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January 17, 1969

LAWRENCE RADIATION LABORATORY
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NOTES ON $\pi\pi$ SCATTERING. III

J PLANE PHENOMENA*

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This is the final part of a series of notes on $\pi\pi$ scattering. The first two parts are contained in UCRL-18637 and UCRL-18664.

TABLE OF CONTENTS - PART III

VII. GENERAL REMARKS

VII.A. Properties of Legendre Functions[1]

The Legendre functions, $P_\nu(z)$ and $Q_\nu(z)$, are solutions of the differential equation:

$$\frac{d}{dz}\left[(1 - z^2)\frac{du}{dz}\right] + \nu(\nu + 1)u = 0 \quad . \tag{7.1}$$

If $\nu = J$ (in this section J will always indicate a nonnegative integer) the $P_J(z)$ reduce to polynomials. The first four P_J's and Q_J's are:

$$P_0(x) = 1 \quad ,$$

$$P_1(x) = x \quad ,$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1) \quad ,$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \quad . \tag{7.2}$$

$$Q_0(x) = \frac{1}{2} \ln \frac{1 + x}{1 - x} \quad ,$$

$$Q_1(x) = \frac{x}{2} \ln \frac{1 + x}{1 - x} - 1 \quad ,$$

$$Q_2(x) = \tfrac{1}{4}(3x^2 - 1) \ln \frac{1 + x}{1 - x} - \frac{3}{2} x \quad ,$$

$$Q_3(x) = \tfrac{1}{4}(5x^3 - 3x) \ln \frac{1 + x}{1 - x} + \frac{2}{3} - \frac{5}{2} x^2 \quad . \tag{7.3}$$

The P_J's and Q_J's have the symmetry properties:

$$P_J(-z) = (-1)^J P_J(z) , \tag{7.4}$$

$$Q_J(-z) = (-1)^{J+1} Q_J(z) . \tag{7.5}$$

For a continuous index, these generalize to

$$Q_\nu(-z) = -e^{-i\pi\nu} Q_\nu(z) \quad (\text{Im } z < 0) , \tag{7.6}$$

$$Q_\nu(-z) = -e^{+i\pi\nu} Q_\nu(z) \quad (\text{Im } z > 0) , \tag{7.7}$$

$$P_\nu(-z) = e^{i\pi\nu} - \frac{2}{\pi} \sin \nu\pi \, Q_\nu(z) \quad (\text{Im } z < 0) , \tag{7.8}$$

$$P_\nu(-z) = e^{-i\pi\nu} - \frac{2}{\pi} \sin \nu\pi \, Q_\nu(z) \quad (\text{Im } z > 0) . \tag{7.9}$$

The sign of the index can be changed using the following relations:

$$Q_\nu(z) - Q_{-\nu-1}(z) = \pi \cot(\pi\nu) \, P_\nu(z) \quad (\sin \pi\nu \neq 0) , \tag{7.10}$$

$$P_\nu(z) = P_{-\nu-1}(z) . \tag{7.11}$$

When the argument is 0 or 1

$$P_\nu(1) = 1 \quad , \tag{7.12}$$

$$P_\nu(0) = -\frac{\sin \pi\nu}{\pi^{3/2}} \Gamma\left(\frac{1+\nu}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \quad , \tag{7.13}$$

$$Q_\nu(0) = \frac{1}{4\pi^{\frac{1}{2}}} (1 - \cos \pi\nu) \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) . \tag{7.14}$$

The Legendre functions can be expressed in terms of the hypergeometric functions, $F(a,b;\, c;\, z)$, by

$$Q_\nu(z) = \frac{\pi^{\frac{1}{2}} \Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})(2z)^{\nu+1}} F\left(\frac{\nu}{2} + 1, \frac{\nu}{2} + \frac{1}{2};\, \nu + \frac{3}{2};\, \frac{1}{z^2}\right) , \tag{7.15}$$

$$P_\nu(z) = \frac{\tan \pi\nu}{\pi^{\frac{1}{2}}(2z)^{\nu+1}} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})} F\left(1 + \frac{\nu}{2}, \frac{\nu+1}{2};\, \nu + \frac{3}{2};\, \frac{1}{z^2}\right)$$

$$+ \frac{(2z)^\nu}{\pi^{\frac{1}{2}}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} F\left(\frac{1-\nu}{2}, -\frac{\nu}{2};\, \frac{1}{2} - \nu;\, \frac{1}{z^2}\right) \quad . \tag{7.16}$$

For $\nu = J$, (7.16) reduces to

$$P_J(z) = \frac{\pi^{\frac{1}{2}} \Gamma(J + 1)}{(2z)^{J+1}} F\left(-J, J+1;\, 1;\, \frac{1-z}{2}\right) \quad . \tag{7.17}$$

In conjunction with the hypergeometric series

$$F(a, b;\, c;\, z) = \sum_{K=0}^{\infty} \frac{T_K(a)\, T_K(b)}{T_K(c)} \frac{z^K}{\Gamma(K + 1)} \quad , \tag{7.18}$$

$$|z| < 1$$

$$T_K(a) = \frac{\Gamma(a + K)}{\Gamma(a)} \quad , \quad \text{(see Section III.A)} \quad , \tag{3.3}$$

(7.15) and (7.16) give the asymptotic behavior in z of the Legendre functions.

$$Q_\nu(z) \underset{z\to\infty}{\sim} \frac{\pi^{\frac{1}{2}}\,\Gamma(\nu + 1)}{(2z)^{\nu+1}\;\Gamma(\nu + \frac{3}{2})} \left\{1 + \frac{(\nu + 1)(\nu + 2)}{(4\nu + 6)}\,\frac{1}{z^2} + O\!\left(\frac{1}{z^4}\right)\right\}, \tag{7.19}$$

$$P_\nu(z) \underset{z\to\infty}{\sim} \frac{\tan \pi\nu\;\Gamma(\nu + 1)}{\pi^{\frac{1}{2}}(2z)^{\nu+1}\;\Gamma(\nu + \frac{3}{2})} \left\{1 + \frac{(\nu + 1)(\nu + 2)}{(4\nu + 6)}\,\frac{1}{z^2} + O\!\left(\frac{1}{z^4}\right)\right\} , \tag{7.20a}$$

when real $\nu < -\frac{1}{2}$

$$P_\nu(z) \underset{z\to\infty}{\sim} \frac{(2z)^\nu\;\Gamma(\nu + \frac{1}{2})}{\pi^{\frac{1}{2}}\;\Gamma(\nu + 1)} \left\{1 - \frac{\nu(\tau - 1)}{(4\nu - 2)}\,\frac{1}{z^2} + O\!\left(\frac{1}{z^4}\right)\right\} , \tag{7.20b}$$

when real $\nu > -\frac{1}{2}$.

Equations (7.20) can be expressed more compactly as

$$P_\nu(z) \underset{z\to\infty}{\sim} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)}\, z^\nu \left[1 - \frac{\nu(\nu - 1)}{4\nu - 2}\, z^{-2} + \cdots\right]$$

$$+ \frac{2^{-\nu-1}}{\sqrt{\pi}}\,\frac{\Gamma(-\frac{1}{2} - \nu)}{\Gamma(-\nu)}\, z^{-\nu-1} \left[1 + \frac{(\nu + 1)(\nu + 2)}{4\nu + 6}\, z^{-2} + \cdots\right] . \tag{7.21}$$

In addition, the large ν behavior of $Q_\nu(z)$ is given by

$$Q_\nu(z) \underset{|\nu|\to\infty}{\sim} \sqrt{\frac{\pi}{2\nu(z^2-1)}} \exp\left[-\text{arct } z(\nu+\tfrac{1}{2})\right]\left[1 + 0(\tfrac{1}{\nu})\right] . \qquad (7.22)$$

The derivative of $P_\nu(z)$ with respect to ν at $\nu = 0$ is:

$$\frac{d}{d\nu} P_\nu(z)\Big|_{\nu=0} = 2\,\ln\sqrt{\frac{z-1}{2}} ,$$

$$(\text{Im } z = 0, \quad -1 \le \text{Re } z \le 1) \quad . \qquad (7.23)$$

Useful recursion relations are:

$$(z^2 - 1)\frac{d}{dz} P_\nu(z) = \nu\, z\, P_\nu(z) - \nu\, P_{\nu-1}(z) ,$$

$$(2\nu + 1) z\, P_\nu(z) = (\nu + 1)\, P_{\nu+1}(z) + \nu\, P_{\nu-1}(z) ,$$

$$(z^2 - 1)\frac{d}{dz} Q_\nu(z) = \nu\, z\, Q_\nu(z) - \nu\, Q_{\nu-1}(z) ,$$

$$(2\nu + 1) z\, Q_\nu(z) = (\nu + 1)\, Q_{\nu+1}(z) + \nu\, Q_{\nu-1}(z) . \qquad (7.24)$$

Heine's series is given by

$$\frac{1}{z - t} = \sum_{K=0}^{\infty} (2K + 1)\, P_K(t)\, Q_K(z) , \qquad (7.25)$$

which converges in z for $|t + \sqrt{t^2 - 1}| < |z + \sqrt{z^2 - 1}|$. Some useful integrals are:

$$\int_{1}^{\infty} P_\nu(z)\, Q_J(z)dz = \frac{1}{(J-\nu)(J+\nu+1)} , \qquad (7.26)$$

$$\frac{1}{2}\int_{-1}^{+1} P_\nu(-z)\, P_J(z)dz = \frac{1}{\pi}\frac{\sin \pi\nu}{(\nu - J)(\nu + J + 1)} , \qquad (7.27)$$

$$\int_{-1}^{+1} P_K(z)\, P_J(z)dz = \frac{2\,\delta_{JK}}{2J+1} , \qquad (7.28)$$

$$Q_J(z) = -\frac{1}{2}\int_{-1}^{+1} \frac{dz'}{z'-z}\, P_J(z') , \qquad (7.29)$$

$$Q_\nu(z) = \frac{1}{2}\int_{-1}^{+1} \frac{dz'}{z-z'}\, P_\nu(z') - \frac{1}{\pi}\int_{-\infty}^{-1} \sin(\pi\nu)\, \frac{Q_\nu(-z')\, dz'}{z-z'} . \qquad (7.30)$$

VII.B. Review of Regge Poles in the Scattering of Spinless Equal-Mass Particles[2]

Consider the partial wave expansion for the amplitude of the scattering of spinless equal-mass particles, [defining $a(J,s) = a_J(s)$ in (1.20)]

$$f(s,\, t) = \sum_{J=0}^{\infty} (2J+1)\, a(J,\, s)\, P_J(z) . \qquad (7.31)$$

We assume that $f(s,t)$ satisfies an unsubtracted dispersion relation, with $D_t(s,\, z)$ and $D_u(s,\, z)$ the discontinuities across the cuts in the t and u channels, respectively. Then

$$f(s, t) = \frac{1}{\pi} \int_{z_{t0}}^{\infty} \frac{D_t(s, z')dz'}{z' - z} + \frac{1}{\pi} \int_{z_{u0}}^{-\infty} \frac{D_u(s, z')dz'}{z' - z} . \tag{7.32}$$

From (7.31) we have

$$a(J, s) = \frac{1}{2} \int_{-1}^{+1} dz\, P_J(z)\, f(s, t) , \tag{7.33}$$

which can be manipulated, using (7.4), (7.29), and (7.32) to yield

$$a(J, s) = \frac{1}{\pi} \int_{z_0}^{\infty} dz'\, Q_J(z')[D_t(s, z') + (-1)^J D_u(s, z')] . \tag{7.34}$$

The factor $(-1)^J$ in (7.34) is ill-suited for analytic continuation so we define the amplitudes of definite signature

$$a^{\pm}(J, s) = \frac{1}{\pi} \int_{z_0}^{\infty} dz'\, Q_J(z')[D_t(s, z') \pm D_u(s, z')] . \tag{7.35}$$

In (7.34) and (7.35) $z_0 = \min(|z_{to}|, |z_{uo}|)$. The physical amplitude is then given in terms of the definite signature amplitudes by

$$f(s, z) = \frac{1}{2}[f^+(s,z) + f^+(s,-z) + f^-(s,z) - f^-(s,-z)] . \tag{7.36}$$

In view of (7.4),

$$a^+(J, s) = a(J, s) \quad \text{for even } J ,$$

$$a^-(J, s) = a(J, s) \quad \text{for odd } J . \tag{7.37}$$

It is convenient to refer to these as the "physical" values of J for the partial wave amplitudes of definite signature.

Taking the analog of (7.31), the partial wave expansion, for the amplitudes of definite signature is

$$f^\pm(s, t) = \sum_{J=0}^{\infty} (2J + 1)\, a^\pm(J, s)\, P_J(z) . \tag{7.38}$$

We write this sum as a contour integral over the contour, C, shown in Fig. 7.1

$$f^\pm(s, t) = -\frac{1}{2i} \int_C dJ(2J + 1) \frac{a^\pm(J, s)\, P_J(-z)}{\sin \pi J} . \tag{7.39}$$

Up to this point $s \geq 4\mu^2$. To get the t-channel asymptotic behavior ($|z_s| \to \infty$) we want $s \leq 0$. (See Fig. 7.2.) If the lowest (u, t) singularities occur at (u_0, t_0) we can rewrite (7.35) as

$$a^\pm(J, s) = \frac{1}{\pi} \int_T^\infty \frac{dt}{2q^2} Q_J(1 + \frac{t}{2q^2})[D_t(s,t) \pm D_u(s,t)] , \tag{7.40}$$

where $q^2 = \frac{1}{4}(s - 4\mu^2)$, $z = 1 + \frac{2t}{s - 4\mu^2}$, $T = \min(u_0, t_0)$. We now define

$$g_J^{\pm}(s) = \frac{a^{\pm}(J, s)}{q^{2J}} \quad . \tag{7.41}$$

The new quantities $g_J^{\pm}$ are real in the interval $4\mu^2 - T < s < 4\mu^2$ because on the cut $-1 \leq \mathrm{Re}\, z \leq +1$ we have from (7.6), (7.7),

$$Q_\nu(z) = \frac{1}{2}[Q_\nu(z + i\epsilon) + Q_\nu(z - i\epsilon)] \quad , \tag{7.42}$$

so that $Q_J(1 + \frac{t}{2q^2})/q^{2J}$ is real on the interval $-\frac{1}{4} T < q^2 < 0$ and $g_J^{\pm}(s)$ is real analytic in s except for cuts: $-\infty < s < 4\mu^2 - T$; $4\mu^2 < s < \infty$. Finally, we have

$$a^{\pm}(J, s)\, P_J\left(1 + \frac{t}{2q^2}\right) = q^{2J}\, g_J^{\pm}(s)\, P_J\left(1 + \frac{t}{2q^2}\right)$$

$$= (-q^2)^J\, g_J^{\pm}(s)\, P_J\left(-1 - \frac{t}{2q^2}\right) , \tag{7.43}$$

and the deformation of the contour in Fig. 7.1 gives contributions from Regge poles in $g_J^{\pm}(s)$ and a background integral along $\mathrm{Re}\, J = -\frac{1}{2}$.

$$f^{\pm}(s, t) = -\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dJ \left\{(2J + 1)\, g_J^{\pm}(s)\, \frac{(-q^2)^J\, P_J\left(-1 - \frac{t}{2q^2}\right)}{\sin \pi J}\right.$$

$$+ \sum_{\substack{K \\ \text{(poles)}}} [2\, \alpha_K^{\pm}(s) + 1]\, \frac{P_{\alpha_K}^{\ \pm}(s)\left(-1 - \frac{t}{2q^2}\right)}{\sin \pi\alpha_K^{\pm}(s)}\, \beta_K^{\pm}(s) \quad . \quad (7.44)$$

So far we have restricted ourselves to the region $\text{Re } J \geqslant -\frac{1}{2}$. The line $\text{Re } J = -\frac{1}{2}$ is a natural boundary as long as we express $f^{\pm}(s, t)$ in terms of P_J's. From (7.21) we see that the asymptotic behavior of $P_J(z)$ changes rather completely for $\text{Re } J < -\frac{1}{2}$, so that it is useless to push the contour further to the left with the integrand in its present form, since the integral will become, asymptotically, very large.

To proceed, we follow Mandelstam[3] and write

$$f(s, t) = \sum_{J=0}^{\infty} (2J + 1)\, a(J, s)\, P_J(z) + \frac{1}{\pi} \sum_{J=0}^{\infty} (-1)^{J-1}\, 2J\, a(J - \tfrac{1}{2}, s)$$

$$\cdot\, Q_{J-\frac{1}{2}}(z) - \frac{1}{\pi} \sum_{J=0}^{\infty} (-1)^{J-1}\, 2J\, a(J - \tfrac{1}{2}, s)\, Q_{J-\frac{1}{2}}(z) \quad . \quad (7.45)$$

This particular addition and subtraction is chosen, because, as observed by Mandelstam, (7.10) implies that the functions

$$R_J(z) \equiv -\frac{1}{\pi} Q_{-J-1}(z) \tan \pi J \quad , \tag{7.46}$$

have the property that $R_J(z) = P_J(z)$ for positive integral J, while the asymptotic expansion of $R_J(z)$ for large z contains only the first term in (7.21).[4]

Now the first two sums in (7.45) can be transformed in the same way as before, with $P_J(-z)/\sin \pi J$ as in (7.44) replaced by $P_J(-z)/\sin \pi J - Q_J(-z)/\pi \cos \pi J = -\frac{R_J(-z)}{\sin \pi J}$. The new result for the amplitude is

$$f(s, t) = -\frac{1}{2i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dJ(2J + 1)\, a(J,s)\, R_J(-z)/\sin \pi J$$

$$+ \sum_{\substack{K \\ \text{(poles)}}} \left(2\alpha_K(s) + 1\right) \beta_K(s)\, R_{\alpha_K(s)}(-z)/\sin \pi\alpha_K(s)$$

$$- \frac{1}{\pi} \sum_{J=0}^{\infty} (-1)^{J-1}\, 2J\, a(J - \tfrac{1}{2}, s)\, Q_{J-\frac{1}{2}}(z) \quad . \tag{7.47}$$

The functions $-R_J(-z)/\sin \pi J$ have poles at negative half-integral $J = -N - \frac{1}{2}$, with residues $(-1)^{N-1}/\pi^2 \cdot Q_{N-\frac{1}{2}}(z)$. As we move the contour back behind $\mathrm{Re}\, J = -\frac{1}{2}$, to $\mathrm{Re}\, J = -L$, where $-N - \frac{1}{2} < -L < -N + \frac{1}{2}$, (7.47) becomes

$$f(s, t) = -\frac{1}{2i} \int_{-L-i\infty}^{-L+i\infty} dJ(2J+1)\, a(J, s)\, R_J(-z)/\sin \pi J$$

$$+ \sum_K (2\alpha_K(s) + 1)\, \beta_K(s)\, R_{\alpha_K(s)}(-z)/\sin \pi\alpha_K(s)$$

[poles - Re $\alpha_K > -L$.]

$$- \frac{1}{\pi} \sum_{J=0}^{\infty} (-1)^{J-1}\, 2J\, a(J - \tfrac{1}{2}, s)\, Q_{J-\frac{1}{2}}(z)$$

$$+ \frac{1}{\pi} \sum_{J=0}^{N-1} (-1)^{J-1}\, 2J\, a(-J - \tfrac{1}{2}, s)\, Q_{J-\frac{1}{2}}(z) \quad , \qquad (7.48)$$

where the last sum arises from the poles at half-integral negative J mentioned above, and we have used $Q_J = Q_{-J-1}$ for half-integral J.

If the partial-wave amplitudes satisfy Mandelstam symmetry[3] for J half integral, $a(-J - \frac{1}{2}, s) = a(J - \frac{1}{2}, s)$ and the last sum cancels against part of the next to last sum leaving $\sum_{J=N}^{\infty}$. (The pole at $J = -\frac{1}{2}$ is cancelled by the $2J + 1$ factor.) If Mandelstam symmetry is not satisfied, there will be fixed poles at negative half-integral $J \leq -\frac{3}{2}$.

A similar phenomenon must occur for the sum over Regge poles in (7.48). Either $\beta_K = 0$ when α_K = half-integer, or trajectories occur in pairs (α, α') about $J = -\frac{1}{2}$, and at $\alpha_K = J_0$ (a half-integer), $\alpha_K' = -J_0 - 1$, with $\beta_K = \beta_K'$ at this point.

As first shown by Gribov and Pomeranchuk,[5,4] the third Mandelstam double spectral function ρ'_{tu} gives rise to fixed singularities at J = wrong signature negative integers in $a(J, t)$. That this is a possibility can be seen from the Froissart-Gribov definition (7.35) and the fact that $Q_J(z)$ has poles at negative integers. As we see easily from (7.10), near $J = -N$,

$$Q_J(z) \approx \frac{P_{N-1}(z)}{J + N} \quad . \tag{7.49}$$

VIII. J-PLANE BEHAVIOR OF VENEZIANO[6] MODEL FOR $\pi\pi \to \pi\pi$

Independent of experimental comparisons, the Veneziano functions have educational value as textbook examples of scattering amplitudes containing infinite families of Regge poles. They can be used to illustrate many of the elementary properties discussed in Section VII. For example, the Froissart-Gribov partial-wave amplitudes associated with Veneziano functions can be calculated explicitly and their properties investigated in detail.

VIII.A. Partial Fraction Expansions

Throughout Section VIII we will remain in the t-channel unless otherwise noted. In order to partial wave analyze our expressions for the t-channel amplitudes we will need the two basic Mittag-Leffler expansions:

$$F_0[x, y] = \sum_{K=1}^{\infty} \frac{\Gamma(K + y)}{\Gamma(K)\,\Gamma(y)} \frac{1}{x - K} , \tag{3.1}$$

valid for $\text{Re } y < 0$, and, defining $\tau = 1 - x - y$ $\nu = \frac{1}{2}(x - y)$,

$$F_0[x, y] = \sum_{K=1}^{\infty} \frac{(-1)^K\,\Gamma(K + \tau)}{\Gamma(\tau)\,\Gamma(K)} \left\{ \frac{1}{\nu + \frac{1}{2}(1 - \tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1 - \tau) - K} \right\}, \tag{8.1}$$

which converges absolutely for $\text{Re } \tau < 1$.

In making a partial fractions expansion of a function as in (3.1) and (8.1), we are simply expressing it in terms of poles. Care must be taken to insure that we are not neglecting an entire function. We will show below that (3.1) and (8.1) are valid as they stand and will indicate briefly how they can be modified to extend their range of validity in y and τ, respectively.

To prove (3.1) we use Gauss' theorem for the hypergeometric function[7]

$$F(a,b;\ c;\ 1) = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)} , \tag{8.2}$$

which holds for $\mathrm{Re}\ [c - a - b] > 0$. We remind the reader that F satisfies the hypergeometric equation

$$z(1-z)u'' + [c - (a+b+1)z]u' - abu = 0 , \tag{8.3}$$

and is defined by the hypergeometric series,

$$F(a,b;\ c;\ z) = \sum_{K=0}^{\infty} \frac{T_K(a)\ T_K(b)}{T_K(c)\ \Gamma(K+1)}\ z^K . \tag{8.4}$$

The series converges for $|z| < 1$. For $|z| = 1$ the series is

(a) divergent for $\mathrm{Re}(a + b - c) \geq 1$

(b) absolutely convergent for $\mathrm{Re}(a + b - c) < 0$

(c) conditionally convergent for $0 \leq \mathrm{Re}(a + b - c) < 1$, the point $z = 1$ being excluded.

To verify (3.1) we note that

$$\sum_{K=1}^{\infty} \frac{\Gamma(K + y)}{\Gamma(K)\ \Gamma(y)}\ \frac{1}{x - K} = \sum_{m=0}^{\infty} \frac{\Gamma(m + 1 + y)}{\Gamma(y)\ \Gamma(m + 1)}\ (-1)\ \frac{\Gamma(m + 1 - x)}{\Gamma(m + 2 - x)}\ , \tag{8.5}$$

$$\text{(by 8.4)} \qquad = -\frac{\Gamma(1 + y)\ \Gamma(1 - x)}{\Gamma(2 - x)\ \Gamma(y)}\ F(1 + y, 1 - x; 2 - x; 1)$$

$$\text{(by 8.2)} \qquad = (-y)\ \frac{\Gamma(1 - x)\ \Gamma(-y)}{\Gamma(1 - x - y)}$$

$$= F_0[x, y]\ ,$$

whenever $\text{Re}\ y < 0$.

Verification of (8.1) is more involved since both sets of poles have been simultaneously exhibited.

In the notation of (8.1) we write

$$F_0[x, y] = \tau\ B[-\nu + \tfrac{1}{2}(1 + \tau),\ + \nu + \tfrac{1}{2}(1 + \tau)]\ . \tag{8.6}$$

Making use of the integral representation of the Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\ dt$$

$$= \int_0^1 (1 + t)^{-x-y} \left\{ t^{x-1} + t^{y-1} \right\} dt\ , \tag{8.7}$$

and of the hypergeometric function

$$F(a,b;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(b)\ \Gamma(c-b)} \int_0^1 dt\ t^{b-1}(1-t)^{c-b-1}\ (1-tz)^{-a}\ , \tag{8.8}$$

we have

$$B[-\nu + \tfrac{1}{2}(1+\tau),\ \nu + \tfrac{1}{2}(1+\tau)]$$

$$= \frac{\Gamma\left(-\nu + \frac{1}{2}(1+\tau)\right)}{\Gamma\left(-\nu + \frac{1}{2}(1+\tau) + 1\right)} F(\tau + 1,\ -\nu + \tfrac{1}{2}(1+\tau);\ -\nu + \tfrac{1}{2}(1+\tau) + 1;\ -1)$$

$$+ \frac{\Gamma\left(\nu + \frac{1}{2}(1+\tau)\right)}{\Gamma\left(\nu + \frac{1}{2}(1+\tau) + 1\right)} F(\tau + 1,\ \nu + \tfrac{1}{2}(1+\tau);\ \nu + \tfrac{1}{2}(1+\tau) + 1;\ -1)$$

$$= \frac{1}{\Gamma(\tau+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\tau+1+n)}{\Gamma(n+1)} (-1)^n \left\{\frac{1}{-\nu + \frac{1}{2}(1+\tau) + n} + \frac{1}{\nu + \frac{1}{2}(1+\tau) + n}\right\}, \tag{8.9}$$

and letting $K = n + 1$, (8.1) follows. From the properties of F at $z = -1$ listed above, we see the series in (8.1) is absolutely convergent for $\mathrm{Re}\ \tau < 0$, and conditionally convergent for $0 < \mathrm{Re}\ \tau < 1$.

Some limitations on the use of partial fraction expansions is now evident from their derivation. When we use the expression (3.1) for $F_0(x,y)$ in terms of the poles in one variable there are no poles explicitly present in the other variable. These poles appear in a region where the series diverges. By examining the asymptotic form of the terms in (3.1), we can isolate the divergence, recapture the first pole and calculate its residue. The large K asymptotic expansion of $\Gamma(x+K)/\Gamma(K)$, [cf (3.14)], can be combined with

$$\frac{1}{x-K} = -\frac{1}{K}\sum_{n=0}^{\infty}\left(\frac{x}{K}\right)^n \quad (|x| < |K|) \quad , \tag{8.10}$$

to give

$$F_0(x,y) \approx -\frac{1}{\Gamma(y)}\sum_{K=1}^{\infty}\left\{K^y + \tfrac{1}{2}y(y-1)K^{y-1} + O(K^{y-2})\right\} \cdot \left\{K^{-1} + xK^{-2} + O(K^{-3})\right\} \quad , \tag{8.11}$$

$$F_0(x,y) \approx -\frac{1}{\Gamma(y)}\left\{\zeta(1-y) + [\tfrac{1}{2}y(y-1) + x]\,\zeta(2-y) + \cdots\right\} \quad . \tag{8.12}$$

The first Riemann zeta function has a pole at $y = 0$ which is cancelled by the zero of $1/\Gamma(y)$ at that point. The second zeta function has a pole at $y = 1$ with the residue

$$\mathrm{res}\left(F_0(x,y)\right)\Big|_{y=1} = x \quad . \tag{8.13}$$

This is exactly the residue we would find for this pole in the Mittag-Leffler expansion (3.1) for $F_0(x,y)$ in the y variable. In order to get a series which does not diverge for $y > 1$ we must subtract this divergence, and therby analytically continue the sum. If we let $y = P + a$ where P is a positive integer, and $a < 1$, we have

$$\frac{\Gamma(1-x)\,\Gamma(1-P-a)}{\Gamma(1-x-P-a)} = c_P(x)\,\frac{\Gamma(1-x)\,\Gamma(1-a)}{\Gamma(1-x-a)} , \tag{8.14}$$

$$= \sum_{K=1}^{\infty} \frac{\Gamma(K+a)}{\Gamma(K)\,\Gamma(a)}\,\frac{c_P(x)}{x-K} , \tag{8.15}$$

where, suppressing the a dependence,

$$c_P(x) = \prod_{n=0}^{P-1} \left(1 + \frac{x}{a+n}\right) , \tag{8.16}$$

$$\frac{\Gamma(1-x)\,\Gamma(1-y)}{\Gamma(1-x-y)} = \sum_{K=1}^{\infty} \left\{\frac{\Gamma(K+y)}{\Gamma(K)\,\Gamma(y)}\,\frac{1}{x-K} + D_K(x)\right\} . \tag{8.17}$$

$$D_K(x) = \frac{\Gamma(a+K)}{\Gamma(a)\,\Gamma(K)} \left\{\frac{c_P(x) - c(K)}{x-K}\right\} , \tag{8.18}$$

a polynomial of degree $P - 1$, is the subtraction term necessary for the series (8.17) to be valid for $\mathrm{Re}\, y < P+1$. In an analogous fashion we analyze $\frac{\Gamma(x)\,\Gamma(y-x)}{\Gamma(y)}$, letting $y = P + a$

$$\frac{\Gamma(x)\,\Gamma(P+a-x)}{\Gamma(P+a)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)}\,\frac{\Gamma(a+n)}{\Gamma(a)}\,\frac{c_P(-x)}{x+n}$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)}\,\frac{\Gamma(a+n)}{\Gamma(a)}\,\frac{c_P(-x)}{x-a-n} . \tag{8.19}$$

The first $P - 1$ terms of the second sum in (8.19) have no poles. If we let

$$E_P(x) = \sum_{n=0}^{P-1} \frac{(-1)^n}{\Gamma(n+1)} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{c_P(-x)}{x-a-n} , \qquad (8.20)$$

we have

$$\frac{\Gamma(x)\,\Gamma(y-x)}{\Gamma(y)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{c_P(-x)}{x+n}$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^{P+m}}{\Gamma(m+P+1)} \frac{\Gamma(a+P+m)}{\Gamma(a)} \frac{c_P(-x)}{x-y-m} + E_P(x) . \qquad (8.21)$$

Writing

$$(-1)^n \frac{\Gamma(a+n)}{\Gamma(n+1)} \frac{c_P(-x)}{x+n}$$

$$= (-1)^n \frac{\Gamma(y+n)}{\Gamma(n+1)\,\Gamma(y)} \frac{1}{x+n} + (-1)^n \frac{\Gamma(a+n)}{\Gamma(n+1)\,\Gamma(a)} \left\{ \frac{c_P(-x) - c(n)}{x+n} \right\}, \qquad (8.21a)$$

and similarly for the second sum in (8.21), we have

$$\frac{\Gamma(x)\,\Gamma(y-x)}{\Gamma(y)} = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{\Gamma(y+n)}{\Gamma(n+1)\,\Gamma(y)} \frac{1}{x+n} + D_n'(x) \right\}$$

$$+ \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{\Gamma(y+n)}{\Gamma(n+1)\,\Gamma(y)} \frac{1}{-x+a+n} + D_n''(x) \right\} + E_P(x) , \qquad (8.22)$$

where D_n', D_n'' are polynomials defined by (8.21a).

Identifying

$$x \to -\nu + \tfrac{1}{2}(1 + \tau) \quad , \tag{8.23a}$$

$$y \to (1 + \tau) \quad , \tag{8.23b}$$

we have the result which extends the validity of (8.1) into the region Re $\tau < P + 1$. For most applications, however, we will be content to use (3.1) and (8.1) as they stand by restricting our attention to regions in which they hold.

VIII.B. Froissart-Gribov Partial Wave Amplitudes

We consider the t-channel amplitudes

$$A_2{}^t(\nu,\tau) = g \sum_{K=1}^{\infty} \frac{(-1)^K T_K(\tau)}{\Gamma(K)} \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right] , \tag{8.31}$$

$$A_1{}^t(\nu,\tau) = g \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} - \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right] , \tag{8.32}$$

$$a(\nu,\tau) = \frac{2}{3} A_0{}^t + \frac{1}{3} A_2{}^t = g \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right] . \tag{8.33}$$

As discussed in VIII.A., we can find regions in τ where these expressions can be used to give us the discontinuities $D_s(\tau,z)$ and $D_u(\tau,z)$ in (7.32). From (7.34) and (7.35) we can then compute the Froissart-Gribov partial wave amplitudes corresponding to (8.31)-(8.33). We have

$$a_2^-(J,\tau) = 0 \quad , \tag{8.34}$$

$$a_2^+(J,\tau) = 2g \sum_{K=1}^{\infty} \frac{(-1)^K T_K(\tau)}{\Gamma(K)} Q_J\left(1 + \frac{2K-1}{\tau}\right) \quad , \tag{8.35}$$

$$a_1^+(J,\tau) = 0 \quad , \tag{8.36}$$

$$a_1^-(J,\tau) = 2g \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} Q_J\left(1 + \frac{2K-1}{\tau}\right) \quad . \tag{8.37}$$

Corresponding to $a(\nu,\tau)$ in (8.33) we find the partial-wave amplitudes $a_0^{\pm}(J,\tau)$.

$$a_0^-(J,\tau) = 0 \quad , \tag{8.38}$$

$$a_0^+(J,\tau) = 2g \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} Q_J\left(1 + \frac{2K-1}{\tau}\right) = a_1^-(J,\tau) \quad . \tag{8.39}$$

In computing $a_i^{\pm}$ we have introduced $z = \frac{2\nu}{\tau}$ which is $\cos\theta_t$ if $\mu = 0$, $b = 1\ \mathrm{BeV}^{-2}$, and $a = \frac{1}{2}$.

VIII.C. Location of Moving Poles in the J-Plane

Regge poles appear as divergences of the sums, (8.35) and (8.37), which define the partial-wave amplitudes. The $I = 1$ Regge poles can be found by examining the large K behavior of the sum and of (8.37). Letting $\alpha \equiv \tau + \frac{1}{2}$, and using (3.14), (7.15, 7.16) we have

$$a_1^-(J,\tau) \sim \frac{g}{2\sqrt{\pi}} \frac{1}{[4/\tau]^J} \frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})\,\Gamma(\tau+\frac{1}{2})} \sum_{K=1}^{\infty} \left\{K^{\alpha} + \frac{1}{2}\alpha(\alpha-1)K^{\alpha-1} + \cdots\right\}$$

$$\cdot \left\{K^{-J-1} - (J+1)\,\frac{1}{2}(\alpha - \tfrac{3}{2})\,K^{-J-2} + \cdots\right\} , \qquad (8.40)$$

which can be summed to give a series of Riemann zeta functions, $\zeta(z)$.

$$\zeta(z) = \sum_{K=1}^{\infty} K^{-z} . \qquad (8.41)$$

We have

$$a_1^-(J,\tau) \sim \frac{g}{2\sqrt{\pi}} \frac{1}{(4/\tau)^J} \frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})\,\Gamma(\alpha)} \sum_{n=1}^{\infty} b_n(J,\alpha)\,\zeta(n+J-\alpha) . \qquad (8.42)$$

Since the only singularity of $\zeta(z)$ is a simple pole of unit residue at $z = 1$, poles occur in (8.42) at $J = \alpha, \alpha - 1, \alpha - 2, \cdots$.

In principle we can compute any number of Regge residues, but in practice the procedure of combining the two asymptotic expansions becomes cumbersome very quickly. The first two residues are

$$(J = \alpha)\ \beta_0(\tau) = \frac{g}{2\sqrt{\pi}} \frac{\alpha(\frac{4}{\tau})^{-\alpha}}{\Gamma(\alpha + \frac{3}{2})} , \quad (8.43)$$

$$(J = \alpha - 1)\ \beta_1(\tau) = \frac{g}{2\sqrt{\pi}} \frac{(\frac{4}{\tau})^{-\alpha+1}}{\Gamma(\alpha + \frac{1}{2})} \left\{ \frac{1}{2}\alpha(\alpha - 1) - \frac{\alpha(\tau - 1)}{2} \right\}$$

$$= \frac{g}{8\sqrt{\pi}} \frac{\alpha(\frac{4}{\tau})^{-\alpha+1}}{\Gamma(\alpha + \frac{1}{2})} . \quad (8.44)$$

These residues are of the form predicted for linearly rising trajectories by Mandelstam.[12] In addition to the threshold factor $(\frac{4}{\tau})^{-\alpha} = q^{2\alpha}$, in (8.43), note the zeros which appear at the negative half-integers as they must, due to Mandelstam symmetry and the absence of compensating trajectories.[3]

If we apply the same methods to (8.35) we find that there are, as expected, no $I = 2$ Regge poles. Although (8.35) converges absolutely only for $J > \tau$ we can use

$$\sum_{K=1}^{\infty} (-1)^K K^{-z} = (2^{1-z} - 1)\ \zeta(z) , \quad (8.45)$$

to verify the absence of moving poles. The conditional convergence of (8.35) caused by the factor $(-1)^K$ eliminates the kind of poles appearing in (8.40).

VIII.D. Fixed Poles in the J Plane

As noted above in (7.49) the $Q_J(z)$ have poles at negative integral J. Let us see whether these appear in our Froissart-Gribov amplitudes.

The $I = 2$ poles, from (7.49) and (8.35) have residues

$$\gamma_2^{(N)} = 2 \sum_{K=1}^{\infty} \frac{(-1)^K T_K(\tau)}{\Gamma(K)} P_{N-1}\left(1 + \frac{2K - 1}{\tau}\right) , \tag{8.50}$$

while the $I = 1$ poles have residues

$$\gamma_1^{(N)} = 2 \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} P_{N-1}\left(1 + \frac{2K - 1}{\tau}\right) . \tag{8.51}$$

In (8.50) and (8.51) the poles have positions $J = -N$. $(N = 1,2,\cdots .)$

In fact, $\gamma_1(N) = 0$. The proof goes as follows. Set $\alpha \equiv \tau + \frac{1}{2}$, and choose $\alpha < -N$. Then (8.51) converges and if it is zero for $\alpha < -N$, analytic continuation tells us it is zero everywhere. (There can never be a barrier of singularities because the zeta function has only a single pole.)

The sum in (8.51) is zero for $\alpha < -N$. Define (for $\alpha < -N$)

$$F(\alpha, N, P) = \sum_{K=1}^{P}{}' \frac{K^{N-1} T_K(\alpha)}{\Gamma(K)} . \tag{8.52}$$

By induction it follows easily that

$$F(\alpha, 1, P) = \frac{T_{P+1}(\alpha)}{(\alpha + 1)\,\Gamma(P)} \quad , \tag{8.53}$$

and we then have

$$\lim_{P\to\infty} F(\alpha, 1, P) = \frac{P^{\alpha+1}}{(\alpha + 1)\,\Gamma(\alpha)} \to 0 \quad . \tag{8.54}$$

Now

$$F(\alpha, 2, P) = \sum_{K=1}^{P} \frac{K\; T_K(\alpha)}{\Gamma(K)} = \alpha[F(\alpha + 1, 0, P) - F(\alpha, 0, P)] \quad , \tag{8.55}$$

and therefore

$$\lim_{P\to\infty} F(\alpha, 2, P) \sim P^{\alpha+2} \to 0 \quad (\alpha + 2 < 0) \quad . \tag{8.56}$$

This procedure can be extended to arbitrary N, which gives us the required result.

Similarly, we define, to study (8.50),

$$G(x, N, P) = \sum_{K=1}^{P} \frac{(-1)^K T_K(x) K^{N-1}}{\Gamma(K)} . \tag{8.57}$$

Just as for the F's above, the study of $G(x, N, P)$ can be reduced to that of $G(x, 1, P)$, but here $G(x, 1, \infty) \neq 0$. In fact we have

$$G(x, 1, \infty) = \frac{x \sin \pi x}{\pi} \sum_{K=1}^{\infty} (-1)^K B(-x, m + 1 + x) , \tag{8.58}$$

$$= \frac{x \sin \pi x}{\pi} \int_0^1 \sum_{m=0}^{\infty} (-1)^m t^{m+x}(1 - t)^{-x-1}$$

$$= \frac{x \sin \pi x}{\pi} \int_0^1 dt\, t^x (1 - t)^{-x-1} (1 + t)^{-1} , \tag{8.59}$$

Using the result

$$\int_0^1 t^{a-1} (1 - t)^{b-1} (1 + t)^{-b-a} dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a + b)} 2^{-a} , \tag{8.591}$$

we get

$$G(x, 1, \infty) = -x\, 2^{-(x+1)} . \tag{8.592}$$

Therefore the $I = 0$ and $I = 2$ amplitudes contain fixed poles, whose residues are functions of τ. The poles manifest themselves in the amplitudes a_0^+ and a_2^+ odd integer (wrong signature) values of J, and hence from (7.36) and the symmetry properties of the $P_K(z)$ we can see they give no contribution to the asymptotic behavior of the physical amplitudes.[18]

VIII.E. Asymptotic Behavior of $\beta(t)$

If we use the Stirling approximation for $\Gamma(z)$, (8.43) yields

$$\beta_0(\tau) \xrightarrow[\tau \to \infty]{} \frac{1}{4\pi} e^{\frac{3}{2}} \left(\frac{4}{e}\right)^{-\alpha(\tau)} \sim e^{-\alpha \ell n(\frac{4}{e})} , \tag{8.60}$$

and as τ goes to ∞ along a wedge near the negative real axis we get an exponential blowup. Since $\beta_0(\tau)$ is an analytic function of τ, and has an infinite string of zeros at $\alpha = -\frac{3}{2}, -\frac{5}{2} \cdots$ Carlson's theorem tells us this exponential blowup must occur, as pointed out by Jones and Teplitz.[13]

Jones and Teplitz remark that in a theory with infinitely rising trajectories one of the following set of assumptions, considered in a related context by Khuri,[14] must fail:

(i) The amplitude $A(s,t)$ is analytic in the cut s plane and is bounded for fixed t by

$$f(s) = c \exp(|s|^{\frac{1}{2}-\epsilon}); \tag{8.61}$$

(ii) $A(s,z)$ is bounded by $f(s)$ for fixed z;

(iii) The Sommerfeld-Watson transformation of the partial-wave amplitudes $a(J,s)$ exists, and $a(J,s)$ is bounded by $f(s)$ for fixed J;

(iv) $\alpha(s)$ and $\beta(s)$ are analytic with a single cut from $s = 4\mu^2$ to ∞, $\alpha(s)$ is polynomial bounded, and $\beta(s)$ is bounded by $f(s)$.

Let us recheck (i), using the partial fractions expansion (3.1). We have with $P < |x| < P + 1$

$$\lim_{\substack{x \to -\infty \\ \text{fixed } y}} \sum_{K=1}^{\infty} \frac{\Gamma(K + x)}{\Gamma(K)\,\Gamma(x)} \frac{1}{y - K}$$

$$= \lim_{P \to \infty} \left(\sum_{K=1}^{P} + \sum_{K=P+1}^{\infty} \right) \frac{\Gamma(K + x)}{\Gamma(K)\,\Gamma(x)} \frac{1}{y - K}$$

$$\cong \lim_{P \to \infty} \sum_{K=1}^{P} \frac{x^K}{\Gamma(K)} \frac{1}{y - K} + \lim_{P \to \infty} \sum_{K=P+1}^{\infty} \frac{K^x}{\Gamma(x)} \frac{1}{y - K}$$

$$\cong \lim_{P \to \infty} \sum_{K=1}^{P} \frac{x^K}{\Gamma(K)} \frac{1}{y - K} . \tag{8.62}$$

The incomplete Γ function satisfies

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha + n)} . \qquad (8.63)$$

If we set $-y = -1 + \alpha$, analytically continuing to $\mathrm{Re}\, \alpha < 0$, we have

$$e^{i\pi y} x^y \gamma(1 - y, -x) = \sum_{K=1}^{\infty} \frac{x^K}{\Gamma(K)} \frac{1}{y - K} , \qquad (8.64)$$

and using $\lim_{x \to -\infty} \gamma(1 - y, -x) = \Gamma(1 - y)$ we get the limit we want

$$\lim_{\substack{x \to -\infty \\ \text{fixed } y}} \sum_{K=1}^{\infty} \frac{\Gamma(K + x)}{\Gamma(K)\, \Gamma(x)} \frac{1}{y - K} \sim e^{-i\pi y} x^y \Gamma(1 - y) , \qquad (8.65)$$

which matches with (3.19).

Let us repeat the calculation, switching x and y

$(P \leqslant x \leqslant P + 1)$ $(y < 0)$

$$\lim_{\substack{x\to\infty \\ \text{fixed } y}} \sum_{K=1}^{P} \frac{\Gamma(K+y)}{\Gamma(K)\Gamma(y)} \frac{1}{x-K} = \lim_{P\to\infty} \left(\sum_{K=1}^{P} + \sum_{K=P+1}^{\infty} \right) \frac{\Gamma(K+y)}{\Gamma(K)\Gamma(y)} \frac{1}{x-K}$$

$$= \lim_{P\to\infty} \sum_{K=1}^{P} \frac{\Gamma(K+y)}{\Gamma(K)\Gamma(y)} \frac{1}{x} \sum_{L=0}^{\infty} \left(\frac{K}{x}\right)^{L} + \lim_{P\to\infty} \sum_{K=P+1}^{\infty} \frac{\Gamma(K+y)}{\Gamma(K)\,\Gamma(y)}$$

$$\left(-\frac{1}{K}\right) \sum_{L=0}^{\infty} \left(\frac{x}{K}\right)^{L} = \lim_{P\to\infty} \sum_{L=0}^{\infty} x^{-L-1}\, F(y,\, L+1,\, P)$$

$$- \lim_{P\to\infty} \sum_{L=0}^{\infty} \sum_{K=P+1}^{\infty} \frac{\Gamma(K+y)}{\Gamma(K)\,\Gamma(y)} \frac{x^{L}}{K^{L+1}} \cong -\lim_{P\to\infty} \sum_{L=0}^{\infty} x^{L} \sum_{K=P+1}^{\infty} \frac{K^{y-L-1}}{\Gamma(y)}$$

$$= \frac{1}{\Gamma(y)} \sum_{L=0}^{\infty} \frac{x^{L}}{y-L} = \frac{1}{y\,\Gamma(y)} F(1,\, -y;\, 1-y;\, x) \xrightarrow[x\to\infty]{} \frac{\pi\, e^{-i\pi y}\, x^{y}}{\Gamma(y)\, \sin \pi y}\,, \tag{8.66}$$

which matches (8.65).[15]

Let us check the fixed z bound (ii).

We have, with $x = \alpha(s)$, $w = \alpha(u)$, $y = \alpha(t)$, $\tau = \frac{2\nu}{z}$,

$$F_0(x, w) \xrightarrow[\substack{\nu\to\infty \\ \text{fixed } z}]{} \frac{(2\pi)^{1/2}}{e}\left(\frac{z}{2\nu}\right)^{1/2} \exp\{\nu[(1+1/z)\ln(1+1/z)$$

$$+(-1+1/z)\ln(-1+1/z) - 2/z\ln(2/z)]\}. \tag{8.67}$$

Therefore the fixed z-bound is violated.

Now for the fixed J bounds of (iii).

Near $z = 1$[16]

$$Q_\nu(z) \cong -\frac{1}{2}\log(\tfrac{1}{2}z - \tfrac{1}{2}) - \gamma - \psi(\nu + 1) \quad , \qquad (\nu \neq -1, -2, \cdots) \tag{8.69}$$

where

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) \quad , \tag{8.70}$$

and the Euler-Mascheroni constant is[17]

$$\gamma = \lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n} - \log m\right) = 0.5772156649\cdots$$

$$= -\psi(1) = \lim_{z\to 1}\left\{\zeta(z) - \frac{1}{z-1}\right\} . \tag{8.71}$$

We rewrite (8.35) and (8.37) as $(P \leqslant \tau \leqslant P + 1)$

$$a_2^+(J, \tau) = 2 \lim_{P\to\infty} \left(\sum_{K=1}^{P} + \sum_{K=P+1}^{\infty} \right) \frac{(-1)^K T_K(\tau)}{\Gamma(K)}$$

$$\cdot \; Q_J\left(1 + \frac{2K-1}{\tau}\right)$$

$$\cong 2 \sum_{K=1}^{\infty} \frac{(-1)^K \tau^K}{\Gamma(K)} \left\{ -\frac{1}{2} \log\left(\frac{2K-1}{2\tau}\right) - \gamma - \psi(J+1) \right\}$$

$$+ \lim_{P\to\infty} 2 \sum_{K=P+1}^{\infty} \frac{(-1)^K K^\tau}{\Gamma(\tau)} \frac{\sqrt{\pi}\,\Gamma(1+J)}{\Gamma(J+\frac{3}{2})} \left(\frac{4K}{\tau}\right)^{-J-1}$$

$$\cong \frac{2\sqrt{\pi}\,\Gamma(1+J)}{\Gamma(J+\frac{3}{2})\,\Gamma(\tau)} \left(\frac{4}{\tau}\right)^{-J-1} \left(2^{\tau-J} - 1\right) \frac{1}{J-\tau}$$

$$+ \text{ terms containing } G(\tau, 1, \infty) \quad . \qquad (8.72)$$

The first term on the RHS of (8.72) gives no trouble with the high τ bound. For the second term, however, we need to consider the asymptotic behavior of

$$g(x) = \sum_{K=1}^{\infty} (-x)^K/\Gamma(K) = -x\,e^{-x} \quad . \qquad (8.73)$$

This violates (iii) for $\tau \to -\infty$. The same difficulty occurs for $a_1^-(J, \tau)$ as $\tau \to +\infty$ in (8.37).

The authors have profited from discussions with S. Mandelstam.

FOOTNOTES AND REFERENCES

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18. M. A. Virasoro (private communication to S. Mandelstam) has also found the additive fixed poles in Veneziano's representation.

FIGURE CAPTIONS

Fig. 7.1. Contour for Sommerfeld-Watson transform.

Fig. 7.2. Analytic continuation of partial-wave amplitudes down to nearest t or u channel singularities.

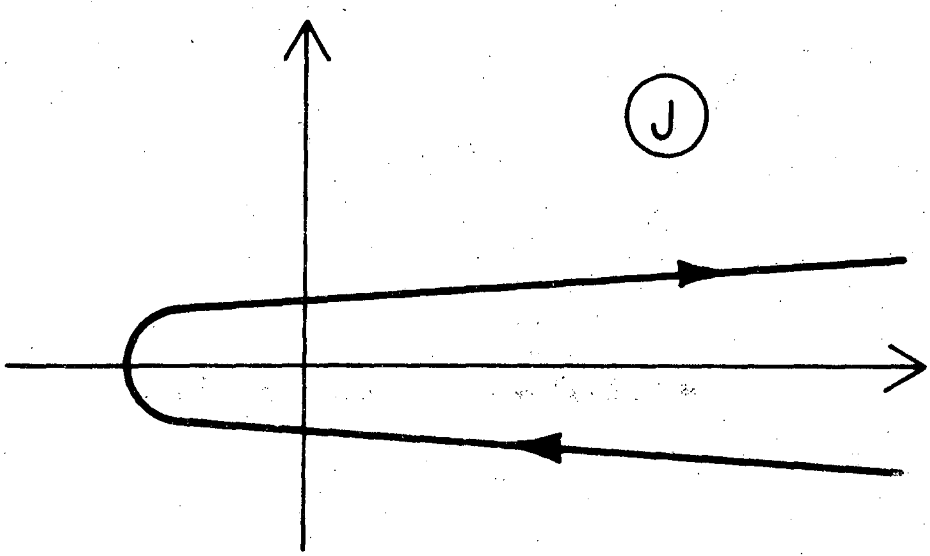

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Fig. 7.1

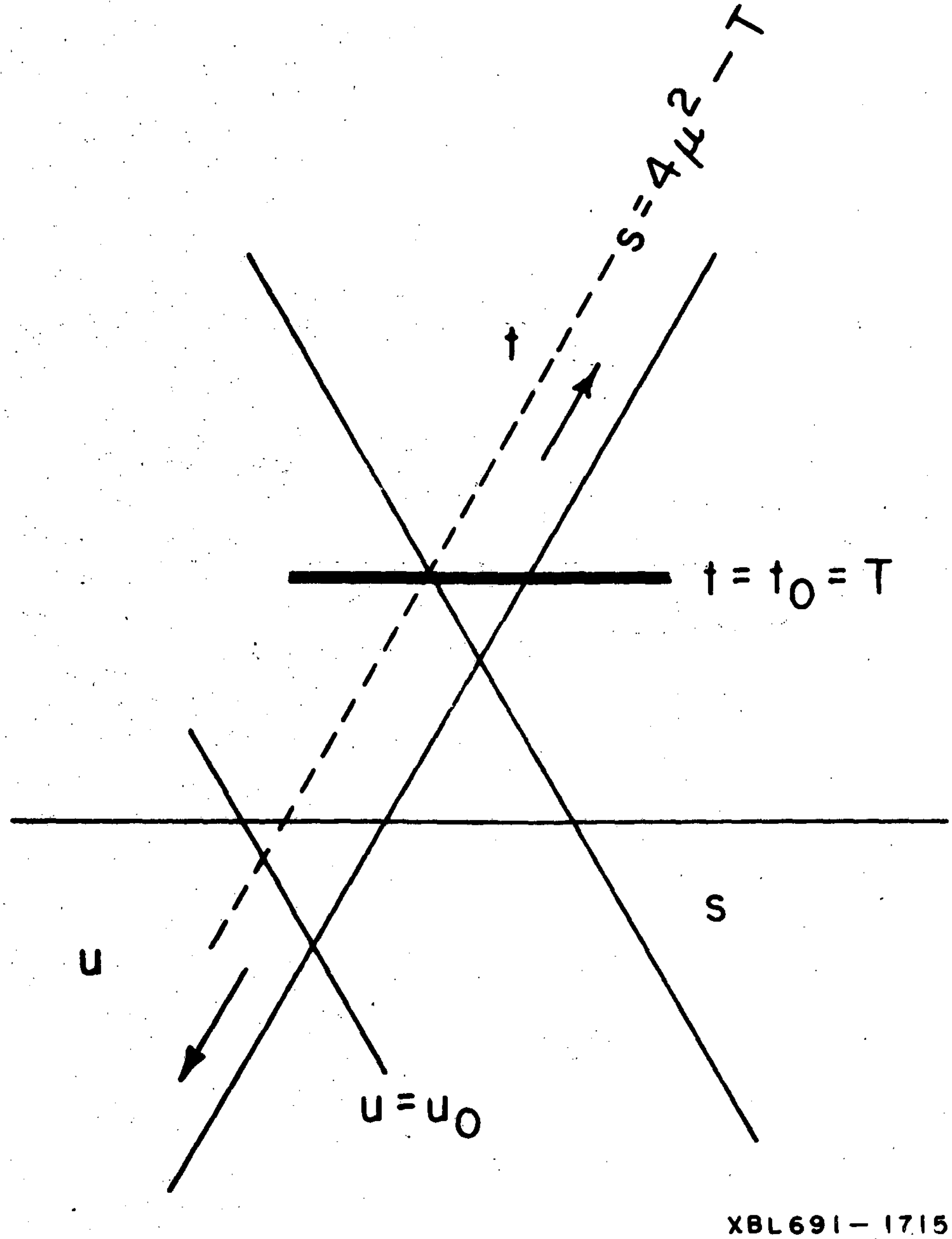

Fig. 7.2

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