mathrm{R}$ can be obtained directly from the knowledge of the spectrum of the Laplace operator: one needs to find the eigen-modes of the Laplace and Dirac operator on the sphere and sum over all the modes. Many of the bosonic modes will cancel out against the fermionic ones, and one finds that only the BPS modes (i.e. the modes that correspond to the kernel of the boson-fermion map) contribute to the index. This is the most direct method to compute the partition function, but it is in practice very difficult to work out on generic supersymmetry preserving manifolds. Another method is to compute the letter index [142] for theories defined on a conformally flat background.

In these cases, it is straightforward to obtain the quantum numbers of the curved space fields because conformal mapping relates them to their flat space counterpart. We can then identify the operators that saturate the BPS inequality. Nevertheless, it is not simple to extend this method to backgrounds that are not conformally flat.

In chapter 2 based on my paper [2], we develop a method to achieve this aim. The essential logic relies on the fact that only the modes that correspond to the kernel of the boson-fermion map contribute to the index and therefore it is enough to identify just these modes. We showed that it is straight forward to do this if we know how SUSY is realized in the given background manifold. Consequently the problem boils down to a set of first order differential equations. As an application of our prescription we calculated the partition function of $\mathcal{N} = 1$ theories on $\mathrm{S}^3_b \times \mathrm{S}^1$. Here S^3_b is a squashed sphere with $U(1)^2$ isometry and b denotes its squashing parameter. The partition function so obtained matches the partition function of $\mathrm{S}^3 \times \mathrm{S}^1$ upon a redefinition of its fugacities. This is not a surprise since, as mentioned previously, the $\mathrm{S}^3 \times \mathrm{R}$ partition function is a topological quantity which therefore should be invariant under the deformation of its background from S^3 to S^3_b. Note that [87, 67] describe a general scheme of reducing 4d superconformal indices to 3d partition functions (on S^3_b background) [130, 106]. However there appears to be no a priori reason why their prescription works. Through our enumeration of the BPS modes on $\mathrm{S}^3_b \times \mathrm{S}^1$ we showed that these are in one to one correspondence with the BPS modes of [106] that contribute to the S^3_b partition function. Therefore the original fugacities entering our expression for the $\mathrm{S}^3 \times \mathrm{S}^1$ partition function define a natural scheme to recover the three-dimensional partition function via dimensional reduction.

In chapter 3 based on work done in [1], we continue our investigation of the superconformal index and study quiver gauge theories describing the worldvolume of D3 branes probing toric Calabi-Yau singularities in the large N limit. In [84], it was noticed that in this limit, the SCI for the Y^{pq} familiy of quiver gauge theories factorizes such that each factor corresponds to the contribution from the so called extremal BPS mesons of the theory [45]. A proof for the conjecture was later provided in [70], where the authors explained the factorization of the index from the properties of the toric geometry, for the case of smooth CY_3. In our work we showed that the factorization also holds in gauge theories dual to geometries where new singularities arise far from the tip of the CY cone (such as the L^{aaa} family, the L^{aba} family and the chiral orbifold L^{264} of L^{123}). In the process we reformulate the factorization of SCI in terms of the factorization over the so called zig-zag paths in the brane tiling of toric gauge theories. This factorization in terms of the zig-zag paths then continues to be well defined in the case of quiver gauge theories dual to geometries with orbifold singularities. We also show that the factorization continues to hold when SCI is written in terms of trial R-charges with the only constraints on the R-charge coming from the vanishing NSVZ beta function and requiring that the superpotential is marginal.

1.4 $3d$ dualities and exact partition function

Chapter 4 is based on my paper [3]. Here we propose a large class of dualities in three-dimensional field theories with different gauge groups. Using the integral identities of [187] we show the analytic matching of S^3 partition functions of the dual phases for any value of the couplings, thereby providing a non-trivial check for our proposal. We consider two classes of models. The first class, motivated by the AdS/CFT conjecture, consists of necklace U(N) quiver gauge theories with non chiral matter fields. We also consider orientifold projections and established dualities among necklace quivers with alternating orthogonal and symplectic groups. The second class consists of theories with tensor matter fields. Such theories were conjectured to be dual to free chiral multiplets [129, 134]. In these theories certain gauge invariant operators hit the unitarity bound and decouple as we flow towards the IR fixed point. This decoupling gives rise to accidental symmetries which mix with the R-symmetry of the theory. Thus, the exact R-charge in the IR can not be straightforwardly obtained by the naive extremization of the partition function [130]. This also implies that the naive partition functions on the two sides of the duality will not match. A prescription to properly account for the mixing with accidental symmetries in 3d theories was hitherto lacking. In order to get over this hurdle, we adapted the technique of [146] to correct the partition function of $3d$ theories and demonstrated that once the accidental symmetries have been taken care of, the partition functions of the dual theories indeed match.

1.5 $\mathcal{N} = 1$ theories of class $\mathcal{S}$

Supersymmetric theories of class $\mathcal{S}$ are $4d$ SUSY theories which can be obtained by compactifying $M5$ branes on a punctured Riemann surface. The punctures on the Riemann surface are associated to the flavor symmetries of the $4d$ SCFT and gluing two Riemann surfaces along their punctures then corresponds to gauging diagonal flavor symmetry of the two punctures. The $4d$ $\mathcal{N} = 2$ theories obtained in this manner have been extensively studied following the work of [88, 90, 181]. In [82], the authors showed that in the realm of $4d$ $\mathcal{N} = 1$ theories, class $\mathcal{S}$ can be used to write the $SU(N)$ gauge theory generalization of the $N_c = 2$, $N_f = 2N_c$ duality of [63]. In [6], we extended the results of [82] to the case of $SO(2N)$, $Sp(N)$ and G_2 gauge theories. In the process we also discovered the analogue of Argyres-Seiberg type duality in $\mathcal{N} = 1$ theories. In the same work we provide a prescription to write the superconformal index of $4d$ theories obtained by wrapping $6d$ (2,0) theories of type D_4 on a three punctured sphere with $\mathbb{Z}_2$ and $\mathbb{Z}_3$ outer-automorphism twist lines. This theory was conjectured by Tachikawa to have flavor symmetry that gets enhanced to E_7 at the IR fixed point [182]. Through our prescription we were able to show that the superconformal index of theories does exhibit such an enhancement, thereby providing a non-trivial check of the above conjecture. This became the basis of chapter 5.

Chapter 6 is based on [4]. In this chapter we study the nilpotent Higgsing in $\mathcal{N} = 1$ linear quivers of class S. In the case of $\mathcal{N} = 2$ theories such Higgsing yields regular punctures that can be associated to quiver tails labelled by partitions of N. Surprisingly, in $\mathcal{N} = 1$ linear quivers, such Higgsing yields a new type of quiver tails dubbed as the Fan. This object is labelled by two integers N and N_0 , and a partition of $N - N_0$. We provide further evidence of the Fan by "Higgsing" the superconformal index. Using the Fan we construct many new $\mathcal{N} = 1$ SCFTs which provide various field theoretic descriptions of M5-branes wrapped on punctured Riemann surfaces.

In chapter 7, based on my work [5], we explore the process of closing some of the punctures on the Riemann surface. In terms of field theory, this is same as completely Higgsing the flavor symmetry associated with that puncture. In doing so we systematically obtain an infinite number of theories that describe the same IR physics as an $SU(N)$ SQCD with four flavors and a quartic superpotential.

Chapter 2

BPS States and Their Reductions

Supersymmetric field theories on curved backgrounds are of great interest due to the fact that they capture the full quantum information about quantities of the corresponding field theory defined on flat space, where the same exact quantum results would be difficult to find.

Different choices of the background manifold correspond to a different information about the flat space theory. One of the first examples has been $T^3 \times \mathbb{R}$ [191], where the supersymmetric partition function counts supersymmetric vacua and has been dubbed index (see also [52]). Because it is an integer number, it cannot depend upon the continuous superpotential and gauge couplings, under mild assumptions. More recently another manifold, the Euclidean $S^3 \times S^1$, has attracted much attention, because in this case the supersymmetric partition function is an index that counts a reduced set of states of the flat space theory, namely the BPS states [168, 142]. The latter are protected by supersymmetry so that a weak coupling computation can be continued to strong coupling and compared in the AdS/CFT framework to the computation of the graviton index in AdS space. The matching of the two indices on the two sides corroborates the conjectured duality between them. This is only one of the calculable exact results. By using localization [192], we can in principle compute the supersymmetric partition function (see [164]) on any manifold that preserves at least one complex supercharge (or, in Euclidean space, two real supercharges), by reducing it to a matrix model, i.e. a finite dimensional ordinary integral.

Turning back to the case of the four-dimensional index, there are many available methods to obtain the matrix model formula for it. In [174, 175, 168] the BPS states on the sphere have been found explicitly from the knowledge of the spectrum of the Laplace operator: one needs to find the eigenmodes of the Laplace and Dirac operator on the sphere and sum over all the modes. Many of the bosonic modes will cancel out against the fermionic ones, and one finds that only the BPS modes contribute to the index. This is the most direct method, but it is in practice very difficult to work out on generic supersymmetry preserving manifolds. Another method is to compute the *letter* index [142] for theories defined on a conformally flat background. In these cases, it is straightforward to obtain the quantum numbers of the curved space fields because conformal mapping relates them to their flat space counterpart. We can then identify the operators that saturate the BPS inequality. Nevertheless it is not simple to extend this method to backgrounds that are not conformally flat. Finally, one can consider using localization. This amounts to picking a Q-exact term, generically related to the supersymmetry transformations, and evaluate the ratio of two determinants, which represents the full quantum corrections to the quantity one is considering [1].

Because of the difficulties of applying the previous methods to other manifolds, it is simpler to identify just the BPS states in the Hilbert space. One of the purposes of this paper is to develop a method to achieve this aim. The essential logic relies on the fact that the non-BPS modes are paired up by supersymmetry and hence the BPS modes correspond to the kernel of the boson-fermion map. The problem boils down to a set of first order differential equations.

We also argue a general relation between the BPS states and the set of states

[1] Recently the $\mathcal{N} = 4$ superconformal index has been computed from localization in [160].

that contribute non trivially to a corresponding partition function in one dimension less. More precisely, we will see that there exists a one-to-one map between these two sets of states, and we identify the energy of each BPS state with the quantum contribution of the dimensionally reduced state to the supersymmetric partition function. This relation has two immediate consequences. The first one is that an index in d dimensions reduces to a supersymmetric partition function for the dimensionally reduced field theory, thus providing an argument which generalizes previous observations for the three-sphere [67, 87, 122].

Another consequence is the following. Since the states contributing to the partition function are the BPS states in one higher dimension, we can uplift the quantum contribution to the partition function to the computation of the energies of BPS states in one higher dimension and use the method outlined above. In this way, we only need to know the uplifted supersymmetry transformations and read from them the pairing map. We believe that this leads to a simplification in the computation of exact partition functions.

The paper is organized as follows. In section 2.1 we review the definitions for the quantities we are interested in, and explain our method to identify the BPS states in a general field theory. We also describe in full detail the relation between d-dimensional BPS states and the $(d-1)$-dimensional physical states, focusing on the $4d/3d$ case for concreteness. In section 2.2 we show how the computations can be worked out for the examples of the round and squashed spheres. We give all the necessary details to explicitly perform the computation, review previous results and discuss the physical meaning of our results applied to the cases at hand. The reduction of these two indices to the corresponding three-dimensional partition functions is shown in section 2.3. In section 2.4 we discuss generalizations of our technique to compute the index to other manifolds and dimensions, while the idea to uplift the computation of the partition function to a higher dimensional index is developed in section 2.5.

2.1 A correspondence between $4d$ and $3d$ states

One of the aims of the present paper is to develop a method to identify the BPS states and to compute the supersymmetric index and the partition function on a general class of manifolds. In doing that, we will point out a connection between these two objects in different dimensions. To be concrete, in this section we focus on the $4d/3d$ case.

Given a three-dimensional manifold $\mathcal{M}_3$ that preserves some supersymmetry, and given a four dimensional supersymmetric theory defined on $X \equiv \mathcal{M}_3 \times S^1$, we can define two different quantities. The first one is the four dimensional superconformal index, defined on X, that only takes contributions from BPS states. It is the supersymmetric partition function

$$\mathcal{I}_{sp}\left(t, y_i\right) = \mathrm{Tr}(-)^F\, e^{-\tau \Xi}\, t^H \prod_i y_i^{\gamma_i} \tag{2.1}$$

where $\Xi \equiv \{Q, Q^\dagger\}$, F is the fermion number and the trace is taken over every state in the theory. H and the γ's form a complete set of operators that commute with the conserved supercharge Q. In the following, we will call H the energy operator and its eigenvalues the energies of the corresponding eigenstates. Moreover, the time direction is identified with the circle and is thus periodic with period τ. The statement that the quantity (2.1) only takes contributions from BPS states means that for each bosonic state with $\Xi \neq 0$ there exists a fermionic state with the same (Ξ, H, γ_i) quantum numbers; thus, the index turns out to be independent of τ due to the boson-fermion cancellations, and the trace can be taken over the Hilbert space of $\Xi = 0$ states.[2] The index in (2.1)

[2]By a "state" of the theory we mean a configuration field which solves the equations of motion.

is the *single particle* index. In the case of a gauge theory one has to sum over all the possible gauge invariant configurations.

On the other hand, we can reduce the given supersymmetric theory on $\mathcal{M}_3$ itself and compute, at least in principle, the exact partition function for this theory via localization. The latter reduces the partition function to the matrix integral

$$Z_{\mathcal{M}_3} \sim \int [\mathrm{d}\sigma]\; e^{-S_*} \frac{\mathrm{Pf} D_F}{\sqrt{\det D_B}} \tag{2.2}$$

where $[\mathrm{d}\sigma]$ represents the measure over the Cartan of the gauge group. We have set the following notation for the two quantities we are interested in. We denote by S_* the classical action evaluated at the saddle points, while the exact quantum contribution from the generic superfield Φ is

$$Z_\Phi = \frac{\mathrm{Pf} D_F}{\sqrt{\det D_B}} \tag{2.3}$$

where D_F and D_B are, respectively, a linear first order and second order differential operator, and Φ labels both the chiral and gauge multiplets.[3] A boson-fermion cancellation manifests itself in the fact that some of the eigenvalues simplify between the numerator and the denominator in (2.3).

We argue that the BPS states that contribute to (2.1) are in one-to-one correspondence to the states contributing to (2.3). More precisely, for each four-dimensional BPS state with eigenvalue E of H there is a three-dimensional state for which E is an eigenvalue of the D_B or of D_F in the case of boson or fermion respectively.

These states can be found by solving a first order differential equation that can be directly read from the supersymmetry transformations of the four dimensional theory.

[3]The three-dimensional action may not be derived by dimensional reduction of a corresponding four-dimensional theory. This is the case, for instance, when a Chern-Simons term is present. The one loop determinants are not sensitive to these contributions and our results also hold in those cases.

Finally, the saddle points in (2.2) correspond to the zero energy states in the BPS spectrum: it follows that, if there is no zero energy solution for a four-dimensional field Φ, the only three-dimensional saddle point corresponds to $\Phi = 0$. We will give more details on this point in section 2.3.

An argument for this correspondence is the following. It is well known that the index (2.1) does not depend on the radius of the compact time direction and thus it does not change even when we shrink the circle to zero size. More precisely, consider a fermionic state ψ of a four-dimensional theory and define a corresponding bosonic state

$$\phi \equiv \zeta\psi \tag{2.4}$$

where ζ is the Killing spinor which commutes with the BPS condition. Then ϕ has the same (Ξ, H, γ_i) quantum numbers and will cancel the contribution of ψ in (2.1), unless $\phi = 0$ or, equivalently, $\psi = \zeta F$, with F a scalar function with the same (Ξ, H, γ_i) quantum numbers of ψ. If ψ is a state of the theory it satisfies the corresponding equation of motion: if we set $\psi \sim \psi_3(\vec{x})e^{Et}$, it is thus easy to recognize that the four-dimensional equation of motion can be interpreted as the eigenvalue equation for a three-dimensional fermion with eigenvalue E.

We now consider the bosonic states that contribute to the index: we set up a map from the bosonic spectrum to the fermionic one by

$$\psi = i\sigma^\mu\tilde{\zeta}D_\mu\phi \tag{2.5}$$

which is an infinitesimal supersymmetry transformation (see below and section 2.2.1). We see that every boson that contributes to the index is given by $\sigma^\mu\tilde{\zeta}D_\mu\phi = 0$. Once again, this can be interpreted as an eigenvalue equation for a three-dimensional bosonic mode that contributes non trivially to the partition function.

The argument above can be cast in the following form. In four dimensions, the supersymmetry transformations for the chiral multiplet are

$$\begin{aligned}\delta\phi &= \zeta\psi \\ \delta\psi &= \zeta F + i\sigma^\mu\tilde{\zeta}\, D_\mu\phi \\ \delta F &= \tilde{\zeta}\tilde{\sigma}^\mu\mathcal{D}_\mu\psi\end{aligned} \tag{2.6}$$

where our conventions are explained in section 2.2.1. Notice that the fermion equation of motion implies $\delta F = 0$. This is a necessary condition that must be satisfied by the fermionic degrees of freedom.

The map that identifies the BPS states can be found to be

$$\begin{aligned}\text{fermion:} &\qquad \psi = \zeta F \quad \text{and} \quad \tilde{\sigma}^\mu\mathcal{D}_\mu\psi = 0 \\ \text{boson:} &\qquad i\sigma^\mu\tilde{\zeta}\, D_\mu\phi = 0\end{aligned} \tag{2.7}$$

We further notice the following. The system (2.7) implies $|\delta\psi|^2 = 0$ (in the absence of F-terms), and when the fields are independent on the time direction, we can dimensionally reduce the latter equation which becomes the three-dimensional saddle point equation used in the localization setting.

We now turn to the vector multiplet. Once again we can set up a map between the bosonic and the fermionic Hilbert space by using the supersymmetry transformations. Analogously to the discussion above, all the contributions will cancel out but those coming from the zero modes of the map.

In four dimensions, the physical fields in the vector multiplet are a gauge field v_μ

and the gaugino λ. The supersymmetry transformations are

$$\begin{aligned} \delta v_\mu &= i\zeta\sigma_\mu\tilde{\lambda} - i\tilde{\zeta}\tilde{\sigma}_\mu\lambda \\ \delta\lambda &= \zeta D + i\sigma^{\mu\nu}F_{\mu\nu}\zeta \\ \delta D &= i\tilde{\zeta}\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda - i\zeta\sigma^\mu\mathcal{D}_\mu\tilde{\lambda} \end{aligned} \tag{2.8}$$

The map that identifies BPS states can be found to be

$$\begin{aligned} &\text{gaugino:} \qquad -i\tilde{\zeta}\tilde{\sigma}_\mu\lambda = \partial_\mu\varphi \quad \text{and} \quad i\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda = 0 \\ &\text{gauge boson:} \qquad i\sigma^{\mu\nu}F_{\mu\nu}\zeta = 0 \end{aligned} \tag{2.9}$$

where once again $\delta D = 0$ is a necessary condition for the gaugino degrees of freedom.

In the first line we had set the gauge field to a pure gauge configuration because any such solution does not give rise to a state in the Hilbert space of the theory and hence the gaugino does not have a superpartner state. Alternatively, we could have considered the map between the field strength $F_{\mu\nu}$ and the gaugino, which leads to the same condition. It is easy to see that the zero energy solutions to (2.8) reduce to the three-dimensional saddle point equations for a three-dimensional Q-exact action. The set of non-trivial solutions for λ and $F_{\mu\nu}$ gives the Hilbert space we have to trace over in equation (2.1), or alternatively the spectrum of eigenvalues contributing to (2.3).

To summarize, we are led to the conclusion that a priori different exact results in different dimensions are related one to the other. The reduction of the four-dimensional index to the three-dimensional partition function follows directly from the proposed connection between the four-dimensional and three-dimensional states. While we will give more details on this point in section 2.3, we stress here that our claim is stronger than the dimensional reduction of the superconformal index to the partition function, because we set up a one-to-one map between states and eigenvalues of different operators.

On the one hand we look for eigenstates of the four-dimensional Hamiltonian, on the other hand we look for eigenstates of the equations of motion derived from a Q-exact three-dimensional Lagrangian, that contributes to the partition function. While the former is a first order differential operator, the latter is in general a second order one.

In the next section we will explicitly check our proposal in two cases: $\mathcal{M}_3 = S^3$, in which case we can compare with known results, and $\mathcal{M}_3 = S_b^3$, with S_b^3 a squashed sphere. In the latter case, because the index is a topological invariant, it can be cast in the same form as the index on a sphere via a redefinition of its arguments. However, we show that one can keep the original definitions and define a natural limit to recover the three-dimensional partition function on the squashed three-sphere computed in [106]. We thus conclude that, although the index does not carry different physical information on different but topologically equivalent manifolds, it contains different information when we reduce the four-dimensional theory to a three-dimensional one by shrinking the time circle. It thus becomes interesting, from a three-dimensional point of view, to compute the four-dimensional index even on topologically equivalent manifolds.

2.2 Examples: sphere and squashed spheres

2.2.1 Review of rigid supersymmetry on a curved manifold

We review here a simple and recent procedure to place an $\mathcal{N} = 1$ supersymmetric theory on a curved four-dimensional manifold [73]. The basic idea is to start with $\mathcal{N} = 1$ supergravity and take an appropriate limit such as to decouple gravity but preserve the classical background configuration. Because a convenient off-shell formulation and its couplings to matter fields are known, the gravitino supersymmetry transformation looks very simple [177, 176]

$$\begin{aligned}
\delta\psi_\mu &= -2\mathcal{D}_\mu\zeta - 2iV^\nu\sigma_{\mu\nu}\zeta \\
\delta\tilde{\psi}_\mu &= -2\mathcal{D}_\mu\tilde{\zeta} + 2iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} \\
D_\mu &\equiv \nabla_\mu - iq_A A_\mu \\
\mathcal{D}_\mu &\equiv D_\mu - iq_V V_\mu
\end{aligned} \tag{2.10}$$

where q_A and q_V are the charges (under the A and V background gauge fields) of the field on which the covariant derivative is acting on. For the Killing spinor ζ, $q_A^\zeta = 1$ and $q_V^\zeta = -1$, and $\tilde{\zeta}$ has opposite quantum numbers. Because gravity is decoupled, one can give an expectation value to the background gauge fields A and V and to the metric without having to take care of their equations of motion.

Once we have found a solution to $\delta\psi_\mu = 0$ and $\delta\tilde{\psi}_\mu = 0$, the supersymmetry transformations of the matter fields are

$$\begin{aligned}
&\delta\phi = \zeta\psi && q_A^\phi = q && q_V^\phi = -1/2 \\
&\delta\psi = \zeta F + i\sigma^\mu\tilde{\zeta}\, D_\mu\phi && q_A^\psi = q-1 && q_V^\psi = 1/2 \\
&\delta F = \tilde{\zeta}\tilde{\sigma}^\mu\mathcal{D}_\mu\psi && q_A^F = q-2 && q_V^F = 3/2
\end{aligned} \tag{2.11}$$

for the chiral multiplet, and, in the Wess-Zumino gauge,

$$\begin{aligned}
&\delta v_\mu = i\zeta\sigma_\mu\tilde{\lambda} - i\tilde{\zeta}\tilde{\sigma}_\mu\lambda && q_A^v = 0 && q_V^v = -1/2 \\
&\delta\lambda = \zeta D + i\sigma^{\mu\nu}F_{\mu\nu}\zeta && q_A^\lambda = 1 && q_V^\lambda = -3/2 \\
&\delta D = i\tilde{\zeta}\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda - i\zeta\sigma^\mu\mathcal{D}_\mu\tilde{\lambda} && q_A^D = 0 && q_V^D = -1/2
\end{aligned} \tag{2.12}$$

for the vector multiplet. An action which is invariant under these supersymmetry transformations is[4]

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}^B + \mathcal{L}^F \qquad (2.13)\\
\frac{\mathcal{L}^B}{\sqrt{g}} &= \left(-\frac{1}{4}\mathcal{R} - \frac{3}{2}V_\mu V^\mu\right) q\phi\bar{\phi} - D_\mu\phi D^\mu\bar{\phi} + F\bar{F}\\
&\quad + iV^\mu\left(\bar{\phi}D_\mu\phi - \phi D_\mu\bar{\phi}\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2\\
\frac{\mathcal{L}^F}{\sqrt{g}} &= -i\tilde{\lambda}\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda - i\tilde{\psi}\tilde{\sigma}^\mu\mathcal{D}_\mu\psi
\end{aligned}$$

2.2.2 Supersymmetry on a general squashed sphere

In this section we present all the necessary results to work out the examples of the sphere and the squashed sphere to be described in full details in the next sections. We give the full expressions in the case of the squashed sphere, while supersymmetry on the sphere is recovered by taking an appropriate limit. Some of the results shown here can be also found in [69, 143].

The squashed sphere S^3_b enjoys a $U(1)^2$ isometry. The latter is made manifest if we choose the Hopf coordinates $x^\mu = \{t, \theta, \alpha, \beta\}$, with $\mu = 1, \ldots, 4$, such as t denotes the Euclidean time coordinate compactified on a circle. The coordinates α and β have range $[0, 2\pi)$ while $\theta \in [0, \pi/2]$. The metric reads

$$ds^2 = dt^2 + f(\theta)^2 d\theta^2 + a^2\cos(\theta)^2 d\alpha^2 + b^2\sin(\theta)^2 d\beta^2 \qquad (2.14)$$

where $f(\theta)$ is regular on $(0, \pi/2)$ and and $f(0) = b$ and $f(\pi/2) = a$. Moreover the manifold even if compact can also be locally hyperbolic. The Ricci tensor is

$$\mathcal{R} = \frac{6f(\theta) + 4\cot(2\theta)f'(\theta)}{f(\theta)^3} \qquad (2.15)$$

[4]We are considering Euclidean signature. The derivatives should be understood to be covariant with respect to the gauge field too, but due to the invariance of the index under continuous transformations, we can switch off the gauge coupling without changing the result.

In principle, we could have introduced two parameters, say R_1 and R_2, multiplying the time and squashed sphere terms respectively in the metric. The gravitino variation then imposes $R_1 = R_2$, and the overall factor can be set to unity by a redefinition of the time period, which does not affect our computations.

The Killing spinor equations in the new minimal formalism are solved by

$$\begin{aligned}
\zeta_\alpha &= -\frac{i}{\sqrt{2}} e^{\frac{i}{2}(\alpha+\beta)} \begin{pmatrix} e^{-\frac{i}{2}\theta} \\ i e^{\frac{i}{2}\theta} \end{pmatrix} \qquad \tilde{\zeta}^{\dot\alpha} = -\frac{i}{\sqrt{2}} e^{-\frac{i}{2}(\alpha+\beta)} \begin{pmatrix} e^{\frac{i}{2}\theta} \\ i e^{-\frac{i}{2}\theta} \end{pmatrix} \\
V_\mu dx^\mu &= -\frac{i}{f(\theta)} dt \\
A_\mu dx^\mu &= -\frac{i}{f(\theta)} dt + \left(\frac{1}{2} - \frac{a}{2f(\theta)}\right) d\alpha + \left(\frac{1}{2} - \frac{b}{2f(\theta)}\right) d\beta
\end{aligned} \tag{2.16}$$

which shows that, for generic squashing parameters a, b there are two supercharges. In the round sphere limit we can find two more Killing spinors, showing that the manifold enjoys four supercharges. Our results only rely on the existence of two real supercharges, and we choose (2.16) which is a convenient choice both for the sphere and the squashed spheres.

With our choice of background fields, the algebra involving the two supercharges above is

$$\begin{aligned}
&[H, Q] = 0 \qquad \{Q, \tilde{Q}\} = H - \frac{R}{2}\left(\frac{1}{a} + \frac{1}{b}\right) + 2J_3 \\
&[R, Q] = -Q \qquad \left[2\tilde{J}_3 + \frac{R}{2}\left(\frac{1}{a} - \frac{1}{b}\right), Q\right] = 0 \\
&H \equiv \partial_t \qquad 2J_3 \equiv -\frac{i}{a}\partial_\alpha - \frac{i}{b}\partial_\beta \qquad 2\tilde{J}_3 \equiv \frac{i}{a}\partial_\alpha - \frac{i}{b}\partial_\beta
\end{aligned} \tag{2.17}$$

From the supersymmetric action

$$
\begin{aligned}
\frac{\mathcal{L}}{\sqrt{g}} &= \left(-\frac{1}{4}\mathcal{R}+\frac{3}{2f^2}\right)q\phi\bar{\phi} - D_\mu\phi D^\mu\bar{\phi} + F\bar{F} \\
&\quad +\frac{1}{f}\left(\bar{\phi}D_t\phi - \phi D_t\bar{\phi}\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 \\
&\quad -i\tilde{\lambda}\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda - i\tilde{\psi}\tilde{\sigma}^\mu\mathcal{D}_\mu\psi
\end{aligned}
\tag{2.18}
$$

we can derive the following equations of motion

$$
\begin{aligned}
\Delta_\phi\phi &\equiv \left(D^\mu D_\mu + \frac{2}{f}D_t + q\left(-\frac{1}{4}\mathcal{R}+\frac{3}{2f^2}\right)\right)\phi = 0 \\
\Delta_\psi\psi &\equiv i\tilde{\sigma}^\mu\mathcal{D}_\mu\psi = 0 \\
\nabla_\mu F^{\mu\nu} &= J^\nu \\
\Delta_\lambda\lambda &\equiv i\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda = 0
\end{aligned}
\tag{2.19}
$$

where J^ν is an appropriate current which vanishes in the $g_{YM}\to 0$ limit.

2.2.3 The three sphere

In this section we apply the proposal explained above to the calculation of the superconformal index on $S^3\times S^1$, and show that it agrees with previous results [168, 142]. We start by reviewing the calculation of the index in terms of the expansion of the field configurations in spherical harmonics. There are two multiplets contributing to the index, the chiral multiplet $\Phi=(\phi,\psi)$ with $R[\phi]=q$ and the vector multiplet $\mathcal{V}=(v,\lambda)$.

The harmonic expansion has first been done in [174, 175] and we report it here with conventions adapted to Euclidean signature. The algebra chosen there coincides with the round sphere limit of our equation (2.17), so the definition of the index works without further changes.

The eigenvalues of the Laplace operator acting on scalars on the three-sphere are

$-j(j+2)$, with j a nonnegative integer. By plugging the expansion scalar field

$$\Phi = \sum_n a_n \Phi_+(n) e^{E_+(n)t} + c_n^\dagger \Phi_-^\dagger(n) e^{E_-(n)t} \tag{2.20}$$

in the equation of motion, one sees that, including the R-charge contribution, the normal modes are

Wave function	E	$(J_3, \tilde{J}_3)$
a_n	$j+q$	$\left(\frac{j}{2}, \frac{j}{2}\right)$
$c_n^\dagger$	$-j-2+q$	$\left(\frac{j}{2}, \frac{j}{2}\right)$

where $j \geq 0$ and in the last column we have indicated the representation of the fields under the Cartan subgroup of the isometry group of the sphere. A field is in the $(j/2, j/2)$ representation means that the j_3 and $\tilde{j}_3$ eigenvalues can range from $-j/2$ to $j/2$ at fixed j.

An analogous expansion holds for the chiral fermion

$$\Psi = \sum_n b_n \Psi_+(n) e^{E_+(n)t} + d_n^\dagger \Psi_-^\dagger(n) e^{E_-(n)t} \tag{2.21}$$

which gives

Wave function	E	$(J_3, \widetilde{J}_3)$
b_n	$j+q$	$\left(\frac{j-1}{2}, \frac{j}{2}\right)$
$d_n^\dagger$	$-j+q-1$	$\left(\frac{j}{2}, \frac{j-1}{2}\right)$

where $-j(j+1)$, $j \geq 1$, are the eigenvalues of the Laplace operator on spinors on the three-sphere.

To compute the index, one has in principle to sum over all these states. However, we know that the index only takes contributions from BPS states, i.e. states that satisfy $\Xi = 0$. It is easy to realize that this constraint fixes the $j_3 = -j/2$ particle state for the

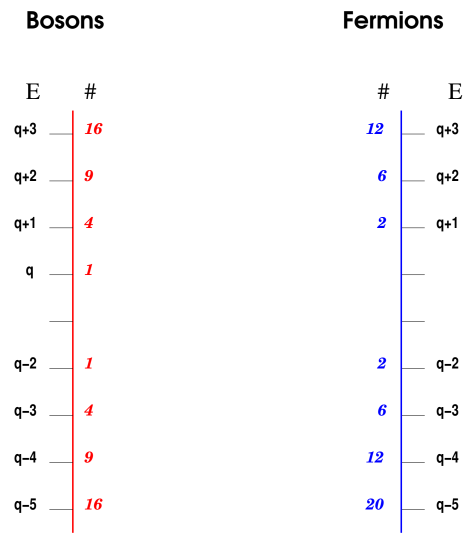


Figure 2.1. A schematic structure of the pairings among the modes. Here E is the energy of the mode and # is the number of bosonic and fermionic modes with a given energy.

scalar field and the $j_3 = j/2$ antiparticle state for the fermion, while $\tilde{j}_3$ is unconstrained because it does not appear in Ξ. By summing over all these states the contribution to the superconformal index of the chiral multiplet is

$$I_\Phi = \sum_{j,\tilde{j}_3} (-1)^F e^{-\tau\Xi} t^H y^{2\tilde{j}_3} = \frac{t^q - t^{2-q}}{(1-ty)(1-t/y)} \tag{2.22}$$

This structure of pairing and un-pairing among the modes is explicitly shown in the Figure 2.1. In general, for $j \geq 0$, we have the following structure

	E_+	Degeneration	E_-	Degeneration
Boson	$j+q$	$(j+1)^2$	$-j-2+q$	$(j+1)^2$
Fermion	$j+q$	$j(j+1)$	$-j-2+q$	$(j+1)(j+2)$

The BPS modes are the modes unpaired in this table, and they are counted by the superconformal index as explained above.

We can repeat the above procedure for the vector multiplet. In the case of the gaugino we have

Wave function	E	$(J_3, \tilde{J}_3)$
b_n	j	$\left(\frac{j-1}{2}, \frac{j}{2}\right)$
$d_n^\dagger$	$-j-1$	$\left(\frac{j}{2}, \frac{j-1}{2}\right)$

with $j \geq 1$. For the vector field one can expand in terms of the spin-1 spherical harmonics and the modes are

Wave function	E	$(J_3, \tilde{J}_3)$
a_n	$j+1$	$\left(\frac{j-1}{2}, \frac{j+1}{2}\right)$
$c_n^\dagger$	$-j-1$	$\left(\frac{j+1}{2}, \frac{j-1}{2}\right)$

with $j \geq 1$. By summing over all these states the contribution of the vector multiplet to the superconformal index is

$$I_V = \sum_{j, j_3 \tilde{j}_3} (-1)^F e^{-\tau \Xi} t^H y^{2\tilde{j}_3} = \frac{2t^2 - t(y + 1/y)}{(1-ty)(1-t/y)} \tag{2.23}$$

In the rest of this section we apply our prescription to obtain the BPS states in a different way, in which it is not necessary to solve for the whole spectrum. We start by considering the metric as in (2.14) with $a = b = 1$. The two angles α and β can be associated to the Cartan subgroup of the $SU(2)^2$ isometry group of the metric.

We start by solving the equation (2.7) for the BPS fermion in the chiral multiplet . Once we write the fermion as $\psi = \zeta F$ and solve the equation $\Delta_\psi (\zeta F) = 0$, we expand F as $F = e^{Et + in\alpha + im\beta} g_\psi(\theta)$, where E is the eigenvalue associated to the S^1 and n and m are integer numbers associated to the two $SU(2)$ in the S^3, parameterized by the periodic coordinates α and β in the metric. We obtain

$$\begin{cases} g'_\psi + i\, g_\psi(2 + E + m + n - q + i\, m \cot\theta - i\, n \tan\theta) &= 0 \\ g'_\psi - i g_\psi(2 + E + m + n - q - i\, m \cot\theta + i\, n \tan\theta) &= 0 \end{cases} \tag{2.24}$$

These two equation can be simultaneously solved for $E = q - 2 - m - n$ and the solution is

$$g_\psi(\theta) = \sin^m\theta\cos^n\theta \qquad \text{for} \qquad \theta \sim 0, \pi/2 \tag{2.25}$$

that is square integrable if $m, n \geq 0$. This represents the contributions of the BPS fermion to the index. Because E is negative, we have found that the corresponding state is an *antiparticle* mode of the fermion. Thus, when we plug its quantum numbers in the index, we have to flip their signs: the energy of the field is $E^{(\psi)} = -E = 2 - q + m + n$. The other operator that commutes with the supercharge is $\tilde{J}_3$ that has eigenvalues $m - n$. The fermionic contribution to the index is then

$$I_\psi = -\sum_{m,n\geq 0} t^{n+m+2-q}y^{m-n} = -\frac{t^{2-q}}{(1-ty)(1-t/y)} \tag{2.26}$$

We parameterize the BPS boson as $\phi = e^{Et+in\alpha+im\beta}g_\phi(\theta)$ and the equation (2.5) becomes

$$\begin{cases} g'_\phi + ig_\phi(E + m + n - q - i\,m\cot\theta + i\,n\tan\theta) &= 0 \\ g'_\phi - ig_\phi(E + m + n - q + i\,m\cot\theta + i\,n\tan\theta) &= 0 \end{cases} \tag{2.27}$$

The two equations are compatible if $E = q - m - n$ and the solution is

$$g_\phi(\theta) = \frac{1}{\sin^m\theta\cos^n\theta} \qquad \text{for} \qquad \theta \sim 0, \pi/2 \tag{2.28}$$

and square integrability imposes $m, n \leq 0$. The BPS boson that contributes to the index is the particle in the expansion in terms of creation and annihilation operators, with energy $E^{(\phi)} = E$. The bosonic index is

$$I_\phi = \sum_{m,n\geq 0} t^{n+m+q}y^{m-n} = \frac{t^q}{(1-ty)(1-t/y)} \tag{2.29}$$

We now turn to the vector multiplet. In the case of the gaugino we read the pairing map from the transformation of $F_{\mu\nu}$. The BPS modes are the solution of the equation

$$\partial_\mu(\tilde{\zeta}\bar{\sigma}_\nu\lambda) - \partial_\nu(\tilde{\zeta}\bar{\sigma}_\mu\lambda) = 0 \tag{2.30}$$

This equation is solved by

$$\tilde{\zeta}\bar{\sigma}_\mu\lambda = \partial_\mu\Phi \tag{2.31}$$

Alternatively, one can require that the SUSY variation for v_μ gives a purely longitudinal field. We then impose the usual ansatz dictated by the $U(1)$ symmetries

$$\Phi = e^{Et+i\alpha n+i\beta m}h_\Phi(\theta) \quad , \quad \lambda = e^{Et+i\alpha\left(n+\frac{1}{2}\right)+i\beta\left(m+\frac{1}{2}\right)}\begin{pmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{pmatrix} \tag{2.32}$$

and we plug it in (2.31). Moreover we impose that λ satisfies its equations of motion. In this way we find

$$\begin{aligned} E &= -m-n \\ \lambda_1(\theta) &= \frac{1}{\sqrt{2}}e^{\frac{i\theta}{2}}h_\Phi(\theta)\left(\frac{m}{\sin\theta}+i\frac{n}{\cos\theta}\right) \\ \lambda_2(\theta) &= -\frac{1}{\sqrt{2}}e^{-\frac{i\theta}{2}}h_\Phi(\theta)\left(i\frac{m}{\sin\theta}+\frac{n}{\cos\theta}\right) \\ h'_\Phi(\theta) &= h_\Phi(\theta)\left(n\tan\theta - m\cot\theta\right) \end{aligned} \tag{2.33}$$

The equation for $h_\Phi(\theta)$ tells us that the solution is square integrable for $m, n \leq 0$, but we exclude the vanishing solution corresponding to $(m,n) = (0,0)$. Thus the energy is positive and the gaugino contribution to the index is

$$I_\lambda = -\left(\sum_{m,n=-\infty}^{0} t^{-m-n}y^{m-n} - 1\right) = \frac{t^2 - t(y+1/y)}{(1-ty)(1-t/y)} \tag{2.34}$$

where the second term comes from subtracting the $(m, n) = (0, 0)$ contribution.

The gauge field works as follows. First we impose that the BPS equation is satisfied

$$\sigma^{\mu\nu}\zeta F_{\mu\nu} = 0 \tag{2.35}$$

We consider the Abelian case and define the components of the EM field as $\mathcal{E}_i = F_{ti}$ and $2\mathcal{B}_i = \epsilon_{ijk}F_{jk}$ where the latin letters label the S^3 coordinates. We parametrize these fields with the ansatz

$$\mathcal{E}_i(t,\theta,\alpha,\beta) = e^{Et+i\alpha n+i\beta m}\mathcal{E}_i(\theta) \quad , \quad \mathcal{B}_i(t,\theta,\alpha,\beta) = e^{Et+i\alpha n+i\beta m}\mathcal{B}_i(\theta) \tag{2.36}$$

From (2.35) we derive the following three equations

$$\begin{aligned} &\csc\theta\sec\theta\mathcal{B}_\theta + \mathcal{E}_\theta \equiv y(\theta) \\ &\csc\theta\mathcal{B}_\alpha + \sec\theta\mathcal{E}_\alpha = -i\sin\theta\, y(\theta) \\ &\sec\theta\mathcal{B}_\beta + \csc\theta\mathcal{E}_\beta = i\cos\theta\, y(\theta) \end{aligned} \tag{2.37}$$

where $y(\theta)$ is arbitrary. The other equations are the Maxwell equation (or equivalently the Bianchi identities and the equations of motion). The equations of motion $\mathcal{D}_\mu F^{\mu\nu} = 0$ are

$$\begin{aligned} -i\left(m\sec^2\theta\mathcal{E}_\alpha + n\csc^2\theta\mathcal{E}_\beta\right) + E_\theta - 2\cot 2\theta\mathcal{E}_\theta - \mathcal{E}'_\theta = 0 \\ in\csc^2\theta\mathcal{B}_\alpha - im\sec^2\theta\mathcal{B}_\beta + E\,\mathcal{E}_\theta = 0 \\ -in\csc^2\theta f^2\mathcal{B}_\theta + \tan\theta\mathcal{B}_\beta + E\,\mathcal{E}_\alpha - \mathcal{B}_\beta + \mathcal{B}'_\beta = 0 \\ -im\sec^2\theta\mathcal{B}_\theta + \cot\theta\mathcal{B}_\alpha + E\,\mathcal{E}_\beta + \mathcal{B}_\alpha\mathcal{B}'_\alpha = 0 \end{aligned} \tag{2.38}$$

and the Bianchi identities $\partial_{[\mu}F_{\nu\rho]} = 0$ are

$$
\begin{aligned}
im\mathcal{B}_\alpha + in\mathcal{B}_\beta + \mathcal{B}'_\theta &= 0 \\
E\mathcal{B}_\beta + im\mathcal{E}_\theta - \mathcal{E}'_\alpha &= 0 \\
E\mathcal{B}_\alpha + in\mathcal{E}_\theta - \mathcal{E}'_\beta &= 0 \\
E\mathcal{B}_\theta + in\mathcal{E}_\alpha - im\mathcal{E}_\beta &= 0
\end{aligned} \tag{2.39}
$$

We then have eleven equations for seven variables (the energy and the non zero components of the electromagnetic fields). Even if the system looks overdetermined these equations are linearly dependent. By expressing every function in terms of $y(\theta)$ and E we obtain

$$
\left(\partial_\theta - (m-1)\cot\theta + (n-1)\tan\theta\right) y(\theta) = 0 \quad , \quad E = -m - n \tag{2.40}
$$

The solution is square integrable for $m, n \geq 1$. In this case the contribution comes from the antiparticle in the mode expansion and the index is

$$
I_B^{(V)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} t^{m+n} y^{n-m} = \frac{t^2}{(1-ty)(1-t/y)} \tag{2.41}
$$

If we consider a non abelian gauge group we must add an extra chemical potential for the gauge symmetry. Indeed since the index is a topological invariant the gauge coupling does not play any role and we only need to take care of the fact that the vector multiplet transforms in the adjoint representation. The gauge invariant combinations are given by the Plethystic exponential after integrating over the Haar measure [10, 42].

2.2.4 Squashed spheres

The superconformal index on the squashed sphere is expected to coincide with the one computed in the round limit, up to a redefinition of the variables. Indeed this manifold preserves the topological properties of S^3 and this guarantees that the index does not change under squashing.

This can be shown with a simple argument based on the definition of the index. Indeed the index on the three sphere is defined as

$$\begin{aligned}\mathcal{I}_{S^3\times S^1}(t,y) &= \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} t^H y^{2\tilde{J}_3} = \sum_{BPS} \mathrm{Tr}(-1)^F t^{R-2\tilde{J}_3} y^{2\tilde{J}_3} \\ &= \sum_{BPS} \mathrm{Tr}(-1)^F t^{R+J_\alpha+J_\beta} y^{J_\alpha-J_\beta} \end{aligned} \tag{2.42}$$

where $J_\alpha \equiv i\partial_\alpha$ and $J_\beta \equiv i\partial_\alpha$ are the generators of the two $U(1)$'s in the Hopf fibration. By defining $p = ty$ and $q = t/y$ the index becomes (this change of coordinates has been first considered in [66])

$$\mathcal{I}_{S^3\times S^1}(p,q) = \sum_{BPS} \mathrm{Tr}(-1)^F p^{R/2+J_\alpha} q^{R/2+J_\beta} \tag{2.43}$$

The same definition of the index on the squashed sphere is

$$\begin{aligned}\mathcal{I}_{S^3\times S^1}(t,y) &= \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} t^H y^{2\tilde{J}_3+\frac{R}{2}\left(\frac{1}{a}-\frac{1}{b}\right)} \\ &= \sum_{BPS} \mathrm{Tr}(-1)^F t^{\frac{R}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{J_\alpha}{a}+\frac{J_\beta}{b}} y^{\frac{J_\alpha}{a}-\frac{J_\beta}{b}+\frac{R}{2}\left(\frac{1}{a}-\frac{1}{b}\right)} \end{aligned} \tag{2.44}$$

By defining $p = (ty)^{\frac{1}{a}}$ and $q = (t/y)^{\frac{1}{b}}$ the index on the squashed sphere is defined as (2.43) and its definition coincides with the one for the round case as expected. Then the index is expected to coincide because the two spaces have the same topology, and the same BPS states contributing to the index in the round case contribute to the index in the squashed case.

In this section we explicitly show this result by exploiting the power of our prescription for the identification of the BPS states. Indeed there are no known result for expansion in terms of harmonics on these spaces and a direct calculation is not at hand.

We start by writing the fermion in the chiral multiplet as $\psi = \zeta F$ and solve the equation $\Delta_\psi (\zeta F) = 0$, where we expand F as $F = e^{Et+in\alpha+im\beta} g_\psi(\theta)$, obtaining the following set

$$\begin{cases} g'_\psi = -\frac{e^{-i\theta} g_\psi}{2ab} \left(2abe^{i\theta}(iEf - (q-2)\cot 2\theta) + f\left(\frac{(a(q-2(m+1))}{\sin\theta} + \frac{ib(2(n+1)-q)}{\cos\theta}\right)\right) \\ g'_\psi = \frac{g_\psi}{2ab}\left(\frac{2ab(q-2)}{\tan 2\theta} + f\left(\frac{ae^{i\theta}(2(m+1)-q)}{\sin\theta} + \frac{ib(2aE+(2(n+1)-q)(1+i\tan\theta))}{\cos\theta}\right)\right) \end{cases}$$

These two equations can be simultaneously solved if

$$E = \frac{q}{2}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{a} - \frac{1}{b} - \frac{n}{a} - \frac{m}{b} \tag{2.45}$$

Square integrability requires the quantum numbers $m, n \geq 0$ as in the case of the sphere. The mode contributing to the index is an *antiparticle* and its energy is $E_\psi = -E$. By summing over the BPS states we have

$$I_\psi = -\frac{(t/y)^{\frac{2-q}{2b}}(ty)^{\frac{2-q}{2a}}}{\left(1-(t/y)^{\frac{1}{b}}\right)\left(1-(ty)^{\frac{1}{a}}\right)} \tag{2.46}$$

The equations for the scalar $\phi = e^{Et+i\alpha m+i\beta m} g_\phi(\theta)$ become

$$\begin{cases} g'_\phi = -\frac{g_\phi}{2ab}\left(\frac{2abq}{\tan 2\theta} + f\left(2iabE + e^{i\theta}\left(\frac{a(2m-q)}{\sin\theta} + \frac{ib(2n-q)}{\cos\theta}\right)\right)\right) \\ g'_\phi = -\frac{g_\phi e^{-i\theta}}{2ab}\left(\frac{2abqe^{i\theta}}{\tan 2\theta} + f\left(\frac{a(2m-q)}{\sin\theta} - ib\left(2ae^{i\theta}E + \frac{(2n-q)}{\cos\theta}\right)\right)\right) \end{cases} \tag{2.47}$$

They can be simultaneously solved if

$$E = \frac{q}{2}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{n}{a} - \frac{m}{b} \tag{2.48}$$

with $m, n \leq 0$. This constraint fixes $E_\phi = E$ and the index for the scalar field in the chiral multiplet is

$$I_\phi = \frac{(t/y)^{\frac{q}{2b}}(ty)^{\frac{q}{2a}}}{\left(1-(t/y)^{\frac{1}{b}}\right)\left(1-(ty)^{\frac{1}{a}}\right)} \tag{2.49}$$

Note that the two single particle indices that we have found only depend on the two parameters $(ty)^{1/a}$ and $(t/y)^{1/b}$. Thus, the following redefinition of the fugacities

$$t \to t^{\frac{a+b}{2}} y^{\frac{a-b}{2}} \quad , \quad y \to t^{\frac{a-b}{2}} y^{\frac{a+b}{2}} \tag{2.50}$$

gives $\mathcal{I}_{\text{sphere}} = \mathcal{I}_{\text{squash}}$. The transformation (2.50) does not modify the physical content of the index, because the fugacities are, a priori, arbitrary parameters. The only constraints come from the requirement of convergence of the index, and are given by $ty < 1$ and $t/y > 1$ [142]. Of course, the latter are preserved by equation (2.50) for positive a and b.

On the squashed sphere, the gaugino equation (2.33) gives

$$\begin{aligned} \lambda_1(\theta) &= \frac{e^{\frac{i\theta}{2}}}{\sqrt{2}} h(\theta)\left(\frac{m}{b\sin\theta} + i\frac{n}{a\cos\theta}\right) \quad , \quad \lambda_2(\theta) = -\frac{e^{-\frac{i\theta}{2}}}{\sqrt{2}} h(\theta)\left(i\frac{m}{b\sin\theta} + \frac{n}{a\cos\theta}\right) \\ E &= -\frac{n}{a} - \frac{m}{b} \quad , \quad h'(\theta) = f(\theta)h(\theta)\left(\frac{n}{a}\tan\theta - \frac{m}{b}\cot\theta\right) \end{aligned}$$

The solution for λ is square integrable if $m, n \leq 0$, but we exclude the mode $(m,n) = (0,0)$ because it is identically vanishing. The sum over the gaugino states gives

$$I_\lambda = -\left(\sum_{m=0}^{-\infty}\sum_{n=0}^{-\infty} t^{-\frac{n}{a}-\frac{m}{b}} y^{\frac{n}{a}-\frac{m}{b}} - 1\right) = -\frac{(t/y)^{\frac{1}{b}}(ty)^{\frac{1}{a}} - (t/y)^{\frac{1}{b}} - (ty)^{\frac{1}{a}}}{\left(1-(t/y)^{\frac{1}{b}}\right)\left(1-(ty)^{\frac{1}{a}}\right)} \tag{2.51}$$

For the gauge bosons the equations (2.35) become

$$\begin{aligned} \frac{\csc\theta\sec\theta\mathcal{B}_\theta}{ab} + \frac{\mathcal{E}_\theta}{f} &\equiv y(\theta) \\ \frac{\csc\theta\mathcal{B}_\alpha}{bf} + \frac{\sec\theta\mathcal{E}_\alpha}{a} &= -i\sin\theta\, y(\theta) \\ \frac{\sec\theta\mathcal{B}_\beta}{af} + \frac{\csc\theta\mathcal{E}_\beta}{b} &= i\cos\theta\, y(\theta) \end{aligned} \tag{2.52}$$

After applying the equations of motions

$$\begin{aligned} -if^2\left(\frac{m\sec^2\theta\mathcal{E}_\alpha}{a^2}+\frac{n\csc^2\theta\mathcal{E}_\beta}{b^2}\right)+\frac{\mathcal{E}_\theta f'}{f}-2\cot 2\theta\mathcal{E}_\theta-\mathcal{E}'_\theta=0 \\ \frac{in\csc^2\theta\mathcal{B}_\alpha}{b^2}-\frac{im\sec^2\theta\mathcal{B}_\beta}{a^2}+E\mathcal{E}_\theta=0 \\ -\frac{in\csc^2\theta f^2\mathcal{B}_\theta}{b^2}+\tan\theta\mathcal{B}_\beta+Ef^2\mathcal{E}_\alpha-\frac{\mathcal{B}_\beta f'}{f}+\mathcal{B}'_\beta=0 \\ -\frac{imf^2\sec^2\theta\mathcal{B}_\theta}{a^2}+\cot\theta\mathcal{B}_\alpha+Ef^2\mathcal{E}_\beta+\frac{\mathcal{B}_\alpha f'}{f}-\mathcal{B}'_\alpha=0 \end{aligned} \tag{2.53}$$

and the Bianchi identities

$$\begin{aligned} im\mathcal{B}_\alpha+in\mathcal{B}_\beta+\mathcal{B}'_\theta=0 \\ E\mathcal{B}_\beta+im\mathcal{E}_\theta-\mathcal{E}'_\alpha=0 \\ E\mathcal{B}_\alpha+in\mathcal{E}_\theta-\mathcal{E}'_\beta=0 \\ E\mathcal{B}_\theta+in\mathcal{E}_\alpha-im\mathcal{E}_\beta=0 \end{aligned} \tag{2.54}$$

we find

$$\left(\frac{\partial_\theta}{f(\theta)}-\left(\frac{m}{b}-\frac{1}{f(\theta}\right)\cot\theta+\left(\frac{n}{a}-\frac{1}{f(\theta)}\right)\tan\theta\right)y(\theta)=0 \quad , \quad E=-\frac{n}{a}-\frac{m}{b} \tag{2.55}$$

and the index is

$$I_B^{(V)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}t^{\frac{n}{a}+\frac{m}{b}}y^{\frac{n}{a}-\frac{m}{b}}=\frac{(t/y)^{\frac{1}{b}}(ty)^{\frac{1}{a}}}{\left(1-(t/y)^{\frac{1}{b}}\right)\left(1-(ty)^{\frac{1}{a}}\right)} \tag{2.56}$$

2.3 Example: reducing $4d$ indices to $3d$ partition functions

In this section we revisit the reduction of the four dimensional superconformal index to the three dimensional partition function [135, 130, 105]. We will show that the reduction follows very easily, and the same argument can be generalized to other dimensions. The example of the round sphere can be found in [67, 87, 122].

For concreteness, we consider the index on $S_b^3 \times S^1$ and show that it reduces to the three-dimensional partition function $\mathcal{Z}_{S_b^3}$ by dimensional reduction. In four dimensions we consider the multi-particle index for a chiral and a vector multiplet, that takes into account all the multi-trace gauge invariant combinations. The multi-particle index can be found by taking the Plethystic exponential of the single particle index (2.1)

$$\mathcal{I}_{m.p.} = \text{Exp}\left[\sum_{k=1}^{\infty} \frac{\mathcal{I}_{s.p.}(t^k, y^k, f^k, g^k)}{k}\right] \tag{2.57}$$

Comparing to equation (2.1), we have added two more parameters to the single particle index: the fugacity f for the internal flavor symmetries and the one g for the gauge symmetry. In the rest of this section we consider only the $f, y \to 1$ limit.

We start by looking at the contribution of the chiral multiplet. As we already pointed out the fields contributing to the index on $S_b^3 \times S^1$ are the particle ϕ for the bosonic component and the antiparticle $\psi^\dagger$ for the fermionic component. If one component is in the ρ representation of the gauge group, than the other component is in the $\bar{\rho}$. By recalling the single particle result

$$\mathcal{I}_\Phi = \mathcal{I}_\phi + \mathcal{I}_\psi = \sum_{BPS} \left(t^{E_\phi} g^\rho - t^{E_\psi} g^{-\rho}\right) \tag{2.58}$$

the multi-trace contribution to the superconformal index from a chiral multiplet in the ρ

representation of the gauge group is

$$\text{Exp}\left[\sum_{k=1}^{\infty}\frac{1}{k}\sum_{BPS}\left(t^{kE_\phi+ik\sigma\rho}-t^{kE_\psi-ik\sigma\rho}\right)\right] \tag{2.59}$$

that becomes

$$\prod_{BPS}\frac{1-t^{E_\phi+i\sigma\rho}}{1-t^{E_\psi-i\sigma\rho}} \xrightarrow{t\to 1} \prod_{BPS}\frac{E_\phi+i\sigma\rho}{E_\psi-i\sigma\rho} \tag{2.60}$$

where we identified the chemical potential for the gauge group g with $t^{i\sigma}$, where σ is the solution to the three-dimensional saddle point equations (or to the four-dimensional zero energy supersymmetry equations), which set σ to a constant [135].[5]

The product in (2.60) ranges over the set of BPS states. As we have seen, this set is labeled by the Cartan subgroup of the three-dimensional isometry group, which in the case at hand consists of the two $U(1)$ symmetries $U(1)_\alpha$ and $U(1)_\beta$ that rotate the Hopf angles independently. If we identify the fugacity t with $e^{-\tau}$, where τ is the period of the time direction, then the limit in (2.60) corresponds to shrinking the time circle to zero size, i.e. to dimensional reduction. Indeed the right hand side of (2.60) is the one loop exact contribution of the chiral multiplet to the three-dimensional partition function found in [106]. The energies E_ϕ and E_ψ of the BPS states in four dimensions, obtained with the procedure explained in section 2.1, become the eigenvalues of the unpaired states in the three dimensional case. An analogous derivation can be performed for the vector multiplet.

We expect that our correspondence and the reduction are more general than shown

[5]The reason for setting $g = t^{i\sigma}$, in our language, is the following. Till now, we solved the BPS equations in a vanishing gauge background, because we know that the gauge representation can be associated to another chemical potential in the index (also see footnote 4). However, we could have solved the BPS equations in the $\sigma \neq 0$ background and obtain that the energies are $E' = E + i\sigma\rho$. A comparison of the two methods shows that g goes as $t^{i\sigma}$ when the time circle shrinks.

here and that they apply generically to $\mathcal{M}_{d-1} \times S^1 \to \mathcal{M}_{d-1}$ [6], provided at least two real supersymmetries are preserved. The result (2.60) should apply to any $(d-1)$-dimensional theory, if its field content may be derived by dimensional reduction of a corresponding d-dimensional model. The $(d-1)$-dimensional saddle points and the quantum corrections may be derived by the d-dimensional analysis, but in the full partition function there may be an additional contribution, denoted S_* in (2.2), due to a classical term which does not have an uplift to d-dimensions. This is the case, for instance, for the Chern-Simons term in the three dimensional case. However, once the $(d-1)$-dimensional action is known, one can plug the saddle point configuration in it and obtain also the classical term.

It is interesting to compare with the known results in the literature. To the best of our knowledge, this is the first time that the superconformal index on a squashed sphere is computed explicitly. Of course, because it is identical to the one on the round sphere up to a redefinition of the fugacities, one can consider reducing the latter to the partition function on the squashed sphere. This is usually done by taking an ad hoc limit instead of the one in (2.60) [67]. Namely, we can reinterpret those results by stating that one can squash the chemical potentials without affecting the physical meaning of the index, and then take the natural limit $t \to 1$ to shrink the time circle. The necessary redefinitions are not known in general, and we believe that our results offer a very clean physical interpretation and can be easily generalized.

2.4 The conjecture in other dimensions and manifolds

From the discussion in section 2.3 we see that our results can be more general than stated until now. We propose that the same one-to-one map described there holds in more general cases, like in other dimensions, manifolds and for extended supersymmetric

[6] It would be interesting to study the same correspondence between the states on $\mathcal{M}_{d-n} \times T^n$ and the ones on $\mathcal{M}_{d-n} \times T^{n-1}$.

theories.

Localization on a three-sphere does not give rise to any non-perturbative (instanton or monopole) contribution, and this is in full agreement with the BPS correspondence we have proposed. However the localizing term in different dimensions can lead to a sum over the instantons as happens, for instance, on the four-sphere. If our argument can be applied also in that case, the five-dimensional BPS equations should contain all the quantum information also about the non-perturbative states.

In section 2.1 we have mostly focused on a three-dimensional manifold whose Cartan subgroup is $U(1)^2$, and thus there are two well-defined quantum numbers, one can break the Cartan to $U(1)$ and still preserve two real supersymmetries. In this case one has only one integer quantum number to sum over, and the BPS conditions will give constraints on its range.

2.5 General partition functions via an uplift to an index

We have observed above that the reduction of the superconformal index on $\mathcal{M}_{d-1} \times S^1$ to the partition function on $\mathcal{M}_{d-1}$ highlights the relation between the BPS states in d dimensions and the $d-1$ dimensional unpaired states. Equivalently one can obtain the three dimensional partition function on $\mathcal{M}_{d-1}$ by uplifting the supersymmetry from $\mathcal{M}_{d-1}$ to $\mathcal{M}_{d-1} \times S^1$. The d-dimensional Killing spinors are independent from the S^1 and the d dimensional unpaired states are preserved by shrinking the circle. Even if this procedure is similar to the reduction explained in section 2.1 it is interesting to investigate the problem in this way because it shows the relation of our construction and localization. Indeed the $d-1$-dimensional saddle point equations coincide with the zero energy equations of the d-dimensional problem. We now exploit this fact to simplify the computation of the exact partition function itself.

Consider a d-dimensional field theory $\mathcal{F}_d$ and its dimensional reduction to $\mathcal{F}_{d-1}$, which preserves the same amount of supersymmetry.[7] We can place $\mathcal{F}_{d-1}$ on a curved manifold $\mathcal{M}_{d-1}$ and localize the corresponding path integral to an at most finite dimensional integral by picking two real conserved supercharges and solving the corresponding equation $|\delta\psi_{d-1}|^2 = 0$, where ψ is any fermion of the theory. This is the same as picking the uplifted supercharges on $\mathcal{M}_d$ and solving for

$$|\delta\Psi_d|^2\Big|_{E=0} = 0 \tag{2.61}$$

where $\Psi_d \sim \Psi_{d-1}e^{Et}$ is the set of fermions in the $\mathcal{F}_d$ theory, and gives the loci that solve the saddle point equations in the path integral. Denote the latter by Φ_* and the classical action $S(\Phi_*) \equiv S_*$. The exact path integral on $\mathcal{M}_{d-1}$ is now given by

$$Z_{\mathcal{M}_{d-1}} \sim \int [d\Phi]\ e^{-S_*}\frac{\mathrm{Pf}D_F}{\sqrt{\det D_B}} \tag{2.62}$$

where $[d\Phi]$ is the measure over the loci Φ_*, and in general D_F and D_B are respectively a first order and second order differential operator derived by a $(d-1)$-dimensional Q-exact action. Notice that we did not compute any Q-exact action, so we do not know the explicit form of D_F and D_B, but we know that Φ_* are their zero modes. In general, we should find the spectrum of their eigenvalues around the solutions of (2.61), and it turns out that many of them simplify between the numerator and the denominator in (2.62) due to supersymmetry. The ones that do not simplify are obtained with the procedure explained in section 2.1.

To summarize we can derive the spectrum of eigenvalues necessary to compute the exact partition function in $d-1$ dimensions (2.62) by finding the energy eigenvalues from a corresponding set of first order differential operators in d dimensions. We do not

[7] Actually, the action for $\mathcal{F}_{d-1}$ may contain terms without an uplift to d dimensions. As we already stressed our results also hold in those cases.

need the Lagrangian giving the equations of motion for $\mathcal{F}_d$, but only the supersymmetry transformations of the matter multiplets that appear there. This means that we only need the uplift of the conserved supercharges, without worrying about the uplift of the Lagrangian.

This chapter is a reprint of the material as it appears in “BPS states and their reductions ”, Prarit Agarwal, Antonio Amariti, Alberto Mariotti, Massimo Siani, JHEP 1308 (2013) 011, of which I was a co-author.

Chapter 3

A Zig-Zag Index

3.1 Introduction

The superconformal index (SCI) of four dimensional superconformal field theories [168, 142] is the supersymmetric partition function of the theory defined on the euclidean space $S^3 \times S^1$. Alternatively, it can be defined as a weighted (over the fermion number) sum of the states of the theory, where the contribution of the long multiplets vanishes. The index counts the short BPS multiplets and it is invariant under marginal deformations of the theory. It has been extensively studied in the recent years, especially to check field theory dualities and the AdS/CFT correspondence [168, 142, 169, 66, 178, 179, 84, 70].

There are many prescriptions for obtaining the functional form of the index [168, 142, 174, 175, 169, 160, 2]. In the large N limit, the computation of the index simplifies and in some cases it can be carried over with matrix model techniques.

In this paper, we focus on a large class of superconformal gauge theories, namely the quiver gauge theories arising as the world volume of D3 branes probing a toric CY_3 singularity. It has been shown that the large N index for such theories can be computed, matches with the dual description, and that it usually factorizes on a specific subset of operators, the so called extremal BPS mesons, corresponding to the edges of the dual cone of the toric fan.

This factorization was first observed in [84] for the SCI of of the Y^{pq} families [43] of quiver gauge theories. By fixing the value of the superconformal R-charge imposed by a-maximization the authors computed the index in the Y^{p0} and Y^{pp} theories and guessed a general behavior for the Y^{pq} case. A proof for the conjecture was later provided in [70], where the authors explained the factorization of the index from the properties of the toric geometry, for the case of smooth CY_3's.

In this paper we show that the factorization property of the SCI for toric quiver gauge theories is more general. First, we observe that the index factorizes without fixing the exact superconformal R-charge, but just by requiring that the NSVZ beta functions vanish and the superpotential is marginal [1]. Second, we show that the factorization holds also in gauge theories dual to geometries with additional singularities.

For this purpose, we reformulate the factorization of the SCI on extremal BPS mesons as a factorization of the SCI over a set of paths in the brane tiling. These paths are called zig-zag paths because they turn maximally left (right) at the black (white) nodes of the bipartite tiling. We conjecture a general factorized formula for the SCI in terms of the zig-zag paths, as a function of a trial R-charge. This expression continues to be well defined in the case of quiver gauge theories dual to geometries with orbifold singularities.

We check the validity of our formula and the factorization of the SCI index over the zig-zag paths in various examples, including infinite families of orbifold singularities. Moreover, we verify the invariance of our formula under Seiberg duality. As a byproduct, the factorization over the zig-zag path allows us to express the SCI directly in terms of the CY geometry and the toric data.

The paper is organized as follows. In section 3.2 we review the relevant aspects of

[1] With a slight abuse of notation we keep on referring to this supersymmetric partition function on $S^3 \times S^1$ as the superconformal index also in this case.

D3 branes at toric CY_3 singularities and of the large N calculation of the superconformal index. In section 3.3 we explain the factorization of the index over the extremal BPS mesons as discovered in [70]. In 3.3.2 we give the prescription to relate the R-charges of the extremal BPS mesons to the ones of the zig-zag paths and we re-formulate the factorization in terms of these paths. In section 3.4 we study the factorization over the zig-zag paths, in some simple examples, for general values of the trial R-charges that satisfy the constraints imposed by marginality. In section 3.5 we prove the factorization in the infinite families of L^{aba} non-chiral singularities. In section 3.6 we show that our formula is preserved by Seiberg duality. In section 3.7 we show the role of the global, non anomalous and non R-symmetries in the factorization. In section 3.8 we translate the index from the zig-zag paths to their geometric counterpart. We conclude in 3.9 with some open problems. In appendix B.1 we compare the zig-zag factorization with the one discovered in [84] for the whole Y^{pq} family

3.2 Review: SCI and toric quivers

3.2.1 D3 branes on toric CY_3

In this section we review some aspects of the world-volume theory describing D3 branes probing a toric CY_3 singularity, that will be useful for the rest of the paper (see [140] and references therein for a comprehensive review).

We start by the definition of a quiver gauge theory. A quiver is a graph made of vertices with directed edges connecting them. The vertices represent the $SU(N)$ gauge groups and the edges represent bifundamental or adjoint matter fields. The direction of the arrow of an edge is associated to the representation of the corresponding matter field under the gauge groups.

Since we study SCFTs there are two classes of constraints imposed by superconformality, both associated to the vanishing of the beta functions.

The first constraint comes from requiring the vanishing of the NSVZ beta function for each gauge group. This corresponds to the requirement of the existence of a non anomalous R-symmetry in the SCFT and hence becomes a constraint on the R-charges. At the k-th node of the quiver we have

$$\sum_{i=1}^{n_k}(r_i - 1) + 2 = 0 \tag{3.1}$$

where the sum is over all the n_k bifundamentals charged under the k-th gauge group. The second constraint comes from imposing the marginality of the superpotential terms.

These two constraints restrict the possible R-charge assignments of the superconformal field theory to a subset named R_{trial}. The extra freedom is fixed through a-maximization [128], that gives eventually the exact R-charge. In the following we refer to the case where R is exact as the *on-shell* case, while the case obtained by just imposing the marginality constraints is referred as the *off-shell* case.

Note that in general the superpotential cannot be read from the quiver, but in the case of toric CY it is possible thanks to the notion of planar quiver. Toric quiver gauge theories have the property that each field appears linearly in the superpotential and in precisely two terms with opposite signs. It can be shown that we can exploit this structure of superpotential terms to transmute the quiver into a planar quiver embedded in T^2. The planar quiver is thus a periodic quiver built from the original one by separating all the possible multiple arrows connecting the nodes such that corresponding to each superpotential term there is a plaquette whose boundaries are given by the arrows, the bifundamental fields appearing in that superpotential term. Plaquettes representing superpotential terms with a common bifundamental are glued together along the corresponding edge. The sign of a superpotential term corresponds to orientation of its plaquette.

Moreover, it is possible to define a set of paths on the planar quiver called zig-zag paths. They are loops on the torus defining the planar quiver. These loops are composed by the arrows. These arrows are chosen such that if a path turns mostly left at one node it turns mostly right at the next one. This notion is not illuminating on the quiver but it becomes more important in the description of the moduli space on the dual graph, called the bipartite tiling or the dimer model.

The dimer model is built from the planar quiver by reversing the role of the faces and of the vertices. The superpotential terms become the vertices of the tiling, and the orientation is absorbed in the color (black or white), i.e. the tiling is bipartite. The edges are mapped to dual edges, and the orientation is lost (all the information is in the vertices). The faces represent the gauge groups.

The zig-zag paths are oriented closed loops on the tiling with non trivial homology along the T^2. Every node of the tiling is surrounded by a closed loop made out of the zig-zag paths, and the orientation of the loops determines the color of the vertices, consistently with the bipartite structure of the tiling.

On the bipartite tiling there are sets of edges, called perfect matchings, that connect black and white nodes, such that every node is covered by exactly one edge. As already mentioned, the tiling is defined on the torus, that possesses two winding cycles γ_ω and γ_z. An intersection number with the homology classes $(1,0)$ and $(0,1)$ of the two winding cycles is associated to each perfect maching.

A monomial in $z^m \omega^n$ is associated to each perfect matching, where m and n represent the intersection number of the perfect matching with the cycles γ_ω and γ_z. A polynomial that counts the perfect matchings in the brane tiling is obtained by summing over these monomials

The convex hull of the exponents of this polynomial is a polyhedral on $\mathbf{Z}^2$, the toric diagram. This rational polyhedral encodes the informations of the moduli space of the D3 probing the toric CY_3.

3.2.2 Large N index in toric quivers

The superconformal index for a four dimensional $\mathcal{N} = 1$ field theory is defined as

$$I = Tr(-1)^F e^{-\beta\Xi} t^{R-2J_3} y^{2\tilde{J}_3} \prod \mu_i^{q_i} \tag{3.2}$$

where $\Xi = \{Q_1, Q_1^\dagger\}$ represents the superconformal algebra on $S^3 \times S^1$. The index gets contributions only from the states with $\Xi = 0$ and hence it is independent from β.

The chemical potentials t, y and μ are associated to the abelian symmetries of the theory that commute with Q_1 and their charges are the exponents, R is the R-symmetry, J_3 and $\tilde{J}_3$ are the Cartan of the $SU(2)_L \times SU(2)_R \in SO(4,2)$ and q_i are the charges of the flavor symmetries. The single particle index receives contributions from both the chiral and the vector multiplet. In the first case we have

$$I_{s.p}(\phi) = \frac{t^{r_\phi}}{(1-ty)(1-t/y)} \quad , \quad I_{s.p}(\psi^\dagger) = -\frac{t^{2-r_\phi}}{(1-ty)(1-t/y)} \tag{3.3}$$

where both ϕ and ψ belong to the chiral multiplet Φ. The contribution of the vector multiplet is

$$I_{s.p.}(\mathbf{V}) = \frac{2t^2 - t(1+1/y)}{(1-ty)(1-t/y)} \tag{3.4}$$

In the case of quiver gauge theories there are only two possible representations, bifundamental and adjoint. A bifundamental superfield X_{ij} contains a scalar in the fundamental for the i-th group and in the antifundamental for the j-th group. The fermion $\psi^\dagger$ is in the opposite representation.

The single particle index $I(t, y, \chi)$ associated to the quiver is the sum of the contributions of the vector multiples and the bifundamental multiplets in the quiver. At each node i there is a contribution $I_{s.p.}(V_i)\chi_i^{adj}$, where χ_i^{adj} is the character of the adjoint representation of the i-th gauge group. For every bifundamental Φ_{ij} there is a contribution

$$I_{i,j}(t,y,\chi) = I_{s.p}(\phi_{ij})\chi_i\bar{\chi_j} + I_{s.p.}(\psi^{\dagger}_{ji})\bar{\chi}_i\chi_j \tag{3.5}$$

where the χ_i and $\bar{\chi}_i$ are the characters of the fundamental and antifundamental representation associated to the $SU(N_i)$-th gauge group. If the matter field is a bifundamental the product $\chi_i\bar{\chi_j}$ in (3.5) must be substituted with χ_i^{adj}.

The single trace index is obtained by taking the plethystic exponential [42]. In order to single out contributions from gauge-invariant states, we also need to integrate over the gauge measure. In formulae

$$I_{m.t.}(x) = \int \prod_{i=1}^{G}[d\alpha_i]PE[I(t,y,\chi(\alpha_i))] \tag{3.6}$$

where the α_i are the Cartan of the i-th gauge group. By taking the large N limit this becomes a Gaussian integral and the index is

$$I_{m.t.}(t,y) = \prod_k \frac{e^{\frac{1}{k}\mathrm{Tr}\, i(t^k,y^k)}}{\det(1-i(t^k,y^k))} \tag{3.7}$$

where

$$1-i(t,y) = \frac{1-m(t)+t^2m^T(t^{-1})-t^2}{(1-ty)(1-t/y)} \equiv \frac{M(t)}{(1-ty)(1-t/y)} \tag{3.8}$$

The matrix $m(t)$ represents the adjacency matrix weighted by the R-charge. For every edge e, connecting the i-th node to the j-th one in the quiver, the matrix picks up a contribution $t^{R(e)}$ such that $m_{ij}(t) = \sum_{e:i\to j} t^{R(e)}$. The index can be further simplified and it becomes

$$I_{s.t.}(t,y) = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \det M(t^k) - \mathrm{Tr}\left(\frac{m(t) - t^2 m(t^{-1})}{(1-ty)(1-t/y)} \right) \tag{3.9}$$

where φ is the Euler-phi function. Observe that the second term in (3.9) vanishes in absence of adjoint matter because $m(t)$ becomes traceless.

3.3 Factorization of the SCI

3.3.1 SCI over the extremal BPS mesons

The factorization of the index was first observed in [84] and then proven in [70] for toric CY_3 without additional singularities away from the tip of the cone.

Consider a toric CY_3 cone probed by a D3 brane. This cone is described by the fan $\mathcal{C}$, a convex polyhedral cone in R^3. The BPS mesons (their vev), up to F-term equivalences, are in 1-1 correspondence with the semigroup of integer points in $\mathcal{C}^*$, the dual cone of $\mathcal{C}$. The three integer numbers defining the points in the dual cone (and equivalently the BPS mesons) are the three $U(1)$ isometries of the CY_3 or equivalently the mesonic symmetries of the field theory $(U(1)_F^2 \times U(1)_R)$. The points in the dual cone can be divided in points on the edges, on the faces and on the internal of the cone itself.

After this geometrical digression we can now report the result of [70] on the factorization of the index. It states that the determinant $\det(M(t))$ factorizes over the extremal BPS mesons [45] that are described by the edges of the dual cone $\mathcal{C}^*$

$$\det(M(t)) = \prod_{i \in E_M} \left(1 - t^{r_i} \mu_1^{F_i} \mu_2^{\widetilde{F}_i} \right) \tag{3.10}$$

where E_M refers to the edges of the dual cone or equivalently to the extremal BPS mesons. The charges appearing in [84, 70] are the exact R-charge of the SCFT and the two $U(1)_F$.

There are some interesting questions following from the factorization. The first regards the exactness of the R-charge. One may wonder if the exact R-charge is a necessary condition for the factorization of the index, or if it possible to relax this assumption, just by imposing the marginality constraints (vanishing of the beta functions), corresponding to the *off-shell* R_{trial} case defined above.

A second question regards theories with extra singularities far from the tip of the cone. These theories are characterized by having extra points on the edges of the toric diagram. In the dual cone these points are not associated to any edge but they live on the faces. These theories have not been investigated in [70] and one may wonder how the factorization formula is modified in these cases.

3.3.2 Extremal BPS mesons and zig-zag paths

In this section we study the two problems discussed above by using the brane tiling instead of the dual cone. By starting from the observation that both the extremal BPS mesons and the zig-zag paths are in 1-1 correspondence with the primitive vectors of the toric diagram we give a prescription to extract the charges of the extremal BPS mesons from the charges of the zig-zag paths. This allows us to define a factorization formula for the SCI in terms of the zig-zag paths.

The BPS mesons, not necessarily extremal, are represented on the tiling as string of operators built by connecting a face with its image by a path. These paths have to cross the edges of the tiling by leaving the nodes of the same color on the same side. A BPS meson is the product of the edges crossed by such paths. Products of operators with the same homology and the same R-charge are F-term equivalent. There is a set of these BPS mesons that have maximal $U(1)$-charge (up to a sign) for a given R-charge. These are the extremal BPS mesons, corresponding to the edges of the dual cone [45].

They can be built (up to degenerations) from the zig-zag paths. First we associate

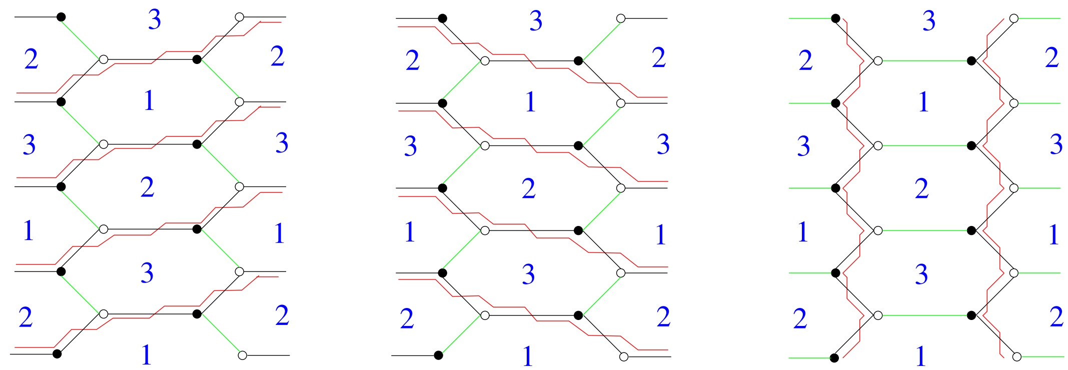


Figure 3.1. Zig-zag paths and extremal BPS mesons for $\mathbb{C}^3/\mathbb{Z}_3$.

an orientation to every zig-zag path such that they leave a black node on the right and a white node on the left. For every black n-valent node [2] the i-th zig-zag path crosses two edges. The i-th extremal BPS meson is obtained by associating the other $n-2$ edges at every black node crossed by the i-th zig-zag path.

For example in the figure 3.1 we highlight in red the three zig-zag paths of $\mathbb{C}^3/\mathbb{Z}_3$ and in green the three extremal BPS mesons. From this definition we obtain a general formula relating the R-charges of the extremal BPS mesons and the R-charges of the zig-zag paths.

At each n-valent black node the condition of marginality of the superpotential implies that

$$\sum_{j=1}^{n} r_j = 2 \tag{3.11}$$

where r_j are the charges of the fields related to the edges connected with the black node that we are considering.

Let us suppose that the first two ($j = 1, 2$) are in the zig-zag paths and the others in the extremal BPS meson. By using the previous relation we have that

$$r_3 + \cdots + r_n = 2 - r_1 - r_2 = (1 - r_1) + (1 - r_2) \tag{3.12}$$

[2]The same correspondence can be obtained by using the white nodes crossed by the i-th zig-zag path

and we have expressed the R-charges of the fields forming the extremal BPS meson in terms of the R-charges of the edges belonging to the zig-zag path.

By summing over all the black nodes crossed by the zig-zag path we obtain the R-charge of the extremal BPS meson associated to the i-th zig-zag path (denoted with Z_i)

$$R_{BPS_i} = \sum_{k \in \{Z_i\}} (1 - r_k^{(i)}) \tag{3.13}$$

where k runs over the set of edges $\{Z_i\}$ belonging to the i-th zig-zag path, and $r_k^{(i)}$ is the R-charge of the k-th field in the i-th zig-zag path.

By using the relation between the R-charges of the extremal BPS mesons and of the zig-zag paths the determinant $\det M(t)$ factorizes over the zig-zag paths as [3]

$$\det M = \prod_{i=1}^{Z} (1 - t^{\sum_{j \in \{Z_i\}} (1 - r_j^{(i)})}) \tag{3.14}$$

where Z is the number of zig-zag paths, and $\{Z_i\}$ and $r_j^{(i)}$ are defined as above.

We conjecture (3.14) to be valid also *off-shell* and in the singular cases. In the rest of the paper we study the validity of this formula with many examples and checks.

3.4 Examples

In this section we study the two simplest examples of quiver gauge theories described by a bipartite graph and associated to a toric CY_3 singularity. They are the $\mathcal{N} = 4$ SYM and the conifold.

In both cases we explicitly show how the Gaussian integral obtained in the large N limit factorizes over the zig-zag paths *off-shell*.

[3] In the following we set $\mu_1 = \mu_2 = 1$, at the end of the paper we will show how to insert these symmetries back in the index.

3.4.1 N=4

We start by considering the $\mathcal{N}=4$ SYM. We study this theory as an $\mathcal{N}=1$ theory. In $\mathcal{N}=1$ notations there is an $SU(N)$ gauge group and three adjoint fields, that we call X_1, X_2 and X_3. The interaction is $W = X_1[X_2, X_3]$ which imposes $r_{X_1}+r_{X_2}+r_{X_3}=2$. The three zig-zag paths correspond to the three products of fields

$$zz_1 = X_1 X_2 \quad , \quad zz_2 = X_2 X_3 \qquad zz_3 = X_3 X_1 \tag{3.15}$$

In this theory the determinant at large N (3.10) is given by

$$\det(M(t)) = 1 - t^2 + \sum_{i=1}^{3} t^{r_i} + \sum_{i=1}^{3} t^{2-r_i} \tag{3.16}$$

We now show that this determinant factorize in a product over the zig-zag path as claimed in (3.14), by manipulating each term in expression (3.16).

The term t^2 generically corresponds to t^{2n_G}, where n_G is the number of gauge groups in the quiver, and it can be re-written from the relation in the dimer as

$$n_{Faces} + n_{Points} - n_{Edges} = 0 \to 2n_{fields} - 2n_W = 2n_G \tag{3.17}$$

By imposing the superpotential constraint we have

$$\sum_{i=1}^{Z} \sum_{j\in\{Z_i\}} (1 - r_j^{(i)}) = 2n_G \tag{3.18}$$

In this case we have $t^2 \to t^{6-2(r_1+r_2+r_3)}$.

The term $\sum t^{r_i}$ can be re-written by using the constraints from the superpotential and it becomes $\sum_{i<j} t^{2-r_i-r_j}$. In the same way the last term becomes

$$2 - r_i = r_j + r_k = (2 - r_i - r_k) + (2 - r_j - r_i) \tag{3.19}$$

By putting everything together the final formula is

$$\det(M(t)) = (1 - t^{2-r_1-r_2})(1 - t^{2-r_1-r_3})(1 - t^{2-r_2-r_3}) \tag{3.20}$$

which corresponds to the expression (3.14), factorized over the three zig-zag paths.

3.4.2 Conifold

The second example is the worldvolume theory of a stack of N D3 branes probing the conifold. This is represented by a quiver gauge theory with two gauge groups $SU(N)_1 \times SU(N)_2$ and two pairs of bifundamental-antibifundamental (a_i, b_i) connecting them. The superpotential is $W = \epsilon_{ij}\epsilon_{lk} a_i b_l a_j b_k$ that imposes

$$r_{a_1} + r_{a_2} + r_{b_1} + r_{b_2} = 2 \tag{3.21}$$

At large N the determinant of $M(t)$ is

$$1 - \sum_{i,j} t^{r_{a_i}+r_{b_j}} + 2t^2 + \sum_{i \neq j} \left(t^{2-a_i+a_j} + t^{2-b_i+b_j} \right) - \sum_{i,j} t^{4-a_i-b_j} + t^4 \tag{3.22}$$

we can reorganize the sum as a sum over the zig-zag paths as follows. There are four zig-zag paths parameterized by

$$zz_1 = a_1 b_1 \quad , \quad zz_2 = a_2 b_1 \quad , \quad zz_3 = a_1 b_2 \quad , \quad zz_3 = a_2 b_2 \tag{3.23}$$

We keep fixed the first term in the sum (3.22). The second one becomes

$$\sum_{i,j} t^{r_{a_i}+r_{b_j}} \to \sum_{i,j} t^{2-r_{a_j}-r_{b_i}} = \sum_{i=1}^{Z} t^{\sum_{j\in\{Z_i\}}(1-r_j^{(i)})} \tag{3.24}$$

the third and the fourth terms can be written together and thanks to the relation (3.21) we have

$$2t^2 + \sum_{i\neq j} \left(t^{2-a_i+a_j} + t^{2-b_i+b_j} \right) \to \sum_{i=1}^{Z} \sum_{j=i+1}^{Z} t^{\sum_{k\in\{Z_i\}}(1-r_k^{(i)})+\sum_{l\in\{Z_j\}}(1-r_l^{(j)})} \tag{3.25}$$

Also in the fifth term of (3.22) we can insert the relation (3.21) and obtain

$$\sum_{i,j} t^{4-a_i-b_j} \to \sum_{i=1}^{Z} \sum_{j=i+1}^{Z} \sum_{k=j+1}^{Z} t^{\sum_{l\in\{Z_i\}}(1-r_l^{(i)})+\sum_{m\in\{Z_j\}}(1-r_m^{(j)})+\sum_{n\in\{Z_k\}}(1-r_n^{(k)})} \tag{3.26}$$

The last term is obtained as already explained in the $\mathcal{N}=4$ case. Finally, by collecting all the terms, we have

$$\det(M(t)) = (1-t^{2-r_{a_1}-r_{b_1}})(1-t^{2-r_{a_1}-r_{b_2}})(1-t^{2-r_{a_2}-r_{b_1}})(1-t^{2-r_{a_2}-r_{b_2}}) \tag{3.27}$$

3.5 The singular cases

The second result that we argue in this paper is that the determinant of the matrix $M(t)$ arising in the large N calculation of the superconformal index (see formula (3.9)) factorizes over the zig-zag paths also in the case where new singularities arise far from the tip of the CY cone.

For example in the L^{pqr} families [45, 78, 50] there are many examples corresponding to orbifolds. Inside these classes of orbifolds there are two infinite families, L^{aaa} and L^{aba},

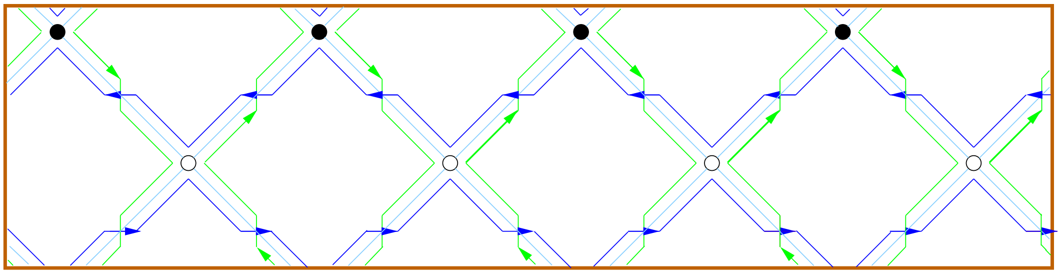


Figure 3.2. Tiling and zig-zag paths for a generic L^{aaa} model. We grouped the zig-zag paths with homology $(\pm 1, 0)$ with the green color while the blue ones have homology $(0, \pm 1)$. We distinguished the sign by specifing the orientation with.

associated to non-chiral theories that can be studied in a unified way. In this section we show that $\det(M(t))$ factorizes in both these cases over the zig-zag paths. Moreover we study a non chiral case, L^{264} corresponding to the $L^{a,b,\frac{b-a}{2}}$ singular family, and observe the factorization.

3.5.1 The L^{aaa} family

In this section we compute the large N index for an infinite class of theories, the L^{aaa} theories. These theories are vector like theories with a bifundamental and an antibifundamental connecting the i-th node and the $i+1$-th one. We start by studying the phase without any adjoint matter field. Subsequently we show that the factorization of the index over the zig-zag paths is maintained even in phases that contain the adjoint fields.

By looking at the tiling there are four classes of zig-zag paths. The first two classes have homology $(1, 0)$ and $(-1, 0)$ respectively and contain $2a$ fields. By imposing the constraints imposed by the marginality we have two possible charge assignations, as in figure 3.3. The two zig-zag paths both contribute to the index with a factor $(1-t^a)$. There are also other a zig-zag paths with homology $(0, 1)$ and a with homology $(0, -1)$. The first class contains only fields with charge r, and every zig-zag of this kind contributes

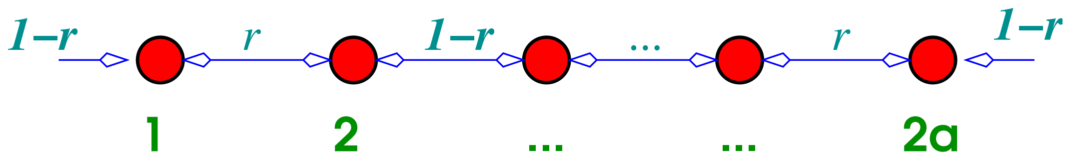


Figure 3.3. Trial R-charge asignation for a generic L^{aaa} model.

with a factor $(1 - t^{2-2r})$. In the second case the charge is $1 - r$ and the contribution is $(1 - t^{2r})$. The final contribution to the index is

$$\det M(t) = (1 - t^a)^2 (1 - t^{2-2r})^a (1 - t^{2r})^a \tag{3.28}$$

We now give a proof of our claimed factorization. We start by writing the matrix

$$M(t) = \begin{pmatrix} a_1 & b_1 & 0 & \dots & \dots & c_a \\ b_1 & a_2 & c_1 & \dots & \dots & 0 \\ 0 & c_1 & 0 & & \dots & 0 \\ \dots & \dots & \dots & \dots & c_{a-1} & 0 \\ \dots & \dots & \dots & c_{a-1} & a_{2a-1} & b_a \\ c_a & 0 & 0 & 0 & b_a & a_{2a} \end{pmatrix} \tag{3.29}$$

where

$$a_i = (1 - t^2) \quad , \quad b_{2i} = c_{2i} = (t^{r+1} - t^{1-r}) \quad , \quad b_{2i+1} = c_{2i+1} = (t^{2-r} - t^r) \tag{3.30}$$

Since (3.29) is a circulant matrix the determinant can be easily computed. Actually here we use a more complicated technique, more useful for the L^{aba} case. The determinant of

(3.29) can be written in an equivalent way by the formula

$$\det M(t) = Tr \prod_{i=1}^{2a} L_i - 2\prod_{i=1}^{a} b_i c_i \; , \; L_{2j} = \begin{pmatrix} a_j & -b_{j-1}^2 \\ 1 & 0 \end{pmatrix} , \; L_{2j+1} = \begin{pmatrix} a_j & -c_{j-1}^2 \\ 1 & 0 \end{pmatrix} \tag{3.31}$$

The trace is easily computed by defining $F = L_i L_{i+1}$ and by observing that

$$Tr F^a = Tr \prod_{i=1}^{2a} L_i \tag{3.32}$$

The trace is computed from the eigenvalues of F. We have

$$Tr F^a = Tr \begin{pmatrix} \lambda_1^a & 0 \\ 0 & \lambda_2^a \end{pmatrix} = (1 + t^{2a})(1 - t^{2-2r})^a (1 - t^{2r})^a \tag{3.33}$$

By adding the extra contribution

$$\prod_{i=1}^{a} b_i \, c_i = t^a (1 - t^{2-2r})^a (1 - t^{2r})^a \tag{3.34}$$

the expected factorization is obtained.

It is interesting to observe the behavior of the index under Seiberg duality. As we will show later the factorization of the determinant is not affected by the duality. Here the problem is that a duality on the n-th node adds two extra adjoints on the $n \pm 1$-th nodes. But as we already observed in section 3.2 the extra adjoints must be subtracted in the computation of the index.

While the $\mathcal{N} = 1$ vector multiplet usually cancels the y dependence of the index, the presence of the extra adjoints fields reintroduces this and in principle one may expect that the index does not match among different phases. However, this extra contribution

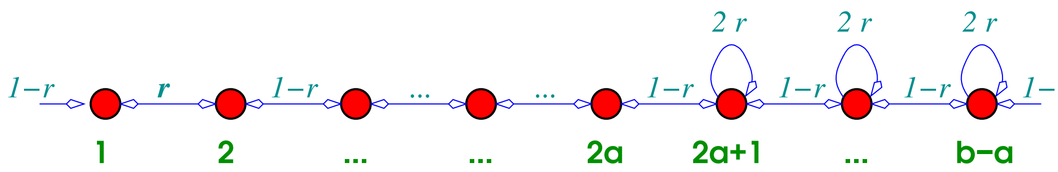


Figure 3.4. Quiver and R-charge parameterization for the L^{aba} theories.

is

$$\frac{1}{(1-ty)(1-t/y)}\left(t^{2r}-t^{2(1-r)}+t^{2(1-r)}-t^{2-2(1-r)}\right) \tag{3.35}$$

and it vanishes in the dual phase.

3.5.2 The L^{aba} family

In this section we generalize the case of the L^{aaa} theories studied above to the whole L^{aba} family. In this case the contributions from the extra adjoint matter fields has to be subtracted, and the index is y dependent. Nevertheless the determinant of the matrix M still factorizes over the zig-zag paths. By parameterizing the fields as in figure 3.4there are four classes of zig-zag paths:

- a paths formed by the pairs of fields $X_{i,i+1}$ and $X_{i+1,i}$ with charge r. They contribute to the index as $(1-t^{2(1-r)})^a$.
- b paths formed by the pairs of fields $X_{i,i+1}$ and $X_{i+1,i}$ with charge $1-r$. They contribute to the index as $(1-t^{2r})^b$.
- One path formed by all the adjoints and all the fields $X_{i,i+1}$. It contributes to the index as $(1-t^{ar+b(1-r)})$.
- One path formed by all the adjoints and all the fields $X_{i+1,i}$. It contributes to the index as $(1-t^{ar+b(1-r)})$.

With the parameterization of the charges in figure 3.4 the matrix M is

$$M = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_b \\ b_1 & a_2 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & \dots & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & \dots & b_a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_a & a_{2a-1} & c_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_a & d_1 & c_{a+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{a+1} & d_2 & c_{a+2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{a+2} & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & c_{b--1} \\ c_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{b-1} & d_{b-a} \end{pmatrix} \tag{3.36}$$

where

$$a_i = 1 - t^2 \qquad b_i = t^{2-r} - t^r$$
$$c_i = t^{r-1} - t^{1-r} \qquad d_i = 1 - t^2 - t^{2r} + t^{2(1-r)} \tag{3.37}$$

As before the determinant of this matrix can be obtained by defining the two dimensional L matrices (3.31). The determinant becomes

$$\det M = \mathrm{Tr} \prod_{i=1}^{a+b} L_i + 2\,(-1)^{b+1} \prod_{i=1}^{a} b_i \prod_{j=1}^{b} c_j \tag{3.38}$$

The first trace can be evaluated by redefining the matrices $L_i L_{i+1} = K$ for $i = 1, \dots, 2a-1$

and $L_j = J$ for $j = 2a+1\ldots,a+b$. The trace becomes TrK^aJ^{b-a} where

$$
\begin{aligned}
K^a =& \frac{\left(\left(1-t^{2(1-r)}\right)\left(1-t^{2r}\right)\right)^{a-1}}{t^{2r}} \times \\
& \begin{pmatrix} \left(1-t^{2r}\right)\left(t^{2r}-t^{2(a+1)}\right) & -t^2\left(1-t^{2a}\right)\left(1-t^{2r}\right)^2 \\ \left(1-t^{2a}\right)t^{2r} & \left(1-t^{2r}\right)\left(t^{2(a+r)}-t^2\right) \end{pmatrix}
\end{aligned}
\tag{3.39}
$$

and

$$
\begin{aligned}
J^{b-a} =& \frac{\left(1-t^{2r}\right)^{-a+b-1}}{t^{2r}-t^2} \times \\
& \begin{pmatrix} \left(t^{2r}-t^{4r}\right)\left(1-t^{2(1-r)(-a+b+1)}\right) & \left(1-t^{2r}\right)^2\left(t^{2(1-r)(b-a)+2}-t^2\right) \\ t^{2r}\left(1-t^{2(1-r)(b-a)}\right) & \left(t^{2r}-t^2\right)\left(1-t^{2(1-r)(-a+b-1)}\right) \end{pmatrix}
\end{aligned}
\tag{3.40}
$$

After plugging (3.39) and (3.40) in (3.38) we have

$$
\det M(t) = \left(1-t^{2-2r}\right)^a\left(1-t^{2r}\right)^b\left(1-t^{ar+b(1-r)}\right)^2 \tag{3.41}
$$

that coincides with the formula computed from the zig-zag paths.

3.5.3 A chiral orbifold

We conclude the analysis of the singular cases by studying a chiral orbifold of L^{pqr}. This model belongs to an infinite class of chiral orbifolds, $L^{a,b,\frac{a+b}{2}}$. We study a single case here, the L^{264} theory, that is an orbifold of L^{132}. We show that $\det(M(t))$ factorizes over the zig-zag paths with an *off shell* R_{trial}. The tiling and the toric diagram are represented in (3.5) The matrix $M(t)$ is

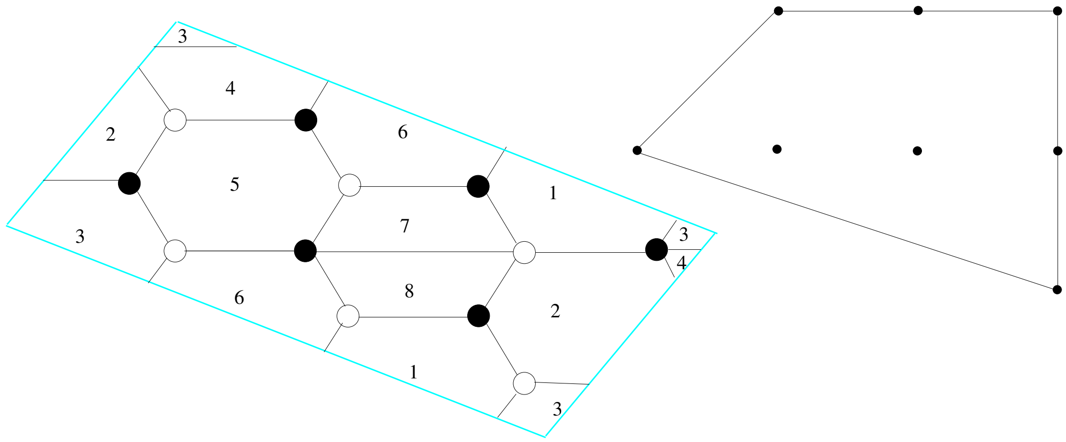


Figure 3.5. Tiling and toric diagram of L^{264}

$$M(t) =$$

$$\begin{pmatrix}
1-t^2 & -t^{r^X_{1,2}}-t^{r^Y_{1,2}} & t^{2-r^X_{3,1}} & 0 & 0 & -t^{r^X_{1,6}} & t^{2-r^X_{7,1}} & t^{2-r^X_{8,1}} \\
t^{2-r^X_{1,2}}+t^{2-r^Y_{1,2}} & 1-t^2 & -t^{r^X_{2,3}} & -t^{r^X_{2,4}} & t^{2-r^X_{5,2}} & 0 & 0 & -t^{r^X_{2,8}} \\
-t^{r^X_{3,1}} & t^{2-r^X_{2,3}} & 1-t^2 & t^{2-r^X_{4,3}} & -t^{r^X_{3,5}} & 0 & 0 & 0 \\
0 & t^{2-r^X_{2,4}} & -t^{r^X_{4,3}} & 1-t^2 & -t^{r^X_{4,5}} & t^{2-r^X_{6,4}} & 0 & 0 \\
0 & -t^{r^X_{5,2}} & t^{2-r^X_{3,5}} & t^{2-r^X_{4,5}} & 1-t^2 & -t^{r^X_{5,6}}-t^{r^Y_{5,6}} & t^{2-r^X_{7,5}} & 0 \\
t^{2-r^X_{1,6}} & 0 & 0 & -t^{r^X_{6,4}} & t^{2-r^X_{5,6}}+t^{2-r^Y_{5,6}} & 1-t^2 & -t^{r^X_{6,7}} & -t^{r^X_{6,8}} \\
-t^{r^X_{7,1}} & 0 & 0 & 0 & -t^{r^X_{7,5}} & t^{2-r^X_{6,7}} & 1-t^2 & t^{2-r^X_{8,7}} \\
-t^{r^X_{8,1}} & t^{2-r^X_{2,8}} & 0 & 0 & 0 & t^{2-r^X_{6,8}} & -t^{r^X_{8,7}} & 1-t^2
\end{pmatrix} \tag{3.42}$$

The zig-zag paths are as

$$zz_1 = X_{1,2}\, X_{2,4}\, X_{4,5}\, X_{5,6}\, X_{6,7}\, X_{7,1}$$

$$zz_2 = X_{1,2}\, X_{1,6}\, X_{2,3}\, X_{2,8}\, X_{3,1}\, X_{4,5}\, X_{5,2}\, X_{6,4}\, X_{6,7}\, X_{7,5}\, X_{8,1}\, Y_{5,6}$$

$$zz_3 = X_{2,8}\, X_{3,1}\, X_{4,3}\, X_{5,6}\, X_{6,4}\, X_{7,5}\, X_{8,7}\, Y_{1,2}$$

$$zz_4 = X_{2,4}\, X_{3,5}\, X_{4,3}\, X_{5,2}$$

$$zz_5 = X_{1,6}\, X_{6,8}\, X_{7,1}\, X_{8,7}$$

$$zz_6 = X_{2,3}\, X_{3,5}\, X_{6,8}\, X_{8,1}\, Y_{1,2}\, Y_{5,6} \tag{3.43}$$

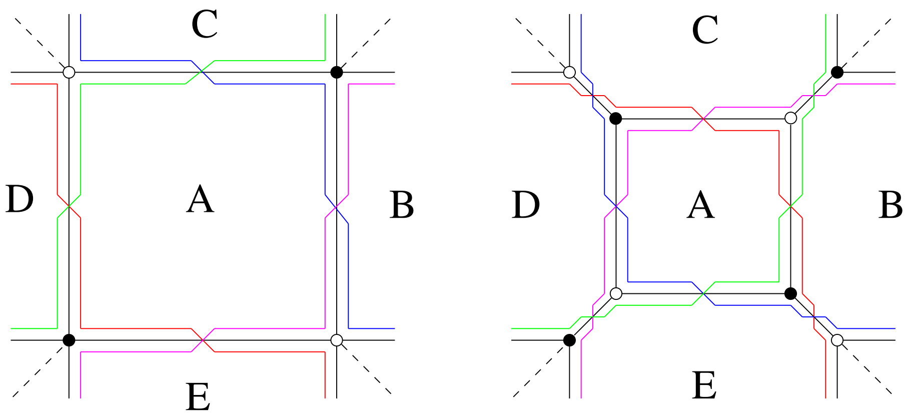


Figure 3.6. Seiberg Duality on the zig-zag paths.

After imposing the NSVZ and the W constraints we have

$$\det M(t) = \prod_{i=1}^{6} (1 - t^{\sum_{j \in \{Z_i\}} (1 - r_j^{(i)})}) \tag{3.44}$$

3.6 Seiberg duality

In this section we study the invariance of the formula (3.14) under Seiberg duality. The duality on the dimer and on the zig-zag paths is shown in figure 3.6. The zig-zag paths involved in the duality are the four represented in the picture, the red (R), green (G), blue (B) and magenta (M). In the *electric* case the zig-zag paths that are involved in the duality are

$$\begin{aligned} zz_R &= X_{AE} \quad X_{DA} \quad \widetilde{zz}_R \\ zz_G &= X_{DA} \quad X_{AC} \quad \widetilde{zz}_G \\ zz_B &= X_{AC} \quad X_{BA} \quad \widetilde{zz}_B \\ zz_M &= X_{BA} \quad X_{AE} \quad \widetilde{zz}_M \end{aligned} \tag{3.45}$$

where $\widetilde{zz}_i$ is the part of the zig-zag part that does not transform under the duality. In the magnetic theory we have

$$\begin{aligned} zz'_R &= Y_{DC} \quad Y_{CA} \quad Y_{AB} \quad Y_{BE} \quad \widetilde{zz}_R \\ zz'_G &= Y_{DE} \quad Y_{EA} \quad Y_{AB} \quad Y_{BC} \quad \widetilde{zz}_G \\ zz'_B &= Y_{BE} \quad Y_{EA} \quad Y_{AD} \quad Y_{DC} \quad \widetilde{zz}_B \\ zz'_M &= Y_{BC} \quad Y_{CA} \quad Y_{AD} \quad Y_{DE} \quad \widetilde{zz}_M \end{aligned} \tag{3.46}$$

The electric and magnetic R-charges are related by

$$\begin{aligned} r^Y_{CA} = 1 - r^X_{AC}, &\qquad r^Y_{DC} = r^X_{DA} + r^X_{AC} \\ r^Y_{AB} = 1 - r^X_{BA}, &\qquad r^Y_{BE} = r^X_{BA} + r^X_{AE} \\ r^Y_{EA} = 1 - r^X_{AE}, &\qquad r^Y_{BC} = r^X_{BA} + r^X_{AC} \\ r^Y_{AD} = 1 - r^X_{DA}, &\qquad r^Y_{DE} = r^X_{DA} + r^X_{AE} \end{aligned} \tag{3.47}$$

It is know easy to check that index calculated in the electric phase coincide with the one of the magnetic phase thanks to (3.47).

3.7 Global symmetries

In this section we show that the chemical potentials of the global symmetries preserve the factorization of the *off-shell* index over the zig-zag paths. There are two kind of global symmetries, baryonic and flavor symmetries. The first class of symmetries may be visualized as a sub set of the $U(1)$ symmetries inside the $U(N)$ at each node. The non anomalous baryonic symmetries are obtained from the trace anomaly $\mathrm{Tr}SU(N)_i^2 U(1)_{B_j}$. This can be visualized with the signed adjacency matrix. The kernel of this operator defines the combinations of baryonic symmetries that decouple in the IR or become anomalous. The zig-zag paths are uncharged under these symmetries, because they are gauge invariant paths, or equivalently they are closed on the quiver. This is consistent with the expectation that the baryonic symmetries do not contribute to the index. On the other hand the flavor symmetries are associated to the homologies of the paths in the tiling and they are expected to contribute. By assuming the factorization of the index over the zig-zag paths

$$\det M(t) = \prod_{i=1}^{Z}(1 - t^{\sum_{j\in\{Z_i\}}(1-r_j^{(i)})}) \tag{3.48}$$

we now prove that

$$\det M(t) = \prod_{z=1}^{Z} 1 - t^{\sum_{j\in\{Z_i\}}(1-r_j^{(i)})}\mu_1^{-\sum_{j\in\{Z_i\}} F_j^{(i)}}\mu_2^{-\sum_{j\in\{Z_i\}} \widetilde{F}_j^{(i)}}) \tag{3.49}$$

The index is a polynomial with three types of contributions t^2 and t^{r_i} and t^{2-r_i}, where r_i is the R-charge of the i-th scalar in the chiral multiplet. Every term in the polynomial is generically a set of disjoint closed loops in the quiver, a gauge invariant string of bosonic and fermionic fields. After adding the flavor symmetries the three possible contributions change as

$$\begin{aligned} t^2 &\to t^2 \\ t^{r_i} &\to t^{r_i}\mu_1^{F_i}\mu_2^{\widetilde{F}_i} \\ t^{2-r_i} &\to t^{2-r_i}\mu_1^{-F_i}\mu_2^{-\widetilde{F}_i} \end{aligned} \tag{3.50}$$

By using the constraints from $NSVZ$ and the superpotential we can convert the charge associated to a fermion ψ_{ij} in the charge associated to a product of bosons $\prod \phi_\alpha$, where $\alpha \in I$ is a set of pairs of labels that parameterizes the fields involved in this relation, we have

$$t^{2-r_{ij}}\mu_1^{-F_{ij}}\mu_2^{-\widetilde{F}_{ij}} = \prod_{\alpha\in I} t^{r_\alpha}\mu_1^{F_\alpha}\mu_2^{\widetilde{F}_\alpha} \tag{3.51}$$

We can also convert the terms in the diagonal entries of $M(t)$, proportional to t^2 in t^{r_W} or t^{r_F}, where the exponent is the sum of the charges of fields in a generic superpotential term or in a face in the tiling. Putting everything together we observe that before considering the flavor symmetries the index is a polynomial in $P(t^{r_i})$ where r_i represents the charge in the i-th scalar, while after we add these symmetries the index is a polynomial in the form $P\left(t^{r_i}\mu_1^{F_i}\mu_2^{\widetilde{F}_i}\right)$. This shows that the mesonic flavor symmetries preserve the factorization.

3.8 Geometric formulation

In this section we translate our formula of the index factorized over the zig-zag in terms of toric geometry. As a standard procedure a set of variables a_i is assigned to every external point of the toric diagram as in [51] [4]. They are constrained by $\sum a_i = 2$, which in the geometry represents the superpotential constraint $R(W) = 2$. A variable b_i can be assigned to the primitive normals, that are $1-1$ with the zig-zag paths, as

$$b_i = \sum_{j=1}^{i} a_i \tag{3.52}$$

such that $b_d = 2$ where d is the number of external point of the diagram. We give a pictorial representation of the toric diagram and the dual primitive vectors for dP_1 in figure 3.7. On the tiling πb_i is the angle of intersection of the zig-zag paths with the rombhi edges in the isoradial embedding [110]. Every edges (fields) is crossed by two

[4] In this case we restrict to the case without points on the edges.

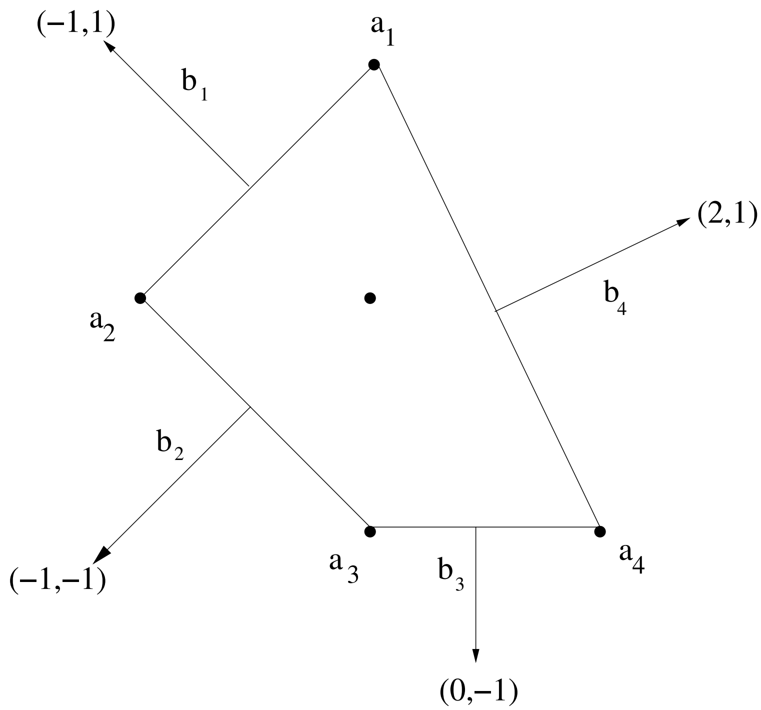


Figure 3.7. Toric diagram of dP_1, primitive normals and charges.

zig-zag paths and their R charges are defined as

$$\begin{cases} R_{ij} = b_i - b_j & i < j \\ R_{ij} = 2 - b_i + b_j & i > j \end{cases} \tag{3.53}$$

If more fields are crossed by the same pair of paths they have the same charge. Once we obtained the formula for the R-charges in terms of the geometry we can guess a formula that expresses the index in terms of the b_i variables.

A geometric formula that reproduces the field theory index is

$$\det M_{geom} = \prod_{i=1}^{d} (1 - t^{\sum_j |\omega_{ij}|(1-R_{ij})}) \tag{3.54}$$

This formula holds in the minimal phase, where the number of intersections between two

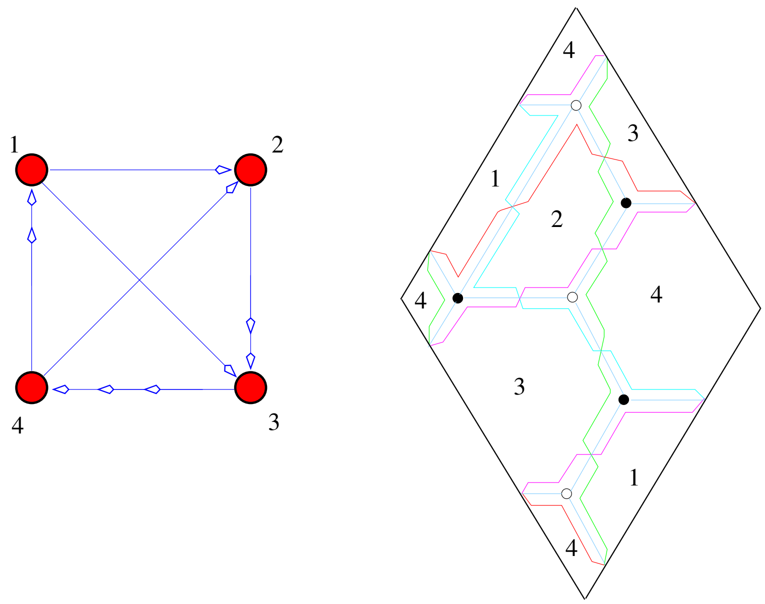


Figure 3.8. Quiver, Tiling zig-zag paths and toric diagram of dP_1

zig-zag paths is fixed by

$$\omega_{ij} = \langle \omega_i, \omega_j \rangle = \det \begin{pmatrix} p_i & q_i \\ p_j & q_j \end{pmatrix} \tag{3.55}$$

where $\omega_i = (p_i, q_i)$ are the primitive normal vectors of the toric diagram. After Seiberg duality one can end up with non-minimal cases, where the number of intersections is just bounded from below by $\langle \omega_i, \omega_j \rangle$. In that case the formula is still valid because the extra intersections always come in pairs with an opposite orientation and they cancel in (3.55) [108].

3.8.1 dP_1

As an example we study the dP_1 model. The quiver the tiling and the toric diagram are shown in figure 3.8. First we write the index in terms of the zig-zag paths, and than we use the geometric formula and show that the two formulas agree. The superpotential is

$$W = \epsilon_{\alpha\beta} X_{23}^{(\alpha)} X_{34}^{(\beta)} X_{42} - \epsilon_{\alpha\beta} X_{12} X_{23}^{(\alpha)} X_{34}^{(3)} X_{41}^{(\beta)} + \epsilon_{\alpha\beta} X_{34}^{(\alpha)} X_{41}^{(\beta)} X_{13} \tag{3.56}$$

The four perfect matchings related to the external points of the toric diagram are

$$\begin{aligned} v_1 &= (0,1) \quad \to X_{13} X_{24} X_{34}^{(3)} \\ v_2 &= (-1,0) \to X_{23}^{(1)} X_{34}^{(1)} X_{41}^{(1)} \\ v_3 &= (0,-1) \to X_{12} X_{34}^{(1)} X_{34}^{(2)} \\ v_4 &= (1,-1) \to X_{34}^{(2)} X_{23}^{(2)} X_{41}^{(2)} \end{aligned} \tag{3.57}$$

$$\begin{aligned} zz_1 &= X_{13} X_{34}^{(1)} X_{42} X_{23}^{(1)} X_{34}^{(3)} X_{41}^{(1)} & \qquad zz_2 &= X_{41}^{(1)} X_{12} X_{23}^{(1)} X_{34}^{(2)} \\ zz_3 &= X_{41}^{(2)} X_{12} X_{23}^{(2)} X_{34}^{(1)} & \qquad zz_4 &= X_{34}^{(2)} X_{13} X_{41}^{(2)} X_{34}^{(3)} X_{23}^{(2)} X_{42} \end{aligned}$$

The index is computed from the matrix

$$M(t)=\begin{pmatrix} 1-t^2 & -t^r x_{12} & -t^r x_{13} & t^{2-r} x_{41}^{(1)} + t^{2-r} x_{41}^{(2)} \\ t^{2-r} x_{12} & 1-t^2 & -t^r x_{23}^{(1)} - t^r x_{23}^{(2)} & t^{2-r} x_{42} \\ t^{2-r} x_{13} & t^{2-r} x_{23}^{(1)} + t^{2-r} x_{23}^{(2)} & 1-t^2 & -t^r x_{34}^{(1)} - t^r x_{34}^{(2)} - t^r x_{34}^{(3)} \\ -t^r x_{41}^{(1)} - t^r x_{41}^{(2)} & -t^r x_{42} & t^{2-r} x_{34}^{(1)} + t^{2-r} x_{34}^{(2)} + t^{2-r} x_{34}^{(3)} & 1-t^2 \end{pmatrix} \tag{3.58}$$

The determinant of this matrix factorizes by imposing the marginality constraints and it is equivalent to

$$(1 - t^{4-r_{zz_1}})(1 - t^{6-r_{zz_2}})(1 - t^{4-r_{zz_3}})(1 - t^{6-r_{zz_4}}) \tag{3.59}$$

We now write the index from the geometric formula. The (p,q) web is parameterized by the four vectors

$$w_1 = (-1,1) \quad , \quad w_2 = (-1,-1) \quad , \quad w_3 = (0,-1) \quad , \quad w_4 = (2,1) \tag{3.60}$$

The R-charges of the fields intersecting on the zig-zag paths can be written in terms of b as

$$\begin{aligned} &R(1,2) = 2(b_2 - b_1), \quad R(1,3) = b_3 - b_1, \quad R(2,3) = b_3 - b_2 \\ &R(2,4) = b_4 - b_2, \quad R(3,4) = 2(b_4 - b_3), \quad R(4,1) = 3(b_1 - b_4 + 2) \end{aligned}$$

In terms of the b variables the determinant is given by (3.54). We have

$$\text{det}M(t) = \left(1 - t^{-3b_1+b_2+2b_3}\right)\left(1 - t^{b_1+b_2-2b_4+4}\right)\left(1 - t^{2b_1-b_3-b_4+4}\right)\left(1 - t^{-2b_2-b_3+3b_4}\right) \tag{3.61}$$

The b are related to the a variables as $b_i = \sum_{j=1}^{i} a_i$. By assigning the a_i variables to the external points we can calculate the R-charge of the fields in terms of the a_i. We have

X_{12}	$X_{23}^{(1)}$	$X_{23}^{(2)}$	$X_{34}^{(1)}$	$X_{34}^{(2)}$	$X_{34}^{(3)}$	$X_{41}^{(1)}$	$X_{41}^{(2)}$	X_{13}	X_{42}
a_3	a_2	a_4	$a_2 + a_3$	$a_3 + a_4$	a_1	a_2	a_4	a_1	a_1

(3.62)

The expression in (3.61) coincides with (3.59) after substituting in the latter (3.62).

3.9 Conclusions

In this paper we observed that the superconformal index factorizes over a set of gauge invariant paths on the dimer, called zig-zag paths.

We showed that this factorization remains valid also for theories with orbifold singularities, and without fixing the exact R-charge but on a generic set of R_{trial} satisfying the marginality constraints.

The zig-zag paths have an important role at geometrical level because they give a mirror dual interpretation of the tiling. Indeed, as observed in [72], the zig-zag paths are both (p, q) winding cycles in the dimer and boundaries of the faces in the tiling of the Riemann surface associated to a punctured region. This allows a dual description in IIA in terms of mirror D6 branes. Our formulation in terms of the zig-zag paths may be interesting for a *mirror* interpretation of the index.

A different duality, called specular duality, has been recently discovered in [109]. This duality exchanges the tiling with its mirror dual, written in terms of the zig-zag paths. Since the zig-zag paths have a crucial role in the factorization of the index, it would be interesting to analyze the relation among the indices in specular dual phases, as done here for the case of the usual Seiberg duality.

Another interesting development regards the relation with the orientifolded theories. Indeed it is known that the orientifold action on the tiling corresponds to a fixed line or fixed point projection [77]. These projections are naturally extended to the zig-zag paths. It would be nice to study the relation between the zig-zag index and the orientifold in the tiling and in the geometry.

A further line of investigation concerns the bipartite field theories recently defined in [75, 195, 118, 76]. Indeed, even if they are not usually conformal, the zig-zag paths are well defined on these theories. It would be interesting to understand if the formula we discussed in this paper has some field theoretical or geometrical interpretation in those cases.

Finally, as discussed in the text, the zig-zag path are in one to one correspondence with extremal BPS mesons. In [45] it has been shown that the extremal BPS mesons correspond to massless geodesics of semiclassical strings moving in the internal geometry.

It would be intriguing to investigate possible connections between this hamiltonian system and the factorization of the superconformal index.

This chapter is a reprint of the material as it appears in “A Zig-Zag Index ”, Prarit Agarwal, Antonio Amariti, Alberto Mariotti, arXiv:1304.6733, of which I was a co-author.

Chapter 4

Refined Checks and Exact Dualities in Three Dimensions

4.1 Introduction

Three dimensional dualities between supersymmetric field theories have been studied since a long time. Some of them are similar to the four dimensional case of Seiberg duality, like the Aharony duality [7] and the Giveon-Kutasov one [100].

More recently, new nonperturbative techniques have been used to gain more insights into aspects of three-dimensional field theories. In particular, the exact partition function of any $\mathcal{N} \geq 2$ superconformal field theory reduces to a matrix model for any value of the coupling constants [135, 130, 105], and gives information about physical quantities of the given model [180, 119, 151, 60, 131, 12, 16, 158, 18, 19, 97, 17, 68, 14] that can be compared with previous results [28, 30, 104, 29, 9, 96, 27, 132, 150, 113, 111, 186, 123, 79, 107, 47, 13, 81, 48, 64].

Moreover, one can also compare the partition functions of two field theories that are conjectured to describe dual phases of the same superconformal fixed point, thus providing a nontrivial check of the duality. Showing that both sides share the same partition function is non trivial . One can consider different limits. Seiberg-like dualities for theories with at least $\mathcal{N} = 3$ supersymmetry have been considered in [137, 136]. With lower supersymmetry, the partition function is considerably more complicated. In the large-N limit, one can use the saddle point approximation and successfully study infinite classes of theories which involve an arbitrary product gauge group [16, 103]. For finite values of the gauge group rank and Chern-Simons (CS) level one can exploit the following observation. Exact results can be also obtained by generalizing the three-dimensional space on which the theory is defined to a squashed three-sphere, which enjoys a $U(1)^2$ subgroup of the isometry group $SU(2)^2$ of S^3. The localized partition function on this space can be written in terms of hyperbolic functions [106]. A review of their properties is given in [187], and in appendix C.1, and they have revealed themselves very useful to give further evidence to a large class of dualities [190, 133, 40, 161].

In most of these cases a single gauge group has been considered, but in principle one can use the same approach to match exact results for physical quantities among dual phases of theories describing generic configurations of M2 branes.

In this paper we are interested in different classes of dualities. Some of these have been considered in the framework of the large-N approximation of the partition function in [16, 103]. However, this limit does not catch an important subtlety of the duality transformation. If one starts with a product of unitary gauge groups $\prod U(N)_{k_i}$ in the electric theory and performs a duality transformation on the group i the resulting dual gauge group contains a factor $U(N+|k_i|)$. At the leading order in a large-N expansion, this dependence upon the CS level k does not play any role. We drop the large-N limit and provide nontrivial evidence for this duality to hold at any value of N and the k's in section 4.4. We also consider other models, which can be derived as the low energy theories living on the worldvolume of intersecting D-branes and orientifold O-planes, their dual phases and match the finite-N partition function for them.

Another interesting set of dualities recently proposed in [129] and extended in [134] can be studied by computing the partition function on the squashed three-sphere.

In these cases we re-derive some of the known results by applying the exact calculations of [187] and we compare with known dualities.

In these cases one has to pay attention to infrared accidental symmetries. Indeed in some cases the exact computation shows that some theories look dual to free theories in which the scaling dimensions of the gauge invariant operators are not consistent with the free theory value. A proper modification of the extremization principle, to account for the mixing of accidental symmetries with the R-symmetry, is necessary for the calculation of the exact R-charge.

The paper is organized as follows. In section 4.2 we review the rules to write the all-loop partition function on a squashed three-sphere, and show how it can be written in terms of hyperbolic functions. We also list a few basic properties of the hyperbolic functions. In section 4.3 we review some of the classes of models we are interested in. We describe how they can be embedded in a type IIB setup, and how the duality transformations follow from this embedding. We consider theories with unitary, orthogonal and symplectic factors in the product gauge group. The dualities are proved for any value of the ranks and CS levels in section 4.4 through the matching of the partition functions on both sides. Models with free field theory duals will be considered in section 4.5, where we also raise the problem of accidental symmetries which we further describe in section 4.6. Open problems and hints for future work are discussed in section 4.7. We include some appendices which contain technical details.

4.2 The partition function on a squashed three sphere

Localization has allowed to reduce the partition function of any $\mathcal{N} = 2$ three dimensional supersymmetric theory on a three sphere S^3 [130, 105]. A further refinement [106] involves two different squashed spheres S^3_b: One of them preserves an $SU(2) \times U(1)$ isometry, but in this case the localization does not give any new result, the other one, which will be very useful in this paper, preserves an $U(1)^2$ isometry. The partition function on the latter squashed sphere S_b^3 for a CS matter theory with gauge group G is

$$\begin{aligned}\mathcal{Z}_{S_b^3} &= \int_{\mathbb{T}^{\mathrm{rk}(G)}} \prod_{i=1}^{\mathrm{rk}(G)} dx_i e^{i\pi k Tr_F x^2} \det{}_{\mathrm{adj}} \left(\sinh\left(\pi b \rho_\alpha(x)\right) \sinh\left(\pi b^{-1} \rho_\alpha(x)\right)\right) \\ &\times \prod_{\rho \in r} S_b\left(\frac{i}{2}(b+b^{-1})(1-\Delta_r) - \rho_r(x)\right) \end{aligned} \tag{4.1}$$

where Δ_r is the scaling dimension (which in three dimensions coincides with the R-charge) of a chiral matter field in the representation r, ρ_r are the weights of the representation r, and ρ_α are the roots of the gauge groups G. The various factors in the integrand in (4.1) correspond to the contribution from the CS term, the vector multiplet and the matter superfields (in the representation r) respectively. The function S_b is the double sine function defined as

$$S_b\left(\frac{i}{2}(b+\frac{1}{b})(1-\Delta_r) - \rho(x)\right) = \prod_{n_1,n_2\geq 0}^{\infty} \frac{n_1 b + n_2 \frac{1}{b} + \frac{b+\frac{1}{b}}{2} + i\rho(x) + \frac{b+\frac{1}{b}}{2}(1-\Delta_r)}{n_1 b + n_2 \frac{1}{b} + \frac{b+\frac{1}{b}}{2} - i\rho(x) - \frac{b+\frac{1}{b}}{2}(1-\Delta_r)} \tag{4.2}$$

The limit $b = 1$ corresponds to the round sphere considered in [130, 105]. In that case the double sine reduces to

$$S_1\left(i(1-\Delta_r) - \rho(x)\right) \equiv S_1(iz) = e^{l(z)} \tag{4.3}$$

where $l(z)$ is defined such that its derivative is $-\pi z \cot(\pi z)$.

The partition function on the squashed sphere is more complicated than the corresponding one on the round sphere. However, since the double sine function can be identified with the hyperbolic Gamma function [170], we can exploit the recent work by mathematicians which provide us with exact results for the integral involved in physical computations [187]. In the following we introduce the basic definitions relevant for this paper, and provide more technical details to appendix C.1.

4.2.1 Hyperbolic functions

We start by introducing the *periods* ω_1 and ω_2, that in this case are identified with

$$\omega_1 = ib \quad , \quad \omega_2 = ib^{-1} \quad , \quad \omega = \frac{\omega_1 + \omega_2}{2} \tag{4.4}$$

The double sine function in terms of ω_1, ω_2 and z becomes

$$S(-iz; -i\omega_1, -i\omega_2) = \prod_{n1,n2 \geq 0}^{\infty} \frac{(n_1+1)\omega_1 + (n_2+1)\omega_2 - z}{n_1\omega_1 + n_2\omega_2 + z} \tag{4.5}$$

This corresponds to the hyperbolic gamma function $\Gamma_h(z; \omega_1, \omega_2) \equiv \Gamma_h(z)$ first defined in [170]. This function satisfies the difference equations

$$\Gamma_h(z+\omega_1) = 2\sin\left(\frac{\pi z}{\omega_2}\right)\Gamma_h(z) \quad , \quad \Gamma_h(z+\omega_2) = 2\sin\left(\frac{\pi z}{\omega_1}\right)\Gamma_h(z) \tag{4.6}$$

and the reflection formula

$$\Gamma_h(z+\psi_1)\Gamma_h(\psi_2 - z) = 1 \qquad \text{if} \qquad \psi_1 + \psi_2 = 2\omega \tag{4.7}$$

Other useful identities are

$$\Gamma_h(\omega) = 1\,,\, \Gamma_h\left(\frac{\omega_1}{2}\right) = \Gamma_h\left(\frac{\omega_2}{2}\right) = \frac{1}{\sqrt{2}}\,,\, \Gamma_h\left(\omega + \frac{\omega_1}{2}\right) = \Gamma_h\left(\omega + \frac{\omega_2}{2}\right) = \sqrt{2} \tag{4.8}$$

and

$$\Gamma_h(2z) = \Gamma_h\left(z\right)\Gamma_h\left(z + \frac{\omega_1}{2}\right)\Gamma_h\left(z + \frac{\omega_2}{2}\right)\Gamma_h(z+\omega) \tag{4.9}$$

By combining (4.6) and (4.7) one has

$$\Gamma_h(\pm z) \equiv \Gamma_h(z)\Gamma_h(-z) = \frac{\Gamma_h(z+\omega_1)\Gamma_h(\omega_2 - z)}{4\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(-\frac{\pi z}{\omega_2}\right)} = -\frac{1}{4\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi z}{\omega_2}\right)} \tag{4.10}$$

which correspfonds to the one loop contribution of the vector multiplet in (4.1). The final expression for the partition function in terms of the hyperbolic gamma function is

$$\mathcal{Z}(\Delta_R, \omega_1, \omega_2) = \frac{1}{\sqrt{(-\omega_1\omega_2)^n}\mathcal{W}} \int \prod_{i=1}^{n} du_i e^{\frac{-i\pi k}{\omega_1\omega_2}x_i^2} \frac{\prod_{\rho_r \in R}\Gamma_h(\rho_r(x) + \omega\Delta_R)}{\prod_{\rho_\alpha \in \alpha^{(+)}}\Gamma_h(\pm\rho_\alpha(x_i))} \tag{4.11}$$

where $\mathcal{W}$ is the dimension of the Weyl subgroup and n is the rank of the gauge group. Many exact results concerning these integrals have been studied in [187]. To deal with the notations there we define the functions $c(x)$ and ζ as

$$c(x) \equiv \exp\left(\frac{i\pi x}{2\omega_1\omega_2}\right) \quad , \quad \zeta = e^{\frac{i\pi(\omega_1^2+\omega_2^2)}{24\omega_1\omega_2}} \tag{4.12}$$

in terms of which the CS contribution at level $k = -\frac{t}{2}$ is

$$\exp\left(\frac{i\pi t}{2\omega_1\omega_2}x_i^2\right) = c(tx_i^2) \tag{4.13}$$

Also notice that in the S^3 limit, $\omega_1 = \omega_2 = i$, we obtain $\log\left(\Gamma_h\left(z\right)\right) = l(1+iz)$ which is the one loop contribution of matter fields computed in [130].

4.3 Families of quiver gauge theories and M2 branes

In this section we survey the classes of models dual to M2 branes on Calabi-Yau fourfold that we will be interested in. These models have been deeply investigated in [132, 150, 113, 111, 186, 123, 79, 107, 13, 81, 64].

Each one can be understood in the framework of type IIB SUGRA compactified on a circle. The low energy brane dynamics is described by the worldvolume theory living in the $2+1$ infinite directions of some D3 brane suspended between pairs of $(1, p_i)$ fivebranes. The latter picture also provides us with a representation in terms of quiver diagrams, according to which we associate a node to each gauge group and an arrow to each matter field. We distinguish two types of arrows: one which connects two distinct nodes is associated to bifundamental matter fields, while one that has both its endpoints on the same node represents a chiral field in the adjoint representation.

In the three-dimensional case, in addition to the above information we also have to provide the CS levels. In the type IIB picture, they are given by the difference $(p_i - p_{i-1})$. From a purely field theoretical point of view, our only constraint will be that they add up to zero.

Finally, we will let the gauge group factors to be either the unitary, orthogonal or symplectic group (i.e. we also consider cases with O3 planes in the brane construction).

4.3.1 Unitary groups

We take type IIB string theory compactified on a circle, which we parametrize with the x_6 coordinate. The worldvolume theory of a stack of N D3 branes wrapped on the circle is described by a $U(N)$ gauge theory in three dimensions. If the D3's intersect g NS5 extended along the D3 worldvolume but not around the circle, the gauge group contains g $U(N)$ factors. The introduction of the CS terms is achieved by replacing the NS5 with a tilted bound state of NS5 and p_i D5, dubbed $(1, p_i)$ fivebrane. We refer to table 4.1 for the precise definition of the embedding. The $(0, 1, 2)$ directions represent the three-dimensional spacetime, with x_6 compact. The α-th NS5 brane, $\alpha = 1, \ldots, a$, combines with the Q_α D5$_\alpha$ branes to give a $(1, Q_\alpha)$-fivebrane stretched along the $012[37]_{\theta_\alpha}45$ direction. The β-th NS5 brane, $\beta = 1, \ldots, b$, combines with the P_β D5$_\beta$ branes to give a $(1, P_\beta)$-fivebrane stretched along the $012[37]_{\theta_\beta}89$ direction. For

Table 4.1. Type IIB embedding of low energy CS field theories.

brane	0	1	2	3	4	5	6	7	8	9
D3	X	X	X				X			
$NS5_\alpha$	X	X	X	X	X	X				
$NS5_\beta$	X	X	X	X					X	X
$D5_\alpha$	X	X	X		X	X		X		
$D5_\beta$	X	X	X					X	X	X

specific values of the angles θ_α and θ_β determined by Q_α and P_β, the supersymmetry is enhanced to $\mathcal{N} \geq 3$. We will consider generic configurations, so our results will be also valid when this enhancement does occur. The fivebranes are chosen to be placed in the following order: first we put $b-a$ $(1, P_\beta)$ fivebranes on the circle and then we alternate the remaining a $(1, P_\beta)$ and the a $(1, Q_\alpha)$.

The N_i D3-branes stretched between each pair of $(1, p_i)$ give rise to a $U(N_i)_{k_i}$ gauge group in the quiver. Each $(1, p_i)$ is associated to a pair of bifundamental chiral fields in the $(N_i, \bar{N}_{i+1})$ representation. In addition, we have an adjoint chiral field for each consecutive pair of $(1, p_i)$ of the same type. The resulting field theory is a $\prod_{i=1}^{g} U(N_i)_{k_i}$ gauge theory, where k_i represents the CS level of the i-th group and $g = b + a$. The levels are given by the relation $k_i = p_i - p_{i+1}$ which also implies $\sum k_i = 0$. In the quiver representation we have the first b nodes with adjoint matter and the last a without adjoints; every pair of consecutive nodes is connected by a pair of bifundamental and anti-bifundamental fields. We also obtain the following superpotential

$$W = X_{1,1}X_{1,a+b}X_{a+b,1} + \sum_{i=1}^{b-a} X_{i,i}X_{i,i+1}X_{i+1,i} + \sum_{i=b-a+1}^{a+b} X_{i,i-1}X_{i-1,i}X_{i,i+1}X_{i+1,i} \quad (4.14)$$

$$(4.15)$$

where $X_{i,j}$ indicates a bifundamental field connecting nodes i and j and $X_{i,i}$ corresponds to an adjoint of the node i. Globally, the brane construction and thus the field theory preserves $\mathcal{N}=2$ supersymmetry.

Duality

The above brane picture allows us to describe Seiberg-like dualities in an unified way, through the Hanany-Witten transition [112]. Consider two consecutive, non-parallel, $(1,p_i)$ fivebranes and move one towards the other until they cross and exchange their positions along the x_6 direction. Quantum charge conservation requires the creation of $|P_{\beta_{b+i}}-Q_{\alpha_i}|=k_{b+i}$ D3 branes on top of the existing N_{b+i} ones.

Correspondingly, in the low energy field theory the i-th gauge factor changes its rank from N_i to $N_i+|k_i|$, and because the fivebrane charges and order determine the CS levels, the latter also undergo the following shift

$$\begin{aligned} k_{i-1} &= p_{i-1}-p_i \;\rightarrow\; k'_{i-1}=p_{i-1}-p_{i+1}=k_{i-1}+k_i \\ k_i &= p_i-p_{i+1} \;\rightarrow\; k'_i=p_{i+1}-p_i=-k_i \\ k_{i+1} &= p_{i+1}-p_{i+2} \;\rightarrow\; k'_{i+1}=p_i-p_{i+2}=k_{i+1}+k_i \end{aligned} \tag{4.16}$$

Note that the sum of all the CS levels is preserved in this process. The local nature of the Hanany-Witten transition is reflected in the field theory by the fact that only one gauge group and its first neighbors go through a change. Finally the superpotential locally changes as[1]

$$\begin{aligned} \widetilde{W} &= Y_{b+i-1,b+i-1}Y_{b+i-1,b+i}Y_{b+i,b+i-1}+Y_{b+i,b+i-1}Y_{b+i-1,b+i}Y_{b+i,b+i+1}Y_{b+i+1,b+i} \\ &+ Y_{b+i+1,b+i+1}Y_{b+i+1,b+i}Y_{b+i,b+i+1} \end{aligned} \tag{4.17}$$

Notice that the dual theory also contains two new adjoint fields. Thanks to the above

[1] Actually also the nodes b-2 and b-2 are involved, because there are two extra terms $X_{b-2,b-1}Y_{b-1,b-1}X_{b-1,b-2}$ and $X_{b+2,b+1}Y_{b+1,b+1}X_{b+1,b+2}$ in the superpotential. We can skip this contribution in our analysis because the R-charges of X fields are not affected.

superpotential, the two models have the same moduli space and are conjectured to be dual to each other in the deep infrared. Also notice that nowhere did we use the fact that in this example the electric ranks of the gauge groups are equal to each other. Thus the same argument can be straightforwardly applied to a product of arbitrary unitary groups.

The duality above extends the Kutasov-Giveon duality [100] for three dimensional supersymmetric gauge theories with CS terms. Nontrivial checks are required in order to validate the whole picture provided above. In fact, there exist two limits where such checks have been given. One is the large N limit [16]. In this case, the dual gauge group can be safely taken to be the original one, because any difference in the ranks due to the CS levels is subleading. Notice that, in general, this is a nontrivial statement.[2]

The second limit corresponds to the case of finite N with $\mathcal{N} \geq 3$. Only partial results have been studied in this limit. For instance, when $a = b = 1$ the model is the ABJM model which enjoys $\mathcal{N} = 6$ supersymmetry. In that case the analysis becomes much simpler and many checks have been provided. In fact, beside the moduli space matching, there is no check for models with $g > 2$ gauge group factors and $\mathcal{N} = 2$ supersymmetry. The main difficulties in this case are due to the nontrivial anomalous dimensions of the fields. We will see how we can identify the scaling dimensions of the fields on the two sides of the duality so that the two partition functions agree even for $g > 2$ and for arbitrary ranks. We will also argue that the map we will describe preserves extremization of the partition function with respect to scaling dimensions themselves.

4.3.2 Orthogonal and symplectic groups: the orientifold

While we focused on unitary gauge groups in the above subsection, more general models can be derived from the same type IIB picture above. An immediate extension

[2] We are grateful to Claudius Klare and Alberto Zaffaroni for discussions on this point.

Table 4.2. O3 planes, their D3 brane charge and the corresponding gauge group.

Type	Charge	Group
$O3^+$	$-\frac{1}{4}$	$SP(2N)$
$O3^-$	$\frac{1}{4}$	$SO(2N)$
$\widetilde{O3}^+$	$\frac{1}{4}$	$SP(2N)$
$\widetilde{O3}^-$	$\frac{1}{4}$	$SO(2N+1)$

includes adding orientifold O3 planes on top of the D3 branes, which we employ in the following. This construction does not break any residual supersymmetry, so we will end up with $\mathcal{N} \geq 2$ theories [121, 8].[3]

For simplicity, we restrict to the class of theories with $a = b$. Under the orientifold projection, the $(1, p_i)$ fivebranes which intersect the O3 are identified with their own image while the projection does not act on the D3 branes. There are four kind of O3 planes, named $O3^{\pm}$ and $\widetilde{O3}^{\pm}$, that differ, among other, by the amount of D3 brane charge they carry, and by the resulting worldvolume theory gauge group they lead to. We summarize the different cases in table 4.2. A $(1, p_i)$ fivebrane which intersects the orientifold plane switches its type according to the following rule: If p_i is even we have $(O3^+, \widetilde{O3}^+) \leftrightarrow (O3^-, \widetilde{O3}^-)$ otherwise if p_i is odd we have $(O3^{\pm} \leftrightarrow \widetilde{O3}^{\mp})$. We restrict to the case of p_i even and make this explicit by considering $(1, 2p_i)$ instead. According to the general discussion on the brane engineering of CS matter theories above, all the CS terms will be even too.

It is then clear that the gauge group will include alternating factors of orthogonal and symplectic groups. Their ranks are given by the choice of the O3 planes, namely we obtain a chain of $SO(2N)_{2k_i} \times SP(2N)_{k_j}$ factors for alternating $O3^+$ and $O3^-$ planes, and of $SO(2N+1)_{2k_i} \times SP(2N)_{k_j}$ factors in the $\widetilde{O3}^{\pm}$ case (with the convention $SP(2) \simeq SU(2)$).[4] An example of such construction is given in figure 4.1 for the case

[3] Other orientifold constructions that break supersymmetry have been investigated in [26, 74].

[4] Observe that the level of the SP group k is integer. For this reason taking an odd number of D5

with O3$^{\pm}$. The fields are projected such that every pair of bifundamental and anti-

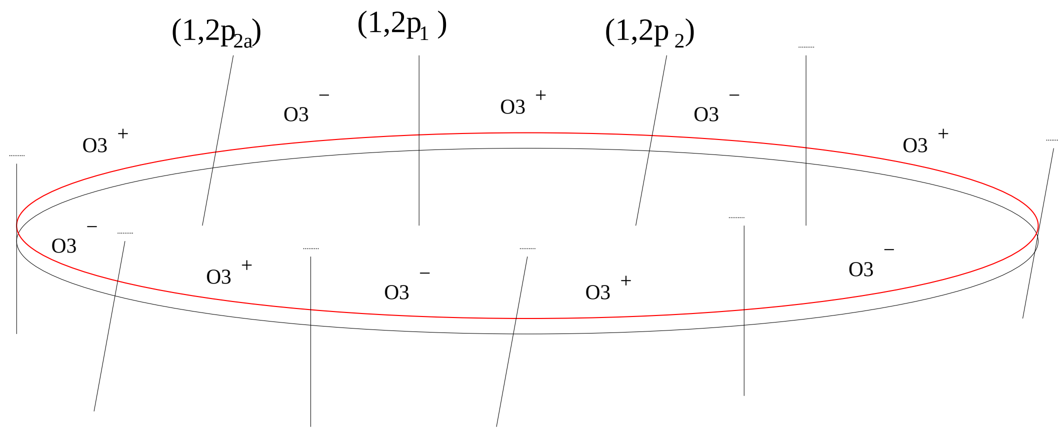


Figure 4.1. A type IIB embedding of orthogonal and symplectic field theories via O3$^{\pm}$ planes on a stack of D3.

bifundamental becomes a single field in the fundamental of both the SP and SO node. By starting with $X_{i-1,i}$ and $X_{i,i-1}$ one ends up with a single field $X_{i-1,i}$ when we fix the left to right convention on the indices. The superpotential is

$$W = (X_{i-1,i} \cdot X_{i,i+1})^2 \tag{4.18}$$

where the products are appropriately taken in the SP and/or SO case.

Duality

We again apply the brane creation effect when two fivebranes cross each other to derive the rules for the low energy field theory duality. The steps are in close analogy with the ones above, with the charge of the O3 plane properly taken into account.

Because the duality only acts locally on the quiver, we can isolate the node over which we perform the fivebrane exchange and collect the changes in the gauge group and

in the fivebrane is quantum mechanically inconsistent, because we would get a semi-integer CS level. Moreover in the CS contribution to the partition function there will be an extra factor of 2 for the SP cases, due to normalization of the generators [190].

CS level of itself and its neighbors as follows: suppose we apply the duality on the node A which locally looks like

A-1	A	A+1
$SO(2N)_{2k_{A-1}}$	$SP(2N)_{k_A}$	$SO(2N)_{2k_{A+1}}$
$SO(2N+1)_{2k_{A-1}}$	$SP(2N)_{k_A}$	$SO(2N+1)_{2k_{A+1}}$
$SP(2N)_{k_{A-1}}$	$SO(2N)_{2k_A}$	$SP(2N)_{k_{A+1}}$
$SP(2N)_{k_{A-1}}$	$SO(2N+1)_{2k_A}$	$SP(2N)_{k_{A+1}}$

with superpotential (4.18). Then the dual theory is locally given by

A-1	A	A+1
$SO(2N)_{2k_{A-1}+2k_A}$	$SP(2(N+\lvert k_A\rvert-1)_{-k_A}$	$SO(2N)_{2k_{A+1}+2k_A}$
$SO(2N+1)_{2k_{A-1}+2k_A}$	$SP(2(N+\lvert k_A\rvert-1))_{-k_A}$	$SO(2N+1)_{2k_{A+1}+2k_A}$
$SP(2N)_{k_{A-1}+k_A}$	$SO(2(N+k_A-1))_{-2k_A}$	$SP(2N)_{k_{A+1}+k_A}$
$SP(2N)_{k_{A-1}+k_A}$	$SO(2(N+\lvert k_A\rvert)+1)_{-k_A}$	$SP(2N)_{k_{A+1}+k_A}$

with all the remaining nodes in the quiver unchanged and dual superpotential given by

$$\tilde{W} = Y_{A-1,A-1}\cdot Y_{A-1,A}^2 + Y_{A-1,A}^2 Y_{A,A+1}^2 + Y_{A+1,A+1}\cdot Y_{A+1,A}^2 \tag{4.19}$$

These dualities fit with the ones proposed in [133] for the case without the quiver structure, and with the ones for the case of two gauge groups and higher supersymmetry [8].

In the following we will show that the partition function is preserved at finite N for all of these dualities.

4.4 Exact results for the dualities

In this section we evaluate the exact partition function on a squashed three sphere of the above models and provide further evidence for the dualities. We review the identities we use in Appendix C.1 and also refer to [187] for more details. Because

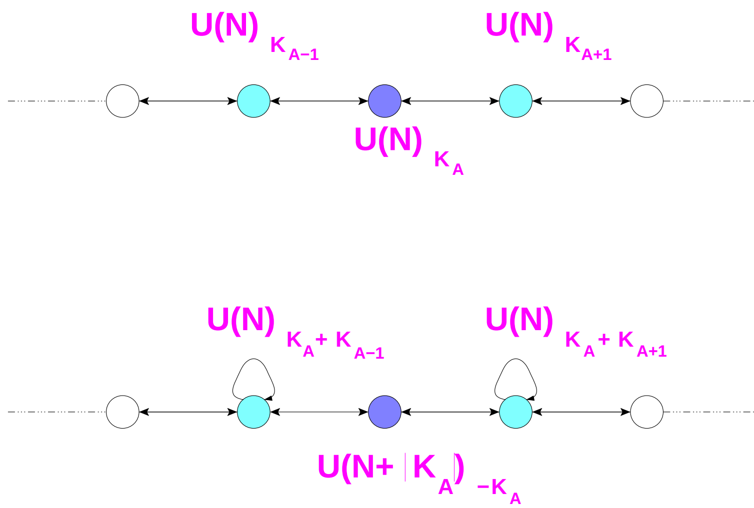


Figure 4.2. Dual phases describing a stack of M2 branes probing a Calabi-Yau fourfold.

the duality only acts on the local structure of the quiver, we can restrict ourselves to the the subset of variables which undergo the duality transformation. In other words, we explicitly write only the integration variables corresponding to the gauge group factor we are performing the duality on.

4.4.1 Duality in $U(N)_k$ non-chiral quivers

In this case the large N partition function have been studied in [68, 119, 151, 131, 60, 180, 16], and the agreement between dual phases have been checked in this limit in [137, 136, 103, 16, 14]. Here we provide the agreement at finite N.

In terms of the hyperbolic functions defined in section 4.2, the partition function for models with only unitary gauge groups can be written in a very compact way. The matter content and local quiver structure are represented in figure 4.2, where we used the letter A to label the gauge group factor over which we perform the duality. From the top figure we read the relevant contribution to the partition function involved in the duality as

$$\mathcal{Z}_e = \frac{1}{\sqrt{(-\omega_1\omega_2)^N}N!}\int \frac{\prod_{J=A-1}^{A}\prod_{i,j=1}^{N}\prod_{\eta=\pm 1}\Gamma_h\left(\eta\left(x_J^{(i)}-x_{J+1}^{(j)}\right)+\omega\Delta_{J,J+1}^{(\eta)}\right)}{\prod_{i<j}^{N}\Gamma_h\left(\pm\left(x_A^{(i)}-x_A^{(j)}\right)\right)}$$
$$\times \prod_{J=A-1}^{A+1}\prod_{i=1}^{N} c\left(-2k_J {x_J^{(i)}}^2\right)\prod_{i=1}^{N} dx_A^{(i)} \tag{4.20}$$

where the round sphere corresponds to the limit $\omega_1 = \omega_2 = \omega = i$. Our aim is to write (4.20) in a form that can be interpreted as the partition function of the dual theory described in section 4.3. We find it is useful to define the following shorthand notation

$$\begin{aligned}
&\Delta_{J,J+1} = \Delta_{J,J+1}^{(+)} && \Delta_{J+1,J} = \Delta_{J,J+1}^{(-)}\\
&\mu_{A+1}^{(i)} = x_{A+1}^{(i)} + \omega\Delta_{A,A+1}^{(-)} && \mu_{A-1}^{(i)} = x_{A-1}^{(i)} + \omega\Delta_{A-1,A}^{(+)}\\
&\nu_{A+1}^{(i)} = -x_{A+1}^{(i)} + \omega\Delta_{A,A+1}^{(+)} && \nu_{A-1}^{(i)} = -x_{A-1}^{(i)} + \omega\Delta_{A-1,A}^{(-)}\\
&\mu_r = \{\mu_{A-1}^{(i)}, \mu_{A+1}^{(i)}\} && \nu_s = \{\nu_{A-1}^{(i)}, \nu_{A+1}^{(i)}\}
\end{aligned} \tag{4.21}$$

which satisfy the superpotential contraint

$$\sum_{r=1}^{2N}\mu_r + \sum_{s=1}^{2N}\nu_s = N\omega(\Delta_{A,A+1}^{(-)} + \Delta_{A,A+1}^{(+)} + \Delta_{A-1,A}^{(-)} + \Delta_{A-1,A}^{(+)}) = 2\omega N \tag{4.22}$$

Here r and s are collective indices for elements of the respective sets. By applying equation (C.5) and fixing $k_A > 0$ we obtain

$$\mathcal{Z}_m = \frac{1}{(-\omega_1\omega_2)^{\frac{N+k_A}{2}}(N+k_A)!}\times$$
$$\int \frac{\prod_{J=A\pm 1}\prod_{i=1}^{N}\prod_{j=1}^{N+k_A}\Gamma_h\left(\omega-\nu_J^{(i)}-x_A^{(j)}\right)\Gamma_h\left(\omega-\mu_J^{(i)}+x_A^{(j)}\right)}{\prod_{i<j}^{N+k_A}\Gamma_h\left(\pm\left(x_A^{(i)}-x_A^{(j)}\right)\right)} \tag{4.23}$$
$$\prod_{i=1}^{N} c\left(-2(k_A+k_J){x_J^{(i)}}^2\right)\prod_{i=1}^{N+k_A} c\left(2k_A {x_A^{(i)}}^2\right)dx_A^{(i)}\times\prod_{r,s=1}^{2N}\Gamma_h\left(\mu_r+\nu_s\right)$$
$$\zeta^{-k_A^2-2}\; c\left(k_A^2\left(2\omega^2-1\right)+2\,N\,k_A\left(\omega^2\left(\Delta_{A-1,A}^2+\Delta_{A,A+1}^2-2\right)-1\right)\right)$$

The denominator can be interpreted as the 1-loop contribution from the vector superfield of the gauge group $U(N+k_A)$ (recall that the duality does not change the ranks of other factors).[5] The numerator in the first term contains the contribution from the (anti)bifundamental fields: it is easy to see that bifundamental fields are mapped to anti-bifundamental fields and viceversa, as required by Seiberg duality. Moreover, we also obtain the offshell map between the scaling dimensions $\tilde{\Delta}$ of the dual fields and the electric ones

$$\widetilde{\Delta}_{A,A\pm1} = 1 - \Delta_{A\pm1,A} \qquad \widetilde{\Delta}_{A\pm1,A} = 1 - \Delta_{A,A\pm1} \tag{4.24}$$

The last factor in the second line of (4.23) gives the contribution from the new adjoint fields. Indeed, it can be written in the form

$$\begin{aligned}\prod_{r,s=1}^{2N} \Gamma_h(\mu_r + \nu_s) \quad &= \quad \prod_{J=A\pm1} \prod_{i,j=1}^{N} \Gamma_h\left(x_J^{(i)} - x_J^{(j)} + \omega \widetilde{\Delta}_{J,J}\right) \times \\ &\prod_{i,j=1}^{N} \Gamma_h\left(x_{A+1}^{(i)} - x_{A-1}^{(j)} + \omega(\Delta_{A,A+1}^{(+)} + \Delta_{A-1,A}^{(-)})\right) \\ &\prod_{i,j=1}^{N} \Gamma_h\left(x_{A-1}^{(i)} - x_{A+1}^{(j)} + \omega(\Delta_{A,A+1}^{(-)} + \Delta_{A-1,A}^{(+)})\right) \end{aligned} \tag{4.25}$$

where $\widetilde{\Delta}_{A\pm1,A\pm1} = \Delta_{A\pm1,A} + \Delta_{A,A\pm1}$ gives the R-charge of the adjoint fields. On the field theory side the dual superpotential is

$$\begin{aligned} W = \ldots \quad &+ \quad Y_{A,A-1}Y_{A-1,A-1}Y_{A-1,A} + Y_{A-1,A}Y_{A,A+1}Y_{A+1,A-1} \\ &+ \quad Y_{A+1,A}Y_{A,A-1}Y_{A-1,A+1} + Y_{A,A+1}Y_{A+1,A+1}Y_{A+1,A} + \ldots \end{aligned} \tag{4.26}$$

[5] In this case we choose all the ranks N_J equal to N. In more general situations, when fractional branes are considered in the electric theory, all the ranks can be different, and the duality preserves the partition function as in this case. Moreover, as explained in the appendix, we are restricting to $k_A > 0$. For a generic k_A the dual rank becomes $N + |k_A|$.

and by integrating out the fields $Y_{A-1,A+1}$ and $Y_{A+1,A-1}$ it becomes

$$\begin{aligned} W =& Y_{A,A-1}Y_{A-1,A-1}Y_{A-1,A} - Y_{A-1,A}Y_{A,A+1}Y_{A+1,A}Y_{A,A-1} \\ &+ Y_{A,A+1}Y_{A+1,A+1}Y_{A+1,A} + \dots \end{aligned} \tag{4.27}$$

where the dual fields $Y_{A\pm1,A\pm1}$ are related to the electric ones as

$$Y_{A\pm1,A\pm1} = X_{A\pm1,A}X_{A,A\pm1} \tag{4.28}$$

Formula (4.25) takes properly into account the contribution of the new mesons $Y_{A+1,A+1}$ and $Y_{A-1,A-1}$. The contribution of the two extra mesons reduces to 1 in (4.25) after using the reflection formula (4.7) and the superpotential constraint

$$\widetilde{\Delta}_{A-1,A+1} + \widetilde{\Delta}_{A+1,A-1} = 2$$

.

We now check that the CS levels shift according to the discussion in section 4.3. For simplicity we gauge fix the complexified Fayet-Iliopoulos (FI) term Δ_m to zero, but the corresponding generalization is straightforward and one can easily map the electric FI in the magnetic one as $\Delta'_m = \Delta'_m\left(\Delta_m, \Delta^{(\pm)}_{J,J+1}\right)$. We stress that we can perform this gauge fixing choice without worrying about extremization with respect to Δ_m because we consider $U(N)$ factors as opposed to $SU(N)$ ones. Below, when we will consider orthogonal and symplectic gauge groups, the FI term will vanish even for simple group factors because of invariance under charge conjugation.

Having fixed the FI term, the linear terms in the function c in (4.23) have to cancel out. Recall that in a vector-like theory with vanishing FI term we also have $\Delta^{(+)}_{J,J+1} = \Delta^{(-)}_{J,J+1}$ and $\sum \mu_r = \sum \nu_s$. We only need these relations here, but they can be easily relaxed if one wishes to introduce a nontrivial FI term in the model. We obtain in (4.23) the shift of the levels $k_{A\pm1}$ by a factor of k_A while the level for the dualized group switches its sign. Finally the last line in (4.23) represents a pure phase factor, which does not spoil the duality.

Adding an adjoint field

We now consider a slightly different model which also contains an adjoint field $X_{A-1,A-1}$ on the electric side. The quiver for the dual phases is depicted in Figure 4.3.

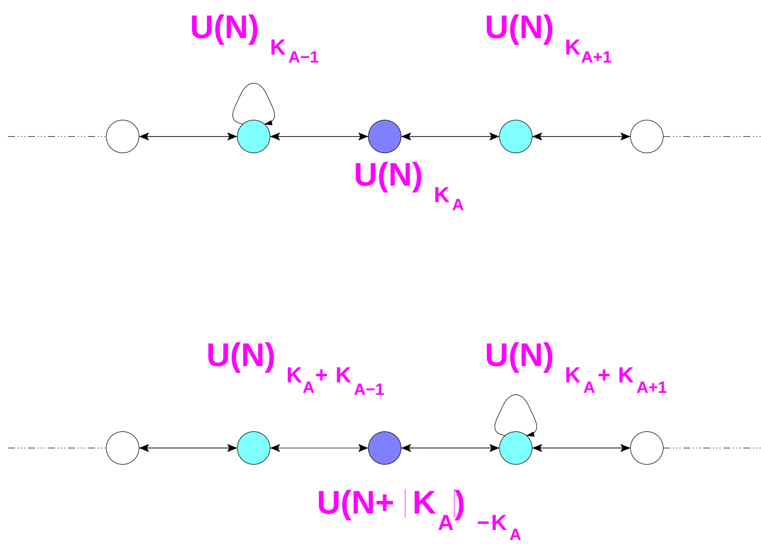


Figure 4.3. Dual phases describing a stack of M2 branes probing Calabi-Yau fourfold with adjoint matter involved in the duality.

The superpotential for the colored nodes of the quiver is

$$W_e = \cdots + X_{A-1,A-1}X_{A-1,A}X_{A,A-1} - X_{A-1,A}X_{A,A+1}X_{A+1,A}X_{A,A-1} + \ldots \quad (4.29)$$

The dual superpotential is

$$W_m = \ldots Y_{A-1,A}Y_{A,A+1}Y_{A+1,A}Y_{A,A-1} - Y_{A+1,A+1}Y_{A+1,A}Y_{A,A+1} + \ldots \quad (4.30)$$

The relevant contribution to the electric partition function on the squashed sphere is:

$$\mathcal{Z}_e = \qquad \frac{1}{\sqrt{\omega_1^N \omega_2^N} N!} \int \frac{\prod_{J=A-1}^{A} \prod_{i,j=1}^{N} \Gamma_h\left(\pm\left(x_J^{(i)} - x_{J+1}^{(j)}\right) + \omega \Delta_{J,J+1}^{(\eta)}\right)}{\prod_{J=A-1}^{A+1} \prod_{i<j}^{N} \Gamma_h\left(\pm\left(x_J^{(i)} - x_J^{(j)}\right)\right)} \tag{4.31}$$

$$\prod_{i,j=1}^{N} \Gamma_h\left(\pm\left(x_i^{(A-1)} - x_i^{(A-1)}\right) + \omega \Delta_{A-1,A-1}\right) \prod_{J=A-1}^{A+1} \prod_{i=1}^{N} c\left(-2k_J {x_J^{(i)}}^2\right) dx_J^{(i)}$$

The duality can be shown by following the same steps as in Subsection 4.4.1. The only difference is that in (4.25) there is an extra constraint $\Delta_{A-1,A-1} + \Delta_{A-1,A} + \Delta_{A,A-1} = 2$. This constraint sets the contribution of the meson $Y_{A-1,A-1}$ to 1 in the dual partition function (in field theory it is integrated out) because of (4.7).

4.4.2 The first class of orientilfolds: O3 planes

In this section we study the duality on the first class of orientifolded models introduced in section 4.3.2 and match the partition function between different phases. Recall that the relevant models are quiver field theories with alternating "a" $SP(2N)_{k_i}$ and "a" $SO(2N)_{2k_i}$ nodes, with $\sum k_i = 0$. The superpotential is

$$W = \sum_{J=1}^{2a-1} \left(X_{J,J+1} \cdot X_{J+1,J+2}\right)^2 \tag{4.32}$$

where $X_{2a,2a+1} = X_{2a,1}$. If $a > 1$ there is always a field connecting two consecutive nodes labeled by J and $J+1$, and we assign to this field the charge $\Delta_{J,J+1}$.[6] The superpotential imposes the constraint $\Delta_{J-1,J} + \Delta_{J,J+1} = 1$.

[6] The case $a = 1$ reduces to the models studied in [8].

Duality on an $SP(2N)_k$ node

We first study the duality on an $SP(2N)_{k_A}$ group. Also in this case we refer to the quiver in Figure 4.2, but we erase the arrows because the groups are real and there is no distinction between fundamental and antifundamental representations. The relevant contribution to the partition function for this model is

$$\mathcal{Z}_{SP(2N)_{k_A}} = \int \frac{\prod_{J=A-1}^{A} \prod_{i,j=1}^{N} \Gamma_h\left(\pm x_J^{(i)} \pm x_{J+1}^{(j)} + \omega\Delta_{J,J+1}\right)}{\prod_{1\le i<j\le N} \Gamma_h\left(\pm x_A^{(i)} \pm x_A^{(j)}\right) \prod_{i=1}^{N} \Gamma_h\left(\pm 2x_A^{(i)}\right)} \prod_{J=A-1}^{A+1} \prod_{i=1}^{N} c\left(-4k_J {x_J^{(i)}}^2\right) \prod_{i=1}^{N} dx_A^{(i)} \tag{4.33}$$

where we used the notation $\Gamma_h(x+a)\Gamma_h(-x+a) = \Gamma_h(\pm x + a)$. In this case we define the μ_r variables as

$$\mu_{i,A-1}^{(\pm)} = \pm x_{A-1}^{(i)} + \omega\Delta_{A-1,A} \quad , \quad \mu_{i,A+1}^{(\pm)} = \pm x_{A+1}^{(i)} + \omega\Delta_{A,A+1} \tag{4.34}$$

Since there are $4N$ different μ the index r runs from 1 to $4N$, such that

$$\mu_r = \{\mu_{i,A-1}^{(+)}, \mu_{i,A-1}^{(-)}, \mu_{i,A+1}^{(+)}, \mu_{i,A+1}^{(-)}\} \tag{4.35}$$

where every i runs from 1 to N. The dual gauge group is

$$SO(2N)_{2(k_{A-1}+k_A)} \times SP(2(N+|k_A|-1))_{-k_A} \times SO(2N)_{2(k_A+k_{A+1})} \tag{4.36}$$

The dual superpotential is

$$W_m = Y_{A\pm1,A\pm1} \cdot Y_{A\pm1,A} \cdot Y_{A,A\pm1} - (Y_{A-1,A} \cdot Y_{A,A+1})^2 \tag{4.37}$$

The partition function of the dual gauge theory corresponds to the RHS of (C.9) by fixing $k_A > 0$. In this case we have

$$I^{N+k_A-1}_{N,2(1+k_A)a}(\mu) = I^{N}_{N+k_A-1,2(1+k_A)b}(\omega-\mu) \prod_{1\le r<s\le 4N} \Gamma_h\left(\mu_r+\mu_s\right)\zeta^{(k_A-1)(1-2k_A)}$$
$$c\left(\omega^2\left(2k_A^2 - k_A\left(3+4N\left(\Delta^2_{A-1,A}+\Delta^2_{A,A+1}-1\right)\right)+1\right) - 4k_A\left(\sum_{i=1}^{N} {x^{(i)}_{A-1}}^2 + \sum_{i=1}^{N} {x^{(i)}_{A+1}}^2\right)\right) \tag{4.38}$$

The case $k_A < 0$ in the electric theory is studied by inverting (C.9) as explained in Appendix C.1. As expected the rank of the dual groups is $\tilde{N} = N + |k_A| - 1$.

It is straightforward to see from the first term in the RHS of (4.38) that the electric R-charge of a bifundamental connecting a pair of nodes in the electric theory is related in the magnetic theory to the R-charge of a bifundamental connecting the same pair of nodes nodes through $\widetilde{\Delta}_{i,j} = 1 - \Delta_{i,j}$.

The second term in the RHS of (4.38) can be expanded in terms of μ_r and it becomes

$$\begin{aligned}\prod_{1\le r<s\le 4N}\Gamma_h\left(\mu_r+\mu_s\right) = &\prod_{1\le i<j\le N}\Gamma_h\left(\pm x^{(i)}_{A-1}\pm x^{(j)}_{A-1}+2\omega\Delta_{A-1,A}\right)\times {\Gamma_h}^N\left(2\omega\Delta_{A-1,A}\right)\\ &\times\prod_{1\le i<j\le N}\Gamma_h\left(\pm x^{(i)}_{A+1}\pm x^{(j)}_{A+1}+2\omega\Delta_{A,A+1}\right)\times {\Gamma_h}^N\left(2\omega\Delta_{A,A+1}\right)\\ &\times\prod_{i,j=1}^{N}\Gamma_h\left(\pm x^{(i)}_{A+1}\pm x^{(j)}_{A-1}+\omega\Delta_{A-1,A}+\omega\Delta_{A,A+1}\right)\end{aligned} \tag{4.39}$$

The first two terms are the mesons of the dual theory while the last one evaluates to 1 because of the superpotential constraint on the R-charges.

We conclude the proof of the duality with the analysis of the CS contributions to the partition function. The CS of the dual SP group switches from k_A to $-k_A$, because the dual theory is a "b" integral (see Appendix C.1 for details). The CS of the SO groups transform in (4.38) as $2k_{A\pm1} \to 2k_A + 2k_{A\pm1}$, as expected.

Similar to the case of unitary theories, (4.38) also has which we ignore.

Duality on an $SO(2N)_{2k}$ node

In the O3$^{\pm}$ orientifolded quiver one can also dualize an $SO(2N)_{2k_A}$ node. The dual gauge group is

$$SP(2N)_{k_{A-1}+k_A} \times SO(2(N+|k|+1))_{-2k_A} \times SP(2N)_{k_A+k_{A+1}} \tag{4.40}$$

and the superpotential is again (4.37) with the proper products. The relevant contribution to the partition function of the electric theory is

$$\mathcal{Z}_{SO(2N)_{k_A}} = \int \frac{\prod_{J=A-1}^{A} \prod_{i,j=1}^{N} \Gamma_h \left(\pm x_J^{(i)} \pm x_{J+1}^{(j)} + \omega \Delta_{J,J+1} \right)}{\prod_{1 \le i < j \le N} \Gamma_h \left(\pm x_A^{(i)} \pm x_A^{(j)} \right)} \prod_{J=A-1}^{A+1} \prod_{i=1}^{N} c\left(-4k_J {x_J^{(i)}}^2 \right) \prod_{i=1}^{N} dx_A^{(i)} \tag{4.41}$$

As in [190, 40] the measure of the $SO(2N)$ gauge group can be converted into that of an $SP(2N)$ group by applying the relation (4.9) and inserting in the partition function the contribution

$$1 = \frac{\prod_{i=1}^{N} \prod_{\alpha=1}^{4} \Gamma_h(\pm x_A^{(i)} + \rho_\alpha)}{\Gamma_h(\pm 2 x_A^{(i)})} \tag{4.42}$$

where $\rho_\alpha = \left(0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \omega\right)$. The μ vector becomes

$$\mu_r = \{\mu_{i,A-1}^{(+)}, \mu_{i,A-1}^{(-)}, \mu_{i,A+1}^{(+)}, \mu_{i,A+1}^{(-)}, \rho_\alpha\} \tag{4.43}$$

where $r = 1, \ldots, 4N+4$ and

$$\mu_{i,A-1}^{(\pm)} = \pm x_{A-1}^{(i)} + \omega \Delta_{A-1,A} \quad , \quad \mu_{i,A+1}^{(\pm)} = \pm x_{A+1}^{(i)} + \omega \Delta_{A,A+1} \tag{4.44}$$

By applying (C.9) with $k_A > 0$ we have

$$\begin{aligned} I^{N+k_A+1}_{N,2(1+k_A)_a}(\mu) &= I^{N}_{N+k_A+1,2(1+k_A)_b}(\omega-\mu) \prod_{1\leq r<s\leq 4N+4} \Gamma_h\left(\mu_r+\mu_s\right)\zeta^{(k_A-1)(1-2k_A)} \\ &\times \; c\left(4k_A\left(\sum_{i=1}^{N} {x_{A-1}^{(i)}}^2 + \sum_{i=1}^{N} {x_{A+1}^{(i)}}^2\right) - \frac{1}{2}k_A\left({\omega_1}^2+{\omega_1}^2\right)\right) \\ &\times \; c\left(\omega^2\left(2{k_A}^2+k_A\left(3-4N\left({\Delta_{A-1,A}}^2+{\Delta_{A,A+1}}^2-1\right)\right)+1\right)\right) \end{aligned} \tag{4.45}$$

The case $k_A < 0$ in the electric theory is studied by inverting (C.9). As expected the rank of the dual groups is $\tilde{N} = N + |k_A| + 1$. Observe that it fits with the proposal of [134], $SO(\tilde{N}_c) = SO(N_f + |K| - 2 - N_c)$. Indeed in our case $N_c = 2N$, $K = 2k_A$, $N_f = 4N + 4$ and $\tilde{N}_c = 2(N + |k_A| + 1)$.

The RHS of (4.45) corresponds to the partition function of the dual theory. By using the relation (4.42) the extra terms in the measure arising in (4.45) become

$$\frac{\prod_{i=1}^{N+|k_a|+1} \prod_{\alpha=1}^{4} \Gamma_h(\omega \pm x_A^{(i)} - \rho_\alpha)}{\Gamma_h(\pm 2x_A^{(i)})} = 1 \tag{4.46}$$

thus giving us the measure of the $SO(2(N + |k_A| + 1))$ dual gauge group.

Upon expanding the $\Gamma_h(\mu_r + \mu_s)$ term in the RHS of (4.45) we find

$$\begin{aligned} \prod_{1\leq r<s\leq 4N+4} \Gamma_h\left(\mu_r+\mu_s\right) = \prod_{1\leq r<s\leq 4N} \Gamma_h\left(\mu_r+\mu_s\right) \prod_{\substack{1\,\leq\, r\,\leq\, 4N \\ 4N\,<\, s\,<\, 4N+4}} \Gamma_h\left(\mu_r+\mu_s\right) \\ \times \prod_{4N<r<s\leq 4N+4} \Gamma_h\left(\mu_r+\mu_s\right) \end{aligned} \tag{4.47}$$

By combining the first two products we obtain

$$\prod_{i,j=1}^{N} \Gamma_h(\pm x_{A+1}^{(i)} \pm x_{A+1}^{(j)} + 2\omega\Delta_{A,A+1}) \times \prod_{i,j=1}^{N} \Gamma_h(\pm x_{A-1}^{(i)} \pm x_{A-1}^{(j)} + 2\omega\Delta_{A-1,A}) \tag{4.48}$$

which represent the massless mesons of dual theory (they are the adjoints of neighbouring SP($2N$)). The extra contributions from the first two terms in the RHS of (4.47) correspond to the massive mesons and evaluate to 1. The last term in the product in (4.47) is

$$\Gamma_h\left(\frac{\omega_1}{2}\right)\Gamma_h\left(\frac{\omega_2}{2}\right)\Gamma_h(\omega)^2\Gamma_h\left(\omega+\frac{\omega_1}{2}\right)\Gamma_h\left(\omega+\frac{\omega_2}{2}\right)=1 \tag{4.49}$$

bacause of (4.8). The rest of the terms in (4.45) give the right transformation on the CS levels and an extra phase as usual.

4.4.3 The second class of orientifolds: duality on $SO(2N+1)_k$

If we consider $\widetilde{\text{O3}}^{\pm}$ orientifold planes, the gauge groups of the necklace quiver involve $SO(2N+1)$ factors instead of $SO(2N)$. We are interested in studying the duality on these nodes. The relevant contribution to the partition function is

$$\begin{aligned}\mathcal{Z}_{SO(2N+1)_{k_A}} &= \int \frac{\prod_{J=A-1}^{A}\prod_{i,j=1}^{N}\Gamma_h\left(\pm x_J^{(i)}\pm x_{J+1}^{(j)}+\omega\Delta_{J,J+1}\right)\prod_{J=A-1}^{A+1}\prod_{i=1}^{N} c\left(-4k_J {x_J^{(i)}}^2\right)}{\prod_{1\le i<j\le N}\Gamma_h\left(\pm x_A^{(i)}\pm x_A^{(j)}\right)\prod_{i=1}^{N}\Gamma_h\left(\pm x_A^{(i)}\right)} \\ &\times \prod_{i=1}^{N}\Gamma_h\left(\pm x_{A-1}^{(i)}+\omega\Delta_{A-1,A}\right)\Gamma_h\left(\pm x_{A+1}^{(i)}+\omega\Delta_{A,A+1}\right)dx_A^{(i)}\end{aligned} \tag{4.50}$$

The measure of the $SO(2N+1)$ group can be converted into the one of an $SP(2N)$ group by applying (4.9). We find

$$\frac{1}{\Gamma_h\left(\pm x_A^{(i)}\right)}=\frac{\Gamma_h\left(\pm x_A^{(i)}+\frac{\omega_1}{2}\right)\Gamma_h\left(\pm x_A^{(i)}+\frac{\omega_2}{2}\right)}{\Gamma_h\left(\pm 2x_A^{(i)}\right)} \tag{4.51}$$

In this case the μ vector is $4N+2$ dimensionful. The first $4N$ elements are the same of the previous orthogonal case while the extra two are $\frac{\omega_1}{2}$ and $\frac{\omega_2}{2}$.

The partition function of the dual theory is obtained by applying (C.9) to (4.50).

By fixing $k_A > 0$ we have

$$
\begin{aligned}
I^{N+k_A}_{N,2(1+k_A)_a}(\mu) &= I^{N}_{N+k_A,2(1+k_A)_b}(\omega-\mu) \prod_{1\le r<s\le 4N+2} \Gamma_h\left(\mu_r+\mu_s\right)\zeta^{(k_A+1)(1-2k_A)} \\
&\times\ c\left(-4k_A\left(\sum_{i=1}^{N} {x^{(i)}_{A-1}}^2+\sum_{i=1}^{N} {x^{(i)}_{A+1}}^2\right)-\frac{1}{2}k_A\left(\omega_1^2+\omega_2^2\right)\right) \\
&\times\ c\left(\omega^2 k_A\left(2k_A+1+4N-4N\left(\Delta^2_{A-1,A}+\Delta^2_{A,A+1}\right)\right)\right)
\end{aligned}
\tag{4.52}
$$

The case $k_A < 0$ in the electric theory is studied by inverting (C.9). As expected the rank of the dual group is $\tilde{N} = N + |k_A|$. Observe that it fits with the proposal of [134], $SO(\tilde{N}_c) = SO(N_f + |K| - 2 - N_c)$. Indeed in our case $N_c = 2N + 1$, $K = 2k_A$, $N_f = 4N + 2$ and $\tilde{N}_c = 2(N + |k_A|) + 1$.

As before we can transform the measure back to $SO(2(N + |k_A|) + 1)$ by applying (4.51). Then we study the mesons: we have to expand the product

$$
\prod_{1\le r<s\le 4N+2} \Gamma_h\left(\mu_r+\mu_s\right)\times \prod_{1\le r\le 4N} \Gamma_h\left(\mu_r\right)
\tag{4.53}
$$

where the first term come from (4.52) and the second one from (4.50). It is not difficult to recognize the contribution of the dual mesons predicted by the duality. The term $1 \le r < s \le 4N$ gives

$$
\prod_{1\le i<j\le N} \Gamma_h(\pm x^{(i)}_{A\pm1} \pm x^{(j)}_{A\pm1} + 2\omega\Delta_{A,A\pm1})
\tag{4.54}
$$

The extra contributions come from

$$
\prod_{1\le r<\le 4N,4N<s\le 4N+2} \Gamma_h\left(\mu_r+\mu_s\right)\times \prod_{1\le r\le 4N} \Gamma_h\left(\mu_r\right)
\tag{4.55}
$$

Explicitly we have

$$\prod_{i=1}^{N}\left(\prod_{\alpha=1,2}\Gamma_h\left(\pm x_{A\pm1}^{(i)}+\omega\Delta_{A\pm1}+\frac{\omega_\alpha}{2}\right)\right)\times\Gamma_h\left(\pm x_{A\pm1}^{(i)}+\omega\Delta_{A\pm1}\right)=$$
$$=\quad\frac{\Gamma_h\left(\pm 2x_{A\pm1}+\omega\Delta_{A\pm1,A}\right)}{\Gamma_h\left(\pm x_{A\pm1}+\omega\Delta_{A\pm1,A}+\omega\right)}\tag{4.56}$$

The numerator in this expression replaces $i<j$ with $i\leq j$ in (4.54) while the denominator can be transformed as

$$\frac{1}{\Gamma_h\left(\pm x_{A\pm1}+\omega\Delta_{A\pm1,A}+\omega\right)}=\Gamma_h\left(\pm x_{A\pm1}+\omega(1-\Delta_{A\pm1,A})\right)\tag{4.57}$$

which corresponds to the dual of the second line of (4.50).

4.5 Duality and free theories: some exact results

Three dimensional dualities are not only important for theories with an AdS dual but also for more general SCFTs. For example in [129] a new duality was proposed between an $SU(2)_1$ CS theory with an adjoint and no superpotential and a free theory. This duality was further studied in [134], in which an interacting CS matter theory without superpotential is dual to a free theory. While many checks have been performed by expanding the superconformal index, and comparing the expansions on both sides of the dualities, a full understanding of the matching of the partition function is still missing. Here we show the matching between the partition functions analytically.

The models considered in this section do not suffer from accidental symmetries. In every case the partition function matrix integral of the electric interacting theory can be worked out exactly and the extremization of the result sets the R-charges of the magnetic fields to the canonical value without any modification. In general, the naive extremization does not give this result, because the R-symmetry mixes with accidental symmetries. We comment on the latter cases in the next section.

A technical comment is in order. In the following we need some relations involving the integrals dubbed as *JI* in appendix C.1. We take them from [187] and mention them in the text when necessary.

4.5.1 $SU(2)_1$ theory with an adjoint field

The first example is an $SU(2)_1$ CS theory with an adjoint, studied in [129]. The authors proposed a general formula for the partition function in this case but they did not prove this formula analytically. Here we use the results of [187] to show the agreement. The partition function on the round sphere is

$$\begin{aligned} Z_{SU(2)_1}(\Delta) &= \int dx \sinh^2(2\pi x)\, e^{2\pi i x^2} e^{l(1-\Delta)+l(1-\Delta+2ix)+l(1-\Delta-2ix)} \\ &= \frac{1}{4}\int d\epsilon \int dx_1 dx_2 \left(-4\sin(\frac{\pi(x_1-x_2)}{i})\sin(\frac{\pi(x_1-x_2)}{i})\right) \\ &\quad e^{\pi i(x_1^2+x_2^2)} e^{l(1+i\tau)+l(1+i\tau+i(x_1-x_2))+l(1+i\tau-i(x_1-x_2))} e^{2\pi i(x_1+x_2)\epsilon} \end{aligned} \tag{4.58}$$

where we used the relation

$$\int d\,x_1 d\,x_2 \delta(x_1+x_2) = \int d\epsilon \int d\,x_1 d\,x_2 e^{2\pi i(x_1+x_2)\epsilon} \tag{4.59}$$

and we set $\tau = i\Delta$, where Δ represents the R-charge of the adjoint field. In terms of the hyperbolic functions the partition function becomes

$$\begin{aligned} Z_{SU(2)_1}(\Delta) = \quad & -\frac{1}{4\Gamma'_h(\tau)}\int d\epsilon \left(\frac{\Gamma'_h(\tau)^2}{2}\int d\,x_1 d\,x_2 \frac{\Gamma'_h(\tau \pm (x_1-x_2))}{\Gamma'_h(\pm(x_1-x_2)}\right. \\ & \left. \times c'(2\lambda(x_1+x_2)-2(x_1^2+x_2^2))\right) \end{aligned} \tag{4.60}$$

where $\lambda = -2\epsilon$. We used the notations Γ'_h and c' to specify that we are considering $\omega_1 = \omega_2 = i$, i.e. this is the partition function on the three sphere. More generally the formula inside the parenthesis can be associated to the partition function on the squashed three sphere, and the resulting integral has been computed in [187]. Here we quote the result

$$\begin{aligned}&\frac{\Gamma_h(\tau)^n}{\sqrt{(-\omega_1\omega_2)^n}n!}\int\prod_{1<\leq i\leq j\leq n}\frac{\Gamma_h(\tau\pm(x_i-x_j))}{\Gamma_h(\pm(x_i-x_j))}\prod_{j=1}^{n}c(2\lambda x_j-2x_j^2)dx_j=\\&\zeta^{-3n}\prod_{j=1}^{n}\Gamma_h\left(j\tau\right)c\left(\frac{n}{2}\left(2\omega^2+\lambda^2+2(n-1)\tau\omega+\frac{1}{3}(n-1)(2n-1)\tau^2\right)\right)\end{aligned}\tag{4.61}$$

By reducing on the three sphere, fixing $n = 2$ and applying (4.61) we have

$$Z_{SU(2)_1} = -\frac{1}{4\Gamma'_h(\tau)}\int d\epsilon\ \zeta'^{-6}\Gamma'_h(\tau)\Gamma'_h(2\tau)c'\left(-2+\lambda^2+2i\tau+\tau^2\right)\tag{4.62}$$

If we substitue $\zeta' = e^{i\pi/12}$ and $\tau = i\Delta$ in (4.62) and perform the gaussian integration

$$\int_{-\infty}^{\infty} d\,\lambda e^{-\frac{i\pi\lambda^2}{2}} = \frac{2}{\sqrt{2}}e^{-\frac{i\pi}{4}}\tag{4.63}$$

the final expression becomes

$$Z_{SU(2)_1} = \frac{1}{2\sqrt{2}}e^{l(1-2\Delta)}e^{\frac{i\pi}{2}(1+\Delta)^2-\frac{i\pi}{4}}\tag{4.64}$$

which coincides with the one proposed by [129].

4.5.2 SO$(4)_1$ with the adjoint field

As discussed in [134], the $SO(4)_1$ with an adjoint reduces to two copies of the [129] duality, because $SO(4) \simeq SU(2) \times SU(2)$. Here we show that the partition function can be exactly computed in the SO cases by using the results of [187] and reproduce the

$SO(4)_1$ case explicitly. In this case we need the relation

$$
\begin{aligned}
&\frac{\Gamma_h(\tau)^n}{\sqrt{(\omega_1\omega_2)}^n 2^n n}\int \frac{\prod_{1\leq i<j\leq n}\Gamma_h(\tau\pm x_i\pm x_j)\prod_{j=1}^{n}\prod_{r=1}^{3}\Gamma_h(\mu_r\pm x_j)}{\prod_{1\leq i<j\leq n}\Gamma_h(\pm x_i\pm x_j)\prod_{j=1}^{n}\Gamma_h(\pm 2x_j)}\prod_{j=1}^{n}c(-2x_j^2)dx_j \\
&=\prod_{j=1}^{n}\Gamma_h(j\tau)\prod_{1\leq r<s\leq 3}\Gamma_h(j\tau+\mu_r+\mu_s) \\
&\times c\left(n\left(2(\mu_0\mu_1+\mu_1\mu_2+\mu_2\mu_0)+2(n-1)\tau\sum_{r=1}^{3}\mu_r+\frac{1}{3}(n-1)(4n-5)\tau^2\right)\right)
\end{aligned}
\tag{4.65}
$$

This equation can be applied to the $\mathrm{SO}(2n)_1$ case after we identify

$$
\mu_1=0 \quad , \quad \mu_2=\frac{\omega_1}{2} \quad , \quad \mu_3=\frac{\omega_2}{2}
\tag{4.66}
$$

because, by applying (4.7) and (4.9), we have

$$
\frac{\Gamma_h(\pm x)\,\Gamma_h\left(\pm x+\frac{\omega_1}{2}\right)\Gamma_h\left(\pm x+\frac{\omega_2}{2}\right)}{\Gamma_h(\pm 2x)}=1
\tag{4.67}
$$

Formula (4.65) then reduces to the partition function of a $\mathrm{SO}(2n)_1$ theory with a field in the adjoint representation. If we reduce to the case $n=2$ and fix $\tau=i\Delta$ and $\omega_1=\omega_2=i$ the partition function on the three sphere is

$$
e^{2l(1-2\Delta)}e^{2\pi i\left(\Delta+\frac{1}{2}\right)^2-\frac{3}{2}\pi i}
\tag{4.68}
$$

which reduces to two copies of $SU(2)_1$ theories with the adjoint and differs from that case just by a phase factor.

4.5.3 $SP(4)_2$ with an absolutely antisymmetric field

In the case of symplectic groups also there are exact relations in [187] that can be applied to obtain a CS matter theory dual to a free theory.

Here we study the irreducible absolutely antisymmetric representation, described by the Dynkin label $\vec{s} = (1,1,0\ldots,0)$ (see Appendix C.2 for details). By using the Schur polynomial in the appendix the character of this irreducible representation is

$$\chi_{(1,1,0,\ldots,0)} = \sum_{i\neq j}\left(z^i z^j + z^i z^{-j} + z^{-i} z^j + z^{-i} z^{-j}\right) + N - 1 \tag{4.69}$$

In this case we need the equality [187]

$$\begin{aligned} \frac{\Gamma_h(\tau)^{n-1}}{\sqrt{(-\omega_1\omega_2)^n}2^n n!}\int \prod_{1\leq i<j\leq n}\frac{\Gamma_h(\tau\pm x_i\pm x_j)}{\Gamma_h(\pm x_i\pm x_j)}\prod_{j=1}^{n}\frac{1}{\Gamma_h(\pm 2x_j)}c(-8x_j^2)dx_j = \\ = \quad \zeta^{-3n}\prod_{j=2}^{n}\Gamma_h(j\tau)\, c\left(n\left(3\omega^2+3(n-1)\tau\omega+\frac{1}{6}(n-1)(2n-7)\tau^2\right)\right) \end{aligned} \tag{4.70}$$

For $n = 2$ it represents the partition function for a $SP(4)_2$ gauge theory with an absolutely antisymmetric two index tensor. By fixing $\tau = i\Delta$ and $\omega_1 = \omega_2 = i$ the partition function on the three sphere becomes

$$\mathcal{Z}_{SP(4)_2} = e^{l(1-2\Delta)}e^{-\frac{i\pi}{2}(\Delta-3)^2} \tag{4.71}$$

This relation suggests that this theory is dual to a free theory with a singlet.

A similar duality appeared in [134], however the antisymmetric representation considered there was not irreducible and contained another singlet. This extra singlet adds a $\Gamma_h(\tau)$ factor on both sides of (4.71), leaving the equality unchanged. In this case the theory is dual to a theory with two singlets with charges Δ and 2Δ. This case contains accidental symmetries which mix wit the R-symmetry and need to be properly accounted in the extremization of the partition function. We will comment on this issue in section 4.6.

The superconformal index

The superconformal index is a Witten like index which counts over the protected BPS states of the theory. The index for three dimensional theories with $\mathcal{N} \geq 3$ SUSY was first proposed in [46] by localizing the theory on on $S^2 \times S^1$. The expression for the index is given by

$$\mathcal{I}(x, y_i) = \text{Tr}(-1)^F x^{E+j_3} \prod_i y_i^{F_i} \tag{4.72}$$

where F is the fermion number, E is the energy, j_3 is the third component of the $SU(2)$ rotational symmetry in the superconformal group. This index was refined to include the monopole contributions in [141]. For theories with $\mathcal{N} = 2$ supersymmetry the R-charge is not constrained to be canonical anymore, and the index for a generic R-charge assignment was found in [124]. It is important to observe that, in the general case, dual theories share the same index only after the contribution from the monopole sectors is included. In some cases the index matches sector by sector but in general one has to sum over all the sectors. For example if an interacting theory is dual to a free theory one has to necessarily include the monopole corrections before matching the indices.

Here we consider the index of the $SP(4)_2$ CS theory with one matter field in the absolutely antisymmetric representation and R-charge Δ. After including the contribution from monopoles with GNO charge $(1,0)$ the superconformal index is[7]

$$\begin{aligned}\mathcal{I} &= \left(1 - x^2 + x^{2\Delta} + x^{4\Delta} + x^{6\Delta} + x^{8\Delta} + x^{10\Delta} + x^{12\Delta} + x^{14\Delta} + \dots\right)_{(0,0)} \\ &\quad + \left(-x^{2-2\Delta} - x^{4-2\Delta} + \dots\right)_{(1,0)} + \dots \\ &= 1 - x^2 - x^{2-2\Delta} - x^{4-2\Delta} + x^{2\Delta} + x^{4\Delta} + x^{6\Delta} + x^{8\Delta} + x^{10\Delta} + x^{12\Delta} + x^{14\Delta} + \dots\end{aligned} \tag{4.73}$$

This coincides with the index of a free multiplet with R-charge 2Δ, corroborating the duality proposed above.

4.6 Comments on accidental symmetries

In this section we briefly comment on a proposal to deal with accidental symmetries in three-dimensional field theories. We will adapt to the three-dimensional case a similar prescription used in the four-dimensional a-maximization [146], with the respective physical meaning [38], which also allows for an extension away from the fixed points [17] based on the four dimensional analogy [144]. For a preliminary discussion, see [159].

Any time the fixed point scaling dimension of a scalar gauge invariant operator drops below the d-dimensional unitary bound $\Delta \geq (d-2)/2$, this signals that the UV description that we are using to extract information about the IR physics is no longer valid, because the theory enjoys new "accidental" symmetries which are not manifest in the UV description. The new symmetries are generated by the gauge invariant operators, which decouple from the rest of the theory in the IR: they retain their canonical scaling dimensions and describe free fields. In these cases we need to modify the UV description in a suitable way, which we describe in the following.

Consider a model where m gauge invariant operators $\mathcal{O}_i$, $i = 1, \dots, m$ hit the

[7] GNO charges are quantum numbers labeling the different monopole sectors of the theory [101] . In the $SP(4)$ case the GNO charge of a sector carrying m unit of magnetic flux is $(m, 0)$.

unitary bound, and consider coupling to that theory m sources L_i and m gauge invariant operators M_i through the superpotential

$$\Delta W = L^i \left(\mathcal{O}_i + \lambda M_i\right) \tag{4.74}$$

where λ is small in the UV. The operators $\mathcal{O}_i$ are, in general, not related to each other, and so are the L's and the M's. Imposing the condition $R(L_i) + R(\mathcal{O}_i) = 2$ we see that when $R(\mathcal{O}_i) > (d-2)/(d-1)$ the last term is indeed relevant and makes the fields L and M massive.[8] Once they are integrated out, we obtain the IR superconformal theory we started with, and no physical quantity has changed.[9] On the other hand, when $R(\mathcal{O}_i) < (d-2)/(d-1)$, the LM coupling is irrelevant and the M's are free decoupled fields in the IR.

In the case of three-dimensional field theories, where $R = \Delta$, a free field contributes a factor $\exp\left(\ell(1/2)\right) = 2^{-1/2}$ to the partition function, or equivalently a term $\log(2)/2$ to the free energy. The R-charge of the L's is fixed by the first term in (4.74), and their contribution to the partition function is $\exp\left(m\ell(-1+\Delta(\mathcal{O})\right)$. Summing everything up, we obtain

$$F = F_0 + \left(m\frac{\log(2)}{2} + \sum_{i=1}^{m} \ell\left(1 - \Delta(\mathcal{O}_i)\right)\right) \tag{4.75}$$

where we also used $\ell(1-\Delta) = -\ell(-1+\Delta)$ for $0 < \Delta < 2$, which is always the case in any sensible theory (see footnote 9). Equation (4.75) has a very clear interpretation: along the RG flow, the R-charges as a function of the RG scale are given by the Lagrange multiplier technique [17]; when a gauge invariant operator hits the unitary bound, one subtracts

[8] Recall that in a superconformal field theory the R-charge and the scaling dimension are related by $R = 2\Delta/(d-1)$, where the superpotential has R-charge 2.

[9] This is a physical requirement on any physical quantity that depends on the exact superconformal R-charges: the contribution from massive fields has to cancel out.

its contribution to the free energy and adds the contribution of the same number of free fields. Because both the correction term to (4.75) and its first derivative vanish at the free field point $\Delta(\mathcal{O}_i) = 1/2$, all the R-charges and the free energy itself are continuous and differentiable functions of the RG scale.

4.6.1 Accidental symmetries in the duality with free theories

In section 4.5 we focused on theories with a free magnetic dual whose partition function can be exactly and consistently computed by localization and extremization without any further modification. We now apply the discussion in the previous subsection and show how the computation of the exact superconformal R-charge can be consistently worked out even when the infrared theory enjoys accidental symmetries. This provides new and stronger checks of three-dimensional dualities.

We start by describing an example in some detail, in which the dual gauge group vanishes and the magnetic theory only contains a tower of non-interacting singlets with naive R-charges different from the canonical ones. The simplest electric theory of this kind has $U(N_c)_1$ gauge group and contains one adjoint X with a vanishing superpotential [134].

The partition function of the $U(N_c)_1$ theory with one adjoint X can be exactly computed [187]

$$\mathcal{Z}_{U(N_c)_1,X} = e^{\frac{i\pi}{12}N\left(3+6\Delta(N_c-1)+\Delta^2\left(2N_c^2-3N_c+1\right)\right)} \times \prod_{j=1}^{N_c} e^{l(1-j\Delta)} \tag{4.76}$$

where $j\Delta$ is the R-charge of $\mathrm{Tr}X^j$. Notice that the $U(1) \subset U(N_c)$ decouples and $\mathrm{Tr}X$ is a free field. However, for the sake of uniform treatment, we keep its R-charge to be Δ instead of $1/2$.

The naive R charges of the N_c free fields of the magnetic theory, given by

u_j =TrX^j, are obtained by extremizing (4.76), which boils down to solving the equation

$$\frac{\partial \log \left|\mathcal{Z}_{U(N_c)_1,X}(\Delta, N_c)\right|}{\partial \Delta} = \sum_{j=1}^{N_c} j\pi(1-j\Delta)\cot(\pi(1-j\Delta)) = 0 \tag{4.77}$$

The solution is

$$\Delta = \frac{1}{N_c+1} \tag{4.78}$$

Proving that (4.78) solves (4.77) is pretty straightforward. Indeed

$$\left.\frac{\partial \log \left|\mathcal{Z}_{U(N_c)_1,X}(\Delta, N_c)\right|}{\partial \Delta}\right|_{\Delta=\frac{1}{N_c+1}} = \sum_{j=1}^{N_c}\left(\frac{j(N_c+1-j)}{N_c+1}\right)\cot\left(\frac{j\,\pi}{N_c+1}\right) =$$

$$= \quad \frac{1}{2}\left(\sum_{j=1}^{N_c}\left(\frac{j(N_c+1-j)}{N_c+1}\right)\cot\left(\frac{j\,\pi}{N_c+1}\right) + \left(j\to N_c+1-j'\right)\right) = 0 \tag{4.79}$$

It follows that the singlets do not have the canonical scaling dimension, and there are $\left[\frac{N_c+1}{2}\right]$ gauge invariant operators with R-charge below or at the unitarity bound; thus, we have to treat them as free fields, and modify the extremization principle according to equation (4.75). We interpret this first step by noticing that along the RG flow, the operators TrX^j with $j < \left[\frac{N_c+1}{2}\right]$ will hit the unitarity bound at higher energies. For high enough N_c, this is not the end of the story: extremization of the modified free energy shows that we did not cure all the accidental symmetries. Again, roughly half of the operators have R-charges below or at the unitarity bound, and we again apply (4.75). [10] The process continues until all but one operator, namely u_{N_c}, remains and we end up

[10]More precisely, the number of operators is $\left[\frac{N_c+2}{4}\right]$ for N_c even and $\left[\frac{N_c+1}{4}\right]$ for N_c odd, and the solution is $\Delta = \frac{2}{3N_c+2}$ and $\Delta = \frac{2}{3(N_c+1)}$ for N_c even and odd respectively. These formulas can be proved by induction. Since a proof would be very marginal to our discussion, we do not include it in this paper.

with the following modified partition function

$$\left|\mathcal{Z}_{U(N_c)_1,X}\right| = 2^{-\frac{N_c-1}{2}} e^{l(1-N_c\Delta)} \tag{4.80}$$

which is extremized at $N_c\Delta = 1/2$. We have then shown that the $U(N_c)_1$ partition function coincides with the one of N_c free fields u_j, and that a proper treatment of the accidental symmetries allows us to identify the duality map as $u_j = \mathrm{Tr}X^j$. The same arguments may be carried over to other models.

4.7 Open questions

We provided some nontrivial evidence for classes of infinite three-dimensional dualities for theories with unitary, orthogonal and symplectic gauge groups. Our results provide support for arbitrary gauge group and CS levels, and extend previous results which were limited either to the large-N limit or to numerical evaluations for low ranks and one factor in the gauge group.

Our main tool has been the exact, all-loop partition function evaluated on a squashed three sphere. Allowing for arbitrary R-charges, it can be written as an integral of hyperbolic functions which have been recently studied by mathematicians.

Exact evaluation of the above quantities, available in the literature for classical gauge groups, allowed to uncover new dualities. In the large-N limit, and for low enough CS levels, they could also be inferred by the AdS/CFT duality, and exact evaluation of the above quantities allows for an extension to arbitrary ranks and levels.

Unitary gauge groups have been extensively studied in the large-N limit, and precise prescriptions for the computation of the partition function in this regime are available in the literature. Because of its simplifying nature, it is much more tractable than the computation of the finite-N partition function and it allows for comparison of physical quantities in the AdS/CFT correspondence. Based on this observation, we tried analyzing the case of the other classical gauge groups, where a similar analysis still lacks. We found that the set of saddle point equations are not consistent with the long range cancellation in these cases. Thus, the continuum limit would require a different approach. A similar situation also holds in chiral-like models for unitary gauge groups [19, 16]. There exist other dualities between quivers with unitary gauge groups and quivers with symplectic/orthogonal gauge groups [8]. These dualities suggest that the theories with symplectic and orthogonal groups also exhibit the $N^{3/2}$ scaling of the free energy at large N. It will be interesting to prove the matching of the partition function for these dualities along the lines of this paper.

Some of the dualities we have studied involve free field theories on the magnetic side. Some comments are in order. Any nontrivial check involving the partition function in this case requires the possibility of an exact evaluation of the full matrix integral, because on the free theory side there is no integral at all. Secondly, when one considers such theories, it turns out that the free theory contains n free fields with charge $j\Delta$, with Δ the smallest charge and $j = 1, \ldots, n$. While this constitutes an offshell check of the duality, we know that a free field has R-charge 1/2, which cannot be obtained by extremization of the naive partition function. If the duality holds, this means that the electric R-symmetry mixes with an accidental symmetry and we showed how to handle this scenario in Section 4.6.

More generally accidental symmetries arise in presence of gauge theories with tensor matter and superpotential [134]. These dualities are three dimensional generalizations of the KSS dualities [145]. It would be interesting to study the matching of the partition functions in these cases, as already proposed in [159], at finite values for the ranks of the gauge groups and CS level.

We conclude by recalling that accidental symmetries are one of the main issue in the proof of a c-theorem.[11] In the three dimensional case the candidate c-function in is the free energy F on the round S^3 ($F = -\log|\mathcal{Z}|$), which has been conjecture to decrease along the RG flow [131]. Relevant deformations break the abelian symmetries which are manifest in the UV description of the theory and once we have a quantity that is maximized by the exact superconformal R-symmetry [12] we can interpret it as the c-function. The c-theorem immediately follows from the two line "almost proof" of [128]. However accidental symmetries constitute a loophole to this argument and a proof of the F-theorem requires more care in this case: the free field value is a maximum for the function $-\ell(1-\Delta)$, thus the infrared correction term in (4.75) is always positive, for any value of the scaling dimensions, in full agreement with the maximization of F. However, the correction term adds a positive contribution to F_{IR}, possibly invalidating the F-theorem $F_{IR} < F_{UV}$.

This chapter is a reprint of the material as it appears in "Refined Checks and Exact Dualities in Three Dimensions " , Prarit Agarwal, Antonio Amariti, Massimo Siani , JHEP 1210 (2012) 178, of which I was a co-author.

[11] See [15] for other subtleties related to them.

[12] A proof of the maximization of F has been given recently in [61].

Chapter 5

New N=1 Dualities from M5-branes and Outer-automorphism Twists

5.1 Introduction

Supersymmetric gauge theories have been extremely fruitful in our endeavor to uncover the rich structure of quantum field theory. One of the most remarkable phenomenon discovered in supersymmetric gauge theory is Seiberg duality [171] where two different UV gauge theories flow to the same fixed-point in the IR . The original example studied by Seiberg was $\mathcal{N} = 1$ SQCD with $SU(N)$ gauge group, which was subsequently generalized to $SO(N)$ gauge groups by Intriligator-Seiberg [126] and to $USp(2N)$ gauge groups[1] by Intriligator-Pouliot [125].

Recently, a new dual description to $SU(N)$ SQCD has been found by Gadde-Maruyoshi-Tachikawa-Yan (GMTY) [82]. Their new dual theory involves coupling two copies of the so-called T_N theory. The new theory can be thought of as a generalization of the (multiple) self-duality of Csaki-Schmaltz-Skiba-Terning [63] from $SU(2)$ to $SU(N)$. The main component they used was the T_N theory which arises from wrapping N coincident M5-branes or A_{N-1} six-dimensional $\mathcal{N} = (2, 0)$ theory on a 3-punctured sphere [88].

[1] In this paper we use the notation $USp(2N) = C_N$ for the symplectic groups so that $USp(2) = SU(2)$.

One of the objectives of this paper is to generalize the GMTY duality to the SO/USp theories thereby adding more dual theories in addition to the ones found in [126, 125]. Moreover, we will show that there is not just one new dual theory but three more dual descriptions to each theory. From this, we argue there are five different theories in the UV that flow to the same superconformal theory in the IR.

We also find new dual theories for the G_2 gauge theory with 8 fundamentals. G_2 is the simplest group with a trivial center and hence QCD with a G_2 gauge group provide us with an opportunity to study the role of the center of a gauge group in confinement [120]. A dual for G_2 QCD with 5 flavors was discussed in [99, 163] while for $5 < N_f < 12$, a magnetic theory with an $SU(N_f - 3)$ gauge group was found by Pouliot [165]. The duality frames discovered in this paper are either non-Lagrangian or based on $Spin(8)$ gauge group and hence constitute a new class of magnetic theories.

Two dual frames among five have Lagrangian descriptions. The 'electric theory' $\mathcal{U}$ is the original SQCD with certain number of flavors and the 'magnetic theory' $\mathcal{U}_{c1}$ is also an SQCD with the same number of flavors[2] but also has mesons coupled through a superpotential. Three non-Lagrangian dual theories can be categorized into 'swap' theories $\mathcal{U}_s$ following the nomenclature of [82], and Argyres-Seiberg type $\mathcal{U}_{as}$ since it can be thought of as $\mathcal{N} = 1$ version of the dualities found in [24], and the crossing type $\mathcal{U}_{c2}$.

Our discussion is motivated from the six-dimensional construction of $\mathcal{N} = 1$ superconformal field theories. It is an extension of the so-called the $\mathcal{N} = 2$ theories of class $\mathcal{S}$ [90, 88]. A class $\mathcal{S}$ theory is constructed by compactifying the six-dimensional $\mathcal{N} = (2, 0)$ theory of type $\Gamma = A, D, E$ on a Riemann surface $\mathcal{C}$ with a partial topological twist. This gives rise to $\mathcal{N} = 2$ theory in 4-dimensions labeled by $\mathcal{C}$ called the UV curve. Since any (negatively curved) Riemann surface can be decomposed in terms of pair of pants or 3-punctured sphere, it is natural to associate a 4d theory to a 3-punctured sphere

[2] Except for the G_2 case where both the gauge group and matter contents changed.

and regard it as a building block for the 4d theory. The 4-dimensional theory associated to $\mathcal{C}$ has to be the same regardless of how we decompose the Riemann surface. The statement of duality is equivalent to saying that the different pair-of-pants decompositions give rise to the same 4-dimensional theory.

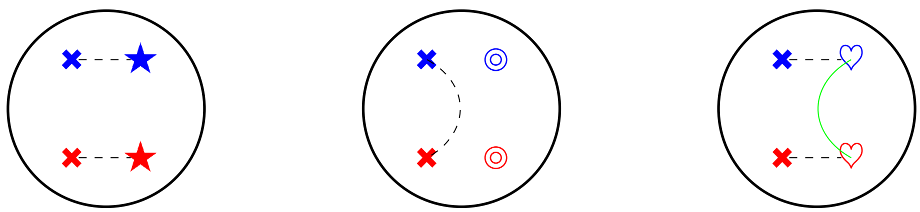
In order to write down various dual theories, one needs to identify the theory corresponding to the various different types of three punctured spheres. This has been extensively studied, for example in [53, 54, 55, 57], from which they find new $\mathcal{N} = 2$ SCFTs and dualities. The class $\mathcal{S}$ construction for the D_N type was first studied in [181] and the effect of outer-automorphism twists has been studied in [182].

One can generalize this construction to $\mathcal{N} = 1$ theory. The simplest way is to give mass to the chiral adjoints in the $\mathcal{N} = 2$ vector multiplets. In the IR, the massive chiral adjoints will be integrated out and we land on a SCFT [153, 41]. One can construct more general theories by requiring non-baryonic $U(1)_{\mathcal{F}}$ to be conserved. This gives rise to a new class of $\mathcal{N} = 1$ SCFTs generically not the same as the mass-deformed $\mathcal{N} = 2$ theories in class $\mathcal{S}$ [37, 34]. This class of theories are subsequently generalized to include the Riemann surface with punctures in [82, 193, 35] so that the theory can have larger global symmetries. Further studies of $\mathcal{N} = 1$ class $\mathcal{S}$ theories have been done in [152, 49, 196, 197, 31]. In this paper, we generalize this construction to the case of $\Gamma = D_N$ series with outer-automorphism twists.

This construction requires extra data beyond the choice of the Riemann surface, namely the degree of the normal bundles $\mathcal{L}(p) \oplus \mathcal{L}(q) \to \mathcal{C}_{g,n}$ with $p + q = 2g - 2 + n$. This stems from the fact that we have one parameter ways to twist the 6d $\mathcal{N} = (2, 0)$ theory while preserving $\mathcal{N} = 1$ SUSY in 4-dimensions.[3] The punctures also have to be more general than the $\mathcal{N} = 2$ counterpart. In our case, we put $\mathbb{Z}_2$ valued 'color' to the

[3]For the purpose of preserving supersymmetry, the rank 2 bundle $E \to \mathcal{C}_{g,n}$ is not necessarily given by a sum of two line bundles. The only necessary condition is to have $\det E$ equal to the canonical bundle $K_{\mathcal{C}_{g,n}}$. But here we restrict ourselves to the case where the rank 2 bundle is given by a direct sum.

punctures in addition to the usual $\mathcal{N}=2$ data. In order to realize SQCD of gauge group $SO(2N)/USp(2N-2)$ with $(4N-4)/(4N)$ fundamental quarks,[4] we put the $\Gamma = D_N$ theory on a 4-punctured sphere with two twisted full punctures with each color and two twisted null punctures with each color and choose the normal bundle to be $(p,q) = (1,1)$. For the case of G_2 theory, start with $\Gamma = D_4$ with 4 punctures, but also with $\mathbb{Z}_3$ twist line running between two $USp(4)$ punctures of each color. We also need two twisted null punctures of each color as well.



(a) $SO(2N)$ theory **(b)** $USp(2N-2)$ theory **(c)** G_2 theory

Figure 5.1. The UV curves realizing SQCDs in this paper. The symbol ✖ denotes twisted null puncture, ◎ the full puncture having $SO(2N)$ flavor symmetry, ★ the twisted full puncture having $USp(2N-2)$ flavor symmetry and ♡ denotes $USp(4)$ puncture. The dashed line and the green solid line denote $\mathbb{Z}_2$ and $\mathbb{Z}_3$ twist line respectively.

The notion of pair-of-pants decomposition needs an extra ingredient because of the normal bundles. It can be realized by putting colors to the pair-of-pants itself. It turns out there are five different colored pair-of-pants decompositions for our setup, thereby giving five dual frames to the SQCD.[5] The list of dual theories we find are summarized as follows. For the $SO(2N)$ theory, the five dual frames are:

- $\mathcal{U}^{SO}$: $SO(2N)$ with $4N-4$ fundamentals (vectors)
- $\mathcal{U}^{SO}_{c1}$: $SO(2N)$ with $4N-4$ fundamentals and mesons [126]
- Three non-Lagrangian duals: $\mathcal{U}^{SO}_{s}, \mathcal{U}^{SO}_{as}, \mathcal{U}^{SO}_{c2}$

[4]The number of flavors here is counted by the number of chiral multiplets. This is in contrast with the $SU(N)$ theory, which has both quarks and anti-quarks.

[5]Actually there is one more in terms of colored pair-of-pants decomposition, but it is identical to one of five upon inverting the color.

and for the $USp(2N-2)$ theory:

- $\mathcal{U}^{Sp}$: $USp(2N-2)$ with $4N$ fundamentals
- $\mathcal{U}^{Sp}_{c1}$: $USp(2N-2)$ with $4N$ fundamentals and mesons [125]
- Three non-Lagrangian duals: $\mathcal{U}^{Sp}_{s}$, $\mathcal{U}^{Sp}_{as}$, $\mathcal{U}^{Sp}_{c2}$

and for the G_2 theory:

- $\mathcal{U}^{G_2}$: G_2 with 8 fundamentals
- $\mathcal{U}^{G_2}_{c1}$: $Spin(8)$ with 6 quarks in 8_V and 8_S and mesons
- Three non-Lagrangian duals: $\mathcal{U}^{G_2}_{s}$, $\mathcal{U}^{G_2}_{as}$, $\mathcal{U}^{G_2}_{c2}$

Three out of five dual theories are non-Lagrangian. We will explain these non-Lagrangian duals in detail in later sections.

We provide evidence to these dualities through computing the anomaly coefficients and the superconformal indices. In order to compute the superconformal index of G_2 theory, we also discuss $\mathcal{N}=2$ index with $\mathbb{Z}_3$ twist line and G_2 puncture. Especially, we find that the theory with UV curve given by three punctured sphere with $USp(6)$, G_2 and twisted null punctures has enhanced E_7 flavor symmetry as expected in [182] where it was identified as the theory of Minahan-Nemeschansky [157].

The paper is organized as follows. In section 5.2, we review construction of the $\mathcal{N}=1$ theories of class $\mathcal{S}$ from which we construct our dual theories. We will also discuss the effect of outer-automorphism twist in the setup. In section 5.3, we propose dualities of $SO(2N)$ gauge theories and check the 't Hooft anomaly coefficients. In section 5.4, we discuss the dualities of $USp(2N-2)$ gauge theories. In the section 5.5, we discuss the dualities of G_2 gauge theory. Finally, in section 5.6, we check our duality proposals by computing the superconformal index. In the appendix, we derive certain chiral ring relations for the $T_{SO(2N)}$ and the twisted $\tilde{T}_{SO(2N)}$ blocks, which are necessary in other sections.

5.2 Constructing $\mathcal{N}=1$ theory from M5-branes

In this section, we review the construction of 4d $\mathcal{N}=1$ theories from 6d perspective due to [153, 41, 37, 34, 82, 193, 35]. From this, we propose several dual theories based on different ways of gluing the 3-punctured spheres.

Setup and Data

In order to obtain an $\mathcal{N}=1$ SCFT from M5-branes dubbed the theories of class $\mathcal{S}$, we need the following data:

- Choice of the 'gauge' group $\Gamma = A_n, D_n, E_{6,7,8}$.
- Riemann surface $\mathcal{C}_{g,n}$ of genus g and n punctures. We call it a UV-curve.
- Choice of two normal bundles $\mathcal{L}_1(p), \mathcal{L}_2(q)$ of degree p, q over $\mathcal{C}_{g,n}$ such that $p+q=2g-2+n$.
- The choice of appropriate boundary condition on each punctures.

The choice of Γ labels the 6-dimensional $\mathcal{N}=(2,0)$ theory and we compactify the 6d theory on $\mathcal{C}_{g,n}$ to obtain the $\mathcal{N}=1$ theory in 4-dimension. When compactifying the theory, we have to perform partial topological twist in order to preserve any supersymmetry. It turns out that there is an integer parameter family of different ways to twist the theory while preserving 4 supercharges. This can be understood as the choice of the normal bundles $\mathcal{L}_1(p) \oplus \mathcal{L}_2(q) \to \mathcal{C}_{g,n}$. The total space of this rank-2 bundle becomes Calabi-Yau 3-fold if it satisfies $p+q=2g-2+n$.

The data on a puncture is specified by the following conditions which are all equivalent:

- $\frac{1}{4}$-BPS boundary condition of $\mathcal{N}=4, d=4$ SYM theory.

- Choice of the singular boundary condition of a generalized Hitchin equation on $\mathcal{C}_{g,n}$

$$
\begin{aligned}
&D_{\bar{z}}\Phi_1 = D_{\bar{z}}\Phi_2 = 0 \ , \\
&[\Phi_1, \Phi_2] = 0 \ , \\
&F_{z\bar{z}} + [\Phi_1, \Phi_1^*] + [\Phi_2, \Phi_2^*] = 0 \ .
\end{aligned} \tag{5.1}
$$

- Choice of the singular boundary condition of a generalized Nahm's equation.

When one of p or q is zero, then we go back to the $\mathcal{N}=2$ theories of class $\mathcal{S}$ [90, 88]. In this case, the data on the puncture is specified by a $\frac{1}{2}$-BPS boundary condition of $\mathcal{N}=4, d=4$ SYM theory, or the embedding of $SU(2)$ group to Γ. Equivalently, one of the Higgs field $\Phi_{1,2}$ vanishes and we get the ordinary Hitchin equation. When $\Gamma = A_{n-1}$ it is labeled by a Young tableau with n boxes.

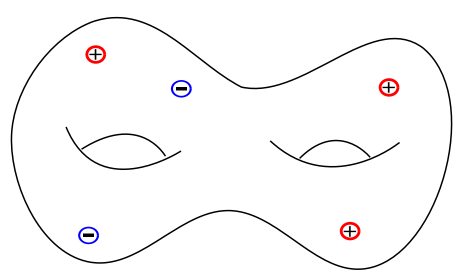


Figure 5.2. A choice of UV curve with colored punctures. Here we suppressed the labeling ρ for each punctures.

Colored $\mathcal{N}=2$ punctures

Generally, $\mathcal{N}=1$ puncture will involve both Φ_1 and Φ_2 in (5.1) developing singularities at the same point where the punctures sits. Throughout the paper we restrict ourselves to the case where only one of them develops a singularity at a given point. In a sense this makes our system $\mathcal{N}=2$-like near the puncture. We will label each puncture by a color $\sigma = \pm$ along with the choice of embedding $\rho : SU(2) \to \Gamma$. We will call them as colored $\mathcal{N} = 2$ punctures.

When the group Γ admits an outer-automorphism (when the corresponding Dynkin diagram is symmetric under a discrete action o), we can twist the punctures accordingly [181, 182, 56]. Once we twist the puncture, the punctures are no longer

Table 5.1. The group Γ changes to G under the outer-automorphism twist o. It is given by the Langlands-dual of the $G^\vee$ which is the subgroup of Γ invariant under o.

Γ	A_{2n-1}	A_{2n}	D_{n+1}	D_4	E_6
o	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
G	B_n	C_n	C_n	G_2	F_4
$G^\vee$	C_n	B_n	C_n	G_2	F_4

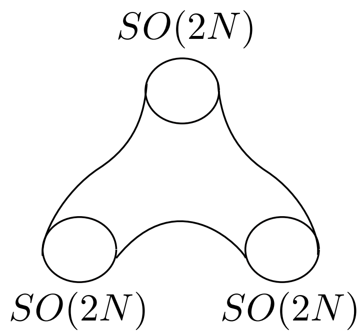
labeled by the $SU(2)$ embedding into Γ but into G, see Table 5.1. Another thing to notice here is that the number of twisted punctures cannot be arbitrary, but is required to be such that the product of monodromies around the punctures should be equal to one.[6] For example, we need to have even numbers of $\mathbb{Z}_2$-twisted punctures. In the case with $\mathbb{Z}_3$ punctures, we could also have odd number of $\mathbb{Z}_3$-twisted punctures as in the figure 5.23.

Colored pair-of-pants decomposition

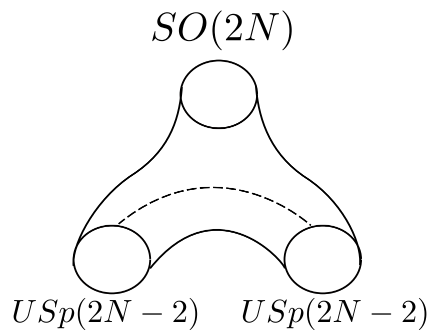
For a given such configuration, we can have various different dual frames by considering different pair-of-pants decompositions. On each pair of pants, we also label it by a color $\sigma = \pm$. The number of the pair-of-pants labelled by $+$ and by $-$ are given by the degree of line bundles p and q respectively. Now, for a given pair of pants, we have the following data:

1. The choice of color σ^p of the pair of pants itself.

[6] We thank Yuji Tachikawa for bringing this to our attention.



(a) D_N theory on sphere with 3 untwisted punctures



(b) D_N theory on sphere with 1 untwisted and 2 twisted punctures

Figure 5.3. By twisting the punctures of D_N theory, we get twisted punctures having the C_{N-1} flavor symmetry.

2. (ρ_i^p, σ_i^p) where $\rho_i^p : SU(2) \to G$ labels the $SU(2)$ embedding in G and σ_i^p denotes a coloring for each punctures $i = 1, 2, 3$.

When we glue two pair of pants, we gauge the flavor symmetry associated to punctures we glue. When the σ^p of two pair of pants are the same, we gauge it using the $\mathcal{N} = 2$ vector multiplets, and when the σ^p are different, we glue it through $\mathcal{N} = 1$ vector multiplet. Note that when we glue two punctures, we can always choose the coloring of the punctures as the same as the pair of pants that we are gluing. See figure 5.4 for an illustration of the construction.

Now, for a given colored pair of pants with color σ^p, we identify the building block as follows. When all the punctures have the same color as the pair of pants itself,

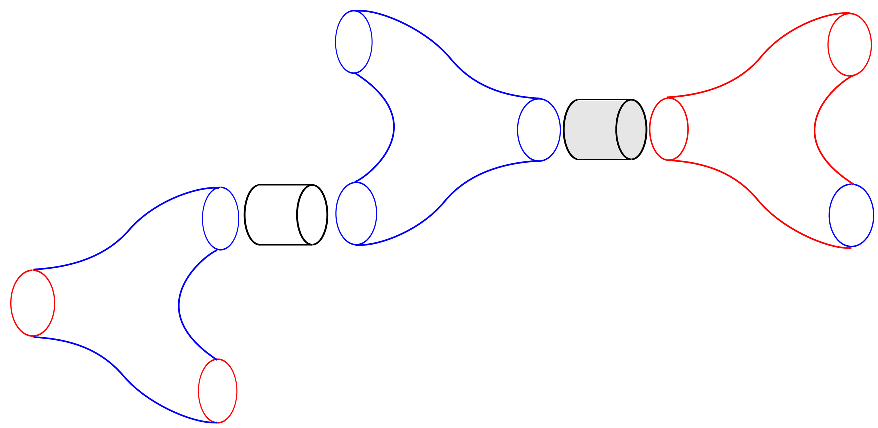


Figure 5.4. An example of colored pair-of-pants decomposition. Here red/blue means $\sigma = \pm$ respectively. Three red punctures and two blue punctures with $p = 1, q = 2$. Grey tube denotes $\mathcal{N} = 1$ vector, white tube denotes $\mathcal{N} = 2$ vector multiplet. We have 3 mesons associated to the blue puncture on the right and two red punctures on the left.

we identify the theory as the same one as $\mathcal{N} = 2$ theory. For example, when all the punctures are (untwisted) full punctures, then we get T_Γ theory.[7] When a full puncture has a different sign from the pair of pants, we add a 'meson' field that transforms as an adjoint of Γ associated to the puncture. Moreover we add a superpotential term for the meson field: $W = \mathrm{Tr}(M\mu)$, where μ is an operator associated to the puncture. The operator μ transforms under the adjoint representation of Γ and has the conformal weight $\Delta = 2$.

For a theory in class $\mathcal{S}$, we have $U(1)_\mathcal{F}$ global symmetry in addition to the $\mathcal{N} = 1$ superconformal symmetry and the global symmetry labeled by the punctures. Suppose we have only maximal punctures meaning ρ is given by the trivial embedding and has the full global symmetry G. We define the $U(1)_\mathcal{F}$ global charge to be

$$\mathcal{F} = \sum_i \sigma_i J_i \ , \tag{5.2}$$

where J_i are the global $U(1)$ charge at each pair of pants. Note that each pair-of-pants or

[7] For $\Gamma = A_{N-1}$, it is usually called as T_N theory.

three punctured sphere describes $\mathcal{N}=2$ superconformal theory. It has $SU(2)_R \times U(1)_R$ R-symmetry which is broken down to $U(1)_R \times U(1)_{J_i}$ upon coupling to $\mathcal{N}=1$ vectors. The coloring σ_i labels the choice of the sign of $U(1)_{J_i}$ charge we can make.

The color σ_i^p of a puncture tells us the charges of the operator μ_i^p. We assign $U(1)_\mathcal{F}$ of μ to be $2\sigma_i^p$. When we have a meson field M_i^p, the $U(1)_\mathcal{F}$ charge for the μ_i^p is reversed to $-2\sigma_i^p$ and the meson has charge $2\sigma_i^p$ instead. In addition to the operators corresponding to the punctures, we also have 'internal' operators μ_i associated to the punctures glued via cylinders in the pair-of-pants decompositions. The $U(1)_\mathcal{F}$ charge for μ_i is given by $2\sigma_i$. When the gluing is done through $\mathcal{N}=2$ vectors, we also have an adjoint chiral multiplet ϕ. The $U(1)_\mathcal{F}$ charge for ϕ is $-2\sigma_i$, so that the $\mathcal{N}=2$ superpotential term $W = \mathrm{Tr}(\phi\mu + \phi\tilde{\mu})$ preserves the $U(1)_\mathcal{F}$ where $\tilde{\mu}$ is the operator corresponds to the other glued puncture. For the $\mathcal{N}=1$ gluing, we can have a superpotential term $W = \mathrm{Tr}(\mu\tilde{\mu})$ which is exactly marginal.

This global symmetry is not anomalous and in general not baryonic. The true R-charge in the IR will mix with $U(1)_\mathcal{F}$ charge in the UV. Therefore one needs to perform a-maximization [128] to obtain the correct R-charge.

Non-maximal punctures via Higgsing

If the labeling of the punctures ρ is non-maximal, we 'Higgs' a maximal puncture down to a non-maximal one in the following ways: For the puncture with the same color as the color of the pair-of-pants σ_p, we give vev to the moment map $\langle\mu\rangle = \rho(\sigma^+)$, and for the puncture with different color σ_p, we give vev to the meson $\langle M\rangle = \rho(\sigma^+)$ where ρ is the embedding of $SU(2)$ into G which labels the puncture itself. For the latter case, this yields the superpotential $W = \mathrm{tr}\rho(\sigma^+)\mu + \mathrm{tr}M'\mu'$ where μ' are the components of μ which commute with ρ^T and M' are the mesonic fluctuations around its vev. Higgsing breaks the global symmetry from G down to the commutant G_F of $\rho(SU(2))$ in G.

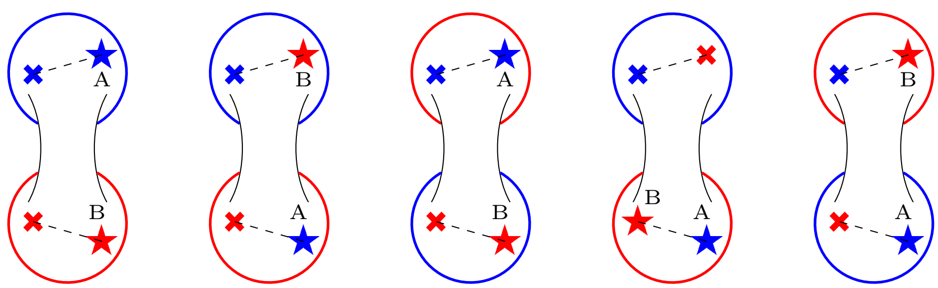
When some of the punctures are non-maximal, it shifts the $U(1)_{\mathcal{F}}$ of (5.2) by a certain amount, if the color of the puncture is different from the pair-of-pants. The shifted $U(1)_{\mathcal{F}}$ is given by

$$\mathcal{F} = \sum_i \left(\sigma_i J_i + 2 \sum_{p, \sigma_i^p = -\sigma_i} \sigma_i^p \rho_i^p(\sigma_3) \right) , \tag{5.3}$$

where (ρ_i^p, σ_i^p) labels the punctures and their colors.

$\mathcal{N} = 1$ Dualities from colored pair-of-pants decompositions

As we discussed above, the different pair of pants decomposition describes different dual frames. Additional ingredient here is the assignment of color σ_i^p for each pair of pants. This adds another choices on the top of the pair of pants decomposition and it makes the duality structure richer than the $\mathcal{N} = 2$ counterpart. We call it colored pair-of-pants decomposition.



(a) $\mathcal{U}^{SO}$ **(b)** $\mathcal{U}^{SO}_{c1}$ **(c)** $\mathcal{U}^{SO}_{s}$ **(d)** $\mathcal{U}^{SO}_{as}$ **(e)** $\mathcal{U}^{SO}_{c2}$

Figure 5.5. Colored pair-of-pants decompositions for a 4-punctured sphere with two twisted full punctures and two twisted null punctures of each color. The degrees of normal bundles are $(p, q) = (1, 1)$. Each subscript stands for: crossing-type 1, swap, Argyres-Seiberg type, crossing-type 2. The first two dual frames have Lagrangian descriptions. The theory $\mathcal{U}^{SO}_{c1}$ turns out to be identical to the dual theory of [126]. The latter three theories are all non-Lagrangian theories. The theory $\mathcal{U}^{SO}_{s}$ is an SO version of [82].

The SQCD with $SO(2N)$ gauge group and $4N-4$ vectors can be realized by choosing the normal bundles and the UV curve to be $\mathcal{L}(1) \oplus \mathcal{L}(1) \to \mathcal{C}_{g=0,n=4}$. Two of the punctures are twisted maximal ones having $USp(2N-2)$ flavor symmetries with each color, and we also put two twisted punctures with no flavor symmetry with each color. Since we have 4 distinct punctures and two distinct pair-of-pants, there are many more dual frames compared to the case of $\mathcal{N}=2$ theory. See figure 5.5.

One can also consider having other type of punctures to realize $USp(2N-2)$ gauge theories or G_2 gauge theory. The colored pair-of-pants decompositions will be almost the same as this example. There are five dual frames, one of them being the electric gauge theory. There is one Lagrangian dual which we denote as crossing 1 and three non-Lagrangian theories which we name as swap, Argyres-Seiberg type and the crossing 2 type. This fact will be universal regardless of the choice of the gauge group, as it can be easily read off from the geometry. In the later sections, we study each theories in more detail.

5.3 Dualities for $SO(2N)$ gauge theory

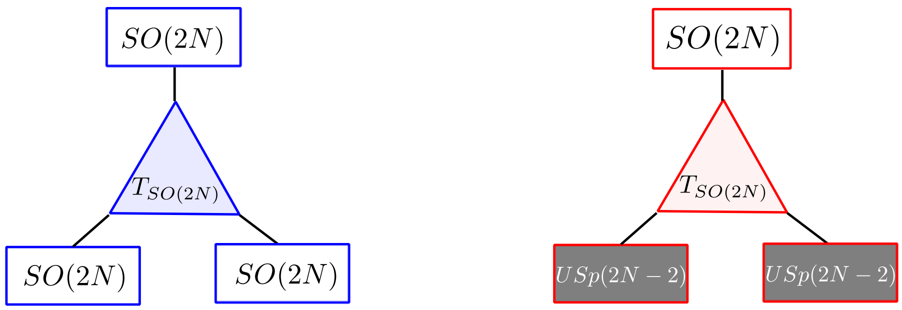
In this section, we study dualities for the $SO(2N)$ gauge theory with $4N-4$ vectors.

5.3.1 $T_{SO(2N)}$ and $\tilde{T}_{SO(2N)}$ theory and Higgsing

For a class $\mathcal{S}$ theory of type Γ, the most basic building block is T_Γ which is given by wrapping the 6d theory on a three punctured sphere with 3 maximal punctures. The theory has $\Gamma_A \times \Gamma_B \times \Gamma_C$ global symmetry, and has special dimension 2 operators $\mu_{A,B,C}$ that transform under the adjoint of $\Gamma_{A,B,C}$ respectively. These operators satisfy a chiral ring relation

$$\mathrm{tr}\mu_A^2 = \mathrm{tr}\mu_B^2 = \mathrm{tr}\mu_C^2. \tag{5.4}$$

This relation is proved in [41] for the $\Gamma = SU(N)$ where the theory is usually called as T_N. We will mainly use the twisted $\tilde{T}_{SO(2N)}$ theory to construct various theories of interest. It has $SO(2N) \times USp(2N-2) \times USp(2N-2)$ global symmetry. We prove the chiral ring relation (5.4) for the T_{D_n} and the twisted $\tilde{T}_{D_n}$ in appendix D.1.



(a) $T_{SO(2N)}$ **(b)** $\tilde{T}_{SO(2N)}$

Figure 5.6. Left: $T_{SO(2N)}$ theory, Right: $\tilde{T}_{SO(2N)}$ theory

The number of effective vector multiplets n_v and hypermultiplets n_h for T_{D_n} and $\tilde{T}_{D_n}$ can be computed using the equations (3.16) and (3.19) of [56]. Each puncture contributes by

$$n_v(SO(2N)) = \frac{1}{3}N(7 - 15N + 8N^2) \ , \tag{5.5}$$

$$n_v(USp(2N-2)) = \frac{1}{6}(-3 + 20N - 30N^2 + 16N^3) \ , \tag{5.6}$$

$$n_h(SO(2N)) = n_h(USp(2N-2)) = \frac{2}{3}N(2N-1)(2N-2) \ . \tag{5.7}$$

There is also a contribution from the bulk

$$n_v(g=0) \ = \ -\frac{4}{3}(2N-2)N(2N-1) - N \ , \tag{5.8}$$

$$n_h(g=0) \ = \ -\frac{4}{3}(2N-2)N(2N-1) \ , \tag{5.9}$$

from which we can compute the n_v, n_h for T_{D_n} and $\tilde{T}_{D_n}$ to get

$$\begin{aligned} n_v(T_{D_n}) &= \frac{1}{3}N(10-21N+8N^2) \ , && (5.10)\\ n_v(\tilde{T}_{D_n}) &= -1+\frac{16}{3}N-7N^2+\frac{8}{3}N^3 \ , && (5.11)\\ n_h(T_{D_n}) &= n_h(\tilde{T}_{D_n}) = \frac{4}{3}n(n-1)(2n-1) \ . && (5.12) \end{aligned}$$

We will use these formula in later sections to compute the anomaly coefficients.

Higgsing the $\tilde{T}_{SO(2N)}$ theory

From the $\tilde{T}_{SO(2N)}$ theory, we can obtain other building blocks by partially closing the full puncture to a one with smaller global symmetries. The $SU(2)$ embedding $\rho : SU(2) \to G$ where $G = SO(2N)$ or $G = USp(2N-2)$ induces a decomposition of adjoint representations into the representations of $SU(2)$ and its commutant G_F

$$\text{adj} = \bigoplus_j R_j \otimes V_j \ , \qquad (5.13)$$

where V_j is the spin-j representation of SU(2) and R_j are the representations of the flavor symmetry G_F associated to the puncture.

For example, when we close one of the twisted puncture having $USp(2N-2)$ completely to have no global symmetry, we obtain a free theory with bifundamental of $SO(2N)$-$USp(2N-2)$. More concretely, we give vev to the operator μ associated to the puncture as

$$\langle \mu \rangle = \rho_{\varnothing}(\sigma^+) = \sum_\alpha E^+_\alpha \ , \qquad (5.14)$$

where α are the simple roots of $USp(2N-2)$ and E^+_α are the corresponding raising operators. [8] The $\rho_\varnothing$ denotes the principal embedding of $SU(2)$ into $USp(2N-2)$, and

[8] We will be cavalier about our notations denoting the Lie groups and Lie algebras.

$\sigma^+ = \sigma_1 + i\sigma_2$ where σ_i are the Pauli matrices. This embedding leaves no flavor symmetry at all. Under this embedding the adjoint representation of $USp(2N-2)$ decomposes as

$$\square\square = \bigoplus_{k=1}^{N-1} V_{2k-1} \ , \tag{5.15}$$

where V_j is the spin-j representation of $SU(2)$. The dimension of the nilpotent orbit of $\rho(\sigma^+)$ then gives us the number of free half-hyper multiplets produced in the process. Thus we find that after Higgsing, the theory flows to an $SO(2N)$-$USp(2N-2)$ bifundamental along with $2(N-1)^2$ free half-hypermultiplets. See for example section 2 of [56].

5.3.2 Dualities for $SO(2N)$-coupled $\tilde{T}_{SO(2N)}$ theories

Before going into the SQCD, let us consider the theory that does not have a known Lagrangian description. Consider a theory realized by the UV curve given by 4 punctured sphere with two red and blue colors each. Choose all the punctures to be the twisted maximal ones having $USp(2N-2)$ flavor symmetries. We decompose it as two pair-of-pants with red and blue colors and arrange all the punctures to lie in the same color as the pair-of-pants. Each pair-of-pants gives $\tilde{T}_{SO(2N)}$ block. Let us call the red punctures to be A, B and blue punctures to be C, D.

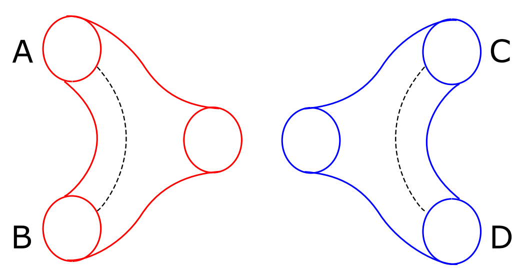


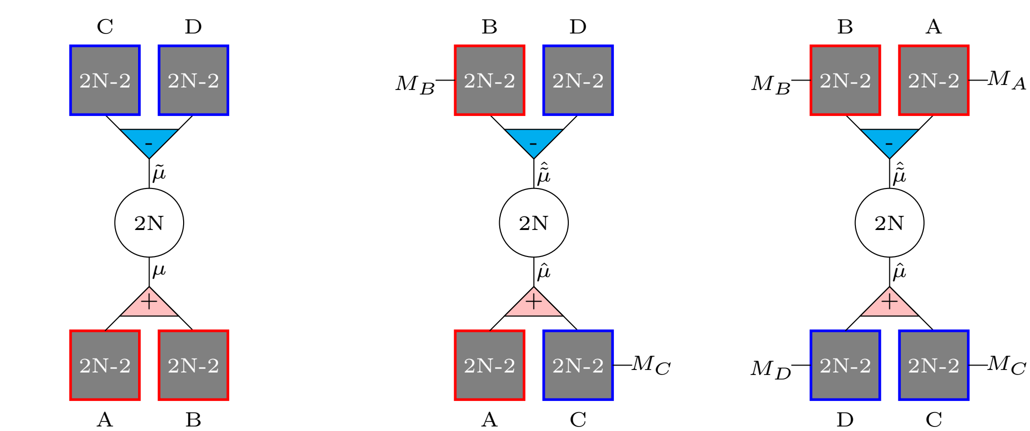
Figure 5.7. Coupling two copies of $\tilde{T}_{SO(2N)}$ theories

This construction realizes two $\tilde{T}_{SO(2N)}$ blocks coupled along their $SO(2N)$ puncture by an $\mathcal{N}=1$ vector multiplet and a superpotential given by

$$W = c\mathrm{tr}\mu\tilde{\mu} \ . \tag{5.16}$$

Here μ is the dimension 2 operator transforming in the adjoint representation of the $SO(2N)$ flavor symmetry of $\tilde{T}_{SO(2N)}$ while $\tilde{\mu}$ is its counterpart coming from the other $\tilde{T}_{SO(2N)}$ block. The $U(1)_{\mathcal{F}}$ charge for μ is $+2$ while $\tilde{\mu}$ has -2. The $U(1)_{\mathcal{F}}$ charges of the operators are determined by the color choice σ for each punctures as described in section 5.2. Diagrammatically we can represent this theory as in figure 5.8a. We will call this theory as $\mathcal{T}^{SO}$.

This theory can also be obtained by starting from two $\tilde{T}_{SO(2N)}$ blocks coupled along with their $SO(2N)$ flavor symmetry by an $\mathcal{N}=2$ vector multiplet and then integrating out the adjoint chiral in the vector multiplet by giving it mass and then flowing to the IR. Since the operators μ and $\tilde{\mu}$ both have R-charge 1, the operator $\mu\tilde{\mu}$ is marginal.



(a) Electric theory $\mathcal{T}^{SO}$ **(b)** Crossing $\mathcal{T}_c^{SO}$ **(c)** Swapped $\mathcal{T}_s^{SO}$

Figure 5.8. Dual frames of the two $\tilde{T}_{SO}$ blocks coupled by SO gauge group. The red/blue color means $\sigma = +/-$ respectively.

A dual of this theory can be obtained by exchanging the punctures labeled B and

C. We will also have to integrate in mesons M_B and M_C that transform in the adjoint representation of $USp(2N-2)_B$ and $USp(2N-2)_C$ respectively [82]. The superpotential in the dual theory is given by

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \mathrm{tr}\ \Omega M_B \Omega \hat{\mu}_B + \mathrm{tr}\ \Omega M_C \Omega \hat{\mu}_C\ , \tag{5.17}$$

where Ω is the $USp(2N-2)$ invariant antisymmetric form. We now have the dual operators $\hat{\mu}_B, \hat{\mu}_C$ for the punctures B, C which have their $U(1)_{\mathcal{F}}$ charges reversed, and also meson operators M_B, M_C which has the same $U(1)_{\mathcal{F}}$ charges as μ_B, μ_C. We depict this theory by figure 5.8b.

We can further exchange punctures A and D to obtain a third theory which is dual to the previous two. The superpotential now becomes

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \mathrm{tr} M_A \Omega \hat{\mu}_A \Omega + \mathrm{tr}\Omega M_B \Omega \hat{\mu}_B + \mathrm{tr}\Omega M_C \Omega \hat{\mu}_C + \mathrm{tr} M_D \Omega \hat{\mu}_D \Omega\ , \tag{5.18}$$

with extra meson fields M_A and M_D. See the figure 5.8c.

One can also derive these dualities starting from $\mathcal{N}=2$ S-duality and giving mass to the adjoint chiral multiplet in the $\mathcal{N}=2$ vector multiplet and integrating it out and then flowing to the IR. Then by using the chiral ring relation derived in the appendix D.1 and integrating in the mesons, one can reproduce the superpotentials (5.17), (5.18). We refer to the section 2.2.4 of [82] for details.

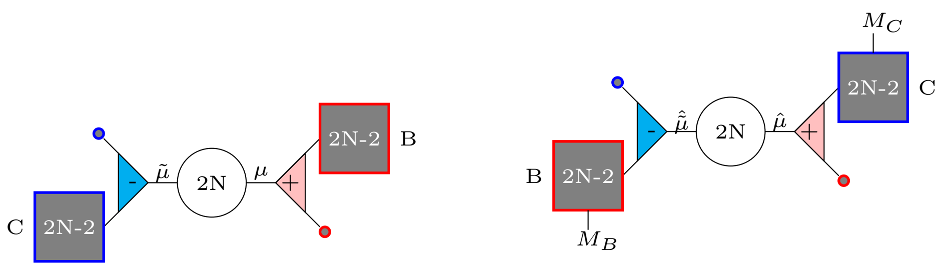
Following the nomenclature used in [82], we refer to the dual theories obtained above as being in the "crossing frame" $\mathcal{T}_c^{SO}$ and the "swapped frame" $\mathcal{T}_s^{SO}$ respectively. These three duality frames will be the basis of the dualities discussed in this section.

5.3.3 Dualities for $SO(2N)$ SQCD

Now, let us move on to discuss dualities for the theory with UV Lagrangian descriptions.

Intriligator-Seiberg duality

By partially closing the punctures A and D in the electric theory $\mathcal{T}^{SO}$, we reduce it to SQCD with gauge group $SO(2N)$ and $N_f = 4N-4$ fundamental (vector) flavors. Partial closing of the puncture is implemented by giving appropriate vevs as in (5.14) to μ_A and μ_D. Closing the punctures changes the dual theories as well. Upon



(a) Electric theory $\mathcal{U}^{SO}$ **(b)** Magnetic theory $\mathcal{U}^{SO}_{c1}$

Figure 5.9. Intriligator-Seiberg duality

Higgsing, the two copies of $\tilde{T}_{SO(2N)}$ become free bifundamentals of $SO(2N) \times USp(2N-2)_B$ and $SO(2N) \times USp(2N-2)_C$. Therefore, the original theory $\mathcal{T}$ becomes $SO(2N)$ gauge group with $4N-4$ fundamental(vector) flavors where only the $USp(2N-2)_B \times USp(2N-2)_C \subset SU(4N-4)$ global symmetry is manifest. This is nothing but the usual SQCD. We also have the marginal superpotential

$$W = c\mathrm{tr}\mu\tilde{\mu} \ , \tag{5.19}$$

where now $\mu_{\alpha\beta} = (Q_{\alpha i}\Omega^{ij}Q_{\beta j})_B$ and $\tilde{\mu}_{\alpha\beta} = (Q_{\alpha i}\Omega^{ij}Q_{\beta j})_C$ with $\alpha, \beta = 1, \ldots, 2N$ denoting the $SO(2N)$ vector indices and $i, j = 1, \ldots, 2N-2$ denoting the USp indices. Here

$(Q_B)_{\alpha i}$ is the quark transforming as the bifundamental of $SO(2N) \times USp(2N-2)_B$ while $(Q_C)_{\alpha i}$ is the bifundamental of $SO(2N) \times USp(2N-2)_C$. This superpotential term breaks the global symmetry to $USp(2N-2)_B \times USp(2N-2)_C$. We will denote this theory as $\mathcal{U}^{SO}$.

Now, let us look at the theory obtained by closing the punctures A and B of crossing frame, $\mathcal{T}_c^{SO}$. The theory so obtained has two meson fields M_B, M_C each transforming under the adjoint of $USp(2N-2)_B$ and $USp(2N-2)_C$. Also we get superpotential terms as

$$W = \hat{c}\,\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \mathrm{tr}M_B\Omega\hat{\mu}_B\Omega + \mathrm{tr}M_C\Omega\hat{\mu}_C\Omega \ . \tag{5.20}$$

We can write $\hat{\mu}_B$ and $\hat{\mu}_C$ in terms of the fundamental dual quarks $\hat{Q}$ as $\mu_B = \hat{Q}_B\hat{Q}_B$ and $\mu_C = \hat{Q}_C\hat{Q}_C$ which are in the adjoint (=symmetric) representations of $USp(2N-2)_{B,C}$. The $\hat{\mu}$ and $\hat{\hat{\mu}}$ are given by the dual quark bilinears as $\hat{\mu} = \hat{Q}_B\Omega\hat{Q}_B$ and $\hat{\hat{\mu}} = \hat{Q}_C\Omega\hat{Q}_C$ which are in the adjoint of $SO(2N)$.

The duality frames obtained through this procedure are depicted in figure 5.9. These two duality frames are related to each other by the Intriligator-Seiberg duality [126]. Applying Intriligator-Seiberg duality to the $SO(2N)$ gauge theory with $4N-4$ fundamentals we find that the magnetic dual is given by the theory with $SO(2N)$ gauge group and $4N-4$ dual quarks $\hat{Q}$ along with mesons and the superpotential term $W = \mathrm{tr}M\hat{Q}\hat{Q}$. In the absence of any other superpotential the global symmetry of this theory would be $SU(4N-4)$ with the mesons transforming in the symmetric representation of $SU(4N-4)$. In terms of $USp(2N-2)_B \times USp(2N-2)_C \subset SU(4N-4)$, the quarks split into bifundamentals of $SO(2N) \times USp(2N-2)_B$ and $SO(2N) \times USp(2N-2)_B$ while the meson splits into the following irreducible representations.

- symmetric tensor of $USp(2N-2)_B$: $(M_B)_{ij}$

- symmetric tensor of $USp(2N-2)_C$: $(M_C)_{ij}$
- bifundamental of $USp(2N-2)_B \times USp(2N-2)_C$: M_{ij}

Note that M_{ij} is dual to the meson of the electric theory formed by $Q_{B\alpha i}Q_{C\alpha j}$. The electric superpotential $\mathrm{tr}\mu\tilde{\mu}$ induces a mass term for M_{ij}. The dual superpotential of the magnetic theory can be written as

$$W = \mathrm{tr}M\Omega M\Omega + \mathrm{tr}M_B\Omega\hat{Q}_B\hat{Q}_B\Omega + M_C\Omega\hat{Q}_C\hat{Q}_C\Omega + \mathrm{tr}M\Omega\hat{Q}_B\hat{Q}_C\Omega \ . \quad (5.21)$$

Integrating out the massive mesons M_{ij} then gives us the superpotential of (5.20). We will denote this theory as $\mathcal{U}_{c1}^{SO}$ since it arises from exchanging the two punctures in the electric theory.

Non-Lagrangian dual 1: Swap

An interesting non-Lagrangian dual (figure 5.10) to the $SO(2N)$ SQCD is obtained by the Higgsing the swapped theory $\mathcal{T}_s^{SO}$ of figure 5.8c. In this frame the Higgsing of $USp(2N-2)_A$ and $USp(2N-2)_D$ is implemented through a vev $\rho_\varnothing(\sigma^+)$ to the meson fields M_A and M_D. The low energy dynamics of this theory can be obtained as follows. With a little abuse of notation, let M_A now represent the fluctuations around the vev $\rho^A(\sigma^+)$. The deformed superpotential now becomes

$$\begin{aligned} W &= \mathrm{tr}\Omega\rho^A(\sigma^+)\Omega\hat{\mu}_A + \mathrm{tr}\Omega M_A\Omega\hat{\mu}_A \\ &= (\hat{\mu}_A)_{1,-1} + \sum_{j,m}(M_A)_{j,-m}(\hat{\mu}_A)_{j,m} \ , \end{aligned} \quad (5.22)$$

where we rewrite the components of $(\mu_A)_{ij}$ and $(M_A)_{ij}$ by decomposing into $SU(2)$ representations as in (5.13). The indices j, m with $m = -j, -j+1, \ldots, j-1, j$ labels the spin-j representations of $SU(2)$ and $k = 1, \ldots, \mathrm{dim}R_k$. Since there is no flavor symmetry left here, we do not have any k dependence.

Since the first term of (5.22) break the $U(1)_{\mathcal{F}}$, we should shift its charge appropriately. Also we want our superpotential term to have $U(1)_R$ charge 2. In order to achieve this, we shift the $U(1)_{\mathcal{F}}$ flavor symmetry and R-symmetry to

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + 2\rho^A(\sigma^3) \ , \\ R &= R_0 - \rho^A(\sigma^3) \ , \end{aligned} \tag{5.23}$$

where $\mathcal{F}_0$ and R_0 are the $U(1)_{\mathcal{F}}$ and R-charges of the fields before Higgsing.

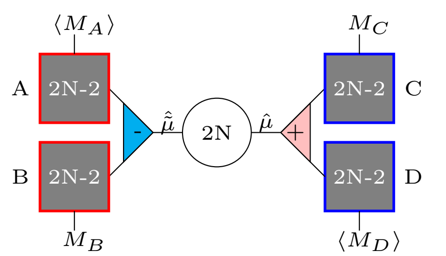


Figure 5.10. Non-Lagrangian dual $\mathcal{U}_s^{SO}$ of $SO(2N)$ SQCD

The $USp(2N-2)_A$ flavor symmetry gets broken. The resulting non-conservation of the associated global currents can be expressed as

$$\bar{D}^2(J_A)_{j,m} = \delta W = (\hat{\mu}_A)_{j,m-1} \ . \tag{5.24}$$

The right-hand side vanishes only if $m = -j$. This implies that the operators $(\hat{\mu}_A)_{j,m-1}$ are no longer BPS and hence the superpotential terms that couples them to mesonic fields become IR-irrelevant. As a result of this, the fields $(M_A)_{j,m}$ for $m \neq -j$ decouple. The number of such free fields is $2(N-1)^2$ which is same as the number of free half-hypers obtained from Higgsing $USp(2N-2)_A$ in figure 5.8a.

Repeating the same analysis for $USp(2N-2)_D$ then leads to the following superpotential for our proposed dual

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\tilde{\mu}} + \mathrm{tr}\Omega M_C \Omega \hat{\mu}_C + \mathrm{tr}\Omega M_B \Omega \hat{\mu}_B + \sum_j (M_A)_{j,-j} (\hat{\mu}_A)_{j,j} + \sum_j (M_D)_{j,-j} (\hat{\mu}_D)_{j,j} \ , \tag{5.25}$$

where $j = 1, 3, \ldots, 2N-3$ from which we see $2(N-1)$ gauge singlets. The charges for the $U(1)_{\mathcal{F}}$ and $U(1)_R$ are shifted to

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + 2\rho^A(\sigma^3) - 2\rho^D(\sigma^3) \ , \\ R &= R_0 - \rho^A(\sigma^3) - \rho^D(\sigma^3) \ . \end{aligned} \tag{5.26}$$

We will denote this theory as U_s^{SO}.

Non-Lagrangian dual 2: Argyres-Seiberg type dual

Another type of dual theory to the SQCD can be obtained from Higgsing punctures B and D of the duality frames in figure 5.8. This is possible since the punctures with the same colors are indistinguishable in the non-Lagrangian theory of figure 5.8 and therefore their labels can be interchanged. In the present case we relabel $A \leftrightarrow B$.

Higgsing the frames $\mathcal{T}^{SO}$ and $\mathcal{T}_s^{SO}$ give us the theories $\mathcal{U}^{SO}$ and $\mathcal{U}_s^{SO}$ respectively. However an Argyres-Seiberg type dual, $\mathcal{U}_{as}^{SO}$, is obtained upon closing the afore mentioned punctures in $\mathcal{T}_c^{SO}$ (see figure 5.11). Firstly, Higgsing the puncture D will make the theory $\tilde{T}_{SO(2N)}$ on the upper sphere to be the theory of bifundamentals. Therefore we have $\tilde{T}_{SO(2N)}$ theory with $SO(2N)$ flavor symmetry gauged and coupled to $2N-2$ fundamentals (vectors).

The punctures A and C have different colors from their pair of pants. Therefore we will have meson field M_A and M_C coupled through

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\tilde{\mu}} + \mathrm{tr}M_A \Omega (QQ)_A \Omega + \mathrm{tr}M_C \Omega \hat{\mu}_C \Omega \ , \tag{5.27}$$

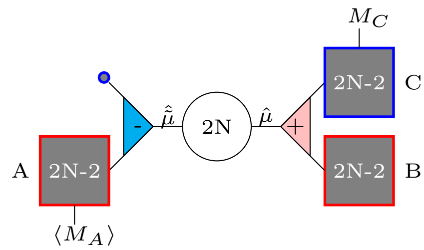


Figure 5.11. Argyres-Seiberg type dual $\mathcal{U}^{SO}_{as}$ of $SO(2N)$ SQCD

where we replaced the operator $\hat{\mu}_B$ by the product of the quarks $(Q_{\alpha i}Q_{\beta i})_A$. In order to Higgs the puncture A, we give vev to the meson field $\langle M_B \rangle = \rho_\varnothing$. We can now consider low energy fluctuations around this vacuum and repeat the analysis of the previous subsection. The vev for the meson gives a mass to one of the quark bifundamentals which should be integrated out. The resulting low energy theory consists of $2N-3$ fundamentals coupled to a $\tilde{T}_{SO(2N)}$ block along with $N-1$ gauge singlets $(M_A)_{j,-j}$ and mesons M_C coupled through the superpotential

$$W = \hat{c}\,\mathrm{tr}\hat{\mu}\hat{\tilde{\mu}} + \sum_j (M_A)_{j,-j}(\hat{\mu}_A)_{j,j} + \mathrm{tr} M_C \Omega \hat{\mu}_C \Omega \ , \tag{5.28}$$

The R- and $\mathcal{F}$-charges are shifted to

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + 2\rho^A(\sigma^3) \ , \\ R &= R_0 - \rho^A(\sigma^3) \ . \end{aligned} \tag{5.29}$$

One interesting aspect of this dual description compared to the $\mathcal{N}=2$ counterpart is that this dual theory has the same gauge group as the electric one. In the $\mathcal{N}=2$ case, this type of duality changes the gauge group to be $SU(2)$ subgroup of T_Γ [24, 53], whereas in the present case the gauge group is still $SO(2N)$ unbroken.

Non-Lagrangian dual 3: Crossing type

One more dual frame can be obtained from Higgsing B and C punctures of $\mathcal{T}_c^{SO}$ of figure 5.8b and relabeling $A \leftrightarrow B$ and $C \leftrightarrow D$. We call this the crossing type dual and denote it by $\mathcal{U}_{c2}^{SO}$ (see figure 5.12). It consists of two $\tilde{T}_{SO(2N)}$ blocks coupled to each other along their $SO(2N)$ flavor symmetry. Also there will be mesons M_A and M_D with a vev $\langle M_A \rangle = \langle M_D \rangle = \rho_\varnothing$. The low energy superpotential for the theory becomes

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \sum_{j,m}(M_A)_{j,-m}(\hat{\mu}_A)_{j,m} + \sum_{j,m}(M_D)_{j,-m}(\hat{\mu}_D)_{j,m} \ , \tag{5.30}$$

and shifted R- and $\mathcal{F}$-charges

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + 2\rho^A(\sigma^3) - 2\rho^D(\sigma^3) \ , \\ R &= R_0 - \rho^A(\sigma^3) - \rho^D(\sigma^3) \ . \end{aligned} \tag{5.31}$$

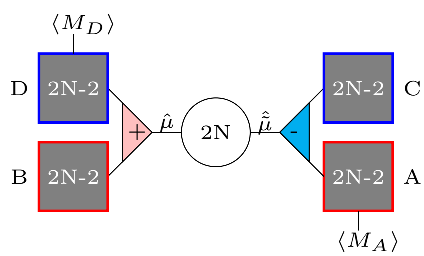


Figure 5.12. Crossing type dual $\mathcal{U}_{c2}^{SO}$ of $SO(2N)$ SQCD

5.3.4 ’t Hooft anomaly matching

Now we test our dualities by computing the anomaly coefficients in different dual frames.

Non-Lagrangian duals

Upon giving a mass to the adjoint chiral superfield in the vector multiplet of an $\mathcal{N}=2$ theory and hence reducing SUSY down to $\mathcal{N}=1$, the residual $U(1)_R$ symmetry that is preserved by this deformation is given by

$$R_{\mathcal{N}=1} = \frac{1}{2}R_{\mathcal{N}=2} + I_3 \ , \tag{5.32}$$

where I_3 is the Cartan of $SU(2)_R$ in the parent theory. Thus we can write the $\mathrm{tr}R_{\mathcal{N}=1}$ and $\mathrm{tr}R^3_{\mathcal{N}=1}$ anomalies in terms of the anomalies of the parent $\mathcal{N}=2$ theory as

$$\mathrm{tr}R_{\mathcal{N}=1} = \frac{1}{2}\mathrm{tr}R_{\mathcal{N}=2} = n_v - n_h \ , \tag{5.33}$$

and

$$\mathrm{tr}R^3_{\mathcal{N}=1} = \frac{1}{8}\mathrm{tr}R^3_{\mathcal{N}=2} + \frac{3}{2}\mathrm{tr}R_{\mathcal{N}=2}I_3^2 = n_v - \frac{1}{4}n_h \ , \tag{5.34}$$

where n_v is the effective number of vector multiplets and n_h is the effective number of hyper-multiplets in the parent theory.

It is now straight-forward to check that the $\mathrm{tr}R$ and $\mathrm{tr}R^3$ anomalies of the duality frames shown in figure 5.8 match. This is because the mesons have R-charge 1 and hence do not contribute to the R-anomalies. Thus all the R-anomalies of these theories are destined to match as a direct consequence of their matching in the $\mathcal{N}=2$ parent theories. This also implies that the flavor central charge given by $K\delta^{ab} = -3\mathrm{tr}RT^aT^b$ will only get contributions from the coupled $\tilde{T}_{SO(2N)}$ blocks and hence match in all the three duality frames.

Let us now consider the matching of $\mathrm{tr}\mathcal{F}T^aT^b$ across the various duality frames.

The global current $\mathcal{F} = \sum_i \sigma_i J_i$ is given by the sum of J_i where the global symmetry J is given by

$$J = R_{\mathcal{N}=2} - 2I_3, \tag{5.35}$$

for the each building block $\tilde{T}_{SO(2N)}$. Note that if the corresponding $\tilde{T}_{SO(2N)}$ block has a $U(1)_{\mathcal{F}}$-charge σ then

$$\mathrm{tr}\mathcal{F}T^aT^b = \sigma \mathrm{tr} R_{\mathcal{N}=2} T^a T^b = -\frac{\sigma}{2} k_{\mathfrak{g}} \ . \tag{5.36}$$

To begin with, consider the anomaly coefficient for $T^a \in \mathfrak{sp}(2N-2)_A$. Note that $k_{\mathfrak{sp}(2N-2)}$ for $\tilde{T}_{SO(2N)}$ is $4N$ as can be checked by comparing the dual theories of figure D.2. Thus for the electric theory, $\mathcal{T}^{SO}$ (figure 5.8a) we have

$$\mathrm{tr}\mathcal{F}T^a_A T^b_A = -2N\delta^{ab} \ . \tag{5.37}$$

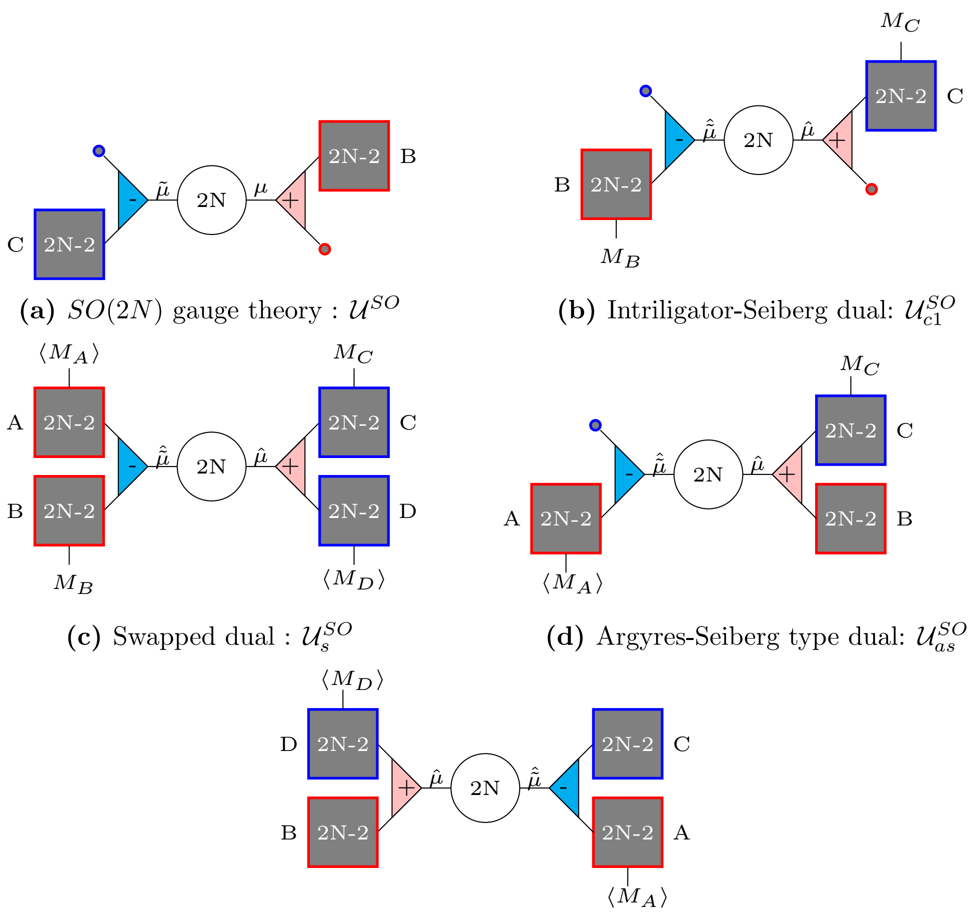
This matches trivially to the anomaly coefficient of the theory, $\mathcal{T}^{SO}_c$ (figure 5.8b). It is much more interesting to compare this with the anomaly coefficient of $\mathcal{T}^{SO}_s$ (figure 5.8c) which, after taking the contributions of the meson M_A into account, becomes

$$\begin{aligned} \mathrm{tr}\mathcal{F}T^a_A T^b_A &= 2N\delta^{ab} - 2\mathrm{tr}_{\mathrm{adj}} T^a_A T^b_A \\ &= 2N\delta^{ab} - 2(2N)\delta^{ab} \\ &= -2N\delta^{ab} \ , \end{aligned} \tag{5.38}$$

which agrees with the original theory. The anomalies of $USp(2N-2)_B$, $USp(2N-2)_C$ and $USp(2N-2)_D$ match in all the duality frames in an analogous manner.

Dual theories of $SO(2N)$ SQCD

The various duality frames obtained after Higgsing some of the $USp(2N-2)$ punctures are shown in figure 5.13. The theories $\mathcal{U}^{SO}$ and $\mathcal{U}^{SO}_{c1}$ are related by Intriligator-



(a) $SO(2N)$ gauge theory : $\mathcal{U}^{SO}$

(b) Intriligator-Seiberg dual: $\mathcal{U}^{SO}_{c1}$

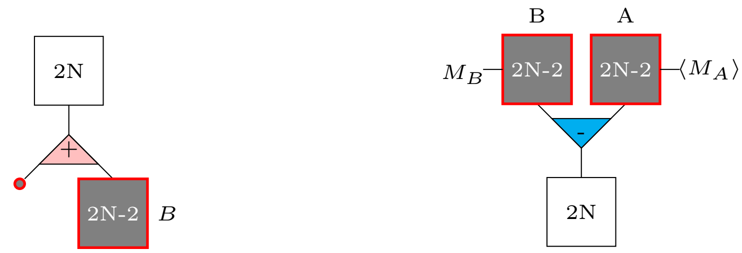
(c) Swapped dual : $\mathcal{U}^{SO}_{s}$

(d) Argyres-Seiberg type dual: $\mathcal{U}^{SO}_{as}$

(e) Crossing type dual: $\mathcal{U}^{SO}_{c2}$

Figure 5.13. Dual frames of $SO(2N)$ SQCD

Seiberg duality and their anomalies match in the usual manner. For the purpose of matching the anomalies between $\mathcal{U}^{SO}$ and $\mathcal{U}^{SO}_{s}$, we observe that we only have to match the anomalies of the $SO(2N) \times USp(2N-2)$ bifundamental to the anomalies of the $\tilde{T}_{SO(2N)}$ block appropriately coupled to mesons (figure 5.14). For the bifundamental we



(a) Bifundamentals **(b)** $\tilde{T}_{SO(2N)}$ with Meson and Higgsing

Figure 5.14. Building blocks used to construct the electric and the swapped frames

have

$$\mathrm{tr}R\big|_{\mathrm{bifund}} = \left(-\frac{1}{2}\right)(2N)(2N-2) = -N(2N-2) \ . \tag{5.39}$$

Note that on the dual side, after giving a vev to the meson M_A, the R-charge gets shifted: $R \to R - \rho(\sigma^3)$. This will not affect the contribution of the $\tilde{T}_{SO(2N)}$ block, since its $\mathrm{tr}\rho(\sigma^3) = 0$. For the mesons, we will only consider the contributions of $(M_A)_{j,-j}$ since the others decouple. This implies

$$\mathrm{tr}R\big|_{\langle M_A \rangle} = \sum_j j = \sum_{n=1}^{N-1}(2n-1) = (N-1)^2 \ . \tag{5.40}$$

The meson M_B does not contribute to the R-anomalies since its R-charge is not shifted and is equal to 1. Putting these together, we find that in this frame

$$\mathrm{tr}R = \mathrm{tr}R\big|_{\tilde{T}_{SO(2N)}} + \mathrm{tr}R\big|_{\langle M_A \rangle} = -N(2N-2) \ , \tag{5.41}$$

which matches with the corresponding anomaly of the bifundamental.

Moving on, we now compare the $\mathrm{tr}R^3$ anomalies on the two sides and find

$$\mathrm{tr}R^3\big|_{\mathrm{bifund}} = \left(-\frac{1}{2}\right)^3(2N)(2N-2) = -\frac{1}{2}N(N-1) \ . \tag{5.42}$$

On the dual side, since $R = R_0 - \rho(\sigma^3)$, where $R_0 = \frac{1}{2}R_{\mathcal{N}=2} + I_3$, therefore

$$\text{tr}R^3 = \text{tr}R_0^3 + 3\text{tr}R\rho^2 \ . \tag{5.43}$$

Also $3\text{tr}R\rho^2\delta^{ab} = \frac{3\mathcal{I}}{2}\text{tr}R_{\mathcal{N}=2}T_A^aT_A^b$, where $\mathcal{I}$ is the $SU(2)$ embedding index. Since our embedding takes $2N-2$ dimensional representation of $USp(2N-2)$ to the $2N-2$ dimensional representation of $SU(2)$, therefore

$$\mathcal{I} = 2\sum_{j_z=1/2}^{N-3/2} j_z^2 = \frac{1}{6}(N-1)(4N^2-8N+3) \ . \tag{5.44}$$

Thus, due to the shift in R-charges the $\tilde{T}_{SO(2N)}$ now contributes

$$\begin{aligned}\text{tr}R^3 &= \text{tr}R_0^3 + 3\text{tr}R\rho^2 \\ &= -1 + \frac{13}{2}N - \frac{23}{2}N^2 + 8N^3 - 2N^4 \ .\end{aligned} \tag{5.45}$$

Also

$$\text{tr}R^3\big|_{\langle M_A\rangle} = \sum_j j^3 = \sum_{n=1}^{N-1}(2n-1)^3 = 1 - 6N + 11N^2 - 8N^3 + 2N^4 \ . \tag{5.46}$$

Adding the contributions of the $\tilde{T}_{SO(2N)}$ block and the mesons we find that the $\text{tr}R^3$ anomalies match with those of the bifundamental. The $\text{tr}RT_B^aT_B^b$ and $\text{tr}\mathcal{F}T_B^aT_B^b$ anomalies for the bifundamental are given by $(-\frac{1}{2})(2N)$ and $(-1)(2N)$ respectively. On the dual side these have the same values as in the scenario before Higgsing. This is because $\text{tr}\rho T_B^aT_B^b = 0$ for the $\tilde{T}_{SO(2N)}$ block. We therefore conclude that these anomalies have the same value in the electric and the swapped theory.

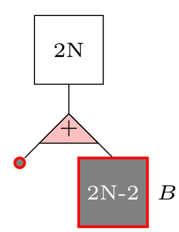
Similarly, we can match the anomaly coefficients of $\mathcal{U}^{SO}$ and $\mathcal{U}^{SO}_{as}$. In $\mathcal{U}^{SO}_{as}$ we have (up to the gaugino-contributions)

$$\mathrm{tr}R = \mathrm{tr}R|_{\tilde{T}_{SO(2N)}} + \mathrm{tr}R|_{\langle M_A\rangle} + 2N\sum_m \Big(-\frac{1}{2} - m\Big) = -2N(2N-2) \qquad (5.47)$$

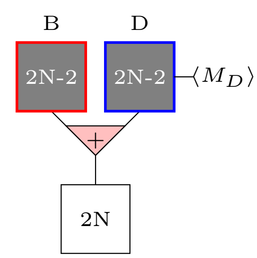
$$\mathrm{tr}R^3 = \mathrm{tr}R^3|_{\tilde{T}_{SO(2N)}} + \mathrm{tr}R^3|_{\langle M_A\rangle} + 2N\sum_m \Big(-\frac{1}{2} - m\Big)^3 = -N(N-1) \qquad (5.48)$$

which match with those in the electric theory. The coefficients of $\mathrm{tr}RT^a_BT^b_B$, $\mathrm{tr}RT^a_CT^b_C$, $\mathrm{tr}\mathcal{F}T^a_BT^b_B$ and $\mathrm{tr}\mathcal{F}T^a_CT^b_C$ are not affected by Higgsing and therefore match with their electric counterparts.

The anomaly coefficients in $\mathcal{U}^{SO}_{c2}$ can also be matched to those in the other duality frames. This follows from the matching between the anomalies of the bifundamental and the $\tilde{T}_{SO(2N)}$ block (with mesons) shown in figure 5.15.



(a) Bifundamentals



(b) $\tilde{T}_{SO(2N)}$ with Meson and Higgsing

Figure 5.15. Building blocks used to construct the electric and the crossing frames

5.4 Dualities for $USp(2N-2)$ gauge theory

We now repeat the same procedure as the previous section for $USp(2N-2)$ gauge theory with $4N$ fundamentals.

5.4.1 Dualities for $USp(2N-2)$-coupled $\tilde{T}_{SO(2N)}$ theories

We begin by considering two $\tilde{T}_{SO(2N)}$ blocks coupled to each other at a $USp(2N-2)$ puncture via an $\mathcal{N}=1$ vector multiplet, giving the electric theory of figure 5.16a. The superpotential for this theory is

$$W = c\text{tr}\mu\Omega\tilde{\mu}\Omega \tag{5.49}$$

We will henceforth denote this theory by $\mathcal{T}^{Sp}$. The frames dual to $\mathcal{T}^{Sp}$ can be obtained by using the rules of section 5.2 to move the punctures around. This gives us the set of theories shown in figure 5.16.

We will call the theory in figure 5.16b as 'crossing frame 1' and denote it by $\mathcal{T}^{Sp}_{c1}$. It is obtained by exchanging punctures B and C. Since these punctures will no longer have the same color as their pants, we will therefore have to integrate in mesons M_B and M_C transforming as the adjoints of the respective symmetries. The superpotential of the theory becomes

$$W = \hat{c}\text{tr}\hat{\mu}\Omega\hat{\tilde{\mu}}\Omega + \text{tr}M_C\hat{\mu}_C + \text{tr}M_B\hat{\mu}_B \ , \tag{5.50}$$

Similarly when we exchange the puncture A and D, we end up with the theory in 'crossing frame 2' (figure 5.16c) which is denoted by $\mathcal{T}^{Sp}_{c2}$. Once again we will have to couple mesons M_A and M_D via the superpotential

$$W = \hat{c}\text{tr}\hat{\mu}\Omega\hat{\tilde{\mu}}\Omega + \text{tr}M_A\Omega\hat{\mu}_A\Omega + \text{tr}M_D\Omega\hat{\mu}_D\Omega \ , \tag{5.51}$$

The theory in 'crossing frame 3' (figure 5.16d) is obtained by exchanging punctures B and D. This will correspond to a pair of pants decomposition where one of the pants has no outer automorphism twists. In other words it consists of an $\tilde{T}_{SO(2N)}$ block coupled to a $T_{SO(2N)}$ block at its $SO(2N)$ puncture. To compensate for the mismatch in the color of the punctures their respective pants, we will have to integrate in mesons M_B and M_D

with the superpotential being

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\tilde{\mu}} + \mathrm{tr}M_B\hat{\mu}_B + \mathrm{tr}M_D\Omega\hat{\mu}_D\Omega \ , \tag{5.52}$$

Interestingly this gives us a duality between an $\mathcal{N}=1$ theory with a $USp(2N-2)$ gauge group and a theory with $SO(2N)$ gauge group. We will denote the theory in this duality frame by $\mathcal{T}_{c3}^{Sp}$.

The theory in figure 5.16e will be called the 'swapped' theory and we will denote it by $\mathcal{T}_s^{Sp}$. It is obtained by moving the 4 punctures around such that none of them have the same color as the pants in which they reside. This will require us to integrate in mesons at each puncture. The superpotential will now become

$$W = \hat{c}\mathrm{tr}\hat{\mu}\Omega\hat{\tilde{\mu}}\Omega + \mathrm{tr}M_A\Omega\hat{\mu}_A\Omega + \mathrm{tr}M_B\hat{\mu}_B + \mathrm{tr}M_C\hat{\mu}_C + \mathrm{tr}M_D\Omega\hat{\mu}_D\Omega \ . \tag{5.53}$$

5.4.2 Dualities for $USp(2N-2)$ SQCD

Now, let us consider the dual theories of SQCD.

The Intriligator-Pouliot Duality

By Higgsing punctures A and D of $\mathcal{T}^{Sp}$ and $\mathcal{T}_{c1}^{Sp}$ (figure 5.16a and 5.16b), with a vev to their adjoint representation operators, we obtain the usual pair of Intriligator-Pouliot dual theories [125]. The electric theory is given by figure 5.17a. We will use the short-hand notation $\mathcal{U}^{Sp}$ to denote this theory. It is a $USp(2N-2)$ gauge theory with $4N$ fundamental quarks. It has an $SO(2N)_B \times SO(2N)_C \subset SU(4N)$ global symmetry. Its superpotential is given by

$$W = c\mathrm{tr}\mu\Omega\tilde{\mu}\Omega \ , \tag{5.54}$$

where now $\mu_{ij} = (Q_{\alpha i} Q_{\alpha j})_B$ and $\tilde{\mu}_{ij} = (Q_{\alpha i} Q_{\alpha j})_C$. Here $(Q_B)_{\alpha i}$ is the quark transforming as the bifundamental of $SO(2N)_B \times USp(2N-2)$ while $(Q_C)_{\alpha i}$ is the bifundamental of $SO(2N)_C \times USp(2N-2)$.

Applying Intriligator-Pouliot duality to the above electric theory, we get a theory with $4N$ quarks $\hat{Q}$ transforming under a $USp(2N-2)$ gauge group. In the absence of any superpotential this theory will enjoy $SU(4N)$ global symmetry. The spectrum of the theory will also include mesons transforming in the anti-symmetric representation of $SU(4N)$. In terms of the $SO(2N)_B \times SO(2N)_C$ subgroup of the flavor symmetry the quarks split into bifundamentals of $SO(2N)_B \times USp(2N-2)$ and $SO(2N)_C \times USp(2N-2)$ while the meson splits into the following irreducible representations.

1. anti-symmetric tensor of $SO(2N)_B$: $M_{B\alpha\beta}$
2. anti-symmetric tensor of $SO(2N)_C$: $M_{C\alpha\beta}$
3. bifundamental of $SO(2N)_B \times SO(2N)_C$: $M_{\alpha\beta}$

Note that $M_{\alpha\beta}$ is dual to the meson of the electric theory formed by $(Q_B)_{\alpha i} \Omega^{ij} (Q_C)_{\beta j}$. The dual superpotential becomes

$$W_m = c\,\mathrm{tr} MM + \mathrm{tr} M_B \hat{Q}_B \Omega \hat{Q}_B + \mathrm{tr} M_C \hat{Q}_C \Omega \hat{Q}_C + \mathrm{tr} M \hat{Q}_B \Omega \hat{Q}_C \tag{5.55}$$

Integrating out the massive mesons $M_{\alpha\beta}$ then gives us the theory of figure 5.17b. We will use $\mathcal{U}^{Sp}_{c1}$ to denote this theory.

Non-Lagrangian dual 1: Swap

A non-Lagrangian dual (figure 5.18) of the electric theory $\mathcal{U}^{Sp}$ is generated upon Higgsing the punctures A and D in $\mathcal{T}^{Sp}_s$. This Higgsing is implemented by giving vev $\rho_{\varnothing}(\sigma^+)$ from eq.(5.14) to the mesons M_A and M_D. Upon considering the mesonic fluctuations around their vev and taking into account the breaking of flavor symmetries and the resulting non-conservation of their currents, we obtain the superpotential of our proposed non-Lagrangian dual:

$$W = \hat{c}\mathrm{tr}\hat{\mu}\Omega\hat{\hat{\mu}}\Omega + \mathrm{tr}M_C\hat{\mu}_C + \mathrm{tr}M_B\hat{\mu}_B + \sum_j (M_A)_{j,-j}(\hat{\mu}_A)_{j,j} + \sum_j (M_D)_{j,-j}(\hat{\mu}_D)_{j,j} \ . \tag{5.56}$$

As usual the R- and $\mathcal{F}$-charges get shifted to:

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + 2\rho^A(\sigma^3) - 2\rho^D(\sigma^3) \ , \\ R &= R_0 - \rho^A(\sigma^3) - \rho^D(\sigma^3) \ . \end{aligned} \tag{5.57}$$

We will denote this theory by $\mathcal{U}_s^{Sp}$.

Non-Lagrangian dual 2: Argyres-Seiberg type dual

A more interesting non-Lagrangian dual is obtained if one considers Higgsing the A and D punctures of $\mathcal{T}_{c3}^{Sp}$ (figure 5.16d). Closing the puncture for $USp(2N-2)_A$ reduces the corresponding $\tilde{T}_{SO(2N)}$ block to a bifundamental of $USp(2N-2) \times SO(2N)$. Giving vev to M_D then gives mass to one of the quarks. We therefore end up with a theory of $2N-3$ fundamentals of $SO(2N)$ coupled to a $T_{SO(2N)}$ block as shown in figure 5.19. The dual superpotential now becomes

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \mathrm{tr}M_B\hat{\mu}_B + \sum_j (M_D)_{j,-j}(\hat{\mu}_D)_{j,j} \ , \tag{5.58}$$

where $(\hat{\mu}_D)_{\alpha\beta} = (\hat{Q}_{m\alpha}\Omega^{ml}\hat{Q}_{l\beta})_D$ and the new $U(1)_{\mathcal{F}}$ and $U(1)_R$ charges are

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 - 2\rho^D(\sigma^3) \ , \\ R &= R_0 - \rho^D(\sigma^3) \ . \end{aligned} \tag{5.59}$$

We will use the short-hand notation $\mathcal{U}_{as}^{Sp}$ for this theory.

Non-Lagrangian dual 3: Crossing type dual

The crossing type dual of $\mathcal{U}^{SO}$ can be obtained by exchanging its (closed) punctures A and D. This will bring A (and similarly D) into a pair pants whose color is opposite to that of A. The statement that these punctures are closed in $\mathcal{U}^{Sp}$ is then equivalent to saying that the puntures are Higgsed by giving a vev to the mesons that we had to couple to the pants. We will denote this theory by $\mathcal{U}_{c2}^{Sp}$. The quiver diagram for $\mathcal{U}_{c2}^{Sp}$ is shown in figure 5.20. Its superpotential is

$$W = \hat{c}\mathrm{tr}\hat{\mu}\Omega\hat{\hat{\mu}}\Omega + \sum_{j,m}(M_A)_{j,-m}(\hat{\mu}_B)_{j,m} + \sum_{j,m}(M_D)_{j,-m}(\hat{\mu}_C)_{j,m} \ , \tag{5.60}$$

while R- and $\mathcal{F}$-charges are

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + 2\rho^B(\sigma^3) - 2\rho^C(\sigma^3) \ , \\ R &= R_0 - \rho^B(\sigma^3) - \rho^C(\sigma^3) \ , \end{aligned} \tag{5.61}$$

where R_0 and $\mathcal{F}_0$ are the charges in the theory without a vev for the mesons i.e. $\mathcal{T}_{c2}^{Sp}$ (see figure 5.16c).

5.4.3 ’t Hooft anomaly matching

Let us go on to put the dualities to test.

Non-Lagrangian duals

It is a simple exercise to check that the tr$R$ and trR^3 anomalies of the electric theory, the theories in the crossing frames 1 and 2, and the theory in the swapped frame match since the mesons have R-charge 1 and hence do not contribute to the R-anomalies.

This also implies that the flavor central charge given by $K\delta^{ab} = -3\text{tr}RT^aT^b$ will also only get contributions from the $\tilde{T}_{SO(2N)}$ blocks and hence will match in all the these frames.

It is instructive to match the tr$R$ and trR^3 anomalies of $\mathcal{T}^{Sp}$ and $\mathcal{T}^{Sp}_{c3}$. Thus in the electric frame these anomalies get contributions from the two $\tilde{T}_{SO(2N)}$ blocks and the gauginos in the $USp(2N-2)$, $\mathcal{N}=1$ vector multiplet. Each $\tilde{T}_{SO(2N)}$ block contributes

$$\text{tr}R\big|_{\tilde{T}_{SO(2N)}} = n_v - n_h = -2(N-1)^2 - N^2 + 1 \ , \tag{5.62}$$

while the gauginos give

$$\text{tr}R\big|_{\text{gaugino}} = 1(N-1)(2N-1) \ . \tag{5.63}$$

This implies

$$\begin{aligned} \text{tr}R\big|_{\mathcal{T}^{Sp}} &= 2\text{tr}R\big|_{\tilde{T}_{SO(2N)}} + \text{tr}R\big|_{\text{gaugino}} \\ &= -4N^2 + 5N - 1 \ . \end{aligned} \tag{5.64}$$

Similarly,

$$\text{tr}R^3\big|_{\mathcal{T}^{Sp}} = 4N^3 - 10N^2 + 7N - 1 \ . \tag{5.65}$$

Let us calculate the above anomalies in $\mathcal{T}^{Sp}_{c3}$. The mesons will not contribute since they have R-charge 1. Thus the contributions come from a $T_{SO(2N)}$ block, a $\tilde{T}_{SO(2N)}$ block and the $SO(2N)$ gauginos. For the $T_{SO(2N)}$ block we find that

$$\text{tr}R\big|_{T_{SO(2N)}} = n_v - n_h = (2-3N)N \ , \tag{5.66}$$

and

$$\mathrm{tr}R^3|_{T_{SO(2N)}} = n_v - \frac{1}{4}n_h = N(3 - 6N + 2N^2) \ . \tag{5.67}$$

The anomalies of $\mathcal{T}_{c3}^{Sp}$ can now be computed:

$$\begin{aligned} \mathrm{tr}R\big|_{\mathcal{T}_{c3}^{Sp}} &= \mathrm{tr}R\big|_{T_{SO(2N)}} + \mathrm{tr}R\big|_{\tilde{T}_{SO(2N)}} + \mathrm{tr}R\big|_{\text{gaugino}} \\ &= -4N^2 + 5N - 1 \ , \end{aligned} \tag{5.68}$$

and

$$\begin{aligned} \mathrm{tr}R^3\big|_{\mathcal{T}_{c3}^{Sp}} &= \mathrm{tr}R^3\big|_{T_{SO(2N)}} + \mathrm{tr}R^3\big|_{\tilde{T}_{SO(2N)}} + \mathrm{tr}R^3\big|_{\text{gaugino}} \\ &= 4N^3 - 10N^2 + 7N - 1 \ . \end{aligned} \tag{5.69}$$

Thus we see the anomalies of the duality frames proposed here match perfectly.

Let us now consider the matching of $\mathrm{tr}\mathcal{F}T^aT^b$ across the various duality frames. For the $\tilde{T}_{SO(2N)}$ block with $U(1)_{\mathcal{F}}$-charge σ, we have

$$\mathrm{tr}\mathcal{F}T^aT^b = \sigma \mathrm{tr}R_{\mathcal{N}=2}T^aT^b = -\frac{\sigma}{2}k_{\mathfrak{g}} \ . \tag{5.70}$$

To begin with, consider the anomaly coefficient for $T^a \in \mathfrak{sp}(2N-2)_A$. Note that $k_{\mathfrak{sp}(2N-2)}$ for $\tilde{T}_{SO(2N)}$ is $4N$ as can be checked by comparing the dual theories of figure D.2. In the electric frame $\mathcal{T}^{Sp}$, we find

$$\mathrm{tr}\mathcal{F}T_A^aT_A^b = -2N\delta^{ab} \tag{5.71}$$

This matches trivially to the anomaly coefficient of $\mathcal{T}_{c1}^{Sp}$ and $\mathcal{T}_{c3}^{Sp}$. It is much more interesting to compare this with the anomaly coefficient of $\mathcal{T}_{c2}^{Sp}$ and $\mathcal{T}_{s}^{Sp}$. After taking the contributions of the meson M_A into account, the anomaly evaluates to

$$\begin{aligned} \mathrm{tr}\mathcal{F}T^a_A T^b_A &= 2N\delta^{ab} - 2\mathrm{tr}_{\mathrm{adj}} T^a_A T^b_A. \\ &= 2N\delta^{ab} - 2(2N)\delta^{ab} \\ &= -2N\delta^{ab} \end{aligned} \tag{5.72}$$

This agrees with the original theory. We can analogously match the anomaly coefficient when $T^a \in \mathfrak{sp}(2N-2)_D$.

We now consider the case when $T^a \in \mathfrak{so}(2N)_B$. This time, by comparing the dual theories of figure D.3, we find that the contribution of $\tilde{T}_{SO(2N)}$ to $k_{\mathfrak{so}(2N)}$ is $4N-4$ and hence in $\mathcal{T}^{Sp}$, the requisite coefficient is

$$\mathrm{tr}\mathcal{F}T^a_B T^b_B = -(2N-2)\delta^{ab} \ . \tag{5.73}$$

After adding the contributions of the meson M_B in the theories corresponding to $\mathcal{T}^{Sp}_{c1}$ and $\mathcal{T}^{Sp}_{s}$ respectively , it is simple to check that their coefficients match the original theory. The above discussion also applies when comparing the anomaly coefficients with $T^a \in \mathfrak{so}(2N)_C$ or $T^a \in \mathfrak{sp}(2N-2)_D$. The matching of these coefficient between $\mathcal{T}^{Sp}$ and $\mathcal{T}^{Sp}_{c2}$ is trivial.

It is much more non-trivial and interesting to match the anomalies of $SO(2N)_B$ and $SO(2N)_C$ in $\mathcal{T}^{Sp}$ and $\mathcal{T}^{Sp}_{c3}$. Let us start by comparing the $SO(2N)_C$ anomalies. In the electric theory we find that

$$\begin{aligned} \mathrm{tr}\mathcal{F}T^a_C T^b_C &= -\mathrm{tr}R_{\mathcal{N}=2} T^a_C T^b_C \\ &= (2N-2)\delta^{ab} \end{aligned} \tag{5.74}$$

Using the linear quiver to evaluate $\mathrm{tr}R_{\mathcal{N}=2}T^a_C T^b_C$ in the $T_{SO(2N)}$ block we find that the anomaly in the magnetic theory matches that in the electric theory. We can then immediately see that the anomalies of $SO(2N)_B$ will match in the electric and magnetic theory after including the contributions of the mesons, M_B.

Dual theories of $USp(2N-2)$ SQCD

The various duality frames obtained after Higgsing some of the $USp(2N-2)$ punctures are summarized in figure 5.21. Since $\mathcal{U}^{Sp}$ and $\mathcal{U}^{Sp}_{c1}$ are related by Intriligator-Pouliot duality and their anomalies match without much ado. For the purpose of matching the anomalies between $\mathcal{U}^{Sp}$ and $\mathcal{U}^{Sp}_{s}$, we observe that we only have to match the anomalies of the $SO(2N) \times USp(2N-2)$ bifundamental to the anomalies of the $\tilde{T}_{SO(2N)}$ block appropriately coupled to mesons (figure 5.22). For the bifundamental we have

$$\mathrm{tr}R\big|_{\mathrm{bifund}} = (-\frac{1}{2})(2N)(2N-2) = -N(2N-2) \ . \tag{5.75}$$

On the dual side, after giving a vev to the mesons, M_A, the R-charge gets shifted: $R \to R - \rho(\sigma^3)$. This will not affect the contribution of the $\tilde{T}_{SO(2N)}$ block, since its $\mathrm{tr}\rho(\sigma^3) = 0$. However for the mesons, we will only consider the contributions of $M_{A,j,-j}$ since the rest decouple. This implies

$$\mathrm{tr}R\big|_{\langle M_A \rangle} = \sum_j j = \sum_{n=1}^{N-1} (2n-1) = (N-1)^2 \ . \tag{5.76}$$

Also M_B does not contribute to the R-anomalies since their R-charge is not shifted and is equal to 1. Putting these together, we find that in this frame

$$\mathrm{tr}R = \mathrm{tr}R|_{\tilde{T}_{SO(2N)}} + \mathrm{tr}R|_{M_A} = -N(2N-2) \ , \tag{5.77}$$

which is identical to the corresponding anomaly of the bifundamental.

Moving on, we now compare the $\mathrm{tr}R^3$ anomalies on the two sides and find

$$\mathrm{tr}R^3|_{\mathrm{bifund}} = \left(-\frac{1}{2}\right)^3 (2N)(2N-2) = -\frac{1}{2}N(N-1)\ . \tag{5.78}$$

On the dual side, since $R = R_0 - \rho(\sigma^3)$, where $R_0 = \frac{1}{2}R_{\mathcal{N}=2} + I_3$, therefore

$$\mathrm{tr}R^3 = \mathrm{tr}R_0^3 + 3\mathrm{tr}R\rho^2\ . \tag{5.79}$$

Adding the contributions of $\tilde{T}_{SO(2N)}$ and the mesons using (5.45) and (5.46), we find that the $\mathrm{tr}R^3$ anomalies match with those of the bifundamental. The $\mathrm{tr}RT^a_BT^b_B$ and $\mathrm{tr}\mathcal{F}T^a_BT^b_B$ anomalies for the bifundamental are given by $(-\frac{1}{2})(2N-2)$ and $(-1)(2N-2)$ respectively. On the dual side these have the same values as in the scenario before Higgsing. This is because $\mathrm{tr}\rho T^a_BT^b_B = 0$ for the $\tilde{T}_{SO(2N)}$block. We therefore conclude that these anomalies have the same value in $\mathcal{U}^{Sp}$ and $\mathcal{U}^{Sp}_s$. The anomalies of $\mathcal{U}^{Sp}$ and $\mathcal{U}^{Sp}_{c2}$ can also be matched in a similar manner by comparing the contributions made by their building blocks shown in figure 5.22a and 5.22c.

We now compare the anomalies of the Argyres-Seiberg type dual, $\mathcal{U}^{Sp}_{as}$. Note that in $\mathcal{U}^{Sp}$

$$\begin{aligned}\mathrm{tr}R &= 2\left(-\frac{1}{2}\right)(2N)(2N-2) + (N-1)(2N-1)\\ &= -2N^2 + N + 1\ ,\end{aligned} \tag{5.80}$$

and

$$\begin{aligned}\mathrm{tr}R^3 &= 2\left(-\frac{1}{2}\right)^3(2N)(2N-2) + (N-1)(2N-1)\\ &= (N-1)^2\ .\end{aligned} \tag{5.81}$$

In the $\mathcal{U}^{Sp}_{as}$, the R-charges are shifted to $R = R_0 - \rho(\sigma^3)$. Also, the meson, M_B, does not get a vev. It therefore has an R-charge 1 and hence does not contribute. In the $T_{SO(2N)}$ block, $\mathrm{tr}\rho(\sigma^3) = 0$, which implies

$$\mathrm{tr}R\big|_{T_{SO(2N)}} = \mathrm{tr}R_0\big|_{T_{SO(2N)}} = (2-3N)N \ . \tag{5.82}$$

The contribution from those components of M_D which continue to stay coupled to the theory after giving a vev is

$$\mathrm{tr}R|_{\langle M_D\rangle} = \sum_j j = \sum_{n=1}^{N-1}(2n-1) = (N-1)^2 \ . \tag{5.83}$$

For the purpose of anomaly matching we can consider the $2N-3$ fundamentals coupled to the $T_{SO(2N)}$ block as a bifundamental of $SO(2N)\times USp(2N-2)$ with shifted R-charges. As usual the shift will correspond to the embedding of $SU(2)$ in $USp(2N-2)$. The shift in the R-charge of the bifundamental does not change its contribution to trR, since $\mathrm{tr}\rho = 0$ for the bifundamental. Thus we find that $\mathrm{tr}R$ in $\mathcal{U}^{Sp}_{as}$ is given by

$$\begin{aligned}\mathrm{tr}R &= \mathrm{tr}R\big|_{T_{SO(2N)}} + \mathrm{tr}R\big|_{M_D} + \mathrm{tr}R\big|_{\mathrm{bifund}} + \mathrm{tr}R\big|_{\mathrm{gaugino}} \\ &= (2-3N)N + (N-1)^2 - N(2N-2) + N(2N-1) \\ &= -2N^2 + N + 1 \ .\end{aligned} \tag{5.84}$$

This shows perfect agreement with the corresponding anomaly in $\mathcal{U}^{Sp}$. Similarly, we find

$$\begin{aligned}\mathrm{tr}R^3|_{T_{SO(2N)}} &= \mathrm{tr}R_0^3|_{T_{SO(2N)}} \\ &= N(3-6N+2N^2) \ .\end{aligned} \tag{5.85}$$

As was mentioned before, the meson, M_B will contribute trivially while the contribution from those modes of M_D that are still coupled to the theory becomes

$$\mathrm{tr}R^3\big|_{\langle M_D\rangle} = \sum_j j^3 = \sum_{n=1}^{N-1}(2n-1) = 1 - 6N + 11N^2 - 8N^3 + 2N^4 \ . \tag{5.86}$$

The contribution of the bifundamental is given by

$$\begin{aligned}\mathrm{tr}R^3\big|_{\mathrm{bifund}} &= \mathrm{tr}R_0^3\big|_{\mathrm{bifund}} + 3\mathrm{tr}R\rho^2 \\ &= -\frac{1}{2}N(N-1) + \frac{1}{2}(N-1)(4N^2-8N+3)(-N) \\ &= -N(N-1)(2N^2-4N+2) \ .\end{aligned} \tag{5.87}$$

Combining all these contributions we find

$$\mathrm{tr}R^3 = \mathrm{tr}R^3\big|_{T_{SO(2N)}} + \mathrm{tr}R^3\big|_{M_D} + \mathrm{tr}R^3\big|_{\mathrm{bifund}} + \mathrm{tr}R^3\big|_{\mathrm{gaugino}} = (N-1)^2 \ , \tag{5.88}$$

hence providing a nontrivial check of our proposal. It can also be checked, via a pretty direct calculation, that the $\mathrm{tr}RT^aT^b$ and $\mathrm{tr}\mathcal{F}T^aT^b$ anomalies also match in these theories.

5.5 Dualities for the G_2 gauge theory

In this section, we study a G_2 gauge theory and its dual frames. The G_2 gauge group can be obtained from $\Gamma = D_4$ theory with $\mathbb{Z}_3$ outer-automorphism twist. Since the D_4 theory allows both $\mathbb{Z}_2$-twisting σ_2 and $\mathbb{Z}_3$-twisting σ_3. We should take the twist lines with slightly more care to go to various different dual frames.

We study the G_2 gauge theory with 8 fundamental quarks in the 7 dimensional representation of G_2. A dual theory for the G_2 gauge theory was first proposed in [165] where the dual theory is given by SU gauge group with anti-symmetric tensors. We find new dual descriptions for the G_2 gauge theory flowing to the same fixed point in the IR. We test the duality via anomaly matching and comparison of superconformal indices.

5.5.1 G_2 gauge theory and its dual from coupled E_7 blocks

To obtain the G_2-dual we propose the following procedure: start with the strongly coupled block of [182] given by D_4 theory on a three punctured sphere with a twisted null puncture, a $USp(6)$ puncture and a G_2 puncture as in figure 5.23. Even though the E_7 flavor symmetry is not manifest, the theory exhibits enhanced E_7 symmetry which is the theory of Minahan-Nemeschansky [157]. We will demonstrate in section 5.6 that the superconformal index of the theory of figure 5.23 agrees with the E_7 theory.

Now prepare two copies of this theory. By gauging the G_2 symmetry common to the two blocks we obtain an $\mathcal{N} = 2$ SCFT with a G_2 gauge group which can be represented by figure 5.24a. We can obtain its S-dual by exchanging the punctures. One of its S-dual can be obtained by exchanging two null punctures. It is given by an $\mathcal{N} = 2$ SCFT with $Spin(8)$ gauge symmetry along with three hypermultiplets in $\mathbf{8}_V$ and three hypermultiplets in $\mathbf{8}_S$ representations which can be represented as in the figure 5.24b. This duality was first found in [25]. Another frame can be found by colliding two null punctures and two $USp(6)$ punctures. This is similar to the Argyres-Seiberg duality, where in this case we partially gauge the theory with $USp(6)^2 \times G_2$ flavor symmetry.

$\mathcal{N} = 1$ duality from E_7 blocks

Let us go to the $\mathcal{N} = 1$ construction. It can be done by giving colors to the punctures and the pair of pants. In figure 5.24a, let's color the two punctures on the bottom to be red, and the other two punctures to be blue. Also color the pair of pants on the bottom to be red and the other to be blue. Since the color of the punctures and the pants are the same, we can identify the 'matter content' to be the same two E_7 blocks as before. Then we glue two G_2 punctures by $\mathcal{N} = 1$ vector multiplet with the superpotential

$$W = c\,\mathrm{tr}\mu\tilde{\mu} \ , \tag{5.89}$$

where μ and $\tilde{\mu}$ transform in the adjoint representation of G_2.

A dual frame is described by a $Spin(8)$ gauge theory, with quarks in the $\mathbf{8}_V \times \mathbf{6}$ of $Spin(8) \times USp(6)_A$ and another in the $\mathbf{8}_S \times \mathbf{6}$ representations of $Spin(8) \times USp(6)_B$. There are also mesons transforming in the adjoint representations of $USp(6)_A$ and $USp(6)_B$ respectively. One can also prove the duality starting from $\mathcal{N} = 2$ construction and then giving mass to the chiral adjoint in the vector multiplet if we assume the chiral ring relation

$$\mathrm{tr}{\mu_{G_2}}^2 = \mathrm{tr}(\mu_{USp(6)}\Omega)^2 \ , \tag{5.90}$$

and then following the procedure of [82]. The dual superpotential is given by

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\tilde{\mu}} + \mathrm{tr}M_A\hat{\mu}_A + \mathrm{tr}M_B\hat{\mu}_B \ , \tag{5.91}$$

where $\hat{\mu}_A = Q_A Q_A$ and $\hat{\mu}_B = Q_B Q_B$.

Upon Higgsing the $USp(6)$ flavor symmetries, in the electric frame, down to $USp(4)$, we obtain two copies of bifundamentals of $G_2 \times USp(4)$ with the G_2 being gauged. Higgsing is achieved by giving a vev to the adjoint of $USp(6)$ along the partition: $6 = [2, 1^4]$. We will use the short-hand notation $\mathcal{U}^{G_2}$ to denote this theory. In the dual frame we will have to give the same vev to the mesons M_A and M_B. This will generate a mass for the dual quarks with $SU(2)$ quantum numbers $(j = \frac{1}{2}, m = -\frac{1}{2})$. We integrate these out and obtain the low energy theory which is described by 5 vectors and 5 spinors of the $Spin(8)$ gauge group and transforming as $\mathbf{4} \oplus \mathbf{1}$ of their respective $USp(4)$ flavor symmetries. The low energy superpotential in the dual frame becomes

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\tilde{\mu}} + \sum_j M_{Aj,-j}\hat{\mu}_{Aj,j} + \sum_j M_{Bj,-j}\hat{\mu}_{Bj,j} \ , \tag{5.92}$$

with $\hat{\mu}_{Aj,j}$ being quadratics $Spin(8)$ invariants. The R-charge in magnetic frame is shifted by $R \to R - \rho^A(\sigma^3) - \rho^B(\sigma^3)$, where as usual ρ specifies the $SU(2)$ embedding in $USp(6)$. The $U(1)_{\mathcal{F}}$ gets shifted to $\mathcal{F} \to \mathcal{F} - 2\rho^A(\sigma^3) + 2\rho^B(\sigma^3)$. Some of the mesons decouple.and we are left with the mesons $M_{j,j,k}$ coupled to the magnetic theory. This theory will be denoted by the symbol $\mathcal{U}_{c1}^{G_2}$.

Non-Lagrangian duals

We can also get several non-Lagrangian duals to the G_2 theory using different colored pair-of-pants decompositions. See the figure 5.26.

Non-Lagrangian dual 1: Argyres-Seiberg type

The Argyres-Seiberg type dual of figure 5.26a is obtained by colliding the punctures A and B on the Riemann surface. This will land us upon a theory consisting of an E_7 block (with $G_2 \times USp(6) \subset E_7$ manifest) coupled to a $USp(6) \times USp(4) \times G_2$ block via an $\mathcal{N} = 1$, G_2 vector multiplet. We will also have to integrate in mesons (with appropriate vevs) to compensate for the mismatch between the colors of the punctures and the pair of pants. Its superpotential is

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \mathrm{tr}\sum_j (M_A)_{j,-j}(\hat{\mu}_A)_{j,j} + \sum_j (M_C)_{j,-j}(\hat{\mu}_C)_{j,j} \tag{5.93}$$

with the shifted charges being

$$R \to R - \rho^A(\sigma^3) - \rho^C(\sigma^3) \ , \tag{5.94}$$

$$\mathcal{F} \to \mathcal{F} - 2\rho^A(\sigma^3) + 2\rho^C(\sigma^3) \ . \tag{5.95}$$

We will use the symbol $\mathcal{U}_{as}^{G_2}$ to denote this theory.

Non-Lagrangian dual 2: Swapped G_2

We arrive at the swapped G_2 frame by permuting all the four punctures such that we exchange A with B and C with D. This is equivalent to coupling two $USp(6)^2 \times G_2$ theories along their G_2 puncture. We will have to integrate in 4 mesons M_A, M_B, M_C and M_D. We give vevs to these mesons such that $USp(6)_C$ and $USp(6)_D$ get completely Higgsed while $USp(6)_A$ and $USp(6)_B$ get Higgsed down to their respective $USp(4)$. This theory will henceforth be denoted by $\mathcal{U}_s^{G_2}$. Its superpotential becomes

$$\begin{aligned} W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \mathrm{tr}\sum_j (M_A)_{j,-j}(\hat{\mu}_A)_{j,j} + \sum_j (M_B)_{j,-j}(\hat{\mu}_B)_{j,j} + \\ \sum_j (M_C)_{j,-j}(\hat{\mu}_C)_{j,j} + \sum_j (M_D)_{j,-j}(\hat{\mu}_D)_{j,j} \ , \end{aligned} \tag{5.96}$$

while the charges get shifted such that

$$R \to R - \rho^A(\sigma^3) - \rho^B(\sigma^3) - \rho^C(\sigma^3) - \rho^D(\sigma^3) \ , \tag{5.97}$$

$$\mathcal{F} \to \mathcal{F} - 2\rho^A(\sigma^3) + 2\rho^B(\sigma^3) + 2\rho^C(\sigma^3) - 2\rho^D(\sigma^3) \ . \tag{5.98}$$

Non-Lagrangian dual 3: Crossing-type

The crossing-type frame is shown in figure 5.26c. It consists of two blocks with $USp(6) \times USp(4) \times SO(8)$ flavor symmetry glued along their $SO(8)$ puncture. The spectrum of the theory also includes mesons M_C and M_D as the punctures C and D lie in pants that are colored oppositely to their own color. We will give a vev to the mesons such that the $USp(6)$ flavor symmetry of these punctures gets completely Higgsed. The superpotential then becomes

$$W = \hat{c}\mathrm{tr}\hat{\mu}\hat{\hat{\mu}} + \textstyle\sum_j (M_C)_{j,-j}(\hat{\mu}_C)_{j,j} + \sum_j (M_D)_{j,-j}(\hat{\mu}_D)_{j,j} \ , \tag{5.99}$$

and the new charges are given by

$$R \to R - \rho^C(\sigma^3) - \rho^D(\sigma^3) \ , \tag{5.100}$$

$$\mathcal{F} \to \mathcal{F} + 2\rho^C(\sigma^3) - 2\rho^D(\sigma^3) \ . \tag{5.101}$$

We will use the symbol $\mathcal{U}_{c2}^{G_2}$ to represent this theory.

5.5.2 Anomaly matching

We now show that the anomalies of our proposed dual frames match.

tr$R$ and trR^3

In the G_2 electric theory, we find that

$$\text{tr}R = 14 + \left(-\frac{1}{2}\right)(8 \times 7) = -14 \ , \tag{5.102}$$

$$\text{tr}R^3 = 14 + \left(-\frac{1}{2}\right)^3 (8 \times 7) = 7 \ . \tag{5.103}$$

After considering the shift in the charges, we find that in $\mathcal{U}_{c1}^{G_2}$ frame

$$\begin{aligned} \text{tr}R &= 28 + 2\sum_{j,m}\left(-\frac{1}{2} - m\right) \times 8 + 2\sum_j j \\ &= 28 - 48 + 6 = -14 \ , \end{aligned} \tag{5.104}$$

which is same as the result obtained for the electric theory. Similarly for the trR^3 anomaly in the $Spin(8)$ theory we obtain

$$\begin{aligned} \text{tr}R^3 &= 28 + 2\sum_{j,m}\left(-\frac{1}{2} - m\right)^3 \times 8 + 2\sum_j j^3 \\ &= 28 - 24 + 3 = 7 \ , \end{aligned} \tag{5.105}$$

which matches with the electric theory.

The effective number of hypers and vectors in a block with $USp(6) \times USp(4) \times SO(8)$ flavor symmetry is 102 and 72 respectively [57]. Using this result we find that the tr$R$ anomaly in the $\mathcal{U}_{c2}^{G_2}$ is

$$\mathrm{tr}R = 2(72 - 102) + 28 + 2 \times 9 = -14\ , \tag{5.106}$$

here the first term on the RHS corresponds to the contribution of the non-Lagrangian blocks to tr$R$, the second term is the contribution from $SO(8)$ gauginos while the last term is the contribution of the mesons used to Higgs the $USp(6)$ flavor symmetry of the blocks. Using (5.43) and (5.44) along with the fact that in $\tilde{T}_{SO(2N)}$, $k_{\mathfrak{sp}(2N-2)} = 4N$ we find that in $\mathcal{U}_{c2}^{G_2}$

$$\mathrm{tr}R^3 = -327 + 28 + 306 = 7\ . \tag{5.107}$$

As before the various terms on the RHS are obtained from the contribution of the non-Lagrangian blocks, the SO(8) gauginos and the mesons respectively.

The effective number of hypers and vectors in the block with $USp(6)^2 \times G_2$ symmetries can be obtained by comparing the $\mathcal{N} = 2$ theory obtained by gluing two such blocks along their G_2 puncture and its S-dual corresponding to two copies of the block with $USp(6)^2 \times SO(8)$ punctures glued along their $SO(8)$ puncture with a $\mathbb{Z}_3$ twist around the cylinder. This will also provide us with the central charges of the various flavor symmetries. Following this procedure we find that in the $USp(6)^2 \times G_2$ block, $n_v = 86$ and $n_h = 112$. Using this and including the contribution of the mesons that stay coupled to the theory (after Higgsing one of the $USp(6)$ down to $USp(4)$ and completely

Higgsing the other $USp(6)$), we find that in $\mathcal{U}_s^{G_2}$

$$\mathrm{tr}R = 2\times(86-112)+14+2\times 9+2\times 3 = -14 \ . \tag{5.108}$$

If we now calculate the $\mathrm{tr}R^3$ anomaly in this theory, we find

$$\mathrm{tr}R^3 = -316\times 2+14+\frac{3}{2}\times 2+153\times 2 = 7 \ , \tag{5.109}$$

This is in agreement with our proposal.

We can use our knowledge of the number of hypers and vectors and central charges in the $USp(6)\times USp(4)\times SO(8)$ block to evaluate this data for the $USp(6)\times USp(4)\times G_2$ block which are: $n_v = 79$ and $n_h = 102$. The $\mathrm{tr}R$ anomaly can now be calculated in $\mathcal{U}_{as}^{G_2}$ and is found to match with that in the other duality frames:

$$\mathrm{tr}R|_{\mathcal{U}_{as}^{G_2}} = (79-102)+(7-24)+14+9+3 = -14 \ , \tag{5.110}$$

here the first term on the RHS is the contribution from the $USp(6)\times USp(4)\times G_2$ block while second term is the contribution from the E_7 theory. The third term is the contribution of G_2 gauginos while the last two terms are the contributions of the mesonic excitations. The coefficient of $\mathrm{tr}R^3$ in this theory is

$$\mathrm{tr}R^3 = -209+\frac{95}{2}+14+153+\frac{3}{2} = 7 \ . \tag{5.111}$$

This is consistent with our expectations.

$\mathrm{tr}R\mathcal{F}^2$

In the G_2 theory, each block contributes

$$\mathrm{tr}R\mathcal{F}^2 = \left(-\frac{1}{2}\right)(4\times 7) = -14\ . \tag{5.112}$$

In the $SO(8)$ theory, the $\mathcal{F}$ charges are shifted such that $\mathcal{F} \to \mathcal{F} - 2\rho^A(\sigma^3) + 2\rho^B(\sigma^3)$. The contribution of the pants with color 'σ' is therefore given by

$$\mathrm{tr}R\mathcal{F}^2 = \sum_{j,m}\Big(-\frac{1}{2} - m\Big)(\sigma + 2\sigma m)^2 \times 8 + \sum_j j(-2\sigma - 2\sigma j)^2 = -14\ . \tag{5.113}$$

This shows a perfect match with the G_2 theory.

In the non-Lagrangian duals, the shifted charges $R = R_0 - \rho(\sigma^3)$ and $\mathcal{F} = \mathcal{F} - 2\sigma\rho(\sigma^3)$, (for pants with color 'σ') give rise to the following expression for $\mathrm{tr}R\mathcal{F}^2$:

$$\mathrm{tr}R\mathcal{F}^2 = \mathrm{tr}(R_0 - \rho)(\mathcal{F}_0 - 2\sigma\rho)^2 = \mathrm{tr}R_0\mathcal{F}_0^2 + 6\mathcal{I}\mathrm{tr}R_{\mathcal{N}=2}T^aT^b \tag{5.114}$$

where we have used the $SU(2)$ embedding index $\mathcal{I}$ to evaluate $\mathrm{tr}R\rho^2$ and $\mathrm{tr}\mathcal{F}\rho^2$. The final expression in (5.114) is independent of the color of pants, as should be the case. Also, on each pair-of-pants $\mathrm{tr}R_0\mathcal{F}_0^2 = -n_h$. Using this and taking the contribution of mesons into account, it can be verified that each pair-of-pants in the decomposition of $\mathcal{U}_{c2}^{G_2}$ and $\mathcal{U}_s^{G_2}$, contributes a -14 to the anomaly, thereby establishing the match with the electric frame. In $\mathcal{U}_{as}^{G_2}$, since the pair-of-pants decomposition is not symmetric thus the pants contribute different amounts to the total anomaly. The pants with $USp(6)\times USp(4)\times SO(8)$ punctures contributes -92 while the other pant contributes 64, thereby bringing the total to -28 which is same as in the electric theory.

trRT^aT^b and tr$\mathcal{F}T^aT^b$

After Higgsing the $USp(6)$ punctures in the G_2-frame of figure 5.24a, we are left with a $USp(4)_A \times USp(4)_B$ flavor symmetry which is enhanced to $USp(8)$ in the electric theory when there is no superpotential. We now match the 'tHooft anomalies of these flavor symmetries in the electric and the magnetic frames. In the G_2 theory we find that

$$\mathrm{tr}RT^a_AT^b_A = 7 \times (-\frac{1}{2}) \times \mathrm{tr}_{\square}T^a_AT^b_A = -\frac{7}{2}\delta^{ab} \ , \tag{5.115}$$

$$\mathrm{tr}\mathcal{F}T^a_AT^b_A = 7 \times (-1) \times \mathrm{tr}_{\square}T^a_AT^b_A = -7\delta^{ab} \ . \tag{5.116}$$

It is straight forward to check that these match with those in the $SO(8)$ theory, once we use the shifted R and $\mathcal{F}$ charges. Thus in the SO(8) theory we have

$$\mathrm{tr}RT^a_AT^b_A = 8 \times \mathrm{tr}\left(-\frac{1}{2} + \rho\right)T^a_AT^b_A + \sum_j \mathrm{tr}jT^a_AT^b_A = -\frac{7}{2}\delta^{ab} \ , \tag{5.117}$$

and

$$\mathrm{tr}\mathcal{F}T^a_AT^b_A = 8 \times \mathrm{tr}(1+2\rho)T^a_AT^b_A + \sum_j \mathrm{tr}(-2-2j)T^a_AT^b_A = -7\delta^{ab} \ , \tag{5.118}$$

which is same as the corresponding anomalies of the G_2 theory. The same discussion will also apply in the case of anomalies for the $USp(4)_B$ flavor symmetries.

In $\mathcal{U}^{G_2}_{c2}$ the anomaly coefficients can be obtained from the flavor central charges: $\mathrm{tr}RT^a_AT^b_A = \frac{1}{2}\mathrm{tr}R_{\mathcal{N}=2}T^a_AT^b_A$ and $\mathrm{tr}\mathcal{F}T^a_AT^b_A = \mathrm{tr}R_{\mathcal{N}=2}T^a_AT^b_A$. Since $k_{\mathfrak{sp}(4)} = 7$, we find that the anomaly coefficients match those in the electric frame. The same holds for the anomalies of $USp(4)_B$.

The anomalies of in $\mathcal{U}^{G_2}_{s}$ can be obtained from the embedding index of $USp(4)$ in

$USp(6)$. Thus for the pair-of-pants containing the puncture A we find

$$\mathrm{tr}R_{\mathcal{N}=2}T^a_A T^b_A = \mathcal{I}\mathrm{tr}R_{\mathcal{N}=2}T^a_{\mathfrak{sp}(6)}T^b_{\mathfrak{sp}(6)} \ . \tag{5.119}$$

Since the **6** of $USp(6)$ becomes $\mathbf{4}\oplus\mathbf{1}\oplus\mathbf{1}$ of $USp(4)$, therefore $\mathcal{I}=1$. We will also have to add the contribution of the mesons. Thus

$$\begin{aligned}\mathrm{tr}RT^a_A T^b_A|_{\mathcal{U}^{G_2}_s} &= \frac{1}{2}\mathcal{I}\mathrm{tr}R_{\mathcal{N}=2}T^a_{\mathfrak{sp}(6)}T^b_{\mathfrak{sp}(6)} + \sum_j trjT^a_A T^b_A \\ &= (-4+\frac{1}{2}\times 1)\delta^{ab} = -\frac{7}{2}\delta^{ab} \ .\end{aligned} \tag{5.120}$$

Similarly we can show that $\mathrm{tr}\mathcal{F}T^a_A T^b_A|_{\mathcal{U}^{G_2}_s} = -7\delta^{ab}$. The anomalies of $USp(4)_B$ match those in the electric frame in an analogous manner.

The anomalies of $\mathcal{U}^{G_2}_{as}$ can also be shown to match after using the fact that $k_{\mathfrak{sp}(4)_B} = 7$ and proceeding in the same way as in $\mathcal{U}^{G_2}_s$ for the anomalies of $USp(4)_A$.

5.6 Superconformal index

In this section, we put our new dualities to test by comparing the superconformal indices for the dual theories. We first review superconformal indices for the $\mathcal{N}=2$ theories of class $\mathcal{S}$ studied in [83, 85, 86, 92] which was extended to the case of type D by [148]. In the process, we close some of the loose ends regarding the $\mathbb{Z}_{2,3}$-twisted punctures of D_n theories. Then we compute the superconformal indices for the $\mathcal{N}=1$ theories studied in section 5.3, 5.4, 5.5 using a similar formalism developed in [39, 82].

5.6.1 $\mathcal{N}=2$ index

The $\mathcal{N}=2$ superconformal index is defined as

$$I = \mathrm{Tr}(-1)^F \left(\frac{t}{pq}\right)^r p^{j_2+j_1} q^{j_2-j_1} t^R \prod_i x_i^{f_i} \ , \tag{5.121}$$

where (j_1, j_2) are the Cartans of the Lorentz group $SU(2)_1 \times SU(2)_2$, r and R are the $U(1)_R$ and $SU(2)_R$ generators respectively. The f_i denote the Cartans for the flavor symmetry group. For any class $\mathcal{S}$ theories, the indices can be thought of as a correlation function for a topological field theory. It turns out that the indices for a class $\mathcal{S}$ theory defined by a Riemann surface $\mathcal{C}$ with genus g and n twisted or untwisted punctures labeled by $\rho_{1,\cdots,n}$ can be written as

$$I = \sum_\lambda \frac{\prod_{I=1}^n K_{\rho_I}(\vec{a}_I) P_\lambda(\vec{a}_{\rho_I})}{(K_\varnothing P_\lambda(t^\varnothing))^{2g-2+n}} \ , \tag{5.122}$$

where the summation is over the representations λ of Γ. Let us explain the meaning of various symbols.

- The function P_λ is some special function defined by requiring the function $f_\lambda(\vec{a}) = K_{\text{full}}(\vec{a}) P_\lambda(\vec{a})$ to be orthonormal under the measure given by the vector multiplet index $I_V(\vec{a})$:

$$\oint [d\vec{z}] I_V(\vec{z}) f_\lambda(\vec{a}) f_{\lambda'}(\vec{a}) = \delta_{\lambda\lambda'} \ . \tag{5.123}$$

  The function P_λ can be Schur function or Macdonald polynomial or related to the wave function of elliptic Ruijsenaars-Schneider model depending on the number of fugacities (p, q, t) we want to keep. The P_λ also depend on the choice of twisted/untwisted puncture.

- The K-factor K_ρ is labeled by a embedding ρ of $SU(2)$ into G, where $G = \Gamma$ for the untwisted puncture and G is the group formed by folding the Dynkin diagram with the choice of outer-automorphism as in the table 5.1. The embedding ρ induces a decomposition of adjoint into the form $\oplus_j R_j \otimes V_j$ where V_j is the spin-j irrep of $SU(2)$ and R_j are representations for the flavor symmetry group associated to the puncture. For the case of the Macdonald index ($p = 0$), the K-factor can be written as [156]

$$K_\Lambda(\vec{a}) = \mathrm{PE}\left[\sum_j \frac{t^{j+1}}{1-q}\mathrm{tr}_{R_j}(\vec{a})\right] \ , \tag{5.124}$$

where PE stands for the plethystic exponential. For example, for the full puncture, it is simply given by

$$K_{\text{full}}(\vec{a}) = \mathrm{PE}\left[\frac{t}{1-q}\chi_{\text{adj}}(\vec{a})\right] \ . \tag{5.125}$$

For the null puncture $\varnothing$, it is given by

$$K_\varnothing = \mathrm{PE}\left[\frac{t^{d_i}}{1-q}\right] = \prod_{i=1}^{\mathrm{rank}(\Gamma)} (t^{d_i}; q)^{-1} \ , \tag{5.126}$$

where d_i are the degrees of invariants of G and $(x;q) = \prod_{i=0}^{\infty}(1 - xq^i)$ is the Pochhammer symbol. The general form of $K_\Lambda(\vec{a}; p, q, t)$ has been conjectured in [82] to be

$$K_\Lambda(\vec{a}) = \mathrm{PE}\left[\sum_j \frac{t^{j+1} - pqt^j}{(1-q)(1-p)}\mathrm{tr}_{R_j}(\vec{a})\right] \ . \tag{5.127}$$

- The argument $\vec{a}_{\rho_I}$ can be determined by looking at the embedding of $\rho(SU(2)) \times G_F$ into G where $\rho(SU(2))$ is image under the map ρ and G_F is the flavor symmetry group associated to the puncture. The fundamental of G can be decomposed in terms of spin-j irreps of $SU(2)$ as $\text{fund}_G = \oplus_j R_j^F \otimes V_j$. One can match the fugacities by using characters. First write down the character for the fundamental of G. And then compare it with the characters of the representations of $SU(2) \times G_F$. By comparing the two, one can map the fugacities for the G_F to the fugacities of G appear in $P_\lambda(\vec{a})$. See the section 4.2.1 of [148] for more details.

Now, let us focus on the examples of twisted D_n-type theories. We will restrict our discussion to the case of Macdonald index $p = 0$.

We implemented computation of Macdonald polynomials using the procedure outlined in appendix B of [156] through direct Gram-Schmidt process using Mathematica and LieART [71]. There is more efficient method of computing Macdonald polynomials for $A, B, C, D, E_{6,7}$ through determinantal construction [188]. We refer to appendix A of [148] for a nice review on the construction of Macdonald and Hall-Littlewood polynomials.

D_n-type theories with $\mathbb{Z}_2$-twist

The function P_λ in our case becomes the normalized Macdonald polynomial of type G where G is either $\Gamma = D_n$ or $G = C_{n-1}$ depending on the choice of untwisted and twisted puncture.

$$P_\lambda(\vec{a}) = N_\lambda^{-1/2} P_{M,G}^\lambda(\vec{a}; q, t) \ , \tag{5.128}$$

where $P_{M,G}$ is the Macdonald polynomial given by the root-system of G.[9] The $N_\lambda(q,t)$ is a normalization factor given by inner product of two Macdonald polynomials

$$N_\lambda = \langle P^\lambda_{M,G}, P^\lambda_{M,G} \rangle = \int [d\vec{z}]_G \mathrm{PE}\left[\frac{-q+t}{1-q}\chi_{\mathrm{adj}}(\vec{z})\right] P^\lambda_{M,G}(\vec{z}) P^\lambda_{M,G}(\vec{z}) \ , \tag{5.129}$$

where $[dx]_G$ stands for the Haar measure of the group G. For the D_4 case, we have two different choice of twisting, namely $\mathbb{Z}_2$ and $\mathbb{Z}_3$ which gives C_3 and G_2. We will treat this special case later in this section.

The superconformal index for the $T_{SO(2n)}$ theory is given by

$$I = \frac{K^{SO}_{\mathrm{full}}(\vec{a}_1) K^{SO}_{\mathrm{full}}(\vec{a}_2) K^{SO}_{\mathrm{full}}(\vec{a}_3)}{K^{SO}_{\varnothing}} \sum_{\lambda \in R_{SO(2n)}} \frac{P^{SO}_\lambda(\vec{a}_1) P^{SO}_\lambda(\vec{a}_2) P^{SO}_\lambda(\vec{a}_3)}{P^{SO}_\lambda(t^{\varnothing})} \ , \tag{5.130}$$

where the P_λ is given by the $SO(2n)$ Macdonald polynomial. One can start from this theory and then by partially closing or Higgsing the punctures, to obtain general theory corresponding to a 3 punctured sphere. In more extreme limit, one can consider completely closing the punctures. Then the index should be trivial, which completely fixes the factor in the denominator which is the structure constant of the TQFT.

More generally, when we have twisted punctures, the structure constant can be fixed by requiring it to become trivial when we close all the three punctures. Therefore we can write the index for the $\tilde{T}(SO(2n))$ theory as

$$I = \frac{K^{SO}_{\mathrm{full}}(\vec{a}) K^{USp}_{\mathrm{full}}(\vec{b}_1) K^{USp}_{\mathrm{full}}(\vec{b}_2)}{K^{SO}_{\varnothing}} \sum_{\lambda \in R_{USp(2n-2)}} \frac{P^{SO}_\lambda(\vec{a}) P^{USp}_\lambda(\vec{b}_1) P^{USp}_\lambda(\vec{b}_2)}{P^{SO}_\lambda(t^{\varnothing})} \ , \tag{5.131}$$

where the sum is over the representations of $USp(2n-2)$ not $SO(2n)$. For the $P^{SO(2n)}$,

[9] In general, P_M is labeled by an affine root system. There is many to one map between the affine root systems and the group G. In our case, only the Macdonald polynomial for G appears. The other ones such as the dual root system $G^\vee$ and the non-reduced affine root system $(C_n^\vee, C_n)$ appear when we consider outer-automorphism twisted index. [156]

we restrict the sum to the case of outer-automorphism invariant representations. In terms of Dynkin labels, they are of the form $[\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_{n-1}]$.

One can completely close one of the USp puncture to obtain the free theory of $SO(2n) \times USp(2n-2)$ bifundamental half-hypermultiplets. It is given by

$$I_{\text{bifund}} = \frac{K^{SO}_{\text{full}}(\vec{a}) K^{USp}_{\text{full}}(\vec{b}) K^{USp}_{\varnothing}}{K^{SO}_{\varnothing}} \sum_{\lambda \in R_{USp(2n-2)}} \frac{P^{SO}_{\lambda}(\vec{a}) P^{USp}_{\lambda}(\vec{b}) P^{USp}_{\lambda}(t^{\varnothing})}{P^{SO}_{\lambda}(t^{\varnothing})} \; . \qquad (5.132)$$

We have checked this relation up to $n = 5$ and to a few orders in q.

When we glue three punctured spheres, we integrate with a vector multiplet measure. From the orthonormality condition (5.123), we arrive at the same result of (5.122). One interesting aspect here is that whenever there is a twisted puncture, summation over the representations of Γ reduces to that of G.

D_4-type theories with $\mathbb{Z}_3$-twist

The $\Gamma = D_4$ theory can be twisted in two different ways because the outer-automorphism group is generated by $\mathbb{Z}_2$ and also $\mathbb{Z}_3$. The $\mathbb{Z}_2$ twisting gives $C_3 = USp(6)$ puncture and the $\mathbb{Z}_3$ twisting gives G_2 puncture. Consider the three punctured sphere given by one $USp(6)$ puncture and one G_2 puncture with twisted null puncture as in the figure 5.23. From the TQFT structure, we can write its index as

$$I_{E_7}(\vec{a}, \vec{b}) = \frac{K^{G_2}_{\text{full}}(\vec{a}) K^{USp}_{\text{full}}(\vec{b}) K^{USp}_{\varnothing}}{K^{SO}_{\varnothing}} \sum_{\lambda \in R_{G_2}} \frac{P^{G_2}_{\lambda}(\vec{a}) P^{USp}_{\lambda}(\vec{b}) P^{USp}_{\lambda}(t^{\varnothing})}{P^{SO}_{\lambda}(t^{\varnothing})} \; . \qquad (5.133)$$

Here the sum is over the representations of G_2. For the $SO(8)$ and $USp(6)$ punctures, this means summing over the representations invariant under the $\mathbb{Z}_3$ action. In terms of the Dynkin labels, they are $[\lambda_1, \lambda_2, \lambda_1, \lambda_1]$ and $[\lambda_1, \lambda_2, \lambda_1]$ for the G_2 representation $[\lambda_2, \lambda_1]$.

The TQFT structure requires S-duality invariance of the index. In our case, it translates to the condition that the indices for the first two frames of G_2-coupled two E_7 theories as in figure 5.24 being equal. We should have

$$\oint [d\vec{\omega}] I_{\text{vec}}^{G_2}(\vec{\omega}) I_{E_7}(\vec{\omega},\vec{a}) I_{E_7}(\vec{\omega},\vec{b}) = \oint [d\vec{z}] I_{\text{vec}}^{SO(8)}(z) I_{\text{bifund}}(\vec{z},\vec{a}) I_{\text{bifund}}(\vec{\tilde{z}},\vec{b}) \ , \qquad (5.134)$$

where I_{vecl}^{G} is the vector multiplet index for the gauge group G and I_{bifund} denotes the index of the $SO(8) \times USp(6)$ bifundamentals (5.132). We represent the G_2 fugacities with $\vec{\omega}$ while the $SO(8)$ fugacities are given by $\vec{z} = (z_1, z_2, z_3, z_4)$ and $\vec{\tilde{z}} = (z_4, z_2, z_3, z_1)$. The transformation of $SO(8)$ fugacities from $\vec{z}$ to $\vec{\tilde{z}}$ implements the $\mathbb{Z}_3$ twist around the $SO(8)$ cylinder in figure 5.24b. Orthogonality of the $SO(8)$ wave-functions upon integration with respect to the $SO(8)$ vector multiplets implies that only those representations that are of the $\mathbb{Z}_3$ invariant form mentioned before, contribute to the RHS of (5.134). This is enough to show the identity of (5.134)

As a remark, we find that the index for the E_7 theory can also be written as

$$I_{E_7}(\vec{a},\vec{b}) = \frac{K_{\varnothing}^{G_2} K_{\text{full}}^{G_2}(\vec{a}) K_{\text{full}}^{USp}(\vec{b})}{K_{\varnothing}^{USp}} \sum_{\lambda \in R_{G_2}} \frac{P_{\lambda}^{G_2}(t^{\varnothing}) P_{\lambda}^{G_2}(\vec{a}) P_{\lambda}^{USp}(\vec{b})}{P_{\lambda}^{USp}(t^{\varnothing})} \ . \qquad (5.135)$$

We can get this form from the identity

$$\left(P_{\lambda}^{USp}(t^{\varnothing})\right)^2 = P_{\lambda}^{SO}(t^{\varnothing}) P_{\lambda}^{G_2}(t^{\varnothing}) \ , \qquad (5.136)$$

where the representations λ are now restricted to belong to the $\mathbb{Z}_3$ invariant form discussed above. We do not have an analytic proof of the identity (5.136), but we were able to check this relation for several low-dimensional representations.

From the form (5.135), the index becomes 1 upon closing all the punctures. For the case of UV curves without twisted punctures, we always get 1 upon closing all the punctures. It is not clear whether it should be the case with twisted punctures, because even after closing a twisted puncture it still carries non-trivial information. Nevertheless, it turns out that the superconformal index is unity for the theory having a UV curve with only null punctures (with or without twist) of type A_n, D_n.

Enhancement of Global symmetry $USp(6) \times G_2$ to E_7

As we have discussed in section 5.5, the theory given by $USp(6)$ and G_2 punctures is expected to have enhanced E_7 global symmetry [182]. Here we check this explicitly through the computation of index. We find that the index of this theory computed by (5.133) can be indeed written in terms of the characters of E_7.

The product algebra $G_2 \times USp(6)$ is embedded into E_7 such that [155]

$$56 \rightarrow (7,6) \oplus (1,14) \ , \tag{5.137}$$

$$133 \rightarrow (7,14) \oplus (14,1) \oplus (1,21) \ , \tag{5.138}$$

$$\begin{aligned} 7371 \rightarrow\ & (27,90) \oplus (14,70) \oplus (64,14) \oplus (7,189) \oplus (77',1) \\ & \oplus (27,14) \oplus (7,70) \oplus (14,21) \oplus (1,126') \oplus (1,90) \\ & \oplus (7,21) \oplus (7,14) \oplus (27,1) \oplus (1,14) \oplus (1,1) \ . \end{aligned} \tag{5.139}$$

We find that the index of the $USp(6) \times G_2$ theory can be written in terns of the E_7 characters. For example, the Schur index $(p = 0, q = t)$ can be written as

$$I_{\text{Schur}} = 1 + \chi^{E_7}_{133}(\vec{a}, \vec{b})q + (\chi^{E_7}_{7371}(\vec{a}, \vec{b}) + \chi^{E_7}_{133}(\vec{a}, \vec{b}) + 1)q^2 + \cdots \ , \tag{5.140}$$

where we used the above decompositions to write as $\chi^{E_7}_{133}(\vec{a}, \vec{b}) = \chi^{G_2}_{7}(\vec{a})\chi^{USp(6)}_{14}(\vec{b}) + \chi^{G_2}_{14}(\vec{a}) \cdot 1 + 1 \cdot \chi^{USp(6)}_{21}(\vec{b})$ and so on.

Especially, the Hall-Littlewood index $(p = 0, q = 0)$ is known to reproduce the

Hilbert Series of the Higgs branch when the UV curve has genus 0 [86]. The Higgs branch of Minahan-Nemeschansky E_7 theory is known to be the moduli space of E_7 instantons with instanton number 1. The Hilbert series of 1 instanton moduli space is entirely given in terms of the characters for the symmetric product of adjoint representations:

$$\text{Hilb}(\mathcal{M}_{G,k=1}) = \sum_{n\geq 0} \chi_{Sym^n(\text{adj})} t^n \ . \tag{5.141}$$

This relation for the exceptional group was proven in [189, 166] and studied in the physics literatures by [44, 138, 139]. We verified that the Hall-Littlewood index for the $USp(6) \times G_2$ theory is indeed written in terms of the characters of the adjoint representations of E_7

$$I_{HL} = \sum_{n\geq 0} \chi[n,0,0,0,0,0,0] t^n \ , \tag{5.142}$$

where we used the Dynkin label here.

Bifundamentals of $G_2 \times USp(4)$ through Higgsing the E_7 theory

As we have discussed in section 5.5, we can obtain a free theory of $G_2 \times USp(4)$ bifundamentals by partially Higgsing the $USp(6)$ global symmetry down to $USp(4)$ of the E_7 theory. We obtain the K-factor from decomposing the adjoint of $USp(6)$ to the representations of $SU(2) \times USp(4)$ which is

$$K_{USp(4)} = \text{PE}\left[\frac{1}{1-q}\left(\chi_{[2,0]} t + \chi_{[1,0]} t^{3/2} + \chi_{[0,0]} t^2\right)\right] \ , \tag{5.143}$$

where we used Dynkin labels to write the representation of $USp(4)$. The fugacities for the $USp(4)$ puncture is $(t, t^{1/2} b_1, t^{1/2} b_2)$ in the α-basis meaning all the weights are given as a linear combination of the simple roots. The fugacities for the null puncture is (t, t^{-1})

for the G_2 and $(t^{5/2}, t^4, t^{9/2})$ for the $USp(6)$ in the α-basis.

5.6.2 $\mathcal{N} = 1$ index

Now, let us move on to the discussion of the superconformal indices of $\mathcal{N} = 1$ class $\mathcal{S}$ theories. The $\mathcal{N} = 1$ superconformal index is defined as

$$I(z; p, q, \xi) = \mathrm{Tr}(-1)^F p^{j_1 - j_2 + R/2} q^{j_1 + j_2 + R/2} \xi^{-\mathcal{F}/2} z^Q \ , \tag{5.144}$$

where $\mathcal{F}$ is the $U(1)_{\mathcal{F}}$ global symmetry preserved in the class $\mathcal{S}$ theory.

The $\mathcal{N} = 1$ index of theories constructed in the present paper can be obtained from the $\mathcal{N} = 2$ index of their building blocks. These building blocks can be classified into the colored T_N^σ blocks ($\sigma = \pm$) and the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ vector multiplets that couple them together. Their contribution to the $\mathcal{N} = 1$ index is given by $I_{\mathcal{N}=1} = I_{\mathcal{N}=2}(p, q, t = \xi^\sigma \sqrt{pq})$, where ξ^σ gives their charge with respect to the $U(1)_{\mathcal{F}}$ flavor symmetry.[10] As mentioned previously, the underlying TQFT structure implies that the $\mathcal{N} = 2$ superconformal index of class $\mathcal{S}$ theories can be written in terms of orthogonal functions $f_\lambda(\vec{a}; p, q, t)$. It is expected that in general $f_\lambda(\vec{a}; p, q, t)$ are related to the wave-functions of elliptic Ruijsenaars-Schneider model. There is some evidence that for theories of type A_N these functions satisfy the identity [92]

$$f_\lambda(\vec{a}; p, q, t) = \mathrm{PE}\left[\frac{t - pq/t}{(1-p)(1-q)} \chi_{\mathrm{adj}}(\vec{a})\right] f_\lambda(\vec{a}; p, q, \frac{pq}{t}) \ . \tag{5.145}$$

We will henceforth assume that this identity continues to hold for theories of type D_N and their outer-automorphism twists. This identity implies that the functions $P_\lambda(p, q, t)$ are invariant under $t \leftrightarrow pq/t$. Upon reducing this to the case of $\mathcal{N} = 1$ index, it ensures

[10] Here for the sake of brevity, we have omitted the fugacities for all flavor symmetries of the three punctured spheres. Nevertheless they are there and will be important for matching the index across various duality frames.

the invariance of $P_\lambda(p,q,\xi^\sigma\sqrt{pq})$ under $\xi \leftrightarrow \xi^{-1}$. The superconformal index of two T_N^σ blocks coupled by an $\mathcal{N}=1$ vector multiplet can be written as

$$I(\vec{a},\vec{b};\vec{c},\vec{d}) = \oint [d\vec{z}] I_{\text{vec}}^{\mathcal{N}=1}(\vec{z}) I_{T_N^+}(\vec{z},\vec{a},\vec{b}) I_{T_N^-}(\vec{z},\vec{c},\vec{d}) \ , \tag{5.146}$$

where $\vec{a},\vec{b},\vec{c},\vec{d}$ are the fugacities for the flavor symmetries of the theory while $\vec{z}$ are the fugacities for the gauge group. $I_{T_N^\sigma}$ is the $\mathcal{N}=1$ index of T_N^σ theory obtained from its $\mathcal{N}=2$ index. Due to orthonormality of the wavefunctions, the index in (5.146) formally simplifies to

$$I(\vec{a},\vec{b};\vec{c},\vec{d}) = \sum_\lambda \frac{f_\lambda^+(\vec{a}) f_\lambda^+(\vec{b}) f_\lambda^-(\vec{c}) f_\lambda^-(\vec{d})}{f_\lambda^+(\varnothing) f_\lambda^-(\varnothing)} \ . \tag{5.147}$$

Here $f_\lambda^\sigma(\vec{a})$ is short-hand for $f_\lambda(\vec{a};p,q,t=\xi^\sigma\sqrt{pq})$ and has to be chosen appropriately according to the flavor symmetry of puncture "a". $f_\lambda^\pm(\varnothing)$ correspond to the structure constants in the $\mathcal{N}=2$ index. The sum in (5.147) is over the set of representations whose Dynkin labels are of the form explained earlier in the paper.

SO dualities

We first compare the superconformal index of the unHiggsed theories across the various duality frames. In the electric theory, $\mathcal{T}^{SO}$, we find that the index can be written as

$$\begin{aligned} I_{\mathcal{T}^{SO}}(\vec{a},\vec{b};\vec{c},\vec{d}) =& \frac{K_+^{USp}(\vec{a}) K_+^{USp}(\vec{b}) K_-^{USp}(\vec{c}) K_-^{USp}(\vec{d})}{K_{\varnothing,+}^{SO} K_{\varnothing,-}^{SO}} \\ & \times \sum_\lambda \frac{P_\lambda^{USp}(\vec{a}) P_\lambda^{USp}(\vec{b}) P_\lambda^{USp}(\vec{c}) P_\lambda^{USp}(\vec{d})}{P_\lambda^{SO}(t^\varnothing) P_\lambda^{SO}(t^\varnothing)} \ . \end{aligned} \tag{5.148}$$

In the crossing frame, $\mathcal{T}_c^{SO}$, the punctures B and C are exchanged with each other. Their $U(1)_\mathcal{F}$ charges switch signs and we had to integrate in mesons M_B and M_C with $U(1)_\mathcal{F}$

charges being -2 and $+2$ respectively. The index of $\mathcal{T}_c^{SO}$ then becomes

$$I_{\mathcal{T}_c^{SO}}(\vec{a},\vec{c};\vec{b},\vec{d}) = M^+(\vec{b})M^-(\vec{c})I_{\mathcal{T}^{SO}}(\vec{a},\vec{c};\vec{b},\vec{d}) \ , \tag{5.149}$$

where $M^\sigma(\vec{x})$ is the contribution of the mesons having $\mathcal{F}$-charge -2σ and flavor fugacities $\vec{x}$

$$M^\sigma(\vec{x}) = \text{PE}\left[\frac{\sqrt{pq}(\xi^\sigma - \xi^{-\sigma})}{(1-p)(1-q)}\chi_{\text{adj}}(\vec{x})\right] . \tag{5.150}$$

The equality of the indices in (5.148) and (5.149) then follows from the identity

$$M^\sigma(\vec{x})K^{-\sigma}(\vec{x}) = K^\sigma(\vec{x}) \ . \tag{5.151}$$

We can repeat this exercise for the index of the theory $\mathcal{T}_s^{SO}$, in the swapped frame wherein we find

$$I_{\mathcal{T}_s^{SO}}(\vec{d},\vec{c};\vec{b},\vec{a}) = M^+(\vec{a})M^+(\vec{b})M^-(\vec{c})M^-(\vec{d})I_{\mathcal{T}^{SO}}(\vec{d},\vec{c};\vec{b},\vec{a}) \ . \tag{5.152}$$

The identity in (5.151) can now be used to match the indices in the various duality frames.

The procedure of Higgsing the $USp(2N-2)$ punctures can be implemented in the index by transmuting the $USp(2N-2)$ fugacities into the fugacities of the partially closed puncture. As has been mentioned earlier this can be achieved by comparing the character of the $USp(2N-2)$ fundamental written in terms of the fugacities of the $USp(2N-2)$ symmetry, to the character written in terms of the $SU(2)\times G_F \subset USp(2N-2)$. The $SU(2)$ here is embedded into $USp(2N-2)$ through the vev we use to Higgs the puncture while G_F is residual flavor symmetry left invariant by the vev. The fugacity for $SU(2)$ characters is required to be $\tau = (\xi^\sigma \sqrt{pq})^{1/2}$. The redundancy in the choice of fugacities corresponds to the Weyl symmetries of $USp(2N-2)$. The prefactor $K_\Lambda(\vec{a}; p, q, t = \xi^\sigma \sqrt{pq})$ is given by (5.127).

Applying this to close the punctures A and D we find that the index for the electric theory $\mathcal{U}^{SO}$ can be written as

$$\begin{aligned} I_{\mathcal{U}^{SO}}(\varnothing, \vec{b}; \vec{c}, \varnothing) =& \frac{K^{USp}_{\varnothing,+} K^{USp}_{\varnothing,-} K^{USp}_{+}(\vec{b}) K^{USp}_{-}(\vec{c})}{K^{SO}_{\varnothing,+} K^{SO}_{\varnothing,-}} \\ &\times \sum_\lambda \frac{P^{USp}_\lambda((\xi\sqrt{pq})^\varnothing) P^{USp}_\lambda((\xi^{-1}\sqrt{pq})^\varnothing) P^{USp}_\lambda(\vec{b}) P^{USp}_\lambda(\vec{c})}{P^{SO}_\lambda((\xi\sqrt{pq})^\varnothing) P^{SO}_\lambda((\xi^{-1}\sqrt{pq})^\varnothing)} \ . \end{aligned} \tag{5.153}$$

In the Intriligator-Seiberg (magnetic) frame $\mathcal{U}^{SO}_{c1}$, the superconformal index is

$$I_{\mathcal{U}^{SO}_{c1}}(\varnothing, \vec{c}; \vec{b}, \varnothing) = M^+(\vec{b}) M^-(\vec{c}) I_{\mathcal{U}^{SO}}(\varnothing, \vec{c}; \vec{b}, \varnothing) \ , \tag{5.154}$$

which matches with the index of the electric theory upon using (5.151).

In the swapped frame it is the mesons that get a vev, leading to a shift in the R-and $\mathcal{F}$-charges. The shift of the charges can be accommodated into the index by the following substitution: in the $\tilde{T}^\sigma_N$ block of the swapped theory, replace the fugacities for $USp(2N-2)$ with those for $SU(2) \times G_F \subset USp(2N-2)$ using $\xi^\sigma/\sqrt{pq}$ as the fugacity for $SU(2)$. The index of the swapped theory, $\mathcal{U}^{SO}_s$, is therefore given by

$$I_{\mathcal{U}^{SO}_s} = M^+_\varnothing M^-_\varnothing M^+(\vec{b}) M^-(\vec{c}) I_{\mathcal{T}^{SO}}((\xi/\sqrt{pq})^\varnothing, \vec{c}; \vec{b}, (\xi^{-1}/\sqrt{pq})^\varnothing) \ , \tag{5.155}$$

where $M^\sigma_\varnothing$ is the contribution from the mesonic excitations $M_{Aj,-j}$ and $M_{Dj,-j}$ that stay coupled to the theory:

$$M^\sigma_\varnothing = \prod_j \mathrm{PE}\left[\frac{(\xi^\sigma\sqrt{pq})^{1+j} - pq/(\xi^\sigma\sqrt{pq})^{1+j}}{(1-p)(1-q)}\right] \ . \tag{5.156}$$

Similarly the index for the theory, $\mathcal{U}_{as}^{SO}$, in the Argyres-Seiberg frame can be written as

$$I_{\mathcal{U}_{as}^{SO}} = M_{\varnothing}^{+} M^{-}(\vec{c}) I_{\mathcal{T}^{SO}}(\vec{c}, \vec{b}; (\xi^{-1}/\sqrt{pq})^{\varnothing}, \varnothing) \ , \tag{5.157}$$

while the index for the theory, $\mathcal{U}_{c2}$, in the crossing frame is given by

$$I_{\mathcal{U}_{c2}^{SO}} = M_{\varnothing}^{+} M_{\varnothing}^{-} I_{\mathcal{T}^{SO}}((\xi/\sqrt{pq})^{\varnothing}, \vec{b}; \vec{c}, (\xi^{-1}/\sqrt{pq})^{\varnothing}) \ . \tag{5.158}$$

The equality of the indices in the various duality frames can be established by using the identity

$$M_{\varnothing}^{\sigma} K_{-\sigma}^{USp}((\xi^{-\sigma}/\sqrt{pq})^{\varnothing}) = K_{\varnothing,\sigma}^{USp} \ . \tag{5.159}$$

along with (5.151) and the invariance of P_{λ}^{USp} under the Weyl symmetries of $USp(2N-2)$.[11]

USp dualities

Following a similar procedure as in the case of the SQCD with $SO(2N)$ gauge group, we can now write down the index of the various duality frames of SQCD with $USp(2N-2)$ gauge group. Before Higgsing some the punctures, we compare the indices of the unHiggsed theories in the various duality frames we obtain by moving the punctures around. The index for the electric theory, $\mathcal{T}^{Sp}$ is

$$\begin{aligned} I_{\mathcal{T}^{Sp}}(\vec{a}, \vec{b}; \vec{c}, \vec{d}) =& \frac{K_{+}^{USp}(\vec{a}) K_{+}^{SO}(\vec{b}) K_{-}^{SO}(\vec{c}) K_{-}^{USp}(\vec{d})}{K_{\varnothing,+}^{SO} K_{\varnothing,-}^{SO}} \\ & \times \sum_{\lambda} \frac{P_{\lambda}^{USp}(\vec{a}) P_{\lambda}^{SO}(\vec{b}) P_{\lambda}^{SO}(\vec{c}) P_{\lambda}^{USp}(\vec{d})}{P_{\lambda}^{SO}(t^{\varnothing}) P_{\lambda}^{SO}(t^{\varnothing})} \ , \end{aligned} \tag{5.160}$$

[11] More specifically we use the fact that $P_{\lambda}^{USp}(\vec{a}) = P_{\lambda}^{USp}(\vec{a}^{-1})$.

where the sum now is over the representations of $USp(2N-2)$, as was explained earlier. In the duality frame $\mathcal{T}_{c1}^{Sp}$ obtained by exchanging punctures B and C, we find

$$I_{\mathcal{T}_{c1}^{Sp}}(\vec{a},\vec{c};\vec{b},\vec{d}) = M^{+}(\vec{b})M^{-}(\vec{c})I_{\mathcal{T}^{Sp}}(\vec{a},\vec{c};\vec{b},\vec{d}) \ , \tag{5.161}$$

Similarly the index of the crossing theory $\mathcal{T}_{c2}^{Sp}$, obtained by exchanging punctures A and D, is

$$I_{\mathcal{T}_{c2}^{Sp}}(\vec{d},\vec{b};\vec{c},\vec{a}) = M^{+}(\vec{a})M^{-}(\vec{d})I_{\mathcal{T}^{Sp}}(\vec{d},\vec{b};\vec{c},\vec{a}) \ . \tag{5.162}$$

In the frame $\mathcal{T}_{c3}^{Sp}$, obtained by exchanging puncture B and D, the index becomes

$$I_{\mathcal{T}_{c3}^{Sp}}(\vec{a},\vec{d};\vec{c},\vec{b}) = M^{+}(\vec{b})M^{-}(\vec{d})I_{\mathcal{T}^{Sp}}(\vec{a},\vec{d};\vec{c},\vec{b}) \ . \tag{5.163}$$

The index for the theory $\mathcal{T}_{s}^{Sp}$ in the swapped frame is

$$I_{\mathcal{T}_{s}^{Sp}}(\vec{d},\vec{c};\vec{b},\vec{a}) = M^{+}(\vec{a})M^{+}(\vec{b})M^{-}(\vec{c})M^{-}(\vec{d})I_{\mathcal{T}^{Sp}}(\vec{d},\vec{c};\vec{b},\vec{a}) \ . \tag{5.164}$$

Equality of the above indices follows from (5.151).

Upon appropriately Higgsing the punctures A and D we find that the index in the electric theory $\mathcal{U}^{Sp}$ can be written as

$$\begin{aligned} I_{\mathcal{U}^{Sp}}(\varnothing,\vec{b};\vec{c},\varnothing) =& \frac{K_{\varnothing,+}^{USp}K_{\varnothing,-}^{USp}K_{+}^{SO}(\vec{b})K_{-}^{SO}(\vec{c})}{K_{\varnothing,+}^{SO}K_{\varnothing,-}^{SO}} \\ & \times \sum_{\lambda} \frac{P_{\lambda}^{USp}((\xi\sqrt{pq})^{\varnothing})P_{\lambda}^{USp}((\xi^{-1}\sqrt{pq})^{\varnothing})P_{\lambda}^{SO}(\vec{b})P_{\lambda}^{SO}(\vec{c})}{P_{\lambda}^{SO}((\xi\sqrt{pq})^{\varnothing})P_{\lambda}^{SO}((\xi^{-1}\sqrt{pq})^{\varnothing})} \ . \end{aligned} \tag{5.165}$$

The index of Intriligator-Pouliot theory $\mathcal{U}_{c1}^{Sp}$ is

$$I_{\mathcal{U}_{c1}^{Sp}}(\varnothing, \vec{c}; \vec{b}, \varnothing) = M^{+}(\vec{b}) M^{-}(\vec{c}) I_{\mathcal{U}^{Sp}}(\varnothing, \vec{c}; \vec{b}, \varnothing) \ . \tag{5.166}$$

For the crossing theory $\mathcal{U}_{c2}^{Sp}$, the index is given by

$$I_{\mathcal{U}_{c2}^{Sp}} = M_{\varnothing}^{+} M^{-}(\varnothing) I_{\mathcal{T}^{Sp}}((\xi/\sqrt{pq})^{\varnothing}, \vec{b}; \vec{c}, (\xi^{-1}/\sqrt{pq})^{\varnothing}) \ . \tag{5.167}$$

Similarly in the swapped frame $\mathcal{U}_{s}^{Sp}$ and the Argyres-Seiberg dual frame $\mathcal{U}_{as}^{Sp}$, the respective superconformal indices are:

$$I_{\mathcal{U}_{s}^{Sp}} = M_{\varnothing}^{+} M_{\varnothing}^{-} M^{+}(\vec{b}) M^{-}(\vec{c}) I_{\mathcal{T}^{Sp}}((\xi/\sqrt{pq})^{\varnothing}, \vec{c}; \vec{b}, (\xi^{-1}/\sqrt{pq})^{\varnothing}) \ , \tag{5.168}$$

$$I_{\mathcal{U}_{as}^{Sp}} = M_{\varnothing}^{+} M^{-}(\vec{c}) I_{\mathcal{T}^{Sp}}(\vec{c}, \vec{b}; (\xi^{-1}/\sqrt{pq})^{\varnothing}, \varnothing) \ . \tag{5.169}$$

The indices in the various duality frames match owing to the identities (5.151) and (5.159) and the Weyl invariance of P_{λ}^{USp}.

G_2 dualities

The index of the theories involved in the G_2 dualities proposed by us can be written in terms of the $\mathcal{N}=1$ index of the theory $\mathcal{T}^{G_2}$ obtained by coupling two $\tilde{T}_{SO(8)}$ blocks with an $\mathcal{N}=1$, $SO(8)$ vector multiplet and a Z_3 twist around the cylinder that couples two spheres. The superconformal index for this theory is

$$\begin{aligned} I_{\mathcal{T}^{G_2}}(\vec{p}, \vec{q}; \vec{r}, \vec{s}) = & \frac{K_{+}^{USp}(\vec{p}) K_{+}^{USp}(\vec{q}) K_{-}^{USp}(\vec{r}) K_{-}^{USp}(\vec{s})}{K_{\varnothing,+}^{SO} K_{\varnothing,-}^{SO}} \\ & \times \sum_{\lambda} \frac{P_{\lambda}^{USp}(\vec{p}) P_{\lambda}^{USp}(\vec{q}) P_{\lambda}^{USp}(\vec{r}) P_{\lambda}^{USp}(\vec{s})}{P_{\lambda}^{SO}(t^{\varnothing}) P_{\lambda}^{SO}(t^{\varnothing})} \ , \end{aligned} \tag{5.170}$$

where the sum is over G_2 representations. The electric theory $\mathcal{U}^{G_2}$ is built from bifundamentals of $G_2 \times USp(4)$ and its index is

$$I_{\mathcal{U}^{G_2}}(\vec{a};\vec{b}) = I_{\mathcal{T}^{G2}}(\varnothing, \vec{a}(\xi\sqrt{pq})^{\heartsuit}; \vec{b}(\xi^{-1}\sqrt{pq})^{\heartsuit}, \varnothing) \ . \tag{5.171}$$

Here $\vec{a}$ and $\vec{b}$ are the fugacities for $USp(4)_A$ and $USp(4)_B$ respectively and $\heartsuit$ represents the embedding of $SU(2)$ in $USp(6)$ that reduces the flavor symmetry of the puncture down to $USp(4)$.

In the $Spin(8)$ frame, the superconformal index of the theory is given by

$$I_{\mathcal{U}_{c1}^{G_2}} = M_{\heartsuit}^{+}(\vec{a}) M_{\heartsuit}^{-}(\vec{b}) I_{\mathcal{T}^{G2}}(\varnothing, \vec{b}(\xi/\sqrt{pq})^{\heartsuit}; \vec{a}(\xi^{-1}/\sqrt{pq})^{\heartsuit}, \varnothing) \ , \tag{5.172}$$

where $M_{\heartsuit}^{\sigma}(\vec{a})$ are the mesons that remain in the theory after Higgsing the corresponding $USp(6)$puncture down to $USp(4)$ which is given by

$$\begin{aligned} M_{\heartsuit}^{\sigma}(\vec{a}) \quad &= \quad \mathrm{PE}\left[\frac{(\xi^{\sigma}\sqrt{pq}) - pq/(\xi^{\sigma}\sqrt{pq})}{(1-p)(1-q)}\chi_{\mathrm{adj}}(\vec{a})\right] \\ &\quad \times \mathrm{PE}\left[\frac{(\xi^{\sigma}\sqrt{pq})^{\frac{3}{2}} - pq/(\xi^{\sigma}\sqrt{pq})^{\frac{3}{2}}}{(1-p)(1-q)}\chi_{\mathrm{f}}(\vec{a})\right] \\ &\quad \times \mathrm{PE}\left[\frac{(\xi^{\sigma}\sqrt{pq})^{2} - pq/(\xi^{\sigma}\sqrt{pq})^{2}}{(1-p)(1-q)}\right] \ . \end{aligned} \tag{5.173}$$

In the crossing-type frame we find

$$I_{\mathcal{U}_{c2}^{G_2}} = M_{\varnothing}^{+} M_{\varnothing}^{-} I_{\mathcal{T}^{G2}}((\xi/\sqrt{pq})^{\varnothing}, \vec{a}(\xi\sqrt{pq})^{\heartsuit}; \vec{b}(\xi^{-1}\sqrt{pq})^{\heartsuit}, (\xi^{-1}/\sqrt{pq})^{\varnothing}) \ . \tag{5.174}$$

The superconformal index for the Argyres-Seiberg type dual can be written as

$$I_{\mathcal{U}_{as}^{G_2}} = M_{\heartsuit}^{+}(\vec{a}) M_{\varnothing}^{-} I_{\mathcal{T}^{G2}}((\xi/\sqrt{pq})^{\varnothing}, \varnothing; \vec{b}(\xi^{-1}\sqrt{pq})^{\heartsuit}, \vec{a}(\xi^{-1}/\sqrt{pq})^{\heartsuit}) \ . \tag{5.175}$$

Similarly the index of the theory in the swapped G_2 frame is

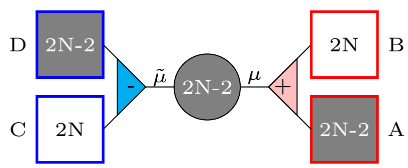
$$\begin{aligned} I_{\mathcal{U}_s^{G_2}} =& M_{\varnothing}^{+} M_{\varnothing}^{-} M_{\heartsuit}^{+}(\vec{a}) M_{\heartsuit}^{-}(\vec{b}) \\ & \times I_{\mathcal{T}^{G_2}}((\xi/\sqrt{pq})^{\varnothing}, \vec{b}(\xi/\sqrt{pq})^{\heartsuit}; (\xi^{-1}/\sqrt{pq})^{\heartsuit}, \vec{a}(\xi^{-1}/\sqrt{pq})^{\heartsuit}) \ . \end{aligned} \tag{5.176}$$

The indices in all these frames match upon using the Weyl invariance of P_{λ}^{USp} along with (5.151) and the generalized form of (5.159) given by

$$M_{\Lambda}^{\sigma} K_{-\sigma}^{USp}((\xi^{-\sigma}/\sqrt{pq})^{\Lambda}) = K_{\Lambda,\sigma}^{USp} \ . \tag{5.177}$$

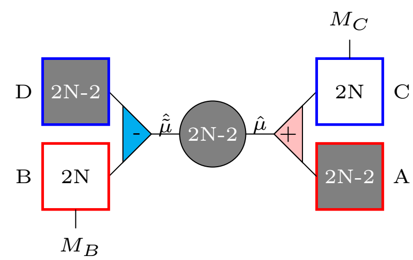
Therefore we find the indices all agree on five dual frames of the G_2 gauge theory.

This chapter is a reprint of the material as it appears in “New $N = 1$ Dualities from $M5$-branes and Outer-automorphism Twists ”, Prarit Agarwal, Jaewon Song, JHEP 1403 (2014) 133, of which I was a co-author.



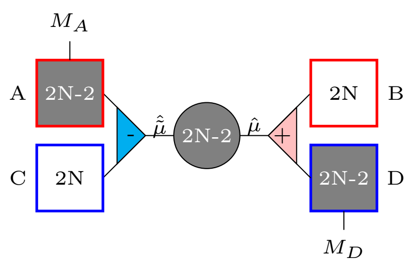
$$W = c\text{tr}\mu\tilde{\mu}$$

(a) Electric theory: $\mathcal{T}^{Sp}$.



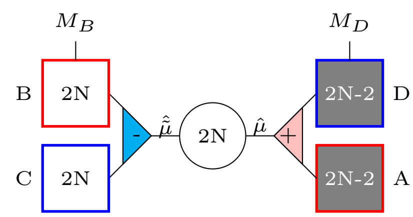
$$W = \hat{c}\text{tr}\hat{\mu}\hat{\tilde{\mu}} + \text{tr}M_C\hat{\mu}_C + \text{tr}M_B\hat{\mu}_B$$

(b) Crossing frame 1: $\mathcal{T}^{Sp}_{c1}$.



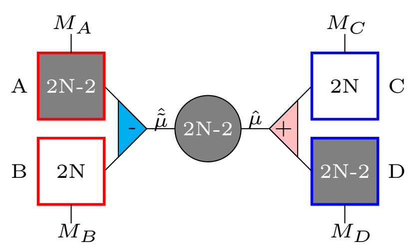
$$W = \hat{c}\text{tr}\hat{\mu}\hat{\tilde{\mu}} + \text{tr}M_A\hat{\mu}_A \; + \text{tr}M_D\hat{\mu}_D$$

(c) Crossing frame 2: $\mathcal{T}^{Sp}_{c2}$



$$W = \hat{c}\text{tr}\hat{\mu}\hat{\tilde{\mu}} + \text{tr}M_B\hat{\mu}_B + \text{tr}M_D\hat{\mu}_D$$

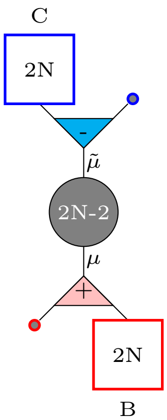
(d) Crossing frame 3: $\mathcal{T}^{Sp}_{c3}$



$$W = \hat{c}\text{tr}\hat{\mu}\hat{\tilde{\mu}} + \text{tr}M_A\hat{\mu}_A + \text{tr}M_B\hat{\mu}_B + \text{tr}M_C\hat{\mu}_C + \text{tr}M_D\hat{\mu}_D$$

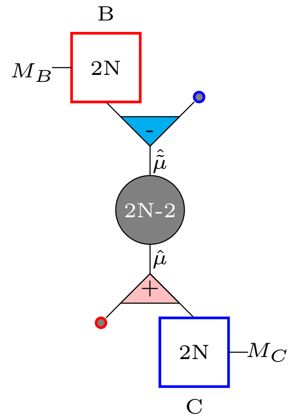
(e) Swapped theory: $\mathcal{T}^{Sp}_{s}$.

Figure 5.16. The $\mathcal{T}^{Sp}$ theory, obtained by coupling two $\tilde{T}_{SO(2N)}$ blocks along a $USp(2N-2)$ puncture with an $\mathcal{N}=1$ vector multiplet, and its duals obtained by moving the punctures around. Here we omit the anti-symmetric forms in the superpotential. The red/blue color means $\sigma = \pm$.



$W = c\text{tr}\mu\tilde{\mu}$

(a) Electric theory $\mathcal{U}^{Sp}$



$W = \hat{c}\text{tr}\hat{\mu}\hat{\tilde{\mu}} + \text{tr}M_C\hat{\mu}_C + \text{tr}M_B\hat{\mu}_B$

(b) Magnetic theory $\mathcal{U}^{Sp}_{c1}$

Figure 5.17. Intriligator-Pouliot duality

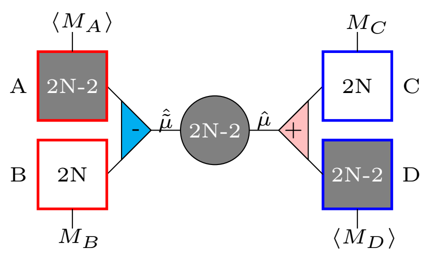


Figure 5.18. Non-Lagrangian dual $\mathcal{U}^{Sp}_s$ of $USp(2N-2)$ SQCD.

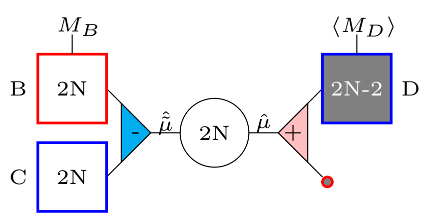


Figure 5.19. Argyres-Seiberg type dual $\mathcal{U}^{Sp}_{as}$ to USp gauge theory.

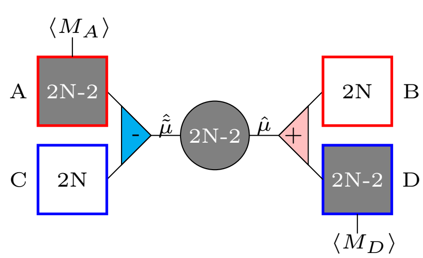
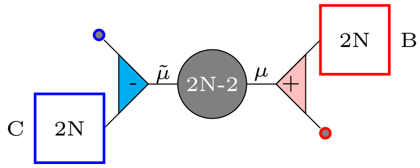
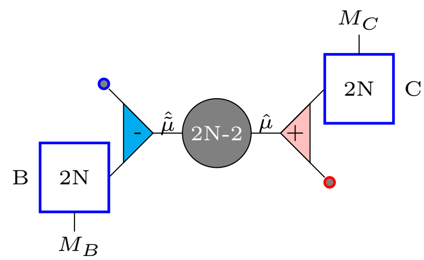


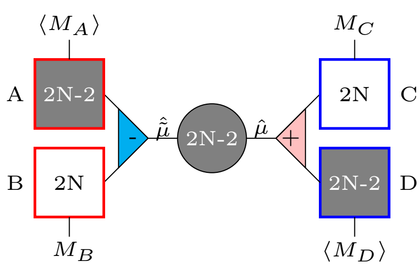
Figure 5.20. The Crossing type dual $\mathcal{U}^{Sp}_{c2}$ of $USp(2N-2)$ SQCD



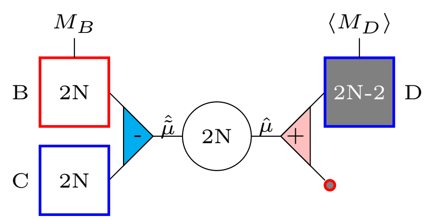
(a) USp gauge theory: $\mathcal{U}^{Sp}$



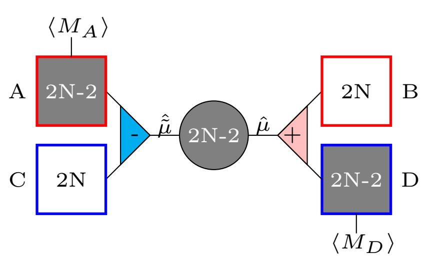
(b) Intriligator-Pouliot dual: $\mathcal{U}^{Sp}_{c1}$



(c) Swapped dual: $\mathcal{U}^{Sp}_{s}$

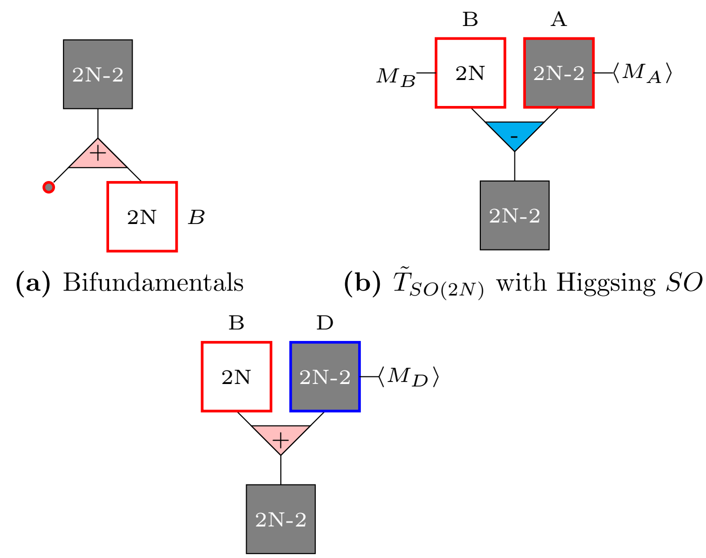


(d) Argyres-Seiberg type dual: $\mathcal{U}^{Sp}_{as}$



(e) The crossing type dual: $\mathcal{U}^{Sp}_{c2}$

Figure 5.21. Dual frames of USp SQCD



(a) Bifundamentals **(b)** $\tilde{T}_{SO(2N)}$ with Higgsing SO

(c) $\tilde{T}_{SO(2N)}$ with Higgsing USp

Figure 5.22. The building blocks of $\mathcal{U}^{Sp}$, $\mathcal{U}_s^{Sp}$ and $\mathcal{U}_{c2}^{Sp}$

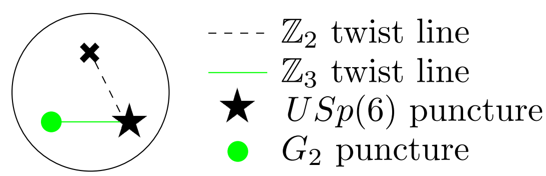
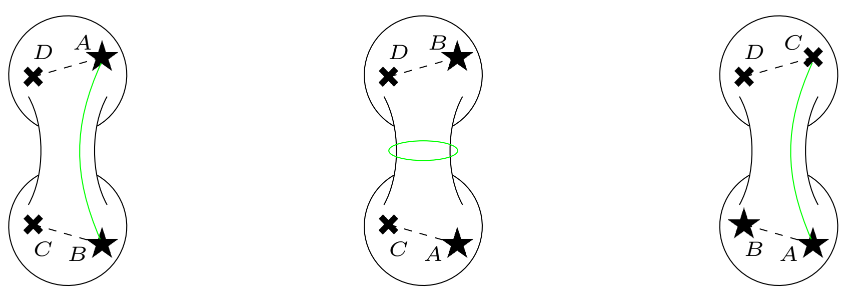
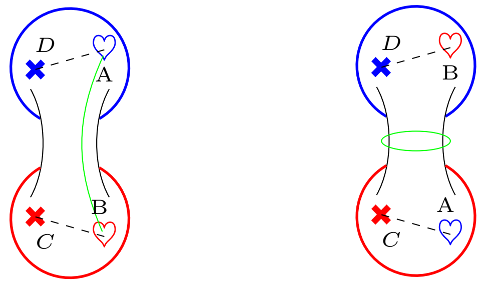


Figure 5.23. Three punctured sphere with $USp(6)$ and G_2 punctures.



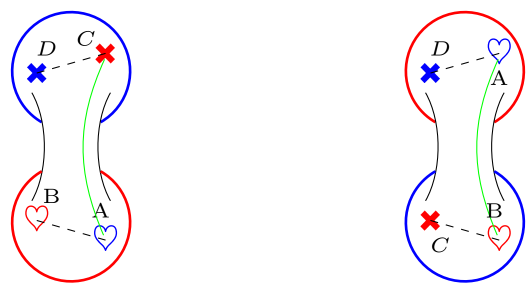
(a) G_2 frame **(b)** $Spin(8)$ frame **(c)** Argyres-Seiberg like frame

Figure 5.24. S-duality for the G_2-coupled two E_7 theories.

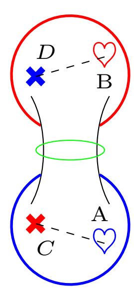


(a) G_2 frame: $\mathcal{U}^{G_2}$ **(b)** $Spin(8)$ frame: $\mathcal{U}^{G_2}_{c1}$

Figure 5.25. Lagrangian duals to the G_2 gauge theory with 8 fundamentals



(a) AS frame: $\mathcal{U}^{G_2}_{as}$ **(b)** Swapped G_2: $\mathcal{U}^{G_2}_{s}$



(c) Crossing-type: $\mathcal{U}^{G_2}_{c2}$

Figure 5.26. Non-Lagrangian dual theories for the $\mathcal{N} = 1$ G_2 gauge theory with 8 fundamentals

Chapter 6

Quiver Tails and $\mathcal{N} = 1$ SCFTs from M-branes

6.1 Introduction

Six-dimensional $(2,0)$ theory, as the low energy effective theory on the M5-brane worldvolume, plays a crucial role in studying lower dimensional supersymmetric gauge theories. In particular, a large class of four-dimensional $\mathcal{N} = 2$ superconformal theories, which are called class $\mathcal{S}$ theories, have been discovered in [88, 90] as a compactification of the $(2,0)$ theory on a Riemann surface with a partial twist. Class $\mathcal{S}$ theories turn out to be related to various objects in different dimensional theories [11, 83], bridged by the (2,0) theory picture.

$\mathcal{N} = 2$ class $\mathcal{S}$ theories are included in a larger class of theories with $\mathcal{N} = 1$ supersymmetry associated to compactifications of the $(2,0)$ theory [34]. This latter class, which we will call $\mathcal{N} = 1$ class $\mathcal{S}$, has been investigated in [37, 33, 34, 39, 82, 193, 35, 6] in field theory and in [33, 34, 31, 36] in AdS/CFT (see [153, 41] for the mass deformed $\mathcal{N} = 2$ class $\mathcal{S}$ theories). The theories in this class flow to superconformal fixed points in the IR. See also [152, 49, 196, 197] for theories in Coulomb and confining phases.

The $\mathcal{N} = 1$ theories of class $\mathcal{S}$ are specified through the following data

- The choice of 'gauge group' $\Gamma = A, D, E$.

- A Riemann surface $\mathcal{C}_{g,n}$ of genus g with n punctures called UV curve.

- Two integers p, q with a constraint $p + q = 2g - 2 + n$.

From the M-theory point of view, this class of theories is obtained by wrapping M5-branes on $\mathcal{C}_{g,n}$ inside the total space of two line bundles over $\mathcal{C}_{g,n}$. Then, p and q are the degrees of the two line bundles.[1]

In addition, we assign data to each puncture. A class of punctures, called the regular colored $\mathcal{N} = 2$ punctures, are specified by the following data:

- For each puncture, the choice of ρ_i which is an embedding of $SU(2)$ into Γ.

- The choice of $\mathbb{Z}_2$-valued 'color' $\sigma_i = \pm$.

When $\Gamma = A_{N-1}$ which we will focus on, the choice of ρ_i is in one-to-one correspondence with the choice of partition of N or a Young diagram of N boxes[2] with $N = \sum_k n_k k$. The monicker 'colored $\mathcal{N} = 2$ puncture' stems from the fact that locally these punctures are the same as those of $\mathcal{N} = 2$ theories except that we have the freedom to choose one of the two normal directions to the M5-branes.[3]

A four-dimensional UV theory can be associated to every pair-of-pants decomposition of $\mathcal{C}_{g,n}$.[4] These UV theories are in the same class, in the sense that the theories corresponding to the different pants decompositions of the same $\mathcal{C}_{g,n}$, flow to fixed points

[1] In general, to preserve supersymmetry, the normal bundle over the Riemann surface needs to be a rank-2 bundle whose determinant line bundle is the canonical bundle. Here we restrict ourselves to a particular case where the normal bundle simply decomposes as a sum of two line bundles.

[2] Punctures can also be twisted by an outer-automorphism group of Γ. This will affect the choice of ρ_i. We will not consider the twist in this paper.

[3] While we will not study in this paper, the $\mathcal{N} = 1$ punctures should be given by the $\frac{1}{4}$-BPS codimension-2 defects of the 6d $\mathcal{N} = (2, 0)$ theory. Upon dimensional reduction these yield the $\frac{1}{4}$-BPS boundary conditions of $\mathcal{N} = 4$ super Yang-Mills theory. This problem has been studied recently by [114, 115] generalizing the work of [95, 94] who studied the $\frac{1}{2}$-BPS boundary conditions.

[4] Here by UV theory or UV description we do not mean the underlying six-dimensional theory. By partial topological twist and dimensional reduction, we are looking at the four-dimensional theory below the Kaluza-Klein scale given by the size of the UV curve. Here we are interested in various different four-dimensional gauge theories (which may also have non-Lagrangian building blocks) that flow to the SCFT in the same conformal manifold. We refer to these gauge theories as UV descriptions or duality frames.

that are connected by exactly marginal deformations. This provides a nice geometric picture of the duality of $\mathcal{N}=1$ class $\mathcal{S}$ theories [34, 39, 6, 82] generalizing the well-known Seiberg duality [171].

Among these theories, linear quiver gauge theories form an important subset describing characteristic features of class $\mathcal{S}$. A linear quiver theory has two tails each of which is composed of a product of gauge groups whose ranks are non-decreasing. In $\mathcal{N}=2$ theories, the quiver tail has been fully understood to be related to a sphere with a maximal puncture ($N = 1+1+\cdots+1$), a number of minimal punctures ($N = 1+(N-1)$), and a generic puncture [88]. The purpose of this paper is to identify the $\mathcal{N}=1$ version of quiver tails associated with a similar sphere but with colors.

It turns out that the $\mathcal{N}=1$ quiver tails have an important ingredient, which we will call the Fan. The Fan is composed of a collection of various chiral multiplets coupled by a specific superpotential that preserve the global symmetry $SU(N)\times SU(N')\times \prod_k U(n_k)\times U(1)$. The quiver tail is constructed by gauging some of the global symmetries. When N' is absent, the Fan is shown to be associated to a pair-of-pants whose three punctures are: one maximal, one minimal, and a third generic puncture specified by a partition of $N=\sum_k kn_k$. (The color of the former two punctures are the same as that of the pair-of-pants, and are different from that of the generic puncture.)

We obtain the $\mathcal{N}=1$ quiver tail, and in particular the Fan, by the nilpotent Higgsing which was first studied in [117] from the different point of view and in [183, 82] from the class $\mathcal{S}$ point of view. We start from the linear quiver theory where all gauge groups are $SU(N)$, and give a nilpotent vev to the quark bilinear at the end of quiver. This produces a quiver tail. In $\mathcal{N}=2$ linear quiver theories, the nilpotent Higgsing propagates to neighboring gauge nodes of the quiver because of the F-term equations [183], which we also discuss in detail in appendix E.2. On the other hand, if there is an $\mathcal{N}=1$ gauge group in the quiver, the Higgsing stops at that node and does not propagate further. This indicates the main characteristic difference of the Higgsing between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories. We will confirm this in different ways by using multiple Seiberg dualities.

The Fan can be used as a new building block to construct not only the quiver tail, but more general $\mathcal{N} = 1$ gauge theories in class $\mathcal{S}$. Moreover, the Fan plays a crucial role in the study of the dualities in class $\mathcal{S}$ theories. As a remarkable example, we find that the Fan coupled to an $\mathcal{N} = 1$ vector multiplet appears as a dual description of the $\mathcal{N} = 1$ supersymmetric QCD with $N_f = 2N$ flavors. The precise description is an $\mathcal{N} = 1$ $SU(N)$ gauge theory coupled to the Fan, a T_N theory [88] and an adjoint chiral multiplet, with a particular superpotential. From the UV curve viewpoint, this duality can be seen as a pair-of-pants decomposition that exchanges maximal and minimal punctures, and therefore is an $\mathcal{N} = 1$ analog of the Argyres-Seiberg duality [24], which was first discussed in [6] for the case of $SO/Sp/G_2$ gauge theories.

The organization of this paper is as follows. In section 6.2, we first review the $\mathcal{N} = 1$ linear quiver gauge theories of class $\mathcal{S}$ [35], and the nilpotent Higgsing. In section 6.3, the Fan is introduced. We will see that the $\mathcal{N} = 1$ quiver tail in which the Fan plays a central role can be obtained by the nilpotent Higgsing of the $\mathcal{N} = 1$ linear quiver gauge theory. In section 6.4, we consider the application of the Fan to dualities. We first show that the Fan appears in an $\mathcal{N} = 1$ quiver theory with an $\mathcal{N} = 2$ quiver tail by successive application of Seiberg duality. We then consider the duality of $\mathcal{N} = 1$ SQCD with $N_f = 2N$ flavors. In section 6.5, we study the 't Hooft anomaly coefficients of the $\mathcal{N} = 1$ class $\mathcal{S}$ theories, in particular the Fan. We then present formulae of the anomalies in terms of the UV curve. In section 6.6, we calculate the superconformal index of the class $\mathcal{S}$ theories involving the Fan. This is the strongest check of the duality conjecture in section 6.4. In appendix E.1, we derive the superpotential of the Fan from nilpotent Higgsing. We also discuss the nilpotent Higgsing in the $\mathcal{N} = 2$ linear quiver theories in appendix E.2.

6.2 $\mathcal{N} = 1$ quiver theories of class $\mathcal{S}$ and nilpotent Higgsing

Our main object is the class of theories, in particular quiver tails, obtained by giving nilpotent vevs to $\mathcal{N} = 1$ linear quiver gauge theories of class $\mathcal{S}$ [35]. We first discuss our criteria for constructing $\mathcal{N} = 1$ class $\mathcal{S}$ theories in section 6.2.1 and then describe $\mathcal{N} = 1$ linear quiver gauge theories of class $\mathcal{S}$ in section 6.2.2. We then study the generic features of nilpotent Higgsing of the quiver theory in section 6.2.3, focusing on the differences between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ quiver theories.

6.2.1 Generic features of $\mathcal{N} = 1$ class $\mathcal{S}$

There is no complete classification of $\mathcal{N} = 1$ class $\mathcal{S}$ field theories from compactifications of the six-dimensional $(2, 0)$ theory. But there are two prevalent features of the existing constructions of class $\mathcal{S}$ theories. In our explorations, we impose these conditions as criteria for class $\mathcal{S}$. They are:

Criterion I: R-symmetry $\mathcal{N} = 1$ class $\mathcal{S}$ theories admit a $U(1)_+ \times U(1)_-$ global symmetry, whose generators will be denoted by (J_+, J_-). This corresponds to the generic subgroup of the $SO(5)$ R-symmetry of the $(2, 0)$ theory that can be preserved after a partial topological twist on a UV curve. From the point of view of M5-branes, this symmetry corresponds to the rotations of the two line bundles fibered over the UV curve. One combination of this symmetry will become the superconformal R-symmetry and the other will be a global symmetry of the four-dimensional $\mathcal{N} = 1$ SCFT.

Another notation for the global symmetry $U(1) \times U(1)$ is $(R_0, \mathcal{F})$ defined as

$$R_0 = \frac{1}{2}(J_+ + J_-)\,, \qquad \mathcal{F} = \frac{1}{2}(J_+ - J_-)\ . \tag{6.1}$$

This latter notation is more convenient when computing central charges and anomalous dimensions. The superconformal R-symmetry is

$$R_{\mathcal{N}=1} = R_0 + \epsilon \mathcal{F}\ , \tag{6.2}$$

where ϵ is fixed by a-maximization [128].

In order to satisfy the R-symmetry criterion, we impose the condition: All additional $U(1)$ symmetries, F_I, are baryonic; i.e., they cannot mix with the R-symmetry. In the class $\mathcal{S}$ theories, there are flavor symmetries associated to the punctures on the UV curve. We assume these are all baryonic symmetries hence do not mix with the R-symmetry; this is the case for all known theories.[5]

Criterion II: Marginal Coupling For every gauge coupling, there is an associated exactly marginal direction. In the construction of class $\mathcal{S}$, the number of gauge groups is given by the dimension of the complex structure moduli space of the UV curve. The addition of gauge groups maps to the addition of punctures or handles on the UV curve and therefore increases the dimension of the conformal manifold [153, 88, 41, 34].

This condition is not entirely correct if the UV curve has an irregular puncture. For example, one can realize $SU(N)$ gauge theory with $N_f < 2N$ flavors by a three-punctured sphere with irregular punctures. This theory flows to a conformal fixed point with no marginal direction. There is no complex structure deformation associated to this

[5] The flavor symmetry associated with a puncture for a Lagrangian theory comes from a pair of chiral multiplets. The axial symmetries are usually anomalous, and we only see the baryonic part of the symmetry. In fact, for a given puncture with global symmetry G_F, we generally expect the theory has $G_F \times G_F$ symmetry at some point in the conformal manifold, which is broken in a general point.

UV curve, nevertheless it has a gauge group. In this paper, we aim to find theories with regular punctures only, where the number of gauge groups is the same as the dimension of complex structure moduli space of the UV curve.

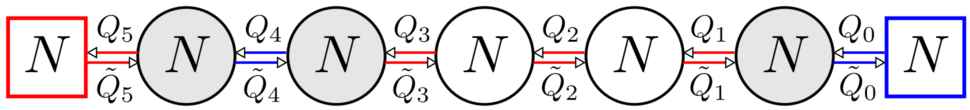
These criteria are surprisingly constraining and generic quiver gauge theories do not satisfy them. They are satisfied in $\mathcal{N}=1$ class $\mathcal{S}$ linear quivers and all theories constructed so far. As we will find, they are always preserved by nilpotent Higgsing.

6.2.2 Linear quiver theory

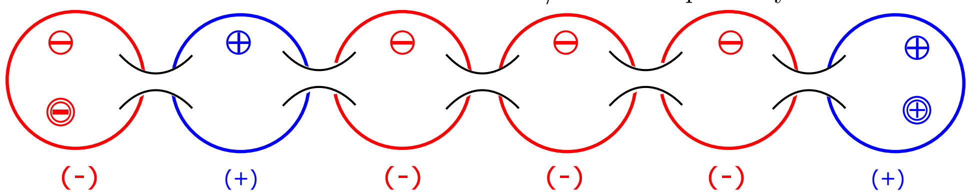
Let us consider a linear quiver theory given as follows. It has ℓ gauge groups labelled as $SU(N)_i$, which can be $\mathcal{N}=2$ or $\mathcal{N}=1$. The former is an $\mathcal{N}=1$ vector multiplet with a chiral multiplet transforming in the adjoint representation of the gauge group. The gauge nodes, $SU(N)_{i+1}$ and $SU(N)_i$, are linked by hypermultiplets, $H_i = (Q_i, \widetilde{Q}_i)$, transforming in the bifundamental representation of $SU(N)_{i+1}$ and $SU(N)_i$. Our conventions are such that $(Q_i, \widetilde{Q}_i)$ transforms in $(\mathbf{N}\otimes\bar{\mathbf{N}}, \bar{\mathbf{N}}\otimes\mathbf{N})$ of $SU(N)_{i+1}\times SU(N)_i$. The right-most and left-most hypermultiplets are denoted by H_0, H_ℓ respectively and they transform in the bifundamental representations of $SU(N)_1 \times SU(N)_0$ and $SU(N)_{\ell+1}\times SU(N)_\ell$ where $SU(N)_0, SU(N)_{\ell+1}$ are flavor symmetries. See figure 6.1a for the $\ell = 5$ case.

As mentioned above, the theory preserves distinguished anomaly-free $U(1)$ symmetries, $U(1)_+ \times U(1)_-$. We denote the charge of fields under this symmetry as (j_+, j_-); the charge of any gaugino is $(1,1)$. We fix the charges of the matter fields and a theory by giving the sequence $(\sigma_{-1}, \sigma_0, \sigma_1, \cdots, \sigma_\ell, \sigma_{\ell+1})$ with $\sigma_i^2 = 1$. Each hypermultiplet H_i also comes with a baryonic $U(1)_i$, whose generators we denote as J_i. The charges of the H_i are given as

$$J_\pm(Q_i) = \frac{1\pm\sigma_i}{2}, \qquad J_j(Q_i) = \delta_{ij}. \tag{6.3}$$



(a) The quiver diagram for a generic class $\mathcal{S}$ linear quiver gauge theory. The black and white node corresponds to $\mathcal{N}=1$ and $\mathcal{N}=2$ gauge nodes respectively. The blue/red arrows denote the bifundamental matter fields with $\sigma = 1/\sigma = -1$ respectively.



(b) The UV curve and its colored pair-of-pants decomposition corresponding to the quiver 6.1a. The symbols $\oplus, \ominus$ denote the minimal punctures of each color, and the ones with extra circle denote the maximal punctures. The $(+), (-)$ below each pair-of-pants denote the coloring of the pair-of-pants itself.

Figure 6.1. An example of a generic $SU(N)$ quiver theory corresponding to the UV curve given by a sphere with two maximal and a number of minimal punctures. Note that the colored pair-of-pants mapped to the bifundamentals, and the tubes mapped to the gauge nodes.

Note that the J_j charge of the anti-fundamental $\widetilde{Q}_i$ has an opposite sign.

Each gauge group can come with an $\mathcal{N}=2$ or with an $\mathcal{N}=1$ vector multiplet. When $\sigma_i = \sigma_{i-1} = \pm 1$, the $SU(N)_i$ gauge group has a chiral field $\phi_i^{\mp}$ transforming in the adjoint representation and we add the superpotential terms

$$W_i = \sigma_i \mathrm{Tr}\left[\phi_i^{\mp}(Q_{i-1}\widetilde{Q}_{i-1} - \widetilde{Q}_i Q_i)\right]. \tag{6.4}$$

For $\sigma_i = -\sigma_{i-1}$, there is no adjoint chiral field. However we can add the quartic superpotential terms

$$W_i = \mathrm{Tr}\left(Q_{i-1}\widetilde{Q}_{i-1}\widetilde{Q}_i Q_i\right) - \frac{1}{N}\mathrm{Tr}(Q_{i-1}\widetilde{Q}_{i-1})\mathrm{Tr}(\widetilde{Q}_i Q_i). \tag{6.5}$$

Let us note that these can be uniformly written as

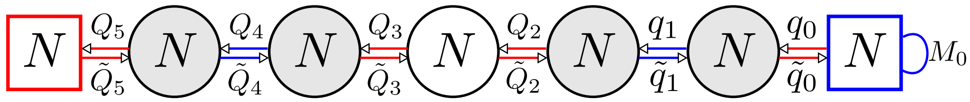
$$W_i = \mathrm{Tr}\left[\widetilde{Q}_i Q_i \left(\frac{1-\sigma_i}{2}\phi_i^+ - \frac{1+\sigma_i}{2}\phi_i^-\right) + Q_{i-1}\widetilde{Q}_{i-1}\left(\frac{1+\sigma_{i-1}}{2}\phi_i^- - \frac{1-\sigma_{i-1}}{2}\phi_i^+\right)\right.$$
$$\left. + m_i \left(\frac{1-\sigma_i}{2}\phi_i^- - \frac{1+\sigma_i}{2}\phi_i^+\right)\left(\frac{1-\sigma_{i-1}}{2}\phi_i^- - \frac{1+\sigma_{i-1}}{2}\phi_i^+\right)\right] , \tag{6.6}$$

where the trace is over the gauge group $SU(N)_i$. Below the energy scale m_i, some of adjoint fields are integrated out, giving (6.4) or (6.5) depending on σ_i and σ_{i-1}. The total superpotential is given as $W = \sum_{i=1}^{\ell} W_i$.

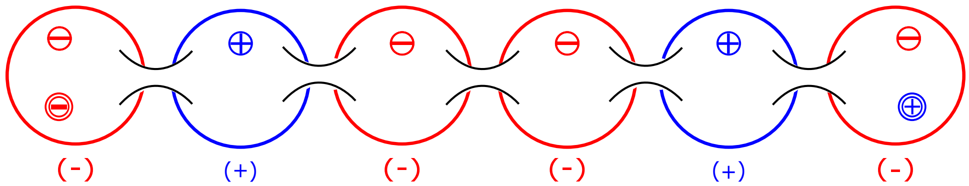
Since the fields $H_{\ell+1}$ and H_{-1} do not exist, and $SU(N)_{\ell+1}$ and $SU(N)_0$ are flavor groups, the choices σ_{-1} and $\sigma_{\ell+1}$ attaches or turns off adjoint chiral multiplets to the end of hypermultiplets. Namely, if $\sigma_{-1} = \sigma_0 = \pm$, we attach the adjoint $\phi_0^{\mp}$ with $W_0 = \mathrm{Tr}\widetilde{Q}_0 Q_0 \phi_0^{\mp}$; if $\sigma_{-1} = -\sigma_0$, we do not have any adjoints. The $U(1)_{\pm}$ charges of the fields are

$$J_{\pm}(\phi_i^{\pm}) = \frac{2+\sigma_i+\sigma_{i-1}}{2} , \qquad J_{\pm}(\phi_i^{\mp}) = \frac{2-\sigma_i-\sigma_{i-1}}{2} . \tag{6.7}$$

Let us now briefly describe the connection with the UV curve picture. The linear quiver gauge theory is in class $\mathcal{S}$ and is associated to the sphere with $\ell+1$ minimal punctures and two maximal punctures [35]. See figure 6.1b for illustration. The sphere is decomposed into $\ell+1$ pairs-of-pants, each of which has a color. Note that the color of pair-of-pants is the same as that of the minimal puncture it contains. Locally each unit preserves $\mathcal{N}=2$ supersymmetry and corresponds to bifundamental hypermultiplet H_i. The σ_i ($i = 0, 1, \ldots, \ell$) is exactly the color of the i-th pair-of-pants. The $\mathcal{N}=1$ vector multiplet appears when two pairs-of-pants with different colors are connected by a tube; the $\mathcal{N}=2$ vector multiplet appears when two pairs-of-pants with the same colors are connected. The σ_{-1} and $\sigma_{\ell+1}$ are associated with the colors of the maximal punctures. If



(a) The quiver diagram corresponding to the Seiberg dual of figure 6.1a.



(b) The UV curve and its colored pair-of-pants decomposition corresponding to the quiver 6.2a.

Figure 6.2. The Seiberg dual of the quiver given by figure 6.1a and its colored pair-of-pants decomposition. Here we dualized the right-most gauge group $SU(N)_1$. Note that the second gauge group $SU(N)_2$ became $\mathcal{N}=1$ because of the meson dual to $Q_1\widetilde{Q}_1$ behaves as an extra adjoint chiral, which generates a mass term for the adjoint chiral. From the UV curve viewpoint, this is represented by that the colors of the second and third pairs-of-pants are different.

the color of the maximal puncture is different form that of the pair-of-pants, an adjoint chiral multiplet is attached. See figures 6.2a and 6.2b.

It is important to consider Seiberg duality in this class of theories. Given a quiver where $SU(N)_i$ gauge group is $\mathcal{N}=1$ with $\sigma_i = -\sigma_{i-1}$, we can dualize at this node. This will map a linear quiver to another linear quiver since each gauge node satisfies $N_f = 2N_c$. Dualizing at $SU(N)_i$ will have the effect $\sigma_i \to -\sigma_i$ and $\sigma_{i-1} \to -\sigma_{i-1}$. From the perspective of the UV curve, this is equivalent to exchanging neighboring two minimal punctures of different colors and at the same time inverting the colors of pair-of-pants, as in figures 6.2a and 6.2b. The Seiberg duality preserves the parameters p and q which correspond to the number of pairs-of-pants or $\sigma_{i=0,1,\cdots,\ell}$'s with $+$ and $-$, respectively.

6.2.3 Nilpotent Higgsing

$\mathcal{N}=2$ **Higgsing** Before discussing nilpotent Higgsing in $\mathcal{N}=1$ theories, we summarize the effect in the case of $\mathcal{N}=2$ theories. We elaborate more in the appendix E.2. This was also discussed in [183].

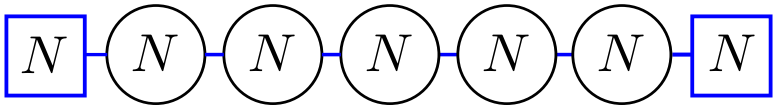


Figure 6.3. An $\mathcal{N}=2$ linear quiver theory.

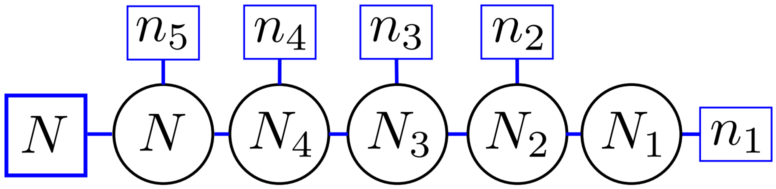


Figure 6.4. An $\mathcal{N}=2$ quiver theory obtained after Higgsing specified by the partition $N = \sum_{k=1}^{5} n_k k$. The ranks of the gauge groups are fixed by conformality condition $2N_i = N_{i-1} + N_{i+1} + n_i$.

Consider a linear quiver theory as in figure 6.3 with gauge group $G = \prod_{i=1}^{\ell} SU(N)_i$. This is the special case of the quiver introduced in the section 6.2.2 by setting all the colors of punctures and pairs-of-pants to be the same. From the superpotential (6.4), we get the F-term equation for the ϕ_i

$$F_{\phi_i} = Q_{i-1}\widetilde{Q}_{i-1} - \widetilde{Q}_i Q_i = 0 \ . \tag{6.8}$$

Now, let us consider a Higgsing of H_0 by giving a nilpotent vev to $\mu_0 = \widetilde{Q}_0 Q_0 - \frac{1}{N}\mathrm{Tr}\widetilde{Q}_0 Q_0$, which partially closes the maximal puncture. For a given partition of $N = \sum_k n_k k$, we give the vev $\langle \mu_0 \rangle = \bigoplus_k J_k^{\oplus n_k}$, where J_k is the Jordan cell of size k

$$J_k = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} . \tag{6.9}$$

The matrix J_k is the k-dimensional representation of the raising operator $\sigma^+ = \sigma^1 + i\sigma^2$

of $SU(2)$. A crucial observation here is that from the F-term for the adjoint chirals (6.8), the vev of Q's are propagated to the neighboring node. As it propagates, the operator $\widetilde{Q}_i Q_i$ will have smaller rank than that of $\widetilde{Q}_{i-1} Q_{i-1}$ until it hits zero at some finite length. From this way, we can explicitly derive the quiver tails corresponding to a given partition of N labeling the puncture, as in figure 6.4.

Before going to $\mathcal{N}=1$ theories, let us make a comment on the Higgsing through a diagonal vev such as $Q_0 = \widetilde{Q}_0 = \text{diag}(v_1, v_2, 0, \cdots, 0)$. It is certainly possible to solve the F-term equation (6.8) by such a diagonal vev for all the bifundamental hypermultiplets $Q_0 = Q_1 = \cdots Q_\ell$. Therefore all the gauge symmetries are broken by the same amount. We will not discuss these cases.

$\mathcal{N}=1$ **Higgsing** Suppose every gauge node we described above is replaced by $\mathcal{N}=1$ gauge nodes. Let us Higgs the theory by giving the vev to μ_0 as before. This time, from the superpotential (6.5), the F-term equation for $Q_i, \widetilde{Q}_i$

$$F_{Q_i} = Q_{i-1}\widetilde{Q}_{i-1}\widetilde{Q}_i + \widetilde{Q}_i\widetilde{Q}_{i+1}Q_{i+1} = 0 \ , \tag{6.10}$$

does not give us a propagating effect to the neighboring node. The F-term can be simply solved by taking all the other $Q_i, \widetilde{Q}_i$ to be zero. Therefore, the Higgsing happens completely locally on the first node. There is no propagation of vev contrary to the case of $\mathcal{N}=2$. Generally if we have a number of $\mathcal{N}=2$ nodes on the right, the propagation continues until it hits the $\mathcal{N}=1$ node and then stop. In the next section, we will describe how Higgsing creates an $\mathcal{N}=1$ version of the quiver tail.

In the case of a diagonal vev, the D-term equations for the quiver theories can be solved. The effect of diagonal Higgsing has been thoroughly studied and has been used to test the consistency of the Seiberg duality in $\mathcal{N}=1$ $SU(N_c)$ SQCD with N_f

flavors [171]: the gauge symmetry and the flavor symmetry go down by a same amount, say k. Then the gauge symmetry will be $SU(N_c - k)$ and the flavor symmetry will be $SU(N_f - k)$. On the dual side, the gauge group remains the same, but only the dual quarks become massive and reduces the number of flavors by the same amount k. From the magnetic theory perspective, mass terms for the dual quarks are generated through the superpotential $W = (\langle M \rangle + \delta M)\, q\tilde{q}$, where $\langle M \rangle$ is of rank k. Once we integrate out the massive (dual) quarks, we generate $M^2 q\tilde{q}$ term in the superpotential which is irrelevant in the IR. The Higgsed theory will have $SU(N_f - k)$ flavor symmetry which is the same as the electric theory.

On the other hand, as we have seen in the $\mathcal{N} = 2$ case, the nilpotent vevs can deform the theory in an interestingly different way. The number of flavors will be reduced, but the superpotential terms generated are quite different from the diagonal Higgsing. Depending on the choice of nilpotent vevs, we can generate various types of flavor symmetry of the form

$$G_F = S\left(\prod_{i=1}^{\ell} U(n_k)\right) \ . \tag{6.11}$$

We will see how the nilpotent Higgsing works for $\mathcal{N} = 1$ theories in detail in the next section. There will be various seemingly irrelevant terms in the superpotential generated through this procedure. But, we will argue that all of these terms become exactly marginal in the IR SCFT. This kind of operators in the superpotential which looks irrelevant in the UV but not in the IR are called dangerously irrelevant operators. See [145] for example.

6.3 Higgsing, Fan and quiver tails

In this section, we give an $\mathcal{N} = 1$ version of the quiver tails. First, we define the Fan in section 6.3.1. Then in section 6.3.2 we describe its Seiberg duality. Then in section

6.3.3, we will summarize the $\mathcal{N}=1$ quiver tail obtained by the nilpotent Higgsing of the linear quiver, where the Fan appears as an important ingredient. Finally in section 6.3.4 we show that the Fan is indeed obtained by Higgsing the linear quiver with the adjoint fields attached to the end.

6.3.1 Description of the Fan

The Fan is a collection of free chiral multiplets with certain global symmetries and superpotential. It is labelled by two integers N, N' with $N > N'$ and an ℓ-partition

$$N - N' = \sum_{k=1}^{\ell} k n_k \ . \tag{6.12}$$

We will refer to ℓ as its size. The matter content is displayed in table 6.1. We also have a choice of a color, σ; that we pick to be $\sigma = -1$ for simplicity. The other choice, $\sigma = 1$, corresponds to swapping J_+ and J_- in table 6.1. It has the global symmetry

$$SU(N) \times SU(N') \times U(1)_B \times \prod_{i=1}^{\ell} U(n_i) \times U(1)_+ \times U(1)_- \ . \tag{6.13}$$

Figure 6.5 is a representation of the Fan with size $\ell = 5$. Each line corresponds to a bifundamental hypermultiplet and each loop corresponds to an adjoint chiral multiplet.

The Fan appears in quiver gauge theories with the $SU(N) \times SU(N')$ symmetries gauged. When the fan is glued, chiral anomalies at the $SU(N) \times SU(N')$ gauge groups of $J_\pm$ must be cancelled. This will restrict the matter content that can appear on either side. The contributions of the Fan to the anomaly coefficient are:

$$SU(N): \qquad \mathrm{Tr} J_+ T^a T^b = -N\delta^{ab} \ , \quad \mathrm{Tr} J_- T^a T^b = 0 \ , \tag{6.14}$$

$$SU(N'): \qquad \mathrm{Tr} J_+ T'^a T'^b = -N'\delta^{ab} \ , \quad \mathrm{Tr} J_- T'^a T'^b = -\sum_{i=1}^{\ell} n_i \delta^{ab} \ , \tag{6.15}$$

Table 6.1. The Fan contains many fields organized in representation of the flavor symmetry. The indices i, j range in the interval $[1, \ell]$ and are ordered as $i < j$. The index p labels a tower of fields in the same representation of the flavor symmetry, its range is $0 \le p \le i-1$.

	$SU(N)$	$SU(N')$	$U(n_i)$	$U(n_j)$	$U(1)_B$	J_+	J_-
$(Q, \widetilde{Q})$	$(\square, \bar{\square})$	$(\bar{\square}, \square)$	$\cdot$	$\cdot$	$(1, -1)$	0	1
$(Z_i, \widetilde{Z}_i)$	$(\square, \bar{\square})$	$\cdot$	$(\bar{\square}, \square)$	$\cdot$	$(1, -1)$	$1-i$	1
$(Y_i, \widetilde{Y}_i)$	$\cdot$	$(\square, \bar{\square})$	$(\bar{\square}, \square)$	$\cdot$	$\cdot$	$i+1$	0
$M_{ii}^{(p)}$	$\cdot$	$\cdot$	adj	$\cdot$	$\cdot$	$2(i-p)$	0
$(M_{ij}^{(p)}, M_{ji}^{(p)})$	$\cdot$	$\cdot$	$(\square, \bar{\square})$	$(\bar{\square}, \square)$	$\cdot$	$i+j-2p$	0

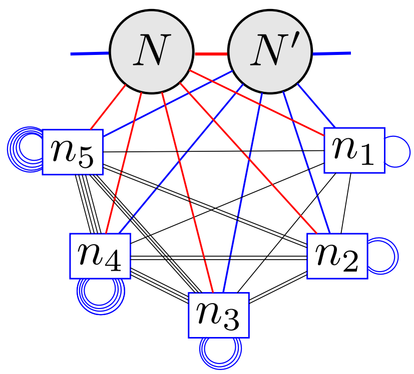
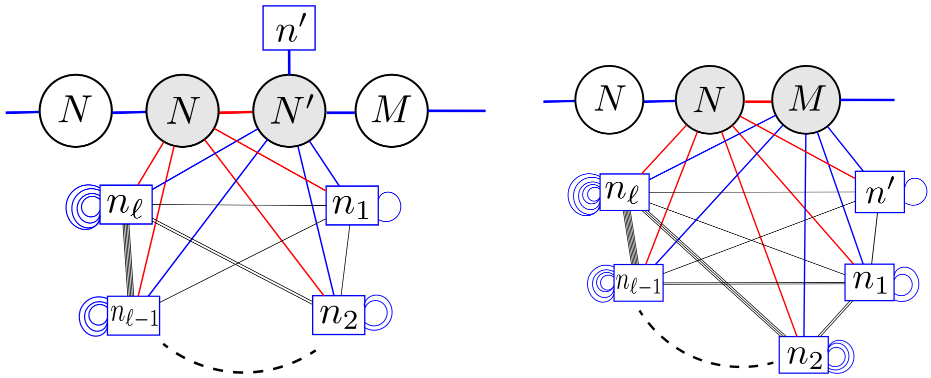


Figure 6.5. A generic form of the Fan given by (N, N') and the partition $N - N' = \sum_{k=1}^{5} k n_k$.

where T^a and T'^a are the generators of $SU(N)$ and $SU(N')$ respectively. The anomaly at $SU(N)$, when it is gauged with an $\mathcal{N}=1$ vector multiplet, can be cancelled by coupling the Fan to N $(1,0)$-fundamental hypermultiplets.[6] When it is gauged with an $\mathcal{N}=2$ vector, the anomaly is cancelled by coupling N $(0,1)$-fundamental hypermultiplets to the Fan. This provides $\mathcal{N}=1$ and $\mathcal{N}=2$ gluing of the Fan at the $SU(N)$ gauge group.

When the $SU(N')$ is gauged with an $\mathcal{N}=1$ vector multiplet, the anomaly at the $SU(N')$ can be cancelled by adding $(N' - \sum_{i=1}^{\ell} n_i)$ $(1,0)$ fundamental hypermultiplets. Unlike the $SU(N)$ side, we cannot gauge $SU(N')$ with an $\mathcal{N}=2$ vector multiplet because

[6] When we say (m,n)-operators/fields, (m,n) are their (J_+, J_-) charges.



(a) A quiver tail with the Fan **(b)** Seiberg dual

Figure 6.6. Seiberg dualizing at $SU(N')$ in 6.6a yields another quiver 6.6b with the new Fan. The $U(n')$ group is absorbed into the new Fan, labelled by (N, M) and the partition $N - M = \sum_k kn'_k$ with $n'_1 = n', n'_{i+1} = n_i$.

the anomaly cannot be cancelled with either $(1, 0)$ or $(0, 1)$ hypermultiplets only. We can glue the Fan to an $\mathcal{N} = 2$ quiver tail labelled by a partition of N' by an $\mathcal{N} = 1$ $SU(N')$ vector multiplet. In figure 6.6a we illustrate the Fan glued to general quivers with $\mathcal{N} = 1$ gluing at the $SU(N)$ gauge group.

Superpotential When the Fan appears in a larger quiver, we can write a superpotential by considering all possible gauge invariant $(2, 2)$-operators that preserve the flavor symmetry. We decompose it into three contributions

$$W_F = W_0 + W_R + W_L \tag{6.16}$$

where W_0 is composed of fields in the Fan only, W_R comes from gluing at $SU(N')$ and W_L comes from gluing at $SU(N)$. Now we describe them.

If we consider the matter content of the Fan, the only superpotential terms we can write are

$$W_0 = \sum_{i=1}^{\ell} \left[\lambda_i^0 \mathrm{Tr}\left(Z_i \widetilde{Q} \widetilde{Y}_i\right) + \tilde{\lambda}_i^0 \mathrm{Tr}\left(\widetilde{Z}_i Q Y_i\right)\right] \tag{6.17}$$

Table 6.2. Charges of the $\mathcal{M}$ and μ operators used in (6.19).

	$SU(N)$	$U(n_i)$	$U(n_j)$	J_+	J_-
$\mu_\alpha^{(p)}$	adj	$\cdot$	$\cdot$	$2p$	0
$\mathcal{M}_{ii}^{(p),\alpha}$	$\cdot$	adj	$\cdot$	$2(i-p)$	0
$\left(\mathcal{M}_{ij}^{(p),\alpha}, \mathcal{M}_{ji}^{(p),\alpha}\right)$	$\cdot$	$(\square, \bar{\square})$	$(\bar{\square}, \square)$	$i+j-2p$	0

where the λ's are complex coupling constants.

The next class of operators comes from the coupling of the quiver tail to the Fan through the $SU(N')$. To write these terms we consider the set of $(2,0)$-operators, μ' and μ_t, constructed from the $U(n')$ and $SU(M)$ quarks in figure 6.6a. The superpotential is

$$W_R = \lambda' \mathrm{Tr}\left(Q\widetilde{Q}\mu'\right) + \lambda_t \mathrm{Tr}\left(Q\widetilde{Q}\mu_t\right). \tag{6.18}$$

The last class of operators come from gluing the Fan at the $SU(N)$. To write these terms, we consider the tower operators, $\mu_\alpha^{(p)}$, $\left(\mathcal{M}_{ij}^{(p),\alpha}, \mathcal{M}_{ji}^{(p),\alpha}\right)$, and $\mathcal{M}_{ii}^{(p),\alpha}$. The μ's are constructed from fields to the left of the Fan. The $\mathcal{M}$'s are constructed from the M_{ij} fields of the Fan. Their charges are written in the table 6.2. When we glue at the $SU(N)$, we obtain the superpotential

$$\begin{aligned} W_L = {}& \lambda^\alpha \mathrm{Tr}\left(\mu_\alpha^{(1)}\widetilde{Q}Q\right) + \sum_{i=1}^{\ell}\sum_{p=0}^{i-1} \lambda_{i,p}^{\alpha,\beta} \mathrm{Tr}\left(\mu_\alpha^{(p)} \widetilde{Z}_i Z_i \mathcal{M}_{ii}^{(p),\beta}\right) \\ &+ \sum_{i=1}^{\ell}\sum_{p=0}^{i-1} \lambda_{ij,p}^{\alpha,\beta} \mathrm{Tr}\left(\mu_\alpha^{(p)} \widetilde{Z}_i Z_j \mathcal{M}_{ji}^{(p),\beta}\right) + \sum_{i=1}^{\ell}\sum_{p=0}^{i-1} \lambda_{ji,p}^{\alpha,\beta} \mathrm{Tr}\left(\mu_\alpha^{(p)} \widetilde{Z}_j Z_i \mathcal{M}_{ij}^{(p),\beta}\right). \end{aligned} \tag{6.19}$$

To illustrate the $\mathcal{M}$ operators, we consider the set $\mathcal{M}_{ij}^{(p),\alpha}$. The simplest examples in this class are $M_{ik}^{(p_1)} M_{kj}^{(p_2)}$ with $p_1 + p_2 - k = p$ where we trace over the $U(n_k)$ group.

In the case of $\mathcal{N}=2$ gluing at $SU(N)$, the $\mu_\alpha^{(p)}$ operators are entirely given by the chiral adjoint ϕ in the $\mathcal{N}=2$ vector multiplet, as $\mu^{(p)} = \phi^p$. The index α is trivial in this case. On the other hand, if we consider $\mathcal{N}=1$ gluing, then the μ operators are more complicated. To illustrate this, we consider gluing the $\mathcal{N}=2$ linear quiver in figure 6.3 with the box N identified with the $SU(N)$ in the Fan gauged with an $\mathcal{N}=1$ vector. In this case, the set $\mu_\alpha^{(p)}$ corresponds to the chain operators that can be constructed from the products of the quarks. To give an explicit example, we label the bifundamentals as $(Q_a, \widetilde{Q}_a)$ with $a=1$ corresponding to the one attached to the Fan. The operators, $\mu_\alpha^{(2)}$ are $(\widetilde{Q}_1 Q_1)^2_{\text{adj}}$ and $(\widetilde{Q}_1\widetilde{Q}_2 Q_2 Q_1)_{\text{adj}}$.

6.3.2 Seiberg duality and Fans

Under the Seiberg duality, a quiver with the Fan maps to another quiver with the Fan. To illustrate this, we consider the quiver in 6.6a and dualize at $SU(N')$ to obtain 6.6b. Under the duality, the $U(n')$ flavor group is absorbed into the new Fan and thereby increasing its size to $\ell+1$. We denote the $U(n')$ and $SU(M)$ hypermultiplets as $(Q', \widetilde{Q}')$ and $(Q_t, \widetilde{Q}_t)$. We also denote the fields of the new Fan as $(q,\tilde{q})$, $(z,\tilde{z})$, $(y,\tilde{y})$ and $(m,\tilde{m})$.

- Firstly we need to replace $SU(N')$ with its magnetic dual, $SU(N_f - N')$. The total number of flavors coming into this gauge group is $N_f = N + N'$; the contributions are N Q's, $\sum_{i=1}^{\ell} n_i$ Y's, and $n'+M$ $(1,0)$ fields where $n'+M = N' - \sum n_i$.

- The superpotential terms in (6.17) and (6.18) become mass terms under the duality. In the magnetic theory, we replace the meson operators QY_i, $\widetilde{Q}\widetilde{Y}_i$, $Q\widetilde{Q}'$, $Q'\widetilde{Q}$, $Q\widetilde{Q}_t$, and $Q_t\widetilde{Q}$ with their dual chiral superfields. The cubic terms in (6.17) become mass terms for the Z's while the quartic terms in (6.18) become mass terms for the new chiral fields. Integrating out the Z's decouples the $SU(N)$ gauge group from the Fan.

- The chiral superfield dual to $\widetilde{Q}Q$ is an adjoint of the first $SU(N)$ group. If we have $\mathcal{N}=2$ gluing, the first term in equation (6.19) will become a mass term for the chiral adjoint in the vector multiplet. Integrating out the massive chirals yields an $\mathcal{N}=1$ vector multiplet. On the other hand if the gluing is $\mathcal{N}=1$, the vector multiplet will become $\mathcal{N}=2$ with the addition of the chiral fields dual to $\widetilde{Q}Q$.

- The cubic superpotential involving the chiral adjoint of $SU(M)$ becomes a mass term when we replace the meson $\widetilde{Q}_tQ_t$ with its dual chiral superfield. Thus the $SU(M)$ gauge group becomes an $\mathcal{N}=1$.

- The fields of the Fan in figure 6.6b come from three different sectors, which are listed as in the table 6.3. The first set of fields is inherited from the old Fan. And

Table 6.3. The set of new fields appears upon dualizing the Fan.

New fields	Electric dual
$m^{(p+1)}_{i+1,i+1}$	$M^{(p)}_{ii}$
$\left(m^{(p+1)}_{i+1,j+1}, m^{(p+1)}_{j+1,i+1}\right)$	$(M^{(p)}_{ij}, M^{(p)}_{ji})$
$(q,\tilde{q})$	$(Q_t,\widetilde{Q}_t)$
$(z_1,\tilde{z}_1)$	$(Q',\widetilde{Q}')$
$(z_{i+1},\tilde{z}_{i+1})$	$(Y_i,\widetilde{Y}_i)$
$(y_1,\tilde{y}_1)$	$\left(\mathrm{Tr}_g(q\widetilde{Q}_t), \mathrm{Tr}_g(\tilde{q}Q_t)\right)$
$(y_{i+1},\tilde{y}_{i+1})$	$\left(\mathrm{Tr}_g(Y_i\widetilde{Q}_t), \mathrm{Tr}_g(\widetilde{Y}_iQ_t)\right)$
$m^{(0)}_{1,1}$	$\mathrm{Tr}_g(q\tilde{q})$
$m^{(0)}_{i+1,i+1}$	$\mathrm{Tr}_g(Y_i\widetilde{Y}_i)$
$(m^{(0)}_{1,j+1}, m^{(0)}_{j+1,1})$	$\left(\mathrm{Tr}_g(q\widetilde{Y}_j), \mathrm{Tr}_g(Y_j\tilde{q})\right)$
$(m^{(0)}_{i+1,j+1}, m^{(0)}_{j+1,i+1})$	$\left(\mathrm{Tr}_g(Y_i\widetilde{Y}_j), \mathrm{Tr}_g(Y_j\widetilde{Y}_i)\right)$

the second set of fields consists of the dual quarks of the $SU(N')$ gauge group. The last set of fields consists of the ones dual to the mesons of the old quiver tail.

- The flavor group $U(n')$ is absorbed into the Fan as the first flavor group $U(n'_1)$, and the labeling of the rest is shifted by 1 to $n'_{i+1}=n_i$. This yields the Fan labelled by (N,M) and the partition $N-M=\sum_k kn'_k$.

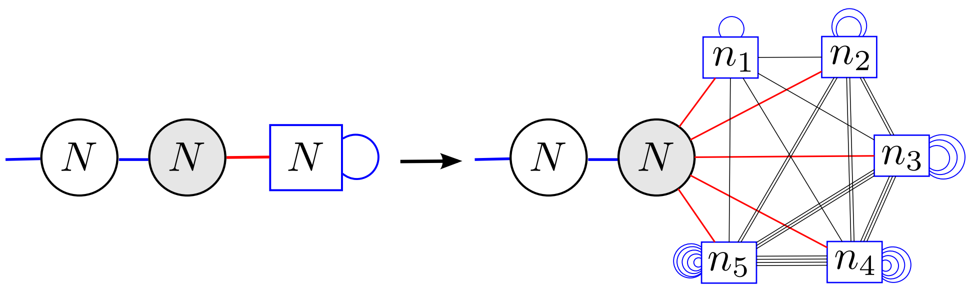


Figure 6.7. A Nilpotent vev to the adjoint chiral gives a Fan attached to the end of the quiver with $N = 1n_1 + 2n_2 + \cdots 5n_5$ and $N' = 0$.

The superpotential of the dual theory is constructed by considering all possible gauge invariant $(2,2)$-operators that preserve the global symmetry. The same superpotential is reproduced under the Seiberg duality.

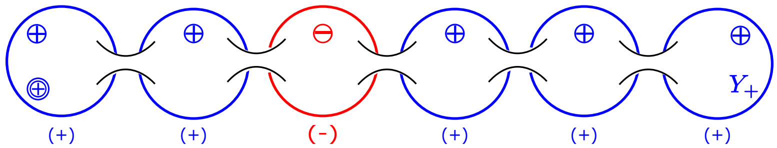
6.3.3 Fan as a quiver tail

In this section, we describe how the Fan and quiver tails appear in class $\mathcal{S}$ theories. A quiver tail associated to the partition Y of N is given by a punctured sphere with one maximal, a number of minimal punctures and a puncture labeled by Y. Here Y corresponds to the partition $N = \sum_{k=1}^{\ell} k n_k$.

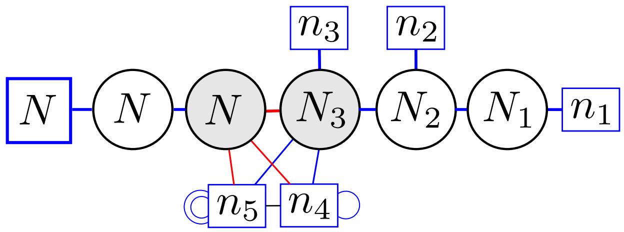
Starting from the linear quiver given in section 6.2.2, we can get the quiver tail by Higgsing one of the maximal punctures to Y. When the puncture has the same color as that of the pair-of-pants, this is same as giving a nilpotent vev to the quark bilinear $\mu_0 = \widetilde{Q}_0 Q_0 - \frac{1}{N}\mathrm{Tr}\widetilde{Q}_0 Q_0$. When the color of the puncture is different from that of the pair-of-pants, we give a vev to the adjoint chiral multiplet. In both cases, the $U(1)_0 \times SU(N)_0$ flavor symmetry of the quiver is broken down to $\left(\prod_{i=1}^{\ell} U(n_i)\right)$.

Now, let us describe the quiver tail associated to the partition above. If the color of the puncture we Higgs is different from that of the pair-of-pants, the theory we obtain is given by attaching the Fan with $(N, N' = 0)$ as in the figure 6.7.

If the color of the puncture is the same as the pair-of-pants, we proceed as follows.



(a) A colored pair-of-pants decomposition corresponding to the quiver tail.



(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

Figure 6.8. The quiver tail given by the partition $N = 1n_1 + 2n_2 + \cdots 5n_5$. $\mathcal{N} = 2$ Higgsing propagated until we hit $k = 3$. Then the gauge group of the next node becomes $SU(N)$, and we have the Fan between $SU(N)$ and $SU(N_3)$. The Fan is given by (N, N_3) and the partition $N - N_3 = n_4 + 2n_5$.

1. When the neighboring gauge node of Q_0 is $\mathcal{N} = 2$, the flavor node becomes n_1 and the gauge node becomes $N_1 = \sum_{i=1}^{\ell} n_i$. If it is $\mathcal{N} = 1$, then go to step 3.

2. When the next neighboring gauge node is again $\mathcal{N} = 2$, the gauge group becomes $N_2 = N_1 + \sum_{i=2}^{\ell} n_i$, and add n_2 fundamental flavors to it. If it is $\mathcal{N} = 1$, then go to step 3.

3. Proceed until we hit an $\mathcal{N} = 1$ gauge node. In this case, the neighboring gauge node remains to be $SU(N)$, since the Higgsing stops propagating. Suppose we hit the $\mathcal{N} = 1$ node at step k. In this case, the remaining flavor boxes n_i with $k < i < \ell$ should be attached to the gauge node of N_k. Therefore we get the Fan labelled by (N, N_k) with partition $N - N_k = \sum_{m=1}^{\ell-k} m n_{m+k}$.

See figure 6.8 for the case with $\ell = 5$ and $k = 3$. We see that the Fan serves as a role of gluing $\mathcal{N} = 1$ nodes with different ranks in the quiver tail.

Let us remark on the flavor symmetry of the quiver tail with the Fan. Even though the Fan itself has the flavor symmetry $U(1) \times \prod_k U(n_k)$, the flavor symmetry of the whole quiver tail does not include the overall $U(1)$ piece of $\prod_k U(n_k)$. The global symmetry of the quiver tail associated to the puncture Y does not contain the extra $U(1)$. We can see this directly in the case of figure 6.7. In this case, we see that the overall $U(1)$ can be identified with $U(1)_B$ symmetry of the Fan.

6.3.4 Nilpotent Higgsing and Fan

In this section, we give a derivation of the Fan for the case when $N' = 0$. Let us now consider the linear quiver theory as in figure 6.2. It has a chiral adjoint M_0 attached at the flavor $SU(N)$ node. The superpotential is $W = \mathrm{Tr} M_0 \mu_0$, where μ_0 is the quark bilinear $\mu_0 = \tilde{q}_0 q_0 - \frac{1}{N}\mathrm{Tr}\tilde{q}_0 q_0$ with $(J_+, J_-) = (0, 2)$. Here we choose the color of the pair-of-pants corresponding to q_0 to be $\sigma = -1$. We Higgs the flavor $SU(N)$ by a nilpotent vev corresponding to the partition $N = \sum_k k n_k$ to M_0. In the following, we omit the subscript of μ and M for simplicity.

Under the $SU(2)$ embedding ρ labelled by the partition of N, the fundamental representation of $SU(N)$ decomposes as follows:

$$\mathbf{N} \to \bigoplus_{i=1}^{\ell} V_{\frac{i-1}{2}} \otimes \mathbf{n}_i \ , \tag{6.20}$$

where V_j is the spin j representation of $SU(2)$ and $\mathbf{n}_i$ is the fundamental representation of $SU(n_i) \subset S[\prod_{i=1}^{\ell} U(n_i)]$. The residual flavor symmetry $S[\prod_{i=1}^{\ell} U(n_i)]$ is given by the commutant of the embedding. The adjoint representation of $SU(N)$ decomposes as

$$\begin{aligned} \mathrm{adj} \ &\to \ \bigoplus_{i,j=1}^{\ell} (V_{\frac{i-1}{2}} \otimes \mathbf{n}_i) \otimes (V_{\frac{j-1}{2}} \otimes \mathbf{n}_j) - V_0 \\ &= \ \bigoplus_{i<j} \bigoplus_{k=1}^{i} V_{\frac{j-i+2k-2}{2}} \otimes (\mathbf{n}_i \otimes \bar{\mathbf{n}}_j \oplus \bar{\mathbf{n}}_i \otimes \mathbf{n}_j) \oplus \bigoplus_{i=1}^{\ell} \bigoplus_{k=1}^{i} V_{k-1} \otimes \mathbf{n}_i \otimes \bar{\mathbf{n}}_i - V_0 \end{aligned} \tag{6.21}$$

This decomposition gives us the quantum numbers of the various elements of the $SU(N)$-

adjoint M.

We now use the decoupling argument of [82]. Due to the vev of M the superpotential is written as

$$W = \mu_{1,-1,1} + \sum_{J,m,f} M_{J,-m,f}\mu_{J,m,f}, \tag{6.22}$$

where $M_{J,m,f}$ is the fluctuation from the vev, and J, m and f labels the spins, σ_3-eigenvalues and the representations of the flavor symmetry $\prod_i SU(n_i)$ appearing in the decomposition (6.21). By the presence of the first term the $SU(N)$ current is not conserved anymore, and becomes non-BPS by absorbing the components of μ except for the $m = J$. The components of M which coupled to the absorbed μ will be decoupled and the remaining components are $M_{J,-J,k}$. Namely the $m = -J$ component of each term of (6.21). Also we should note that due to the first term of the superpotential the $U(1)_\pm$ symmetries are shifted as

$$J_+ \to J_+ - 2\rho(\sigma^3), \qquad J_- \to J_- \ , \tag{6.23}$$

(or $R_0 \to R_0 - \rho(\sigma^3)$ and $\mathcal{F} \to \mathcal{F} - \rho(\sigma^3)$) in order to keep the first term to be $J_+ = J_- = 2$ ($R_0 = 2$, $\mathcal{F} = 0$).

This gives us the gauge neutral components of the Fan in the low energy theory. We saw that there are i gauge neutral chiral multiplets $(M_{ij}^{(p)}, M_{ji}^{(p)})$, $0 \leq p < i$ transforming as bifundamentals of $U(n_i) \times U(n_j)$, $i \leq j$. We identify these chirals with the component of M (6.21) with $m = -J$ (and $k = i - p$). As a consequence of (6.23), the (J_+, J_-) charges of $(M_{ij}^{(p)}, M_{ji}^{(p)})$ become $(i + j - 2p, 0)$, which indeed match with table 6.1.

Some elements of the (anti-)quark multiplet transforming in the (anti-)fundamental representation of the $SU(N)$ flavor symmetry become massive due to the Higgsing and will be integrated out. Since $\langle M\rangle = \rho(\sigma^+)$ which is $J=1$, $m=1$ component, it implies that the (anti-)quarks Z_i ($\widetilde{Z}_i$) that remain massless are the components with $m=\frac{i-1}{2}$ in $V_{\frac{i-1}{2}}\otimes \mathbf{n}_i$ ($V_{\frac{i-1}{2}}\otimes \bar{\mathbf{n}}_i$). Z_i and $\widetilde{Z}_i$ together form a hypermultiplet whose (J_+, J_-) charges are $(1-i, 1)$ by using (6.23).

In addition we have the superpotential (6.16). We give a derivation of it in Appendix E.1.

The Goldstone multiplets In any field theory we expect the spontaneous breaking of global symmetries to be accompanied by the presence of massless Goldstone bosons whose number is equal to the number of broken generators of the global symmetry. In supersymmetric theories these Goldstone bosons will form the scalar components of massless chiral multiplets which we will call Goldstone multiplets.

However, the number of Goldstone multiplets is not necessarily equal to the number of broken generators of the global symmetry. For example, consider the linear quiver of figure 6.2 with gauge group being $SU(3)$. Upon nilpotent Higgsing (giving a nilpotent vev to M_0) of the $SU(3)$ linear quiver by the partition $3=2+1$, the $SU(3)$ symmetry gets broken down to $U(1)$. The chiral fields that decouple from the low energy theory are expected to be the Goldstone multiplets. But there are only 4 such chiral multiplets while the number of broken generators is 7.

The reason behind the discrepancy in this counting is that the scalar in a Goldstone multiplet is complex. Thus it might be that a Goldstone multiplet is either made up of two Goldstone bosons or a single Goldstone boson that gets paired up with a non-Goldstone scalar. In view of this we see that the number of Goldstone multiplets will always be less than or equal to the number broken generators of the global symmetry. The correct number of Goldstone multiplets is obtained by observing that the superpotential is holomorphic. This implies we should count the number of broken generators of the complexified global symmetry [149]. Using this we now show that the number of decoupled chirals indeed matches with the number of expected Goldstone multiplets.

In the theories of interest here, we want to consider the breaking of $G = SU(N)$ down to $H = S[\prod_{i=1}^{\ell} U(n_i)]$. The complexification of G is $\bar{G} = SL(N, \mathbb{C})$. Since the breaking of global symmetries is achieved through $\langle M \rangle = \rho^+$, we therefore look for generators X of $SL(N, \mathbb{C})$ which satisfy

$$[\rho^+, X] \neq 0 \ . \tag{6.24}$$

Note that any generator of $SL(N, \mathbb{C})$ can be thought of as a complex matrix transforming in the adjoint representation of $SU(N)$. We can therefore label each element of X by its $SU(2) \hookrightarrow SU(N)$ quantum numbers. In fact we can also simultaneously label them by the $S\Big(\prod_{i=1}^{\ell} U(n_i)\Big)$ symmetries that commute with the $SU(2)$ embedding. The components of X are therefore classified as in (6.21). In terms of $X_{J,m,k}$, we see that (6.24) is satisfied if X has a non-zero component with $m \neq J$. The Goldstone multiplet corresponding to such an X will be the quantum fluctuation proportional to $[\rho^+, X]$. These fluctuations therefore correspond to the components in (6.21) that have σ_3-egenvalues, $m \neq -J$. This is same as the quantum numbers of the decoupled chiral multiplets. We thus establish a one-to-one correspondence between the expected Goldstone multiplets and the decoupled chirals.

6.4 $\mathcal{N} = 1$ dualities

In this section, we discuss various duality frames for an SCFT associated to a UV curve. In order to give a UV description of the theory, we need to specify a colored pair-of-pants decomposition. Any Riemann surface with negative Euler number can be decomposed in terms of pairs-of-pants. We assign $\mathbb{Z}_2$-valued colors to each pairs-of-pants so that the number of $(+,-)$-colored pants are the degrees of the normal bundles (p,q). Different colored pair-of-pants decompositions give rise to different UV descriptions of the same SCFT in the IR. See figure 6.9 for an example.

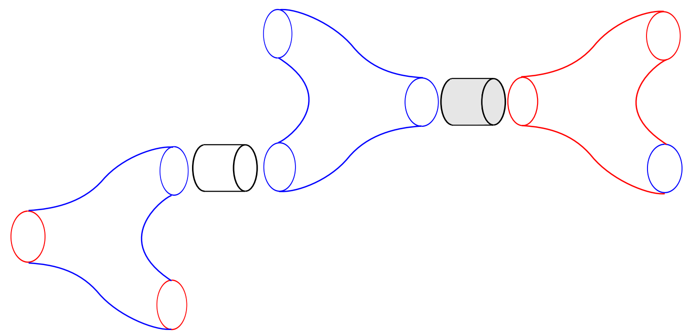


Figure 6.9. An example of colored pair-of-pants decomposition for $(p,q)=(2,1)$. The shaded cylinder corresponds to an $\mathcal{N}=1$ vector multiplet and unshaded one correspond to an $\mathcal{N}=2$ vector multiplet. We have 3 punctures of opposite color. There is an adjoint chiral attached to each of them.

Let us assume all the punctures to be maximal for the moment. For a given colored pair-of-pants, we associate the T_N theory found in [88] which we will review in 6.4.2. For each puncture, we have an operator μ_i transforms as the adjoint of $SU(N)_i$. When the puncture has a different color from the pair-of-pants itself, we add chiral field M_i transforming as the adjoint of $SU(N)_i$ and also a superpotential $W = \text{Tr}(M_i\mu_i)$. When we glue two pair-of-pants with the same color, we gauge the flavor symmetry with an $\mathcal{N}=2$ vector multiplet. When gluing two different colored pair-of-pants, we gauge the flavor symmetry by an $\mathcal{N}=1$ vector multiplet. See figure 6.10, which is the UV description corresponding to the pair-of-pants decomposition of figure 6.9.

Non-maximal punctures can be obtained by Higgsing or partially closing the puncture. Let us call ρ_i to be the $SU(2)$ embedding into Γ that is used to label the punctures. For a puncture having the same color as the pair-of-pants, Higgsing is implemented through giving a nilpotent vev $\rho_i(\sigma^+)$ to the operator μ_i, and for an opposite colored puncture, we give a vev to M_i instead. For example, consider the UV description

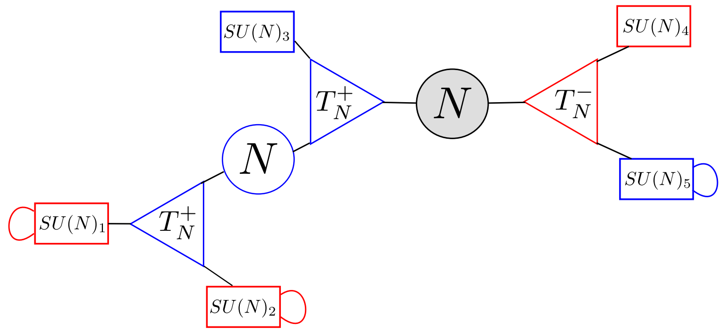


Figure 6.10. The UV description corresponding to the colored pair-of-pants description of figure 6.9. Here we assumed all punctures to be maximal.

of figure 6.10. When we Higgs $SU(N)_3$ and $SU(N)_4$ to minimal punctures, we get the theory as in figure 6.11. Since we closed the punctures that have the same color as

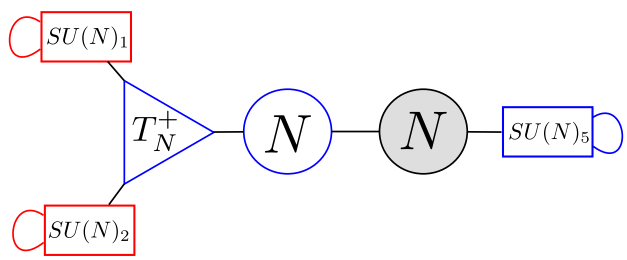


Figure 6.11. A UV description obtained from partially closing $SU(N)_{3,4}$ punctures to the minimal punctures.

the pair-of-pants, we can simply use $\mathcal{N}=2$ results of [88, 181, 53, 54, 55, 57, 58, 56] to identify the theory corresponding to the pair-of-pants. This is really the same as choosing $\mathcal{N}=2$ building block and gluing through the $\mathcal{N}=1$ or $\mathcal{N}=2$ vector multiplets.

Things are different when we close the punctures with opposite colors. When we close $SU(N)_1$ to minimal puncture, the theory (in this duality frame) is still non-Lagrangian, but we can identify decoupled operators and global symmetry [82]. When we close $SU(N)_5$, we give a vev $\rho_5(\sigma^+)$ to the chiral superfield M_5, from which the quarks acquire nilpotent masses. This theory has a Lagrangian description. As we have seen, this kind of Higgsing yields the Fan labelled by $(N, N'=0)$ and the partition corresponding to ρ_5.

We see that there are many different colored pair-of-pants decompositions for a given UV curve. From the six-dimensional perspective, four-dimensional physics in the IR has to be independent from the specific choice of colored pair-of-pants. Therefore we can give equivalent descriptions for the same IR theory from the UV curve and its colored pair-of-pants decompositions. This generalizes the usual Seiberg duality for the $\mathcal{N}=1$ theories and also Argyres-Seiberg-Gaiotto duality of $\mathcal{N}=2$ class $\mathcal{S}$ theories.

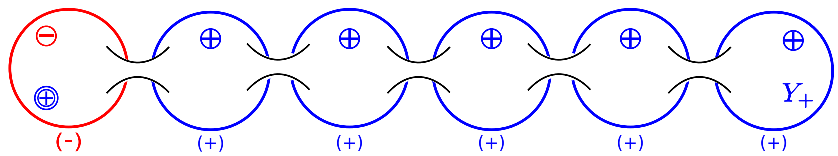
In the rest of this section, we discuss two particular examples. In section 6.4.1, we study successive application of Seiberg duality on the $\mathcal{N}=2$ quiver tail connected by an $\mathcal{N}=1$ gauge node. This illustrates the appearance of the Fan in $\mathcal{N}=1$ quiver tail. In section 6.4.2, we discuss duality of $SU(N)$ SQCD with $2N$ fundamental flavors. We find a dual frame involving the T_N theory and the Fan, which is similar to the strong coupling dual of $\mathcal{N}=2$ SQCD discovered by Argyres and Seiberg [24].

6.4.1 $\mathcal{N}=1$ quiver tails

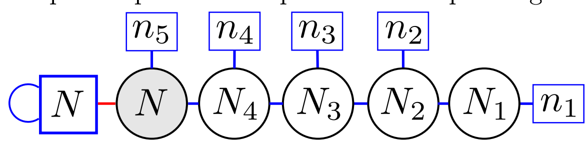
Let us consider a UV curve with 5 minimal punctures of + color, 1 minimal puncture of − color, one + colored maximal puncture and one + colored generic puncture labelled by a partition $N=\sum_k kn_k$. We also pick the degrees of normal bundles to be $(p,q)=(5,1)$. This theory has many different dual frames. We start with a dual frame which resembles the more familiar $\mathcal{N}=2$ quiver tail and then dualize multiple times to see the various dual frames for the $\mathcal{N}=1$ quiver tail.

Consider the dual frame given by the colored pair-of-pants decomposition of figure 6.12a. This is essentially the same as the $\mathcal{N}=2$ quiver tail, so that we get the 6.12b. Only the very last node is gauged via an $\mathcal{N}=1$ vector multiplet.

Now, if we Seiberg dualize the $\mathcal{N}=1$ node, we get the quiver as shown in figure 6.13. We see that there is a chiral multiplet dual to the meson formed from the quarks attached at node n_5. The dual quarks will have the opposite $\mathcal{F}$ charge which is depicted

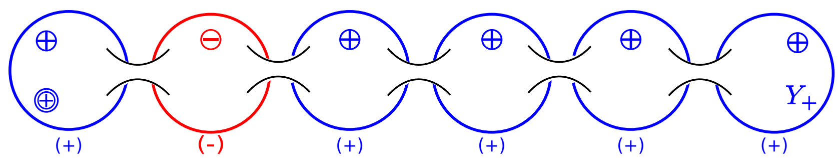


(a) A colored pair-of-pants decomposition corresponding to the quiver.

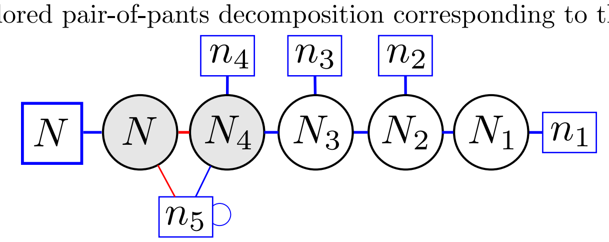


(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

Figure 6.12. The quiver tail obtained from $\mathcal{N} = 2$ Higgsing for the partition $N = 1n_1 + 2n_2 + \ldots + 5n_5$. The rank of gauge group is fixed by $2N_i = N_{i+1} + N_{i-1} + n_i$.



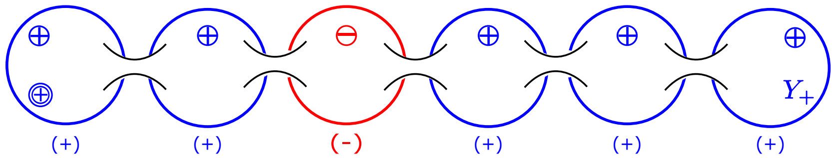
(a) A colored pair-of-pants decomposition corresponding to the quiver.



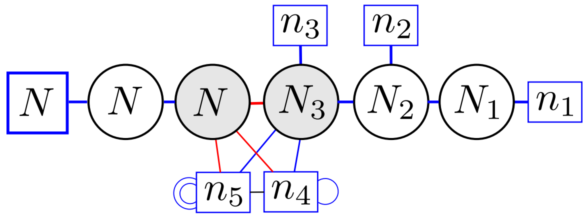
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

Figure 6.13. The quiver tail consists of the $\mathcal{N} = 2$ tail of length 4 and the Fan labelled by (N, N_4) and the partition $N - N_4 = 1n_5$.

by red. Also, there is an additional blue edge connecting N_4 and n_5 which is the dual to the quark bilinear formed from the $SU(N) \times SU(N_4)$ bifundamental and the fundamental attached at n_5 node in figure 6.12b. The rest of the dual mesons become massive from the superpotential. In this frame, we see that there is the Fan labelled by (N, N_4) and the partition $N - N_4 = 1 \cdot n_5$, connecting a shorter $\mathcal{N} = 2$ quiver tail of length 4 and the left-hand segment of the quiver. In terms of nilpotent Higgsing of the linear quiver, the propagation of vev is terminated at the $\mathcal{N} = 1$ node N_4, giving us the Fan that glues to the $SU(N)$ gauge node.



(a) A colored pair-of-pants decomposition corresponding to the quiver.



(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

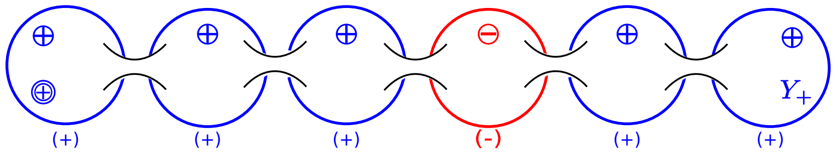
Figure 6.14. The quiver tail consists of the $\mathcal{N}=2$ tail of length 3 and the Fan labelled by (N, N_3) and the partition $N - N_3 = 1n_4 + 2n_5$.

Now, we dualize the gauge group $SU(N_4)$ node to get the quiver depicted in figure 6.14. The flavor node n_4 becomes part of the new Fan, which is labelled by (N, N_3) and the partition $N - N_3 = 1n_4 + 2n_5$. We see that there is an extra dual meson attached to the n_5 node.

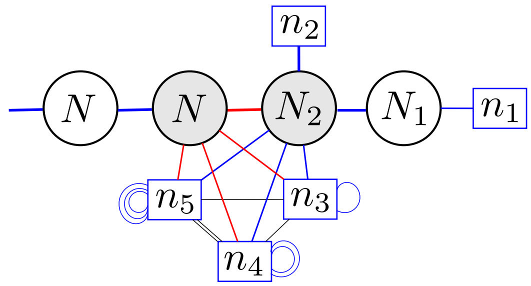
Further dualizing the $SU(N_3)$ node, we get the quiver of figure 6.15. The flavor node n_3 now becomes the part of the Fan, and we get extra dual mesons for each of the preexisting nodes in the Fan. Note that we also have additional chiral multiplets transforming as the bifundamental of $U(n_4) \times U(n_5)$.

Dualizing once again, we get the quiver tail of figure 6.16. Once again, the flavor node n_2 becomes a part of the Fan, and chiral multiplets get added. This quiver tail can also be obtained from starting with the linear quiver and Higgsing $\mu_0 = (\widetilde{Q}_0 Q_0)_{\text{adj}}$ directly by a nilpotent vev associated to the partition $N = \sum_k k n_k$. We see that the Higgsing does not propagate beyond N_1. All the flavor nodes are attached to N_1 and its neighbor N.

Now finally, upon dualizing the $SU(N_1)$ gauge node, we get the theory as in the figure 6.17. This gives us the Fan of size $\ell = 5$ labelled by $(N, 0)$ and the partition



(a) A colored pair-of-pants decomposition corresponding to the quiver.



(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

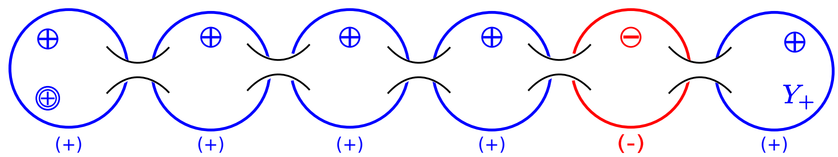
Figure 6.15. The quiver tail consists of the $\mathcal{N}=2$ tail of length 2 and the Fan labelled by (N, N_2) and the partition $N - N_2 = 1n_3 + 2n_4 + 3n_5$.

$N = \sum_{k=1}^{5} kn_k$ attached to the right end of the quiver.

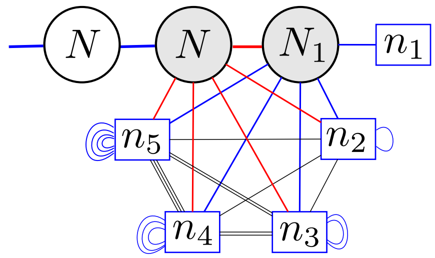
We see that there are many different quiver tail descriptions for a given choice of punctures in $\mathcal{N}=1$ class $\mathcal{S}$ theories. In the above example, we have only described UV frames that have Lagrangian descriptions. For these cases, all the pairs-of-pants have the same color as the minimal puncture inside. In general, one can also consider a dual frame which has a different colored puncture inside its pair-of-pants. Then the dual frame has a sector with no Lagrangian description. We will discuss such a case in the next section.

6.4.2 $\mathcal{N}=1$ analog of Argyres-Seiberg duality

In this section we use the Fan to provide a new dual description of $\mathcal{N}=1$ $SU(N)$ SQCD with $2N$ flavors with the quartic coupling (6.5) with $i=1$. This is the $(\sigma_{-1}, \sigma_0, \sigma_1, \sigma_2) = (-1, 1, -1, 1)$ linear quiver as described in section 6.2.2. The flavor symmetry of the theory is $SU(N)_1 \times SU(N)_2 \times U(1)_A \times U(1)_B$. We summarize the matter content in table 6.4 and quiver in figure 6.18. In this section it is more convenient to use the symmetries R_0 and $\mathcal{F}$ defined in (6.1).



(a) A colored pair-of-pants decomposition corresponding to the quiver.



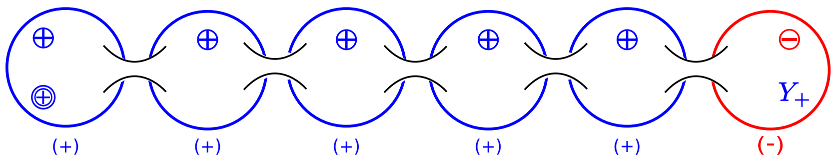
(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

Figure 6.16. The quiver tail consists of the $\mathcal{N}=2$ tail of length 1 and the Fan labelled by (N, N_1) and the partition $N - N_1 = 1n_2 + 2n_3 + 3n_4 + 4n_5$.

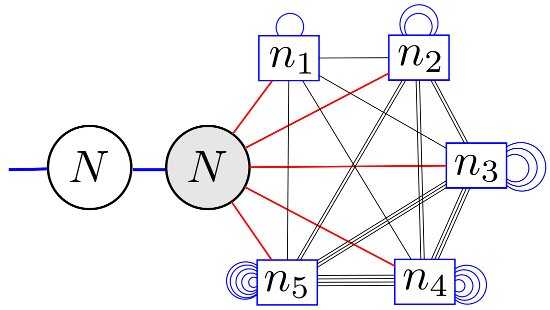
Table 6.4. Charges of matter multiplets in SQCD.

	$SU(N)_g$	$SU(N)_1$	$SU(N)_2$	$U(1)_{R_0}$	$U(1)_{\mathcal{F}}$	$U(1)_A$	$U(1)_B$
$(Q_0, \widetilde{Q}_0)$	$(\square, \bar{\square})$	$(\bar{\square}, \square)$	$\cdot$	$1/2$	$-1/2$	$(1,-1)$	$\cdot$
$(Q_1, \widetilde{Q}_1)$	$(\bar{\square}, \square)$	$\cdot$	$(\square, \bar{\square})$	$1/2$	$1/2$	$\cdot$	$(1,-1)$

It has been pointed out in [82] that there are two dual descriptions of the SQCD. Let us shortly explain these here. One of them is $\mathcal{N}=1$ $SU(N)$ SQCD with $2N$ flavors with a chiral multiplet in the adjoint representation of $SU(N)_1$ and a chiral multiplet in the adjoint of $SU(N)_2$ coupled by the cubic interaction with quarks. This is indeed the Seiberg dual theory of the original SQCD with the quartic coupling. In terms of the Riemann surface this is understood as the exchange of the maximal punctures as in figures 6.19b. Other dual description whose Lagrangian is not known corresponds to the exchange of the minimal punctures as in figure 6.19c. To obtain this theory, we first consider an $\mathcal{N}=1$ $SU(N)$ gauge theory coupled to two T_N theories [88] (which will be reviewed below) and to two chiral multiplets, which are the adjoints of $SU(N)_A$ and $SU(N)_B$ flavor symmetries of the two T_N theories respectively. This is associated to the Riemann surface where all the punctures are maximal, but the color assignment is same



(a) A colored pair-of-pants decomposition corresponding to the quiver.



(b) The quiver tail corresponding to the above colored pair-of-pants decomposition.

Figure 6.17. The quiver tail consists of the maximal Fan of size $\ell = 5$, labelled by $(N, 0)$ and the partition $N = \sum_k k n_k$.

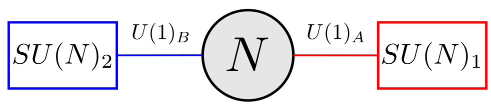


Figure 6.18. The quiver diagram of $SU(N)$ SQCD with $2N$ flavors.

as in 6.19c. Then the dual description is obtained by Higgsing of $SU(N)_A$ and $SU(N)_B$ symmetries down to $U(1)_A$ and $U(1)_B$.

In this section we will find a third dual description of the SQCD corresponding to the figure 6.19d. Since the UV description involves the T_N theory, we will review relevant details first.

The T_N theory is obtained by compactifying N coincident M5-branes, with $\mathcal{N} = 2$ twist, on a sphere with three maximal punctures. Each puncture carries an $SU(N)$ global symmetry, thereby leading to an $SU(N)^3$ flavor symmetry. It is an $\mathcal{N} = 2$ SCFT and it admits $U(1)_{\mathcal{N}=2} \times SU(2)_R$ R-symmetry. When we describe it as an $\mathcal{N} = 1$ SCFT, we use the $\mathcal{N} = 2$ R-symmetry to write R_0 and $\mathcal{F}$ as

$$R_0 = \frac{1}{2} R_{\mathcal{N}=2} + I_3, \quad \mathcal{F} = -\frac{1}{2} R_{\mathcal{N}=2} + I_3 \tag{6.25}$$

where $R_{\mathcal{N}=2}$ and I_3 are generators of $U(1)_{\mathcal{N}=2}$ and the diagonal $U(1)$ of the $SU(2)_R$ respectively. This theory has chiral operators μ_i (i labels the three $SU(N)$ flavor symmetries) which are the moment maps of the $SU(N)$ flavor symmetries. It also has operators $Q^{(k)}$ transforming in the k-th antisymmetric representation of all three $SU(N)$ symmetries [91, 86, 152]. Their R_0 and $\mathcal{F}$ charges are

$$R_0(\mu_i) = \mathcal{F}(\mu_i) = 1, \quad R_0(Q^{(k)}) = \mathcal{F}(Q^{(k)}) = \frac{k(N-k)}{2}. \tag{6.26}$$

The results of section 6.3 tell us that figure 6.19d represents an $SU(N)$ gauge theory coupled to the Fan with $\sigma = -1$ labelled by $(N, 0)$ and a partition $N = 1 + (N-1)$, i.e., $\ell = N-1$, $n_1 = n_{N-1} = 1$ and $n_i = 0$ otherwise. It is coupled to the T_N theory by gauging an $SU(N)$ flavor symmetry. Furthermore, a chiral field X transforming in the adjoint representation of $SU(N)_1$ flavor symmetry of the T_N theory is added. $SU(N)_{1,2}$ are the flavor symmetries of the T_N theory which are not gauged.

The dual theory is described by the quiver in figure 6.20. The matter content is summarized in table 6.5. For convenience of the discussion, we write fields from the Fan as $M_1 := M^{(0)}_{1,1}$, $M^{(k)}_{N-1} := M^{(N-1-k)}_{N-1,N-1}$ and $z := Z_{N-1}$.

The important data needed in including the T_N in these quivers is its contribution to the anomalies. These are described in section 6.5. For the purpose of the quiver in 6.20, the contribution of T_N to the chiral anomalies $(R_0 SU(N)^2, \mathcal{F}SU(N)^2)$ is the same as N fundamental $(J_+, J_-) = (1, 0)$ hypermultiplets.

Finally, one linear combination of M_1 and $M^{(1)}_{N-1}$ must be projected out. We denote the combination that survives as $\hat{M}_1$. We can then write the superpotential as

$$\begin{aligned} W_m \quad = \quad & \hat{M}_1(\mathrm{tr} z\mu_g^{N-2}\tilde{z} + \mathrm{tr} Z_1 \widetilde{Z}_1) + \sum_\alpha \sum_{k=2}^{N-1} \mathcal{M}^{(k),\alpha}_{N-1} \mathrm{tr} z \mu_g^{N-1-k}\tilde{z} \\ & + M_{1,N-1}\mathrm{tr} Z_1 \tilde{z} + M_{N-1,1}\mathrm{tr} z \widetilde{Z}_1 + \mathrm{tr}\mu_1 X + \mathrm{tr} Z_1 \mu_g \widetilde{Z}_1 + \mathrm{tr} z \mu_g^{N-1}\tilde{z}, \end{aligned} \tag{6.27}$$

Table 6.5. Charges of matter multiplets in the dual theory, where $M_1 := M_{1,1}^{(0)}$ and $M_{N-1}^{(k)} := M_{N-1,N-1}^{(N-1-k)}$ and $z = Z_{N-1}$.

	$SU(N)_g$	$SU(N)_1$	$SU(N)_2$	$U(1)_{R_0}$	$U(1)_{\mathcal{F}}$	$U(1)_1$	$U(1)_{N-1}$
$(Z_1, \widetilde{Z}_1)$	$(\square, \bar{\square})$	$\cdot$	$\cdot$	$1/2$	$-1/2$	$(-1, 1)$	$\cdot$
$(z, \tilde{z})$	$(\square, \bar{\square})$	$\cdot$	$\cdot$	$\frac{3-N}{2}$	$\frac{1-N}{2}$	$\cdot$	$(-1, 1)$
M_1	$\cdot$	$\cdot$	$\cdot$	1	1	$\cdot$	$\cdot$
$(M_{1,N-1}, M_{N-1,1})$	$\cdot$	$\cdot$	$\cdot$	$N/2$	$N/2$	$(1, -1)$	$(-1, 1)$
$M_{N-1}^{(k=1,\cdots,N-1)}$	$\cdot$	$\cdot$	$\cdot$	k	k	$\cdot$	$\cdot$
X	$\cdot$	adj	$\cdot$	1	-1	$\cdot$	$\cdot$

where μ_1 and μ_g are the moment maps of $SU(N)_1$ and $SU(N)_g$ symmetries respectively. The set of operators, $\mathcal{M}_{N-1}^{(k),\alpha}$ correspond to all possible composite operators with charge $(2k, 0)$.

Note that this is reminiscent of the Argyres-Seiberg duality [24] of $\mathcal{N} = 2$ $SU(3)$ SQCD with six flavors. Indeed, if we consider the analogous UV curve in the $\mathcal{N} = 2$ setting without color assignments, this dual frame is exactly that of Argyres-Seiberg when $N = 3$. The duality presented here is an $\mathcal{N} = 1$ analog of that. It will be interesting to derive this duality through the technique of inherited duality [23, 22].

We identify $U(1)_A$ and $U(1)_B$ of the SQCD as

$$U(1)_A = U(1)_1 + U(1)_{N-1}, \qquad U(1)_B = (N-1)U(1)_1 - U(1)_{N-1}. \tag{6.28}$$

It is a straightforward calculation to show that all the anomaly coefficients of the flavor symmetries agree on both sides of the duality. We will see this in section 6.5. In section 6.6, we will also see the agreement of the superconformal index of both theories. This will be the strongest check of the duality.

6.5 Anomalies and central charges

In this section we compute the 't Hooft anomaly coefficients of various objects. In section 6.5.1, we start with computing those of the Fan introduced in section 6.3.1. We then interpret the results in terms of a sphere with punctures and give a concise expression for the anomaly coefficients of the class $\mathcal{S}$ theories in general, in section 6.5.2.

6.5.1 Anomalies of the Fan

The matter content of the Fan labelled by $(N, N' = 0)$ and a partition $N = \sum_{k=1}^{\ell} kn_k$ with $\sigma = -1$ is given in the table 6.1. One can choose $\sigma = +1$ by swapping J_+ and J_- charges. In evaluating the anomalies, it is useful to write them in terms of

$$N_i = \sum_{k=1}^{i} n_k k + i \sum_{k=i+1} n_k, \tag{6.29}$$

and to notice the following identity

$$N^2 = 2\sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j - \sum_{i=1}^{\ell} N_i n_i. \tag{6.30}$$

We find the 't Hooft anomaly coefficients of $SU(n_i)$ and $U(1)_i$ for $i \geq 2$ are

$$\begin{aligned} &\mathrm{Tr}T_i^2 R_0 = \sigma \mathrm{Tr}T_i^2 \mathcal{F} = -\frac{1}{2}N_i, \\ &\mathrm{Tr}U_i^2 R_0 = -N_i - i(i-1)n_i, \qquad \sigma \mathrm{Tr}U_i^2 \mathcal{F} = -N_i + i(i+1)n_i, \end{aligned} \tag{6.31}$$

where T_i and U_i are the generators of $SU(n_i)$ and $U(1)_i$ respectively. The other anomaly coefficients are given by

$$\mathrm{Tr}R_0 \;=\; -\sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j, \qquad \sigma \mathrm{Tr}\mathcal{F} = \sum_{i=1}^{\ell} N_i \sum_{j=i+1}^{\ell} n_j + 1, \tag{6.32}$$

$$\mathrm{Tr}\mathcal{F}^a R_0^{3-a} = -\frac{(-\sigma)^a}{4}\sum_{i=1}^{\ell} n_i \left[\sum_{j=1}^{i} n_j \left(i^3 j - f_a(i,j)\right) + \sum_{j=i+1}^{\ell} n_j \left(i^3 j - f_a(j,i)\right)\right], \quad (6.33)$$

where

$$f_a(i,j) = \frac{1}{2}\sum_{p=0}^{j-1} (i+j-2p-2)^{3-a} (i+j-2p)^a \,. \quad (6.34)$$

Writing explicitly,

$$\mathrm{Tr}R_0\mathcal{F}^2 = \frac{1}{4}\sum_{i=1}^{\ell}\left(N^2 - N_i^2\right) - \frac{1}{4}\sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j, \quad (6.35)$$

$$\sigma\mathrm{Tr}R_0^2\mathcal{F} = \frac{1}{4}\sum_{i=1}^{\ell}\left(N^2 - N_i^2\right) + \frac{1}{2}N^2 - \frac{1}{4}\sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j. \quad (6.36)$$

We found that the rest can be obtained from

$$\mathrm{Tr}\mathcal{F}^3 = \mathrm{Tr}\mathcal{F} - 3\mathrm{Tr}\mathcal{F}R_0^2, \qquad \mathrm{Tr}R_0^3 = \mathrm{Tr}R_0 - 3\mathrm{Tr}\mathcal{F}^2 R_0. \quad (6.37)$$

From Linear quiver The above anomalies can also be obtained by a rather indirect way. The idea is to use the duality: as we saw in section 6.4.1, the Fan was obtained by taking various Seiberg dualities to the linear quiver theories with $\mathcal{N}=1$ $SU(N)$ gauge theory coupled to $\mathcal{N}=2$ quiver tail labelled by partitions of N: $N=\sum_{k=1}^{\ell} n_k k$. Thus, let us first focus on this original theory. This theory has gauge symmetry $G=\prod_{k=1}^{\ell} SU(N_k)$ with (6.29). Notice that $N_\ell = N$. All the gauge groups except for the ℓ-th one are $\mathcal{N}=2$. In addition to the bifundamentals, there are n_i fundamental hypermultiplets attached to the $SU(N_i)$ gauge group. The tail has a label $\sigma=\pm 1$ depending on the $\mathcal{F}$-charge of the matter fields $\mathcal{F}=\sigma/2$. (The $\mathcal{F}$-charge of the chiral adjoint multiplets of the gauge symmetry is $-\sigma$.) We end the quiver by adding N fundamental hypers with $R_0 = 1/2$ and $\mathcal{F}=-\sigma/2$ to $SU(N_\ell)$ gauge group. We further attach a chiral multiplet ($R_0=1$ and $\mathcal{F}=\sigma$) in the adjoint representation of the $SU(N)$ flavor symmetry of N hypers.

Then, the 't Hooft anomaly coefficients of R_0 and $\mathcal{F}$ of this theory are given as

$$\mathrm{Tr}R_0 = -\ell - \sum_{i=1}^{\ell} N_i \sum_{j=i}^{\ell} n_j, \quad \mathrm{Tr}R_0^3 = \frac{1}{4}\mathrm{Tr}R_0 + \frac{3}{4}\sum_{i=1}^{\ell}\left(N_i^2 - 1\right), \tag{6.38}$$

$$\mathrm{Tr}\mathcal{F} = -\sigma(2 + \mathrm{Tr}R_0), \quad \mathrm{Tr}\mathcal{F}^3 = \frac{1}{4}\mathrm{Tr}\mathcal{F} - \frac{3\sigma}{4}\left[\sum_{i=1}^{\ell}\left(N_i^2 - 1\right) - 2\left(N^2 - 1\right)\right], \tag{6.39}$$

$$\mathrm{Tr}R_0\mathcal{F}^2 = \frac{1}{4}\mathrm{Tr}R_0 - \frac{1}{4}\sum_{i=1}^{\ell}(N_i^2 - 1), \quad \mathrm{Tr}R_0^2\mathcal{F} = -\sigma\left(\mathrm{Tr}R_0 F^2 + \frac{1}{2}N^2\right). \tag{6.40}$$

Again we note that they satisfy (6.37).

After repeatedly applying the Seiberg dualities, we end up with an $\mathcal{N} = 2$ linear quiver attached to the Fan. The quiver has ℓ gauge nodes with $SU(N)$ gauge groups linked together by bifundamentals with $R_0 = 1/2$ and $\mathcal{F} = \sigma/2$. All gauge groups are $\mathcal{N} = 2$ vector multiplets except for the one at $k = 1$. The Fan (with $-\sigma$) is attached to this $k = 1$ $\mathcal{N} = 1$ gauge node. By subtracting the contribution of this quiver except for the Fan from (6.38), (6.39) and (6.40), we reproduce the anomaly coefficients (6.32) and (6.33). Note that for $\mathrm{Tr}\mathcal{F}$, $\mathrm{Tr}\mathcal{F}^3$ and $\mathrm{Tr}R_0^2\mathcal{F}$, there are overall sign differences from (6.32) and (6.33). This is because the Fan appeared here is specified by $-\sigma$.

6.5.2 Anomalies of class $\mathcal{S}$ theories

So far we have computed the anomaly coefficients of the Fan. In the class $\mathcal{S}$ point of view, the Fan with $\sigma = +1$ is associated to a sphere ($p = 1, q = 0$) with a maximal puncture with $\sigma = +1$, a minimal puncture with $\sigma = +1$ and a puncture labeled by Y with $\sigma = -1$ or the opposite choice. Here we will show that the anomaly coefficients can be given in terms of the data of the Riemann sphere and the punctures. By generalizing this observation, we will conjecture that the anomaly coefficients of the class-$\mathcal{S}$ theories can be written down as a sum of contributions from the following:

- Background contribution from the curve: $\mathcal{C}_{g,n}$ with normal bundle $\mathcal{L}(p) \oplus \mathcal{L}(q)$ specified. Here $p + q = 2g - 2 + n$ is imposed.
- Local contributions from each puncture $(\rho, \sigma)_{i=1,\ldots n}$.

If we write the number of punctures with color σ to be n_σ, $n = n_+ + n_-$ is the total number of punctures. We will first summarize the case of $\mathcal{N} = 2$ theories, which have been worked out in full generality by [56], and then give a generalization to the $\mathcal{N} = 1$ theories.

In the $\mathcal{N} = 2$ case, we always set $q = 0$ and $n_- = 0$ so that the total space becomes the cotangent bundle of the Riemann surface $\mathcal{C}_{g,n}$. All the punctures have the same color, thus they are specified entirely by the embedding of $SU(2)$ into Γ labeling the class $\mathcal{S}$ theory. For these $\mathcal{N} = 2$ theories, the number of effective vector multiplets n_v and hypermultiplets n_h can be used to determine the anomaly coefficients of the $\mathcal{N} = 2$ R-symmetries:

$$\mathrm{Tr} R_{\mathcal{N}=2} = \mathrm{tr} R^3_{\mathcal{N}=2} = 2(n_v - n_h), \qquad \mathrm{Tr} R_{\mathcal{N}=2} I_3^2 = \frac{n_v}{4}. \tag{6.41}$$

The quantities n_v, n_h are well-defined in the case of Lagrangian theories, but it is useful book-keeping device to use for non-Lagrangian theories as well.

For a given punctured Riemann surface, we can separate the contribution from the background Riemann surface and the punctures. For $\Gamma = A_{N-1}$, the background contribution for a genus g Riemann surface with n punctures is given by

$$n_h(\mathcal{C}_{g,n}) = \frac{2}{3}(2g-2+n)N(N^2-1), \tag{6.42}$$

$$n_v(\mathcal{C}_{g,n}) = \frac{1}{6}(2g-2+n)(4N^3-N-3). \tag{6.43}$$

Note that the definition of the background contribution is slightly different from the one in the literature by the terms including n. The factor $2g-2+n$ is the number of the pairs of pants, and this definition is more convenient to proceed to $\mathcal{N}=1$ class $\mathcal{S}$ theories.

For a puncture labeled by a Young diagram Y, (called regular punctures)

$$n_h(Y) = \frac{1}{2}\sum_r l_r^2 + \sum_{k=2}^{N}(2k-1)p_k - \frac{1}{6}(4N^3-N), \tag{6.44}$$

$$n_v(Y) = \sum_{k=2}^{N}(2k-1)p_k - \frac{1}{6}(4N^3-N-3), \tag{6.45}$$

where p_k labels the structure of the poles at the puncture (which can be read off from Y) [88] and l_r is the length of the r-th row of Y. For example, the maximal puncture has the pole structure $p_{\text{max}} = (0,1,2,\cdots,N-1)$ and the minimal puncture has $p_{\text{min}} = (0,1,1,\cdots,1)$. Note again that the last terms are absent in the definition in the literature. These compensate the changes in the background contributions. In general, one can also have irregular punctures as well, but we will not consider them here. For example, the maximal puncture has

$$n_h(Y_{\text{max}}) = 0, \quad n_v(Y_{\text{max}}) = -\frac{1}{2}(N^2-1), \tag{6.46}$$

and the minimal puncture has

$$n_h(Y_{\text{min}}) = -\frac{1}{6}(4N^3 - 6N^2 - 4N), \quad (6.47)$$

$$n_v(Y_{\text{min}}) = -\frac{1}{6}(4N^3 - 6N^2 - N + 3). \quad (6.48)$$

By summing altogether, n_h and n_v are

$$n_h = n_h(\mathcal{C}_{g,n}) + \sum_i n_h(Y_i), \qquad n_v = n_v(\mathcal{C}_{g,n}) + \sum_i n_v(Y_i). \quad (6.49)$$

Also, the flavor central charge of an $\mathcal{N} = 2$ theory is defined by

$$k\delta^{ab} = -2\text{tr}R_{\mathcal{N}=2}T^aT^b \quad (6.50)$$

where T^a is the generator of the flavor symmetry.

We now define the $\mathcal{N} = 1$ version of n_h and n_v. Let σ_i be the sign of the i-th puncture. They are given by

$$\hat{n}_h = \hat{n}_h(\mathcal{L}^{p,q}) + \sum_i \sigma_i n_h(Y_i) \ , \qquad \hat{n}_v = \hat{n}_v(\mathcal{L}^{p,q}) + \sum_i \sigma_i n_v(Y_i) \ , \quad (6.51)$$

where

$$\hat{n}_h(\mathcal{L}^{p,q}) = \frac{2}{3}(p - q)N(N^2 - 1) \ , \quad (6.52)$$

$$\hat{n}_v(\mathcal{L}^{p,q}) = \frac{1}{6}(p - q)(4N^3 - N - 3) \ . \quad (6.53)$$

Since we are considering $\mathcal{N} = 1$ theories, $\hat{n}_h$ and $\hat{n}_v$ do not have the interpretation of the effective numbers of hyper and vector multiplets. However, we continue to use these letters. In terms of these, our proposal for the 't Hooft anomaly coefficients are as follows:

$$\mathrm{Tr}R_0 = n_v - n_h, \quad \mathrm{Tr}R_0^3 = n_v - \frac{n_h}{4}, \tag{6.54}$$

$$\mathrm{Tr}\mathcal{F} = -(\hat{n}_v - \hat{n}_h), \quad \mathrm{Tr}\mathcal{F}^3 = -\hat{n}_v + \frac{\hat{n}_h}{4}, \tag{6.55}$$

$$\mathrm{Tr}R_0\mathcal{F}^2 = -\frac{n_h}{4}, \quad \mathrm{Tr}R_0^2\mathcal{F} = \frac{\hat{n}_h}{4}, \tag{6.56}$$

where n_h and n_v are (6.49) with $2g - 2 + n = p + q$.

In an $\mathcal{N} = 1$ theory which can be obtained from the $\mathcal{N} = 2$ one, we identify the R-symmetries as [185]

$$R_0 = \frac{1}{2}R_{\mathcal{N}=2} + I_3, \qquad \mathcal{F} = -\frac{1}{2}R_{\mathcal{N}=2} + I_3. \tag{6.57}$$

With these, the above anomaly coefficients (without hats) can be obtained by using (6.41). Then we changed n_v and n_h into $\hat{n}_v$ and $\hat{n}_h$ for the anomalies involving odd power of $\mathcal{F}$. We are proposing these formulae, however, for the theories which do not necessarily have the $\mathcal{N} = 2$ origin, like the Fan.

Let us check these formulae are indeed correct for a few theories.

Fan The Fan with $\sigma = +1$ is associated with a sphere with $p = 1$ and $q = 0$ and three punctures, maximal, minimal and the one specified by Y. Therefore, we get from (6.49) and (6.51),

$$\begin{aligned} n_v &= \sum_{k=2}^{N}(2k-1)p_k - \frac{1}{6}(4N^3 - 3N^2 - N), \quad n_v - n_h = -\frac{1}{2}(N^2 + \sum_r l_r^2), \\ \hat{n}_v &= -\sum_{k=2}^{N}(2k-1)p_k + \frac{1}{6}(4N^3 + 3N^2 - N - 6), \quad \hat{n}_v - \hat{n}_h = -\frac{1}{2}(N^2 + 2 - \sum_r l_r^2). \end{aligned}$$

It is straightforward to see that the anomaly coefficients obtained by substituting these into (6.56) agree with the ones from the direct computation (6.32) and (6.33), by using the identities $\sum_r l_r^2 = \sum_{i=1}^{\ell} N_i n_i$, $\sum_{k=2}^{N}(2k-1)p_k = \frac{N}{6}(4N^2 - 3N - 1) - \sum_i (N^2 - N_i^2)$, and (6.30).

$SU(N)$ **SQCD with** $N_f = 2N$ Now let us try to see how the formulae work in other class $\mathcal{S}$ theories. A simple example is SQCD with $N_f = 2N$ considered in section 6.4.2 which is associated with a sphere with two maximal punctures with $\sigma = +1$ and $\sigma = -1$ and two minimal punctures with $\sigma = +1$ and $\sigma = -1$ and also with $p = q = 1$. The anomalies are given by

$$\mathrm{Tr}R_0 = -N^2 - 1, \quad \mathrm{Tr}R_0^3 = \frac{N^2}{2} - 1, \quad \mathrm{Tr}R_0\mathcal{F}^2 = -\frac{N^2}{2}, \tag{6.58}$$

$$\mathrm{Tr}\mathcal{F} = \mathrm{Tr}\mathcal{F}^3 = \mathrm{Tr}R_0^2\mathcal{F} = 0. \tag{6.59}$$

These can also be computed directly from the matter content of the SQCD as in the table 6.4.

For completeness, let us compute the anomaly coefficients of non-Abelian symmetry. For the gauge symmetry, we have $\mathrm{Tr}R_0 T_g^2 = \mathrm{Tr}\mathcal{F}T_g^2 = 0$ indicating the vanishing exact beta function and anomaly-free $U(1)_{\mathcal{F}}$. The anomalies which involves $SU(N)$ flavor symmetries are as follows:

$$\mathrm{tr}R_0 T_1^2 = \mathrm{tr}R_0 T_2^2 = \mathrm{tr}\mathcal{F}T_1^2 = -\mathrm{tr}\mathcal{F}T_2^2 = -\frac{N}{2}, \tag{6.60}$$

where $T_{1,2}$ are the generators of $SU(N)_{1,2}$. Since there is no non-baryonic $U(1)$ symmetry, the $U(1)_{R_0}$ is the true R-symmetry in the IR.

Linear quiver We have computed in the previous section the 't Hooft anomaly coefficients of the linear quiver with $\mathcal{N}=2$ tail (6.38), (6.39) and (6.40). Let us reproduce these results from our formulae. The quiver (we fix $\sigma=1$) is associated with a sphere with $p=\ell$ and $q=1$ and $\ell+1$ minimal punctures with $\sigma=+1$, one maximal puncture with $\sigma=+1$ and a puncture specified by Y with $\sigma=-1$. It is easy to get

$$n_v = \sum_{i=1}^{\ell}(N_i^2-1), \quad n_v-n_h=-\ell-\sum_{i=1}^{\ell}N_i\sum_{j=i}^{\ell}n_j, \tag{6.61}$$

$$\hat{n}_v = \sum_{i=1}^{\ell}(N_i^2-1)-2N^2+2, \quad \hat{n}_h=-\ell+2-\sum_{i=1}^{\ell}N_i\sum_{j=i}^{\ell}n_j. \tag{6.62}$$

These reproduce (6.38), (6.39) and (6.40).

$\mathcal{N}=1$ gauging Let us consider a pair of class $\mathcal{S}$ theories, $\mathcal{T}_1$ and $\mathcal{T}_2$, each of which has an $SU(N)$ flavor symmetry. Let the colors of the maximal punctures be different and $\mathcal{T}_1$ and $\mathcal{T}_2$ be associated to a pair-of-pants decompositions where each color of the maximal puncture is the same as that of the pair-of-pants to which the puncture attached. Then let us think of gluing these punctures. This corresponds to the $\mathcal{N}=1$ gauging of the diagonal $SU(N)$ symmetry of two $SU(N)$ flavor symmetries of $\mathcal{T}_1$ and $\mathcal{T}_2$. The resulting theory is again in class $\mathcal{S}$.

The 't Hooft anomaly coefficients of the resulting theory are written as the sum of those of $\mathcal{T}_1$ and $\mathcal{T}_2$, and of $\mathcal{N}=1$ vector multiplet. The anomalies of the latter can be computed as

$$\mathrm{Tr}R_0 = \mathrm{Tr}R_0^3=N^2-1, \quad \mathrm{Tr}\mathcal{F}=\mathrm{Tr}\mathcal{F}^3=\mathrm{Tr}R_0^2\mathcal{F}=\mathrm{Tr}R_0\mathcal{F}^2=0. \tag{6.63}$$

These can be obtained from our formulae. Indeed from the Riemann surface point of view, the $\mathcal{N}=1$ gauging corresponds to subtracting two maximal punctures with different signs. Thus, we have $\delta n_v = N^2 - 1$, $\delta \hat{n}_v = \delta n_h = \delta \hat{n}_h = 0$. These reproduce (6.63).

$\mathcal{N} = 2$ **gauging** Instead, let us consider the gauging by an $\mathcal{N} = 2$ vector multiplet. Namely, consider $\mathcal{T}_1$ and $\mathcal{T}_2$ with maximal punctures whose colors are the same. $\mathcal{T}_i$ is associated to pants decomposition where the colors of the maximal puncture and of the pair-of-pants to which the puncture is attached are the same. Let us suppose the color is $\sigma = +$. In this case the 't Hooft anomalies are the sum of those of $\mathcal{T}_1$, $\mathcal{T}_2$ and of $\mathcal{N} = 2$ vector multiplet where the gauge adjoint chiral field has $R_0 = -\mathcal{F} = 1$. The latter contributes to the anomalies as

$$\mathrm{Tr}R_0 \quad = \quad \mathrm{Tr}R_0^3 = -\mathrm{Tr}\mathcal{F} = -\mathrm{Tr}\mathcal{F}^3 = N^2 - 1, \quad \mathrm{Tr}R_0^2\mathcal{F} = \mathrm{Tr}R_0\mathcal{F}^2 = 0. \quad (6.64)$$

Again this can be obtained from the formulae with $\delta n_v = \delta \hat{n}_v = N^2 - 1$ and $\delta n_h = \delta \hat{n}_h = 0$.

A theory coupled to an adjoint Let us consider the Riemann surface with a maximal puncture such that $\sigma_{Y_{max}}$ is different from the sign of the background. In [82], it was noticed that this represents a theory (associated to the same Riemann surface where the maximal puncture has the same sign as the bulk) coupled to a chiral multiplet M transforming in the adjoint representation of the $SU(N)$ flavor symmetry associated to the maximal puncture, by the superpotential $W = \mathrm{tr}\mu M$ where μ is the moment map of the $SU(N)$. The charges of M are $R_0 = 1$ and $\mathcal{F} = \sigma_{Y_{max}}$. (When the Riemann surface is a sphere with two maximal and a minimal punctures, this boils down to the Fan with Y is maximal.) Let us see this is consistent with our formula.

Suppose that the sign of the background is $+1$ and $\sigma_{Y_{max}} = -1$. Compared to the case where $\sigma_{Y_{max}} = +1$, $\hat{n}_v$ increases by $\delta \hat{n}_v = N^2 - 1$, while n_v, n_h and $\hat{n}_h$ are kept intact. Therefore the contribution of changing $\sigma_{Y_{max}}$ from $+1$ to -1 to the anomaly coefficients is

$$\delta \text{Tr}\mathcal{F} = 1 - N^2, \quad \delta \text{Tr}\mathcal{F}^3 = 1 - N^2. \tag{6.65}$$

while other coefficients remain to be the same. These are exactly the contribution of a chiral multiplet in the adjoint representation of $SU(N)$ with $\mathcal{F} = -1$ and $R_0 = 1$.

$\mathcal{N} = 1$ **Argyres-Seiberg dual theory** The dual theory is an $\mathcal{N} = 1$ SU(N) gauge theory coupled to the Fan specified by Y_{min} and to the T_N theory where a adjoint chiral multiplet is attached to a maximal puncture. By the class $\mathcal{S}$ interpretation of the anomaly coefficients, it is almost trivial to see that the anomalies of this dual theory agree with those of the SQCD, because they are represented by the same Riemann surface. Actually, we already show above that the anomaly coefficients of the Fan satisfies the formulae, and that attaching an adjoint field is interpreted as changing the sign of the puncture. Also, the anomalies of the T_N theory itself are written by using (6.49): $n_v^{T_N} = \frac{2N^3}{3} - \frac{3N^2}{2} - \frac{N}{6} + 1$, $n_h^{T_N} = \frac{2N^3}{3} - \frac{2N}{3}$.

For the anomaly coefficients of non-Abelian symmetries, we use the result of the contribution of the Fan and the T_N theory ($k^{T_N} = 2N$) to the flavor central charge (6.50). It is easy to show that these cancel for $\text{Tr}RT_g^2$ and $\text{Tr}\mathcal{F}T_g^2$. For the anomalies involving the flavor $SU(N)$, the Fan part does not contribute, thus reproducing (6.60) upon using (6.50).

6.6 Superconformal index

In this section, we compute the superconformal indices of the $\mathcal{N} = 1$ class $\mathcal{S}$ theories. The two-dimensional generalized TQFT structure ensures the invariance of index under various dualities we described. The generalized TQFT structure of the

$\mathcal{N}=1$ class $\mathcal{S}$ theories has been shown in [39], generalizing the $\mathcal{N}=2$ case studied in [83, 85, 86, 92]. In [82, 6], the prescription of adding adjoint chiral field for oppositely colored punctures has been shown to be consistent with the generalized TQFT structure of the $\mathcal{N}=1$ class $\mathcal{S}$ theories. In this section, we show that the matter content and charges of the Fan can be obtained by assuming the TQFT structure.

6.6.1 Review

The superconformal index for $\mathcal{N}=1$ theories is defined as

$$I(\mathfrak{p},\mathfrak{q},\xi;\vec{z}) = \mathrm{Tr}(-1)^F \mathfrak{p}^{j_1+j_2+R_0/2}\mathfrak{q}^{j_2-j_1+R_0/2}\xi^{\mathcal{F}}\prod_i z_i^{F_i} \ , \tag{6.66}$$

where j_1, j_2 are the Cartans of the Lorentz group $SU(2)_1\times SU(2)_2$ and F_i are generators of flavor symmetries. Strictly speaking, R_0 in the index has to be the exact R-charge in the IR. However in our case, we can simply keep it as R_0, as long as we keep the fugacity ξ turned on. After determining the amount of mixing through a-maximization, we can simply replace $\xi\to\xi(\mathfrak{p}\mathfrak{q})^{\epsilon/2}$ to get the correct R-charge $R=R_0+\epsilon\mathcal{F}$.

A good thing about the superconformal index is that it can be computed purely in terms of the matter content in the UV. The contribution for a chiral multiplet in a representation Λ of certain flavor or gauge group is given by

$$I_{\mathrm{chi}}(\mathfrak{p},\mathfrak{q},\xi;\vec{z}) = \mathrm{PE}\left[\frac{(\mathfrak{p}\mathfrak{q})^{R_0/2}\xi^{\mathcal{F}}\chi_\Lambda(\vec{z}) - (\mathfrak{p}\mathfrak{q})^{1-R_0/2}\xi^{-\mathcal{F}}\chi_{\bar{\Lambda}}(\vec{z})}{(1-\mathfrak{p})(1-\mathfrak{q})}\right] \ , \tag{6.67}$$

where $\chi_\Lambda(z)$ is the character of the representation Λ. The R_0 is the R_0-charge of the scalar in the chiral multiplet.

The chiral multiplet index (6.67) can be written in terms of elliptic Gamma function as

$$I_{\text{chi}}(\mathfrak{p},\mathfrak{q},\xi;\vec{z}) = \prod_{v\in\Lambda} \Gamma((\mathfrak{p}\mathfrak{q})^{R/2}\xi^{\mathcal{F}}\vec{z}^{v};\mathfrak{p},\mathfrak{q}) \ , \tag{6.68}$$

where Λ is the weight lattice of the representation. We use the notation $\vec{z}^{v} = \prod_i z_i^{v_i}$. Also,

$$\Gamma(z;\mathfrak{p},\mathfrak{q}) = \prod_{m,n\geq 0} \frac{1 - z^{-1}\mathfrak{p}^{m+1}\mathfrak{q}^{n+1}}{1 - z\mathfrak{p}^{m}\mathfrak{q}^{n}} \ , \tag{6.69}$$

is the elliptic Gamma function. For a vector multiplet, it contributes

$$I_{\text{vec}}^{\mathcal{N}=1}(\mathfrak{p},\mathfrak{q};\vec{z}) = ((\mathfrak{p};\mathfrak{p})(\mathfrak{q};\mathfrak{q}))^{r} \prod_{\alpha\in\Delta(G)} \Gamma(\vec{z}^{\alpha};\mathfrak{p},\mathfrak{q})^{-1} \ , \tag{6.70}$$

where $\Delta(G)$ is the set of all roots for the gauge group G, r being the rank of G and $(z;q) \equiv \prod_{m=0}^{\infty}(1 - zq^{m})$ is the q-Pochammer symbol.[7]

Generalized TQFT structure of the index The superconformal index of a class $\mathcal{S}$ theory given by a UV curve can be written in terms of pair-of-pants (or three-punctured sphere) and cylinders connecting them. For a pair-of-pants (or three-punctured sphere) with maximal punctures with colors σ_i, we can write the index as

$$I_{(\sigma,\sigma_i)}(\vec{a_1},\vec{a_2},\vec{a_3}) = \sum_{\vec{\lambda}} C_{\vec{\lambda}}^{\sigma}(\mathfrak{p},\mathfrak{q},\xi)\psi_{\vec{\lambda}}^{\sigma_1}(\vec{a_1})\psi_{\vec{\lambda}}^{\sigma_2}(\vec{a_2})\psi_{\vec{\lambda}}^{\sigma_3}(\vec{a_3}) \ , \tag{6.71}$$

for the σ-colored pair-of-pants. Here the sum is over the representations $\vec{\lambda}$ of $\Gamma \in ADE$ labeling the class $\mathcal{S}$ theory. The functions $C_{\vec{\lambda}}^{\sigma}(\mathfrak{p},\mathfrak{q},\xi)$ and $\psi_{\vec{\lambda}}^{\sigma_i}(\vec{a_i})$ are called the 'structure constant' and the wave function of the TQFT respectively. We omitted its dependence

[7] Here we also included the Haar measure factor to the $I_{\text{vec}}^{\mathcal{N}=1}(\mathfrak{p},\mathfrak{q};\vec{z})$.

on $(\mathfrak{p}, \mathfrak{q}, \xi)$. One of the key relation we use for the wave function is

$$\psi^{\sigma}_{\vec{\lambda}}(\vec{a}) = M^{\sigma}(\vec{a})\psi^{-\sigma}_{\vec{\lambda}}(\vec{a}) \ , \tag{6.72}$$

where

$$M^{\sigma}(\vec{a}) = \mathrm{PE}\left[\frac{(\xi^{\sigma} - \xi^{-\sigma})\sqrt{\mathfrak{pq}}}{(1-\mathfrak{p})(1-\mathfrak{q})}\chi_{\mathrm{adj}}(\vec{a})\right] = \Gamma(t_{\sigma}; \mathfrak{p}, \mathfrak{q})^{r} \prod_{i \neq j} \Gamma(t_{\sigma} a_i a_j^{-1}; \mathfrak{p}, \mathfrak{q}) \ , \tag{6.73}$$

where $t_{\sigma} = \xi^{\sigma}\sqrt{\mathfrak{pq}}$ and r is the rank of group Γ. It was shown in [39] that this wave function is essentially determined by the $\mathcal{N} = 2$ counterpart, which is given by an eigenfunction of elliptic Ruijsenaars-Schneider model [92]. More precisely, the relation between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ version of the wave function is given as

$$\psi^{\sigma}_{\vec{\lambda}}(\vec{a}; \mathfrak{p}, \mathfrak{q}, \xi) = \psi_{\vec{\lambda}}(\vec{a}; \mathfrak{p}, \mathfrak{q}, t = t_{\sigma}). \tag{6.74}$$

Also, the structure constant can be simply fixed from that of the $\mathcal{N} = 2$ couterpart as $C^{\sigma}_{\vec{\lambda}}(\mathfrak{p}, \mathfrak{q}, \xi) = C_{\vec{\lambda}}(\mathfrak{p}, \mathfrak{q}, t = \xi^{\sigma}\sqrt{\mathfrak{pq}})$.

The wave function can be written as $\psi^{\sigma}_{\vec{\lambda}}(\vec{a}) = K^{\sigma}(\vec{a})\Psi_{\vec{\lambda}}(\vec{a})$ where $K^{\sigma}(\vec{a})$ is a prefactor which does not depend on $\vec{\lambda}$ and $\Psi_{\vec{\lambda}}(\vec{a})$ is another function which depends on the representations $\vec{\lambda}$ of the group Γ. The prefactor is given by

$$K^{\sigma}(\vec{a}) = \mathrm{PE}\left[\frac{\xi^{\sigma}\sqrt{\mathfrak{pq}} - \mathfrak{pq}}{(1-\mathfrak{p})(1-\mathfrak{q})}\chi_{\mathrm{adj}}(\vec{a})\right] \ . \tag{6.75}$$

Note that the function $\Psi_{\vec{\lambda}}$ does not depend on color σ. In terms of these functions, we can write the index for a three-punctured sphere as

$$I_{(\sigma,\sigma_i)}(\vec{a_1}, \vec{a_2}, \vec{a_3}) = \frac{K^{\sigma_1}(\vec{a_1})K^{\sigma_2}(\vec{a_2})K^{\sigma_3}(\vec{a_3})}{K^{\sigma}(t^{\rho}_{\sigma})} \sum_{\vec{\lambda}} \frac{\Psi_{\vec{\lambda}}(\vec{a_1})\Psi_{\vec{\lambda}}(\vec{a_2})\Psi_{\vec{\lambda}}(\vec{a_3})}{\Psi_{\vec{\lambda}}(t^{\rho}_{\sigma})} \ , \tag{6.76}$$

where $t_\sigma^\rho = ((t_\sigma)^{\rho_1}, (t_\sigma)^{\rho_2}, \cdots, (t_\sigma)^{\rho_r})$ with ρ being the Weyl vector of the group Γ. When we glue the pair-of-pants by a gauge group, we integrate over the gauge fugacities with vector multiplet measure. Since we have

$$\int [da] I_{\text{vec}}^{\sigma\sigma'}(p,q;\vec{z})\psi_{\vec{\lambda}}^{\sigma}(\vec{z})\psi_{\vec{\lambda}'}^{\sigma'}(\vec{z}) = \delta_{\vec{\lambda}\vec{\lambda}'} \ , \tag{6.77}$$

where $I_{\text{vec}}^{\sigma\sigma'}(\mathfrak{p},\mathfrak{q};\vec{z})$ is $\mathcal{N}=2$ vector multiplet when $\sigma=\sigma'$ and $\mathcal{N}=1$ otherwise, we can write the superconformal index for any UV curve with colored full punctures as

$$I(\vec{a_i},\vec{b_j};\mathfrak{p},\mathfrak{q},\xi) = \sum_{\vec{\lambda}} \frac{\prod_{i=1}^{n_+}\psi_{\vec{\lambda}}^{+}(\vec{a_i})\prod_{j=1}^{n_-}\psi_{\vec{\lambda}}^{-}(\vec{b_j})}{\left(\psi_{\vec{\lambda}}^{+}(t_+^\rho)\right)^p\left(\psi_{\vec{\lambda}}^{-}(t_-^\rho)\right)^q} \ , \tag{6.78}$$

where (n_+,n_-) are the number of punctures of each color, and (p,q) are the degrees of the normal bundles satisfying $2g-2+(n_++n_-)=p+q$ and $\vec{a_i},\vec{b_j}$ are the flavor fugacities. As we see clearly, the index only depends on the topological data.

Now, if we choose the punctures to be non-maximal, we replace the fugacities in the wave function appropriately. The prescription is to replace

$$\Psi_{\vec{\lambda}}^{\sigma}(\vec{a}) \to \Psi_{\vec{\lambda}}^{\sigma}(\vec{u}t_\sigma^\Lambda) \ , \quad K^\sigma(\vec{a}) \to K_\Lambda^\sigma(\vec{u}) = \text{PE}\left[\sum_j \frac{t_\sigma^{1+j}-\mathfrak{p}\mathfrak{q}t_\sigma^j}{(1-\mathfrak{p})(1-\mathfrak{q})}\chi_{R_j}(\vec{u})\right] \ , \tag{6.79}$$

for a puncture labelled by the $SU(2) \hookrightarrow \Gamma$ embedding Λ which decomposes adj $\to \bigoplus_j R_j \otimes V_j$ where R_j is the representation of the commutant of $\Lambda(SU(2))$ in Γ and V_j is the spin-j representation of $SU(2)$. The notation $\vec{u}t^\Lambda$ means replacing the flavor fugacity appropriately in accordance with the broken flavor symmetry. See [156] for a detailed discussion on this notation and its physical meaning. We will give an example in the section 6.6.2, and then the full expression in 6.6.3.

6.6.2 $\mathcal{N}=1$ Argyres-Seiberg duality

The agreement of the index for the Argyres-Seiberg duality can be checked using the TQFT language as done in [82, 6]. In the SQCD frame as in figure 6.18 or figure 6.19a, the index can be written as

$$I(\vec{x}_1, \vec{x}_2, a, b) = \frac{K_\star^-(a)K^-(\vec{x}_1)K_\star^+(b)K^+(\vec{x}_2)}{K_\varnothing^- K_\varnothing^+} \sum_{\vec{\lambda}} \frac{\Psi_{\vec{\lambda}}(at_-^\star)\Psi_{\vec{\lambda}}(\vec{x}_1)\Psi_{\vec{\lambda}}(bt_+^\star)\Psi_{\vec{\lambda}}(\vec{x}_2)}{\Psi_{\vec{\lambda}}(t_-^\varnothing)\Psi_{\vec{\lambda}}(t_+^\varnothing)}, \quad (6.80)$$

where $\star$ denotes the embedding associated to the minimal puncture. Here, all the $+$ colored contributions are coming from the functions with $+$ labels and vice versa since the color of the pair-of-pants is the same as the punctures. Here we denote fugacities of the flavor symmetry $SU(3)_1 \times SU(3)_2 \times U(1)_A \times U(1)_B$ to be $\vec{x}_1, \vec{x}_2, a, b$ respectively.

Now, we need to show that this index is the same in the Argyres-Seiberg frame as in the figure 6.19d. There, we have punctures with different color from the pair-of-pants. On the left-side of the figure, we have maximal punctures with each color and thus an adjoint chiral field N. On the right-hand side, we have two minimal punctures with each color, corresponding to an adjoint field but with a nilpotent vev imposed, giving a number of components that survive according to the $SU(2)$ embedding labelled by the puncture. In the case of generic ρ being adj $\to \bigoplus_j R_j \otimes V_j$, this contribution to the index is

$$M_\rho^\sigma(\vec{u}) = \mathrm{PE}\left[\sum_j \frac{t_\sigma^{1+j} - \mathfrak{p}\mathfrak{q}t_\sigma^{-1-j}}{(1-\mathfrak{p})(1-\mathfrak{q})}\chi_{R_j}(\vec{u})\right] . \quad (6.81)$$

This is coming from the shift of R-charge $R_0 \to R_0 + 2\rho(\sigma_3)$ under the Higgsing. Thus $M_\star^+$ represents the components appearing in the dual frame.

The index in the Argyres-Seiberg dual frame can be written as

$$M^-(\vec{x}_1)M^+_\star(b)\frac{K^-_\star(a)K^-(bt^\star_+)K^+(\vec{x}_1)K^+(\vec{x}_2)}{K^+_\varnothing K^-_\varnothing}\sum_{\vec{\lambda}}\frac{\Psi_{\vec{\lambda}}(at^\star_-)\Psi_{\vec{\lambda}}(bt^\star_-)\Psi_{\vec{\lambda}}(\vec{x}_1)\Psi_{\vec{\lambda}}(\vec{x}_2)}{\Psi_{\vec{\lambda}}(t^\varnothing_+)\Psi_{\vec{\lambda}}(t^\varnothing_-)}\ . \tag{6.82}$$

The first two terms are coming from the additional fields in the dual theory and the signs of K^σs are determined by the color of the pair-of-pants. From the identity [82]

$$M^\sigma_\Lambda(\vec{u})K^{-\sigma}(\vec{u}t^\Lambda_\sigma) = K^\sigma_\Lambda(\vec{u})\ , \tag{6.83}$$

we see that the (6.80) and (6.82) are equal. This shows consistency of the TQFT description of the superconformal index of class $\mathcal{S}$ theories.

This agreement from the TQFT was quite formal, and works for any kind of puncture. We should be able to calculate this index in the Argyres-Seiberg dual frame using the matter content we found in the previous section. This can be done by looking at the index of the unhiggsed theory and Higgsing to get the Argyres-Seiberg frame. Let us consider the Argyres-Seiberg frame before Higgsing the dual meson M, which is realized by two maximal punctures on the left with each color, and one maximal puncture with + color and one minimal puncture with − color as in the figure 6.21. This realizes T_N theory with one of $SU(N)$ gauged by $\mathcal{N}=1$ vector multiplet, and N fundamentals attached to it. We also have an adjoint field N associated to one of $SU(N)$ flavor symmetries on the T_N side, and another adjoint field M attached to the fundamentals. The index of this theory can be written as

$$\begin{aligned} I(\vec{x}_1,\vec{x}_2,\vec{y},b) &= \oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \Delta(\vec{z}) I_{\rm vec}^{\mathcal{N}=1}(\vec{z}) I_{T_N}^{+}(\vec{x}_1,\vec{x}_2,\vec{z}) \prod_{i,j=1}^{N} \Gamma(t_-^{\frac{1}{2}}(z_i y_j b)^{\pm}) \\ &\quad \times \left(\Gamma(t_-)^{N-1} \prod_{i\neq j} \Gamma(t_- x_{1,i} x_{1,j}^{-1})\right) \left(\Gamma(t_+)^{N-1} \prod_{i\neq j} \Gamma(t_+ y_i y_j^{-1})\right) , \end{aligned} \quad (6.84)$$

where we used the short-hand notation $\Gamma(z) = \Gamma(z;\mathfrak{p},\mathfrak{q})$, and $\prod_{i=1}^{N} z_i = 1$ is assumed. The symbol I_{T_N} refers to the index of the T_N block and $I_{\rm vec}^{\mathcal{N}=1}$ is the $\mathcal{N}=1$ vector multiplet contribution to the index and $\Delta(\vec{z})$ is the Haar measure of the gauge group. The last term in the first line is the contribution from the fundamental quarks with $R_0 = \frac{1}{2}, \mathcal{F} = -\frac{1}{2}$. The second line corresponds to the contributions from the fields N and M respectively.

Now, upon Higgsing, we specialize the fugacity $\vec{y}$ to the ones determined from the $SU(2)$ embedding $N \to (N-1)+1$. For our case, we will have to substitute $\vec{y} = (at_+^{\frac{N}{2}-1}, at_+^{\frac{N}{2}-2}, \cdots, at_+^{1-\frac{N}{2}}, a^{-N+1})$. Then the last term in the first line of (6.84) becomes

$$\prod_{i=1}^{N} \left[\left(\prod_{m=1}^{N-1} \Gamma((\mathfrak{pq})^{\frac{1}{4}} \xi^{-\frac{1}{2}} (z_i a (\xi\sqrt{\mathfrak{pq}})^{\frac{N}{2}-m} b)^{\pm}) \right) \Gamma((\mathfrak{pq})^{\frac{1}{4}} \xi^{-\frac{1}{2}} (z_i a^{-N+1} b)^{\pm}) \right] , \quad (6.85)$$

where the terms in the parenthesis can be written as

$$\prod_{m=1}^{N-1} \Gamma((\mathfrak{pq})^{\frac{1+N-2m}{4}} \xi^{\frac{-1+N-2m}{2}} z_i ab) \prod_{m'=1}^{N-1} \Gamma((\mathfrak{pq})^{\frac{1-N+2m'}{4}} \xi^{\frac{-1-N+2m'}{2}} (z_i ab)^{-1}) . \quad (6.86)$$

Due to the identity $\Gamma(z;\mathfrak{p},\mathfrak{q})\Gamma(\frac{\mathfrak{pq}}{z};\mathfrak{p},\mathfrak{q}) = 1$, all the terms with $m = m'-1$ are cancelled. The only remaining terms are the ones with $(m,m') = (N-1,1)$. Therefore, (6.85) can be written as

$$\prod_{i=1}^{N} \Gamma((\mathfrak{pq})^{\frac{3-N}{4}} \xi^{\frac{1-N}{2}} (z_i ab)^{\pm}) \Gamma((\mathfrak{pq})^{\frac{1}{4}} \xi^{-\frac{1}{2}} (z_i a^{-N+1} b)^{\pm}) , \quad (6.87)$$

which is the contribution from the quarks of $(J_+, J_-) = (2-N, 1), (0,1)$ or $(R_0, \mathcal{F}) = (\frac{3-N}{2}, \frac{1-N}{2}), (1,-1)$. We see that the index can be used to extract the matter content and the charges of the Higgsed theory.

Contribution from M upon Higgsing is determined through the minimal $SU(2)$ embedding

$$\text{adj} \quad \to \quad \left(\bigoplus_{m=1}^{N-1} V^0_{m-1} \right) \oplus V^{-N}_{\frac{N-2}{2}} \oplus V^{N}_{\frac{N-2}{2}} \ , \tag{6.88}$$

where the supersubscript means the charge of the commuting $U(1)$. From this, we get

$$M^+_\star(a) \quad = \quad \text{PE}\left[\sum_{m=1}^{N-1} \frac{(\mathfrak{pq})^{\frac{m}{2}} \xi^m - (\mathfrak{pq})^{1-\frac{m}{2}} \xi^{-m}}{(1-\mathfrak{p})(1-\mathfrak{q})} + \frac{(\mathfrak{pq})^{\frac{N}{4}} \xi^{\frac{N}{2}} - (\mathfrak{pq})^{1-\frac{N}{4}} \xi^{-\frac{N}{2}}}{(1-\mathfrak{p})(1-\mathfrak{q})} (a^N + a^{-N}) \right] . \tag{6.89}$$

From here, we see that we have mesons with $(J_+, J_-) = (2m, 0)$ or $(R_0, \mathcal{F}) = (m, m)$ with $m = 1, \cdots, N-1$ and two mesons with $(J_+, J_-) = (N, 0)$ or $(R_0, \mathcal{F}) = (N/2, N/2)$ which are exactly the same as that of our result in the section 6.4.2.

6.6.3 Index of the Fan

We can repeat the similar procedure for a generic Fan as in the section 6.6.2. Consider a partition of N given by $\sum_{k=1}^{\ell} k n_k$ labelled by a Young diagram Y. For this partition, the flavor fugacity for the puncture is given as

$$\begin{aligned} \vec{u} t_\sigma^{\Lambda} \quad &= \quad (\vec{u}_1 t_\sigma^{\Lambda_1}, \vec{u}_2 t_\sigma^{\Lambda_2}, \cdots, \vec{u}_\ell t_\sigma^{\Lambda_\ell}) \ , \\ \vec{u}_k t_\sigma^{\Lambda_k} \quad &= \quad (\vec{u}_k t_\sigma^{\frac{k-1}{2}}, \vec{u}_k t_\sigma^{\frac{k-3}{2}}, \cdots \vec{u}_k t_\sigma^{\frac{1-k}{2}}) \ , \end{aligned} \tag{6.90}$$

where $\vec{u}_k = (u_{k,1}, u_{k,2}, \cdots, u_{k,n_k})$ is an n_k-dimensional vector for the $U(n_k)$ fugacities. We also impose the condition $\prod_{k=1}^{\ell} \prod_{i=1}^{n_k} u_{k,i} = 1$. This implies that the flavor symmetry is given by $S\left[\prod_{k=1}^{\ell} U(n_k)\right]$.

Plugging (6.90) into the index formula for N fundamental quarks, we get

$$\prod_{\alpha,\beta=1}^{N} \Gamma(\xi^{-1/2}(\mathfrak{pq})^{1/4}(z_\alpha y_\beta b)^{\pm}) \to \prod_{\alpha=1}^{N}\prod_{k=1}^{\ell}\prod_{i=1}^{n_k}\prod_{m=1}^{k} \Gamma(\xi^{-1/2}(\mathfrak{pq})^{1/4}(z_\alpha u_{k,i} t_+^{\frac{k-2m+1}{2}} b)^{\pm}) \ , \tag{6.91}$$

where we assumed that the Fan is of the type $\sigma = -$ as in the previous example for simplicity. As in the section 6.6.2, we see cancellations among upon Higgsing. The above equation can be written as

$$\prod_{\alpha=1}^{N}\prod_{k=1}^{\ell}\prod_{i=1}^{n_k}\left[\prod_{m=1}^{k} \Gamma(\xi^{\frac{k-2m}{2}}(\mathfrak{pq})^{\frac{2+k-2m}{4}} z_\alpha u_{k,i}) \prod_{m'=1}^{k} \Gamma(\xi^{\frac{-2+2m'-k}{2}}(\mathfrak{pq})^{\frac{2m'-k}{4}}(z_\alpha u_{k,i})^{-1})\right] . \tag{6.92}$$

We can see that the terms with $m' = m+1$ are cancelled so that only terms with $m = k$, $m' = 1$ contribute. Therefore, we get

$$I^{\text{quarks}}(\vec{z}, \vec{u}) = \prod_{\alpha=1}^{N}\prod_{k=1}^{\ell}\prod_{i=1}^{n_k} \Gamma(\xi^{-\frac{k}{2}}(\mathfrak{pq})^{\frac{2-k}{4}}(z_\alpha u_{k,i} b)^{\pm}) \ , \tag{6.93}$$

which is the contribution from the quarks of desired charges $(J_+, J_-) = (1-k, 1)$ or $(R_0, \mathcal{F}) = (\frac{2-k}{2}, -\frac{k}{2})$.

The contribution from the adjoint fields are given as (6.81). In the current case, the adjoint representation will decompose in to the form written as (6.21). Therefore, the index for the resulting components can be written as

$$
\begin{aligned}
M_Y^\sigma(\vec{u}) \;=\; & \prod_{i<j}\prod_{k=1}^{i} \mathrm{PE}\left[\frac{t_\sigma^{\frac{1}{2}(j-i+2k)} - (\mathfrak{pq})t_\sigma^{-\frac{1}{2}(j-i+2k)}}{(1-\mathfrak{p})(1-\mathfrak{q})}\Big(\chi_{R_i}(\vec{u}_i)\chi_{\bar{R}_j}(\vec{u}_j)+\chi_{\bar{R}_i}(\vec{u}_i)\chi_{R_j}(\vec{u}_j)\Big)\right] \\
& \times\left(\prod_{i=1}^{\ell}\prod_{k=1}^{i}\mathrm{PE}\left[\frac{t_\sigma^k-(\mathfrak{pq})t_\sigma^{-k}}{(1-\mathfrak{p})(1-\mathfrak{q})}\chi_{\mathrm{adj}}^{U(n_i)}(\vec{u}_i)\right]\right)\times \mathrm{PE}\left[\frac{t_\sigma - \mathfrak{pq}t_\sigma^{-1}}{(1-\mathfrak{p})(1-\mathfrak{q})}\right]^{-1}, \qquad (6.94)
\end{aligned}
$$

where the first term is coming from the bifundamentals of $U(n_i)\times U(n_j)$ and the second term is coming from the adjoints of $U(n_i)$ and the last piece takes care of the traceless condition. One can rearrange the first term by taking $i\to k-p$ so that the R-charges are given by $(R_0,\mathcal{F})=\frac{1}{2}(i+j-2p, i+j-2p)$ with $p=0,\cdots,\min(i,j)-1$. Likewise, the second term gives the adjoint fields of charge $(R_0,\mathcal{F})=(i-p,i-p)$ with $p=0,\cdots,i-1$ which agrees with the charges of the table 6.1.

Therefore, we find all the matter fields and charges as given in the table 6.1 for $N'=0$ case. Now, the index can be written in a contour integral form as

$$
I(\vec{x}_1,\vec{x}_2,\vec{y},\vec{u}) = M^-(\vec{x}_1)\int\prod_{i=1}^{N-1}\frac{dz_i}{2\pi i z_i}\Delta(\vec{z})I_{\mathrm{vec}}(\vec{z})I_{T_N}(\vec{x}_1,\vec{x}_2,\vec{z})I^{\mathrm{quarks}}(\vec{z},\vec{u})M_Y^\sigma(\vec{u}), \qquad (6.95)
$$

where $I^{\mathrm{quarks}}(\vec{z},\vec{u})$ and $M_Y^\sigma(\vec{u})$ are given by (6.93) and (6.94) respectively, representing the components of the Fan.

Contour of the index integral Let us make a comment on the integration contour of equation (6.82) and (6.95). Normally, for the purpose of evaluating the superconformal index, it is assumed that $|\mathfrak{p}|,|\mathfrak{q}|<1$ and all the flavor fugacities to be unimodular $|a|=1$. Typically, the poles are of the form $z=a(\mathfrak{pq})^{r/2}\mathfrak{p}^m\mathfrak{q}^n$ with $m,n\in\mathbb{Z}$ and R-charge of the chiral multiplet being $r>0$. Therefore we pick all the poles with $m,n\geq 0$. But if we use this prescription in the current case, we may hit a pole along the contour of integration.

Therefore we need to find a good contour to get away with this situation, because the usual contour of integration is not well-defined.

In order to understand the situation, let us go back to the procedure of evaluating the index. When we evaluate the index, we first count all the (gauge non-invariant) operators satisfying certain shortening condition formed out of elementary quarks and various matter multiplets in the theory. Then, we impose the gauge invariance condition or the Gauss law constraint by integrating over the gauge group with the Haar measure. From this perspective, we have to include contributions from every elementary field regardless of its R-charges. This Gauss law constraint should be imposed after rescaling a such that $|a(\mathfrak{pq})^{r/2-1}| = 1$ for any chiral multiplet of R-charge r with global symmetry fugacity a.

Higgsing procedure is consistent with this prescription. Prior to Higgsing, all the flavor fugacities were assumed to be unimodular. But when we Higgs, the flavor fugacities are dressed with $\mathfrak{p}, \mathfrak{q}$ and quite often it contributes negative powers in $\mathfrak{pq}$. Superficially, this makes us think that some of the poles with $m = n = 0$ are outside of the unit circle. As we have seen in the previous paragraph, due to the cancellation among the integrands, some of the poles are gone and the remaining poles under the Higgsing are those coming from the quarks in the Fan. But, note that all the Higgsed flavor fugacities $\vec{u}t_\sigma^\Lambda$ have to be unimodular. Even though superficially the poles appear to be outside of the unit circle, it is actually $a(\mathfrak{pq})^{\frac{r-1}{2}}$ that has modulus 1 with a being the flavor fugacity. Therefore, we have to include all the poles of the form $z = a(\mathfrak{pq})^{r/2}$ even for negative or zero r.

6.7 Conclusion

We studied nilpotent Higgsing in $\mathcal{N} = 1$ linear quiver gauge theories of class $\mathcal{S}$. In the case of $\mathcal{N} = 2$ theories such Higgsing yields regular punctures that can be associated to quiver tails labelled by partitions of N. Surprisingly, in $\mathcal{N} = 1$ linear quiver gauge theories, such Higgsing yields a new type of quiver dubbed as the Fan. This object is labelled by two integers N and N', and a partition of $N - N'$. We provided further evidence of the Fan by "Higgsing" the superconformal index.

Armed with the Fan, we constructed many new SCFTs. These provide various field theoretic descriptions of M5-branes wrapped on punctured Riemann surfaces. Under Seiberg duality, quivers with Fans will transform to new quivers with different Fans. Geometrically, this corresponds to different colored pair-of-pants decomposition of Riemann surface. Using the Fan, we find a new dual frame of $\mathcal{N} = 1$ $SU(N)$ SQCD with $2N$ flavors which is analogous to the Argyres-Seiberg duality. This dual frame is described by a T_N theory coupled to the Fan and chiral multiplets.

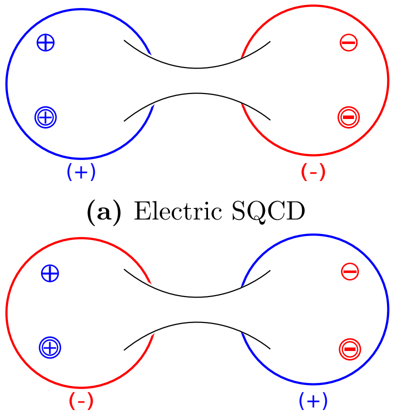
In our discussion, we only considered the UV curve with locally $\mathcal{N} = 2$ regular punctures. In $\mathcal{N} = 1$ class $\mathcal{S}$ theories, one could have much more general punctures. In terms of generalized Hitchin system [193, 49], we only considered the case where only one of two Hitchin fields become singular at a given puncture. It should be possible to consider the case where two Hitchin fields have singularities at the same point. This will yield genuinely $\mathcal{N} = 1$ punctures that we expect to be associated with a variation of the Fans. This is a work in progress.

We hope to find an intersecting brane realization of these new SCFTs in type IIA string theory, which can be uplifted to M-theory. It will also be interesting to find a gravity dual description of the Fan and its variations in M-theory by using the system of [31]. This is also a work in progress.

In this paper, we have not studied the detailed phase structure of the theory. The spectral curve approach from the generalied Hitchin system as done in [49, 196] should be useful. It would be also interesting to identify the Fan for the D, E-type theories of class $\mathcal{S}$, also possibly with outer-automorphism twists.

This chapter is a reprint of the material as it appears in "Quiver tails and $N = 1$

SCFTs from $M5$-branes ", Prarit Agarwal, Ibrahima Bah, Kazunobu Maruyoshi, Jaewon Song, JHEP 1503 (2015) 049, of which I was co-author.



(a) Electric SQCD **(b)** Magnetic SQCD **(c)** Swapped SQCD **(d)** Argyres-Seiberg dual

Figure 6.19. Colored pair-of-pants decompositions of the UV curve corresponding to the SQCD with $SU(N)$ gauge group and $2N$ flavors and its dual descriptions.

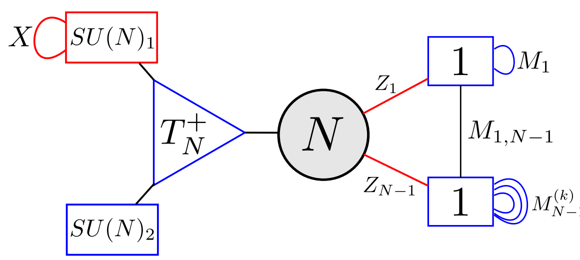


Figure 6.20. Analog of Argyres-Seiberg dual to the $\mathcal{N} = 1$ $SU(N)$ SQCD with $2N$ flavors.

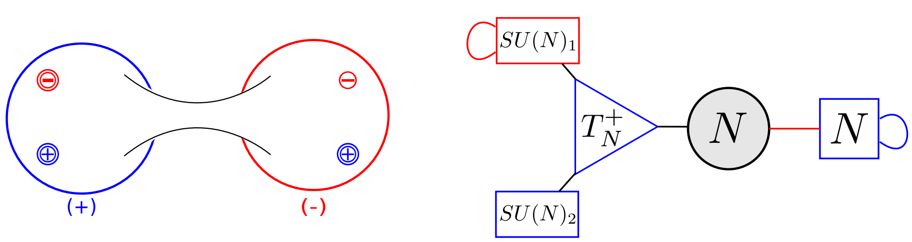


Figure 6.21. Unhiggsed SQCD in the Argyres-Seiberg frame

Chapter 7

Infinitely many $\mathcal{N} = 1$ dualities from $m + 1 - m = 1$

7.1 Introduction

Different 4d $\mathcal{N} = 1$ supersymmetric theories can RG flow to the same IR SCFT [171]. Such dual descriptions are not merely two similar UV completions of the same IR physics, but rather encode the IR physics quite differently, exchanging strong and weak coupling effects such as Higgsing and mass terms. The original duality of [171] relates the electric $SU(N_c)$ SQCD theory, with N_f flavors, to a magnetic $SU(N_f - N_c)$ theory, with N_f flavors and added meson singlets and superpotential.

We will be focussing on $SU(N_c)$ SQCD with $N_f = 2N_c$, where the gauge group is self-dual[1]. In [82], a new dual of $N_f = 2N_c$ SQCD was found, involving two copies of the T_N theory of [88] (see [184] for a nice, recent review), along with $2N^2 + 2N$ gauge singlets and a specific superpotential. In [4], another new dual of $N_f = 2N_c$ SQCD was found, involving a single T_N theory, two quarks/anti-quarks, $N^2 + N$ gauge singlets, and an intricate superpotential. For $N = 2$, the T_2 theory reduces to eight free chiral multiplets,

[1]Upon adding a quartic W_{tree} on the electric side, the theory is completely self-dual, as the meson singlets of the magnetic theory get a mass and can be integrated out. This theory can be obtained from the self-dual $\mathcal{N} = 2$ SQCD superconformal field theory with $N_f = 2N_c$, upon breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by an added mass term for the adjoint chiral superfield; see [147, 23] for discussion of the $\mathcal{N} = 1$ duality from this perspective.

the gauging can then be written as a standard Lagrangian, and the duals in this case reduces to ones analyzed in [125, 63].

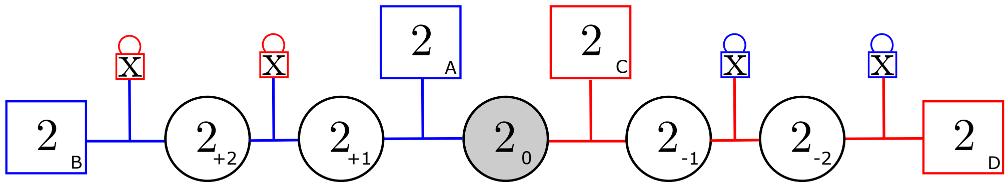
In this paper, we argue for the existence of two infinite classes of 4d $\mathcal{N} = 1$ theories, $T_N^{(m)}$ and $\mathcal{U}_N^{(m)}$, labelled by an arbitrary integer $m \geq 0$. $T_N^{(m)}$ theories are superconformal theories that have several duality frame representations. We argue that, for all m, $\mathcal{U}_N^{(m)}$ RG flow to the same IR fixed point SCFT as SQCD with $N_f = 2N_c \equiv 2N$ fundamentals and quartic superpotential

$$W = \lambda^{i\tilde{j};k\tilde{\ell}} M_{i\tilde{j}} M_{k\tilde{\ell}} \ , \tag{7.1}$$

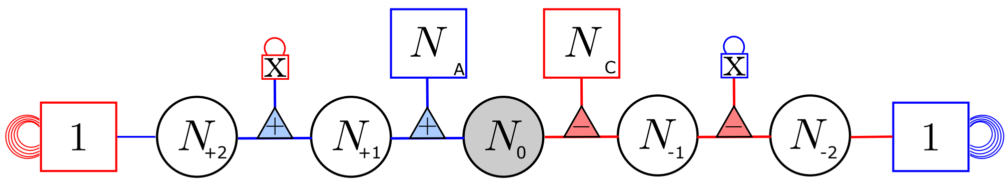
where $M_{i\tilde{j}} = Q_i \tilde{Q}_{\tilde{j}}$, and $\lambda^{i\tilde{j};k\tilde{\ell}}$ are chosen to preserve a $SU(N_c) \times SU(N_c) \times U(1) \times U(1)_B \subset SU(2N_c)_D \times U(1)_B \subset SU(N_f)_L \times SU(N_f)_R \times U(1)_B$; this is a one-complex dimensional conformal manifold of SCFTs. The $\mathcal{U}_N^{(m)}$ is a quiver gauge theory consisting of $2m+1$ gauge nodes and components constructed from T_N, along with a specific superpotential. The $m = 2$ case is illustrated in the the generalized quiver diagram of figure 7.1.

The $\mathcal{U}_N^{(m)}$ can be obtained by gluing (via gauging) two copies of the $T_N^{(m)}$ theories (when $N > 2$, we glue partially Higgsed $T_N^{(m)}$). The $T_N^{(m)}$ theories are new $\mathcal{N} = 1$ SCFTs, which like the $\mathcal{N} = 2$ T_N theories only have a Lagrangian description in the $N = 2$ case. Nevertheless, for all N, results can be obtained via holomorphy [173, 127], much as in [82, 152] for the T_N case. Also, a-maximization [128] enables us to determine exact R-charges of the chiral operators and the central charges. We thus compute the exact R charges, the anomaly coefficients, and the superconformal index [168, 142] of the $T_N^{(m)}$ and the $\mathcal{U}_N^{(m)}$ theories.

All of these theories have a natural description as being of class $\mathcal{S}$: the low-energy limit of the six-dimensional $\mathcal{N} = (2, 0)$ theory of type $\Gamma = A_{N-1}$, compactified on



(a) Quiver diagram for $\mathcal{U}_2^{(2)}$. The edges connecting the nodes denote bifundamental chiral multiplets. A small box with an 'x'-mark denotes a singlet chiral multiplet coupled to the bifundamental.



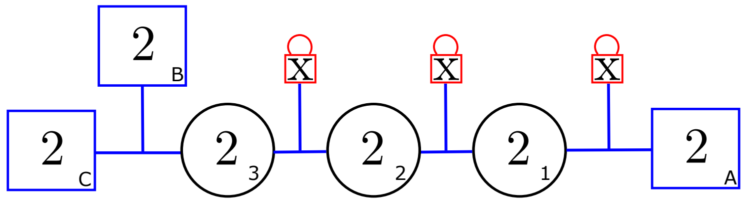
(b) Quiver diagram for $\mathcal{U}_N^{(2)}$. The triangle refers to the T_N theory. Here a small box with 'x'-mark refers to a certain deformation or Higgsing of the theory which breaks one of the $SU(N) \subset SU(N)^3$ global symmetries in T_N. There are gauge/flavor singlets as well.

Figure 7.1. Dual descriptions $\mathcal{U}_N^{(m)}$ of $SU(N)$ SQCD with $2N$ flavors. Here $m = 2$, where m refers to the number of white nodes on both sides of the black node in the middle. Black circular nodes denote $\mathcal{N} = 1$ vector multiplets, and white circular nodes denote $\mathcal{N} = 2$ vector multiplets. As usual, square nodes denote global symmetries.

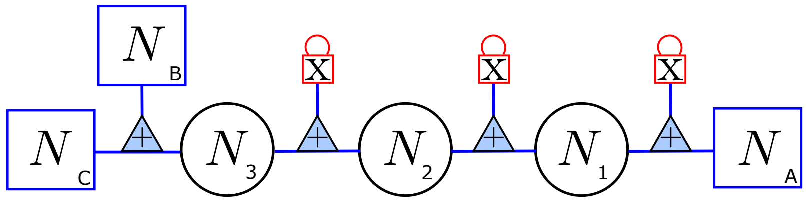
punctured Riemann surfaces $\mathcal{C}_{g,n}$, generalizing the 4d $\mathcal{N} = 2$ theories of [90, 88]. For the 4d $\mathcal{N} = 1$ theories, in addition to $\mathcal{C}_{g,n}$ (called the UV curve) we need to assign a pair of integers (p, q)

$$\mathcal{C}_{g,n}^{(p,q)} \equiv \mathcal{L}(p) \oplus \mathcal{L}(q) \to \mathcal{C}_{g,n}, \quad \text{with} \quad p + q = -\chi(\mathcal{C}_{g,n}) = 2g - 2 + n, \tag{7.2}$$

where $p \equiv c_1(\mathcal{L}(p))$ and $q \equiv c_1(\mathcal{L}(q))$ and the condition is to preserve $\mathcal{N} = 1$ supersymmetry [41, 33, 34, 193] (as discussed in these references, there are more general possibilities). From the 6d perspective, various dualities can be understood as arising from different choices of the (generalized) pair-of-pants decompositions of the same Riemann surface [153, 41, 37, 34, 39, 193, 35, 6, 4, 154]. For $\mathcal{N} = 1$ theories, when we decompose $\mathcal{C}_{g,n}$ into pants, the (p, q) integers are also decomposed into sums over the pants, with each pair of pants also satisfying (7.2), with $g = 0$ and $n = 3$.



(a) A quiver diagram describing the $T_2^{(3)}$ theory.



(b) A quiver diagram describing the $T_N^{(3)}$ theory.

Figure 7.2. Some examples of the quiver diagram describing the $T_N^{(m)}$ theories. In general, there is a number of dual descriptions for the $T_N^{(m)}$ theory itself.

Previous works on class $\mathcal{S}$ field theories restricted to $(p,q) \geq 0$, whereas here we consider cases with negative degree. In particular, our $T_N^{(m)}$ theory arises from reducing the 6d A_{N-1} $\mathcal{N} = (2,0)$ theory on the three-punctured sphere $\mathcal{C}_{0,3}$, with the line bundle degrees

$$T_N^{(m)}: \qquad \mathcal{L}(p) \oplus \mathcal{L}(q) \to \mathcal{C}_{g=0,n=3}, \qquad \text{with} \qquad (p,q) = (m+1, -m) \tag{7.3}$$

Some perspectives or expressions that are compatible with negative degree include gravity duals [33, 34, 31, 36, 32], superconformal indices [39] and generalilzed Hitchin system associated to the UV curve [193, 49, 196, 197, 194]. A possible objection to combining positive and negative degree pairs of bundles as in (7.3) is that they are unstable[2] to transitions $m \to m-1$, eventually reducing down to $m = 0$. We find that the $T_N^{(m)}$ theories are stable, but the $\mathcal{U}_N^{(m)}$ exhibit $m \to m-1$ cascade processes, via renormalization group flows in the associated 4d QFTs.

The 6d A_{N-1}, $\mathcal{N} = (2,0)$ theory on a 4-punctured sphere (with punctures being

[2]We thank Edward Witten for this remark.

appropriately decorated) gives

$$SU(N) \text{ SQCD with } N_f = 2N_c \text{ via } \quad \mathcal{L}(1) \oplus \mathcal{L}(1) \to \mathcal{C}_{g=0,n=4} \tag{7.4}$$

with the $SU(N)^2 \times U(1) \times U(1)$-preserving superpotential (7.1). Upon decomposing $\mathcal{C}_{g=0,n=4}^{p=1,q=1}$ into two pairs-of-pants, one can assign degrees as in (7.3), $(m+1, -m)$ to one and $(-m, m+1)$ to the other. This suggests $N_f = 2N_c$ SQCD is dual to theories labeled by general m, with a RG flow down to $m = 0$, leading to an infinite set of duals. We will flesh out this relation, and provide a number of checks. Among the checks is a matching of the superconformal index [66], which can be seen easily via the generalized TQFT structure studied in [39] and in [82, 4].

The outline of this paper is as follows. In section 7.2, we review the 4d $\mathcal{N} = 1$ SCFT in class $\mathcal{S}$ and show how to obtain the theories corresponding to general (p, q) through the nilpotent Higgsing. In section 7.3, we will discuss the construction of $T_2^{(m)}$ theory in detail. For the case of $\Gamma = A_1$, we always get a Lagrangian theory with $SU(2)$ gauge groups. From these building blocks, we show how to obtain the dual theories of $SU(2)$ SQCD. In section 7.4, we generalize the construction to $T_N^{(m)}$ which involves multiple copies of T_N theory. Using these building blocks, we construct dual theories of $SU(N)$ SQCD. In section 7.5, we compute the superconformal indices of the $T_N^{(m)}$ theory as further checks of our proposed dualities.

7.2 Four-dimensional $\mathcal{N} = 1$ SCFTs and dualities from M5-branes

In this section, we briefly review the $\mathcal{N} = 1$ class $\mathcal{S}$ theories, and our particular constructions.

7.2.1 Review of class $\mathcal{S}$ theories

For more detail, we refer to the papers [33, 34, 82, 193, 6, 4].

Data The $\mathcal{N}=1$ class $\mathcal{S}$ theories we consider are labelled by:

1. The choice of a 'gauge group' $\Gamma \in ADE$ of the 6d, $\mathcal{N}=(2,0)$ theory.
2. The choice of a Riemann surface $\mathcal{C}_{g,n}$ (UV curve) of genus g and n punctures.
3. The choice of the degree of line bundles (p,q) over $\mathcal{C}_{g,n}$ satisfying (7.2).
4. We decorate each of the punctures $i=1,\cdots n$ with an $SU(2)$ embedding ρ_i into Γ and a $\mathbb{Z}_2$-valued color σ_i.

We will here focus on $\Gamma = A_{N-1}$, though much of the discussion is valid for general Γ. The total space $\mathcal{C}_{g,n}^{(p,q)} \equiv \mathcal{L}(p)\oplus\mathcal{L}(q)\to\mathcal{C}_{g,n}$ in (7.2) is a local Calabi-Yau 3-fold, so M5-branes wrapped on the base $\mathcal{C}_{g,n}$ preserves 4 supercharges in the 11-dimensional M-theory. The fourth data labels the punctures that specify the global symmetry of the theory. Here we restrict to the class of punctures that we call the 'colored $\mathcal{N}=2$ punctures', since locally they are of the same type that appear in $\mathcal{N}=2$ class $\mathcal{S}$ theories [90, 88]. For $\Gamma=A_{N-1}$, the choice of ρ_i is in one-to-one correspondence with the choice of a partition of N, or equivalently a Young diagram of N boxes. The commutant of the $SU(2)$ embedding ρ_i gives the flavor symmetry associated with the i-th puncture.

Such $\mathcal{N}=1$ class $\mathcal{S}$ theories admit a $U(1)_+\times U(1)_-$ global symmetry [33], with generators (J_+,J_-), from those Cartans of the $SO(5)$ R-symmetry of the $\mathcal{N}=(2,0)$ theory that can be preserved after a partial topological twist on the UV curve. Defining

$$R_0 \equiv \frac{1}{2}\left(J_+ + J_-\right), \qquad \mathcal{F}\equiv \frac{1}{2}\left(J_+ - J_-\right) \tag{7.5}$$

R_0 is a $U(1)_R$ symmetry and $\mathcal{F}$ is a non-R global $U(1)$ symmetry. The exact superconformal R-symmetry is a linear combination

$$R_{\mathcal{N}=1} = R_0 + \epsilon\mathcal{F} \; = \frac{1+\epsilon}{2}J_+ + \frac{1-\epsilon}{2}J_-, \tag{7.6}$$

where ϵ is fixed by a-maximization [128]. For the case $p = q$, this gives $\epsilon = 0$.

Pair-of-pants decomposition and duality The pair-of-pants decomposition of (hyperbolic) $\mathcal{C}_{g,n}$ yields a way to build the theory, and find duals. One decomposes the total space $\mathcal{C}_{g,n}^{(p,q)}$, including the normal bundle degrees, with $p + q = 1$ for each pant ($g = 0$, $n = 3$). If one restricts to (p, q) both non-negative, the two options for each pant are $(1, 0)$ or $(0, 1)$, which are denoted by a coloring $\sigma = \pm$, with $\mathcal{C}_{g,n}^{(p,q)}$ then decomposed into p pants of color $\sigma = +$ and q pants with $\sigma = -$. Two pants of same color are glued with an $\mathcal{N} = 2$ vector multiplet, while pants of opposite colors are glued with an $\mathcal{N} = 1$ vector multiplet. See figure 7.3 for an illustration of the construction. Figure 7.4 gives the theory corresponding to the pair-of-pants decomposition in figure 7.3. Different pair-of-pants decompositions of $\mathcal{C}_{g,n}$ give IR dual theories.

Each puncture has a $SU(N)$ symmetry, which is unbroken if the puncture is maximal. In addition to the $\mathcal{N} = 1$ $SU(N)$ current multiplet, there is a $SU(N)$ adjoint-valued chiral superfield multiplet, μ (often called the "moment-map" operator). The $\mathcal{N} = 1$ current multiplet and μ combine to form the $\mathcal{N} = 2$ $SU(N)$ current multiplet when $\mathcal{N} = 2$ supersymmetry is preserved. When the two pants of the same color are glued, the diagonal combination of these $\mathcal{N} = 2$ $SU(N)$ currents is gauged. When there is an oppositely colored puncture on the pants, we also have extra chiral multiplet M in the adjoint of $SU(N)$, with a superpotential coupling $W = \mathrm{Tr}M\mu$, so M effectively replaces the role of μ via a Legendre transform.

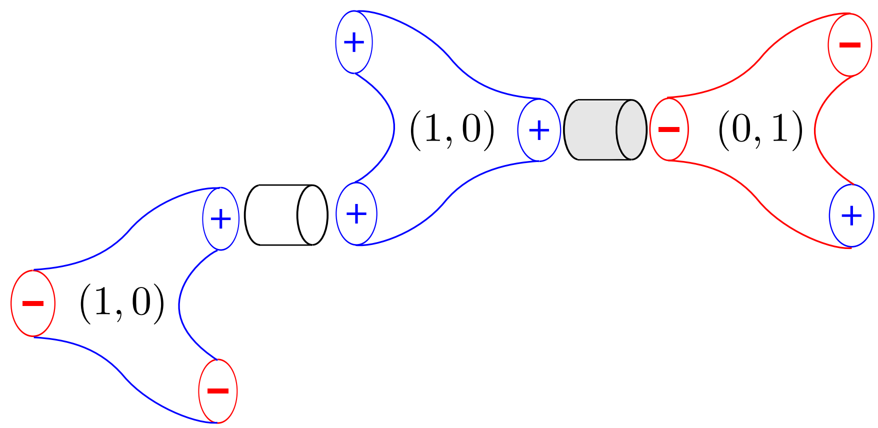


Figure 7.3. An example of colored pair-of-pants decomposition. Here red/blue means $\sigma = \pm$ respectively. Three red punctures and two blue punctures with $p = 2, q = 1$. Grey tube denotes $\mathcal{N} = 1$ vector, white tube denotes $\mathcal{N} = 2$ vector multiplet. There are 3 punctures of opposite color. There is an adjoint chiral multiplet attached to each of them.

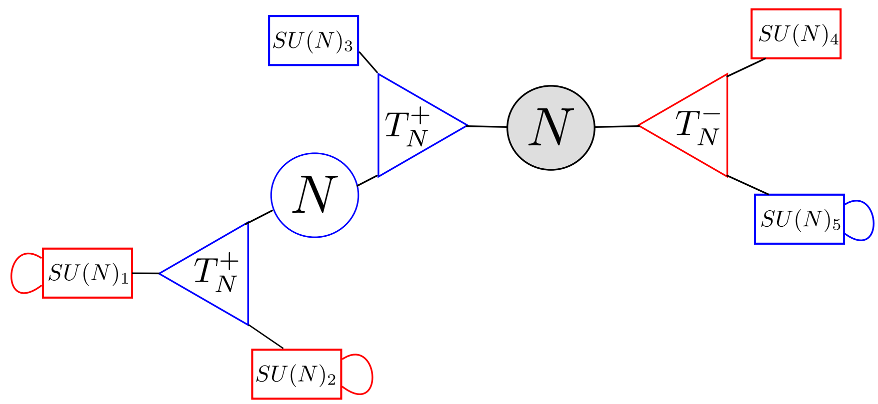


Figure 7.4. The UV description corresponding to the colored pair-of-pants description of figure 7.3. Here we assumed all punctures to be maximal.

Non-maximal punctures are labelled by an $SU(2)$ embedding ρ. We then partially close, or Higgs, the puncture by giving a nilpotent vev $\rho(\sigma^+)$ to μ if the color of puncture is the same as the pants, and to M if the puncture has the opposite color. This breaks the global symmetry associated to the puncture from $SU(N)$ to the commutant of the $\rho(SU(2))$ inside $SU(N)$. The building blocks corresponding to a sphere with generic three punctures can be identified from the previous works [53, 56] for the case of same colored puncture, and [82, 4] for the oppositely colored puncture.

7.2.2 General (p, q) class $\mathcal{S}$ theories from nilpotent Higgsing

We aim to find $\mathcal{N} = 1$ class $\mathcal{S}$ theories corresponding to $\mathcal{C}_{g,n}^{(p,q)}$ satisfying (7.2), here allowing for negative p or q. The idea is to start with a theory with positive degrees, $(p', q') \geq 0$, and obtain negative degrees via nilpotent Higgsing of the puncture. Following the prescription in [82, 4], for the case $\Gamma = A_{n-1}$, we can identify the Higgsed theory. For example, to get the three punctured sphere with degree $(m+1, -m)$, we start with a sphere with $m+3$ punctures, and line bundles of degree $(m+1, 0)$, with 3 $+$ punctures and m $-$ punctures. If we Higgs all m of the $-$ punctures, we are left with three $+$ punctures with degrees $(m+1, -m)$.

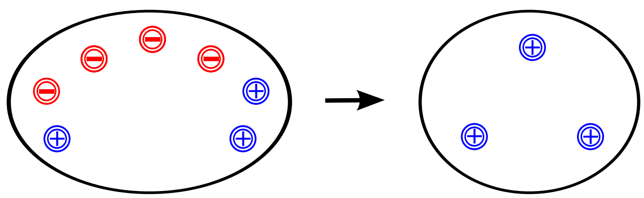
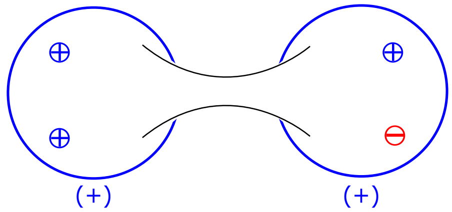
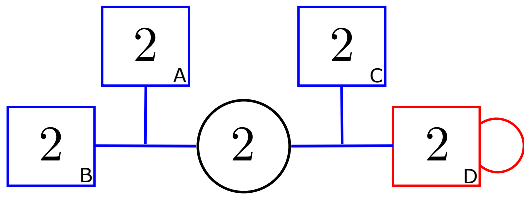


Figure 7.5. Higgsing the punctures to get the UV curve with lower degrees.

This procedure allows us to identify the theory corresponding to non-positive (p, q). In the following, we mainly focus on the three ($+$ colored) maximal punctured sphere with normal bundle degrees $(m+1, -m)$, which yields the $\mathcal{N} = 1$ theories that we denote by $T_N^{(m)}$, The $m = 0$ case reduces to the T_N theory of [88]. As we discuss, the $T_N^{(m)}$, theory can be constructed from gluing $m+1$ copies of the T_N theory with a number of singlet chiral multiplets and then Higgsing/closing the punctures. The closure of the puncture is implemented via giving a nilpotent vev to associated chiral adjoints M. This can thought of as a nilpotent mass deformation when $\Gamma = A_1$, i.e. for $N = 2$. We will discuss this in detail in later sections.



(a) A colored pair-of-pants decomposition for the 4-punctured sphere.



(b) The quiver diagram corresponding to the UV curve and the colored pair-of-pant decomposition on the left.

Figure 7.6. A colored pair-of-pants decomposition of $\mathcal{C}_{0,4}^{(2,0)}$, with $(n_+, n_-) = (3, 1)$ and its corresponding quiver diagram, see also [88]. Each node denotes $SU(2)$ global/gauge symmetries.

7.3 $SU(2)$ theories

Let us start with the $SU(2)$ case, coming from the 6d $\Gamma = A_1$ theory, and recall that the T_2 theory of [88] reduces to 8 free chiral multiplets. Likewise, there is a Lagrangian description for every (p, q). We first consider the $T_2^{(m)}$ theories, and then obtaining duals of $\mathcal{N} = 1$ $SU(2)$ SQCD with $N_f = 4$ flavors by gluing two copies of $T_2^{(m)}$.

7.3.1 The simplest example: $T_2^{(m=1)}$

To obtain the 3-punctured sphere with normal bundle degrees $(m + 1, -m) = (2, -1)$, we start with the UV curve $\mathcal{C}_{0,4}^{(2,0)}$ with $(n_+, n_-) = (3, 1)$ where $n_\pm$ denotes the number of $\pm$ punctures. Upon closing the $-$ puncture, we will obtain the UV curve $\mathcal{C}_{0,3}^{(2,-1)}$ with all $+$ punctures. Before closing the puncture, the Lagrangian description of the 4d $\mathcal{N} = 1$ theory is given as in figure 7.6. The field content of the theory is given as in the table 7.1.

Table 7.1. The field content of the theory corresponding to the curve $\mathcal{C}_{0,4}^{(2,0)}$

	$SU(2)_g$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	$SU(2)_D$	R_0	$\mathcal{F}$	(J_+, J_-)
ϕ	adj					1	-1	$(0,2)$
q_1	$\square$	$\square$	$\square$			$\frac{1}{2}$	$\frac{1}{2}$	$(1,0)$
q_2	$\square$			$\square$	$\square$	$\frac{1}{2}$	$\frac{1}{2}$	$(1,0)$
M'					adj	1	-1	$(0,2)$

Here $J_\pm$ are combinations of $R_0, \mathcal{F}$ defined so that $R_0 = \frac{1}{2}(J_+ + J_-)$ and $\mathcal{F} = \frac{1}{2}(J_+ - J_-)$. They are the 'candidate R-charges' which were used in [4]. The exact R-charge is given by a linear combination of the two, which is determined by a-maximization [128]. In terms of the quiver diagram 7.6b, $SU(2)_{A,B}$ refers to the blue flavor nodes on the left, and $SU(2)_C$ refers to the blue flavor node on the right, and $SU(2)_D$ corresponds to the red flavor node on the right. The theory has a superpotential $W = \text{Tr}\phi(q_1 q_1 + q_2 q_2) + \text{Tr}M' q_2 q_2$.

We now close the red puncture corresponding to $SU(2)_D$ by giving a nilpotent vev, $M' \sim \sigma^+$. This triggers a relevant RG flow, giving a mass to some components of the q_2 matter multiplet. Upon integrating them out, we obtain an IR SCFT described by the quiver diagram of figure 7.7. It can also be understood as the Fan corresponding to the partition $2 \to 2$ [4]. The matter content is given as in table 7.2 :[3] The remaining

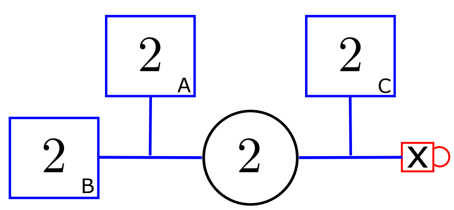


Figure 7.7. The quiver diagram for the $T_2^{(1)}$ theory. The 'x'-marked box denotes a closed puncture. It also means there is a singlet coupled to the quarks connected.

[3]It was shown in [82] that upon Higgsing a puncture labelled by $\rho : SU(2) \to \Gamma$ in the above manner, the (J_+, J_-) charges shift to $(J_+, J_- - \rho(\sigma^3))$, where ρ in this case is given by the identity map. This explains the charge assignments of 7.2.

Table 7.2. The field content of $T_2^{(m=1)}$

	$SU(2)_g$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	R_0	$\mathcal{F}$	(J_+, J_-)
ϕ	adj				1	-1	$(0, 2)$
q_1	$\square$	$\square$	$\square$		$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
q_2	$\square$			$\square$	0	1	(1, -1)
M					2	-2	(0, 4)

theory has superpotential

$$W = \mathrm{Tr}\phi q_1 q_1 + M\mathrm{Tr}q_2 q_2 \ , \tag{7.7}$$

which is generic for the global symmetry with $(J_+, J_-) = (2, 2)$ charges.[4]

The charged matter is that of $\mathcal{N} = 2$ $SU(2)$ with $N_f = 3$, but the theory is $\mathcal{N} = 1$ supersymmetric because one of the flavors does not couple to the adjoint, instead coupling to the gauge singlet M. This theory has a quantum moduli space of vacua, with several branches. The M field can have arbitrary expectation value, and $\langle M \rangle$ gives a mass to the q_2 field. The low-energy theory for $\langle M \rangle \neq 0$ thus has an accidental $\mathcal{N} = 2$ supersymmetry, given by $\mathcal{N} = 2$ with $N_f = 2$ flavors, with global symmetry $SU(2)_A \times SU(2)_B \times SU(2)_R \times U(1)_{\mathcal{R}}$. That theory has [172] a Coulomb branch, with modulus $u = \mathrm{Tr}\phi^2$, and two Higgs branches, emanating from the massless monopole and dyon points on the Coulomb branch, at $u \sim \pm\Lambda_L^2 \sim \pm M\Lambda$. Each Higgs branch is a copy of $\mathbf{C}^2/\mathbf{Z}_2$, and either $SU(2)_A$ or $SU(2)_B$ is spontaneously broken, depending on which branch. For $M \to 0$, the two Higgs branches meet at the origin of the Coulomb branch, with additional moduli from q_2, subject to the F-term $\mathrm{Tr}q_2 q_2 = 0$. It would be interesting to interpret this moduli space via geometric construction.

The IR theory at the origin of the moduli space is an $\mathcal{N} = 1$ interacting SCFT. It has a manifest $SU(2)^3$ flavor symmetry, with three $(J_+, J_-) = (2, 0)$ moment map chiral

[4]There are no terms of the form $\phi^2 q_2 q_2$, because $(\phi^2)_{\alpha\beta}(q_2)^{\alpha i}(q_2)^{\beta j}$ is identically zero and $\mathrm{tr}(\phi^2)\mathrm{Tr}(q_2 q_2)$ is not in the chiral ring due to the F-term for M.

operators, in the adjoint representations of $SU(2)_{A,B,C}$, given by

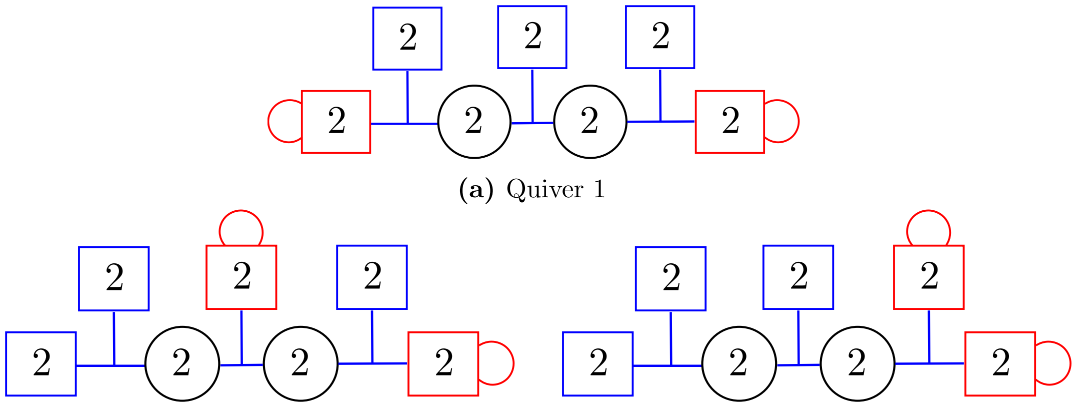
$$(\mu_A)_i{}^j = (q_1)_{\alpha ik}(q_1)^{\alpha jk}, \quad (\mu_B)_i{}^j = (q_1)_{\alpha ki}(q_1)^{\alpha kj}, \quad (\mu_C)_i{}^j = (q_2)_{\alpha i}\phi^{\alpha}{}_{\beta}(q_2)^{\beta j}. \tag{7.8}$$

The operator μ_C is dressed with the adjoint chiral multiplet ϕ to have the correct charges, $(J_+, J_-) = (2, 0)$. Despite the apparent difference between $\mu_{A,B}$ vs μ_C, the IR SCFT is expected to be S_3 permutation symmetric under permutation of the $SU(2)_{A,B,C}$ symmetries. Because the theory is $\mathcal{N} = 1$ supersymmetric and not $\mathcal{N} = 2$, these chiral operators are not in the $SU(2)_{A,B,C}$ current multiplets, and they receive anomalous dimension. The exact superconformal R-charge is as in (7.6), $R = R_0 + \epsilon\mathcal{F}$, and then chiral scalar operator dimensions are given by $\Delta(\mathcal{O}) = \frac{3}{2}R(\mathcal{O})$, e.g. $\Delta(\mu_{A,B,C}) = \frac{3}{2}(1+\epsilon)$, $\Delta(\mathrm{Tr}\phi^2) = 3(1-\epsilon)$, $\Delta(M) = 3(1-\epsilon)$, with ϵ determined via a-maximization to be[5] $\epsilon \simeq 0.52$. We find that the superconformal index computed from this gauge theory description agrees with the TQFT prediction of [39]. The index is compatible with the S_3 permutation symmetry.

7.3.2 $T_2^{(m=2)}$

We start from the theory corresponding $\mathcal{C}_{0,5}^{(3,0)}$ with $(n_+, n_-) = (3, 2)$ (unhiggsed theory) and then close the two $-$ punctures to obtain $\mathcal{C}_{0,3}^{(3,-2)}$. There are three different ways to do this, starting from the three dual frames of the unhiggsed theory as in the figure 7.8. The unHiggsed theory has $SU(2) \times SU(2)$ gauge group with bifundamental hypermultiplets and two more fundamentals attached to each of the gauge groups. The blue parts of the quiver are $\mathcal{N} = 2$ supersymmetric, with chiral adjoints ϕ for each gauge group and $\mathcal{N} = 2$ matter couplings. The red nodes are $\mathcal{N} = 1$ supersymetric, given by two chiral multiplets transforming as adjoints of the flavor groups, coupled via a

[5] It is outside of the bound $|\epsilon| \leq \frac{1}{3}$ found in [35], but here the operator dimensions are above the unitarity bound.



(a) Quiver 1

(b) Quiver 2

(c) Quiver 3

Figure 7.8. Three dual frames for the UV curve $\mathcal{C}_{0,5}^{(3,0)}$ and $(n_+, n_-) = (3, 2)$ where $n_\pm$ denotes the number of $\pm$ punctures respectively.

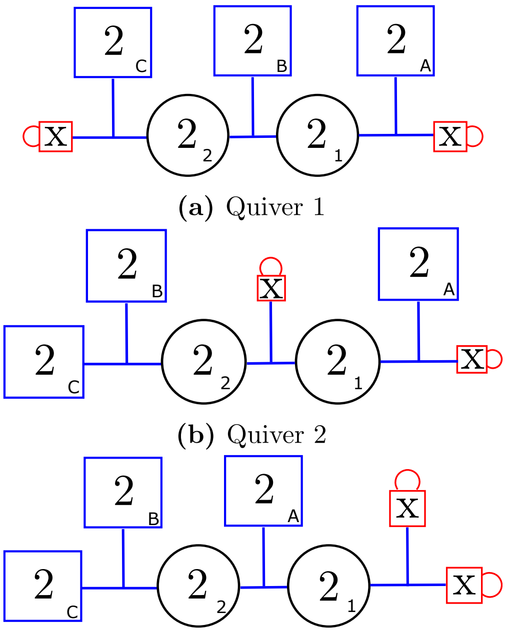
superpotential of the form

$$W_m = \sum_{a \in \text{red nodes}} \text{Tr} M_a \mu_a \ , \tag{7.9}$$

where μ_a is the gauge invariant bilinear of chiral multiplets, in the adjoint of the $SU(2)_a$ global symmetry. We then close the $-$ punctures by giving nilpotent vevs to the two chiral multiplets M_a attached to the $-$ punctures. This triggers a relevant deformation of the theory which leads to a new SCFT in the IR. Since the three different quivers are dual to each other before Higgsing, they all flow to the same SCFT in the IR.

The nilpotent M_a vev in quivers 1 and 2 gives rise to mass terms for some of the quarks, which we integrate out. Figure 7.9 describes the quiver after Higgsing. In the figure, an 'x'-marked box denotes the remnant of a closed puncture, where a gauge / flavor singlet component of M_a remains, with coupling to the remaining quarks in the theory. Quiver 3 requires a special treatment since the second nilpotent vev does not introduce a mass term.

Consider first quiver 1. The nilpotent M_a on the right/left-hand side gives the same type of the matter content as in the figure 7.7, with matter and charges as in table



(a) Quiver 1

(b) Quiver 2

(c) Quiver 3

Figure 7.9. Three dual frames corresponding to the UV curve $\mathcal{C}_{0,3}^{(3,-2)}$ and $(n_+, n_-) = (3, 0)$.

7.3 : The singlet field attached to the 'x'-marked box couples to the neighboring quarks, which gives rise to a cubic superpotential term similar to that in (7.7). In addition, there is a quintic coupling between the quarks and the adjoint chiral multiplets:

$$W_{\text{quiver 1}} = M_1 q_1 q_1 + M_3 q_3 q_3 + \phi_1 q_2 q_2 + \phi_1 q_1 q_1 q_2 q_2 + \phi_2 q_2 q_2 q_3 q_3 \tag{7.10}$$

Quiver 2 can be understood by considering a decoupling limit of the $SU(2)$ gauge group

Table 7.3. The matter content and charges of the quiver shown in Figure 7.9a

	$SU(2)_1$	$SU(2)_2$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	R_0	$\mathcal{F}$	(J_+, J_-)
ϕ_1	adj					1	-1	(0, 2)
ϕ_2		adj				1	-1	(0, 2)
q_1	$\square$		$\square$			0	1	(1, -1)
q_2	$\square$	$\square$		$\square$		$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
q_3		$\square$			$\square$	0	1	(1, -1)
$M_{1,2}$						2	-2	(0, 4)

Table 7.4. The matter content and charges of the quiver shown in Figure 7.9b

	$SU(2)_1$	$SU(2)_2$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	R_0	$\mathcal{F}$	(J_+, J_-)
ϕ_1	adj					1	-1	(0, 2)
ϕ_2		adj				1	-1	(0, 2)
q_1	$\square$		$\square$			0	1	(1, -1)
q_2	$\square$	$\square$				0	1	(1, -1)
q_3		$\square$		$\square$	$\square$	$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
$M_{1,2}$						2	-2	(0, 4)

corresponding to the rightmost gauge node. The left-hand side of the quiver is then the same as the $T_2^{(1)}$ theory. We list the matter content and charges of the theory in the table below: The superpotential for the quiver 2 is

$$W_{\text{quiver 2}} = M_1 q_1 q_1 + M_2 q_2 q_2 + \phi_1 q_2 \phi_2 q_2 + \phi_2 q_3 q_3 \ , \tag{7.11}$$

where we suppress gauge and flavor indices, which are as determined by the symmetry. The superpotential is generic given the $(J_+, J_-) = (2, 2)$ or $R_0 = 2$ and $\mathcal{F} = 0$ symmetry.

Non-mass deformation Let us consider quiver 3. When we close one of the $-$ punctures, we get a similar description as quiver 1 and 2. Now, we need to further close the $-$(red) $SU(2)$ puncture by giving a vev to the chiral flavor adjoint of say $SU(2)_0$. Before closing the last puncture, we have a superpotential term $\text{Tr} M_0 \phi_1 (q_0 q_0)$ where q_0 is the quark transforming as a fundamental of $SU(2)_0$, and ϕ_1 is the chiral adjoint of $SU(2)_0$. The nilpotent vev $\langle M_0 \rangle = \sigma^+$ then gives the deformation term $\text{Tr}\sigma^+ \phi_1 (q_0 q_0)$. Though not a mass term for the quarks, it nevertheless turns out to be a relevant deformation, breaking the $SU(2)_0$ global symmetry. To see that $\text{Tr}\sigma^+ \phi_1 (q_0 q_0)$ is relevant, note that it has charge $(J_+, J_-) = (2, 0)$ which means the exact R-charge (before the deformation) is $R = 1 + \epsilon$, which is relevant, $R < 2$, since a-maximization gives $\epsilon \simeq 0.46$. This gives $a \simeq 1.55$ before the deformation.

Table 7.5. The matter content of the quiver shown in Figure 7.9c

	$SU(2)_1$	$SU(2)_2$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	R_0	$\mathcal{F}$	(J_+, J_-)
ϕ_1	adj					1	-1	(0, 2)
ϕ_2		adj				1	-1	(0, 2)
q_0	$\square$					$-\frac{1}{2}$	$\frac{3}{2}$	(1, -2)
$\tilde{q}_0$	$\square$					$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
q_1	$\square$	$\square$	$\square$			$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
q_2		$\square$		$\square$	$\square$	$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
M_1, M_2						2	-2	(0, 4)

The $SU(2)_0$ breaking $\langle M_0 \rangle = \sigma^+$ yields a superpotential with terms

$$W \supset \mu_{\mathfrak{m}=-1} + \sum_{\mathfrak{m}=-1,0,1} \mu_{\mathfrak{m}} M_{-\mathfrak{m}} \ , \tag{7.12}$$

where $\mu_{\mathfrak{m}=-1,0,1} = \mathrm{Tr}\sigma_m \phi_1 q_0 q_0$ is in the adjoint of $SU(2)_0$. Much as in [82], the first term in (7.12) leads to $SU(2)_0$ current non-conservation for the $m = 0,\ 1$ components:

$$(\bar{D}^2 J)_{\mathfrak{m}} = \delta_{\mathfrak{m}} W = \mu_{\mathfrak{m}-1} \ , \tag{7.13}$$

so, for $m = 0, 1$, J_m and μ_{m-1} pair up to become long multiplets. The remaining superpotential is

$$W = \phi_1 \tilde{q}_0 \tilde{q}_0 + M_2(\phi_1 q_0 q_0) + M_1(q_0 \tilde{q}_0) + \phi_1 q_1 q_1 + \phi_2 q_1 q_1 + \phi_2 q_2 q_2 \ . \tag{7.14}$$

The charges (J_+, J_-) must be shifted to be conserved and unbroken

$$J_+ \to J_+, \qquad J_- \to J_- - 2\mathfrak{m} \ . \tag{7.15}$$

The matter content after Higgsing is as in Figure 7.9, with charges being as in table 7.5. We will consider similar type of deformations in section 7.4.

’t Hooft Anomalies The anomaly coefficients of $T_2^{(2)}$, in all three dual frames, are:

$$
\begin{array}{c|c}
\mathrm{Tr}J_+,\ \mathrm{Tr}J_+^3 & -2 \\
\mathrm{Tr}J_-,\, J_-^3 & -6 \\
\mathrm{Tr}J_+^2 J_- & 18 \\
\mathrm{Tr}J_+ J_-^2 & -18
\end{array}
\tag{7.16}
$$

a-maximization yields $\epsilon \simeq 0.534$ and $a \simeq 1.45$ for the $T_2^{(2)}$ theory in all three dual frames.

7.3.3 $T_2^{(m)}$

We can generalize previous subsection to construct a general $T_2^{(m)}$ theory. Start with the UV curve $\mathcal{C}_{0,m+3}^{(m+1,0)}$ with $(n_+, n_-) = (3, m)$. By closing all the $-$ punctures, we arrive at the sphere with 3 $+$ punctures and normal bundle degree $(m+1, -m)$. We can consider a number of different dual frames, but let us consider the analog of quiver 2 in figure 7.9. The resulting theory will be a quiver gauge theory, with $SU(2)^m$ gauge symmetry, bifundamental chiral multiplets for the neighboring nodes, and 2 fundamental chirals at the end nodes. In addition, we have adjoint chiral multiplets for each gauge nodes, and m gauge/flavor singlet chiral multiplets. We summarize the matter contents

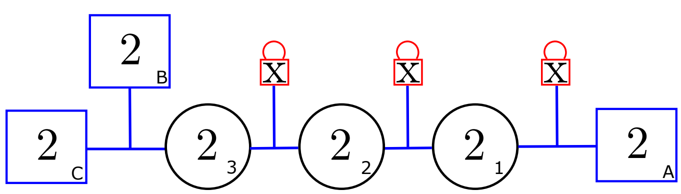


Figure 7.10. One of the dual frames describing the $T_2^{(3)}$ theory.

and their charges in the table 7.6. The superpotential is (with indices, and their contractions, suppressed)

$$
W = \sum_{i=1}^{m} M_i q_i q_i + \sum_{i=1}^{m-1} (\phi_i q_{i+1} \phi_{i+1} q_{i+1}) + \phi_m q_{m+1} q_{m+1} \ . \tag{7.17}
$$

Table 7.6. The matter content of $T_2^{(m)}$. Here $SU(2)_0$ is the flavor symmetry $SU(2)_A$.

	$SU(2)_{i-1}$	$SU(2)_i$	$SU(2)_B$	$SU(2)_C$	R_0	$\mathcal{F}$	(J_+, J_-)
ϕ_i $(1 \leq i \leq m)$		adj			1	-1	(0, 2)
q_i $(1 \leq i \leq m)$	$\square$	$\square$			0	1	(1, -1)
q_{m+1} $(i = m)$		$\square$	$\square$	$\square$	$\frac{1}{2}$	$\frac{1}{2}$	(1, 0)
M_i					2	-2	(0, 4)

The 't Hooft anomaly coefficients for this theory are

$$\begin{array}{c|c} J_+, J_+^3 & -m \\ J_-, J_-^3 & m-8 \\ J_+^2 J_- & 9m \\ J_+ J_-^2 & -9m \\ J_+ SU(2)^2_{A,B,C} & 0 \\ J_- SU(2)^2_{A,B,C} & -2 \end{array} \tag{7.18}$$

The trial R-charge $R = R_0 + \epsilon\mathcal{F} = \frac{1+\epsilon}{2} J_+ + \frac{1-\epsilon}{2} J_-$ yields the trial a-function

$$a(\epsilon) = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R) = \frac{1}{32}\left(3 + 3(19m+5)\epsilon - 27\epsilon^2 + (9-63m)\epsilon^3\right) \ . \tag{7.19}$$

The value of ϵ is fixed, by maximizing $a(\epsilon)$, to be

$$\epsilon(m) = \frac{-3 + \sqrt{133m^2 + 16m + 4}}{21m - 3} \lesssim 0.5492 \ . \tag{7.20}$$

As a check, $\epsilon(m=0) = \frac{1}{3}$ which is the value of the free field theory T_2. The central charge $a(\epsilon(m))$ grows linearly in m, which is not surprising from the quiver gauge theory perspective.

The $T_2^{(m)}$ theories do not have any exactly marginal deformations: there are

$m + (m-1) + 1 + m = 3m$ couplings from the terms in the superpotential (7.17), and the gauge couplings, and there is no linear relation among their beta functions. The conformal manifold is an isolated point; this is consistent with geometric construction, since the three punctured sphere has no complex structure modulus.

7.3.4 Infinitely namy $\mathcal{N} = 1$ duals for $SU(2)$ SQCD with 4 flavors

$\mathcal{N} = 1$ $SU(2)$ SQCD with 4 flavors can be realized by choosing the UV curve $\mathcal{C}_{0,4}^{(1,1)}$ with $(n_+, n_-) = (2, 2)$. The theory enjoys multiple dualities [125, 63] which also has a class $\mathcal{S}$ interpretation [82]. Moreover, this theory is known to have 72 dual frames [178, 65]. We now argue that gluing two copies of $T_2^{(m)}$ with an $\mathcal{N} = 1$ vector multiplet, for any integer $m \in \mathbb{Z}_{\geq 0}$, flows to the same SCFT as $SU(2)$ SQCD with 4 flavors. In the class $\mathcal{S}$ language, we have chosen two pairs-of-pants labelled by an integer m which gives the same 4-punctured sphere.

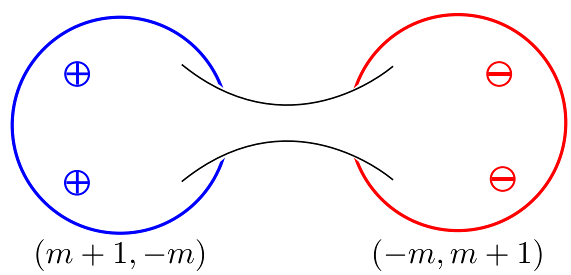


Figure 7.11. The 4-punctured sphere, with $(p, q) = (1, 1)$, via gluing two pair-of-pants of degrees $(m+1, -m)$ and $(-m, m+1)$. When $m = 0$, we get $SU(2)$ SQCD with 4 flavors. The pair-of-pants on the right gives $T_2^{(m)}$, but with reversed (J_+, J_-) charge assignments.

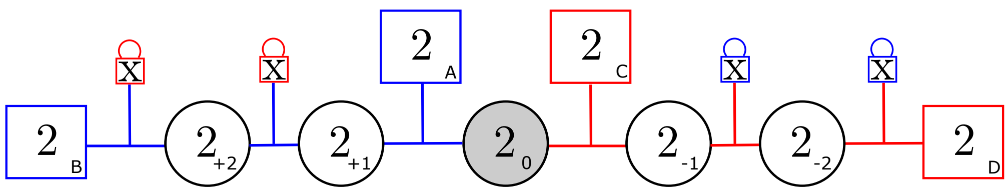
For $m = 0$, upon gauging an $SU(2)$, each $T_2^{(m=0)}$ factor contributes $N_f = 2$ flavors, and the resulting theory is $SU(2)$ with $N_f = 4$. More generally, for all m, the effective number of flavors contributed by each $T_2^{(m)}$ theory upon gauging $SU(2)_{X=A,B,C}$

global symmetries is given by the 't Hooft anomaly

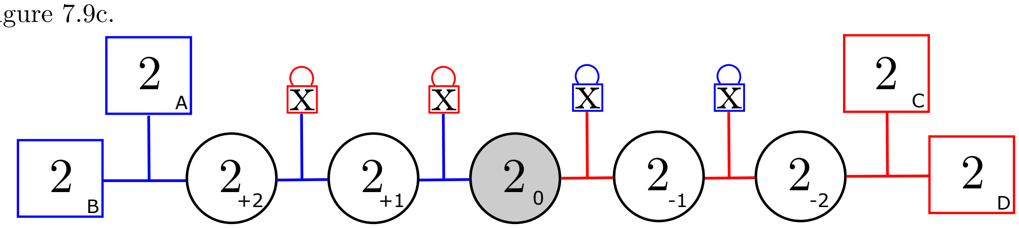
$$k = -3\mathrm{Tr}RSU(2)_X^2 = 3(1-\epsilon) \tag{7.21}$$

e.g. $\epsilon(m=0) = 1/3$ gives $k=2$; the gauged $SU(2)$ will be asymptotically free if $2k < 3N_c = 6$, which is satisfied for all m in (7.20).

There are several, dual descriptions of the resulting theory, corresponding to the dual descriptions of each pair-of-pants discussed in section 7.3.2. Let us pick the dual frame referred to there as quiver 2. As we claimed in section 7.3.2, there is a non-manifest S_3 permutation symmetry among the $SU(2)_{A,B,C}$ global symmetries. Correspondingly, there are two dual ways to gauge the the $SU(2)$ flavor group; see figure 7.12. Let us



(a) The $\mathcal{U}_2^{(2)}$ quiver, obtained by gauging the $SU(2)$ flavor group on the left-hand side of figure 7.9c.



(b) The $\widehat{\mathcal{U}}_2^{(2)}$ quiver, obtained by gauging the $SU(2)$ flavor group on the right-hand side of figure 7.9c.

Figure 7.12. Two different quivers obtained by gluing two copies of $T_2^{(2)}$. These quiver theories all flow to the same SCFT as $SU(2)$ SQCD with 4 flavors.

pick the dual frame shown in figure 7.12a. We will label duality frames of this type as $\mathcal{U}_2^{(m)}$. The matter content and their charges are given by two copies of $T_2^{(m)}$ where one copy has flipped (J_+, J_-) charges, as listed in the table. In addition to the added gauge

Table 7.7. The $\mathcal{U}_2^{(m)}$ matter content. $SU(2)_0^{\pm}$ is the gauge group at the center of the figure 7.12.

	$SU(2)_{i-1}^{\pm}$	$SU(2)_i^{\pm}$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	$SU(2)_D$	(J_+, J_-)
ϕ_i^+ $(1 \leq i \leq m)$		adj					(0, 2)
q_1^+ $(i = 1)$	$\square$	$\square$	$\square$				(1, 0)
q_i^+ $(2 \leq i \leq m)$	$\square$	$\square$					(1, -1)
q_{m+1}^+ $(i = m)$		$\square$		$\square$			(1, -1)
M_i^+							(0, 4)
ϕ_i^- $(1 \leq i \leq m)$		adj					(2, 0)
q_i^- $(i = 1)$	$\square$	$\square$			$\square$		(0, 1)
q_i^- $(2 \leq i \leq m)$	$\square$	$\square$					(-1, 1)
q_{m+1}^- $(i = m)$		$\square$				$\square$	(-1, 1)
M_i^-							(4, 0)

multiplet, we have a superpotential term

$$W = W_+ + W_- + \lambda_0 \mathrm{Tr} \mu_+ \mu_- \ , \tag{7.22}$$

where $\mu_{\sigma=\pm} = q_1^\sigma q_1^\sigma$ is the operator, with $(J_+, J_-) = (2,0)$ or $(0,2)$, associated to the glued punctures and superpotential (with gauge indices contracted and coupling constants λ)

$$W_\sigma = \sum_{i=1}^{m} \lambda_i^\sigma M_i^\sigma (q_{i+1}^\sigma q_{i+1}^\sigma) + \sum_{i=1}^{m-1} \tilde{\lambda}_i^\sigma \left(\phi_i^\sigma q_{i+1}^\sigma \phi_{i+1}^\sigma q_{i+1}^\sigma\right) + \lambda'_\sigma \phi_1^\sigma q_1^\sigma q_1^\sigma \ . \tag{7.23}$$

We argue that the $\mathcal{U}_2^{(m)}$ theories RG flow to the same IR fixed point as $N_f = 4$ $SU(2)$ SQCD, which is the $m = 0$ case of $\mathcal{U}_2^{(m)}$. As a first check, we find that the 't Hooft anomaly coefficients of the $\mathcal{U}_2^{(m)}$ quiver theory are m-independent:

$$\begin{array}{c|c} J_+, J_+^3, J_-, J_-^3 & -5 \\ J_+^2 J_-, J_+ J_-^2 & 3 \\ J_+ SU(2)^2_{A,B}, J_- SU(2)^2_{C,D} & 0 \\ J_- SU(2)^2_{A,B}, J_+ SU(2)^2_{C,D} & -2 \end{array} \tag{7.24}$$

The $U(1)_R$ at the superconformal fixed point is thus determined by a-maximization to be $R = R_0 = \frac{1}{2}(J_+ + J_-)$.

Matching of operators Among the single trace, gauge invariant operators of $\mathcal{U}_2^{(m)}$ are

$$\mu_A = q_1^+ q_1^+, \quad \mu_B = \phi_m^+ q_{m+1}^+ q_{m+1}^+, \quad \mu_C = q_1^- q_1^-, \quad \mu_D = \phi_m^- q_{m+1}^- q_{m+1}^- \tag{7.25}$$

in the adjoints of $SU(2)_{A,B,C,D}$ respectively, all with superconformal R-charge $R = 1$. These map to meson operators of $N_f = 4$ $SU(2)$ SQCD. The $N_f = 4$ $SU(2)$ SQCD theory has an $SU(8)$ global symmetry (though it is broken by (7.1) to $SU(2)^4$) with meson / baryon operators in the $\binom{8}{2}$ and the remaining meson/baryon operators are in the $(2, 2, 2, 2)$ of the $SU(2)_A \times SU(2)_B \times SU(2)_C \times SU(2)_D$ subgroup; these operators map to the $R = 1$ operators

$$q_{m+1}^- q_m^- \dots q_2^- q_1^- q_1^+ q_2^+ \dots q_m^+ q_{m+1}^+ \tag{7.26}$$

However, there initially appears to be a mismatch in our proposed duality between $\mathcal{U}_2^{(m)}$ and $N_f = 4$ $SU(2)$ SQCD: each of the white circle quiver nodes of $\mathcal{U}_2^{(m)}$ seems to contribute extra gauge singlet operators, $M_i^{\pm}$ and $u_i = \mathrm{tr}(\phi_i^{\pm})^2$, for $i = 1 \dots m$. Classically, these would lead to a mismatch with $N_f = 4$ $SU(2)$ SQCD, not only in the spectrum of operators, but also in the moduli space of vacua. Actually, as we now discuss, the quantum theory does not have the M_i and u_i classical moduli. They are quantum-lifted in a way similar to what happens in magnetic SQCD, where the classical electric condition $rank(M) \leq N$ arises from non-perturbative dynamics in the dual [171]. A vev of the would-be moduli would induce a dynamically generated superpotential, which is inconsistent with the F-term constraints.

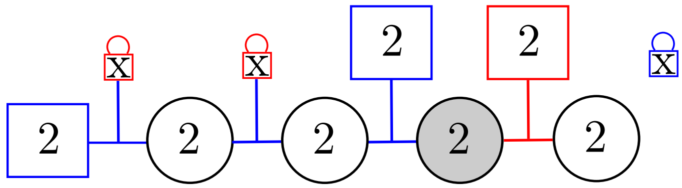


Figure 7.13. The effective theory after giving a vev to M_1^- or $\mathrm{tr}(\phi_2^-)^2$.

To see this in our setup, suppose first that some M_{n-1}^- has a non-zero vev, which spontaneously breaks J_+ and gives a mass to the quarks q_n^- from the first term of (7.23). This effectively decouples the side of the $\mathcal{U}_2^{(m)}$ quiver in with gauge group $SU(2)_{i\geq n}^-$, as in the figure 7.13. This gives $\mathrm{Tr}J_+(SU(2)_{n-1}^-)^2 \neq 0$, so the low-energy $SU(2)_{n-1}^-$ instanton factor $(\Lambda_{n-1,L}^-)^{b_L} \sim M_{n-1}^-$ has J_+ charge 4, which allows for superpotential terms

$$W_{\rm dyn} \supset \frac{M_{n-1}^-}{q_j^+ q_j^+} \tag{7.27}$$

consistent with the symmetries for all j. This would lead to a q_j^- runaway that is incompatible with $F_{M_i^-} = 0$, so the apparent M_{n-1}^- flat direction is actually lifted. Likewise, if u_n^- gets an expectation value, the associated non-zero ϕ_n^- spontaneously breaks J_+ and gives a relevant deformation from the second term of (7.23) (since $q_i^- \phi_{i-1}^- q_i^-$ has R-charge 1 or $(J_+, J_-) = (0, 2)$). In order to preserve J_+ symmetry in the IR, the charge of q_n^- becomes $(J_+, J_-) = (0, 1)$ and the $SU(2)_{n-1}$ instanton factor gets J_+ charged, $(\Lambda_{n-1,L}^-)^{b_L} \sim u_n^-$ so the theory admits

$$W_{\rm dyn} \supset \frac{u_n^-}{q_j^+ q_j^+} \ , \tag{7.28}$$

which has a runaway for $q_i^\pm$ that is incompatible with $F_{M_j^\pm}$, so the u_n flat direction is lifted. The superpotentials (7.27), (7.28) involves only the quarks on the other (+) side of the quiver, so this quantum effect is present when we couple two $T_N^{(m)}$ theories via $\mathcal{N} = 1$ vector multiplet, but not in the $T_N^{(m)}$ theory itself or when they are coupled via

$\mathcal{N}=2$ vector multiplet.

We give a refined check of operator matching through computing the superconformal index in section 7.5. The index of the $\mathcal{U}_2^{(m)}$ theory agrees with that of the SQCD, which provides a strong check of the duality. Therefore we conjecture that for every choice of m, the $\mathcal{U}_2^{(m)}$ theory flow to the same SCFT as SQCD in the IR.

Exactly marginal deformations $\mathcal{N}=1$ $SU(2)$ SQCD with 4 flavors has a large conformal manifold of exactly marginal deformations

$$W_{SQCD} = \lambda_{[ij];[kl]} M^{[ij]} M^{[kl]}, \qquad M^{[ij]} = Q^i Q^j, \qquad i,j = 1 \ldots 8, \tag{7.29}$$

including a one-complex dimensional line of fixed points which preserve $SU(2)^4$ flavor symmetry. This line of fixed points can also be seen in the $\mathcal{U}_2^{(m)}$ theory via the method of [147]. The exact NSVZ beta functions for the gauge couplings of $SU(2)_0$ and $SU(2)_i^{\pm}$ are (with g_i^σ the gauge couplings for $SU(2)_i^\sigma$)

$$\begin{aligned}
&\beta_{g_0} \propto -(2 + 2\gamma_{q_1^+} + 2\gamma_{q_1^-}) \ , \\
&\beta_{g_1^\sigma} \propto -(1 + 2\gamma_{\phi_1^\sigma} + 2\gamma_{q_1^\sigma} + \gamma_{q_2^\sigma}) \ , \\
&\beta_{g_i^\sigma} \propto -(2 + 2\gamma_{\phi_i^\sigma} + \gamma_{q_i^\sigma} + \gamma_{q_{i+1}^\sigma}) \ , \quad (i = 2, \cdots, m) \ .
\end{aligned} \tag{7.30}$$

The exact beta functions for the superpotential couplings are

$$\begin{aligned}
&\beta_{\lambda_0} \propto 1 + \gamma_{q_1^+} + \gamma_{q_1^-} \ , & &\beta_{\lambda_i^\sigma} \propto \frac{1}{2}\gamma_{M_i^\sigma} + \gamma_{q_i^\sigma} \ , \\
&\beta_{\tilde{\lambda}_i^\sigma} \propto 1 + \frac{1}{2}\gamma_{\phi_i^\sigma} + \frac{1}{2}\gamma_{\phi_{i+1}^\sigma} + \gamma_{q_{i+1}^\sigma} \ , & &\beta_{\lambda'_\sigma} \propto \frac{1}{2}\gamma_{\phi_1^\sigma} + \gamma_{q_1^\sigma} \ ,
\end{aligned} \tag{7.31}$$

where the anomalous dimension $\gamma_{\mathcal{O}}$ is given by $\Delta(\mathcal{O}) \equiv \Delta_{\text{classical}}(\mathcal{O}) + \frac{1}{2}\gamma_{\mathcal{O}}$. Since

$$\beta_{g_0} \propto \beta_{\lambda_0} \ , \tag{7.32}$$

Table 7.8. Matter contents of the $\widehat{\mathcal{U}}_2^{(m)}$ theory; $SU(2)_0^{\pm}$ is the shaded node in figure 7.12.

	$SU(2)_{i-1}^{\pm}$	$SU(2)_i^{\pm}$	$SU(2)_A$	$SU(2)_B$	$SU(2)_C$	$SU(2)_D$	(J_+, J_-)
ϕ_i^+ $(1 \leq i \leq m)$		adj					(0, 2)
q_i^+ $(1 \leq i \leq m)$	$\square$	$\square$					(1, -1)
q_{m+1}^+ $(i = m)$		$\square$	$\square$	$\square$			(1, 0)
M_i^+							(0, 4)
ϕ_i^- $(1 \leq i \leq m)$		adj					(2, 0)
q_i^- $(1 \leq i \leq m)$	$\square$	$\square$					(-1, 1)
q_{m+1}^- $(i = m)$		$\square$			$\square$	$\square$	(0, 1)
M_i^-							(4, 0)

the $\mathcal{U}_2^{(m)}$ theory has a one complex dimensional conformal manifold. This can also be seen via the the method of [102]. There are $6m + 2$ couplings, which break $U(1)^{(6m+2)-1}$ global symmetries (the -1 is because we preserve $U(1)_{\mathcal{F}}$), so there is a one-complex dimensional conformal manifold that preserves the $SU(2)^4 \times U(1)_{\mathcal{F}} \times U(1)_R$ global symmetry.

Cascading RG flow to SQCD The duality frame of figure 7.12b is the $\widehat{\mathcal{U}}_2^{(m)}$ theory, which we claim is dual to the $\mathcal{U}_2^{(m)}$ theory, giving another description of the theory obtained by gluing two copies of $T_2^{(m)}$. The $\widehat{\mathcal{U}}_m$ theory has superpotential term

$$W = W_+ + W_- + \mathrm{Tr}\mu_+\mu_- \ , \tag{7.33}$$

where $\mu_{\sigma=\pm} = \phi_1^\sigma q_1^\sigma q_1^\sigma$ is the operator with $(J_+, J_-) = (2,0)$ or $(0,2)$ associated to the punctures that we are gluing and (with implicit gauge index contractions)

$$W_{\sigma=\pm} = \sum_{i=1}^{m} M_i^\sigma (q_i^\sigma q_i^\sigma) + \sum_{i=1}^{m-1} \left(\phi_i^\sigma q_{i+1}^\sigma \phi_{i+1}^\sigma q_{i+1}^\sigma\right) + \phi_m^\sigma q_{m+1}^\sigma q_{m+1}^\sigma \ . \tag{7.34}$$

In this dual frame, the $SU(2)_0$ gauge group has $N_f = N_c$ and no adjoint, so it confines, with a quantum deformed moduli space constraint as in [173]. At energies below the $SU(2)_0$ dynamical scale, the $SU(2)_0$ node is eliminated, and its adjoining fundamentals are replaced with the $SU(2)_0$ neutral composites

$$V^+ = q_1^+ q_1^+, \quad V^{+-} = q_1^+ q_1^- \quad \text{and} \quad V^- = q_1^- q_1^- \ , \tag{7.35}$$

where V^+ and V^- (the $SU(2)$ analog of baryons) are gauge singlets, while the mesons V^{+-} transform as a bifundamental of $SU(2)_{+1} \times SU(2)_{-1}$, with the constraint [173]

$$\det(V^{+-}) - V^+ V^- = \Lambda_0^4 \ . \tag{7.36}$$

The superpotential (7.33) becomes (with implicit trace over gauge and flavor indices)

$$\begin{aligned} W =& \phi_1^+ V^{+-} \phi_1^- V^{+-} \\ &+ \sum_{\sigma=\pm} \left(M_1^\sigma V^\sigma + \sum_{i=2}^{m} M_i^\sigma q_i^\sigma q_i^\sigma + \sum_{i=1}^{m-1} \phi_i^\sigma q_{i+1}^\sigma \phi_{i+1}^\sigma q_{i+1}^\sigma + \phi_m^\sigma q_{m+1}^\sigma q_{m+1}^\sigma \right) . \end{aligned} \tag{7.37}$$

We see that $V^\pm$ combine with $M_1^\pm$ to become massive, so they can all be integrated out, setting $V^\pm = M_1^\pm = 0$. The quantum constraint on the moduli space (7.33) then implies that $V^{+-} \neq 0$. The non-zero $V^\pm$ bifundamental vev Higgses $SU(2)_{+1} \times SU(2)_{-1}$ to the diagonal $SU(2)$ subgroup. It follows from the superpotential (7.37) that $\phi_1^\pm$ become massive, and are integrated out. The resulting low-energy theory is thus similar to the original theory (shown in figure 7.14) with $m \to m-1$, i.e. it is $\widehat{\mathcal{U}}_2^{(m-1)}$. The above analysis applies to that theory, again reducing m, giving a cascading RG flow that eventually ends up at the $m = 0$ theory, $\widehat{\mathcal{U}}_2^{(0)}$, which is simply $SU(2)$ SQCD with $N_f = 4$.

The $\mathcal{U}_2^{(m)}$ dual to $\widehat{\mathcal{U}}_2^{(m)}$ thus also flows to the same IR SCFT as SQCD.

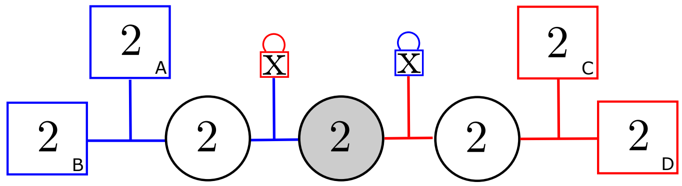


Figure 7.14. The low energy description of the theory in figure 7.12a at scales below Λ_0

7.4 $SU(N)$ theories

We here generalize the discussion in section 7.3 to $\mathcal{N}=1$ $SU(N)$ SQCD with $2N$ flavors. The new element is that we have to replace each bifundamental or trifundamental chiral multiplet, in the links of the quiver, by the T_N theory and its deformations. We first construct the $\mathcal{N}=1$ $T_N^{(m)}$ theories, which have $SU(N)_A \times SU(N)_B \times SU(N)_C$ flavor symmetry. We then glue two such theories with $\mathcal{N}=1$ vector multiplets to construct gauged $T_N^{(m)}$ theories. We argue that this flows to the same theory as obtained from gluing two T_N theories. Then we construct the $\widetilde{T}_N^{(m)}$ theory via partially Higgsing one of the punctures in $T_N^{(m)}$ theory so that we have $SU(N)^2 \times U(1)$ flavor symmetry. We then glue two such theories to obtain $\mathcal{U}_N^{(m)}$, and other dual versions, which give new dual descriptions of $SU(N)$ SQCD with $2N$ flavors.

7.4.1 Review of the T_N theory

Recall that the T_N theory is an $\mathcal{N}=2$ SCFT with $SU(N)_A \times SU(N)_B \times SU(N)_C$ flavor symmetry. The theory also has $\Delta = 2$ "moment-map" chiral operators, $\mu_{A,B,C}$, in the adjoint of the $SU(N)_{A,B,C}$ respectively. These operators satisfy the chiral ring relation [152]

$$\mathrm{tr}\mu_A^k = \mathrm{tr}\mu_B^k = \mathrm{tr}\mu_C^k \ , \tag{7.38}$$

for $k = 2, 3, \cdots N$. There are also operators $Q_{ijk}, \tilde{Q}_{ijk}$ which transform as the trifundamental and anti-trifundamental of $SU(N)_A \times SU(N)_B \times SU(N)_C$ with scaling dimension $N-1$. The T_N theory has a Coulomb branch of complex dimension $(N-2)(N-3)/2$, and a Higgs branch, which meet at the origin. See [152, 184] for more detailed discussion on the chiral ring operators and their relations of the T_N theory.

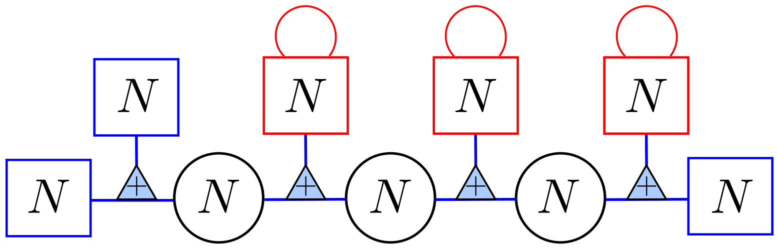
Since the T_N theory at the origin is a $\mathcal{N}=2$ SCFT, it has $U(1)_{R_{\mathcal{N}=2}} \times SU(2)_R$ symmetry. When we couple this theory to an $\mathcal{N}=1$ theory, we preserve $(J_+, J_-) = (2I_3, R_{\mathcal{N}=2})$, where I_3 is the Cartan generator of $SU(2)_R$. As in the previous section, one linear combination of J_+, J_- will become exact R-charge, and $\mathcal{F} = \frac{1}{2}(J_+ - J_-)$ will be a charge of the global symmetry of the theory. The $\mu_{A,B,C}$ operators have the charge $(J_+, J_-) = (2, 0)$, and $Q_{ijk}, \tilde{Q}_{ijk}$ have $(J_+, J_-) = (N-1, 0)$. The 't Hooft anomaly coefficients of the T_N theory are:

$$\begin{array}{c|c} J_+, J_+^3 & 0 \\ J_-, J_-^3 & -(N-1)(3N+2) \\ J_+^2 J_- & \frac{1}{3}(N-1)(N-2)(4N+3) \\ J_+ J_-^2 & 0 \\ J_+ SU(N)_{A,B,C}^2 & 0 \\ J_- SU(N)_{A,B,C}^2 & -N \end{array} \tag{7.39}$$

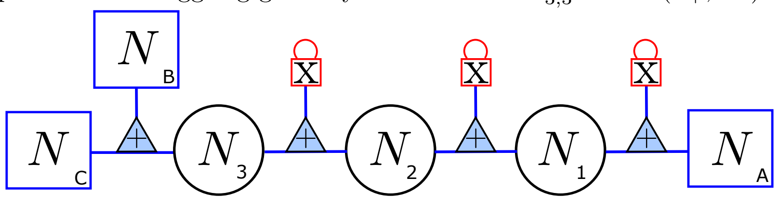
7.4.2 $T_N^{(m)}$ theory

We start with a $m+3$-punctured sphere with 3 $+$ punctures and m $-$ punctures and degrees $(p, q) = (m+1, 0)$. Here we assume all the punctures to be the maximal one carrying $SU(N)$ global symmetry. Let us choose the colored pair-of-pants decomposition so that we get the quiver as described in the figure 7.15a.

The theory is composed of $m+1$ copies of T_N theory that are connected via



(a) A quiver before Higgsing given by the UV curve $\mathcal{C}^{(3,0)}_{3,3}$ with $(n_+, n_-) = (3, 3)$.



(b) A quiver diagram for the $T^{(3)}_N$ theory, obtained by Higgsing three $-$ punctures above.

Figure 7.15. Quiver diagrams for the $T^{(3)}_N$ theory.

$\mathcal{N}=2$ vector multiplets and m extra chiral multiplets $M^{(i)}$ $(i = 1, \cdots, m)$ transforming under the adjoint of the $SU(N)_i$ global symmetry associated to the $-$ punctures. We denote the moment map operators of the $+$ colored operators by $\mu_{A,B,C}$ and those of $-$ colored operators by $\mu^{(i)}$ $(i = 1, \cdots m)$. We use ϕ_i for the adjoint chiral multiplets in the $\mathcal{N}=2$ vector multiplet and μ_k, $\tilde{\mu}_k$ for the moment map operators for the symmetry group $SU(N)_k$ that are being gauged. The superpotential is

$$W = \sum_{k=1}^{m} \mathrm{Tr}\phi_k(\mu_k - \tilde{\mu}_k) + \sum_{i=1}^{m} \mathrm{Tr}\mu^{(i)} M^{(i)} \ . \tag{7.40}$$

Now, we close the punctures by giving a nilpotent vev to M_i's as

$$\langle M^{(i)} \rangle = \rho(\sigma^+) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} , \tag{7.41}$$

where ρ is the principal embedding of $SU(2)$ into $SU(N)$. This will induce a relevant deformation to the theory which we name as $T_N^{(m)}$. Here we closely follow the discussion of [82]. We can decompose the adjoint representation of $SU(N)$ in terms of sum of the spin-j irreducible representation V_j of $SU(2)$ as $\text{adj} = \bigoplus_{j=1}^{N-1} V_j$. Using this, one can write each components of the adjoint of $SU(N)$ in terms of $(\mathfrak{j}, \mathfrak{m})$ with $\mathfrak{m} = -\mathfrak{j}, -\mathfrak{j}+1, \cdots, \mathfrak{j}-1, \mathfrak{j}$. After giving the vev, the superpotential can be written as

$$W = \sum_{k=1}^{m} \mathrm{Tr}\phi_k(\hat{\mu}_k - \hat{\mu}'_k) + \sum_{i=1}^{m} \left(\mu^{(i)}_{1,-1} + \sum_{\mathfrak{j},\mathfrak{m}} \mu^{(i)}_{\mathfrak{j},\mathfrak{m}} M^{(i)}_{\mathfrak{j},-\mathfrak{m}} \right) \ . \tag{7.42}$$

This superpotential preserves $(J_+, J_-) = (2, 2)$ upon the shift

$$J_+ \to J_+ \ , \qquad J_- \to J_- - \sum_i 2\mathfrak{m}^{(i)} \ , \tag{7.43}$$

where $\mathfrak{m}^{(i)}$ are the weights of the $SU(2)$ representations or the image of $J_3 = \sigma^3/2$ under ρ_i associated to each puncture (i) being closed. The vev breaks the original $SU(N)$ global symmetry, with the non-conservation of the current given by

$$(\bar{D}^2 J^{(i)})_{\mathfrak{j},\mathfrak{m}} = \delta_{\mathfrak{j},\mathfrak{m}} W = \mu^{(i)}_{\mathfrak{j},\mathfrak{m}-1} \ . \tag{7.44}$$

The semi-short multiplet $(J^{(i)})_{\mathfrak{j},\mathfrak{m}}$ and the chiral multiplet $\mu^{(i)}_{\mathfrak{j},\mathfrak{m}-1}$ combine into a long-multiplet. Therefore all the operators $M^{(i)}_{\mathfrak{j},-\mathfrak{m}}$ coupled to $\mu^{(i)}_{\mathfrak{j},\mathfrak{m}}$ decouple, except for $\mathfrak{m} = \mathfrak{j}$. Finally, the remaining superpotential is

$$W = \sum_{k=1}^{m} \mathrm{Tr}\phi_k(\hat{\mu}_k - \hat{\mu}'_k) + \sum_{i=1}^{m} \sum_{\mathfrak{j}=1}^{N-1} \mu^{(i)}_{\mathfrak{j},\mathfrak{j}} M^{(i)}_{\mathfrak{j},-\mathfrak{j}} \ . \tag{7.45}$$

We summarize the ‘matter content’ of the theory in the table 7.9.

Table 7.9. The 'matter content' of the $T_N^{(m)}$ theory.

	$SU(N)_i$	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	(J_+, J_-)
ϕ_i $(1 \leq i \leq m)$	adj				(0, 2)
μ_i $(1 \leq i \leq m)$	adj				(2, 0)
$\tilde{\mu}_i$ $(1 \leq i \leq m)$	adj				(2, 0)
μ_A		adj			(2, 0)
μ_B			adj		(2, 0)
μ_C				adj	(2, 0)
$\mu^{(i)}_{\mathrm{j,j}}$ $(1 \leq \mathrm{j} \leq N-1)$					$(2, -2\mathrm{j})$
$M^{(i)}_{\mathrm{j,-j}}$ $(1 \leq \mathrm{j} \leq N-1)$					$(0, 2\mathrm{j}+2)$

Anomaly coefficients To compute the 't Hooft anomaly coefficients of the $T_N^{(m)}$ theory, we need to compute effect of the Higgsed T_N block, with the nilpotent vev. Accounting for the above shifts, we find that we simply need to add the contributions from $M_{\mathrm{j},-\mathrm{j}}$ to that of the T_N theory. This gives, for the single puncture Higgsed T_N or equivalently the theory corresponding to the UV curve $\mathcal{C}_{0,2}^{(1,-1)}$:

$$
\begin{array}{c|c}
J_+, J_+^3 & 1-N \\
J_-, J_-^3 & (1-N)(2N+1) \\
J_+^2 J_- & \frac{1}{3}(N-1)(4N^2-2N-3) \\
J_+ J_-^2 & \frac{1}{3}(1-N)(4N^2+4N+3) \\
J_+ SU(N)^2_{Z,Z'} & 0 \\
J_- SU(N)^2_{Z,Z'} & -N
\end{array}
\tag{7.46}
$$

Combining this with the known results of the T_N theory and the quiver description depicted in figure 7.15 and the charges of the singlets as given in (7.9), we obtain the anomaly coefficients of the $T_N^{(m)}$ as follows:

$$\begin{array}{c|c} J_+, J_+^3 & m(1-N) \\ J_-, J_-^3 & (N-1)(m-3N-2) \\ J_+^2 J_- & \frac{1}{3}(N-1)(4N^2-5N-6+m(4N^2+4N+3)) \\ J_+ J_-^2 & \frac{1}{3}m(3+N-4N^3) \\ J_+ SU(N)^2_{A,B,C} & 0 \\ J_- SU(N)^2_{A,B,C} & -N \end{array} \tag{7.47}$$

Note that the anomalies involving the $SU(N)_{A,B,C}$ are the same as that of T_N theory. These coefficients can also be obtained from the formula given in the section 5.2 of [4] by extrapolating all the formulas to the negative p or q.

The trial a-function is

$$\begin{aligned} a(\epsilon) \;\; = \;\; & \frac{3}{64}(N-1)(1-\epsilon)\left(3N^2(\epsilon+1)^2 - 3N\left(2\epsilon^2+\epsilon+1\right) - 2\left(3\epsilon^2+3\epsilon+2\right)\right) \\ & + \frac{3}{32}m\epsilon\left(3N^3\left(\epsilon^2-1\right)+2N-3\epsilon^2+1\right) \ , \end{aligned} \tag{7.48}$$

and the value of ϵ is fixed by a-maximization to be

$$\begin{aligned} \epsilon = & \frac{-N^2-N}{3(2m(N^2+N+1)+N^2-2N-2)} \\ & + \frac{\sqrt{4m^2(N^2+N+1)(3N^2+N+1)+4m(3N^4-5N^2-5N-2)+(2N^2-N-2)^2}}{3(2m(N^2+N+1)+N^2-2N-2)} . \end{aligned}$$

For $m=0$, we find $\epsilon = \frac{1}{3}$, which is the expected value for the $\mathcal{N}=2$ T_N theory. The value of a increases linearly with respect to m and grows cubically with respect to N. We can also determine the $SU(N)$ flavor central charge $k_{SU(N)}$ [21, 20] to be

$$k_{SU(N)}\delta^{ab} = -3\mathrm{Tr}RT^aT^b = \frac{3}{2}(1-\epsilon)N\delta^{ab} \ . \tag{7.49}$$

Table 7.10. The 'matter content' of the gauged T_N theory. The $SU(N)$ in the first column denotes the gauge group.

	$SU(N)$	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	$SU(N)_D$	$U(1)_R$	$U(1)_{\mathcal{F}}$	(J_+, J_-)
μ^+	adj					1	1	$(2,0)$
μ^-	adj					1	-1	$(0,2)$
μ_A		adj				1	1	$(2,0)$
μ_B			adj			1	1	$(2,0)$
μ_C				adj		1	-1	$(0,2)$
μ_D					adj	1	-1	$(0,2)$

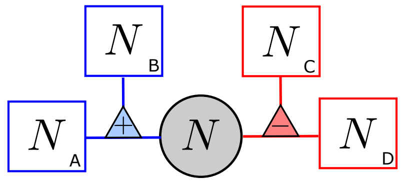
When $\epsilon = \frac{1}{3}$, $k_{SU(N)} = N$ which agrees with the known result of T_N theory. Since $\frac{1}{3} < \epsilon < \frac{1}{\sqrt{3}}$ for $m > 0$, we see the flavor central charge is less than N for $m > 0$. In many respect, the T_N theory behaves as N fundamental flavors [152] since it contributes the same amount to the beta function of the gauge coupling. For the $T_N^{(m)}$ case, it contributes to the beta function as that of $N_f < N$.

7.4.3 Infinitely many $\mathcal{N} = 1$ duals for gauged T_N theories

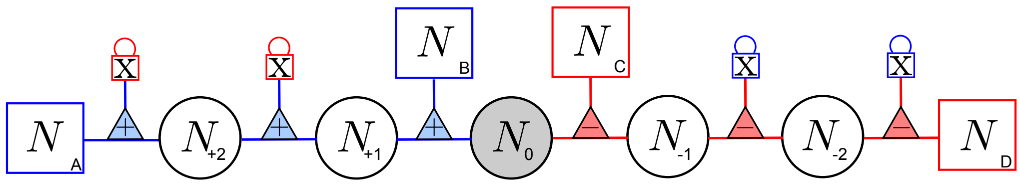
As a preparation of the SQCD, let us first consider the theory obtained by gluing two copies of T_N theory by gauging one of the $SU(N)$ flavor groups on each of T_N. It can be obtained from choosing the UV curve to be the 4-punctured (all maximal, 2 +, and 2 − colored) sphere with $(p, q) = (1, 1)$. See the figure 7.16a. This theory and its dualities have been studied in [41, 82] which we review here. This theory has $SU(N)_A \times SU(N)_B \times SU(N)_C \times SU(N)_D \times U(1)_{\mathcal{F}} \times U(1)_R$ global symmetry with the 'matter content' as given in the table 7.10.

For this theory, the superconformal R-charge is given by $R_0 = \frac{1}{2}(J_+ + J_-)$. The $\mu_{A,B,C,D}$'s are the operators present in the T_N theory, which are associated to the punctures on the UV curve. The operators $\mu^{\pm}$ are the operators corresponding to the punctures that we are gluing/gauging. We can write a superpotential term

$$W = \mathrm{tr}\mu^+\mu^- \ , \tag{7.50}$$



(a) Two T_N theories coupled by gauging the $SU(N)$ flavor symmetry subgroup with an $\mathcal{N}=1$ vector multiplet.



(b) A quiver description obtained by gauging the $SU(N)$ flavor group of two copies of the $T_N^{(2)}$ theory.

Figure 7.16. Different quiver descriptions for the 4 maximal-punctured sphere theory with $(p,q)=(1,1)$. Shaded circular nodes denote the $\mathcal{N}=1$ vector multiplets and unshaded nodes denote the $\mathcal{N}=2$ vector multiplets.

which preserves all the global symmetries of the theory.

Now let us describe the dual theories of the coupled T_N. We couple two copies of $T_N^{(m)}$ with an $\mathcal{N}=1$ vector multiplet to get the theory corresponding to the same 4-punctured (all maximal, 2 + and 2 − colored) sphere with $(p,q)=(1,1)$. When gluing the two theory with an $\mathcal{N}=1$ vector, the (J_+,J_-) charge assignment of one of the $T_N^{(m)}$ has to be flipped in order to write the superpotential term (7.50). See figure 7.16. The 'matter content' of the theory is given in the table 7.11.

The theory has a superpotential

$$W = W_+ + W_- + \mathrm{tr}\mu_0^+\mu_0^- \ , \tag{7.51}$$

where

$$W_\sigma = \sum_{k=1}^{m} \mathrm{Tr}\phi_k^\sigma(\mu_k^\sigma - \tilde{\mu}_k^\sigma) + \sum_{i=2}^{m+1}\sum_{\mathrm{j}=1}^{N-1} \mu_{\mathrm{j},\mathrm{j}}^{\sigma,(i)} M_{\mathrm{j},-\mathrm{j}}^{\sigma,(i)} \ . \tag{7.52}$$

Table 7.11. Matter contents of the quiver obtained by gluing two copies of $T_N^{(m)}$. Here $SU(N)_0^{\pm}$ is identified as the $SU(2)$ gauge group at the center of the figure 7.16b. The operators $\mu_{\mathrm{j},-\mathrm{j}}^{\pm,(i)}$ are the ones in the i-th T_N block in the quiver. Here $\mathrm{j} = 1, 2, \cdots, N-1$.

	$SU(N)_i^{\pm}$	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	$SU(N)_D$	(J_+, J_-)
ϕ_i^+ $(1 \leq i \leq m)$	adj					(0, 2)
μ_i^+ $(0 \leq i \leq m)$	adj					(2, 0)
$\tilde{\mu}_i^+$ $(1 \leq i \leq m)$	adj					(2, 0)
μ_A		adj				(2, 0)
μ_B			adj			(2, 0)
$\mu_{\mathrm{j},\mathrm{j}}^{+,(i)}$ $(2 \leq i \leq m+1)$						$(2, -2\mathrm{j})$
$M_{\mathrm{j},-\mathrm{j}}^{+,(i)}$ $(2 \leq i \leq m+1)$						$(0, 2\mathrm{j}+2)$
ϕ_i^- $(1 \leq i \leq m)$	adj					(2, 0)
μ_i^- $(0 \leq i \leq m)$	adj					(0, 2)
$\tilde{\mu}_i^-$ $(1 \leq i \leq m)$	adj					(0, 2)
μ_C				adj		(0, 2)
μ_D					adj	(0, 2)
$\mu_{\mathrm{j},\mathrm{j}}^{-,(i)}$ $(2 \leq i \leq m+1)$						$(-2\mathrm{j}, 2)$
$M_{\mathrm{j}}^{-,(i)}$ $(2 \leq i \leq m+1)$						$(2\mathrm{j}+2, 0)$

Since the coupled theory for any m comes from the same UV curve, we expect they all flow to the same SCFT in the IR.

Let us compute the anomaly coefficients of the quiver theory. We can use the anomaly coefficients we computed for the $T_N^{(m)}$ and add up with that of $T_N^{(m)}$ with flipped J_+ and J_- in addition to the gaugino contributions at the center node. Then we obtain:

$$\begin{array}{c|c} J_+, J_+^3, J_-, J_-^3 & (2N+1)(1-N) \\ J_+^2 J_-, J_+ J_-^2 & \frac{1}{3}(N-1)(4N^2-2N-3) \\ J_+ SU(N)_{A,B}^2, J_- SU(N)_{C,D}^2 & 0 \\ J_- SU(N)_{A,B}^2, J_+ SU(N)_{C,D}^2 & -N \end{array} \tag{7.53}$$

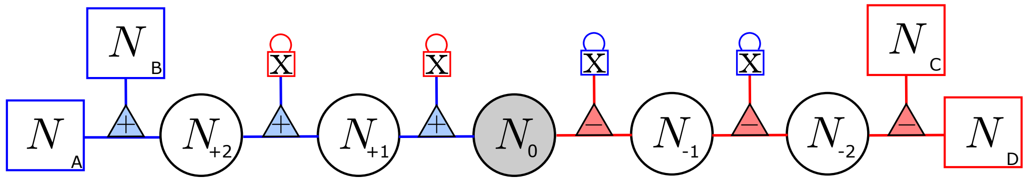
We see that the anomaly coefficients are independent of m, therefore it agrees with the gauged T_N which corresponds to the case with $m = 0$.

We will match the set of supersymmetric operators by computing the superconformal index in section 7.5.

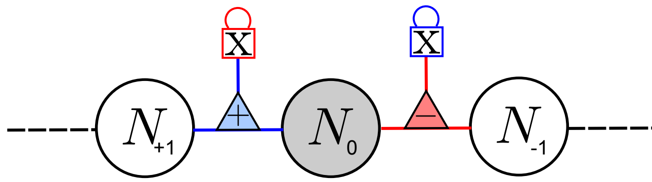
Cascading RG flows to the gauged T_N theory In section 7.3.4 we saw that in the dual frame of the form figure 7.12b, the central gauge node $SU(2)_0$ confines and we get a cascade of RG flows which ultimately reduces the whole system to $SU(2)$ SQCD with 4 flavors. Here, we will argue that a similar mechanism occurs when two $T_N^{(m)}$ blocks are glued to each other to give the duality frame of figure 7.17a. Guided by the $SU(2)$ case, we claim that the $\mathcal{N} = 1$ node in the sub-quiver shown in figure 7.17b undergoes confinement with a quantum deformed moduli space. At energies below confinement-scale, the spectrum of the quiver will include operators that transform as bifundamentals of the ± 1-th nodes of the original quiver. The quantum deformation of the moduli space will imply that these bifundamentals have a non-zero expectation value, breaking the product gauge group $SU(N)_{+1} \times SU(N)_{-1}$ down to the diagonal $SU(N)$. The expectation value will also make the adjoint chiral fields coupled to the ± 1-th nodes massive, which will therefore get integrated out. The upshot will be a reduction of $m \to m - 1$: at low energies, the quiver shown in figure 7.17b reduces to that shown in figure 7.17c. This process triggers a cascade of RG flows which reduces the quiver of figure 7.17a down to that shown in figure 7.16a.

As an evidence to support our claim about figure 7.17b, we consider the theory obtained by gluing two $T_N^{(1)}$ blocks via an $\mathcal{N} = 1$ vector multiplet along one of their full punctures. The other full puncture of each block is glued (via an $\mathcal{N} = 2$ vector) to an $\mathcal{N} = 2$ quiver tail corresponding to the minimal puncture, giving the quiver in figure 7.18. If our claim is correct then the central $\mathcal{N} = 1$ node of this quiver should also exhibit confinement, and the theory will then flow to the quiver of figure 7.19. We now argue that this is indeed the case.

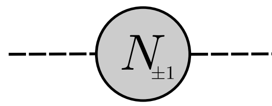
Note that the quiver of figure 7.18 is dual to the linear quiver shown in figure 7.21. When the 'x'-marked punctures of the figure 7.18 are not closed, as in figure 7.20a, the theory is dual to the linear quiver of figure 7.20b [89]. The only difference here is



(a) Another quiver description obtained by gluing two copies of the $T_N^{(2)}$ theory. This quiver has a cascade of RG flows which reduces it to the quiver of figure 7.16a in the IR.



(b) The $\mathcal{N} = 1$ node shown here undergoes confinement, triggering a cascade of RG flows in figure 7.17a. The dynamics that lead to this behavior are local to this section of the quiver and do not depend upon the rest of the quiver.



(c) Due to confinement at the $\mathcal{N} = 1$ node in quiver of figure 7.17b, it reduces to the quiver shown here at low energies.

Figure 7.17. The quiver in figure 7.17a gives an interesting duality frame of the theory obtained by gluing two copies of $T_N^{(2)}$. The sub-quiver shown in figure 7.17b undergoes confinement at the $\mathcal{N} = 1$ node reducing it to the sub-quiver of figure 7.17c. This process triggers a cascade of RG flows in figure 7.17a reducing it to the quiver of figure 7.16a.

that we added gauge singlets to the punctures. From here, we close the punctures at each ends by a nilpotent Higgsing to get the linear quiver theory as given in the figure 7.21 [4]. We have also shown the (J_+, J_-) charges of the various fields in the same figure. The superpotential terms of this quiver are given by all the single trace gauge singlet local operators with charges $(J_+, J_-) = (2, 2)$.

Let us now dualize the central $\mathcal{N} = 1$ node of figure 7.21, followed by dualizing the ± 1-st nodes, then dualize the ± 2-nd nodes and so on until we finally dualize the $\pm(N-2)$-th nodes of the quiver. This will land us on a linear quiver which has an $\mathcal{N} = 1$ vector multiplet at the 0-th, $\pm(N-2)$-th and $\pm(N-1)$-th nodes while the rest of the nodes have an $\mathcal{N} = 2$ vector multiplet as shown in figure 7.22. Notice that the $\mathcal{N} = 1$ node at either ends of the quiver in the current duality frame is equivalent to an

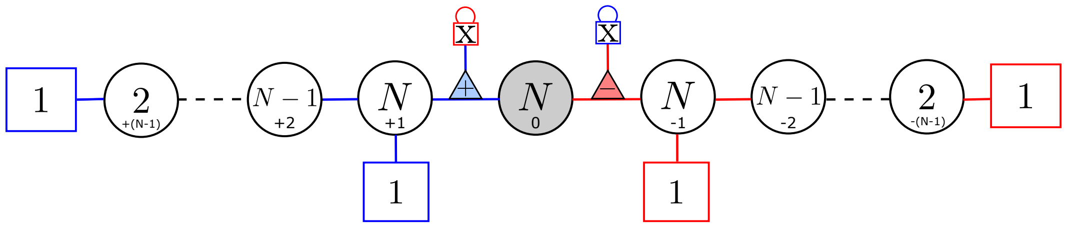


Figure 7.18. The quiver obtained by gluing two $T_N^{(1)}$ blocks and $\mathcal{N}=2$ quiver tails corresponding to the minimal puncture. The $T_N^{(1)}$ blocks are glued to each other via an $\mathcal{N}=1$ vector multiplet along one of their full punctures. The other full puncture of each block is glued, via an $\mathcal{N}=2$ vector multiplet, to an $\mathcal{N}=2$ tail corresponding to the minimal puncture.

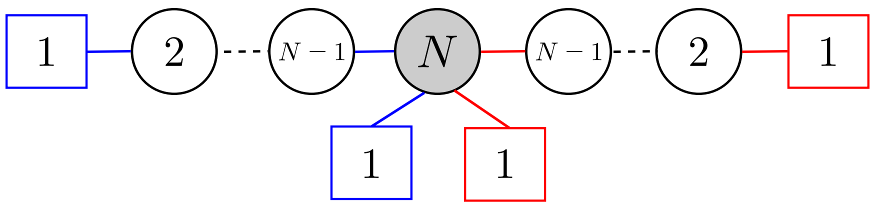
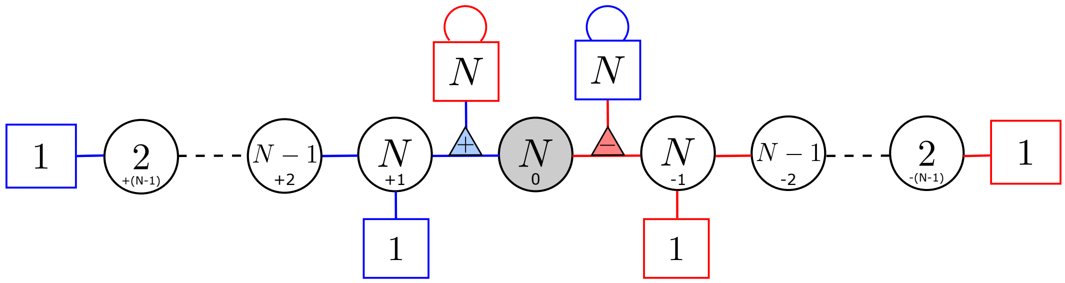


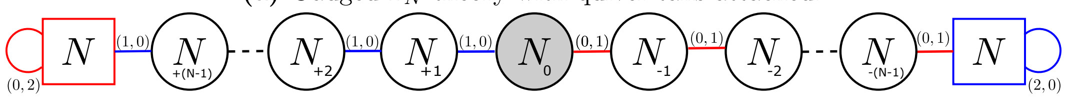
Figure 7.19. The expected low energy theory if the central $\mathcal{N}=1$ node in figure 7.18 undergoes confinement.

SQCD with $N_f = N_c + 1$ flavors. These nodes will therefore undergo s-confinement. The low energy theory of this quiver will then be given by fields describing the mesonic and baryonic fluctuations of the end nodes. Equivalently, we can Seiberg dualize this node to get the theory of free chiral multiplets. This corresponds to the quiver of figure 7.23. Once again the superpotential of this quiver can be written down by considering all the chiral gauge invariant operators which have charges $(J_+, J_-) = (2, 2)$. This will include the low energy superpotential of $N_f = N_c + 1$ SQCD that is expected to be there after s-confinement of the edge nodes in figure 7.22.

In order to proceed we will first have to go through the following series of dualities: dualize the 0-th node in the quiver of figure 7.23 followed by the $\pm$1st nodes, then the $\pm$2nd nodes and so on until we finally dualize $\pm(N-3)$-th nodes. This series of dualities will produce a quiver whose central and last two nodes on either sides are gauged using an $\mathcal{N}=1$ vector multiplet while the rest of the nodes are gauged using an $\mathcal{N}=2$ vector



(a) Gauged T_N theory with quiver tails attached.



(b) Linear quiver dual of the above quiver.

Figure 7.20. Quiver theory of figure 7.18 before closing the punctures. It is given by a gauged T_N theory with quiver tails attached.

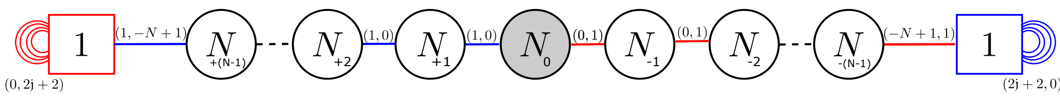


Figure 7.21. The linear quiver dual to the duality frame of figure 7.18. We have $N - 1$ singlets attached to each ends. Here $\mathrm{j} = 1, \cdots, N - 1$.

multiplet. This quiver is depicted in figure 7.24.

If we now dualize the nodes at the left and the right ends of the quiver in figure 7.24, we obtain the quiver of figure 7.25.

We will now have to again go through the series of dualities mentioned in the previous paragraph, this time stopping when we dualize the $\pm(N-4)$-th nodes. This gives us the quiver of figure 7.26. Dualizing the penultimate nodes on either sides of this quiver gives the quiver that can be represented by figure 7.27. We can now repeat the series of dualities outlined earlier (starting by dualizing the 0-th node, followed by dualizing the (± 1)-st node and so on) multiple times such that we ultimately land on a linear quiver that corresponds to figure 7.28. Dualizing the 0-th node of this quiver then lands us on the duality frame of figure 7.19 which is the result we sought.

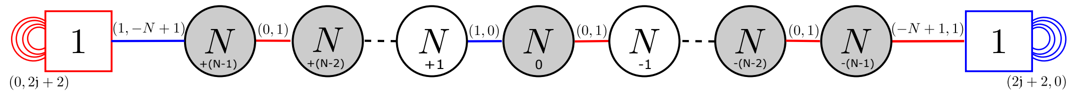


Figure 7.22. A duality frame of figure 7.21 obtained by dualizing, the 0-th node, then the ±1-st nodes, followed by ±2-nd nodes and so on until we finally dualize the $\pm(N-2)$-th nodes.

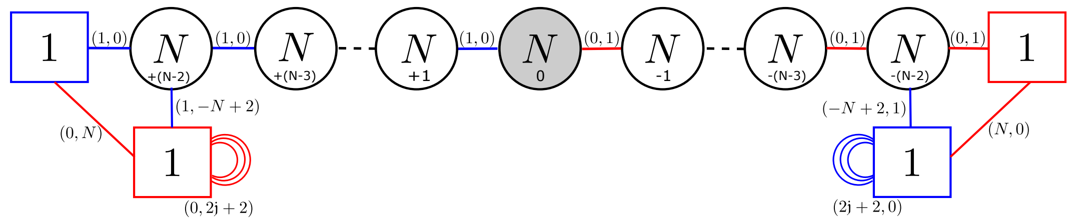


Figure 7.23. The low energy theory of the quiver in figure 7.22 obtained by noticing that the nodes at its left and the right ends undergo s-confinement. Here $\mathrm{j} = 1, \cdots, N-2$.

7.4.4 Infinitely many $\mathcal{N} = 1$ duals for $SU(N)$ SQCD with $2N$ flavors

Let us now consider the case of SQCD with $SU(N)$ gauge group and $2N$ flavors. From the class $\mathcal{S}$ point of view, what we need to do is to start with 4-punctured (all maximal, 2 + and 2− color) sphere with $(p,q) = (1,1)$ as in the section 7.4.3, and then partially close the two maximal punctures of each color. This will result in replacing the T_N block we glued to the end of the quivers by bifundamental hypermultiplets of $SU(N) \times SU(N)$. See the figure 7.29.

The matter content for the theory $\mathcal{U}_N^{(m)}$ similar to the figure 7.29b is given in the table 7.12. The superpotential is given by

$$W = W'_+ + W'_- + \mathrm{tr}\mu_0^+ \mu_0^- \ , \tag{7.54}$$

where

$$W'_\sigma = \sum_{k=1}^{m} \mathrm{Tr}\phi_k^\sigma (\mu_k^\sigma - \tilde{\mu}_k^\sigma) + \sum_{i=2}^{m+1} \sum_{\mathrm{j}=1}^{N-1} \mu_{\mathrm{j},\mathrm{j}}^{\sigma,(i)} M_{\mathrm{j},-\mathrm{j}}^{\sigma,(i)} \ , \tag{7.55}$$

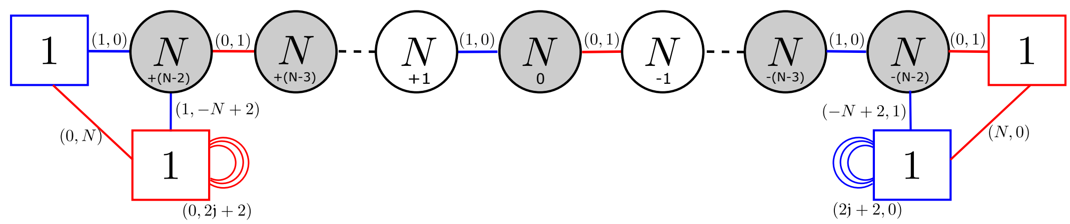


Figure 7.24. The duality frame of the theory in figure 7.23 obtained by dualizing its 0-th node, followed by the ±1-th nodes and so on until we dualize the ±(N − 3)-th nodes.

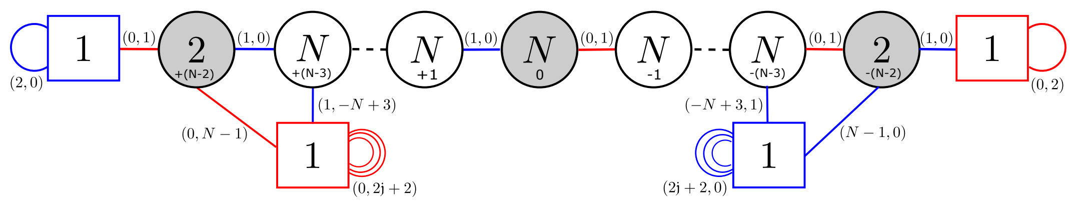


Figure 7.25. The quiver obtained by dualizing the end nodes of the quiver in figure 7.24. Here $\mathrm{j} = 1, \cdots, N-3$.

with

$$\mu_m^\sigma = q^\sigma \tilde{q}^\sigma - \frac{1}{N}\mathrm{tr}(q^\sigma \tilde{q}^\sigma) \ , \quad \hat{\mu}_{\mathrm{j,j}}^{\sigma,(m+1)} = \mathrm{tr}\tilde{q}^\sigma q^\sigma (\phi_m^\sigma)^{N-j-1} \ . \tag{7.56}$$

Anomaly coeffecients As an intermediate step, let us consider the Higgsed $T_N^{(m)}$ theory by Higgsing one of the punctures. Let us call it $\widetilde{T}_N^{(m)}$. This theory is given by the UV curve $\mathcal{C}_{0,3}^{(m+1,-m)}$ with $n_+ = 3$ where 2 of the punctures are maximal the other is minimal. The quiver diagram of the theory is the left half of figure 7.30 with central gauge group ungauged. When $m = 0$, it becomes a theory of free $SU(N)_A \times SU(N)_G$ bifundamental hypermultiplets with $U(1)_B$ baryonic symmetry. The anomalies of this

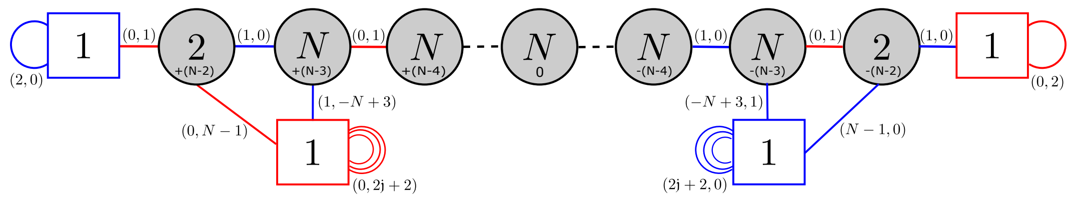


Figure 7.26. The quiver of figure 7.25 can be dualized to the one shown in this figure.

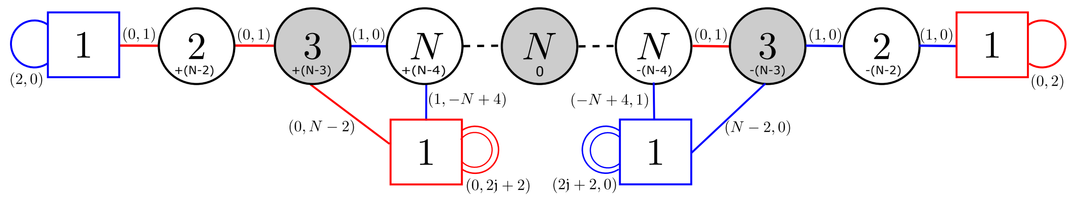


Figure 7.27. The quiver obtained by dualizing the penultimate nodes on either sides of the quiver in figure 7.26. Here $\mathrm{j} = 1, \cdots, N-4$.

theory are given as:

$$\begin{array}{c|c} J_+, J_+^3 & m(1-N) \\ J_-, J_-^3 & m(N-1) - 2N^2 \\ J_+^2 J_- & \frac{1}{3}(4N^3 - N - 3) \\ J_+ J_-^2 & -\frac{1}{3}(4N^3 - N - 3) \\ J_- SU(N)_A^2, J_- SU(N)_G^2 & -N \\ J_+ SU(N)_A^2, J_+ SU(N)_G^2 & 0 \\ J_+ U(1)_B^2 & 0 \\ J_- U(1)_B^2 & -2N^2 \\ J_+^2 U(1)_B, J_-^2 U(1)_B & 0 \end{array} \tag{7.57}$$

Here A and G are the two maximal punctures while B is the name we used for the minimal puncture. The anomalies of the $T_N^{(m)}$ theory with all its colors inverted can be obtained by interchanging the roles of J_+ and J_- in the above table.

We now compare the anomaly coefficients of our proposed dual theories. For

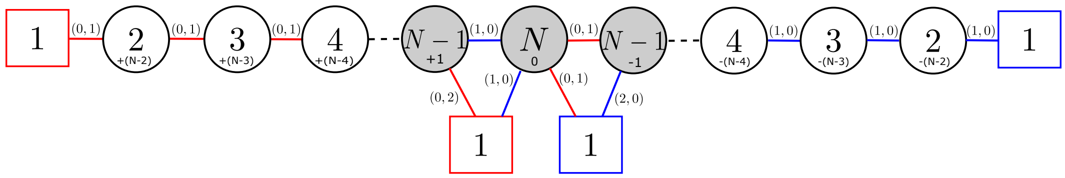
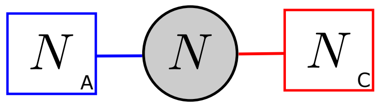
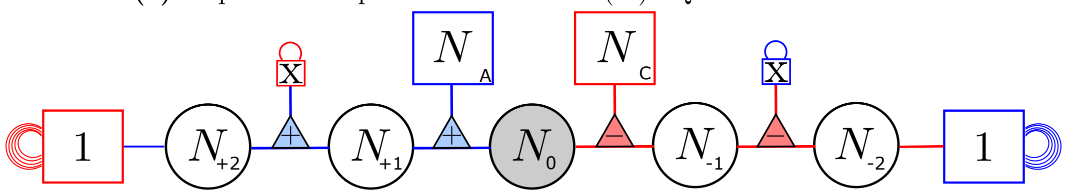


Figure 7.28. Repeated action of Seiberg duality on the quiver in figure 7.27 can mutate it into the quiver shown here. All the singlets become massive and integrated out.



(a) A quiver description dual to the $SU(N)$ SQCD with $2N$ flavors.



(b) A quiver description of $\mathcal{U}_N^{(2)}$ obtained by gluing the two copies of $\widetilde{T}_N^{(2)}$.

Figure 7.29. Some of the dual descriptions for the 4-punctured sphere theory with $(p, q) = (1, 1)$. Here we have maximal punctures of each color and minimal punctures of each color.

$\mathcal{U}_N^{(m)}$, we find:

$$
\begin{array}{c|c}
J_+, J_+^3, J_-, J_-^3 & -N^2 - 1 \\
J_+^2 J_-, J_+ J_-^2 & N^2 - 1 \\
J_+ SU(N)_A^2, J_- SU(N)_C^2 & 0 \\
J_- SU(N)_A^2, J_+ SU(N)_C^2 & -N \\
J_+ U(1)_B^2, J_- U(1)_D^2 & 0 \\
J_+ U(1)_D^2, J_- U(1)_B^2 & -2N^2 \\
J_+^2 U(1)_{B,D}, J_-^2 U(1)_{B,D} & 0
\end{array}
\tag{7.58}
$$

As before we find that these coefficients are independent of m and match perfectly with those of $SU(N)$ SQCD with $2N$ flavors.

Table 7.12. 'Matter content' of the $\mathcal{U}_N^{(m)}$ theory. Here $1 \leq \mathrm{j} \leq N-1$.

	$SU(N)_i^{\pm}$	$SU(N)_A$	$U(1)_B$	$SU(N)_C$	$U(1)_D$	(J_+, J_-)
$\phi_i^+ \ (1 \leq i \leq m)$	adj					$(0,2)$
$q^+, \tilde{q}^+ \ (i=m)$	$\square, \bar{\square}$		$1,-1$			$(1,-N+1)$
$\mu_i^+ \ (0 \leq i \leq m)$	adj					$(2,0)$
$\tilde{\mu}_i^+ \ (1 \leq i \leq m-1)$	adj					$(2,0)$
μ_A		adj				$(2,0)$
$\mu_{\mathrm{j,j}}^{+,(i)} \ (2 \leq i \leq m+1)$						$(2,-2\mathrm{j})$
$M_{\mathrm{j,-j}}^{+,(i)} \ (2 \leq i \leq m+1)$						$(0,2\mathrm{j}+2)$
$\phi_i^- \ (1 \leq i \leq m)$	adj					$(2,0)$
$q^-, \tilde{q}^- \ (i=m)$	$\square, \bar{\square}$				$1,-1$	$(-N+1,1)$
$\mu_i^- \ (0 \leq i \leq m)$	adj					$(0,2)$
$\tilde{\mu}_i^- \ (1 \leq i \leq m-1)$	adj					$(0,2)$
μ_C				adj		$(0,2)$
$\mu_{\mathrm{j,j}}^{-,(i)} \ (2 \leq i \leq m+1)$						$(-2\mathrm{j},2)$
$M_{\mathrm{j,-j}}^{-,(i)} \ (2 \leq i \leq m+1)$						$(2\mathrm{j}+2,0)$

Cascading RG flows to SQCD As in the case of the section 7.4.3, let us consider a dual description for the $\widetilde{T}_N$ theory itself to show that it flows to the same theory as the $SU(N)$ SQCD with $2N$ flavors. The 'matter content' of the theory $\mathcal{U}_N^{(m)}$ (figure 7.30)

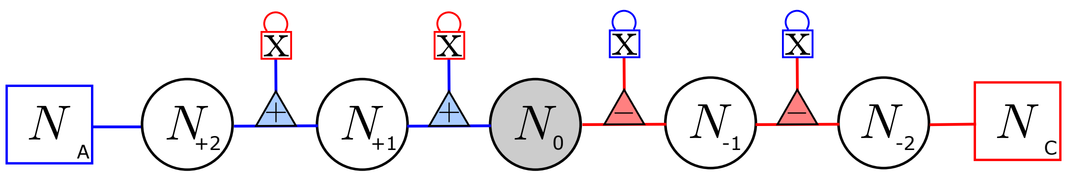


Figure 7.30. Another quiver description obtained by gluing two copies of $\widetilde{T}_N^{(2)}$. We call this as $\widehat{\mathcal{U}}_N^{(2)}$. The theory will undergo cascading RG glow to the SQCD.

is quite similar as in section 7.4.3, but we get $SU(N)_A \times U(1)_B \times SU(N)_C \times U(1)_D \times U(1)_R \times U(1)_{\mathcal{F}}$ global symmetry instead. It is described in the table 7.13.

The set of chiral operators in the T_N theory contains (anti-)trifundamental operator Q_{ijk} and $\tilde{Q}^{ijk}$. When an oppositely colored puncture of the T_N block is closed, the operators $Q_{ijk}, \tilde{Q}^{ijk}$ split into N bifundamental operators $Q_{ij(\ell)}, \tilde{Q}^{ij(\ell)}$ with $-\frac{N-1}{2} \leq \ell \leq \frac{N-1}{2}$, and the corresponding charges being $(J_+, J_-) = (N-1, -2\ell)$ or $(-2\ell, N-1)$ depending on the choice of color. These operators will be important to our

Table 7.13. The 'matter content' of the $\widehat{\mathcal{U}}_N^{(m)}$ theory.

	$SU(N)_i^{\pm}$	$SU(N)_A$	$U(1)_B$	$SU(N)_C$	$U(1)_D$	(J_+, J_-)
ϕ_i^+ $(1 \leq i \leq m)$	adj					$(0, 2)$
$q^+, \tilde{q}^+$ $(i = m)$	$\square, \bar{\square}$	$\square, \bar{\square}$	$1, -1$			$(1, 0)$
μ_i^+ $(0 \leq i \leq m)$	adj					$(2,\ 0)$
$\tilde{\mu}_i^+$ $(1 \leq i \leq m)$	adj					$(2,\ 0)$
$\mu_{\mathrm{j,j}}^{+,(i)}$ $(1 \leq i \leq m)$						$(2, -2\mathrm{j})$
$M_{\mathrm{j,-j}}^{+,(i)}$ $(1 \leq i \leq m)$						$(0, 2\mathrm{j}+2)$
ϕ_i^- $(1 \leq i \leq m)$	adj					$(2,\ 0)$
$q^-, \tilde{q}^-$ $(i = m)$	$\square, \bar{\square}$			$\square, \bar{\square}$	$1, -1$	$(0,\ 1)$
μ_i^- $(0 \leq i \leq m)$	adj					$(0,\ 2)$
$\tilde{\mu}_i^-$ $(1 \leq i \leq m)$	adj					$(0,\ 2)$
$\mu_{\mathrm{j,j}}^{-,(i)}$ $(1 \leq i \leq m)$						$(-2\mathrm{j}, 2)$
$M_{\mathrm{j}}^{-,(i)}$ $(1 \leq i \leq m)$						$(2\mathrm{j}+2, 0)$

analysis and we will label those coming from the i-th block in figure 7.30 as $Q_\ell^{\sigma,(i)}, \tilde{Q}_\ell^{\sigma,(i)}$ suppressing indices.

The superpotential for the theory is given as

$$W = W'_+ + W'_- + \mathrm{tr}\mu_0^+\mu_0^- + \sum_{k=1}^{m} \mathrm{tr}\hat{\mu}_k^+\hat{\mu}_k^- \ , \tag{7.59}$$

where

$$W'_\sigma = \sum_{k=1}^{m} \mathrm{Tr}\phi_k^\sigma(\mu_k^\sigma - \tilde{\mu}_k^\sigma) + \sum_{i=1}^{m}\sum_{\mathrm{j}=1}^{N-1} \mu_{\mathrm{j,j}}^{\sigma,(i)} M_{\mathrm{j,-j}}^{\sigma,(i)} \ , \tag{7.60}$$

with

$$\mu_m^\sigma = q^\sigma\tilde{q}^\sigma - \frac{1}{N}\mathrm{tr}(q^\sigma\tilde{q}^\sigma) \ , \tag{7.61}$$

and

$$\hat{\mu}_k^\sigma = \left(\prod_{i=1}^{k} Q^{\sigma,(i)}_{\frac{N-1}{2}} \right) \phi_k^\sigma \left(\prod_{i=1}^{k} \widetilde{Q}^{\sigma,(i)}_{\frac{N-1}{2}} \right) \ . \tag{7.62}$$

Here we formed the gauge invariant operators μ_m^σ so as to transform as the adjoint of $SU(N)_{\pm m}$ according to whether $\sigma = \pm$ while $\hat{\mu}_k^\sigma$ is constructed such that it transforms as the adjoint of $SU(N)_0$.

By applying a sequence of dualities, we have showed earlier that the central $SU(N)_0$-node confines. From this, we conjecture that the $SU(N)_0$-node undergoes confinement with N^2 mesonic operators $\tilde{Q}^{\pm,(1)}_{\frac{N-1}{2}} Q^{\mp,(1)}_{\frac{N-1}{2}}$ and quantum deformed moduli space given by

$$\det\big(\tilde{Q}^{\pm,(1)}_{\frac{N-1}{2}} Q^{\mp,(1)}_{\frac{N-1}{2}}\big) - \text{``}(\mu^{+,(1)}_{\mathrm{j,j=1}} \mu^{-,(1)}_{\mathrm{j,j=1}})^{\frac{1}{2}N(N-1)}\text{''} = \Lambda_0^{b(N-1)} \ , \tag{7.63}$$

where Λ_0^b is the $SU(N)_0$ instanton factor, with the exponent b determined by

$$b = 3N - 2k = 3\epsilon_{UV} N, \qquad \text{where} \quad k = -3\mathrm{Tr} R_{UV} SU(N)_0^2 = \frac{3}{2}(1-\epsilon_{UV})N. \tag{7.64}$$

The scaling dimensions of the two sides of (7.63) agree, upon using $\Delta = \frac{3}{2} R_{UV}$, where R_{UV} is the superconformal R-charge before gauging $SU(N)_0$. Gauging $SU(N_0)$ breaks the separate $U(1)_{\mathcal{F}_\pm}$ to $U(1)_{\mathcal{F}} = U(1)_{\mathcal{F}_+} - U(1)_{\mathcal{F}_-}$, with $U(1)_A = U(1)_{\mathcal{F}_+} + U(1)_{\mathcal{F}_-}$ anomalous. The $\mathrm{Tr} U(1)_A SU(N)_0^2 = N$ anomaly implies that Λ_0^b carries charge $+2N$ under $U(1)_A$, which is consistent with the $U(1)_A$ charge of the product of operators on the LHS of (7.63). The operators on the LHS of (7.63) carry $U(1)_{R_{IR}}$ charge zero, as required for a quantum deformed chiral ring relation (and that is why other $Q_\ell^{\pm,(i)}$, $\tilde{Q}_\ell^{\pm,(i)}$ do not appear in (7.63)).

The first and second term in the LHS of (7.63) are analogs of $\det\mathcal{M}$ and $\mathcal{B}\tilde{\mathcal{B}}$

in SQCD with $N_f = N_c$. We put the second term in quotes because we have not fully determined the dependence on the $\mu^{\pm}_{\mathrm{j,j}}$ beyond what is fixed by the symmetries. In any case, the F terms of superpotential (7.59) sets the operators $\mu^{\pm,(i)}_{\mathrm{j,j}}$ to zero, setting the terms in quotes to zero in (7.63). On the deformed space (7.63), the $Q^{\pm,(1)}_{\frac{N-1}{2}}$ and $\tilde{Q}^{\pm,(1)}_{\frac{N-1}{2}}$ thus have non-zero expectation value. Then ϕ_1^+ and ϕ_1^- will become massive via the last term of (7.59) with $k = 1$. Moreover, the $SU(N)_{+1} \times SU(N)_{-1}$ gauge symmetry is broken down to the diagonal $SU(N)$, which will again undergo confinement. This is an iterative cascade of RG flows, reducing m in each step, eventually flowing to $SU(N)$ SQCD with $2N$ flavors with a quartic superpotential in the IR.

7.5 Superconformal index

The superconformal index for a $\mathcal{N} = 1$ superconformal field theory is defined as

$$I(\mathfrak{p}, \mathfrak{q}, \xi; \vec{x}) = \mathrm{Tr}(-1)^F \mathfrak{p}^{j_1+j_2+\frac{R_0}{2}} \mathfrak{q}^{j_2-j_1+\frac{R_0}{2}} \xi^{\mathcal{F}} \prod_i x_i^{F_i} \ . \tag{7.65}$$

where we introduced the fugacity ξ for the $U(1)_{\mathcal{F}}$ which is present for generic class $\mathcal{S}$ theories. For the theory having a Lagrangian description in the UV, the index can be simply computed by multiplying the contributions from each matter multiplets in the UV and then by integrating over the gauge group. The contribution of each matter multiplets is calculated using the exact R-charge in the IR [168]. In our case, the only possible non-anomalous $U(1)$ symmetry that can mix with R-symmetry in the IR is $U(1)_{\mathcal{F}}$. Therefore we can obtain the index using the UV R-charge as long as we keep the fugacity ξ turned on. Once we know the exact R-charge $R = R_0 + \epsilon\mathcal{F}$, we can simply redefine $\xi \to \xi(\mathfrak{p}\mathfrak{q})^{\epsilon/2}$ to obtain the true superconformal index.

7.5.1 Topological field theory and superconformal index

For an $\mathcal{N}=1$ SCFT in class $\mathcal{S}$, the superconformal index can be written in terms of a correlation function of the 2d (generalized) topological field theory living on the UV curve. This topological field theory is related to a deformation of 2d Yang-Mills theory [83, 85, 86, 92, 39, 167]. The index can be written as

$$I(\mathfrak{p},\mathfrak{q},\xi;\vec{a}_i)=\sum_{\lambda}(C_\lambda^+)^p(C_\lambda^-)^q\prod_{i=1}^{n}\psi_\lambda^{\rho_i,\sigma_i}(\vec{a}_i)\ , \tag{7.66}$$

where (p,q) are the degrees of the line bundles and n is the number of punctures, which should satisfy the relation $p+q=2g-2+n$. Here we suppressed the $\mathfrak{p},\mathfrak{q},\xi$ dependence and the sum is over the representations λ of Γ labelling the six-dimensional $(2,0)$ theory.

The basis function $\psi_\lambda^{\rho,\sigma}(\vec{a})$ corresponding to the puncture labelled by the embedding $\rho: SU(2)\to\Gamma$ and color σ can be written in the following form

$$\psi_\lambda^{\rho,\sigma}(\vec{a})=K_\rho(\vec{a};t_\sigma)P_\lambda(\vec{a}t_\sigma^\rho)\ , \tag{7.67}$$

where $t_\sigma=\xi^\sigma\sqrt{\mathfrak{p}\mathfrak{q}}$ and we suppressed the $\mathfrak{p},\mathfrak{q}$ dependence. The K-factor does not depend on λ, but the form of the function depends on the type of puncture. P_λ is a symmetric function of $\vec{a}$ which in certain limit reduces to the Macdonald polynomial. The argument $\vec{a}t_\sigma^\rho$ is determined by the embedding ρ of $SU(2)$ into Γ labelling the puncture (see [156]). The structure constant can be written as $C_\lambda^\sigma=(\psi_\lambda^{\varnothing,\sigma})^{-1}$ in terms of the basis function ψ's.

Let us compute the index of the $T_N^{(m)}$ starting from the theory given by the UV curve $\mathcal{C}_{0,m+3}^{(m+1,0)}$ with $(n_+,n_-)=(3,m)$ where we know how to write the index from the

TQFT:

$$I[\mathcal{C}_{0,m+3}^{(m+1,0)}] = \sum_\lambda (C_\lambda^+)^{m+1} \prod_{i=1}^{3} \psi_\lambda^+(\vec{a}_i) \prod_{j=1}^{m} \psi_\lambda^-(\vec{b}_i) \ . \tag{7.68}$$

Now, we want to Higgs all the $-$ punctures. Complete Higgsing or closing of a puncture is implemented via replacing the wave function $\psi_\lambda^{\rho,\sigma}(\vec{b})$ corresponding to the puncture to close by $\psi_\lambda^{\varnothing,\sigma}(t_\sigma^\rho)$. From the relation $C_\lambda^\sigma = (\psi_\lambda^{\varnothing,\sigma})^{-1}$, we see that the degree of the normal bundle corresponding to the color σ reduces upon Higgsing. We get

$$I[T_N^{(m)}](\mathfrak{p}, \mathfrak{q}, \xi; \vec{a}_i) = \sum_\lambda \frac{(C_\lambda^+)^{m+1}}{(C_\lambda^-)^m} \psi_\lambda^+(\vec{a}_1)\psi_\lambda^+(\vec{a}_2)\psi_\lambda^+(\vec{a}_3) \ , \tag{7.69}$$

where we suppressed ρ_i to denote full punctures. One can also flip all the colors $\pm$ in the components to get the same index with $\xi \to \xi^{-1}$. This is of the same form as the equation (7.66), from which we can plug in $(p,q) = (m+1, -m)$ with 3 + colored punctures.

Once we have the equation (7.69), it is a piece of cake to show that the index is the same for the dual theories, independent of m. Gluing two copies of $T_N^{(m)}$ with opposite color by a cylinder to form the theory corresponding to the 4-punctured sphere with $(p,q) = (1,1)$, the index can be written as

$$\begin{aligned} I(\vec{a}, \vec{b}, \vec{c}, \vec{d}) &= \sum_{\lambda,\mu} \frac{(C_\lambda^+)^{m+1}}{(C_\lambda^-)^m} \psi_\lambda^+(\vec{a})\psi_\lambda^+(\vec{b}) \left(\oint [d\vec{z}] I_{\text{vec}}(\vec{z}) \psi_\lambda^+(\vec{z}) \psi_\mu^-(\vec{z}) \right) \qquad (7.70) \\ &\quad \times \frac{(C_\mu^-)^{m+1}}{(C_\mu^+)^m} \psi_\mu^-(\vec{c})\psi_\mu^-(\vec{d}) \\ &= \sum_\lambda C_\lambda^+ C_\lambda^- \psi_\lambda^+(\vec{a})\psi_\lambda^+(\vec{b})\psi_\lambda^-(\vec{c})\psi_\lambda^-(\vec{d}) \ . \qquad (7.71) \end{aligned}$$

We here used the fact that wave functions are orthonormal:

$$\oint [d\vec{z}] I_{\text{vec}}(\vec{z}) \psi_\lambda^+(\vec{z}) \psi_\mu^-(\vec{z}) = \delta_{\lambda\mu} \ , \tag{7.72}$$

where $I_{\text{vec}}(\vec{z})$ is the contribution to the index from a $\mathcal{N}=1$ vector multiplet. Therefore for any choice of $m \in \mathbb{Z}$ the gluing gives us the same index as that of the theory described by 2 full + punctures and 2 full - punctures and $(p,q)=(1,1)$. It describes the two copies of T_N theory glued by $\mathcal{N}=1$ vector multiplet. The same argument goes through when we Higgs or partially close the full punctures of each color to minimal punctures to get the SQCD.

In the paper [39], the superconformal index for the generic (p,q) was proposed from the structure of the (generalized) topological field theory, initially without concrete SCFTs that realize the indices. The SCFT that we discuss here gives such a concrete realization.

7.5.2 Direct computation for the $SU(2)$ theories

The proof of the previous section holds as long as the index of the T_N theory can be written in terms of the basis wave function $\psi_\lambda(\vec{a})$. Here, we confirm the TQFT formula for $T_2^{(m)}$ theories (7.69) by directly computing the index using the matter content of section 7.3.

The index for a chiral multiplet with (J_+, J_-) charge is given as

$$I_{\text{chi}}^{(J_+,J_-)}(\mathfrak{p},\mathfrak{q},\xi;\vec{z}) = \prod_{\vec{v}\in\mathcal{R}} \Gamma((\mathfrak{p}\mathfrak{q})^{\frac{R_0}{2}}\xi^{\mathcal{F}}\vec{z}^{\vec{v}};\mathfrak{p},\mathfrak{q}) = \prod_{\vec{v}\in\mathcal{R}} \Gamma((\mathfrak{p}\mathfrak{q})^{\frac{J_++J_-}{4}}\xi^{\frac{J_+-J_-}{2}}\vec{z}^{\vec{v}};\mathfrak{p},\mathfrak{q}) \ , \tag{7.73}$$

where $\vec{v}$ are the weight vectors of the representation $\mathcal{R}$ of the symmetry group the chiral multiplet is charged under. Here the notation $\vec{z}^{\vec{v}}$ is a short-hand for $\prod_i z_i^{v_i}$. Here, we used the elliptic gamma function which is defined as

$$\Gamma(z;\mathfrak{p},\mathfrak{q}) = \prod_{m,n=0}^{\infty} \frac{1-z^{-1}\mathfrak{p}^{m+1}\mathfrak{q}^{n+1}}{1-z\mathfrak{p}^m\mathfrak{q}^n} \ , \tag{7.74}$$

to write the index in a concise form. We will suppress the $\mathfrak{p},\mathfrak{q}$ dependence of $\Gamma(\vec{z};\mathfrak{p},\mathfrak{q})$

whenever possible.

The vector multiplet contribution to the index is given by

$$I_{\rm vec}(\mathfrak{p},\mathfrak{q};\vec{z}) = \frac{1}{|\mathcal{W}|}\prod_{\vec{\alpha}\in\Delta_G}\Gamma(\vec{z}^{\vec{\alpha}})^{-1} \ , \tag{7.75}$$

where $\mathcal{W}$ is the Weyl group of G and Δ_G is the set of root lattices of G. We also included the Haar measure for the gauge group G to the vector multiplet index for convenience. For the $SU(N)$ gauge group, we get

$$I_{\rm vec}(\mathfrak{p},\mathfrak{q};\vec{z}) = \frac{(\mathfrak{p};\mathfrak{p})^{N-1}(\mathfrak{q};\mathfrak{q})^{N-1}}{N!}\prod_{i\neq j}\frac{1}{\Gamma(z_i/z_j)} \ , \tag{7.76}$$

where $i,j=1,\cdots N$ and $\prod_i z_i = 1$. Here $(z;q)$ is the q-Pochhammer symbol which is defined to be $(z;q)=\prod_{m=0}^{\infty}(1-zq^m)$.

$T_2^{(m)}$ **theory** Let us compute the superconformal index of the $T_2^{(1)}$ theory discussed in section 7.3.1. We would like to compute the index in the UV using the description given as in figure 7.7 and show that it agrees with the TQFT formula. The index on the electric side can be written as

$$\begin{aligned} I(\mathfrak{p},\mathfrak{q},\xi;a,b,c) &= \oint\frac{dz}{2\pi iz}I_{\rm vec}(z)I_{\rm chi}^{(0,2)}(z^{\pm2,0})I_{\rm chi}^{(1,0)}(z^{\pm}a^{\pm}b^{\pm})I_{\rm chi}^{(1,-1)}(z^{\pm}c^{\pm})I_{\rm chi}^{(0,4)}(1) \\ &= \kappa\oint\frac{dz}{2\pi iz}\frac{\Gamma(z^{\pm2,0}(\mathfrak{p}\mathfrak{q})^{\frac{1}{2}}\xi^{-1})}{2\Gamma(z^{\pm2})}\Gamma(z^{\pm}a^{\pm}b^{\pm}(\mathfrak{p}\mathfrak{q})^{1/4}\xi^{-\frac{1}{2}})\Gamma(z^{\pm}c^{\pm}\xi)\Gamma(\mathfrak{p}\mathfrak{q}\xi^{-2}) \ , \end{aligned}$$

where $\kappa=(\mathfrak{p};\mathfrak{p})(\mathfrak{q};\mathfrak{q})$. We use a short-hand notation of $\pm$ to denote multiple products involving each sign. For example $f(a^{\pm}b^{\pm})\equiv f(ab)f(ab^{-1})f(a^{-1}b)f(a^{-1}b^{-1})$. Also, $f(z^{\pm2,0})$ means $f(z^2)f(z^{-2})f(z^0)$.

One tricky part here is choosing the correct contour for this integral. Usually, one picks the contour to be the unit circle and assumes $|\mathfrak{p}|,|\mathfrak{q}|<1$ and $|\xi|=|a|=|b|=|c|=1$

so that we pick up the poles only inside the unit circle. This works as long as there is no chiral multiplet with R_0 or R charge less than equal to zero. But if there is a chiral multiplet having $R_0 \leq 0$, some of the poles may lie along the unit circle. In [4], it was argued that one should take $|\xi^f (\mathfrak{pq})^{r/2}| < 1$ for the chiral multiplet with R_0-charge r and $\mathcal{F}$-charge f. Therefore, we need to include all the poles of the form $x\xi^f \mathfrak{p}^{\frac{r}{2}+m}\mathfrak{q}^{\frac{r}{2}+n}$ with x being products of the fugacities corresponding to the gauge/flavor symmetries.

In our case, we have the poles of the form $z = (a^{\pm}b^{\pm}\xi^{1/2}(\mathfrak{pq})^{1/4}\mathfrak{p}^m\mathfrak{q}^n)^{\pm}$ with $m, n \in \mathbb{Z}_{\geq 0}$ from the chiral multiplets with $(J_+, J_-) = (1, 0)$ and poles of the form $z = (c^{\pm}\xi^{-1}\mathfrak{p}^m\mathfrak{q}^n)^{\pm}$ from the chirals with $(J_+, J_-) = (1, -1)$. Among the first set of poles, $z = a^{\pm}b^{\pm}\xi^{1/2}(\mathfrak{pq})^{1/4}\mathfrak{p}^m\mathfrak{q}^n$ are the ones inside the unit circle and the other half of the poles are outside the contour. For the second set of poles, $z = c^{\pm}\xi^{-1}\mathfrak{p}^m\mathfrak{q}^n$ are the ones inside the contour.

The index for the $T_2^{(m)}$ can be written as

$$I^{(m)} = \oint \prod_{i=1}^{m} \left(\frac{dz_i}{2\pi i z_i} I_{\text{vec}}(z_i) I_{\text{chi}}^{(0,2)}(z^{\pm 2,0}) I_{\text{chi}}^{(1,-1)}(z_{i-1}^{\pm} z_i^{\pm}) I_{\text{chi}}^{(0,4)}(1) \right) I_{\text{chi}}^{(1,0)}(z_m^{\pm} a^{\pm} b^{\pm}) \text{ ,} \tag{7.77}$$

where $z_0 = c$. We confirmed that this indeed gives us the same index as the TQFT prediction of (7.69) at the first few leading orders in $\mathfrak{p}$ and $\mathfrak{q}$ for $m = 1, 2$. If the dualities hold, we have the identity

$$\oint \frac{dz}{2\pi i z} I_{\text{vec}}(z) I^{(m)}(\xi) I^{(m)}(\xi^{-1}) = \oint \frac{dz}{2\pi i z} I_{\text{vec}}(z) I_{\text{chi}}^{(1,0)}(z^{\pm} a^{\pm} b^{\pm}) I_{\text{chi}}^{(0,1)}(z^{\pm} c^{\pm} d^{\pm}) \text{ ,} \tag{7.78}$$

where we glued two $T_2^{(m)}$ with opposite $\mathcal{F}$ charges. We have verified this identity to hold for $m = 1, 2$ at the leading orders in $\mathfrak{p}$ and $\mathfrak{q}$.

SQCD vs $\widehat{\mathcal{U}}_N^{(m)}$ theory Let us compute the index in the dual frame $\widehat{\mathcal{U}}_N^{(m)}$. In this frame, we should be able to see $SU(8)$ flavor symmetry since it cascades to the SQCD in the IR. In order to see this from the index, first we refine the index 7.77 as

$$\tilde{I}^{(m)}(\vec{a}) = \oint \prod_{i=1}^{m} \left(\frac{dz_i}{2\pi i z_i} I_{\text{vec}}(z_i) I_{\text{chi}}^{(0,2)}(z^{\pm 2,0}) I_{\text{chi}}^{(1,-1)}(z_{i-1}^{\pm} z_i^{\pm}) I_{\text{chi}}^{(0,4)}(1) \right) \prod_{n=1}^{4} I_{\text{chi}}^{(1,0)}(z_m^{\pm} a_n), \quad (7.79)$$

where $\prod_{i=1}^{4} a_i = 1$. Here we introduced the fugacities for the $SU(4)$ flavor symmetry $a_{i=1,2,3}$. And then we find

$$\oint \frac{dz}{2\pi i z} I_{\text{vec}}(z) \tilde{I}^{(m)}(\vec{a}, \xi) \tilde{I}^{(m)}(\vec{b}, \xi^{-1}) = \oint \frac{dz}{2\pi i z} I_{\text{vec}}(z) \prod_{m=1}^{4} I_{\text{chi}}^{(1,0)}(z^{\pm} a_m) I_{\text{chi}}^{(0,1)}(z^{\pm} b_m), \quad (7.80)$$

where we also refined the index for the SQCD. One can easily check the index preserves $SU(8)$ flavor symmetry by relabelling the fugacities.

We should keep in mind that $\tilde{I}^{(m)}$ in (7.79) is not a genuine index of the theory, since $T_2^{(m)}$ itself does not have the $SU(4)$ symmetry. There is a cubic coupling which breaks $SU(4) \to SU(2)^2$, and this coupling cannot be tuned to zero as we have discussed in section 7.3.3. But after gluing two copies of $T_2^{(m)}$, we have exactly marginal deformations which includes the point with enhanced symmetry.

7.6 Conclusion and outlook

Guided by the construction of 4d QFTs from M5 branes wrapping Riemann surfaces, we constructed an infinite set of dual theories of $4d$ $\mathcal{N}=1$ $SU(N)$ SQCD with $2N$ flavors. These theories are parametrized by an integer $m \in \mathbb{Z}_{\geq 0}$ and involve $2m$ copies of the T_N theory of [88], $2N$ quarks/anti-quarks along with $2m(N-1)$ singlet chiral superfields as their building blocks. As a check of the dualities we compared their central charges, anomaly coefficients and superconformal indices. Along the way, we constructed a family of new $\mathcal{N}=1$ SCFTs with $SU(N)^3$ flavor symmetries, which generalize the $\mathcal{N}=2$ T_N theory.

The dual theories discussed here can be used to construct more duals, for example by applying them to the magnetic dual of [171]. This will result in adding extra chiral multiplets transforming as adjoints of global symmetries $SU(N)_{A,C}$ and cubic superpotential terms. We can also consider the swapped dual of [82], and also Argyres-Seiberg type duals of [6, 4]. Moreover, as we have discussed in the section 7.3.2, even the building block $T_N^{(m)}$ itself has many different dual descriptions, so the number of duals grows rapidly with m.

One question is how to generalize our dualities to $N_f \neq 2N$. This may be possible e.g. by considering a mass deformation of the $T_N^{(m)}$ theory, as was done in the T_N case [116]. From the class $\mathcal{S}$ perspective, this involves understanding dualities in the presence of irregular punctures. Another direction would be a more detailed study of phase structure and chiral ring of the new theories. The spectral curve of the generalized Hitchin system associated to the $\mathcal{N} = 1$ theories [49, 196, 197, 98, 194] will be useful. It will be also interesting to generalize our construction of $T_N^{(m)}$ to D and E type theories and also with outer-automorphism twists using the $\mathcal{N} = 2$ results [181, 182, 54, 55, 57, 58, 59], as well as possible generalizations using the theories of [93, 80], which will provide analogous infinitely many duals for other gauge groups.

This chapter is a reprint of the material as it appears in "Infinitely many $N = 1$ dualities from $m + 1 - m = 1$", Prarit Agarwal, Kenneth Intriligator, Jaewon Song, arXiv:1505.00255, of which I was a co-author.

Appendices

Appendix A

BPS States and Their Reductions

A.1 More on the gauge field contribution

In this Appendix we show a different approach to compute the contribution of the gauge field to the index. We focus on the four-dimensional case of $S^1 \times S^3_b$, and the round sphere can be obtained by taking the $a, b \to 1$ limit.

As explained in the main text, our method relies on finding a map from the bosonic modes to the fermionic ones such that their contributions to the index cancel out, and the unpaired modes are identified with the BPS states. We have seen that the normal modes of the gauge field strength can be mapped to the gaugino modes. Among the unpaired modes of the gauge field strength, those which also satisfy the Maxwell equations are the BPS modes. Here we offer another interpretation for the latter.

Besides the map between the gauge and the gaugino, we can find another map that relates a mode of the gauge field strength to a fermion with the same $(\Xi, H, \tilde{J}_3)$ quantum numbers

$$\delta\chi = \tilde{\zeta}\tilde{\sigma}^{\mu} v_{\mu} \tag{A.1}$$

The field χ is a pure supergauge field, i.e. it is set to zero in the Wess-Zumino gauge.

For this reason it does not belong to the Hilbert space and every gauge field such that

$$\tilde{\zeta}\tilde{\sigma}^{\mu}v_{\mu} = 0 \tag{A.2}$$

can contribute to the index. The solution to this equation is

$$\begin{aligned}
v_1 &= 0 \\
v_2 &= Y(\theta)e^{Et+i(n\alpha+m\beta)} \\
v_3 &= ia\frac{\sin\theta\cos\theta}{f(\theta)}Y(\theta)e^{Et+i(n\alpha+m\beta)} \\
v_4 &= -ib\frac{\sin\theta\cos\theta}{f(\theta)}Y(\theta)e^{Et+i(n\alpha+m\beta)}
\end{aligned} \tag{A.3}$$

where the first line is a gauge choice and $Y(\theta)$ is to be determined. Then the two equations $\sigma^{\mu\nu}F_{\mu\nu}\zeta = 0$ for $Y(\theta)$ give

$$E = -\frac{n}{a} - \frac{m}{b} \tag{A.4}$$

for $n, m \leq 1$. This is the same result that we obtained in section 2.2. Notice that equation (A.2) is satisfied by the pure gauge configuration $v_{\mu} = \partial_{\mu}\Phi$ that appears in equation (2.31).

This appendix is a reprint of the material as it appears in “BPS states and their reductions ”, Prarit Agarwal, Antonio Amariti, Alberto Mariotti, Massimo Siani, JHEP 1308 (2013) 011, of which I was a co-author.

Appendix B

A Zig-Zag Index

B.1 Y^{pq} theories

In [84] the on shell superconformal index has been computed for a generic Y^{pq} theory [43], and the authors guessed a generic formula by looking at different cases. Here we show that by applying our formula in terms of the zig-zag paths we can match their result on shell, but off shell the factorization takes place over a different set of operators.

A Y^{pq} theory is a quiver gauge theory with $2p$ gauge groups. In figure B.1 we show the dimer and the four kind of fields distinguished by their representation under the global symmetries. From the figure one can extract the number of fields and their charges. They are given in the table

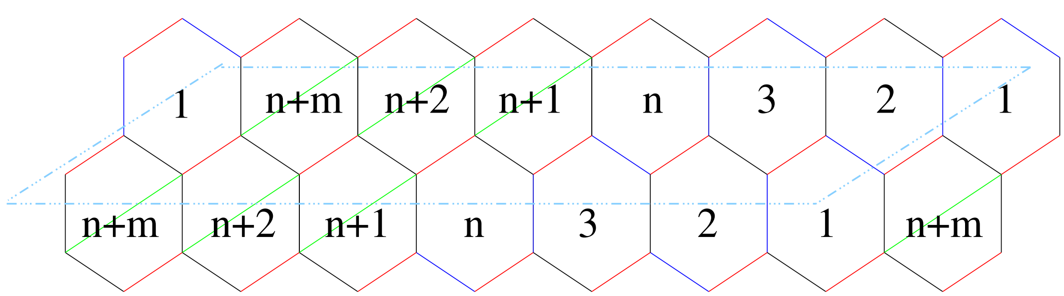


Figure B.1. Tiling for the Y^{pq} theories. The different colors represent the fields U (black), V (blue), Y (red) and Z (green).

Field	Multiplicity	Charge
Z-green	$p-q$	x
Y-red	$p+q$	y
V-blue	$2q$	$1+\frac{1}{2}(x-y)$
U-black	$2p$	$1-\frac{1}{2}(x+y)$

The charges x and y are determined by a-maximization.

$$\begin{aligned} x &= \frac{(-4p^2-2pq+3q^2+(2p+q)\sqrt{4p^2-3q^2})}{(3q^2)} \\ y &= \frac{-4p^2+2pq+3q^2+(2p-q)\sqrt{4p^2-3q^2}}{3q^2} \end{aligned} \quad \text{(B.1)}$$

There are four kind of zig-zag paths. Two of them involve all the Z and $p(q)$ $U(V)$ fields. The other zig-zag paths exchange Z with Y. The contribution of these four paths to the index are

$$\begin{aligned} \sum_{j=1}^{Z_1}(1-r_j^{(1)}) = \sum_{j=1}^{Z_2}(1-r_j^{(2)}) & \\ &= 2p-((p-q)r_Z+qr_V+pr_U) \\ &= \frac{(p-q)(2-x)+(p+q)y}{2} \sum_{j=1}^{Z_3}(1-r_j^{(3)}) \\ &= \sum_{j=1}^{Z_4}(1-r_j^{(4)}) \\ &= 2(p+q)-((p+q)r_Y+qr_V+pr_U \\ &= \frac{(p+q)(2-y)+(p-q)x}{2} \end{aligned} \quad \text{(B.2)}$$

By comparing the formula obtained in [84] with our formula we find that the two agree once the exact R-charge is imposed. If instead we just fix the constraints from the marginality of the couplings, i.e. we keep x and y as generic variables parameterizing a trial R-charge, we have

$$\det(M(t)) = \prod_{i=1}^{4}(1 - t^{\sum_{j=1}^{Z_i}(1-r_j^{(i)})}) \neq (1 - t^{p(1+(x-y)/2)})^2(1 - t^{p+1/2q(1-1/2(x+y))})^2 \tag{B.3}$$

and the *off-shell* index still factorizes over the zig-zag paths.

This appendix is a reprint of the material as it appears in “A Zig-Zag Index ”, Prarit Agarwal, Antonio Amariti, Alberto Mariotti, arXiv:1304.6733, of which I was a co-author.

Appendix C

Refined Checks and Exact Dualities in Three Dimensions

C.1 Relations among hyperbolic integrals

In this appendix we review the equivalence among the hyperbolic integrals necessary to match the dual phases in the quiver gauge theories that we studied in the paper. We refer to [187] for more details.

C.1.1 The unitary case

The partition function for a $U(n)$ gauge theory with CS level $2t$, s_1 fundamentals, s_2 anti-fundamentals and one adjoint matter field corresponds to the integral dubbed as $JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau)$ in [187]. The original integral is defined as

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int \prod_{i\leq j<k\leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^{n}\prod_{r=1}^{s_1}\Gamma_h(\mu_r - x_j)\prod_{s=1}^{s_2}\Gamma_h(\nu_s + x_j)c(2\lambda x_j + t x_j^2)dx_j$$

The variables τ, ν and μ are linear combinations of the chemical potentials for the global symmetries under which the adjoint, fundamental and anti-fundamental fields are charged respectively.

In the cases studied in section 4.4 the theory does not contain an adjoint. This corresponds to identifying the parameter τ with ω. In the hyperbolic function analysis, setting $\tau = \omega$, removes the adjoint field contributions from the above integral because of (4.7) and (4.8). The new integral is defined as

$$J_{n,(s_1,s_2),t}(\mu;\nu;\lambda) = JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\omega) \tag{C.1}$$

The field theory duality is translated in an equivalence between the integrals in (C.1). These equivalences are derived from the transformation properties of certain integrals named *degenerations* in [187]

$$I^m_{n,\xi}(\mu;\nu;\lambda) = J_{n,(s_1,s_2),t}(\mu;\nu;\lambda) \tag{C.2}$$

where ξ labels the integrals on the LHS of (C.2). The value taken by ξ is either (p,q)a or (p,q)b and it can be fixed by using the following table

condition	type	m	p	q
$t < -\lvert s_1 - s_2\rvert$	(p,q)a	$\frac{s_1+s_2-t-2n}{2}$	$\frac{s_1-s_2-t+4}{2}$	$\frac{s_2-s_1-t+4}{2}$
$t > \lvert s_1 - s_2\rvert$	(p,q)b	$\frac{s_1+s_2+t-2n}{2}$	$\frac{s_2-s_1+t+4}{2}$	$\frac{s_1-s_2+t+4}{2}$

Even if the definition of $I^m_{n,\xi}$ looks like a re-parametrization of $J_{n,(s_1,s_2),t}$, the equality (C.2) is valid only under certain very *broad* conditions on the μ, ν and τ variables [1] . At this point of the discussion we prefer to switch to more physical notations, that involve the usual terminology for the gauge group ranks, the CS level and the number of flavors. Thus the quantities n, m, s_1, s_2 and t are redefined as

$$n = N_c \quad , \quad m = \widetilde{N}_c \quad , \quad s_1 = N_f \quad , \quad s_1 = \widetilde{N}_f \quad , \quad t = -2k \quad , \tag{C.3}$$

[1]We can always suppose that the values of μ, ν and τ are quite generic and that this does not spoil the relations between the integrals.

We are only interested in non chiral like theories and therefore fix $\widetilde{N}_f = N_f$. In terms of these variables the table becomes

condition	type	$\widetilde{N}_c$	p	q
$k > 0$	(p,q)a	$N_c + k$	$2 + k$	$2 + k$
$k < 0$	(p,q)b	$N_c - k$	$2 - k$	$2 - k$

Eventually the most useful result of [187], for our applications, is that the a and b type integrals are related as [2]

$$\begin{aligned}
I^{m}_{n,(p,q)a}(\mu;\nu;\lambda) = I^{n}_{m,(p,q)b}&(\omega-\nu;\omega-\mu;(p-q)\omega-\lambda)\prod_{r,s}\Gamma_h(\mu_r+\nu_s)\zeta^{(-6+2p+2q-pq)} \\
&\times c((\frac{1}{2}(p-q)^2+pq+(4-p-q)(m+2)-4)\omega^2+\frac{1}{2}\lambda^2) \\
&\times c((2-p)\sum_r \mu_r^2+(2-q)\sum_s \nu_s^2+\frac{1}{2}(2m\omega-\sum_r\mu_r-\sum_s\nu_s)^2) \\
&\times c(\lambda(\sum_r\mu_r-\sum_s\nu_s+(p-q)\omega)+(p+q-4)(\sum_r\mu_r+\sum_s\nu_s)\omega)
\end{aligned} \tag{C.4}$$

where $r = 1,\ldots,m+n+2-q\,(\equiv s_1)$ and $s = 1,\ldots,m+n-2+p\,(\equiv s_2)$. Upon substituting (C.3) and fixing $\widetilde{N}_f = N_f$ this becomes

$$\begin{aligned}
I^{\widetilde{N}_c}_{N_c,(2+k,2+k)a}(\mu;\nu;\lambda) =& I^{N_c}_{\widetilde{N}_c,(2+k,2+k)b}(\omega-\nu;\omega-\mu;-\lambda)\times \\
&\prod_{r,s=1}^{N_f}\Gamma_h(\mu_r+\nu_s)\zeta^{-k^2-2}\times \\
&c\left(k\left(\sum_{r=1}^{N_f}\mu_r^2+\sum_{s=1}^{N_f}\nu_r^2\right)+k(k-2m)\omega^2+\frac{1}{2}\lambda^2-2k\left(\sum_{r=1}^{N_f}\mu_r+\sum_{s=1}^{N_f}\nu_s\right)\omega\right) \\
&\times c\left(\lambda\left(\sum_{r=1}^{N_f}\mu_r-\sum_{s=1}^{N_f}\nu_s\right)+\frac{1}{2}(2m\omega-\sum_{r=1}^{N_f}\mu_r-\sum_{s=1}^{N_f}\nu_s)^2\right)
\end{aligned} \tag{C.5}$$

A few comments are in order. First the difference between the case a and b is in the sign

[2] As observed in [40] this result slightly differs from the one on [187]. We are grateful to the authors of [40] for discussions on this point.

of the CS level k. In this case we fixed $k > 0$ but the same equality can be reversed if one starts with $k < 0$ and use the equation (5.5.7) in [187]. This identifies $\tilde{N}_c$ with $N_c + |k|$. Moreover, as discussed in [187], $t + s_1 + s_2$ is always even for the above *degenerations*. This corresponds to requiring $|k| + \frac{N_f + \tilde{N}_f}{2}$ to be integer. This is the same as the parity anomaliy condition of three dimensional field theories [162].

C.1.2 The symplectic case

The second class of integral that we need from [187] is associated with the symplectic group $SP(2N_c)_k$. The integrals have been dubbed as $JI_{n,s,t}(\mu;\tau)$ in [187]. Explicitly they are

$$JI_{n,s_1,t}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int \frac{\prod_{i\le j<k\le n}\Gamma_h(\tau \pm x_j \pm x_k)\prod_{j=1}^{n}\prod_{r=1}^{s_1}\Gamma_h(\mu_r \pm x_j)}{\prod_{i\le j<k\le n}\Gamma_h(\pm x_j \pm x_k)\prod_{j=1}^{n}\Gamma_h(\pm 2x_j)}\prod_{j=1}^{n} c(2tx_j^2)dx_j \tag{C.6}$$

In this case τ labels the fields in the antisymmetric representation while μ is the label for fields in the fundamental representation. In the absence any anti-symmetric representations τ gets identified with ω. In this case the integral (C.6) becomes

$$\begin{aligned} I^m_{n,\mathrm{p}a}(\mu) &= JI_{n,2n+2m+4-\mathrm{p},2-\mathrm{p}}(\mu;\omega) \\ I^m_{n,\mathrm{p}b}(\mu) &= JI_{n,2n+2m+4-\mathrm{p},\mathrm{p}-2}(\mu;\omega) \end{aligned} \tag{C.7}$$

where where pa or pb are fixed as

condition	type	m	p
$t < 0$	pa	$\frac{s_1 - t - 2n - 2}{2}$	$2 - t$
$t > 0$	pb	$\frac{s_1 - t - 2n - 2}{2}$	$2 + t$

As in the case of unitary groups we switch to more physical parameters

$$t = -2k \quad n = N_c \quad s_1 = 2N_f \quad m = \widetilde{N}_c \tag{C.8}$$

In temrs of these parameters the table becomes

condition	type	$\widetilde{N}_c$	p
$k > 0$	pa	$N_f + k - N_c - 1$	$2(1+k)$
$k < 0$	pb	$N_f - k - N_c - 1$	$2(1-k)$

The transformation properties of these integrals, given in [187], become (we fix $k > 0$)

$$\begin{aligned} I^{\widetilde{N}_c}_{N_c,2(1+k)a}(\mu) = I^{N_c}_{\widetilde{N}_c,2(1+k)b}(\omega-\mu) \prod_{1\leq r<s\leq 2N_f} \Gamma_h(\mu_r+\mu_s)\zeta^{(k-1)(1-2k)} \\ \times c\left(-2k\sum_{r=1}^{2N_f}(\mu_r-\omega)^2+\left(\left(2\widetilde{N}_c+1\right)\omega-\sum_{r=1}^{2N_f}\mu_r\right)^2+2k\left(2N_c-\frac{2k-1}{2}\right)\omega^2\right) \end{aligned} \tag{C.9}$$

As in the unitary case the difference between the case a and b is in the sign of the CS level k, and the case with $k < 0$ is obtained from (C.9) after using relation (5.5.2) of [187] .

C.2 Characters

In the paper we studied different representations for the orthogonal, symplectic and unitary groups. In this appendix we list the formula for the characters of the representation of these groups. As usual we identify a representation of a simple group of rank n by its Dynkin labels, a set of n integers $(s_1, \ldots, s_n)$ which are assigned to the simple roots of the group by the Dynkin diagrams. Then the characters of the representations are associated to the Schur polynomials as functions of the eigenvalues of the group G, parameterizing the maximal abelian torus. In the cases we investigated the Schur polynomials are

- $\underline{U(n)}$

$$P_{\vec{s}} = \frac{\det z_i^{s_j+n-j}}{\det z_i^{n-j}} \qquad i,j=1,\ldots,n \tag{C.10}$$

- $\underline{SP(2n)}$

$$P_{\vec{s}} = \frac{\det\left(z_i^{s_j+n-j+1} - z_i^{-(s_j+n-j+1)}\right)}{\det\left(z_i^{n-j+1} - z_i^{-(n-j+1)}\right)} \qquad i,j=1,\ldots,n \tag{C.11}$$

- $\underline{SO(2n)}$

$$P_{\vec{s}} = \frac{\det\left(z_i^{s_j+n-j} + z_i^{-(s_j+n-j)}\right) + \det\left(z_i^{s_j+n-j} - z_i^{-(s_j+n-j)}\right)}{2\det\left(z_i^{n-j+1} - z_i^{-(n-j+1)}\right)} \prod_{i=1}^{n}\left(z_i - \frac{1}{z_i}\right) \tag{C.12}$$

with $i,j=1,\ldots,n$

- $\underline{SO(2n+1)}$

$$P_{\vec{s}} = \frac{\det\left(z_i^{s_j+\frac{1}{2}+n-j} + z_i^{-(s_j+\frac{1}{2}+n-j)}\right)}{2\det\left(z_i^{n-j+1} - z_i^{-(n-j+1)}\right)} \frac{\prod_{i=1}^{n}\left(z_i - \frac{1}{z_i}\right)}{\prod_{i=1}^{n}\left(z_i^{\frac{1}{2}} - \frac{1}{z_i^{\frac{1}{2}}}\right)} \tag{C.13}$$

In the computation of the partition function we actually used the substitution

$$z_i = e^{ix_i} \tag{C.14}$$

and we studied the characters to respect to the x_i variables. For example in the adjoint representation we have

Group	Dynkin Label	Non Zero Roots ($i < j$)
$U(n)$	$s = (2, 1, \ldots, 1, 0)$	$\pm(x_i - x_j)$
$SP(2n)$	$s = (2, 0, \ldots, 0, 0)$	$\pm x_i \pm x_j, \quad \pm 2x_i$
$SO(2n)$	$s = (1, 1, 0, \ldots, 0, 0)$	$\pm x_i \pm x_j$
$SO(2n+1)$	$s = (1, 1, 0, \ldots, 0, 0)$	$\pm x_i \pm x_j, \quad \pm x_i$

In addition in every case there are n zero roots associated to the adjoint of the four cases. By applying the same formulas we can obtain the characters for the other representations.

This appendix is a reprint of the material as it appears in "Refined Checks and Exact Dualities in Three Dimensions " , Prarit Agarwal, Antonio Amariti, Massimo Siani , JHEP 1210 (2012) 178, of which I was a co-author.

Appendix D

New N=1 Dualities from M5-branes and Outer-automorphism Twists

D.1 Chiral ring relations of $T_{SO(2N)}$ and $\tilde{T}_{SO(2N)}$ theories

D.1.1 $T_{SO(2N)}$

Consider the $\mathcal{N}=2$ superconformal quiver gauge theory with the gauge groups

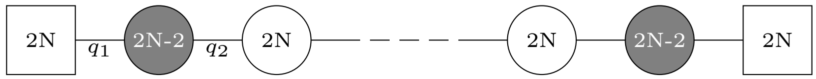
$$USp(2N-2) \times SO(2N) \times \cdots \times SO(2N) \times USp(2N-2) ,$$

with a total of $2N-3$ gauge factors and also N fundamentals at the two end of the quiver, from which we realize the $SO(2N)$ flavor symmetry at each ends. This is dual to a T_N block with $SO(2N)^3$ flavor symmetry, coupled to a superconformal tail given by

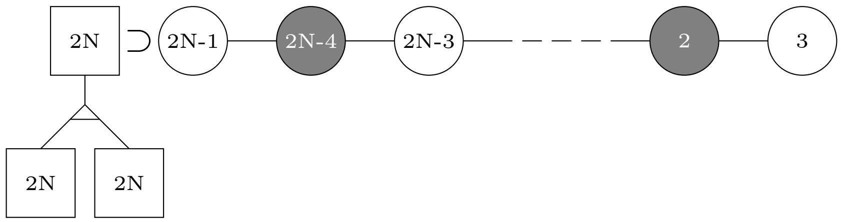
$$SO(2N-1) \times USp(2N-4) \times SO(2N-2) \times \cdots \times USp(2) \times SO(3) .$$

Pictorially we can represent the two dual theories by figure D.1.

Note that in the dual frame the $SO(2N-1)$ sub-group of one of the three $SO(2N)$ flavor symmetries of the T_N block is gauged while the other two $SO(2N)$ flavor symmetries are in one to one correspondence with flavor symmetries at the ends of the



(a) Linear quiver with $SO(2N)$ ends



(b) Dual frame with $T_{SO(2N)}$ block

Figure D.1. The linear quiver dual to $T_{SO(2N)}$ coupled to a superconformal tail

linear quiver. We thus expect the operator $\mu_{1\alpha\beta}$ transforming in the adjoint representation of $SO(2N)_1$ to be identified with $\Omega_{ij}q_{1\alpha}^{\ i}q_{1\beta}^{\ j}$ in the linear quiver. Here Ω is the invariant anti-symmetric form of the $USp(2N-2)$ group. Similarly we can also identify the operator that corresponds to the dual of $\mu_{2\alpha\beta}$. We now want to establish the chiral ring relation

$$\mathrm{tr}\mu_1^2 = \mathrm{tr}\mu_2^2 \ . \tag{D.1}$$

To see this note that the F-term equation of motion of the linear quiver are given by

$$\begin{aligned} q_{1\alpha}^{\ i}q_{1\alpha}^{\ j} + q_{2\beta}^{\ i}q_{2\beta}^{\ j} &= 0 \ , \\ \Omega_{ij}(q_{2\alpha}^{\ i}q_{2\beta}^{\ j} + q_{3\alpha}^{\ i}q_{3\beta}^{\ j}) &= 0 \ , \\ q_{3\alpha}^{\ i}q_{3\alpha}^{\ j} + q_{4\beta}^{\ i}q_{4\beta}^{\ j} &= 0 \ , \\ &\vdots \end{aligned} \tag{D.2}$$

Using these relations we find that

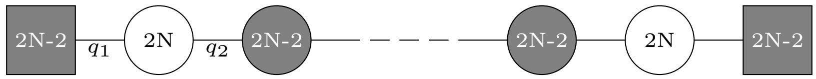
$$\begin{aligned}
\mathrm{tr}\mu_1^2 &= \mu_{1\alpha\beta}\mu_{1\beta\alpha} \\
&= \Omega_{ij}\Omega_{lm} q_{1\alpha}^{\ i} q_{1\beta}^{\ j} q_{1\beta}^{\ l} q_{1\alpha}^{\ m} \\
&= \Omega_{ij}\Omega_{lm} q_{1\alpha}^{\ i} q_{1\alpha}^{\ m} q_{1\beta}^{\ j} q_{1\beta}^{\ l} \\
&= \Omega_{ij}\Omega_{lm} q_{2\alpha}^{\ i} q_{2\alpha}^{\ m} q_{2\beta}^{\ j} q_{2\beta}^{\ l} \\
&= (\Omega_{ij} q_{2\alpha}^{\ i} q_{2\beta}^{\ j})(\Omega_{lm} q_{2\beta}^{\ l} q_{2\alpha}^{\ m}) \\
&= \Omega_{ij}\Omega_{lm} q_{3\alpha}^{\ i} q_{3\beta}^{\ j} q_{3\beta}^{\ l} q_{3\alpha}^{\ m} \\
&= \Omega_{ij}\Omega_{lm} q_{4\alpha}^{\ i} q_{4\beta}^{\ j} q_{4\beta}^{\ l} q_{4\alpha}^{\ m} \\
&= \mathrm{tr}\dot{\mu}^2 \ ,
\end{aligned} \tag{D.3}$$

where $\dot{\mu}_{\alpha\beta}$ is the operator transforming in the adjoint of the $SO(2N)$ gauge group in the linear quiver. Propagating this relation across the quiver we then establish that $\mathrm{tr}\mu_1^2 = \mathrm{tr}\mu_2^2$. By symmetry we thus expect that in the strongly coupled $T_{SO(2N)}$ block the following chiral ring relation holds

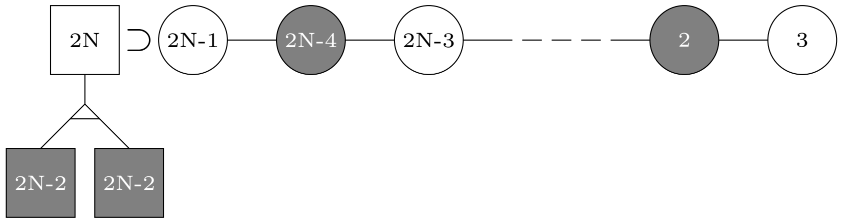
$$\mathrm{tr}\mu_1^2 = \mathrm{tr}\mu_2^2 = \mathrm{tr}\mu_3^2 \ . \tag{D.4}$$

D.1.2 $\tilde{T}_{SO(2N)}$

We now consider the linear quiver given by gauge groups $SO(2N) \times USp(2N-2) \times \cdots \times USp(2N-2) \times SO(2N)$. There are a total of $2N-3$ gauge groups and each end has $USp(2N-2)$ flavor symmetry. This is dual to a $\tilde{T}_{SO(2N)}$ block coupled to a superconformal tail $SO(2N-1) \times USp(2N-4) \times \cdots \times USp(2) \times SO(3)$ where the $SO(2N-1)$ node of the tail is a sub-group of the $SO(2N)$ flavor symmetry of $\tilde{T}_{SO(2N)}$. See figure D.2. The two $USp(2N-2)$ flavor symmetries of the $\tilde{T}_{SO(2N)}$ block can then



(a) Linear quiver with $USp(2N-2)$ ends



(b) Dual frame with $\tilde{T}_{SO(2N)}$ block

Figure D.2. The linear quiver dual to $\tilde{T}_{SO(2N)}$ coupled to a superconformal tail

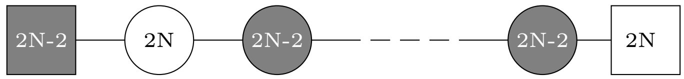
be identified with the flavor symmetry at either end of the linear quiver. It is then straight forward to use the F-term relations of the linear quiver to establish the chiral ring relation

$$\mathrm{tr}\Omega\mu_1\Omega\mu_1 = \mathrm{tr}\Omega\mu_2\Omega\mu_2 \ , \tag{D.5}$$

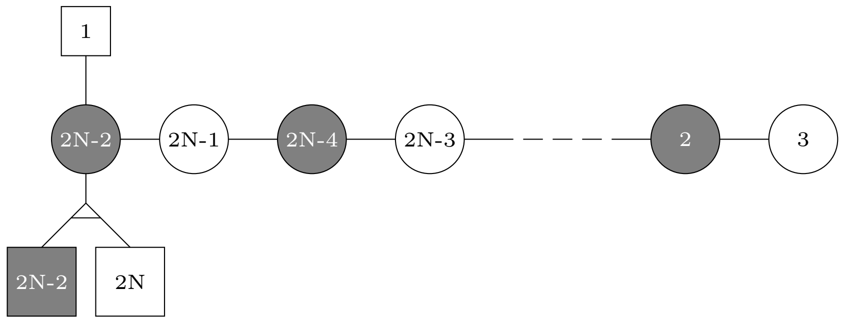
where μ_1 and μ_2 are the dimension 2 operators transforming as the adjoint of $USp(2N-2)$ flavor symmetries of $\tilde{T}_{SO(2N)}$.

We can also consider the superconformal linear quiver of $2N-2$ nodes given by $SO(2N) \times USp(2N-2) \times \cdots \times USp(2N-2)$. The quiver then ends in a $USp(2N-2)$ flavor symmetry on the left and a $SO(2N)$ symmetry on the right. This theory can be shown to be S-dual to a $\tilde{T}_{SO(2N)}$ block coupled to a superconformal tail whose nodes are $USp(2N-2) \times SO(2N-1) \times USp(2N-4) \times \cdots \times SO(3)$. The $USp(2N-2)$ node of the tail is obtained by gauging one of the two $USp(2N-2)$ flavor symmetries of the $\tilde{T}_{SO(2N)}$ block. We will also need to couple a half-hyper to this node in order to ensure that its β-function vanishes. These theories can be visualized as in figure D.3.

Now if μ_1^{ij} is the dimension 2 operator of $\tilde{T}_{SO(2N)}$ theory transforming in adjoint representation of $USp(2N-2)$ flavor symmetry while $\mu_{3\alpha\beta}$ is the dim. 2 operator transforming as the adjoint of the $SO(2N)$ flavor symmetry then we identify their duals



(a) Linear quiver with $USp(2N-2)$ and $SO(2N)$ ends



(b) Dual frame with $\tilde{T}_{SO(2N)}$ block

Figure D.3. The linear quiver dual to $\tilde{T}_{SO(2N)}$ coupled to a superconformal tail

in the linear quiver to be such that

$$\mu_1^{ij} = q_{1\alpha}^{\;i} q_{1\alpha}^{\;j} \ , \tag{D.6}$$

$$\mu_{3\alpha\beta} = \Omega_{ij} q_{2N-1,\alpha}^{\;i} q_{2N-1,\beta}^{\;j} \ . \tag{D.7}$$

The F-term relations of the linear quiver are

$$\begin{aligned} \Omega_{ij}(q_{1\alpha}^{\;i} q_{1\beta}^{\;j} + q_{2\alpha}^{\;i} q_{2\beta}^{\;j}) &= 0 \ , \\ q_{2\alpha}^{\;i} q_{2\alpha}^{\;j} + q_{3\beta}^{\;i} q_{3\beta}^{\;j} &= 0 \ , \\ \Omega_{ij}(q_{3\alpha}^{\;i} q_{3\beta}^{\;j} + q_{4\alpha}^{\;i} q_{4\beta}^{\;j}) &= 0 \ , \\ &\vdots \end{aligned} \tag{D.8}$$

Using these we can then write

$$\begin{aligned}
\mathrm{tr}\Omega\mu_1\Omega\mu_1 &= \Omega_{ij} q_{1\alpha}^{\ j} q_{1\alpha}^{\ k} \Omega_{kl} q_{1\beta}^{\ l} q_{1\beta}^{\ i} \\
&= \Omega_{kl}\Omega_{ij} q_{2\beta}^{\ i} q_{2\alpha}^{\ j} q_{2\alpha}^{\ k} q_{2\beta}^{\ l} \\
&= \Omega_{kl}\Omega_{ij} q_{3\beta}^{\ i} q_{3\alpha}^{\ j} q_{3\alpha}^{\ k} q_{3\beta}^{\ l} \\
&= (\Omega_{ij} q_{3\beta}^{\ i} q_{3\alpha}^{\ j})(\Omega_{kl} q_{3\alpha}^{\ k} q_{3\beta}^{\ l}) \qquad \text{(D.9)} \\
&\vdots \\
&= (\Omega_{ij} q_{2N-1\beta}^{\ i} q_{2N-1\alpha}^{\ j})(\Omega_{kl} q_{2N-1\alpha}^{\ k} q_{2N-1\beta}^{\ l}) \\
&= \mathrm{tr}\mu_3^2 \ .
\end{aligned}$$

Thus we establish that for $\tilde{T}_{SO(2N)}$ theories, the following chiral ring relation holds

$$\mathrm{tr}\Omega\mu_1\Omega\mu_1 = \mathrm{tr}\Omega\mu_2\Omega\mu_2 = \mathrm{tr}\mu_3^2 \ . \qquad \text{(D.10)}$$

This appendix is a reprint of the material as it appears in “New $N = 1$ Dualities from $M5$-branes and Outer-automorphism Twists ”, Prarit Agarwal, Jaewon Song, JHEP 1403 (2014) 133, of which I was a co-author.

Appendix E

Quiver Tails and $\mathcal{N} = 1$ SCFTs from M-branes

E.1 The superpotential for the Fan

We now derive the superpotential that is obtained after integrating out the massive modes in section 6.3.4. Before integrating these out, the superpotential is given by

$$W_1 = \mathrm{tr} q_0 \rho^+ \tilde{q}_0 + \mathrm{tr} q_0 M \tilde{q}_0 + \mathrm{tr} \tilde{\mu}_1 q_0 \tilde{q}_0 \ , \tag{E.1}$$

where $\rho^\pm = \rho(\sigma^\pm)$. Here we write only those terms in the superpotential that are relevant to Higgsing. Recall that ρ^+ here is also the raising operator for the $SU(2)$ embedding specified by the partition of N. Also, $\tilde{\mu}_1$ is the quark bilinear given by $\tilde{\mu}_1 = \tilde{q}_1 q_1 - \frac{1}{N} \mathrm{tr} q_1 \tilde{q}_1$.

Let P and $\widetilde{P}$ be the projection matrices that project on to the massive modes of q_0 and $\tilde{q}_0$ respectively i.e.

$$\begin{aligned} \chi &= q_0 P \ , \\ \widetilde{\chi} &= \widetilde{P} \tilde{q}_0 \ , \end{aligned} \tag{E.2}$$

where χ and $\widetilde{\chi}$ represent the massives chiral fields. It is easy to check that

$$\begin{aligned}\widetilde{P} &= \rho^-\rho^+ \ , \\ P &= \rho^+\rho^- \ .\end{aligned} \tag{E.3}$$

These projection operators satisfy $\widetilde{P}\widetilde{P} = \widetilde{P}$ and $PP = P$ as is expected. The massless modes are given by $Z = q_0(\mathbb{1} - P)$ and $\widetilde{Z} = q_0(\mathbb{1} - \widetilde{P})$.

The superpotential in (E.1) can now be expanded in terms of the massive and massless modes, such that the equation of motion for χ can be written as

$$\rho^+\widetilde{\chi} + M\widetilde{\chi} + \widetilde{\chi}\tilde{\mu}_1 + M\widetilde{Z} + \widetilde{Z}\tilde{\mu}_1 = 0 \ . \tag{E.4}$$

Note that since $\tilde{\mu}_1$ in the above equation is contracted through the color indices, therefore it can be treated as a scalar multiplier in the above equation. This equation of motion can be simplified by multiplying it on the left with ρ^- reducing it to the following form

$$\widetilde{\chi} + \rho^- M\widetilde{\chi} + \rho^-\widetilde{\chi}\tilde{\mu}_1 + \rho^- M\widetilde{Z} + \rho^-\widetilde{Z}\tilde{\mu}_1 = 0 \ . \tag{E.5}$$

The solution for $\widetilde{\chi}$ is

$$\widetilde{\chi} = (\mathbb{1} - A)^{-1} A\widetilde{Z} \ , \tag{E.6}$$

where

$$A = -(\rho^- M + \tilde{\mu}_1\rho^-) \ . \tag{E.7}$$

Recall that here we are treating $\tilde{\mu}_1$ as a scalar multiplier and will appropriately contract it using its color indices at a later stage. Notice that A is a nilpotent matrix such that

$A^\ell = 0$. This follows from the fact that $A^\ell \propto (\rho^-)^\ell (M + \tilde{\mu}_1 \not{K})^\ell$ and $(\rho^-)^\ell = 0$ since it is the lowering operator of $SU(2) \hookrightarrow SU(N)$. Here we have also used the commutation relation $[\rho^-, M] = 0$ which is due to the elements of M being in the lowest weight state of their respective $SU(2)$ representations. Thus

$$\widetilde{\chi} = \sum_{n=1}^{\ell-1} A^n \widetilde{Z} \ . \tag{E.8}$$

Substituting this back in (E.1) we find that the low energy superpotential is

$$W_{\text{eff}} \quad = \quad \text{Tr} Z \widetilde{Z} \tilde{\mu}_1 + \text{Tr} Z M \widetilde{Z} + \sum_{n=1}^{\ell-1} \text{Tr} Z M A^n \widetilde{Z} + \sum_{n=1}^{\ell-1} \text{Tr} Z A^n \widetilde{Z} \tilde{\mu}_1 \ . \tag{E.9}$$

An example for the $SU(6)$ quiver As an example of our previous derivation, let us study the nilpotent Higgsing of the linear quiver with $SU(6)$ symmetries. Consider the partition $6 \to 3 + 2 + 1$. This implies

$$\langle M_0 \rangle = \rho^+ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{E.10}$$

The components of the (anti-)quark matrices can be written as

$$q_0 = \begin{pmatrix} \chi_1 & \chi_2 & Z_3 & \chi_3 & Z_2 & Z_1 \end{pmatrix} \quad \text{and} \quad \tilde{q}_0 = \begin{pmatrix} \widetilde{Z}_3 \\ \widetilde{\chi}_1 \\ \widetilde{\chi}_2 \\ \widetilde{Z}_2 \\ \widetilde{\chi}_3 \\ \widetilde{Z}_1 \end{pmatrix} .$$

with $\widetilde{\chi}_1$, $\widetilde{\chi}_2$, $\widetilde{\chi}_3$, $\widetilde{Z}_1$, $\widetilde{Z}_2$, and $\widetilde{Z}_3$ being row vectors, each of which corresponds to an anti-fundamental of $SU(6)_1$; similarly, χ_1, χ_2, χ_3, Z_1, Z_2, and Z_3 are column vectors, each of which corresponds to a fundamental of $SU(6)_1$. The vev for M gives mass to $\widetilde{\chi}_1$, $\widetilde{\chi}_2$, $\widetilde{\chi}_3$, χ_1, χ_2 and χ_3. The fluctuations M (around the vev ρ^+) that stay coupled to the theory are found by using the argument in [82]. These are

$$M = \begin{pmatrix} M^2_{33} & 0 & 0 & 0 & 0 & 0 \\ M^1_{33} & M^2_{33} & 0 & M^1_{32} & 0 & 0 \\ M^0_{33} & M^1_{33} & M^2_{33} & M^0_{32} & M^1_{32} & M^0_{31} \\ M^1_{23} & 0 & 0 & M^1_{22} & 0 & 0 \\ M^0_{23} & M^1_{23} & 0 & M^0_{22} & M^1_{22} & M^0_{21} \\ M^0_{13} & 0 & 0 & M^0_{12} & 0 & -(3M^2_{33} + 2M^1_{22}) \end{pmatrix} . \tag{E.11}$$

Upon integrating out the massive chiral fields and including the fluctuations (E.11), the effective superpotential becomes

$$\begin{aligned} W_{\text{eff}} =& \text{tr}\tilde{\mu}_1 Z_1 \widetilde{Z}_1 - 3\text{tr}Z_1 M^2_{33} \widetilde{Z}_1 - 2\text{tr}Z_1 M^1_{22} \widetilde{Z}_1 + \text{tr}Z_2 M^0_{22} \widetilde{Z}_2 + \text{tr}Z_3 M^0_{33} \widetilde{Z}_3 \\ & - \text{tr}Z_2 \widetilde{Z}_2 (\tilde{\mu}_1)^2 - 2\text{tr}Z_2 M^1_{22} \widetilde{Z}_2 \tilde{\mu}_1 - \text{tr}Z_2 (M^1_{22})^2 \widetilde{Z}_2 + \text{tr}Z_3 (M^2_{33})^3 \widetilde{Z}_3 \\ & + 3\text{tr}Z_3 (M^2_{33})^2 \widetilde{Z}_3 \tilde{\mu}_1 + 3\text{tr}Z_3 M^2_{33} \widetilde{Z}_3 (\tilde{\mu}_1)^2 + \text{tr}Z_3 \widetilde{Z}_3 (\tilde{\mu}_1)^3 - 2\text{tr}Z_3 M^1_{33} \widetilde{Z}_3 \tilde{\mu}_1 \\ & - 2\text{tr}Z_3 M^1_{33} M^2_{33} \widetilde{Z}_3 - \text{tr}Z_3 M^1_{23} M^1_{32} \widetilde{Z}_3 + \text{tr}Z_1 M^0_{12} \widetilde{Z}_2 + \text{tr}Z_1 M^0_{13} \widetilde{Z}_3 \\ & + \text{tr}Z_2 M^0_{21} \widetilde{Z}_1 + \text{tr}Z_2 M^0_{23} \widetilde{Z}_3 - \text{tr}Z_2 M^1_{22} M^1_{23} \widetilde{Z}_3 - \text{tr}Z_2 M^2_{33} M^1_{23} \widetilde{Z}_3 \\ & - 2\text{tr}Z_2 M^1_{23} \widetilde{Z}_3 \tilde{\mu}_1 + \text{tr}Z_3 M^0_{31} \widetilde{Z}_1 + \text{tr}Z_3 M^0_{32} \widetilde{Z}_2 - \text{tr}Z_3 M^1_{22} M^1_{32} \widetilde{Z}_2 \\ & - \text{tr}Z_3 M^2_{33} M^1_{32} \widetilde{Z}_2 - 2\text{tr}Z_3 M^1_{32} \widetilde{Z}_2 \tilde{\mu}_1 + \text{tr}\mu_2 \phi + \text{tr}\tilde{\mu}_2 \phi \ . \end{aligned} \tag{E.12}$$

This matches exactly with what one would write for the Fan corresponding to the partition $6 \to 3 + 2 + 1$.

E.2 Higgsing $\mathcal{N} = 2$ quiver theories

Consider the linear quiver in $\mathcal{N} = 2$ class $\mathcal{S}$ theories of type A_{N-1} with the gauge group

$$G = \prod_{i=1}^{N-1} SU(N)_i \ . \tag{E.13}$$

The matter content of the theory consist of hypermultiplets $H_i = (Q_i, \widetilde{Q}_i)$ of $SU(N)_i \times SU(N)_{i+1}$. In addition to this we also have N hypermulitplets $H_0 = (Q_0, \widetilde{Q}_0)$ transforming in the fundamental representation of $SU(N)_1$ and N hypermultiplets $H_{N-1} = (Q_{N-1}, \widetilde{Q}_{N-1})$ transforming in the fundamental representation of $SU(N)_{N-1}$. Thus at each of the quiver there is an SU(N) flavor symmetry acting on the hypermultiplets H_0 and H_{N-1} respectively. We denote the flavor symmetry of H_0 by $SU(N)_0$ and that of

H_{N-1} by $SU(N)_N$.

In order to avoid introducing too many indices labeling the symmetries under which Q_i and $\widetilde{Q}_i$ transform, we will treat them as $N \times N$ matrices such that $Q_i\widetilde{Q}_i$ will be an invariant of $SU(N)_i$ while $\widetilde{Q}_iQ_i$ will be an invariant of $SU(N)_{i+1}$. Thus the superpotential of this quiver will be given by

$$W = \sqrt{2}\sum_{i=1}^{N-1} \mathrm{Tr}\left(\widetilde{Q}_{i-1}\Phi_i Q_{i-1} - Q_i\Phi_i\widetilde{Q}_i\right) \ . \tag{E.14}$$

We now wish to consider an $SU(N)$ linear quiver and Higgsing its leftmost full puncture down to a puncture given by the Young's tableau corresponding to the following partition of N

$$N = n_1 + 2n_2 + \ldots + \ell n_\ell \ . \tag{E.15}$$

This breaks $SU(N)_0$ down to $S[U(n_1) \times U(n_2) \times \ldots U(n_\ell)]$. The corresponding vev for $\mu_0 = \widetilde{Q}_0Q_0 - \frac{1}{N}\mathrm{tr}\widetilde{Q}_0Q_0$ that does the job for us is given by

$$\langle\mu_0\rangle = J_1^{\oplus n_1} \oplus J_2^{\oplus n_2} \oplus \ldots \oplus J_\ell^{\oplus n_\ell} \ , \tag{E.16}$$

where J_k is the Jordan cell of size k. This can then be decomposed into the following vevs for Q_0 and $\widetilde{Q}_0$:

$$\langle\widetilde{Q}_0\rangle = J_1^{\oplus n_1} \oplus J_2^{\oplus n_2} \oplus \ldots \oplus J_\ell^{\oplus n_\ell} \ , \tag{E.17}$$

and

$$\langle Q_0\rangle = J_1^{\oplus n_1} \oplus (J_1 \oplus I_1)^{\oplus n_2} \oplus \ldots \oplus (J_1 \oplus I_{\ell-1})^{\oplus n_\ell} \ . \tag{E.18}$$

Here I_k is the identity matrix of size k. It is straight forward to see that this breaks $SU(N)_1$ down to $SU(n_1 + n_2 + \ldots + n_k)$. The D-term constraints are trivially satisfied while the F-term for Φ_1 gives us

$$Q_0\widetilde{Q}_0 - \frac{1}{N}\mathrm{tr}Q_0\widetilde{Q}_0 = \widetilde{Q}_1 Q_1 - \frac{1}{N}\mathrm{tr}Q_1\widetilde{Q}_1 \ . \tag{E.19}$$

This chiral ring relation then forces us to have

$$\langle \widetilde{Q}_1 Q_1 \rangle = J_1^{\oplus(n_1+2n_2)} \oplus (J_1 + J_2)^{\oplus n_3} \oplus \ldots \oplus (J_1 \oplus J_{\ell-1})^{\oplus n_\ell} \ , \tag{E.20}$$

which decomposes into

$$\langle \widetilde{Q}_1 \rangle = J_1^{\oplus(n_1+2n_2)} \oplus (J_1 + J_2)^{\oplus n_3} \oplus \ldots \oplus (J_1 \oplus J_{\ell-1})^{\oplus n_\ell} \ , \tag{E.21}$$

and

$$\langle Q_1 \rangle = J_1^{\oplus(n_1+2n_2)} \oplus (J_1 \oplus J_1 \oplus I_1)^{\oplus n_3} \oplus \ldots \oplus (J_1 \oplus J_1 \oplus I_{\ell-2})^{\oplus n_\ell} \ , \tag{E.22}$$

thereby breaking $SU(N)_2$ down to $SU(n_1 + 2n_2 + 2n_3 + \ldots + 2n_k)$. Application of chiral ring relation at each node then gives us the general pattern of the vevs, which are found to be

$$\begin{aligned} \langle \widetilde{Q}_{i-1} Q_{i-1} \rangle =& J_1^{\oplus(n_1+2n_2+\ldots+in_i)} \oplus \ldots \oplus (J_1^{\oplus(i-1)} \oplus J_{k-i+1})^{\oplus n_k} \\ & \oplus \ldots \oplus (J_1^{\oplus(i-1)} \oplus J_{\ell-i+1})^{\oplus n_\ell} \ , \end{aligned} \tag{E.23}$$

such that

$$\begin{aligned}\langle \widetilde{Q}_{i-1} \rangle =& J_1^{\oplus(n_1+2n_2+\ldots+in_i)} \oplus \ldots \oplus (J_1^{\oplus(i-1)} \oplus J_{k-i+1})^{\oplus n_k} \\ & \oplus \ldots \oplus (J_1^{\oplus(i-1)} \oplus J_{\ell-i+1})^{\oplus n_\ell} \ ,\end{aligned} \tag{E.24}$$

and

$$\begin{aligned}\langle Q_{i-1} \rangle =& J_1^{\oplus(n_1+2n_2+\ldots+in_i)} \oplus \ldots \oplus (J_1^{\oplus i} \oplus I_{k-i})^{\oplus n_k} \\ & \oplus \ldots \oplus (J_1^{\oplus i} \oplus I_{\ell-i})^{\oplus n_\ell} \ .\end{aligned} \tag{E.25}$$

To check that these vevs do satisfy (E.23) we use the rules that $J_k \cdot (J_1 \oplus I_{k-1}) = J_k$ and $(J_1 \oplus I_{k-1}) \cdot J_k = J_1 \oplus J_{k-1}$. The structure of these vevs imply that $SU(N)_i$ gets broken down to $SU(n_1 + 2n_2 + 3n_3 + \ldots + in_i + in_{i+1} \ldots + in_k)$. Also $SU(N)_{\ell-1}$ gets broken down to $SU(N - n_l)$ while all the gauge groups from $SU(N)_\ell$ onwards remain unbroken. Thus we see that the gauge symmetry of the low energy theory obtained after Higgsing is given by

$$G' = \prod_{i=1}^{\ell-1} SU(N_i) \times \prod_{j=1}^{N-\ell} SU(N)_j \ , \tag{E.26}$$

where $N_i = n_1 + 2n_2 + 3n_3 + \ldots + in_i + in_{i+1} \ldots + in_\ell$. Apart from hypermultiplets H_i transforming as the bifundamental of $SU(N_{i-1}) \times SU(N_i)$, there will be m_i fundamentals at the gauge group $SU(N_i)$. Superconformality requires that

$$m_i + N_{i-1} + N_{i+1} = 2N_i \ , \tag{E.27}$$

which then leads to $m_i = n_i$. This is coherent with the fact that the flavor symmetry of the Higgsed puncture corresponds to the symmetry associated with the additional n_i fundamentals attached to $SU(N_i)$.

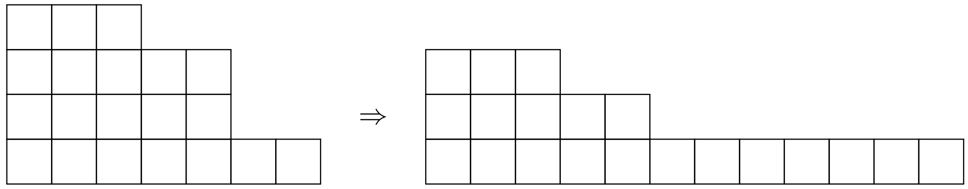


Figure E.1. Collapsing of a Young tableau

Notice that the vev $\langle\mu_i\rangle = \langle\widetilde{Q}_i Q_i\rangle - \frac{1}{N}\langle \mathrm{tr}\widetilde{Q}_i Q_i\rangle$ can be understood as the vev corresponding to partitioning N as $N = (N_{i-1} + n_i) + 2n_{i+1} + \ldots + (\ell - i + 1)n_\ell$. The section of the quiver tail from the i-th node onward can then be thought of as being obtained from a linear $SU(N)$-quiver whose left puncture has been Higgsed according to this partition. This implies that the propagation of vevs along the tail can also be neatly encoded into the process of collapsing the Young's tableau at each step. Thus if we start with the partition $N = n_1 + 2n_2 + \ldots + \ell n_\ell$, then the Young's tableau at the next step in the quiver tail is obtained in the following manner: We remove the highest box from each column of boxes in the tableau. The boxes that were removed are stacked against the residual tableau in a single row. For example if we consider the partition $20 = 1 + 1 + 3 + 3 + 4 + 4 + 4$, then at the next step in the quiver tail, its tableau collapses into the partition as described in figure E.1.

The massive and massless matter fields In order to obtain the number of fundamentals at the i-th node of the tail, we had invoked superconformality of the low energy theory, however, we should be able to derive this without resorting to an a priori assumption that the low energy theory is superconformal. To do this we now focus on the various matter fields that get massive in the process of giving vevs. Once again we consider the case of partial Higgsing (given by the partition of N, as in (E.15)) of a full-puncture of the $SU(N)$ linear quiver. We will make use of the following rules of decomposition:

$$\begin{aligned} SU(N)_i &\to SU(N_i) \\ N &\to N_i \oplus 1^{\oplus(N-N_i)} \ , \\ \text{adj} &\to \text{adj} \oplus N_i{}^{\oplus(N-N_i)} \oplus \bar{N}_i^{\oplus(N-N_i)} \oplus 1^{\oplus(N-N_i)^2} \ . \end{aligned} \tag{E.28}$$

Also note that H_{i-1} transforms as a bifundamental of $SU(N)_{i-1} \times SU(N)_i$ and can be decomposed into irreducible representations of $SU(N_{i-1}) \times SU(N_i)$ as

$$\begin{aligned} SU(N)_{i-1} \times SU(N)_i &\to SU(N_{i-1}) \times SU(N_i) \\ Q_{i-1} : (\bar{N}, N) &\to (\bar{N}_{i-1}, N_i) \oplus (\bar{N}_{i-1}, 1)^{\oplus(N-N_i)} \oplus (1, N_i)^{\oplus(N-N_{i-1})} \\ &\quad \oplus (1,1)^{\oplus(N-N_i)(N-N_{i-1})} \ , \\ \widetilde{Q}_{i-1} : (N, \bar{N}) &\to (N_{i-1}, \bar{N}_i) \oplus (N_{i-1}, 1)^{\oplus(N-N_i)} \oplus (1, \bar{N}_i)^{\oplus(N-N_{i-1})} \\ &\quad \oplus (1,1)^{\oplus(N-N_i)(N-N_{i-1})} \ . \end{aligned}$$

From (E.28) we see that upon Higgsing $SU(N)_i \to SU(N_i)$ via vevs for H_{i-1} and H_i, the vector multiplets of $SU(N)_i$ that end up getting a mass will need to eat $2(N-N_i)$ chiral multiplets transforming as the N_i-dimensional representation of $SU(N_i)$. There are $(N-N_{i-1})$ such chirals in H_{i-1} and $(N-N_i)$ such chirals in H_i. Thus we are left behind with $2(N-N_i)-(N-N_{i-1})-(N-N_i)=n_i$ chiral super fields that transform as fundamentals of $SU(N_i)$. We will similarly be left with n_i chiral multiplets transforming as the anti-fundamental of $SU(N_i)$. These will together give us n_i hypers transforming in the fundamental of $SU(N_i)$. We also end up eating some of the singlets. The number of singlet hypers that are left behind (these are the hypers that decouple from the rest of the quiver) is then given by

$$\sum_{i=1}^{k}(N-N_i)(N_i-N_{i-1}) \qquad \text{where } N_0=0\ . \tag{E.29}$$

These decoupled hypers are the Goldstone multiplets that we expect upon spontaneously breaking the global symmetry. It can be easily checked that the number of the Goldstone chiral superfields in these hypers is same as the number of generators of the complexified $SU(N)$ that are broken by $\langle\mu\rangle$ i.e. the Goldstone chiral superfields are in one-to-one correspondence with the generators X of $SL(N,\mathbb{C})$ which obey

$$[X,\langle\mu_0\rangle]\neq 0\ . \tag{E.30}$$

Apart from these there will of course be massless hypers that transform as bifundamentals of $SU(N_{i-1})\times SU(N_i)$. We thus obtained the desired low energy quiver.

As an explicit example of the above pattern of massive and massless matter fields, we consider an $SU(4)$ linear quiver and Higgs its left full-puncture down to a simple puncture. We give appropriate vevs to H_0 and H_1, Higgsing $SU(4)_1\times SU(4)_2$ down to $SU(2)\times SU(3)$. The decomposition of vector multiplets into irreps. of the low energy gauge symmetry is given by

$$\begin{aligned}
&SU(4)_1\times SU(4)_2\to SU(2)\times SU(3)\\
&\qquad V_1:(\text{adj},1)\to(\text{adj},1)\oplus(2,1)\oplus(2,1)\oplus(\bar{2},1)\oplus(\bar{2},1)\oplus(1,1)^{\oplus 4}\ ,\\
&\qquad V_2:(1,\text{adj})\to(1,\text{adj})\oplus(1,3)\oplus(1,\bar{3})\oplus(1,1)\ ,
\end{aligned} \tag{E.31}$$

while the hypers H_0 and H_1 decompose as

$$\begin{aligned} & SU(4)_1 \times SU(4)_2 \to SU(2) \times SU(3) \\ & \quad (Q_0)_i : (4,1) \to (2,1) \oplus (1,1)^{\oplus 2} \ , \\ & \quad (\widetilde{Q}_0)_i : (\bar{4},1) \to (\bar{2},1) \oplus (1,1)^{\oplus 2} \ , \\ & \qquad Q_1 : (\bar{4},4) \to (\bar{2},3) \oplus (1,3)^{\oplus 2} \oplus (\bar{2},1) \oplus (1,1)^{\oplus 2} \ , \\ & \qquad \widetilde{Q}_1 : (4,4) \to (2,\bar{3}) \oplus (1,\bar{3})^{\oplus 2} \oplus (2,1) \oplus (1,1)^{\oplus 2} \ , \end{aligned} \tag{E.32}$$

The various chiral multiplets that get eaten via Higgsing are: 4 copies transforming as $(2,1)$, 4 copies of $(\bar{2},1)$, 2 copies each of $(1,3)$ and $(1,\bar{3})$ and 10 copies of $(1,1)$. We are thus left behind with a chiral multiplet for each of $(2,1)$, $(\bar{2},1)$, $(2,\bar{3})$ and $(\bar{2},3)$ along with 10 chirals which are singlets and hence decouple from the rest of the theory. These can then be organized as a hyper transforming in the fundamental of $SU(2)$, another hyper transforming as the bifundamental of $SU(2) \times SU(3)$ and 5 decoupled hypers.

This appendix is a reprint of the material as it appears in "Quiver tails and $N = 1$ SCFTs from $M5$-branes ", Prarit Agarwal, Ibrahima Bah, Kazunobu Maruyoshi, Jaewon Song, JHEP 1503 (2015) 049, of which I was a coauthor.

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