## Title

# Initial Data Construction in General Relativity



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Initial Data Construction in General Relativity

by

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Abstract

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The subject of this thesis is the *initial data sets* on $\mathbb{R}^3$ for the vacuum Einstein equation, i.e., pairs $(g,k) = (g_{ij}, k_{ij})$ of symmetric tensor fields on $\mathbb{R}^3$ (where $g$ is positive definite) solving the *vacuum constraint equation*

$$\begin{cases} R[g] + (\operatorname{tr}_g k)^2 - |k|_g^2 = 0, \\ \operatorname{div}_g k - \mathrm{d} \operatorname{tr}_g k = 0. \end{cases} \tag{0.0.1}$$

Indeed, any triple $(\mathcal{M}^3, g, k)$ of a 3-manifold, Riemannian metric $g$ and a covariant symmetric 2-tensor $k$ satisfying (0.0.1) is called an *initial data set* for the vacuum Einstein equation

$$\operatorname{Ric}[\mathbf{g}] - \frac{1}{2}\mathbf{g} \operatorname{tr}_{\mathbf{g}} \operatorname{Ric}[\mathbf{g}] = 0. \tag{0.0.2}$$

Various applications of initial data construction include global stability of Minkowski spacetime [CK93], weak cosmic censorship and naked singularity formation [Chr94], disproving the third law of black hole thermodynamics [KU22], and formation of black holes ([LY15] as one example), etc.

To understand the nature of the nonlinear PDE system (0.0.1), it is instructive to consider the direct linearization of this system around the simplest solution – i.e., the flat solution $(\delta, 0)$ – that takes the form, respectively,

$$\begin{cases} \partial^i \partial^j (\dot{g}_{ij} - \frac{1}{2}\delta_{ij} \operatorname{tr} \dot{g}) = 0, \\ \partial^j (\dot{k}_{ij} - \delta_{ij} \operatorname{tr} \dot{k}) = 0, \end{cases} \tag{0.0.3}$$

This system is comprised of *underdetermined* PDEs for a symmetric tensor $\dot{g}$ and a symmetric tensor $\dot{k}$. In analogy with vector fields $v$ obeying the divergence-free condition $\partial_j v^j = 0$ – the

simplest such equation – one should expect quite a bit of flexibility for the set of solutions to (3.1.1).

The author has developed with his collaborators *explicit integral formulas* for solving the linearized system 0.0.3 that enjoy special localization properties so that they can be used to study the nonlinear system 0.0.1.

Although being underdetermined, the system 0.0.1 is known to be rigid: one famous example is the celebrated positive mass theorem first proved by [SY79] which states that only Minkowski initial data can have rapid decay. Thus, an interesting question is in what special unbounded regions the initial data can be localized. This will be the main topic of Chapter 2.

Another way to construct initial data sets is through *gluing*. Gluing means, by definition, given two initial data sets $(g_1, k_1)$ and $(g_2, k_2)$ defined on disjoint regions $A_1, A_2$, respectively, finding a region $A \supset A_1 \cup A_2$ and initial data set $(g, k)$ defined on $A$ such that $(g, k)|_{A_i} = (g_i, k_i), i = 1, 2$. Because the divergence-type operators have nontrivial cokernel (especially on the flat background), we expect conservation laws that these linear equations should satisfy, called linear obstructions. One can ask under what conditions two initial data sets can be glued together, especially if they violate the linear obstructions. The answer might be surprising: Czimek–Rodnianski [CR22] first proved obstruction-free gluing of characteristic data defined on a null hypersurface, as well as spacelike obstruction-free gluing but under a non-sharp positivity condition. The author and his collaborators have developed a method using the aforementioned integral solution operators and manipulating the nonlinearity to achieve obstruction-free gluing on spacelike hypersurfaces in the asymptotically flat regime under a sharp positivity condition. This will be the topic of Chapter 3.

It is then natural to ask if the integral solution operators can also be constructed for curved geometry. The author and his collaborators have developed a general abstract theory for a wide class of underdetermined (and their adjoint) linear differential operators where an integral solution (representation) formula can be constructed. This will be the topic of Chapter 4. Some applications of this theory in geometry and physics are included in Appendix B.

To my parents.

# Contents

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# Chapter 1

# Introduction

The major goal of this thesis is to illustrate a collection of mathematical tools developed and used by the author and his collaborators to construct relativistic initial data.

## 1.1 Constraint Equation and Initial Data Construction: An Overview

Let $(\mathcal{M}, \mathbf{g})$ be a 4-dimensional Lorentzian manifold. We say $(\mathcal{M}, \mathbf{g})$ solves the *Einstein vacuum equation* if

$$\mathrm{Ric}_{\mathbf{g}} = 0. \tag{1.1.1}$$

The equation (1.1.1) models spacetime in a vacuum.

A 3-dimensional Riemannian manifold $(\Sigma, g)$ together with a covariant symmetric 2-tensor $k$ is a initial data set of the Einstein vacuum equation (1.1.1) if $(\Sigma, g, k)$ satisfies the *constraint equation*

$$\begin{cases} R[g] + (\mathrm{tr}_g\, k)^2 - |k|_g^2 = 0, \\ \mathrm{div}_g\, k - \mathrm{d}\,\mathrm{tr}_g\, k = 0. \end{cases} \tag{1.1.2}$$

where $R[g]$ is the scalar curvature of $(\Sigma, g)$. The system (1.1.2) is derived from Gauss and Codazzi equations on a hypersurface. $(\mathcal{M}, \mathbf{g})$ is a solution to (1.1.1) with initial data $(\Sigma, g, k)$ if $\mathrm{Ric}_{\mathbf{g}} = 0$ and $(\Sigma, g)$ is isometrically embedded into $(\mathcal{M}, \mathbf{g})$ and $k$ is the second fundamental form on $\Sigma$ in $\mathcal{M}$ with fixed future unit normal vector field along $\Sigma$.

In this thesis, we are mainly concerned with constructing initial data –i.e. finding special solutions to the system of nonlinear PDEs (1.1.2). Unlike other evolution equations (e.g. wave equations) for which one is free to prescribe well localized data, the constrain equation (1.1.2), although underdetermined and thus granting some freedom for construction, poses rigidity to its solutions. The most well known result regarding rigidity of the system (1.1.2) is the positive mass theorem first proved by Schoen–Yau [SY79] that states nontrivial (i.e. non-flat) initial data cannot have rapid decay. Thus one interesting question is what kind of localized data one can construct.

Another way of constructing initial data is gluing. Here gluing means, given two initial data sets defined on disjoint regions, finding an initial data set defined on a larger region containing both regions and coincides with the given initial data sets on each small region where they are defined. At the linear level, if the background geometry has symmetries, this is almost impossible due to conservation laws of the linearized system. For simplicity, let us examine the linearized system near the flat solution $(\delta, 0)$ to (1.1.2):

$$\begin{cases} \partial^i\partial^j(\dot{g}_{ij} - \frac{1}{2}\delta_{ij}\operatorname{tr}\dot{g}) = 0, \\ \quad \partial^j(\dot{k}_{ij} - \delta_{ij}\operatorname{tr}\dot{k}) = 0. \end{cases} \tag{1.1.3}$$

This is a system of divergence-type equations, and similar to the divergence equation whose solutions must have zero spherical integrals due to divergence theorem, solutions to (1.1.3) have conserved quantities, called charges, linked to symmetries of the background. This is a main difficulty when gluing initial data sets that violate the linear obstruction of conserved charges.

Solving the nonlinear system (1.1.2) usually starts from studying the linearized system (1.1.3), so in order to study (1.1.2) near curved background, one needs a robust method of solving underdetermined linear elliptic systems. Although a generic underdetermined linear elliptic differential operator may have only trivial cokernel, many interesting examples, especially those from linearized constraint equations near special solutions, have nontrivial cokernels and thus causing the aforementioned linear obstruction. From the perspective of finding a solution operator (i.e. a right inverse to the differential operator up to the cokernel), one needs to understand how the possibly nontrivial cokernel can be characterized in the general method. Moreover, since an important motivation to this kind of analysis is solving the nonlinear constraint equation (1.1.2), one also hopes that the solution operators enjoy certain localization property.

The author together with his collaborators has established a robust method of finding explicit integral solution operators with different localization properties to the linearized constraint equations that can be applied to constructing solutions to (1.1.2). We summarize the main results in the next section.

## 1.2 Main Results

### Notation and Preliminaries

We first define

$$\Omega_{y,\omega,\theta} := \{x \in \mathbb{R}^d : \angle(x-y,\omega) \le \theta\}, \Omega_{\omega,\theta} := \Omega_{0,\omega,\theta}. \tag{1.2.1}$$

For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is the Sobolev space. For $s, \delta \in \mathbb{R}$, $H^s_{\mathrm{b}}(\mathbb{R}^d)$ and $H^{s,\delta}_{\mathrm{b}}(\mathbb{R}^d)$ are b-Sobolev spaces that are defined in Definition 2.3.1. Moreover, for $\Omega \subset \mathbb{R}^d$, we denote $H^{s,\delta}_{\mathrm{b}}(\Omega) := \{u \in H^{s,\delta}_{\mathrm{b}} : \operatorname{supp} u \subset \overline{\Omega}\}$ the distributions supported in $\overline{\Omega}$. The anisotropic b-Sobolev space

$H^s_\alpha = H^s_\alpha(\Omega_\alpha)$ is defined in Definition 2.5.1 for a specific region $\Omega_\alpha = \{(x', x_d) \in \mathbb{R}^d : |x'| \le x_d^\alpha, x_d \ge 1\}$.

We define $B_r(\xi)$ to be the ball of radius $r$ centered at $\xi$, $A_r(\xi) = B_{2r}(\xi) \setminus \overline{B_r(\xi)}$ for an annulus of radii $\simeq r$ and $\widetilde{A}_r(\xi) = B_{4r}(\xi) \setminus \overline{B_{\frac{r}{2}}(\xi)}$ for an enlargement of $A_r$. When $\xi = 0$, we shall omit $(\xi)$ and write $B_r = B_r(0)$, $A_r = A_r(0)$ etc.

Given a Banach space $X = X(\mathbb{R}^3)$ of (possibly vector-valued) functions on $\mathbb{R}^3$ and an open subset $\Omega \subseteq \mathbb{R}^3$, we define the space of extendible functions on $\Omega$ as $X(\Omega) = X(\mathbb{R}^3)/\{u \in X(\mathbb{R}^3) : u|_\Omega = 0\}$, equipped with the $\|u\|_{X(\Omega)} = \inf_{\widetilde{u} \in X(\mathbb{R}^3) : \widetilde{u}|_\Omega = u} \|\widetilde{u}\|_X$. We also define the space of functions supported in $\Omega$, denoted by $X_0(\Omega)$, to be the completion of $C^\infty_c(\Omega)$ with respect to $\|\cdot\|_{X(\Omega)}$.

The charges of the linearized constraint near the flat data (1.1.3) form a 10-vector, denoted by

$$\mathbf{Q}[(g,k); \partial B_r] = (\mathbf{E}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)^\dagger[(g,k); \partial B_r].$$

A $\mathcal{Q}$-admissible family of annular initial data sets is a family of solutions to (1.1.2) supported on an annulus parametrized by $Q \in \mathcal{Q}$ where $\mathcal{Q}$ is an open bounded subset of $\mathbb{R}^{10}$. For the precise definitions of charges and $\mathcal{Q}$-admissible families, see 3.1.

In Chapter 4 and [Ise+25a], one of the main techniques that we use to construct explicit integral solution/representation operators for under/overdetermined elliptic operators is the *recovery on curves condition*. For simplicity, in this section we use a qualitative version of this condition: Let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator with $C^\infty_c$ coefficients on an open subset $U \subseteq \mathbb{R}^d$ where $r_0 \le s_0$.

(RC) Given any $\varphi \in C^\infty_c(U)$, there exists a linear way to continuously recover $\varphi(\mathbf{x}(0))$ from (the jet of) $\mathcal{P}^*\varphi$ on $\mathbf{x}$ and (the jet of) $\varphi$ at $\mathbf{x}(1)$.

For the precise formulation of this condition, see Section 4.5. We also give an algebraic sufficient condition of (RC), called the *finite-dimensional cokernel condition*: Let $\mathcal{P}^*$ denote the formal $L^2$-adjoint of $\mathcal{P}$ and $p(x,\xi), p^*(x,\xi)$ denote the symbol of $\mathcal{P}, \mathcal{P}^*$, respectively. Let $\mathcal{P}_{\text{prin}}$ denote the principal part of $\mathcal{P}$.

(FC) For all $x \in U$ and $\xi \in \mathbb{C}^d \setminus \{0\}$, $p^*(x,\xi)$ is injective (or equivalently, $p^*(x,\xi)$ is full rank, or $p(x,\xi)$ is surjective, or $p(x,\xi)$ is full rank).

Finally, we define the formal cokernel of $\mathcal{P}$:

$$\ker \mathcal{P}^*(U) \coloneqq \{\mathbf{Z} \in C^\infty(\overline{U}) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } \mathcal{D}'(U)\}, \tag{1.2.2}$$

## Main Results and Organization of the Thesis

In Chapter 2, we prove the following theorems from [MT22]:

First result is construction of initial data supported in cones with optimal decay.

**Theorem 1.2.1** (Initial data with conic support, [MT22])**.** *Let $d \geq 3$. For $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$, there exists a nontrivial asymptotically flat solution $(g,k) \in C^\infty(\mathbb{R}^d)$ of equation (1.1.2) on $\mathbb{R}^d$ supported in the cone $\Omega_{\omega,\theta}$ defined in (1.2.1), i.e.* $\operatorname{supp}(g_{ij} - \delta_{ij}, k_{ij}) \subset \Omega_{\omega,\theta}$ *and the decay rate of $(g,k)$ is given by*

$$\partial^\ell(g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x\rangle^{2-d-\ell}), \quad \partial^\ell k_{ij}(x) = \mathcal{O}(\langle x\rangle^{1-d-\ell}), \quad \ell \in \mathbb{N}. \tag{1.2.3}$$

*Moreover, for $s > d/2$ and $-d/2 < \delta < d/2 - 2$, there is a smooth map*

$$F : H^{s,\delta}_{\mathrm{b}}(\Omega_{\omega,,\theta}) \times H^{s-1,\delta+1}_{\mathrm{b}}(\Omega_{\omega,,\theta}) \to H^{s,\delta}_{\mathrm{b}}(\Omega_{\omega,,\theta}) \times H^{s-1,\delta+1}_{\mathrm{b}}(\Omega_{\omega,,\theta})$$

*such that $F$ gives a one-to-one correspondences between solutions $(h,\pi)$ to (1.1.3) and solutions $(g-\delta,k)$ of (1.1.2) in a neighbourhood of $0$. The differential of $F$ at $0$ is the identity.*

As a corollary, we have the following gluing result of [CS16a]:

**Theorem 1.2.2** (Gluing flat data to data in a cone, [CS16a], [MT22])**.** *Let $d \geq 3$, $0 < \theta_0 < \theta < \pi/2$ and $\omega \in \mathbb{S}^{d-1}$. Suppose $(g_0,k_0) \in C^\infty(\mathbb{R}^d)$ solves (1.1.2) inside $\Omega_{\omega,\theta}$ and satisfies (1.2.3). Then there exists $y_0 \in \Omega_{\omega,\theta_0}$, $|y_0| \gg 1$ and an asymptotically flat solution $(g,k) \in C^\infty(\mathbb{R}^d)$ of equation (2.1.1) on $\mathbb{R}^d$ such that*

$$(g,k) = \begin{cases} (g_0,k_0), & \Omega_{y_0,\omega,\theta_0} \setminus B_1(y_0), \\ (\delta,0), & \mathbb{R}^d \setminus (\Omega_{y_0,\omega,\theta} \cup B_1(y_0)), \end{cases}$$

*and $(g,k)$ also has decay rate in (1.2.3) on $\mathbb{R}^d$.*

We can also construct data supported in a *degenerate sector*, an example of a regular unbounded region that is strictly contained in a cone near infinity, as conjectured in [Car21].

**Theorem 1.2.3** (Initial data supported in a degenerate sector, [MT22])**.** *Let $d \geq 3$, $0 < \theta_0 < \theta < \pi/2$ and $\omega \in \mathbb{S}^{d-1}$. Suppose $(g_0,k_0) \in C^\infty(\mathbb{R}^d)$ solves (1.1.2) inside $\Omega_{\omega,\theta}$ and satisfies (1.2.3). Then there exists $y_0 \in \Omega_{\omega,\theta_0}$, $|y_0| \gg 1$ and an asymptotically flat solution $(g,k) \in C^\infty(\mathbb{R}^d)$ of equation (1.1.2) on $\mathbb{R}^d$ such that*

$$(g,k) = \begin{cases} (g_0,k_0), & \Omega_{y_0,\omega,\theta_0} \setminus B_1(y_0), \\ (\delta,0), & \mathbb{R}^d \setminus (\Omega_{y_0,\omega,\theta} \cup B_1(y_0)), \end{cases}$$

*and $(g,k)$ also has decay rate in (1.2.3) on $\mathbb{R}^d$.*

In Chapter 3, we prove the following theorems from [MOT23]: We start with the gluing result that respects the linear obstructions:

**Theorem 1.2.4** (Gluing up to linear obstructions at unit scale, [MOT23])**.** *For each $s > \frac{3}{2}$, there exist $\epsilon_c = \epsilon_c(s) > 0$ and $M_c = M_c(s) > 0$ such that the following holds. Let $(\mathring{g}, \mathring{k})$ be a solution to (3.1.1) in $H^s \times H^{s-1}(\widetilde{A}_1)$ such that*

$$\|(\mathring{g} - \delta, \mathring{k})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq \epsilon. \tag{1.2.4}$$

*Define* $\mathring{Q} \in \mathbb{R}^{10}$ *by* $\mathring{Q} = \mathbf{Q}[(\mathring{g}, \mathring{k}); A_1]$*. Let* $\mathcal{Q} \subseteq \mathbb{R}^{10}$ *be a bounded open set such that*

$$\mathring{Q} \in \mathcal{Q}, \quad B_{M_c\epsilon^2}(\mathring{Q}) \subseteq \mathcal{Q}. \tag{1.2.5}$$

*and consider an* $\mathcal{Q}$*-admissible family of annular initial data sets* $\{(g_Q, k_Q)\}_{Q\in\mathcal{Q}}$ *on* $\widetilde{A}_1$ *with Sobolev regularity* $s$ *such that, for all* $Q, Q' \in \mathcal{Q}$*,*

$$\|(g_Q - \delta, k_Q)\|_{H^s\times H^{s-1}(\widetilde{A}_1)} \leq \epsilon, \tag{1.2.6}$$

$$\|(g_Q - g_{Q'}, k_Q - k_{Q'})\|_{H^s\times H^{s-1}(\widetilde{A}_1)} \leq K|Q - Q'|. \tag{1.2.7}$$

*Then the following holds:*

1. **Exterior gluing.** *If* $\epsilon < \epsilon_c$ *and* $K\epsilon < \epsilon_c$*, then there exists a solution* $(g, k) \in H^s \times H^{s-1}(\widetilde{A}_1)$ *to* (1.1.2) *and* $Q \in \mathcal{Q}$ *such that* $\|(g - \delta, k)\|_{H^s\times H^{s-1}(\widetilde{A}_1)} \lesssim \epsilon$*,* $|Q - \mathring{Q}| \leq M_c\epsilon^2$*, and*
$$(g, k) = \begin{cases} (\mathring{g}, \mathring{k}) & \text{in } A_{\frac{1}{2}}, \\ (g_Q, k_Q) & \text{in } A_2. \end{cases}$$

2. **Interior gluing.** *If* $\epsilon < \epsilon_c$ *and* $K\epsilon < \epsilon_c$*, then there exists a solution* $(g, k) \in H^s \times H^{s-1}(\widetilde{A}_1)$ *to* (1.1.2) *and* $Q \in \mathcal{Q}$ *such that* $\|(g - \delta, k)\|_{H^s\times H^{s-1}(\widetilde{A}_1)} \lesssim \epsilon$*,* $|Q - \mathring{Q}| \leq M_c\epsilon^2$*, and*
$$(g, k) = \begin{cases} (g_Q, k_Q) & \text{in } A_{\frac{1}{2}}, \\ (\mathring{g}, \mathring{k}) & \text{in } A_2. \end{cases}$$

*In both cases, the following additional statements hold as well:*

- *(Lipschitz continuity) The map* $(\mathring{g}, \mathring{k}) \mapsto (g, k, Q)$ *is Lipschitz as a map from the subset of* $H^s \times H^{s-1}$ *restricted by* (1.2.4)*–*(1.2.5) *into* $H^s \times H^{s-1} \times \mathbb{R}^{10}$*.*
- *(Persistence of regularity) If* $(\mathring{g}, \mathring{k}) \in H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)$ *and the family* $\{(g_Q, k_Q)\}_{Q\in\mathcal{Q}}$ *is of Sobolev regularity* $s + m$ *for* $m \in \mathbb{N}$*, then* $(g, k) \in H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)$ *and*
$$\begin{aligned}\|(g - \delta, k)\|_{H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)} \leq{}& C\|(\mathring{g} - \delta, \mathring{k})\|_{H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)} \\ &+ C\|(g_Q - \delta, k_Q)\|_{H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)},\end{aligned}$$
*where* $C = C(s, m, \|(\mathring{g} - \delta, \mathring{k})\|_{H^{s+1}\times H^s(\widetilde{A}_1)}, \|(g_Q - \delta, k_Q)\|_{H^{s+1}\times H^s(\widetilde{A}_1)})$*.*

Then we state the obstruction-free gluing result for almost flat on unit annuli:

**Theorem 1.2.5** (Obstruction-free gluing for almost flat data on annuli, [MOT23])**.** *Given* $s > \frac{3}{2}$ *and* $\Gamma > 1$*, there exist* $\epsilon_o = \epsilon_o(s, \Gamma) > 0$*,* $\mu_o = \mu_o(s, \Gamma) > 0$ *and* $C_o = C_o(s, \Gamma) > 0$ *such that the following holds. Let* $(g_{in}, k_{in}) \in H^s \times H^{s-1}(A_1)$*,* $(g_{out}, k_{out}) \in H^s \times H^{s-1}(A_{32})$ *be pairs solving* (1.1.2)*. Define* $\Delta Q = (\mathbf{\Delta E}, \ldots, \mathbf{\Delta J}_3) \in \mathbb{R}^{10}$ *by*

$$\Delta Q = \mathbf{Q}[(g_{out}, k_{out}); A_{32}] - \mathbf{Q}[(g_{in}, k_{in}; A_1)], \tag{1.2.8}$$

*and assume that*

$$\Delta\mathbf{E} > |\Delta\mathbf{P}|, \tag{1.2.9}$$
$$\frac{\Delta\mathbf{E}}{\sqrt{(\Delta\mathbf{E})^2 - |\Delta\mathbf{P}|^2}} < \Gamma, \tag{1.2.10}$$
$$\Delta\mathbf{E} < \epsilon_o^2, \tag{1.2.11}$$
$$|\Delta\mathbf{C}| + |\Delta\mathbf{J}| < \mu_o \Delta\mathbf{E}, \tag{1.2.12}$$
$$\|(g_{in} - \delta, k_{in})\|^2_{H^s \times H^{s-1}(A_1)} + \|(g_{out} - \delta, k_{out})\|^2_{H^s \times H^{s-1}(A_{32})} < \mu_o \Delta\mathbf{E}. \tag{1.2.13}$$

*Then there exists* $(g,k) \in H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})$ *solving* (1.1.2) *such that*

$$(g,k) = (g_{in}, k_{in}) \quad \text{in } A_1, \quad (g,k) = (g_{out}, k_{out}) \quad \text{in } A_{32}, \tag{1.2.14}$$
$$\|(g - \delta, k)\|^2_{H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})} < C_o \Delta\mathbf{E}. \tag{1.2.15}$$

We remark that the inequality condition (1.2.9) is sharp–i.e. it is a necessary condition for this kind of gluing to be possible.

After a scaling argument we can prove obstruction-free gluing for asymptotically flat data:

**Theorem 1.2.6** (Obstruction-free gluing for asymptotically flat data, [MOT23])**.** *Let* $s > \frac{3}{2}$ *and* $\alpha > \frac{1}{2}$*. Let* $(g_{in}, k_{in})$*,* $(g_{out}, k_{out}) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ *be* $\alpha$*-asymptotically flat pairs solving* (1.1.2)*. Define, for those components that are well-defined,*

$$\Delta Q := \mathbf{Q}^{ADM}[(g_{out}, k_{out})] - \mathbf{Q}^{ADM}[(g_{in}, k_{in})],$$

*and assume that*

$$\Delta\mathbf{E} > |\Delta\mathbf{P}|.$$

*Then there exists an asymptotically flat pair* $(g,k) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ *solving* (1.1.2) *and* $r \geq R_0$ *such that* $(g,k) = (g_{in}, k_{in})$ *on* $B_{2r}$ *and* $(g,k) = (g_{out}, k_{out})$ *on* $B^c_{32r}$*.*

Again, here the condition $\Delta\mathbf{E} > |\Delta\mathbf{P}|$ is sharp.

In Chapter 4, we prove the following theorems from [Ise+25a]: Recall the conditions *(RC)* and *(FC)*. We first state the theorem of conic-type integral operators:

**Theorem 1.2.7** (Conic-type solution operators, representation formulas, and Poincaré-type inequalities, [Ise+25a])**.** *Let* $U$ *be an open subset of* $\mathbb{R}^d$*. Let* $\mathcal{P}$ *satisfy (RC) for an admissible family of curves* $\mathbf{x} = \mathbf{x}(y, y_1, s)$ *for all* $y$ *in a neighborhood of* $\overline{U}$ *and* $y_1 \in U_1$ *for some open subset* $U_1$ *of* $\mathbb{R}^d$ *(see Section 4.5 for the precise formulation). Assume, moreover, that the curves are* nontrapped in $U$ *(i.e., the curve eventually exits* $U$*) in the sense that*

$$\overline{U} \cap \overline{U}_1 = \emptyset.$$

*Then the following holds.*

1. **Cokernel in $\widetilde{W}^{-s,p'}(U)$.** *For any $1 < p < +\infty$ and $s \in \mathbb{R}$, define the cokernel in $\widetilde{W}^{-s,p'}(U)$ to be:*

$$\ker_{\widetilde{W}^{-s,p'}(U)} \mathcal{P}^* := \{\mathbf{Z} \in \widetilde{W}^{-s,p'}(U) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } \mathcal{D}'(\widetilde{U})\}, \tag{1.2.16}$$

*where $\widetilde{U}$ is an open subset of $\mathbb{R}^d$ such that $\overline{U} \subseteq \widetilde{U}$, and the coefficients of $\mathcal{P}^*$ are extended in a smooth way to $\widetilde{U}$ (since $\mathbf{Z} \in \widetilde{W}^{-s,p'}(U)$, this definition is independent of these choices). Under the assumptions of this theorem, we have*

$$\ker_{\widetilde{W}^{-s,p'}(U)} \mathcal{P}^* = \{0\}.$$

2. **Integral solution formula.** *There exists a locally integrable integral kernel $K : U \times U \to \mathbb{C}^{s_0 \times r_0}$ with the support property*

$$\operatorname{supp} K(\cdot, y) \subseteq \overline{\bigcup_{y_1 \in U_1} \mathbf{x}(y, y_1, [0, 1])} \quad \text{for every } y \in U,$$

*such that the integral operator $\mathcal{S}f(x) := \int_U K(x,y)f(y)\,\mathrm{d}y$ for $f \in C_c^\infty(U; \mathbb{C}^{s_0})$ satisfies*

$$\mathcal{P}\mathcal{S}f = f \quad \text{for all } f \in C_c^\infty(U; \mathbb{C}^{s_0}),$$

*Moreover, for any $1 < p < +\infty$ and $s \in \mathbb{R}$, $\mathcal{S}$ extends to a bounded operator from $\widetilde{W}^{s,p}(U; \mathbb{C}^{r_0})$ to $W^{s+m_1,p}(U) \times \cdots \times W^{s+m_{s_0},p}(U)$.*

3. **Integral representation formula & Friedrich-type inequality.** *Dually, we have*

$$\varphi = \mathcal{S}^*\mathcal{P}^*\varphi \quad \text{for all } \varphi \in C_c^\infty(U; \mathbb{C}^{r_0}).$$

*For any $1 < p < +\infty$ and $s \in \mathbb{R}$, $\mathcal{S}^*$ extends to a bounded operator from $\widetilde{W}^{-s-m_1,p'}(U) \times \cdots \times \widetilde{W}^{-s-m_{s_0},p'}(U)$ to $W^{-s,p'}(U; \mathbb{C}^{r_0})$. Moreover, the following* Friedrich-type inequality *holds:*

$$\|\varphi\|_{\widetilde{W}^{-s,p'}(U;\mathbb{C}^{r_0})} \lesssim \|\mathcal{P}^*\varphi\|_{\widetilde{W}^{-s-m_1,p'}(U)\times\cdots\times\widetilde{W}^{-s-m_{s_0},p'}(U)} \quad \text{for all } \varphi \in \widetilde{W}^{-s,p'}(U; \mathbb{C}^{r_0}).$$

The following theorem describes the Bogovskii-type integral operators:

**Theorem 1.2.8** (Bogovskii-type solution operators, representation formulas, and Poincaré-type inequalities, [Ise+25a])**.** *Let $U$ be a connected bounded open subset of $\mathbb{R}^d$. Let $\mathcal{P}$ satisfy (RC) for an admissible family of curves $\mathbf{x} = \mathbf{x}(y, y_1, s)$ (see Section 4.5 for the precise formulation). Assume that $\overline{U}$ is* $\mathbf{x}$-star-shaped *with respect to $\overline{U_1}$, where $U_1$ is an open subset of $U$ such that $\overline{U_1} \subseteq U$, in the sense that*

$$\overline{\bigcup_{y \in \overline{U},\, y_1 \in \overline{U_1}} \mathbf{x}(y, y_1, [0, 1])} \subseteq \overline{U}. \tag{1.2.17}$$

*Then the following holds.*

1. **Cokernel in $W^{-s,p'}(U)$.** *For any $1 < p < +\infty$ and $s \in \mathbb{R}$, define the cokernel in $W^{-s,p'}(U)$ to be:*
$$\ker_{W^{-s,p'}(U)} \mathcal{P}^* := \{\mathbf{Z} \in W^{-s,p'}(U) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } \mathcal{D}'(U)\}. \tag{1.2.18}$$
*We have the* invariance property
$$\ker_{W^{-s,p'}(U)} \mathcal{P}^* = \ker \mathcal{P}^*,$$
*and the* finite-dimensional property
$$\dim \ker \mathcal{P}^*(U) < +\infty. \tag{1.2.19}$$
*where $\ker \mathcal{P}^*$ is the formal cokernel of $\mathcal{P}$ defined in (1.2.2). Moreover, for any open subset $V \subseteq U$, the restriction of $\ker \mathcal{P}^*$ to $V$, i.e.,*
$$\ker \mathcal{P}^*|_V := \{\mathbf{Z}|_V : \mathbf{Z} \in \ker \mathcal{P}^*\}$$
*has the same dimension as $\ker \mathcal{P}^*$.*

2. **Integral solution formula under orthogonality conditions.** *Consider a family $w_{\mathbf{A}}(x) \in C_c^\infty(U_1; \mathbb{C}^{r_0})$ ($\mathbf{A} \in \{1, \ldots, \dim \ker \mathcal{P}^*\}$) satisfying $\langle w_{\mathbf{A}}, \mathbf{Z}^{\mathbf{A}'}\rangle = \delta_{\mathbf{A}}^{\mathbf{A}'}$ for some basis $\{\mathbf{Z}^{\mathbf{A}'}\}$ of $\ker \mathcal{P}^*$. Then there exists a locally integrable integral kernel $\widetilde{K} : U \times U \to \mathbb{C}^{s_0 \times r_0}$ with the support property*
$$\operatorname{supp} \widetilde{K}(\cdot, y) \subseteq \overline{\bigcup_{y_1 \in \overline{U_1}} \mathbf{x}(y, y_1, [0,1])} \subseteq \overline{U} \quad \text{for every } y \in U,$$
*(where the last inclusion follows simply from (1.2.17)) such that the integral operator $\widetilde{\mathcal{S}} f(x) \coloneqq \int_U \widetilde{K}(x,y) f(y)\, \mathrm{d}y$ for $f \in C_c^\infty(U; \mathbb{C}^{s_0})$ satisfies*
$$\mathcal{P}\widetilde{\mathcal{S}} f = f - \sum_{\mathbf{A} \in \{1, \ldots, \dim \ker \mathcal{P}^*\}} w_{\mathbf{A}} \langle \mathbf{Z}^{\mathbf{A}}, f\rangle \quad \text{for all } f \in C_c^\infty(U; \mathbb{C}^{s_0}).$$
*Moreover, for any $1 < p < +\infty$ and $s \in \mathbb{R}$, $\widetilde{\mathcal{S}}$ extends to a bounded operator from $\widetilde{W}^{s,p}(U; \mathbb{C}^{r_0})$ to $\widetilde{W}^{s+m_1}(U) \times \cdots \times \widetilde{W}^{s+m_{s_0}}(U)$.*

3. **Integral representation formula & Poincaré-type inequality under orthogonality conditions.** *Dually, we have*
$$\varphi = \widetilde{\mathcal{S}}^* \mathcal{P}^* \varphi + \sum_{\mathbf{A} \in \{1, \ldots, \dim \ker \mathcal{P}^*\}} \mathbf{Z}^{\mathbf{A}} \langle w_{\mathbf{A}}, \varphi\rangle \quad \text{for all } \varphi \in C^\infty(\overline{U}; \mathbb{C}^{r_0}).$$
*Furthermore, for any $1 < p < +\infty$ and $s \in \mathbb{R}$, $\widetilde{\mathcal{S}}^*$ defines a bounded operator from $W^{-s-m_1,p}(U) \times \cdots \times W^{-s-m_{s_0},p'}(U)$ to $W^{-s,p'}(U; \mathbb{C}^{r_0})$, and the following* Poincaré-type inequality *holds:*
$$\|\varphi - \sum_{\mathbf{A}} \mathbf{Z}^{\mathbf{A}} \langle w_{\mathbf{A}}, \varphi\rangle\|_{W^{-s,p'}(U;\mathbb{C}^{r_0})} \lesssim \|\mathcal{P}^*\varphi\|_{W^{-s-m_1,p'}(U) \times \cdots \times W^{-s-m_{s_0},p'}(U)} \quad \text{for all } \varphi \in W^{-s,p'}(U; \mathbb{C}^{r_0}).$$

Finally, we state the theorem of (FC) as a sufficient condition of (RC):

**Theorem 1.2.9** ((FC) $\Longrightarrow$ (RC), [Ise+25a])**.** *Let $U$ be a connected open subset of $\mathbb{R}^d$, and let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on $U$ with smooth coefficients.*

1. *Condition (FC) implies the existence of a maximal graded augmented system $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$. Hence, (FC) implies (RC) for any admissible family of curves, and Theorems 1.2.7–1.2.8 apply.*

2. *If $p^*$ is independent of $x$ (i.e., $\mathcal{P}_{\text{prin}}$ has constant coefficients), then the following are equivalent:*

   *a) $p^*$ satisfies (FC),*

   *b) any $r_0 \times s_0$-matrix-valued differential operator $\mathcal{P}'$ on $U$ with principal symbol $p^*$ satisfies (RC) for any admissible family of curves, and*

   *c) the formal cokernel of $\mathcal{P}_{\text{prin}}$ (i.e., $\ker \mathcal{P}^*_{\text{prin}} := \{\mathbf{Z} \in C^\infty(\overline{U}) : \mathcal{P}^*_{\text{prin}}\mathbf{Z} = 0$ in $\mathcal{D}'(U)\}$) is finite dimensional.*

In addition, we give some examples in Appendix B that satisfies (FC) coming from geometry and physics as applications of Theorem 1.2.8.

## 1.3 Related works

In this section we discuss some related works to our main theorems in Section 1.2. The reader should note that this section is not intended to be a complete survey on works in relativistic data construction, but rather a discussion of some pioneering works and more recent developments in this area at the time this thesis is composed. The reader is referred to Sections 2.1, 3.1, 4.1 for a more comprehensive review of literature related to each chapter's main results.

The first gluing results were proved by Corvino [Cor00], Corvino–Schoen [CS06], and Chruściel–Delay [CD03] on compact regions using elliptic theory. Carlotto–Schoen proved a gluing result with data supported in a cone [CS16a] (see also Theorem 1.2.3). Aretakis–Czimek–Rodnianski [ACR25] [ACR23b] [ACR23a] proved the first results of characteristic gluing–i.e. gluing of data on null hypersurfaces. A breakthrough in gluing problems was the obstruction-free gluing of characteristic data by Czimek–Rodnianski [CR22]. While their proof is fundamentally characteristic, they also proved a spacelike obstruction-free gluing result under a non-sharp positivity condition. Kehle–Unger [KU22] used characteristic gluing to glue a Minkowski piece to an extremal Reissner–Nordström black hole solving the Einstein-Maxwell-charged scalar field equation, disproving the third law of thermodynamics of black holes. Reshetnyak [Res83] [Res94] introduced the idea of *completely integrable graded augmented systems* when studying some overdetermined operators, which heavily influenced our framework in Theorem 1.2.7 and Theorem 1.2.8.

More recently, Shen–Wan [SW25] constructed Cauchy data containing no trapped surfaces but leading to formation of multiple black holes by combining Christodoulou's short pulse framework, stability analysis in transition region connecting the short pulse piece and the gluing region, vacuum geometrostatic manifolds, and obstruction-free gluing in [MOT23] (see also Theorem 1.2.2). Shen–Wan [SW26a] gave a simple construction of Cauchy data leading to the formation of multiple black holes with prescribed ADM charges using conic gluing of [CS16a] (see also [MT22] and Theorem 1.2.2) as well as obstruction-free gluing of [MOT23] (see also Theorem 1.2.5). Shen–Wan [SW26b] gave a simple construction of vacuum initial data with minimal decay by applying the conic solution operator in [MT22] (see also Chapter 2 and Theorem 1.2.7). Giorgi–Shen–Wan [GSW25] constructed initial data for the Einstein-Maxwell-Klein-Gordon equation leading to multiple collapsing boson stars using an approach similar to [SW25]. Chen–Klainerman [CK25] constructed vacuum initial data near Schwarzschild parametrized by 4 scalar functions and 6 small charges. Nützi [Nüt26] constructed vacuum initial data by adding Kerr or Schwarzschild quadratic corrections to small and decaying prescribed solutions to the linearized constraint.

# Chapter 2

# Localized Initial Data for the Einstein Vacuum Equation

## Notation

- For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is the Sobolev space. For $s, \delta \in \mathbb{R}$, $H^s_{\mathrm{b}}(\mathbb{R}^d)$ and $H^{s,\delta}_{\mathrm{b}}(\mathbb{R}^d)$ are b-Sobolev spaces that are defined in Definition 2.3.1. The anisotropic b-Sobolev space $H^s_\alpha = H^s_\alpha(\Omega_\alpha)$ is defined in Definition 2.5.1 for a specific region $\Omega_\alpha = \{(x', x_d) \in \mathbb{R}^d : |x'| \le x_d^\alpha, x_d \ge 1\}$. We define $H^{\infty,\delta}_{\mathrm{b}} := \cap_s H^{s,\delta}_{\mathrm{b}}$ and $H^{\infty,\delta}_\alpha := \cap_s H^{s,\delta}_\alpha$.

- For a Banach space $X = X(\mathbb{R}^d)$ that is contained in the space of distributions $\mathscr{D}'(\mathbb{R}^d)$ and a subset $\Omega \subset \mathbb{R}^d$, $X(\Omega)$ always means the supported $X$-space, i.e. functions in $X(\mathbb{R}^d)$ that are supported inside $\overline{\Omega}$. If $\Omega \subset \mathbb{R}^d$ is open, $X_{comp}(\Omega)$ means the spaces of functions/distributions in $X(\mathbb{R}^d)$ that are supported in a compact subset in $\Omega$.

- In this chapter, there are scalar functions, vector-valued functions, and functions valued in symmetric two tensors $\mathrm{Sym}^2\,\mathbb{R}^d$. The function spaces are naturally extended to functions valued in vectors and symmetric two tensors, by considering all the components.

- We define the cones in $\mathbb{R}^d$ as follows. For $y \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$, let
$$\Omega_{y,\omega,\theta} := \{x \in \mathbb{R}^d : \angle(x - y, \omega) \le \theta\} \tag{2.0.1}$$
be the cone in $\mathbb{R}^d$ with center at $y$, center vector $\omega$ and angle $\theta$. When $y = 0$, we introduce the abbreviated notation $\Omega_{\omega,\theta} := \Omega_{0,\omega,\theta}$.

- An operator $K : C^\infty_c(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ is identified with its Schwartz kernel $K(x, y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$.

## 2.1 Introduction

In this chapter, we provide a simple way to construct asymptotically flat initial data of the Einstein equation with new localization properties. The vacuum Einstein equation reads

$$\mathrm{Ric}_{\mathbf{g}} = 0$$

where $\mathbf{g}$ is a Lorentzian metric and $\mathrm{Ric}_{\mathbf{g}}$ is the Ricci curvature. When we restrict to a spacelike hypersurface, we get the Einstein constraint equation

$$\begin{cases} R_g + (tr_g k)^2 - |k|_g^2 = 0 \\ \mathrm{div}_g(k - (\mathrm{tr}_g k)g) = 0 \end{cases} \tag{2.1.1}$$

where $g$ is the restriction of $\mathbf{g}$ to the spacelike hypersurface, $k$ is the second fundamental form, and $R_g$ is the scalar curvature of $g$. It is a system of nonlinear underdetermined PDEs for initial data $(g, k)$. When $k = 0$, it specializes to a problem in Riemannian geometry, namely, the vanishing of scalar curvature. In particular, we are interested in the following question.

**Question 1.** *What localization of asymptotically flat solutions to the Einstein constraint equation* (2.1.1) *is possible?*

This question has surprisingly nontrivial answers. The famous positive mass theorem [SY79; SY81; Wit81] says localization to a compact set is impossible. On the positive direction, Carlotto–Schoen [CS16b] gives a gluing construction which gives a localized solution inside a cone. Aretakis–Czimek–Rodnianski [ACR25; ACR23b; ACR23a] gives an alternative gluing construction based on the characteristic gluing.

An interesting problem that was left open in [CS16b] was whether the localized solution $g_{ij}(x)$ decays to the flat metric $\delta_{ij}$ as $|x| \to \infty$ with the rate $\mathcal{O}(|x|^{2-d})$, which is ideal in view of the positive mass theorem (see also Proposition 2.1.4 below); see Carlotto [Car21, Open Problem 3.18]. The construction in [CS16b] achieves $\mathcal{O}(|x|^{2-d+\epsilon})$ for arbitrary $\epsilon > 0$. The characteristic gluing approach of Aretakis–Czimek–Rodnianski [ACR25] gives an affirmative answer to [Car21, Open Problem 3.18], but only for finite regularity. Here we give an alternative spacelike proof, which can also achieve $C^\infty$ regularity. Moreover, we also construct nontrivial solutions localized in an even smaller domain, namely in a degenerate sector $\{(x', x_d) \in \mathbb{R}^d : |x'| \le x_d^\alpha, x_d \ge 0\}$ for $\frac{3}{d+1} < \alpha < 1$, with sharp decay rate consistent with the positive mass theorem. We state our main results below.

### Main results

We propose a new method of solving the constraint equation (2.1.1) by constructing solution operators for the corresponding linearized system. The linearized equation of (2.1.1) at the trivial metric $g_{ij} = \delta_{ij}$ and second fundamental form $k_{ij} = 0$ can be written in the following form after a change of variables (see §2.2)

$$\partial_i \partial_j h^{ij} = 0, \quad \partial_i \pi^{ij} = 0 \tag{2.1.2}$$

where $h^{ij} = h^{ji}$ and $\pi^{ij} = \pi^{ji}$ are symmetric 2-tensors. Let $P(h,\pi) = (\partial_i\partial_j h^{ij}, \partial_i \pi^{ij})$, we construct in §2.2 a right inverse $S$ of $P$, which we call a solution operator, with nice properties in regularity, support, and asymptotic behavior of functions.

As the first application of our method, we give a simple spacelike construction of localized solutions in a cone with optimal decay, answering [Car21, Open Problem 3.18] with $C^\infty$ regularity.

**Theorem 2.1.1** ([MT22]). *Let $d \geq 3$. For $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$, there exists a nontrivial asymptotically flat solution $(g,k) \in C^\infty(\mathbb{R}^d)$ of equation (2.1.1) on $\mathbb{R}^d$ supported in the cone $\Omega_{\omega,\theta}$ defined in (2.0.1), i.e. $\operatorname{supp}(g_{ij} - \delta_{ij}, k_{ij}) \subset \Omega_{\omega,\theta}$ and the decay rate of $(g,k)$ is given by*

$$\partial^\ell (g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{2-d-\ell}), \quad \partial^\ell k_{ij}(x) = \mathcal{O}(\langle x \rangle^{1-d-\ell}), \quad \ell \in \mathbb{N}. \tag{2.1.3}$$

*Moreover, for $s > d/2$ and $-d/2 < \delta < d/2 - 2$, there is a smooth map*

$$F : H_{\mathrm{b}}^{s,\delta}(\Omega_{\omega,,\theta}) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega_{\omega,,\theta}) \to H_{\mathrm{b}}^{s,\delta}(\Omega_{\omega,,\theta}) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega_{\omega,,\theta})$$

*such that $F$ gives a one-to-one correspondences between solutions $(h,\pi)$ to (2.1.2) and solutions $(g-\delta,k)$ of (2.1.1) in a neighbourhood of $0$. The differential of $F$ at $0$ is the identity.*

For any solution $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ of the linearized constraint equation (2.1.2) supported in $\Omega_{\omega,\theta}$ with the same decay rate

$$\partial^\ell h_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{2-d-\ell}), \quad \partial^\ell \pi_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{1-d-\ell}), \quad \ell \in \mathbb{N} \tag{2.1.4}$$

such that $(h_0,\pi_0)$ is small in certain b-Sobolev spaces (see §2.3), we can find such a solution to the nonlinear equation (2.1.1) as a perturbation of $(h_0,\pi_0)$, and will obey bounds of the form

$$\|(h - h_0, \pi - \pi_0)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim \|(h_0,\pi_0)\|^2_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}.$$

So a small solution of the linearized equation (2.1.2) with nontrivial linearized curvature will lead to a nontrivial solution of (2.1.1) that is not isometric to the Euclidean space. For more details, see the proof in §2.4.

We can also prove a similar gluing result as in [CS16b] following the same strategy.

**Theorem 2.1.2** ([MT22]). *Let $d \geq 3$, $0 < \theta_0 < \theta < \pi/2$ and $\omega \in \mathbb{S}^{d-1}$. Suppose $(g_0,k_0) \in C^\infty(\mathbb{R}^d)$ solves (2.1.1) inside $\Omega_{\omega,\theta}$ and satisfies (2.1.3). Then there exists $y_0 \in \Omega_{\omega,\theta_0}$, $|y_0| \gg 1$ and an asymptotically flat solution $(g,k) \in C^\infty(\mathbb{R}^d)$ of equation (2.1.1) on $\mathbb{R}^d$ such that*

$$(g,k) = \begin{cases} (g_0,k_0), & \Omega_{y_0,\omega,\theta_0} \setminus B_1(y_0), \\ (\delta, 0), & \mathbb{R}^d \setminus (\Omega_{y_0,\omega,\theta} \cup B_1(y_0)), \end{cases}$$

*and $(g,k)$ also has decay rate in (2.1.3) on $\mathbb{R}^d$.*

Another natural conjecture that Carlotto made in [Car21, Open Problem 3.14] is that whether we can construct solutions localized in a smaller region as long as we do not violate the constraint of the positive mass theorem. We give a partial answer for the case of a degenerate sector.

**Theorem 2.1.3** ([MT22]). *Let $d \geq 3$, $\frac{3}{d+1} < \alpha < 1$. Consider the degenerate sector*

$$\Omega_\alpha = \{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha, x_d \geq 1\}.$$

*Then there exists a nontrivial asymptotically flat solution $(g, k) \in C^\infty(\mathbb{R}^d)$ of (2.1.1) supported in $\Omega_\alpha$, i.e.*

$$\operatorname{supp}(g_{ij} - \delta_{ij}, k_{ij}) \subset \Omega_\alpha,$$

*and the decay rate of the solution is given by*

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} (g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{2.1.5}$$

*and*

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} k_{ij}(x) = \mathcal{O}(\langle x \rangle^{1-\alpha d-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{2.1.6}$$

Again the solutions to the nonlinear equation (2.1.1) are obtained as perturbations of solutions $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ of the linearized constraint equation (2.1.2) supported in $\Omega_\alpha$ with the decay rate

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} h_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{2.1.7}$$

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} \pi_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{1-\alpha d-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{2.1.8}$$

When $\alpha = 1$, this reduces to Theorem 2.1.1. The constraint $\alpha > \frac{3}{d+1}$ comes from the proof and we do not expect it to be optimal. The decay rate (2.1.5) is optimal for localized solutions in degenerate sectors with this type of anisotropic decay, in the following sense:

**Proposition 2.1.4.** *Suppose $g \in C^2(\mathbb{R}^d)$ is a metric such that $g_{ij}(x) = \delta_{ij}$ for $x \notin \Omega = \{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha,\ x_d \geq 0\}$ with $0 < \alpha \leq 1$. Suppose that for some $\epsilon > 0$ we have*

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} (g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{1-\epsilon-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad |\beta'| + \beta_d \leq 2.$$

*Then the ADM energy vanishes:*

$$E := \lim_{R\to\infty} \frac{1}{2} \int_{|x|=R} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) \frac{x^j}{|x|} dS = 0.$$

*Proof.* We claim $\int_{|x|=R} \partial_i g(x) \frac{x^j}{|x|} dS = \mathcal{O}(R^{-\epsilon})$. This is because

- For $i = j = d$,
$$\left|\int_{|x|=R} \partial_d g(x) \frac{x_d}{|x|} dS\right| \lesssim R^{-\alpha(d-1)-\epsilon} R^{\alpha(d-1)} = R^{-\epsilon}.$$

- For $i \neq d$ and $j \neq d$,
$$\left|\int_{|x|=R} \partial_i g(x) \frac{x^j}{|x|} dS\right| \lesssim R^{1-\alpha d-\epsilon} R^{\alpha-1} R^{\alpha(d-1)} = R^{-\epsilon}.$$

- Since
$$\int_{|x|=R} \partial_j g(x) \frac{x_d}{|x|} dS = \int_{|x|\leq R} \partial_d \partial_j g(x) dx = \int_{|x|=R} \partial_d g(x) \frac{x^j}{|x|} dS,$$
we have for $j \neq d$,
$$\left|\int_{|x|=R} \partial_j g(x) \frac{x_d}{|x|} dS\right| = \left|\int_{|x|=R} \partial_d g(x) \frac{x^j}{|x|} dS\right| \lesssim R^{-\alpha(d-1)-\epsilon} R^{\alpha-1} R^{\alpha(d-1)} = R^{\alpha-1-\epsilon}. \quad \square$$

If the positive mass theorem holds and $R_g = 0$, then $g_{ij}(x)$ satisfying the conditions in Proposition 2.1.4 must be isometric to the flat Euclidean space. In this case, Theorem 2.1.3 has the optimal decay. We note that the positive mass theorem (see Carlotto [Car21, Appendix B]) is usually stated without considering the anisotropic behavior. For example, [Car21, Theorem B.9] works for $\alpha > \frac{d+4}{2(d+1)}$, which does not include the full range $\alpha > \frac{3}{d+1}$. We expect that the positive mass theorem would actually hold for a larger region of $\alpha$ if one considers solutions supported in a degenerate sector with anisotropic asymptotic behavior like (2.1.5)(2.1.6).

We remark that our proof also gives a finite regularity version of Theorem 2.1.1-2.1.3, which can deal with conic gluings for $(g_{ij} - \delta_{ij}, k_{ij}) \in H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}$ when $s > d/2$, which is optimal modulo the end point. We choose to state the smooth version here since it is easier to understand.

The gluing technique in studying (2.1.1) was first studied by Corvino [Cor00] and Corvino–Schoen [CS06], and was generalized by Chruściel–Delay [CD03]. They studied the gluing problem in compact regions, in which case the linearized equation (2.1.2) has cokernel. For an account of gluing with cokernels using similar solution operators, see Mao–Oh–Tao [MOT23] (see also Chapter 3) and Chruściel–Cogo–Nützi [CCN24]. In the pioneering paper of Carlotto–Schoen [CS16b], which was discussed earlier, the gluing technique was extended to noncompact regions and surprising first examples of localized solutions to (2.1.1) were exhibited. Aretakis–Czimek–Rodnianski [ACR23a; ACR25; ACR23b] (see also [Are15; Are17]) introduced and studied the characteristic gluing problem, and related it to the spacelike gluing problems to show the optimal decay. We refer to Chruściel [Chr19] and Carlotto [Car21] for reviews on the conic gluing method and some open problems. See also [Hin22; Hin23c] for a microlocal approach.

### Main ideas

Our construction is different from that of [CS16b]. Recall the linearized constraint equation (2.1.2) is a divergence type equation $Pu = 0$. In [CS16b], Carlotto–Schoen use $PP^*$ to reduce to an elliptic system, which makes it hard to control the support of solutions. The key idea of our construction is to find a right inverse $S$ of $P$ that controls the support of the output. In particular, if $f$ is supported inside a cone, the solution $Sf$ of the equation $Pu = f$ is also supported in the same cone. Such solution operators of divergence type operators with good support properties is motivated by Oh–Tataru [OT19, Section 4], in which they constructed nice solution operators for the divergence equation $\partial_j u^j = 0$. We generalize their construction here to (2.1.2). Then we use Picard iteration in appropriate Sobolev spaces to get the solution of the nonlinear equation (2.1.1). In the case of the degenerate sector (Theorem 2.1.3), we develop a new fundamental solution and introduce anisotropic Sobolev spaces adapted to the degenerate sector. The sharp decay rate is obtained by representing the solution as the solution operator applied to the nonlinearity.

### Organization of the chapter

In §2.2 we give the construction of the solution operator adapted to a cone. In §2.3 we recall several estimates for the b-Sobolev spaces. In §2.4 we prove Theorem 2.1.1 and 2.1.2 using our solution operator. In §2.5 we adapt our method to the degenerate sector to give a proof of Theorem 2.1.3.

## 2.2 Construction of the solution operator for the linearized equation

The crux of our argument is an explicit solution operator to (2.1.2). In this section we will show how to construct a solution operator $S : C_c^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ ($d \geq 3$) for the linearized equation (2.1.2) such that for a cone $\Omega = \Omega_{\omega,\theta}$,

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} Sf \subset \Omega. \tag{2.2.1}$$

Unlike in Corvino [Cor00], the linearized equation (2.1.2) does not have cokernel (on appropriate weighted Sobolev spaces) since we consider (2.1.2) on a noncompact region $\Omega$. The Schwartz kernel of $S$ will have an appropriate decay property.

### Linearized problem

We begin by reformulating (2.1.1) in the perturbative regime of the flat data $(g, k) = (\delta_{ij}, 0)$. We introduce new variables $(h, \pi)$, defined as follows:

$$(h_{ij}, \pi_{ij}) = (g_{ij} - \delta_{ij} - \delta_{ij} \operatorname{tr}_\delta(g - \delta), k_{ij} - \delta_{ij} \operatorname{tr}_\delta k) \tag{2.2.2}$$

Observe that the transformation is obviously invertible with the formulae

$$(g_{ij}, k_{ij}) = (\delta_{ij} + h_{ij} - \frac{1}{d-1}\delta_{ij} \operatorname{tr}_\delta h, \pi_{ij} - \frac{1}{d-1}\delta_{ij} \operatorname{tr}_\delta \pi). \tag{2.2.3}$$

With respect to the new variables, the left-hand sides of (2.1.1) may be written as

$$R[g] = \partial_i \partial_j h^{ij} - M_h^{(2)}(h, \partial^2 h) - M_h^{(1)}(\partial h, \partial h), \tag{2.2.4}$$

$$(\operatorname{tr}_g k)^2 - |k|_g^2 = -M_h^{(0)}(\pi, \pi), \tag{2.2.5}$$

$$g^{jj'}(g^{ii'}\nabla_{g;i} k_{i'j'} - \partial_{j'} \operatorname{tr}_g k) = \partial_i \pi^{ij} - N_h^{(1)j}(h, \partial\pi) - N_h^{(0)j}(\partial h, \pi), \tag{2.2.6}$$

where each of $M_h^{(n)}(u, v)$ or $N_h^{(n)j}(u, v)$ is a linear combination of contraction of $u$ and $v$ with a smooth tensor field (of the appropriate order) on $\mathbb{R}^d$ that depends only on $h$. Here the index $(n)$ is just a label and does not have any particular meaning.

In conclusion, (2.1.1) takes the form

$$\partial_i \partial_j h^{ij} = M_h^{(2)}(h, \partial^2 h) + M_h^{(1)}(\partial h, \partial h) + M_h^{(0)}(\pi, \pi), \tag{2.2.7}$$

$$\partial_i \pi^{ij} = N_h^{(1)j}(h, \partial\pi) + N_h^{(0)j}(\partial h, \pi). \tag{2.2.8}$$

The right hand side is viewed as the nonlinearity and its precise form will not matter in this chapter. In the following, we study how to solve the linearized equations given by the left hand side of (2.2.7)(2.2.8), which we call the *double divergence equation* and the *symmetric divergence equation*, respectively.

## Solution operator for the divergence equations

The construction of the following solution operators is the key for our proof of Theorem 2.1.1 and 2.1.2. In the following $\delta_0$ means Dirac delta function at 0.

**Lemma 2.2.1.** *Suppose that $\kappa \in C^\infty(\mathbb{S}^{d-1})$ and $\int_{\mathbb{S}^{d-1}} \kappa = 1$, then*

$$\partial_j \left( \kappa \left( \frac{x}{|x|} \right) \frac{x^j}{|x|^d} \right) = \delta_0(x). \tag{2.2.9}$$

*Proof.* It is easy to verify (2.2.9) by definition. Here we give a derivation of this formula.

We want to construct a fundamental solution $u$ which satisfies

$$\partial_i u^i = \delta_0. \tag{2.2.10}$$

For a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$, (2.2.10) is equivalent to

$$-\int_{\mathbb{R}^d} u^i(x) \partial_i \varphi(x) dx = \langle \partial_i u^i, \varphi \rangle = \varphi(0).$$

Thus, a natural fundamental solution supported on the ray $\mathbb{R}_{\geq 0}\omega$ is defined by

$$\langle T_\omega, \psi \rangle = \int_0^\infty \omega \psi(t\omega) dt, \quad \psi \in C_c^\infty(\mathbb{R}^d).$$

From this we can construct a smooth fundamental solution by averaging in $\omega$. Indeed, let $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \kappa(\omega) d\omega = 1$. Then

$$\begin{aligned} \left\langle \int_{\mathbb{S}^{d-1}} \kappa(\omega) T_\omega d\omega, \psi \right\rangle &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \omega \psi(t\omega)\kappa(\omega) d\omega dt \\ &= \int_{\mathbb{R}^d} \psi(x) \kappa\left(\frac{x}{|x|}\right) \frac{x}{|x|^d} dx. \end{aligned}$$

Thus we have a fundamental solution

$$\int_{\mathbb{S}^{d-1}} \kappa(\omega) T_\omega(x) d\omega = \kappa\left(\frac{x}{|x|}\right) \frac{x}{|x|^d} \tag{2.2.11}$$

which is homogeneous of degree $1-d$ and smooth outside the origin. □

The same idea can be applied to the linearized constraint equations

$$\partial_i \partial_j h^{ij} = f, \tag{2.2.12}$$
$$\partial_i \pi^{ij} = g. \tag{2.2.13}$$

**Theorem 2.2.2.** *Let $v \in \mathbb{S}^{d-1}$ and $\kappa \in C^\infty(\mathbb{S}^{d-1})$ such that $\int_{\mathbb{S}^{d-1}} \kappa = 1$. We define $K_\kappa, L_{\kappa,v} \in \mathscr{D}'(\mathbb{R}^d)$ by*

$$K_\kappa^{ij}(x) = \kappa\left(\frac{x}{|x|}\right) \frac{x^i x^j}{|x|^d}, \tag{2.2.14}$$
$$L_{\kappa,v}^{jk}(x) = \partial_\ell \left( \kappa\left(\frac{x}{|x|}\right) \frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d} \right). \tag{2.2.15}$$

*Then $K_\kappa^{ij}$ and $L_{\kappa,v}^{ij}$ are fundamental solutions of the linearized constraint equations (2.2.12)(2.2.13), respectively, i.e.,*

$$\partial_i \partial_j K_\kappa^{ij} = \delta_0, \quad \partial_i L_{\kappa,v}^{ij} = \delta_0 v^j.$$

*Moreover, they satisfy the following properties*

- *$K_\kappa$ and $L_{\kappa,v}$ are symmetric 2-tensors;*
- *The supports of $K_\kappa, L_{\kappa,v}$ lie inside the convex hull of the cone $\overline{\{x \in \mathbb{R}^d : \frac{x}{|x|} \in \operatorname{supp} \kappa\}}$;*
- *$K_\kappa$ is homogeneous of degree $2-d$, $L_{\kappa,v}$ is homogeneous of degree $1-d$.*

- $K_\kappa, L_{\kappa,v}$ *are smooth on* $\mathbb{R}^d \setminus \{0\}$.

*Remark* 2.2.3. For a cone $\Omega_{\omega,\theta}$, we fix $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \kappa = 1$ and $\operatorname{supp} \kappa \subset \Omega_{\omega,\theta}$. Then for $f \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ and $g = (g_j)_{j=1}^d \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$, we define

$$S_{\Omega_{\omega,\theta}}(f, g) = (K_\kappa^{ij} * f, \sum_k L_{\kappa,\mathbf{e}_k}^{ij} * g_k) \in C_c^\infty(\mathbb{R}^d; \operatorname{Sym}^2 \mathbb{R}^d) \times C_c^\infty(\mathbb{R}^d; \operatorname{Sym}^2 \mathbb{R}^d)$$

where $\mathbf{e}_k$ is the unit vector in the $x_k$ direction. By Theorem 2.2.2, for $P(h, \pi) = (\partial_i \partial_j h^{ij}, \partial_i \pi^{ij})$, the operator $S$ is a right inverse of $P$, i.e., $PS = I$. Moreover, the support property of $K_\kappa$ and $L_{\kappa,v}$ implies that $S$ satisfies (2.2.1).

*Proof.* It is not hard to verify the properties from the formulas (2.2.14)(2.2.15). We give the derivation below.

For the first equation (2.2.12), in order to solve for $\partial_i \partial_j h^{ij} = \delta_0$, we take a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$. The equation is equivalent to

$$\int_{\mathbb{R}^d} h^{ij}(x) \partial_i \partial_j \varphi(x) dx = \langle \partial_i \partial_j h^{ij}, \varphi \rangle = \varphi(0).$$

Thus, a natural fundamental solution supported on the ray $\mathbb{R}_{\geq 0}\omega$ is given by

$$\langle K_\omega^{ij}, \psi \rangle = \int_0^\infty t \omega^i \omega^j \psi(t\omega) dt, \quad \psi \in C_c^\infty(\mathbb{R}^d).$$

Averaging $\omega$, we have

$$K_\kappa^{ij}(x) := \int_{\mathbb{S}^{d-1}} \kappa(\omega) K_\omega^{ij}(x) d\omega = \kappa\left(\frac{x}{|x|}\right) \frac{x^i x^j}{|x|^d}.$$

For the second equation (2.2.13), we need to first find singular fundamental solutions as before. For $v, \omega \in \mathbb{S}^{d-1}$, let

$$\pi^{jk} = \partial_\ell \phi (\omega^j \omega^\ell v^k + \omega^\ell \omega^k v^j - v^\ell \omega^k \omega^j), \tag{2.2.16}$$

the equation becomes

$$\partial_j \pi^{jk} = \partial_j \partial_\ell \phi \omega^j \omega^\ell v^k = \delta_0 v^k.$$

Define $\phi \in \mathscr{D}'(\mathbb{R}^d)$ so that for a test function $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\langle \phi, \psi \rangle = \int_0^\infty t \psi(t\omega) dt.$$

Then $\partial_j\partial_\ell\phi\omega^j\omega^\ell = \delta_0$ and (2.2.16) gives a fundamental solution $L^{jk}_{\omega,v}$ such that $\partial_j L^{jk}_{\omega,v} = \delta_0 v^k$ and $L^{jk}_{\omega,v} = L^{kj}_{\omega,v}$. Averaging along $\omega$ as before, we get

$$\begin{aligned}
\left\langle \int_{\mathbb{S}^{d-1}} \kappa(\omega) L^{jk}_{\omega,v} d\omega, \psi \right\rangle &= -\int_{\mathbb{S}^{d-1}} \kappa(\omega) \int_0^\infty t(\partial_\ell\psi)(t\omega)(\omega^j\omega^\ell v^k + \omega^\ell\omega^k v^j - v^\ell\omega^k\omega^j) dt d\omega \\
&= -\int_{\mathbb{R}^d} \kappa\left(\frac{x}{|x|}\right)(\partial_\ell\psi)(x)\frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d} dx \\
&= \left\langle \partial_\ell\left(\kappa\left(\frac{x}{|x|}\right)\frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d}\right), \psi \right\rangle.
\end{aligned}$$

So the fundamental solution reads

$$L^{jk}_{\kappa,v}(x) = \partial_\ell\left(\kappa\left(\frac{x}{|x|}\right)\frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d}\right). \qquad \square$$

## 2.3 Estimates on the b-Sobolev spaces

In order to capture the decay property of functions and get optimal regularity, we recall basic estimates for b-Sobolev spaces in this section, and prove our solution operators are bounded on b-Sobolev spaces.

### The b-Sobolev space

**Definition 2.3.1.** For $s \in \mathbb{N}_0$, the b-Sobolev space $H^s_{\mathrm{b}} = H^s_{\mathrm{b}}(\mathbb{R}^d)$ is defined by the norm

$$\|u\|^2_{H^s_{\mathrm{b}}} := \sum_{k\le s} \|\langle x\rangle^k \partial^k u\|^2_{L^2(\mathbb{R}^d)}.$$

We extend the definition to $s \in \mathbb{R}$ by duality and (complex) interpolation. We further define for $\delta \in \mathbb{R}$, $H^{s,\delta}_{\mathrm{b}} := \langle x\rangle^{-\delta} H^s_{\mathrm{b}}$. Moreover, for $\Omega \subset \mathbb{R}^d$, we denote $H^{s,\delta}_{\mathrm{b}}(\Omega) := \{u \in H^{s,\delta}_{\mathrm{b}} : \operatorname{supp} u \subset \overline{\Omega}\}$ the distributions supported in $\overline{\Omega}$.

The b-Sobolev space captures the property that the decay rate of a function improves by $\langle x\rangle^{-1}$ after taking a derivative. When $s$ is an integer, the b-Sobolev space is nothing but the weighted Sobolev space, which already appears in the classical work of Fischer and Marsden [FM75]. We use noninteger $s \in \mathbb{R}$ because we want to solve the problem with optimal regularity ($H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}$ for $s > d/2$ in our case). Later we will use those spaces for vector valued functions, but we shall use the same notation $H^{s,\delta}_{\mathrm{b}}$.

We recall some properties of the b-Sobolev space below. The most important one is the Littlewood–Paley decomposition (2.3.2), which can serve as an alternative definition of the b-Sobolev space and is what we actually use in this chapter.

**Proposition 2.3.2.** *We have the following properties.*

*(a) Littlewood–Paley decomposition: let $\mathcal{B}_r = B_r(0)$ be the ball with radius $r > 0$ and $\mathcal{D}_j = \{2^{j-2} < |x| < 2^{j+2}\} \subset \mathbb{R}^d$ be dyadic annuli for $j \geq 0$, and*

$$\phi_j : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto 2^j x, \quad j \geq 0.$$

*Let $\sum_{j=0}^{\infty} \chi_j = 1$ be a dyadic partition of unity such that*

$$\chi_0 \in C_c^\infty(\mathcal{B}_1; [0,1]), \quad \chi \in C_c^\infty(\mathcal{D}_0; [0,1]), \quad \chi_j = \phi_{j*}(\chi) := \chi \circ \phi_j^{-1}, \ j \geq 1. \tag{2.3.1}$$

*Then*

$$\|u\|^2_{H_{\mathrm{b}}^{s,\delta}} \approx \sum_{j=0}^{\infty} 2^{2j(\delta+d/2)} \|\phi_j^*(\chi_j u)\|^2_{H^s}. \tag{2.3.2}$$

*(b) Sobolev embedding: for $s > d/2$,*

$$\|\langle x\rangle^{d/2+\delta} u\|_{L^\infty} \lesssim \|u\|_{H_{\mathrm{b}}^{s,\delta}}. \tag{2.3.3}$$

*(c) Bilinear estimate: the multiplication is bounded on the following spaces*

$$(u,v) \mapsto uv : H_{\mathrm{b}}^{s_1,\delta_1} \times H_{\mathrm{b}}^{s_2,\delta_2} \to H_{\mathrm{b}}^{s,\delta} \tag{2.3.4}$$

*for $\delta_1 + \delta_2 = \delta - d/2$, $s_1 + s_2 > 0$,*

$$s = \begin{cases} \min(s_1, s_2), & \max(s_1,s_2) > \frac{d}{2}, \\ s_1 + s_2 - \frac{d}{2}, & \max(s_1,s_2) < \frac{d}{2}. \end{cases}$$

*Proof.* We refer to [Hin22, §2.4] for a discussion on basic properties of b-Sobolev spaces.

(a) The Littlewood–Paley decomposition (2.3.2) is [Hin22, Lemma 2.3]. By interpolation and duality, it suffices to check this for $s \in \mathbb{N}_0$, and it is direct to check (2.3.2) inductively for $s$. Note our convention on $\delta$ is shifted by $d/2$ compared to [Hin22, §2.4] due to a different choice of density near the boundary.

(b) The Sobolev embedding is essentially [Hin22, Corollary 2.4]. It is proved as a corollary of (2.3.2):

$$\|\langle x\rangle^{d/2+\delta} u\|_{L^\infty} \lesssim \sup_j 2^{j(d/2+\delta)} \|\chi_j u\|_{L^\infty} \lesssim \sup_j 2^{j(d/2+\delta)} \|\phi_j^*(\chi_j u)\|_{H^s} \lesssim \|u\|_{H_{\mathrm{b}}^{s,\delta}}.$$

(c) The bilinear estimate (2.3.4) is again a corollary of (2.3.2). We introduce a new cutoff $\tilde{\chi}_0 \in C_c^\infty(\mathcal{B}_1; [0,1])$ such that $\tilde{\chi}_0(x) = 1$ on $\operatorname{supp}\chi_0$. Similarly, let $\tilde{\chi}_j = \phi_{j*}\tilde{\chi}$ for $j \geq 1$ where $\tilde{\chi} \in C_c^\infty(\mathcal{D}_0; [0,1])$ and $\tilde{\chi}(x) = 1$ for $x \in \operatorname{supp}\chi$. We have

$$\begin{aligned}
\|uv\|^2_{H_{\mathrm{b}}^{s,\delta}} &\lesssim \sum_j 2^{2j(\delta+d/2)} \|\phi_j^*(\chi_j uv)\|^2_{H^s} \\
&\lesssim \sum_j 2^{2j(\delta+d/2)} \|\phi_j^*(\chi_j u)\|^2_{H^{s_1}} \|\phi_j^*(\tilde{\chi}_j v)\|^2_{H^{s_2}} \\
&\lesssim \left(\sum_j 2^{2j(\delta_1+d/2)} \|\phi_j^*(\chi_j u)\|^2_{H^{s_1}}\right)\left(\sum_j 2^{2j(\delta_2+d/2)} \|\phi_j^*(\tilde{\chi}_j u)\|^2_{H^{s_2}}\right) \\
&\lesssim \|u\|^2_{H_{\mathrm{b}}^{s_1,\delta_1}} \|v\|^2_{H_{\mathrm{b}}^{s_2,\delta_2}}. \qquad \square
\end{aligned}$$

We now show our solution operator is bounded on $H_{\mathrm{b}}^{s,\delta}$. One important property of our solution operator is that it is outgoing in the following sense.

**Definition 2.3.3.** Let $\Omega \subset \mathbb{R}^d$ be a subset of $\mathbb{R}^d$. $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ be the Schwartz kernel of an operator which we also denote by $K$. Suppose $K$ has the property

$$u \in C_c^\infty(\mathbb{R}^d),\ \operatorname{supp} u \subset \overline{\Omega} \Longrightarrow \operatorname{supp} Ku \subset \overline{\Omega}.$$

We say $K$ is outgoing (incoming, respectively) on $\Omega$ if there exists a constant $C > 0$ such that $K(x,y) = 0$ for $x, y \in \Omega$ and $|y| > C|x| + C$ (for $|x| > C|y| + C$, respectively). We also say $K$ is diagonal if $K$ is both incoming and outgoing.

**Example 2.3.4.** Let $\Omega = \Omega_{\omega,\theta}$ for some $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$. Let $\chi \in C_c^\infty(\Omega \cap \mathbb{S}^{d-1})$. The solution operator $S$ constructed in Theorem 2.2.2 with Schwartz kernel $(K_\chi^{ij}(x-y), L_{\chi,\mathbf{e}_k}^{ij}(x-y))$ is outgoing on $\Omega$, where $\mathbf{e}_k$ is the unit vector in the $x_k$ direction. Indeed we have the following stronger property: $S(x,y) = 0$ for $|x| < |y|$, i.e. the value of $Sf(x)$ only depends on $f(y)$ for $|y| \leq |x|$.

We give some criteria for the boundedness of diagonal/incoming/outgoing operators on b-Sobolev spaces. The main idea is to decompose the space into dyadic regions and use the decay of the operator norm away from the diagonal. It will be applied to our solution operators constructed in §2.2 and their variants.

**Lemma 2.3.5.** *Suppose $K$ is a diagonal operator on $\Omega$ such that for some $s_1, s_2, k \in \mathbb{R}$, $K$ is bounded on $H^{s_1}_{comp} \to H^{s_2}_{\mathrm{loc}}$, and for any $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, there exists $C > 0$ such that for all $j \in \mathbb{N}$, the operator with Schwartz kernel*

$$K(2^j x, 2^j y)\chi(x)\chi(y) \tag{2.3.5}$$

*is bounded by $C2^{j(k-d)}$ on $H^{s_1} \to H^{s_2}$. Then*

$$K : H_{\mathrm{b}}^{s_1,\delta}(\Omega) \to H_{\mathrm{b}}^{s_2,\delta-k}(\Omega)$$

*is bounded for any $\delta \in \mathbb{R}$. In particular, if*

$$|\partial_{x,y}^{\ell} K(x,y)| \lesssim \langle x\rangle^{k-d-\ell}, \quad \ell \in \mathbb{N},$$

*then $K : H_{\mathrm{b}}^{-N,\delta} \to H_{\mathrm{b}}^{N,\delta-k}$ is bounded for any $N \in \mathbb{N}$, $\delta \in \mathbb{R}$.*

*Proof.* Recall from Proposition 2.3.2 that for $u \in C_c^\infty(\Omega)$ and a partition of unity as in (2.3.1),

$$\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} \approx \sum_j 2^{2j(\delta+d/2-k)} \|\phi_j^*(\chi_j Ku)\|^2_{H^{s_2}}$$

Let

- $\tilde{\chi}_0 \in C_c^\infty(\mathbb{R}^d; [0,1])$;
- $\tilde{\chi} \in C_c^\infty(\{2^{-C_0} < |x| < 2^{C_0}\}; [0,1])$ and $\tilde{\chi}_j(x) = \phi_{j*}\tilde{\chi}(x) = \tilde{\chi}(2^{-j}x)$ for $j \geq 1$

be another family of dyadic cutoffs such that for any $(x,y) \in \operatorname{supp} K(x,y)$, $x \in \operatorname{supp} \chi_j$ implies $\tilde{\chi}_j(y) = 1$ (this is possible because $K$ is diagonal). Therefore by the bound for (2.3.5), we have

$$\|\phi_j^*(\chi_j Ku)\|_{H^{s_2}} = \|\phi_j^*(\chi_j K(\tilde{\chi}_j u))\|_{H^{s_2}} \lesssim 2^{jk} \|\phi_j^*(\tilde{\chi}_j u)\|_{H^{s_1}}$$

and

$$\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} \lesssim \sum_j 2^{2j(\delta+d/2-k)} 2^{2jk} \|\phi_j^*(\tilde{\chi}_j u)\|^2_{H^{s_1}} \approx \|u\|^2_{H_{\mathrm{b}}^{s_1,\delta}}. \qquad \square$$

**Lemma 2.3.6.** *Suppose $K$ is an outgoing (incoming, respectively) operator on $\Omega$ such that for some $s_1, s_2, k \in \mathbb{R}$, $K$ is bounded on $H^{s_1}_{comp} \to H^{s_2}_{\mathrm{loc}}$, and for any $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and $\widetilde{\chi} \in C_c^\infty(\mathbb{R}^d)$, there exists $C > 0$ such that for any $j, j' \in \mathbb{N}$, the operator with Schwartz kernel*

$$K(2^j x, 2^{j'} y)\chi(x)\widetilde{\chi}(y) \quad (K(2^j x, 2^{j'} y)\chi(y)\widetilde{\chi}(x), \textit{respectively})$$

*is bounded by $C2^{j(k-d)}$ ($C2^{j'(k-d)}$, respectively) on $H^{s_1} \to H^{s_2}$. Then*

$$K : H_{\mathrm{b}}^{s_1,\delta}(\Omega) \to H_{\mathrm{b}}^{s_2,\delta-k}(\Omega)$$

*is bounded for $\delta < d/2$ ($\delta > -d/2 + k$, respectively).*

*Proof.* We will only show the outgoing case since the incoming case is similar. For $K$ outgoing, there exists a constant $C_0 > 0$ such that $(x,y) \in \operatorname{supp} K(x,y)$, $x \in \operatorname{supp} \chi_j$, $y \in \operatorname{supp} \chi_{j'}$ implies that $j' \leq j + C_0$. Thus for $\beta \in \mathbb{R}$ be to determined later and $u \in C_c^\infty(\Omega)$, we fix a dyadic partition of unity as in (2.3.1) and

$$\begin{aligned}
\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} &\approx \sum_j 2^{2j(\delta+d/2-k)}\|\phi_j^*(\chi_j Ku)\|^2_{H^{s_2}} \\
&\approx \sum_j 2^{2j(\delta+d/2-k)}\left\|\sum_{j'\le j+C_0}\phi_j^*(\chi_j K\chi_{j'}u)\right\|^2_{H^{s_2}} \\
&\lesssim \sum_j 2^{2j(\delta+d/2-k)}\left(\sum_{j'\le j+C_0} 2^{j(k-d)}2^{j'd}\|\phi_{j'}^*(\chi_{j'}u)\|_{H^{s_1}}\right)^2 \\
&\lesssim \sum_j 2^{2j(\delta+d/2-k)}\left(\sum_{j'\le j+C_0} 2^{j(k-d)}2^{j'd}2^{\beta(j-j')}\right)\left(\sum_{j'\le j+C_0} 2^{j(k-d)}2^{j'd}2^{\beta(j'-j)}\|\phi_{j'}^*(\chi_{j'}u)\|^2_{H^{s_1}}\right).
\end{aligned}$$

We use the bound on $\phi_j^*(\chi_j K\chi_{j'}u)$ in the second to last step. Since $\delta < d/2$, we may choose $\beta \in \mathbb{R}$ such that $2\delta < \beta < d$, so (in the first term one sums over $j'$, while in the second term one sums over $j$)

$$\sum_{j'\le j+C_0} 2^{j(k-d)}2^{j'd}2^{\beta(j-j')} \lesssim 2^{jk}, \quad \sum_{j'\le j+C_0} 2^{2j(\delta+d/2-k)}2^{jk}2^{j(k-d)}2^{j'd}2^{\beta(j'-j)} \lesssim 2^{2j'(\delta+d/2)}.$$

So we conclude that

$$\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} \lesssim \sum_{j'} 2^{2j'(\delta+d/2)}\|\phi_{j'}^*(\chi_{j'}u)\|^2_{H^{s_1}} \approx \|u\|^2_{H_{\mathrm{b}}^{s_1,\delta}}. \qquad \square$$

*Remark* 2.3.7. The threshold $\delta < d/2$ is natural. Since for an outgoing operator $K$ with Schwartz kernel $\langle x-y\rangle^{k-d}$ and compactly supported input $f \in C_c^\infty$, the best decay one can hope is

$$|Kf(x)| \lesssim \langle x\rangle^{k-d} \notin H_{\mathrm{b}}^{s,\delta-k}, \quad \delta \ge d/2.$$

**Proposition 2.3.8.** *Let $K \in C^\infty(\mathbb{R}^d\setminus\{0\})\cap\mathscr{D}'(\mathbb{R}^d)$ be a homogeneous distribution of degree $k-d$, $0<k<d$. For $-d/2<\delta<d/2-k$, the map*

$$u \mapsto K*u(x) := \int_{\mathbb{R}^d} K(x-y)u(y)dy \tag{2.3.6}$$

*is bounded on $H_{\mathrm{b}}^{s-k,\delta+k} \to H_{\mathrm{b}}^{s,\delta}$. Indeed, the operator $u \mapsto K*u$ can be decomposed into three parts $K_{\mathrm{diag}}+K_{\mathrm{in}}+K_{\mathrm{out}}$ such that*

*(a) $K_{\mathrm{diag}}(x,y)$ is supported in $\{2^{-1}|y|\le|x|\le 2|y|\}\cup\mathcal{B}_1\times\mathcal{B}_1$ and $K_{\mathrm{diag}} : H_{\mathrm{b}}^{s-k,\delta+k} \to H_{\mathrm{b}}^{s,\delta}$ is bounded for $s\in\mathbb{R}$ and $\delta\in\mathbb{R}$.*

*(b) $K_{\mathrm{out}}(x,y)$ is supported in $\{|x|\ge(1+c)|y|\}\setminus\mathcal{B}_c\times\mathcal{B}_c$ for some $c>0$, and $K_{\mathrm{out}} : H_{\mathrm{b}}^{-N,\delta+k} \to H_{\mathrm{b}}^{N,\delta}$ is bounded for any $N\in\mathbb{N}$ and $\delta<d/2-k$.*

(c) *$K_{\mathrm{in}}(x,y)$ is supported in $\{|x| \le (1-c)|y|\} \backslash \mathcal{B}_c \times \mathcal{B}_c$ for some $c > 0$, and $K_{\mathrm{in}} : H_{\mathrm{b}}^{-N,\delta+k} \to H_{\mathrm{b}}^{N,\delta}$ is bounded for any $N \in \mathbb{N}$ and $\delta > -d/2$.*

*We have the estimate*

$$|\partial_{x,y}^{\ell} K_{\mathrm{out}}(x,y)| + |\partial_{x,y}^{\ell} K_{\mathrm{in}}(x,y)| \lesssim (1+|x|+|y|)^{k-d-\ell}, \quad \ell \in \mathbb{N}. \tag{2.3.7}$$

*Moreover, if we suppose $K(x-y)$ is outgoing (incoming, respectively) on some $\Omega \subset \mathbb{R}^d$, then for $\delta < d/2 - k$ (for $\delta > -d/2$, respectively), the map (2.3.6) is bounded on $H_{\mathrm{b}}^{s-k,\delta+k}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega)$.*

*Proof.* We decompose $K(x-y) = K_{\mathrm{d}iag}(x,y) + K_{\mathrm{in}}(x,y) + K_{out}(x,y)$, where

$$K_{\mathrm{d}iag}(x,y) = K(x-y)\varphi(x,y)$$

where $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a cutoff function such that $\varphi(x,y) = 1$ near $\{|x| = |y|\}$ and

$$\varphi(x,y) = \psi\left(\frac{|x|-|y|}{|y|}\right), \quad |x|+|y| > 1 \tag{2.3.8}$$

for some other cutoff $\psi \in C_c^\infty((-1/2,1/2))$ such that $\psi = 1$ near 0; $K_{\mathrm{in}} = (K - K_{\mathrm{d}iag})\mathbb{1}_{|x|<|y|}$ is incoming part of $K$, and $K_{out} = (K - K_{\mathrm{d}iag})\mathbb{1}_{|x|>|y|}$ is the outgoing part of $K$. One can check $K_{in}$ is incoming, $K_{out}$ is outgoing and $K_{\mathrm{d}iag}$ is diagonal in the sense of Definition 2.3.3.

(a) For the diagonal part $K_{\mathrm{d}iag}$, for any constant $C_0 > 0$ and $C_0^{-1} < |x|+|y| < C_0$, we have
$$K_{\mathrm{d}iag}(2^j x, 2^j y) = K(2^j(x-y))\varphi(2^j x, 2^j y) = 2^{j(k-d)} K(x-y)\varphi(x,y)$$
is a singular integral operator bounded by $C2^{j(k-d)}$ on $H^{s-k} \to H^s$. So the result follows from Lemma 2.3.5.

(b) For the outgoing part $K_{out}$, for any constant $C_0 > 0$ with $C_0^{-1} < |x|+|y| < C_0$ and $j > j'$,
$$\begin{aligned} K_{out}(2^j x, 2^{j'} y) &= K(2^j x - 2^{j'} y)(1 - \varphi(2^j x, 2^{j'} y))\mathbb{1}_{2^j|x|>2^{j'}|y|} \\ &= 2^{j(k-d)} K(x - 2^{j'-j} y)(1 - \varphi(x, 2^{j'-j} y))\mathbb{1}_{|x|>2^{j'-j}|y|} \end{aligned}$$
satisfies
$$|\partial_{x,y}^{\ell} K_{out}(2^j x, 2^{j'} y)| \lesssim 2^{j(k-d)}, \quad \ell \in \mathbb{N}$$
and is thus bounded on $H^{-N} \to H^N$ for any $N \in \mathbb{N}$ by Lemma 2.3.6.

(c) Similarly for the incoming parting $K_{\mathrm{in}}$, for any constant $C_0 > 0$ with $C_0^{-1} < |x|+|y| < C_0$ and $j' > j$,
$$\begin{aligned} K_{\mathrm{in}}(2^j x, 2^{j'} y) &= K(2^j x - 2^{j'} y)(1 - \varphi(2^j x, 2^{j'} y))\mathbb{1}_{2^j|x|<2^{j'}|y|} \\ &= 2^{j'(k-d)} K(2^{j-j'} x - y)(1 - \varphi(2^{j-j'} x, y))\mathbb{1}_{|y|>2^{j-j'}|x|} \end{aligned}$$

satisfies

$$|\partial^\ell_{x,y} K_{\text{in}}(2^j x, 2^{j'} y)| \lesssim 2^{j'(k-d)}, \quad \ell \in \mathbb{N}.$$

It is bounded on $H^{-N} \to H^N$ for any $N \in \mathbb{N}$ by Lemma 2.3.6.

The estimate (2.3.7) follows from the explicit expression and we note that the derivative falls on $\varphi(x,y)$ only if $|x| \approx |y|$.

Finally, if $K(x-y)$ is outgoing (incoming, respectively), the same proof works without dealing with the incoming (outgoing, respectively) part. □

## Smoothness of curvature

As a corollary of the bilinear estimate in Proposition 2.3.2(c), we can now conclude the maps we consider are smooth on b-Sobolev spaces.

**Lemma 2.3.9.** *Let $s > d/2$, $\delta > -d/2$. In a small neighbourhood of $\delta_{ij}$, the inverse matrix map*

$$(g_{ij}) \mapsto (g^{ij}) : (\delta_{ij}) + H^{s,\delta}_{\mathrm{b}} \to (\delta_{ij}) + H^{s,\delta}_{\mathrm{b}}$$

*is a smooth map.*

*Proof.* By Cramer's rule, the inverse matrix map can be obtained as compositions of multiplications and the following map (which appears in $\det(g_{ij})^{-1}$ if we write $\det(g_{ij}) = 1 - f$):

$$T : f \mapsto \frac{1}{1-f} - 1 = f + f^2 + f^3 + \cdots$$

Since multiplication is bounded by Proposition 2.3.2(c), it suffices to prove $T : H^{s,\delta}_{\mathrm{b}} \to H^{s,\delta}_{\mathrm{b}}$ is smooth for $\|f\|_{H^{s,\delta}_{\mathrm{b}}} \ll 1$. The boundedness is a corollary of the bilinear estimate (2.3.4) again. To prove the boundedness of the derivatives, just observe

$$DT_{f_0}(f) = (1 + 2f_0 + 3f_0^2 + \cdots)f, \quad D^2T_{f_0}(f_1, f_2) = (2 + 6f_0 + 12f_0^2 + \cdots)f_1 f_2, \cdots$$

are all continuous multilinear maps in $H^{s,\delta}_{\mathrm{b}}$. □

Recall from (2.2.4)-(2.2.6) that $M^{(\ell)}_h$ and $N^{(\ell)j}_h$ are bilinear expressions which depend smoothly on $h$.

**Proposition 2.3.10.** *For $s > d/2$, $\delta > -d/2$, the operator*

$$(h, \pi) \mapsto (M^{(2)}_h(h, \partial^2 h), M^{(1)}_h(\partial h, \partial h), M^{(0)}_h(\pi, \pi), N^{(1)j}_h(h, \partial\pi), N^{(0)j}_h(\partial h, \pi))$$

*is smooth $H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}} \to H^{s-2,\delta+2}_{\mathrm{b}}$ in a small neighbourhood of $(0,0)$.*

*Proof.* The dependence of $M^{(\ell)}_h$ and $N^{(\ell)j}_h$ on $h$ only involves multiplication and the map in Lemma 2.3.9, which are both smooth by Proposition 2.3.2(c) and Lemma 2.3.9. To get the correct exponents, one also uses that $\partial : H^{s,\delta}_{\mathrm{b}} \to H^{s-1,\delta+1}_{\mathrm{b}}$ is bounded. □

## 2.4 Solving the nonlinear equation

In this section we use our solution operators from Theorem 2.2.2 to prove Theorem 2.1.1 and 2.1.2. The basic idea is to solve the linearized equation (2.1.2) first and then use the Banach fixed point theorem (or Picard iteration) to upgrade to a solution of the nonlinear system (2.1.1).

In §2.4, we give a simple construction of solutions of the linearized equation (2.1.2), which shows the linearized operator has a big kernel. In particular, one can choose a solution so that the linearized Riemann curvature does not vanish, which will ensure that the final solution to (2.1.1) we get is not isometric to Euclidean space. In §2.4, we solve the nonlinear system (2.1.1) and prove it is actually smooth and has the decay (2.1.3). In §2.4 we prove Theorem 2.1.2 following a similar strategy by also using the Bogovskii-type operator introduced in [MOT23] (see also Chapter 3).

### Construction of solutions to the linearized equation

Our construction of solutions of the constraint equation (2.1.1) relies on solutions of the linearized equation (2.1.2). There are many compactly supported solutions to the homogeneous linearized equations (2.1.2). A basic observation is that to solve the double divergence equation, we only need to solve the symmetric divergence equation

$$\partial_j \pi^{jk} = 0 \tag{2.4.1}$$

since this would give $\partial_j \partial_k \pi^{jk} = 0$. For (2.4.1), we may take an arbitrary $\phi \in C_c^\infty(\mathbb{R}^d)$ and

$$\pi^{11} = \partial_2\partial_2\phi,\ \pi^{12} = \pi^{21} = -\partial_1\partial_2\phi,\ \pi^{22} = \partial_1\partial_1\phi,\ \pi^{jk} = 0 \text{ for } \{j,k\} \not\subset \{1,2\}.$$

Then we get a solution to $\partial_j \pi^{jk} = 0$.

### Proof of Theorem 2.1.1

In this section we prove Theorem 2.1.1 by the Banach fixed point theorem (or Picard iteration).

*Proof of Theorem 2.1.1.* The proof goes by three steps.

**Step 1: Existence.** Let

$$\begin{aligned} P(h,\pi) &= (\partial_i\partial_j h^{ij}, \partial_i \pi^{ij}) \\ \Phi(h,\pi) &= (M_h^{(2)}(h,\partial^2 h) + M_h^{(1)}(\partial h,\partial h) + M_h^{(0)}(\pi,\pi), N_h^{(1)j}(h,\partial\pi) + N_h^{(0)j}(\partial h,\pi)). \end{aligned}$$

Then the equations (2.1.1) become

$$P(h,\pi) = \Phi(h,\pi). \tag{2.4.2}$$

Let $\Omega = \Omega_{\omega,\theta}$ be the cone in the statement of Theorem 2.1.1 and $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ be a solution of the linearized equation $P(h_0, \pi_0) = 0$ supported in $\Omega$ with decay rate (2.1.4) (such solutions exist even with compact support, as discussed in §2.4). Let $s > d/2$, $-d/2 < \delta < d/2 - 2$ and

$$S(f, g) = S_\Omega(f, g) = (K_\kappa^{ij} * f, \sum_k L_{\kappa,\mathbf{e}_k}^{ij} * g_k) \in C_c^\infty(\mathbb{R}^d; \operatorname{Sym}^2 \mathbb{R}^d) \times C_c^\infty(\mathbb{R}^d; \operatorname{Sym}^2 \mathbb{R}^d)$$

be the solution operator given by Theorem 2.2.2 and Remark 2.2.3 (see also Example 2.3.4). Then

$$f \in \mathscr{D}'(\mathbb{R}^d),\ \operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} Sf \subset \Omega,\ PSf = f$$

due to the convexity of the cone $\Omega$. We consider the following fixed point problem

$$(h_1, \pi_1) = S\Phi(h_0 + h_1, \pi_0 + \pi_1). \tag{2.4.3}$$

By Proposition 2.3.10, $\Phi : H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \to H_{\mathrm{b}}^{s-2,\delta+2}(\Omega)$ is a smooth map with the differential at zero $D\Phi_0 = 0$. By choosing $\|(h, \pi)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \le \epsilon := \|(h_0, \pi_0)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}$ for some sufficiently small $\epsilon > 0$ we get

$$\|\Phi(h_0 + h, \pi_0 + \pi)\|_{H_{\mathrm{b}}^{s-2,\delta+2}} \lesssim \epsilon^2,$$
$$\|\Phi(h_0 + h, \pi_0 + \pi) - \Phi(h_0 + \tilde{h}, \pi_0 + \tilde{\pi})\|_{H_{\mathrm{b}}^{s-2,\delta+2}} \lesssim \epsilon \|(h, \pi) - (\tilde{h}, \tilde{\pi})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}.$$

By Proposition 2.3.8, $S : H_{\mathrm{b}}^{s-2,\delta+2}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$ is bounded. For $\epsilon$ sufficiently small, the map

$$G(h, \pi) = S\Phi(h_0 + h, \pi_0 + \pi) : H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \tag{2.4.4}$$

is a contraction on $\{\|(h, \pi)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \le \epsilon\}$. By the Banach fixed point theorem, there exists a unique fixed point $(h_1, \pi_1) \in H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$ such that

$$(h_1, \pi_1) = S\Phi(h_0 + h_1, \pi_0 + \pi_1), \quad \|(h_1, \pi_1)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim \|(h_0, \pi_0)\|^2_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}.$$

This implies that $(h_0 + h_1, \pi_0 + \pi_1)$ solves (2.4.2):

$$P(h_0 + h_1, \pi_0 + \pi_1) = P(h_1, \pi_1) = \Phi(h_0 + h_1, \pi_0 + \pi_1).$$

An alternative way is to consider the following operator:

$$F : H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$$

defined by

$$F(h, \pi) = (h, \pi) - S\Phi(h, \pi).$$

Then $(h, \pi)$ solves the constraint equation (2.4.2) if and only if $F(h, \pi)$ solves the linearized constraint equation $P(F(h, \pi)) = 0$. Since $F$ is a smooth operator and the differential of $F$ at 0 is the identity, we know $F$ is invertible in a neighbourhood of 0.

As explained at the beginning of this section, choosing $h_0$ with nonzero linearized curvature gives a metric with nonzero curvature since $\|h_1\|_{H_{\mathrm{b}}^{s,\delta}} \lesssim \|h_0\|^2_{H_{\mathrm{b}}^{s,\delta}}$.

**Step 2: Smoothness.** In the previous step, we construct a solution to (2.1.1) for fixed regularity $s$. Now we show that if $(h_0, \pi_0)$ is taken to be smooth with the decay assumptions (2.1.4), the solution we get is actually smooth. For simplicity, we take $s = d + 2$ (this is not essential) and $-d/2 < \delta < d/2 - 2$. We claim for $\epsilon$ sufficiently small,

$$(h_1, \pi_1) \in H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}, \quad \forall s \geq d + 2. \tag{2.4.5}$$

We prove it inductively: suppose $(h_1, \pi_1) \in H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ for some integer $s \geq d+2$, we will show $(h_1, \pi_1) \in H_{\mathrm{b}}^{s+1,\delta} \times H_{\mathrm{b}}^{s,\delta+1}$.

We decompose $S = S_{\mathrm{diag}} + S_{out}$ as in the proof of Proposition 2.3.8 such that $S_{\mathrm{diag}}(x, y)$ is supported near the diagonal and $S_{out}$ is supported in $\{|x| \geq (1 + c)|y|\} \setminus \mathcal{B}_c \times \mathcal{B}_c$ for some $c > 0$. Note there $S_{\mathrm{in}} = 0$ due to the outgoing property of $S$. Recall $(h_1, \pi_1)$ solves the equation (2.4.3), which can be written as

$$(h_1, \pi_1) = S_{\mathrm{diag}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)) + S_{\mathrm{out}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)). \tag{2.4.6}$$

By bilinear estimate (2.3.4), $\Phi(h_0 + h_1, \pi_0 + \pi_1) \in H_{\mathrm{b}}^{s-2,\delta+2}$. By Proposition 2.3.8(b), $S_{out}(\Phi(h_0 + h_1, \pi_0 + \pi_1)) \in H_{\mathrm{b}}^{\infty,\delta} \times H_{\mathrm{b}}^{\infty,\delta+1}$. Now we differentiate (2.4.6),

$$\partial(h_1, \pi_1) = [\partial, S_{\mathrm{diag}}](\Phi(h_0+h_1, \pi_0+\pi_1))+S_{\mathrm{diag}}(\partial\Phi(h_0+h_1, \pi_0+\pi_1))+\partial S_{out}(\Phi(h_0+h_1, \pi_0+\pi_1)).$$

Recall

$$S_{\mathrm{diag}}(x, y) = (S^{(1)}_{\mathrm{diag}}(x, y), S^{(2)}_{\mathrm{diag}}(x, y)) = (K^{(1)}(x - y), K^{(2)}(x - y))\varphi(x, y)$$

where $\varphi$ is defined as in (2.3.8) and $K^{(j)}$ is homogeneous of degree $k_j - d$ with $k_1 = 2$, $k_2 = 1$. The Schwartz kernel of $[\partial, S^{(j)}_{\mathrm{diag}}]$ is given by

$$K^{(j)}(x - y)(\partial_x + \partial_y)\varphi(x, y)$$

which vanishes near $\{|x| = |y|\}$. Moreover, we have the following bound

$$\partial^{\ell}_{x,y}(K^{(j)}(x - y)(\partial_x + \partial_y)\varphi(x, y)) \lesssim \langle x\rangle^{k_j-d-\ell-1}, \quad \ell \in \mathbb{N}, \ j = 1, 2.$$

By Lemma 2.3.5, this implies boundedness of (we actually have better regularity improvement but we write a weaker estimate here which works for other cases we consider later in Theorem 2.1.2)

$$[\partial, S_{\mathrm{diag}}] : H_{\mathrm{b}}^{s-2,\delta+2} \to H_{\mathrm{b}}^{s,\delta+1} \times H_{\mathrm{b}}^{s-1,\delta+2}, \quad s \in \mathbb{R}, \ \delta \in \mathbb{R}.$$

So $[\partial, S_{\mathrm{d}iag}](\Phi(h_0+h_1,\pi_0+\pi_1)) \in H_{\mathrm{b}}^{s,\delta+1}\times H_{\mathrm{b}}^{s-1,\delta+2}$. We are left with $S_{\mathrm{d}iag}(\partial\Phi(h_0+h_1,\pi_0+\pi_1))$. Note we can differentiate (2.4.3) $s$ times and get

$$\partial^s(h_1,\pi_1) = S_{\mathrm{d}iag}(\partial^s\Phi(h_0+h_1,\pi_0+\pi_1)) + e_1, \quad e_1 \in H_{\mathrm{b}}^{1,\delta+s}\times H_{\mathrm{b}}^{0,\delta+s+1}.$$

Now we note

$$\partial^s\Phi(h_0+h_1,\pi_0+\pi_1) = (M_{h_0,h_1}(h_0+h_1,\partial^{s+2}h_1), N_{h_0,h_1}(h_0+h_1,\partial^{s+1}\pi_1))+e_2, \quad e_2 \in H_{\mathrm{b}}^{-1,\delta+s+2}$$

where $M, N$ are bilinear forms depending on $h_0, h_1$. So

$$\|\partial^s\Phi(h_0+h_1,\pi_0+\pi_1)\|_{H_{\mathrm{b}}^{-1,\delta+s+2}} \lesssim \|h_0+h_1\|_{H_{\mathrm{b}}^{d/2+1,\delta}}(\|h_1\|_{H_{\mathrm{b}}^{s+1,\delta}}+\|\pi_1\|_{H_{\mathrm{b}}^{s,\delta+1}}) + \|e_2\|_{H_{\mathrm{b}}^{-1,\delta+s+2}}.$$

Since $S_{\mathrm{d}iag}: H_{\mathrm{b}}^{-1,\delta+s+2}\to H_{\mathrm{b}}^{1,\delta+s}\times H_{\mathrm{b}}^{0,\delta+s+1}$ is bounded by Proposition 2.3.8(a), this implies that

$$\begin{aligned}\|(h_1,\pi_1)\|_{H_{\mathrm{b}}^{s+1,\delta}\times H_{\mathrm{b}}^{s,\delta+1}} \lesssim{}& \|h_0+h_1\|_{H_{\mathrm{b}}^{d/2+1,\delta}}(\|h_1\|_{H_{\mathrm{b}}^{s+1,\delta}}+\|\pi_1\|_{H_{\mathrm{b}}^{s,\delta+1}}) \\ &+ \|(h_1,\pi_1)\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}} + \|e_1\|_{H_{\mathrm{b}}^{1,\delta+s}\times H_{\mathrm{b}}^{0,\delta+s+1}} + \|e_2\|_{H_{\mathrm{b}}^{-1,\delta+s+2}}.\end{aligned}$$

Since $\|h_0+h_1\|_{H_{\mathrm{b}}^{d/2+1,\delta}}$ is small, this shows

$$(h_1,\pi_1)\in H_{\mathrm{b}}^{s+1,\delta}\times H_{\mathrm{b}}^{s,\delta-1}.$$

This finishes the induction and proves (2.4.5).

**Step 3: Tail estimate.** Now we prove the last part of Theorem 2.1.1, namely the decay rate estimate (2.1.3). Roughly speaking, if the nonlinearity is integrable, then the decay rate comes from applying the solution operator to the nonlinearity, and should be the same as the decay rate of the Schwartz kernel of the solution operator. For simplicity of the proof, we choose $\delta = d/2-2.1$. Since $(h_0,\pi_0)$ satisfies (2.1.4), in order to show (2.1.3) it suffices to show the following

$$|\partial^\ell h_1(x)| \lesssim \langle x\rangle^{2-d-\ell}, \quad |\partial^\ell \pi_1(x)| \lesssim \langle x\rangle^{1-d-\ell}, \quad \ell\in\mathbb{N}. \tag{2.4.7}$$

First we note the bilinear estimate (2.3.4) actually implies

$$\Phi(h_0+h_1,\pi_0+\pi_1)\in H_{\mathrm{b}}^{\infty,2\delta+2+d/2}\subset H_{\mathrm{b}}^{\infty,d/2+1/2}.$$

As in the previous step, we decompose $S = S_{\mathrm{d}iag}+S_{\mathrm{o}ut}$. Recall from (2.4.6) that

$$(h_1,\pi_1) = S_{\mathrm{d}iag}(\Phi(h_0+h_1,\pi_0+\pi_1)) + S_{\mathrm{o}ut}(\Phi(h_0+h_1,\pi_0+\pi_1)).$$

Since $S_{\mathrm{d}iag}: H_{\mathrm{b}}^{s-2,\delta+2}\to H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}$ is bounded for any $s,\delta\in\mathbb{R}$, we have $S_{\mathrm{d}iag}(\Phi(h_0+h_1,\pi_0+\pi_1))\in H_{\mathrm{b}}^{\infty,d/2-3/2}\times H_{\mathrm{b}}^{\infty,d/2-1/2}$. By Sobolev embedding (2.3.3), this implies

$$|\partial^\ell S_{\mathrm{d}iag}^{(j)}(\Phi(h_0+h_1,\pi_0+\pi_1))| \lesssim \langle x\rangle^{k_j-1/2-d-\ell}, \quad \ell\in\mathbb{N},\ j=1,2. \tag{2.4.8}$$

Let $S_{\text{out}}^{(1)}$, $S_{\text{out}}^{(2)}$ be the first and second component of $S_{out}$, and $k_1 = 2$, $k_2 = 1$. By (2.3.7),

$$|\partial_x^\ell S_{\text{out}}^{(j)}(x,y)| \lesssim \langle x\rangle^{k_j-d-\ell}, \quad \ell \in \mathbb{N}.$$

This implies

$$|\partial_x^\ell S_{\text{out}}^{(j)}(\Phi(h_0+h_1,\pi_0+\pi_1))| \lesssim \langle x\rangle^{k_j-d-\ell} \int_{\mathbb{R}^d} |\Phi(h_0+h_1,\pi_0+\pi_1)(y)|dy. \tag{2.4.9}$$

Since $\Phi(h_0+h_1,\pi_0+\pi_1) \in H_{\text{b}}^{\infty,d/2+1/2}$, the integral on the right hand side is finite by Sobolev embedding (2.3.3). From (2.4.8) and (2.4.9), we conclude (2.4.7), which finishes the proof. □

## Gluing construction of the solution

In this section we provide the proof of the gluing result in Theorem 2.1.2.

Let $\Omega_{y_0,\omega,\theta_0} \subset \Omega_{y_0,\omega,\theta}$ be the two cones centered at $y_0 \in \mathbb{R}^d$ in Theorem 2.1.2. Let $\chi \in C^\infty(\mathbb{R}^d;[0,1])$ be a cutoff such that

$$\chi(x) = 1 \text{ near } \Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0), \quad \chi(x) = 0 \text{ in } \mathbb{R}^d \setminus (\Omega_{y_0,\omega,\theta} \cup B_1(y_0))$$

and $\chi(x) = \kappa_0((x-y_0)/|x-y_0|)$ for $x \in \Omega_{y_0,\omega,\theta} \setminus B_{10}(y_0)$ for some $\kappa_0 \in C^\infty(\mathbb{S}^{d-1})$. After cutting off we only need to solve the constraint equation

$$P(\chi h_0 + h, \chi\pi_0 + \pi) = \Phi(\chi h_0 + h, \chi\pi_0 + \pi) \tag{2.4.10}$$

with $(h,\pi)$ supported inside the region $(\Omega_{y_0,\omega,\theta} \cup B_1(y_0)) \setminus (\Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0))$. From now on, all the weighted norms are defined with respect to the center $y_0$. For example, let $\Omega_{\text{int}} := (\Omega_{y_0,\omega,\theta} \cup B_1(y_0)) \setminus \overline{(\Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0))}$, when $s \in \mathbb{N}$, we define

$$\|u\|^2_{H_{\text{b}}^{s,\delta}(\Omega_{\text{int}})} = \sum_{|\alpha|\le s} \|\langle x-y_0\rangle^{|\alpha|}\partial^\alpha u(x)\|^2_{L^2}.$$

By asymptotic flatness, we may choose $|y_0| \gg 1$ so that $\|(\chi h_0, \chi\pi_0)\|_{H_{\text{b}}^{s,\delta}\times H_{\text{b}}^{s-1,\delta+1}}$ is sufficiently small, and then we can apply the following solution operator to get a solution.

**Proposition 2.4.1.** *Let $\Omega_{\text{int}} = (\Omega_{y_0,\omega,\theta} \cup B_1(y_0)) \setminus \overline{(\Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0))}$, then there is a solution operator $S_{\text{int}}$ bounded on the following spaces*

$$S_{\text{int}} : H_{\text{b}}^{s-2,\delta+2}(\Omega_{\text{int}}) \to H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}})$$

*for $s \in \mathbb{R}$ and $\delta < d/2-2$, such that for any $f \in C_c^\infty(\Omega_{\text{int}})$, $\operatorname{supp} S_{\text{int}} f \subset \Omega_{\text{int}}$ and $PS_{\text{int}} f = f$. Moreover, $S_{\text{int}}$ has a decomposition*

$$S_{\text{int}} = S_{\text{int},diag} + S_{\text{int},out}$$

*where $S_{\text{int},diag}$ is diagonal operator and $S_{\text{int},out}$ is an outgoing operator in the sense of Definition 2.3.3. The mappings*

$$S_{\text{int},diag} : H_{\mathrm{b}}^{s-2,\delta+2}(\Omega_{\text{int}}) \to H_{\mathrm{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega_{\text{int}}), \quad s, \delta \in \mathbb{R}, \tag{2.4.11}$$

$$S_{\text{int},out} : H_{\mathrm{b}}^{-N,\delta+2}(\Omega_{\text{int}}) \to H_{\mathrm{b}}^{N,\delta}(\Omega_{\text{int}}) \times H_{\mathrm{b}}^{N,\delta+1}(\Omega_{\text{int}}), \quad N \in \mathbb{N},\ \delta < \frac{d}{2} - 2, \tag{2.4.12}$$

$$[\partial, S_{\text{int},diag}] : H_{\mathrm{b}}^{s-2,\delta+2}(\Omega_{\text{int}}) \to H_{\mathrm{b}}^{s,\delta+1}(\Omega_{\text{int}}) \times H_{\mathrm{b}}^{s-1,\delta+2}(\Omega_{\text{int}}), \quad s, \delta \in \mathbb{R} \tag{2.4.13}$$

*are bounded, and $S_{\text{int},out} = (S^{(1)}_{\text{int},out}, S^{(2)}_{\text{int},out})$ satisfies the following bound*

$$|\partial_x^{\ell} \partial_y^{\ell'} S^{(j)}_{\text{int},out}(x,y)| \lesssim \langle x \rangle^{k_j - d - \ell} \langle y \rangle^{-\ell'}, \quad k_1 = 2,\ k_2 = 1,\ \ell, \ell' \in \mathbb{N}. \tag{2.4.14}$$

*Proof.* Let $f \in H_{\mathrm{b}}^{s-2,\delta+2}(\Omega_{\text{int}})$ and we would like to solve $Pu = f$. Since $\Omega_{\text{int}}$ is not a conic region, we need to use the Bogovskii-type solution operator in [MOT23] (see also Chapter 3) to modify it. Let $\Omega_3 := \Omega_{\text{int}} \cap B_3(y_0)$ and $\Omega_2 := \Omega_{\text{int}} \cap B_2(y_0)$. By Lemma 3.2.1, 3.2.2 in Chapter 3, there exists a pseudo-differential operator $S_0$ with the following properties for $f \in C_c^\infty(\Omega_3)$:

- For $f \in C_c^\infty(\Omega_3)$, $\operatorname{supp} S_0 f \subset \Omega_3$.
- For $f \in C_c^\infty(\Omega_3)$, $PS_0(f) = f$ inside $\Omega_2$; moreover, $I - PS_0$ has smooth Schwartz kernel.
- $S_0 : H^{s-2}(\Omega_3) \to H^s(\Omega_3) \times H^{s-1}(\Omega_3)$ is bounded for any $s \in \mathbb{R}$.
- $[\partial, S_0] : H^{s-2}(\Omega_3) \to H^s(\Omega_3) \times H^{s-1}(\Omega_3)$ is bounded for any $s \in \mathbb{R}$.

We remark that 3.2.2 constructs such operators for a star-shaped region with respect to a ball. However, a similar procedure (see also 3.5.8) can give such Bogovskii-type operators on a finite union of star-shaped regions with respect to balls. We emphasize that we only require $PS_0(f) = f$ on $\Omega_2$ instead of the whole $\Omega_3$ since this cannot be achieved unless an integral condition 3.2.1 (S2)(T2) is satisfied.

Let $\chi_0 \in C_c^\infty(B_3(y_0))$ such that $\chi_0 = 1$ on $\Omega_2$. Then $S_0(\chi_0 f)$ solves $Pu = f$ in $\Omega_2$. Now the outside region is conic and we can use our conic solution operators from Theorem 2.2.2. Since $\Omega_{\text{int}} \setminus \Omega_2$ is not convex, we make a partition of unity in angular variables and use the conic solution operator on each piece.

Let $\mathbb{S}^{d-1}_{y_0} = \mathbb{S}^{d-1} + y_0$. Then we can decompose $(\Omega_{y_0,\omega,\theta} \setminus \Omega_{y_0,\omega,\theta_0}) \cap \mathbb{S}^{d-1}_{y_0} = \cup U_i$ into a finite union of open sets $U_i$, where each $U_i$ is star-shaped with respect to an open subset $V_i \subset U_i$. Let $\kappa_i \in C_c^\infty(V_i)$ be a cutoff function supported in $V_i$, then as in Theorem 2.2.2 we have solution operators $S_{\kappa_i}$ with respect to $U_i$ so that $\operatorname{supp} f \subset y_0 + \mathbb{R}_{>1}(U_i - y_0)$ implies $\operatorname{supp} S_{\kappa_i} f \subset y_0 + \mathbb{R}_{>1}(U_i - y_0)$ and $PS_{\kappa_i} f = f$. We can take $\mathbb{R}_{>1}$ because of the strong outgoing property of the solution operator. We take a partition of unity $\tilde{\chi}_j$ with respect to the covering $\{U_i\}$ and extend them conically. Define

$$S_1 = \sum_j S_{\kappa_j} \tilde{\chi}_j.$$

The final solution operator $S_{int}$ is defined to be

$$S_{\text{int}} f = S_1(f - PS_0(\chi_0 f)) + S_0(\chi_0 f) \in H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}}).$$

One can check it is bounded on $H_{\text{b}}^{s-2,\delta+2}(\Omega_{\text{int}}) \to H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}})$ and

$$PS_{\text{int}} f = PS_1(f - PS_0(\chi_0 f)) + PS_0(\chi_0 f) = f - PS_0(\chi_0 f) + PS_0(\chi_0 f) = f.$$

The estimates (2.4.11)(2.4.12)(2.4.13)(2.4.14) follow from Proposition 2.3.8 applied to $S_{\kappa_j}$ since the Schwartz kernel of $S_0 \chi_0$ is compactly supported in $\overline{\Omega_{\text{int}} \times \Omega_{\text{int}}}$. □

*Proof of Theorem 2.1.2.* Following the procedure of the previous section, we can get a gluing solution from the solution operator $S_{int}$ as follows. Consider the fixed point problem

$$(h, \pi) = G(h, \pi), \quad (h, \pi) \in H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}}) \tag{2.4.15}$$

where $G(h, \pi) = S_{int}(\Phi(\chi h_0 + h, \chi \pi_0 + \pi) - P(\chi h_0, \chi \pi_0))$. We choose $|y_0| \gg 1$ so that $\epsilon := \|\chi h_0, \chi \pi_0)\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}}$ is sufficiently small. Since $\Phi(\chi h_0, \chi \pi_0) - P(\chi h_0, \chi \pi_0)$ is supported in $\Omega_{\text{int}}$ and $S_{int}$ is bounded, we have

$$\|S_{\text{int}}(\Phi(\chi h_0, \chi \pi_0) - P(\chi h_0, \chi \pi_0))\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}} \le C_0 \epsilon$$

and

$$\|G(h, \pi)\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}} \le C_0 \epsilon + C \epsilon^2, \quad \text{for } \|(h, \pi)\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}} \le 2C_0 \epsilon.$$

So $G$ maps $\{\|(h, \pi)\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}} \le 2C_0 \epsilon\}$ to itself when $\epsilon$ is sufficiently small. Moreover,

$$\|G(h, \pi) - G(\tilde{h}, \tilde{\pi})\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}} \lesssim \epsilon \|(h, \pi) - (\tilde{h}, \tilde{\pi})\|_{H_{\text{b}}^{s,\delta} \times H_{\text{b}}^{s-1,\delta+1}}.$$

So for $\epsilon$ sufficiently small, $G$ is a contraction. By the Banach fixed point theorem we conclude that there is a unique solution $(h_1, \pi_1) \in H_{\text{b}}^{s,\delta}(\Omega_{int}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{int})$ solving the fixed point problem (2.4.15). The proof for the smoothness and decay rate is similar to the arguments in Theorem 2.1.1. □

## 2.5 Solving the problem in a degenerate sector

In this section we construct solutions of the Einstein constraint equations in the degenerate sector

$$\Omega = \Omega_\alpha := \{(x', x_d) \in \mathbb{R}^d \,:\, |x'| \le x_d^\alpha,\, x_d \ge 1\}.$$

## Anisotropic Sobolev space

In order to analyze the regularity of the solution, we introduce the *anisotropic weighted Sobolev space* defined as follows.

**Definition 2.5.1.** Let $\alpha \in (0, 1]$, $s \in \mathbb{R}$ and $\delta \in \mathbb{R}$, $H^{s,\delta}_\alpha(\Omega)$ is defined as the completion of $C^\infty_c(\Omega)$ with respect to the norm

$$\|u\|^2_{H^{s,\delta}_\alpha} := \sum_{j=0}^{\infty} 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}\right)} \|\phi_j^*(\chi_j u)\|^2_{H^s}$$

where

$$\phi_0 = \mathrm{id}, \quad \phi_j(x) = (2^{\alpha j} x', 2^j x_d), \quad j \geq 1$$

and $\sum_{j=0}^{\infty} \chi_j = 1$ be a dyadic partition of unity such that

$$\chi_0 \in C^\infty_c(\mathcal{B}_1; [0, 1]), \quad \chi \in C^\infty_c(\mathcal{D}_0; [0, 1]), \quad \chi_j = \phi_{j*}(\chi) := \chi \circ \phi_j^{-1}, \ j \geq 1.$$

It is easy to check that when $s \in \mathbb{N}$, we have an alternative description

$$\|u\|^2_{H^{s,\delta}_\alpha} \approx \sum_{|\beta'|+\beta_d \leq s} \|\langle x\rangle^{|\beta'|\alpha+\beta_d+\delta} \partial^{\beta'}_{x'} \partial^{\beta_d}_{x_d} u\|^2_{L^2}.$$

We provide a few properties of this norm.

**Proposition 2.5.2.** *Let $u, v \in C^\infty_c(\Omega)$, then*

*(a) For $s > d/2$ and $\delta \in \mathbb{R}$,*

$$\|\langle x\rangle^{\frac{(d-1)\alpha+1}{2}+\delta} u\|_{L^\infty} \lesssim \|u\|_{H^{s,\delta}_\alpha}. \tag{2.5.1}$$

*(b) For $s_1 + s_2 > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$, we have*

$$\|uv\|_{H^{s,\delta}_\alpha} \lesssim \|u\|_{H^{s_1,\delta_1}_\alpha} \|v\|_{H^{s_2,\delta_2}_\alpha}, \tag{2.5.2}$$

*where $\delta = \delta_1 + \delta_2 + \frac{(d-1)\alpha+1}{2}$ and*

$$s = \begin{cases} \min(s_1, s_2), & \max(s_1, s_2) > \frac{d}{2}, \\ s_1 + s_2 - \frac{d}{2}, & \max(s_1, s_2) < \frac{d}{2}. \end{cases}$$

*Proof.* The Sobolev embedding estimate (a) follows from the definition. The bilinear estimate (b) is proved similarly to Proposition 2.3.2(c). We omit the details since the proof is identical. □

There are also analogues of Lemma 2.3.5 and Lemma 2.3.6. The proof is identical but with different exponents. We omit the details here.

**Lemma 2.5.3.** *Suppose $K$ is a diagonal operator on $\Omega$ such that for some $s_1, s_2, k \in \mathbb{R}$, $K$ is bounded on $H^{s_1}_{comp} \to H^{s_2}_{\mathrm{loc}}$, and for any $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, there exists $C > 0$ such that for any $j \in \mathbb{N}$, the operator with the Schwartz kernel*

$$K(\phi_j(x), \phi_j(y))\chi(x)\chi(y) \tag{2.5.3}$$

*is bounded by $C2^{j(k-\alpha(d-1)-1)}$ on $H^{s_1} \to H^{s_2}$. Then*

$$K : H^{s_1,\delta}_\alpha(\Omega) \to H^{s_2,\delta-k}_\alpha(\Omega)$$

*is bounded for any $\delta \in \mathbb{R}$.*

**Lemma 2.5.4.** *Suppose $K$ is an outgoing (incoming, respectively) operator on $\Omega$ such that for some $s_1, s_2, k \in \mathbb{R}$, $K$ is bounded on $H^{s_1}_{comp} \to H^{s_2}_{\mathrm{loc}}$, and for any $\chi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and $\widetilde{\chi} \in C_c^\infty(\mathbb{R}^d)$, there exists $C > 0$ such that for any $j, j' \in \mathbb{N}$, the operator with Schwartz kernel*

$$K(\phi_j(x), \phi_{j'}(y))\chi(x)\widetilde{\chi}(y) \quad (K(\phi_j(x), \phi_{j'}(y))\chi(y)\widetilde{\chi}(x), \text{respectively})$$

*is bounded by $C2^{j(k-\alpha(d-1)-1)}$ ($C2^{j'(k-\alpha(d-1)-1)}$, respectively) on $H^{s_1} \to H^{s_2}$. Then*

$$K : H^{s_1,\delta}_\alpha(\Omega) \to H^{s_2,\delta-k}_\alpha(\Omega)$$

*is bounded for $\delta < \frac{(d-1)\alpha+1}{2}$ ($\delta > -\frac{(d-1)\alpha+1}{2} + k$, respectively).*

## Solution operators on a degenerate sector

Now we turn to the construction of solution operators for the linearized constraint equation (2.1.2). We start with the divergence equation $\partial_j u^j = f$ as before. We can find fundamental solutions of the divergence equation supported on a curve.

**Lemma 2.5.5.** *For any smooth curve $\gamma : [0, \infty) \to \mathbb{R}^d$ with $\gamma(0) = y$ and $\lim_{t\to\infty} \gamma(t) = \infty$, the distribution $\delta_\gamma$ defined as*

$$(\delta_\gamma, \psi) = \int_0^\infty \psi(\gamma(t))\gamma'(t)dt$$

*is a fundamental solution for the divergence equation*

$$\partial_j u^j = \delta_y.$$

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, then

$$(\partial_j \delta^j_\gamma, \varphi) = -\int_0^\infty \partial_j \varphi(\gamma(t))\gamma'^j(t)dt = -\int_0^\infty \partial_t \varphi(\gamma(t))dt = \varphi(\gamma(0)) = \varphi(y). \qquad \square$$

As before, we average a family of fundamental solutions to get a more regular solution supported in $\Omega$. Note the solution operators will no longer be convolution operators due to the geometry of the degenerate sector $\Omega$. In other words, the shape of the curves we are taking average on will depend on the initial point $y$ (previously they were just translations of straight lines).

**Lemma 2.5.6.** *We have a solution operator $\tilde{K}_0$ for the divergence equation on $\Omega = \Omega_\alpha$, i.e. $\partial_j \tilde{K}_0^j f = f$ such that for $\delta < \frac{\alpha(d-1)-1}{2}$,*

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} \tilde{K}_0 f \subset \Omega, \quad \textit{and } \tilde{K}_0 : H_\alpha^{s-1,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega) \textit{ is bounded.}$$

*Moreover, $\tilde{K}_0 = \tilde{K}_{0,\mathrm{diag}} + \tilde{K}_{0,\mathrm{out}}$ such that*

$$\tilde{K}_{0,\mathrm{diag}} : H_\alpha^{s-1,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R}, \tag{2.5.4}$$

$$[\partial, \tilde{K}_{0,\mathrm{diag}}] : H_\alpha^{s-1,\delta}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R} \tag{2.5.5}$$

$$\tilde{K}_{0,\mathrm{out}} : H_\alpha^{-N,\delta+1}(\Omega) \to H_\alpha^{N,\delta}(\Omega), \quad N \in \mathbb{N},\ \delta < \frac{\alpha(d-1)-1}{2} \tag{2.5.6}$$

*are all bounded, and the Schwartz kernel of $\tilde{K}_{0,\mathrm{out}}$ satisfies*

$$|\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} \partial_{y'}^{\tau'} \partial_{y_d}^{\tau_d} \tilde{K}_{0,\mathrm{out}}(x,y)| \lesssim \langle x \rangle^{-\alpha(d-1)-|\beta'|\alpha-\beta_d} \langle y \rangle^{-|\tau'|\alpha-\tau_d}, \quad \beta, \tau \in \mathbb{N}^d. \tag{2.5.7}$$

*Proof.* We construct the solution operator in two steps.

Let $\gamma_{y,\omega}^{(1)}(t) = y + (\omega y_d^\alpha, y_d)t$, $\kappa_1 \in C_c^\infty(\mathbb{R}^{d-1})$ with $\int_{\mathbb{R}^{d-1}} \kappa_1 = 1$. We define

$$K_1(x,y) = \int_{\mathbb{R}^{d-1}} \kappa_1(\omega) \delta_{\gamma_{y,\omega}^{(1)}}(x) d\omega = \kappa_1 \left( \frac{(x'-y')/y_d^\alpha}{(x_d - y_d)/y_d} \right) \frac{(x-y)}{y_d^{\alpha(d-1)+1} |\frac{x_d - y_d}{y_d}|^d}. \tag{2.5.8}$$

Then $\operatorname{div} K_1 = \delta(x-y)$. Let $\chi_2 \in C_c^\infty(\mathbb{R})$ be a cutoff with $\chi_2 = 1$ near 0 and

$$\tilde{K}_1(x,y) = \chi_2 \left( \frac{x_d - y_d}{y_d} \right) K_1(x,y).$$

Then $\operatorname{div} \tilde{K}_1 f = f + \frac{1}{y_d} \chi_2' \left( \frac{x_d - y_d}{y_d} \right) K_1$. Let $\gamma_{y,\omega}^{(2)}(t) = (y' + \omega((1+t)^\alpha - 1), y_d + t)$ and

$$\begin{aligned} K_2(x,y) &= \int_{\mathbb{R}^{d-1}} \kappa_1(\omega) \delta_{\gamma_{y,\omega}^{(2)}}(x) d\omega \\ &= \kappa_1 \left( \frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) ((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right). \end{aligned}$$

We define $\tilde{K}_0(x,y) = \tilde{K}_1(x,y) + \tilde{K}_2(x,y)$ where

$$\tilde{K}_2(x,y) = -\int K_2(x,z) \frac{1}{y_d} \chi_2' \left( \frac{z_d - y_d}{y_d} \right) K_1(z,y) dz.$$

Then

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} Kf \subset \Omega, \quad \operatorname{div} Kf = f.$$

It is straightforward to verify

- $\tilde{K}_1$ is diagonal, $\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}\tilde{K}_1(x,y) \lesssim y_d^{(1-\alpha)(d-1)}|x_d - y_d|^{1-d-|\beta'|\alpha-\beta_d}$;
- $\tilde{K}_2$ is outgoing and smooth, with the property

$$\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}\partial_{y'}^{\tau'}\partial_{y_d}^{\tau_d}\tilde{K}_2(x,y) \lesssim |x_d|^{-\alpha(d-1)-|\beta'|\alpha-\beta_d}|y_d|^{-|\tau'|\alpha-\tau_d}, \quad \beta,\tau \in \mathbb{N}^d. \tag{2.5.9}$$

We then define $\tilde{K}_{0,diag}$ to be the operator with Schwartz kernel $\tilde{K}_1$ and $\tilde{K}_{0,out}$ to be the operator with Schwartz kernel $\tilde{K}_2$. It suffices to show their boundedness.

(a) First we show (2.5.4), i.e. $\tilde{K}_1 : H_\alpha^{s-1,\delta+1} \to H_\alpha^{s,\delta}$ is bounded for any $s,\delta \in \mathbb{R}$. Let $\tilde{K}_1 = (\tilde{K}_1', \tilde{K}_1^d)$, then

$$\tilde{K}_1(2^{j\alpha}x', 2^j x_d, 2^{j\alpha}y', 2^j y_d) = 2^{-(d-1)\alpha-1}(2^{j\alpha}\tilde{K}_1', 2^j\tilde{K}_1^d).$$

By Lemma 2.5.3, it suffices to show $\tilde{K}_1$ is bounded on $H_{comp}^{s-1} \to H_{\rm loc}^s$. Indeed, $\tilde{K}_1(x,y)$ is a pseudodifferential operator of order $-1$. Let

$$b(y_d,z) = \kappa_1\left(\frac{z'/y_d^\alpha}{z_d/y_d}\right)\frac{(z'/y_d^\alpha, z_d/y_d)}{y_d^{\alpha(d-1)+1}|\frac{z_d}{y_d}|^d}, \quad a(\xi) = \mathcal{F}(b(1,\cdot)).$$

Then $a$ is homogeneous of degree $-1$ and

$$(y_d^{-\alpha}\tilde{K}_1'(x,y), y_d^{-1}\tilde{K}_1^d(x,y)) = \frac{1}{(2\pi)^d}\chi_2\left(\frac{x_d-y_d}{y_d}\right)\int e^{i(x-y)\cdot\xi}a(y_d^\alpha\xi', y_d\xi_d)d\xi.$$

Let $\chi$ be a cutoff near $\xi = 0$, then $(1-\chi(\xi))a(y_d^\alpha\xi', y_d\xi_d)$ is a symbol in the sense that

$$|\partial_y^\beta\partial_\xi^\gamma((1-\chi(\xi))a(y_d^\alpha\xi', y_d\xi_d))| \lesssim_{\beta,\gamma} \langle\xi\rangle^{-1-\gamma}.$$

Thus the pseudodifferential operator maps from $H_{comp}^{s-1}$ to $H_{\rm loc}^s$. Moreover, the rest part $\int e^{i(x-y)\cdot\xi}\chi(\xi)a(y_d^\alpha\xi', y_d\xi_d)d\xi \in C^\infty$ has smooth Schwartz kernel, thus gives a smoothing operator.

Now the bound (2.5.5) is similar, since from the expression (2.5.8) one sees that the commutator only comes from differentiating in the $y_d$ terms, which will give a similar expression with improvement of decay by $y_d^{-1}$.

(b) For the outgoing part $\tilde{K}_2$, the bounds (2.5.6)(2.5.7) follow from (2.5.9). Lemma 2.5.4 implies that $\tilde{K}_2 : H_\alpha^{-N,\delta+1} \to H_\alpha^{N,\delta}$ is bounded for any $N \in \mathbb{N}$ and $\delta < \frac{(d-1)\alpha+1}{2} - 1 = \frac{(d-1)\alpha-1}{2}$. □

The construction of solution operators for the double divergence equation and the symmetric divergence equation is similar.

**Proposition 2.5.7.** *There is a solution operator $\tilde{K}$ for the double divergence equation, i.e. $\partial_i\partial_j\tilde{K}^{ij}f = f$ and $\tilde{K}^{ij} = \tilde{K}^{ji}$ such that for $\delta < \frac{\alpha(d-1)-3}{2}$,*

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp}\tilde{K}f \subset \Omega, \quad \textit{and } \tilde{K} : H_\alpha^{s-2,\delta+2}(\Omega) \to H_\alpha^{s,\delta}(\Omega) \textit{ is bounded.}$$

*Moreover, $\tilde{K} = \tilde{K}_{\mathrm{diag}} + \tilde{K}_{\mathrm{out}}$ such that*

$$\tilde{K}_{\mathrm{diag}} : H_\alpha^{s-2,\delta+2}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R}, \tag{2.5.10}$$

$$\tilde{K}_{\mathrm{out}} : H_\alpha^{-N,\delta+2}(\Omega) \to H_\alpha^{N,\delta}(\Omega), \quad N \in \mathbb{N},\ \delta < \frac{\alpha(d-1)-3}{2}, \tag{2.5.11}$$

$$[\partial, \tilde{K}_{\mathrm{diag}}] : H_\alpha^{s-2,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R} \tag{2.5.12}$$

*are all bounded. Moreover, the Schwartz kernel $\tilde{K}_{\mathrm{out}}(x, y)$ of $\tilde{K}_{\mathrm{out}}$ satisfies*

$$|\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}\partial_{y'}^{\tau'}\partial_{y_d}^{\tau_d}\tilde{K}_{\mathrm{out}}(x, y)| \lesssim \langle x\rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}\langle y\rangle^{-|\tau'|\alpha-\tau_d}, \quad \beta, \tau \in \mathbb{N}^d. \tag{2.5.13}$$

*Proof.* Given a smooth curve $\gamma(t) : [0, \infty) \to \mathbb{R}^d$ such that $\gamma(0) = y$ and $\lim_{t\to\infty}\gamma(t) = \infty$. We want to find $K_\gamma \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that for a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\varphi(y) = \langle \partial_i\partial_j K_\gamma^{ij}(\cdot, y), \varphi(\cdot)\rangle = \langle K_\gamma^{ij}(\cdot, y), \partial_i\partial_j\varphi(\cdot)\rangle.$$

In order to recover $\varphi(y)$ for $\partial_i\partial_j\varphi$, it suffices to integrate twice:

$$\begin{aligned}
\varphi(y) &= \int_0^\infty \gamma'(t)^i \int_t^\infty \gamma'(s)^j\partial_i\partial_j\varphi(\gamma(s))dsdt \\
&= \int_0^\infty \left(\int_0^s \gamma'(t)^i dt\right)\gamma'(s)^j\partial_i\partial_j\varphi(\gamma(s))ds \\
&= \int_0^\infty (\gamma(s) - \gamma(0))^i\gamma'(s)^j\partial_i\partial_j\varphi(\gamma(s))ds.
\end{aligned}$$

So the fundamental solution $K_\gamma^{ij}$ supported on the curve $\gamma$ is given by

$$\langle K_\gamma^{ij}, \psi_{ij}\rangle = \int_0^\infty (\gamma(s) - \gamma(0))^i\gamma'(s)^j\psi_{ij}(\gamma(s))ds.$$

We then average along curves as in Lemma 2.5.6. For $\gamma_{y,\omega}^{(1)}(t) = y + (\omega y_d^\alpha, y_d)t$, $\omega \in \mathbb{R}^{d-1}$, $\kappa_1 \in C_c^\infty(\mathbb{R}^{d-1})$, we average among the solution operator $K_{\gamma_{y,\omega}^{(1)}}(x, y)$ we get from the curve $\gamma_{y,\omega}^{(1)}$:

$$\begin{aligned}
K_1(x, y) :&= \int_{\mathbb{R}^{d-1}} \kappa_1(\omega)K_{\gamma_{y,\omega}^{(1)}}(x, y)d\omega \\
&= \kappa_1\left(\frac{(x'-y')/y_d^\alpha}{(x_d - y_d)/y_d}\right)\frac{(x-y)^i(x-y)^j}{y_d^{1+\alpha(d-1)}|\frac{x_d-y_d}{y_d}|^d}.
\end{aligned}$$

For $\gamma^{(2)}_{y,\omega}(t) = (y' + \omega((1+t)^\alpha - 1), y_d + t)$, we average among $K_{\gamma^{(2)}_{y,\omega}}(x,y)$ we get form the curve $\gamma^{(2)}_{y,\omega}$:

$$K_2(x,y) := \int_{\mathbb{R}^{d-1}} \kappa_1(\omega) K^{(2)}_{y,\omega}(x,y)d\omega$$
$$= \frac{1}{2}\kappa_1\left(\frac{x'-y'}{(1+x_d-y_d)^\alpha - 1}\right)((1+x_d-y_d)^\alpha - 1)^{-(d-1)}$$
$$\left((x-y)^i\left(\alpha(x'-y')\frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha-1}, 1\right)^j + (x-y)^j\left(\alpha(x'-y')\frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha-1}, 1\right)^i\right).$$

We then define $\tilde{K}$ from $K_1$ and $K_2$ as in Lemma 2.5.6. The bounds (2.5.10)-(2.5.13) follow from a similar proof. $\square$

**Proposition 2.5.8.** *There exists a solution operator $\tilde{L}$ for the symmetric divergence equation, i.e. $\partial_i \tilde{L}^{ij}_k f_j = f_k$ and $\tilde{L}^{ij}_k = \tilde{L}^{ji}_k$ such that for $\delta < \frac{\alpha(d+1)-3}{2}$,*

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} \tilde{L}f \subset \Omega, \quad \textit{and } \tilde{L} : H^{s-1,\delta+2-\alpha}_\alpha(\Omega) \to H^{s,\delta}_\alpha(\Omega) \textit{ is bounded.}$$

*Moreover, $\tilde{L} = \tilde{L}_{\text{diag}} + \tilde{L}_{\text{out}}$ such that*

$$\tilde{L}_{\text{diag}} : H^{s-1,\delta+2-\alpha}_\alpha(\Omega) \to H^{s,\delta}_\alpha(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R}, \tag{2.5.14}$$

$$\tilde{L}_{\text{out}} : H^{-N,\delta+2-\alpha}_\alpha(\Omega) \to H^{N,\delta}_\alpha(\Omega), \quad N \in \mathbb{N},\ \delta < \frac{\alpha(d+1)-3}{2}, \tag{2.5.15}$$

$$[\partial, \tilde{L}_{\text{diag}}] : H^{s-1,\delta+1-\alpha}_\alpha(\Omega) \to H^{s,\delta}_\alpha(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R} \tag{2.5.16}$$

*are all bounded. Moreover, the Schwartz kernel $\tilde{L}_{\text{out}}(x,y)$ of $\tilde{L}_{\text{out}}$ satisfies*

$$|\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d}\partial^{\tau'}_{y'}\partial^{\tau_d}_{y_d}\tilde{L}_{\text{out}}(x,y)| \lesssim \langle x\rangle^{1-\alpha d-|\beta'|\alpha-\beta_d}\langle y\rangle^{-|\tau'|\alpha-\tau_d}, \quad \beta,\tau \in \mathbb{N}^d. \tag{2.5.17}$$

*Proof.* Fix a smooth curve $\gamma(t) : [0,\infty) \to \mathbb{R}^d$ such that $\gamma(0) = y$ and $\lim_{t\to\infty}\gamma(t) = \infty$. We want to find $L_\gamma \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\varphi_k(y) = \langle \partial_i L^{ij}_{\gamma,k}(\cdot,y), \varphi_j(\cdot)\rangle = -\frac{1}{2}\langle L^{ij}_{\gamma,k}(\cdot,y), \partial_i\varphi_j + \partial_j\varphi_i\rangle, \quad \varphi_j \in C^\infty_c(\mathbb{R}^d).$$

In order to recover $\varphi_k(y)$ from $\zeta_{ij} = -\frac{1}{2}(\partial_i\varphi_j + \partial_j\varphi_i)$, we let $\eta_{ij} = \frac{1}{2}(\partial_i\varphi_j - \partial_j\varphi_i)$ so that

$$\partial_i\varphi_j = \eta_{ij} - \zeta_{ij}. \tag{2.5.18}$$

Since

$$\partial_j\zeta_{ik} = -\frac{1}{2}\partial^2_{ij}\varphi_k - \frac{1}{2}\partial^2_{jk}\varphi_i, \quad \partial_k\zeta_{ij} = -\frac{1}{2}\partial^2_{ik}\varphi_j - \frac{1}{2}\partial^2_{jk}\varphi_i,$$

we have

$$\partial_i\eta_{jk} = \frac{1}{2}(\partial^2_{ij}\varphi_k - \partial^2_{ik}\varphi_j) = \partial_k\zeta_{ij} - \partial_j\zeta_{ik}. \tag{2.5.19}$$

Now we can integrate (2.5.19) along $\gamma$ and get

$$\eta_{jk}(\gamma(t)) = -\int_t^\infty (\gamma'(s)^i\partial_k\zeta_{ij}(\gamma(s)) - \gamma'(s)^i\partial_j\zeta_{ik}(\gamma(s)))ds.$$

Then we can integrate (2.5.18),

$$\begin{aligned}\varphi_j(y) &= -\int_0^\infty (\gamma'(t)^i\eta_{ij}(\gamma(t)) - \gamma'(t)^i\zeta_{ij}(\gamma(t)))dt \\ &= \int_0^\infty (\gamma'(s)^l\partial_j\zeta_{li}(\gamma(s)) - \gamma'(s)^l\partial_i\zeta_{lj}(\gamma(s)))\left(\int_0^s \gamma'(t)^i dt\right)ds + \int_0^\infty \gamma'(t)^i\zeta_{ij}(\gamma(t))dt \\ &= \int_0^\infty (\gamma'(t)^l\partial_j\zeta_{li}(\gamma(t)) - \gamma'(t)^l\partial_i\zeta_{lj}(\gamma(t)))(\gamma(t)^i - \gamma(0)^i)dt + \int_0^\infty \gamma'(t)^i\zeta_{ij}(\gamma(t))dt.\end{aligned}$$

So the fundamental solution $L^{ij}_{\gamma,k}$ supported on the curve $\gamma$ is given by

$$\begin{aligned}&\langle L^{ij}_{\gamma,k}(\cdot,y),\zeta_{ij}(\cdot)\rangle \\ =&\int_0^\infty (\gamma'(t)^j\partial_k\zeta_{ij}(\gamma(t)) - \gamma'(t)^j\partial_i\zeta_{jk}(\gamma(t)))(\gamma(t)^i - \gamma(0)^i)dt + \int_0^\infty \gamma'(t)^i\zeta_{ik}(\gamma(t))dt.\end{aligned}$$

We then average along curves as in Lemma 2.5.6. For $\gamma^{(1)}_{y,\omega}(t) = y + (\omega y_d^\alpha, y_d)t$, $\omega \in \mathbb{R}^{d-1}$, $\kappa_1 \in C^\infty_c(\mathbb{R}^{d-1})$, we average among the solution operators $L_{\gamma^{(1)}_{y,\omega}}(x,y)$ we get for the curve $\gamma^{(1)}_{y,\omega}$:

$$\begin{aligned}L_1(x,y) :=& \int_{\mathbb{R}^{d-1}} \kappa_1(\omega)L_{\gamma^{(1)}_{y,\omega}}(x,y)d\omega = -\partial_k\left(\kappa_1\left(\frac{(x'-y')/y_d^\alpha}{(x_d-y_d)/y_d}\right)\frac{(x^j-y^j)(x^i-y^i)}{y_d^{\alpha(d-1)+1}|x_d-y_d|^d/y_d^d}\right) \\ &+ \frac{1}{2}\partial_l\left(\kappa_1\left(\frac{(x'-y')/y_d^\alpha}{(x_d-y_d)/y_d}\right)\frac{(x_l-y_l)((x^j-y^j)\delta_{ik} + (x^i-y^i)\delta_{jk})}{y_d^{\alpha(d-1)+1}|x_d-y_d|^d/y_d^d}\right) \\ &+ \frac{1}{2}\kappa_1\left(\frac{(x'-y')/y_d^\alpha}{(x_d-y_d)/y_d}\right)\frac{(x^i-y^i)\delta_{jk} + (x^j-y^j)\delta_{ik}}{y_d^{\alpha(d-1)+1}|x_d-y_d|^d/y_d^d}.\end{aligned} \tag{2.5.20}$$

For $\gamma^{(2)}_{y,\omega}(t) = (y' + \omega((1+t)^\alpha - 1), y_d + t)$, we average among $L_{\gamma^{(2)}_{y,\omega}}(x,y)$ from the curve $\gamma^{(2)}_{y,\omega}$:

$$
\begin{aligned}
L_2(x,y) :=& \int_{\mathbb{R}^{d-1}} \kappa_1(\omega) L_{\gamma^{(2)}_{y,\omega}}(x,y) d\omega \\
=& -\frac{1}{2}\partial_k \left( \kappa_1 \left( \frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) ((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \right. \\
& \left. \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^j (x^i - y^i) + \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^i (x^j - y^j) \right) \\
& + \frac{1}{2}\partial_l \left( \kappa_1 \left( \frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) (x_l - y_l)((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \right. \\
& \left. \left( \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^j \delta_{ik} + \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^i \delta_{jk} \right) \right) \\
& + \frac{1}{2}\kappa_1 \left( \frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) ((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \\
& \left( \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^i \delta_{jk} + \left( \alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^j \delta_{ik} \right).
\end{aligned}
$$

We then define the solution operator $\tilde{L}$ from $L_1$ and $L_2$ as in Lemma 2.5.6 and (2.5.14)-(2.5.17) follow from a similar proof. □

*Remark* 2.5.9. In Proposition 2.5.8, the mapping property $\tilde{L} : H^{s-1,\delta+2-\alpha}_\alpha(\Omega) \to H^{s,\delta}_\alpha(\Omega)$ looks different from Proposition 2.5.7. This can be seen easily from the expressions. For example, for the first term in $L_1$:

$$
L^{ij}_{1,k}(x,y) = \partial_{x^k} \left( \kappa_1 \left( \frac{(x'-y')/y_d^\alpha}{(x_d-y_d)/y_d} \right) \frac{(x^j-y^j)(x^i-y^i)}{y_d^{\alpha(d-1)+1} |x_d-y_d|^d / y_d^d} \right),
$$

where we compute the rescaling as in Lemma 2.5.3 for $i = j = d$ and $1 \le k \le d-1$, we get

$$
L^{dd}_{1,k}(\phi_j(x), \phi_j(y)) = 2^{-j((d-1)\alpha+1)} 2^{(2-\alpha)j} L^{dd}_{1,k}(x,y).
$$

So the $\delta$ index is shifted by $2-\alpha$.

From Proposition 2.5.7 and Proposition 2.5.8, $\tilde{S} := (\tilde{K}, \tilde{L})$ is a right inverse of $P(h,\pi) = (\partial_i\partial_j h^{ij}, \partial_i \pi^{ij})$. Moreover, $\tilde{S}$ has the decomposition into the diagonal part and outgoing part

$$
\tilde{S} = \tilde{S}_{\text{diag}} + \tilde{S}_{\text{out}}, \qquad \text{where } \tilde{S}_{\text{diag}} = (\tilde{K}_{\text{diag}}, \tilde{L}_{\text{diag}}) \text{ and } \tilde{S}_{\text{out}} = (\tilde{K}_{\text{out}}, \tilde{L}_{\text{out}}).
$$

We can now give the proof of Theorem 2.1.3 following the same strategy as the proof of Theorem 2.1.1.

*Proof of Theorem 2.1.3.* Let $d \geq 3$ and $\frac{3}{d+1} < \alpha < 1$ be as in Theorem 2.1.3. Take $s \geq d+2$ and $\frac{3-(d+3)\alpha}{2} < \delta < \frac{\alpha(d-1)-3}{2}$. We start with solutions $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ of the linearized equation $P(h_0, \pi_0) = 0$ supported in $\Omega$ with decay (2.1.7)(2.1.8) (again such solutions exist even with compact support, as discussed in §2.4). We want to solve the fixed point problem

$$(h, \pi) = \tilde{S}\Phi(h_0 + h, \pi_0 + \pi). \tag{2.5.21}$$

on the space $H^{s,\delta}_\alpha(\Omega) \times H^{s-1,\delta+\alpha}_\alpha(\Omega)$. The condition $\delta > \frac{3-(d+3)\alpha}{2}$ ensures the multiplication is bounded on $H^{s,\delta}_\alpha \times H^{s-2,\delta+2\alpha}_\alpha \to H^{s-2,\delta+2}_\alpha$, $H^{s-1,\delta+\alpha}_\alpha \times H^{s-1,\delta+\alpha}_\alpha \to H^{s-2,\delta+2}_\alpha$ (see Proposition 2.5.2(b)) and the following bilinear estimate holds (note $\partial : H^{s,\delta}_\alpha \to H^{s-1,\delta+\alpha}_\alpha$ in general).

$$\|\Phi(h_0 + h, \pi_0 + \pi)\|_{H^{s-2,\delta+2}_\alpha} \lesssim_{\|h_0+h\|_{H^{s,\delta}_\alpha}} \|(h_0 + h, \pi_0 + \pi)\|^2_{H^{s,\delta}_\alpha \times H^{s-1,\delta+\alpha}_\alpha}.$$

The condition $\delta < \frac{\alpha(d-1)-3}{2}$ ensures the solution operators

$$\tilde{K} : H^{s-2,\delta+2}_\alpha \to H^{s,\delta}_\alpha, \quad \tilde{L} : H^{s-2,\delta+2}_\alpha \to H^{s-1,\delta+\alpha}_\alpha$$

are bounded (note this index is shifted from Proposition 2.5.8).

By the Banach fixed point theorem, for $\|(h_0, \pi_0)\|_{H^{s,\delta}_\alpha \times H^{s-1,\delta+\alpha}_\alpha}$ sufficiently small, we get a unique solution to (2.5.21):

$$(h_1, \pi_1) \in H^{s,\delta}_\alpha(\Omega) \times H^{s-1,\delta+\alpha}_\alpha(\Omega), \quad \|(h_1, \pi_1)\|_{H^{s,\delta}_\alpha \times H^{s-1,\delta+\alpha}_\alpha} \lesssim \|(h_0, \pi_0)\|^2_{H^{s,\delta}_\alpha \times H^{s-1,\delta+\alpha}_\alpha}. \tag{2.5.22}$$

Applying $P$ to (2.5.21), we conclude $(h_0 + h_1, \pi_0 + \pi_1)$ is a solution to the constraint equation (2.4.2). The estimate (2.5.22) ensures that the solution is nontrivial.

Now we claim indeed

$$(h_1, \pi_1) \in H^{\infty,\delta}_\alpha \times H^{\infty,\delta+\alpha}_\alpha \tag{2.5.23}$$

where $H^{\infty,\delta}_\alpha := \cap_s H^{s,\delta}_\alpha$. In order to show the smoothness, we inductively prove the following statement: if $(h_1, \pi_1) \in H^{s,\delta}_\alpha \times H^{s-1,\delta+\alpha}_\alpha$ for some integer $s \geq d+2$, then $(h_1, \pi_1) \in H^{s+1,\delta}_\alpha \times H^{s,\delta+\alpha}_\alpha$.

We write (2.5.21) as

$$(h_1, \pi_1) = \tilde{S}_{\mathrm{d}iag}\Phi(h_0 + h_1, \pi_0 + \pi_1) + \tilde{S}_{out}\Phi(h_0 + h_1, \pi_0 + \pi_1). \tag{2.5.24}$$

Differentiate it we get

$$\begin{aligned}\partial(h_1, \pi_1) = {} & \tilde{S}_{\mathrm{d}iag}\partial\Phi(h_0 + h_1, \pi_0 + \pi_1) \\ & + [\partial, \tilde{S}_{\mathrm{d}iag}]\Phi(h_0 + h_1, \pi_0 + \pi_1) + \partial\tilde{S}_{out}\Phi(h_0 + h_1, \pi_0 + \pi_1).\end{aligned}$$

By (2.5.11) and (2.5.15), we have

$$\tilde{S}_{out}\Phi(h_0 + h_1, \pi_0 + \pi_1) \in H^{\infty,\delta}_\alpha \times H^{\infty,\delta+\alpha}_\alpha.$$

By (2.5.12) and (2.5.16), we have

$$[\partial, \tilde{S}_{\mathrm{diag}}]\Phi(h_0+h_1,\pi_0+\pi_1) \in H^{s,\delta+1}_\alpha \times H^{s-1,\delta+1+\alpha}_\alpha.$$

So we conclude

$$\partial^\beta(h_1,\pi_1) = \tilde{S}_{\mathrm{diag}}\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1) + e_{1,\beta}, \quad |\beta| = s$$

with $e_{1,\beta} \in H^{1,\delta+|\beta'|\alpha+\beta_d}_\alpha \times H^{0,\delta+|\beta'|\alpha+\beta_d+\alpha}_\alpha$. We note as before, for $|\beta| = s$,

$$\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1) = (M_{h_0,h_1}(h_1+h_0,\partial^{\beta+2}h_1), N_{h_0,h_1}(h_1+h_0,\partial^{\beta+1}\pi_1)) + e_{2,\beta}$$

where $e_{2,\beta} \in H^{-1,\delta+|\beta'|\alpha+\beta_d+2}_\alpha$ and $M, N$ are bilinear forms depending smoothly on $h_0, h_1$. The bilinear estimate (2.5.2) shows

$$\begin{aligned}
&\|\tilde{S}_{\mathrm{diag}}\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1)\|_{H^{1,\delta+|\beta'|\alpha+\beta_d}_\alpha \times H^{0,\delta+|\beta'|\alpha+\beta_d+\alpha}_\alpha} \\
&\lesssim \|\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1)\|_{H^{-1,\delta+|\beta'|\alpha+\beta_d+2}_\alpha} \\
&\lesssim \|h_0+h_1\|_{H^{d/2+1,\delta}_\alpha}(\|h_1\|_{H^{s+1,\delta}_\alpha} + \|\pi_1\|_{H^{s,\delta+\alpha}_\alpha}) + \|e_{2,\beta}\|_{H^{-1,\delta+|\beta'|\alpha+\beta_d+2}_\alpha}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|(h_1,\pi_1)\|_{H^{s+1,\delta}_\alpha \times H^{s,\delta+\alpha}_\alpha} \lesssim{}& \|h_0+h_1\|_{H^{d/2+1,\delta}_\alpha}(\|h_1\|_{H^{s+1,\delta}_\alpha} + \|\pi_1\|_{H^{s,\delta+\alpha}_\alpha}) + \|(h_1,\pi_1)\|_{H^{s,\delta}_\alpha \times H^{s-1,\delta+\alpha}_\alpha} \\
&+ \sum_{|\beta|=s} \|e_{1,\beta}\|_{H^{1,\delta+|\beta'|\alpha+\beta_d}_\alpha \times H^{0,\delta+|\beta'|\alpha+\beta_d+\alpha}_\alpha} + \sum_{|\beta|=s} \|e_{2,\beta}\|_{H^{-1,\delta+|\beta'|\alpha+\beta_d+2}_\alpha}.
\end{aligned}$$

For $\|h_0+h_1\|_{H^{d/2+1,\delta}_\alpha}$ sufficiently small, we conclude $(h_1,\pi_1) \in H^{s+1,\delta}_\alpha \times H^{s,\delta+\alpha}_\alpha$. This finishes the induction and proves (2.5.23).

Finally we check the decay rate (2.1.5)(2.1.6). We estimate the two terms on the right hand side of (2.5.24) separately. For $(h,\pi) \in H^{\infty,\delta}_\alpha \times H^{\infty,\delta+\alpha}_\alpha$, the bilinear estimate (2.5.2) implies

$$\Phi(h_0+h_1,\pi_0+\pi_1) \in H^{\infty,2\delta+2\alpha+\frac{(d-1)\alpha+1}{2}}_\alpha.$$

Since $\alpha > \frac{3}{d+1}$ and $\delta < \frac{(d-1)\alpha-3}{2}$ can be taken to be arbitrarily close to $\frac{(d-1)\alpha-3}{2}$, we have $\Phi(h_0+h_1,\pi_0+\pi_1) \in H^{\infty,\frac{(d-1)\alpha+1}{2}+\epsilon}_\alpha$ for some $\epsilon > 0$ and is thus integrable. Thus the tail estimates (2.5.13)(2.5.17) implies $\tilde{S}_{out}\Phi(h_0+h_1,\pi_0+\pi_1)$ has the decay rate (2.1.5)(2.1.6). For the diagonal part $\tilde{S}_{\mathrm{diag}}\Phi(h_0+h_1,\pi_0+\pi_1)$ we recall $\Phi(h_0+h_1,\pi_0+\pi_1) \in H^{\infty,\frac{(d-1)\alpha+1}{2}+\epsilon}_\alpha$ and thus

$$\tilde{S}_{\mathrm{diag}}\Phi(h_0+h_1,\pi_0+\pi_1) \in H^{\infty,\frac{(d-1)\alpha-3}{2}+\epsilon}_\alpha \times H^{\infty,\frac{(d+1)\alpha-3}{2}+\epsilon}_\alpha.$$

The Sobolev embedding (2.5.1) then implies the diagonal part has even stronger decay than (2.1.5)(2.1.6). $\square$

# Chapter 3

# Gluing of Asymptotically Flat Initial Data

## 3.1 Introduction

Let $(\mathcal{M}^3, g)$ be a Riemannian manifold and $k$ be a symmetric (contravariant) 2-tensor on $\mathcal{M}^3$. The *Einstein vacuum constraint equation* reads

$$\begin{cases} R[g] + (\operatorname{tr}_g k)^2 - |k|_g^2 = 0, \\ \qquad \operatorname{div}_g k - \mathrm{d}\operatorname{tr}_g k = 0, \end{cases} \tag{3.1.1}$$

where $R[g]$ is the scalar curvature of $g$, $\operatorname{div}_g k_j = D_i k_{i'j}(g^{-1})^{ii'}$ with $D_i$ the Levi-Civita connection associated with $g$, $\operatorname{tr}_g k = (g^{-1})^{ij} k_{ij}$ and $|k|_g^2 = k_{ij} k_{i'j'} (g^{-1})^{ii'} (g^{-1})^{jj'}$. By the fundamental theorem of Choquet-Bruhat [Fou52], to any (sufficiently regular) triple $(\mathcal{M}^3, g, k)$ solving (3.1.1) corresponds a (geometrically) unique spacetime $(\mathcal{M}^{1+3}, \mathbf{g})$ solving the Einstein vacuum equation

$$\operatorname{Ric}[\mathbf{g}] - \frac{1}{2} \mathbf{g} \operatorname{tr}_{\mathbf{g}} \operatorname{Ric}[\mathbf{g}] = 0, \tag{3.1.2}$$

into which $(\mathcal{M}^3, g)$ isometrically embeds with $k$ as the second fundamental form. In this sense, a pair $(g, k)$ solving (3.1.1) constitutes an initial data set for the Einstein vacuum equation.

While underdetermined, the nonlinear nature of (3.1.1) imposes nontrivial constraints on the class of initial data sets for the Einstein vacuum equation, as exemplified by the celebrated positive mass theorem [SY79; Wit81]. Understanding the flexibility of initial data solving (3.1.1) is often an indispensable part of the study of solutions to the Einstein vacuum equation.

The subject of this chapter is *initial data gluing*, which asks: given two initial data sets, find – if possible – another initial data set which contains the two. We focus on initial data sets $(\mathcal{M}^3, g, k)$ such that either (1) $(\mathcal{M}^3, g)$ is bounded and $(g, k)$ is almost flat (i.e., close to the flat data $(\delta, 0)$) or (2) $\mathcal{M}^3$ is unbounded but $(g, k)$ is asymptotically flat (i.e., $\mathcal{M}^3$

is diffeomorphic to the exterior of a ball in $\mathbb{R}^3$ and $(g,k) \to (\delta, 0)$ as $|x| \to \infty$). These two cases are closely related as the latter is typically reduced to the former via rescaling. Since the pioneering work of Corvino [Cor00], many results have appeared on initial data gluing in this setting, such as (1) gluing a general asymptotically flat initial data set to the exterior of initial data for one of the Kerr spacetimes [CCI11; CS06; CD03], and (2) gluing an interior region to a general asymptotically flat initial data defined in the exterior of a ball to produce a globally defined initial data [Chr17; BC17], and so on. See also the recent works [Hin22; Hin23c; Hin23a] adopting a geometric microlocal approach. For further discussion and additional references, we refer the reader to the excellent review article [Car21].

The first contribution of this chapter is a new short proof of a basic gluing result on an almost flat annulus at unit scale (Theorem 3.1.3), which leads to the above gluing results (1) and (2) for asymptotically flat initial data after suitable rescaling arguments; see also Theorem 3.1.6 and Remark 3.1.11. The main simplification comes from the use of explicit Bogovskii-type solution operators to solve the linearization of (3.1.1) around the flat case $(g_{ij}, k_{ij}) = (\delta_{ij}, 0)$ on an annular domain with zero boundary condition. As a byproduct of its simple and explicit nature, our proof readily handles the optimal (modulo the endpoint) Sobolev regularity.

Another contribution of this chapter is a proof of a new optimal version of *obstruction-free gluing* à la Czimek–Rodnianski [CR22]; see Theorems 3.1.7 and 3.1.10 for the precise formulation. This novel type of gluing theorem, which relies on remarkable properties of the nonlinearity of (3.1.1), was originally proved in [CR22] using the theory of *null* (or *characteristic*) *data gluing*, pioneered by Aretakis [Are15; Are17] and Aretakis–Czimek–Rodnianski [ACR25; ACR23b; ACR23a] (see also [CC22; KU22; KU23]). Our direct proof, independent of null gluing, optimizes the positivity, regularity, size and spatial decay requirements in obstruction-free gluing for asymptotically flat initial data; see Remarks 3.1.8 and 3.1.11. It is based on the first main theorem (Theorem 3.1.3), as well as the construction of multi-bump initial data sets with prescribed charges and support properties that rely on several ingredients: (1) conic-type solution operators for the linearized constraint equations around the flat case as in [MT22] (see also Chapter 2), which can be used to construct initial data sets localized in conic regions (cf. Carlotto–Schoen [CS16a] as well as [ACR25; ACR23b; ACR23a]); (2) a *nonlinear* computation similar to that of Bartnik in [Bar86] concerning the positivity of mass in the time-symmetric, almost flat case; and (3) a localized boost argument based on the fundamental theorem of Choquet-Bruhat [Fou52] and the computation of Chruściel [Chr85] (see also [CD03]). A more detailed sketch of the proof is in Section 3.5 below, which can be read after Sections 3.1 and 3.2.

*Remark* 3.1.1 (Solution operators to divergence-type equations with prescribed support). As indicated above, the approach in this chapter is based on the use of solution operators for the linearized constraint equations around the flat case whose support properties are prescribed (in a bounded set or in a cone); see Lemmas 3.2.2 and 3.2.4 below. In [MT22] (see also Chapter 2), a similar approach was employed to construct of localized initial data sets à la Carlotto–Schoen [CS16a]. In fact, construction of such solution operators may be extended to other divergence-type equations with variable coefficients and/or on different manifolds; this is addressed in Chapter 4 (see also [Ise+25a]).

## Notation and conventions

Before stating the main results, we introduce some notation.

- As usual, $A \lesssim B$ is the shorthand for $|A| \leq CB$ for some $C > 0$ (implicit constant), which may differ from line to line. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.
- $\mathbb{N}$ is the set of (positive) natural integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- We denote by $(x^1, x^2, x^3)$ the rectangular coordinates on $\mathbb{R}^3$. We equip $\mathbb{R}^3$ with the Euclidean metric, given by the identity matrix $\delta_{ij}$ with respect to $(x^1, x^2, x^3)$. We use latin indices $i, j, \dots$ for tensors on $\mathbb{R}^3$; all latin indices are raised and lowered using $\delta$. Also, $\operatorname{tr}_\delta T = \sum_i T_{ii}$.
- We denote by $(x^0, x^1, x^2, x^3)$ the rectangular coordinates on $\mathbb{R}^{1+3}$. We equip $\mathbb{R}^{1+3}$ with the Minkowski metric, given by the diagonal matrix $\boldsymbol{\eta}_{\mu\nu}$ with entries $-1, +1, +1, +1$ with respect to $(x^0, x^1, x^2, x^3)$. We use greek indices $\mu, \nu, \lambda, \dots$ for tensors on $\mathbb{R}^{1+3}$; all greek indices are raised and lowered using $\boldsymbol{\eta}$. Also, $\operatorname{tr}_{\boldsymbol{\eta}} \mathbf{T} = \boldsymbol{\eta}^{\mu\nu}\mathbf{T}_{\mu\nu}$.
- We define $B_r(\xi)$ to be the ball of radius $r$ centered at $\xi$, $A_r(\xi) = B_{2r}(\xi) \setminus \overline{B_r(\xi)}$ for an annulus of radii $\simeq r$ and $\widetilde{A}_r(\xi) = B_{4r}(\xi) \setminus \overline{B_{\frac{r}{2}}(\xi)}$ for an enlargement of $A_r$. When $\xi = 0$, we shall omit $(\xi)$ and write $B_r = B_r(0)$, $A_r = A_r(0)$ etc. We introduce two pieces of notation for cones in $\mathbb{R}^3$: first, given $\omega \subseteq \mathbb{S}^2$, we define $C_\omega = \{x \in \mathbb{R}^3 : \frac{x}{r} \in \omega\}$, and second, given $\theta \in (0, \pi)$ and $\boldsymbol{\omega} \in \mathbb{S}^2$, we define $C_\theta(\boldsymbol{\omega}) = \{x \in \mathbb{R}^3 : \angle(x, \boldsymbol{\omega}) < \theta\}$. We write $\Omega^c$ for the complement of $\Omega$ in $\mathbb{R}^3$, $-\Omega = \{x \in \mathbb{R}^3 : -x \in \Omega\}$, and $\Omega + \xi = \{x + \xi \in \mathbb{R}^3 : x \in \Omega\}$.
- For $F_{ij}$ defined on a symmetric subset $\Omega$ of $\mathbb{R}^3$ (i.e., $\Omega = -\Omega$), we denote its decomposition into the even and odd parts by $F_{ij} = F^+_{ij} + F^-_{ij}$, where $F^\pm(x) = \frac{1}{2}(F(x) \pm F(-x))$.
- We introduce $\mathbf{e}_i = \partial_i$, $\boldsymbol{\nu} = \frac{x^j}{|x|}\mathbf{e}_j$ (the outward unit normal to $\partial B_r$) and $\mathbf{Y}_i = \epsilon_{ij}{}^k x^j \mathbf{e}_k$ (the infinitesimal generator of rotation about the $x^i$-axis), where $\epsilon_{ijk}$ is the Levi-Civita symbol.
- Given a Banach space $X = X(\mathbb{R}^3)$ of (possibly vector-valued) functions on $\mathbb{R}^3$ and an open subset $\Omega \subseteq \mathbb{R}^3$, we define the space of extendible functions on $\Omega$ as $X(\Omega) = X(\mathbb{R}^3)/\{u \in X(\mathbb{R}^3) : u|_\Omega = 0\}$, equipped with the $\|u\|_{X(\Omega)} = \inf_{\widetilde{u} \in X(\mathbb{R}^3) : \widetilde{u}|_\Omega = u} \|\widetilde{u}\|_X$. We also define the space of functions supported in $\Omega$, denoted by $X_0(\Omega)$, to be the completion of $C^\infty_c(\Omega)$ with respect to $\|\cdot\|_{X(\Omega)}$.

## Scaling invariance and charges in the almost flat regime

In this chapter, we shall consider solutions $(g, k)$ to (3.1.1) equipped with global coordinates $x^1, x^2, x^3$ – or more concretely, defined on a subset $\Omega$ of $\mathbb{R}^3$. For any such initial data $(g, k)$ and any $r > 0$, define $(g^{(r)}, k^{(r)})$ by

$$(g, k) \mapsto (g^{(r)}(x), k^{(r)}(x)) := (g(rx), rk(rx)). \tag{3.1.3}$$

Observe that (3.1.1) is invariant under (3.1.3), which we shall call the *invariant scaling transformation*.

We also introduce the following charges of $(g, k)$ measured on the sphere $\partial B_r$, which are conserved (i.e., independent of $r$) if $(g, k)$ solves the linearization of (3.1.1) around $(\delta, 0)$ (see Lemma 3.2.6 below):

$$\mathbf{E}[(g, k); \partial B_r] = \frac{1}{2} \int_{\partial B_r} \sum_i (\partial_i g_{ij} - \partial_j g_{ii})\, \boldsymbol{\nu}^j \, \mathrm{d}S, \tag{3.1.4}$$

$$\mathbf{P}_i[(g, k); \partial B_r] = \int_{\partial B_r} (k_{ij} - \delta_{ij} \operatorname{tr}_\delta k)\, \boldsymbol{\nu}^j \, \mathrm{d}S, \tag{3.1.5}$$

$$\mathbf{C}_k[(g, k); \partial B_r] = \frac{1}{2} \int_{\partial B_r} \sum_i (x_k \partial_i g_{ij} - x_k \partial_j g_{ii} - \delta_{ik}(g_{ij} - \delta_{ij}) + \delta_{jk}(g_{ii} - \delta_{ii}))\, \boldsymbol{\nu}^j \, \mathrm{d}S, \tag{3.1.6}$$

$$\mathbf{J}_k[(g, k); \partial B_r] = \int_{\partial B_r} (k_{ij} - \delta_{ij} \operatorname{tr}_\delta k)\, Y_k^i \boldsymbol{\nu}^j \, \mathrm{d}S. \tag{3.1.7}$$

We put these together to form a 10-vector,

$$\mathbf{Q}[(g, k); \partial B_r] = (\mathbf{E}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)^\dagger [(g, k); \partial B_r].$$

Taking (formally) the limit as $r \to \infty$ leads to the usual ADM energy, linear momenta, center of mass and angular momenta under a suitable asymptotic flatness assumption (see Definition 3.1.5 and Lemma 3.4.1 below). For these quantities, we introduce the notation

$$\mathbf{Q}^{ADM}[(g, k)] = (\mathbf{E}^{ADM}, \ldots, \mathbf{J}_3^{ADM})^\dagger [(g, k)] := (\lim_{r \to \infty} \mathbf{E}[(g, k); \partial B_r], \ldots, \lim_{r \to \infty} \mathbf{J}_3[(g, k); \partial B_r])^\dagger.$$

Since we will be working with annuli, it is also convenient to introduce the following (smoothly) *averaged charges*. Fix $\eta \in C_c^\infty(0, \infty)$ such that $\operatorname{supp} \eta \subseteq (1, 2)$, $\int \eta(r')\, \mathrm{d}r' = 1$ and $\eta_r(r') := r^{-1} \eta(r^{-1} r')$. We define

$$\mathbf{Q}[(g, k); A_r] = (\mathbf{E}, \ldots, \mathbf{J}_3)[(g, k); A_r] = \int \eta_r(r') (\mathbf{E}, \ldots, \mathbf{J}_3)[(g, k); \partial B_{r'}]\, \mathrm{d}r'. \tag{3.1.8}$$

In accordance with this notation, we shall denote the components of $Q \in \mathbb{R}^{10}$ by

$$Q = (\mathbf{E}(Q), \mathbf{P}_1(Q), \mathbf{P}_2(Q), \mathbf{P}_3(Q), \mathbf{C}_1(Q), \mathbf{C}_2(Q), \mathbf{C}_3(Q), \mathbf{J}_1(Q), \mathbf{J}_2(Q), \mathbf{J}_3(Q))^\dagger.$$

These quantities are conserved for solutions to the linearization of (3.1.1) around $(\delta, 0)$; see Lemma 3.2.6. They are *not* invariant under (3.1.3), but transform as follows:

$$(\mathbf{E}, \mathbf{P})[(g^{(r)}, k^{(r)}); A_{R_0}] = r^{-1} (\mathbf{E}, \mathbf{P})[(g, k); A_{rR_0}], \quad (\mathbf{C}, \mathbf{J})[(g^{(r)}, k^{(r)}); A_{R_0}] = r^{-2} (\mathbf{C}, \mathbf{J})[(g, k); A_{rR_0}],$$

and similarly for $\mathbf{Q}[(g, k); \partial B_r]$.

## Gluing up to linear obstructions

Suppose that we are given two disjoint dyadic annuli $A_{R_{in}}$ and $A_{R_{out}}$ (i.e., $2R_{in} < R_{out}$) and initial data sets $(g_{in}, k_{in})$ and $(g_{out}, k_{out})$ on the annuli $A_{R_{in}}$ and $A_{R_{out}}$, respectively, that are almost flat (i.e., close to $(\delta, 0)$ in some suitable sense). Consider the following gluing problem: find an almost flat initial data set $(g, k)$ on $B_{2R_{out}} \setminus \overline{B_{R_{in}}}$ that agrees with $(g_{in}, k_{in})$ and $(g_{out}, k_{out})$ on $A_{R_{in}}$ and $A_{R_{out}}$, respectively.

If all the initial data sets solve not (3.1.1) but rather its linearization around $(\delta, 0)$, then an obvious necessary condition for the solvability of this problem is $\mathbf{Q}[(g_{in}, k_{in}); A_{R_{in}}] = \mathbf{Q}[(g_{out}, k_{out}); A_{R_{out}}]$, in view of the conservation laws (see Lemma 3.2.6 below). In fact, this condition turns out to be sufficient as well[1]; in this sense, (the failure of) the condition $\mathbf{Q}[(g_{in}, k_{in}); A_{R_{in}}] = \mathbf{Q}[(g_{out}, k_{out}); A_{R_{out}}]$ is precisely the obstruction for linearized initial data gluing.

Our first main result generalizes the above linear considerations to the nonlinear setting. To state it, we first need to formulate the notion of *admissible initial data sets* (cf. [CS06]) on an annulus.

**Definition 3.1.2** ($\mathcal{Q}$-admissible initial data sets)**.** Let a bounded open subset $\mathcal{Q} \subseteq \mathbb{R}^{10}$ and $s \in \mathbb{R}$ be given. We say that a 10-parameter family $\{(g_Q, k_Q)\}_{\mathcal{Q}}$ is a *$\mathcal{Q}$-admissible family of annular initial data sets* on $\widetilde{A}_r$ with Sobolev regularity $s$ if:

1. for each $Q \in \mathcal{Q}$, $(g_Q, k_Q) \in (H^1 \cap C^0) \times L^2(\widetilde{A}_r)$ and solves (3.1.1) in $\widetilde{A}_r$;
2. for each $Q \in \mathcal{Q}$, $(g_Q, k_Q) \in H^s \times H^{s-1}(\widetilde{A}_r)$;
3. the map $\mathcal{Q} \to H^s \times H^{s-1}(\widetilde{A}_r)$ defined by $Q \mapsto (g_Q, k_Q)$ is Lipschitz;
4. for each $Q \in \mathcal{Q} \subseteq \mathbb{R}^{10}$, we have $Q = \mathbf{Q}[(g_Q, k_Q); A_r]$.

**Theorem 3.1.3** (Gluing up to linear obstructions at unit scale,[MOT23])**.** *For each $s > \frac{3}{2}$, there exist $\epsilon_c = \epsilon_c(s) > 0$ and $M_c = M_c(s) > 0$ such that the following holds. Let $(\mathring{g}, \mathring{k})$ be a solution to (3.1.1) in $H^s \times H^{s-1}(\widetilde{A}_1)$ such that*

$$\|(\mathring{g} - \delta, \mathring{k})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq \epsilon. \tag{3.1.9}$$

*Define $\mathring{Q} \in \mathbb{R}^{10}$ by $\mathring{Q} = \mathbf{Q}[(\mathring{g}, \mathring{k}); A_1]$. Let $\mathcal{Q} \subseteq \mathbb{R}^{10}$ be a bounded open set such that*

$$\mathring{Q} \in \mathcal{Q}, \quad B_{M_c \epsilon^2}(\mathring{Q}) \subseteq \mathcal{Q}. \tag{3.1.10}$$

*and consider an $\mathcal{Q}$-admissible family of annular initial data sets $\{(g_Q, k_Q)\}_{Q \in \mathcal{Q}}$ on $\widetilde{A}_1$ with Sobolev regularity $s$ such that, for all $Q, Q' \in \mathcal{Q}$,*

$$\|(g_Q - \delta, k_Q)\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq \epsilon, \tag{3.1.11}$$

$$\|(g_Q - g_{Q'}, k_Q - k_{Q'})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq K|Q - Q'|. \tag{3.1.12}$$

[1]Sufficiency of $\mathbf{Q}[(g_{in}, k_{in}); A_{R_{in}}] = \mathbf{Q}[(g_{out}, k_{out}); A_{R_{out}}]$ for linearized gluing can be established using Lemma 3.2.1 along with simple extension procedures; we leave the details to the interested reader.

*Then the following holds:*

1. **Exterior gluing.** *If $\epsilon < \epsilon_c$ and $K\epsilon < \epsilon_c$, then there exists a solution $(g,k) \in H^s \times H^{s-1}(\widetilde{A}_1)$ to (3.1.1) and $Q \in \mathcal{Q}$ such that $\|(g-\delta,k)\|_{H^s\times H^{s-1}(\widetilde{A}_1)} \lesssim \epsilon$, $|Q-\mathring{Q}| \leq M_c\epsilon^2$, and*
$$(g,k) = \begin{cases} (\mathring{g},\mathring{k}) & \text{in } A_{\frac{1}{2}}, \\ (g_Q,k_Q) & \text{in } A_2. \end{cases}$$

2. **Interior gluing.** *If $\epsilon < \epsilon_c$ and $K\epsilon < \epsilon_c$, then there exists a solution $(g,k) \in H^s \times H^{s-1}(\widetilde{A}_1)$ to (3.1.1) and $Q \in \mathcal{Q}$ such that $\|(g-\delta,k)\|_{H^s\times H^{s-1}(\widetilde{A}_1)} \lesssim \epsilon$, $|Q-\mathring{Q}| \leq M_c\epsilon^2$, and*
$$(g,k) = \begin{cases} (g_Q,k_Q) & \text{in } A_{\frac{1}{2}}, \\ (\mathring{g},\mathring{k}) & \text{in } A_2. \end{cases}$$

*In both cases, the following additional statements hold as well:*

- *(Lipschitz continuity) The map $(\mathring{g},\mathring{k}) \mapsto (g,k,Q)$ is Lipschitz as a map from the subset of $H^s \times H^{s-1}$ restricted by (3.1.9)–(3.1.10) into $H^s \times H^{s-1} \times \mathbb{R}^{10}$.*
- *(Persistence of regularity) If $(\mathring{g},\mathring{k}) \in H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)$ and the family $\{(g_Q,k_Q)\}_{Q\in\mathcal{Q}}$ is of Sobolev regularity $s+m$ for $m \in \mathbb{N}$, then $(g,k) \in H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)$ and*
$$\begin{aligned}\|(g-\delta,k)\|_{H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)} \leq{}& C\|(\mathring{g}-\delta,\mathring{k})\|_{H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)} \\ &+C\|(g_Q-\delta,k_Q)\|_{H^{s+m}\times H^{s+m-1}(\widetilde{A}_1)},\end{aligned}$$
*where $C = C(s,m,\|(\mathring{g}-\delta,\mathring{k})\|_{H^{s+1}\times H^{s}(\widetilde{A}_1)}, \|(g_Q-\delta,k_Q)\|_{H^{s+1}\times H^{s}(\widetilde{A}_1)})$.*

Theorem 3.1.3 is proved in Section 3.3.

*Remark* 3.1.4. 1. In view of the fact that $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ is invariant under (3.1.3), the regularity requirement $s > \frac{3}{2}$ is optimal modulo the endpoint. Moreover, using different Moser and product estimates on $\mathbb{R}^3$ in lieu Lemma 3.3.1 below, it is not difficult to extend the result to the case when $H^s$ is replaced by $B^{\frac{3}{2},2}_1$ (scaling invariant Besov space) or $W^{s,p}$ with $s > \frac{3}{p}$ and $1 < p < 3$.

2. We formulated Theorem 3.1.3 for initial data sets on $\widetilde{A}_1$ for technical convenience. Nonetheless, by the extension lemmas proved below (Lemmas 3.5.8 and 3.5.9), the theorem may be immediately reformulated in terms of initial data in two annuli $A_{\frac{1}{2}}$ and $A_2$ with averaged charges measured in the respective annuli. As alluded to before, this formulation provides a sufficient condition for solving the gluing problem with $R_{in} = \frac{1}{2}$ and $R_{out} = 2$ analogous to the linear problem. The result may also be easily extended to any other pair of concentric annuli whose closures are disjoint.

Using Theorem 3.1.3, we may retrieve the celebrated gluing result of Corvino–Schoen [CS06] (see also Chruściel–Delay [CD03] and Chruściel–Corvino–Isenberg [CCI11]) for asymptotically flat initial data sets, under (essentially) optimal assumptions on the regularity and the spatial decay. We adopt the following definition of asymptotic flatness:

**Definition 3.1.5.** Let $(g, k) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ for $s > \frac{3}{2}$ and $R_0 \in 2^{\mathbb{N}}$. We say that $(g, k)$ is $\alpha$*-asymptotically flat* if, for some $\alpha \in \mathbb{R}$ and $D_0 > 0$, we have

$$\|(g - \delta, k)\|_{\dot{\mathcal{H}}^s \times \dot{\mathcal{H}}^{s-1}(\widetilde{A}_r)} \leq D_0 r^{-s+\frac{3}{2}-\alpha} \quad \text{for all } r \in 2^{\mathbb{N}} \cap B^c_{R_0}, \tag{3.1.13}$$

where[2] $\|u\|_{\dot{\mathcal{H}}^s(\widetilde{A}_R)} := R^{-s+\frac{3}{2}}\|u(Rx)\|_{H^s(\widetilde{A}_1)}$.

By the Sobolev embedding, $\alpha$ corresponds to the pointwise decay rate of $g - \delta$ in $|x|^{-1}$.

**Theorem 3.1.6** (Corvino–Schoen [CS06] and Chruściel–Delay [CD03]; see also [CCI11])**.** *Let $s > \frac{3}{2}$ and $\alpha > \frac{1}{2}$. Given any $\alpha$-asymptotically flat initial data $(g_{in}, k_{in}) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ solving (3.1.1) and satisfying $|\mathbf{E}^{ADM}[(g, k)]| > |\mathbf{P}^{ADM}[(g, k)]|$, there exist $(g, k) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ solving (3.1.1) and $r \geq R_0$ such that $(g, k) = (g_{in}, k_{in})$ on $B_r$ and $(g, k)$ equals an initial data set for one of the Kerr spacetimes on $B^c_{2r}$.*

Theorem 3.1.6 is proved in Section 3.4. We remark that $\alpha > \frac{1}{2}$ is the sharp (up to endpoint) threshold for the ADM energy-momentum to be well-defined [Bar86]; see also Lemma 3.4.1 for sufficiency.

## Obstruction-free gluing

We return to the problem of gluing two initial data sets $(A_{R_{in}}, g_{in}, k_{in})$ and $(A_{R_{out}}, g_{out}, k_{out})$ that are almost flat. Recall from the discussion in Section 3.1 that the *linearized* gluing problem is solvable (if and) only if the 10 charges $\mathbf{Q}$ are identical. Remarkably, the original *nonlinear* problem turns out to be solvable if these identities among charges (obstructions) are replaced by mere positivity conditions! This novel type of gluing – *obstruction-free gluing* – has been recently introduced and established by Czimek–Rodnianski[3] [CR22].

We provide a new purely spacelike proof of obstruction-free gluing (the original approach of [CR22] involves null gluing), and furthermore sharpen the positivity, regularity and size requirements – see Remark 3.1.8 below. Our main result for almost flat data on two annuli reads as follows.

**Theorem 3.1.7** (Obstruction-free gluing for almost flat data on annuli, [MOT23])**.** *Given $s > \frac{3}{2}$ and $\Gamma > 1$, there exist $\epsilon_o = \epsilon_o(s, \Gamma) > 0$, $\mu_o = \mu_o(s, \Gamma) > 0$ and $C_o = C_o(s, \Gamma) > 0$*

[2] When $s \in \mathbb{N}$, observe that $\|u\|_{\dot{\mathcal{H}}^s(\widetilde{A}_R)} \simeq \|\partial_x^{(s)} u\|_{L^2(\widetilde{A}_R)} + R^{-s}\|u\|_{L^2(\widetilde{A}_R)}$.

[3] We note that, instead of annular data, [CR22] work with data on two spheres.

*such that the following holds. Let* $(g_{in}, k_{in}) \in H^s \times H^{s-1}(A_1)$, $(g_{out}, k_{out}) \in H^s \times H^{s-1}(A_{32})$ *be pairs solving* (3.1.1)*. Define* $\Delta Q = (\Delta\mathbf{E}, \dots, \Delta\mathbf{J}_3) \in \mathbb{R}^{10}$ *by*

$$\Delta Q = \mathbf{Q}[(g_{out}, k_{out}); A_{32}] - \mathbf{Q}[(g_{in}, k_{in}; A_1)], \tag{3.1.14}$$

*and assume that*

$$\Delta\mathbf{E} > |\Delta\mathbf{P}|, \tag{3.1.15}$$

$$\frac{\Delta\mathbf{E}}{\sqrt{(\Delta\mathbf{E})^2 - |\Delta\mathbf{P}|^2}} < \Gamma, \tag{3.1.16}$$

$$\Delta\mathbf{E} < \epsilon_o^2, \tag{3.1.17}$$

$$|\Delta\mathbf{C}| + |\Delta\mathbf{J}| < \mu_o \Delta\mathbf{E}, \tag{3.1.18}$$

$$\|(g_{in} - \delta, k_{in})\|^2_{H^s \times H^{s-1}(A_1)} + \|(g_{out} - \delta, k_{out})\|^2_{H^s \times H^{s-1}(A_{32})} < \mu_o \Delta\mathbf{E}. \tag{3.1.19}$$

*Then there exists* $(g, k) \in H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})$ *solving* (3.1.1) *such that*

$$(g, k) = (g_{in}, k_{in}) \quad \text{in } A_1, \quad (g, k) = (g_{out}, k_{out}) \quad \text{in } A_{32}, \tag{3.1.20}$$

$$\|(g - \delta, k)\|^2_{H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})} < C_o \Delta\mathbf{E}. \tag{3.1.21}$$

*Remark* 3.1.8 (Sharpness of positivity, regularity and size assumptions). As observed in [CR22], the positivity requirement $\Delta\mathbf{E} > |\Delta\mathbf{P}|$ is a key necessary condition for the validity of obstruction-free gluing in general (see also Remark 3.1.11 below). In Theorem 3.1.7, $|\Delta\mathbf{P}|$ is allowed to be arbitrarily close to $\Delta\mathbf{E}$ (i.e., $\Gamma$ is arbitrarily large in view of (3.1.16)) provided that the bounds (3.1.17)–(3.1.19) hold with sufficiently small $\epsilon_o$ and $\mu_0$ (depending on $\Gamma$).

As before, the regularity requirement $s > \frac{3}{2}$ is sharp (up to endpoint) in relation to the scaling critical exponent. In view of the conservation law for $\mathbf{E}$ (see Lemma 3.2.6 below), for a solution $(g, k) \in H^s \times H^{s-1}(B_{64\Gamma} \setminus \overline{B_1})$ ($s > \frac{3}{2}$) satisfying (3.1.1), (3.1.14) and (3.1.20), it is necessary that

$$\Delta\mathbf{E} \lesssim_{s,\Gamma} \|(g - \delta, k)\|^2_{H^s \times H^{s-1}(B_{64\Gamma} \setminus \overline{B_1})}.$$

Thus (3.1.21) is sharp up to a constant, and hence so are the assumptions (3.1.18) and (3.1.19). Finally, we remark that the choice of the radii $R_{in} = 1$ and $R_{out} = 32$ is arbitrary, and the result may be easily extended to any other pair of concentric annuli whose closures are disjoint.

*Remark* 3.1.9. By a minor modification of the proof of Theorem 3.1.7, the following statements may also be established (where the domains are omitted for simplicity):

- (Persistence of regularity) If $(g_{in}, k_{in}, g_{out}, k_{out}) \in (H^{s+m} \times H^{s+m-1}) \times (H^{s+m} \times H^{s+m-1})$ for $m \in \mathbb{N}$, then $(g, k) \in H^{s+m} \times H^{s+m-1}$,
- (Lipschitz continuity) The correspondence $(g_{in}, k_{in}, g_{out}, k_{out}) \mapsto (g, k)$ in Theorem 3.1.7 may be put together to define a locally Lipschitz map from the subset of $(H^s \times H^{s-1}) \times (H^s \times H^{s-1})$ restricted by (3.1.15)–(3.1.19) into $H^s \times H^{s-1}$.

The corresponding result for asymptotically flat initial data sets has a more elegant hypothesis, at the expense of performing the gluing procedure in an annulus sufficiently afar (as in Theorem 3.1.6).

**Theorem 3.1.10** (Obstruction-free gluing for asymptotically flat data, [MOT23])**.** *Let $s > \frac{3}{2}$ and $\alpha > \frac{1}{2}$. Let $(g_{in}, k_{in})$, $(g_{out}, k_{out}) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ be $\alpha$-asymptotically flat pairs solving (3.1.1). Define, for those components that are well-defined,*

$$\Delta Q := \mathbf{Q}^{ADM}[(g_{out}, k_{out})] - \mathbf{Q}^{ADM}[(g_{in}, k_{in})],$$

*and assume that*

$$\Delta \mathbf{E} > |\Delta \mathbf{P}|.$$

*Then there exists an asymptotically flat pair $(g, k) \in H^s_{loc} \times H^{s-1}_{loc}(B^c_{R_0})$ solving (3.1.1) and $r \geq R_0$ such that $(g, k) = (g_{in}, k_{in})$ on $B_{2r}$ and $(g, k) = (g_{out}, k_{out})$ on $B^c_{32r}$.*

Theorems 3.1.7 and 3.1.10 are proved in Section 3.5.

*Remark* 3.1.11. Taking $(g_{in}, k_{in}) = (\delta, 0)$ shows that the condition $\Delta \mathbf{E} > |\Delta \mathbf{P}|$ is sharp in view of the positive mass theorem [SY79; Wit81]. Note that this case recovers the interior gluing of Bieri–Chruściel [BC17].

The requirements on the regularity and spatial decay exponents $s > \frac{3}{2}$ and $\alpha > \frac{1}{2}$ are sharp (up to endpoints) as discussed in Section 3.1; in particular, $\Delta \mathbf{E}$ and $\Delta \mathbf{P}$ are well-defined. On the contrary, $\mathbf{C}^{ADM}, \mathbf{J}^{ADM}$ need not be well-defined for either $(g_{in}, k_{in})$ or $(g_{out}, k_{out})$ to apply Theorem 3.1.10.

## Organization of the chapter

The remainder of the chapter is structured as follows. **Section 3.2** collects some preliminary facts concerning the linearization of (3.1.1) and (3.1.2) around the flat case. More specifically, after rewriting (3.1.1) as a quasilinear perturbation of divergence-type equations on $(\mathbb{R}^3, \delta)$ (i.e., the linearization of (3.1.1) around the flat case) in **Section 3.2**, we write down Bogovskii- and conic-type operators for these equations (see also Chapter 4 and [Ise+25a]) in **Sections 3.2** and **3.2**, respectively. We discuss the conservation laws for the linearization of (3.1.1) (which involve **Q**) in **Section 3.2**, and for the linearization of (3.1.2) (which will be used in the proof of obstruction free gluing) in **Section 3.2**. Then in **Section 3.3**, we prove Theorem 3.1.3. In **Section 3.4**, we collect some basic facts about asymptotic flatness and establish Theorem 3.1.6, with some technical details concerning Kerr initial data sets deferred to **Appendix A**. Finally, in **Section 3.5**, we prove Theorems 3.1.7 and 3.1.10; an outline of the proof is provided in **Section 3.5**.

## 3.2 Linearization around the flat case

### Schematic notation and reformulation of (3.1.1)

We begin by reformulating (3.1.1) in a sufficiently flat region. We introduce new variables $(h, \pi)$, defined as follows:

$$(h_{ij}, \pi_{ij}) = (g_{ij} - \delta_{ij} - \delta_{ij} \operatorname{tr}_\delta (g - \delta), k_{ij} - \delta_{ij} \operatorname{tr}_\delta k). \tag{3.2.1}$$

Observe that the transformation is obviously invertible with the formulae

$$(g_{ij}, k_{ij}) = (\delta_{ij} + h_{ij} - \frac{1}{2}\delta_{ij} \operatorname{tr}_\delta h, \pi_{ij} - \frac{1}{2}\delta_{ij} \operatorname{tr}_\delta \pi). \tag{3.2.2}$$

With respect to the new variables, we write the left-hand sides of (3.1.1) schematically as

$$R[g] = \partial_i \partial_j h^{ij} - M_h^{(2)}(h, \partial^2 h) - M_h^{(1)}(\partial h, \partial h), \tag{3.2.3}$$

$$(\operatorname{tr}_g k)^2 - |k|_g^2 = -M_h^{(0)}(\pi, \pi), \tag{3.2.4}$$

$$g^{jj'}(g^{ii'} D_{g;i} k_{i'j'} - \partial_{j'} \operatorname{tr}_g k) = \partial_i \pi^{ij} - N_h^{(1)j}(h, \partial \pi) - N_h^{(0)j}(\partial h, \pi). \tag{3.2.5}$$

In conclusion, (3.1.1) takes the form

$$\partial_i \partial_j h^{ij} = M_h^{(2)}(h, \partial^2 h) + M_h^{(1)}(\partial h, \partial h) + M_h^{(0)}(\pi, \pi), \tag{3.2.6}$$

$$\partial_i \pi^{ij} = N_h^{(1)j}(h, \partial \pi) + N_h^{(0)j}(\partial h, \pi), \tag{3.2.7}$$

where the indices are raised and lowered using the Euclidean metric $\delta$. Introducing

$$\vec{P}(h, \pi) = (\partial_i \partial_j h^{ij}, \partial_i \pi^{ij}), \quad \vec{N}(h, \pi) = (M(h, \pi), N(h, \pi)) := (\text{RHS of } (3.2.6), \text{RHS of } (3.2.7)), \tag{3.2.8}$$

we may abbreviate (3.2.6), (3.2.7) as $\vec{P}(h, \pi) = \vec{N}(h, \pi)$.

### Bogovskii-type operators

We now state our main tool for inverting the LHS of (3.2.6)–(3.2.7) while preserving the annular support property.

**Lemma 3.2.1.** *Let $\Omega = A_1$. There exists a linear operator $S$ defined for $f \in C_c^\infty(\Omega)$ and taking values in symmetric 2-tensor fields (i.e., symmetric $3 \times 3$ matrix-valued functions) such that*

(*S1*) $\operatorname{supp} Sf \subseteq \Omega$ *(recall $f \in C_c^\infty(\Omega)$);*

(*S2*) $\partial_i \partial_j (Sf)^{ij} = f$ *if* $\int_\Omega f \cdot (1, x_1, x_2, x_3)^\dagger \, \mathrm{d}x = 0$*;*

(*S3*) $\|Sf\|_{H^{s'}} \lesssim_{s'} \|f\|_{H^{s'-2}}$ *for any* $s' \in \mathbb{R}$*;*

*($S4$) $\|[S,\partial_j]f\|_{H^{s'}} \lesssim_{s'} \|f\|_{H^{s'-2}}$ for any $s' \in \mathbb{R}$ and $j = 1, \ldots, d$.*

*Moreover, there exists a linear operator $T$ defined for $\mathbf{f} \in C_c^\infty(\Omega; \mathbb{R}^3)$ and taking values in symmetric 2-tensor fields such that*

*($T1$) $\operatorname{supp} T\mathbf{f} \subseteq \Omega$ (recall $\mathbf{f} \in C_c^\infty(\Omega; \mathbb{R}^3)$);*

*($T2$) $\partial_i (T\mathbf{f})^{ij} = \mathbf{f}^j$ if $\int_\Omega \mathbf{f} \cdot (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)^\dagger \, \mathrm{d}x = 0$;*

*($T3$) $\|T\mathbf{f}\|_{H^{s'}} \lesssim_{s'} \|\mathbf{f}\|_{H^{s'-1}}$ for any $s' \in \mathbb{R}$;*

*($T4$) $\|[T,\partial_j]\mathbf{f}\|_{H^{s'}} \lesssim_{s'} \|\mathbf{f}\|_{H^{s'-1}}$ for any $s' \in \mathbb{R}$ and $j = 1, \ldots, d$.*

Concerning the integral conditions in ($S2$) (resp. ($T2$)), observe that $1$, $x_1$, $x_2$, $x_3$ span the kernel of the formal $L^2$-adjoint of the double divergence operator $h \mapsto \partial_i \partial_j h^{ij}$ (resp. $\mathbf{e}_1, \ldots, \mathbf{Y}_3$ span the kernel of the formal $L^2$-adjoint of the symmetric divergence operator $\pi \mapsto \partial_i \pi^{ij}$).

The basic ingredient for the proof of Lemma 3.2.1 is the following explicit operators in the case $\Omega$ is star-shaped with respect to a ball (see Lemma 3.2.2 for the definition), which is analogous to the classical Bogovskii operator [Bog79] for the divergence operator $u \mapsto \partial_j u^j$.

**Lemma 3.2.2** (Bogovskii-type operators)**.** *Let $\Omega$ be an open set in $\mathbb{R}^3$ that is star-shaped with respect to a ball $B \subset \Omega$, i.e., for every $x \in U$ and $y \in B$, the line segment $[\overline{x,y}]$ is contained in $\Omega$. Fix $C_c^\infty(B)$ such that $\int \eta \, \mathrm{d}x = 1$, and $\chi_\Omega \in C_c^\infty(\mathbb{R}^3)$ such that $\chi_\Omega = 1$ on $\Omega$.*

*Then the operator defined for all $f \in C_c^\infty(\mathbb{R}^3)$ by*

$$(Sf)^{ij}(x) = \int \mathbf{\Psi}_\eta^{ij}(x,y) f(y) \, \mathrm{d}y, \tag{3.2.9}$$

$$\mathbf{\Psi}_\eta^{ij}(z+y,y) = \left( \int_{|z|}^\infty \eta \left( r \frac{z}{|z|} + y \right) r^2 \, \mathrm{d}r \right) \frac{z^i z^j}{|z|^3}. \tag{3.2.10}$$

*satisfies ($S1$) (when $\operatorname{supp} f \subseteq \Omega$) and ($S2$). Moreover, $f \mapsto S(\chi_\Omega f)$ is a classical pseudodifferential operator of order $-2$, which also implies ($S3$) and ($S4$) when $\operatorname{supp} f \subseteq \Omega$.*

*Moreover, the operator defined by*

$$(T\mathbf{f})^{ij}(x) = \int (\mathbf{\Psi}_\eta)^{ij}_k(x,y) \mathbf{f}^k(y) \, \mathrm{d}y, \tag{3.2.11}$$

$$\begin{aligned}
(\mathbf{\Psi}_\eta)^{ij}_k(z+y,y) = & -\frac{1}{2} \left( \int_{|z|}^\infty \eta \left( r \frac{z}{|z|} + y \right) r^2 \, \mathrm{d}r \right) |z|^{-3} \left( z^i \boldsymbol{\delta}_k^j + \boldsymbol{\delta}_k^i z^j \right) \\
& + \frac{1}{2} \partial_{z^m} \left( \left( \int_{|z|}^\infty \eta \left( r \frac{z}{|z|} + y \right) r^2 \, \mathrm{d}r \right) |z|^{-3} \left( z^i \boldsymbol{\delta}_k^j z^m + \boldsymbol{\delta}_k^i z^j z^m \right) \right) \\
& - \partial_{z^k} \left( \left( \int_{|z|}^\infty \eta \left( r \frac{z}{|z|} + y \right) r^2 \, \mathrm{d}r \right) |z|^{-3} z^i z^j \right).
\end{aligned} \tag{3.2.12}$$

*satisfies ($T1$) (when $\operatorname{supp} \mathbf{f} \subseteq \Omega$) and ($T2$). Moreover, $\mathbf{f} \mapsto T(\chi_\Omega \mathbf{f})$ is a classical pseudodifferential operator of order $-1$, which also implies ($T3$) and ($T4$) when $\operatorname{supp} \mathbf{f} \subseteq \Omega$.*

*Proof.* That $(S1)$ and $(T1)$ hold may be verified by computation using the explicit formulae. To prove $(S1)$, we first recall the computation $\partial_{z^i}\left[\frac{z^i}{|z|^3}\int_{|z|}^{\infty}\eta(r\frac{z}{|z|}+y)r^2\,\mathrm{d}r\right]=\delta_0(z)-\eta(z+y)$, where the expression inside the parentheses is the classical Bogovskii operator for the divergence operator [Bog79]. We may then compute

$$\begin{aligned}\partial_{z^i}\partial_{z^j}\mathbf{\Psi}_\eta(z+y,y)&=\partial_{z^i}\left[\partial_{z^j}\left(\frac{z^iz^j}{|z|^3}\right)\int_{|z|}^{\infty}\eta\left(r\frac{z}{|z|}+y\right)r^2\,\mathrm{d}r+\frac{z^iz^j}{|z|^3}\partial_{z^j}\int_{|z|}^{\infty}\eta\left(r\frac{z}{|z|}+y\right)r^2\,\mathrm{d}r\right]\\&=\partial_{z^i}\left[\frac{z^i}{|z|^3}\int_{|z|}^{\infty}\eta\left(r\frac{z}{|z|}+y\right)r^2\,\mathrm{d}r\right]-\partial_{z^i}\left[z^i\eta\left(z+y\right)\right]\\&=\delta_0(z)-4\eta(z+y)-z^i(\partial_i\eta)(z+y),\end{aligned}$$

where we used the preceding identity on the last line. From this computation, $(S2)$ follows. On the other hand, a proof of $(T2)$ can be found in [IO16], where the same operator was introduced and used (alternatively, $(T2)$ can also be verified by computation as above). Next, following an argument similar to [OT19], it may be verified that $f\mapsto S(\chi_\Omega f)$ and $\mathbf{f}\mapsto T(\chi_\Omega\mathbf{f})$ are pseudodifferential operators of order $-2$ and $-1$, respectively. Then $(S3)$, $(S4)$, $(T3)$ and $(T4)$ follow. □

*Proof of Lemma 3.2.1.* We shall call $S$ (resp. $T$) satisfying $(S1)$–$(S4)$ (resp. $(T1)$–$(T4)$) a *Bogovskii-type* operator on $\Omega$. Lemma 3.2.2 says that such operators exist for any open set star-shaped with respect to a ball. We claim that a Bogovskii-type operator can be defined on the union of any two open sets on each of which such an operator is defined. By a simple recursion argument, this observation would allow us to define a Bogovskii-type operator on any finite union of such sets. Since $A_1$ is clearly a finite union of open star-shaped sets with respect to balls, Lemma 3.2.1 would then follow.

It remains to verify the claim. We focus on the case of $S$, the case of $T$ being similar. Let $U_1$ and $U_2$ be open sets on which Bogovskii-type operators $S_1$ and $S_2$, respectively, exist. When $U_1\cap U_2\neq\emptyset$, let $\{\chi_1,\chi_2\}$ be a smooth partition of unity on $U$ subordinate to $\{U_1,U_2\}$ and fix $\eta\in C_c^\infty(U_1\cap U_2)$. Write $g_0=1$, $g_j=x_j$ $(j=1,2,3)$ and define $G_{\mu\nu}=\int g_\mu g_\nu\eta^2\,\mathrm{d}x$. If $\eta\neq 0$, note that $G_{\mu\nu}$ is positive-definite and in particular invertible. Now, we define $\theta^\mu=(G^{-1})^{\mu\nu}g_\nu\eta^2$, so that $\int\theta^\mu g_{\mu'}\,\mathrm{d}x=\delta^\mu_{\mu'}$ and $\theta^\mu\in C_c^\infty(U_1\cap U_2)$. We decompose $f$ into

$$f=f_1+f_2+\left(\int f\,\mathrm{d}x\right)\theta^0-\sum_{j=1}^{3}\left(\int fx_j\,\mathrm{d}x\right)\theta^j,$$

where $f_k=f\chi_k-\left(\int f\chi_k\,\mathrm{d}x\right)\theta^0-\sum_{j=1}^3\left(\int f\chi_kx_j\,\mathrm{d}x\right)\theta^j$. Then $f=f_1+f_2$ if and only if $\int f(1,x_1,x_2,x_3)^\dagger\,\mathrm{d}x=0$. Moreover, $\operatorname{supp} f_k\subseteq U_k$ with $\int f_k(1,x_1,x_2,x_3)^\dagger\,\mathrm{d}x=0$. We claim that the following defines a desired Bogovskii-type operator on $U$:

$$Sf:=S_1f_1+S_2f_2.$$

That $(S1)$–$(S3)$ hold is straightforward. To check $(S4)$, note that

$$[S, \partial_j]f = [S_1, \partial_j]f_1 + [S_2, \partial_j]f_2 + S_1((\partial_j f)_1 - \partial_j f_1) + S_2((\partial_j f)_2 - \partial_j f_2). \tag{3.2.13}$$

That the $H^{s'}$ norm of the first two terms on the RHS is bounded by $\|f\|_{H^{s'-2}}$ is clear by $(S4)$ for $S_k$ and the definition of $f_k$. To estimate the $H^{s'}$ norm of the third term, in view of $(S3)$ for $S_1$, we write

$$\|S_1((\partial_j f)_1 - \partial_j f_1)\|_{H^{s'}} \lesssim \|(\partial_j f)_1 - \partial_j f_1\|_{H^{s'-2}} \lesssim \|\partial_j f \chi_1 - \partial_j(f\chi_1)\|_{H^{s'-2}} + \|f\|_{H^{s'-2}},$$

where, for the last inequality, we used the observation that all terms involving $\theta^\mu$ may be bounded by $\|f\|_{H^{s'-2}}$. But $\|\partial_j f \chi_1 - \partial_j(f\chi_1)\|_{H^{s'-2}} = \|f \partial_j \chi_1\|_{H^{s'-2}} \lesssim \|f\|_{H^{s'-2}}$, which is acceptable. The fourth term in (3.2.13) is handled similarly. □

## Conic operators

Here we state the main tool for inverting the LHS of (3.2.6)–(3.2.7) while preserving the conic support property. We first define the relevant weighted Sobolev space.

**Definition 3.2.3.** For $s \in \mathbb{N}_0$, the b-Sobolev space $H^s_{\mathrm{b}}(\mathbb{R}^3)$ is defined by the norm

$$\|u\|^2_{H^s_{\mathrm{b}}} := \sum_{k \leq s} \|\langle x \rangle^k \nabla^k u\|^2_{L^2(\mathbb{R}^3)}.$$

We extend the definition to $s \in \mathbb{R}$ by duality and (complex) interpolation. We further define for $\delta \in \mathbb{R}$, $H^{s,\delta}_{\mathrm{b}} := \langle x \rangle^{-\delta} H^s_{\mathrm{b}}$.

In [MT22] (see also Chapter 2 and [OT19]), the following was proved.

**Lemma 3.2.4.** *Let $\omega \subseteq \mathbb{S}^2$ be a convex open subset, and consider $\kappa \in C^\infty(\mathbb{S}^2)$ with $\int \kappa = 1$ and $\operatorname{supp} \kappa \subseteq \omega$. Consider the linear translation-invariant operator $S_c$ defined for $f \in C^\infty_c(C_\omega)$ and taking values in symmetric 2-tensor fields (i.e., symmetric $3 \times 3$ matrix-valued functions) given by*

$$(S_c f)^{ij} = \int K^{ij}_\kappa(x - y) f(y)\,\mathrm{d}y, \quad K^{ij}_\kappa(z) = \kappa(\tfrac{z}{|z|}) \frac{z^i z^j}{|z|^3}.$$

*Then $S_c$ satisfies*

$(S_c1)$ $\operatorname{supp} S_c f \subseteq C_\omega$;

$(S_c2)$ $\partial_i \partial_j (S_c f)^{ij} = f$;

$(S_c3)$ $\|Sf\|_{H^{s',\delta}_{\mathrm{b}}} \lesssim_{s',\delta} \|f\|_{H^{s'-2,\delta+2}_{\mathrm{b}}}$ *for any $s' \in \mathbb{R}$ and $\delta < -\frac{1}{2}$.*

*Moreover, consider the linear translation-invariant operator $T_c$ defined for $\mathbf{f} \in C_c^\infty(C_\omega;\mathbb{R}^3)$ and taking values in symmetric 2-tensor fields given by*

$$T_c\mathbf{f} = \int (K_\kappa)^{ij}_k(x-y)\mathbf{f}^k(y)\,\mathrm{d}y, \quad (K_\kappa)^{ij}_k(z) = \partial_m\left(\kappa(\tfrac{z}{|z|})\frac{z^i\boldsymbol{\delta}^j_k z^m + \boldsymbol{\delta}^i_k z^j z^m - z^i z^j \boldsymbol{\delta}^m_k}{|z|^3}\right).$$

*Then $T_c$ satisfies*

($T_c1$) $\operatorname{supp} T_c\mathbf{f} \subseteq C_\omega$;

($T_c2$) $\partial_i(T_c\mathbf{f})^{ij} = \mathbf{f}^j$;

($T_c3$) $\|T_c\mathbf{f}\|_{H_{\mathrm{b}}^{s',\delta}} \lesssim_{s'} \|\mathbf{f}\|_{H_{\mathrm{b}}^{s'-1,\delta+1}}$ *for any $s' \in \mathbb{R}$ and $\delta < \frac{1}{2}$.*

*Remark* 3.2.5. The definition of $K_\kappa$ is not exactly the same as [MT22] (see also Chapter 2), but the boundedness assertion ($S_c3$) for the solution operator was proved for any homogeneous distribution with outgoing property, which clearly applies here. The fact ($S_c1$) is obvious from the defintion and convexity of the cone and ($S_c2$) can be checked by direct computation:

$$\partial_i\partial_j\left(\kappa\left(\frac{z}{|z|}\right)\frac{z^iz^j}{|z|^3}\right) = \partial_i\left(\partial_j\kappa\left(\frac{z}{|z|}\right)\frac{z^iz^j}{|z|^3} + \kappa\left(\frac{z}{|z|}\right)\partial_j\left(\frac{z^iz^j}{|z|^3}\right)\right) = \partial_i\left(\kappa\left(\frac{z}{|z|}\right)\frac{z^i}{|z|^3}\right) = \delta_0.$$

## Conservation laws for the linearized constraint equation

We now state a tool for controlling the variation of charges $\mathbf{Q}$ (both averaged or unaveraged). For $0 < r_0 < r_1$, and $\eta_r$ is as in (3.1.8), we introduce

$$\chi_{r_0,r_1}(r) = \int_{-\infty}^{r} (\eta_{r_0}(r') - \eta_{r_1}(r'))\,\mathrm{d}r'. \tag{3.2.14}$$

**Lemma 3.2.6.** *Let $(g,k)$ and $(h,\pi)$ be related by (3.2.1). For any $0 < r_0 < r_1$, we have*

$$\int_{B_{r_1}\setminus B_{r_0}} \tfrac{1}{2}\partial_i\partial_j h^{ij}\begin{pmatrix}1\\x_1\\x_2\\x_3\end{pmatrix}\mathrm{d}x = \begin{pmatrix}\mathbf{E}\\\mathbf{C}_1\\\mathbf{C}_2\\\mathbf{C}_3\end{pmatrix}[(g,k);\partial B_{r_1}] - \begin{pmatrix}\mathbf{E}\\\mathbf{C}_1\\\mathbf{C}_2\\\mathbf{C}_3\end{pmatrix}[(g,k);\partial B_{r_0}],$$

$$\int_{B_{r_1}\setminus B_{r_0}} \sum_j \partial_i\pi^{ij}\begin{pmatrix}\mathbf{e}_1^j\\\mathbf{e}_2^j\\\mathbf{e}_3^j\\\mathbf{Y}_1^j\\\mathbf{Y}_2^j\\\mathbf{Y}_3^j\end{pmatrix}\mathrm{d}x = \begin{pmatrix}\mathbf{P}_1\\\mathbf{P}_2\\\mathbf{P}_3\\\mathbf{J}_1\\\mathbf{J}_2\\\mathbf{J}_3\end{pmatrix}[(g,k);\partial B_{r_1}] - \begin{pmatrix}\mathbf{P}_1\\\mathbf{P}_2\\\mathbf{P}_3\\\mathbf{J}_1\\\mathbf{J}_2\\\mathbf{J}_3\end{pmatrix}[(g,k);\partial B_{r_0}],$$

*and in terms of averaged charges, we have*

$$\int \chi_{r_0,r_1}(r)\tfrac{1}{2}\partial_i\partial_j h^{ij}\begin{pmatrix}1\\x_1\\x_2\\x_3\end{pmatrix}\mathrm{d}x = \begin{pmatrix}\mathbf{E}\\\mathbf{C}_1\\\mathbf{C}_2\\\mathbf{C}_3\end{pmatrix}[(g,k);A_{r_1}] - \begin{pmatrix}\mathbf{E}\\\mathbf{C}_1\\\mathbf{C}_2\\\mathbf{C}_3\end{pmatrix}[(g,k);A_{r_0}],$$

$$\int \sum_j \chi_{r_0,r_1}(r)\partial_i\pi^{ij}\begin{pmatrix}\mathbf{e}_1^j\\\mathbf{e}_2^j\\\mathbf{e}_3^j\\\mathbf{Y}_1^j\\\mathbf{Y}_2^j\\\mathbf{Y}_3^j\end{pmatrix}\mathrm{d}x = \begin{pmatrix}\mathbf{P}_1\\\mathbf{P}_2\\\mathbf{P}_3\\\mathbf{J}_1\\\mathbf{J}_2\\\mathbf{J}_3\end{pmatrix}[(g,k);A_{r_1}] - \begin{pmatrix}\mathbf{P}_1\\\mathbf{P}_2\\\mathbf{P}_3\\\mathbf{J}_1\\\mathbf{J}_2\\\mathbf{J}_3\end{pmatrix}[(g,k);A_{r_0}].$$

This lemma follows immediately by integration by parts, (3.2.1) and the fact that $1, x_1, \dots, x_3$ (resp. $\mathbf{e}_1, \dots, \mathbf{Y}_3$) belong to the kernel of the $L^2$-adjoint of $h \mapsto \partial_i\partial_j h^{ij}$ (resp. $\pi \mapsto \partial_i\pi^{ij}$).

## Conservation laws for the linearized Einstein vacuum equations

In this subsection, we work in an oriented spacetime domain $\mathbf{\Omega}$ equipped with a flat (i.e., Minkowski) metric $\boldsymbol{\eta}$. We use greek indices $\alpha, \beta, \dots$ to refer to spacetime tensorial indices. We always raise and lower indices using the flat metric $\boldsymbol{\eta}$, and denote by $\nabla$ the (trivial) covariant derivative with respect to $\boldsymbol{\eta}$.

Consider the Einstein tensor $\mathbf{G}[\mathbf{g}] = \mathrm{Ric}[\mathbf{g}] - \frac{1}{2}\mathbf{g}\,\mathrm{tr}_{\mathbf{g}}\,\mathrm{Ric}[\mathbf{g}]$. Its linearization around the flat metric $\boldsymbol{\eta}$, which trivially satisfies $\mathbf{G}[\boldsymbol{\eta}] = 0$, takes the form

$$\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta} = \frac{1}{2}\left(-\nabla^\gamma\nabla_\gamma\mathbf{H}_{\alpha\beta} + \nabla_\alpha\nabla^\gamma\mathbf{H}_{\gamma\beta} + \nabla_\beta\nabla^\gamma\mathbf{H}_{\gamma\alpha} - \boldsymbol{\eta}_{\alpha\beta}\nabla^\gamma\nabla^\delta\mathbf{H}_{\gamma\delta}\right), \tag{3.2.15}$$

where $\mathbf{H} = \dot{\mathbf{g}} - \frac{1}{2}\boldsymbol{\eta}\,\mathrm{tr}_{\boldsymbol{\eta}}\,\dot{\mathbf{g}}$. Note that, by linearizing the second Bianchi identity $\mathbf{D}^\beta\mathbf{G}[\mathbf{g}]_{\alpha\beta} = 0$, we have $\nabla^\beta\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta} = 0$.

Let $\mathbf{X}$ be a Killing vector field with respect to $\boldsymbol{\eta}$, i.e., $(\mathcal{L}_{\mathbf{X}}\boldsymbol{\eta})_{\alpha\beta} = \nabla_\alpha\mathbf{X}_\beta + \nabla_\beta\mathbf{X}_\alpha = 0$. By the linearized second Bianchi identity, observe that $\nabla^\beta(\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}\mathbf{X}^\alpha) = 0$; hence, the 1-form $\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}\mathbf{X}^\alpha$ is co-closed. In fact, there exists 2-form whose co-differential is this 1-form; following [CD03, Appendix E] (see also [Chr85]), we introduce

$${}^{(\mathbf{X})}\mathbb{U}_{\alpha\beta}[\dot{\mathbf{g}}] = \frac{1}{2}\left[\left(-\nabla_\alpha\mathbf{H}_{\gamma\beta} + \nabla_\beta\mathbf{H}_{\gamma\alpha} + \boldsymbol{\eta}_{\gamma\alpha}\nabla^\delta\mathbf{H}_{\beta\delta} - \boldsymbol{\eta}_{\gamma\beta}\nabla^\delta\mathbf{H}_{\alpha\delta}\right)\mathbf{X}^\gamma + \mathbf{H}_{\gamma\alpha}\nabla^\gamma\mathbf{X}_\beta - \mathbf{H}_{\gamma\beta}\nabla^\gamma\mathbf{X}_\alpha\right]. \tag{3.2.16}$$

By a straightforward computation, it may be verified that

$$\nabla^\alpha({}^{(\mathbf{X})}\mathbb{U}_{\alpha\beta}[\dot{\mathbf{g}}]) = \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}\mathbf{X}^\alpha. \tag{3.2.17}$$

Equivalently, $\mathrm{d}(\star^{(\mathbf{X})}\mathbb{U}[\dot{\mathbf{g}}]) = -\star \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}](\mathbf{X},\cdot)$ using the Hodge star operator $\star$ associated to $\boldsymbol{\eta}$. By the Stokes theorem,

$$\int_{\partial U} \star^{(\mathbf{X})}\mathbb{U}[\dot{\mathbf{g}}] = \int_U \mathrm{d}(\star^{(\mathbf{X})}\mathbb{U}[\dot{\mathbf{g}}]) = -\int_U \star \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}](\mathbf{X},\cdot). \tag{3.2.18}$$

Consider a system of coordinates $\{x^0,\dots,x^3\}$ with respect to which $\boldsymbol{\eta} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2$ and $\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$ has positive orientation – such coordinates shall be referred to as *canonical*. On $\Sigma_0 = \{x^0 = 0\}$, define

$$\dot{g}_{ij} = \dot{\mathbf{g}}_{ij}\Big|_{\{x^0=0\}}, \quad \dot{k}_{ij} = \frac{1}{2}\left(\partial_0\dot{\mathbf{g}}_{ij} - \partial_i\dot{\mathbf{g}}_{j0} - \partial_j\dot{\mathbf{g}}_{i0}\right)\Big|_{\{x^0=0\}}. \tag{3.2.19}$$

It may be checked that (cf. (3.2.1) and (3.2.8))

$$\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{00} = \tfrac{1}{2}\partial_j\partial_k(\dot{g}^{jk} - \delta^{jk}\operatorname{tr}_\delta \dot{g}), \quad \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{0j} = \partial^\ell(\dot{k}_{j\ell} - \delta_{j\ell}\operatorname{tr}_\delta \dot{k}_{j\ell}). \tag{3.2.20}$$

Indeed, these may be proved by linearizing the nonlinear relations $\mathbf{G}[\mathbf{g}]_{00} = \frac{1}{2}(R[\overline{g}] - (\operatorname{tr}_{\overline{g}}\overline{k})^2 + |\overline{k}|^2_{\overline{g}})$, $\mathbf{G}[\mathbf{g}]_{0j} = \overline{D}^\ell\overline{k}_{j\ell} - \partial_j\operatorname{tr}\overline{k}$ on $\Sigma_0$, where $\overline{g}$ is the induced metric, $\overline{D}$ is the induced connection and $\overline{k}$ is the second fundamental form on $\Sigma_0$ (see (3.5.10) below for the formulae for $(\overline{g},\overline{k})$).

The space of Killing vector fields with respect to $\boldsymbol{\eta}$ are spanned by $\partial_{x^\mu}$ and $x_\mu\partial_{x^\nu} - x_\nu\partial_{x^\mu}$ $(\mu,\nu = 0,1,2,3)$. For $r>0$, we define the associated charges by

$$\begin{aligned}
\mathbb{P}_\mu[\dot{\mathbf{g}};\{x^0=0,\,|x|=r\}] &= \int_{\{x^0=0,\,|x|=r\}} \star^{(\partial_{x^\mu})}\mathbb{U}[\dot{\mathbf{g}}],\\
\mathbb{J}_{\mu\nu}[\dot{\mathbf{g}};\{x^0=0,\,|x|=r\}] &= \int_{\{x^0=0,\,|x|=r\}} \star^{(x_\mu\partial_{x^\nu} - x_\nu\partial_{x^\mu})}\mathbb{U}[\dot{\mathbf{g}}],
\end{aligned} \tag{3.2.21}$$

where $\{x^0=0,\,|x|=r\}$ is oriented with $\star(\mathrm{d}x^0\wedge\mathrm{d}|x|)$. We have the following relationship between $\mathbb{P}_\mu$, $\mathbb{J}_{\mu\nu}$ and the charges for $(g,k)$.

**Lemma 3.2.7.** *For any $r>0$ and canonical coordinates $x^\mu$ on $\mathbf{\Omega}$ containing $\{x^0=0,\,|x|<r\}$,*

$$(\mathbb{P}_0,\mathbb{P}_i,\mathbb{J}_{i0},\mathbb{J}_{jk})[\dot{\mathbf{g}};\{x^0=0,\,|x|=r\}] = (\mathbf{E},\mathbf{P}_i,\mathbf{C}_i,\epsilon^i{}_{jk}\mathbf{J}_i)[(\delta+\dot{g},\dot{k});\partial B_r],$$

*where $(\dot{g},\dot{k})$ is given in terms of $\dot{\mathbf{g}}$ by (3.2.19) in the coordinates $(x^0,x^1,x^2,x^3)$.*

Indeed, these identities may be quickly verified by comparing (3.2.18) with $U = \{x^0 = 0,\,|x|<r\}$ and (3.2.20) with Lemma 3.2.6.

We conclude this discussion with some formulae concerning Poincaré transformations. Let $\Lambda^\mu{}_\nu \in SO^+(1,3)$, i.e., $\boldsymbol{\eta}_{\mu\nu}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'} = \boldsymbol{\eta}_{\mu'\nu'}$ (isometry), $\det\Lambda = 1$ (proper) and $\Lambda^0{}_0 > 0$

(orthochronous) and $\boldsymbol{\xi} \in \mathbb{R}^{1+3}$. If $\{x^\mu\}$ is a system of canonical coordinates, then so is $\{y^\mu = \Lambda^\mu{}_{\mu'} x^{\mu'} + \boldsymbol{\xi}^\mu\}$. Observe the following transformation laws for the Killing vector fields:

$$\begin{aligned}\partial_{y^\mu} &= \Lambda_\mu{}^{\mu'} \partial_{x^{\mu'}}, \\ y_\mu \partial_{y^\nu} - y_\nu \partial_{y^\mu} &= \Lambda_\mu{}^{\mu'} \Lambda_\nu{}^{\nu'} (x_{\mu'} \partial_{x^{\nu'}} - x_{\nu'} \partial_{x^{\mu'}}) + \boldsymbol{\xi}_\mu \Lambda_\nu{}^{\nu'} \partial_{x^{\nu'}} - \boldsymbol{\xi}_\nu \Lambda_\mu{}^{\mu'} \partial_{x^{\mu'}},\end{aligned} \tag{3.2.22}$$

where $\Lambda_\mu{}^\nu$ (defined via index raising and lowering using $\boldsymbol{\eta}$) is identical to $(\Lambda^{-1})^\nu{}_\mu$ in view of the isometry property. Using (3.2.18), (3.2.21), (3.2.22) and Lemma (3.2.7), as well as the linearity of ${}^{(\mathbf{X})}\mathbb{U}$ in $\mathbf{X}$, transformation properties of the charges $(\mathbf{E}, \mathbf{P}, \mathbf{C}, \mathbf{J})$ under isometries of $\boldsymbol{\eta}$ may be derived.

## 3.3 Gluing up to linear obstructions

*Proof of Theorem 3.1.3.* **Step 1.** We begin by forming the first trial for the desired initial data set. Let $\chi(x)$ be a smooth radial function which equals 0 for $|x| < 1$ and 1 for $|x| > 2$. We introduce

$$(g_{in;Q}, k_{in;Q}, g_{out;Q}, k_{out;Q}) = \begin{cases} (\mathring{g}, \mathring{k}, g_Q, k_Q) & \text{for Statement (1)}, \\ (g_Q, k_Q, \mathring{g}, \mathring{k}) & \text{for Statement (2)}. \end{cases}$$

Let $(h_{in;Q}, \pi_{in;Q})$ and $(h_{out;Q}, \pi_{out;Q})$ be defined by (3.2.1) with $(g, k)$ replaced by $(g_{in;Q}, k_{in;Q})$ and $(g_{out;Q}, k_{out;Q})$, respectively. We introduce the first guess for the glued initial data, namely,

$$(\overline{h}_Q, \overline{\pi}_Q) = (1 - \chi)(h_{in;Q}, \pi_{in;Q}) + \chi(h_{out;Q}, \pi_{out;Q}). \tag{3.3.1}$$

Of course, $(\overline{h}_Q, \overline{\pi}_Q)$ would not solve the constraint equations (3.2.6)–(3.2.7); our aim is to find a correction $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ supported in $A_1$ and a choice of $Q \in \mathcal{Q}$ such that $(h, \pi) = (\overline{h}_Q + \widetilde{h}_Q, \overline{\pi}_Q + \widetilde{\pi}_Q)$ solves (3.2.6)–(3.2.7).

A quick algebraic computation shows that we want $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ to satisfy

$$\begin{aligned}\partial_i \partial_j \widetilde{h}^{ij}_Q = F_Q &+ M^{(2)}_{\overline{h}_Q + \widetilde{h}_Q}(\overline{h}_Q + \widetilde{h}_Q, \partial^2(\overline{h}_Q + \widetilde{h}_Q)) - M^{(2)}_{\overline{h}_Q}(\overline{h}_Q, \partial^2 \overline{h}_Q) \\ &+ M^{(1)}_{\overline{h}_Q + \widetilde{h}_Q}(\partial(\overline{h}_Q + \widetilde{h}_Q), \partial(\overline{h}_Q + \widetilde{h}_Q)) - M^{(1)}_{\overline{h}_Q}(\partial \overline{h}_Q, \partial \overline{h}_Q) \\ &+ M^{(0)}_{\overline{h}_Q + \widetilde{h}_Q}(\overline{\pi}_Q + \widetilde{\pi}_Q, \overline{\pi}_Q + \widetilde{\pi}_Q) - M^{(0)}_{\overline{h}_Q}(\overline{\pi}_Q, \overline{\pi}_Q),\end{aligned} \tag{3.3.2}$$

$$\begin{aligned}\partial_i \widetilde{\pi}^{ij}_Q = G^j_Q &+ N^{(1)j}_{\overline{h}_Q + \widetilde{h}_Q}(\overline{h}_Q + \widetilde{h}_Q, \partial(\overline{\pi}_Q + \widetilde{\pi}_Q)) - N^{(1)j}_{\overline{h}_Q}(\overline{h}_Q, \partial \overline{\pi}_Q) \\ &+ N^{(0)j}_{\overline{h}_Q + \widetilde{h}_Q}(\partial(\overline{h}_Q + \widetilde{h}_Q), \overline{\pi}_Q + \widetilde{\pi}_Q) - N^{(0)j}_{\overline{h}_Q}(\partial \overline{h}_Q, \partial \overline{\pi}_Q),\end{aligned} \tag{3.3.3}$$

where $F_Q$ and $G_Q$ are the errors incurred by plugging $(\overline{h}_Q, \overline{\pi}_Q)$ into (3.2.6)–(3.2.7), i.e.,

$$\begin{aligned}F_Q &= -\partial_i \partial_j \overline{h}^{ij}_Q + M^{(2)}_{\overline{h}_Q}(\overline{h}_Q, \partial^2 \overline{h}_Q) + M^{(1)}_{\overline{h}_Q}(\partial \overline{h}_Q, \partial \overline{h}_Q) + M^{(0)}_{\overline{h}_Q}(\overline{\pi}_Q, \overline{\pi}_Q), \\ G^j_Q &= -\partial_i \overline{\pi}^{ij}_Q + N^{(1)j}_{\overline{h}_Q}(\overline{h}_Q, \partial \pi_Q) + N^{(0)j}_{h}(\partial \overline{h}_Q, \overline{\pi}_Q).\end{aligned}$$

There is, of course, considerable flexibility in specifying $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ due to the underdetermined nature of the problem, which we now fix. Let $(S, T)$ be defined as in Lemma 3.2.1 with $\Omega = A_1$. For each $Q \in \mathcal{Q}$, we look for $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ solving the fixed point problems

$$(\widetilde{h}_Q, \widetilde{\pi}_Q) = \vec{S}(\widetilde{M}_Q, \widetilde{N}_Q), \tag{3.3.4}$$

where $\widetilde{M}_Q$ and $\widetilde{N}_Q^j$ are our shorthands for the RHS of (3.3.2) and (3.3.3), respectively, and $\vec{S}(F, G) = (SF, TG)$. Next, we look for $Q \in \mathcal{Q}$ such that

$$\int \tfrac{1}{2}\widetilde{M}_Q(1, x_1, x_2, x_3)^\dagger \, \mathrm{d}x = 0, \tag{3.3.5}$$

$$\int \widetilde{N}_Q \cdot (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)^\dagger \, \mathrm{d}x = 0, \tag{3.3.6}$$

which would ensure that $\widetilde{h}_Q$, $\widetilde{\pi}_Q$ solve (3.3.2) and (3.3.3) by $(S2)$, $(T2)$.

**Step 2: Finding $(\widetilde{h}_Q, \widetilde{\pi}_Q)$.** We shall use the following standard estimates:

**Lemma 3.3.1.** *Let $u, v$ be (possibly vector-valued) Schwartz functions on $\mathbb{R}^3$.*

1. **Moser estimates.** *Let $F$ be a $C^\infty$ function with bounded derivatives. Then for $s > 0$, we have*
$$\|F(u) - F(0)\|_{H^s} \lesssim_{s,F,\|u\|_{L^\infty}} \|u\|_{H^s}. \tag{3.3.7}$$
2. **Moser difference estimates.** *Let $F$ be a $C^\infty$ function with bounded derivatives. Then for $s > 0$, we have*
$$\|F(u) - F(v)\|_{H^s} \lesssim_{s,F,\|u\|_{L^\infty},\|v\|_{L^\infty}} \|u - v\|_{H^s} + \|u - v\|_{L^\infty}\left(\|u\|_{H^s} + \|v\|_{H^s}\right). \tag{3.3.8}$$
3. **Product estimates.** *For $s_0, s_1, s_2$ such that $s_0 + s_1 + s_2 \geq \frac{3}{2}$, $s_0 + s_1 + s_2 \geq \max\{s_0, s_1, s_2\}$ with at least one of the inequalities strict, we have*
$$\|u \cdot v\|_{H^{-s_0}} \lesssim_{s_0,s_1,s_2} \|u\|_{H^{s_1}}\|v\|_{H^{s_2}}. \tag{3.3.9}$$

See, for instance, [BCD11, Sec. 2.8]; the results therein easily imply Lemma 3.3.1. In what follows, we shall apply (3.3.7) and (3.3.8) with the given $s$, and (3.3.9) with $(s_0, s_1, s_2) = (2 - s, s, s - 2)$ and $(2 - s, s - 1, s - 1)$, all of which are valid thanks to $s > \frac{3}{2}$.

For each $Q \in \mathcal{Q}$, we now find $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ satisfying (3.3.4). We first claim that,

$$\mathrm{supp}(F_Q, G_Q) \subseteq A_1, \quad \|F_Q\|_{H^{s-2}} + \|G_Q\|_{H^{s-2}} \lesssim \epsilon. \tag{3.3.10}$$

Indeed, from (3.3.1) it is clear that $(\overline{h}_Q, \overline{\pi}_Q)$ solves (3.2.6)–(3.2.7) outside $A_1$, from which the support property follows. Using the definition of localized norms, the support property and Lemma 3.3.1, the Sobolev norm bound follows[4].

[4]We note that, in fact, $F_Q$ and $G_Q$ enjoy better Sobolev regularities, but this gain is useless in view of (3.3.11), which is sharp.

Next, we claim that if $\mathrm{supp}(\widetilde{h}_Q, \widetilde{\pi}_Q) \subseteq A_1$ and $\|(\widetilde{h}_Q, \widetilde{\pi}_Q)\|_{H^s \times H^{s-1}} \leq M_c \epsilon$ for $M_c > 0$, then

$$\mathrm{supp}(\widetilde{M}_Q - F_Q, \widetilde{N}_Q - G_Q) \subseteq A_1, \quad \|\widetilde{M}_Q - F_Q\|_{H^{s-2}} + \|\widetilde{N}_Q - G_Q^j\|_{H^{s-2}} \lesssim_{M_c} \epsilon^2. \tag{3.3.11}$$

Indeed, the support property follows from that of $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ and the structure of the terms $\widetilde{M}_Q - F_Q$ and $\widetilde{N}_Q - G_Q$. The Sobolev norm bound follows from the assumptions on $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ and $(\overline{h}_Q, \overline{\pi}_Q)$, as well as Lemma 3.3.1. By the same lemma, if, in addition to the previous assumptions for $(\widetilde{h}_Q, \widetilde{\pi}_Q)$, we have $\mathrm{supp}(\widetilde{h}'_Q, \widetilde{\pi}'_Q) \subseteq A_1$ and $\|(\widetilde{h}'_Q, \widetilde{\pi}'_Q)\|_{H^s \times H^{s-1}} \leq M_c \epsilon$, then

$$\begin{aligned} &\|\widetilde{M}_q[(\widetilde{h}_Q, \widetilde{\pi}_Q)] - \widetilde{M}_Q[(\widetilde{h}'_Q, \widetilde{\pi}'_Q)]\|_{H^{s-2}} + \|\widetilde{N}_Q[(\widetilde{h}_Q, \widetilde{\pi}_Q)] - \widetilde{N}_Q[(\widetilde{h}'_Q, \widetilde{\pi}'_Q)]\|_{H^{s-2}} \\ &\lesssim \epsilon \|(\widetilde{h}_Q - \widetilde{h}'_Q, \widetilde{\pi}_Q - \widetilde{\pi}'_Q)\|_{H^s \times H^{s-1}}. \end{aligned} \tag{3.3.12}$$

Therefore, by a standard Picard iteration argument, if $0 < \epsilon < \epsilon_c$ and $\epsilon_c > 0$ is sufficiently small, then for every $Q \in \mathcal{Q}$ we may find a unique $(\widetilde{h}_Q, \widetilde{\pi}_Q) \in H^s \times H^{s-1}$. By the support-preserving property of $S$ and $T$, it follows that $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ vanish outside of $A_1$. Finally, by varying $Q$ and using a similar argument as before, one may verify that $Q \mapsto (\widetilde{h}_Q, \widetilde{\pi}_Q) \in H^s \times H^{s-1}$ is locally Lipschitz, whose details we omit.

**Step 3: Finding $Q$.** To conclude the proof of existence, it only remains to find $Q \in \mathcal{Q}$ such that (3.3.5)–(3.3.6) hold. Since $(\overline{h}_Q, \overline{\pi}_Q)$ and $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ are $\epsilon$-small on $\widetilde{A}_1$, it is sensible to isolate the linear terms $-\partial_i \partial_j \overline{h}^{ij}_Q$ and $-\partial_i \overline{\pi}^{ij}_Q$ on the RHS of (3.3.2) and (3.3.3), respectively. Introduce $\chi_{\frac{1}{2},2}$ as in (3.2.14). Observe that $\mathrm{supp}\, \chi_{\frac{1}{2},2}(r) \subseteq \widetilde{A}_1$, and $\chi_{\frac{1}{2},2} = 1$ on $A_1$, which contains the support of the RHS of (3.3.2). Hence, by Lemma 3.2.6, the LHS of (3.3.5) equals

$$\begin{aligned} &\int -\chi_{\frac{1}{2},2}(r) \tfrac{1}{2} \partial_i \partial_j \overline{h}^{ij}_Q \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \, \mathrm{d}x + \int_{\widetilde{A}_1} \chi_{\frac{1}{2},2}(r) \tfrac{1}{2} \widetilde{M}^{nonlin}_Q \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \, \mathrm{d}x \\ &= \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_{\frac{1}{2}}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_2] + \int_{\widetilde{A}_1} \chi_{\frac{1}{2},2}(r) \tfrac{1}{2} \widetilde{M}^{nonlin}_Q \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \, \mathrm{d}x, \end{aligned}$$

where $\widetilde{M}^{nonlin}_Q = (\text{RHS of } (3.3.2)) + \partial_i \partial_j \overline{h}^{ij}_Q$. To compute further the first two terms on the last line, we multiply (3.2.6) for $(h_{in}, \pi_{in})$ by $\chi_{\frac{1}{2},1}(r)$ and integrate by parts; by which we obtain

$$\begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_{\frac{1}{2}}] = \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] + \int_{\widetilde{A}_1} \chi_{\frac{1}{2},1}(r) \tfrac{1}{2} M^{nonlin}_{in;Q} \begin{pmatrix} 1 \\ \vdots \\ x_3 \end{pmatrix} \, \mathrm{d}x,$$

where $M^{nonlin}_{in;Q} =$ (RHS of (3.2.6)) with $(h,\pi) = (h_{in;Q}, \pi_{in;Q})$. Carrying out a similar computation for $(h_{out;Q}, \pi_{out;Q})$ using $\chi_{1,2}$ and $M^{nonlin}_{out;Q} =$ (RHS of (3.2.6)) with $(h,\pi) = (h_{out;Q}, \pi_{out;Q})$, then going back to the previous computation, we arrive at

$$\text{(LHS of (3.3.5))} = \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_1] \quad + \begin{pmatrix} n[Q]_{\mathbf{E}} \\ \vdots \\ n[Q]_{\mathbf{C}_3} \end{pmatrix} \tag{3.3.13}$$

where

$$\begin{pmatrix} n[Q]_{\mathbf{E}} \\ \vdots \\ n[Q]_{\mathbf{C}_3} \end{pmatrix} = \int_{\widetilde{A}_1} \tfrac{1}{2} \left( \chi_{\frac{1}{2},2}(r) \widetilde{M}^{nonlin}_{Q} + \chi_{\frac{1}{2},1}(r) M^{nonlin}_{in;Q} + \chi_{1,2}(r) M^{nonlin}_{out;Q} \right) \begin{pmatrix} 1 \\ \vdots \\ x_3 \end{pmatrix} \, \mathrm{d}x. \tag{3.3.14}$$

Similarly, working with (3.3.3), we may show that

$$\text{(LHS of (3.3.6))} = \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{J}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{J}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_1] + \begin{pmatrix} n[Q]_{\mathbf{P}_1} \\ \vdots \\ n[Q]_{\mathbf{J}_3} \end{pmatrix} \tag{3.3.15}$$

with

$$\begin{pmatrix} n[Q]_{\mathbf{P}_1} \\ \vdots \\ n[Q]_{\mathbf{J}_3} \end{pmatrix} = \int_{\widetilde{A}_1} \left( \chi_{\frac{1}{2},2}(r) \widetilde{N}^{nonlin}_{Q} + \chi_{\frac{1}{2},1}(r) N^{nonlin}_{in;Q} + \chi_{1,2}(r) N^{nonlin}_{out;Q} \right) \cdot \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{Y}_3 \end{pmatrix} \, \mathrm{d}x, \tag{3.3.16}$$

where $\widetilde{N}^{nonlin}_{Q} = \widetilde{N}^{nonlin}[(\overline{h}_Q, \overline{\pi}_Q), (\widetilde{h}_Q, \widetilde{\pi}_Q)]$ and $N^{nonlin}_{\square;Q} =$ (RHS of (3.2.7)) with $(h,\pi) = (h_{\square;Q}, \pi_{\square;Q})$ with $\square = in$ or $out$.

For the nonlinear contribution $n[Q]$, we claim that

$$|n[Q]| \lesssim \epsilon^2, \quad \|n[\cdot]\|_{Lip} \lesssim K\epsilon.$$

We begin by recalling the expressions (3.3.14) and (3.3.16). The terms that do not involve $\partial_{i'}\partial_{j'} h^{ij}_Q$ and $\partial_{i'} \pi^{ij}_Q$, where $(h_Q, \pi_Q)$ may be $(\widetilde{h}_Q, \widetilde{\pi}_Q)$, $(\overline{h}_Q, \overline{\pi}_Q)$, $(h_{in;Q}, \pi_{in;Q})$, or $(h_{out;Q}, \pi_{out;Q})$, can be handled using the $L^\infty$ and $H^1 \times L^2$ norms of $(h_Q, \pi_Q)$ (which are all $\epsilon$-small), as well as the trivial observation that $1, x^1, \ldots, \mathbf{e}_1, \ldots, \mathbf{Y}_3$ and their derivatives are uniformly bounded on $\widetilde{A}_1$. For the terms containing such a factor, observe that this factor is always linear and we may integrate one derivative by parts off of it, after which the $L^\infty$ and $H^1$ norms of $(h_Q, \pi_Q)$ again suffice. The Lipschitz bound is proved similarly, using in addition (3.1.12).

We are now ready to conclude this step. For the sake of concreteness, we focus on the case of exterior gluing from this point on; the case of interior gluing is similar. Then

$\mathbf{Q}[(g_{in;Q}, k_{in;Q}); A_1] = \mathring{Q}$, whereas $\mathbf{Q}[(g_{out;Q}, k_{out;Q}); A_1] = Q$. Hence, (3.3.13)–(3.3.15) reduce to a fixed point problem

$$Q = \mathring{Q} + n[Q]. \tag{3.3.17}$$

Moreover, we have shown that $|n[Q]| \lesssim \epsilon^2$ and $\|n[Q]\|_{Lip} \lesssim K\epsilon$. **We take $M_c$ to be the implicit constant in the first inequality** so that $|n[Q]| \leq M_c\epsilon^2$. By hypothesis, $\mathrm{dist}(\mathring{Q}, \partial\mathcal{Q}) > M_c\epsilon^2$. Hence, for $\epsilon_c$ sufficiently small (recall that $\epsilon < \epsilon_c$), the RHS of (3.3.17) is thus a contraction on $\{Q : |Q - \mathring{Q}| \leq M_c\epsilon^2\} \subseteq \mathcal{Q}$. It follows that a unique $Q \in \mathcal{Q}$ satisfying (3.3.17) and $|Q - \mathring{Q}| \leq M_c\epsilon^2$ exists by the Banach fixed point theorem.

**Step 4: Conclusion of the proof.** It remains to verify that, taking $\epsilon_c > 0$ smaller if necessary, the Lipschitz dependence and persistence of regularity properties hold. The Lipschitz dependence property is immediate from the Picard iteration (or Banach fixed point) schemes above; we omit the details. In case of persistence of regularity, we need to differentiate (3.3.4) and estimate $(\partial^\alpha \widetilde{h}_Q, \partial^\alpha \widetilde{\pi}_Q)$ in $H^s \times H^{s-1}$ for $|\alpha| \leq m$. While achieving this bound is straightforward if $\epsilon_c$ is allowed to depend on $m$, we need to show that a single choice of $\epsilon_c$ works for all $m$. We sketch the necessary argument below.

From now on, we omit the subscript $Q$ since it remains fixed. We first consider a-priori bounds under the assumption that all objects are smooth. We take $\partial^\alpha$ of (3.3.4) and rearrange the equations as

$$\begin{aligned}
&(\partial^\alpha \widetilde{h}, \partial^\alpha \widetilde{\pi}) - \vec{S}\left(\mathrm{D}_{(\widetilde{h},\widetilde{\pi})}(\widetilde{M}, \widetilde{N})[(\partial^\alpha \widetilde{h}, \partial^\alpha \widetilde{\pi})]\right) \\
&= \vec{S}[\partial^\alpha (F, G)] + [\partial^\alpha, \vec{S}](\widetilde{M}, \widetilde{N})(\widetilde{h}, \widetilde{\pi}) + (E_{\alpha,\widetilde{M}}, E_{\alpha,\widetilde{N}})[(\widetilde{h}, \widetilde{\pi})],
\end{aligned} \tag{3.3.18}$$

where $\mathrm{D}_{(\widetilde{h},\widetilde{\pi})}(\widetilde{M}, \widetilde{N})[(\dot{h}, \dot{\pi})]$ is the linearization of $\widetilde{M}$ around $(\widetilde{h}, \widetilde{\pi})$ applied to $(\dot{h}, \dot{\pi})$. (Here, $\overline{h}_{ij}(x)$, $\overline{\pi}_{ij}(x)$ are regarded as coefficients). Observe that we have put all highest order derivatives of $(\widetilde{h}, \widetilde{\pi})$ on the LHS. Proceeding as in Step 2, but also using $(S4)$ and $(T4)$, as well as Gagliardo–Nirenberg, we may show that

$$\begin{aligned}
\|\text{RHS of (3.3.18)}\|_{H^s \times H^{s-1}} &\lesssim (1+\epsilon)\|(\overline{h}, \overline{\pi})\|_{H^{s+m} \times H^{s+m-1}} \\
&\quad + \left[\|(\overline{h}, \overline{\pi})\|_{H^{s+1} \times H^s} + \|(\widetilde{h}, \widetilde{\pi})\|_{H^{s+1} \times H^s}\right] \|(\widetilde{h}, \widetilde{\pi})\|_{H^{s+m-1} \times H^{s+m-2}},
\end{aligned}$$

where $m = |\alpha|$. On the other hand, observe that the LHS of (3.3.18) defines the same linear operator for $(\partial^\alpha \widetilde{h}, \partial^\alpha \widetilde{\pi})$ independent of $\alpha$. Proceeding as in Step 2, for $\epsilon$ sufficiently small *independent of* $\alpha$, we obtain

$$\|\text{LHS of (3.3.18)}\|_{H^s \times H^{s-1}} \gtrsim \|(\partial^\alpha \widetilde{h}, \partial^\alpha \widetilde{\pi})\|_{H^s \times H^{s-1}}.$$

**We fix $\epsilon_c > 0$ so that this estimate holds.** At this point, it is straightforward to set up an induction scheme involving difference quotients to prove the persistence of regularity. □

## 3.4 Asymptotic flatness

In this section, we collect some facts concerning asymptotic flat initial data sets, which will be useful in the remainder of this chapter. As an application, we also establish Theorem 3.1.6.

**Lemma 3.4.1** (Annular restriction of asymptotically flat data)**.** *Let $(g, k)$ be an $\alpha$-asymptotically flat pair on $B^c_{R_0}$ solving (3.1.1), and let $r \in 2^{\mathbb{Z}}$. Then the following holds.*

1. *The pair $(g^{(r)}, k^{(r)})$ solves (3.1.1) on $\{|x| > r^{-1}R_0\}$ and obeys*

$$\|(g^{(r)} - \delta, k^{(r)})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq D_0 r^{-\alpha},$$

*where $s$, $D_0$ and $\alpha$ are from (3.1.13).*

2. *For $\alpha > \frac{1}{2}$, $\mathbf{E}^{ADM}$ and $\mathbf{P}^{ADM}$ are well-defined for $(g, k)$. Moreover, for $r \geq R_0$ such that $D_0 r^{-\alpha} \leq 1$, we have*

$$|\mathbf{E}[(g^{(r)}, k^{(r)}); A_1] - r^{-1}\mathbf{E}^{ADM}| + |\mathbf{P}[(g^{(r)}, k^{(r)}); A_1] - r^{-1}\mathbf{P}^{ADM}| \lesssim D_0^2 r^{-2\alpha}, \quad (3.4.1)$$

*where $\mathbf{E}^{ADM}$ and $\mathbf{P}^{ADM}$ are evaluated for $(g, k)$. For the remaining averaged charges, for $r \geq r_0 \geq R_0$ such that $D_0 r_0^{-\alpha} \leq 1$, we have*

$$|(\mathbf{C}, \mathbf{J})[(g^{(r)}, k^{(r)}); A_1]| \lesssim r^{-2}|(\mathbf{C}^{r_0}, \mathbf{J}^{r_0})| + D_0^2 r^{-2\min\{\alpha,1\}} r_0^{2\min\{0,1-\alpha\}} \log^{\delta_1(\alpha)}(\tfrac{r}{r_0}), \quad (3.4.2)$$

*where $\delta_1(\alpha) = 1$ when $\alpha = 1$ and $0$ otherwise, and $(\mathbf{C}^{r_0}, \mathbf{J}^{r_0}) := (\mathbf{C}, \mathbf{J})[(g, k); A_{r_0}]$.*

3. *Assume furthermore that $(g, k)$ also satisfies the $\alpha_-$-*parity *(or Regge–Teitelbaum)* condition,

$$\|g^-_{ij}(x)\|_{\dot{\mathcal{H}}^s(\widetilde{A}_r)} + \|\partial_i(g^-_{jk}(x))\|_{\dot{\mathcal{H}}^{s-1}(\widetilde{A}_r)} + \|k^+_{ij}(x)\|_{\dot{\mathcal{H}}^{s-1}(\widetilde{A}_r)} \leq D_0 r^{-s+\frac{3}{2}-\alpha_-} \quad (3.4.3)$$

*for all $r \in 2^{\mathbb{Z}} \cap B^c_{R_0}$. If $\alpha_- + \alpha > 2$, then $\mathbf{C}^{ADM}_i$ and $\mathbf{J}^{ADM}_i$ $(i = 1, 2, 3)$ are well-defined for $(g, k)$. Moreover, for $r \geq R_0$ such that $D_0 r^{-\alpha} \leq 1$, we have*

$$\sum_i |\mathbf{C}_i[(g^{(r)}, k^{(r)}); A_1] - r^{-2}\mathbf{C}^{ADM}_i| + \sum_i |\mathbf{J}_i[(g^{(r)}, k^{(r)}); A_1] - r^{-2}\mathbf{J}^{ADM}_i| \lesssim D_0^2 r^{-\alpha-\alpha_-}.$$

*where $\mathbf{C}^{ADM}_i$ and $\mathbf{J}^{ADM}_i$ are evaluated for $(g, k)$.*

Part (1) is trivial, Part (2) is proved using Lemma 3.2.6 and proceeding as in Step 3 in Section 3.3, and Part (3) follows from the observation that the contribution of $(h^-, \pi^+)$ in Lemma 3.2.6 for $\mathbf{C}, \mathbf{J}$ vanishes due to parity considerations. We omit the details as they are straightforward.

We also state the following quantitative facts concerning (exterior) Kerr initial data sets.

**Lemma 3.4.2.** *Let $\mathcal{E} := \{Q \in \mathbb{R}^{10} : |\mathbf{E}(Q)| > |\mathbf{P}(\mathbf{Q})|\}$. For each $Q \in \mathcal{E}$, there exists an exterior region of an initial data set for one of the Kerr spacetimes, denoted by $(g_Q^{Kerr}, k_Q^{Kerr})$, such that*

$$\mathbf{Q}^{ADM}[(g_Q^{Kerr}, k_Q^{Kerr})] = Q, \tag{3.4.4}$$

$$|x|^n|\partial^{(n)}(g_Q^{Kerr} - \delta)| + |x|^{n+1}|\partial^{(n)}k_Q^{Kerr}| \leq C_D(n,\gamma)|\mathbf{M}||x|^{-1}, \tag{3.4.5}$$

$$|x|^n|\partial^{(n)}g_Q^{Kerr,-}| + |x|^{n+1}|\partial^{(n)}k_Q^{Kerr,+}| \leq C_D(n,\gamma)|\mathbf{M}||x|^{-2}, \tag{3.4.6}$$

$$|x|^n|\partial^{(n)}\partial_{\mathbf{E},\mathbf{P}}g_Q^{Kerr}| + |x|^{n+1}|\partial^{(n)}\partial_{\mathbf{E},\mathbf{P}}k_Q^{Kerr}| \leq C_D(n,\gamma)|x|^{-1}, \tag{3.4.7}$$

$$|x|^n|\partial^{(n)}\partial_{\mathbf{E},\mathbf{P}}g_Q^{Kerr,-}| + |x|^{n+1}|\partial^{(n)}\partial_{\mathbf{E},\mathbf{P}}k_Q^{Kerr,+}| \leq C_D(n,\gamma)(|\mathbf{M}|^{-1}|(\mathbf{C},\mathbf{J})| + 1)|x|^{-2}, \tag{3.4.8}$$

$$|x|^n|\partial^{(n)}\partial_{\mathbf{C},\mathbf{J}}g_Q^{Kerr}| + |x|^{n+1}|\partial^{(n)}\partial_{\mathbf{C},\mathbf{J}}k_Q^{Kerr}| \leq C_D(n,\gamma)|x|^{-2}, \tag{3.4.9}$$

*for $|x| \geq C_R(\gamma)(|\mathbf{M}| + |\mathbf{M}|^{-1}|(\mathbf{C},\mathbf{J})|)$, where $\mathbf{M} = \operatorname{sgn}\mathbf{E}\sqrt{\mathbf{E}^2 - |\mathbf{P}|^2}$ and $\gamma = \frac{|\mathbf{E}|}{|\mathbf{M}|}$.*

An elegant construction of such a family has been given in [CD03] (via application of isometries of the background Minkowski metric to Kerr initial data sets), although the bounds (3.4.7)–(3.4.9) are not explicitly established there. We sketch the proof of Lemma 3.4.2 in Appendix A.

*Proof of Theorem 3.1.6.* Assume, without loss of generality, that $\frac{1}{2} < \alpha < 1$. Let $(\mathring{g}, \mathring{k}) = (g_{in}^{(r)}, k_{in}^{(r)})$, where $r > 0$ will be fixed at the end. By Lemma 3.4.1.(1)–(2),

$$\begin{aligned} &\|(g_{in}^{(r)} - \delta, k_{in}^{(r)})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq D_0 r^{-\alpha}, \\ &(\mathbf{E},\mathbf{P})[(g_{in}^{(r)}, k_{in}^{(r)}); A_1] = r^{-1}(\mathbf{E}^{ADM}, \mathbf{P}^{ADM})[(g_{in}, k_{in})] + \mathcal{O}(D_0^2 r^{-2\alpha}), \\ &|(\mathbf{C},\mathbf{J})[(g_{in}^{(r)}, k_{in}^{(r)}); A_1]| \lesssim D_0 r^{-2} r_0^{2-\alpha} + D_0^2 r^{-2\alpha}, \end{aligned} \tag{3.4.10}$$

where $r_0 = D_0^{\frac{1}{\alpha}}$ (so that $D_0 r_0^{-\alpha} = 1$). Define $\epsilon = D_0 r^{-\alpha}$ and $\mathcal{Q} = B_{M_c\epsilon^2}(\mathring{Q})$ with $\mathring{Q} = \mathbf{Q}[(g_{in}^{(r)}, k_{in}^{(r)}); A_1]$ so that (3.1.9) and (3.1.10) trivially hold. We shall construct a $\mathcal{Q}$-admissible family from Lemma 3.4.2 by rescaling and reparametrizing $Q$. Observe that, by (3.4.5)–(3.4.6) and Lemma 3.4.1.(1)–(3),

$$\begin{aligned} &\|(g_Q^{Kerr(r)} - \delta, k_Q^{Kerr(r)})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq C_{s,\gamma}|\mathbf{M}(Q)|r^{-1}, \\ &(\mathbf{E},\mathbf{P})[(g_Q^{Kerr(r)}, k_Q^{Kerr(r)}); A_1] = r^{-1}(\mathbf{E},\mathbf{P})(Q) + \mathcal{O}_\gamma(|\mathbf{M}(Q)|^2 r^{-2}), \\ &(\mathbf{C},\mathbf{J})[(g_Q^{Kerr(r)}, k_Q^{Kerr(r)}); A_1] = r^{-2}(\mathbf{C},\mathbf{J})(Q) + \mathcal{O}_\gamma(|\mathbf{M}(Q)|^2 r^{-3}), \end{aligned} \tag{3.4.11}$$

where $\gamma = \frac{|\mathbf{E}(Q)|}{|\mathbf{M}(Q)|}$, as long as

$$r \geq C_\gamma(|\mathbf{M}(Q)| + |\mathbf{M}(Q)|^{-1}|(\mathbf{C},\mathbf{J})(Q)|), \tag{3.4.12}$$

Similarly, by (3.4.5)–(3.4.9) and Lemma 3.4.1.(1)–(3),

$$\begin{aligned}
&\mathbf{Q}[(g_Q^{Kerr(r)}, k_Q^{Kerr(r)}); A_1] - \mathbf{Q}[(g_{Q'}^{Kerr(r)}, k_{Q'}^{Kerr(r)}); A_1] && (3.4.13)\\
&= (r^{-1}(\mathbf{E}, \mathbf{P})(Q - Q'), r^{-2}(\mathbf{C}, \mathbf{J})(Q - Q')) && (3.4.14)\\
&+ \mathcal{O}_{\gamma_{\max}}\left(M_{\max} r^{-1} |(r^{-1}(\mathbf{E}, \mathbf{P})(Q - Q'), r^{-2}(\mathbf{C}, \mathbf{J})(Q - Q'))|\right)
\end{aligned}$$

for $\gamma_{\max} \geq \max\{\gamma(Q), \gamma(Q')\}$, $M_{\max} \geq \max\{|\mathbf{M}(Q)|, |\mathbf{M}(Q')|\}$ and $r$ satisfying (3.4.12) with respect to $Q, Q'$.

Consider

$$\begin{aligned}
\mathcal{E}_1^{(r)} = \{Q \in \mathcal{E} : \tfrac{1}{2}\mathbf{E}(Q) < \mathbf{E}^{ADM}[(g_{in}, k_{in})] < 2\mathbf{E}(Q),\\
\gamma(Q) < 2\gamma^{ADM}[(g_{in}, k_{in})],\ |(\mathbf{C}, \mathbf{J})|(Q) \leq C D_0^2 r^{2-2\alpha}\},
\end{aligned}$$

where $\gamma^{ADM} = ((\mathbf{E}^{ADM})^2 - |\mathbf{P}^{ADM}|^2)^{-\frac{1}{2}} \mathbf{E}^{ADM}$ and $C$ is larger than the implicit constant in the last bound in (3.4.10). Since $\alpha > \frac{1}{2}$, there exists $R_1 \geq R_0$ such that (3.4.12) is satisfied for all $r \geq R_1$ and $Q \in \mathcal{E}_1^{(r)}$. In particular, by (3.4.13), the map $T^{(r)} : \mathcal{E}_1^{(r)} \to \mathbb{R}^{10}$, $Q \mapsto \mathbf{Q}[(g_Q^{Kerr(r)}, k_Q^{Kerr(r)}); A_1]$ is one-to-one and bi-Lipschitz if $r \geq R_1$. Moreover, in view of (3.4.10) and (3.4.11), as well as the inverse function theorem, we may ensure that $T^{(r)}(\mathcal{E}_1^{(r)})$ contains $\mathcal{Q} = B_{M_c \epsilon^2}(\mathring{Q})$ if $r$ is sufficiently large. Hence, inverting $T^{(r)}$ and composing with $(g_Q^{Kerr(r)}, k_Q^{Kerr(r)})$, we produce a $\mathcal{Q}$-admissible family on $A_1$. Finally, taking $r$ even larger if necessary, we obtain $\epsilon < \epsilon_c$ and $C_D(\lceil s \rceil, \gamma)\epsilon < \epsilon_c$, so Theorem 3.1.3 can be applied. $\square$

## 3.5 Obstruction-free gluing

In this section, we prove the obstruction-free gluing theorems stated in Section 3.1.

### Outline of the proof

Our proof consists of the following steps.

#### Construction of a single localized bump

At the heart of our proof is the following construction. Given $\theta \in (0, \frac{\pi}{2})$, $\boldsymbol{\omega} \in \mathbb{S}^2$, $b > 0$, $t > 0$, $\ell \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ such that $\overline{B_b(\xi)} \subseteq C_\theta(\boldsymbol{\omega}) \cap A_8$, we construct a smooth solution $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ to (3.1.1) on $\mathbb{R}^3$ that

(i) equals $(\delta, 0)$ outside $C_\theta(\boldsymbol{\omega}) \cap \{|x| > 8\}$, and

(ii) attains the following (averaged) charges $\Delta Q \in \mathbb{R}^{10}$ in an outer annulus (say, $A_{16}$):

$$(\mathbf{E}, \mathbf{P}_i, \mathbf{C}_i, \mathbf{J}_i)[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] = (\gamma(\ell)t^2, \gamma(\ell)t^2\ell_i, \gamma(\ell)t^2\xi_i, \gamma(\ell)t^2\epsilon_i{}^{jk}\xi_j\ell_k) + \cdots,$$

where $\gamma(\ell) = (1 - |\ell|^2)^{-\frac{1}{2}}$ and we omitted terms that are smaller than $t^2$ provided that $t \ll b \ll 1$.

Intuitively, $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ may be thought of as bump (or particle) with the mass, velocity and position $t^2$, $\ell$ and $\xi$, respectively, with characteristic scale $b$.

A key difficulty in this construction is the fact that any smooth solution $(\alpha, \beta)$ on $\mathbb{R}^3$ to the linearization of (3.1.1) around $(\delta, 0)$ has zero charges (measured on any $A_r$) due to the conservation laws, Lemma 3.2.6. Hence, (ii) must arise from the properties of the nonlinearity of (3.1.1). Another difficulty lies in ensuring the support property (i) while achieving (ii).

With these difficulties in mind, let us first consider the zero velocity (or time-symmetric) case $\ell = 0$. Let $\mathring{\alpha}_{ij}$ be a smooth symmetric 2-tensor that satisfies $\mathring{\alpha} = \mathcal{O}(t)$, $\operatorname{supp} \mathring{\alpha} \subseteq B_b(0)$ and

$$\partial^i(\mathring{\alpha}_{ij} - \delta_{ij} \operatorname{tr}_\delta \mathring{\alpha}) = 0,$$

which is stronger than $(\mathring{\alpha}, 0)$ solving the linearized constraint equation (i.e., $\partial^i\partial^j(\mathring{\alpha}_{ij} - \delta_{ij} \operatorname{tr}_\delta \mathring{\alpha}) = 0$). Suppose that $(g_{t,0,0}, k_{t,0,0})$ is a solution to (3.1.1) such that $(g_{t,0,0}, k_{t,0,0}) = (\delta + \mathring{\alpha}, 0) + \mathcal{O}(t^2)$. A key *nonlinear* computation, which is similar to that of Bartnik in [Bar86], shows that the charges of $(g_{t,0,0}, k_{t,0,0})$ are determined by an explicit expression in $\mathring{\alpha}$ up to leading order:

$$(\mathbf{E}, \mathbf{P}_i, \mathbf{C}_i, \mathbf{J}_i)[(g_{t,0,0}, k_{t,0,0}); A_{r_0}(\xi_0)] = \Big(\tfrac{1}{16} \int \sum_{j,k,\ell} (\partial_k \mathring{\alpha}_{j\ell} - \partial_\ell \mathring{\alpha}_{jk})^2 \, \mathrm{d}x, 0, 0, 0\Big) + \cdots,$$

where $\operatorname{supp} \mathring{\alpha} \subseteq B_{r_0}(\xi_0)$. By normalizing $\mathring{\alpha}$, we may ensure that $\mathbf{E}[(g_{t,0,0}, k_{t,0,0}); A_r] = t^2$.

To construct such a solution $(g_{t,0,0}, k_{t,0,0})$ with a good support property, we use the conic solution operator $\vec{S}_c$ from Chapter 2 (see Section 3.2) to set up a Picard iteration argument. By selecting the convolution kernel of $\vec{S}_c$ to be supported in $C_\theta(\boldsymbol{\omega})$, we may ensure that $\operatorname{supp}(g_{t,0,0}, k_{t,0,0}) \subseteq \cup_{x \in \operatorname{supp} \mathring{\alpha}} (C_\theta(\boldsymbol{\omega}) + x)$. Applying the spatial translation $x \mapsto x - \xi$ (with $b$ and $\xi$ as above), we obtain $(g_{t,0,\xi}, k_{t,0,\xi})$ with properties (i) and (ii).

To handle the case $\ell \neq 0$, the idea is to utilize Lorentz boosts, which now requires thinking about objects defined on spacetime (as opposed to spacelike hypersurface). For solutions to the linearization of (3.1.2) around $\boldsymbol{\eta}$, an elegant proof of the transformation properties of the charges under Lorentz boosts can be given using the Hamiltonian (3.2.16) associated with (continuous) isometries of $\boldsymbol{\eta}$ introduced by Chruściel [Chr85]. One way to extend this property to our nonlinear setting is to first consider the Cauchy development of $(g_{t,0,\xi}, k_{t,0,\xi})$ with respect to (3.1.2) using the Choquet-Bruhat theorem [Fou52], then controlling the nonlinear error terms in the proof of the transformation property (cf. [CD03, Appendix E] in the case of the ADM charges). The upside is that this procedure produces a boosted pair $(\widetilde{g}_{t,\ell,\xi}, \widetilde{k}_{t,\ell,\xi})$ attaining the desired charges (ii). The downside, however, is that the support property (i) is difficult to keep, particularly for $|\ell|$ close to 1.

To avoid this issue, we instead consider the Cauchy development $\boldsymbol{\alpha}$ of $(\mathring{\alpha}, 0)$ with respect to the *linearized* Einstein vacuum equation (around $\boldsymbol{\eta}$), obtain the boosted pair $(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$ solving the linearized constraint equation (around $(\delta, 0)$), then upgrade it using the conic solution operator $\vec{S}_c$ to a solution $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ to (3.1.1) with the desired support property (i). Next, we apply the above Lorentz boost argument in reverse for the Cauchy development of $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ with respect to (3.1.2). The key observation is that the resulting un-boosted (also un-translated by $\xi$) pair still obeys $(\widetilde{g}_{t,0,0}, \widetilde{k}_{t,0,0}) = (\delta + \mathring{\alpha}, 0) + \mathcal{O}(t^2)$, which is sufficient for the charge computations to go through.

### Multi-bump configurations with prescribed charges

The next step of the proof is to put together multiple bumps obtained in §3.5 – which may be arranged to have pairwise disjoint supports – to construct a smooth solution $(g_{\Delta Q}^{bump;\Gamma}, k_{\Delta Q}^{bump;\Gamma})$ to (3.1.1) on $\mathbb{R}^3$ that attains the (averaged) charges $\Delta Q \in \mathbb{R}^{10}$ as measured in the annulus $A_{16}$ for any $\Delta Q = (\Delta\mathbf{E}, \dots, \Delta\mathbf{J}_3)$ satisfying

$$\Delta\mathbf{E} > |\Delta\mathbf{P}|,\ \frac{\Delta\mathbf{E}}{\sqrt{(\Delta\mathbf{E})^2 - |\Delta\mathbf{P}|^2}} < 2\Gamma,\ \Delta\mathbf{E} < \epsilon_b^2,\ |\Delta\mathbf{C}| + |\Delta\mathbf{J}| < \mu_b \Delta\mathbf{E},$$

with $\epsilon_b$, $\mu_b$ sufficiently small depending on $\Gamma$. Importantly, this range is larger than (3.1.15)–(3.1.18) ($\epsilon_o, \mu_o$ will be chosen small compared to $\epsilon_b, \mu_b$).

There are many possible ways to achieve this goal. Our configuration consists of six bumps – see Figure 3.1. Two of these bumps are placed symmetrically (relative to 0) on the $x^3$ axis and carry most of $\mathbf{E}$ and all of $\mathbf{P}$ while contributing zero $\mathbf{C}$ and $\mathbf{J}$. The remaining four are placed roughly symmetrically (relative to 0) on the $x^1$ and $x^2$ axes, are almost time-symmetric (i.e., small $|\ell|$) and attain the desired values of $\mathbf{C}$ and $\mathbf{J}$ while contributing zero $\mathbf{P}$. We remark that the $\mathbf{P}$, $\mathbf{C}$ and $\mathbf{J}$ of the latter quadruple is well-approximated by the total linear momentum, center of mass and angular momentum of the system of 4 point particles in Newtonian mechanics in the same configuration.

### Extension procedures and proof of Theorem 3.1.7

Extending $(g_{out}, k_{out})$ to $\widetilde{A}_{16}$ (which is straightforward; see Lemma 3.5.9) and applying Theorem 3.1.3 (gluing up to linear obstructions), the proof of Theorem 3.1.7 is reduced to constructing an admissible family of extensions $(g_{\Delta Q}, k_{\Delta Q})$ (in the outward direction) of $(g_{in}, k_{in})$ with

$$\mathbf{Q}[(g_{\Delta Q}, k_{\Delta Q}]; A_{16}) \approx \mathbf{Q}[(g_{in}, k_{in}]; A_1) + \Delta Q \approx \mathbf{Q}[(g_{out}, k_{out}]; A_{32})$$

up to errors of order $\mathcal{O}(c\Delta\mathbf{E})$ with $c = c(\Gamma)$ small. To achieve this goal, we first extend $(g_{in} - \delta, k_{in})$ outward to $(\widetilde{g}_{in} - \delta, \widetilde{k}_{in})$ using Carlotto–Schoen-type techniques (but based on conic- and Bogovskii-type solution operators as in Chapter 2), which can be arranged to have disjoint supports from $(g_{\Delta Q}^{bump;\Gamma}, k_{\Delta Q}^{bump;\Gamma})$; see Figure 3.1. We then simply superpose $(\widetilde{g}_{in} - \delta, \widetilde{k}_{in})$ on $(g_{\Delta Q}^{bump;\Gamma}, k_{\Delta Q}^{bump;\Gamma})$ to produce the desired pair $(g_{\Delta Q}, k_{\Delta Q})$.

### Theorem 3.1.7 ⇒ Theorem 3.1.10

This is a simple rescaling argument (cf. Section 3.4).

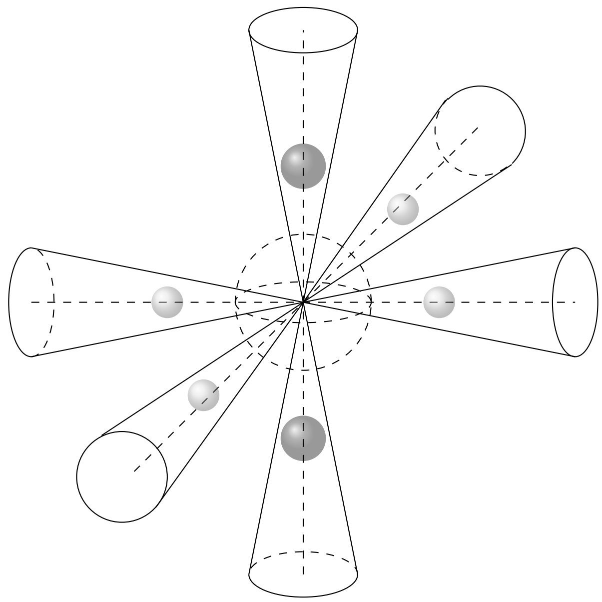

Figure 3.1: Six-bump configuration. The two darker balls represent the two bumps carrying most of $\mathbf{E}$ and all of $\mathbf{P}$, while the four lighter balls represent the four bumps that give $\mathbf{C}$ and $\mathbf{J}$. The six-bump solution is supported inside the union of the six cones and away from the inner dashed ball. The extension $(\widetilde{g}_{in} - \delta, \widetilde{k}_{in})$ of $(g_{in} - \delta, k_{in})$ will be designed to have disjoint support from the six-bump solution. The gluing annulus $A_{16}$ sits outside of all the bumps.

*Remark* 3.5.1 (Comparison with [CR22]). Apart from the apparent differences – such as null vs. spacelike gluing, spherical vs. annular data and the use of a Lorentz boost to optimize the range of $\Delta\mathbf{P}$ – perhaps the biggest difference between our approach and [CR22] is the absence of extra oscillations in the bumps, which correspond to, in the context of [CR22], the "nonlinear corrections" for "gluing" the charges. Roughly speaking, instead of separation in frequencies as in [CR22], we rely on separation in physical space to preclude dangerous interactions. Our use of conic solution operators (also Bogovskii operators for extension procedures) to ensure that the multi-bump configuration and $(\widetilde{g}_{in} - \delta, \widetilde{k}_{in})$ have disjoint supports is an instance of this idea. Placing the gluing annulus $A_{16}$ outside the support of the leading order part $(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$ of all bumps is another (at a more technical level, note the improved bounds (3.5.31), (3.5.32) in this region).

In the rest of this section, we carry out the proof of Theorems 3.1.7 and 3.1.10 outlined above.

### Construction of a single localized bump

The main goal of this subsection is to prove the following.

**Proposition 3.5.2.** *Fix any $-1 < \delta < -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2})$ and $\boldsymbol{\omega} \in \mathbb{S}^2$. For any $s > \frac{3}{2}$, $b > 0$ and $\Gamma > 1$, there exists $\epsilon_{b,0} = \epsilon_{b,0}(b, \Gamma) > 0$ such that the following holds. Given $t > 0$, $\ell \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ obeying*

$$t < \epsilon_{b,0}, \quad 1 \leq \gamma(\ell) := (1 - |\ell|^2)^{-\frac{1}{2}} < \Gamma, \quad \overline{B_b(\xi)} \subseteq C_\theta(\boldsymbol{\omega}) \cap A_8,$$

*there exists a solution $(g_{t,\ell,\xi}, k_{t,\ell,\xi}) \in C^\infty(\mathbb{R}^3)$ to (3.1.1) satisfying*

$$\operatorname{supp}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi}) \subseteq C_\theta(\boldsymbol{\omega}) \cap \{|x| > 8\}, \tag{3.5.1}$$

*and, for $n = 0, 1$,*

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}} \lesssim_{s,b,\Gamma} t, \tag{3.5.2}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{E}[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\right] = \mathcal{O}_{b,\Gamma}(t^3), \tag{3.5.3}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{P}_i[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\ell_i\right] = \mathcal{O}_{b,\Gamma}(t^3), \tag{3.5.4}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{C}_i[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\xi_i\right] = \mathcal{O}(bt^2) + \mathcal{O}_{b,\Gamma}(t^3), \tag{3.5.5}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{J}_i[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\epsilon_i^{\ jk}\xi_j\ell_k\right] = \mathcal{O}(bt^2) + \mathcal{O}_{b,\Gamma}(t^3). \tag{3.5.6}$$

*Moreover, the following improved bound holds in $\{|x| > 16\}$ (for $n = 0, 1$):*

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}(\{|x|>16\})} \lesssim_{s,b,\Gamma} t^2. \tag{3.5.7}$$

We remark that $B_b(\xi)$ contains the support of the seed bump, or more precisely the $\mathcal{O}(t)$ part of $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$; the reason behind the improvement (3.5.7) is that $\{|x| > 16\}$ is disjoint from $B_b(\xi)$.

To establish Proposition 3.5.2, we need some preliminary lemmas.

**Lemma 3.5.3** (Charge computation for special initial data)**.** *Given $r_0 \in 2^{\mathbb{N}_0}$, $\xi_0 \in \mathbb{R}^3$, $t, b_0, A > 0$ and $\tau \in (-\delta_0, \delta_0)$ for some $\delta_0 > 0$, let $(g, k) = (g(\tau), k(\tau)) \in C^2 \times C^1(B_{2r_0}(\xi_0))$ be a solution to (3.1.1) satisfying, for $n = 0, 1$,*

$$\|\partial_\tau^n(g_{ij} - \delta_{ij} - \mathring{\alpha}_{ij}, k_{ij})\|_{C^2 \times C^1(B_{2r_0}(\xi_0))} \leq At^2,$$

*where $\mathring{\alpha}_{ij} = \mathring{\alpha}(\tau)_{ij}$ obeys the special condition*

$$\partial^i(\mathring{\alpha}_{ij} - \delta_{ij} \operatorname{tr}_\delta \mathring{\alpha}) = 0, \tag{3.5.8}$$

*as well as the size and support properties (for $n = 0, 1$ and $n' = 0, 1, 2$)*

$$|\partial_\tau^n(b_0\partial_x)^{(n')}\mathring{\alpha}| \leq Ab_0^{\frac{3}{2}}t, \quad \operatorname{supp} \mathring{\alpha} \subseteq B_{b_0}(0). \tag{3.5.9}$$

*Provided that $\overline{B_{b_0}(0)} \subseteq B_{r_0}(\xi_0)$, we have, for $n = 0, 1$,*

$$\begin{aligned}
\partial_\tau^n \Big[\mathbf{E}[(g,k); A_{r_0}(\xi_0)] - \tfrac{1}{16}\int \sum_{j,k,\ell} (\partial_k \mathring{\alpha}_{j\ell} - \partial_\ell \mathring{\alpha}_{jk})^2 \, \mathrm{d}x\Big] &= \mathcal{O}_{A,b_0}(t^3),\\
\partial_\tau^n \mathbf{P}_i[(g,k); A_{r_0}(\xi_0)] &= \mathcal{O}_{A,b_0}(t^3),\\
\partial_\tau^n \mathbf{C}_i[(g,k); A_{r_0}(\xi_0)] &= \mathcal{O}(A^2 b_0 t^2) + \mathcal{O}_{A,b_0}(t^3),\\
\partial_\tau^n \mathbf{J}_i[(g,k); A_{r_0}(\xi_0)] &= \mathcal{O}_{A,b_0}(t^3).
\end{aligned}$$

*Proof.* For simplicity, we drop the $\tau$-dependence; we leave the similar proof of the $C^1$-dependence on $\tau$ to the reader. Define $(h, \pi) := ((g-\delta) - \delta \operatorname{tr}_\delta (g-\delta), k - \delta \operatorname{tr}_\delta k)$ and $(h_1, \pi_1) := (\mathring{\alpha} - \delta \operatorname{tr}_\delta \mathring{\alpha}, 0)$. Using the notation in (3.2.8), the charges are given by (where $\chi_{r_0}(r) = \int_r^\infty \eta_{r_0}(r')\,\mathrm{d}r'$ as in (3.2.14))

$$\begin{aligned}
&\begin{pmatrix}\mathbf{E}\\ \mathbf{C}_k\end{pmatrix}[(g,k); A_{r_0}(\xi_0)] = \frac{1}{2}\int \chi_{r_0}(x-\xi_0)\partial_i\partial_j h^{ij}\begin{pmatrix}1\\ x_k\end{pmatrix}\mathrm{d}x = \frac{1}{2}\int \chi_{r_0}(x-\xi_0) M(h,\pi)\begin{pmatrix}1\\ x_k\end{pmatrix}\mathrm{d}x\\
&= \frac{1}{2}\int M(h_1,\pi_1)\begin{pmatrix}1\\ x_k\end{pmatrix}\mathrm{d}x + \mathcal{O}_{A,b_0}(t^3) = -\frac{1}{2}\int R[\delta + \mathring{\alpha}]\begin{pmatrix}1\\ x_k\end{pmatrix}\mathrm{d}x + \mathcal{O}_{A,b_0}(t^3),
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix}\mathbf{P}_k\\ \mathbf{J}_k\end{pmatrix}[(g,k); A_{r_0}(\xi_0)] = \int \sum_j \chi_{r_0}(x-\xi_0)\partial_i \pi^{ij}\begin{pmatrix}\mathbf{e}_k^j\\ \mathbf{Y}_k^j\end{pmatrix}\mathrm{d}x = \int \sum_j \chi_{r_0}(x-\xi_0) N^j(h,\pi)\begin{pmatrix}\mathbf{e}_k^j\\ \mathbf{Y}_k^j\end{pmatrix}\mathrm{d}x\\
&= \int \sum_j N^j(h_1,\pi_1)\begin{pmatrix}\mathbf{e}_k^j\\ \mathbf{Y}_k^j\end{pmatrix}\mathrm{d}x + \mathcal{O}_{A,b_0}(t^3) = \mathcal{O}_{A,b_0}(t^3),
\end{aligned}$$

where we used the fact that $\partial_i\partial_j h_1^{ij} = 0$, $\pi_1 = 0$ and $\operatorname{supp}(h_1,\pi_1) \subseteq B_{r_0}(\xi_0)$.

The above already implies the desired assertions for $\mathbf{P}_k$ and $\mathbf{J}_k$. For $\mathbf{E}$ and $\mathbf{C}_k$, recall the general formulae

$$R[g] = (g^{-1})^{ij}(\partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{i\ell}^k \Gamma_{jk}^\ell), \quad \Gamma_{ij}^k = \frac{1}{2}(g^{-1})^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Let $(g_1)_{ij} := \delta_{ij} + \mathring{\alpha}_{ij}$ and write $(g_1^{-1})^{ij} = \delta^{ij} + \widetilde{\alpha}^{ij}$, where we observe that $\widetilde{\alpha}^{ij} = -\mathring{\alpha}_{ij} + \mathcal{O}_{A,b_0}(t^2)$. We will abbreviate terms that are linear in $\mathring{\alpha}$ by $lin$ – note that they will all vanish since $(\mathring{\alpha}, 0)$ is chosen to solve the linearized constraint equation. We have

$$\begin{aligned}
R[g_1] &= \frac{1}{2}(g_1^{-1})^{ij}\partial_k \widetilde{\alpha}^{k\ell}(\partial_i \mathring{\alpha}_{j\ell} + \partial_j \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ij}) + \frac{1}{2}(g_1^{-1})^{ij}(g_1^{-1})^{k\ell}\partial_k(\partial_i \mathring{\alpha}_{j\ell} + \partial_j \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ij})\\
&\quad - \frac{1}{2}(g_1^{-1})^{ij}\partial_j \widetilde{\alpha}^{k\ell}\partial_i \mathring{\alpha}_{k\ell} - \frac{1}{2}(g_1^{-1})^{ij}(g_1^{-1})^{k\ell}\partial_j\partial_i \mathring{\alpha}_{k\ell} + (g_1^{-1})^{ij}(\Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{i\ell}^k \Gamma_{jk}^\ell)\\
&= \frac{1}{2}\partial_k(\widetilde{\alpha}^{k\ell}(2\partial_i \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ii})) + \frac{1}{2}\widetilde{\alpha}^{ij}\partial_k(2\partial_i \mathring{\alpha}_{jk} - \partial_k \mathring{\alpha}_{ij}) - \frac{1}{2}\partial_i(\widetilde{\alpha}^{k\ell}\partial_i \mathring{\alpha}_{k\ell}) - \frac{1}{2}\widetilde{\alpha}^{ij}\partial_i\partial_j \mathring{\alpha}_{kk}\\
&\quad + \frac{1}{4}\partial_\ell \mathring{\alpha}_{kk}(2\partial_i \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ii}) - \frac{1}{4}(\partial_i \mathring{\alpha}_{k\ell})^2 + \frac{1}{4}(\partial_k \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ik})^2 + \mathrm{l}in + \mathcal{O}_{A,b_0}(t^3).
\end{aligned}$$

To simplify the nonlinear terms, we use the special condition (3.5.8), which implies

$$\partial^i \mathring{\alpha}_{ij} = \partial_j \operatorname{tr}_\delta \mathring{\alpha}, \quad \partial^i \partial^j \mathring{\alpha}_{ij} = \Delta \operatorname{tr}_\delta \mathring{\alpha}.$$

Recall also that $lin = \mathrm{D}_\delta R[\mathring{\alpha}] = 0$, $g_1 = \delta + \mathring{\alpha}$ and $\widetilde{\alpha}^{ij} = -\alpha_{ij} + \mathcal{O}_{A,b_0}(t^2)$. Thus

$$\begin{aligned} R[\delta + \mathring{\alpha}] &= \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{2}\right) (\partial_j \operatorname{tr}_\delta \mathring{\alpha})^2 + \left(-\frac{1}{2} - \frac{1}{4} + \frac{1}{2}\right) (\partial_k \mathring{\alpha}_{ij})^2 + \partial(\mathring{\alpha}, \partial\mathring{\alpha}) + \mathcal{O}_{A,b_0}(t^3) \\ &= \frac{1}{4}(\partial_j \operatorname{tr}_\delta \mathring{\alpha})^2 - \frac{1}{4}(\partial_k \mathring{\alpha}_{ij})^2 + \partial(\mathring{\alpha}, \partial\mathring{\alpha}) + \mathcal{O}_{A,b_0}(t^3) \\ &= -\frac{1}{8}(\partial_k \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ik})^2 + \partial(\mathring{\alpha}, \partial\mathring{\alpha}) + \mathcal{O}_{A,b_0}(t^3). \end{aligned}$$

Plugging in this formula into the above formulae for $\mathbf{E}$ and $\mathbf{C}_k$, and using the fact that $|x_k| \leq b_0$ in $\operatorname{supp} \mathring{\alpha} \subseteq B_{b_0}(0)$, the desired assertions for $\mathbf{E}$ and $\mathbf{C}_k$ follow. □

To state the next lemma, consider a Lorentzian metric $\mathbf{g} \in C^2(\mathbf{\Omega})$, where $\mathbf{\Omega} \subseteq \mathbb{R}^{1+3}$. Given $(\Lambda, \boldsymbol{\xi}) \in SO^+(1,3) \times \mathbb{R}^{1+3}$, let $y^\mu = \Lambda^\mu{}_{\mu'} x^{\mu'} + \boldsymbol{\xi}^\nu$. Then, with respect to the coordinates $(y^0, \ldots, y^3)$, define

$$g_{ij}^{(\Lambda,\boldsymbol{\xi})} = \mathbf{g}_{ij}\Big|_{\{y^0=0\}\cap\mathbf{\Omega}}, \quad k_{ij}^{(\Lambda,\boldsymbol{\xi})} = \frac{-(\mathbf{g}^{-1})^{0\mu}}{2\sqrt{-(\mathbf{g}^{-1})^{00}}} (\partial_\mu \mathbf{g}_{ij} - \partial_i \mathbf{g}_{j\mu} - \partial_j \mathbf{g}_{i\mu})\Big|_{\{y^0=0\}\cap\mathbf{\Omega}}. \tag{3.5.10}$$

Motivated by Lemma 3.2.7, we introduce the notation

$$(\mathbb{P}_0, \mathbb{P}_i, \mathbb{J}_{i0}, \mathbb{J}_{jk})[(g,k); A_{r_0}(\xi_0)] = (\mathbf{E}, \mathbf{P}_i, \mathbf{C}_i, \epsilon^i{}_{jk} \mathbf{J}_i)[(g,k); A_{r_0}(\xi_0)]. \tag{3.5.11}$$

**Lemma 3.5.4** (Effect of boost and translation)**.** *Given $t, A > 0$ and $\tau \in (-\delta_0, \delta_0)$ for some $\delta_0 > 0$, let $\mathbf{g} = \mathbf{g}(\tau) \in C^2(\mathbf{\Omega})$ be a solution to (3.1.2) satisfying, for $n = 0, 1$,*

$$\|\partial_\tau^n (\mathbf{g}_{\mu\nu} - \boldsymbol{\eta}_{\mu\nu} - \boldsymbol{\alpha}_{\mu\nu})\|_{C^2(\mathbf{\Omega})} \leq At^2,$$

*where $\boldsymbol{\alpha}_{\mu\nu} = \boldsymbol{\alpha}(\tau)_{\mu\nu}$ obeys*

$$\mathrm{D}_{\boldsymbol{\eta}} \mathbf{G}[\boldsymbol{\alpha}] = 0, \quad \|\partial_\tau \boldsymbol{\alpha}\|_{C^2(\mathbf{\Omega})} \leq At.$$

1. *For any $C^1$-curve $(\Lambda(\tau), \boldsymbol{\xi}(\tau)) \in SO^+(1,3) \times \mathbb{R}^{1+3}$ with $|\partial_\tau(\Lambda, \boldsymbol{\xi})| \leq A$, we have, for $n = 0, 1$,*
$$\|\partial_\tau^n (g_{ij}^{(\Lambda,\boldsymbol{\xi})} - \delta_{ij} - \alpha^{(\Lambda,\boldsymbol{\xi})}, k_{ij}^{(\Lambda,\boldsymbol{\xi})} - \beta^{(\Lambda,\boldsymbol{\xi})})_{ij})\|_{C^2(\mathbf{\Omega})} = \mathcal{O}_A(t^2),$$
*where*
$$\alpha_{ij}^{(\Lambda,\boldsymbol{\xi})} = \boldsymbol{\alpha}_{ij}\Big|_{\{y^0=0\}}, \quad \beta_{ij}^{(\Lambda,\boldsymbol{\xi})} = \frac{1}{2}(\partial_0 \boldsymbol{\alpha}_{ij} - \partial_i \boldsymbol{\alpha}_{j0} - \partial_j \boldsymbol{\alpha}_{i0})\Big|_{\{y^0=0\}}$$
*with respect to $y^\mu = \Lambda^\mu{}_\nu x^\nu + \boldsymbol{\xi}^\mu$.*

2. *Let $(g,k) = (g^{(I,0)}, k^{(I,0)})$. Assume that, for each $\sigma \in [0,1]$, there exists hypersurface $U_\sigma \subseteq \mathbf{\Omega}$ such that $\partial U_\sigma = \{x^0 = 0, |x| = (1+\sigma)r_0\} \cup \{y^0 = 0, |y| = (1+\sigma)r\}$, $U_\sigma \cap \operatorname{supp} \boldsymbol{\alpha} = \emptyset$, and $\mathrm{Vol}(U_\sigma) \leq A$, with the volume induced from the auxiliary Riemannian metric $(\mathrm{d}x^0)^2 + \cdots (\mathrm{d}x^3)^2$. Then for $n = 0, 1$,*

$$\partial_\tau^n \left[\mathbb{P}_\mu[(g^{(\Lambda,\boldsymbol{\xi})}, k^{(\Lambda,\boldsymbol{\xi})}); A_r] - \Lambda_\mu{}^{\mu'} \mathbb{P}_{\mu'}[(g,k); A_{r_0}]\right] = \mathcal{O}_A(t^4),$$
$$\partial_\tau^n \left[\mathbb{J}_{\mu\nu}[(g^{(\Lambda,\boldsymbol{\xi})}, k^{(\Lambda,\boldsymbol{\xi})}); A_r] - \left(\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\mathbb{J}_{\mu'\nu'} + \boldsymbol{\xi}_\mu \Lambda_\nu{}^{\nu'}\mathbb{P}_{\nu'} - \boldsymbol{\xi}_\nu \Lambda_\mu{}^{\mu'}\mathbb{P}_{\mu'}\right)[(g,k); A_{r_0}]\right] = \mathcal{O}_A(t^4).$$

*Proof.* For simplicity, we drop the $\tau$-dependence; we leave the similar proof of the $C^1$-dependence on $\tau$ to the reader. Statement (1) is a simple consequence of extracting the terms of linear order in (3.5.10) around $\boldsymbol{\eta}$. For Statement (2), observe first that $\mathbf{g} = \mathcal{O}_A(t^2)$ on $U_\sigma$ since $\overline{U_\sigma} \cap \operatorname{supp} \boldsymbol{\alpha} = \emptyset$, and hence $\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\mathbf{g}] = \mathcal{O}_A(t^4)$ on $U_\sigma$. Applying (3.2.18) to $U = U_\sigma$, we obtain

$$\int_{\{y^0=0, |y|=(1+\sigma)r\}} \star^{(X)}\mathbb{U}[\mathbf{g}] - \int_{\{x^0=0,|x|=(1+\sigma)r_0\}} \star^{(X)}\mathbb{U}[\mathbf{g}] = \mathcal{O}_A(t^4),$$

for any Killing vector field $X$ with respect to $\boldsymbol{\eta}$. In view of (3.2.22), we obtain

$$\mathbb{P}_\mu[\mathbf{g}; \{y^0 = 0,\, |y| = (1+\sigma)r\}] = \Lambda_\mu{}^{\mu'}\mathbb{P}_{\mu'}[\mathbf{g}; \{x^0 = 0,\, |x| = (1+\sigma)r_0\}] + \mathcal{O}_A(t^4).$$

Using Lemma 3.2.7 and smoothly averaging in $\sigma$, we obtain the desired assertion for $\mathbb{P}_\mu$. The case of $\mathbb{J}_{\mu\nu}$ is similar. □

**Lemma 3.5.5.** *Consider an open subset $\mathbf{\Omega} \subseteq \mathbb{R}^{1+3}$ such that $\Sigma_0 := \{x^0 = 0\} \cap \mathbf{\Omega}$ is a Cauchy hypersurface in $\mathbf{\Omega}$. Consider a symmetric covariant two-tensor field $\mathbf{F}_{\alpha\beta}$ satisfying $\nabla^\alpha \mathbf{F}_{\alpha\beta} = 0$ and a 1-form $\mathbf{J}_\beta$ defined on $\mathbf{\Omega}$, as well as a pair of symmetric covariant two-tensor fields $(\alpha_{ij}, \beta_{ij})$, a function $n$ and a 1-form $N_j$ defined on $\Sigma_0$. The following initial value problem posed on $\mathbf{\Omega}$*

$$\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha\beta} = \mathbf{F}_{\alpha\beta}, \quad \nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = \mathbf{J}_\beta,$$
$$(\boldsymbol{\alpha}_{ij}, \partial_0\boldsymbol{\alpha}_{ij})|_{\Sigma_0} = (\alpha_{ij}, 2\beta_{ij} + \partial_i N_j + \partial_j N_i), \quad \boldsymbol{\alpha}_{00}|_{\Sigma_0} = n, \quad \boldsymbol{\alpha}_{0j}|_{\Sigma_0} = N_j,$$

*is equivalent to the following reduced problem posed on $\mathbf{\Omega}$:*

$$\Box\boldsymbol{\alpha}_{\alpha\beta} = -2(\mathbf{F}_{\alpha\beta} - \tfrac{1}{2}\boldsymbol{\eta}_{\alpha\beta}\operatorname{tr}_{\boldsymbol{\eta}}\mathbf{F}) + \nabla_\alpha \mathbf{J}_\beta + \nabla_\beta \mathbf{J}_\alpha,$$
$$(\boldsymbol{\alpha}_{ij}, \partial_0\boldsymbol{\alpha}_{ij})|_{\Sigma_0} = (\alpha_{ij}, 2\beta_{ij} + \partial_i N_j + \partial_j N_i), \quad \boldsymbol{\alpha}_{00}|_{\Sigma_0} = n, \quad \boldsymbol{\alpha}_{0j}|_{\Sigma_0} = N_j,$$
$$(\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} - \mathbf{J}_\beta)|_{\Sigma_0} = 0, \quad (\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha 0} - \mathbf{F}_{\alpha 0})|_{\Sigma_0} = 0.$$

We remark that $\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = 0$ is the linearized wave coordinate condition, and that $(\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} - \mathbf{J}_\beta)|_{\Sigma_0}$ induces initial conditions for $\partial_0 \boldsymbol{\alpha}_{0\beta}|_{\Sigma_0}$. The reduced problem, being an inhomogeneous initial value problem for the classical wave equation, is well-posed (in, say, $H^s \times H^{s-1}(\{x^0 = \tau\} \cap \mathbf{\Omega})$) by our hypothesis on $\mathbf{\Omega}$.

*Proof.* This is a well-known computation essentially dating back to [Fou52]. From (3.2.15), it follows that

$$\Box\boldsymbol{\alpha}_{\alpha\beta} = -2(\mathbf{F}_{\alpha\beta} - \tfrac{1}{2}\boldsymbol{\eta}_{\alpha\beta}\operatorname{tr}_{\boldsymbol{\eta}}\mathbf{F}) + \nabla_\alpha \mathbf{J}_\beta + \nabla_\beta \mathbf{J}_\alpha$$
$$\Leftrightarrow \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha\beta} = \mathbf{F}_{\alpha\beta} + \frac{1}{2}\left(\nabla_\alpha(\nabla^\gamma \mathbf{H}[\boldsymbol{\alpha}]_{\gamma\beta} - \mathbf{J}_\beta) + \nabla_\beta(\nabla^\gamma \mathbf{H}[\boldsymbol{\alpha}]_{\gamma\alpha} - \mathbf{J}_\alpha) - \boldsymbol{\eta}_{\alpha\beta}\nabla^\delta(\nabla^\gamma \mathbf{H}[\boldsymbol{\alpha}]_{\gamma\delta} - \mathbf{J}_\delta)\right).$$

From this equivalence, the derivation of the reduced problem from the first problem is immediate. To show that the solution $\boldsymbol{\alpha}$ to the reduced problem solves the first problem, it suffices to show that $\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = \mathbf{J}_\beta$. Observe that the second equation combined with $\nabla^\alpha \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha\beta} = 0$ (linearized Bianchi) and $\nabla^\alpha \mathbf{F}_{\alpha\beta} = 0$ implies $\Box(\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} - \mathbf{J}_\beta) = 0$. Moreover, the same equation combined with $(\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha 0} - \mathbf{F}_{\alpha 0})|_{\Sigma_0} = 0$ implies $\partial_0(\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} - \mathbf{J}_\beta)|_{\Sigma_0} = 0$. Since $(\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} - \mathbf{J}_\beta)|_{\Sigma_0} = 0$ as well, it follows from uniqueness of the initial value problem for the classical wave equation that $\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = \mathbf{J}_\beta$. ◻

*Proof of Proposition 3.5.2.* **Step 1. Choice of $\mathring{\alpha}$.** We begin by choosing $\mathring{\alpha}_{ij} \in C_c^\infty(\mathbb{R}^3)$, to which we shall apply Lemma 3.5.3 in the end (in particular, for our motivation behind the choices $b_0 = \frac{1}{2}\Gamma^{-2}b$ and (3.5.12), see Steps 2 and 5 below, respectively). Fix a function $\chi \in C_c^\infty(B_1)$ such that

$$\frac{1}{16}\int |\partial(\partial_{11}^2 + \partial_{22}^2)\chi|^2 \,\mathrm{d}x = 1. \tag{3.5.12}$$

Following 2.4, we set

$$\mathring{\alpha} = t\begin{pmatrix} \frac{1}{2}(\partial_{22}^2\chi_{b,\Gamma} - \partial_{11}^2\chi_{b,\Gamma}) & -\partial_{12}^2\chi_{b,\Gamma} & 0 \\ -\partial_{12}^2\chi_{b,\Gamma} & \frac{1}{2}(\partial_{11}^2\chi_{b,\Gamma} - \partial_{22}^2\chi_{b,\Gamma}) & 0 \\ 0 & 0 & -\frac{1}{2}(\partial_{11}^2 + \partial_{22}^2)\chi_{b,\Gamma} \end{pmatrix}$$

where $\chi_{b,\Gamma}(x) = (\frac{1}{2}\Gamma^{-2}b)^{\frac{3}{2}}\chi(2\Gamma^2 b^{-1}x)$ so that $\|\partial_{ijk}^3\chi_{b,\Gamma}\|_{L^2}^2 = \|\partial_{ijk}^3\chi\|_{L^2}^2$. From the definition, it is straightforward to verify that $\partial^i\mathring{\alpha}_{ij} = \partial_j \operatorname{tr}_\delta \mathring{\alpha}$ and (3.5.9) hold with $b_0 = \frac{1}{2}\Gamma^{-2}b$.

**Step 2. Construction of $\boldsymbol{\alpha}$ and $(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$.** We use Lemma 3.5.5 to define $\boldsymbol{\alpha} \in C^\infty(\mathbb{R}^{1+3})$ to be the unique solution to the problem

$$\begin{gathered} \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha\beta} = 0, \quad \nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = 0, \\ (\boldsymbol{\alpha}_{ij}, \partial_0\boldsymbol{\alpha}_{ij})|_{\{x^0=0\}} = (\mathring{\alpha}_{ij}, 0), \quad \boldsymbol{\alpha}_{00}|_{\{x^0=0\}} = 0, \quad \boldsymbol{\alpha}_{0j}|_{\{x^0=0\}} = 0, \end{gathered} \tag{3.5.13}$$

where $\partial_0\boldsymbol{\alpha}_{00}|_{\{x^0=0\}}, \partial_0\boldsymbol{\alpha}_{0j}|_{\{x^0=0\}}$ are induced by $\nabla^\alpha \mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = 0$. Recall also from Lemma 3.5.5 that, in fact, $\Box\boldsymbol{\alpha}_{\mu\nu} = 0$ and $\operatorname{supp}(\boldsymbol{\alpha}_{\mu\nu}, \partial_t\boldsymbol{\alpha}_{\mu\nu})|_{\{x^0=0\}} \subseteq \{x^0 = 0,\, |x| < \frac{1}{2}\Gamma^{-2}b\}$ for each $\mu, \nu$. Hence, by finite speed of propagation,

$$\operatorname{supp}\boldsymbol{\alpha} \subseteq \boldsymbol{\Omega}_{\boldsymbol{\alpha}} := \{|x| < |x^0| + \tfrac{1}{2}\Gamma^{-2}b\}. \tag{3.5.14}$$

Using the parameters $\ell$ and $\xi$, define $\Lambda_\ell \in SO^+(1,3)$ (Lorentz boost by velocity $\ell$) and $\boldsymbol{\xi} \in \mathbb{R}^{1+3}$ by

$$(\Lambda_\ell x)^0 = \gamma(\ell)(x^0 - \ell\cdot P_\ell x), \quad (\Lambda_\ell x)^j = \gamma(\ell)(-\ell^j x^0 + (P_\ell x)^j) + (P_\ell^\perp x)^j, \quad \boldsymbol{\xi}^0 = 0, \quad \boldsymbol{\xi}^j = \xi^j,$$

where $\gamma(\ell) = (1-|\ell|^2)^{-\frac{1}{2}}$ (Lorentz contraction factor), and $P_\ell, P_\ell^\perp : \mathbb{R}^3 \to \mathbb{R}^3$ are orthogonal projections to $(\ell)$ and $(\ell)^\perp$, respectively. Define

$$y^\mu = (\Lambda_\ell)^\mu{}_\nu x^\nu + \boldsymbol{\xi}^\mu,$$

which constitute another set of canonical coordinates. With respect to the coordinates $(y^0, \dots, y^3)$, we define $(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$ on $\{y^0 = 0\}$ as (cf. (3.2.19))

$$(\alpha_{t,\ell,\xi})_{ij} = \boldsymbol{\alpha}_{ij}\big|_{\{y^0=0\}}, \quad (\beta_{t,\ell,\xi})_{ij} = \tfrac{1}{2}\left(\partial_0 \boldsymbol{\alpha}_{ij} - \partial_i \boldsymbol{\alpha}_{j0} - \partial_j \boldsymbol{\alpha}_{i0}\right)\big|_{\{y^0=0\}}. \tag{3.5.15}$$

Note that this pair solves the linearized constraint equation, in the sense that $(h_{t,\ell,\xi}, \pi_{t,\ell\xi}) = (\alpha_{t,\ell,\xi} - \delta \operatorname{tr}_\delta \alpha_{t,\ell,\xi}, \alpha_{t,\ell,\xi} - \delta \operatorname{tr}_\delta \beta_{t,\ell,\xi})$, as in (3.2.1), solves $\vec{P}(h_{t,\ell,\xi}, \pi_{t,\ell,\xi}) = 0$. We claim that

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^n (\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})\|_{H^s \times H^{s-1}(\mathbb{R}^3)} + \|\nabla (t\partial_t, \partial_\ell, \partial_\xi)^n \boldsymbol{\alpha}_{0\mu}\|_{H^{s-1}(\mathbb{R}^3)} \lesssim_{n,s,b,\Gamma} t, \tag{3.5.16}$$
$$\operatorname{supp}(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi}) \subseteq B_b(\xi). \tag{3.5.17}$$

The $H^s \times H^{s-1}$ bound (3.5.16) follows from the standard energy estimate applied to $\Box (t\partial_t, \partial_\ell, \partial_\xi)^n \boldsymbol{\alpha}_{\mu\nu} = 0$ in the region between the two planes $\{x^0 = 0\}$ and $\{y^0 = 0\}$. To verify (3.5.17), observe first that, since $|x^0| \leq |\ell||x|$ on $\{y^0 = 0\}$, we have

$$\{y^0 = 0\} \cap \{|x| > \tfrac{1}{2}(1-|\ell|)^{-1}\Gamma^{-2} b\} \subseteq \{|x| > |x^0| + \tfrac{1}{2}\Gamma^{-2} b\} = \mathbb{R}^{1+3} \setminus \overline{\boldsymbol{\Omega}_{\boldsymbol{\alpha}}}.$$

Note also that, on $\{y^0 = 0\}$, the coordinates $(y^1, y^2, y^3)$ see length contraction by $\gamma^{-1}$ in the direction of $\ell$ relative to $(x^1, x^2, x^3)$, i.e.,

$$(y^j - \xi^j)\big|_{\{y^0=0\}} = \gamma^{-1}(P_\ell x)^j + (P_\ell^\perp x)^j\big|_{\{y^0=0\}}. \tag{3.5.18}$$

In particular, $|x| \geq |y - \xi|$. Putting these facts together, and using $\frac{1}{2}(1-|\ell|)^{-1} \leq \gamma^2 < \Gamma^2$, we have

$$\{y^0 = 0,\, |y-\xi| \geq b\} \subseteq \{y^0 = 0,\, |x| \geq b\} \subseteq \{y^0 = 0,\, |x| > \tfrac{1}{2}(1-|\ell|)^{-1}\Gamma^{-2} b\} \subseteq \mathbb{R}^{1+3} \setminus \overline{\boldsymbol{\Omega}_{\boldsymbol{\alpha}}}.$$

In view of (3.5.14), $\boldsymbol{\alpha} = 0$ in an open neighborhood of $\{y^0 = 0,\, |y-\xi| \geq b\}$, which implies the desired support statement.

**Step 3. Construction of $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$.** We work on the hyperplane $\{y^0 = 0\}$ using the coordinates $(y^1, y^2, y^3)$. By (3.5.16), (3.5.17) and hypothesis, note that

$$\operatorname{supp}(h_{t,\ell,\xi}, \pi_{t,\ell,\xi}) \subseteq C_\theta(\boldsymbol{\omega}) \cap A_8, \quad \|(t\partial_t, \partial_\ell, \partial_\xi)(h_{t,\ell,\xi}, \pi_{t,\ell,\xi})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}} = \mathcal{O}_{s,b,\Gamma}(t),$$

where the weights do not play any role in the second assertion by the support property. Let $\vec{S}_c = (S_c, T_c)$ be the conic solution operators in Lemma 3.2.4 adapted to $C_\theta(\boldsymbol{\omega})$. By the computation in Section 3.2, in order to construct a solution to (3.1.1) of the form $(g_{t,\ell,\xi}, k_{t,\ell,\xi}) = (\delta + \alpha_{t,\ell,\xi} + \widetilde{\alpha}, \beta_{t,\ell,\xi} + \widetilde{\beta})$, it suffices to solve

$$(\widetilde{h}, \widetilde{\pi}) = \vec{S}_c \vec{N}(h_{t,\ell,\xi} + \widetilde{h}, \pi_{t,\ell,\xi} + \widetilde{\pi}),$$

where $(\widetilde{h}, \widetilde{\pi}) = (\widetilde{\alpha} - \delta \operatorname{tr}_\delta \widetilde{\alpha}, \widetilde{\beta} - \delta \operatorname{tr}_\delta \widetilde{\beta})$. By standard Picard iteration, there is a unique solution $(\widetilde{h}, \widetilde{\pi}) \in (H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}})_0(C_\theta(\boldsymbol{\omega}))$ as long as $t$ is sufficiently small depending on $s, b, \Gamma$, such that

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^n (\widetilde{h}, \widetilde{\pi})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}} \lesssim_{n,s,b,\Gamma} t^2.$$

By (3.2.2), assertions (3.5.1) and (3.5.2) immediately follow. Moreover, since $(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi}) = (\widetilde{\alpha}, \widetilde{\beta})$ outside of $\operatorname{supp}(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$, (3.5.7) follows as well.

**Step 4. Construction of $\mathbf{g}$ and identification with $\boldsymbol{\alpha}$ up to $\mathcal{O}(t^2)$.** In preparation for computation of the charges for $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ on $\{y^0 = 0\}$, we consider its Cauchy development $\mathbf{g}$. More precisely, we solve the vacuum Einstein equation $\mathbf{G}(\mathbf{g}) = 0$ in the domain $\boldsymbol{\Omega} = \{(y^0, y^1, y^2, y^3) \in \mathbb{R}^{1+3} : |y^0| < 64, |y| + 2|y^0| < 256\}$ under the wave coordinate gauge condition $\Box_{\mathbf{g}} y^\mu = 0$ with the following initial data:

$$\mathbf{g}_{00}|_{\{y^0=0\}} = -1 + \boldsymbol{\alpha}_{00}|_{\{y^0=0\}}, \quad \mathbf{g}_{0j}|_{\{y^0=0\}} = \boldsymbol{\alpha}_{0j}|_{\{y^0=0\}}, \tag{3.5.19}$$

as well as $\mathbf{g}_{ij}|_{\{y^0=0\}}$, $\partial_0 \mathbf{g}_{ij}|_{\{y^0=0\}}$, $\partial_0 \mathbf{g}_{0j}|_{\{y^0=0\}}$ and $\partial_0 \mathbf{g}_{00}|_{\{y^0=0\}}$ determined (in order) from the initial data set $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ using (3.5.10) and $\Box_{\mathbf{g}} y^\mu = 0$. Assuming that $t < \epsilon_{b,0}$ with $\epsilon_{b,0}$ sufficiently small depending on $b, \Gamma$, we may ensure the existence of $\mathbf{g}$ in $\boldsymbol{\Omega}$ by the standard local well-posedness theory [Rin09, Ch. 14–15] (see also [Fou52]). Observe also that $\partial \boldsymbol{\Omega}$ is spacelike with respect to $\boldsymbol{\eta}$, and hence with respect to $\mathbf{g}$ provided that $t$ is sufficiently small. By (3.5.16), boundedness, persistence of regularity, Cauchy stability and Sobolev embedding, we have

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^n (\mathbf{g} - \boldsymbol{\eta})\|_{C^s(\boldsymbol{\Omega})} \lesssim_{n,s,b,\Gamma} t. \tag{3.5.20}$$

To conclude this step, it remains to identify $\mathbf{g}$ with $\boldsymbol{\alpha}$ up to terms of order $t^2$. Observe, from (3.5.15) and our choice of initial data for $\mathbf{g}$, that $\widetilde{\boldsymbol{\alpha}} = \mathbf{g} - \boldsymbol{\eta} - \boldsymbol{\alpha}$ solves

$$\begin{gathered}
\mathrm{D}_{\boldsymbol{\eta}} \mathbf{G}[\widetilde{\boldsymbol{\alpha}}]_{\alpha\beta} = \mathcal{O}(t^2), \quad \nabla^\alpha \mathbf{H}[\widetilde{\boldsymbol{\alpha}}]_{\alpha\beta} = \mathcal{O}(t^2), \\
(\widetilde{\boldsymbol{\alpha}}_{ij}, \partial_0 \widetilde{\boldsymbol{\alpha}}_{ij})|_{\{y^0=0\}} = \mathcal{O}(t^2), \quad (\widetilde{\boldsymbol{\alpha}}_{00}, \widetilde{\boldsymbol{\alpha}}_{0j})|_{\{y^0=0\}} = 0, \quad (\partial_0 \widetilde{\boldsymbol{\alpha}}_{00}, \partial_0 \widetilde{\boldsymbol{\alpha}}_{0j})|_{\{y^0=0\}} = \mathcal{O}(t^2),
\end{gathered}$$

where $g = \mathcal{O}(t^2)$ is a shorthand for $\|(t\partial_t, \partial_\ell, \partial_\xi)^n g\|_{C^s(\boldsymbol{\Omega})} \lesssim_{n,s,b,\Gamma} t^2$. It follows from Lemma 3.5.5 (and Sobolev embedding) that

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^n (\mathbf{g} - \boldsymbol{\eta} - \boldsymbol{\alpha})\|_{C^s(\boldsymbol{\Omega})} \lesssim_{n,s,b,\Gamma} t^2. \tag{3.5.21}$$

**Step 5. Computation of charges.** Thanks to (3.5.21), we may now use Lemma 3.5.4 to compute the charges of $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$, which equals $(g^{(\Lambda_\ell, \boldsymbol{\xi})}, k^{(\Lambda_\ell, \boldsymbol{\xi})})$ in the notation of Lemma 3.5.4, in terms of those of $(g, k)$, which is the induced data on $\{x^0 = 0\}$. Moreover, in view of (3.5.10) with $(\Lambda, \boldsymbol{\xi}) = (I, 0)$, (3.5.13) and (3.5.21), we may appeal to Lemma 3.5.3

with $\mathring{\boldsymbol{\alpha}}$ as in Step 1 to compute the charges of $(g,k)$. By (3.5.12), we have

$$\begin{aligned}\frac{1}{16}\int\sum_{j,k,\ell}(\partial_k\mathring{\alpha}_{j\ell}-\partial_\ell\mathring{\alpha}_{jk})^2\,\mathrm{d}x &= \frac{1}{8}\int\left[\sum_{j,k,\ell}(\partial_j\mathring{\alpha}_{k,\ell})^2-\sum_j(\partial_j\operatorname{tr}_\delta\mathring{\alpha})^2\right]\mathrm{d}x\\ &= t^2\int\left[\frac{1}{4}|\partial\partial_{12}^2\chi_{b,\Gamma}|^2+\frac{1}{16}|\partial(\partial_{11}^2-\partial_{22}^2)\chi_{b,\Gamma}|^2\right]\mathrm{d}x\\ &= \frac{t^2}{16}\int|\partial(\partial_{11}^2+\partial_{22}^2)\chi|^2\,\mathrm{d}x = t^2.\end{aligned}$$

Since $\{x^0=0,\,16\le|x|\le 32\}$ and $\{y^0=0,\,16\le|y|\le 32\}$ are contained in $\boldsymbol{\Omega}$ yet disjoint from $\overline{\boldsymbol{\Omega}_{\boldsymbol{\alpha}}}$, for each $\sigma$ we may find a hypersurface $U_\sigma$ such that $\partial U_\sigma=\{x^0=0,\,|x|=16(1+\sigma)\}\cup\{y^0=0,\,16(1+\sigma)\}$, $\overline{U_\sigma}\cap\operatorname{supp}\boldsymbol{\alpha}=\emptyset$ while $\mathrm{Vol}(U_\sigma)=\mathcal{O}(1)$. By Lemma 3.5.3, we have

$$(\mathbb{P}_0,\mathbb{P}_i,\mathbb{J}_{\mu\nu})[(g,k);A_{16}]=(t^2,0,\mathcal{O}(\Gamma^{-2}bt^2))+\mathcal{O}_{b,\Gamma}(t^3),$$

and similarly after applying of $(t\partial_t,\partial_\ell,\partial_\xi)$. Applying Lemma 3.5.4, (3.5.11) and $\|\Lambda_\ell\|_{\mathbb{R}^4\to\mathbb{R}^4}\lesssim\gamma\lesssim\Gamma$, we immediately arrive at (3.5.3)–(3.5.4) for the charges of $(g_{t,\ell,\xi},k_{t,\ell,\xi})$ on $\{y^0=0\}$. Moreover, (3.5.5)–(3.5.6) follow as well by estimating $(\Lambda_\ell)_\mu{}^{\mu'}(\Lambda_\ell)_\nu{}^{\nu'}\mathcal{O}(\Gamma^{-2}bt^2)=\mathcal{O}(bt^2)$. ☐

## Multi-bump configurations with prescribed charges

Our first goal is to establish the following.

**Proposition 3.5.6** (Almost time-symmetric four-bump configuration)**.** *Fix $-1<\delta<-\frac{1}{2}$ and $\theta\in(0,\frac{\pi}{2})$, and define $C_\theta^{(4)}=C_\theta(\mathbf{e}_1)\cup C_\theta(-\mathbf{e}_1)\cup C_\theta(\mathbf{e}_2)\cup C_\theta(-\mathbf{e}_2)$. For $s>\frac{3}{2}$, there exists $\epsilon_{b,1}=\epsilon_{b,1}(s)>0$ such that the following holds. Consider*

$$\mathcal{U}=\{Q\in\mathbb{R}^{10}:0<\mathbf{E}<2\epsilon_{b,1}^2,\ |\mathbf{C}|+|\mathbf{P}|+|\mathbf{J}|<2\epsilon_{b,1}^2\mathbf{E}\}.$$

*For any $Q\in\mathcal{U}$, there exists $(g_Q^{bump},k_Q^{bump})\in C^\infty(\mathbb{R}^3)$ such that*

$$\operatorname{supp}(g_Q^{bump}-\delta,k_Q^{bump})\subseteq C_\theta^{(4)}\cap\{|x|>8\}\quad\textit{and}\quad\mathbf{Q}[(g_Q^{bump},k_Q^{bump});A_{16}]=Q.$$

*Moreover, the map $Q\mapsto(g_Q^{bump}-\delta,k_Q^{bump})$ is $C^1$ in the $H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}$ topology, and*

$$\|(g_Q^{bump}-\delta,k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}}\lesssim_s\sqrt{\mathbf{E}},\tag{3.5.22}$$

$$\|\partial_Q(g_Q^{bump},k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}}\lesssim_s\frac{1}{\sqrt{\mathbf{E}}}.\tag{3.5.23}$$

*Moreover, in $\{|x|>16\}$, we have the improved bounds*

$$\|(g_Q^{bump}-\delta,k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})}\lesssim_s\mathbf{E},\tag{3.5.24}$$

$$\|\partial_Q(g_Q^{bump},k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})}\lesssim_s 1.\tag{3.5.25}$$

*Proof.* We would like to consider a superposition of 4 bumps with disjoint supports and use the inverse function theorem around $\mathbf{E} > 0, \mathbf{C} = \mathbf{P} = \mathbf{J} = 0$. For $\overline{B_b(\xi)} \subseteq C_\theta(\mathbf{e}_1) \cap A_8$, $\overline{B_b(\eta)} \subseteq C_\theta(\mathbf{e}_2) \cap A_8$, let

$$\operatorname{supp}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi}) \subseteq C_\theta(\mathbf{e}_1), \quad \operatorname{supp}(g_{t,m,\eta} - \delta, k_{t,m,\eta}) \subseteq C_\theta(\mathbf{e}_2)$$

be the single bumps constructed in Proposition 3.5.2. We define

$$\begin{aligned}(g^*_{t,\ell^*,\xi^*}(x), k^*_{t,\ell^*,\xi^*}(x)) &:= (g_{t,\ell^*,-\xi^*}(-x), -k_{t,\ell^*,-\xi^*}(-x)) \in C^\infty(C_\theta(-\mathbf{e}_1)),\\ (g^*_{t,m^*,\eta^*}(x), k^*_{t,m^*,\eta^*}(x)) &:= (g_{t,m^*,-\eta^*}(-x), -k_{t,m^*,-\eta^*}(-x)) \in C^\infty(C_\theta(-\mathbf{e}_2))\end{aligned}$$

be the dual bumps defined by inversion symmetry. Since the solutions have disjoint supports, we define the superposition of 4 bumps

$$(g^{(4)} - \delta, k^{(4)}) := (g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi}) + (g_{t,m,\eta} - \delta, k_{t,m,\eta}) + (g^*_{t,\ell^*,\xi^*} - \delta, k^*_{t,\ell^*,\xi^*}) + (g^*_{t,m^*,\eta^*} - \delta, k^*_{t,m^*,\eta^*}), \tag{3.5.26}$$

which solves (3.1.1). Now we get a family with 25 parameters and we are interested in the charges

$$\mathbf{Q} : (t, \ell, \ell^*, \xi, \xi^*, m, m^*, \eta, \eta^*) \mapsto \mathbf{Q}[(g^{(4)}, k^{(4)}); A_{16}]. \tag{3.5.27}$$

We need to choose 10 of them with a uniform non-degenerate Jacobian. Intuitively, we will choose $t$ for the energy, $\xi$ for the central charge, $\ell$ for the momentum and three parameters among $(\ell^*, m)$ for the angular momentum.

We note at $\ell = 0$ we have $k_{t,\ell,\xi} = 0$ and thus

$$\mathbf{P}[(g_{t,0,\xi}, k_{t,0,\xi}); A_{16}] = \mathbf{J}[(g_{t,0,\xi}, k_{t,0,\xi}); A_{16}] = 0.$$

Moreover, by the inversion symmetry, for $\xi + \xi^* = 0$, $\eta + \eta^* = 0$, we have

$$\mathbf{C}[(g^{(4)}, k^{(4)}); A_{16}] = 0.$$

Now we fix $\xi^*, \eta, \eta^*, m^*$ and compute the Jacobian of the family $(t, \xi, \ell, \ell^*, m) \mapsto \mathbf{Q}$ with 13 parameters at the point $\ell = \ell^* = m = m^* = 0$ and $\xi + \xi^* = \eta + \eta^* = 0$:

$$\begin{pmatrix} t\partial_t \mathbf{E} & \partial_\xi \mathbf{E} & \partial_\ell \mathbf{E} & \partial_{\ell^*} \mathbf{E} & \partial_m \mathbf{E} \\ t\partial_t \mathbf{C} & \partial_\xi \mathbf{C} & \partial_\ell \mathbf{C} & \partial_{\ell^*} \mathbf{C} & \partial_m \mathbf{C} \\ t\partial_t \mathbf{P} & \partial_\xi \mathbf{P} & \partial_\ell \mathbf{P} & \partial_{\ell^*} \mathbf{P} & \partial_m \mathbf{P} \\ t\partial_t \mathbf{J} & \partial_\xi \mathbf{J} & \partial_\ell \mathbf{J} & \partial_{\ell^*} \mathbf{J} & \partial_m \mathbf{J} \end{pmatrix} = \begin{pmatrix} 8t^2 & 0 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & t^2 & t^2 & t^2 \\ 0 & 0 & t^2 \epsilon_i^{\ jk} \xi_j & t^2 \epsilon_i^{\ jk} \xi_j^* & t^2 \epsilon_i^{\ jk} \eta_j \end{pmatrix} + \begin{pmatrix} \mathcal{O}_b(t^3) \\ \mathcal{O}(bt^2) + \mathcal{O}_b(t^3) \\ \mathcal{O}_b(t^3) \\ \mathcal{O}(bt^2) + \mathcal{O}_b(t^3) \end{pmatrix}. \tag{3.5.28}$$

If we subtract the $\ell$ columns from the $\ell^*$ columns, and then half the $\ell^*$ column and add to the $\ell$ column, and finally subtract the $\ell$ column from the $m$ column, we would get

$$\begin{pmatrix} 8t^2 & 0 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 & 0 \\ 0 & 0 & 0 & t^2 \epsilon_i^{\ jk} \xi_j^* & t^2 \epsilon_i^{\ jk} \eta_j \end{pmatrix} + \begin{pmatrix} \mathcal{O}_b(t^3) \\ \mathcal{O}(bt^2) + \mathcal{O}_b(t^3) \\ \mathcal{O}_b(t^3) \\ \mathcal{O}(bt^2) + \mathcal{O}_b(t^3) \end{pmatrix}.$$

Let $\xi = (12, 0, 0)$ so that $\xi^* = (-12, 0, 0)$, and $\eta = (0, 12, 0)$, then the $3 \times 6$ minor in the bottom-right corner of the first matrix is

$$t^2(\epsilon_i^{\ jk}\xi_j^*, \epsilon_i^{\ jk}\eta_j) = t^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & -12 & 0 & -12 & 0 & 0 \end{pmatrix}$$

We can then select the variables $\ell_2^*, \ell_3^*, m_3$ so that $\dfrac{\partial(\mathbf{E}, \mathbf{C}, \mathbf{P}, \mathbf{J})}{\partial(t, \xi, \ell, \ell_2^*, \ell_3^*, m_3)}$ is non-degenerate once we choose $t \ll_b 1$ and $b \ll 1$.

Now we fix a sufficiently small $b > 0$ and use the inverse function theorem to conclude the proposition. We denote $\Theta = (\xi, \ell, \ell_2^*, \ell_3^*, m_3)$ and fix $\xi^* = (-12, 0, 0)$, $\eta = (0, 12, 0)$, $\eta^* = (0, -12, 0)$, $\ell_1^* = m_1 = m_2 = 0$ and $m^* = 0$, and apply the inverse function theorem over the region

$$\mathcal{U}_{\epsilon_1} := \{(1 - \epsilon_{b,1}^2)\epsilon_1^2 < \mathbf{E} < (1 + \epsilon_{b,1}^2)\epsilon_1^2,\ |\mathbf{C}| + |\mathbf{P}| + |\mathbf{J}| < 10\epsilon_{b,1}^2\epsilon_1^2\}.$$

By (3.5.28), for $\epsilon_1$ sufficiently small, the map $(\tau, \Theta) \mapsto \epsilon_1^{-2}\mathbf{Q}(\epsilon_1\tau, \Theta)$ is a diffeomorphism with uniformly non-degenerate Jacobian near the point

$$\epsilon_1^{-2}\mathbf{E} = 1, \quad \epsilon_1^{-2}\mathbf{C} = \epsilon_1^{-2}\mathbf{P} = \epsilon_1^{-2}\mathbf{J} = 0, \quad \epsilon_1 \ll_b 1, b \ll 1.$$

More precisely, there is a uniform (i.e. independent of $\epsilon_1$) neighbourhood $\mathcal{V}$ near $\tau = 1/2, \Theta = 0$ such that

$$(\tau, \Theta) \mapsto \epsilon_1^{-2}\mathbf{Q}(\epsilon_1\tau, \Theta) : \mathcal{V} \to \epsilon_1^{-2}\mathcal{U}_{\epsilon_1}$$

is a diffeomorphism with uniformly bounded inverse in $C^1$. Let $\widetilde{Q} \mapsto (\tau, \Theta)$ be the inverse map so that

$$\widetilde{Q} \mapsto (\tau, \Theta) \mapsto \epsilon_1^{-2}\mathbf{Q}(\epsilon_1\tau, \Theta) = \widetilde{Q}, \quad \widetilde{Q} \in \epsilon_1^{-2}\mathcal{U}_{\epsilon_1}.$$

Then we have

$$\mathbf{Q}[(g, k)_{(\epsilon_1\tau, \Theta)_{\widetilde{Q}}}; A_{16}] = \epsilon_1^2\widetilde{Q}, \quad \widetilde{Q} \in \epsilon_1^{-2}\mathcal{U}_{\epsilon_1}.$$

Taking $Q = \epsilon_1^2\widetilde{Q}$, we conclude the construction of a reparametrized family of $(g_Q^{bump}, k_Q^{bump})$ such that $\mathbf{Q}[(g_Q^{bump}, k_Q^{bump}); A_{16}] = Q$ for any $Q \in \mathcal{U}_{\epsilon_1}$. It also follows from the construction that $Q \mapsto (g_Q^{bump} - \delta, k_Q^{bump})$ is smooth in the $H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ topology. Note that by varying $\epsilon_1$, we can cover $\mathcal{U}$ with

$$\mathcal{U} \subset \bigcup_{0 < \epsilon_1 < 10\epsilon_{b,1}} \mathcal{U}_{\epsilon_1}.$$

By the uniqueness of the inverse function theorem, the $\epsilon_1$-family $\{g_Q : Q \in \mathcal{U}_{\epsilon_1}\}$ glues together to a smooth family $g_Q^{bump}$ for $Q \in \mathcal{U}$.

Moreover, by Proposition 3.5.2, we have for $n = 0, 1$ (recall $t = \epsilon_1 \tau \simeq \epsilon_1$),

$$\|(t\partial_t, \partial_\Theta)^{(n)}(g_Q^{bump} - \delta, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} = \mathcal{O}(t) = \mathcal{O}(\sqrt{\mathbf{E}}),$$

as well as the following chain of estimates:

$$\begin{aligned}
\|\partial_Q(g_Q^{bump}, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} &= \epsilon_1^{-2}\|\partial_{\widetilde{Q}}(g_Q^{bump}, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \\
&\lesssim \epsilon_1^{-2}\|\partial_{(\tau,\Theta)}(g_Q^{bump}, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \\
&= \mathcal{O}(\epsilon_1^{-1}) = \mathcal{O}(\sqrt{\mathbf{E}}^{-1}).
\end{aligned}$$

We are left to check the uniform $C^1$-dependence on $Q$ in $\{|x| > 16\}$. By (3.5.7), for $n = 0, 1$,

$$\|(t\partial_t, \partial_\Theta)^{(n)}(g_Q^{bump} - \delta, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})} = \mathcal{O}(t^2) = \mathcal{O}(\mathbf{E}).$$

Thus

$$\begin{aligned}
\|\partial_Q(g_Q^{bump}, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})} &= \epsilon_1^{-2}\|\partial_{\widetilde{Q}}(g_Q^{bump}, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})} \\
&\lesssim \epsilon_1^{-2}\|\partial_{(\tau,\Theta)}(g_Q^{bump}, k_Q^{bump})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})} = \mathcal{O}(1). \qquad \square
\end{aligned}$$

To extend the range of allowed $\mathbf{P}$, we adjoin to the four-bump configuration two more bumps that carry most of the total linear momentum.

**Proposition 3.5.7** (Six-bump configuration)**.** *Fix $-1 < \delta < -\frac{1}{2}$ and $\theta \in (0, \frac{\pi}{2})$, and define $C_\theta^{(6)} = C_\theta^{(4)} \cup C_\theta(\mathbf{e}_3) \cup C_\theta(-\mathbf{e}_3)$. Given $s > \frac{3}{2}$ and $\Gamma \geq 1$, there exist $\epsilon_b = \epsilon_b(s, \Gamma) > 0$ and $\mu_b = \mu_b(s, \Gamma)$ such that the following holds. Consider*

$$\mathcal{U}_\Gamma = \left\{ Q \in \mathbb{R}^{10} : \mathbf{E} > |\mathbf{P}|, \quad \tfrac{\mathbf{E}}{\sqrt{\mathbf{E}^2 - |\mathbf{P}|^2}} < 2\Gamma, \quad \mathbf{E} < \epsilon_b^2, \quad |\mathbf{C}| + |\mathbf{J}| < \mu_b \mathbf{E} \right\}.$$

*For each $Q \in \mathcal{U}_\Gamma$, there exists $(g_Q^{bump;\Gamma}, k_Q^{bump;\Gamma}) \in C^\infty(\mathbb{R}^3)$ such that*

$$\mathrm{supp}(g_Q^{bump;\Gamma} - \delta, k_Q^{bump;\Gamma}) \subseteq C_\theta^{(6)} \cap \{|x| > 8\} \quad \textit{and} \quad \mathbf{Q}[(g_Q^{bump;\Gamma}, k_Q^{bump;\Gamma}); A_{16}] = Q.$$

*Moreover,*

$$\|(g_Q^{bump;\Gamma} - \delta, k_Q^{bump;\Gamma})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim_s \sqrt{\mathbf{E}}, \tag{3.5.29}$$

$$\|\partial_Q(g_Q^{bump;\Gamma}, k_Q^{bump;\Gamma})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim_s \frac{1}{\sqrt{\mathbf{E}}}. \tag{3.5.30}$$

*Moreover, in $\{|x| > 16\}$, we have the improved bounds*

$$\|(g_Q^{bump;\Gamma} - \delta, k_Q^{bump;\Gamma})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})} \lesssim_s \mathbf{E}, \tag{3.5.31}$$

$$\|\partial_Q(g_Q^{bump;\Gamma}, k_Q^{bump;\Gamma})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\{|x|>16\})} \lesssim_s 1. \tag{3.5.32}$$

*Proof.* We first apply Proposition 3.5.2 to place two boosted bumps in $C_\theta(\mathbf{e}_3)\cup C_\theta(-\mathbf{e}_3)$ that carries most of $\mathbf{E}$ and all of $\mathbf{P}$, then use Proposition 3.5.6 to correct the remaining charges. To be more precise, let $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ be the boosted bump supported in $C_\theta(\mathbf{e}_3)\cap\{|x|>8\}$ constructed in Proposition 3.5.2 and define the dual bump by

$$(g^*_{t,\ell^*,\xi^*}(x), k^*_{t,\ell^*,\xi^*}(x)) := (g_{t,\ell^*,-\xi^*}(-x), -k_{t,\ell^*,-\xi^*}(-x)) \in C^\infty(C_\theta(-\mathbf{e}_3)).$$

Since the two bumps have disjoint supports, the superposition

$$(g^{(2)}_{t,\ell,\xi}-\delta, k^{(2)}_{t,\ell,\xi}) := (g_{t,\ell,\xi}-\delta, k_{t,\ell,\xi}) + (g^*_{t,\ell,-\xi}-\delta, k^*_{t,\ell,-\xi}) \tag{3.5.33}$$

solves (3.1.1). By inversion symmetry, we have

$$\mathbf{C}[(g^{(2)}_{t,\ell,\xi}, k^{(2)}_{t,\ell,\xi}); A_{16}] = \mathbf{J}[(g^{(2)}_{t,\ell,\xi}, k^{(2)}_{t,\ell,\xi}); A_{16}] = 0.$$

Moreover, we have

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}((\mathbf{E},\mathbf{P})[(g^{(2)}_{t,\ell,\xi}, k^{(2)}_{t,\ell,\xi}); A_{16}] - 2\gamma(\ell)t^2(1,\ell)) = \mathcal{O}_{b,\Gamma}(t^3).$$

Using inverse function theorem over the region

$$\mathcal{U}^{\mathbf{E},\mathbf{P}}_\Gamma := \left\{(\mathbf{E},\mathbf{P})\in\mathbb{R}^4 : |\mathbf{P}|<\mathbf{E}<10\epsilon_b^2, \tfrac{\mathbf{E}}{\sqrt{\mathbf{E}^2-|\mathbf{P}|^2}}<10\Gamma\right\},$$

it is easy to see we have a 4-parameter family $(g^{(2)}_{\mathbf{E},\mathbf{P}}, k^{(2)}_{\mathbf{E},\mathbf{P}})$ for $(\mathbf{E},\mathbf{P})\in\mathcal{U}^{\mathbf{E},\mathbf{P}}_\Gamma$ with

$$(\mathbf{E},\mathbf{P})[(g^{(2)}_{\mathbf{E},\mathbf{P}}, k^{(2)}_{\mathbf{E},\mathbf{P}}); A_{16}] = (\mathbf{E},\mathbf{P}), \quad (\mathbf{C},\mathbf{J})[(g^{(2)}_{\mathbf{E},\mathbf{P}}, k^{(2)}_{\mathbf{E},\mathbf{P}}); A_{16}] = 0. \tag{3.5.34}$$

Now we want to add the almost time-symmetric four-bump configuration in Proposition 3.5.6 to $(g^{(2)}, k^{(2)})$:

$$(g^{(6)}-\delta, k^{(6)}) := (g^{(2)}_{\widetilde{E},\widetilde{P}}-\delta, k^{(2)}_{\widetilde{E},\widetilde{P}}) + (g^{bump}_Q-\delta, k^{bump}_Q), \tag{3.5.35}$$

which again solves (3.1.1). The charges are given by

$$(\mathbf{E},\mathbf{P})[(g^{(6)}-\delta, k^{(6)}); A_{16}] = (\widetilde{E}+E, \widetilde{P}+P), \quad (\mathbf{C},\mathbf{J})[(g^{(6)}-\delta, k^{(6)}); A_{16}] = (C,J), \tag{3.5.36}$$

for $(\widetilde{E},\widetilde{P})\in\mathcal{U}^{\mathbf{E},\mathbf{P}}_\Gamma$ and $Q\in\mathcal{U}$. For $(\mathbf{E},\mathbf{P},\mathbf{C},\mathbf{J})\in\mathcal{U}_\Gamma$, we take

$$\widetilde{E} = \mathbf{E}\sqrt{1-(10\Gamma)^{-2}},\ E = \mathbf{E}-\widetilde{E},\ \widetilde{P} = \mathbf{P},\ P=0,\ C=\mathbf{C},\ J=\mathbf{J}. \tag{3.5.37}$$

Then it is easy to check $(\widetilde{E},\widetilde{P})\in\mathcal{U}^{\mathbf{E},\mathbf{P}}_\Gamma$ and $Q=(E,P,C,J)\in\mathcal{U}$ for $\epsilon_b>0, \mu_b>0$ small enough. The estimates follow from Proposition 3.5.2 and Proposition 3.5.6. $\square$

## Extension procedures

In this subsection, we prove extension lemmas for $(g_{in}, k_{in})$ and $(g_{out}, k_{out})$.

**Lemma 3.5.8.** *Let $(g_{in}, k_{in}) \in H^s \times H^{s-1}(A_1)$ be a pair solving (3.1.1). Fix any $-1 < \delta < -\frac{1}{2}$, $\theta > 0$ and $\boldsymbol{\omega}_0 \in \mathbb{S}^2$. Provided that $\|(g_{in} - \delta, k_{in})\|_{H^s \times H^{s-1}(A_1)}$ is sufficiently small, there exists $(\widetilde{g}_{in}, \widetilde{k}_{in})$ solving (3.1.1) on $\mathbb{R}^3 \setminus \overline{B_1}$ such that*

$$(\widetilde{g}_{in}, \widetilde{k}_{in}) = (g_{in}, k_{in}) \text{ in } A_1, \quad \operatorname{supp}(\widetilde{g}_{in} - \delta, \widetilde{k}_{in}) \subseteq B_4 \cup C_\theta(\boldsymbol{\omega}_0)$$
$$\|(\widetilde{g}_{in} - \delta, \widetilde{k}_{in})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim_{s,\delta,\theta} \|(g_{in} - \delta, k_{in})\|_{H^s \times H^{s-1}(A_1)}.$$

*Proof.* **Step 1.** The first step is to construct a solution operator $\vec{S}_{out}$ for $\vec{P}$ (i.e., $\vec{P}\vec{S}_{out} = I$) with the mapping property

$$\vec{S}_{out} : (H_{\mathrm{b}}^{s-2,\delta+2})_0(A_2 \cup C_\theta(\boldsymbol{\omega}_0)) \to (H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1})_0(A_2 \cup C_\theta(\boldsymbol{\omega}_0)).$$

We begin by constructing a Bogovskii-type operator $\vec{S}_B$ for the region $\Omega := (A_2 \cup C_\theta(\boldsymbol{\omega}_0)) \cap \{|x| < 8\}$ such that $\vec{P}\vec{S}_B f = f$ in $\Omega \setminus C_\theta(\boldsymbol{\omega}_0)$ for $f \in C_c^\infty(\Omega)$. To achieve this, we carry out an argument similar to the proof of Lemma 3.2.1 above. We begin with covering $\Omega$ by finitely many regions $\Omega_j, j = 0, 1, \cdots, J$ star-shaped with respect to small balls $B_j \subset \Omega_j$ such that $B_0 \subset C_\theta(\boldsymbol{\omega}_0)$ and the following joins of consecutive balls lie in $\Omega$:

$$ch(B_j, B_{j+1}) := \{tx + (1-t)y : x \in B_j, y \in B_{j+1}, t \in [0,1]\} \subset \Omega, \quad j = 0, 1, \cdots, J-1.$$

Let $\vec{S}_j$ be the Bogovskii-type operator on $\Omega_j$ defined in Lemma 3.2.2 and $\chi_j$ be a partition of unity with respect to $\Omega_j$. First of all, $u_j = \vec{S}_j(\chi_j f)$ solves the equation $\vec{P}u_j = \chi_j f$ outside $B_j$. Note $\vec{S}_j$ is also well-defined on the star-shaped region $ch(B_j, B_{j+1})$, we correct the errors inductively:

$$v_J := 0, \quad w_J := 0, \quad v_{j-1} := \vec{S}_{j-1}(\chi_j f - \vec{P}u_j + w_j) \in C_c^\infty(ch(B_{j-1}, B_j)),$$
$$w_{j-1} := (\chi_j f - \vec{P}u_j + w_j) - \vec{P}v_{j-1} \in C_c^\infty(B_{j-1}), \quad j = 1, 2, \cdots, J.$$

Once we define $u := \sum_{j=0}^{J}(u_j + v_j)$, it is easy to see $\vec{P}v_k = 0$ and $\vec{P}u = f$ outside $\cup B_j$. For $x \in B_k, k = 1, 2, \cdots, J$, we note $\vec{P}v_j(x) = 0$ for $j \neq k, k-1$ and thus

$$\begin{aligned}
\vec{P}u(x) &= \sum_{j\neq k} \chi_j f + \vec{P}u_k(x) + \vec{P}v_k(x) + \vec{P}v_{k-1}(x) \\
&= \sum_{j\neq k} \chi_j f + \vec{P}u_k(x) + \vec{P}v_k(x) + \chi_k f - \vec{P}u_k(x) + w_k(x) \\
&= f(x) + (\chi_{k+1} f - \vec{P}u_{k+1} + w_{k+1})(x) = f(x).
\end{aligned}$$

We conclude that $\vec{P}u = f$ outside $B_0 \subset C_\theta(\boldsymbol{\omega}_0)$.

Next, let $\vec{S}_c$ be the conic solution operator adapted to the cone $C_\theta(\boldsymbol{\omega}_0)$ so that $\vec{P}\vec{S}_c f = f$ for $f \in C_c^\infty(C_\theta(\boldsymbol{\omega}_0))$. Let $\chi \in C_c^\infty(\{|x| < 8\})$ be a cutoff function such that $\chi(x) = 1$ near $A_2$. We then define

$$\vec{S}_{out} f := \vec{S}_c(f - P\vec{S}_B(\chi f)) + \vec{S}_B(\chi f) : (H_{\mathrm{b}}^{s-2,\delta+2})_0(A_2 \cup C_\theta(\boldsymbol{\omega}_0)) \to (H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1})_0(A_2 \cup C_\theta(\boldsymbol{\omega}_0)),$$

which obeys $\vec{P}\vec{S}_{out} f = f$ for any $f \in C_c^\infty(A_2 \cup C_\theta(\boldsymbol{\omega}_0))$ as desired.

**Step 2.** Let $(h_{in}, \pi_{in})$ be the initial data corresponding to $(g_{in}, k_{in})$ under the transformation (3.2.1), and we can freely extend $(h_{in}, \pi_{in})$ to $H_0^s \times H_0^{s-1}(\{1 < |x| < 4\})$ (with control on the norm). We then consider the following fixed point problem:

$$(\widetilde{h}, \widetilde{\pi}) = \vec{S}_{out}(\vec{N}(h_{in} + \widetilde{h}, \pi_{in} + \widetilde{\pi}) - \vec{P}(h_{in}, \pi_{in})).$$

By the Banach fixed point theorem, there exists a unique solution $(\widetilde{h}, \widetilde{\pi}) \in (H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1})_0(A_2 \cup C_\theta(\boldsymbol{\omega}_0))$ such that $\|(\widetilde{h}, \widetilde{\pi})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim \|(h_{in}, \pi_{in})\|_{H^s \times H^{s-1}(A_1)}$. Define $(\widetilde{g}_{in}, \widetilde{k}_{in})$ to be the solution corresponding to $(h_{in} + \widetilde{h}, \pi_{in} + \widetilde{\pi})$. Since $(\widetilde{h}, \widetilde{\pi})$ is supported outside $A_1$, we have $(\widetilde{g}_{in}, \widetilde{k}_{in}) = (g_{in}, k_{in})$ in $A_1$. $\square$

**Lemma 3.5.9.** *Let $(g_{out}, k_{out}) \in H^s \times H^{s-1}(A_{2R})$ be a pair solving (3.1.1). If $\|(g_{out} - \delta, k_{out})\|_{H^s \times H^{s-1}(A_{2R})}$ is sufficiently small, there exists $(\widetilde{g}_{out}, \widetilde{k}_{out})$ solving (3.1.1) on $\widetilde{A}_R$ such that*

$$(\widetilde{g}_{out}, \widetilde{k}_{out}) = (g_{out}, k_{out}) \text{ in } A_{2R}, \quad \|(\widetilde{g}_{out} - \delta, \widetilde{k}_{out})\|_{\dot{\mathcal{H}}^s \times \dot{\mathcal{H}}^{s-1}(\widetilde{A}_R)} \lesssim_s \|(g_{out} - \delta, k_{out})\|_{H^s \times H^{s-1}(A_{2R})}.$$

*Proof.* We may assume $R = 1$ by rescaling. First we may freely extend $(g_{out}, k_{out})$ to $H^s(\{|x| \le 4\})$ (with control on norm). Let $\vec{S}_{in}$ be the Bogovskii operator on $B_2$ as in Lemma 3.2.2 with $\eta \in C_c^\infty(\{|x| < 1/4\})$. We consider the following fixed point problem:

$$(\widetilde{h}, \widetilde{\pi}) = \vec{S}_{in}(\vec{N}(h_{in} + \widetilde{h}, \pi_{in} + \widetilde{\pi}) - \vec{P}(h_{in}, \pi_{in})).$$

By the Banach fixed point theorem, there exists a unique solution $(\widetilde{h}, \widetilde{\pi}) \in (H^s \times H^{s-1})_0(B_2)$. Define $(\widetilde{g}_{out}, \widetilde{k}_{out})$ to be the solution corresponding to $(h_{out} + \widetilde{h}, \pi_{out} + \widetilde{\pi})$. By the support of $\eta$, $(\widetilde{g}_{out}, \widetilde{k}_{out})$ solves the (3.1.1) on $\widetilde{A}_1$. Since $\mathrm{supp}(\widetilde{h}, \widetilde{\pi}) \subseteq B_2$, we have $(\widetilde{g}_{out}, \widetilde{k}_{out}) = (g_{out}, k_{out})$ in $A_2$. $\square$

## Proof of Theorem 3.1.7

Fix $-1 < \delta < -\frac{1}{2}$ and $\theta \in (0, \frac{\pi}{8})$. Let $\mathcal{U}_\Gamma(\Delta\mathbf{E}) = \{\Delta Q' \in \mathcal{U}_\Gamma : \frac{1}{2}\Delta\mathbf{E} \le \mathbf{E}(\Delta Q') \le 2\Delta\mathbf{E}\}$. Given $\Delta Q' \in \mathcal{U}_\Gamma(\Delta\mathbf{E})$, by Proposition 3.5.7, there exists a six-bump initial data $(g_{\Delta Q'}^{bump;\Gamma} - \delta, k_{\Delta Q'}^{bump;\Gamma})$ supported in $C_\theta^{(6)} \setminus \overline{B_4}$. By Lemma 3.5.8, given any $\boldsymbol{\omega}_0 \in \mathbb{S}^2$, there also exists $(\widetilde{g}_{in} - \delta, \widetilde{k}_{in}) \in H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ extending $(g_{in} - \delta, k_{in})$ which is supported in

$B_4 \cup C_\theta(\boldsymbol{\omega}_0)$. By an appropriate choice of $\boldsymbol{\omega}_0$, we may ensure that $C_\theta^{(6)}$ and $C_\theta(\boldsymbol{\omega}_0)$ have disjoint closures, so that the same holds for $C_\theta^{(6)} \setminus \overline{B_4}$ and $C_\theta(\boldsymbol{\omega}_0) \cup B_4$. We may thus define

$$(g_{\Delta Q'} - \delta, k_{\Delta Q'}) := (\widetilde{g}_{in} - \delta, \widetilde{k}_{in}) + (g_{\Delta Q'}^{bump;\Gamma} - \delta, k_{\Delta Q'}^{bump;\Gamma}),$$

which extends $(g_{in} - \delta, k_{in})$, belongs to $H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}(\mathbb{R}^3)$ and solves (3.1.1) by the disjoint support property. Moreover, by Lemma 3.2.6, we have

$$\begin{aligned}\mathbf{Q}[(g_{\Delta Q'}, k_{\Delta Q'}); A_{16}] &= \mathbf{Q}[(g_{\Delta Q'}^{bump;\Gamma}, k_{\Delta Q'}^{bump;\Gamma}); A_{16}] + \mathbf{Q}[(\widetilde{g}_{in}, \widetilde{k}_{in}); A_{16}] \\ &= \Delta Q' + \mathbf{Q}[(g_{in}, k_{in}); A_1] + \mathcal{O}_\Gamma(\mu_o \Delta\mathbf{E}),\end{aligned}$$

while

$$\mathbf{Q}[(g_{\Delta Q'}, k_{\Delta Q'}); A_{16}] - \mathbf{Q}[(g_{\Delta Q''}, k_{\Delta Q''}); A_{16}] = \Delta Q' - \Delta Q''.$$

In particular, the map $T : \mathcal{U}_\Gamma(\Delta\mathbf{E}) \to \mathbb{R}^{10}$, $\Delta Q' \mapsto \mathbf{Q}[(g_{\Delta Q'}, k_{\Delta Q'}); A_{16}]$ is bi-Lipschitz.

Next, by Lemma 3.5.9, there exists $(\widetilde{g}_{out}, \widetilde{k}_{out})$ on $\widetilde{A}_{16\Gamma}$ extending $(g_{out}, k_{out})$. Again by Lemma 3.2.6,

$$\mathbf{Q}[(\widetilde{g}_{out}, \widetilde{k}_{out}); A_{16}] = \mathbf{Q}[(g_{out}, k_{out}); A_{32}] + \mathcal{O}_\Gamma(\mu_o \Delta\mathbf{E}).$$

By the preceding identities and the definition of $\Delta Q$, we have

$$T(\Delta Q') = \mathbf{Q}[(g_{\Delta Q'}, k_{\Delta Q'}); A_{16}] = \mathbf{Q}[(\widetilde{g}_{out}, \widetilde{k}_{out}); A_{16}] + (\Delta Q' - \Delta Q) + \mathcal{O}_\Gamma(\mu_o \Delta\mathbf{E}).$$

Choosing $\mu_o$ even smaller depending on $\mu_b$ and $\Gamma$, we may apply the inverse function theorem to ensure that the image of the bi-Lipschitz map $T(\mathcal{U}_\Gamma)$ covers a ball of radius $\mathcal{O}_\Gamma(\mu_o \Delta\mathbf{E})$ around $\mathbf{Q}[(\widetilde{g}_{out}, \widetilde{k}_{out}); A_{16}]$; here, it is important that $\mathcal{U}_\Gamma$ in Proposition 3.5.7 is larger than the subset of $\mathbb{R}^{10}$ defined by (3.1.15)–(3.1.18). By inverting this bi-Lipschitz map and composing with $(g_{\Delta Q'}, k_{\Delta Q'})$, we may produce an admissible family on $A_{16}$ verifying (3.1.10)–(3.1.12) with $\epsilon = \mathcal{O}_\Gamma(\mu_o^{\frac{1}{2}} (\Delta\mathbf{E})^{\frac{1}{2}})$ and $K = \mathcal{O}_\Gamma(1)$; that $(\widetilde{g}_{out}, \widetilde{k}_{out})$ verifies (3.1.9) is also clear. Choosing $\epsilon_o$ sufficiently small, we can apply Theorem 3.1.3 and conclude the proof. □

## Proof of Theorem 3.1.10

Given the two asymptotically flat solutions, we first rescale them using (3.1.3). Arguing as in the proof of (3.4.10) in the proof of Theorem 3.1.6, we have

$$\begin{aligned}&(\mathbf{E}, \mathbf{P})[(g_\square^{(r)}, k_\square^{(r)}); A_{R_\square}] = r^{-1}(\mathbf{E}, \mathbf{P})[(g_\square, k_\square); A_{rR_\square}] = r^{-1}(\mathbf{E}^{ADM}, \mathbf{P}^{ADM})[(g_\square, k_\square)] + \mathcal{O}(r^{-2\alpha}), \\ &(\mathbf{C}, \mathbf{J})[(g_\square^{(r)}, k_\square^{(r)}); A_{R_\square}] = r^{-2}(\mathbf{C}, \mathbf{J})[(g_\square, k_\square); A_{rR_\square}] = \mathcal{O}(r^{-2\alpha}),\end{aligned}$$

for $\square = in$ or $out$, where $R_{in} = 1$ and $R_{out} = 32$. Therefore, for sufficient large $r \geq R_0$, we verify the conditions (3.1.15)–(3.1.18). The asymptotic flatness condition (3.1.13) implies (3.1.19), since

$$\|(g_{in}^{(r)} - \delta, k_{in}^{(r)})\|_{H^s \times H^{s-1}(A_1)}^2 + \|(g_{out}^{(r)} - \delta, k_{out}^{(r)})\|_{H^s \times H^{s-1}(A_{32})}^2 \lesssim r^{-2\alpha} < \mu_o \Delta\mathbf{E}$$

for sufficiently large $r \geq R_0$. We can then apply Theorem 3.1.7 to conclude the proof. □

# Appendix A

# Proof of Lemma 3.4.2

For simplicity, we omit the superscript ${}^{Kerr}$ below. Following [CCI11], we use the Kerr-Schild coordinates $(x^0, x^1, x^2, x^3)$, with respect to which

$$(\mathbf{g}_{M,a})_{\mu\nu} = \boldsymbol{\eta}_{\mu\nu} + \frac{2Mr^3}{r^4 + a^2(x^3)^2}\mathbf{k}_\mu \mathbf{k}_\nu, \quad \mathbf{k} = \mathrm{d}x^0 + \frac{rx^1 + ax^2}{r^2 + a^2}\mathrm{d}x^1 + \frac{rx^2 - ax^1}{r^2 + a^2}\mathrm{d}x^2 + \frac{x^3}{r}\mathrm{d}x^3. \tag{A.0.1}$$

where $r$ is implicitly defined by $r^2((x^1)^2 + (x^2)^2) + (r^2 + a^2)(x^3)^2 = r^2(r^2 + a^2)$.

According to [CD03, Appendix F], there exists a map

$$\mathcal{E} \to \mathbb{R}\times SO^+(1,3)\times\mathbb{R}\times SO(3)\times\mathbb{R}^3, \quad Q = (\mathbf{E}, \mathbf{P}, \mathbf{J}, \mathbf{C}) \mapsto (M(\mathbf{E}, \mathbf{P}), \Lambda(\mathbf{E}, \mathbf{P}), a(Q), R(Q), \xi(Q))$$

such that $(g_Q, k_Q) = (g_{M,a}^{(R,\xi,\Lambda)}, k_{M,a}^{(R,\xi,\Lambda)})$, where the right-hand side is the initial data set induced from $\mathbf{g}_{M,a}$ on the hypersurface $\{y^0 = 0\}$, where $y^\mu = \Lambda^\mu{}_\nu(R^\nu{}_\lambda x^\lambda + \boldsymbol{\xi}^\nu)$ (i.e., rotate, translate, then boost). In fact,

$$M = \operatorname{sgn}\mathbf{E}\sqrt{\mathbf{E}^2 - |\mathbf{P}|^2}, \quad \Lambda = \Lambda_\ell \text{ with } \ell = \mathbf{E}^{-1}\mathbf{P}, \quad a = M^{-1}|\mathbf{J}(\Lambda_\ell^{-1}Q)|, \quad \xi = -M^{-1}\mathbf{C}(\Lambda_\ell^{-1}Q) \tag{A.0.2}$$

where $Q' = \Lambda(Q)$ is defined by the relations

$$\mathbb{P}_\mu(Q') = \Lambda_\mu{}^{\mu'}\mathbb{P}_{\mu'}(Q), \quad \mathbb{J}_{\mu\nu}(Q') = \Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\mathbb{J}_{\mu'\nu'}(Q),$$

where $(\mathbb{P}_0, \mathbb{P}_i, \mathbb{J}_{i0}, \mathbb{J}_{jk})(Q) = (\mathbf{E}, \mathbf{P}_i, \mathbf{C}_i, \epsilon^i{}_{jk}\mathbf{J}_i)(Q)$ (cf. (3.5.11), Lemma 3.5.4 and [CD03, Prop. E.1]). Moreover, $R$ is a rotation in the plane spanned by $\sum_j \mathbf{J}(\Lambda_\ell^{-1}Q)_j\partial_{x^j}$ and $\partial_{x^3}$ that maps the former to the latter. Note also that the Lorentz factor $\gamma$ of $\Lambda$ equals $(\mathbf{E}^2 - |\mathbf{P}|^2)^{-\frac{1}{2}}|\mathbf{E}|$.

In what follows, we shall freely make use of the following basic facts:

$$\gamma^{-1} \lesssim |\Lambda|_{\mathbb{R}^4\to\mathbb{R}^4} \lesssim \gamma, \quad |(a,\xi)| \simeq \gamma^{-2}M^{-1}(\mathbf{C}, \mathbf{J}), \quad ||x|^{-\alpha} - |x-\xi|^{-\alpha}| \lesssim_\alpha |x|^{-\alpha-1} \text{ for } |x| \geq 10|\xi|. \tag{A.0.3}$$

Moreover, to simplify the exposition, we now adopt the following conventions: (1) $M, a \geq 0$ (the general case is similar), (2) all bounds we state in the variables $y$ are to be satisfied for $|y| \geq C_R(M + M^{-1}|(\mathbf{C}, \mathbf{J})|)$ for some $C_R = C_R(\gamma)$, which may vary line by line, and (3) unless otherwise specified, all implicit constants may depend on $n$ and $\gamma$.

We begin with

$$\partial_M \mathbf{g}_{M,a} = \frac{2r^3}{r^4 + a^2(x^3)^2}\mathbf{k}_\mu \mathbf{k}_\nu, \quad \partial_a \mathbf{g}_{M,a} = \partial_a \left( \frac{2Mr^3}{r^4 + a^2(x^3)^2} \right) \mathbf{k}_\mu \mathbf{k}_\nu + \frac{2Mr^3}{r^4 + a^2(x^3)^2}(\partial_a \mathbf{k}_\mu \mathbf{k}_\nu + \mathbf{k}_\mu \partial_a \mathbf{k}_\nu).$$

Notice that $|\partial^{(n)}\mathbf{k}_\mu| \lesssim |x|^{-n}$, so we have $|x|^n|\partial^{(n)}\partial_M \mathbf{g}_{M,a}| \lesssim |x|^{-1}$. Moreover, notice that

$$\partial_a r = -\frac{r((x^1)^2 + (x^2)^2)}{(r^2 + a^2)^2 - a((x^1)^2 + (x^2)^2)} = \mathcal{O}(|x|^{-1}),$$

from which it follows that

$$|x|^n \left| \partial^{(n)}\partial_a \left( \frac{2Mr^3}{r^4 + a^2(x^3)^2} \right) \right| \lesssim \frac{Ma}{|x|^3} + \frac{M}{|x|^2}, \quad |x|^n \left| \partial^{(n)}\partial_a \mathbf{k}_\mu \right| \lesssim \mathcal{O}(|x|^{-1}).$$

We conclude that

$$|x^n||\partial^{(n)}\partial_a(\mathbf{g}_{M,a})| \lesssim \frac{Ma}{|x|^3} + \frac{M}{|x|^2}.$$

Recall $y^\mu = \Lambda^\mu{}_\nu(R^\nu{}_\lambda x^\lambda + \boldsymbol{\xi}^\nu)$, as well as (3.5.10). Using the preceding identities, (A.0.1) and (A.0.3), we have

$$|y|^n \left| \partial_y^{(n)} g_{M,a}^{(R,\xi,\Lambda)} \right| \lesssim M|y|^{-1}, \quad |y|^n \left| \partial_y^{(n)} \partial_M g_{M,a}^{(R,\xi,\Lambda)} \right| \lesssim |y|^{-1}, \quad |y|^n \left| \partial_y^{(n)} \partial_a g_{M,a}^{(R,\xi,\Lambda)} \right| \lesssim M|y|^{-2}.$$

We may also compute[1]

$$|y|^n \left| \partial_y^{(n)} \partial_\xi g_{M,a}^{(R,\xi,\Lambda)} \right| \lesssim M|y|^{-2}, \quad |y|^n \left| \partial_y^{(n)} \partial_\Lambda g_{M,a}^{(R,\xi,\Lambda)} \right| \lesssim M|y|^{-1}, \quad |y|^n \left| \partial_y^{(n)} \partial_R g_{M,a}^{(R,\xi,\Lambda)} \right| \lesssim Ma|y|^{-2},$$

where we used the rotation invariance of the leading term of the Kerr metric for the last bound. Moreover, for $g_{M,A}^{(R,\xi,\Lambda),-}$, the leading term is cancelled and $|y|^{-1}$ improve to $|y|^{-2}$ in the above bounds. Finally, by (3.5.10), similar bounds follow for $k_{M,a}^{(R,\xi,\Lambda)}$ and $k_{M,a}^{(R,\xi,\Lambda),+}$.

From the preceding assertions, (3.4.5)–(3.4.6) immediately follow. To establish the remaining bounds, observe also that, by (A.0.2) and the definition of $R$ (which is locally well-defined),

$$\left| \frac{\partial \Lambda}{\partial(\mathbf{E}, \mathbf{P})} \right| \lesssim_\gamma \frac{1}{M}, \left| \frac{\partial R}{\partial(\mathbf{C}, \mathbf{J})} \right| \lesssim_\gamma \frac{1}{Ma}, \left| \frac{\partial R}{\partial(\mathbf{E}, \mathbf{P})} \right| \lesssim_\gamma \frac{|(\mathbf{C}, \mathbf{J})|}{M} \frac{1}{Ma},$$
$$\left| \frac{\partial(a, \xi)}{\partial(\mathbf{C}, \mathbf{J})} \right| \lesssim_\gamma \frac{1}{M}, \left| \frac{\partial(a, \xi)}{\partial(\mathbf{E}, \mathbf{P})} \right| \lesssim_\gamma \frac{|(\mathbf{C}, \mathbf{J})|}{M^2}.$$

[1]The exact choices of the coordinates $\Lambda$ and $R$ on $SO^+(1,3)$ and $SO(3)$, respectively, are not too important. For instance, they may be regarded as local coordinates near given $\Lambda(\mathbf{E}, \mathbf{P})$ and $R(Q)$ obtained by left-translation of normal coordinates near the identity in $SO^+(1,3)$ and $SO(3)$, respectively.

For (3.4.7), we have

$$\begin{aligned}
&|y|^n|\partial^{(n)}\partial_{\mathbf{E},\mathbf{P}}g_{M,a}^{(R,\xi,\Lambda)}| \lesssim \left|\tfrac{\partial M}{\partial(\mathbf{E},\mathbf{P})}\right| |y|^n|\partial^{(n)}\partial_M g_{M,a}^{(R,\xi,\Lambda)}| + |\tfrac{\partial \Lambda}{\partial(\mathbf{E},\mathbf{P})}||y|^n|\partial^{(n)}\partial_\Lambda g_{M,a}^{(R,\xi,\Lambda)}| \\
&+ |\tfrac{\partial a}{\partial(\mathbf{E},\mathbf{P})}||y|^n|\partial^{(n)}\partial_a g_{M,a}^{(R,\xi,\Lambda)}| + |\tfrac{\partial R}{\partial(\mathbf{E},\mathbf{P})}||y|^n|\partial^{(n)}\partial_R g_{M,a}^{(R,\xi,\Lambda)}| + |\tfrac{\partial \xi}{\partial(\mathbf{E},\mathbf{P})}||y|^n|\partial^{(n)}\partial_\xi g_{M,a}^{(R,\xi,\Lambda)}| \\
&\lesssim |y|^{-1} + M^{-1}M|y|^{-1} + \tfrac{|(\mathbf{C},\mathbf{J})|}{M^2}M|y|^{-2} + \tfrac{|(\mathbf{C},\mathbf{J})|}{aM^2}aM|y|^{-2} \lesssim |y|^{-1}
\end{aligned}$$

and we have a similar bound for $k_{M,a}^{(R,\xi,\Lambda)}$. The proof of (3.4.8) is similar. For (3.4.9), we have

$$\begin{aligned}
&|y|^n|\partial^{(n)}\partial_{\mathbf{C},\mathbf{J}}g_Q^{Kerr}| \\
&\lesssim |\tfrac{\partial a}{\partial(\mathbf{C},\mathbf{J})}||y|^n|\partial^{(n)}\partial_a g_{M,a}^{(R,\xi,\Lambda)}| + |\tfrac{\partial R}{\partial(\mathbf{C},\mathbf{J})}||y|^n|\partial^{(n)}\partial_R g_{M,a}^{(R,\xi,\Lambda)}| + |\tfrac{\partial \xi}{\partial(\mathbf{C},\mathbf{J})}||y|^n|\partial^{(n)}\partial_\xi g_{M,a}^{(R,\xi,\Lambda)}| \\
&\lesssim M^{-1}M|y|^{-2} + (Ma)^{-1}Ma|y|^{-2} + M^{-1}M|y|^{-2} \lesssim |y|^{-2}
\end{aligned}$$

and we have a similar bound for $k_{M,a}^{(R,\xi,\Lambda)}$. $\square$

# Chapter 4

# Integral Formulas for Under/Overdetermined Differential Operators

## 4.1 Introduction

Underdetermined partial differential operators $\mathcal{P}$ resembling the divergence operator appear naturally in various fields of physics and geometry. Take, for instance, the Gauss law in electromagnetism, the divergence-free condition for incompressible fluids, the linearized scalar curvature operator in Riemannian geometry, the constraint equations in general relativity and gauge theories, and so on. The duals $\mathcal{P}^*$ of such underdetermined operators, which are overdetermined, also play a significant role. Examples include the gradient operator (dual to divergence), the Hessian operator (dual to linearized scalar curvature, up to lower order terms), the Killing operator (symmetric part of the covariant gradient of a vector field, dual to divergence of symmetric 2-tensor), the conformal Killing operator (trace-free symmetric part of the covariant gradient of a vector field, dual to divergence of trace-free symmetric 2-tensor), and many others.

In this chapter, we describe a new versatile method for obtaining solution operators (i.e., right-inverses up to a finite rank operator) for such underdetermined operators $\mathcal{P}$ and, by duality, representation formulas (i.e., left-inverses up to a finite rank operator) for such overdetermined operators $\mathcal{P}^*$. The method is based on a direct derivation of integral formulas (i.e., Green's functions) for these operators based on a property we dub *recovery on curves* (see (RC) below). The operators we construct are *regularizing of optimal order* (i.e., they gain $m$ derivatives, where $m$ is the order of $\mathcal{P}$) and, more interestingly, their integral kernels have *prescribed support properties.* This latter feature means that the support of the solutions can be prescribed, provided the data satisfy appropriate assumptions.

Furthermore, we demonstrate that our method is applicable (even in variable coefficient situations) as soon as a simple algebraic condition on the principal symbol $p(x, \xi)$ of the operator $\mathcal{P}$ is satisfied:

(FC) $p(x,\xi)$ is full rank for all $x \in U$ and $\xi \in \mathbb{C}^d \setminus \{0\}$.

At a glance, (FC) is a (strict) strengthening of the familiar notion of ellipticity, which is the same condition but only for real covectors $\xi \in \mathbb{R}^d \setminus \{0\}$. At a deeper level, it is a suitable variable-coefficient generalization of the condition that the *formal cokernel* of $\mathcal{P}$ on $U$,

$$\ker \mathcal{P}^*(U) := \{\mathbf{Z} \in C^\infty(\overline{U}) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } \mathcal{D}'(U)\}, \tag{4.1.1}$$

is finite dimensional (which is implied by (RC); see Theorem 4.1.2). In fact, the three conditions (FC), (RC), and $\dim \ker \mathcal{P}^* < +\infty$ are *equivalent* if each row of $p^*$ is a homogeneous vector-valued polynomial (see Theorem 4.1.11). For this reason, our new condition is dubbed the *finite-dimensional cokernel condition* (FC). All operators mentioned above (and more) satisfy (FC), as the short proof of Theorem 4.1.14 below shows (see also Appendix B).

Our results provide a fresh approach to solving a wide range of linear and nonlinear problems with operators that satisfy (FC) (and hence (RC)): we may now *design* integral operators tailored to each unique problem. We give the following applications of this strategy:

- *L*inear problems: general solvability results for operators $\mathcal{P}$ satisfying (FC) under optimal (up to endpoints) assumptions on the regularity and decay properties of the coefficients; and
- *N*onlinear problems: new sharp results concerning the flexibility of general relativistic initial data sets (on a compact or asymptotically flat background), such as localization, gluing, extension, and parametrization, with or without the constant mean curvature (gauge) condition.

As an example, see Section 4.1 below for the precise statement of our sharp solvability result on bounded Lipschitz domains for $\mathcal{P}$ with rough coefficients from [Ise+25b]. For applications to general relativistic initial data sets, we refer to [MOT25]. We expect this approach to have numerous additional applications.

We are primarily motivated by the investigation of the *flexibility* of solutions to *underdetermined* linear and nonlinear PDEs arising in physics and geometry. Our approach generalizes classical work on the divergence operator by Bogovskii [Bog79] (which may be more familiar in fluid dynamics than general relativity) and clarifies explicit integral solution formulas found in recent studies of Yang-Mills initial data sets [OT19], convex integration in fluid dynamics [IO16], and asymptotically flat general relativistic initial data sets [MT22; MOT23] (see also Chapter 2 and Chapter 3).

By duality, the flexibility of underdetermined PDEs corresponds to the *rigidity* of solutions to *overdetermined* PDEs. In this way, our work also connects to classical investigations of rigidity, notably Reshetnyak's integral representations for solutions of certain overdetermined linear differential operators (e.g., the Killing and conformal Killing operators) arising in geometric rigidity problems [Res83]; see Remark 4.1.13 for more discussion. Our method also provides a unified proof of Poincaré- or Friedrich-type (or rigidity) inequalities, including Korn's inequality [Kor09; KO89], for various overdetermined operators on a broad class of backgrounds, thereby bringing together previously disparate proofs (see Remark 4.1.17).

## Summary of the main results

### Explicit integral formulas for the divergence operator

Before we describe our results, we first exhibit known explicit integral formulas for solving the prescribed divergence equation $\operatorname{div} u = f$ on $\mathbb{R}^d$, which are the main inspiration for our work.

In [Bog79], Bogovskii wrote down a remarkable explicit integral formula for a compactly supported solution to $\operatorname{div} u = f$, where $f$ is a given scalar function with compact support and integral zero. Concretely, it takes the form

$$u(x) := \int_{\mathbb{R}^d} K_{\eta_1}(x,y) f(y)\,\mathrm{d}y, \quad K_{\eta_1}(x,y) := \frac{(x-y)^j}{|x-y|^d} \left( \int_{|x-y|}^{\infty} \eta_1(r\tfrac{x-y}{|x-y|} + y) r^{d-1}\,\mathrm{d}r \right) \text{ for } x \neq y, \tag{4.1.2}$$

where $\eta_1 \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \eta_1(y)\,\mathrm{d}y = 1$. This integral formula turns out to satisfy the following properties:

1. (Green's function) We have $\operatorname{div} u(x) = f(x) - \left(\int_{\mathbb{R}^d} f(y)\,\mathrm{d}y\right) \eta_1(x)$ for all $f \in C_c^\infty(\mathbb{R}^d)$;
2. (Prescribed support) $\operatorname{supp} u \subseteq \cup_{y \in \operatorname{supp} f,\, y_1 \in \operatorname{supp} \eta_1}$(line segment from $y$ to $y_1$);
3. (Optimal regularization) $K_{\eta_1}(x,y)$ is a locally integrable function such that $\partial_{x^j} K_{\eta_1}(x,y)$ is a Calderón–Zygmund integral kernel (and hence $f \mapsto \partial_j u$ is bounded on $L^p$ for any $1 < p < +\infty$).

In view of (2), $u$ is indeed compactly supported if $f$ is, and we may manipulate its support property by varying $\eta_1$. The presence of an extra term involving $\int f\,\mathrm{d}y$ in (1) is natural in view of the following necessary condition for the existence of a compactly supported solution $u$ (via the divergence theorem): $\int f\,\mathrm{d}y = \int \operatorname{div} u\,\mathrm{d}y = \lim_{R\to\infty} \int_{\partial B_R} u \cdot \boldsymbol{\nu}\,\mathrm{d}S = 0$. More abstractly, it is a manifestation of the fact that the formal cokernel of div (which is the pre-annihilator of the image of compactly supported distributions under div), or simply the space of $C^\infty(\mathbb{R}^d)$ functions with zero gradient, consists of constant functions.

In [OT19], another explicit integral formula for a solution to $\operatorname{div} u = f$ was written down, where $f$ is a given scalar function with compact support (but not necessarily integral zero):

$$u(x) := \int_{\mathbb{R}^d} K_{\eta\!\!\!/}(x,y) f(y)\,\mathrm{d}y, \quad (K_{\eta\!\!\!/})^j(x,y) = \frac{(x-y)^j}{|x-y|^d} \eta\!\!\!/(\tfrac{x-y}{|x-y|}) \quad \text{for } x \neq y, \tag{4.1.3}$$

where $\eta\!\!\!/ \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \eta\!\!\!/(\omega)\,\mathrm{d}S(\omega) = 1$. The following properties hold:

1. (Green's function) We have $\operatorname{div} u(x) = f(x)$ for all $f \in C_c^\infty(\mathbb{R}^d)$;

2. (Prescribed support) $\operatorname{supp} u \subseteq \cup_{y \in \operatorname{supp} f, \omega \in \operatorname{supp} \not\eta}$(ray from $y$ in the direction $\omega$);

3. (Optimal regularization) $K_{\not\eta}(x, y)$ is a locally integrable function such that $\partial_{x^j} K_{\not\eta}(x, y)$ is a Calderón–Zygmund integral kernel.

By (2), $u$ is supported in the union of cones over $\operatorname{supp} \not\eta \subseteq \mathbb{S}^{d-1}$; for this reason, we call the operator $f \mapsto u$ in (4.1.3) a *conic solution operator*. In this case, $u$ directly solves $\operatorname{div} u = f$ since it is allowed to have a non-compact support. Indeed, by $\int f \, \mathrm{d}y = \int \operatorname{div} u \, \mathrm{d}y = \liminf_{R \to \infty} \int_{\partial B_R} u \cdot \boldsymbol{\nu} \, \mathrm{d}S$, it necessarily has a non-compact support if $\int f \, \mathrm{d}y \neq 0$. Abstractly, this is a manifestation of the fact that the cokernel of div in $C_c^\infty(\mathbb{R}^d)$ (which is the pre-annihilator of the image of distributions under div), or simply the space of $C_c^\infty(\mathbb{R}^d)$ functions with zero gradient, is trivial.

Thanks to their simplicity and flexibility, these explicit integral formulas have proven useful in many applications: Bogovskii's operator has been extensively used in fluid dynamics (see §4.1 for further discussions), and [OT19] used the conic operator to manipulate initial data sets for the Yang–Mills equation. See also Chapter 2 and Chapter 3, in which these ideas were applied to the study of initial data sets in general relativity.

The results of this chapter *generalize such integral formulas to a large class of underdetermined differential operators* (and simultaneously, *their adjoints to overdetermined differential operators*) *arising from geometry and physics*. In the remainder of this subsection, we explain each component of our approach in more detail. For a systematic derivation of (4.1.2) and (4.1.3) from our viewpoint, as well as a justification of the properties stated above, we refer the reader already to Section 4.3 below.

**Integral solution and representation formulas from (RC)**

Let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on an open subset $U \subseteq \mathbb{R}^d$ where $r_0 \leq s_0$. For simplicity, assume for now that $\mathcal{P}$ has $C^\infty(\overline{U})$ coefficients (for the case of rough coefficients, see Theorem 4.1.15). We first introduce (in a simplified form) the *recovery on curves condition* for $\mathcal{P}$, which plays a basic role in this chapter. For a curve $\mathbf{x} : [0, 1] \to \mathbb{R}^d$, it says (roughly speaking):

(RC) Given any $\varphi \in C_c^\infty(U)$, there exists a linear way to continuously recover $\varphi(\mathbf{x}(0))$ from (the jet of) $\mathcal{P}^* \varphi$ on $\mathbf{x}$ and (the jet of) $\varphi$ at $\mathbf{x}(1)$.

Note that (RC) obviously holds on any curve for the divergence operator $\mathcal{P} u = \partial_j u^j$, in the sense that $\mathcal{P}^* \varphi = -\mathrm{d}\varphi$ (gradient operator) and thus $\varphi(\mathbf{x}(0)) = \int_0^1 (-\mathrm{d}\varphi)(\dot{\mathbf{x}}(t)) \, \mathrm{d}t + \varphi(\mathbf{x}(1))$. For the precise version of this condition, see Sections 4.5 and 4.5 below.

Simply speaking, our first set of results says that (RC) is all we need to construct integral formulas analogous to (4.1.2) and (4.1.3) solving $\mathcal{P} u = f$. More specifically, but still informally, (RC) implies the existence of a solution operator (i.e., right-inverse) for $\mathcal{P}$ that is *regularizing of optimal order* such that, for every $y$, its integral kernel $K(x, y)$ is supported (in the $x$-variable) in a union of curves emanating from $y$ that can be *prescribed* in the sense we will explain below. By duality, (RC) also implies the existence of integral representation formulas (i.e., left-inverse) for $\mathcal{P}^*$ with analogous properties, which in turn imply Poincaré-type inequalities that control $u$ in terms of $\mathcal{P}^*u$ under suitable additional conditions.

We now formulate the results more precisely. We employ the fractional Sobolev spaces $W^{s,p}(U)$ and $\widetilde{W}^{s,p}(U)$ on domains, which are precisely defined in Section 4.2. Here, we simply point out that when $s$ is a nonnegative integer and $1 < p < +\infty$, $W^{s,p}(U)$ agrees with the usual definition (see, e.g., [Eva10]) and $\widetilde{W}^{s,p}(U) = W_0^{s,p}(U)$, the closure of $C_c^\infty(U)$ in $W^{s,p}(U)$. For any open subset $U \subseteq \mathbb{R}^d$, $s \in \mathbb{R}$, and $p \in (1,\infty)$, the following duality relations hold (for details, see Lemma 4.2.1):

$$W^{-s,p'}(U) \equiv (\widetilde{W}^{s,p}(U))^*, \quad \widetilde{W}^{s,p}(U) \equiv (W^{-s,p'}(U))^*,$$

where the isomorphisms (denoted by $\equiv$) are induced by the unique pairing $W^{-s,p'}(U) \times \widetilde{W}^{s,p}(U) \to \mathbb{R}$, $(f,g) \mapsto \langle f, g\rangle$ that coincides with $\int_U \mathrm{Re}(\overline{f}g)\,\mathrm{d}x$ for $(f,g) \in C^\infty(\overline{U}) \times C_c^\infty(U)$.

For each $K \in \{1,\dots,s_0\}$, we write $m_K$ for the order of the operator $\varphi \mapsto (\mathcal{P}^*\varphi)_K$. Furthermore, consider a family of curves $\mathbf{x}(y,y_1,s)$ ($s \in [0,1]$), where $y$ and $y_1$ denote the two endpoints at $s=0$ and $s=1$, respectively. We assume that $\mathbf{x}(y,y_1,s)$ is *admissible* in the sense that it behaves like (or coincides with) straight line segments $\underline{\mathbf{x}}(y,y_1,s) = y + s(y_1 - y)$ when $y$ and $y_1$ are close (in fact, the precise conditions for admissibility consist of ($\mathbf{x}$-1)–($\mathbf{x}$-2) in Section 4.5, and ($\mathbf{x}$-3) in Section 4.6).

**Theorem 4.1.1** (Conic-type solution operators, representation formulas, and Poincaré-type inequalities,[Ise+25a])**.** *Let $U$ be an open subset of $\mathbb{R}^d$. Let $\mathcal{P}$ satisfy (RC) for an admissible family of curves $\mathbf{x} = \mathbf{x}(y,y_1,s)$ for all $y$ in a neighborhood of $\overline{U}$ and $y_1 \in U_1$ for some open subset $U_1$ of $\mathbb{R}^d$ (see Section 4.5 for the precise formulation). Assume, moreover, that the curves are* nontrapped in $U$ *(i.e., the curve eventually exits $U$) in the sense that*

$$\overline{U} \cap \overline{U}_1 = \emptyset.$$

*Then the following holds.*

1. **Cokernel in $\widetilde{W}^{-s,p'}(U)$.** *For any $1 < p < +\infty$ and $s \in \mathbb{R}$, define the cokernel in $\widetilde{W}^{-s,p'}(U)$ to be:*

$$\ker_{\widetilde{W}^{-s,p'}(U)} \mathcal{P}^* := \{\mathbf{Z} \in \widetilde{W}^{-s,p'}(U) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } \mathcal{D}'(\widetilde{U})\}, \tag{4.1.4}$$

*where $\widetilde{U}$ is an open subset of $\mathbb{R}^d$ such that $\overline{U} \subseteq \widetilde{U}$, and the coefficients of $\mathcal{P}^*$ are extended in a smooth way to $\widetilde{U}$ (since $\mathbf{Z} \in \widetilde{W}^{-s,p'}(U)$, this definition is independent of these choices). Under the assumptions of this theorem, we have*

$$\ker_{\widetilde{W}^{-s,p'}(U)} \mathcal{P}^* = \{0\}.$$

2. **Integral solution formula.** *There exists a locally integrable integral kernel* $K : U \times U \to \mathbb{C}^{s_0 \times r_0}$ *with the support property*

$$\operatorname{supp} K(\cdot, y) \subseteq \overline{\bigcup_{y_1 \in U_1} \mathbf{x}(y, y_1, [0,1])} \quad \text{for every } y \in U,$$

*such that the integral operator* $\mathcal{S}f(x) := \int_U K(x,y) f(y)\,\mathrm{d}y$ *for* $f \in C_c^\infty(U; \mathbb{C}^{s_0})$ *satisfies*

$$\mathcal{P}\mathcal{S}f = f \quad \text{for all } f \in C_c^\infty(U; \mathbb{C}^{s_0}),$$

*Moreover, for any* $1 < p < +\infty$ *and* $s \in \mathbb{R}$, $\mathcal{S}$ *extends to a bounded operator from* $\widetilde{W}^{s,p}(U; \mathbb{C}^{r_0})$ *to* $W^{s+m_1,p}(U) \times \cdots \times W^{s+m_{s_0},p}(U)$.

3. **Integral representation formula & Friedrich-type inequality.** *Dually, we have*

$$\varphi = \mathcal{S}^* \mathcal{P}^* \varphi \quad \text{for all } \varphi \in C_c^\infty(U; \mathbb{C}^{r_0}).$$

*For any* $1 < p < +\infty$ *and* $s \in \mathbb{R}$, $\mathcal{S}^*$ *extends to a bounded operator from* $\widetilde{W}^{-s-m_1,p'}(U) \times \cdots \times \widetilde{W}^{-s-m_{s_0},p'}(U)$ *to* $W^{-s,p'}(U; \mathbb{C}^{r_0})$. *Moreover, the following* Friedrich-type inequality *holds:*

$$\|\varphi\|_{\widetilde{W}^{-s,p'}(U;\mathbb{C}^{r_0})} \lesssim \|\mathcal{P}^*\varphi\|_{\widetilde{W}^{-s-m_1,p'}(U) \times \cdots \times \widetilde{W}^{-s-m_{s_0},p'}(U)} \quad \text{for all } \varphi \in \widetilde{W}^{-s,p'}(U; \mathbb{C}^{r_0}).$$

**Theorem 4.1.2** (Bogovskii-type solution operators, representation formulas, and Poincaré-type inequalities,[Ise+25a])**.** *Let* $U$ *be a connected bounded open subset of* $\mathbb{R}^d$. *Let* $\mathcal{P}$ *satisfy (RC) for an admissible family of curves* $\mathbf{x} = \mathbf{x}(y, y_1, s)$ *(see Section 4.5 for the precise formulation). Assume that* $\overline{U}$ *is* $\mathbf{x}$-star-shaped *with respect to* $\overline{U_1}$, *where* $U_1$ *is an open subset of* $U$ *such that* $\overline{U_1} \subseteq U$, *in the sense that*

$$\overline{\bigcup_{y \in \overline{U},\, y_1 \in \overline{U_1}} \mathbf{x}(y, y_1, [0,1])} \subseteq \overline{U}. \tag{4.1.5}$$

*Then the following holds.*

1. **Cokernel in** $W^{-s,p'}(U)$**.** *For any* $1 < p < +\infty$ *and* $s \in \mathbb{R}$, *define the cokernel in* $W^{-s,p'}(U)$ *to be:*

$$\ker_{W^{-s,p'}(U)} \mathcal{P}^* := \{\mathbf{Z} \in W^{-s,p'}(U) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } \mathcal{D}'(U)\}. \tag{4.1.6}$$

*We have the* invariance property

$$\ker_{W^{-s,p'}(U)} \mathcal{P}^* = \ker \mathcal{P}^*,$$

*and the* finite-dimensional property

$$\dim \ker \mathcal{P}^*(U) < +\infty. \tag{4.1.7}$$

*where* $\ker \mathcal{P}^*$ *is the formal cokernel of* $\mathcal{P}$ *defined in* (4.1.1)*. Moreover, for any open subset* $V \subseteq U$*, the restriction of* $\ker \mathcal{P}^*$ *to* $V$*, i.e.,*

$$\ker \mathcal{P}^*|_V := \{\mathbf{Z}|_V : \mathbf{Z} \in \ker \mathcal{P}^*\}$$

*has the same dimension as* $\ker \mathcal{P}^*$*.*

2. **Integral solution formula under orthogonality conditions.** *Consider a family* $w_{\mathbf{A}}(x) \in C_c^\infty(U_1; \mathbb{C}^{r_0})$ $(\mathbf{A} \in \{1, \dots, \dim \ker \mathcal{P}^*\})$ *satisfying* $\langle w_{\mathbf{A}}, \mathbf{Z}^{\mathbf{A}'}\rangle = \delta_{\mathbf{A}}^{\mathbf{A}'}$ *for some basis* $\{\mathbf{Z}^{\mathbf{A}'}\}$ *of* $\ker \mathcal{P}^*$*. Then there exists a locally integrable integral kernel* $\widetilde{K} : U \times U \to \mathbb{C}^{s_0 \times r_0}$ *with the support property*

$$\operatorname{supp} \widetilde{K}(\cdot, y) \subseteq \overline{\bigcup_{y_1 \in \overline{U_1}} \mathbf{x}(y, y_1, [0,1])} \subseteq \overline{U} \quad \textit{for every } y \in U,$$

*(where the last inclusion follows simply from* (4.1.5)*) such that the integral operator* $\widetilde{\mathcal{S}} f(x) := \int_U \widetilde{K}(x,y) f(y) \, \mathrm{d}y$ *for* $f \in C_c^\infty(U; \mathbb{C}^{s_0})$ *satisfies*

$$\mathcal{P}\widetilde{\mathcal{S}} f = f - \sum_{\mathbf{A} \in \{1,\dots,\dim \ker \mathcal{P}^*\}} w_{\mathbf{A}} \langle \mathbf{Z}^{\mathbf{A}}, f \rangle \quad \textit{for all } f \in C_c^\infty(U; \mathbb{C}^{s_0}).$$

*Moreover, for any* $1 < p < +\infty$ *and* $s \in \mathbb{R}$*,* $\widetilde{\mathcal{S}}$ *extends to a bounded operator from* $\widetilde{W}^{s,p}(U; \mathbb{C}^{r_0})$ *to* $\widetilde{W}^{s+m_1}(U) \times \cdots \times \widetilde{W}^{s+m_{s_0}}(U)$*.*

3. **Integral representation formula & Poincaré-type inequality under orthogonality conditions.** *Dually, we have*

$$\varphi = \widetilde{\mathcal{S}}^* \mathcal{P}^* \varphi + \sum_{\mathbf{A} \in \{1,\dots,\dim \ker \mathcal{P}^*\}} \mathbf{Z}^{\mathbf{A}} \langle w_{\mathbf{A}}, \varphi \rangle \quad \textit{for all } \varphi \in C^\infty(\overline{U}; \mathbb{C}^{r_0}).$$

*Furthermore, for any* $1 < p < +\infty$ *and* $s \in \mathbb{R}$*,* $\widetilde{\mathcal{S}}^*$ *defines a bounded operator from* $W^{-s-m_1,p}(U) \times \cdots \times W^{-s-m_{s_0},p'}(U)$ *to* $W^{-s,p'}(U; \mathbb{C}^{r_0})$*, and the following* Poincaré-type inequality *holds:*

$$\|\varphi - \sum_{\mathbf{A}} \mathbf{Z}^{\mathbf{A}} \langle w_{\mathbf{A}}, \varphi \rangle\|_{W^{-s,p'}(U;\mathbb{C}^{r_0})} \lesssim \|\mathcal{P}^* \varphi\|_{W^{-s-m_1,p'}(U) \times \cdots \times W^{-s-m_{s_0},p'}(U)}$$
$$\textit{for all } \varphi \in W^{-s,p'}(U; \mathbb{C}^{r_0}).$$

In Section 4.3 below, we give a more detailed description of the structure of the integral kernels $K(x,y)$ and $\widetilde{K}(x,y)$, and summarize the proofs of Theorems 4.1.1 and 4.1.2.

*Remark* 4.1.3 (On the regularity of $U$). The reader may find it amusing that neither theorem requires any regularity assumptions on the boundary of $U$. For Theorem 4.1.1, the nontrapping assumption is crucial. For Theorem 4.1.2, the **x**-star-shaped assumption in fact embodies some notion of regularity of $\partial U$. For instance, if **x** is the straight line segment $\underline{\mathbf{x}}(y, y_1, s) = y + s(y_1 - y)$, then this assumption implies the uniform cone condition for $U$, which in turn implies that $\partial U$ is Lipschitz [Gri11, Section 1.2]. See also Theorem 4.1.15 for a result that applies to Lipschitz domains rather than those with (4.1.5).

### Tools for verifying (RC): Graded augmented system and (FC)

Our second set of results provides tools for verifying (RC) for a variety of under/overdetermined partial differential operators.

Our basic device is the notion of a (*graded*) *augmented system*, which generalizes a basic procedure for verifying (RC) for the divergence operator on $\mathbb{R}^d$; see Section 4.3 and Remark 4.3.6 below. It is also a generalization of the procedure used by Retshenyak [Res83] to construct integral representation formulas for some overdetermined linear operators (see Remark 4.1.13). Its precise formulation requires us to introduce some conventions and definitions. In what follows, we adopt the following index notation (which is consistent with Section 4.2):

- $J \in \{1, \dots, r_0\}$ (and its variants such as $J'$, etc.): index for components of $\varphi = (\varphi_J)_{J=1,\dots,r_0}$
- $K \in \{1, \dots, s_0\}$ (and its variants such as $K'$, etc.): index for components of $\mathcal{P}^*\varphi = ((\mathcal{P}^*\varphi)_K)_{K=1,\dots,s_0}$
- $\mathbf{A} \in \mathcal{A}$ (and its variants such as $\mathbf{A}'$, etc.): index for components of the augmented variables $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$ (to be defined below).

As usual, we adopt the convention of summing up repeated upper and lower indices, unless otherwise stated. We also make the provision that **a multi-index $\gamma$ in $\partial^\gamma$ is considered a lower index** (hence, $c^\gamma \partial^\gamma = \sum_\gamma c^\gamma \partial^\gamma$).

**Definition 4.1.4** (Graded augmented system)**.** Let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on an open subset $U \subseteq \mathbb{R}^d$, with $m_K$ denoting the order of $(\mathcal{P}^*\varphi)_K$ for each $K \in \{1, \dots, s_0\}$. Given $m'_K \in \mathbb{Z}_{\geq 0}$ for each $K \in \{1, \dots, s_0\}$ and an $\mathbb{C}^{r_0}$-valued function $(\varphi_J)_{J\in\{1,\dots,r_0\}}$ on $U$, consider $(\Phi_{\mathbf{A}} = \Phi_{\mathbf{A}}(y))_{\mathbf{A}\in\mathcal{A}}$ (called *augmented variables*) satisfying the following properties:

(Φ-1) **$(\Phi_{\mathbf{A}})$ is an augmentation of $(\varphi_J)$.** The index set $\mathcal{A}$ contains $\{1, \dots, r_0\}$; moreover, $\Phi_J = \varphi_J$ for $J \in \{1, \dots, r_0\} \subseteq \mathcal{A}$.

(Φ-2) **$(\varphi_J) \mapsto (\Phi_{\mathbf{A}})$ is a differential operator.** There exist functions $c[\Phi_{\mathbf{A}}]^{(\alpha,J)}$ on $U$, where $\alpha$ is a multi-index and $J \in \{1, \dots, r_0\}$, such that

$$\Phi_{\mathbf{A}}(y) = c[\Phi_{\mathbf{A}}]^{(\alpha,J)}(y)\partial^\alpha \varphi_J(y),$$

for all $\mathbf{A} \in \mathcal{A}$ and $y \in U$.

(Φ-3) **$\Phi$ and $\mathcal{P}^*\varphi$ solve a first-order PDE (augmented PDE system).** There exist matrix-valued 1-forms $(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}$ and $(\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}$ on $U$, where $\gamma$ is a multi-index and $K \in \{1, \dots, s_0\}$, such that

$$\partial_i \Phi_{\mathbf{A}}(y) = (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}(y)\Phi_{\mathbf{A}'}(y) + (\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}(y)\partial^\gamma(\mathcal{P}^*\varphi)_K(y), \tag{4.1.8}$$

for all $\mathbf{A} \in \mathcal{A}$, $i \in \{1, \dots, d\}$ and $y \in U$. Furthermore,

$$(\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} = 0 \quad \text{if } |\gamma| > m'_K.$$

($\Phi$-4) **Graded structure.** Define the *degree* $d_{\mathbf{A}}$ of $\Phi_{\mathbf{A}}$ by

$$d_{\mathbf{A}} \coloneqq -\max\{|\alpha| : c[\Phi_{\mathbf{A}}]^{(\alpha,J)} \neq 0 \text{ for some } J\}.$$

In particular, $\varphi_J$ has degree 0, i.e., $d_J = 0$ for $J \in \{1, \dots, r_0\}$. We assume that:

- **Graded structure for the augmented system.**

$$\begin{aligned}(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'} &= 0 \qquad \text{if } d_{\mathbf{A}} > d_{\mathbf{A}'} + 1,\\ (\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} &= 0 \qquad \text{if } d_{\mathbf{A}} > -m_K - |\gamma| + 1.\end{aligned}$$

We write $N_0 \coloneqq \max\{|d_{\mathbf{A}}|\}_{\mathbf{A}\in\mathcal{A}} + 1$ for the maximal degree[1] that occurs in (4.1.8).

We call a collection $(\mathcal{A}, (\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}, (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}, (\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)})$ satisfying ($\Phi$-1)–($\Phi$-4) a (*graded*) *augmented system* for $\mathcal{P}$.

*Remark* 4.1.5 (Graded structure). The degree $d_{\mathbf{A}}$ is (minus) the number of derivatives falling on $\varphi$ in $\Phi_{\mathbf{A}}$. The graded structure for $\mathbf{B}_i$ is the natural requirement that, in the equation for one derivative of $\Phi_{\mathbf{A}} = c[\Phi_{\mathbf{A}}]^{(\alpha,J)}\partial^\alpha\varphi_J$, we do not see derivatives of $\varphi$ that are two orders higher. The graded structure for $\mathbf{C}_i$ is an analogous requirement, where we assign degree $-m_K - |\gamma|$ to $\partial^\gamma(\mathcal{P}^*\varphi)_K$.

Given a graded augmented system $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$, ($\Phi$-3) implies that each $\Phi_{\mathbf{A}}$ satisfies an ODE on each curve $\mathbf{x}(y, y_1, \cdot)$ of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_{\mathbf{A}} \circ \mathbf{x}) = \dot{\mathbf{x}}^i \left((\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'} \circ \mathbf{x}\right)(\Phi_{\mathbf{A}'} \circ \mathbf{x}) + \dot{\mathbf{x}}^i \left((\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} \circ \mathbf{x}\right)(\partial^\gamma(\mathcal{P}^*\varphi)_K \circ \mathbf{x}), \tag{4.1.9}$$

where $\dot{\mathbf{x}}(y, y_1, s) = \partial_s \mathbf{x}(y, y_1, s)$. In particular, by Duhamel's principle (or variation of constants), we may express $\varphi_J$ at $y = \mathbf{x}(y, y_1, 0)$ as follows:

$$\begin{aligned}\varphi_J(y) = &-\int_0^1 {}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_J{}^{\mathbf{A}}(0,s)\dot{\mathbf{x}}^i \left((\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} \circ \mathbf{x}\right)((\partial^\gamma\mathcal{P}^*\varphi)_K \circ \mathbf{x})(y, y_1, s)\,\mathrm{d}s\\ &+ {}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_J{}^{\mathbf{A}}(0,1)\Phi_{\mathbf{A}}(y_1),\end{aligned} \tag{4.1.10}$$

where ${}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_{\mathbf{A}}{}^{\mathbf{A}'}(s,t)$ is the fundamental matrix for $\frac{\mathrm{d}}{\mathrm{d}s} - \dot{\mathbf{x}}^i(\mathbf{B}_i \circ \mathbf{x})$. This formula immediately implies (RC); it also gives a representation of any element $\mathbf{Z} \in \ker \mathcal{P}^*$ (i.e., $\mathcal{P}^*\mathbf{Z} = 0$) in terms of the values of $\Phi_{\mathbf{A}}$ at a point. In fact, we have the following result.

[1] Here, $+1$ accounts for the derivative $\partial_i$ on the LHS of (4.1.8).

**Proposition 4.1.6.** *Assume that $\mathcal{P}$ possesses a graded augmented system $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$, and that $\mathbf{B}_i$ and $\mathbf{C}_i$ and their derivatives are uniformly bounded on $U$. Then $\mathcal{P}$ satisfies (RC) for any admissible family of curves $\mathbf{x}$; more precisely, Theorems 4.1.1 and 4.1.2 are applicable. Moreover, for every $\mathbf{Z} \in \ker \mathcal{P}^*$ and $y_1 \in U$, we have*

$$\mathbf{Z}_J(x) = {}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_J{}^{\mathbf{A}}(0,1)\Phi_{\mathbf{A}}(y_1), \tag{4.1.11}$$

*In particular,* $\dim\ker\mathcal{P}^* \leq \#\mathcal{A}$.

For the precise formulation and proof, see Section 4.6, in particular, Proposition 4.6.6 and Remark 4.6.5.

In view of the bound $\dim\mathcal{P}^* \leq \#\mathcal{A}$, it is of interest to ask when equality holds. The following result answers this question under reasonable assumptions:

**Proposition 4.1.7.** *Let $U$ be a simply connected open subset of $\mathbb{R}^d$. Assume that $\mathcal{P}$ possesses a graded augmented system $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$, and that $\mathbf{B}_i$ and $\mathbf{C}_i$ and their derivatives are uniformly bounded on $U$. Then* $\dim\ker\mathcal{P}^* = \#\mathcal{A}$ *if and only if the following condition (called the* zero curvature condition*) holds for all $x \in U$, $i, j = 1, \ldots, d$, and $\mathbf{A}, \mathbf{A}' \in \mathcal{A}$:*

$$\left(\partial_i(\mathbf{B}_j)_{\mathbf{A}}{}^{\mathbf{A}'} - \partial_j(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'} + (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}''}(\mathbf{B}_j)_{\mathbf{A}''}{}^{\mathbf{A}'} - (\mathbf{B}_j)_{\mathbf{A}}{}^{\mathbf{A}''}(\mathbf{B}_i)_{\mathbf{A}''}{}^{\mathbf{A}'}\right)(x) = 0. \tag{4.1.12}$$

For a proof, see Section 4.6, where we give a geometric interpretation of (4.1.12) in terms the curvature of a connection on a vector bundle, a viewpoint that is of interest on its own. This motivates the following:

**Definition 4.1.8** (Completely integrable augmented systems)**.** We say that an augmented system is *complete integrable* if the zero curvature condition (4.1.12) is satisfied for all $x \in U$, $i, j = 1, \ldots, d$, and $\mathbf{A}, \mathbf{A}' \in \mathcal{A}$.

For examples of completely integrable augmented systems, we refer to Appendix B, as well as Reshetnyak [Res83; Res94] (see also Remark 4.1.13).

Complete integrability, or more precisely $\#\mathcal{A} = \dim\ker\mathcal{P}^*$, leads to a simplification of the derivation of integral solution and representation formulas; see Remark 4.3.3, Remark 4.3.4, and Theorem 4.5.15 below. However, it is not necessary for the validity of Theorems 4.1.1 and 4.1.2. In fact, to handle a larger class of operators $\mathcal{P}$, it turns out to be useful to consider the other extreme case, namely, graded augmented systems with the *maximal* number of augmented variables with a given maximal degree $N_0$.

**Definition 4.1.9** (Maximal graded augmented system)**.** Given an integer $N_0 \geq \max_K m_K$, a graded augmented system with augmented variables $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}} = (\partial^\alpha\varphi_J)_{(\alpha,J):1\leq J\leq r_0,\,|\alpha|\leq N_0-1}$ consisting of all partial derivatives of $\varphi$ up to order $N_0 - 1$ (i.e., $\mathcal{A} = \{(\alpha, J) : 1 \leq J \leq r_0,\, |\alpha| \leq N_0 - 1\}$) is called a *maximal graded augmented system.*

A useful property of a maximal graded augmented system is that it is *stable under lower order perturbations*, i.e., the same variables constitute a graded augmented system for $\widetilde{\mathcal{P}}$ as long as $((\widetilde{\mathcal{P}}-\mathcal{P})^*\varphi)_K$ is of order less than $m_K$. Observe that such a stability property is not evident for (RC), nor for completely integrable augmented systems.

We are now ready to formulate an algebraic sufficient condition for (RC) in terms of the principal symbol $p^*(x,\xi)$ of $\mathcal{P}^*$, which greatly facilitates the applicability of our theory (see, for instance, Theorem 4.1.14 below). To formulate this result, we begin with a suitable definition of the principal symbol of the matrix-valued operator $\mathcal{P}^*$:

**Definition 4.1.10.** Let $U$ be an open subset of $\mathbb{R}^d$, and let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on $U$. Suppose that $\mathcal{P}$ and its adjoint $\mathcal{P}^*$ can be written out in the form[2]

$$(\mathcal{P}u)^J(x) = \sum_\alpha c[\mathcal{P}]^{(\alpha,J)}{}_K(x)\partial_x^\alpha u^K(x), \qquad (\mathcal{P}^*\varphi)_K(x) = \sum_\alpha c[\mathcal{P}^*]_K{}^{(\alpha,J)}(x)\partial_x^\alpha \varphi_J(x).$$

We define the *principal parts* of $\mathcal{P}$ and $\mathcal{P}^*$, respectively, to be

$$(\mathcal{P}_{\mathrm{prin}}u)^J(x) \coloneqq \sum_{\alpha:|\alpha|=m_K} c[\mathcal{P}]^{(\alpha,J)}{}_K(x)\partial_x^\alpha u^K(x), \qquad (\mathcal{P}^*_{\mathrm{prin}}\varphi)_K(x) = \sum_{|\alpha|=m_K} c[\mathcal{P}^*]_K{}^{(\alpha,J)}(x)\partial_x^\alpha \varphi_J(x),$$

where we recall that $m_K$ is the order of the operator $\varphi \mapsto (\mathcal{P}^*\varphi)_K$. Accordingly, we define the *principal symbols* $p(x,\xi)$ and $p^*(x,\xi)$ of $\mathcal{P}$ and $\mathcal{P}^*$, respectively, to be

$$p^J{}_K(x,\xi) \coloneqq \sum_{\alpha:|\alpha|=m_K} c[\mathcal{P}]^{(\alpha,J)}{}_K(x) i^{|\alpha|}\xi^\alpha, \qquad (p^*)_K{}^J(x,\xi) \coloneqq \sum_{\alpha:|\alpha|=m_K} c[\mathcal{P}^*]_K{}^{(\alpha,J)}(x) i^{|\alpha|}\xi^\alpha.$$

In terms of this definition, we formulate the following algebraic condition:

(FC) For all $x \in U$ and $\xi \in \mathbb{C}^d \setminus \{0\}$, $p^*(x,\xi)$ is injective (or equivalently, $p^*(x,\xi)$ is full rank, or $p(x,\xi)$ is surjective, or $p(x,\xi)$ is full rank).

Observe that (FC) is stronger than over/underdetermined *ellipticity*, which would be the same condition but only for $\xi \in \mathbb{R}^d \setminus \{0\}$. Our key result is:

**Theorem 4.1.11** ([Ise+25a])**.** *Let $U$ be a connected open subset of $\mathbb{R}^d$, and let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on $U$ with smooth coefficients.*

1. *Condition (FC) implies the existence of a maximal graded augmented system $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$. Hence, (FC) implies (RC) for any admissible family of curves, and Theorems 4.1.1–4.1.2 apply.*

2. *If $p^*$ is independent of $x$ (i.e., $\mathcal{P}_{\mathrm{prin}}$ has constant coefficients), then the following are equivalent:*

[2] Note that $c[\mathcal{P}^*]_K{}^{(\alpha,J)}(x) = (-1)^{|\alpha|}\overline{c[\mathcal{P}]}^{(\alpha,J)}{}_K(x)$.

*a) $p^*$ satisfies (FC),*

*b) any $r_0 \times s_0$-matrix-valued differential operator $\mathcal{P}'$ on $U$ with principal symbol $p^*$ satisfies (RC) for any admissible family of curves, and*

*c) the formal cokernel of $\mathcal{P}_{\mathrm{prin}}$ (i.e., $\ker \mathcal{P}^*_{\mathrm{prin}} \coloneqq \{\mathbf{Z} \in C^\infty(\overline{U}) : \mathcal{P}^*_{\mathrm{prin}}\mathbf{Z} = 0$ in $\mathcal{D}'(U)\}$) is finite dimensional.*

In light of Theorem 4.1.11.(2), we refer to (FC) as the *finite-dimensional cokernel* condition. In order for (FC) to hold, we necessarily have $r_0 \leq s_0$.

A key ingredient in our proof of Theorem 4.1.11 is a basic result in algebraic geometry – namely, Hilbert's Nullstellensatz (Proposition 4.7.1) – which we use to construct a maximal graded augmented system with a sufficiently high maximal degree $N_0$ for an operator satisfying the algebraic condition (FC). We refer to Section 4.7 for a proof of Theorem 4.1.11.

*Remark* 4.1.12 (Further generalization)*.* Our setup assumes that each component of $(\varphi_J)_{J\in\{1,\ldots,r_0\}}$ (or equivalently, $(f^J)_{J\in\{1,\ldots,r_0\}}$) has the same degree (see, in particular, ($\Phi$-4)); accordingly, we look at $(\mathcal{P}^*\varphi)_K$ for each $K \in \{1, \ldots, r_0\}$ to define the order $m_K$. This setup is sufficient for our applications in Theorem 4.1.14 and Appendix B. Nevertheless, we note that it is possible to develop the theory under the assumption that $\varphi_J$ have different degrees (i.e., $d_J$ are not all equal). In this case, one needs to introduce ${m_K}^J \in \mathbb{Z}_{\geq 0}$ in place of $m_K$ and alter Definition 4.1.10 and ($\Phi$-4). We will not pursue this more generalized setup in more detail.

*Remark* 4.1.13 (Comparison with Reshetnyak's approach)*.* The ideas presented in this part owes much to the work of Reshetnyak [Res83; Res94]. In our terminology, Reshetnyak introduced completely integrable graded augmented systems for certain overdetermined operators $\mathcal{P}^*$ (including the Killing and conformal Killing operators on Euclidean space), and utilized them to derive integral representation formulas involving $\mathcal{P}^*$ based on line segments $\underline{\mathbf{x}}(y, y_1, s) = y + s(y_1 - y)$. Of this procedure, Retshenyak [Res94, p. 27] remarks: "The general scheme of constructing such representations is apparently beyond formalisation."

Among others, the most important departure of our approach from that of Reshetnyak is the relaxation of Reshetnyak's completely integrable system to Definition 4.1.4, and then furthermore considering *maximal* systems with a possibly large $N_0$. This idea plays a crucial role in our proof of Theorem 4.1.11, which in turn is key to the wide applicability of our method. Indeed, while the question of which $\mathcal{P}^*$ admits completely integrable graded augmented systems seems difficult to answer in general, our result implies that, at least for homogeneous constant-coefficient $\mathcal{P}^*$ (which includes all examples considered in [Res83]), the existence of a maximal graded augmented system is *equivalent* to the algebraic condition (FC).

## Applications I: examples of $\mathcal{P}$

With the help of (FC), one may easily check that our method applies to a variety of operators that arise naturally from physics and geometry, as well as their lower-order variable-coefficient perturbations (which include their covariant versions on Riemannian manifolds;

see Appendix B). For the next theorem, we adopt the following conventions: on an open subset $U$ of $\mathbb{R}^d$, we use $\mathbf{g}_{jk}$ to refer to a metric (i.e., positive definite symmetric 2-tensor); $(\mathbf{g}^{-1})^{jk}$ its inverse; $\varphi$ a real-valued function; $\mathbf{u}^j$ a vector field; $\boldsymbol{\omega}_j$ a one-form, $\mathbf{h}^{jk}$ and $\boldsymbol{\pi}^{jk}$ symmetric 2-tensors; and $\widehat{\mathbf{h}}^{jk}$ and $\widehat{\boldsymbol{\pi}}^{jk}$ symmetric 2-tensors that are trace-free (with respect to $\mathbf{g}$).

**Theorem 4.1.14** ([Ise+25a]). *Matrix-valued differential operators $\mathcal{P}$ on $U$ with smooth coefficients that have the following principal parts satisfy (FC):*

1. **Divergence & (adjoint) gradient operator, $d \geq 1$.**

$$\mathcal{P}_{\text{prin}}\mathbf{u} = \partial_j \mathbf{u}^j \quad \textit{or equivalently} \quad \mathcal{P}^*_{\text{prin}}\varphi = -\partial_j \varphi,$$

*in which case* $(p^*)_j = -i\xi_j$.

2. **Double divergence (or linearized scalar curvature) & (adjoint) Hessian operator, $d \geq 1$.**

$$\mathcal{P}_{\text{prin}}u = \partial_j \partial_k \mathbf{h}^{jk} \quad \textit{or equivalently} \quad \mathcal{P}^*_{\text{prin}}\varphi = \partial_j \partial_k \varphi,$$

*in which case* $(p^*)_{jk} = -\xi_j \xi_k$.

3. **Trace-free double divergence & (adjoint) trace-free Hessian operator, $d \geq 2$.**

$$\mathcal{P}_{\text{prin}}\widehat{\mathbf{h}} = \partial_j \partial_k \widehat{\mathbf{h}}^{jk} \quad \textit{or equivalently} \quad \mathcal{P}^*_{\text{prin}}\varphi = \partial_j \partial_k \varphi - \frac{1}{d}\mathbf{g}_{jk}(x)(\mathbf{g}^{-1})^{\ell m}(x)\partial_\ell \partial_m \varphi,$$

*in which case* $(p^*)_{jk} = -\xi_j \xi_k + \frac{1}{d}\mathbf{g}_{jk}(x)(\mathbf{g}^{-1})^{\ell m}(x)\xi_\ell \xi_m$.

4. **Symmetric divergence & (adjoint) Killing operator, $d \geq 1$.**

$$(\mathcal{P}_{\text{prin}}\mathbf{h})_j = \partial_k \mathbf{h}^{jk} \quad \textit{or equivalently} \quad (\mathcal{P}^*_{\text{prin}}\boldsymbol{\omega})_{jk} = -\frac{1}{2}(\partial_j \boldsymbol{\omega}_k + \partial_k \boldsymbol{\omega}_j),$$

*in which case* ${(p^*)_{jk}}^{\ell} = -\frac{i}{2}(\xi_j \delta^\ell_k + \xi_k \delta^\ell_j)$.

5. **Symmetric trace-free divergence & (adjoint) conformal Killing operator, $d \geq 3$.**

$$(\mathcal{P}_{\text{prin}}\widehat{\mathbf{h}})_j = \partial_k \widehat{\mathbf{h}}^{jk} \quad \textit{or equivalently} \quad (\mathcal{P}^*_{\text{prin}}\boldsymbol{\omega})_{jk} = -\frac{1}{2}(\partial_j \boldsymbol{\omega}_k + \partial_k \boldsymbol{\omega}_j) + \frac{1}{d}\mathbf{g}_{jk}(\mathbf{g}^{-1})^{\ell m}\partial_\ell \omega_m,$$

*in which case* ${(p^*)_{jk}}^{\ell} = -\frac{i}{2}(\xi_j \delta^\ell_k + \xi_k \delta^\ell_j) + \frac{i}{d}\mathbf{g}_{jk}(x)(\mathbf{g}^{-1})^{\ell m}(x)\xi_m$.

6. **Linearized Einstein vacuum constraint & (adjoint) Killing Initial Data operator, $d \geq 1$.**

$$\mathcal{P}_{\text{prin}}(\mathbf{h}, \boldsymbol{\pi}) = (\partial_j \partial_k \mathbf{h}^{jk}, \partial_{k'} \boldsymbol{\pi}^{j'k'}) \textit{ or equivalently } \mathcal{P}^*_{\text{prin}}(\varphi, \boldsymbol{\omega}) = \left(-\partial_j \partial_k \varphi, -\frac{1}{2}(\partial_{j'} \boldsymbol{\omega}_{k'} + \partial_{k'} \boldsymbol{\omega}_{j'})\right)$$

*in which case $p^*$ is $2 \times 2$-block-diagonal with the principal symbols of the Hessian and Killing operators as blocks.*

7. **Linearized Einstein vacuum constraint operator with constant mean curvature, $d \geq 3$.**

$$\mathcal{P}_{\mathrm{prin}}(\mathbf{h}, \widehat{\boldsymbol{\pi}}) = (\partial_j \partial_k \mathbf{h}^{jk}, \partial_{k'} \widehat{\boldsymbol{\pi}}^{j'k'}),$$

*in which case $p^*$ is $2 \times 2$-block-diagonal with the principal symbols of the Hessian and conformal Killing operators as blocks.*

The proof of this theorem, which we immediately provide, consists of short algebraic computations.

*Proof.* For (1), it is clear that, for each $\xi \in \mathbb{C}^d \setminus \{0\}$, $(p^*)_j(\xi)\varphi = \xi_j \varphi = 0$ implies $\varphi = 0$; similarly for (2).

To prove (3), fix $x \in U$ and assume (by passing to the normal coordinates) that $\mathbf{g}^{\ell m}(x)\mathbf{g}_{jk}(x) = \delta^{\ell m}\delta_{jk}$. For each $\xi \in \mathbb{C}^d \setminus \{0\}$, we need to show that if $\varphi \in \mathbb{R}$ satisfies

$$(p^*)_{jk}(x, \xi)\varphi = \left( -\xi_j \xi_k + \frac{1}{d} \xi_\ell \xi_m \delta^{\ell m} \delta_{jk} \right) \varphi = 0 \quad \text{for all } j, k \in \{1, \ldots, d\},$$

then $\varphi = 0$. In view of (2), it suffices to show that $\xi_\ell \xi_m \delta^{\ell m} \varphi = \sum_\ell \xi_\ell^2 \varphi = 0$. For each $j$, we have

$$\begin{aligned}\xi_j \sum_\ell \xi_\ell^2 \varphi = \xi_j^3 \varphi + \sum_{\ell : \ell \neq j} \xi_\ell \xi_\ell \xi_j \varphi &= \xi_j \left( -(p^*)_{jj}(x, \xi)\varphi + \frac{1}{d} \sum_\ell \xi_\ell^2 \varphi \right) + \sum_{\ell : \ell \neq j} \xi_\ell \left( -(p^*)_{\ell j}(x, \xi)\varphi \right) \\ &= \frac{1}{d} \xi_j \sum_\ell \xi_\ell^2 \varphi,\end{aligned}$$

which implies $\sum_\ell \xi_\ell^2 \varphi = 0$ as long as $d > 1$. .

For (4), we need to show that if $\omega \in T_x^* \mathcal{M}$ (identified with $\mathbb{R}^d$) and $\xi \in \mathbb{C}^d \setminus \{0\}$ satisfies

$$(p^*)_{jk}{}^{\ell}(\xi)\omega_\ell = -\frac{i}{2}(\xi_j \omega_k + \xi_k \omega_j) = 0 \quad \text{for all } j, k \in \{1, \ldots, d\},$$

then $\omega_\ell = 0$ for all $\ell \in \{1, \ldots, d\}$. Note that the previous condition implies $\xi_j \omega_k = -\xi_k \omega_j$, and in particular, $\xi_j \omega_j = 0$ for any $j, k \in \{1, \ldots, d\}$. Thus, for any $j, k \in \{1, \ldots, d\}$, we have

$$\xi_j^2 \omega_k = -\xi_j \xi_k \omega_j = \xi_k \xi_j \omega_j = 0,$$

which implies $\omega_k = 0$ as desired.

The proof of (5) requires a bit more computation compared to the previous cases. Fix $x \in U$ and assume (by passing to the normal coordinates) that $\mathbf{g}^{\ell m}(x)\mathbf{g}_{jk}(x) = \delta^{\ell m}\delta_{jk}$. Fix $\xi \in \mathbb{C}^d \setminus \{0\}$, and assume that $\omega \in T_x^* \mathcal{M}$ (identified with an element in $\mathbb{R}^d$) satisfies

$$(p^*)_{jk}{}^{\ell}(x, \xi)\omega_\ell = -\frac{i}{2}(\xi_j \omega_k + \xi_k \omega_j) + \frac{i}{d} \delta^{\ell m} \xi_m \omega_\ell \delta_{jk} = 0 \quad \text{for all } j, k \in \{1, \ldots, d\}.$$

We need to show that $\omega_\ell = 0$ for all $\ell$. In view of (4), it suffices to show that $w := \frac{1}{d}\delta^{\ell m}\xi_m\omega_\ell = \frac{1}{d}\sum_\ell \xi_\ell\omega_\ell = 0$. The above condition implies

$$\xi_j\omega_k = -\xi_k\omega_j + 2w\delta_{jk} \quad \text{for any } j,k \in \{1,\dots,d\}.$$

and in particular, $\xi_j\omega_j = w$ for any $j \in \{1,\dots,d\}$. We first compute $\xi_j w$ (for any $J \in \{1,\dots,d\}$):

$$\xi_j w = \frac{1}{d}\xi_j\xi_j\omega_j + \frac{1}{d}\sum_{\ell:\ell\neq j}\xi_\ell(\xi_j\omega_\ell) = \frac{1}{d}\xi_j w - \frac{1}{d}\sum_{\ell:\ell\neq j}\xi_\ell^2\omega_j,$$

which shows that $\xi_j w = -\frac{1}{d-1}\sum_{\ell:\ell\neq j}\xi_\ell^2\omega_j$. Multiplying by $\xi_j$, we arrive at

$$\xi_j^2 w = -\frac{1}{d-1}\sum_{\ell:\ell\neq j}\xi_\ell^2 w.$$

On the one hand, by summing up in $j$, we conclude that $\sum_\ell \xi_\ell^2 w = 0$. On the other hand, using $\sum_{\ell:\ell\neq j}\xi_\ell^2 w = -\xi_j^2 w$, we arrive at $\xi_j^2 w = \frac{1}{d-1}\xi_j^2 w$, which implies $\xi_j^2 w = 0$ for any $j \in \{1,\dots,d\}$, since $d > 2$. Thus $w = 0$.

Finally, (6) follows from combining (2) and (4); similarly, (7) follows from (2) and (5). $\square$

By Theorem 4.1.11, a maximal graded augmented system exists for each example, (RC) thus holds, and Theorems 4.1.1 and 4.1.2 are applicable. Alternatively, in Appendix B, each example is revisited with a more geometric viewpoint, and we provide a special graded augmented system that is (i) stable under the addition of lower order terms, and (ii) completely integrable on backgrounds with constant sectional curvature. Using this augmented system, we also compute explicitly the Bogovskii and conic integral kernels on the flat space, which also recovers the known results from [IO16; MT22; MOT23; Res83].

While the results in Appendix B are of independent interest – which is why we have worked them out – we point out the contrast between the simplicity of the proof of Theorem 4.1.14 versus the case-by-case ingenuity required in the direct derivation of the special graded augmented systems in Appendix B.

## Applications II: sharp solvability results (from [Ise+25b])

Next, we discuss an application of our method to the study of differential operators satisfying (FC) under optimal (up to endpoints) assumptions on the coefficients. Our overall approach is to:

1. first use our method to *design* appropriate integral formulas for $\mathcal{P}$ that have constant coefficients, are homogeneous (i.e., $\mathcal{P} = \mathcal{P}_{\text{prin}}$ in the sense of Definition 4.1.10); and

2. handle the general case via *local perturbation* (or freezing-coefficients) techniques.

We restrict our attention to $L^2$-based Sobolev spaces, but extensions to other function spaces that behave well under singular integral operators (e.g., Hölder or $L^p$-based Sobolev spaces with $1 < p < +\infty$) should be possible. The main result in the bounded domain case in [Ise+25b] is as follows (we refer again to Section 4.2 for our notation and conventions concerning function spaces, and to (4.1.4) and (4.1.6) for the precise definitions of $\ker_{\widetilde{H}^{-s}(U)} \mathcal{P}^*$ and $\ker_{H^{-s}(U)} \mathcal{P}^*$, respectively):

**Theorem 4.1.15** (Solvability on a bounded Lipschitz domain,[Ise+25b])**.** *Fix an exponent $s_{\mathcal{P},0}$ such that $s_{\mathcal{P},0} > \frac{d}{2}$ and $s_{\mathcal{P},0} \geq \frac{1}{2}\max_K m_K$. Let $U$ be a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary, i.e., $\partial U$ is compact and can be covered by finite balls, in each of which $\partial U$ can be written as the graph of a Lipschitz function after suitably relabeling and rotating the coordinate axes. Let $\mathcal{P}$ be a differential operator on $U$ satisfying (FC) and, for some $s_{\mathcal{P}} \geq s_{\mathcal{P},0}$, assume that*

$$c[\mathcal{P}]^{(\alpha,J)}{}_K(x) \in H^{s_{\mathcal{P}}-(m_K-|\alpha|)}(U) \qquad \text{for all } |\alpha| \leq m_K.$$

*Then the following statements hold:*

1. *(Cokernel in $H^{-s}(U)$) For every $s$ satisfying $-s_{\mathcal{P}} \leq s \leq s_{\mathcal{P}} - \max_K m_K$, we have the* finite-dimensional property

$$\dim \ker_{H^{-s}(U)} \mathcal{P}^* < +\infty,$$

*and the* invariance property

$$\ker_{H^{-s}(U)} \mathcal{P}^* = \ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*,$$

*Moreover, for any open subset $V \subseteq U$, the restriction of $\ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*$ to $V$, i.e.,*

$$\ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*|_V \coloneqq \{\mathbf{Z}|_V : \mathbf{Z} \in \ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*\}$$

*has the same dimension as $\ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*$.*

2. *(Solution operators & representation formulas associated with cokernel in $H^{-s}(U)$) Given $s_1 \geq -s_{\mathcal{P}}$, consider a family $w_{\mathbf{A}}(x) \in \widetilde{H}^{s_1}(U)$ ($\mathbf{A} \in \{1, \dots, \dim \ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*\}$) satisfying $\langle w_{\mathbf{A}}, \mathbf{Z}^{\mathbf{A}'}\rangle = \delta_{\mathbf{A}}^{\mathbf{A}'}$ for some basis $\{\mathbf{Z}^{\mathbf{A}'}\}$ of $\ker_{H^{s_{\mathcal{P}}}(U)} \mathcal{P}^*$. Then there exists an operator $\widetilde{\mathcal{S}} : C_c^\infty(U; \mathbb{C}^{r_0}) \to \mathcal{D}'(U; \mathbb{C}^{s_0})$ independent of $s_{\mathcal{P}}$ such that, for $s$ satisfying*

$$-s_{\mathcal{P}} \leq s \leq s_{\mathcal{P}} - \max_K m_K, \quad s \leq s_1, \tag{4.1.13}$$

*we have $\widetilde{\mathcal{S}} : \widetilde{H}^s(U) \to \widetilde{H}^{s+m_1}(U) \times \cdots \times \widetilde{H}^{s+m_{s_0}}(U)$ and*

$$\mathcal{P}\widetilde{\mathcal{S}} f = f - \sum_{\mathbf{A} \in \{1,\dots,\dim \ker \mathcal{P}^*\}} w_{\mathbf{A}} \langle \mathbf{Z}^{\mathbf{A}}, f\rangle \quad \text{for all } f \in \widetilde{H}^s(U),$$

*and by duality,*

$$\varphi = \widetilde{\mathcal{S}}^* \mathcal{P}^* \varphi + \sum_{\mathbf{A} \in \{1,\dots,\dim\ker \mathcal{P}^*\}} \mathbf{Z}^{\mathbf{A}} \langle w_{\mathbf{A}}, \varphi \rangle \quad \text{for all } \varphi \in H^{-s}(U).$$

*In particular, the following* Poincaré-type inequality *holds:*

$$\|\varphi - \sum_{\mathbf{A}} \mathbf{Z}^{\mathbf{A}} \langle w_{\mathbf{A}}, \varphi \rangle\|_{H^{-s}(U)} \lesssim \|\mathcal{P}^* \varphi\|_{H^{-s-m_1}(U) \times \cdots \times H^{-s-m_{s_0}}(U)} \quad \text{for all } \varphi \in H^{-s}(U).$$

3. *(Cokernel in $\widetilde{H}^{-s}(U)$) For every $s$ satisfying $-s_{\mathcal{P}} \le s \le s_{\mathcal{P}} - \max_K m_K$, we have*

$$\ker_{\widetilde{H}^{-s}(U)} \mathcal{P}^* = \{0\}.$$

4. *(Solution operators & representation formulas associated with cokernel in $\widetilde{H}^{-s}(U)$) There exists an operator*[3] *$\mathcal{S} : C^\infty(\overline{U}; \mathbb{C}^{r_0}) \to \mathcal{D}'(U; \mathbb{C}^{s_0})$ independent of $s_{\mathcal{P}}$ such that, for $s$ satisfying*

$$-s_{\mathcal{P}} \le s \le s_{\mathcal{P}} - \max_K m_K, \tag{4.1.14}$$

*we have $\mathcal{S} : H^s(U) \to H^{s+m_1}(U) \times \cdots \times H^{s+m_{s_0}}(U)$ and*

$$\mathcal{P}\mathcal{S} f = f \quad \text{for all } f \in \widetilde{H}^s(U),$$

*and by duality,*

$$\varphi = \mathcal{S}^* \mathcal{P}^* \varphi \quad \text{for all } \varphi \in \widetilde{H}^{-s}(U).$$

*In particular, the following* Friedrich-type inequality *holds:*

$$\|\varphi\|_{\widetilde{H}^{-s}(U)} \lesssim \|\mathcal{P}^* \varphi\|_{\widetilde{H}^{-s-m_1}(U) \times \cdots \times \widetilde{H}^{-s-m_{s_0}}(U)} \quad \text{for all } \varphi \in \widetilde{H}^{-s}(U).$$

*Remark* 4.1.16 (Comparison with elliptic operators). Theorem 4.1.15 – especially Parts (1) and (2) – is analogous to the standard solvability result for the Dirichlet boundary value problem (BVP) for an elliptic equation $\widetilde{\mathcal{P}}\widetilde{u} = f$ on a bounded domain (see, e.g., [Eva10, Chapter 6]). However, some interesting differences stand out:

1. Given $f \in C_c^\infty(U)$, the solution $\widetilde{\mathcal{S}} f$ in Part (2) vanishes to all possible orders at the boundary, while the solution to the elliptic BVP necessarily only vanishes to order one unless it is trivial.

[3] Unlike in Theorem 4.1.1, we are able to define $\mathcal{S}$ for $f \in H^s(U)$ (not $\widetilde{H}^s(U)$) via the existence of a Sobolev extension operator under the Lipschitz boundary regularity assumption. This feature is useful for, say, setting up a Picard iteration scheme to solve nonlinear problems; see [Ise+25b; MOT25].

2. As opposed to $\widetilde{\mathcal{S}} : \widetilde{H}^s(U;\mathbb{C}^{r_0}) \to \widetilde{H}^{s+m_1}(U) \times \cdots \times \widetilde{H}^{s+m_{s_0}}(U)$ in Part (2), it is well-known that boundary elliptic regularity, or more precisely an estimate of the form $\|\widetilde{u}\|_{W^{s+m,p}(U)} \lesssim \|f\|_{W^{s,p}(U)}$ (where $m$ is the order of $\widetilde{\mathcal{P}}$) fails in general on Lipschitz domains, the simplest case being $\widetilde{\mathcal{P}} = -\Delta$ and $U \subseteq \mathbb{R}^2$ has a corner; see [Gri11, Chapters 4 and 5].

Another basic but important distinction is that operators considered in Theorem 4.1.15 are *not* Fredholm in most cases of interest, as the kernel of $\mathcal{P}$ may be infinite dimensional (take, for instance, the divergence operator). Our approach therefore does *not* rely on the Fredholm alternative theorem as in the usual proof of elliptic solvability results (see, e.g., [Eva10, Chapter 6]).

*Remark* 4.1.17 (Applications to Poincaré- and Friedrich-type estimates). Simply combining Theorem 4.1.14 with the Poincaré- and Friedrich-type inequalities in Theorem 4.1.15.(2) and (4), respectively, recovers and generalizes (to the rough-coefficients, Lipschitz domain setting) various standard inequalities in analysis, such as the standard Poincaré and Friedrich inequalities (for $\mathcal{P}^*$ equal to the gradient), Korn's first and second inequalities (for $\mathcal{P}^*$ equal to the symmetric gradient) [Kor09; KO89], Korn's inequalities for the trace-free symmetric gradient [Dai06], etc.

*Remark* 4.1.18. A natural question one might ask is what happens on domains with boundary less regular than Lipschitz. In the case of the divergence operator, Acosta–Duran–Muschietti [ADM06] showed that the existence of a Bogovskii-type operator that is bounded from $L^p$ to $W^{1,p}$ for all $1 < p < \infty$ is equivalent to $U$ being a John domain. For more discussion on this literature, see §4.1. It would be interesting to generalize this to general differential operators satisfying (FC).

In fact, in [Ise+25b], we also establish an analog of Theorem 4.1.15 in the case of unbounded Lipschitz domains using weighted Sobolev spaces, under the additional assumptions that (i) $\mathcal{P}$ asymptotes (in a suitable way) to a homogeneous constant-coefficient operator at infinity, and (ii) $U$ is non-degenerate (in a suitable way) towards infinity. In this case, the cokernel varies according to the weight in the function space. We refer the interested reader to that paper for more details.

## Related works

### Bogovskii-type operators in fluid dynamics

We use the term Bogovskii-type operator to refer to the type of operator that yields compactly supported solutions to the underdetermined PDE in question. The original Bogovskii operator (4.1.2) has become a basic tool in fluid dynamics [Gal11], where the divergence free constraint plays a fundamental role in the study of incompressible fluids. The way the operator is often used is to construct divergence free vector fields by starting with a vector field that $v$ that is approximately divergence-free and then correcting $v$ to obtain a truly divergence free vector field $V = v + u$ by solving $\operatorname{div} u = -\operatorname{div} v$. Applying the Bogovskii operator to solve this equation does not disturb the compact support property. A recent and profound example where the Bogovskii operator is used in this way is the recent breakthrough paper [ABC22], which constructs nonunique solutions of the Leray-Hopf class to the forced Navier Stokes equations on $\mathbb{R}^3$. To give just a few other examples, see [BM90; BS87; Gal91; GS89; Iwa89; Shu92; Tan91].

In [IO16], the authors construct a Bogovskii-type operator for the equation $\operatorname{div} R = F$, where $R$ is a symmetric 2 tensor field and $F$ is a compactly supported vector field that is orthogonal to the Killing vector fields of Euclidean space, which are spanned by translations and rotations. This operator is key to the construction of nonunique and energy non-conserving continuous weak solutions to $3D$ incompressible Euler defined on Euclidean space. The same operator turned out surprisingly to be crucial for the construction in [Ise24] of weak (periodic) solutions to $3D$ Euler of class $\bigcap_{\epsilon>0} C^{1/3-\epsilon}$ that fail to conserve energy. This latter result is the best result towards the endpoint case of the famous Onsager conjecture. We expect that the general class of Bogovskii-type operators obtained in this chapter will continue to be useful for related and future applications.

**Underdetermined problems in general relativity**

Due to the divergence structure of the Einstein constraint equation (on a spacelike Cauchy hypersurface), divergence equations have applications in general relativity, especially in initial data construction. Corvino [Cor00] in his pioneering work proved rigidity estimates for the dual linearized scalar curvature operator on a compact region via variational methods and applied them to prove a gluing result for the prescribed scalar curvature problem, which corresponds to time-symmetric initial data sets. Corvino–Schoen [CS06] generalized this gluing technique to the full Einstein constraint equation, and constructed a large class of initial data sets which coincide with Kerr initial data outside a large ball. Chruściel–Delay[CD03] extended the aforementioned gluing results by establishing mapping properties of linearized constraint operator on a large class of weighted Sobolev spaces, and Delay [Del12] used a similar method to study underdetermined elliptic operators and proved various gluing results for those operators. Carlotto–Schoen [CS16a] were the first to construct initial data with conic support by developing a gluing scheme using variational techniques. Hintz [Hin23b; Hin24a; Hin24b], with a geometric microlocal approach, generalized Corvino–Schoen-type gluing by considering generic initial data sets outside of a compact region.

In [MT22] and [MOT23] (see also Chapter 2 and Chapter 3), the authors constructed conic- and Bogovskii-type solution operators, respectively, for the linearized Einstein vacuum constraint operator around flat initial data. Moreover, these operators were applied to simplify the proofs of existing gluing results and obtain new results, such as the existence of nontrivial initial data sets localized to degenerate cones [MT22] (see also Chapter 2), and obstruction-free gluing with a sharp positivity condition [MOT23] (see also Chapter 3), improving upon the seminal work of Czimek–Rodnianski [CR22]. The ideas introduced in

[MT22; MOT23] (see also Chapter 2 and Chapter 3) served as a precursor for the present chapter.

We also mention a recent work of Chruściel–Cogo–Nützi [CCN24], in which a Bogovskii-type solution operator for the linearized constant scalar curvature operator was constructed for the hyperbolic metric near infinity.

### Overdetermined problems in geometry

The study of rigidity problems in geometry and elasticity has a long and rich history. By linearization, such problems often lead to overdetermined differential operators $\mathcal{P}^*$, many of which turn out to satisfy (FC). We refer the interested reader to [Res94; DM05; CDM24] (in geometry), [FJM02] (in elasticity) and references therein. Among these works, Reshetnyak's approach to the derivation of integral representation formulas has been a major influence on this chapter, as highlighted above (see, e.g., Remark 4.1.13).

### Divergence operator on rough domains

There is a substantial body of work on divergence operators and Poincaré-type inequalities on rough domains. In the aforementioned work [ADM06], Acosta–Duran–Muschietti constructed an explicit solution operator for the divergence equation on John domains using singular integral techniques. Duran–Garcia [DG08][DG10] proved existence of bounded right inverse of the divergence operators on planar simply-connected Hölder-$\alpha$ domains and domains with an external cusp, using singular integrals and $A_p$-weights. Duran–Muschietti–Russ–Tchamitchian [Dur+10] gave a general sufficient condition on invertibility of divergence operators on weighted $L^p$ spaces on an arbitrary domain via Calderon-Zygmund type arguments. An interesting question would be to generalize these investigations to the setting of differential operators satisfying (FC).

### Other related problems

There is a strong resemblance between our construction of the rough integral kernel (see Section 4.3) and a well-known proof of Poincaré's lemma on star-shaped domains via the construction of a chain homotopy to the De Rham chain complex over the base point (see, for instance, [Spi65, Theorem 4.11]). Indeed, Bogovskii-type chain homotopy for the De Rham complexes has been found by Takahashi [Tak92]; since the last map in the De Rham complex is the divergence operator, this chain homotopy generalizes the original Bogovskii operator. We also note the recent work of Nützi [Nüt24] on the construction of a Bogovskii-type chain homotopy for an elliptic complex whose last homomorphism corresponds to the symmetric trace-free divergence operator. In this case, the chain homotopy specializes to a Bogovskii-type solution operator for the symmetric trace-free divergence operator.

An approach akin to ours for De Rham complex may be of utility in the study of (non-linear) pullback equation for differential forms, which is the generalization of the well-known theorems of Darboux (see, for instance, [Can01, Chapter 8]) and Dacorogna–Moser [DM90] on finding a diffeomorphism that pulls back a given differential form to another given differential form (symplectic in the case of Darboux, and volume in the case of Dacorogna–Moser). We refer to the monograph of Csato–Dacorogna–Kneuss [CDK12] for more on this topic.

## Structure of the Chapter

the chapter is structured as follows.

- In Section 4.2, we collect the notation, conventions and preliminary facts from analysis and geometry used in this chapter.
- In Section 4.3, we summarize the ideas behind our construction of solution operators and representation formulas via (RC), and demonstrate our approach in the simple special case of the divergence operator with a lower order term, i.e., $\mathcal{P}u = \partial_j \mathbf{u}^j + B_j \mathbf{u}^j$ (which already leads to results that are, to the best of our knowledge, new). This discussion will serve as a motivation for the remainder of the chapter concerning the general case.
- In Section 4.4, we prove a proposition that will allow us to show that the integral kernel produced by our method defines a singular integral operator with good boundedness properties.
- In Section 4.5, we give a precise formulation of (RC) and describe how it leads to our construction of solution operators and representation formulas with prescribed support properties, thereby proving (precise versions of) Theorems 4.1.1 and 4.1.2.
- In Section 4.6, we study graded augmented systems and establish Propositions 4.1.6 and 4.1.7.
- In Section 4.7, we prove Theorem 4.1.11.
- Finally, in Appendix B, we write down special graded augmented systems for operators (2)–(7) in Theorem 4.1.14 in the geometric context, which turn out to be completely integrable on constant sectional curvature backgrounds. On such backgrounds, we explicitly compute the fundamental matrix $\mathbf{\Pi_A}^{\mathbf{A}'}(y, y_1, s)$ on geodesic segments. Specializing further to the flat background, we also explicitly compute the Bogovskii and conic integral kernels.

## 4.2 Preliminaries

### Notation and conventions for the operator $\mathcal{P}$

Some important objects used throughout this chapter are as follows. Let $U$ be an open subset of $\mathbb{R}^d$, and let $W$ be an open subset of $\mathbb{R}^d \times \mathbb{R}^d$ whose $\mathbb{R}^d_y$-projection contains $U$, i.e.,

$$U \subseteq \{y \in \mathbb{R}^d : \exists y_1 \text{ such that } (y, y_1) \in W\}.$$

We denote by $\mathbf{x} = \mathbf{x}(y, y_1, s)$ a family of curves defined for $(y, y_1) \in W$ and $s \in [0, 1]$ with $\partial_s \mathbf{x}(y, y_1, s) \neq 0$ for every $s \in [0, 1]$, such that $\mathbf{x}(y, y_1, 0) = y$ and $\mathbf{x}(y, y_1, 1) = y_1$.

Let $\mathcal{P}$ be an $r_0 \times s_0$ (possibly complex-)matrix-valued differential operator on $U$. Unless otherwise specified, we follow the following conventions:

- $u = (u^K)_{K=1,\dots,s_0}$ is an $\mathbb{C}^{s_0}$-valued function, $f = (f^J)_{J=1,\dots,r_0}$ is an $\mathbb{C}^{r_0}$-valued function, $\varphi = (\varphi_J)_{J=1,\dots,r_0}$ is an $\mathbb{C}^{r_0}$-valued function, and $\psi = (\psi_K)_{K=1,\dots,s_0}$ is an $\mathbb{C}^{s_0}$-valued function.
- We employ the natural $L^2$-inner products on $(U, \mathrm{d}x)$, which are

$$\langle \psi, u \rangle \coloneqq \mathrm{Re} \int_U \overline{\psi}_K u^K \, \mathrm{d}x, \quad \langle \varphi, f \rangle \coloneqq \mathrm{Re} \int_U \overline{\varphi}_J f^J \, \mathrm{d}x.$$

- $\mathcal{P}^*$ denotes the adjoint of $\mathcal{P}$ with respect to the above $L^2$-inner products on $(U, \mathrm{d}x)$.
- Finally, we will often omit writing out the identity matrix in equations, e.g., $\delta_0(x-y) = \delta_0(x-y) I_{r_0 \times r_0}$ in (4.5.4).

### Notation and conventions for geometry and analysis

Throughout the chapter, we adopt the following conventions.

- We write $A \lesssim B$ if there exists a positive constant $C > 0$ (that may differ from expression to expression) such that $A \leq CB$, and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. We specify the dependencies of $C$ by a subscript, e.g., $A \lesssim_d B$.
- We write $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.
- We adopt the Einstein summation convention, i.e., repeated upper and indices are summed.
- $\alpha, \beta, \gamma, \dots$ usually denote multi-indices, i.e., elements of $\mathbb{Z}^d_{\geq 0}$ ($\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers). As usual, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.
- We write $X \Subset U$ for a compact subset $X$ of some topological space $U$.

- We also use the following notation for geometric objects:
  - $B_R(x)$: Ball of radius $R$ centered at $x$ in $\mathbb{R}^d$.
  - $C_\Omega$: Given $\Omega \subseteq \mathbb{S}^{d-1}$, the cone over $\Omega$ is defined as $C_\Omega = \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \Omega\}$.
  - $\mathrm{d}S(\omega)$: the $(d-1)$-dimensional surface measure on $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ (set of unit directions)
- Fix $m_{<1} \in C_c^\infty(\mathbb{R}^d)$ that is nonnegative, equals 1 on $B_1(0)$, and vanishes outside $B_2(0)$. Given $N \in 2^{\mathbb{Z}}$, define $m_{<N}(\xi) \coloneqq m_{<1}(N^{-1}\xi)$ and $m_N(\xi) \coloneqq m_{<2N}(\xi) - m_{<N}(\xi)$. These functions form a smooth partition of unity subordinate to dyadic annuli in $\mathbb{R}^d$, i.e., $m_{<1} + \sum_{N \in 2^{\mathbb{Z}_{\geq 0}}} m_N = 1$ on $\mathbb{R}^d$ and $\operatorname{supp} m_N \subseteq \{N < |\xi| < 4N\}$.
- We use the following convention for the Fourier and inverse Fourier transforms on $\mathbb{R}^d$:
$$\mathcal{F}[f](x) = \int f(y) e^{-i\xi\cdot y} \, \mathrm{d}y, \quad \mathcal{F}^{-1}[F](x) = \int F(\xi) e^{i\xi\cdot x} \, \frac{\mathrm{d}\xi}{(2\pi)^d}.$$
Given $m : \mathbb{R}^d \to \mathbb{C}$, the *Fourier multiplier operator with symbol $m$* is defined as
$$m(D)f = \mathcal{F}^{-1}[m(\xi)\mathcal{F}[f](\xi)]$$
for every Schwartz function $f$ on $\mathbb{R}^d$.
- To discuss the boundedness properties of the solution operators below, it will be convenient to also use the language of *pseudodifferential operators*. Given $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, the *right-quantized pseudodifferential operator associated with the symbol $a$* (or simply the *right-quantization of $a$*) is defined as
$$a(D, x)f = \int\int a(\xi, y) f(y) e^{-i\xi\cdot(x-y)} \, \frac{\mathrm{d}\xi}{(2\pi)^d} \, \mathrm{d}y$$
for every Schwartz function $f$ on $\mathbb{R}^d$. Note that a Fourier multiplier is a particular instance of a (right-quantized) pseudodifferential operator with $a(\xi, y) = m(\xi)$. The integral kernel $K(x, y)$ of $a(D, x)$ is related to the symbol $a(\xi, y)$ by
$$\int K(z + y, y) e^{-i\xi\cdot z} \, \mathrm{d}z = a(\xi, y). \tag{4.2.1}$$

## Preliminaries on Sobolev spaces

The classical references on this subject are Lions–Magenes [LM72a; LM72b; LM73], Grisvard [Gri11], and Triebel [Tri10]. Our definitions and notation, however, follow closely those of McLean [McL00]; see also [CHM17].

Given $s \in \mathbb{Z}_{\geq 0}$ (nonnegative integers) and $p \in (1, \infty)$, we define

$$\|f\|_{W^{s,p}} := \Big( \sum_{\alpha : |\alpha| \leq s} \|\partial^\alpha f\|_{L^p}^p \Big)^{\frac{1}{p}}, \qquad \text{where } f \in \mathcal{D}'(\mathbb{R}^d),$$

and write $W^{s,p}$ for the space of all distributions $f$ on $\mathbb{R}^d$ with $\|f\|_{W^{s,p}} < +\infty$. For $s \geq 0$, $W^{s,p}$ is defined by (complex) interpolation (see [Tri10, §2.4.2][4])), and $W^{-s,p'} := (W^{s,p})^*$, where $\frac{1}{p'} = 1 - \frac{1}{p}$. It turns out that $C_c^\infty(\mathbb{R}^d)$ is a dense subspace of $W^{s,p}$ [Tri10, §2.3.2] (in fact, $C_c^\infty(\mathbb{R}^d) \subseteq W^{s,p}$ is dense even for $s < 0$ once $W^{s,p}$ is identified with a space of distributions as below). Hence, by the duality pairing

$$W^{-s,p'} \times W^{s,p} \to \mathbb{R}, (f, g) \mapsto \langle f, g \rangle$$

we may **identify** $f \in W^{-s,p'}$ with a distribution on $\mathbb{R}^d$ (extended from $g \in C_c^\infty(\mathbb{R}^d)$ to $g \in W^{s,p}$ uniquely via continuity). Moreover, $W^{s,p}$ is reflexive, which implies that the above pairing induces the isomorphism $W^{-s,p'} \equiv (W^{s,p})^*$ holds for all $s \in \mathbb{R}$ [Tri10, §2.6.1].

Given an open subset $U$ of $\mathbb{R}^d$, $s \in \mathbb{R}$ and $p \in (1, \infty)$, we introduce the following spaces:

- $W^{s,p}(U)$, which consists of distributions on $U$ which arise by restricting elements of $W^{s,p}$ to $U$, i.e.,

$$W^{s,p}(U) := \{f \in \mathcal{D}'(U) : f = \widetilde{f}|_U \text{ for some } \widetilde{f} \in W^{s,p}\} \tag{4.2.2}$$

  equipped with the norm

$$\|f\|_{W^{s,p}(U)} := \inf_{\widetilde{f} \in W^{s,p} : \widetilde{f}|_U = f} \|\widetilde{f}\|_{W^{s,p}}.$$

- $\widetilde{W}^{s,p}(U)$, which is the closure of $C_c^\infty(U)$ viewed as a subspace of $W^{s,p}$, i.e.,

$$\widetilde{W}^{s,p}(U) := \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{s,p}}}. \tag{4.2.3}$$

Note the fundamental distinction that $\widetilde{W}^{s,p}(U) \subseteq \mathcal{D}'(\mathbb{R}^d)$ (i.e., distributions on $\mathbb{R}^d$), while $W^{s,p}(U) \subseteq \mathcal{D}'(U)$ (i.e., distributions on $U$). Elements in $\widetilde{W}^{s,p}(U)$ may be identified with elements in $W^{s,p}(U)$, by the natural maps $\widetilde{W}^{s,p}(U) \to W^{s,p} \to W^{s,p}(U)$. In particular, for $\varphi \in \widetilde{W}^{s,p}(U)$, we have $\|\varphi\|_{W^{s,p}(U)} \leq \|\varphi\|_{\widetilde{W}^{s,p}(U)}$. Note, however, that the map $\widetilde{W}^{s,p}(U) \to W^{s,p}(U)$ may *not* be one-to-one; indeed, for $s < 0$, note that $\widetilde{W}^{s,p}(U)$ may contain distributions (on $\mathbb{R}^d$) supported in $\partial U$, while such objects do not even correspond to non-trivial elements in $\mathcal{D}'(U)$, and thus in $W^{s,p}(U)$.

[4]In [Tri10], $W^{s,p}(U)$, $\widetilde{W}^{s,p}(U)$, and $W_0^{s,p}(U)$ are denoted $H_p^s(U)$, $\widetilde{H}_p^s(U)$, and $\mathring{H}_p^s(U)$, respectively. Note also that $H_p^s = F_{p,2}^s$.

Intuitively, for $s > 0$, $\widetilde{W}^{s,p}(U)$ consists of elements that vanish to all possible orders on $\partial U$, while those in $W^{s,p}(U)$ are allowed to be nontrivial near $\partial U$. However, when $s < 0$, one must be careful of the distinctions discussed in the preceding paragraph. For more on these points, see Remarks 4.2.3 and 4.2.4 below.

Given a closed subset $F \subseteq \mathbb{R}^d$, we introduce the closed subspace $W^{s,p}_F$ of $W^{s,p}$ that consists of elements that are supported in $F$, i.e.,

$$W^{s,p}_F := \{f \in W^{s,p}(\mathbb{R}^d) : \operatorname{supp} f \subseteq F\}. \tag{4.2.4}$$

Clearly, we have $\widetilde{W}^{s,p}(U) \subseteq W^{s,p}_{\overline{U}}$; equality holds under additional assumptions on $\partial U$ (see Remark 4.2.3). Given an open subset $U$ of $\mathbb{R}^d$, note that we have the Banach space isomorphism

$$W^{s,p}(U) \equiv W^{s,p}/W^{s,p}_{\mathbb{R}^d \setminus U}, \tag{4.2.5}$$

where the isomorphism is given by the quotient of the restriction map $W^{s,p} \to W^{s,p}(U)$, $\widetilde{f} \mapsto \widetilde{f}|_U$ by its kernel (which is precisely $W^{s,p}_{\mathbb{R}^d \setminus U}$). Using $W^{-s,p'} = (W^{s,p})^*$, it is also clear that we have the identity

$$W^{-s,p'}_{\mathbb{R}^d \setminus U} = (\widetilde{W}^{s,p}(U))^{\perp}, \tag{4.2.6}$$

where for a subspace $Y$ of a Banach space $X$, $Y^{\perp} := \{\ell \in X^* : \langle \ell, g\rangle = 0 \text{ for all } g \in Y\}$ (space of annihilators). The following duality statement, which generalizes $(W^{s,p})^* = W^{-s,p'}$, is now obvious:

**Lemma 4.2.1** (Duality). *For any open subset $U \subseteq \mathbb{R}^d$, $s \in \mathbb{R}$ and $p \in (1, \infty)$, the bilinear map*

$$W^{-s,p'}(U) \times \widetilde{W}^{s,p}(U) \to \mathbb{R},\ (f, g) \mapsto \langle f, g\rangle,$$

*which coincides with the usual pairing $\langle f, g\rangle$ for $f \in W^{-s,p'}(U) \subseteq \mathcal{D}'(U)$ and $g \in C^\infty_c(U)$, is well-defined, continuous, and induces the Banach space isomorphisms*

$$(\widetilde{W}^{s,p}(U))^* \equiv W^{-s,p'}(U), \quad (W^{-s,p'}(U))^* \equiv \widetilde{W}^{s,p}(U). \tag{4.2.7}$$

*Proof.* The first isomorphism follows from the natural identification $Y^* \equiv X^*/Y^{\perp}$ for any closed subspace $Y$ of a Banach space $X$. The second statement then follows from the fact that $\widetilde{W}^{s,p}(U)$ is a reflexive Banach space, being a closed subspace of the Banach space $W^{s,p}$, which is reflexive. □

The Rellich–Kondrachov theorem holds for both scales of Sobolev spaces $W^{s,p}(U)$ and $\widetilde{W}^{s,p}(U)$:

**Lemma 4.2.2** (Rellich–Kondrachov). *For any bounded open subset $U \subseteq \mathbb{R}^d$, $s \in \mathbb{R}$, $p \in (1, \infty)$, and $\delta > 0$, the natural embeddings*

$$W^{s,p}(U) \to W^{s-\delta,p}(U), \quad \widetilde{W}^{s,p}(U) \to \widetilde{W}^{s-\delta,p}(U)$$

*are compact.*

*Proof.* For $\widetilde{W}^{s,p}(U)$, this result is a quick consequence of Rellich–Kondrachov for $W^{s,p}_{\overline{U}}$ (see, e.g., McLean [McL00, Theorem 3.27]). For $W^{s,p}(U)$, the result follows by duality (Lemma 4.2.1). □

When $p = 2$ (and $s \in \mathbb{R}$), we write $H^s(U) = W^{s,2}(U)$ and $\widetilde{H}^s(U) = \widetilde{W}^{s,2}(U)$, etc.

*Remark* 4.2.3 (Consequences of regularity of $\partial U$). So far, we have not used any regularity assumptions on $\partial U$. Under some regularity assumptions on $\partial U$, the spaces introduced above may be given alternative characterizations as follows.

1. If $\partial U$ is $C^0$, then $W^{s,p}_{\overline{U}} = \widetilde{W}^{s,p}(U)$.
2. If $\partial U$ is Lipschitz, $s = 1, 2, \ldots$, and $p \in (1, \infty)$, we have the equivalence

$$\|f\|_{W^{s,p}(U)} \simeq_{s,p} \Big( \sum_{\alpha : |\alpha| \leq s} \|\partial^\alpha f\|^p_{L^p(U)} \Big)^{\frac{1}{p}}.$$

For $p = 2$, the proofs of these assertions can be found in McLean [McL00, Chapter 3]; the case $p \neq 2$ can also be handled in a similar way. See also [Tri10, §4.3.2, §4.2.4] for the case $\partial U$ is smooth.

*Remark* 4.2.4 (The space $W^{s,p}_0(U)$). Closely related to $\widetilde{W}^{s,p}(U)$ is the space $W^{s,p}_0(U)$, which is the closure of $C^\infty_c(U)$ with respect to $\|\cdot\|_{W^{s,p}(U)}$ (as opposed to $\|\cdot\|_{W^{s,p}}$ as in the case of $\widetilde{W}^{s,p}(U)$). An alternative characterization for $W^{s,p}_0(U)$ is in terms of vanishing boundary trace [Tri10, §4.7.1]: for $p \in (1, \infty)$, and $s = m + \alpha + \frac{1}{p}$ with $m \in \mathbb{Z}_{\geq 0}$ and $\alpha \in [0, 1)$, we have

$$W^{s,p}_0(U) = \{f \in W^{s,p}(U) : f|_{\partial U} = \tfrac{\partial}{\partial \boldsymbol{\nu}} f|_{\partial U} = \cdots = \tfrac{\partial^m}{\partial \boldsymbol{\nu}^m} f|_{\partial U} = 0\}.$$

There are subtle differences between the two spaces $\widetilde{W}^{s,p}(U)$ and $W^{s,p}_0(U)$. For simplicity, assume that $\partial U$ is smooth and bounded. Then for any $p \in (1, \infty)$, we have (see, e.g., [Tri10, §4.3.2])

$$\widetilde{W}^{s,p}(U) \subseteq W^{s,p}_0(U) \quad \text{with equality if } \frac{1}{p} - 1 < s < \infty,\ s + \frac{1}{p} \notin \mathbb{Z},$$

where $\widetilde{W}^{s,p}(U)$ is realized as a (non-closed) subspace of $W^{s,p}(U)$ under the map $\widetilde{W}^{s,p}(U) \to W^{s,p} \to W^{s,p}(U)$.

The inclusion $\widetilde{W}^{s,p}(U) \subseteq W^{s,p}_0(U)$ may not be strict. Indeed, for $U = (0, \infty) \subseteq \mathbb{R}$, for any $m \in \mathbb{Z}_{\geq 0}$ and $f \in C^\infty_c((-1, 1))$ with $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$ but $f^{(m)}(0) \neq 0$ (where the condition is $f(0) \neq 0$ if $m = 0$), we have $f|_U \in H^{m+\frac{1}{2}}_0(U) \setminus \widetilde{H}^{m+\frac{1}{2}}(U)$. We also note that the inclusion fails in general if $s < \frac{1}{p} - 1$. Indeed, for $U = (0, \infty) \subseteq \mathbb{R}$ and $s < \frac{1}{2}$, $\delta_0 \in \widetilde{H}^s(U)$, while $\delta_0$ is not even a nontrivial element of $\mathcal{D}'(U)$.

We note that $W^{s,p}_0(U)$ is heavily used in the classical treatments of Lions–Magenes [LM72a] and Grisvard [Gri11]. In this chapter, we prefer to use $\widetilde{W}^{s,p}(U)$ as it behaves better under standard operations such as duality (see (4.2.7)) and interpolation (see [Tri10, §4.3.2, Theorem 2]).

## 4.3 A preview of the recovery on curves method

The goal of this expository section is to provide a summary and a basic example for our derivation of integral formulas (i.e., solution operators and representation formulas) from (RC), which is carried out in Section 4.5 in the general case. In Section 4.3, we summarize the proof of Theorems 4.1.1 and 4.1.2, and in Section 4.3, we work out the basic example of the divergence operator $\mathcal{P}u = \partial_j u^j + B_j u^j$ (for $u : U \to \mathbb{R}^d$) with possibly variable coefficients $B : U \to \mathbb{R}^d$ according to our general method developed in Sections 4.4 and 4.5 below. As a byproduct, we provide a self-contained derivation of the Bogovskii and conic operators, (4.1.2) and (4.1.3), respectively.

### Summary of the derivation of the integral formulas

Here, we summarize the proof of Theorems 4.1.1 and 4.1.2, which concern the derivation of integral solution operators and representation formulas for $\mathcal{P}$ (with $C^\infty(\overline{U})$ coefficients) satisfying (RC). The precise results and arguments are in Section 4.5 below.

*Step* 1: Constructing a rough integral kernel supported on curves. Consider curves $\mathbf{x}(y, y_1, s) \in \mathbb{R}^d$ with $s \in [0, 1]$, where $y$ and $y_1$ are the two end points (i.e., $\mathbf{x}(y, y_1, 0) = y$ and $\mathbf{x}(y, y_1, 1) = y_1$). Recall that the (RC) condition posits that we are able to linearly recover the value of a function $\phi$ at $y$ in terms of its jet at $y_1$ and the jet of $\mathcal{P}^*\phi$ along $\mathbf{x}(y, y_1, \cdot)$. By duality, (RC) on the curve $\mathbf{x}(y, y_1, \cdot)$ amounts to the existence of distributions $K_{y_1}(\cdot, y)$ and $b_{y_1}(\cdot, y)$ on $\mathbb{R}^d$ with the following properties:

1. (Green's function) We have

$$\mathcal{P}K_{y_1}(x, y) = \delta_0(x - y) - b_{y_1}(x, y) \quad \text{in } U,$$

   where $\mathcal{P}$ acts in the $x$-variable; and

2. (Prescribed support) $K_{y_1}(\cdot, y)$ is supported on the image of the curve $\mathbf{x}(y, y_1, [0, 1])$, and $b_{y_1}(\cdot, y)$ is supported in $\{y_1\}$.

See Step 1 in Section 4.3 for a concrete example, and Proposition 4.5.1 for a precise formulation. One case where this observation is immediately useful is when $y_1$ lies outside of the open set $U$ where we wish to solve $\mathcal{P}u = f$. Then $\mathcal{P}K_{y_1}(x, y) = \delta_0(x - y)$ in $U$, so the integral operator $\mathcal{S}_{y_1} f(x) := \int K_{y_1}(x, y) f(y)\, \mathrm{d}y$ (for $f \in C^\infty_c(U)$) is already a solution operator (i.e., right-inverse) for $\mathcal{P}$!

*Remark* 4.3.1. For the construction in the case $y_1 \notin U$, note that the following weaker version of the recovery on curves condition is sufficient:

(wRC) Given any $\varphi \in C^\infty_c(U)$ that vanishes in a neighborhood of $\mathbf{x}(1)$, there exists a linear way to continuously recover $\varphi(\mathbf{x}(0))$ from (the jet of) $\mathcal{P}^*\varphi$ on $\mathbf{x}$.

It is conceptually interesting to observe that the apparently weaker version (wRC), in fact, essentially implies (RC) by a truncation argument; see Remark 4.5.3. However, we also note that the stronger version (RC) directly follows from an augmented system; see Section 4.3 below, as well as Sections 4.6 and B.

While we have already succeeded in finding a solution operator with an integral kernel $K_{y_1}(x,y)$ having prescribed support properties, it is unfortunately very singular (indeed, $K_{y_1}(\cdot,y)$ is merely a distribution). In particular, $\mathcal{S}_{y_1}$ does not obey good boundedness properties in standard function spaces (e.g., Sobolev spaces), and is unsuitable for many applications (e.g., nonlinear analysis).

*S*tep 2: Smooth averaging. To remedy the issue of the singularity of $K_{y_1}(\cdot,y)$, we introduce a smooth function $\eta(y,y_1)$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\int \eta(y,y_1)\,\mathrm{d}y_1 = 1$ and define a new *smoothly averaged integral kernel*

$$K_\eta(x,y) \coloneqq \int K_{y_1}(x,y)\eta(y,y_1)\,\mathrm{d}y_1.$$

Under suitable assumptions, the smoothly averaged kernel $K_\eta(x,y)$ has the following properties:

1. (Green's function) We have

$$\mathcal{P}K_\eta(x,y) = \delta_0(x-y) - b_\eta(x,y) \quad \text{in } U, \qquad \text{where } b_\eta(x,y) \coloneqq \int b_{y_1}(x,y)\eta(y,y_1)\,\mathrm{d}y_1;$$

2. (Prescribed support) $\operatorname{supp} K_\eta(\cdot,y) \subseteq \cup_{y_1 \in \operatorname{supp}\eta(y,\cdot)} \operatorname{ran}\mathbf{x}(y,y_1,\cdot)$;

3. (Optimal regularization) $K_\eta$ is a singular integral kernel such that $\mathcal{S}_\eta f \coloneqq \int K_\eta(x,y)f(y)\,\mathrm{d}y$ defines a pseudodifferential operator of order $-m$ (where $m$ is the order of $\mathcal{P}$).

In particular, we may now summarize the proof of Theorem 4.1.1. Assuming that the $y_1$-support of $\eta(y,y_1)$ lies outside of $U$, i.e.,

$$\overline{U} \cap \overline{\cup_{y\in U} \operatorname{supp}\eta(y,\cdot)} = \emptyset, \tag{4.3.1}$$

the integral operator $\mathcal{S}_\eta$ again defines a solution operator for $f \in C_c^\infty(U)$ in the sense that

$$\mathcal{P}\mathcal{S}_\eta f = f \quad \text{for all } f \in C_c^\infty(U),$$

but this time, has desirable boundedness properties (i.e., regularizing to optimal order) in standard function spaces. We emphasize that $\mathcal{S}_\eta f$ is *not* compactly supported in $U$ in general (cf. Step 3 below). By duality, we also obtain the representation formula

$$\mathcal{S}_\eta^* \mathcal{P}^* \varphi = \varphi \quad \text{for all } \varphi \in C_c^\infty(U)$$

where we emphasize that the compact support assumption on $\varphi$ is necessary (cf. Step 3 below). We call $\mathcal{S}_\eta$ a *conic-type solution operator.*

*Remark* 4.3.2. As we will see in Section 4.3 (see also Example 4.5.13), this construction generalizes the conic solution operator introduced by Oh–Tataru [OT19] for the divergence operator $\mathcal{P}u = \partial_j u^j$ on $\mathbb{R}^d$, which explains its name. The conic solution operator, in turn, has been generalized to the case of the linearized Einstein constraint equations around the flat space by the author and Tao [MT22] (see also Chapter 2), who used it to simplify and improve the nonlinear construction of localized asymptotically flat initial data sets by Carlotto–Schoen [CS16a] (see also Section B).

*S*tep 3: Solutions with compact support I: completely integrable case. Without the simplifying assumption (4.3.1), $\mathcal{S}_\eta$ does not directly define a right-inverse of $\mathcal{P}$. Nevertheless, under suitable assumptions, $b_\eta(x,y)$ turns out to be smooth and supported (in the $x$-variable) in $\operatorname{supp}\eta(y,\cdot)$. Therefore,

$$\mathcal{P}\mathcal{S}_\eta f = f - \mathcal{B}_\eta f \quad \text{ where } \mathcal{B}_\eta : C_c^\infty(U) \to C_c^\infty(U) \text{ is smoothing.}$$

Observe that if $f \in C_c^\infty(U)$ then $u = \mathcal{S}_\eta f \in C_c^\infty(U)$ (under suitable assumptions on $\mathbf{x}$, $\eta$ and using the optimal regularization property). The observation that $u$ does *not* solve $\mathcal{P}u = f$ is now not surprising: In order for $C_c^\infty(U)$ solution to exist, $f$ must be orthogonal to the formal cokernel of $\mathcal{P}$ (i.e., $\ker \mathcal{P}^*$, which consists of solutions to $\mathcal{P}^*\mathbf{Z} = 0$ with $\mathbf{Z} \in C^\infty(\overline{U})$), which is often nontrivial; see Section 4.3, as well as Appendix B, for examples. Specifically, if $\mathcal{P}\tilde{u} = f$, and $\tilde{u}$ has compact support, then for any cokernel element $\mathbf{Z} \in \ker \mathcal{P}^*$ one has $\int \langle f, \mathbf{Z}\rangle dx = \int \langle \mathcal{P}\tilde{u}, \mathbf{Z}\rangle dx = \int \langle \tilde{u}, \mathcal{P}^* Z\rangle dx = 0$.

Let us first discuss an important special case, namely, when the point distribution $b_{y_1}$ from Step 1 is already of the form

$$b_{y_1}(x,y) = \sum_{\mathbf{A} \in \{1,\dots,\dim \ker \mathcal{P}^*\}} \mathbf{Z}^A(y) \zeta_{\mathbf{A}}(x, y_1) \tag{4.3.2}$$

where $\{\mathbf{Z}^{\mathbf{A}}(x)\}_{\mathbf{A} \in \{1,\dots,\dim \ker \mathcal{P}^*\}} \subseteq C^\infty(\overline{U})$ is a basis for $\ker \mathcal{P}^*$ and $\zeta_{\mathbf{A}}(\cdot, y_1)$ is a distribution supported in $\{y_1\}$. As we will see in Section 4.3, the divergence operator $\mathcal{P}u = \partial_j u^j$ falls into this case [Bog79], and in fact, so do any operators with a *completely integrable* graded augmented system by Proposition 4.1.7 (see also Appendix B for many concrete examples). In this case, choosing $\eta(y, y_1) = \eta(y_1)$ in Step 2 immediately leads to

$$\mathcal{P}\mathcal{S}_\eta f = f - \mathcal{B}_\eta f, \quad \text{ with } \mathcal{B}_\eta f(x) = \sum_{\mathbf{A} \in \{1,\dots,\dim \ker \mathcal{P}^*\}} \left( \int \mathbf{Z}^{\mathbf{A}}(y) f(y) \, \mathrm{d}y \right) \langle \zeta_{\mathbf{A}}(x, \cdot), \eta(\cdot)\rangle. \tag{4.3.3}$$

In particular, if $f$ is orthogonal to $\ker \mathcal{P}^*$, then $\mathcal{B}_\eta f = 0$ and thus $\mathcal{P}\mathcal{S}_\eta f = f$, i.e., $S_\eta$ is a solution operator. By duality, we also have the representation formula $\varphi = \mathcal{S}_\eta^* \mathcal{P}^* \varphi + \mathcal{B}_\eta^* \varphi$ that holds for all $\varphi \in C^\infty(\overline{U})$ (i.e., without the compact support assumption).

*Remark* 4.3.3. The construction described so far in the completely integrable case extends to a general setting the classical Bogovskii operator [Bog79] for $\mathcal{P}u = \partial_j u^j$ on $\mathbb{R}^d$, as well as the integral representation formulas of Reshetnyak [Res83; Res94] for $\mathcal{P}^*$ the Killing and conformal Killing operators, etc. It has been carried out for the linearized Einstein constraint equations around the flat space in [MOT23] (see also Chapter 3), and it was used to simplify and advance (nonlinear) initial data gluing results in the asymptotically flat setting.

*S*tep 4: Solutions with compact support II: general case. We now consider Theorem 4.1.2 in the general case, when $b_\eta$ is not necessarily of the form (4.3.2). Our result is the existence of a smoothing operator $\mathcal{Q} : C_c^\infty(U) \to C_c^\infty(U)$ (which preserves the compact support property) that deforms the integral solution operator $\mathcal{S}_\eta$ to a correct solution operator $\widetilde{\mathcal{S}}$ (i.e., $\widetilde{\mathcal{S}} = \mathcal{S}_\eta - \mathcal{Q}$) that satisfies

$$\mathcal{P}\widetilde{\mathcal{S}}f = f - \sum_{\mathbf{A}\in\{1,\dots,\dim\ker\mathcal{P}^*\}} \langle f, \mathbf{Z}^{\mathbf{A}}\rangle w_{\mathbf{A}}. \tag{4.3.4}$$

Here, $\{\mathbf{Z}^{\mathbf{A}}\}_{\mathbf{A}\in\{1,\dots,\dim\ker\mathcal{P}^*\}}$ is a basis of $\ker\mathcal{P}^*$ and $w_{\mathbf{A}} \in C_c^\infty(U)$ are *prescribable* functions satisfying $\langle \mathbf{Z}^{\mathbf{A}'}, w_{\mathbf{A}}\rangle = \delta_{\mathbf{A}}^{\mathbf{A}'}$. By duality, we also have the representation formula $\varphi = \widetilde{\mathcal{S}}^*\mathcal{P}^*\varphi + \sum_{\mathbf{A}\in\{1,\dots,\dim\ker\mathcal{P}^*\}} \mathbf{Z}^{\mathbf{A}}\langle w_{\mathbf{A}}, \varphi\rangle$ for any $\varphi \in C^\infty(\overline{U})$ (i.e., without the compact support assumption). In particular, if $\ker\mathcal{P}^* = \{0\}$, then the last term in (4.3.4)is dropped and $\widetilde{\mathcal{S}}$ is a *b*ona fide right-inverse of $\mathcal{P}$; by duality, $\widetilde{\mathcal{S}}^*$ is a left-inverse of $\mathcal{P}^*$. We call $\widetilde{\mathcal{S}}$ a *Bogovskii-type operator*.

A key ingredient for this argument is a Poincaré-type (or rigidity) inequality

$$\|\varphi\|_{H^{-s}(U)} \lesssim \|\mathcal{P}^*\varphi\|_{H^{-s-m}(U)} \quad \text{for all } \varphi \in H^{-s}(U) \text{ with } \langle w_{\mathbf{A}}, \varphi\rangle = 0 \quad (\mathbf{A} \in \{1,\dots,\dim\ker\mathcal{P}^*\}). \tag{4.3.5}$$

Using a standard contradiction argument, (4.3.5) follows from the weaker inequality

$$\|\varphi\|_{H^{-s}(U)} \lesssim \|\mathcal{P}^*\varphi\|_{H^{-s-m}(U)} + \|\varphi\|_{H^{-s-\delta}(U)} \quad \text{for all } \varphi \in H^s(U), \tag{4.3.6}$$

where $\delta > 0$ may be arbitrary. Inequality (4.3.6), in turn, can be proved using the representation formula obtained via duality, and also using the optimal regularization property of $\mathcal{S}_\eta$ and the smoothing property of $\mathcal{B}_\eta$. On the other hand, by another duality argument, (4.3.5) is equivalent to the existence of a (special) solution $u \in \widetilde{H}^{s+m}(U)$ to $\mathcal{P}u = f$ for any $f \in \widetilde{H}^s(U)$ with $f \perp \ker\mathcal{P}^*$. In fact, that the latter statement may then be upgraded to the existence of the desired linear operator $\mathcal{Q}$, and thus of $\widetilde{\mathcal{S}}$; see Step 4 of Section 4.3 for a simple version of this argument, and §4.5 for our actual proof.

*Remark* 4.3.4. In the non-completely integrable case, the bound for $\mathcal{Q}$ is *non-effective* in general (i.e., we know that $\mathcal{Q}$ is a smoothing operator but have no quantitative relationship between its bound and other constants). But it is only because the argument relies on (4.3.5) whose implicit constant is non-effective due to our use of a contradiction argument. In specific situations where adequate special solutions are already known, the bound for $\mathcal{Q}$ may be made quantitative.

*Remark* 4.3.5. On the other hand, we note that the implicit constant in the second Poincaré-type inequality (4.3.6) can be easily made effective. Hence, our method provides a way to establish effective Poincaré-type inequalities (akin to (4.3.6)) for a large class of *over*determined operators $\mathcal{P}^*$, including the Killing operator (Section B) and the conformal Killing operator (Section B) on curved domains. See also §4.1.

## A basic example: the divergence operator with variable coefficients

To illustrate our method with a simple concrete example, we consider the divergence operator with variable zeroth-order coefficients:

$$\mathcal{P}u = (\partial_j + B_j)u^j \quad \text{in } U, \tag{4.3.7}$$

where $U$ is an open subset of $\mathbb{R}^d$ ($d \geq 2$), $u^j$ is a vector field on $U$ and $B_j$ is a 1-form on $U$ ($j = 1, \ldots, d$). Its adjoint is given by

$$(\mathcal{P}^* \varphi)_j = -\partial_j \varphi + B_j \varphi \quad \text{in } U,$$

where $\varphi$ is a function on $U$.

*S*tep 1: Constructing integral kernel supported on a curve. We begin by verifying (RC) on an arbitrary (smooth) curve $\mathbf{x} : [0,1] \to \mathbb{R}^d$. In view of the formula for $\mathcal{P}^*$, we immediately obtain

$$\partial_i \varphi(x) = B_i(x)\varphi(x) - (\mathcal{P}^*\varphi)_i(x) \tag{4.3.8}$$

Restricting to the curve $\mathbf{x}$ and contracting with $\dot{\mathbf{x}}$, we obtain the ODE

$$\frac{\mathrm{d}}{\mathrm{d}s}(\varphi(\mathbf{x}(s))) = \partial_s \mathbf{x}^i(s)\partial_i\varphi(\mathbf{x}(s)) = \partial_s\mathbf{x}^i(B_i \circ \mathbf{x})(s)\varphi(\mathbf{x}(s)) - \partial_s \mathbf{x}^i((\mathcal{P}^*\varphi)_i \circ \mathbf{x})(s),$$

whose integration leads to

$$\begin{aligned} \varphi(\mathbf{x}(0)) = &\int_0^1 \exp\left(-\int_0^s \partial_s \mathbf{x}^j (B_j \circ \mathbf{x})(s')\,\mathrm{d}s'\right) \partial_s \mathbf{x}^j \left((\mathcal{P}^*\varphi)_j \circ \mathbf{x}\right)(s)\,\mathrm{d}s \\ &+ \exp\left(-\int_0^1 \partial_s \mathbf{x}^j (B_j \circ \mathbf{x})(s')\,\mathrm{d}s'\right)\varphi(\mathbf{x}(1)), \end{aligned} \tag{4.3.9}$$

which verifies (RC) on $\mathbf{x}$.

*Remark* 4.3.6. In this example, (4.3.8) is a (graded) *augmented system* for $\mathcal{P}$ in the sense of Definition 4.1.4. The immediate verification of (RC) on every smooth curve segment $\mathbf{x}(s)$ (more precisely, (4.3.9)) via (4.3.8) is a special instance of Proposition 4.1.6.

Accordingly, given a smooth family of curves $\mathbf{x}(y, y_1, s) \in \mathbb{R}^d$ with endpoints $y$ and $y_1$, if we define the distributions $K_{y_1}(\cdot, y)$ and $b_{y_1}(\cdot, y)$ by (for $\psi \in C_c^\infty(U; \mathbb{R}^d)$ and $\varphi \in C_c^\infty(U)$)

$$\langle K_{y_1}(\cdot, y), \psi\rangle = \int_0^1 \exp\left(-\int_0^s \partial_s \mathbf{x}^j (B_j \circ \mathbf{x})(y, y_1, s')\,\mathrm{d}s'\right) \partial_s \mathbf{x}^j \left(\psi_j \circ \mathbf{x}\right)(y, y_1, s)\,\mathrm{d}s, \tag{4.3.10}$$

$$\langle b_{y_1}(\cdot, y), \varphi\rangle = \exp\left(-\int_0^1 \partial_s \mathbf{x}^j (B_j \circ \mathbf{x})(y, y_1, s')\,\mathrm{d}s'\right)\varphi(y_1), \tag{4.3.11}$$

then (4.3.9) is equivalent to the identity $\langle \mathcal{P} K_{y_1}(\cdot, y), \varphi\rangle = \langle \delta_0(x-y) - b_{y_1}(\cdot, y), \varphi\rangle$ (for $\varphi \in C^\infty_c(U)$). Clearly, $\operatorname{supp} K_{y_1}(\cdot, y) \subseteq \mathbf{x}(y, y_1, [0,1])$ and $\operatorname{supp} b_{y_1}(\cdot, y) \subseteq \{y_1\}$. In conclusion, $K_{y_1}(\cdot, y)$ and $b_{y_1}(\cdot, y)$ satisfy properties (1) and (2) in Step 1 of Section 4.3.

*S*tep 2: Smooth averaging. To illustrate the effect of smooth averaging, we consider the following special case (with a slightly modified construction[5] for simplicity). Take $U = \mathbb{R}^d$, and for every $y \in \mathbb{R}^d$ and $\omega \in \mathbb{S}^{d-1}$, consider the family of curves

$$\mathbf{x}(y, \omega, s) \coloneqq y + s\omega.$$

Following (a slight modification of) Step 1, we define $K_\omega(\cdot, y)$ by (for $\psi \in C^\infty_c(U; \mathbb{R}^d)$)

$$\langle K_\omega(\cdot, y), \psi\rangle = \int_0^\infty \exp\left( - \int_0^s \omega^j B_j(y + s'\omega) \,\mathrm{d}s' \right) \omega^j \psi_j(y + s\omega) \,\mathrm{d}s,$$

which satisfies $\mathcal{P} K_\omega(\cdot, y) = \delta_0(x-y)$ and $\operatorname{supp} K_\omega(\cdot, y) \subseteq \mathbf{x}(y, \omega, [0, \infty))$. Next, given a smooth averaging kernel $\not\eta \in C^\infty_c(\mathbb{S}^{d-1})$ with $\int \not\eta \,\mathrm{d}S(\omega) = 1$, we define the smoothly averaged kernel $K_{\not\eta}$ by (for $\psi \in C^\infty_c(U; \mathbb{R}^d)$)

$$\langle K_{\not\eta}(\cdot, y), \psi\rangle = \int_{\mathbb{S}^{d-1}} \int_0^\infty \exp\left( - \int_0^s \omega^j B_j(y + s'\omega) \,\mathrm{d}s' \right) \omega^j \psi_j(y + s\omega) \not\eta(\omega) \,\mathrm{d}s \mathrm{d}S(\omega).$$

By construction, $\mathcal{P} K_{\not\eta}(\cdot, y) = \delta_0(x-y)$ and $\operatorname{supp} K_{\not\eta}(\cdot, y) \subseteq \cup_{\omega \in \operatorname{supp} \not\eta} \mathbf{x}(y, \omega, [0, \infty))$, which forms a cone over the angular set $\operatorname{supp} \not\eta$ with its tip at $y$. Hence, the operator $\mathcal{S}_{\not\eta}$ with integral kernel $K_{\not\eta}$ satisfies $\mathcal{P}\mathcal{S}_{\not\eta} = I$ and has the property

$$\operatorname{supp} f \subseteq C_\Omega \quad \Rightarrow \quad \operatorname{supp} \mathcal{S}_{\not\eta} f \subseteq C_\Omega$$

for any cone $C_\Omega \subseteq \mathbb{R}^d$ over an angular set $\Omega$ containing $\operatorname{supp} \not\eta$.

Moreover, using the polar integration formula, we can explicitly compute $K_{\not\eta}$. Indeed,

$$\begin{aligned} \langle K_{\not\eta}(\cdot, y), \psi\rangle &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \exp\left( - \int_0^s \omega^j B_j(y + s'\omega) \,\mathrm{d}s' \right) s^{-(d-1)} \omega^j \psi_j(y + s\omega) \not\eta(\omega) s^{d-1} \,\mathrm{d}s \mathrm{d}S(\omega) \\ &= \int_{\mathbb{R}^d} \exp\left( - \int_0^1 (x-y)^j B_j(y + s(x-y)) \,\mathrm{d}s \right) \tfrac{(x-y)^j}{|x-y|^d} \not\eta(\tfrac{x-y}{|x-y|}) \psi_j(x) \,\mathrm{d}x. \end{aligned}$$

In conclusion, $(K_{\not\eta})^j(x, y)$ coincides with a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$(K_{\not\eta})^j(x, y) = \exp\left( - \int_0^1 (x-y)^j B_j(y + s(x-y)) \,\mathrm{d}s \right) \frac{(x-y)^j}{|x-y|^d} \not\eta(\tfrac{x-y}{|x-y|}) \quad \text{for } x \neq y.$$

When $B_j = 0$, this is precisely the *conic operator* of Oh–Tataru [OT19] for the divergence operator. Moreover, when $B_j = 0$ and $\not\eta$ is constant (i.e., $\not\eta = |\mathbb{S}^{d-1}|^{-1}$), we have $\mathcal{S}_{\not\eta} f =$

[5] In Example 4.5.13 below, the same operator is constructed following the method in Section 4.3 more faithfully.

$-\nabla(-\Delta)^{-1}f$; in particular, $\mathcal{S}_{\not\eta}f$ coincides with the *gradient of a harmonic function* outside of $\operatorname{supp} f$. From this expression, it follows that $\mathcal{S}_{\not\eta}$ is a singular integral operator of order $-1$ for a suitably regular $B_j$.

*S*tep 3: Solutions with compact support I: completely integrable case. For this step, consider an open subset $U$ of $\mathbb{R}^d$, which we assume to be connected. Then $\ker \mathcal{P}^*$ consists of $\mathbf{Z} \in C^\infty(\overline{U})$ satisfying $\partial_j \mathbf{Z} = B_j \mathbf{Z}$ in $U$. It is not difficult to see that any nontrivial $\mathbf{Z} \in \ker \mathcal{P}^*$ (which is then nonzero everywhere on $U$ by the equation) must satisfy $\partial_j \log \mathbf{Z} = B_j$. Hence,

$$\ker \mathcal{P}^* = \begin{cases} \{0\} & \text{if } B \text{ is not exact,} \\ (e^z) & \text{if } B_j = \partial_j z. \end{cases}$$

Let us first consider the case $B_j = \partial_j z$, which corresponds to the complete integrability of the graded augmented system (4.3.8). We introduce the shorthand $\mathbf{Z} := e^z$, which generates $\ker \mathcal{P}^*$. In this case, the integral inside the exponential in $b_{y_1}(\cdot, y)$ in Step 1 may be computed, and we obtain

$$\langle b_{y_1}(\cdot, y)\rangle = \mathbf{Z}(y)(\mathbf{Z}^{-1}\varphi)(y_1).$$

Let $K_{\eta_1}(x, y)$ be the smoothly averaged kernel defined with respect to a smooth function $\eta_1 = \eta_1(y_1)$ with $\int \eta_1(y_1)\,\mathrm{d}y_1 = 1$, and let $\mathcal{S}_{\eta_1}$ be the operator with integral kernel $K_{\eta_1}$. A quick computation shows that

$$\mathcal{P}\mathcal{S}_{\eta_1} f = \mathcal{P}\left(\int K_{\eta_1}(x, y) f(y)\,\mathrm{d}y\right) = f(x) - (\mathbf{Z}^{-1}\eta_1)(x)\left(\int \mathbf{Z}(y) f(y)\,\mathrm{d}y\right).$$

Moreover, by construction, $\operatorname{supp} K_{\eta_1}(\cdot, y) \subseteq \cup_{y_1 \in \operatorname{supp}\eta_1} \mathbf{x}(y, y_1, [0, 1])$. In particular, if $U$ is $\mathbf{x}$-*star-shaped with respect to* $\operatorname{supp}\eta_1$ in the sense that

$$\bigcup_{y\in U,\, y_1 \in \operatorname{supp}\eta_1} \mathbf{x}(y, y_1, [0, 1]) \subseteq U,$$

then $\mathcal{S}_{\eta_1}$ has the support property

$$\operatorname{supp} f \subseteq U \quad \Rightarrow \quad \operatorname{supp} \mathcal{S}_{\eta_1} f \subseteq U.$$

To illustrate the optimal regularization property, let us consider the special case

$$\mathbf{x}(y, y_1, s) = y + s(y_1 - y),$$

i.e., $\mathbf{x}(y, y_1, \cdot)$ is the line segment from $y$ to $y_1$. Then, as in Step 2, we can explicitly compute $K_{\eta_1}$. Indeed,

$$\begin{aligned}\langle K_{\eta_1}(\cdot, y), \psi\rangle &= \int_0^1 \int_{\mathbb{R}^d} \exp\left(-\int_0^s (y_1 - y)^j \partial_j z(y + s'(y_1 - y))\,\mathrm{d}s'\right)(y_1 - y)^j \psi_j(y + s(y_1 - y))\eta_1(y_1)\,\mathrm{d}y_1\mathrm{d}s \\ &= \int_0^1 \int_{\mathbb{R}^d} \frac{\mathbf{Z}(y)}{\mathbf{Z}(x)}(x - y)^j \psi_j(x)\eta_1(y + s^{-1}(x - y))s^{-d-1}\,\mathrm{d}x\mathrm{d}s \\ &= \int_{\mathbb{R}^d} \frac{\mathbf{Z}(y)}{\mathbf{Z}(x)} \frac{(x - y)^j}{|x - y|^d}\psi_j(x) \int_{|x-y|}^\infty \eta_1(y + r\tfrac{x-y}{|x-y|})r^{d-1}\mathrm{d}r\,\mathrm{d}x,\end{aligned}$$

where we made the change of variables $x = y + s(y_1 - y)$ and $r = s^{-1}|x-y|$. In conclusion, $(K_{\eta_1})^j(x, y)$ coincides with a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$(K_{\eta_1})^j(x,y) = \frac{\mathbf{Z}(y)}{\mathbf{Z}(x)} \frac{(x-y)^j}{|x-y|^d} \int_{|x-y|}^{\infty} \eta_1(r\tfrac{x-y}{|x-y|} + y) r^{d-1} \,\mathrm{d}r \text{ for } x \neq y. \tag{4.3.12}$$

When $B_j = 0$, we have $\mathbf{Z} \equiv 1$ and this is precisely the classical *Bogovskii operator* [Bog79]; cf. (4.1.2). From this expression, it follows that $\mathcal{S}_{\eta_1}$ is a singular integral operator of order $-1$ for a suitably regular $B_j = \partial_j z$ (see also the proof of Theorem 4.1.15 in [Ise+25b] for an alternative construction, which works for a rough $B_j$).

*S*tep 4: Solutions with compact support II. Finally, consider the case when $B$ is not exact, or equivalently, $\ker \mathcal{P}^* = \{0\}$; this corresponds to the non-completely integrable case. Given $s \in \mathbb{R}$, we now show[6] the existence of a right-inverse $\widetilde{\mathcal{S}} : \widetilde{H}^s(U) \to \widetilde{H}^{s+1}(U)$ of $\mathcal{P}$ that preserves the compact support property in $U$. To illustrate the ideas, we focus on the special case $\mathbf{x}(y, y_1, s) = y + s(y_1 - y)$ and $\eta_1 = \eta_1(y_1)$ as before. In this case,

$$\mathcal{P}\mathcal{S}_{\eta_1} f(x) = f(x) - \int \eta_1(x) Z(y,x) f(y) \,\mathrm{d}y,$$

where

$$Z(y,x) = \exp\left(-\int_0^1 \frac{(x-y)^j}{|x-y|} B_j(y + s'(x-y)) \,\mathrm{d}s'\right).$$

We begin by approximating $\eta_1(x)Z(y,x)$ by a finite sum of tensor products (or possibly, a single tensor product). For instance, we may simply write

$$\mathcal{P}\mathcal{S}_{\eta_1} f(x) = f(x) - \mathcal{E}_0 f(x) - \eta_1(x)\left(\int Z(y,0) f(y) \,\mathrm{d}y\right),$$

where $\mathcal{E}_0 f(x) \coloneqq \eta_1(x) \int (Z(y,x) - Z(y,0)) f(y) \,\mathrm{d}y$. From this expression, it is clear that we may arrange $\mathcal{E}_0$ to have operator norm on, say, $L^1(U)$ (which is bounded by $\sup_y \int |\eta_1(x)(Z(y,x) - Z(y,0))| \,\mathrm{d}x$) less than 1 by making $\operatorname{supp} \eta_1$ sufficiently small depending on $\|\partial Z\|_{L^\infty(U)}$. Then $I - \mathcal{E}_0 : L^1(U) \to L^1(U)$ is invertible and we have

$$\mathcal{P}\mathcal{S}_{\eta_1}(I - \mathcal{E}_0)^{-1} f(x) = f(x) - \eta_1(x)\left(\int Z(y,0)(I-\mathcal{E}_0)^{-1} f(y) \,\mathrm{d}y\right).$$

Next, we find a special solution $u \in \widetilde{H}^{s+1}(U)$ with $\operatorname{supp} u \subseteq U$ to $\mathcal{P}u = \eta_1$. A key ingredient is the following Poincaré-type inequality:

$$\|\varphi\|_{H^{-s}(U)} \lesssim \|\mathcal{P}^*\varphi\|_{H^{-s-1}(U)} \quad \text{for all } \varphi \in H^{-s}(U),$$

[6] In fact, our argument in §4.5 is a slight variant of the present argument, where we construct $\widetilde{\mathcal{S}}$ that is independent of the order $s$.

which follows from (4.3.6) by a standard contradiction argument (see Proposition 4.5.19), the key point being that, in this case, there does not exist any nontrivial solutions $\varphi \in H^{-s}(U)$ to $\mathcal{P}^*\varphi = 0$ in $\mathcal{D}'(U)$. Then from the Poincaré-type inequality, by a duality argument involving the Hahn–Banach theorem (see Corollary 4.5.20), the existence of a special solution $u \in \widetilde{H}^{s+1}(U)$ to $\mathcal{P}u = \eta_1$ follows.

With the special solution $u \in \widetilde{H}^{s+1}(U)$ at hand, we may conclude the construction as follows. Note that

$$\widetilde{\mathcal{S}}f(x) := (\mathcal{S}_{\eta_1} - \mathcal{Q})(I - \mathcal{E}_0)^{-1}f(x), \text{ where } \mathcal{Q}f(x) = u(x)\left(\int Z(y,0)f(y)\,\mathrm{d}y\right),$$

defines a right-inverse of $\mathcal{P}$. Moreover, observe that $(I-\mathcal{E}_0)^{-1}f = f + \mathcal{E}_0(I-\mathcal{E}_0)^{-1}f$. Hence,

$$\operatorname{supp}\mathcal{Q}f \subseteq \operatorname{supp} u, \quad \operatorname{supp}(I-\mathcal{E}_0)^{-1}f \subseteq \operatorname{supp}\eta_1 + \operatorname{supp} f,$$

and the support preserving property of $\mathcal{S}_{\eta_1}$ (under the assumption that $U$ is $\mathbf{x}$-star-shaped with respect to $\operatorname{supp}\eta_1$), it follows that $\widetilde{\mathcal{S}}$ also preserves the compact support property in $U$. Finally, since $u \in \widetilde{H}^{s+1}$, it follows that $\mathcal{Q}$ maps into $\widetilde{H}^{s+1}(U)$, and hence $\widetilde{\mathcal{S}} : \widetilde{H}^s(U) \to \widetilde{H}^{s+1}(U)$.

## 4.4 Singular integral kernels

In this section, we perform a computation that will show that the general smoothly averaged integral kernel $K_\eta(x,y)$ as in Step 2 in Section 4.3 (see Section 4.5 below for the precise construction) define adequate singular integral operators.

### Assumptions

Recall the setup in Steps 1 and 2 of Section 4.3. Our aim here is to formulate the precise assumptions on the family of curves $\mathbf{x}(y,y_1,s)$, rough integral kernels $K_{y_1}(\cdot,y)$ and smooth averaging kernel $\eta(y,y_1)$, which guarantees that the smoothly averaged integral kernel $K_\eta(x,y)$ defines a singular integral operator (or more precisely, a classical pseudodifferential operator) of suitable order.

For the purpose of this section, it is more convenient to work with the following spatial variables

$$z_1 := y_1 - y, \quad z(y,z_1,s) := \mathbf{x}(y, z_1 + y, s) - y.$$

*Assumptions on the family of curves.* Let $R_y > 0$, $N_0, M_0 \in \mathbb{Z}_{\geq 0}$, $A_{\mathbf{z}} \geq 0$ be parameters to be used below. Let $U$ and $\mathrm{V}$ be open subsets of $\mathbb{R}^d$, and $\mathrm{W}$ an open subset of $\mathbb{R}^d \times \mathbb{R}^d$, such that $U$ contains the projection of $\mathrm{W}$ to the first $\mathbb{R}^d$, i.e.,

$$\{y \in \mathbb{R}^d : (y,z_1) \in \mathrm{W} \text{ for some } z_1 \in \mathbb{R}^d\} \subseteq U.$$

We assume that $\mathbf{z} : \mathrm{W} \times [0,1] \to \mathrm{V}$, $\mathbf{z} = \mathbf{z}(y,z_1,s)$, which is a smooth family of curves in $\mathrm{V}$ parametrized by $(y,z_1) \in \mathrm{W}$, satisfies the following properties:

- We have $\mathbf{z}(y, z_1, 0) = 0$, $\mathbf{z}(y, z_1, 1) = z_1$. Moreover, $\mathbf{z}(y, z_1, s)$ obeys

$$(1 + A_{\mathbf{z}})^{-1}|z_1| \leq |\partial_s \mathbf{z}(y, z_1, s)| \leq (1 + A_{\mathbf{z}})|z_1| \quad \text{for every } s \in (0, 1).$$

- The map $z_1 \mapsto \mathbf{z}$ is invertible for each fixed $y$ and $s \in (0, 1]$; we denote the inverse by $\mathbf{z}_1(y, z, s)$. We assume that $\frac{\partial \mathbf{z}_1(y,z,s)}{\partial z}$ obeys

$$\left| \frac{\partial \mathbf{z}_1(y, z, s)}{\partial z} \right| \leq (1 + A_{\mathbf{z}}) s^{-1},$$

  where we used the operator norm in $\mathbb{R}^d$.

- For higher derivatives, we have

$$\left| R_y^{|\alpha|} |z|^{|\beta|} \partial_y^{\alpha} \partial_z^{\beta} \mathbf{z}_1(y, z, s) \right| \leq A_{\mathbf{z}} s^{-1} |z| \quad \text{for } |\alpha| \leq N_0,\ 1 \leq |\beta| \leq 1 + |\gamma| + M_0,\ |\alpha| > 0 \text{ or } |\beta| > 1.$$

*Remark* 4.4.1 (Straight line segments). The simplest (yet useful) example of such a family of curves is the *straight line segments*,

$$\underline{\mathbf{z}}(y, z_1, s) \coloneqq s z_1,$$

which indeed obeys the assumptions with $A_{\mathbf{z}} = 0$ and any $R_y > 0$, $N_0, M_0 \in \mathbb{Z}_{\geq 0}$.

*A*ssumptions on the smooth averaging kernel. Let $R_{z_1} > 0$ and $A_{\eta} > 0$ be parameters to be used below. We assume that $\eta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following properties:

- $\operatorname{supp} \eta \subseteq \mathrm{W}$.
- $\eta(y, z_1) = 0$ if $|z_1| \geq R_{z_1}$.
- We have

$$|R_y^{|\alpha|} |z_1|^{|\beta|} \partial_y^{\alpha} \partial_{z_1}^{\beta} \eta(y, z_1)| \leq A_{\eta} R_{z_1}^{-d} \quad \text{for } |\alpha| \leq N_0,\ |\beta| \leq |\gamma| + M_0.$$

*Remark* 4.4.2. In practice, we will take $\eta$ of the form $\eta(y, z_1) = \chi_1(y)\chi_2(z_1)\eta(y, z_1 + y)$, where $\eta$ is a smooth averaging kernel satisfying ($\eta$-1)–($\eta$-4) below and $\chi_1, \chi_2$ are additional smooth functions inserted to make $R_y$ and $R_{z_1}$ constant.

*A*ssumption on the rough integral kernel. Let $m > 0$ and $A_{\mathrm{S}} > 0$ be parameters to be used below. Instead of $K_{y_1}(\cdot, y)$, we work with a rough integral kernel $\mathrm{K}_{z_1}(\cdot, y)$ of the following form: for every $(y, z_1) \in \mathrm{W}$, $\mathrm{K}_{z_1}(\cdot, y) \in \mathcal{D}'(\mathbb{R}^d)$ with

$$\langle \mathrm{K}_{z_1}(\cdot, y), \varphi \rangle \coloneqq \int_0^1 \mathrm{S}^{\gamma}(y, z_1, s) \partial^{\gamma} \varphi(y + z(y, z_1, s)) \, \mathrm{d}s \quad \text{for every } \varphi \in C_c^{\infty}(\mathbb{R}^d),$$

for some multi-index $\gamma$ (which could be 0) and $\mathrm{S}^\gamma : \mathrm{W} \times [0,1] \to \mathbb{C}$. Each component of the rough integral kernel $K_{y_1}(\cdot, y)$ in Section 4.3 will be a linear combination of such distributions; see Section 4.5.

We assume that the function $\mathrm{S}^\gamma$ satisfies the following bound: for every $(y, z_1) \in \operatorname{supp} \eta$ and $s \in [0,1]$,

$$\left| R_y^{|\alpha|} |z_1|^{|\beta|} \partial_y^\alpha \partial_{z_1}^\beta \mathrm{S}^\gamma(y, z_1, s) \right| \leq A_{\mathrm{S}} |z_1|^{m+|\gamma|} s^{m+|\gamma|-1} \quad \text{for } |\alpha| \leq N_0,\ |\beta| \leq |\gamma| + M_0.$$

*Remark* 4.4.3 (Discussion of the parameters $A_{\mathbf{z}}$, $A_\eta$, $A_{\mathrm{S}}$, $R_y$, and $R_{z_1}$). Note that $A_{\mathbf{z}}$, $A_\eta$, and $A_{\mathrm{S}}$ are dimensionless, whereas $R_y$ and $R_{z_1}$ have the dimension of length. The parameter $A_{\mathbf{z}}$ quantifies how for $\mathbf{z}$ is from being the straight line segments $\underline{\mathbf{z}}$; $A_\eta$ and $A_{\mathrm{S}}$ quantify the sizes of $\eta$ and S. The length parameter $R_y$ is the $y$-characteristic scale of $\mathbf{z}$, $\eta$, and $\mathrm{S}^\gamma$, i.e., these objects vary slowly as $y$ varies within scale $R_y$. Finally, $\eta(y, \cdot)$ is supported in $B_{R_{z_1}}(0)$ and is bounded by $O(R_{z_1}^{-d})$; this is consistent with the unit mean property ($\eta$-1) below. The power of $R_{z_1}$ in the assumption for $\mathrm{S}^\gamma$ is consistent with Example 4.4.4 below.

With the above objects, we define the *averaged integral kernel* $\mathrm{K}_\eta(\cdot, y)$ by the following relation for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$:

$$\langle \mathrm{K}_\eta(\cdot, y), \varphi \rangle := \int \int_0^1 \mathrm{S}^\gamma(y, z_1, s) \eta(y, z_1) (\partial^\gamma \varphi)(y + \mathbf{z}(y, z_1, s)) \mathrm{d}s \, \mathrm{d}z_1 \quad \text{for every } \varphi \in C_c^\infty(U) \tag{4.4.1}$$

Informally, $\mathrm{K}_\eta(\cdot, y) = \int \mathrm{K}_{z_1}(\cdot, y) \eta(y, z_1) \, \mathrm{d}z_1$. Note that, thanks to $\operatorname{supp} \eta \subseteq \mathrm{W}$, the right-hand side is well-defined for any $y \in \mathbb{R}^d$, although it is trivial unless $y$ lies in $U$.

The above setup generalizes (modulo some technical modifications) the conic and Bogovskii integral kernels for $\mathcal{P}u = \partial_j u^j + B_j u^j$ (see Section 4.3) constructed using straight line segments, as the following example shows.

**Example 4.4.4** (Conic- and Bogovskii integral kernels)**.** Let $U = \mathrm{V} = \mathbb{R}^d$ and $\mathrm{W} = \mathbb{R}^d \times \mathbb{R}^d$. Define, for $\gamma = 0$,

$$\mathbf{z}(y, z_1, s) = \underline{\mathbf{z}}(y, z_1, s) := s z_1, \quad \mathrm{S}^0(y, z_1, s) = \exp\left( \int_s^1 \partial_s \mathbf{z}(y, z_1, s') \cdot B(y + \mathbf{z}(y, z_1, s')) \, \mathrm{d}s' \right) \partial_s \mathbf{z}(y, z_1, s).$$

As discussed in Remark 4.4.1, the assumptions for $\mathbf{z}$ are satisfied for any $N_0, M_0 \in \mathbb{Z}_{\geq 0}$ with $A_{\mathbf{z}} = 0$ and an arbitrarily large $R_y$, $N_0$ and $M_0$. With any choice of $\eta$ satisfying the above requirements, the assumption for $\mathrm{S}^0$ is satisfied on W for any $N_0, M_0 \in \mathbb{Z}_{\geq 0}$ with $m = 1$, an arbitrarily large $R_y$, and

$$A_{\mathrm{S}} \lesssim_{N_0, M_0, \|B\|} 1,$$

where[7]

$$\|B\| = \sup_{(y, y_1 - y) \in \mathrm{W}} \sum_{\alpha : |\alpha| \leq N_0 + M_0} \int_0^1 |\partial^\alpha B(y + s(y_1 - y))| |y_1 - y|^{1+|\alpha|} s^{|\alpha|} \, \mathrm{d}s.$$

[7] The norm $\|B\|$ coincides with $\|B\|_{\dot{G}^{N_0+M_0,1}(\underline{\mathbf{x}}, \mathrm{W})}$ (with $\underline{\mathbf{x}} = y + \underline{\mathbf{z}}$), which will be properly introduced in Section 4.6 below.

The conic and Bogovskii kernels correspond to the following choices of $\eta$ (and a parameter $R_\eta > 0$):

1. conic case. $\eta(y, z_1) = \psi(|z_1|)\not\eta(\frac{z_1}{|z_1|})$ with $\int_{\mathbb{S}^{d-1}} \not\eta \, \mathrm{d}S = 1$, $\int \psi(r) r^{d-1} \, \mathrm{d}r = 1$, $\operatorname{supp} \psi \subseteq (\frac{1}{2} R_\eta, R_\eta)$ and $|\partial^\alpha \psi| \lesssim_\alpha R_\eta^{-d-|\alpha|}$ for arbitrarily large $R_\eta$. Then the above assumptions are satisfied for any $N_0, M_0 \in \mathbb{Z}_{\geq 0}$ with $R_{z_1}(y) = R_\eta$ (independent of $y$) and $A_\eta \lesssim_{N_0, M_0, \psi, \not\eta} 1$ (independent of $R_\eta$).

2. Bogovskii case. $\eta(y, z_1) = \chi(y) \eta_1(z_1 + y)$ with $\int \eta_1 = 1$, $\operatorname{supp} \eta_1 \subseteq B_{R_\eta}(0)$, and $|\partial^\alpha \eta_1| \lesssim_{|\alpha|} R_\eta^{-d-|\alpha|}$, and an auxiliary cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$. Let $R_\chi = \sup_{y \in \operatorname{supp} \chi} |y|$. Then the assumptions for $\eta$ are satisfied for any $N_0, M_0 \in \mathbb{Z}_{\geq 0}$ with $R_{z_1} = 1 + R_\chi + R_\eta$ and $A_\eta \lesssim_{N_0, M_0, \eta} \left(\frac{R_{z_1}}{R_\eta}\right)^d$; observe that such constants exist thanks to the presence of $\chi$.

Indeed, in the first case, note that the conic kernel agrees with $\mathrm{K}_\eta(z + y, y)$ for $|z| \leq \frac{1}{2} R_\eta$, and hence globally in the limit $R_\eta \to +\infty$. In the second case, the Bogovskii kernel agrees with $\mathrm{K}_\eta(z + y, y)$ for every $y \in \mathbb{R}^d$ such that $\chi(y) = 1$ (hence, in practice, we will choose $\chi$ to be equal to 1 on the domain $U$ under consideration).

## Singular integral kernel and symbol bounds

The main result of this section is as follows.

**Proposition 4.4.5** (Singular integral kernel bounds)**.** *Suppose that the assumptions for $\mathbf{z}$, $\eta$, and $\mathrm{S}^\gamma$ in Section 4.4 hold, and let $\mathrm{K}_\eta$ be defined as in (4.4.1). If $N_0 \geq 0$ and $M_0 \geq 0$, then $\mathrm{K}_\eta(x, y) \in L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, for all $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \setminus \{0\}$, we have the representation*

$$\mathrm{K}_\eta(z + y, y) = \partial_z^\gamma \int_0^1 (-1)^{|\gamma|} \mathrm{S}^\gamma(y, \mathbf{z}_1(y, z, s), s) \eta(y, \mathbf{z}_1(y, z, s)) \left| \det \frac{\partial \mathbf{z}_1}{\partial z} \right| \mathrm{d}s. \tag{4.4.2}$$

*In fact, we have*

$$|R_y^{|\alpha|} |z|^{|\beta|} \partial_y^\alpha \partial_z^\beta \mathrm{K}_\eta(z + y, y)| \leq A_{\alpha, \beta+\gamma} |z|^{-d+m} \tag{4.4.3}$$

*where*

$$A_{\alpha, \beta+\gamma} \leq C_{m, \alpha, \beta+\gamma} A_{\mathrm{S}} A_\eta (1 + A_{\mathbf{z}})^{2(d+|\alpha|+|\beta+\gamma|)+m+|\gamma|} \quad \text{for } |\alpha| \leq N_0,\ |\alpha| + |\beta| \leq M_0. \tag{4.4.4}$$

*Moreover,*

$$\operatorname{supp} \mathrm{K}_\eta(\cdot + y, y) \subseteq B_{(1+A_{\mathbf{z}}) R_{z_1}}(0). \tag{4.4.5}$$

Let $\mathrm{S}_\eta$ be the linear operator with integral kernel $\mathrm{K}_\eta$:

$$\mathrm{S}_\eta f(x) = \int \mathrm{K}_\eta(x, y) f(y) \, \mathrm{d}y \quad \text{for } f \in C_c^\infty(\mathbb{R}^d). \tag{4.4.6}$$

A convenient way to establish the mapping properties of $\mathrm{S}_\eta$ is to show that it is a pseudodifferential operator of order $m$ with a classical (or Kohn–Nirenberg) symbol.

**Proposition 4.4.6** (Symbol bounds)**.** *Suppose that the assumptions for* $\mathbf{z}$*,* $\eta$*, and* $\mathrm{S}^\gamma$ *in Section 4.4 hold, and let* $\mathrm{K}_\eta$ *be defined as in* (4.4.1)*. If* $M_0 \geq \max\{N_0, m+1\}$*, the symbol* $\mathrm{a}_\eta(\xi, y) := \int \mathrm{K}_\eta(y+z, y) e^{-i\xi\cdot z}\,\mathrm{d}z$ *obeys the bound*

$$|\partial_y^\alpha \partial_\xi^\beta \mathrm{a}_\eta(\xi, y)| \leq C_{m,\alpha,\beta,\gamma} A_{\mathrm{S}} A_\eta (1+A_{\mathbf{z}})^{10(d+|\alpha|+|\beta|+|\gamma|+m)} R_y^{-|\alpha|} \left( ((1+A_{\mathbf{z}}) R_{z_1})^{-1} + |\xi| \right)^{-m-|\beta|} \tag{4.4.7}$$

*for* $|\alpha| \leq N_0$ *and* $|\alpha| + |\beta| \leq M_0 - m - 1$*.*

**Corollary 4.4.7.** *Suppose that the assumptions for* $\mathbf{z}$*,* $\eta$*, and* $\mathrm{S}^\gamma$ *in Section 4.4 hold, and let* $\mathrm{K}_\eta$ *and* $\mathrm{S}_\eta$ *be defined as in* (4.4.1) *and* (4.4.6)*, respectively. Then there exists a constant* $c_{m,d} > 0$ *such that, for every* $1 < p < \infty$ *and* $|s| \leq \min\{N_0, M_0\} - c_{m,d}$*, we have* $\mathrm{S}_\eta : W^{s,p}(\mathbb{R}^d) \to W^{s+m,p}(\mathbb{R}^d)$*.*

This immediately follows from Proposition 4.4.6 and standard boundedness results for pseudodifferential operators (see, e.g., [Ste93]), since $\mathrm{S}_\eta$ is the left-quantization of the classical symbol $a_\eta$ of order $-m$.

*Remark* 4.4.8. Alternatively, one may attempt to directly verify that $\partial_x^\alpha \mathrm{K}_\eta(x, y)$ with $|\alpha| = m$ are Calderón–Zygmund kernels. In this case, it is an interesting question to ask what are the minimal regularity assumptions for $\mathbf{z}$ and $\mathrm{S}^\gamma$ for this property to hold. In Acosta–Durán–Muschietti [ADM06], it was shown that for the divergence operator on a John domain, there exists a Bogovskii-type integral kernels $K_{\eta_1}$ such that $\partial_{x^j} K_{\eta_1}$ is a Calderón–Zygmund kernel for every $j$ (and in fact, it characterizes John domains).

We now prove Propositions 4.4.5 and 4.4.6 in a sequence of lemmas. We begin with the formula (4.4.2).

**Lemma 4.4.9.** *Under the hypotheses of Proposition 4.4.5, we have (for every* $\varphi \in C_c^\infty(\mathbb{R}^d)$*)*

$$\langle K_\eta(\cdot, y), \varphi \rangle = \int \int_0^1 \mathrm{S}^\gamma(y, \mathbf{z}_1(y,z,s), s) \eta(y, \mathbf{z}_1(y,z,s)) \left| \det \frac{\partial \mathbf{z}_1}{\partial z} \right| (\partial^\gamma \varphi)(y+z) \mathrm{d}s\,\mathrm{d}z.$$

*Proof.* This is a simple change-of-variables computation. By the definition of $\mathrm{K}_\eta$, we have

$$\begin{aligned}
\langle K_\eta(\cdot, y), \varphi \rangle &= \int \int_0^1 \mathrm{S}^\gamma(y, z_1, s) \eta(y, z_1) (\partial^\gamma \varphi)(y + \mathbf{z}(y, z_1, s)) \mathrm{d}s\,\mathrm{d}z_1 \\
&= \int \int_0^1 \mathrm{S}^\gamma(y, \mathbf{z}_1(y,z,s), s) \eta(y, \mathbf{z}_1(y,z,s)) (\partial^\gamma \varphi)(y+z) \left| \det \frac{\partial \mathbf{z}_1}{\partial z} \right| \mathrm{d}s\,\mathrm{d}z. \qquad \square
\end{aligned}$$

Lemma 4.4.9 already shows (interpreted in the sense of distributions)

$$\mathrm{K}_\eta(z+y, y) = (-1)^{|\gamma|} \partial_z^\gamma \int_0^1 \mathrm{S}^\gamma(y, \mathbf{z}_1(y,z,s), s) \eta(y, \mathbf{z}_1(y,z,s)) \left| \det \frac{\partial \mathbf{z}_1}{\partial z} \right| \mathrm{d}s.$$

The following lemma then completes the proof of Proposition 4.4.5:

**Lemma 4.4.10.** *Under the hypotheses of Proposition 4.4.5, define*

$$\mathrm{I}_{\alpha,\beta'}(z+y,y) = \int_0^1 \partial_y^\alpha \partial_z^{\beta'} \left[ (-1)^{|\gamma|} \mathrm{S}^\gamma(y, \mathbf{z}_1(y,z,s), s) \eta(y, \mathbf{z}_1(y,z,s)) \left| \det \frac{\partial \mathbf{z}_1}{\partial z} \right| \right] \mathrm{d}s$$

*The integral on the RHS is well-defined for every $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \setminus \{0\}$, and we have*

$$|\mathrm{I}_{\alpha,\beta'}(z+y,y)| \leq A_{\alpha,\beta'} R_y^{-|\alpha|} |z|^{-d+m+|\gamma|-|\beta'|}$$

*where $A_{\alpha,\beta'}$ satisfies (4.4.4). Moreover,* $\operatorname{supp} I_{\alpha,\beta'}(\cdot + y, y) \subseteq B_{(1+A_{\mathbf{z}})R_{z_1}}(0)$.

*Proof.* We begin by estimating each factor in the definition of $\mathrm{I}_{\alpha,\beta'}$. By the hypothesis on $\mathbf{z}$, we have

$$\left| R_y^{|\alpha|} |z|^{|\beta'|} \partial_y^\alpha \partial_z^{\beta'} \left| \det \frac{\partial \mathbf{z}_1}{\partial z} \right| \right| \lesssim (1+A_{\mathbf{z}})^{d+|\alpha|} s^{-d}. \tag{4.4.8}$$

for $|\alpha| \leq N_0$ and $|\beta'| \leq |\gamma| + M_0$. For the other factors, we claim that

$$\left| R_y^{|\alpha|} |z|^{|\beta'|} \partial_y^\alpha \partial_z^{\beta'} \eta(y, \mathbf{z}_1(y,z,s)) \right| \lesssim A_\eta (1+A_{\mathbf{z}})^{2|\alpha|+2|\beta'|} R_{z_1}^{-d} \tag{4.4.9}$$

$$\left| R_y^{|\alpha|} |z|^{|\beta'|} \partial_y^\alpha \partial_z^{\beta'} \mathrm{S}^\gamma(y, \mathbf{z}_1(y,z,s), s) \right| \lesssim A_{\mathrm{S}} (1+A_{\mathbf{z}})^{2|\alpha|+2|\beta'|+m+|\gamma|} |z|^{m+|\gamma|} s^{-1}, \tag{4.4.10}$$

as long as $|\alpha| \leq N_0$ and $|\alpha| + |\beta'| \leq M_0 + |\gamma|$.

To establish (4.4.9) and (4.4.10), we need to study compositions of the form $F(y, \mathbf{z}_1(y,z,s))$ (for an appropriate function $F$). First, using the hypothesis on $\partial_s \mathbf{z}$, observe that

$$(1+A_{\mathbf{z}})^{-1} |z| \leq s |\mathbf{z}_1(y,z,s)| \leq (1+A_{\mathbf{z}}) |z|. \tag{4.4.11}$$

Consider a $C^1$ function $F = F(y, z_1)$. For $M_0 + |\gamma| \geq 1$, we have

$$\begin{aligned} |\partial_z F(y, \mathbf{z}_1(y,z,s))| &\leq \left| (\partial_{z_1} F)(y, z_1)|_{z_1 = \mathbf{z}_1(y,z,s)} \right| |\partial_z \mathbf{z}_1(y,z,s)| \\ &\leq (1+A_{\mathbf{z}})^2 \left| (|z_1| \partial_{z_1} F)(y, z_1)|_{z_1 = \mathbf{z}_1(y,z,s)} \right|. \end{aligned}$$

Similarly, for $N_0 \geq 1$ and $M_0 + |\gamma| \geq 1$, we have

$$\begin{aligned} |\partial_y F(y, \mathbf{z}_1(y,z,s))| &\leq |(\partial_y F)(y, z_1)|_{z_1 = \mathbf{z}_1(y,z,s)}| + \left| (\partial_{z_1} F)(y, z_1)|_{z_1 = \mathbf{z}_1(y,z,s)} \right| |\partial_y \mathbf{z}_1(y,z,s)| \\ &\leq |(\partial_y F)(y, z_1)|_{z_1 = \mathbf{z}_1(y,z,s)}| + A_{\mathbf{z}}(1+A_{\mathbf{z}}) \left| (|z_1| \partial_{z_1} F)(y, z_1)|_{z_1 = \mathbf{z}_1(y,z,s)} \right|. \end{aligned}$$

Now, using the hypotheses on $\eta$, and $\mathrm{S}^\gamma$, (4.4.9) and (4.4.10) in the case $|\alpha| + |\beta'| = 1$ follows. The general higher order case follows by a routine induction argument.

Putting together (4.4.8), (4.4.9), and (4.4.10), we arrive at

$$|\mathrm{I}_{\alpha,\beta'}| \lesssim (1+A_{\mathbf{z}})^{d+2(|\alpha|+|\beta'|)} |z|^{m+|\gamma|} R_{z_1}^{-d} \int_0^1 \mathbf{1}_{\operatorname{supp}_{z_1} \eta(y,z_1)}(\mathbf{z}_1(y,z,s)) s^{-d-1} \, \mathrm{d}s$$

To estimate the integral on RHS, we make the change of variables $s = \frac{|z|}{r}$, so that

$$\int_0^1 s^{-d-1} \mathbf{1}_{\mathrm{supp}_{z_1} \eta(y,z_1)}(\mathbf{z}_1(y,z,s))\,\mathrm{d}s = |z|^{-d} \int_{|z|}^{\infty} r^{d-1} \mathbf{1}_{\mathrm{supp}_{z_1} \eta(y,z_1)}(\mathbf{z}_1(y,z,\tfrac{|z|}{r}))\,\mathrm{d}r. \quad (4.4.12)$$

By (4.4.11), it follows that $(1 + A_{\mathbf{z}})^{-1} r \leq |\mathbf{z}_1(y,z,\frac{|z|}{r})|$. Recalling also that $\mathrm{supp}\,\eta(y,\cdot) \subseteq B_{R_{z_1}}(0)$, we have

$$\mathbf{1}_{\mathrm{supp}_{z_1} \eta(y,z_1)}(\mathbf{z}_1(y,z,\tfrac{|z|}{r})) \leq \mathbf{1}_{[0,(1+A_{\mathbf{z}})R_{z_1}]}(r).$$

Thus,

$$\int_{|z|}^{\infty} r^{d-1} \mathbf{1}_{\mathrm{supp}_{z_1} \eta(y,z_1)}(\mathbf{z}_1(y,z,\tfrac{|z|}{r}))\,\mathrm{d}r \leq (1 + A_{\mathbf{z}})^d R_{z_1}^d, \quad (4.4.13)$$

and it vanishes for $z$ in a neighborhood of $\{z \in \mathbb{R}^d : |z| \geq (1 + A_{\mathbf{z}})R_{z_1}\}$. This completes the proof. $\square$

Finally, the symbol bounds (Proposition 4.4.6) follows from Proposition 4.4.5 and the following lemma, which is of independent interest.

**Lemma 4.4.11.** *Assume that* $\mathrm{K}(x,y) \in L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^d)$ *satisfies*

$$|\partial_y^\alpha \partial_z^\beta \mathrm{K}(z+y,y)| \leq A R_y^{-|\alpha|} |z|^{-d+m-|\beta|} \quad \textit{for } |\alpha| \leq N,\ |\beta| \leq M,$$
$$\mathrm{supp}\,\mathrm{K} \subseteq \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < R_z\},$$

*for some* $M, N \in \mathbb{Z}_{\geq 0}$, $m > 0$, $A > 0$, $R_y, R_z > 0$. *Then for* $|\alpha| \leq N$ *and* $|\beta| \leq M - m - 1$, $\mathrm{a}(\xi,y) := \int \mathrm{K}(z+y,y) e^{-i\xi\cdot z}\,\mathrm{d}z$ *satisfies*

$$|\partial_y^\alpha \partial_\xi^\beta \mathrm{a}(\xi,y)| \leq C_{m,\beta} A R_y^{-|\alpha|} (R_z^{-1} + |\xi|)^{-m-|\beta|}. \quad (4.4.14)$$

*Proof.* Let $m_R(z)$ denote a smooth partition of unity subordinate to dyadic annuli $A_R = \{z \in \mathbb{R}^d : R < |z| < 4R\}$ as in Section 4.2. We split $\mathrm{K}(z+y,y) = \sum_{R \in 2^{\mathbb{Z}}} \mathrm{K}_R(z+y,y)$, where

$$\mathrm{K}_R(z+y,y) := m_R(z)\mathrm{K}(z+y,y).$$

We note that this sum is finite thanks to the support property of K. Clearly, the following holds for each $R$:

$$|\partial_y^\alpha \partial_z^\beta \mathrm{K}_R(y+z,y)| \lesssim_\beta A R_y^{-|\alpha|} R^{-d+m-|\beta|} \quad \text{for } |\alpha| \leq N,\ |\beta| \leq M. \quad (4.4.15)$$

Correspondingly, $\mathrm{a}(\xi,y)$ may be split into

$$\mathrm{a}(\xi,y) = \sum_{R \in 2^{\mathbb{Z}}} \mathrm{a}_R(\xi,y) := \sum_{R \in 2^{\mathbb{Z}}} \int \mathrm{K}_R(z+y,y) e^{-i\xi\cdot z}\,\mathrm{d}z.$$

We now estimate $\partial_y^\alpha \partial_\xi^{\beta'} \mathrm{a}_R = \partial_y^\alpha \partial_\xi^{\beta'} \int \mathrm{K}_R(y+z,y) e^{i\xi\cdot z}\,\mathrm{d}z$ for each $R \in 2^{\mathbb{Z}}$. Observe that:

(i) Using the identity $\partial_{\xi_j} e^{-i\xi\cdot z} = -iz^j e^{-i\xi\cdot z}$ and (4.4.15),

$$\left| \partial_y^\alpha \partial_\xi^{\beta'} \int \mathrm{K}_R(z+y,y) e^{i\xi\cdot z} \,\mathrm{d}z \right| \lesssim_{\beta'} A R_y^{-|\alpha|} R^{m+|\beta'|}.$$

(ii) Using the identity $e^{i\xi\cdot z} = (i\xi_j)^{-1} \partial_{z^j} e^{i\xi\cdot z}$, integration by parts and (4.4.15),

$$\left| \partial_y^\alpha \partial_\xi^{\beta'} \int \mathrm{K}_R(z+y,y) e^{i\xi\cdot z} \,\mathrm{d}z \right| \lesssim_{\beta'} A R_y^{-|\alpha|} R^{m-|\beta'|} |\xi|^{-|\beta'|}.$$

(iii) For $R > R_z$,

$$\int \mathrm{K}_R(z+y,y) e^{i\xi\cdot z} \,\mathrm{d}z = 0.$$

For (i) and (ii), we need $|\alpha| \leq N$ and $|\beta'| \leq M$.

We now sum up the above bounds in $R$ to estimate a. By (i) and (iii), we immediately obtain

$$|\partial_y^\alpha \partial_\xi^{\beta'} \mathrm{a}| \lesssim_{\beta'} A R_y^{-|\alpha|} R_z^{m+|\beta'|}$$

On the other hand, by optimizing (i) and (ii) (which requires $M > |\beta'| + m$), we obtain

$$|\partial_y^\alpha \partial_\xi^{\beta'} \mathrm{a}| \lesssim_{\beta'} A R_y^{-|\alpha|} |\xi|^{-m-|\beta'|},$$

Combining the two bounds, (4.4.14) follows. □

Finally, Proposition 4.4.6 follows from Proposition 4.4.5 and Lemma 4.4.11 with appropriate choices of $A$, $N$, $M$, the same $R_y$, and $R_z = (1 + A_{\mathbf{z}}) R_{z_1}$, where we are being loose with the power of $(1 + A_{\mathbf{z}})$ in (4.4.7).

## 4.5 From (RC) to integral formulas

In this section, we carry out in detail the construction outlined in Section 4.3. In particular, the conditions needed for our construction, including the key *recovery on curves* condition, are precisely formulated here; see Section 4.5 (simple qualitative version), Sections 4.5–4.5 (detailed quantitative version), and Section 4.5 (additional quantitative assumptions for obtaining solutions with compact support) below.

### Construction of a rough integral kernel supported on curves

Our aim in this subsection is to formulate *qualitative* conditions (including recovery on curves) that lead to the construction of a rough integral kernel $K_{y_1}(x,y)$ with properties outlined in Step 1 of Section 4.3.

**Recovery on curves and duality argument, qualitative versions**

Let $W$ be an open subset of $\mathbb{R}^d \times \mathbb{R}^d$, and let $\mathbf{x} : W \times [0,1] \to \mathbb{R}^d$ be a smooth family of curves with $\mathbf{x}(y, y_1, 0) = y$ and $\mathbf{x}(y, y_1, 1) = y_1$. We state the precise (but qualitative) formulation of (RC):

(RC$^\vee$) **Recovery on Curves (with endpoint).** For each $(y, y_1) \in W$ and multi-index $\gamma$, and there exists an $r_0 \times s_0$-matrix-valued continuous function $s \mapsto {S_J}^{(\gamma,K)}(y, y_1, s)$ ($s \in [0,1]$) and an $r_0 \times r_0$-matrix-valued distribution ${(b_{y_1})^{J'}}_J(\cdot, y) \in \mathcal{D}'(U)$ such that the following holds. For every ($\mathbb{C}^{r_0}$-valued) $\varphi \in C_c^\infty(U)$ (*without* the vanishing condition near $y_1$), we have

$$\varphi_J(y) = \int_0^1 \sum_\gamma {S_J}^{(\gamma,K)}(y, y_1, s)(\partial_x^\gamma \mathcal{P}^* \varphi)(\mathbf{x}(y, y_1, s))\, \mathrm{d}s + \langle {(b_{y_1})^{J'}}_J(\cdot, y), \varphi_{J'} \rangle, \quad (4.5.1)$$

where ${S_J}^{(\gamma,K)} = 0$ except for finitely many multi-indices $\gamma$, and

$$\operatorname{supp} {(b_{y_1})^{J'}}_J(\cdot, y) \subseteq \{y_1\}. \quad (4.5.2)$$

As outlined in Step 1 of Section 4.3, (RC$^\vee$) leads to (in fact, is equivalent to) the existence of a (distributional) Green's function for $\mathcal{P}$ supported on the curve $\mathbf{x}(y, y_1, [0,1])$.

**Proposition 4.5.1** (Duality argument)**.** *Let $U$, $W$, $\mathbf{x} = \mathbf{x}(y, y_1, s)$, $\mathcal{P}$, ${S_J}^{(\gamma,K)}(y, y_1, s)$, and ${(b_{y_1})_J}^{J'}(\cdot, y)$ satisfy (RC$^\vee$). For each $(y, y_1) \in W$, define the $s_0 \times r_0$-matrix-valued distribution $K_{y_1}(\cdot, y)$ (on $\mathbb{R}^d$) by*

$$\langle K_{y_1}(\cdot, y), \psi \rangle = \langle {(K_{y_1})^K}_J(\cdot, y), \psi_K \rangle = \int_0^1 \sum_\gamma {S_J}^{(\gamma,K)}(y, y_1, s) \partial_x^\gamma \psi_K(\mathbf{x}(y, y_1, s))\, \mathrm{d}s. \quad (4.5.3)$$

*Then we have, for each $(y, y_1) \in W$ with both $y \in U$ and $y_1 \in U$,*

$$\mathcal{P} K_{y_1}(x, y) = \delta_0(x - y) - b_{y_1}(x, y), \quad (4.5.4)$$

*in the sense of distributions on $U$ (where $\mathcal{P}$ acts on the $x$-variable). Moreover, we have*

$$\operatorname{supp} K_{y_1}(\cdot, y) \subseteq \mathbf{x}(y, y_1, [0,1]). \quad (4.5.5)$$

*Proof.* The claim (4.5.5) concerning the support of $K_{y_1}(\cdot, y)$ is clear from (4.5.3); hence it only remains to verify (4.5.4). Indeed, note that (4.5.4) is equivalent to

$$\langle K_{y_1}(\cdot, y), \mathcal{P}^* \varphi \rangle = \varphi(y) - \langle b_{y_1}(\cdot, y), \varphi \rangle \quad \text{for all } \varphi \in C_c^\infty(U),$$

which in turn is equivalent to (4.5.1). $\square$

**Example 4.5.2.** For $\mathcal{P}u = (\partial_j + B_j)u^j$, $\mathbf{x} = \underline{\mathbf{x}}$ (straight line segments), we have (4.3.9), which leads to $K_{y_1}$ and $b_{y_1}$ given by (4.3.10) and (4.3.11), respectively.

*Remark* 4.5.3 (Recovery on curves without endpoint). In view of the support property of $b_{y_1}(\cdot, y)$ in (4.5.2), (RC$^\vee$) implies the following weaker version:

(wRC) **Recovery on Curves (without endpoint).** For each $(y, y_1) \in W$ and multi-index $\gamma$, there exists an $r_0 \times s_0$-matrix-valued continuous function $s \mapsto S_J{}^{(\gamma,K)}(y, y_1, s)$ ($s \in [0,1]$) such that the following holds. For every ($\mathbb{C}^{r_0}$-valued) $\varphi \in C_c^\infty(U)$ with $\varphi$ vanishing in a neighborhood of $y_1$, we have

$$\varphi_J(y) = \int_0^1 \sum_\gamma S_J{}^{(\gamma,K)}(y, y_1, s)(\partial_x^\gamma(\mathcal{P}^*\varphi)_K)(\mathbf{x}(y, y_1, s))\, \mathrm{d}s, \tag{4.5.6}$$

where $S_J{}^{(\gamma,K)} = 0$ except for finitely many multi-indices $\gamma$.

Amusingly, the reverse implication also holds under a mild additional assumption. Specifically, assume that (wRC) and the following holds:

(R-$y_1$) **Regularity at** $y_1$ (or **endpoint regularity**). For each $\gamma$ and $(y, y_1) \in W$, $S_J{}^{(\gamma,K)}(y, y_1, s)$ is $(m_K + |\gamma| - 1)$-times continuously differentiable in $s$ at $s = 1$.

Then, for each $(y, y_1) \in W$, there exists an $r_0 \times r_0$-matrix-valued distribution $b_{y_1}(\cdot, y) \in \mathcal{D}'(U)$ such that (4.5.2) holds and, for every $\varphi \in C_c^\infty(U)$ (without the vanishing condition near $y_1$), (4.5.1) holds. In fact, $b_{y_1}(\cdot, y)$ is given by

$$\langle (b_{y_1})_J{}^{J'}(\cdot, y), \varphi_{J'}\rangle := \lim_{\epsilon \to 0} \int_0^1 \sum_\gamma S_J{}^{(\gamma,K)}(y, y_1, s)([\partial_x^\gamma \mathcal{P}^*, h_\epsilon]\varphi)_K(\mathbf{x}(y, y_1, s))\, \mathrm{d}s, \tag{4.5.7}$$

where $\boldsymbol{\nu}(y, y_1) := \frac{\partial_s \mathbf{x}(y,y_1,1)}{|\partial_s \mathbf{x}(y,y_1,1)|}$, $h_\epsilon(x) := \chi_{>1}(\epsilon^{-1}(\boldsymbol{\nu}(y, y_1) \cdot (y_1 - x)))$, and $\chi_{>1}(s)$ is a smooth nonnegative and nondecreasing function that equals 1 for $s > 1$ and 0 for $s < \frac{1}{2}$. We omit the details, as this fact will not be used in the remainder of the chapter.

## Quantitative formulation of (RC)

We now give quantitative version of the assumptions on the basic objects needed for carrying out our method outlined in Section 4.3.

### Conditions on x, quantitative version

Let $L_y, L_b > 0$, $M_{\mathbf{x}}, M'_{\mathbf{x}} \in \mathbb{Z}_{\geq 0}$ and $A_{\mathbf{x}}, A'_{\mathbf{x}} > 0$ be parameters to be used below (note that $L_b$, $A'_{\mathbf{x}}$, and $M'_{\mathbf{x}}$ are only used in (x-3)). In accordance with our multi-index notation, $(\partial_y + \partial_{y_1})^\alpha = \prod_{j=1}^d (\partial_{y^j} + \partial_{y_1^j})^{\alpha_j}$ and $(\partial_y + \partial_x)^\alpha = \prod_{j=1}^d (\partial_{y^j} + \partial_{x^j})^{\alpha_j}$.

Fix an open set $W \subseteq \mathbb{R}^d_y \times \mathbb{R}^d_{y_1}$. We will consider a family of curves $\mathbf{x} : W \times [0,1] \to \mathbb{R}^d$ satisfying (possibly a subset of) the following properties:

($\mathbf{x}$-1) The map $\mathbf{x}(y, y_1, s)$ is continuous and obeys

$$\mathbf{x}(y, y_1, 0) = y, \quad \mathbf{x}(y, y_1, 1) = y_1,$$
$$(1 + A_{\mathbf{x}})^{-1}|y_1 - y| \le |\partial_s \mathbf{x}(y, y_1, s)| \le (1 + A_{\mathbf{x}})|y_1 - y| \quad \text{for every } s \in (0, 1).$$

For higher derivatives, we have

$$\left|L_y^{|\alpha|}|y_1 - y|^{|\beta|}(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \partial_s \mathbf{x}(y, y_1, s)\right| \le A_{\mathbf{x}}|y_1 - y| \quad \text{for } |\alpha|+|\beta| \le M_{\mathbf{x}},\ |\alpha| > 0 \text{ or } |\beta| > 1.$$

($\mathbf{x}$-2) The map $y_1 \mapsto \mathbf{x}$ is invertible for each fixed $y$ and $s \in (0, 1]$; we denote the inverse by $\mathbf{y}_1(y, x, s)$. We assume that $\frac{\partial \mathbf{y}_1(y,x,s)}{\partial x}$ obeys

$$\left|\frac{\partial \mathbf{y}_1(y, x, s)}{\partial x}\right| \le (1 + A_{\mathbf{x}})s^{-1},$$

where we used the operator norm in $\mathbb{R}^d$. For higher derivatives, we have

$$\left|L_y^{|\alpha|}|x - y|^{|\beta|}(\partial_y + \partial_x)^\alpha \partial_x^\beta \mathbf{y}_1(y, x, s)\right| \le A_{\mathbf{x}} s^{-1}|x - y| \quad \text{for } |\alpha|+|\beta| \le M_{\mathbf{x}},\ |\alpha| > 0 \text{ or } |\beta| > 1.$$

($\mathbf{x}$-3) The map $\mathbf{x}(y, y_1, s)$ obeys

$$\left|L_y^{|\alpha|} L_b^{|\beta|}(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \partial_s \mathbf{x}(y, y_1, s)\right| \le A'_{\mathbf{x}} L_b \quad \text{for } |\beta| > 0,\ |\alpha| + |\beta| \le M'_{\mathbf{x}},$$

for every $(y, y_1, s) \in W \times (0, 1]$.

Hypotheses ($\mathbf{x}$-1) and ($\mathbf{x}$-2) are motivated by the assumptions on $\mathbf{z}$ in Section 4.4; for ($\mathbf{x}$-3), see Remark 4.5.5.

*Remark* 4.5.4 (($\mathbf{x}$-1)–($\mathbf{x}$-3) for straight line segments and other $\mathbf{x}$). As in Remark 4.4.1, a basic example of $\mathbf{x}$ is the *straight line segments*, $\underline{\mathbf{x}}(y, y_1, s) := y + (y_1 - y)s$, for which ($\mathbf{x}$-1)–($\mathbf{x}$-3) hold with $A_{\mathbf{x}} = 1$ and any $L_y, L_b > 0$.

Other interesting examples of $\mathbf{x}$ satisfying ($\mathbf{x}$-1)–($\mathbf{x}$-3) can be obtained by keeping $\mathbf{x}(y, y_1, s)$ close to straight line segments for small $s$ (e.g., $s < \frac{\delta}{|y_1 - y|}$ for some $0 < \delta \ll 1$), but letting it curve for large $s$. The solution operator in [MT22] (see also Chapter 2) adapted to degenerate cones may be constructed using such an $\mathbf{x}$.

*Remark* 4.5.5 (On the hypothesis ($\mathbf{x}$-3)). Note that ($\mathbf{x}$-3) improves upon ($\mathbf{x}$-1) for $|y_1 - y| \lesssim L_b$. Observe also that, since $(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \mathbf{x}(y, y_1, 0) = 0$ for $|\beta| > 0$ (indeed, $\mathbf{x}(y, y_1, 0) = y$), it follows from ($\mathbf{x}$-3) that

$$\left|L_y^{|\alpha|} L_b^{|\beta|}(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \mathbf{x}(y, y_1, s)\right| \le s A_{\mathbf{x}} L_b \quad \text{for } |\alpha| + |\beta| \le M'_{\mathbf{x}}, \tag{4.5.8}$$

for every $(y, y_1, s) \in W \times (0, 1]$.

At a technical level, we remark that ($\mathbf{x}$-3) is *not* needed in the proof of Theorem 4.5.11, our core analytic result. Its only use is to derive ($b_{y_1}$-2) below from a graded augmented system; see Proposition 4.6.6.(2).

**Recovery on curves condition, quantitative version**

In addition to (RC$^\vee$), we assume that the following assumptions on ${S_J}^{(\gamma,K)}$ hold for some $m'_K \in \mathbb{Z}_{\geq 0}$ ($K \in \{1, \dots, s_0\}$), $M_S \in \mathbb{Z}_{\geq 0}$ and $A_S > 0$:

(RC-q) **Recovery on curves, quantitative.** (RC$^\vee$) holds for ${S_J}^{(\gamma,K)}$ with the following bounds:

$$\left| L_y^{|\alpha|} |y_1 - y|^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta {S_J}^{(\gamma,K)}(y, y_1, s) \right| \leq A_S |y_1 - y|^{m_K + |\gamma|} s^{m_K + |\gamma| - 1} \quad \text{for } |\alpha| + |\beta| \leq M_S$$

for all $(y, y_1, s) \in W \times (0, 1]$, where $m_K$ is the order of the operator $(\mathcal{P}^* \varphi)_K$. Moreover, ${S_J}^{(\gamma,K)} = 0$ if $|\gamma| > m'_K$.

We also make (possibly a subset of) the following quantitative assumptions on $b_{y_1}$ for some $M_g, M_Z, M'_Z \in \mathbb{Z}_{\geq 0}$ and $A_g, A_Z, A'_Z > 0$:

($b_{y_1}$-1) **Structure of $b_{y_1}$, quantitative.** For every $(y, y_1) \in W$, there exists a distribution $b_{y_1}(\cdot, y)$ of the form

$$(b_{y_1})^{J'}{}_J(x, y) = \sum_{\mathbf{A} \in \mathcal{A}} (Z^{\mathbf{A}})_J(y, y_1)(g_{\mathbf{A}})^{J'}(x, y_1) \tag{4.5.9}$$

for some finite index set $\mathcal{A}$ (independent of $y, y_1$), which satisfies the relation (4.5.1) for every $(y, y_1) \in W$ with $y \in U$. Moreover, each $Z^{\mathbf{A}}$ obeys

$$|L_y^{|\alpha|} |y - y_1|^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta Z^{\mathbf{A}}(y, y_1)| \leq A_Z \quad \text{for } |\alpha| + |\beta| \leq M_Z,$$

and each $g_{\mathbf{A}}$ takes the form (for $J' = 1, \dots, r_0$)

$$(g_{\mathbf{A}})^{J'}(x, y_1) = \sum_\alpha c[g_{\mathbf{A}}]^{(\alpha, J')}(y_1) \partial^\alpha \delta_0(x - y_1),$$

where the coefficients $c[g_{\mathbf{A}}]^{(\alpha, J')}$ are zero except if $|\alpha| \leq \max_K (m_K + m'_K) - 1$ and obey

$$|\partial_{y_1}^\beta c[g_{\mathbf{A}}]^{(\alpha, J')}| \leq A_g \quad \text{for } |\beta| \leq M_g.$$

($b_{y_1}$-2) **Local regularity of $Z^{\mathbf{A}}$.** ($b_{y_1}$-1) holds with $Z^{\mathbf{A}}$ obeying the following additional bounds:

$$|L_y^{|\alpha|} L_b^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta Z^{\mathbf{A}}(y, y_1)| \leq A_Z \quad \text{for } |\alpha| + |\beta| \leq M'_Z$$

Some remarks concerning these quantitative assumptions are in order.

*Remark* 4.5.6 (On the hypotheses (RC-q) and ($b_{y_1}$-1)). Although these assumptions look complicated, we will see in Section 4.6 that (RC-q) and ($b_{y_1}$-1) may be derived directly on any curves $\mathbf{x}$ satisfying ($\mathbf{x}$-1)–($\mathbf{x}$-2) from the existence of a graded augmented system (with appropriate bounds); see Proposition 4.6.6.(1). Furthermore, this direct derivation will provide us with more concrete expressions for $Z^{\mathbf{A}}$ and $g_{\mathbf{A}}$, as well as a bound for $\#\mathcal{A}$.

*Remark* 4.5.7 (On the hypothesis ($b_{y_1}$-2)). Hypothesis ($b_{y_1}$-2) improves upon the bounds for $Z^{\mathbf{A}}$ in ($b_{y_1}$-1) when $|y_1 - y| \leq L_b$. Given a graded augmented system, ($b_{y_1}$-2) on $\mathbf{x}$ follows from an additional local (i.e., for $y_1$ close to $y$) regularity assumption ($\mathbf{x}$-3) for the curves $\mathbf{x}$.

At a technical level, we note that ($b_{y_1}$-2) is *not* used in the proof of Theorem 4.5.11, our core analytic result. It is only used in Section 4.5 below.

*Remark* 4.5.8 (Recovery on curves condition without endpoint, quantitative version). We note that ($b_{y_1}$-1) can be derived from (RC-q) and a quantitative version of the endpoint regularity condition. Like Remark 4.5.3, the following statement only plays a conceptual role and will not be used elsewhere in the chapter.

For simplicity, take $\mathbf{x}$ to be the straight line segments $\underline{\mathbf{x}}(y, y_1, s) = y + (y_1 - y)s$ (so that ($\mathbf{x}$-1)–($\mathbf{x}$-2) hold), and let $U$ and ${S_J}^{(\gamma,K)}(y, y_1, s)$ satisfy (RC-q). Assume also that

$$\bigcup_{y \in U,\, (y,y_1) \in W} \mathbf{x}(y, y_1, [0,1]) \subseteq U.$$

Assume also that $\mathcal{P}$ takes the form $\sum_{\alpha':|\alpha'| \leq m} {(c_{\mathcal{P}})^{(\alpha',J)}}_K(x)\partial^{\alpha'}$ and obeys, for some $M' \geq \mathbb{Z}_{\geq 0}$ and $A'_{\mathcal{P}} > 0$,

$$|\partial_x^\beta {(c_{\mathcal{P}})^{(\alpha',J)}}_K| \leq A'_{\mathcal{P}} \quad \text{for } |\beta| \leq m'_K + M',$$

and that, for some $A'_S > 0$,

$$\left| |y_1 - y|^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \partial_s^\ell {S_J}^{(\gamma,K)}(y, y_1, 1) \right| \leq A'_S \quad \text{for } |\alpha| + |\beta| \leq M',\ 1 \leq \ell \leq m_K + |\gamma|.$$

Then (4.5.1) holds with $b_{y_1}(\cdot, y)$ given by (4.5.7). Moreover, this $b_{y_1}(\cdot, y)$ satisfies ($b_{y_1}$-1) with $\mathcal{A} = \{(\alpha, J) : |\alpha| \leq \max_K(m_K + m'_K) - 1,\ J = 1, \ldots, r_0\}$ (where each $\alpha$ is a multi-index), $c[g_{(\alpha,J)}]^{(\beta,J')} = \delta_J^{J'} \delta_\alpha^\beta$ and

$$A_Z \leq A'_S, \quad A_g = 1, \quad M_Z = M_g = M' - C_{m,m_1}.$$

The proof proceeds by noting that $b_{y_1}$ defined by (4.5.7) in fact takes the form

$$(b_{y_1})^{J'}{}_J(x, y) = \sum_{\alpha:|\alpha| \leq m + m_1 - 1} (Z^{(\alpha,J')})_J(y, y_1) \partial^\alpha \delta_0(x - y_1),$$

where

$$(Z^{(\alpha,J')})_J(y, y_1) \coloneqq \left\langle (b_{y_1})^{J'}{}_J(x, y), \tfrac{(-1)^{|\alpha|}}{\alpha!}(x - y_1)^\alpha \right\rangle$$

$$= \sum_{\substack{\alpha',\beta,\gamma: \\ |\alpha'+\beta| \leq m + |\gamma|, \\ \alpha' \leq \alpha,\, |\beta| \geq 1,\, |\gamma| \leq m_1}} (-1)^{|\alpha+\beta|} \frac{(\alpha' + \beta)!}{(\alpha - \alpha')!\alpha'!\beta!} \frac{\partial_s \underline{\mathbf{x}}(y, y_1, 1)^\beta}{|\partial_s \underline{\mathbf{x}}(y, y_1, 1)|^{2|\beta|}}$$

$$\times \partial_s^{|\beta|} \left( {S_J}^{(\gamma,K')}(y, y_1, s) {c[\partial_x^\gamma \mathcal{P}^*]_{K'}}^{(\alpha+\beta,J')}(\underline{\mathbf{x}}(y, y_1, s))(\underline{\mathbf{x}}(y, y_1, s) - y_1)^{\alpha-\alpha'} \right) \Big|_{s=1}.$$

Again, we omit the details of the proof, as we will not use this in the remainder of the chapter.

**Conditions on the smooth averaging weight $\eta$, quantitative version**

Given $M_\eta, M'_\eta \in \mathbb{Z}_{\geq 0}$, $A_\eta > 0$, and $L_\eta > 0$, we will consider a smooth function $\eta : W \to \mathbb{R}$ satisfying the following properties:

($\eta$-1) $\int \eta(y, y_1)\, \mathrm{d}y_1 = 1$ for all $y$.

($\eta$-2) $\operatorname{supp} \eta \subseteq W$ and $\operatorname{supp} \eta(y, \cdot) \subseteq B_{L_\eta}(y)$.

($\eta$-3) We have

$$|L_y^{|\alpha|} |y_1 - y|^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \eta(y, y_1)| \leq A_\eta L_\eta^{-d} \text{ for } |\alpha| + |\beta| \leq M_\eta.$$

($\eta$-4) We have

$$|L_y^{|\alpha|} L_b^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \eta(y, y_1)| \leq A_\eta L_\eta^{-d} \quad \text{ for } |\alpha| + |\beta| \leq M'_\eta.$$

*Remark* 4.5.9 (On the hypotheses ($\eta$-1)–($\eta$-3)). Hypotheses ($\eta$-1)–($\eta$-3) are motivated by the assumptions on $\eta$ in Section 4.4. Note that ($\eta$-2) precludes the choice of smooth averaging weight that occurs in the conic operator in Section 4.3. However, it can be easily worked around; see Examples 4.4.4 and 4.5.13.

*Remark* 4.5.10 (On the hypothesis ($\eta$-4)). Note that ($\eta$-4) improves upon the bounds for $\eta$ in ($\eta$-3) when $|y_1 - y| \leq L_b$. At a technical level, we note that ($\eta$-4) is *not* used in the proof of Theorem 4.5.11, our core analytic result. It is only used in Section 4.5 below.

## Smooth averaging

We now carry out the construction of a smoothly averaged integral kernel $K_\eta$ outlined in Step 2 of Section 4.3, which is at the heart of our approach.

### Construction of smoothly averaged kernels

The *smoothly averaged kernel* is defined by the equation

$$\int (K_\eta)^K{}_J(x, y) \psi_K(x)\, \mathrm{d}x = \int \int_0^1 \sum_\gamma S_J{}^{(\gamma, K)}(y, y_1, s) \eta(y, y_1) \partial_x^\gamma \psi_K(\mathbf{x}(y, y_1, s))\, \mathrm{d}s\, \mathrm{d}y_1, \tag{4.5.10}$$

which may be (somewhat informally) also written as $K_\eta(x, y) = \int K_{y_1}(x, y) \eta(y, y_1)\, \mathrm{d}y_1$. Note that, in view of $\operatorname{supp} \eta \subseteq W$, the right-hand side of (4.5.10) is well-defined for every $y \in \mathbb{R}^d$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$, although it is trivial unless $y \in \{y \in \mathbb{R}^d : \exists y_1 \in \mathbb{R}^d \text{ s.t. } (y, y_1) \in W\}$.

**Theorem 4.5.11** (Smoothly averaged integral kernels)**.** *Let* $\mathbf{x} : W \times [0, 1] \to \mathbb{R}^d$ *and* $\eta : W \to \mathbb{R}$ *satisfy* ($\mathbf{x}$-1)–($\mathbf{x}$-2) *and* ($\eta$-1)–($\eta$-3)*, respectively, and assume that* $\mathcal{P}$ *and* $S_J{}^{(\gamma,K)}(y, y_1, s)$ *satisfy* (RC-q)*. Assume also that, either*

1. *$\overline{U} \cap \overline{\cup_{y \in U} \operatorname{supp}_{y_1} \eta(y, y_1)} = \emptyset$; or*

2. *$\mathcal{P}$, ${S_J}^{(\gamma,K)}(y, y_1, s)$ and ${(b_{y_1})^{J'}}_J(\cdot, y)$ satisfy ($b_{y_1}$-1).*

*Then the integral kernel $K_\eta = {(K_\eta)^K}_J$ defined by (4.5.10) satisfies*

$$\mathcal{P} K_\eta(\cdot, y) = \delta_0(\cdot - y) - b_\eta(\cdot, y) \quad \text{on } U,$$

*where ${(b_\eta)^{J'}}_J(x, y) = 0$ in Case (1), and in Case (2), ${(b_\eta)^{J'}}_J(x, y)$ is a distribution on $\mathbb{R}^d_x \times \mathbb{R}^d_y$ defined by*

$$\begin{aligned}
{(b_\eta)^{J'}}_J(x, y) &:= \sum_{\mathbf{A} \in \mathcal{A}} \int (Z^{\mathbf{A}})_J(y, y_1) \eta(y, y_1) (g_{\mathbf{A}})^{J'}(x, y_1) \, \mathrm{d}y_1 \\
&= \sum_{\mathbf{A} \in \mathcal{A}} \sum_\alpha (-1)^{|\alpha|} \partial_x^\alpha \left( c[g_{\mathbf{A}}]^{(\alpha, J')}(x) (Z^{\mathbf{A}})_J(y, x) \eta(y, x) \right),
\end{aligned}$$

*which moreover satisfies*

$$\begin{aligned}
|L_y^{|\alpha|} |x - y|^{|\beta|} (\partial_y + \partial_x)^\alpha \partial_x^\beta {(b_\eta)^{J'}}_J(x, y)| &\le C_{\#\mathcal{A}, m_K + m'_K} A_g A_Z A_\eta \quad \text{for all } x \neq y, \\
\operatorname{supp}_x {(b_\eta)^{J'}}_J(x, y) &\subseteq \operatorname{supp}_{y_1} \eta(y, y_1), \quad \text{for all } y \in \mathbb{R}^d,
\end{aligned}$$

*where $|\alpha| + |\beta| \le \min\{M_{\mathbf{x}}, M_Z, M_g, M_\eta\} - C_{m_K, m'_K}$.*

*In both cases, the integral kernel $K_\eta(x, y)$ is a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying*

$$\begin{aligned}
|{(K_\eta)^K}_J(x, y)| &\le C_{m_K, m'_K} A_S A_\eta (1 + A_{\mathbf{x}})^{2(d + m_{K'})} |x - y|^{-d + m_K} \quad \text{for all } x \neq y, \\
\operatorname{supp} {(K_\eta)^K}_J(\cdot, y) &\subseteq \overline{\bigcup_{y_1 : (y, y_1) \in \operatorname{supp} \eta} \mathbf{x}(y, y_1, [0, 1])}, \\
\operatorname{supp} {(K_\eta)^K}_J(x, \cdot) &\subseteq \overline{\{y \in \mathbb{R}^d : \exists y_1 \in \mathbb{R}^d \ s.t. \ (y, y_1) \in \operatorname{supp} \eta, \ x \in \mathbf{x}(y, y_1, [0, 1])\}},
\end{aligned}$$

*and the symbol ${(a_\eta)^K}_J(\xi, y) = \int {(K_\eta)^K}_J(y + z, y) e^{-i\xi \cdot z} \, \mathrm{d}z$ obeys*

$$|\partial_y^\alpha \partial_\xi^\beta {(a_\eta)^K}_J(\xi, y)| \le C_{m_K, m'_K, \alpha, \beta} A_S A_\eta (1 + A_{\mathbf{x}})^{10(d + |\alpha| + |\beta| + m_K + m'_K)} L_y^{-|\alpha|} (((1 + A_{\mathbf{x}}) L_\eta)^{-1} + |\xi|)^{-m - |\beta|}$$

*where $y \in \mathbb{R}^d$, $|\alpha| + |\beta| \le \min\{M_{\mathbf{x}}, M_S, M_\eta\} - C_{m_K, m'_K}$.*

*Proof.* The expressions for $K_\eta$ and $b_\eta$, as well as the estimates for $b_\eta(x, y)$, follow from the defining relation (4.5.1) and hypothesis ($b_{y_1}$-1). To obtain the claimed properties of $K_\eta$ and $a_\eta$, we apply Proposition 4.4.5 with $\mathrm{W} = \{(y, z_1) \in \mathbb{R}^d \times \mathbb{R}^d : (y, y + z_1) \in W\}$, $z(y, z_1, s) = \mathbf{x}(y, z_1 + y, s) - y$, $\mathrm{k}_\gamma(y, z_1, s) = {S_J}^{(\gamma, K)}(y, z_1 + y, s)$, and $\boldsymbol{\eta}(y, z_1) = \eta(y, z_1 + y)$. Indeed, observe that ($\mathbf{x}$-1)–($\mathbf{x}$-2), (RC-q) and ($\eta$-1)–($\eta$-3) imply that the assumptions in Section 4.4 are all satisfied (with appropriate parameters). $\square$

**Generalization of the conic operator and proof of Theorem 4.1.1**

From Theorem 4.5.11 we immediately obtain the following result, which generalizes the construction of conic solution operators.

**Theorem 4.5.12** (Full solution operator and Friedrich-type inequality, non-compact support)**.** *Assume that Case (1) of the hypothesis of Theorem 4.5.11 holds, i.e.,*

$$\overline{U} \cap \overline{\bigcup_{y\in U} \operatorname{supp}_{y_1} \eta(y, y_1)} = \emptyset. \tag{4.5.11}$$

*Then the following statements hold.*

1. **Full solution operator.** *For every $1 < p < \infty$ and $|s| < \min\{M_{\mathbf{x}}, M_S, M_\eta\} - C_{d,m_K,m'_K}$, the operator $(\mathcal{S}_\eta)^K{}_J$ with integral kernel $(K_\eta)^K{}_J$ defines a bounded operator $\widetilde{W}^{s,p}(U) \to W^{s+m_K,p}(U)$ with*

$$\mathcal{P}\mathcal{S}_\eta f = f \quad \textit{for all } f \in \widetilde{W}^{s,p}(U).$$

2. **Representation formula and Poincaré-type inequality.** *For every $1 < p < \infty$ and $|s| < \min\{M_{\mathbf{x}}, M_S, M_\eta\} - C_{d,m_K,m'_K}$, we have the representation formula*

$$\varphi_J = (\mathcal{S}^*_\eta)_J{}^K (\mathcal{P}^*\varphi)_K \quad \textit{for all } \varphi \in \widetilde{W}^{-s,p'}(U),$$

*where $\mathcal{P}^*\varphi$ is defined in the sense of $\mathcal{D}'(\mathbb{R}^d)$, so that $(\mathcal{P}^*\varphi)_K \in \widetilde{W}^{-s-m_K,p'}(U)$. Moreover, we have the Friedrich-type inequality*

$$\|\varphi\|_{W^{-s,p'}(U)} \lesssim \sum_K \|(\mathcal{P}^*\varphi)_K\|_{\widetilde{W}^{-s-m_K,p'}(U)} \quad \textit{for all } \varphi \in \widetilde{W}^{-s,p'}(U).$$

3. **Cokernel in $\widetilde{W}^{-s,p'}$.** *For any open subset $V$ such that $\overline{V} \subseteq U$, if $\mathbf{Z} \in \widetilde{W}^{-s,p'}(V)$ with $\mathcal{P}^*\mathbf{Z} = 0$ in $\mathcal{D}'(U)$, then $\mathbf{Z} = 0$; in short,*

$$\ker_{\widetilde{W}^{-s,p'}(V)} \mathcal{P}^* = \{0\}.$$

*Here, $\|(\mathcal{S}_\eta)^K{}_J\|_{\widetilde{W}^{s,p}(U)\to W^{s+m_K,p}(U)}$, $\|(\mathcal{S}^*_\eta)_J{}^K\|_{\widetilde{W}^{-s-m_K,p'}(U)\to W^{-s,p'}(U)}$ and the implicit constant in (2) are all bounded by $C_{d,m_K,m'_K,s,p} A_S A_\eta (1 + A_{\mathbf{x}})^{C(d+m_K+m_{K'}+s)}$.*

Note carefully that even for $f \in C^\infty_c(U)$ vanishing near $\partial U$, the theorem does *not* ensure that $\mathcal{S}_\eta f$ vanishes near $\partial U$; correspondingly, the representation formula require $\varphi \in \widetilde{W}^{s,p}(U)$, leading to a Friedrich-type inequality for $\mathcal{P}^*$. In order to ensure that $\mathcal{S}_\eta f$ vanishes near $\partial U$ (correspondingly, to prove a representation formula and Poincaré-type inequality for $\varphi$ in $W^{s,p}(U)$), we need to take into account the cokernel of $\mathcal{P}$ in $W^{s,p}(U)$ into account, as we will in Section 4.5 below.

*Proof.* Part (1) is an immediate consequence of Theorem 4.5.11, Case (1). Indeed, the bounds for the symbol $a_\eta$ in Theorem 4.5.11 implies that

$$\|(\mathcal{S}_\eta)^K{}_J\|_{W^{s,p}\to W^{s+m_K,p}} \leq C_{d,m_K,m'_K,s,p} A_S A_\eta (1+A_{\mathbf{x}})^{C(d+m_K+m_{K'}+s)}$$

by the standard theory of pseudodifferential operators [Ste93]. Restricting to inputs $f \in \widetilde{W}^{s,p}(U)$, and composing with the surjection $W^{s+m_K,p} \to W^{s+m_K,p}(U)$, the same bound for $\|(\mathcal{S}_\eta)^K{}_J\|_{\widetilde{W}^{s,p}\to W^{s+m_K,p}(U)}$ follows.

To prove Part (2), note that we have, by duality (4.2.7),

$$\|(\mathcal{S}^*_\eta)_J{}^K\|_{\widetilde{W}^{-s-m_K,p'}(U)\to W^{-s,p'}(U)} = \|(\mathcal{S}_\eta)^K{}_J\|_{W^{s,p}\to W^{s+m_K,p}} \leq C_{d,m_K,m'_K,s,p} A_S A_\eta (1+A_{\mathbf{x}})^{C(d+m_K+m_{K'}+s)}.$$

Moreover, $\mathcal{P}K_\eta(\cdot,y) = \delta_0(\cdot - y)$ for $y \in U$ is equivalent to

$$\varphi = \mathcal{S}^*_\eta \mathcal{P}^* \varphi \text{ for all } \varphi \in C^\infty_c(U),$$

which extends to all $\varphi \in \widetilde{W}^{-s,p'}(U)$ by approximation. The Friedrich-type inequality for $\|\varphi\|_{W^{-s,p'}(U)}$ now follows.

For $\mathbf{Z} \in \widetilde{W}^{s,p}(U)$ with $\mathcal{P}^*\mathbf{Z} = 0$ in $\mathcal{D}'(U)$, Part (2) only implies that $\mathbf{Z}$ corresponds to a zero element in $W^{s,p}(U)$, which still leaves the possibility that $\mathbf{Z} \neq 0$ as an element of $\widetilde{W}^{s,p}(U)$ (i.e., it may happen that $\operatorname{supp}\mathbf{Z} \subseteq \partial U$; see the discussion in Section 4.2). However, since we assume in addition that $\mathbf{Z} \in \widetilde{W}^{s,p}(V)$ for an open subset $V$ with $\overline{V} \subseteq U$, $\mathbf{Z} = 0$ in $W^{s,p}(U)$ is sufficient to conclude that $\mathbf{Z} = 0$ in $\widetilde{W}^{s,p}(V)$, which proves Part (3). □

**Example 4.5.13.** For $\mathcal{P}u = (\partial_j + B_j)u^j$, $\mathbf{x} = \underline{\mathbf{x}}$ (straight line segments), and a suitable choice of $\eta$, Theorem 4.5.12 specializes to the conic integral kernel $K_{\not\eta}(x,y)$ in Section 4.3, Step 2. Indeed, given $\not\eta \in C^\infty(\mathbb{S}^{d-1})$ with $\int \not\eta \,\mathrm{d}S = 1$, take $\eta(y,y_1) = \psi(y_1 - y)\not\eta(\frac{y_1-y}{|y_1-y|})$, where $\psi$ is as in Example 4.4.4. Then ($\eta$-1)–($\eta$-3) are clearly satisfied (with $L_\eta = R_{\eta}$, which can be taken to be arbitrarily large), and $K_\eta(x,y) = K_{\not\eta}(x,y)$ for $|x-y| \leq \frac{1}{2}R_{\eta}$.

We are now ready to give a precise formulation and proof of Theorem 4.1.1.

*Precise formulation and proof of Theorem 4.1.1.* To make Theorem 4.1.1 precise, we replace "(RC)" and "an admissibility family of curves $\mathbf{x}$" in the statement of the theorem by (RC-q) and ($\mathbf{x}$-1)–($\mathbf{x}$-2), respectively, on an open subset $\widetilde{U}$ that contains $\overline{U}$.

Then Parts (2) and (3), except for the Friedrich-type inequality, follow from Theorem 4.5.12 by simply choosing $\eta(y,y_1) = \eta_1(y_1)$ with $\eta_1 \in C^\infty_c(U_1)$ satisfying $\int \eta_1(y_1)\,\mathrm{d}y_1 = 1$. To prove the Friedrich-type inequality, we apply Theorem 4.1.1.(2) with $U$ replaced by $\widetilde{U}$ (the larger open subset that contains $\overline{U}$ on which the hypotheses hold), and observe that $\|\varphi\|_{\widetilde{W}^{-s,p'}(U)} \lesssim \|\varphi\|_{W^{-s,p'}(\widetilde{U})}$ while $\|(\mathcal{P}^*\varphi)_K\|_{\widetilde{W}^{-s-m_K,p'}(\widetilde{U})} \leq \|(\mathcal{P}^*\varphi)_K\|_{\widetilde{W}^{-s-m_K,p'}(U)}$. Similarly, for Part (1), i.e., the triviality of $\ker_{\widetilde{W}^{-s,p'}(U)}\mathcal{P}^*$, we apply Theorem 4.1.1.(3) with $U$ and $V$ replaced by $\widetilde{U}$ (the larger open subset that contains $\overline{U}$ on which the hypotheses hold) and $U$, respectively. □

## Solutions with compact support

Finally, we carry out the procedure outlined in Step 3 of Section 4.3 for constructing an operator that produces solutions $u$ to $\mathcal{P}u = f$ with $\operatorname{supp} u$ compact (provided, of course, that $f$ has compact support).

### Local regularity conditions for $Z^{\mathbf{A}}$ and $\eta$, and a simplifying assumption

Given an extra parameter $L_b > 0$ (with the dimension of length) and $M'_Z, M'_\eta \in \mathbb{Z}_{\geq 0}$, we **assume ($b_{y_1}$-2) holds** for $Z^{\mathbf{A}}(y, y_1)$. Furthermore, for simplicity, we will make the following smoothness assumption:

($C^\infty$) **Smoothness assumption.** $U$, $W$, $\mathbf{x}$, $\mathcal{P}$, ${S_J}^{(\gamma,K)}$ and ${(b_{y_1})^{J'}}_J$ satisfy (**x**-1)–(**x**-3), ($\eta$-1)–($\eta$-4), (RC-q), ($b_{y_1}$-1)–($b_{y_1}$-2) for arbitrary $M_{\mathbf{x}}$, $M_\eta$, $M'_\eta$, $M_S$, $M_Z$, $M'_Z$, $M_g$ (where the constants $A_{\mathbf{x}}$, $A'_{\mathbf{x}}$ $A_\eta$, $A_S$, $A_Z$, $A'_Z$ and $A_g$ may depend on $M_{\mathbf{x}}$, $M_\eta$, $M'_\eta$, $M_S$, $M_Z$, $M'_Z$, $M_g$).

This simplifying assumption allows us not to worry about the number of derivatives in the ensuing discussion. Without ($C^\infty$), the proofs below may be modified in a straightforward manner to cover the appropriate finite regularity case.

### $(Z^A)_{A\in\mathcal{A}}$, $\ker \mathcal{P}^*$ and generalization of the Bogovskii operator

We define the *formal kernel* of $\mathcal{P}^*$ (or equivalently, *formal cokernel* of $\mathcal{P}$) on $U$ to be

$$\ker \mathcal{P}^* := \{\mathbf{Z} \in C^\infty(\overline{U}) : \mathcal{P}^*\mathbf{Z} = 0 \text{ in } U\}.$$

**Lemma 4.5.14.** *Assume that $U$, $W$, $\mathbf{x}$, $\mathcal{P}$, $S_\gamma$ and $b_{y_1}$ satisfy ($C^\infty$)[8].*

1. *For every $\mathbf{Z} \in \ker \mathcal{P}^*$ and $y, y_1 \in U$ such that $\mathbf{x}(y, y_1, [0,1]) \subseteq U$, we have*

$$\langle b_{y_1}(\cdot, y), \mathbf{Z}\rangle = \mathbf{Z}(y).$$

   *In particular, we have*

$$\dim \ker \mathcal{P}^* \leq \#\mathcal{A}.$$

2. *If $\dim \ker \mathcal{P}^* = \#\mathcal{A}$, then for every basis $\{\mathbf{Z}^{\mathbf{A}}\}_{\mathbf{A}\in\mathcal{A}}$ of $\ker \mathcal{P}^*$ and $y, y_1 \in U$ such that $\mathbf{x}(y, y_1, [0,1]) \subseteq U$, we have the decomposition*

$${(b_{y_1})^{J'}}_J(x, y) = \sum_{\mathbf{A}\in\mathcal{A}} (\mathbf{Z}^{\mathbf{A}})_J(y)(\zeta_{\mathbf{A}})^{J'}(x, y_1).$$

   *Here, $\zeta_{\mathbf{A}}$ and $g_{\mathbf{A}}$ in ($b_{y_1}$-1) are related by the identity*

$$\langle (g_{\mathbf{A}})^{J'}(\cdot, y_1), (\mathbf{Z}^{\mathbf{A}'})_{J'}\rangle (\zeta_{\mathbf{A}'})^J(x, y_1) = (g_{\mathbf{A}})^J(x, y_1).$$

[8] We point out that there is no need for any assumptions on $\eta$ for this lemma.

*Proof.* Part (1) follows by testing $\mathbf{Z}(x) \in C^\infty(\overline{U})$ against (4.5.4), which is possible thanks to (4.5.5) and $\mathbf{x}(y, y_1, [0,1]) \subseteq U$. The claim $\dim\ker\mathcal{P}^* \leq \#\mathcal{A}$ then follows. If $\dim\ker\mathcal{P}^* = \#\mathcal{A}$, then given a basis $\{\mathbf{Z}^{\mathbf{A}}\}_{\mathbf{A}\in\mathcal{A}}$ of $\ker\mathcal{P}^*$, Part (1) implies that

$$\sum_{\mathbf{A}'} Z^{\mathbf{A}'}(y, y_1)\langle g_{\mathbf{A}'}(\cdot, y_1), \mathbf{Z}^{\mathbf{A}}\rangle = \mathbf{Z}^{\mathbf{A}}(y)$$

for every $\mathbf{A} \in \mathcal{A}$. It follows that the square matrix $\langle g_{\mathbf{A}'}(\cdot, y_1), \mathbf{Z}^{\mathbf{A}}\rangle$ is full rank, and therefore is invertible. □

From Theorem 4.5.11 and Lemma 4.5.14, we already obtain the following construction of the solution operator in an important special case, namely, when $\dim\ker\mathcal{P}^* = \#\mathcal{A}$. Motivated by Proposition 4.1.7 and Definition 4.1.8, we call this the *completely integrable* case. This procedure generalizes the construction of the Bogovskii operator for the divergence operator on $\mathbb{R}^d$.

**Theorem 4.5.15** (Full solution operator and Poincaré-type inequality, completely integrable case)**.** *Assume that $U$, $W$, $\mathbf{x}$, $\eta$, $\mathcal{P}$, $S$ and $b_{y_1}$ satisfy ($C^\infty$). Assume furthermore that*

$$\eta = \eta(y_1),$$

*and that $\overline{U}$ is* $\mathbf{x}$-star-shaped *with respect to* $\operatorname{supp}\eta$*, i.e.,*

$$\bigcup_{y\in\overline{U},\, y_1\in\operatorname{supp}\eta} \mathbf{x}(y, y_1, [0,1]) \subseteq \overline{U}.$$

*If the formal cokernel of $\mathcal{P}$ has the maximal dimension, i.e.,*

$$\dim\ker\mathcal{P}^* = \#\mathcal{A},$$

*then the following holds.*

1. **Full solution operator.** *The operator $\mathcal{S}_\eta$ in Theorem 4.5.11 defines a bounded operator $(\mathcal{S}_\eta)^K{}_J : \widetilde{W}^{s,p}(U) \to \widetilde{W}^{s+m_K,p}(U)$ for every $1 < p < \infty$ and $s \in \mathbb{R}$. Moreover, we have*

$$\mathcal{P}\mathcal{S}_\eta f = f - \sum_{\mathbf{A}\in\mathcal{A}} (\zeta_\eta)_{\mathbf{A}}\langle \mathbf{Z}^{\mathbf{A}}, f\rangle \quad \text{for all } f \in \widetilde{W}^{s,p}(U),$$

*where $(\zeta_\eta)_{\mathbf{A}}$ is a smooth function with $\operatorname{supp}(\zeta_\eta)_{\mathbf{A}} \subseteq \operatorname{supp}\eta$ characterized by*

$$\langle(\zeta_\eta)_{\mathbf{A}}, \varphi\rangle = \langle\zeta_{\mathbf{A}}(x, y_1), \varphi(x)\eta(y_1)\rangle \quad \text{for all } \varphi \in C_c^\infty(U),$$

*with $\zeta_{\mathbf{A}}$ and $\mathbf{Z}^{\mathbf{A}}$ as in Lemma 4.5.14. We also have the following support property:*

$$\operatorname{supp}\mathcal{S}_\eta f \subseteq \overline{\bigcup_{y\in\operatorname{supp} f,\, y_1\in\operatorname{supp}\eta} \mathbf{x}(y, y_1, [0,1])} \subseteq \overline{U}.$$

2. **Representation formula and Poincaré-type inequality.** *For* $1 < p < \infty$ *and* $s \in \mathbb{R}$, *we have the representation formula*

$$\varphi_J = (\mathcal{S}_\eta^*)_J{}^K (\mathcal{P}^*\varphi)_K + \sum_{\mathbf{A}\in\mathcal{A}} \langle (\zeta_\eta)_{\mathbf{A}}, \varphi\rangle \mathbf{Z}^{\mathbf{A}} \quad \text{for all } \varphi \in W^{-s,p'}(U),$$

*where* $\mathcal{P}^*\varphi$ *is defined in the sense of* $\mathcal{D}'(U)$, *so that* $(\mathcal{P}^*\varphi)_K \in W^{-s-m_K,p'}(U)$. *Moreover, we have the Poincaré-type inequality*

$$\left\| \varphi - \sum_{\mathbf{A}\in\mathcal{A}} \langle (\zeta_\eta)_{\mathbf{A}}, \varphi\rangle \mathbf{Z}^{\mathbf{A}} \right\|_{W^{-s,p'}(U)} \lesssim \sum_K \|(\mathcal{P}^*\varphi)_K\|_{W^{-s-m_K,p'}(U)} \text{ for all } \varphi \in W^{-s,p'}(U).$$

3. **Cokernel in** $W^{-s,p'}(U)$**.** *If* $\mathbf{Z} \in \widetilde{W}^{-s,p'}(U)$ *with* $\mathcal{P}^*\mathbf{Z} = 0$ *in* $\mathcal{D}'(U)$, *then* $\mathbf{Z} \in \ker \mathcal{P}^*$; *in short,*

$$\ker_{\widetilde{W}^{-s,p'}(U)} \mathcal{P}^* = \ker \mathcal{P}^*.$$

*Here,* $\|(\mathcal{S}_\eta)^K{}_J\|_{\widetilde{W}^{s,p}(U)\to W^{s+m_K,p}(U)}$, $\|(\mathcal{S}_\eta^*)_J{}^K\|_{\widetilde{W}^{-s-m_K,p'}(U)\to W^{-s,p'}(U)}$ *and the implicit constant in (2) are all bounded by* $C_{d,m_K,m_K',s,p} A_S A_\eta (1+A_{\mathbf{x}})^{C(d+m_K+m_{K'}+s)}$, *with* $A_S$, $A_\eta$, $A_{\mathbf{x}}$ *corresponding to* $M_{\mathbf{x}}, M_\eta, M_S \geq s + C_{d,m_K,m_{K'}}$.

In particular, in Part (1), $\mathcal{P}\mathcal{S}_\eta f = f$ if $f \in \widetilde{W}^{s,p}(U)$ is orthogonal to $\ker \mathcal{P}^*$, and in Part (2), we have $\|\varphi\|_{W^{-s,p'}(U)} \lesssim \sum_K \|(\mathcal{P}^*\varphi)_K\|_{W^{-s-m_K,p'}(U)}$ if $\varphi \in W^{-s,p'}(U)$ is orthogonal to $((\zeta_\eta)_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$.

*Proof.* Part (1) is an immediate consequence of Lemma 4.5.14 and Theorem 4.5.11, Case (2). We remark that the operator bounds on $(\mathcal{S}_\eta)^K{}_J$ are obtained as sketched in the proof of Theorem 4.5.12, but we have the additional mapping property $C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ in view of the support property of $K_\eta(\cdot, y)$; by completion, we obtain $\widetilde{W}^{s,p}(U) \to \widetilde{W}^{s+m_K,p}(U)$. We also remark that the smoothness of $\zeta_\eta$ follows from the algebraic formulas in Lemma 4.5.14 and the structure of $g_{\mathbf{A}}$ in ($b_{y_1}$-1).

To prove Part (2), the key observation is that $\mathcal{S}_\eta^*\mathcal{P}^*\varphi$ is well-defined for any $\varphi \in W^{-s,p'}(U)$ thanks to the obvious mapping property $(\mathcal{P}^*\varphi)_K \in W^{-s-m_K,p'}$ and the duality property (4.2.7). The desired identity and inequality are now immediate consequences of Part (1).

Finally, Part (3) is an immediate consequence of Part (2): indeed, if $\mathbf{Z} \in \widetilde{W}^{-s,p'}(U)$ with $\mathcal{P}^*\mathbf{Z} = 0$, then by Part (2), we have $\mathbf{Z} = \sum_{\mathbf{A}\in\mathcal{A}} \langle (\zeta_\eta)_{\mathbf{A}}, \mathbf{Z}\rangle \mathbf{Z}^{\mathbf{A}} \in \ker \mathcal{P}^*$. □

**Example 4.5.16.** For $\mathcal{P}u = (\partial_j + B_j)u^j$, $\mathbf{x} = \underline{\mathbf{x}}$ (straight line segments), and $\eta(y, y_1) = \eta_1(y_1)$, $\mathcal{S}_\eta$ in Theorem 4.5.15 specializes to the Bogovskii solution operator $\mathcal{S}_{\eta_1}(x, y)$ in Section 4.3, Step 3 (completely integrable case), with $\#\mathcal{A} = 1$, $\mathbf{Z} = e^z$, and $\zeta_\eta = \mathbf{Z}^{-1}\eta_1$.

### Soft arguments for the general case and proof of Theorem 4.1.2

We introduce the following assumption (in addition to the objects that have been already introduced), which is a slight generalization of one of the hypotheses of Theorem 4.5.15:

($\overline{U}\star$) $\overline{U}$ **is x-star-shaped with respect to** $\operatorname{supp}\eta$**.** We have

$$\overline{\bigcup_{(y,y_1)\in\operatorname{supp}\eta} \mathbf{x}(y,y_1,[0,1])} \subseteq \overline{U}.$$

In view of Theorem 4.5.11, this assumption ensures that the integral kernel $K_\eta(\cdot,y)$ is supported in $\overline{U}$ for $y\in\overline{U}$.

We begin with a more detailed study of the properties of $\mathcal{B}_\eta$ under our assumptions (in particular, ($b_{y_1}$-2)).

**Lemma 4.5.17.** *Assume that $U$, $W$, $\mathbf{x}$, $\eta$, $\mathcal{P}$, $S$ and $b_{y_1}$ satisfy ($C^\infty$). Assume furthermore that ($\overline{U}\star$) is satisfied for $U$, $\mathbf{x}$ and $\eta$.*

1. $\mathcal{B}_\eta$ **is smoothing.** *The integral kernel $(b_\eta)^{J'}{}_J(x,y)$ for $\mathcal{B}_\eta$ is smooth and satisfies*

$$|L_y^{|\alpha|}L_b^{|\beta|}(\partial_y+\partial_x)^\alpha\partial_x^\beta(b_\eta)^{J'}{}_J(x,y)| \leq C_{\#\mathcal{A},m_K+m'_K}A_gA_ZA_\eta,$$
$$\operatorname{supp}_x(b_\eta)^{J'}{}_J(x,y)\subseteq\operatorname{supp}_{y_1}\eta(y,y_1),\qquad \text{for all } y\in U,$$

*where $A_g$, $A_Z$, $A_\eta$ correspond to $M_g, M_Z, M_\eta, M'_Z, M'_\eta \geq |\alpha|+|\beta|+C_{d,m+m_1}$.*

2. **Approximation of $\mathcal{B}_\eta$ by finite rank operators.** *Assume, in addition, that $U$ is a bounded open subset of $\mathbb{R}^d$. For every $\epsilon_0>0$ and an open set $W_0$ such that*

$$\operatorname{supp} b_\eta\subseteq W_0\subseteq\overline{W_0}\subseteq U\times U,$$

*there exists a linear operator ${}^{(\epsilon_0)}\mathcal{E}_0$ with a smooth integral kernel ${}^{(\epsilon_0)}e_0 : U\times U\to\mathbb{C}^{r_0\times r_0}$, as well as an index set ${}^{(\epsilon_0)}\widetilde{\mathbf{A}}$ and smooth functions ${}^{(\epsilon_0)}(g_0)_{\widetilde{\mathbf{A}}} : U\to\mathbb{C}^{r_0}$ and ${}^{(\epsilon_0)}Z_0^{\widetilde{\mathbf{A}}} : U\to\mathbb{C}^{r_0}$ such that, for all $f\in C_c^\infty(U)$,*

$$\mathcal{P}\mathcal{S}_\eta f = f - {}^{(\epsilon_0)}\mathcal{E}_0 f - \sum_{\widetilde{\mathbf{A}}\in{}^{(\epsilon_0)}\widetilde{\mathcal{A}}} {}^{(\epsilon_0)}(g_0)_{\widetilde{\mathbf{A}}}\langle{}^{(\epsilon_0)}Z_0^{\widetilde{\mathbf{A}}}, f\rangle.$$

*Moreover, it can be arranged so that kernel ${}^{(\epsilon_0)}e_0$ satisfies*

$$\sup_{x\in U}\int_U\left|{}^{(\epsilon_0)}e_0(x,y)\right|\,\mathrm{d}y+\sup_{y\in U}\int_U\left|{}^{(\epsilon_0)}e_0(x,y)\right|\,\mathrm{d}x\leq\epsilon_0, \tag{4.5.12}$$
$$|\partial_x^\alpha\partial_y^\beta({}^{(\epsilon_0)}e_0(x,y))|\leq C_{d,\epsilon_0,U,\operatorname{supp}b_\eta}A_gA_ZA_\eta,$$
$$\operatorname{supp}{}^{(\epsilon_0)}e_0(x,y)\subseteq W_0;$$

*the functions* ${}^{(\epsilon_0)}(g_0)_{\widetilde{\mathbf{A}}}$, ${}^{(\epsilon_0)}Z_0^{\widetilde{\mathbf{A}}}$ *satisfy*

$$\left|\partial_x^\alpha({}^{(\epsilon_0)}(g_0)_{\widetilde{\mathbf{A}}})(x)\right| \le C_{d,\epsilon_0,U,\operatorname{supp} b_\eta} A_g A_\eta, \quad \left|\partial_y^\beta({}^{(\epsilon_0)}Z_0^{\widetilde{\mathbf{A}}})(y)\right| \le C_{d,\epsilon_0,U,\operatorname{supp} b_\eta} A_Z,$$
$$\operatorname{supp}\left(\sum_{\widetilde{\mathbf{A}}\in{}^{(\epsilon_0)}\widetilde{\mathcal{A}}} {}^{(\epsilon_0)}(g_0)_{\widetilde{\mathbf{A}}}(x){}^{(\epsilon_0)}Z_0^{\widetilde{\mathbf{A}}}(y)\right) \subseteq W_0.$$

*and* $\sharp({}^{(\epsilon_0)}\widetilde{\mathcal{A}})$ *is bounded by* $C_{d,\epsilon_0,U,\operatorname{supp} b_\eta}$. *Here,* $A_g$, $A_Z$, $A_\eta$ *correspond to* $M_g, M_Z, M_\eta, M_Z', M_\eta' \ge |\alpha| + |\beta| + C_{d,m_K+m_K'}$

*Proof.* Recall from Theorem 4.5.11 that

$$b_\eta(x,y) = \sum_{\mathbf{A}\in\mathcal{A}}\sum_\alpha (-1)^{|\alpha|}\partial_x^\alpha\left(c[g_{\mathbf{A}}]^{(\alpha,J')}(x)(Z^{\mathbf{A}})_J(y,x)\eta(y,x)\right).$$

Part (1) immediately follows from this expression, ($b_{y_1}$-1), ($b_{y_1}$-2), ($\eta$-3) and ($\eta$-4) (which are a part of ($C^\infty$)). Part (2) is, of course, a direct consequence of Part (1). In what follows, we describe a construction of ${}^{(\epsilon_0)}\mathcal{E}_0$, ${}^{(\epsilon_0)}(g_0)_{\widetilde{\mathbf{A}}}$ and ${}^{(\epsilon_0)}Z_0^{\widetilde{\mathbf{A}}}$.

To ease the notation, we suppress the superscript ${}^{(\epsilon_0)}$ in what follows. Let $(\chi_{x_{\mathbf{G}}}(x)\chi_{y_{\mathbf{G}}}(y))_{\mathbf{G}\in\mathcal{G}_0}$ be a finite partition of unity on $\operatorname{supp} b_\eta$ but vanishing outside of $W_0$ which will be fixed at the end of the construction (see also Remark 4.5.18 below). We have

$$(b_\eta)^{J'}{}_J(x,y) = \sum_{\mathbf{G}\in\mathcal{G}_0}\sum_{\mathbf{A}\in\mathcal{A}}\sum_\alpha (-1)^{|\alpha|}\partial_x^\alpha\left(c[g_{\mathbf{A}}]^{(\alpha,J')}(x)(Z^{\mathbf{A}})_J(y,x)\eta(y,x)\chi_{x_{\mathbf{G}}}(x)\chi_{y_{\mathbf{G}}}(y)\right).$$

Defining $\widetilde{\mathcal{A}} \coloneqq \mathcal{A}\times\mathbf{G}_0$ and

$$\begin{aligned}
(e_0)^{J'}{}_J(x,y) \coloneqq{}& \sum_{\widetilde{\mathbf{A}}\in\widetilde{\mathcal{A}}}\sum_\alpha (-1)^{|\alpha|}\partial_x^\alpha\left(c[g_{\mathbf{A}}]^{(\alpha,J')}(x)Z^{\mathbf{A}}(y,x)\left[\eta(y,x)-\eta(y_{\mathbf{G}},x)\right]\chi_{x_{\mathbf{G}}}(x)\chi_{y_{\mathbf{G}}}(y)\right)\\
&+\sum_{\widetilde{\mathbf{A}}\in\widetilde{\mathcal{A}}}\sum_\alpha (-1)^{|\alpha|}\partial_x^\alpha\left(c[g_{\mathbf{A}}]^{(\alpha,J')}(x)\left[Z^{\mathbf{A}}(y,x)-Z^{\mathbf{A}}(y,x_{\mathbf{G}})\right]\eta(y_{\mathbf{G}},x)\chi_{x_{\mathbf{G}}}(x)\chi_{y_{\mathbf{G}}}(y)\right),\\
(g_0)^{J'}{}_{\widetilde{\mathbf{A}}}(x) \coloneqq{}& \sum_\alpha (-1)^{|\alpha|}\partial_x^\alpha\left(c[g_{\mathbf{A}}]^{(\alpha,J')}(x)\eta(y_{\mathbf{G}},x)\chi_{x_{\mathbf{G}}}(x)\right),\\
Z_0^{\widetilde{\mathbf{A}}}(y) \coloneqq{}& Z^{\mathbf{A}}(y,x_{\mathbf{G}})\chi_{y_{\mathbf{G}}}(y),
\end{aligned}$$

where $\widetilde{\mathbf{A}} = (\mathbf{A},\mathbf{G})$, we obtain the desired decomposition and support properties. Moreover, in view of the presence of the differences in each sum in the definition of $e_0(x,y)$, as well as ($b_{y_1}$-2) and ($\eta$-4), (4.5.12) holds if we ensure that the supports of each $\chi_{x_{\mathbf{G}}}$ and $\chi_{y_{\mathbf{G}}}$ sufficiently small depending on $\epsilon_0$, $d$, $L_y$, $L_b$, $A_Z$ and $A_\eta$; at this point, we fix the choice of these functions. The remaining bounds then follow from ($b_{y_1}$-1), ($b_{y_1}$-2), ($\eta$-3), and ($\eta$-4). □

*Remark* 4.5.18. The finite partition of unity $(\chi_{x_{\mathbf{G}}}(x)\chi_{y_{\mathbf{G}}}(y))_{\mathbf{G}\in\mathcal{G}_0}$ used in the proof always exists since $U$ is bounded (hence $\operatorname{supp} b_\eta$ is compact). The precise quantitative bounds on the functions $\chi_{x_{\mathbf{G}}}$ and $\chi_{y_{\mathbf{G}}}$, which determines the constant $C_{d,\epsilon_0,U,\operatorname{supp} b_\eta}$, would depend on the regularity assumptions on $U$ (alternatively, on $\operatorname{supp}\eta$ or $\operatorname{supp} b_\eta$).

Combining Lemma 4.5.17.(1) with a standard contradiction argument, we obtain the following Poincaré-type inequality for $\mathcal{P}^*$ with optimal orthogonality conditions (i.e., formulated with respect to $\ker \mathcal{P}^*$), but with a non-effective constant.

**Proposition 4.5.19** (Optimal Poincaré-type inequality with a non-effective constant)**.** *Let $U$ be a connected bounded open subset of $\mathbb{R}^d$, and assume that there exist $W$, $\mathbf{x}$, $\eta$, $\mathcal{P}$, $S$ and $b_{y_1}$ satisfying ($C^\infty$). Assume furthermore that ($\overline{U}\star$) is satisfied for $U$, $\mathbf{x}$, and $\eta$.*

1. **Cokernel in $W^{-s,p'}(U)$.** *For any $1 < p_0, p_1 < \infty$ and $s_0, s_1 \in \mathbb{R}$, we have*

$$\ker_{\widetilde{W}^{-s_0,p_0'}(U)} \mathcal{P}^* = \ker_{\widetilde{W}^{-s_1,p_1'}(U)} \mathcal{P}^*.$$

*Both are finite-dimensional and, in fact, coincide with $\ker \mathcal{P}^*$.*

2. **Poincaré-type inequality.** *For $1 < p < \infty$ and $s \in \mathbb{R}$, consider a family $w_{\mathbf{A}}(x) \in \widetilde{W}^{s,p}(U)$ ($\mathbf{A} = \{1, \dots, \dim\ker_{W^{-s,p'}(U)} \mathcal{P}^*\}$) satisfying $\langle w_{\mathbf{A}}, \mathbf{Z}^{\mathbf{A}'}\rangle = \delta_{\mathbf{A}}^{\mathbf{A}'}$ for some basis $\{\mathbf{Z}^{\mathbf{A}'}\}$ of $\ker_{W^{-s,p'}(U)} \mathcal{P}^*$. Then there exists $C > 0$ such that*

$$\|\varphi\|_{W^{-s,p'}(U)} \leq C \sum_K \|(\mathcal{P}^*\varphi)_K\|_{W^{-s-m_K,p'}(U)} \quad \text{for all } \varphi \in W^{-s,p'}(U) \text{ with } \langle w_{\mathbf{A}}, \varphi\rangle = 0,$$

*where $\mathbf{A} = 1, \dots, \dim\ker_{W^{-s,p'}(U)} \mathcal{P}^*$.*

By *non-effective*, we mean that there is no quantitative relationship between the constant $C$ and the parameters of our construction (e.g., $A_{\mathbf{x}}$, $A_S$ etc.). This feature is due to the use of a compactness argument in the proof below.

*Proof.* We begin by observing that, as in the proof of Theorem 4.5.15, we have the mapping properties $(\mathcal{S}_\eta)^K{}_J : \widetilde{W}^{s,p}(U) \to \widetilde{W}^{s+m_K,p}(U)$, and thus $(\mathcal{S}_\eta^*)_J{}^K : W^{-s-m_K,p'}(U) \to W^{-s,p'}(U)$. Thus, for every $\varphi \in W^{-s,p'}(U)$, we have

$$\varphi = \mathcal{S}_\eta^*\mathcal{P}^*\varphi + \mathcal{B}_\eta^*\varphi,$$

where $\mathcal{B}_\eta$ is a smoothing operator according to Lemma 4.5.17. As a consequence, for any $\delta > 0$, $1 < p < \infty$ and $s \in \mathbb{R}$,

$$\|\mathcal{B}_\eta\varphi\|_{W^{-s,p'}(U)} \lesssim_{\delta,s,p} \|\varphi\|_{W^{-s-\delta,p'}(U)}.$$

From this estimate, it immediately follows that for $\mathbf{Z} \in \ker_{W^{-s,p'}(U)} \mathcal{P}^*$, we have $\mathbf{Z} = \mathcal{B}_\eta^* \mathbf{Z}$, and thus the cokernels in different Sobolev spaces are the same; this implies Part (1). Moreover, combined with Theorem 4.5.11, we obtain the elliptic estimate

$$\|\varphi\|_{W^{-s,p'}(U)} \lesssim_{s,p,\delta} \sum_K \|(\mathcal{P}^*\varphi)_K\|_{W^{-s-m_K,p'}(U)} + \|\varphi\|_{W^{-s-\delta,p'}(U)}, \tag{4.5.13}$$

for every $\varphi \in W^{-s,p'}(U)$.

We are ready to start the proof of Part (2). Suppose that the conclusion does not hold; then there exists a sequence $\varphi^{(n)} \in W^{-s,p'}(U)$ such that

$$\|\varphi^{(n)}\|_{W^{-s,p'}(U)} = 1, \quad \|(\mathcal{P}^*\varphi^{(n)})_K\|_{W^{-s-m_K,p'}(U)} \leq \frac{1}{n}, \quad \langle w_{\mathbf{A}}, \varphi^{(n)}\rangle = 0 \text{ for } \mathbf{A} = 1, \dots, \dim \ker_{W^{-s,p'}(U)} \mathcal{P}^*.$$

By Rellich–Kondrachov (Lemma 4.2.2), there exists $\varphi \in W^{-s,p'}(U)$ such that, after passing to a subsequence,

$$\varphi^{(n)} \rightharpoonup \varphi \text{ in } W^{-s,p'}(U), \quad \varphi^{(n)} \to \varphi \text{ in } W^{-s-\delta,p'}(U).$$

By the weak $W^{-s,p'}(U)$-convergence, we have $\mathcal{P}^*\varphi = 0$ (i.e., $\varphi \in \ker_{W^{-s,p'}(U)} \mathcal{P}^*$) and $\langle w_{\mathbf{A}}, \varphi\rangle = 0$ for all $\mathbf{A} = 1, \dots, \dim \ker_{W^{-s,p'}(U)} \mathcal{P}^*$. By the properties of $w_{\mathbf{A}}$, we have $\varphi = 0$. But, by the strong $W^{s-\delta,p}(U)$-convergence and (4.5.13), we have $\varphi \neq 0$, which is a contradiction. □

By a standard duality argument, Proposition 4.5.19 may be turned into an existence statement for the equation $\mathcal{P}u = f$.

**Corollary 4.5.20.** *Let $U$ be a connected bounded open subset of $\mathbb{R}^d$, and assume that there exist $W$, $\mathbf{x}$, $\eta$, $\mathcal{P}$, $S$ and $b_{y_1}$ satisfying ($C^\infty$). Assume furthermore that ($\overline{U}\star$) is satisfied for $U$, $\mathbf{x}$, and $\eta$. For every $f \in \widetilde{H}^s(U)$ satisfying $f \perp \ker_{H^{-s}(U)} \mathcal{P}^*$, there exists $u = (u^K)_{K\in\{1,\dots,s_0\}}$ with $u^K \in \widetilde{H}^{s+m_K}(U)$ such that $\mathcal{P}u = f$ and $\|u^K\|_{\widetilde{H}^{s+m_K}(U)} \leq C\|f\|_{\widetilde{H}^s(U)}$, where $C$ is the constant in Proposition 4.5.19 with $p = 2$.*

*Proof.* By $\widetilde{H}^{s+m_K}(U) = (H^{-s-m_K}(U))^*$ from (4.2.7), it suffices to construct bounded linear functionals $u^K$ on $H^{-s-m_K}(U)$ such that $\langle u^K, (\mathcal{P}^*\varphi)_K\rangle = \langle f^J, \varphi_J\rangle$ for all $\varphi_J \in H^{-s}(U)$. We will use the preceding proposition and the Hahn–Banach theorem.

Let $(\mathbf{Z}^{\mathbf{A}})_J$ and $(w_{\mathbf{A}})^{J'}$ be as in Proposition 4.5.19, and let $\varphi_J \in H^{-s}(U)$. Since $\langle f^J, (\mathbf{Z}^{\mathbf{A}})_J\rangle = 0$ for all $\mathbf{A} = 1, \dots, \dim\ker \mathcal{P}^*$, we have

$$\begin{aligned}|\langle f^J, \varphi_J\rangle| &= \left|\langle f^J, \varphi_J - \sum_{\mathbf{A}} (\mathbf{Z}^{\mathbf{A}})_J \langle (w_{\mathbf{A}})^{J'}, \varphi_{J'}\rangle\rangle\right| \leq \sum_J \|f^J\|_{\widetilde{H}^s(U)} \left\|\varphi_J - \sum_{\mathbf{A}} (\mathbf{Z}^{\mathbf{A}})_J \langle (w_{\mathbf{A}})^{J'}, \varphi_{J'}\rangle\right\|_{H^{-s}(U)} \\ &\leq C \sum_{J,K} \|f^J\|_{\widetilde{H}^s(U)} \|(\mathcal{P}^*\varphi)_K\|_{H^{-s-m_K}(U)},\end{aligned}$$

where $C$ is the constant in Proposition 4.5.19 with $p = 2$. It follows that the linear functional $\langle u^K, (\mathcal{P}^*\varphi)_K\rangle \coloneqq \langle f^J, \varphi_J\rangle$ is well-defined and bounded on the vector subspace $M = \mathcal{P}^*(H^{-s}(U;\mathbb{C}^{r_0}))$ of $H^{-s-m_1} \times \cdots \times H^{-s-m_{s_0}}(U)$. By the Hahn–Banach theorem, it may be extended to an element (with an abuse of notation) $u \in (H^{-s-m_1} \times \cdots \times H^{-s-m_{s_0}}(U))^* = \widetilde{H}^{s+m_1} \times \cdots \times \widetilde{H}^{s+m_{s_0}}(U)$ with $\|u^K\|_{\widetilde{H}^{s+m_K}(U)} \leq C\|f\|_{\widetilde{H}^s(U)}$, as desired. □

Finally, we upgrade Corollary 4.5.20 to the existence of a linear operator $\mathcal{Q}$ that completes $\mathcal{S}_\eta$ to a full solution operator, which is moreover smoothing and has a prescribed support property, but with non-effective bounds on the operator norms.

**Theorem 4.5.21** (Full solution operator and representation formula)**.** *Let $U$ be a bounded open subset of $\mathbb{R}^d$, and assume that $W$, $\mathbf{x}$, $\eta$, $\mathcal{P}$, $S$ and $b_{y_1}$ satisfy ($C^\infty$). Assume furthermore that ($\overline{U}\star$) is satisfied for $U$, $\mathbf{x}$, and $\eta$. Consider a family $w_{\mathbf{A}}(x) \in C_c^\infty(U)$ ($\mathbf{A} = \{1, \ldots, \dim\ker\mathcal{P}^*\}$) satisfying $\langle w_{\mathbf{A}}, \mathbf{Z}^{\mathbf{A}'}\rangle = \delta_{\mathbf{A}}^{\mathbf{A}'}$ for some basis $\{\mathbf{Z}^{\mathbf{A}'}\}$ of $\ker\mathcal{P}^*$. Consider also an open subset $V$ of $U$ satisfying, for all $\mathbf{A} \in \{1, \ldots, \dim\ker\mathcal{P}^*\}$ and $y \in U$,*

$$\operatorname{supp} w_{\mathbf{A}} \subseteq V, \quad \operatorname{supp}_x \eta(y,x) \subseteq V.$$

*Then there exists a linear operator $\mathcal{Q}$ such that*

$$\mathcal{P}(\mathcal{S}_\eta - \mathcal{Q})f = f - \sum_{\mathbf{A}\in\{1,\ldots,\dim\ker\mathcal{P}^*\}} w_{\mathbf{A}}\langle \mathbf{Z}^{\mathbf{A}}, f\rangle \quad \text{for all } f \in C_c^\infty(U),$$

*where the integral kernel $q(x,y)$ of $\mathcal{Q}$ has uniformly bounded derivatives of all order and, for every $y \in U$,*

$$\operatorname{supp}_x q(x,y) \subseteq V \cup \overline{\bigcup_{y'\in V,\,(y',y_1)\in\operatorname{supp}\eta} \mathbf{x}(y', y_1, [0,1])}.$$

*By duality, we also have the representation formula*

$$\varphi = (\mathcal{S}_\eta - \mathcal{Q})^*\mathcal{P}^*\varphi + \sum_{\mathbf{A}\in\{1,\ldots,\dim\ker\mathcal{P}^*\}} \mathbf{Z}^{\mathbf{A}}\langle w_{\mathbf{A}}, \varphi\rangle \quad \text{for all } \varphi \in C^\infty(\overline{U}).$$

As a consequence, for any $1 < p < \infty$ and $s \in \mathbb{R}$, the full solution operator $\mathcal{S}_\eta - \mathcal{Q}$ extends to a bounded operator $\widetilde{W}^{s,p}(U) \to \widetilde{W}^{s+m_1,p} \times \cdots \widetilde{W}^{s+m_{s_0},p}(U)$. Hence, the representation formula holds for all $\varphi \in W^{-s,p'}(U)$, for any $1 < p < \infty$ and $s \in \mathbb{R}$. The operator norm of $\mathcal{Q}$ (and thus that of $\mathcal{S}_\eta - \mathcal{Q}$) produced in our proof below is non-effective, but only through the application of Corollary 4.5.20.

*Proof.* Let $\epsilon_0, \epsilon_1 > 0$ be small parameters to be fixed later. Our starting point is Lemma 4.5.17.(2), which provides us with an approximation of $\mathcal{B}_\eta$ by a finite rank operator $\sum_{\widetilde{\mathbf{A}}}(g_0)_{\widetilde{\mathbf{A}}}\langle Z_0^{\widetilde{\mathbf{A}}}, \cdot\rangle$ up to an error operator $\mathcal{E}_0$ obeying, in particular, (4.5.12) with $\epsilon_0$ on the right-hand side (to ease the notation, we omit the superscript $^{(\epsilon_0)}$). Let us project each $(g_0)_{\widetilde{\mathbf{A}}}$ to $^\perp(\ker\mathcal{P}^*)$ using $w_{\mathbf{A}}$; i.e., we introduce

$$g_{\widetilde{\mathbf{A}}} \coloneqq (g_0)_{\widetilde{\mathbf{A}}} - \sum_{\mathbf{A}} \langle \mathbf{Z}^{\mathbf{A}}, (g_0)_{\widetilde{\mathbf{A}}}\rangle w_{\mathbf{A}},$$

where, here and throughout the proof, the index $\mathbf{A}$ is summed over $\{1, \ldots, \dim\ker\mathcal{P}^*\}$. Then we arrive at

$$\mathcal{P}\mathcal{S}_\eta f = f - \mathcal{E}_0 f - \sum_{\mathbf{A}} w_{\mathbf{A}} \ell^{\mathbf{A}}(f) - \sum_{\widetilde{\mathbf{A}} \in \widetilde{\mathcal{A}}} g_{\widetilde{\mathbf{A}}} \langle Z_0^{\widetilde{\mathbf{A}}}, f \rangle, \tag{4.5.14}$$

where $\ell^{\mathbf{A}}(f)$ is a linear functional for each $\mathbf{A}$ (it may be computed explicitly, but it does not matter) and $g_{\widetilde{\mathbf{A}}} \perp \ker\mathcal{P}^*$ for each $\widetilde{\mathbf{A}} \in \widetilde{\mathcal{A}}$.

For each such $g_{\widetilde{\mathbf{A}}}$, we now look for $u_{\widetilde{\mathbf{A}}} \in C_c^\infty(V)$ that solves the approximate equation

$$\mathcal{P} u_{\widetilde{\mathbf{A}}} = g_{\widetilde{\mathbf{A}}} + e_{\widetilde{\mathbf{A}}}, \tag{4.5.15}$$

with $\operatorname{supp} u_{\widetilde{\mathbf{A}}}, \operatorname{supp} e_{\widetilde{\mathbf{A}}} \subseteq V$ and an error bound (in terms of $\epsilon_1$) for $e_{\widetilde{\mathbf{A}}}$. For this purpose, we first fix an open subset $V_1$ such that

$$\bigcup_{\widetilde{\mathbf{A}} \in \widetilde{\mathcal{A}}} \operatorname{supp} g_{\widetilde{\mathbf{A}}} \subseteq V_1 \subseteq \overline{V_1} \subseteq V,$$

and apply Corollary 4.5.20 to find a solution $v_{\widetilde{\mathbf{A}}}$ in, say, $\widetilde{H}^{1+m_1} \times \cdots \times \widetilde{H}^{1+m_{s_0}}(V_1)$ to $\mathcal{P} v_{\widetilde{\mathbf{A}}} = g_{\widetilde{\mathbf{A}}}$. We fix $\psi_1 \in C_c^\infty(B_1(0))$ such that $\int \psi_1 = 1$ and define

$$u_{\widetilde{\mathbf{A}}} \coloneqq \psi_{\epsilon_1} * v_{\widetilde{\mathbf{A}}},$$

where $v_{\widetilde{\mathbf{A}}}$ is viewed as an element of $H^{1+m_1} \times \cdots \times H^{1+m_{s_0}}(\mathbb{R}^d)$ that is supported in $\overline{V_1}$ (recall the definition of $\widetilde{H}^s(V_1)$ as a subspace of $H^s$) and $\psi_\epsilon(\cdot) \coloneqq \epsilon^{-d}\psi_1(\epsilon^{-1}(\cdot))$. Then (4.5.15) holds with

$$e_{\widetilde{\mathbf{A}}} = -(g_{\widetilde{\mathbf{A}}} - \psi_{\epsilon_1} * g_{\widetilde{\mathbf{A}}}) - [\psi_{\epsilon_1} *, \mathcal{P}] v_{\widetilde{\mathbf{A}}}.$$

We now verify that such $u_{\widetilde{\mathbf{A}}}$ and $e_{\widetilde{\mathbf{A}}}$ have desirable properties. First, as long as $\epsilon_1 < \operatorname{dist}(\overline{V_1}, \partial V)$, we have

$$\operatorname{supp} u_{\widetilde{\mathbf{A}}}, \operatorname{supp} e_{\widetilde{\mathbf{A}}} \subseteq V. \tag{4.5.16}$$

Moreover, rewriting $[\psi_{\epsilon_1} *, \mathcal{P}] v_{\widetilde{\mathbf{A}}} = \psi_{\epsilon_1} * \mathcal{P} v_{\widetilde{\mathbf{A}}} - \mathcal{P} v_{\widetilde{\mathbf{A}}} + \mathcal{P}(v_{\widetilde{\mathbf{A}}} - \psi_{\epsilon_1} * v_{\widetilde{\mathbf{A}}})$ and using the property of convolution, we obtain

$$\|e_{\widetilde{\mathbf{A}}}\|_{L^2} \lesssim \|g_{\widetilde{\mathbf{A}}} - \psi_{\epsilon_1} * g_{\widetilde{\mathbf{A}}}\|_{L^2} + \|\mathcal{P} v_{\widetilde{\mathbf{A}}} - \psi_{\epsilon_1} * \mathcal{P} v_{\widetilde{\mathbf{A}}}\|_{L^2} + \|v_{\widetilde{\mathbf{A}}} - \psi_{\epsilon_1} * v_{\widetilde{\mathbf{A}}}\|_{H^{m_1} \times \cdots \times H^{m_{s_0}}} \lesssim \epsilon_1 \|g_{\widetilde{\mathbf{A}}}\|_{H^1}, \tag{4.5.17}$$

where the implicit constants depend on that in Corollary 4.5.20 with $s_0 = 1$ and $\|\mathcal{P}\|_{H^{s+m_1} \times \cdots \times H^{s+m_{s_0}} \to H^s}$ with $s = 0, 1$. Finally, we clearly have, for every $N \geq 0$,

$$\|u_{\widetilde{\mathbf{A}}}\|_{H^{1+N+m_1} \times \cdots \times H^{1+N+m_{s_0}}} \leq C_N \epsilon_1^{-N} \|g_{\widetilde{\mathbf{A}}}\|_{H^1}, \quad \|e_{\widetilde{\mathbf{A}}}\|_{H^{1+N}} \leq C \|g_{\widetilde{\mathbf{A}}}\|_{H^{1+N}} + C_N \epsilon_1^{-N} \|g_{\widetilde{\mathbf{A}}}\|_{H^1}. \tag{4.5.18}$$

We are now ready to conclude the proof. Introducing the operators

$$\mathcal{Q}_1 f \coloneqq \sum_{\widetilde{\mathbf{A}} \in \widetilde{\mathcal{A}}} u_{\widetilde{\mathbf{A}}} \langle Z_0^{\widetilde{\mathbf{A}}}, f \rangle, \quad \mathcal{E}_1 f \coloneqq \sum_{\widetilde{\mathbf{A}} \in \widetilde{\mathcal{A}}} e_{\widetilde{\mathbf{A}}} \langle Z_0^{\widetilde{\mathbf{A}}}, f \rangle,$$

we arrive at

$$\mathcal{P}(\mathcal{S}_\eta - \mathcal{Q}_1)f = f - (\mathcal{E}_0 + \mathcal{E}_1)f - \sum_{\mathbf{A}\in\{1,\ldots,\dim\ker\mathcal{P}^*\}} w_{\mathbf{A}}\ell^{\mathbf{A}}(f).$$

We define the operator $\mathcal{Q}$ by

$$\mathcal{S}_\eta - \mathcal{Q} = (\mathcal{S}_\eta - \mathcal{Q}_1)(I-\mathcal{E})^{-1}, \quad \text{where } \mathcal{E} \coloneqq \mathcal{E}_0 + \mathcal{E}_1.$$

To finish the proof, it remains to verify the desired properties of $\mathcal{Q}$ (including that it is well-defined). First, by Lemma 4.5.17.(2) and (4.5.18), we see that the integral kernels $e(x,y)$ and $q_1(x,y)$ of $\mathcal{E}$ and $\mathcal{Q}_1$, respectively, have uniformly bounded derivatives of all order and $\operatorname{supp} e(\cdot,y)$, $\operatorname{supp} q_1(\cdot,y) \subseteq V$ for all $y \in U$. Next, in view of Lemma 4.5.17.(2) (especially (4.5.12)) and (4.5.17), we may choose $\epsilon_0$ and $\epsilon_1$ small enough so that $\|\mathcal{E}\|_{L^2(U)\to L^2(U)} < 1$. As a consequence, $I - \mathcal{E}$ is invertible on $L^2(U)$. Moreover, in view of the identities

$$(I-\mathcal{E})^{-1} = I + \mathcal{E} + \mathcal{E}^2 + \cdots = I + \mathcal{E}(1-\mathcal{E})^{-1}, \quad \left[(I-\mathcal{E})^{-1}\right]^* = I + \mathcal{E}^*\left[(1-\mathcal{E})^{-1}\right]^*,$$

it follows that the integral kernel $K(x,y)$ of $(I-\mathcal{E})^{-1} - I$ also has uniformly bounded derivatives of all order and $\operatorname{supp} K(\cdot,y) \subseteq V$ for all $y \in U$. Now writing

$$\mathcal{Q} = -(\mathcal{S}_\eta - \mathcal{Q}_1)\left((I-\mathcal{E})^{-1} - I\right) + \mathcal{Q}_1,$$

the desired properties of the integral kernel $q(x,y)$ of $\mathcal{Q}$ follow from those for $\mathcal{S}_\eta$, $\mathcal{Q}_1$ and $(I-\mathcal{E})^{-1} - I$. □

**Example 4.5.22.** For $\mathcal{P}u = (\partial_j + B_j)w^j$, $\mathbf{x} = \underline{\mathbf{x}}$ (straight line segments), and $\eta(y,y_1) = \eta_1(y_1)$, Theorem 4.5.21 is the precise version of the result outlined in Step 4 (non-completely integrable case).

*Remark* 4.5.23 (Comparison with Theorem 4.5.15). Other than the non-effective nature of the operator bounds, another difference between Theorems 4.5.15 and 4.5.21 is the latter's ability to prescribe the dual family $w_{\mathbf{A}}$ to $\ker\mathcal{P}^*$. This feature allow us to have more freedom in prescribing the support properties of the solution operator.

We are ready to give a precise formulation and proof of Theorem 4.1.2.

*Precise formulation and proof of Theorem 4.1.2.* Theorem 4.1.2.(2) and (3) – with the admissibility of $\mathbf{x}$ and (RC) replaced by ($\mathbf{x}$-1)–($\mathbf{x}$-2) and (RC-q) on $U$ – follows from Theorem 4.5.21 by simply choosing $\eta(y,y_1) = \eta_1(y_1)$ with $\eta_1 \in C_c^\infty(U_1)$ that satisfies $\int \eta_1(y_1)\,\mathrm{d}y_1 = 1$. The full solution operator $\widetilde{\mathcal{S}}$ is precisely $\mathcal{S}_\eta + \mathcal{Q}$. For the properties of $\ker_{W^{-s,p'}(U)}\mathcal{P}^*$, we apply Proposition 4.5.19 (for the invariance and finite-dimensional properties), and Lemma 4.5.14.(1) for paths that connect $y_1 \in V$ and $y \in U$ for the invariance of the dimension under restrictions. □

## 4.6 From graded augmented system to (RC)

In Section 4.5, we identified some abstract conditions needed to construct the operator $\mathcal{S}_\eta$ with prescribed support properties. In this section, we formulate and prove a precise version of Proposition 4.1.6 (see Proposition 4.6.6 below), i.e., that these conditions follow from the existence of a graded augmented system with adequate quantitative bounds. We also provide a proof of Proposition 4.1.7.

This section is structured as follows. In Section 4.6, we record the precise formulas for the objects ${S_J}^{(\gamma,K)}$ and ${(b_{y_1})^{J'}}_J$ arising in (RC$^\vee$) in terms of a graded augmented system. To prove Proposition 4.1.6, it remains to establish the quantitative bounds in (RC-q), ($b_{y_1}$-1) and ($b_{y_1}$-2). In Section 4.6, we formulate and prove an abstract ODE lemma for this purpose. Then in Section 4.6, we prove the main result of this section (Proposition 4.6.6). Finally, in Section 4.6, we give a geometric interpretation of cokernel elements as parallel vectors with respect to a suitable connection on a vector bundle constructed from the augmented system, and prove Proposition 4.1.7 as a simple byproduct.

### Structure of $S(y, y_1, t)$ and $b_{y_1}$ for a given augmented system

Let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on an open subset $U \subseteq \mathbb{R}^d$, with $m_K$ denoting the order of $(\mathcal{P}^*\varphi)_K$ for each $K \in \{1, \dots, s_0\}$. Given $m'_K \in \mathbb{Z}_{\geq 0}$ for each $K \in \{1, \dots, s_0\}$ and an $\mathbb{C}^{r_0}$-valued function $(\varphi_J)_{J\in\{1,\dots,r_0\}}$ on $U$, let $(\Phi_{\mathbf{A}} = \Phi_{\mathbf{A}}(y))_{\mathbf{A}\in\mathcal{A}}$ be (graded) augmented variables as in Definition 4.1.4 (see also the discussion above this definition for our conventions for indices). The aim of this short subsection is to record the formulas for $S(y, y_1, t)$ and $b_{y_1}$ in (RC$^\vee$) in terms of the augmented system.

Let ${(\mathbf{B}_i)_{\mathbf{A}}}^{\mathbf{A}''}$ and ${(\mathbf{C}_i)_{\mathbf{A}}}^{(\gamma,K)}$ be the coefficients of the system of first-order PDEs in ($\Phi$-3). Let ${}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_{\mathbf{A}}}^{\mathbf{A}'}(s,t)$ be the fundamental matrix solving the ODE

$$\partial_s \left[{}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_{\mathbf{A}}}^{\mathbf{A}'}(s,t)\right] = \dot{\mathbf{x}}^i_{y,y_1}\left({(\mathbf{B}_i)_{\mathbf{A}}}^{\mathbf{A}''} \circ \mathbf{x}_{y,y_1}\right)(y, y_1, s)\, {}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_{\mathbf{A}''}}^{\mathbf{A}'}(s,t),$$
$${}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_{\mathbf{A}}}^{\mathbf{A}'}(t,t) = \delta^{\mathbf{A}'}_{\mathbf{A}},$$

or equivalently,

$${}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_{\mathbf{A}}}^{\mathbf{A}'}(s,t) = \delta^{\mathbf{A}'}_{\mathbf{A}} - \int_s^t \dot{\mathbf{x}}^i_{y,y_1}\left( {(\mathbf{B}_i)_{\mathbf{A}}}^{\mathbf{A}''} \circ \mathbf{x}_{y,y_1}\right)(y, y_1, s')\, {}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_{\mathbf{A}''}}^{\mathbf{A}'}(s',t)\, \mathrm{d}s'. \quad (4.6.1)$$

Recall that, by Duhamel's principle, ($\Phi$-1) and ($\Phi$-3), we have

$$\varphi_J(y) = -\int_0^1 {}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_J}^{\mathbf{A}}(0,s)\dot{\mathbf{x}}^i_{y,y_1}\left({(\mathbf{C}_i)_{\mathbf{A}}}^{(\gamma,K)} \circ \mathbf{x}_{y,y_1}\right)\left((\partial^\gamma \mathcal{P}^*\varphi)_K \circ \mathbf{x}_{y,y_1}\right)(y, y_1, s)\, \mathrm{d}s$$
$$+ {}^{(\mathbf{x}_{y,y_1})}{\mathbf{\Pi}_J}^{\mathbf{A}}(0,1)\Phi_{\mathbf{A}}(y_1). \quad (4.1.10)$$

From this expression, we immediately obtain expressions for ${S_J}^{(\gamma,K)}(y, y_1, s)$ and $b_{y_1}(x, y)$ in (RC$^\vee$), as well as $K_{y_1}(x, y)$ in (4.5.3). We record them in the following proposition.

**Proposition 4.6.1.** *Let* $(\varphi_J)_{J\in\{1,\dots,r_0\}} \mapsto (\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$ *satisfy* (Φ-1)–(Φ-3) *with smooth* $c[\Phi_{\mathbf{A}}]^{(\alpha,J)}$, $(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}(y)$ *and* $(\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}(y)$*. Then* ($\mathrm{RC}^\vee$) *is satisfied with the following objects:*

$$S_J{}^{(\gamma,K)}(y,y_1,t) := -{}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_J{}^{\mathbf{A}}(0,t)\dot{\mathbf{x}}^i_{y,y_1}\left((\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}\circ\mathbf{x}_{y,y_1}\right)(y,y_1,t), \tag{4.6.2}$$

$$(b_{y_1})^{J'}{}_J(x,y) := \sum_{\mathbf{A}\in\mathcal{A}}(Z^{\mathbf{A}})_J(y,y_1)(g_{\mathbf{A}})^{J'}(x,y_1), \tag{4.6.3}$$

$$(Z^{\mathbf{A}})_J(y,y_1) := {}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_J{}^{\mathbf{A}}(0,1), \tag{4.6.4}$$

$$\langle (g_{\mathbf{A}})^J(\cdot,y_1),\varphi_J\rangle := \Phi_{\mathbf{A}}(y_1) \quad (i.e.,\ c[g_{\mathbf{A}}]^{(\alpha,J)}(y_1) = c[\Phi_{\mathbf{A}}]^{(\alpha,J)}(y_1)), \tag{4.6.5}$$

*In particular,* $b_{y_1}$ *clearly satisfies* (4.5.2)*. Moreover, defining the kernel* $(K_{y_1})^K{}_J(x,y)$ *by*

$$\langle (K_{y_1})^K{}_J(x,y),\psi_K(x)\rangle = \sum_\gamma\int_0^1 S_J{}^{(\gamma,K)}(y,y_1,s)\partial_x^\gamma\psi_K(\mathbf{x}(y,y_1,s))\,\mathrm{d}s,$$

*we have* (4.5.4)*, i.e.,* $\mathcal{P}K_{y_1}(x,y) = \delta_0(x-y) - b_{y_1}(x,y)$*.*

## Abstract ODE estimates

Let $\mathcal{A}$ be a finite index set of size $N$, which we label $\{1,\dots,N\}$ without loss of generality. Consider the following $N\times N$ system of ODEs for $t\in[0,T]$ for some $T>0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\mathbf{A}}(t) = \sum_{\mathbf{A}'}B_{\mathbf{A}}{}^{\mathbf{A}'}(t)\psi_{\mathbf{A}'}(t) + g_{\mathbf{A}}(t), \tag{4.6.6}$$

where $\mathbf{A},\mathbf{A}'\in\{1,\dots,N\}$. As in (Φ-4), we associate to each index $\mathbf{A}\in\{1,\dots,N\}$ a *degree* $d_{\mathbf{A}}\in\mathbb{Z}$, and without loss of generality, we assume that

$$\max_{\mathbf{A}\in\{1,\dots,N\}} d_{\mathbf{A}} = 0. \tag{4.6.7}$$

We assume that $B$ obeys the following vanishing condition:

$$B_{\mathbf{A}}{}^{\mathbf{A}'}(s)\equiv 0 \quad \text{if } d_{\mathbf{A}} > d_{\mathbf{A}'}+1. \tag{4.6.8}$$

*Remark* 4.6.2. In our applications, we take $B_{\mathbf{A}}{}^{\mathbf{A}'}(s) = \dot{\mathbf{x}}^i\left((\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}\circ\mathbf{x}\right)(y,y_1,s)$ for a fixed $(y,y_1)\in W$. Hence, (4.6.8) follows from the properties of degree introduced in Definition 4.1.4.

The solution to (4.6.6) with the final data $\psi_{\mathbf{A}}(t)=0$ is given by

$$\psi_{\mathbf{A}}(t) = -\sum_{\mathbf{A}'}\int_t^T \Pi_{\mathbf{A}}{}^{\mathbf{A}'}(t,s)g_{\mathbf{A}'}(s)\,\mathrm{d}s, \tag{4.6.9}$$

where $\Pi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t)$ is the fundamental solution solving,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s,t) = B(s)\Pi(s,t), \quad \Pi(t,t) = I,$$

or equivalently,

$$\Pi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t) = \delta_{\mathbf{A}}{}^{\mathbf{A}'} - \int_s^t \sum_{\mathbf{A}''} B_{\mathbf{A}}{}^{\mathbf{A}''}(s')\Pi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)\,\mathrm{d}s'. \tag{4.6.10}$$

Eventually, the kernel $S_\gamma(y,y_1,s)$ in (wRC) will be constructed out of $\Pi_{\mathbf{A}}{}^{\mathbf{A}'}(0,s)$ for $s \in [0,T]$ (see Proposition 4.6.1 and Remark 4.6.2). Our goal in this subsection is to estimate the size of $\Pi(t,s)$ in terms of the coefficients $B$ in (4.6.6).

For this purpose, we first formulate and prove a result for an abstract ODE (or more precisely, an integral equation; see (4.6.11)). We introduce the following scale of norms for $b : [0,T] \to \mathbb{R}$ and $m \in \mathbb{Z}_{\leq 0}$:

$$\begin{aligned} &|||b|||_0 := \|b\|_{L^\infty[0,T]} && \text{if } m = 0, \\ &|||b|||_m := \int_0^T |b(s)| s^{-m-1}\,\mathrm{d}s && \text{if } m < 0. \end{aligned}$$

We introduce

$$\begin{aligned} |||B|||_{(\infty)} &:= \sum_{\mathbf{A},\mathbf{A}':d_{\mathbf{A}}=d_{\mathbf{A}'}+1} \left|\left|\left| B_{\mathbf{A}}{}^{\mathbf{A}'} \right|\right|\right|_{d_{\mathbf{A}}-d_{\mathbf{A}'}-1} = \sum_{\mathbf{A},\mathbf{A}':d_{\mathbf{A}}=d_{\mathbf{A}'}+1} \|B_{\mathbf{A}}{}^{\mathbf{A}'}\|_{L^\infty[0,T]}, \\ |||B|||_{(1)} &:= \sum_{\mathbf{A},\mathbf{A}':d_{\mathbf{A}}\leq d_{\mathbf{A}'}} \left|\left|\left| B_{\mathbf{A}}{}^{\mathbf{A}'} \right|\right|\right|_{d_{\mathbf{A}}-d_{\mathbf{A}'}-1} = \sum_{\mathbf{A},\mathbf{A}':d_{\mathbf{A}}\leq d_{\mathbf{A}'}} \int_0^T |B_{\mathbf{A}}{}^{\mathbf{A}'}(s)| s^{-(d_{\mathbf{A}}-d_{\mathbf{A}'})}\,\mathrm{d}s, \end{aligned}$$

and define

$$|||B||| := \sum_{\mathbf{A},\mathbf{A}'} \left|\left|\left| B_{\mathbf{A}}{}^{\mathbf{A}'} \right|\right|\right|_{d_{\mathbf{A}}-d_{\mathbf{A}'}-1} = |||B|||_{(\infty)} + |||B|||_{(1)}.$$

The main result of this subsection is the following.

**Proposition 4.6.3** (Abstract ODE estimates)**.** *Consider any $T > 0$ and degrees $d_{\mathbf{A}}$ ($\mathbf{A} \in \{1,\ldots,N\}$) and an $N \times N$ matrix-valued function $B$ on $[0,T]$ satisfying* (4.6.7)*,* (4.6.8) *and $|||B||| < +\infty$. Let $\Psi_{\mathbf{A}}{}^{\mathbf{A}'} = \Psi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t)$ ($\mathbf{A},\mathbf{A}' \in \{1,\ldots,N\}$) satisfy the equation*

$$\Psi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t) = \int_s^t G_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)\,\mathrm{d}s' + \int_s^t \sum_{\mathbf{A}''} B_{\mathbf{A}}{}^{\mathbf{A}''}(s')\Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)\,\mathrm{d}s' \tag{4.6.11}$$

*for $0 < s < t < T$, where*

$$\int_s^t (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |G_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)|\,\mathrm{d}s' \leq A_0 \tag{4.6.12}$$

*for some* $A_0 > 0$*. Then for all* $0 < s < t < T$*, we have*

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t)| \leq CA_0 \qquad \text{if } d_{\mathbf{A}} \leq 0, \tag{4.6.13}$$

$$\int_s^t (s')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq CA_0 \qquad \text{if } d_{\mathbf{A}} < 0, \tag{4.6.14}$$

*where we have the following bound for the constant* $C$ *in* (4.6.13)–(4.6.14)*:*

$$C \leq C_{N,\max_{\mathbf{A}}(-d_{\mathbf{A}})} (2 + \|\!|B|\!\|_{(\infty)})^{C_N[1+(1+\|\!|B|\!\|_{(\infty)})^N \|\!|B|\!\|_{(1)}]}.$$

*Proof.* In this proof, we suppress the dependence of constants on $N$ and $\max_{\mathbf{A}}(-d_{\mathbf{A}})$. Exceptions to this rule are the constants denoted by $C_N$, which depend only on $N$ but not on $\max_{\mathbf{A}}(-d_{\mathbf{A}})$. Moreover, some constants that are independent of $N$ and $\max_{\mathbf{A}}(-d_{\mathbf{A}})$ (i.e., absolute) will be pointed out in the argument.

*S*tep 1. We first work under an additional assumption

$$\|\!|B|\!\|_{(1)} < \epsilon, \tag{4.6.15}$$

where $\epsilon < 1$ will be specified at the end of the step. Let $t \in (0,T)$ be fixed, and consider $s \in (0,t)$. Define

$$\begin{aligned} A_{\mathbf{A}}{}^{\mathbf{A}'}(s,t) &:= \left[ \sup_{s' \in [s,t]} (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| + \int_s^t (s')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \right], \\ A(s,t) &:= \sum_{\mathbf{A},\mathbf{A}'} A_{\mathbf{A}}{}^{\mathbf{A}'}(s,t). \end{aligned}$$

By the assumptions on $B$, it easily follows that $\Psi(s,t)$ is continuous for $s \in (0,t]$, and that $A(s,t) < +\infty$ for $s \in (0,t]$. The goal of this step is to prove that, under (4.6.15), we have, for all $s \in (0,t)$,

$$A(s,t) \lesssim (1 + \|\!|B|\!\|_{(\infty)})^N A_0.$$

Let us use (4.6.11) to estimate $\Psi(s,t)$. We begin by estimating the contribution of $G$:

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \int_s^t |G_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq A_0 \qquad \text{if } d_{\mathbf{A}} \leq 0, \tag{4.6.16}$$

$$\int_s^t (s'')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} \int_{s''}^t |G_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \, \mathrm{d}s'' \leq \frac{1}{(-d_{\mathbf{A}})} A_0 \qquad \text{if } d_{\mathbf{A}} < 0. \tag{4.6.17}$$

Indeed, recalling that $d_{\mathbf{A}}, d_{\mathbf{A}'} \leq 0$ by (4.6.7), bound (4.6.16) follows simply from (4.6.12) and the obvious inequality $s^{-d_{\mathbf{A}}} \leq (s')^{-d_{\mathbf{A}}}$ for $s' \in [s,t]$. To prove (4.6.17), note that its LHS is bounded (via Fubini) by

$$\int_s^t \left( \int_s^{s'} (s'')^{-d_{\mathbf{A}}-1} \, \mathrm{d}s'' \right) t^{d_{\mathbf{A}'}} |G_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq \frac{1}{(-d_{\mathbf{A}})} \int_s^t (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |G_{\mathbf{A}}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq \frac{1}{(-d_{\mathbf{A}})} A_0,$$

where we used $d_{\mathbf{A}} < 0$ in the first inequality.

Next, we claim that

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \int_s^t |B_{\mathbf{A}}{}^{\mathbf{A}''} \Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq \begin{cases} \|\!|B|\!\|_{(\infty)} A_{\mathbf{A}''}{}^{\mathbf{A}'}(s,t) & \text{if } d_{\mathbf{A}} \leq 0,\ d_{\mathbf{A}} = d_{\mathbf{A}''} + 1, \\ \|\!|B|\!\|_{(1)} A(s,t) & \text{if } d_{\mathbf{A}} \leq 0,\ d_{\mathbf{A}} \leq d_{\mathbf{A}''}, \end{cases} \tag{4.6.18}$$

$$\int_s^t (s'')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} \int_{s''}^t |B_{\mathbf{A}}{}^{\mathbf{A}''} \Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \, \mathrm{d}s'' \leq \begin{cases} \frac{1}{(-d_{\mathbf{A}})} \|\!|B|\!\|_{(\infty)} A_{\mathbf{A}''}{}^{\mathbf{A}'}(s,t) & \text{if } d_{\mathbf{A}} < 0,\ d_{\mathbf{A}} = d_{\mathbf{A}''} + 1, \\ \frac{1}{(-d_{\mathbf{A}})} \|\!|B|\!\|_{(1)} A(s,t) & \text{if } d_{\mathbf{A}} < 0,\ d_{\mathbf{A}} \leq d_{\mathbf{A}''}. \end{cases} \tag{4.6.19}$$

Indeed, for (4.6.18) in the case $d_{\mathbf{A}} = d_{\mathbf{A}''} + 1$, we have

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \int_s^t |B_{\mathbf{A}}{}^{\mathbf{A}''} \Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq \|\!|B|\!\|_{(\infty)} \int_s^t (s')^{-d_{\mathbf{A}''}-1} t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq \|\!|B|\!\|_{(\infty)} A_{\mathbf{A}''}{}^{\mathbf{A}'}(s,t),$$

and in the case $d_{\mathbf{A}} \leq d_{\mathbf{A}''}$, we have

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \int_s^t |B_{\mathbf{A}}{}^{\mathbf{A}''} \Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \leq \int_s^t (s')^{-d_{\mathbf{A}}+d_{\mathbf{A}''}} |B_{\mathbf{A}}{}^{\mathbf{A}''}| A(s,t) \, \mathrm{d}s' \leq \|\!|B|\!\|_{(1)} A(s,t).$$

Similarly, for (4.6.19) in the case $d_{\mathbf{A}} = d_{\mathbf{A}''} + 1$, we have (by Fubini and $-d_{\mathbf{A}} > 0$)

$$\begin{aligned} \int_s^t (s'')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} \int_{s''}^t |B_{\mathbf{A}}{}^{\mathbf{A}''} \Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \, \mathrm{d}s'' &\leq \|\!|B|\!\|_{(\infty)} \int_s^t (s'')^{-d_{\mathbf{A}}-1} \int_{s''}^t t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \, \mathrm{d}s'' \\ &\leq \frac{1}{(-d_{\mathbf{A}})} \|\!|B|\!\|_{(\infty)} \int_s^t (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |\Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \, \mathrm{d}s'' \end{aligned}$$

and in the case $d_{\mathbf{A}} \leq d_{\mathbf{A}''}$, we have (again by Fubini and $-d_{\mathbf{A}} > 0$)

$$\begin{aligned} \int_s^t (s'')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} \int_{s''}^t |B_{\mathbf{A}}{}^{\mathbf{A}''} \Psi_{\mathbf{A}''}{}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' \, \mathrm{d}s'' &\leq \int_s^t (s'')^{-d_{\mathbf{A}}-1} \int_{s''}^t (s')^{d_{\mathbf{A}''}} |B_{\mathbf{A}}{}^{\mathbf{A}''}| A(s,t) \, \mathrm{d}s' \, \mathrm{d}s'' \\ &\leq \frac{1}{(-d_{\mathbf{A}})} \|\!|B|\!\|_{(1)} A(s,t). \end{aligned}$$

Using (4.6.11), (4.6.17)–(4.6.19), we arrive at the inequality

$$A_{\mathbf{A}}{}^{\mathbf{A}'}(s,t) \lesssim A_0 + \|\!|B|\!\|_{(1)} A(s,t) + \sum_{\mathbf{A}'' : d_{\mathbf{A}''} = d_{\mathbf{A}} - 1} \|\!|B|\!\|_{(\infty)} A_{\mathbf{A}''}{}^{\mathbf{A}'}(s,t).$$

The term with $\|\!|B|\!\|_{(1)}$ will eventually be absorbed into the left-hand side using (4.6.15), but the term with $\|\!|B|\!\|_{(\infty)}$ needs care, since we are *not* assuming smallness. Nevertheless, there is a reductive (or nilpotent) structure for this term. More precisely, it is not present when $d_{\mathbf{A}} = \min_{\mathbf{A}''} d_{\mathbf{A}''}$, and for $d_{\mathbf{A}} > \min_{\mathbf{A}''} d_{\mathbf{A}''}$, it only involves ${A_{\mathbf{A}''}}^{\mathbf{A}'}$ with $d_{\mathbf{A}''} = d_{\mathbf{A}} - 1$. Therefore, iterating this bound (no more than $N$ times), we arrive at

$$A(s,t) \leq C_1 (1 + \|B\|_{(\infty)})^N (A_0 + \|B\|_{(1)} A(s,t)).$$

Taking $\epsilon = c C_1^{-1} (1 + \|B\|_{(\infty)})^{-N}$ for a sufficiently small absolute constant $c > 0$, we may absorb the contribution of $\|B\|_{(1)} A(s,t)$ into the LHS and obtain the desired estimate for $A(s,t)$.

*S*tep 2. Next, we consider the general case when $\|B\|_{(1)}$ is finite but possibly large. Let us split $[0,t] = [t_m, t_{m-1}] \cup \cdots \cup [t_1, t_0]$ with $t_m = 0$ and $t_0 = t$ so that

$$\left\|\mathbf{1}_{[t_i, t_{i-1}]} B\right\|_{(1)} < \epsilon \qquad \text{for each } i = 1, \ldots, m, \tag{4.6.20}$$

where $\epsilon$ is as in Step 1. Since $\|\cdot\|_{(1)}$ consists of $L^1$-type norms, such a splitting with $m \leq C_N \epsilon^{-1} \|B\|_{(1)}$ exists. For $t_i \leq s < t_{i-1}$, define

$$\begin{aligned} A_{(i)}(s, t_{i-1}) \coloneqq \sum_{\mathbf{A}, \mathbf{A}'} \Big[ &\sup_{s' \in [s, t_{i-1}]} (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s', t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i-1}, t)| \\ &+ \int_s^{t_{i-1}} (s')^{-d_{\mathbf{A}} - 1} t^{d_{\mathbf{A}'}} |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s', t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s', t_{i-1})| \, \mathrm{d}s' \Big]. \end{aligned}$$

For $t_i \leq s < t_{i-1}$, we claim that

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t)| \leq A_{(i)}(s, t_{i-1}) + \sum_{i' \leq i-1} A_{(i')}(t_{i'}, t_{i'-1}), \tag{4.6.21}$$

$$\int_s^t (s')^{-d_{\mathbf{A}} - 1} t^{d_{\mathbf{A}'}} |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s', t)| \, \mathrm{d}s' \lesssim A_{(i)}(s, t_{i-1}) + \sum_{i' \leq i-1} (i - i' + 1) A_{(i')}(t_{i'}, t_{i'-1}), \tag{4.6.22}$$

where we take the sum to be vacuous if $i = 1$, and the latter bound is only for $\mathbf{A}, \mathbf{A}'$ such that $d_{\mathbf{A}} < 0$. Indeed, estimate (4.6.21) follows easily by writing ${\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t)$ as a telescopic sum

$${\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t) = {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i-1}, t) + \sum_{i' \leq i-1} ({\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i'}, t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i'-1}, t)). \tag{4.6.23}$$

For the proof of (4.6.22), let us assume that $s = t_i$ for simplicity; the general case is a minor modification of this case. We split the integration domain into $[t_i, t_{i-1}] \cup \cdots \cup [t_1, t_0]$ and also the integrand using a telescopic sum akin to (4.6.23) as follows:

$$\begin{aligned} &\int_s^t (s')^{-d_{\mathbf{A}} - 1} t^{d_{\mathbf{A}'}} |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s', t)| \, \mathrm{d}s' \\ &\leq \sum_{i' \leq i} \int_{t_{i'}}^{t_{i'-1}} (s')^{-d_{\mathbf{A}} - 1} t^{d_{\mathbf{A}'}} \left( |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s', t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i'-1}, t)| + \sum_{i'' \leq i'-1} |{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i''}, t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_{i''-1}, t)| \right) \mathrm{d}s'. \end{aligned}$$

Estimating the contribution of $\Psi(s', t) - \Psi(t_{i'-1}, t)$ on $[t_{i'}, t_{i'-1}]$ by $A_{(i')}(t_{i'}, t_{i'-1})$, and the rest using (4.6.21) and the obvious integral

$$\int_{t_{i'}}^{t_{i'-1}} (s')^{-d_{\mathbf{A}}-1} \, \mathrm{d}s' = \frac{1}{(-d_{\mathbf{A}})} t_{i'-1}^{-d_{\mathbf{A}}},$$

the desired estimate (4.6.22) follows.

For $s \in [t_{i+1}, t_i)$, we rewrite (4.6.11) as

$$\begin{aligned}
{\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_i,t) &= \int_s^{t_i} {G_{\mathbf{A}}}^{\mathbf{A}'}(s',t) \, \mathrm{d}s' + \int_s^{t_i} \sum_{\mathbf{A}''} {B_{\mathbf{A}}}^{\mathbf{A}''}(s') {\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t) \, \mathrm{d}s' \\
&\quad + \int_s^{t_i} \sum_{\mathbf{A}''} {B_{\mathbf{A}}}^{\mathbf{A}''}(s') ({\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(s',t) - {\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t)) \, \mathrm{d}s' \\
&= \int_s^t \mathbf{1}_{[t_{i+1},t_i]}(s') \left( {G_{\mathbf{A}}}^{\mathbf{A}'}(s',t) + \sum_{\mathbf{A}''} {B_{\mathbf{A}}}^{\mathbf{A}''}(s') {\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t) \right) \mathrm{d}s' \\
&\quad + \int_s^t \sum_{\mathbf{A}''} \mathbf{1}_{[t_{i+1},t_i]}(s') {B_{\mathbf{A}}}^{\mathbf{A}''}(s') ({\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(s',t) - {\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t)) \, \mathrm{d}s'.
\end{aligned}$$

Using the last identity, we may trivially extend ${\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_i,t)$ to all $s \in (0,t)$. Then, in view of $\left|\!\left|\!\left| \mathbf{1}_{[t_{i+1},t_i]} B \right|\!\right|\!\right|_{(\infty)} \leq |\!|\!| B |\!|\!|_{(\infty)}$ and (4.6.20), the argument in Step 1 is applicable to ${\Psi_{\mathbf{A}}}^{\mathbf{A}'}(s,t) - {\Psi_{\mathbf{A}}}^{\mathbf{A}'}(t_i,t)$. We claim that for every $\mathbf{A}, \mathbf{A}'$, we have

$$\begin{aligned}
\int_s^t (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \mathbf{1}_{[t_{i+1},t_i]}(s') |{G_{\mathbf{A}}}^{\mathbf{A}'}(s',t)| \, \mathrm{d}s' &\leq A_0, \\
\int_s^t (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \mathbf{1}_{[t_{i+1},t_i]}(s') \sum_{\mathbf{A}''} |{B_{\mathbf{A}}}^{\mathbf{A}''}(s') {\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t)| \, \mathrm{d}s' &\lesssim \left|\!\left|\!\left| \mathbf{1}_{[t_{i+1},t_i]} B \right|\!\right|\!\right| \sum_{i' \leq i} A_{(i')}(t_{i'}, t_{i'-1}).
\end{aligned}$$

The first bound is an immediate consequence of the hypothesis for $G$. To establish the second bound, note that

$$\begin{aligned}
\int_s^t (s')^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} \mathbf{1}_{[t_{i+1},t_i]}(s') |{B_{\mathbf{A}}}^{\mathbf{A}''}(s') {\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t)| \, \mathrm{d}s' &\leq \int_s^{t_i} (s')^{-d_{\mathbf{A}}} t_i^{d_{\mathbf{A}''}} |{B_{\mathbf{A}}}^{\mathbf{A}''}(s')| \, \mathrm{d}s' \left( t_i^{-d_{\mathbf{A}''}} t^{d_{\mathbf{A}'}} |{\Psi_{\mathbf{A}''}}^{\mathbf{A}'}(t_i,t)| \right) \\
&\lesssim \left|\!\left|\!\left| \mathbf{1}_{[t_{i+1},t_i]} B \right|\!\right|\!\right| \sum_{i' \leq i} A_{(i')}(t_{i'}, t_{i'-1}),
\end{aligned}$$

where we used (4.6.21). We remark that the integral can be bounded easily by dividing into cases $d_{\mathbf{A}} = d_{\mathbf{A}''} + 1$ and $d_{\mathbf{A}} \leq d_{\mathbf{A}''}$. Now, applying Step 1, we derive

$$A_{(i+1)}(s, t_i) \lesssim (1 + |\!|\!| B |\!|\!|_{(\infty)})^N \Big( A_0 + \left|\!\left|\!\left| \mathbf{1}_{[t_{i+1},t_i]} B \right|\!\right|\!\right| \sum_{i' \leq i} A_{(i')}(t_{i'}, t_{i'-1}) \Big),$$

where the last term inside the parentheses does not exist for $i = 0$. By a simple induction argument on $i \in \{1, \dots, m-1\}$ (observe also that $\left\|\left|\mathbf{1}_{[t_{i+1},t_i]}B\right|\right\| \leq \|B\|_{(\infty)} + \epsilon$), we obtain

$$\sum_{i \leq m} A_{(i)}(t_i, t_{i-1}) \lesssim (2 + \|B\|_{(\infty)})^{C_N(1+m)} A_0.$$

Combined with (4.6.21)–(4.6.22), the desired estimates (4.6.13)–(4.6.14) follow. □

The following result for the fundamental matrix $\Pi$ is an immediate corollary of Proposition 4.6.3:

**Corollary 4.6.4.** *Consider degrees $d_{\mathbf{A}}$ ($\mathbf{A} \in \{1, \dots, N\}$) and an $N \times N$ matrix-valued function $B$ on $[0, T]$ satisfying (4.6.7), (4.6.8) and $\|B\| < +\infty$. Let $\Pi_{\mathbf{A}}{}^{\mathbf{A}'} = \Pi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t)$ be given by (4.6.10) for $0 < s < t < T$. Then, for $0 < s < t < T$, we have*

$$s^{-d_{\mathbf{A}}} t^{d_{\mathbf{A}'}} |\Pi_{\mathbf{A}}{}^{\mathbf{A}'}(s,t) - \delta_{\mathbf{A}}{}^{\mathbf{A}'}| \leq C_N (1 + \|B\|_{(\infty)})^N \exp(C_N \|B\|_{(1)}) \|B\| \quad \text{if } d_{\mathbf{A}} \leq 0, \tag{4.6.24}$$

$$\int_s^t (s')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} |\Pi_{\mathbf{A}}{}^{\mathbf{A}'}(s',t) - \delta_{\mathbf{A}}{}^{\mathbf{A}'}| \, \mathrm{d}s' \leq C_N (1 + \|B\|_{(\infty)})^N \exp(C_N \|B\|_{(1)}) \|B\| \quad \text{if } d_{\mathbf{A}} < 0. \tag{4.6.25}$$

This corollary will be sufficient to verify (wRC) for many divergence-type equations Appendix B. However, to establish the higher derivative bounds in (RC-q), we will directly apply Proposition 4.6.3 to a suitably differentiated ODE system.

## From augmented system to quantitative (RC)

We now show that graded augmented system in the sense of ($\Phi$-1)–($\Phi$-4) leads to the quantitative version of (RC), namely, (RC-q), ($b_{y_1}$-1) and ($b_{y_1}$-2).

Let $\mathbf{x} : W \times [0, 1] \to \mathbb{R}^d$ satisfy assumptions (**x**-1), (**x**-2), and (**x**-3) in Sections 4.5. Given $M \in \mathbb{Z}_{\geq 0}$ and $\delta \geq 0$, we introduce the following norm adapted to $(\mathbf{x}, W)$ ($G$ is for the underlying Geometry):

$$\|\mathbf{b}\|_{\dot{G}^{M,\delta}(\mathbf{x},W)} := \sup_{(y,y_1) \in W} \sum_{\alpha : |\alpha| \leq M} \int_0^1 |(\partial^\alpha \mathbf{b})(\mathbf{x}(y, y_1, s))| |y_1 - y|^{\delta + |\alpha|} s^{\delta + |\alpha| - 1} \, \mathrm{d}s, \quad \delta > 0,$$

$$\|\mathbf{b}\|_{\dot{G}^{M,\delta}_\infty(\mathbf{x},W)} := \sup_{(y,y_1) \in W} \sum_{\alpha : |\alpha| \leq M} \sup_{0 \leq s \leq 1} \left[ |(\partial^\alpha \mathbf{b})(\mathbf{x}(y, y_1, s))| |y_1 - y|^{\delta + |\alpha|} s^{\delta + |\alpha|} \right].$$

When $\delta = 0$, we will only use the $L^\infty$-based norm; hence we introduce the shorthand

$$\|\mathbf{b}\|_{\dot{G}^{M,0}(\mathbf{x},W)} := \|\mathbf{b}\|_{\dot{G}^{M,0}_\infty(\mathbf{x},W)} := \sup_{(y,y_1) \in W} \sum_{\alpha : |\alpha| \leq M} \sup_{0 \leq s \leq 1} \left[ |(\partial^\alpha \mathbf{b})(\mathbf{x}(y, y_1, s))| |y_1 - y|^{|\alpha|} s^{|\alpha|} \right].$$

Given $L_b > 0$, we also consider the following *inhomogeneous* norms:

$$\|\mathbf{b}\|_{G^{M,\delta}(\mathbf{x},W;L_b)} := \sup_{(y,y_1)\in W} \sum_{\alpha:|\alpha|\le M} \int_0^1 |(\partial^\alpha \mathbf{b})(\mathbf{x}(y,y_1,s))| \max\{L_b, |y_1 - y|\}^{\delta+|\alpha|} s^{\delta-1}\, \mathrm{d}s, \quad \delta > 0,$$

$$\|\mathbf{b}\|_{G^{M,\delta}_\infty(\mathbf{x},W;L_b)} := \sup_{(y,y_1)\in W} \sum_{\alpha:|\alpha|\le M} \sup_{0\le s\le 1} \left[|(\partial^\alpha \mathbf{b})(\mathbf{x}(y,y_1,s))| \max\{L_b, |y_1 - y|\}^{\delta+|\alpha|} s^{\delta}\right].$$

As before, we write $\|\mathbf{b}\|_{G^{M,0}(\mathbf{x},W;L_b)} := \|\mathbf{b}\|_{G^{M,0}_\infty(\mathbf{x},W;L_b)}$.

*Remark* 4.6.5. Observe that all these norms are easily bounded by (appropriate) standard $C^k$-norms. Indeed, for an open set $V \subseteq \mathbb{R}^d$ with $\mathbf{x}(W \times [0,1]) \subseteq V$, we have (for any $\delta \ge 0$ and $M \in \mathbb{Z}_{\ge 0}$)

$$\|\mathbf{b}\|_{\dot{G}^{M,\delta}(\mathbf{x},W)}, \|\mathbf{b}\|_{\dot{G}^{M,\delta}_\infty(\mathbf{x},W)} \lesssim_M \sup_{(y,y_1)\in W} |y_1 - y|^{\delta+M} \|\mathbf{b}\|_{C^M(V)},$$
$$\|\mathbf{b}\|_{G^{M,\delta}(\mathbf{x},W;L_b)}, \|\mathbf{b}\|_{G^{M,\delta}_\infty(\mathbf{x},W;L_b)} \lesssim_M \sup_{(y,y_1)\in W} \max\{L_b, |y_1 - y|\}^{\delta+M} \|\mathbf{b}\|_{C^M(V)}.$$

We are now ready to formulate the main result of this subsection. For $\mathbf{B} = ((\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'})$ and $\mathbf{C}_i = ((\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)})$ as in (Φ-3) and (Φ-4), we define

$$\|\mathbf{B}\|_{\dot{G}^M(\mathbf{x},W)} = \sum_{i,\mathbf{A},\mathbf{A}'} \|(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}\|_{\dot{G}^{M,d_{\mathbf{A}'}+1-d_{\mathbf{A}}}(\mathbf{x},W)},$$
$$\|\mathbf{B}\|_{G^M(\mathbf{x},W;L_b)} = \sum_{i,\mathbf{A},\mathbf{A}'} \|(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}\|_{G^{M,d_{\mathbf{A}'}+1-d_{\mathbf{A}}}(\mathbf{x},W;L_b)},$$
$$\|\mathbf{C}\|_{\dot{G}^M_\infty(\mathbf{x},W)} = \sum_{i,\mathbf{A},K,\gamma} \|(\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}\|_{\dot{G}_\infty^{M,d_K-|\gamma|+1-d_{\mathbf{A}}}(\mathbf{x},W)}.$$

In view of Remark 4.6.5, the following statement holds: If $\mathbf{B} = ((\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'})$ and $\mathbf{C}_i = ((\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)})$ as in (Φ-3) and (Φ-4) belong to $C^M(V)$ for some bounded open set $V \subseteq \mathbb{R}^d$ with $\mathbf{x}(W \times [0,1]) \subseteq V$, then

$$\|\mathbf{B}\|_{\dot{G}^M(\mathbf{x},W)}, \|\mathbf{B}\|_{G^M(\mathbf{x},W;L_b)}, \|\mathbf{C}\|_{\dot{G}^M_\infty(\mathbf{x},W)}, \le C(\|\mathbf{B}\|_{C^M(V)} + \|\mathbf{C}\|_{C^M(V)}), \tag{4.6.26}$$

where $C$ depends on $L_b$, $\operatorname{diam} V$, $d$, $\#\mathcal{A}$, $s_0$, $\max_{\mathbf{A}} |d_{\mathbf{A}}|$, $\max_K |d_K|$, $m'_K$, and $M$.

**Proposition 4.6.6** (From graded augmented system to (RC-q), ($b_{y_1}$-1), and ($b_{y_1}$-2)). *Given an $\mathbb{C}^{r_0}$-valued function $\varphi$ on $U_0$, consider augmented variables $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$ satisfying (Φ-1), (Φ-2), (Φ-3), and (Φ-4) (with $N_0 = \max_{\mathbf{A}\in\mathcal{A}} |d_{\mathbf{A}}| + 1$). Let $\mathbf{x} : W \times [0,1] \to U$ satisfy (x-1) and (x-2). Then the following holds.*

1. *If, for some $M, M_g \in \mathbb{Z}_{\ge 0}$ sufficiently large, we have*

$$\|\mathbf{B}\|_{\dot{G}^M(\mathbf{x},W)} + \|\mathbf{C}\|_{\dot{G}^M_\infty(\mathbf{x},W)} < +\infty, \quad \sup_{\mathbf{A},\alpha,J} \|c[\Phi_{\mathbf{A}}]^{(\alpha,J)}\|_{C^{M_g}(U)} < +\infty,$$

*then (RC-q) and ($b_{y_1}$-1) hold with $M_S, M_Z \leq \min\{M_{\mathbf{x}}, M\} - C_d$ and*

$$A_S \leq C\left(d, A_{\mathbf{x}}, M_{\mathbf{x}}, M, \|\mathbf{B}\|_{\dot{G}^M(\mathbf{x},W)}, \|\mathbf{C}\|_{\dot{G}^M_\infty(\mathbf{x},W)}\right),$$
$$A_Z \leq C\left(d, A_{\mathbf{x}}, M_{\mathbf{x}}, M, \|\mathbf{B}\|_{\dot{G}^M(\mathbf{x},W)}\right) \sup_{(y,y_1)\in W} |y-y_1|^{N_0-1},$$
$$A_g \leq \sup_{\mathbf{A},\alpha,J} \|c[\Phi_{\mathbf{A}}]^{(\alpha,J)}\|_{C^{M_g}(U)}.$$

2. *Assume in addition that ($\mathbf{x}$-3) holds for some $0 \leq M'_{\mathbf{x}} \leq M_{\mathbf{x}}$. If, for some $M' \geq 0$ and $L_b > 0$,*
$$\|\mathbf{B}\|_{G^M(\mathbf{x},W;L_b)} < +\infty,$$
*then ($b_{y_1}$-2) holds with the same $L_b$, $M'_Z \leq \min\{M'_{\mathbf{x}}, M'\} - C_d$ and*
$$A_Z \leq C\left(d, A_{\mathbf{x}}, M'_{\mathbf{x}}, M', \|\mathbf{B}\|_{G^M(\mathbf{x},W;L_b)}\right) \sup_{(y,y_1)\in W} \max\{L_b, |y-y_1|\}^{N_0-1}.$$

*Proof.* In view of Proposition 4.6.1, we may express $S$ and $Z$ in terms of the coefficients $\mathbf{B}_i$ and $\mathbf{C}_i$ in (4.1.8); we review this process as follows. From (4.1.8), we obtain the following ODE system along each curve $\mathbf{x}(y, y_1, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_{\mathbf{A}} \circ \mathbf{x}) = \dot{\mathbf{x}}^i \left((\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'} \circ \mathbf{x}\right)(\Phi_{\mathbf{A}'} \circ \mathbf{x}) + \dot{\mathbf{x}}^i \left((\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} \circ \mathbf{x}\right)(\partial^\gamma(\mathcal{P}^*\varphi)_K \circ \mathbf{x}). \quad (4.1.9)$$

In what follows, we use the shorthand $\mathbf{\Pi}_{\mathbf{A}}{}^{\mathbf{A}'}(y, y_1; s, t) := {}^{(\mathbf{x}_{y,y_1})}\mathbf{\Pi}_{\mathbf{A}}{}^{\mathbf{A}'}(s, t)$ for the fundamental matrix. By (4.6.1), it solves

$$\mathbf{\Pi}_{\mathbf{A}}{}^{\mathbf{A}'}(y, y_1; s, t) = \delta_{\mathbf{A}}^{\mathbf{A}'} - \int_s^t B(y, y_1, s')\mathbf{\Pi}_{\mathbf{A}''}{}^{\mathbf{A}'}(y, y_1; s', t)\,\mathrm{d}s', \quad (4.6.27)$$

where $B(y, y_1, s') := \dot{\mathbf{x}}^i \left((\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}''} \circ \mathbf{x}\right)(y, y_1, s')$.

We now make the following claims:

(i) Under the hypothesis of part (1), we have
$$|L_y^{|\alpha|}|y-y_1|^{|\beta|}(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \mathbf{\Pi}_J{}^{\mathbf{A}}(y, y_1; s, t)| \lesssim |y_1 - y|^{-d_{\mathbf{A}}} t^{-d_{\mathbf{A}}} \quad (4.6.28)$$
for $|\alpha| + |\beta| \leq \min\{M_{\mathbf{x}}, M\} - C_d$, where the implicit constant depends on $d$, $A_{\mathbf{x}}$, $M_{\mathbf{x}}$, $M$, and $\|\mathbf{B}\|_{\dot{G}^M(\mathbf{x},W)}$.

(ii) Under the hypothesis of part (2), we have
$$|L_y^{|\alpha|} L_b^{|\beta|}(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta \mathbf{\Pi}_J{}^{\mathbf{A}}(y, y_1; s, t)| \lesssim L_b^{-d_{\mathbf{A}}} t^{-d_{\mathbf{A}}} \quad \text{for } |y_1 - y| \leq L_b \quad (4.6.29)$$
and for $|\alpha| + |\beta| \leq \min\{M'_{\mathbf{x}}, M'\} - C_d$, where the implicit constant depends on $d$, $A_{\mathbf{x}}$, $M'_{\mathbf{x}}$, $M'$, and $\|\mathbf{B}\|_{G^{M'}(\mathbf{x},W;L_b)}$.

Assuming these claims, let us first complete the proof. Applying Proposition 4.6.1 to the present case, we may express ${S_J}^{(\gamma,K)}$ and $Z^{\mathbf{A}}$ ($\mathbf{A} \in \mathcal{A}$) as follows:

$$
\begin{aligned}
{S_J}^{(\gamma,K)}(y, y_1, t) &= -{\mathbf{\Pi}_J}^{\mathbf{A}}(y, y_1; 0, t)\, \dot{\mathbf{x}}^i \left( {(\mathbf{C}_i)_{\mathbf{A}}}^{(\gamma,K)} \circ \mathbf{x} \right)(y, y_1, t), \\
(Z^{\mathbf{A}})(y, y_1) &= {\mathbf{\Pi}_\varphi}^{\mathbf{A}}(y, y_1; 0, 1), \\
c[g_{\mathbf{A}}]^{(\alpha,K)}(y_1) &= c[\Phi_{\mathbf{A}}]^{(\alpha,K)}(y_1).
\end{aligned}
$$

In view of these expressions, as well as (x-1) (to estimate $\dot{\mathbf{x}}^i = \partial_s \mathbf{x}^i$), the assertions in part (1) concerning $A_S$, $A_Z$, $M_S$, and $M_Z$ follow from the above claims. The assertion concerning $A_g$ is obvious from the last identity. Finally, under the hypothesis of part (2), which is stronger than that of part (1), we may combine (4.6.28) and (4.6.29) to verify ($b_{y_1}$-2) with the above bounds on $M'_Z$ and $A_Z$.

It remains to verify the claims. For (i), we set up an induction on $M_{\mathbf{\Pi}} \in \mathbb{Z}_{\geq 0}$ with the following induction hypothesis for $0 \leq |\alpha| + |\beta| \leq M_{\mathbf{\Pi}}$:

$$
|L_y^{|\alpha|} |y - y_1|^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta {\mathbf{\Pi}_{\mathbf{A}}}^{\mathbf{A}'}(y, y_1; s, t)| \leq C |y_1 - y|^{d_{\mathbf{A}} - d_{\mathbf{A}'} + 1} s^{d_{\mathbf{A}}} t^{-d_{\mathbf{A}'}}, \tag{4.6.30}
$$

$$
\int_s^t (s')^{-d_{\mathbf{A}}-1} t^{d_{\mathbf{A}'}} |L_y^{|\alpha|} |y - y_1|^{|\beta|} (\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta {\mathbf{\Pi}_{\mathbf{A}}}^{\mathbf{A}'}(y, y_1; s', t)| \, \mathrm{d}s' \leq C |y_1 - y|^{d_{\mathbf{A}} - d_{\mathbf{A}'} + 1}, \quad d_{\mathbf{A}} < 0. \tag{4.6.31}
$$

where

$$
C = C\left(d, \alpha, \beta, A_{\mathbf{x}}, M_{\mathbf{x}}, M, \|\mathbf{B}\|_{\dot{G}^M(\mathbf{x}, W)}\right).
$$

To carry out the induction, we introduce the renormalized parameter $\tilde{t} = |y_1 - y| t$ and write

$$
\widetilde{\mathbf{\Pi}}(y, y_1; \tilde{s}, \tilde{t}) = \mathbf{\Pi}(y, y_1; |y_1 - y|^{-1} \tilde{s}, |y_1 - y|^{-1} \tilde{t}).
$$

Then we have

$$
\widetilde{\mathbf{\Pi}}(y, y_1; \tilde{s}, \tilde{t}) = I - \int_{\tilde{s}}^{\tilde{t}} \widetilde{B}(y, y_1, s') \widetilde{\mathbf{\Pi}}(y, y_1; \tilde{s}', \tilde{t}) \, \mathrm{d}\tilde{s}', \tag{4.6.32}
$$

where $\widetilde{B}(y, y_1, \tilde{s}) \coloneqq |y_1 - y|^{-1} B(y_1, y, |y_1 - y|^{-1} \tilde{s})$. Thanks to the renormalization, we obtain $|y_1 - y|^{-1}$ in $\widetilde{B}$, which offsets the factor $|y_1 - y|$ in (x-1) for $\dot{\mathbf{x}} = \partial_s \mathbf{x}$. Indeed, a direct computation using (x-1) and the definition of $\|\cdot\|_{\dot{G}^{M,\delta}(\mathbf{x}, W)}$ shows

$$
\left\| L_y^{|\alpha|} |y - y_1|^{|\beta|} ((\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta B)^{\sim}(y, y_1, \cdot) \right\| \lesssim 1 + \|\mathbf{B}\|_{\dot{G}^M(\mathbf{x}, W)}, \tag{4.6.33}
$$

for $|\alpha| + |\beta| \leq \min\{M_{\mathbf{x}}, M\} - C_d$, where the implicit constant depends on $d$, $\alpha$, $\beta$, $A_{\mathbf{x}}$. We carefully note that the variable change $t \mapsto \tilde{t}$ on the left-hand side is done after taking $(\partial_y + \partial_{y_1})^\alpha \partial_{y_1}^\beta$.

The base case $M_{\mathbf{\Pi}} = 0$ follows from Proposition 4.6.3 applied to the rescaled ODE system (4.6.32).

Now we assume (4.6.30) (4.6.31) holds for all $|\alpha|+|\beta| \leq M_{\mathbf{\Pi}}-1$, we would like to show (4.6.30) (4.6.31) for $|\alpha|+|\beta| = M_{\mathbf{\Pi}}$. We differentiate (4.6.27):

$$\begin{aligned}
&L_y^{|\alpha|}|y-y_1|^{|\beta|}(\partial_y+\partial_{y_1})^{\alpha}\partial_{y_1}^{\beta}\mathbf{\Pi}(y,y_1;s,t)\\
&= -L_y^{|\alpha|}|y-y_1|^{|\beta|}\sum_{\alpha=\alpha'+\alpha'',\beta=\beta'+\beta''}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\int_s^t(\partial_y+\partial_{y_1})^{\alpha'}\partial_{y_1}^{\beta'}B(y,y_1,s')(\partial_y+\partial_{y_1})^{\alpha''}\partial_{y_1}^{\beta''}\mathbf{\Pi}(y,y_1;s',t)\,\mathrm{d}s'\\
&= -\int_s^t G(s',t)\,\mathrm{d}s' - \int_s^t L_y^{|\alpha|}|y-y_1|^{|\beta|}B(y,y_1,s')(\partial_y+\partial_{y_1})^{\alpha}\partial_{y_1}^{\beta}\mathbf{\Pi}(y,y_1;s',t)\,\mathrm{d}s'
\end{aligned}$$

where

$$G(s,t) = L_y^{|\alpha|}|y-y_1|^{|\beta|}\sum_{\alpha=\alpha'+\alpha'',\beta=\beta'+\beta'',|\alpha'|+|\beta'|>0}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}(\partial_y+\partial_{y_1})^{\alpha'}\partial_{y_1}^{\beta'}B(y,y_1,s)(\partial_y+\partial_{y_1})^{\alpha''}\partial_{y_1}^{\beta''}\mathbf{\Pi}(y,y_1;s,$$

At this point, we pass to the renormalized variable $\tilde{t} = |y-y_1|t$ (and similarly use $\tilde{s}$, $\tilde{s}'$). By our induction hypothesis and (4.6.33),

$$\int_{\tilde{s}}^{\tilde{t}}(\tilde{s}')^{-d_{\mathbf{A}}}\tilde{t}^{d_{\mathbf{A}'}}\widetilde{G}_{\mathbf{A}}{}^{\mathbf{A}'}(\tilde{s}',\tilde{t})\,\mathrm{d}\tilde{s}' \leq C.$$

Thus, applying Proposition 4.6.3, we prove the induction hypothesis (4.6.30) (4.6.31) for $|\alpha|+|\beta| = M_{\mathbf{\Pi}}$ as long as $M_{\mathbf{\Pi}} \leq \min\{M_{\mathbf{x}}, M\} - C_d$. Then from the case $\mathbf{A} = J$ of (4.6.30), we obtain (4.6.28).

Estimate (4.6.29) is proved in the same way where $|y_1-y|$ is replaced by $L_b$. More precisely, under the restriction $|y_1-y| \leq L_b$, we use (**x**-3) instead of (**x**-1) and renormalized parameter is defined as $\tilde{t} = L_b t$. We omit the details. □

## Connection and holonomy

The graded augmented system as in Definition 4.1.4 gives a connection on the trivial bundle $U \times \mathbb{C}^{\#\mathcal{A}}$:

$$D_i\Phi_{\mathbf{A}}\mathrm{d}x^i := \Big(\partial_i\Phi_{\mathbf{A}} - (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}(x)\Phi_{\mathbf{A}'}(x)\Big)\,\mathrm{d}x^i.$$

A parallel section $\Phi_{\mathbf{A}}$ of this connection gives an element in $\ker\mathcal{P}^*$. Given a point $y \in U$, let $\mathbf{x}(t) : [0,1] \to U$ be a closed curve with $\mathbf{x}(0) = y$. The parallel transport along this curve induces a linear map on the fiber over $y$, $\alpha_{\mathbf{x}} : \mathbb{C}^{\#\mathcal{A}} \to \mathbb{C}^{\#\mathcal{A}}$, called a *holonomy map based at* $y$. If $D_i\Phi_{\mathbf{A}} = 0$ for all $i$ for some $\Phi_{\mathbf{A}}$ on $U$, then one must have $\alpha_{\mathbf{x}}\Phi_{\mathbf{A}}(y) = \Phi_{\mathbf{A}}(y)$. On the other hand, if there is a point $y \in U$ at which $\phi_{\mathbf{A}} \in \mathbb{C}^{\#\mathcal{A}}$ is invariant under all holonomy maps $\alpha_{\mathbf{x}}$ based at $y$, then the section $\Phi_{\mathbf{A}}$ defined by solving ODEs along (arbitrary) paths with $\Phi_{\mathbf{A}}(y) = \phi_{\mathbf{A}}$ (for all $\mathbf{A} \in \mathcal{A}$) is well-defined and parallel, i.e., $\Phi_{\mathbf{A}} \in \ker\mathcal{P}^*$. Therefore we conclude the following.

**Proposition 4.6.7.** *Let $U$ be a connected subset of $\mathbb{R}^d$, and $\mathcal{P}$ an $r_0 \times s_0$-matrix valued differential operator with $C^\infty(\overline{U})$ coefficients. Let $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$ be graded augmented variables as in Definition 4.1.4, with $C^\infty(\overline{U})$ coefficients $\mathbf{B}_i, \mathbf{C}_i$ in ($\Phi$-3). Then the cokernel $\ker \mathcal{P}^*$ may be identified with the space of vectors $\phi_{\mathbf{A}}$ at a point $y$ invariant under all holonomy maps $\alpha_{\mathbf{x}}$ associated with $D_i := \partial_i - \mathbf{B}_i$ based at $y$.*

The infinitesimal behavior of the holonomy is controlled by the curvature of the connection:

$$\mathbf{R}_{ij\mathbf{A}}{}^{\mathbf{A}'}\mathrm{d}x^i\wedge\mathrm{d}x^j = -\frac{1}{2}\left(\partial_i(\mathbf{B}_j)_{\mathbf{A}}{}^{\mathbf{A}'} - \partial_j(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'} + (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}''}(\mathbf{B}_j)_{\mathbf{A}''}{}^{\mathbf{A}'} - (\mathbf{B}_j)_{\mathbf{A}}{}^{\mathbf{A}''}(\mathbf{B}_i)_{\mathbf{A}''}{}^{\mathbf{A}'}\right)\mathrm{d}x^i\wedge\mathrm{d}x^j$$

The Ambrose–Singer theorem [AS53] states that the Lie algebra of the holonomy group is spanned by the curvature forms $\{\mathbf{R}_{ij\mathbf{A}}{}^{\mathbf{A}'}\}_{i,j=1,\dots,d}$. Hence, if $\mathbf{R}_{ij\mathbf{A}}{}^{\mathbf{A}'}(y)$ has trivial kernel for some $i,j$ and $y \in U$ (which is the generic case), then $\ker \mathcal{P}^* = \{0\}$. On the other hand, Proposition 4.1.7 now admits a simple proof:

*Proof of Proposition 4.1.7.* Note that (4.1.12) is nothing but the condition that $\mathbf{R}_{ij\mathbf{A}}{}^{\mathbf{A}'} = 0$ vanishes in $U$. If $\dim\ker \mathcal{P}^* = \#\mathcal{A}$, then the holonomy group must be trivial, which implies that the curvature vanishes. Conversely, if the curvature vanishes identically and $U$ is simply connected, then the holonomy group is trivial, and $\dim\ker\mathcal{P}^* = \#\mathcal{A}$ (this also follows directly from the Frobenius theorem, as in [Res94, §3.1]). ☐

## 4.7 (FC) implies (RC)

In this section, we give a precise formulation and proof of Theorem 4.1.11. A basic ingredient is Hilbert's Nullstellensatz (see, e.g., [AM69, p. 85]), which gives a correspondence between the set of common zeros of a family of polynomials and the ideal they generate. We recall its statement below:

**Proposition 4.7.1** (Hilbert's Nullstellensatz)**.** *Suppose $f_1(x_1,\cdots,x_d),\cdots,f_N(x_1,\cdots,x_d) \in \mathbb{C}[x_1,\cdots,x_d]$ are polynomials with the set of common zeros*

$$Z = \{(z_1,\cdots,z_d) \in \mathbb{C}^d : f_j(z_1,\cdots,z_d) = 0,\ j = 1,2,\cdots,N\}.$$

*Then for any polynomial $h(x_1.\cdots,x_d) \in \mathbb{C}[x_1,\cdots,x_d]$, if $h$ vanishes on $Z$, then there exist $M > 0$ and polynomials $g_j(x_1,\cdots,x_d) \in \mathbb{C}[x_1,\cdots,x_d]$, $j = 1,2,\cdots,N$ such that*

$$h(x_1,\cdots,x_d)^M = \sum_j g_j(x_1,\cdots,x_d)f_j(x_1,\cdots,x_d).$$

Our main result – which is the precise version of Theorem 4.1.11.(1) – is the following.

**Theorem 4.7.2.** *Let $U$ be an open subset of $\mathbb{R}^d$. Suppose that $\mathcal{P}$ has smooth coefficients. Assume that the $s_0\times r_0$-matrix-valued principal symbol $p^*(x,\xi)$ of $\mathcal{P}^*$ (where $s_0 \geq r_0$) satisfies*

*(FC)* $p^*(x,\xi)$ *is injective for all* $x \in U$ *and* $\xi \in \mathbb{C}^d \setminus \{0\}$.

*Then there exist augmented variables* $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$ *on* $U$ *satisfying* *(Φ-1)–(Φ-4)* *with index set* $\mathcal{A} = \{((\alpha, J) : 1 \le J \le r_0,\ |\alpha| \le N_0 - 1\}$ *(which is* maximal *in the sense of Definition 4.1.9), degree* $d_{(\alpha,J)} = -|\alpha|$, *and coefficients* $\mathbf{B}_i, \mathbf{C}_i \in C^\infty(U)$, *which have the following properties:*

1. *We have* ${(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)} = 0$ *unless* $m_K + |\gamma| = |\alpha| + 1$.
2. *If* $\mathcal{P}$ *has* constant coefficients, *then* $\mathbf{B}_i, \mathbf{C}_i$ *are constant. In addition, if* $\mathcal{P}$ *is also* homogeneous *in the sense that* $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ *(see Definition 4.1.10), then*

$$ {(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')} = \begin{cases} 1 & \text{if } J = J',\ \alpha' = \alpha + \mathbf{e}_i \\ 0 & \text{otherwise.} \end{cases} \tag{4.7.1} $$

3. *In general, on any compact set* $X \Subset U$, *we have*

$$ \|{(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')}\|_{C^k(X)} \lesssim_{\|c[p^*]\|_{C^k(X)}} \begin{cases} 1, & d_{(\alpha,J)} = d_{(\alpha',J')} + 1, \\ \sum\limits_{\alpha,J,K} \|{c[\mathcal{P}^*]_K}^{(\alpha,J)}\|_{C^{k+|\alpha|-m_K+d_{\mathbf{A}'}-d_{\mathbf{A}}+1}(X)}, & d_{(\alpha,J)} < d_{(\alpha',J')} + 1, \end{cases} \tag{4.7.2} $$

*and*

$$ \|{(\mathbf{C}_i)_{(\alpha',J')}}^{(\gamma,K)}\|_{C^k(X)} \lesssim_{\|c[p^*]\|_{C^k(X)}} \sum_{|\alpha|=m_K} \|{c[\mathcal{P}^*]_K}^{(\alpha,J)}\|_{C^k(X)}, \tag{4.7.3} $$

*where* $\|c[p^*]\|_{C^k(X)} := \sum_{|\alpha|=m_K} \|{c[\mathcal{P}^*]_K}^{(\alpha,J)}\|_{C^k(X)}$.

*Proof.* **Step 1:** For any $x_0 \in U$, there exist $N_0 > 0$ and polynomial valued $r_0 \times s_0$ matrices $g_\alpha(\xi)$ for $|\alpha| = N_0$ such that

$$ \xi^\alpha I_{r_0} = g_\alpha(\xi) p^*(x_0, \xi), \quad |\alpha| = N_0. \tag{4.7.4} $$

Moreover, ${(p^*)_K}^J(\xi)$ is homogeneous of degree $m_K$ and ${(g_\alpha)_J}^K(\xi)$ is homogeneous of degree $N_0 - m_K$.

The claim follows from applying Hilbert's Nullstellensatz (Proposition 4.7.1) to $\det(M_j(\xi))$ where $M_j(\xi)$ goes through all the $r_0 \times r_0$ minors of $p^*(x_0,\xi)$. By assumption, the common zeros of $\det(M_j(\xi))$ in $\mathbb{C}^d$ is contained in $\{0\}$. Since $\xi_\ell$ vanishes at 0, there exist $N_1, \cdots, N_d$ and polynomials $h^j_\ell(\xi)$ such that

$$ \xi_\ell^{N_\ell} = \sum_\ell h^j_\ell(\xi) \det(M_j(\xi)). $$

We notice that $\det(M_j(\xi)) I_{r_0}$ can be written as the product of its adjugate matrix $\mathrm{a}dj(M_j(\xi))$ and itself:

$$ \det(M_j(\xi)) I_{r_0} = \mathrm{a}dj(M_j(\xi)) M_j(\xi). $$

Moreover, $M_j(\xi)$ is the product of a constant $0-1$ matrix and $p^*(x_0,\xi)$. Therefore we conclude that there are polynomial-valued $r_0 \times s_0$ matrices $\widetilde{g}^\ell(\xi)$ so that

$$\xi_\ell^{N_\ell} I_{r_0} = \widetilde{g}^\ell(\xi) p^*(x_0,\xi).$$

Taking $N_0 > N_1 + \cdots + N_d$, we conclude (4.7.4). We may assume ${(g_\alpha)_J}^K(\xi)$ is homogeneous of degree $N_0 - m_K$ since terms of other degrees do not contribute.

**Step 2:** There exist $\xi$-polynomial-valued $r_0 \times s_0$ matrices $g_\alpha(x,\xi)$ for $|\alpha| = N_0$ such that

$$\xi^\alpha I_{r_0} = g_\alpha(x,\xi) p^*(x,\xi), \quad |\alpha| = N_0. \tag{4.7.5}$$

The matrix $g_\alpha(x,\xi)$ depends smoothly on the coefficients of $p^*(x,\xi)$ and some smooth cutoff functions. Moreover, ${(p^*)_K}^J(\xi)$ is homogeneous of degree $m_K$ and ${(g_\alpha)_J}^K(\xi)$ is homogeneous of degree $N_0 - m_K$.

When $\mathcal{P}$ has constant coefficients, then it suffices to take $g_\alpha(x,\xi) = g_\alpha(\xi)$ (from Step 1); hence, it suffices to only consider the case when $\mathcal{P}$ does not have constant coefficients. After a partition of unity, it suffices to prove the claim in a small neighborhood of a given point $x_0$. By Step 1, we have

$$\xi^\alpha I_{r_0} = g_\alpha(x_0,\xi) p^*(x_0,\xi), \quad |\alpha| = N_0. \tag{4.7.6}$$

Thus

$$\xi^\alpha I_{r_0} = g_\alpha(x_0,\xi) p^*(x,\xi) + g_\alpha(x_0,\xi)(p^*(x_0,\xi) - p^*(x,\xi)).$$

Since each entry of $g_\alpha(x_0,\xi)(p^*(x_0,\xi) - p^*(x,\xi))$ is a homogeneous polynomial of degree $N_0$ in $\xi$, we apply (4.7.6) again:

$$\xi^\alpha I_{r_0} = g_\alpha(x_0,\xi) p^*(x,\xi) + \sum_\beta \epsilon^\alpha_\beta(x) g^\beta(x_0,\xi) p^*(x_0,\xi)$$

where $\epsilon^\alpha_\beta(x)$ are $r_0 \times r_0$ matrices depending smoothly on coefficients of $p^*(x,\xi)$ such that $\epsilon(x,\xi) \to 0$ as $x \to x_0$. This can be done repeatedly and we get

$$\begin{aligned}\xi^\alpha I_{r_0} &= g_\alpha(x_0,\xi) p^*(x,\xi) + \sum_\beta \epsilon^\beta_\alpha(x) g_\beta(x_0,\xi) p^*(x,\xi) + \sum_{\beta,\gamma} \epsilon^\beta_\alpha(x) \epsilon^\gamma_\beta(x) g_\gamma(x_0,\xi) p^*(x,\xi) + \cdots \\ &= \left( g_\alpha(x_0,\xi) + \sum_\beta \epsilon^\beta_\alpha(x) g_\beta(x_0,\xi) + \sum_{\beta,\gamma} \epsilon^\beta_\alpha(x) \epsilon^\gamma_\beta(x) g_\gamma(x_0,\xi) + \cdots \right) p^*(x,\xi).\end{aligned}$$

For $x$ in a small neighborhood of $x_0$, the Neumann series converges and we conclude (4.7.5).

*S*tep 3. We claim that taking $\Phi_{\mathbf{A}} = \partial^\alpha \varphi_J$ to be the jet of $\varphi_J$ up to order $N_0 - 1$ indexed by $\mathbf{A} = (\alpha, J)$ in the set $\mathcal{A} := \{(\alpha,J) : 1 \le J \le r_0, |\alpha| \le N_0 - 1\}$ gives an augmented system satisfying (Φ-1)-(Φ-4). First, (Φ-1) and (Φ-2) are obvious from the definition. For (Φ-3), we take the trivial relation $\partial_i \partial^\alpha \varphi_J(y) = \partial^{\alpha + \mathbf{e}_i} \varphi_J(y)$ to be the equation if $|\alpha| < N_0 - 1$, i.e.,

$${(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')} = \begin{cases} 1 & \text{if } J = J',\ \alpha' = \alpha + \mathbf{e}_i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad {(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)} = 0 \quad \text{for } |\alpha| < N_0 - 1.$$

It remains to check that, for $|\alpha| = N_0 - 1$, there exist $(\mathbf{B}_i)_{(\alpha,J)}{}^{(\alpha',J')}(y)$ and $(\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}(y)$ such that

$$\partial_i \partial^\alpha \varphi_J(y) = (\mathbf{B}_i)_{(\alpha,J)}{}^{(\alpha',J')}(y)\partial^{\alpha'}\varphi_{J'}(y) + (\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}(y)\partial^\gamma(\mathcal{P}^*\varphi)_K(y). \tag{4.7.7}$$

This follows from (4.7.5) and putting $(\mathcal{P}^*_{\mathrm{prin}} - \mathcal{P}^*)\varphi$ into the first term on the right-hand side. Indeed, $\mathbf{C}_i$ is determined by rewriting (4.7.5) in the form

$$\xi_i \xi^\alpha (I_{r_0})_J{}^{J'} = (\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}(y)\xi^\gamma (p^*)_K{}^{J'}(y,\xi) \quad \text{for } |\alpha| = N_0 - 1, \tag{4.7.8}$$

(i.e., $(g_{\alpha+\mathbf{e}_i})_J{}^K(y,\xi) = (\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}(y)\xi^\gamma$) and $\mathbf{B}_i$ is determined by

$$(\mathbf{B}_i)_{(\alpha,J)}{}^{(\alpha',J')}(y)\partial^{\alpha'}\varphi_{J'}(y) = (\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}(y)(\mathcal{P}^*_{\mathrm{prin}}\partial^\gamma - \partial^\gamma \mathcal{P}^*)_K{}^{J'}\varphi_{J'}(y) \quad \text{for } |\alpha| = N_0 - 1. \tag{4.7.9}$$

It remains to check (Φ-4). We define the degree of $\partial^\alpha \varphi_J$ to be $d_{(\alpha,J)} \coloneqq -|\alpha|$. The condition (Φ-4) for terms with degree $d_{(\alpha,J)} > -N_0 + 1$ is obvious. It suffices to check $(\mathbf{B}_i)_{(\alpha,J)}{}^{(\alpha',J')}$ and $(\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}(y)$ which corresponds to $d_{(\alpha,J)} = -N_0 + 1$ in (Φ-4). Since $d_{(\alpha',J')} \geq -N_0 + 1$, the condition for $\mathbf{B}$ is automatic. Moreover, $m_K + |\gamma| \leq N_0$, so $d_{(\alpha,J)} \leq -m_K - |\gamma| + 1$ and the condition for $\mathbf{C}$ follows.

It remains to verify (1), (2), and (3). The first statement follows from (4.7.8). For (2) and (3), the only nontrivial case to consider is $|\alpha| = N_0 - 1$. When $\mathcal{P}$ has constant coefficients, (4.7.5) holds with a polynomial $g_\alpha(\xi)$. It follows that each coefficient $(\mathbf{C}_i)_{(\alpha,J)}{}^{(\gamma,K)}$ in (4.7.8) is constant. The claim about $\mathbf{B}_i$ (both when $\mathcal{P}$ is homogeneous or not) then follows from (4.7.9). In general, estimate (4.7.3) holds because $g_\alpha(x,\xi)$ depends smoothly on the coefficients $c[\mathcal{P}^*]_K{}^{(\alpha,J)}$ for $|\alpha| = m_K$. Estimate (4.7.2) follows from inspecting the lower order terms in (4.7.7). □

We may now give a precise formulation and proof of Theorem 4.1.11.(2) as well.

**Proposition 4.7.3.** *Let $U$ be an connected open subset of $\mathbb{R}^d$, and $\mathcal{P}$ an $r_0 \times s_0$-matrix-valued operator on $U$ such that $\mathcal{P}_{\mathrm{prin}}$ has constant coefficients. Then the following are equivalent:*

1. *$p^*(\xi)$ satisfies (FC).*

2. *There exists a maximal graded augmented system $\{\Phi_{\mathbf{A}}\}_{\mathbf{A}\in\mathcal{A}}$ for $\mathcal{P}$ on $U$.*

3. *$\mathcal{P}_{\mathrm{prin}}$ satisfies (RC-q) for $\underline{\mathbf{x}}(y, y_1, s) \coloneqq y + s(y_1 - y)$ on any convex open subset $U'$ of $U$.*

4. *$\dim \ker \mathcal{P}^*_{\mathrm{prin}}(U) < +\infty$.*

*Moreover, if any (thus all) of the above conditions holds, then $\mathcal{P}$ satisfies (RC-q) on any $\mathbf{x}(y, y_1, s)$ satisfying ($\mathbf{x}$-1), ($\mathbf{x}$-2), and ($\mathbf{x}$-3).*

*Proof.* The implication (1) $\Rightarrow$ (2) is exactly Theorem 4.7.2. Moreover, (2) $\Rightarrow$ (3), as well as the last assertion, follows from Proposition 4.6.6 and Definition 4.1.9, and (3) $\Rightarrow$ (4) is a consequence of Lemma 4.5.14 applied on the convex open subset $U'$ with $\mathbf{x} = \underline{\mathbf{x}}$, and the obvious observation that $\ker \mathcal{P}^*(U)$ is a subspace of $\ker \mathcal{P}^*(U')$ (by restriction). It remains to prove (4) $\Rightarrow$ (1), or equivalently, its contrapositive. Indeed, if $p^*$ does *not* satisfy (FC), then there exists a constant vector $\{\varphi_J\}_{J\in\{1,\ldots,r_0\}} \neq 0$ and $\xi \in \mathbb{C}^d\setminus\{0\}$ such that $p^*(\xi)\varphi = 0$, which means that $e^{i\lambda\xi x}\varphi \in \ker \mathcal{P}^*_{\text{prin}}$ for every $\lambda \in \mathbb{C}$. Thus $\ker \mathcal{P}^*_{\text{prin}}$ is infinite dimensional. $\square$

# Appendix B

# Examples of $\mathcal{P}$ from geometry and physics

In Section 4.3 (see also Examples 4.5.13, 4.5.16, and 4.5.22), we have demonstrated how our method in Section 4.5 applies to the operator $\mathcal{P}u = (\partial_j + B_j)u^j$. In this appendix, we write down graded augmented systems that are *completely integrable* on a constant sectional curvature background. For each example, we also give explicit computations of (i) the rough integral kernel $K_{y_1}(x,y)$, as well as the corresponding point distribution $b_{y_1}(x,y)$, on a geodesic segment in a constant sectional curvature background; and (ii) the integral kernels for the conic and Bogovskii-type operators on $\mathbb{R}^d$.

## Preliminaries

Before reading any of the following subsections, we advise the reader to go over the preliminaries below.

### Notation and conventions for this appendix

In this appendix, it is conceptually (and algebraically) advantageous to work with the notation and conventions of differential geometry. We work on a $d$-dimensional smooth manifold $\mathcal{M}$ equipped with a Riemannian metric $\mathbf{g}$. We denote by $\nabla$ and $\mathrm{d}V$ the corresponding Levi-Civita connection and the Riemannian volume form, respectively. Given tensor fields $g$ and $\psi$ on (an open subset of) $\mathcal{M}$ of types $(r,s)$ and $(s,r)$, respectively, the duality pairing with respect to $\mathrm{d}V$ is defined as

$$\langle g, \psi \rangle = \int g \cdot \psi \, \mathrm{d}V, \tag{B.0.1}$$

where $g \cdot \psi$ is the natural pointwise contraction of an $(r,s)$-tensor field $g = g^{j_1 \cdots j_r}{}_{k_1 \cdots k_s}$ and an $(s,r)$-tensor field $\psi = \psi_{j_1 \cdots j_r}{}^{k_1 \cdots k_s}$. We also use the standard convention of lowering and raising indices using $\mathbf{g}_{ij}$ and $\mathbf{g}^{ij} = (\mathbf{g}^{-1})^{ij}$. We write $|\cdot|_{\mathbf{g}}$ for the induced norm on tensors, and $\operatorname{tr}_{\mathbf{g}} \mathbf{h} = \mathbf{g}_{ij}\mathbf{h}^{ij}$.

In this appendix, unlike the main body of the chapter, **the formal adjoint $\mathcal{P}^*$ of a differential operator $\mathcal{P}$ acting on tensor fields on (an open subset of) $\mathcal{M}$ is defined with respect to** $\mathrm{d}V$. This convention leads to simpler formulas. Since $\mathrm{d}V = \sqrt{\det \mathbf{g}}\mathrm{d}x$ in local coordinates, the two different definitions are related by the formula

$$\mathcal{P}^{*_{\mathrm{d}V}}\varphi = \frac{1}{\sqrt{\det \mathbf{g}}}\mathcal{P}^{*_{\mathrm{d}x}}(\sqrt{\det \mathbf{g}}\varphi) \tag{B.0.2}$$

In particular, observe that *the principal symbols are identical.*

### Preliminaries for explicit computations in the constant curvature case

In Sections B, B, B, we obtain explicit covariant formulas for the rough integral kernel $K_{y_1}(x,y)$, as well as the corresponding point distribution $b_{y_1}(x,y)$, on a geodesic segment in a constant sectional curvature background (i.e., space form). These furnish a starting point for the construction of various smoothly averaged integral kernels, such as the conic and Bogovskii-type operators on $\mathbb{R}^d$ (cf. Examples 4.5.13 and 4.5.16).

Under the convention $\mathbf{R}_{ij}{}^{k}{}_{\ell}u^{\ell} = (\nabla_i\nabla_j - \nabla_j\nabla_i)u^k$ for the Riemann curvature tensor, recall that the Riemann, Ricci, and scalar curvature tensors on a Riemannian manifold $(\mathcal{M}, \mathbf{g})$ with constant sectional curvature $\kappa$ are given by

$$\mathbf{R}_{ijk\ell} = \kappa(\mathbf{g}_{ik}\mathbf{g}_{j\ell} - \mathbf{g}_{i\ell}\mathbf{g}_{jk}), \quad \mathrm{Ric} = (d-1)\kappa\mathbf{g}, \quad R = d(d-1)\kappa,$$

where our convention is $\mathbf{R}_{ij}{}^{k}{}_{\ell}u^{\ell} = (\nabla_i\nabla_j - \nabla_j\nabla_i)u^k$.

We also introduce the *generalized sine function* that is defined for $\kappa \in \mathbb{R}$ as

$$s_\kappa(t) = \begin{cases} t & \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}} & \kappa < 0, \\ \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}} & \kappa > 0. \end{cases}$$

The generalized cosine function is defined as

$$c_\kappa(t) = s_\kappa'(t) = \begin{cases} 1 & \kappa = 0, \\ \cosh(\sqrt{-\kappa}t) & \kappa < 0, \\ \cos(\sqrt{\kappa}t) & \kappa > 0. \end{cases}$$

Fix a point $y \in \mathcal{M}$, and consider the polar coordinate system $(r, \omega)$ (where $\omega = (\omega^1, \dots \omega^{d-1})$ is a parametrization of $\mathbb{S}^{d-1}$) centered at $y$ (i.e., $r$ is the geodesic distance from $y$). The metric can be written as

$$\mathbf{g} = \mathrm{d}r^2 + s_\kappa(r)^2\mathbf{g}_{\mathbb{S}^{d-1}}.$$

Using ordinary capital latin letters $A, B, \dots$ to denote the angular variables $\omega^A$ (so $A, B, \dots \in \{1, \dots, d-1\}$), the Christoffel symbols take the form

$$\Gamma^A_{rB} = \Gamma^A_{Br} = \frac{s_\kappa'(r)}{s_\kappa(r)}\delta^A_B, \quad \Gamma^r_{AB} = -s_\kappa'(r)s_\kappa(r)(\mathbf{g}_{\mathbb{S}^{d-1}})_{AB}, \quad \Gamma^A_{BC} = {}^{(\mathbf{g}_{\mathbb{S}^{d-1}})}\Gamma^A_{BC},$$

where all the other components vanish. The normalized angular forms and vector fields are defined as $d\widetilde{\omega}^A = s_\kappa(r)d\omega^A$ and $\partial^A_{\widetilde{\omega}} = s_\kappa(t)^{-1}\partial_{\omega^A}$.

When performing explicit computations in the constant curvature case, it is convenient to work with $(r,s)$-*tensor-valued distributions* on an open subset $U$ of $(\mathcal{M},\mathbf{g})$, which have the advantage of being covariant. They are defined to be continuous linear functionals on the space of smooth and compactly supported $(s,r)$-tensor-valued densities of the form ${\psi_{j_1\cdots j_r}}^{k_1\cdots k_s}\mathrm{d}V$. A smooth $(r,s)$-tensor field ${g^{j_1\cdots j_r}}_{k_1\cdots k_s}$ is identified with an $(r,s)$-tensor-valued distribution via the pairing ${\psi_{j_1\cdots j_r}}^{k_1\cdots k_s}\mathrm{d}V \mapsto \langle g,\psi\rangle$ as in (B.0.1).

## Double divergence operator (or linearized scalar curvature operator)

We consider

$$\mathcal{P}\mathbf{h} = \nabla_j\nabla_k\mathbf{h}^{jk} - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj}, \tag{B.0.3}$$

for $d \geq 1$. The operator $\mathcal{P}$ is obtained by making a simple change of variables to the linearization of the scalar curvature operator. In fact, the *linearized scalar curvature operator* is directly given by

$$\mathrm{D}R(\mathbf{g})\dot{\mathbf{g}} = -\nabla^\ell\nabla_\ell \operatorname{tr}_\mathbf{g}\dot{\mathbf{g}} + \nabla^j\nabla^k\dot{\mathbf{g}}_{jk} - \mathrm{Ric}^{jk}\dot{\mathbf{g}}_{jk},$$

where $\operatorname{tr}_\mathbf{g}\dot{\mathbf{g}} = \mathbf{g}^{jk}\dot{\mathbf{g}}_{jk}$. We introduce[1]

$$\mathbf{h}^{jk} = \dot{\mathbf{g}}^{jk} - \mathbf{g}^{jk}\operatorname{tr}_\mathbf{g}\dot{\mathbf{g}}.$$

Then since $\operatorname{tr}_\mathbf{g}\mathbf{h} = (1-d)\operatorname{tr}_\mathbf{g}\dot{\mathbf{g}}$, this change of variables is invertible and we have

$$\dot{\mathbf{g}}^{jk} = \mathbf{h}^{jk} - \frac{1}{d-1}\mathbf{g}^{jk}\operatorname{tr}_\mathbf{g}\mathbf{h}.$$

Under this change of variables, we have

$$\mathrm{D}R(\mathbf{g})\dot{\mathbf{g}} = \nabla_j\nabla_k\mathbf{h}^{jk} - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\mathbf{h}^{jk}$$

where the RHS is equal to $\mathcal{P}(\mathbf{h})$ defined above.

We first compute the formal adjoint operator $\mathcal{P}^*$ (with respect to $\mathrm{d}V$). For a smooth compactly supported symmetric 2-tensor $\mathbf{h}$ on $U$ and $\varphi \in C_c^\infty(U)$, we have

$$\begin{aligned}\int_U \mathcal{P}(\mathbf{h})\varphi\,\mathrm{d}x &= \int_U \left(\nabla_j\nabla_k\mathbf{h}^{jk} - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\mathbf{h}^{jk}\right)\right)\varphi\,\mathrm{d}x\\ &= \int_U \mathbf{h}^{jk}\left(\nabla_j\nabla_k - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\right)\varphi\,\mathrm{d}x.\end{aligned}$$

[1] We carefully note that $\dot{\mathbf{g}}^{jk}$, according to our conventions in Section B, is the metric dual of $\dot{\mathbf{g}}_{jk}$, *not* the first order variation of $(\mathbf{g}^{-1})^{jk}$. These two objects differ by a sign.

So the formal adjoint is

$$(\mathcal{P}^*\varphi)_{jk} = \nabla_j \partial_k \varphi - \left( \mathrm{Ric}_{jk} - \frac{1}{d-1} R \mathbf{g}_{jk} \right) \varphi. \tag{B.0.4}$$

Its principal symbol is

$$(p^*)_{jk}(\xi) = -\xi_j \xi_k, \tag{B.0.5}$$

which clearly satisfies (FC) for all $d \geq 1$.

### Covariant graded augmented system and $\ker \mathcal{P}^*$

Given a function $\varphi$, define

$$\boldsymbol{\omega}_j = \partial_j \varphi. \tag{B.0.6}$$

Then

$$\begin{cases} \nabla_i \varphi = \boldsymbol{\omega}_i, \\ \nabla_i \boldsymbol{\omega}_j = \left( \mathrm{Ric}_{jk} - \frac{1}{d-1} R \mathbf{g}_{jk} \right) \varphi + (\mathcal{P}^*\varphi)_{ij}. \end{cases} \tag{B.0.7}$$

As we will see in Section B, this system of covariant first-order PDEs (B.0.7) is useful for performing explicit computations on space forms.

We note that (B.0.7) immediately leads to graded augmented variables in the sense of Definition 4.1.4. Indeed, if we specialize (B.0.7) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}^{*\mathrm{d}x}_{\mathrm{prin}}$), then we see that $\Phi_\varphi \coloneqq \varphi$, $\Phi_{\boldsymbol{\omega}_j} \coloneqq \boldsymbol{\omega}_j$ define augmented variables for $\mathcal{P}_{\mathrm{prin}}$ that satisfy ($\Phi$-1)–($\Phi$-4), where $\mathcal{A} = \{\varphi, \boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_d\}$ (in particular, $\#\mathcal{A} = d+1$), $d_\varphi = 0$, $d_{\boldsymbol{\omega}_j} = -1$, $m_{ij} = 2$, and $m'_{ij} = 0$ for all $i, j \in \{1, \dots, d\}$. These augmented variables also satisfy ($\Phi$-1)–($\Phi$-4) for any lower order perturbations of $\mathcal{P}_{\mathrm{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

In view of (B.0.7), we see that $\dim \ker \mathcal{P}^* \leq \#\mathcal{A} = d+1$. This bound is optimal, and the maximal dimension is reached (i.e., the augmented system is completely integrable) on space forms:

- For $\mathbb{R}^d$, $\mathcal{P}^*\varphi = \partial_j \partial_k \varphi$ and $\ker \mathcal{P}^* = \mathrm{span}\,\{1, x^1, \dots, x^d\}$.
- For the sphere $\mathbb{S}^d$, $\mathcal{P}^*\varphi = \nabla_j \partial_k \varphi + \mathbf{g}_{jk}\varphi$. If we embed the sphere into $\mathbb{R}^{d+1}$ by $\{(x^0)^2 + \cdots + (x^d)^2 = 1\} \subseteq \mathbb{R}^d$, then $\ker \mathcal{P}^*$ is spanned by the restrictions on $\mathbb{S}^d$ of the ambient coordinates $x^0, x^1, \cdots, x^d$, which are linearly independent. Indeed, for each $\mu \in \{0, 1, \dots, d\}$, that $\mathcal{P}^* x^\mu = 0$ can be seen by a direct computation using the fact that the second fundamental form $\mathrm{II}(X, Y)$ of the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$, which proceeds as follows:

$$(\mathcal{P}^* x^\mu)(X, Y) = X(Y x^\mu) - (\overline{\nabla}_X Y)^{\parallel} x^\mu + \mathbf{g}(X, Y) x^\mu = (\mathrm{II}(X, Y) + \mathbf{g}(X, Y)) x^\mu = 0,$$

where $\overline{\nabla}$ is the covariant derivative in the ambient coordinates, $X, Y \in T\mathbb{S}^d$, and $(\overline{\nabla}_X Y)^{\|}$ is the tangential part of $\overline{\nabla}_X Y$. The expressions $X(Yx^\mu)$ and $(\overline{\nabla}_X Y)^{\|}$ are computed using any extension of $Y$ as a smooth vector field; as is well-known, the identity holds independently of this choice. The linear independence claim is clear. Since $\dim \mathcal{P}^* \leq d+1$, $\{x^0, x^1, \dots, x^d\}$ indeed also spans $\mathcal{P}^*$.

- For the hyperbolic space $\mathbb{H}^d$, $\mathcal{P}^*\varphi = \nabla_j \partial_k \varphi - \mathbf{g}_{jk}\varphi$. We embed the hyperbolic space into the Minkowski space $\mathbb{R}^{1,d}$ with metric $-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^d)^2$ by $\{-(x^0)^2 + (x^1)^2 + \cdots + (x^d)^2 = -1\}$. Then $\ker \mathcal{P}^*$ is again spanned by the restrictions to $\mathbb{H}^d$ of the ambient coordinates $x^0, x^1, \cdots, x^d$. This statement can be verified in a similar manner to the case of $\mathbb{S}^d$. In particular, for $\mu \in \{0, 1, \dots, d\}$ and any $X, Y \in T\mathbb{H}^d$, we have

$$(\mathcal{P}^* x^\mu)(X, Y) = X(Yx^\mu) - (\overline{\nabla}_X Y)^{\|} x^\mu - \mathbf{g}(X, Y)x^\mu = (\Pi(X, Y) - \mathbf{g}(X, Y))x^\mu = 0,$$

where we used that the second fundamental form $\Pi(X, Y)$ of the embedding $\mathbb{H}^d \hookrightarrow \mathbb{R}^{1,d}$ equals $\mathbf{g}$.

**Explicit computations in the constant curvature case**

Let $\kappa$ be the constant sectional curvature of $(\mathcal{M}, \mathbf{g})$. Fix $y, y_1 \in \mathcal{M}$ and a geodesic segment $\mathbf{x}$ from $y$ to $y_1$. We may write $\mathbf{x}(t) = (t, \omega_0)$ in polar coordinates at $y$ for some $\omega_0 \in \mathbb{S}^{d-1}$. From (B.0.7) and the formulas in Section B, we have

$$\partial_r \varphi = \boldsymbol{\omega}_r, \quad \partial_r \boldsymbol{\omega}_r = -\kappa\varphi + (\mathcal{P}^*\varphi)_{rr}.$$

So

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(\mathbf{x}(t)) = \left(-\kappa\varphi + (\mathcal{P}^*\varphi)_{rr}\right).$$

The solution is given by

$$\varphi(\mathbf{x}(0)) = \int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\mathcal{P}^*\varphi)_{ij} \circ \mathbf{x}(s)\,\mathrm{d}s + \varphi \circ \mathbf{x}(d(y_1, y))c_\kappa(d(y_1, y)) - \dot{\mathbf{x}}^j(\partial_j\varphi) \circ \mathbf{x}(d(y_1, y))s_\kappa(d(y_1, y)).$$

Thus,

$$\begin{aligned}
\langle K_{y_1}(\cdot, y), \psi\rangle &= \int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j\psi_{ij} \circ \mathbf{x}(s)\,\mathrm{d}s, \\
\langle b_{y_1}(\cdot, y), \varphi\rangle &= \varphi(y_1)c_\kappa(d(y_1, y)) - \dot{\mathbf{x}}^j(y, y_1, d(y_1, y))(\partial_j\varphi)(y_1)s_\kappa(d(y_1, y)),
\end{aligned}$$

which are tensor-valued distributions in the sense of Section B.

### Explicit formulas on flat spaces

For the readers' convenience, we record the explicit formulas for the Bogovskii-type and conic solution operators on $\mathbb{R}^d$. We average over straight line segments $\mathbf{x}(y, y_1, s) = y + s(y_1 - y)$ for the Bogovski-type solution operator and over straight rays $\mathbf{x}(y, \omega, s) = y + s\omega$ for the conic solution operator.

1. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \eta = 1$. The Bogovskii-type solution operator for $\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$K_\eta^{ij}(z + y, y) = \left( \int_{|z|}^\infty \eta\left( r\frac{z}{|z|} + y \right) r^{d-1} \mathrm{d}r \right) \frac{z^i z^j}{|z|^d}$$

   with

$$b_\eta(x, y) = (d + 1)\eta(x) + (x - y)^i \partial_i \eta(x).$$

2. Let $\not\eta \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \not\eta = 1$. The conic solution operator for $\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$K_{\not\eta}^{ij}(z + y, y) = \frac{z^i z^j}{|z|^d} \not\eta \left( \frac{z}{|z|} \right).$$

## Trace-free double divergence operator

We consider

$$\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj} \text{ and } \operatorname{tr}_{\mathbf{g}} \mathbf{h} = 0, \tag{B.0.8}$$

for $d \geq 2$. We will soon show that the adjoint operator $\mathcal{P}^*$ is the traceless part of the Hessian of a scalar function. Geometrically, a function in the kernel of $\mathcal{P}^*$ has level sets that locally give rise to a warped product decomposition of the manifold on which they are defined[2]. Also, both $\mathcal{P}$ and $\mathcal{P}^*$ arise naturally in the study of the 2D Euler equations in vorticity form and the surface quasi-geostrophic equation. For example, on $\mathbb{R}^2$ the kernel of the adjoint $\mathcal{P}^*$ is spanned by $1, x, y$ and $x^2 + y^2$. Each of these functions can be integrated against a solution to 2D Euler or SQG to define the mean, impulse and angular momentum of a solution, all of which are conserved by the corresponding evolution. The operator $\mathcal{P}$ itself then arises naturally in the construction of weak solutions to these equations [IM21; IL24].

We now compute the formal adjoint operator $\mathcal{P}^*$ (with respect to $\mathrm{d}V$). For a smooth compactly supported trace-free symmetric 2-tensor $\mathbf{h}$ on $U$ and $\varphi \in C_c^\infty(U)$, we have

$$\begin{aligned} \int_U \mathcal{P}(\mathbf{h})\varphi \, \mathrm{d}x &= \int_U \nabla_j \nabla_k \mathbf{h}^{jk} \varphi \, \mathrm{d}x \\ &= \int_U \mathbf{h}^{jk} \left( \nabla_j \nabla_k - \frac{1}{d} \mathbf{g}_{jk} \nabla^\ell \nabla_\ell \right) \varphi \, \mathrm{d}x. \end{aligned}$$

[2] See for instance https://www.math.ucla.edu/~petersen/233.1.10s/BLWformulas.pdf.

So the formal adjoint is

$$(\mathcal{P}^*\varphi)_{jk} = \nabla_j \partial_k \varphi - \frac{1}{d}\mathbf{g}_{jk}\nabla^\ell \partial_\ell \varphi. \tag{B.0.9}$$

Its principal symbol is

$$(p^*)_{jk}(x,\xi) = -\xi_j\xi_k + \frac{1}{d}\xi_\ell\xi_m \mathbf{g}^{\ell m}(x)\mathbf{g}_{jk}(x). \tag{B.0.10}$$

**Covariant graded augmented system and** $\ker \mathcal{P}^*$

Given a function $\varphi$, define

$$\boldsymbol{\omega}_j = \partial_j \varphi, \quad w = \frac{1}{d}\nabla^\ell \boldsymbol{\omega}_\ell. \tag{B.0.11}$$

Then

$$\begin{cases} \nabla_i \varphi = \boldsymbol{\omega}_i, \\ \nabla_i \boldsymbol{\omega}_j = (\mathcal{P}^*\varphi)_{ij} + w\mathbf{g}_{ij}, \\ \nabla_i w = \dfrac{1}{d-1}\nabla^\ell (\mathcal{P}^*\varphi)_{i\ell} - \dfrac{1}{d-1}\mathrm{Ric}_i{}^j \boldsymbol{\omega}_j. \end{cases} \tag{B.0.12}$$

All of these identities are obvious except for the last one, which follows from:

$$\nabla_i w = \frac{1}{d}\nabla^\ell \nabla_i \boldsymbol{\omega}_\ell - \frac{1}{d}\mathbf{R}_i{}^{\ell j}{}_\ell \boldsymbol{\omega}_j = \frac{1}{d}\nabla^\ell (\mathcal{P}^*\varphi)_{i\ell} + \frac{1}{d}\nabla_i w - \frac{1}{d}\mathrm{Ric}_i{}^j \boldsymbol{\omega}_j.$$

We note that (B.0.12) immediately leads to graded augmented variables in the sense of Definition 4.1.4. Indeed, if we specialize (B.0.12) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}_{\mathrm{prin}}^{*\mathrm{d}x}$), then we see that $\Phi_\varphi := \varphi$, $\Phi_{\boldsymbol{\omega}_j} := \boldsymbol{\omega}_j$, $\Phi_w := w$ define augmented variables for $\mathcal{P}_{\mathrm{prin}}$ that satisfy (Φ-1)–(Φ-4), where $\mathcal{A} = \{\varphi, \boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_d, w\}$ (in particular, $\#\mathcal{A} = d+2$), $d_\varphi = 0$, $d_{\boldsymbol{\omega}_j} = -1$, $d_w = -2$, $m_{ij} = 2$, and $m'_{ij} = 1$ for all $i, j \in \{1, \dots, d\}$. These augmented variables also satisfy (Φ-1)–(Φ-4) for any lower order perturbations of $\mathcal{P}_{\mathrm{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

In view of (B.0.12), we see that $\dim\ker\mathcal{P}^* \leq \#\mathcal{A} = d+2$. The maximal dimension is reached (i.e., the augmented system is completely integrable) on space forms:

- For $\mathbb{R}^d$, $\mathcal{P}^*\varphi = \partial_j\partial_k\varphi - \frac{1}{d}\delta_{jk}\Delta\varphi$ and $\ker\mathcal{P}^* = \mathrm{span}\,\{1, x^1, \dots, x^d, |x|^2\}$.
- For the sphere $\mathbb{S}^d$, $\mathcal{P}^*\varphi = \nabla_j\partial_k\varphi - \frac{1}{d}\mathbf{g}_{jk}\Delta_{\mathbf{g}}\varphi$. If we embed the sphere into $\mathbb{R}^{d+1}$ by $\{(x^0)^2 + \cdots + (x^d)^2 = 1\} \subseteq \mathbb{R}^d$ as in §B, then $\ker\mathcal{P}^*$ is spanned by the ambient coordinates $x^0, x^1, \cdots, x^d$ and constant 1, which are linearly independent. This is because our computation in §B shows that $\nabla_j\partial_k x^\mu = -\mathbf{g}_{jk}x^\mu$.
- For the hyperbolic space $\mathbb{H}^d$, $\mathcal{P}^*\varphi = \nabla_j\partial_k\varphi - \frac{1}{d}\mathbf{g}_{jk}\Delta_{\mathbf{g}}\varphi$. We embed the hyperbolic space into the Minkowski space $\mathbb{R}^{1,d}$ with metric $-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^d)^2$ by $\{-(x^0)^2 + (x^1)^2 + \cdots + (x^d)^2 = -1\}$ as in §B. Then $\ker\mathcal{P}^*$ is again spanned by the ambient coordinates $x^0, x^1, \cdots, x^d$ and constant 1, which are linearly independent. This is because our computation in §B shows that $\nabla_j\partial_k x^\mu = \mathbf{g}_{jk}x^\mu$.

We remark that one can also give a derivation of the kernel of $\mathcal{P}^*$ that is independent of §B as follows. Motivated by the fact that functions in the kernel of $\mathcal{P}^*$ induce warped product decompositions, we first look for a solution that is radial in geodesic normal coordinates. Integrating the system of ODE's in §B below gives a two-dimensional space of solutions spanned by constants and $\phi_\kappa(r) := \int s_\kappa(r)\mathrm{d}r$, where $\kappa$ is the constant sectional curvature of the manifold. To find $d$ other linearly independent solutions, the next idea is to consider infinitesimal translations of the base point for the polar coordinates. Namely, we consider the functions $\mathcal{L}_K\phi_\kappa$ where $\mathcal{L}_K$ denotes the Lie derivative with respect to a Killing vector field $K$. Because Lie derivatives with respect to Killing vector fields commute with contraction with the metric and with covariant derivatives (see, e.g., [CK93, Lemma 7.1.3]), we find that $\mathcal{L}_K\phi_\kappa$ also has vanishing traceless Hessian. Choosing a set of Killing vector fields whose flow maps translate the base point in $d$ linearly independent directions gives the desired $d$ remaining linearly independent solutions.

- On $\mathbb{R}^d$, $\phi_0(r) = r^2/2$ and this operation is the observation that $x^i = \partial_i \frac{|x|^2}{2}$.
- On $\mathbb{S}^d$, $\phi_1(r) = x^0$ and this operation is the observation that $x^i = (x^i\partial_0 - x^0\partial_i)x^0$.
- On $\mathbb{H}^d$, $\phi_{-1}(r) = x^0$ and this operation is the observation that $x^i = (x^i\partial_0 + x^0\partial_i)x^0$.

### Explicit computations in the constant curvature case

Let $\kappa$ be the constant sectional curvature of $(\mathcal{M}, \mathbf{g})$. Fix $y, y_1 \in \mathcal{M}$ and a geodesic segment $\mathbf{x}$ from $y$ to $y_1$. We may write $\mathbf{x}(t) = (t, \omega_0)$ in polar coordinates at $y$ for some $\omega_0 \in \mathbb{S}^{d-1}$. From (B.0.12) and the formulas in Section B, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbf{x}(t)) = \boldsymbol{\omega}_r(\mathbf{x}(t)), \quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_r(\mathbf{x}(t)) = (\mathcal{P}^*\varphi)_{rr} + w, \quad \frac{\mathrm{d}}{\mathrm{d}t}w(\mathbf{x}(t)) = \frac{1}{d-1}\nabla^\ell(\mathcal{P}^*\varphi)_{r\ell} - \kappa\boldsymbol{\omega}_r,$$

The solution is given by

$$\begin{aligned}\varphi(\mathbf{x}(0)) = {}& \varphi\circ\mathbf{x}(d(y_1,y)) - s_\kappa(d(y_1,y))(\boldsymbol{\omega}_r\circ\mathbf{x})(d(y_1,y)) - \kappa^{-1}(c_\kappa(d(y_1,y)) - 1)(w\circ\mathbf{x})(d(y_1,y)) \\ & + \int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\mathcal{P}^*\varphi)_{ij}\circ\mathbf{x}(s)\mathrm{d}s + \frac{1}{(d-1)\kappa}\int_0^{d(y_1,y)}(c_\kappa(s)-1)\dot{\mathbf{x}}^i\nabla^j(\mathcal{P}^*\varphi)_{ij}\circ\mathbf{x}(s)\mathrm{d}s\end{aligned}$$

Thus

$$\begin{aligned}\langle K_{y_1}(\cdot,y),\psi\rangle &= \int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j\psi_{ij}\circ\mathbf{x}(s)\mathrm{d}s + \frac{1}{(d-1)\kappa}\int_0^{d(y_1,y)}(c_\kappa(s)-1)\dot{\mathbf{x}}^i\nabla^j\psi_{ij}\circ\mathbf{x}(s)\mathrm{d}s, \\ \langle b_{y_1}(\cdot,y),\varphi\rangle &= \varphi(y_1) - s_\kappa(d(y_1,y))\dot{\mathbf{x}}^j(y,y_1,d(y_1,y))(\partial_j\varphi)(y_1) - (d\kappa)^{-1}(c_\kappa(d(y_1,y))-1)(\Delta\varphi)(y_1).\end{aligned}$$

### Explicit formulas on flat spaces

For the readers' convenience, we record the explicit formulas for the Bogovskii-type and conic solution operators on $\mathbb{R}^d$. We average over straight line segments $\mathbf{x}(y, y_1, s) = y + s(y_1 - y)$

for the Bogovski-type solution operator and over straight half-lines $\mathbf{x}(y,\omega,s) = y + s\omega$ for the conic solution operator.

To state our results, we introduce

$$(\mathcal{T}^* f)_{ij} := \frac{1}{2}(f_{ij} + f_{ji}) - \frac{1}{d}(\operatorname{tr} f)\delta_{ij}.$$

1. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \eta = 1$. The Bogovskii-type solution operator for $\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ and $\operatorname{tr}_{\mathbf{g}} \mathbf{h} = 0$ on $\mathbb{R}^d$ with flat metric is given by

$$\begin{aligned} K_\eta^{ij}(z+y,y) =& \mathcal{T}^* \left( \left( \int_{|z|}^\infty \eta \left( r\frac{z}{|z|} + y \right) r^{d-1} \mathrm{d}r \right) \frac{z^i z^j}{|z|^d} \right) \\ &+ \frac{1}{2(d-1)} \mathcal{T}^* \left( \partial_{z^j} \left( \left( \int_{|z|}^\infty \eta \left( r\frac{z}{|z|} + y \right) r^{d-1} \mathrm{d}r \right) \frac{z^i}{|z|^{d-2}} \right) \right) \end{aligned}$$

   with

$$b_\eta(x,y) = \eta(x) + \partial_j((x-y)^j \eta(x)) + \frac{1}{2d}\Delta(|x-y|^2 \eta(x)).$$

2. Let $\eta\!\!\!/ \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \eta\!\!\!/ = 1$. The conic solution operator for $\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ and $\operatorname{tr}_{\mathbf{g}} \mathbf{h} = 0$ on $\mathbb{R}^d$ with flat metric is given by

$$K_{\eta\!\!\!/}^{ij}(z+y,y) = \mathcal{T}^* \left( \frac{z^i z^j}{|z|^d} \eta\!\!\!/ \left( \frac{z}{|z|} \right) \right) + \frac{1}{2(d-1)} \mathcal{T}^* \left( \partial_{z^j} \left( \frac{z^i}{|z|^{d-2}} \eta\!\!\!/ \left( \frac{z}{|z|} \right) \right) \right).$$

## Symmetric divergence operator (or adjoint Killing operator)

We consider the *symmetric divergence operator*

$$\mathcal{P}\mathbf{h} = \nabla_j \mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj}, \tag{B.0.13}$$

for $d \geq 1$.

We first compute the formal adjoint operator $\mathcal{P}^*$ (with respect to $\mathrm{d}V$). For a smooth compactly supported symmetric 2-tensor $\mathbf{h}$ and 1-form $\boldsymbol{\omega}$ on $U$, we have

$$\int_U (\mathcal{P}\mathbf{h})^k \boldsymbol{\omega}_k \, \mathrm{d}V = \int_U (\nabla_j \mathbf{h}^{jk}) \boldsymbol{\omega}_k \, \mathrm{d}V = -\int_U \mathbf{h}^{jk} \nabla_j \boldsymbol{\omega}_k \, \mathrm{d}V = -\int_U \mathbf{h}^{jk} \frac{1}{2} \left( \nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j \right) \mathrm{d}V.$$

The formal $L^2$-adjoint is

$$(\mathcal{P}^* \boldsymbol{\omega})_{jk} = -\frac{1}{2} \left( \nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j \right). \tag{B.0.14}$$

Observe that $\mathcal{P}^* \boldsymbol{\omega} = 0$ is precisely the condition that the vector field $\boldsymbol{\omega}_\sharp$ is a Killing vector field of $(\mathcal{M}, \mathbf{g})$; for this reason, we will call $\mathcal{P}^*$ the *Killing operator*. Its principal symbol is

$$(p^*)_{jk}{}^\ell(\xi) = -\frac{i}{2}(\xi_j \delta_k^\ell + \xi_k \delta_j^\ell). \tag{B.0.15}$$

**Covariant graded augmented system and** $\ker \mathcal{P}^*$

Given a 1-form $\boldsymbol{\omega}_j$, define

$$\boldsymbol{\eta}_{jk} = \tfrac{1}{2}(\mathrm{d}\boldsymbol{\omega})_{jk} = \tfrac{1}{2}(\nabla_j \boldsymbol{\omega}_k - \nabla_k \boldsymbol{\omega}_j). \tag{B.0.16}$$

Then

$$\begin{cases} \nabla_i \boldsymbol{\omega}_j = \boldsymbol{\eta}_{ij} - (\mathcal{P}^* \boldsymbol{\omega})_{ij}, \\ \nabla_i \boldsymbol{\eta}_{jk} = -\mathbf{R}_{jki}{}^{\ell} \boldsymbol{\omega}_\ell + \nabla_k (\mathcal{P}^* \boldsymbol{\omega})_{ij} - \nabla_j (\mathcal{P}^* \boldsymbol{\omega})_{ki}. \end{cases} \tag{B.0.17}$$

We postpone the proof of (B.0.17) and discuss its consequences first.

We note that (B.0.17) immediately leads to graded augmented variables in the sense of Definition 4.1.4. Indeed, if we specialize (B.0.17) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}_{\mathrm{prin}}^{*\mathrm{d}x}$), then we see that $\Phi_{\boldsymbol{\omega}_j} := \boldsymbol{\omega}_j$, $\Phi_{\boldsymbol{\eta}_{jk}} := \boldsymbol{\eta}_{jk}$ define augmented variables for $\mathcal{P}_{\mathrm{prin}}$ that satisfy (Φ-1)–(Φ-4), where $\mathcal{A} = \{\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_d, \boldsymbol{\eta}_{12}, \dots, \boldsymbol{\eta}_{(d-1)d}\}$ (in particular, $\#\mathcal{A} = d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$), $d_{\boldsymbol{\omega}_{jk}} = 0$, $d_{\boldsymbol{\eta}_j} = -1$, $m_{ij} = 1$, and $m'_{ij} = 1$ for all $i, j \in \{1, \dots, d\}$. These augmented variables also satisfy (Φ-1)–(Φ-4) for any lower order perturbations of $\mathcal{P}_{\mathrm{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

As is well-known, $\mathcal{P}^*$ has a finite dimensional kernel with $\dim \ker \mathcal{P}^* \leq \#\mathcal{A} = \frac{d(d+1)}{2}$. The maximal dimension is reached (i.e., the augmented system is completely integrable) on space forms:

- For $\mathbb{R}^d$, $\ker \mathcal{P}^*$ consists of the metric duals of the Killing vector fields, which are

  $$\mathrm{span}\left(\{\mathbf{e}_J\}_{J=1,\dots,d} \cup \{x_K \mathbf{e}_J - x_J \mathbf{e}_K\}_{1 \leq J < K \leq d}\right).$$

  Geometrically, these correspond to translation and rotation vector fields on $\mathbb{R}^d$.

- For the sphere $\mathbb{S}^d$, $\ker \mathcal{P}^*$ consists of the metric duals of

  $$\ker \mathcal{P}^* = \mathrm{span}\left(\{x_K \mathbf{e}_J - x_J \mathbf{e}_K\}_{0 \leq J < K \leq d}\right).$$

  Geometrically, these correspond to the rotation vector fields on the ambient Euclidean space $\mathbb{R}^{d+1}$.

- For the hyperbolic space $\mathbb{H}^d$, $\ker \mathcal{P}^*$ consists of the metric duals of

  $$\ker \mathcal{P}^* = \mathrm{span}\left(\{x^0 \mathbf{e}_J + x^J \mathbf{e}_0\}_{1 \leq J \leq d} \cup \{x_K \mathbf{e}_J - x_J \mathbf{e}_K\}_{1 \leq J < K \leq d}\right).$$

  Geometrically, these correspond to the Lorentz boost and rotation vector fields on the ambient Minkowski space $\mathbb{R}^{1,d}$.

Finally, we give a proof of (B.0.17), which is a covariant generalization of [Res83]. Indeed, the first identity is obvious. To prove the second identity, we begin by computing $\nabla_i(\mathcal{P}^*\boldsymbol{\omega})_{jk}$ and cycling the indices $i, j, k$:

$$\begin{aligned}
-2\nabla_i(\mathcal{P}^*\boldsymbol{\omega})_{jk} &= \nabla_i\nabla_j\boldsymbol{\omega}_k + \nabla_i\nabla_k\boldsymbol{\omega}_j,\\
-2\nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{ki} &= \nabla_j\nabla_k\boldsymbol{\omega}_i + \nabla_j\nabla_i\boldsymbol{\omega}_k = \nabla_k\nabla_j\boldsymbol{\omega}_i + \nabla_i\nabla_j\boldsymbol{\omega}_k - \mathbf{R}_{jk}{}^{\ell}{}_{i}\boldsymbol{\omega}_\ell + \mathbf{R}_{ij}{}^{\ell}{}_{k}\boldsymbol{\omega}_\ell,\\
-2\nabla_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} &= \nabla_k\nabla_i\boldsymbol{\omega}_j + \nabla_k\nabla_j\boldsymbol{\omega}_i = \nabla_i\nabla_k\boldsymbol{\omega}_j + \nabla_k\nabla_j\boldsymbol{\omega}_i - \mathbf{R}_{ki}{}^{\ell}{}_{j}\boldsymbol{\omega}_\ell.
\end{aligned}$$

Then we subtract the second equation from the third equation. We obtain

$$\begin{aligned}
\nabla_i\boldsymbol{\eta}_{jk} &= \tfrac{1}{2}(\mathbf{R}_{jk}{}^{\ell}{}_{i} - \mathbf{R}_{ij}{}^{\ell}{}_{k} - \mathbf{R}_{ki}{}^{\ell}{}_{j})\boldsymbol{\omega}_\ell + \nabla_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} - \nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{ki}\\
&= \mathbf{R}_{jk}{}^{\ell}{}_{i}\boldsymbol{\omega}_\ell + \nabla_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} - \nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{ki},
\end{aligned}$$

where we used the first Bianchi identity for the last equality.

### Explicit computations in the constant curvature case

Let $\kappa$ be the constant sectional curvature of $(\mathcal{M}, \mathbf{g})$. Fix $y, y_1 \in \mathcal{M}$ and a geodesic segment $\mathbf{x}$ from $y$ to $y_1$. We may write $\mathbf{x}(t) = (t, \omega_0)$ in polar coordinates at $y$ for some $\omega_0 \in \mathbb{S}^{d-1}$. From (B.0.17) and the formulas in Section B, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_r(\mathbf{x}(t)) &= -(\mathcal{P}^*\boldsymbol{\omega})_{rr},\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_A(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\omega}_B + \boldsymbol{\eta}_{rA} - (\mathcal{P}^*\boldsymbol{\omega})_{rA},\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_{rA}(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\eta}_{rB} - \kappa\boldsymbol{\omega}_A + \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr} - \nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA}.
\end{aligned}$$

For the first equation we can solve directly:

$$\boldsymbol{\omega}_r(\mathbf{x}(t)) = \boldsymbol{\omega}_r(y_1) + \int_t^{d(y_1,y)} (\mathcal{P}^*\boldsymbol{\omega})_{rr}(\mathbf{x}(s))\,\mathrm{d}s.$$

The equation for $\boldsymbol{\omega}_A$ can be reduced:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(s_\kappa(t)^{-1}\boldsymbol{\omega}_A(\mathbf{x}(t)) = -\kappa s_\kappa(t)^{-1}\boldsymbol{\omega}_A + s_\kappa(t)^{-1}\nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr} - 2s_\kappa(t)^{-1}\nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA}.$$

Therefore,

$$\begin{aligned}
s_\kappa(t)^{-1}\boldsymbol{\omega}_A(\mathbf{x}(t)) = &\int_t^{d(y_1,y)} s_\kappa(t-s)s_\kappa(s)^{-1}(\nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA} - \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr})(\mathbf{x}(s))\mathrm{d}s\\
&+ \int_t^{d(y_1,y)} c_\kappa(t-s)s_\kappa(s)^{-1}(\mathcal{P}^*\boldsymbol{\omega})_{rA}\mathrm{d}s\\
&+ \boldsymbol{\omega}_A(y_1)c_\kappa(t-d(y_1,y))/s_\kappa(d(y_1,y)) + \boldsymbol{\eta}_{rA}(y_1)s_\kappa(t-d(y_1,y))/s_\kappa(d(y_1,y)).
\end{aligned}$$

Thus

$$\begin{aligned}\langle K_{y_1}(\cdot,y),\psi\rangle &= \left(\int_0^{d(y_1,y)} \dot{\mathbf{x}}^i\dot{\mathbf{x}}^j\psi_{ij}(\mathbf{x}(s))\,\mathrm{d}s\right)\mathrm{d}r + \left(\int_0^{d(y_1,y)} c_\kappa(s)\dot{\mathbf{x}}^j\psi_{j\widetilde{A}}(\mathbf{x}(s))\mathrm{d}s\right)\mathrm{d}\widetilde{\omega}^A \\ &\quad - \left(\int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\nabla_i\psi_{j\widetilde{A}} - \nabla_{\widetilde{A}}\psi_{ij})(\mathbf{x}(s))\,\mathrm{d}s\right)\mathrm{d}\widetilde{\omega}^A\end{aligned}$$

(we recall that $\mathrm{d}\widetilde{\omega}^A = s_\kappa(r)\,\mathrm{d}\omega^A$ is the normalized angular 1-form and $\psi_{j\widetilde{A}} = s_\kappa(r)^{-1}\psi_{jA}$ since $\partial_{\widetilde{\omega}^A} = s_\kappa^{-1}\partial_{\omega^A}$) and

$$\langle b_{y_1}(\cdot,y),\varphi\rangle = \varphi_r(y_1)\,\mathrm{d}r + \left(\varphi_{\widetilde{A}}(y_1)c_\kappa(d(y_1,y)) - \frac{1}{2}\dot{\mathbf{x}}^j(d(y_1,y))(\nabla_j\varphi_{\widetilde{A}} - \nabla_{\widetilde{A}}\varphi_j)(y_1)s_\kappa(d(y_1,y))\right)\mathrm{d}\widetilde{\omega}^A.$$

**Explicit formulas on flat spaces**

For the readers' convenience, we record the explicit formulas for the Bogovskii-type and conic solution operators on $\mathbb{R}^d$. We average over straight line segments $\mathbf{x}(y,y_1,s) = y + s(y_1 - y)$ for the Bogovski-type solution operator and over straight half-lines $\mathbf{x}(y,\omega,s) = y + s\omega$ for the conic solution operator.

1. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\eta = 1$. The Bogovskii-type solution operator for $\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$\begin{aligned}(K_\eta)^{ij}_k(z+y,y) =& \frac{1}{2}\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^{d-1}\,\mathrm{d}r\right)\frac{z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k}{|z|^d} \\ &+ \frac{1}{2}\partial_{z^m}\left(\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^{d-1}\,\mathrm{d}r\right)\frac{z^m(z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k)}{|z|^d}\right) \\ &- \partial_{z^k}\left(\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^{d-1}\,\mathrm{d}r\right)\frac{z^iz^j}{|z|^d}\right)\end{aligned}$$

   with

$$(b_\eta)^j_k(x,y) = \frac{d+1}{2}\eta(x)\boldsymbol{\delta}^j_k + \frac{1}{2}(x-y)^\ell\partial_\ell\eta(x)\boldsymbol{\delta}^j_k - \frac{1}{2}(x-y)^j\partial_k\eta(x).$$

2. Let $\not\eta \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}\not\eta = 1$. The conic solution operator for $\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$(K_{\not\eta})^{ij}_k(z+y,y) = \frac{1}{2}\frac{z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k}{|z|^d}\not\eta\left(\frac{z}{|z|}\right) + \frac{1}{2}\partial_{z^m}\left(\frac{z^m(z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k)}{|z|^d}\not\eta\left(\frac{z}{|z|}\right)\right) - \partial_{z^k}\left(\frac{z^iz^j}{|z|^d}\not\eta\left(\frac{z}{|z|}\right)\right).$$

## Trace-free symmetric divergence operator (or adjoint conformal Killing operator)

We consider the *trace-free symmetric divergence operator*

$$\mathcal{P}\mathbf{h} = \nabla_j \mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj} \text{ and } \operatorname{tr}_{\mathbf{g}} \mathbf{h} = 0, \tag{B.0.18}$$

for $d \geq 3$.

We first compute the formal adjoint $\mathcal{P}^*$ (with respect to $\mathrm{d}V$). For a smooth compactly supported trace-free symmetric 2-tensor $\mathbf{h}$ and 1-form $\boldsymbol{\omega}$ on $U$, we have

$$\begin{aligned}\int_U (\mathcal{P}\mathbf{h})^k \boldsymbol{\omega}_k \,\mathrm{d}V &= \int_U (\nabla_j \mathbf{h}^{jk}) \boldsymbol{\omega}_k \,\mathrm{d}V = -\int_U \mathbf{h}^{jk} \nabla_j \boldsymbol{\omega}_k \,\mathrm{d}V \\ &= -\int_U \mathbf{h}^{jk} \left[\frac{1}{2}\left(\nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j\right) - \frac{1}{d} \nabla^\ell \boldsymbol{\omega}_\ell \mathbf{g}_{jk}\right] \mathrm{d}V.\end{aligned}$$

The formal $L^2$-adjoint is

$$(\mathcal{P}^* \boldsymbol{\omega})_{jk} = -\frac{1}{2}\left(\nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j\right) + \frac{1}{d} \nabla^\ell \boldsymbol{\omega}_\ell \mathbf{g}_{jk}. \tag{B.0.19}$$

Observe that $\mathcal{P}^* \boldsymbol{\omega} = 0$ is precisely the condition that the vector field $\boldsymbol{\omega}_\sharp$ is a conformal Killing vector field of $(\mathcal{M}, \mathbf{g})$ (i.e., the infinitesimal generator of a one-parameter family of conformal isometries); for this reason, we will call $\mathcal{P}^*$ the *conformal Killing operator*. Its principal symbol is

$$(p^*)_{jk}{}^{\ell}(x, \xi) = -\frac{i}{2}(\xi_j \delta_k^\ell + \xi_k \delta_j^\ell) + \frac{i}{d} \xi_m \mathbf{g}_{jk}(x) (\mathbf{g}^{-1})^{\ell m}(x) \tag{B.0.20}$$

### Covariant graded augmented system and $\ker \mathcal{P}^*$

Given a 1-form $\boldsymbol{\omega}_j$, define

$$\boldsymbol{\eta}_{jk} = \tfrac{1}{2}(\mathrm{d}\boldsymbol{\omega})_{jk} = \tfrac{1}{2}(\nabla_j \boldsymbol{\omega}_k - \nabla_k \boldsymbol{\omega}_j), \quad w = \frac{1}{d} \nabla^\ell \boldsymbol{\omega}_\ell, \quad \boldsymbol{\zeta}_j = \partial_j w. \tag{B.0.21}$$

Note that $\frac{1}{2}(\nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j) = -(\mathcal{P}^* \boldsymbol{\omega})_{jk} + w \mathbf{g}_{jk}$. Then

$$\left\{\begin{aligned}
\nabla_i \boldsymbol{\omega}_j &= \boldsymbol{\eta}_{ij} + w \mathbf{g}_{ij} - (\mathcal{P}^* \boldsymbol{\omega})_{ij}, \\
\nabla_i \boldsymbol{\eta}_{jk} &= -\mathbf{R}_{jki}{}^{\ell} \boldsymbol{\omega}_\ell + \boldsymbol{\zeta}_j \mathbf{g}_{ik} - \boldsymbol{\zeta}_k \mathbf{g}_{ij} + \nabla_k (\mathcal{P}^* \boldsymbol{\omega})_{ij} - \nabla_j (\mathcal{P}^* \boldsymbol{\omega})_{ki}, \\
\nabla_i w &= \boldsymbol{\zeta}_i, \\
\nabla_i \boldsymbol{\zeta}_j &= -\frac{1}{d-2} \nabla^m \left(\mathrm{Ric}_{ij} - \frac{1}{2(d-1)} R \mathbf{g}_{ij}\right) \boldsymbol{\omega}_m \\
&\quad -\frac{1}{d-2}\left[\mathrm{Ric}_i{}^m \boldsymbol{\eta}_{jm} + \mathrm{Ric}_j{}^m \boldsymbol{\eta}_{im} + 2\left(\mathrm{Ric}_{ij} - \frac{1}{2(d-1)} R \mathbf{g}_{ij}\right) w\right] \\
&\quad + \frac{1}{d-2} \mathcal{C}(\mathcal{P}^* \boldsymbol{\omega})_{ij}.
\end{aligned}\right. \tag{B.0.22}$$

where

$$\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{ij} = -\nabla^\ell\nabla_i(\mathcal{P}^*\boldsymbol{\omega})_{\ell j} - \nabla^\ell\nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{\ell i} + \nabla^\ell\nabla_\ell(\mathcal{P}^*\boldsymbol{\omega})_{ij} + \frac{1}{d-1}\nabla^\ell\nabla^m(\mathcal{P}^*\boldsymbol{\omega})_{\ell m}\mathbf{g}_{ij}$$
$$+\mathrm{Ric}_i{}^\ell(\mathcal{P}^*\boldsymbol{\omega})_{j\ell} + \mathrm{Ric}_j{}^\ell(\mathcal{P}^*\boldsymbol{\omega})_{i\ell} - \frac{1}{d-1}\mathrm{Ric}^{\ell m}(\mathcal{P}^*\boldsymbol{\omega})_{\ell m}\mathbf{g}_{ij}.$$

We postpone the proof of (B.0.22) and discuss its consequences first. We note that (B.0.22) immediately leads to graded augmented variables in the sense of Definition 4.1.4. Indeed, if we specialize (B.0.22) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}^{*\mathrm{dx}}_{\mathrm{prin}}$), then we see that $\Phi_{\boldsymbol{\omega}_j} := \boldsymbol{\omega}_j$, $\Phi_{\boldsymbol{\eta}_{jk}} := \boldsymbol{\eta}_{jk}$, $\Phi_w := w$ and $\Phi_{\boldsymbol{\zeta}_j} := \boldsymbol{\zeta}_j$ define augmented variables for $\mathcal{P}_{\mathrm{prin}}$ that satisfy (Φ-1)–(Φ-4), where $\mathcal{A} = \{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_d, \boldsymbol{\eta}_{12}, \ldots, \boldsymbol{\eta}_{(d-1)d}, w, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_d\}$ (in particular, $\#\mathcal{A} = d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)(d+2)}{2}$), $d_{\boldsymbol{\omega}_j} = 0$, $d_{\boldsymbol{\eta}_{jk}} = d_w = -1$, $d_{\boldsymbol{\zeta}_j} = -2$, $m_{ij} = 1$, and $m'_{ij} = 2$ for all $i, j \in \{1, \ldots, d\}$. These augmented variables also satisfy (Φ-1)–(Φ-4) for any lower order perturbations of $\mathcal{P}_{\mathrm{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

The operator $\mathcal{P}^*$ has a finite dimensional kernel with $\dim\ker\mathcal{P}^* \leq \#\mathcal{A} = \frac{(d+1)(d+2)}{2}$. This bound is optimal, and the maximal dimension is reached (i.e., the augmented system is completely integrable) on space forms:

- For $\mathbb{R}^d$, $\ker\mathcal{P}^*$ consists of the metric duals of the conformal Killing vector fields, which are
$$\mathrm{span}\ \left(\{\mathbf{e}_J\}_{J=1,\ldots,d} \cup \{x_K\mathbf{e}_J - x_J\mathbf{e}_K\}_{1\leq J<K\leq d} \cup \{x^j\mathbf{e}_j\} \cup \{x_Jx^j\mathbf{e}_j - |x|^2\mathbf{e}_J\}_{1\leq J\leq d}\right). \tag{B.0.23}$$
- For $\mathbb{S}^d$, the stereographic projection
$$y^j = \frac{x^j}{1-x^0}, \quad x^0 = \frac{\sum(y^j)^2 - 1}{\sum(y^j)^2+1}, \quad x^k = \frac{2y^k}{\sum(y^j)^2+1},\ k = 1, \cdots, d$$
gives $\mathrm{d}s^2 = \frac{4((dy^1)^2+\cdots+(dy^d)^2)}{1+\sum(y^j)^2}$, so the conformal Killing vector fields are given by (B.0.23) in the coordinate $y^j$.
- For $\mathbb{H}^d$, in the upper half space model, the metric on $\mathbb{H}^d$ is $\frac{(dx^0)^2+(dx^1)^2+\cdots+(dx^d)^2}{(x^0)^2}$, which is confromal to the Euclidean metric, so the conformal Killing vector fields are also given by (B.0.23).

Finally, we give a proof of (B.0.22), which is a covariant generalization of [Res83]. The first identity is obvious, whereas the second identity follows from Section B. The third identity is again a restatement of the definition of $\boldsymbol{\zeta}_i$. It remains to establish the last identity.

To simplify the computation, we introduce the notation

$$\mathbf{A}_{ijk\ell} := -\nabla^2_{ij}(\mathcal{P}^*\boldsymbol{\omega})_{k\ell} = \frac{1}{2}\left(\nabla^3_{ijk}\boldsymbol{\omega}_\ell + \nabla^3_{ij\ell}\boldsymbol{\omega}_k\right) - \frac{1}{d}\nabla^2_{ij}\nabla^m\boldsymbol{\omega}_m\mathbf{g}_{k\ell},$$

where $\nabla^2_{ij} = \nabla_i \nabla_j$ and $\nabla^3_{ijk} = \nabla_i \nabla_j \nabla_k$. Indeed,

$$\begin{aligned}
\mathbf{A}_{ijk\ell} &= \frac{1}{2}\left(\nabla^3_{ijk}\boldsymbol{\omega}_\ell + \nabla^3_{ij\ell}\boldsymbol{\omega}_k\right) - \frac{1}{d}\nabla^2_{ij}\nabla^m \boldsymbol{\omega}_m \mathbf{g}_{k\ell} \\
&= \frac{1}{2}\left(\nabla^3_{ikj}\boldsymbol{\omega}_\ell + \nabla^3_{j\ell i}\boldsymbol{\omega}_k\right) - \frac{1}{d}\nabla^2_{ij}\nabla^m \boldsymbol{\omega}_m \mathbf{g}_{k\ell} \\
&\quad -\frac{1}{2}\nabla_i(\mathbf{R}_{jk}{}^m{}_\ell \boldsymbol{\omega}_m) - \frac{1}{2}(\mathbf{R}_{ij}{}^m{}_\ell \nabla_m \boldsymbol{\omega}_k + \mathbf{R}_{ij}{}^m{}_k \nabla_\ell \boldsymbol{\omega}_m) - \frac{1}{2}\nabla_j(\mathbf{R}_{i\ell}{}^m{}_k \boldsymbol{\omega}_m),
\end{aligned}$$

so after contracting $i$ and $k$ using the inverse metric, we have

$$\begin{aligned}
\mathbf{A}^k{}_{jk\ell} &= \frac{1}{2}\left(\Delta \nabla_j \boldsymbol{\omega}_\ell + \nabla^2_{j\ell}\nabla^k \boldsymbol{\omega}_k\right) - \frac{1}{d}\nabla^2_{\ell j}\nabla^m \boldsymbol{\omega}_m \\
&\quad -\frac{1}{2}\nabla^k(\mathbf{R}_{jk}{}^m{}_\ell \boldsymbol{\omega}_m) - \frac{1}{2}(\mathbf{R}^k{}_j{}^m{}_\ell \nabla_m \boldsymbol{\omega}_k + \mathbf{R}^k{}_j{}^m{}_k \nabla_\ell \boldsymbol{\omega}_m) - \frac{1}{2}\nabla_j(\mathbf{R}^k{}_\ell{}^m{}_k \boldsymbol{\omega}_m) \\
&= \frac{1}{2}\Delta \nabla_j \boldsymbol{\omega}_\ell + \frac{d-2}{2d}\nabla^2_{j\ell}\nabla^k \boldsymbol{\omega}_k \\
&\quad -\frac{1}{2}\nabla^k \mathbf{R}_{jk}{}^m{}_\ell \boldsymbol{\omega}_m - \frac{1}{2}(-\mathbf{R}_j{}^k{}_\ell{}^m \nabla_k \boldsymbol{\omega}_m + \mathbf{R}_j{}^k{}_\ell{}^m \nabla_m \boldsymbol{\omega}_k - \mathrm{Ric}_j{}^m \nabla_\ell \boldsymbol{\omega}_m) \\
&\quad +\frac{1}{2}\nabla_j \mathrm{Ric}_\ell{}^m \boldsymbol{\omega}_m + \frac{1}{2}\mathrm{Ric}_\ell{}^m \nabla_j \boldsymbol{\omega}_m \\
&= \frac{1}{2}\Delta \nabla_j \boldsymbol{\omega}_\ell + \frac{d-2}{2d}\nabla^2_{j\ell}\nabla^k \boldsymbol{\omega}_k \\
&\quad +\frac{1}{2}(\nabla^m \mathrm{Ric}_{j\ell} - \nabla_\ell \mathrm{Ric}_j{}^m + \nabla_j \mathrm{Ric}_\ell{}^m)\boldsymbol{\omega}_m + \mathbf{R}_j{}^k{}_\ell{}^m \boldsymbol{\eta}_{km} + \frac{1}{2}(\mathrm{Ric}_j{}^m \nabla_\ell \boldsymbol{\omega}_m + \mathrm{Ric}_\ell{}^m \nabla_j \boldsymbol{\omega}_m).
\end{aligned}$$

where, for the last equality, we used the second Bianchi identity to write

$$-\nabla^k \mathbf{R}_{jk}{}^m{}_\ell = \nabla^m \mathbf{R}_{jk\ell}{}^k + \nabla_\ell \mathbf{R}_{jk}{}^{km} = \nabla^m \mathrm{Ric}_{j\ell} - \nabla_\ell \mathrm{Ric}_j{}^m.$$

Thus

$$\begin{aligned}
\mathbf{A}^k{}_{jk\ell} + \mathbf{A}^k{}_{\ell kj} &= \frac{1}{2}\Delta(\nabla_j \boldsymbol{\omega}_\ell + \nabla_\ell \boldsymbol{\omega}_j) + \frac{d-2}{d}\nabla^2_{j\ell}\nabla^k \boldsymbol{\omega}_k \\
&\quad +\nabla^m \mathrm{Ric}_{j\ell}\boldsymbol{\omega}_m + \mathrm{Ric}_j{}^m \nabla_\ell \boldsymbol{\omega}_m + \mathrm{Ric}_\ell{}^m \nabla_j \boldsymbol{\omega}_m.
\end{aligned}$$

Contracting also $j$ and $\ell$ using the inverse metric, we have

$$\mathbf{A}^{k\ell}{}_{k\ell} = \frac{d-1}{d}\Delta \nabla^k \boldsymbol{\omega}_k + \frac{1}{2}\nabla^m R \boldsymbol{\omega}_m + \mathrm{Ric}^{\ell m}\nabla_\ell \boldsymbol{\omega}_m.$$

Since

$$\mathbf{A}^\ell{}_{\ell jk} = \frac{1}{2}\left(\Delta \nabla_j \boldsymbol{\omega}_k + \Delta \nabla_k \boldsymbol{\omega}_j\right) - \frac{1}{d}\Delta \nabla^m \boldsymbol{\omega}_m \mathbf{g}_{jk},$$

we have

$$\begin{aligned}\mathbf{A}^{\ell}{}_{\ell jk}+\frac{1}{d-1}\mathbf{A}^{\ell m}{}_{\ell m}\mathbf{g}_{jk}&=\mathbf{A}^{\ell}{}_{\ell jk}+\frac{1}{d}\Delta\nabla^m\boldsymbol{\omega}_m\mathbf{g}_{jk}+\frac{1}{2(d-1)}\nabla^m R\boldsymbol{\omega}_m\mathbf{g}_{jk}+\frac{1}{d-1}\mathrm{Ric}^{\ell m}\nabla_\ell\boldsymbol{\omega}_m\mathbf{g}_{jk}\\&=\frac{1}{2}\Delta\left(\nabla_j\boldsymbol{\omega}_k+\nabla_k\boldsymbol{\omega}_j\right)+\frac{1}{2(d-1)}\nabla^m R\boldsymbol{\omega}_m\mathbf{g}_{jk}+\frac{1}{2(d-1)}\mathrm{Ric}^{\ell m}(\nabla_\ell\boldsymbol{\omega}_m+\nabla_m\boldsymbol{\omega}_\ell)\mathbf{g}_{jk}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{A}^{k}{}_{jk\ell}+\mathbf{A}^{k}{}_{\ell kj}-\mathbf{A}^{k}{}_{kj\ell}-\frac{1}{d-1}\mathbf{A}^{km}{}_{km}\mathbf{g}_{j\ell}\\&=\frac{d-2}{d}\nabla^2_{j\ell}\nabla^k\boldsymbol{\omega}_k+\nabla^m\mathrm{Ric}_{j\ell}\boldsymbol{\omega}_m-\frac{1}{2(d-1)}\nabla^m R\boldsymbol{\omega}_m\mathbf{g}_{j\ell}\\&\quad+\mathrm{Ric}_j{}^m\nabla_\ell\boldsymbol{\omega}_m+\mathrm{Ric}_\ell{}^m\nabla_j\boldsymbol{\omega}_m-\frac{1}{2(d-1)}\mathrm{Ric}^{km}(\nabla_k\boldsymbol{\omega}_m+\nabla_m\boldsymbol{\omega}_k)\mathbf{g}_{j\ell}.\end{aligned}$$

Note also that

$$\begin{aligned}\mathrm{Ric}_j{}^m\nabla_\ell\boldsymbol{\omega}_m+\mathrm{Ric}_\ell{}^m\nabla_j\boldsymbol{\omega}_m&=\mathrm{Ric}_j{}^m\boldsymbol{\eta}_{\ell m}+\mathrm{Ric}_\ell{}^m\boldsymbol{\eta}_{jm}+2\mathrm{Ric}_{j\ell}w\\&\quad-\mathrm{Ric}_j{}^m(\mathcal{P}^*\boldsymbol{\omega})_{\ell m}-\mathrm{Ric}_\ell{}^m(\mathcal{P}^*\boldsymbol{\omega})_{jm},\\-\frac{1}{2(d-1)}\mathrm{Ric}^{km}(\nabla_k\boldsymbol{\omega}_m+\nabla_m\boldsymbol{\omega}_k)&=-\frac{1}{(d-1)}Rw+\frac{1}{d-1}\mathrm{Ric}^{km}(\mathcal{P}^*\boldsymbol{\omega})_{km}.\end{aligned}$$

We arrive at

$$\begin{aligned}\nabla_j\boldsymbol{\zeta}_\ell&=\frac{1}{d}\nabla^2_{j\ell}\nabla^k\boldsymbol{\omega}_k\\&=-\frac{1}{d-2}\left(\nabla^m\mathrm{Ric}_{j\ell}-\frac{1}{2(d-1)}\nabla^m R\mathbf{g}_{j\ell}\right)\boldsymbol{\omega}_m\\&\quad-\frac{1}{d-2}\left[\mathrm{Ric}_j{}^m\boldsymbol{\eta}_{\ell m}+\mathrm{Ric}_\ell{}^m\boldsymbol{\eta}_{jm}+2\left(\mathrm{Ric}_{j\ell}-\frac{1}{2(d-1)}R\mathbf{g}_{j\ell}\right)w\right]\\&\quad+\frac{1}{d-2}\mathcal{C}_{j\ell}\end{aligned}$$

where

$$\begin{aligned}\mathcal{C}_{j\ell}&=\mathbf{A}^{k}{}_{jk\ell}+\mathbf{A}^{k}{}_{\ell kj}-\mathbf{A}^{k}{}_{kj\ell}-\frac{1}{d-1}\mathbf{A}^{km}{}_{km}\mathbf{g}_{j\ell}\\&\quad+\mathrm{Ric}_j{}^m(\mathcal{P}^*\boldsymbol{\omega})_{\ell m}+\mathrm{Ric}_\ell{}^m(\mathcal{P}^*\boldsymbol{\omega})_{jm}-\frac{1}{d-1}\mathrm{Ric}^{km}(\mathcal{P}^*\boldsymbol{\omega})_{km}\mathbf{g}_{j\ell},\end{aligned}$$

which completes the proof.

**Explicit computations in the constant curvature case**

Let $\kappa$ be the constant sectional curvature of $(\mathcal{M}, \mathbf{g})$. Fix $y, y_1 \in \mathcal{M}$ and a geodesic segment $\mathbf{x}$ from $y$ to $y_1$. We may write $\mathbf{x}(t) = (t, \omega_0)$ in polar coordinates at $y$ for some $\omega_0 \in \mathbb{S}^{d-1}$. From (B.0.22) and the formulas in Section B, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_r(\mathbf{x}(t)) &= w - (\mathcal{P}^*\boldsymbol{\omega})_{rr},\\
\frac{\mathrm{d}}{\mathrm{d}t}w(\mathbf{x}(t)) &= \boldsymbol{\zeta}_r,\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\zeta}_r(\mathbf{x}(t)) &= -\kappa w + \frac{1}{d-2}\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{rr}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_A(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\omega}_B + \boldsymbol{\eta}_{rA} - (\mathcal{P}^*\boldsymbol{\omega})_{rA},\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_{rA}(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\eta}_{rB} - \kappa\boldsymbol{\omega}_A - \boldsymbol{\zeta}_A + \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr} - \nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA},\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\zeta}_A(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\zeta}_A + \frac{1}{d-2}\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{rA}.
\end{aligned}$$

From the first three equations we get

$$\begin{aligned}
w(\mathbf{x}(t)) = {} & w(y_1)c_\kappa(t - d(y_1,y)) + \boldsymbol{\zeta}_r(y_1)s_\kappa(t - d(y_1,y))\\
& - \frac{1}{d-2}\int_t^{d(y_1,y)} s_\kappa(t-s)\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{rr}(\mathbf{x}(s))\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\omega}_r(\mathbf{x}(0)) = {} & \boldsymbol{\omega}_r(y_1) - w(y_1)s_\kappa(d(y_1,y)) + \frac{1}{\kappa}\boldsymbol{\zeta}_r(y_1)(1 - c_\kappa(d(y_1,y))\\
& + \int_0^{d(y_1,y)} (\mathcal{P}^*\boldsymbol{\omega})_{rr}(\mathbf{x}(s))\,\mathrm{d}s - \frac{1}{(d-2)\kappa}\int_0^{d(y_1,y)} (1 - c_\kappa(s))\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{rr}(\mathbf{x}(s))\,\mathrm{d}s.
\end{aligned}$$

From the last three equations, we get

$$s_\kappa(t)^{-1}\boldsymbol{\zeta}_A(\mathbf{x}(t)) = s_\kappa(d(y_1,y))^{-1}\boldsymbol{\zeta}_A(y_1) - \frac{1}{d-2}\int_t^{d(y_1,y)} s_\kappa(s)^{-1}\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{rA}(\mathbf{x}(s))\,\mathrm{d}s$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(s_\kappa(t)^{-1}\boldsymbol{\omega}_A(\mathbf{x}(t))\right) = -\kappa s_\kappa(t)^{-1}\boldsymbol{\omega}_{rA} + s_\kappa(t)^{-1}(-\boldsymbol{\zeta}_A + \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr} - 2\nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA}).$$

Therefore

$$\begin{aligned}
s_\kappa(t)^{-1}\boldsymbol{\omega}_A(\mathbf{x}(t)) &= \int_t^{d(y_1,y)} s_\kappa(t-s)s_\kappa(s)^{-1}(\nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA} - \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr})(\mathbf{x}(s))\,\mathrm{d}s \\
&+ \int_t^{d(y_1,y)} c_\kappa(t-s)s_\kappa(s)^{-1}(\mathcal{P}^*\boldsymbol{\omega})_{rA}\mathrm{d}s \\
&+ \frac{1}{(d-2)\kappa}\int_t^{d(y_1,y)} (1-c_\kappa(t-s))s_\kappa(s)^{-1}\mathcal{C}(\mathcal{P}^*\boldsymbol{\omega})_{rA}(\mathbf{x}(s))\,\mathrm{d}s \\
&- \kappa^{-1}(1-c_\kappa(t-d(y_1,y))s_\kappa(d(y_1,y))^{-1}\boldsymbol{\zeta}_A(y_1) \\
&+ \boldsymbol{\omega}_A(y_1)c_\kappa(t-d(y_1,y))s_\kappa(d(y_1,y))^{-1} + \boldsymbol{\eta}_{rA}(y_1)s_\kappa(t-d(y_1,y))s_\kappa(d(y_1,y))^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle K_{y_1}(\cdot,y),\psi\rangle &= \left(\int_0^{d(y_1,y)} \dot{\mathbf{x}}^i\dot{\mathbf{x}}^j\psi_{ij}(\mathbf{x}(s))\mathrm{d}s - \frac{1}{(d-2)\kappa}\int_0^{d(y_1,y)}(1-c_\kappa(s))\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\mathcal{C}\psi)_{ij}(\mathbf{x}(s))\mathrm{d}s\right)\mathrm{d}r \\
&+ \left(\int_0^{d(y_1,y)} c_\kappa(s)\dot{\mathbf{x}}^j\psi_{j\widetilde{A}}(\mathbf{x}(s))\mathrm{d}s + \frac{1}{(d-2)\kappa}\int_0^{d(y_1,y)}(1-c_\kappa(s))\dot{\mathbf{x}}^j(\mathcal{C}\psi)_{j\widetilde{A}}(\mathbf{x}(s))\mathrm{d}s\right)\mathrm{d}\widetilde{\omega}^A \\
&+ \left(\int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\nabla_{\widetilde{A}}(\mathcal{P}^*\boldsymbol{\omega})_{ij} - \nabla_i(\mathcal{P}^*\boldsymbol{\omega})_{j\widetilde{A}})(\mathbf{x}(s))\mathrm{d}s\right)\mathrm{d}\widetilde{\omega}^A
\end{aligned}$$

and

$$\begin{aligned}
\langle b_{y_1}(\cdot,y),\boldsymbol{\omega}\rangle &= \left(\boldsymbol{\omega}_r(y_1) - w(y_1)s_\kappa(d(y_1,y)) + \frac{1}{\kappa}\boldsymbol{\zeta}_r(y_1)(1-c_\kappa(d(y_1,y))\right)\mathrm{d}r \\
&+ \left(-\kappa^{-1}(1-c_\kappa(d(y_1,y))\boldsymbol{\zeta}_{\widetilde{A}}(y_1) + \boldsymbol{\omega}_{\widetilde{A}}(y_1)c_\kappa(d(y_1,y)) - \boldsymbol{\eta}_{r\widetilde{A}}(y_1)s_\kappa(d(y_1,y))\right)\mathrm{d}\widetilde{\omega}^A.
\end{aligned}$$

**Explicit formulas on flat spaces**

For the readers' convenience, we record the explicit formulas for the Bogovskii-type and conic solution operators on $\mathbb{R}^d$. We average over straight line segments $\mathbf{x}(y,y_1,s) = y + s(y_1 - y)$ for the Bogovski-type solution operator and over straight half-lines $\mathbf{x}(y,\omega,s) = y + s\omega$ for the conic solution operator.

To state our results, we introduce

$$(\mathcal{C}^*f)_{ij} \coloneqq -\partial^\ell\partial_i f_{\ell j} - \partial^\ell\partial_i f_{j\ell} + \partial^\ell\partial_\ell f_{ij} + \frac{1}{d-1}\partial_i\partial_j \operatorname{tr} f, \quad (\mathcal{T}^*f)_{ij} \coloneqq \frac{1}{2}(f_{ij}+f_{ji}) - \frac{1}{d}(\operatorname{tr} f)\delta_{ij}.$$

1. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\eta = 1$. The Bogovskii-type solution operator for $\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk}$

where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ and $\mathrm{tr}_{\mathbf{g}}\,\mathbf{h} = 0$ on $\mathbb{R}^d$ with flat metric is given by

$$\begin{aligned}(K_\eta)^{ij}_k(z+y,y) =& \mathcal{T}^*\left(\int_{|z|}^\infty \eta\left(y + r\frac{z}{|z|}\right) r^{d-1}\frac{z^i}{|z|^d}\boldsymbol{\delta}^j_k\,\mathrm{d}r\right)\\ &+ \frac{1}{2(d-2)}\mathcal{T}^*\mathcal{C}^*\left(\int_{|z|}^\infty \eta\left(y + r\frac{z}{|z|}\right) r^{d-1}\frac{z^i}{|z|^{d-2}}\boldsymbol{\delta}^j_k\,\mathrm{d}r\right)\\ &- \frac{1}{(d-2)}\mathcal{T}^*\mathcal{C}^*\left(\int_{|z|}^\infty \eta\left(y + r\frac{z}{|z|}\right) r^{d-1}\frac{z^iz^jz_k}{|z|^d}\,\mathrm{d}r\right)\\ &- (\mathcal{T}^*\boldsymbol{\delta}^j_\ell\nabla_k - \mathcal{T}^*\boldsymbol{\delta}^j_k\nabla_\ell)\left(\int_{|z|}^\infty \eta\left(y + r\frac{z}{|z|}\right) r^{d-1}\frac{z^iz^\ell}{|z|^d}\,\mathrm{d}r\right)\end{aligned}$$

with

$$\begin{aligned}(b_\eta)^j_k(x,y) =& \eta(x)\boldsymbol{\delta}^j_k + \frac{1}{2}\partial_\ell(\eta(x)(x-y)^\ell)\boldsymbol{\delta}^j_k - \frac{1}{2}\partial_k(\eta(x)(x-y)^j) + \frac{1}{d}\partial^j(\eta(x)(x-y)_k)\\ &- \frac{1}{2d}\partial^j\partial_k(\eta(x)|x-y|^2) + \frac{1}{d}\partial^j\partial_\ell(\eta(x)(x-y)^\ell(x-y)_k).\end{aligned}$$

2. Let $\eta\!\!\!/ \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}\eta\!\!\!/ = 1$. The conic solution operator for $\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ and $\mathrm{tr}_{\mathbf{g}}\,\mathbf{h} = 0$ on $\mathbb{R}^d$ with flat metric is given by

$$\begin{aligned}(K_{\eta\!\!\!/})^{ij}_k(z+y,y) =& \mathcal{T}^*\left(\boldsymbol{\delta}^j_k\frac{z^i}{|z|^d}\eta\!\!\!/\left(\frac{z}{|z|}\right)\right) + \frac{1}{2(d-2)}\mathcal{T}^*\mathcal{C}^*\left(\left(\boldsymbol{\delta}^j_k\frac{z^i}{|z|^{d-2}} - 2\frac{z^iz^jz_k}{|z|^d}\right)\eta\!\!\!/\left(\frac{z}{|z|}\right)\right)\\ &- (\mathcal{T}^*\boldsymbol{\delta}^i_\ell\nabla_k - \mathcal{T}^*\boldsymbol{\delta}^i_k\nabla_\ell)\left(\frac{z^\ell z^j}{|z|^d}\eta\!\!\!/\left(\frac{z}{|z|}\right)\right).\end{aligned}$$

## Linearized Einstein constraint equation

We consider the linearized Einstein constraint equation, first by itself and second under the constant mean curvature condition.

### Linearized Einstein constraint operator

The vacuum *Einstein constraint equation* on $(\mathcal{M},\mathbf{g})$ is a nonlinear underdetermined system of PDEs for $\mathbf{g}$ and a symmetric 2-tensor $\mathbf{k}$ of the form

$$\begin{cases} R_{\mathbf{g}} + (\mathrm{tr}_{\mathbf{g}}\,\mathbf{k})^2 - |\mathbf{k}|^2_{\mathbf{g}} = 0,\\ \nabla^i\mathbf{k}_{ij} - \partial_j\,\mathrm{tr}_{\mathbf{g}}\,\mathbf{k} = 0.\end{cases} \tag{B.0.24}$$

The linearization of the operator on the left-hand side (B.0.24) takes the form

$$\begin{pmatrix} \mathrm{D}R(\mathbf{g})\dot{\mathbf{g}} + 2(\mathrm{tr}_{\mathbf{g}}\,\mathbf{k})\mathbf{k}^{ij}\dot{\mathbf{g}}_{ij} - 2\mathbf{k}^{ii'}\mathbf{k}^{jj'}\mathbf{g}_{i'j'}\dot{\mathbf{g}}_{ij} + 2(\mathrm{tr}_{\mathbf{g}}\,\mathbf{k})\mathbf{g}^{ij}\dot{\mathbf{k}}_{ij} - 2\mathbf{g}^{ii'}\mathbf{g}^{jj'}\mathbf{k}_{i'j'}\dot{\mathbf{k}}_{ij}\\ \nabla^i\left(\dot{\mathbf{k}}_{ij} - \mathbf{g}_{ij}\,\mathrm{tr}_{\mathbf{g}}\,\dot{\mathbf{k}}\right) - \dot{\boldsymbol{\Gamma}}^\ell{}_{ii'}\mathbf{g}^{ii'}\mathbf{k}_{\ell j} - \dot{\boldsymbol{\Gamma}}^\ell{}_{i'j}\mathbf{g}^{ii'}\mathbf{k}_{i\ell} - \partial_j(\mathbf{k}^{i\ell}\dot{\mathbf{g}}_{i\ell})\end{pmatrix}. \tag{B.0.25}$$

As in Section B, the following change of variables simplify the principal terms (recall footnote 1 in Section B for our convention for $\dot{\mathbf{g}}^{ij}$):

$$\mathbf{h}^{ij} = \dot{\mathbf{g}}^{ij} - \mathbf{g}^{ij} \operatorname{tr}_{\mathbf{g}} \dot{\mathbf{g}}, \quad \boldsymbol{\pi}^{ij} = \dot{\mathbf{k}}^{ij} - \mathbf{g}^{ij} \operatorname{tr}_{\mathbf{g}} \dot{\mathbf{k}}. \tag{B.0.26}$$

In terms of $(\mathbf{h}, \boldsymbol{\pi})$, we may rewrite (B.0.25) as

$$\mathcal{P}(\mathbf{h}, \boldsymbol{\pi}) \coloneqq \left(\nabla_i \nabla_j \mathbf{h}^{ij} + (\mathbf{C}^{(1)})_{ij} \boldsymbol{\pi}^{ij} + (\mathbf{R}^{(1)})_{ij} \mathbf{h}^{ij}, \nabla_i \boldsymbol{\pi}^{ij} + \nabla_i ((\mathbf{C}^{(2)})^{ij}{}_{k\ell} \mathbf{h}^{k\ell}) + (\mathbf{R}^{(2)})^{j}{}_{k\ell} \mathbf{h}^{k\ell}\right),$$

for some tensor fields $\mathbf{C}^{(1)}$, $\mathbf{C}^{(2)}$, $\mathbf{R}^{(1)}$, $\mathbf{R}^{(2)}$ determined by $(\mathbf{g}, \mathbf{k})$, which are all zero if $(\mathbf{g}, \mathbf{k}) = (\boldsymbol{\delta}, 0)$. The adjoint $\mathcal{P}^*$ is then given by

$$\begin{aligned}\mathcal{P}^*(\varphi, \boldsymbol{\omega}) &= \left(\mathcal{P}^*(\varphi, \boldsymbol{\omega})_{[1]ij}, \mathcal{P}^*(\varphi, \boldsymbol{\omega})_{[2]ij}\right) \\ &\coloneqq \left(\nabla_i \partial_j \varphi + (\mathbf{R}^{(1)})_{ij} \varphi + (\mathbf{R}^{(2)})^{k}{}_{ij} \boldsymbol{\omega}_k - (\mathbf{C}^{(2)})^{k\ell}{}_{ij} \nabla_k \boldsymbol{\omega}_\ell, -\frac{1}{2}(\nabla_i \boldsymbol{\omega}_j + \nabla_j \boldsymbol{\omega}_i) + (\mathbf{C}^{(1)})_{ij} \varphi\right).\end{aligned}$$

As is well-known, the kernel of $\mathcal{P}^*$ consist of Killing Initial Data sets (KIDs) [CD03].

The principal symbol of $\mathcal{P}^*$ is given by

$$p^*(x, \xi) = \begin{pmatrix} p^*_{\mathrm{d}div}(x, \xi) & 0 \\ 0 & p^*_{\mathrm{s}div}(x, \xi) \end{pmatrix}$$

where $p^*_{\mathrm{d}div}(x, \xi)$ is given by (B.0.5) and $p^*_{\mathrm{s}div}(x, \xi)$ is given by (B.0.15). By the computations in Sections B and B, we can write down an augmented system with the augmented variables

$$\boldsymbol{\alpha}_i \coloneqq \partial_i \varphi, \quad \boldsymbol{\eta}_{ij} \coloneqq \frac{1}{2}(\mathrm{d}\boldsymbol{\omega})_{ij} = \frac{1}{2}(\nabla_i \boldsymbol{\omega}_j - \nabla_j \boldsymbol{\omega}_i).$$

Since the augmented system is stable under lower order perturbations, the same ODE system also works here with extra lower order terms.

Proceeding as in Sections B and B, we obtain the following system:

$$\begin{cases} \nabla_i \varphi = \boldsymbol{\alpha}_i, \\ \nabla_i \boldsymbol{\alpha}_j = (\tilde{\mathbf{R}}^{(1)})_{ij} \varphi + (\tilde{\mathbf{R}}^{(2)})^{k}{}_{ij} \boldsymbol{\omega}_k + (\tilde{\mathbf{C}}^{(1)})^{k\ell}{}_{ij} \boldsymbol{\eta}_{k\ell} + \mathcal{P}^*_{[1]}(\varphi, \boldsymbol{\omega})_{ij} + (\tilde{\mathbf{C}}^{(4)})^{k\ell}{}_{ij} \mathcal{P}^*_{[2]}(\varphi, \boldsymbol{\omega})_{k\ell}, \\ \nabla_i \boldsymbol{\omega}_j = \boldsymbol{\eta}_{ij} + (\tilde{\mathbf{C}}^{(2)})_{ij} \varphi - \mathcal{P}^*_{[2]}(\varphi, \boldsymbol{\omega})_{ij}, \\ \nabla_i \boldsymbol{\eta}_{jk} = (\tilde{\mathbf{R}}^{(3)})_{ijk} \varphi + (\tilde{\mathbf{R}}^{(4)})^{\ell}{}_{ijk} \boldsymbol{\omega}_\ell + (\tilde{\mathbf{C}}^{(3)})^{\ell}{}_{ijk} \boldsymbol{\alpha}_\ell + \nabla_k \mathcal{P}^*_{[2]}(\varphi, \boldsymbol{\omega})_{ij} - \nabla_j \mathcal{P}^*_{[2]}(\varphi, \boldsymbol{\omega})_{ki}. \end{cases} \tag{B.0.27}$$

As before, (B.0.27) immediately leads to graded augmented variables in the sense of Definition 4.1.4. Indeed, if we specialize (B.0.27) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}^{*\mathrm{d}x}_{\mathrm{prin}}$), then we see that $\Phi_\varphi = \varphi$, $\Phi_{\boldsymbol{\alpha}_i} = \boldsymbol{\alpha}_i$, $\Phi_{\boldsymbol{\omega}_i} \coloneqq \boldsymbol{\omega}_i$, and $\Phi_{\boldsymbol{\eta}_{ij}} \coloneqq \boldsymbol{\eta}_{ij}$ define augmented variables for $\mathcal{P}_{\mathrm{prin}}$ that satisfy (Φ-1)–(Φ-4), where $\mathcal{A} = \{\varphi, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_d, \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_d, \boldsymbol{\eta}_{12}, \ldots, \boldsymbol{\eta}_{(d-1)d}\}$ (in particular, $\#\mathcal{A} = 1 + d + d + \frac{d(d-1)}{2} = \frac{(d+1)(d+2)}{2}$), $d_\varphi = d_{\boldsymbol{\omega}_i} = 0$, $d_{\boldsymbol{\alpha}_i} = d_{\boldsymbol{\eta}_{ij}} = -1$, $m_{[1]ij} = 2$, and $m'_{[1]ij} = 0$, $m_{[2]ij} = 1$, and $m'_{[2]ij} = 1$. These augmented variables also satisfy (Φ-1)–(Φ-4) for any lower order perturbations of $\mathcal{P}_{\mathrm{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

**Linearized Einstein constraint operator under constant mean curvature gauge**

The method of this chapter is also applicable to the linearized Einstein vacuum constraint equation under constant mean curvature gauge $\operatorname{tr}_{\mathbf{g}} \mathbf{k} = c$.

Introduce the new variables

$$\mathbf{h}^{ij} = \dot{\mathbf{g}}^{ij} - \mathbf{g}^{ij} \operatorname{tr}_{\mathbf{g}} \dot{\mathbf{g}}, \quad \widehat{\boldsymbol{\pi}}^{ij} = \dot{\mathbf{k}}^{ij} - \frac{1}{d}\mathbf{g}_{ij} \operatorname{tr}_{\mathbf{g}} \dot{\mathbf{k}}, \quad \rho = \operatorname{tr}_{\mathbf{g}} \dot{\mathbf{k}}.$$

The Linearized Einstein constraint operator under constant mean curvature gauge can be written as

$$\begin{aligned}\mathcal{P}(\mathbf{h}, \widehat{\boldsymbol{\pi}}, \rho) = \Big(&\nabla_i\nabla_j \mathbf{h}^{ij} + (\mathbf{C}^{(1)})_{ij}\widehat{\boldsymbol{\pi}}^{ij} + (\mathbf{R}^{(1)})_{ij}\mathbf{h}^{ij} + C^{(2)}\rho,\\ &\nabla_i\widehat{\boldsymbol{\pi}}^{ij} + (\mathbf{R}^{(2)})^j{}_{k\ell}\mathbf{h}^{k\ell} + \nabla_i((\mathbf{C}^{(3)})^{ij}{}_{k\ell}\mathbf{h}^{k\ell}) - \frac{d-1}{d}\nabla^j\rho, \rho\Big),\end{aligned}$$

for some tensor fields $\mathbf{C}^{(1)}, C^{(2)}, \mathbf{C}^{(3)}, \mathbf{R}^{(1)}, \mathbf{R}^{(2)}$ determined by $(\mathbf{g}, \mathbf{k})$.

We may use the last component $\rho$ to eliminate $\rho$ in the equations and only consider (by an abuse of notation)

$$\widehat{\mathcal{P}}(\mathbf{h}, \widehat{\boldsymbol{\pi}}) = (\nabla_i\nabla_j \mathbf{h}^{ij} + (\mathbf{C}^{(1)})_{ij}\widehat{\boldsymbol{\pi}}^{ij} + (\mathbf{R}^{(1)})_{ij}\mathbf{h}^{ij}, \nabla_i\widehat{\boldsymbol{\pi}}^{ij} + (\mathbf{R}^{(2)})^j{}_{k\ell}\mathbf{h}^{k\ell} + \nabla_i((\mathbf{C}^{(2)})^{ij}{}_{k\ell}\mathbf{h}^{k\ell})).$$

The adjoint is given by

$$\begin{aligned}&\widehat{\mathcal{P}}^*(\varphi, \boldsymbol{\omega}) = (\mathcal{P}^*(\varphi, \boldsymbol{\omega})_{[1]ij}, \mathcal{P}^*(\varphi, \boldsymbol{\omega})_{[2]ij})\\ &= \left(\nabla_i\partial_j\varphi + (\mathbf{R}^{(1)})_{ij}\varphi + (\mathbf{R}^{(2)})^k{}_{ij}\boldsymbol{\omega}_k - (\mathbf{C}^{(2)})^{k\ell}{}_{ij}\nabla_k\boldsymbol{\omega}_\ell, -\frac{1}{2}(\nabla_i\boldsymbol{\omega}_j + \nabla_j\boldsymbol{\omega}_i) + \frac{1}{d}\nabla^\ell\boldsymbol{\omega}_\ell\mathbf{g}_{ij} + (\mathbf{C}^{(1)})_{ij}\varphi\right).\end{aligned}$$

As before, the principal symbol of $\widehat{\mathcal{P}}^*$ is given by

$$p^*(x,\xi) = \begin{pmatrix} p^*_{\text{ddiv}}(x,\xi) & 0 \\ 0 & p^*_{\text{tsdiv}}(x,\xi)\end{pmatrix}$$

where $p^*_{\text{ddiv}}(x,\xi)$ is given by (B.0.5) and $p^*_{\text{tsdiv}}(x,\xi)$ is given by (B.0.20).

Let $\boldsymbol{\alpha}_i = \nabla_i\varphi$, $\boldsymbol{\eta}_{ij} = \frac{1}{2}(\nabla_j\boldsymbol{\omega}_j - \nabla_j\boldsymbol{\omega}_i)$, $w = \frac{1}{d}\nabla^\ell\boldsymbol{\omega}_\ell$ and $\boldsymbol{\zeta}_j = \partial_j w$, we have

$$\begin{aligned}\nabla_i\varphi &= \boldsymbol{\alpha}_i,\\ \nabla_i\boldsymbol{\alpha}_j &= (\tilde{\mathbf{R}}^{(1)})_{ij}\varphi + (\tilde{\mathbf{R}}^{(2)})^k{}_{ij}\boldsymbol{\omega}_k + (\tilde{\mathbf{C}}^{(1)})^{k\ell}{}_{ij}(\boldsymbol{\eta}_{k\ell} + w\mathbf{g}_{k\ell} - (\widehat{\mathcal{P}}^*(\varphi,\boldsymbol{\omega})_{[2]k\ell}) + (\widehat{\mathcal{P}}^*(\varphi,\boldsymbol{\omega}))_{[1]ij},\\ \nabla_i\boldsymbol{\omega}_j &= \boldsymbol{\eta}_{ij} + w\mathbf{g}_{ij} + (\tilde{\mathbf{C}}^{(2)})_{ij}\varphi - (\widehat{\mathcal{P}}^*(\varphi,\boldsymbol{\omega}))_{[2]ij},\\ \nabla_i\boldsymbol{\eta}_{jk} &= (\tilde{\mathbf{R}}^{(3)})_{ijk}\varphi + (\tilde{\mathbf{R}}^{(4)})^\ell{}_{ijk}\boldsymbol{\omega}_\ell + (\tilde{\mathbf{C}}^{(3)})^\ell{}_{ijk}\boldsymbol{\alpha}_\ell + \boldsymbol{\zeta}_j\mathbf{g}_{ik} - \boldsymbol{\zeta}_k\mathbf{g}_{ij} + \nabla_k(\widehat{\mathcal{P}}^*\boldsymbol{\omega})_{[2]ij} - \nabla_j(\widehat{\mathcal{P}}^*\boldsymbol{\omega})_{[2]ki},\\ \nabla_i w &= \boldsymbol{\zeta}_j,\\ \nabla_i\boldsymbol{\zeta}_j &= (\tilde{\mathbf{A}}^{(1)})_{ij}\varphi + (\tilde{\mathbf{R}}^{(5)})^k{}_{ij}\boldsymbol{\alpha}_k + (\tilde{\mathbf{A}}^{(2)})^k{}_{ij}\boldsymbol{\omega}_k + (\tilde{\mathbf{R}}^{(6)})^{k\ell}{}_{ij}\boldsymbol{\eta}_{k\ell} + (\tilde{\mathbf{R}}^{(7)})_{ij}w\\ &\quad + (\tilde{\mathbf{C}}^{(4)})^{k\ell}{}_{ij}(\widehat{\mathcal{P}}^*(\varphi,\boldsymbol{\omega}))_{[1]k\ell} + (\tilde{\mathbf{R}}^{(8)})^{k\ell}{}_{ij}(\widehat{\mathcal{P}}^*(\varphi,\boldsymbol{\omega}))_{[2]k\ell} + \frac{1}{d-1}\mathcal{C}(\widehat{\mathcal{P}}^*\boldsymbol{\omega})_{[2]ij}.\end{aligned} \tag{B.0.28}$$

As before, (B.0.28) gives graded augmented variables in the sense of Definition 4.1.4. Indeed, if we specialize (B.0.28) to the Euclidean space in rectangular coordinates (so that $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}_{\text{prin}}$ and $\widehat{\mathcal{P}}^* = \widehat{\mathcal{P}}^{*_{\mathrm{d}x}}_{\text{prin}}$), then we see that $\Phi_\varphi = \varphi$, $\Phi_{\boldsymbol{\alpha}_i} = \boldsymbol{\alpha}_i$, $\Phi_{\boldsymbol{\omega}_i} \coloneqq \boldsymbol{\omega}_i$, $\Phi_{\boldsymbol{\eta}_{ij}} \coloneqq \boldsymbol{\eta}_{ij}$, $\Phi_w = w$, and $\Phi_{\boldsymbol{\zeta}_j} = \boldsymbol{\zeta}_j$ define augmented variables for $\widehat{\mathcal{P}}_{\text{prin}}$ that satisfy ($\Phi$-1)–($\Phi$-4), where

$$\mathcal{A} = \{\varphi, \boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_d, \boldsymbol{\omega}_1, \cdots, \boldsymbol{\omega}_d, \boldsymbol{\eta}_{12}, \cdots, \boldsymbol{\eta}_{(d-1)d}, w, \boldsymbol{\zeta}_1, \cdots, \boldsymbol{\zeta}_d\}$$

and $\#\mathcal{A} = 1+d+d+\frac{d(d-1)}{2}+1+d = \frac{(d+1)(d+4)}{2}$. We have $d_\varphi = d_{\boldsymbol{\omega}_i} = 0$, $d_{\alpha_i} = d_{\boldsymbol{\eta}_{ij}} = d_w = -1$, $d_{\boldsymbol{\zeta}_j} = -2$, $m_{[1]ij} = 2$, and $m'_{[1]ij} = 0$, $m_{[2]ij} = 1$, and $m'_{[2]ij} = 2$. When $\mathbf{k} = 0$, the equations decouple to the linearized scalar curvature equation and symmetric divergence equation in the maximal gauge, studied in Section B and Section B, respectively.

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