LBL-33286
UC-419

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

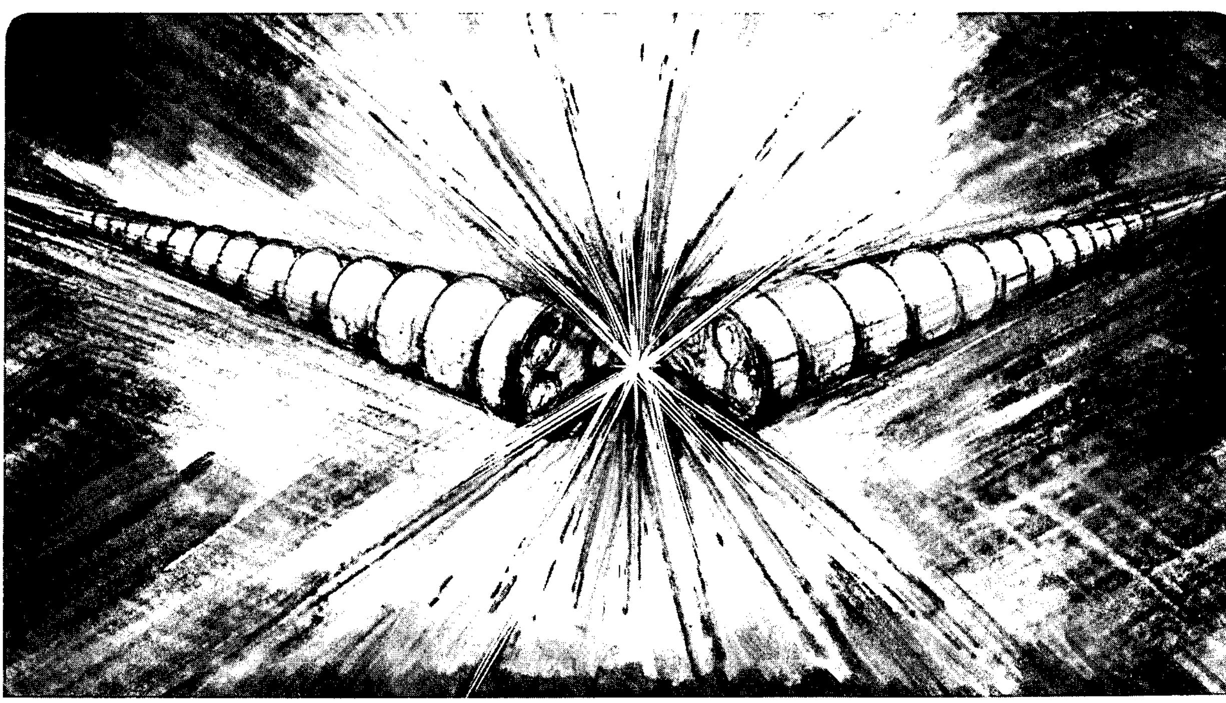
Accelerator & Fusion Research Division

Presented at the IEEE Particle Accelerator Conference,
Washington, D.C., May 17–20, 1993, and to be published
in the Proceedings

Three Dimensional Multipole Decomposition of Fields

K. Hahn

May 1993



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

LBL-33286
HIFAN-571

Three Dimensional Multipole Decomposition of Fields

K. Hahn
Accelerator & Fusion Research Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Submitted to the 1993 Particle Accelerator Conference
Washington, D. C., May 17 - 20, 1993

*This work was supported by the Office of Energy Research, Office of Fusion Energy, U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

Three Dimensional Multipole Decomposition of Fields*

Kyoung Hahn
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720, USA

Abstract

A new method to generate the general multipole representation of the three dimensional static field, electric or magnetic, is obtained via a scalar potential evaluated from the arbitrary specified source. As an application of this formulation, a previously described 3-D electric field decomposition method has been further generalized to the magnetic field.

1. INTRODUCTION

Representing an arbitrary three-dimensional vector field requires enormous amount of information. Multipole expansion is the natural and efficient way of representing a field with symmetry. A good example is the field from a quadrupole magnet which consists of a large quadrupole component with relatively small fringe fields. Then the multipole expansion converges rapidly and from the symmetry of the magnet geometry it can be easily seen that certain multipoles does not occur.

For a static field, electric or magnetic, the Green's function is well known, and the multipole coefficients can be determined from the source of the field. For a electrostatic problems the potential at the electrode is usually given and the charge density can be obtained by the capacity matrix technique[1] without solving for the field everywhere. For the magnetostatic problem, the current source is usually given.

In this report general multipole decomposition method for the static vacuum field from an arbitrary source is presented. In section II, the multipole expansion of the field is defined and the method of generating its coefficients from the Green's function is described. Section III shows the result from its application to a simple magnet geometry. A summary and conclusion is given in Sec. IV.

2. MULTIPOLE EXPANSION

Static vacuum fields, electric or magnetic, can be represented by a scalar potential. The scalar potential can be expressed in terms of multipoles which exploit the polar symmetry of the system. The convergence of the expansion depends on the system of interest, however, most of simple designs such as quadrupoles or sextupoles have a single dominant component in addition to the many small other multipole terms. Then the field can be accurately represented by keeping a few leading terms.

*Work supported by the Director, Office of Energy Research, Office of Fusion Energy, U.S. Dept. of Energy, under Contract No. DE-AC03-76SF00098.

The multipole coefficients $M_{k,l}(z)$ of the potential ϕ are defined in cylindrical coordinates system by

$$\phi(\rho,\theta,z)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} M_{k,l}(z)\,\rho^k\cos(l\theta) \qquad (1)$$

for the system of up-down symmetry. No z-axis expansion is performed and $M_{k,l}(z)$ is calculated at numerous locations in z.

The source-free vacuum potential ϕ satisfies the Laplace equation ($\nabla^2\phi=0$) and thus the $M_{k,l}$ observe the following recursion relation:

$$M_{k,l}=M''_{k-2,l}/(l^2-k^2), \qquad (2)$$

where double prime denotes the second derivative with respect to z. In order for ϕ be analytic near the origin, the relation $k\geq l\geq 0$ and $k-l=even$ must be true for non-zero coefficients. The entire ensemble of multipole coefficients can then be determined from $M_{l,l}$ and its $\hat{z}$−derivatives.

Since the field can be determined from the Green's function which is analytic away from the source, it is possible to decompose the Green's function into multipoles and the total multipole coefficients are obtained by integration over the source.

Electric potential ϕ from the charge distribution $Q(\mathbf{x}')$ is given by (setting $4\pi\epsilon_o\rightarrow 1$)

$$\phi=\int d\mathbf{x}'G_e(\mathbf{x},\mathbf{x}')Q(\mathbf{x}') \qquad (3)$$

where

$$G_e(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}. \qquad (4)$$

Away from the charges the Green's function G is infinitely differentiable, and it is in principle possible to compute the multipole coefficients $M_{k,j}$ by differentiating equation (1). Hence the multipole coefficient of the electrostatic field at the origin has the form

$$M_{k,l}=\int d\mathbf{x}\,K_{k,l}(\mathbf{x})\,Q(\mathbf{x}) \qquad (5)$$

and the explicit expression of $K_{k,l}$ is given in the Table 1.

The magnetic field is determined from the current source I by Biot-Savart's law (setting $\mu_o/4\pi\rightarrow 1$),

$$\mathbf{B}=-\int d\mathbf{x}'\,\mathbf{G}_m(\mathbf{x},\mathbf{x}')\times\mathbf{I}(\mathbf{x}')$$

Table 2. Magnetic multipole coefficient $K^i_{k,l}$ at origin from a unit current source I_i at x. Here $r = \sqrt{x^2+y^2+z^2}$.

$$K^x_{0,0} = \frac{y\,z}{(x^2+y^2)\,r}$$
$$K^x_{1,1} = K^x_{3,1} = K^x_{5,1} = K^x_{7,1} = K^x_{9,1} = 0$$
$$K^x_{2,2} = \frac{3\,y\,z}{4\,r^5}$$
$$K^x_{3,3} = \frac{5\,x\,y\,z}{4\,r^7}$$
$$K^x_{4,4} = \frac{35\,y\,(3\,x^2-y^2)\,z}{64\,r^9}$$
$$K^x_{5,5} = \frac{63\,x\,y\,(x^2-y^2)\,z}{32\,r^{11}}$$
$$K^x_{6,6} = \frac{231\,y\,(5\,x^4-10\,x^2\,y^2+y^4)\,z}{512\,r^{13}}$$
$$K^x_{2,0} = \frac{3\,y\,z}{4\,r^5}$$

$$K^y_{0,0} = -\frac{x\,z}{(x^2+y^2)\,r}$$
$$K^y_{1,1} = \frac{z}{r^3}$$
$$K^y_{2,2} = \frac{3\,x\,z}{4\,r^5}$$
$$K^y_{3,3} = \frac{5\,(x^2-y^2)\,z}{8\,r^7}$$
$$K^y_{4,4} = \frac{35\,x\,(x^2-3\,y^2)\,z}{64\,r^9}$$
$$K^y_{5,5} = \frac{63\,(x^4-6\,x^2\,y^2+y^4)\,z}{128\,r^{11}}$$
$$K^y_{6,6} = \frac{231\,x\,(x^4-10\,x^2\,y^2+5\,y^4)\,z}{512\,r^{13}}$$
$$K^y_{2,0} = -\frac{3\,x\,z}{4\,r^5}$$

$$K^z_{0,0} = K^z_{2,0} = K^z_{4,0} = K^z_{6,0} = K^z_{6,0} = K^z_{10,0} = 0$$
$$K^z_{1,1} = -\frac{y}{r^3}$$
$$K^z_{2,2} = \frac{-3\,x\,y}{2\,r^5}$$
$$K^z_{3,3} = \frac{5\,y\,(-3\,x^2+y^2)}{8\,r^7}$$
$$K^z_{4,4} = \frac{35\,x\,y\,(-x^2+y^2)}{16\,r^9}$$
$$K^z_{5,5} = \frac{63\,y\,(-5\,x^4+10\,x^2\,y^2-y^4)}{128\,r^{11}}$$
$$K^z_{6,6} = \frac{231\,x\,y\,(-3\,x^4+10\,x^2\,y^2-3\,y^4)}{256\,r^{13}}$$
$$K^z_{3,1} = \frac{3\,y\,(-x^2-y^2+4\,z^2)}{8\,r^7}$$

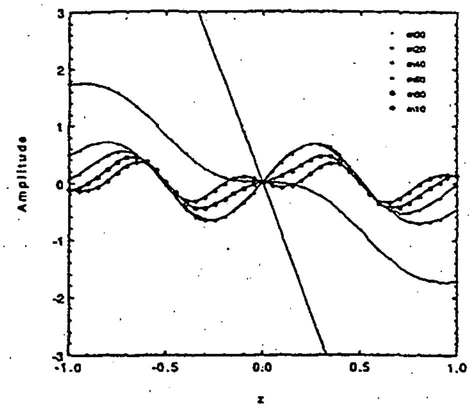


Fig. 1 - Multipole coefficients of the Helmholtz coil up to 10^{th} order. All none-axisymmetric coefficients are zero. In Helmholtz coil, axial separation of the identical rings is equal to their radius (a).

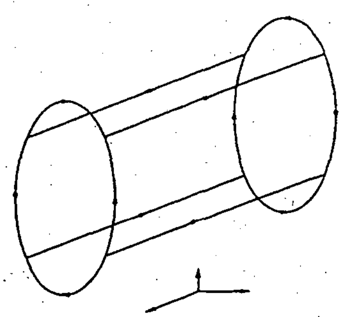


Fig. 2 - Current distribution in a simple quadrupole geometry. A single unit of current is on the arc shaped elememts and two units of current is on the straight section in order to prevent charge accumulation.

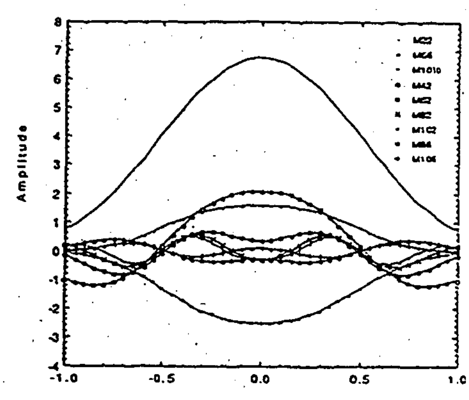


Fig. 3 - Multipole coefficients $M_{k,l}$ for the simple quadrupole geometry in Figure 2. The length of the straight section element is chosen to be the same as the aperture radius. From the symmetry, coefficients of l= 0, 1, 3, 4, 5, 7, 8, and 9 are zero.

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720