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NONLINEAR RESONANCE FOR DUFFING'S DIFFERENTIAL EQUATION

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Loren P. Meissner
(Ph. D. thesis)
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ABSTRACT

Some sufficient conditions are given for the existence and uniqueness of solutions to the problem $Py = Qy - \lambda \cdot y = 0$ when a solution y_0 has been found for $\lambda = 0$: i.e., $Qy_0 = 0$. These conditions are applied to the Duffing problem,

$$\gamma^2 \cdot D^2y(\theta) + y(\theta) + \beta \cdot y^3(\theta) = \cos \theta,$$

$$y(\pi/2) = 0, \quad Dy(0) = 0,$$

in two separate areas: first to prove the existence of a sequence of solutions tending to the solution of the "reduced" problem, $\gamma = 0$, in spite of the failure of the standard singular perturbation approach, and second to rigorously show the nature of the principal solutions near several low-order resonances (γ near 1), including the divergence of the solutions into two separate branches in a region of resonance.

Detailed quantitative information is presented concerning solutions of the Duffing problem for various values of β and γ (particularly in the region $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$) and the numerical procedure for obtaining these results is described. Data is given to substantiate the theoretical results by (a) illustrating the behavior of solutions for small values of γ, and (b) exhibiting the solution and the divergence into branches in low-order resonance regions.

I undertook a scrutiny of those intervals, and succeeded in describing the behavior of the solutions along the fringes of resonance. These descriptive results became the point of departure for the theoretical investigation described here.

The most recent studies of the Duffing problem which have been reported in the literature are those of Struble and his associates (References 1,2). Struble writes

$$(1.2) \qquad \ddot{x} + n^2 x - \bar{\beta} x^3 = \bar{\beta} F_o \cos \lambda t$$

which is the same as Eq. (1.1) when the following substitutions are made:

$$(1.3) \qquad \theta = \lambda t; \quad \gamma = \lambda/n; \quad y = n^2 x/\bar{\beta} F_o; \ \beta = -\bar{\beta}^3 \cdot F_o^2/n^6.$$

The work of Struble is directed particularly to the periodic and almost-periodic solutions in the "nearly linear" case, $|\bar{\beta}|$ small. Here, on the other hand, we shall consider the more general nonlinear problem although we shall be interested in finding out how large $|\beta|$ can be before our hypotheses fail.

Results in two separate areas are reported here; both of these are based on the general theorem of Chapter II: We show the existence of a sequence of values of γ tending to zero such that the corresponding sequence of solutions tends to the solution for $\gamma = 0$, in spite of the fact that $\gamma = 0$ is an accumulation point for the resonant regions. We also show rigorously the nature of the principal solutions near the low-order resonances (γ near 1) including the divergence of the solutions into two separate branches in a region of resonance.

II. NONLINEAR PROBLEMS CONTAINING A PARAMETER

A. A General Theorem

Consider the problem $Py = 0$, where y is an element (to be found) of a Banach space $\mathcal{F}$, and P is an operator (in general nonlinear) which maps some subset of $\mathcal{F}$ into $\mathcal{F}$. Suppose that P has the following form:

$$P = Q - \delta \cdot I \tag{2.1}$$

where I is the identity on $\mathcal{F}$, and δ is a scalar. Suppose that a solution y_o has been found for $\delta = 0$: i.e., $Qy_o = 0$.

The following theorem gives some (sufficient) conditions for the existence and uniqueness of solutions to the problem $Py = 0$ where δ is not necessarily zero. In the statement of this theorem, $\mathcal{D}P(y_o)$ is the Frechet derivative (see Appendix I or Liusternik, Reference 3, page 183) of P at y_o; and we write $S(y_o,r) = \{y : \|y - y_o\| \leq r\}$.

Theorem 1: Given the problem $Py = 0$, with $P = Q - \delta \cdot I$, let y_o be the solution for $\delta = 0$ (i.e., $Qy_o = 0$), and let λ be the eigenvalue of $\mathcal{D}Q(y_o)$ which is closest to δ. Assume that $\mathcal{D}Q(y_o)^{-1}$ exists, and that for all $\tilde{y} \in S(y_o,r)$

$$\|\mathcal{D}Q(y_o)^{-1} \circ \mathcal{D}^2Q(\tilde{y})\| \leq A_o \tag{2.2}$$

where $A_o = 1/(2 \cdot r)$; assume also that δ is such that

$$A_o \cdot \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right)^2 \cdot |\delta| \cdot \|\mathcal{D}Q(y_o)^{-1}\, y_o\| \leq \frac{1}{4}\,. \tag{2.3}$$

Then there exists one and only one solution of $Py = 0$ in $S(y_o,r)$.

The remainder of this chapter is devoted to a proof of Theorem 1. In the following chapters, this theorem is applied to the study of the harmonic resonances of the Duffing problem. Two separate areas of application are given: for resonance of low-order, the theorem is used as the basis for estimates of the allowable variation in δ under which the existence of connected solutions can be asserted; for the high-order cases, the theorem is used to derive the existence of a sequence of "non-resonant" solutions tending to y_o.

B. Proof of Theorem 1

1. Preliminary Lemmas

In all of the following lemmas, $\mathcal{F}$ is a Banach space and $\mathcal{E}$ is a subset of $\mathcal{F}$ which contains $S(y_o,r)$.

Lemma 1: Let T be an operator which maps $\mathcal{E}$ into $\mathcal{F}$. For all $y^{(1)}$, $y^{(2)} \in S(y_o,r)$, it is true that

$$\|Ty^{(2)} - Ty^{(1)}\| \leq \sup\{\|\mathcal{D}T(\tilde{y})\| : \tilde{y} \in S(y_o,r)\} \cdot \|y^{(2)} - y^{(1)}\|. \tag{2.4}$$

Proof: This lemma is based on the "Mean Value Theorem for Banach Space" (see Appendix I or Liusternik, Ref. 3, page 186 f.) which asserts that

$$\|Ty^{(2)} - Ty^{(1)}\| \leq \sup\{\|\mathcal{D}T(\bar{y})\| : \bar{y} \in C(y^{(2)},y^{(1)})\} \cdot \|y^{(2)} - y^{(1)}\|, \tag{2.5}$$

where $C(y^{(2)},y^{(1)}) = \{c \cdot y^{(2)} + (1 - c) \cdot y^{(1)} : 0 \leq c \leq 1\}$. The lemma follows immediately since $C(y^{(2)},y^{(1)}) \subset S(y_o,r)$.

Lemma 2: If $\|\mathcal{D}^2T(\tilde{y})\| \leq L/r$ for all $\tilde{y} \in S(y_o,r)$, and if $\mathcal{D}T(y_o) = 0$, then $\|Ty^{(2)} - Ty^{(1)}\| \leq L \cdot \|y^{(2)} - y^{(1)}\|$ for all $y^{(1)}$, $y^{(2)} \in S(y_o,r)$.

Proof: Comparing Lemma 1, we see that it is sufficient to show that

$$\sup\{\|\mathcal{D}T(\tilde{y})\| : \tilde{y} \in S(y_o,r)\} \leq L. \tag{2.6}$$

Applying Lemma 1 to $\mathcal{D}T$, we obtain in particular for $\tilde{y} \in S(y_o,r)$

$$\|\mathcal{D}T(\tilde{y}) - \mathcal{D}T(y_o)\| \leq \sup\{\|\mathcal{D}^2T(\tilde{\tilde{y}})\| : \tilde{\tilde{y}} \in S(y_o,r)\} \cdot \|\tilde{y} - y_o\| \tag{2.7}$$

but by hypothesis, the first factor on the right does not exceed L/r and the second does not exceed r. Since $\mathcal{D}T(y_o) = 0$, Lemma 2 is proved.

Lemma 3: If $T = A \circ B + C$, where A is a linear operator, then $\mathcal{D}T(y) = A \circ \mathcal{D}B(y) + \mathcal{D}C(y)$. If C is linear, then $\mathcal{D}C(y) = C$. If C is a constant operator then $\mathcal{D}C(y) = 0$.

Proof: All of these statements follow directly from the definition of the Frechet derivative (see Appendix I or Kantorovich, Ref. 4, pages 160 ff.).

Lemma 4: (Fixed Point Theorem): Let T be an operator which maps $\mathcal{E}$ into $\mathcal{F}$, with $S(y_o,r) \subset \mathcal{E}$. If, for all $y^{(1)}, y^{(2)} \in S(y_o,r)$, it is true that

(2.8) $$\|Ty^{(2)} - Ty^{(1)}\| \leq L \cdot \|y^{(2)} - y^{(1)}\|$$

with $L < 1$, and if

(2.9) $$\|Ty_o - y_o\| \leq (1 - L) \cdot r,$$

then there is one and only one point $\bar{y}$ in $S(y_o,r)$ such that $T\bar{y} = \bar{y}$.

Proof: This theorem depends upon the "Contraction mapping principle" applied to $S(y_o,r)$ (see Appendix I or Liusternik, Ref. 3, page 27).

We shall apply the foregoing lemmas to the operator T which is defined as follows:

(2.10) $$Ty = - \mathcal{D}P(y_o)^{-1} \circ Py + y.$$

The sequence $\{y_o, Ty_o, T^2y_o, \ldots, T^ny_o, \ldots\}$, where T is given by Eq. (2.10), is called the "Abbreviated Newton Iteration method" (Kantorovich, Ref. 4, pages 180 ff). Since $\mathcal{D}P(y_o)^{-1}$ is a linear operator, it is clear that $T\bar{y} = \bar{y}$ implies $P\bar{y} = 0$. This scheme differs from the ordinary Newton method in that $\mathcal{D}P^{-1}$ is taken at y_0 throughout instead of being redefined for each step at $T^n y_0$. This makes the Abbreviated method somewhat simpler to analyze, although not necessarily easier to compute.

If the conditions of Lemma 4 are satisfied, it is further true that the iterative application of Eq. (2.10) gives a sequence which in fact converges to the fixed point of T. However, the actual construction of solutions is not essential in the argument of this chapter.

The following lemma applies the Fixed Point Theorem to the Abbreviated Newton Iteration operator defined by Eq. (2.10).

Lemma 5: Assume that $\Gamma_0 = \mathcal{D}P(y_0)^{-1}$ exists, and that for all $\tilde{y} \in S(y_0,r)$ it is true that

$$\|\Gamma_0 \circ \mathcal{D}^2 P(\tilde{y})\| \leq A, \tag{2.11}$$

where $A = 1/(2r)$. Assume also that $A \cdot \eta_0 \leq \frac{1}{4}$, where

$$\eta_0 = \|\Gamma_0 \circ Py_0\|. \tag{2.12}$$

Then there is one and only one solution of $Py = 0$ in $S(y_0,r)$.

Proof: The first hypothesis of the Fixed Point Theorem follows from Lemmas 2 and 3: $\|\mathcal{D}^2 T(\tilde{y})\| = \|\Gamma_0 \circ \mathcal{D}^2 P(\tilde{y})\| \leq A$; hence, taking $L/r = A$ with $A = 1/(2r)$ we have $L = \frac{1}{2}$. The second hypothesis also holds since

$$\|Ty_0 - y_0\| = \|\Gamma_0 \circ Py_0\| = \eta_0 \leq \frac{1}{4A} = \frac{1}{2} \cdot r = (1 - L) \cdot r \tag{2.13}$$

since $(1 - L) = \frac{1}{2}$. Hence, there is a unique fixed point $\bar{y}$ of T in $S(y_0,r)$. But Γ_0 is linear so $P\bar{y} = 0$.

Remark: This Lemma may be compared with certain theorems in the literature (especially Kantorovich, Ref. 4, page 167), which give conditions for the convergence of Newton's method. I have chosen a slightly weaker but considerably simpler formulation which is adequate here. Note especially that the first assumption involves an estimate of $\|\Gamma_o \circ D^2P(y)\|$ instead of $\|\Gamma_o\| \cdot \|D^2P(y)\|$. In the Duffing problem, it is sometimes necessary to take advantage of the inequality $\|\Gamma_o \circ D^2P(y)\| \leq \|\Gamma_o\| \cdot \|D^2P(y)\|$.

Lemma 5 is actually a simple special case of the following lemma which gives closer existence conditions and wider uniqueness conditions when y_o is a good approximation to the solution: i.e., when η_o is small.

<u>Lemma 5a</u>: Assume that $\Gamma_o = DP(y_o)^{-1}$ exists, and that for all $\tilde{y} \in S(y_o,r)$ it is true that

$$\|\Gamma_o \circ D^2P(\tilde{y})\| = A. \tag{2.14}$$

Assume also that $A \cdot \eta_o \leq \frac{1}{4}$, where $\eta_o = \|\Gamma_o \circ Py_o\|$. If

$$1 - \sqrt{(1 - 4 A\eta_o)} < 2 A r < 1 + \sqrt{(1 - 4 A\eta_o)}, \tag{2.15}$$

then there is one and only one solution of $Py = 0$ in $S(y_o,r)$.

<u>Proof</u>: Take $L = Ar$ in Lemma 2. Since $\frac{1}{4} \geq A\eta_o \geq 0$, clearly $Ar < 1$ by the last hypothesis and so the first condition of the Fixed Point Theorem (Lemma 4) is satisfied.

But the last hypothesis also gives

$$\begin{aligned} &|2Ar - 1| < \sqrt{(1 - 4 A\eta_o)} \\ &4 A^2r^2 - 4 Ar + 1 < 1 - 4 A\eta_o \\ &Ar^2 - r < - \eta_o \\ &(1 - Ar) \cdot r > \eta_o \end{aligned} \tag{2.16}$$

so that the second condition of Lemma 4 also holds inasmuch as $L = A r$

and $\eta_o = \|Ty_o - y_o\|$.

2. Effect of the Parameter

One important assumption in Theorem 1 is the existence of the inverse of $\mathcal{D}P(y_o)$. Thus, we must see for which values of δ, if any, the operator $\mathcal{D}P(y_o)$ is singular. We note (by Lemma 3) that

$$\mathcal{D}P(y_o) = \mathcal{D}Q(y_o) - \delta \cdot I. \tag{2.17}$$

Now let λ be an eigenvalue of $\mathcal{D}Q(y_o)$, so that for some eigenfunction y (which must also satisfy the boundary conditions)

$$\begin{aligned} \mathcal{D}Q(y_o)\, y &= \lambda\, y \\ \mathcal{D}P(y_o)\, y &= (\lambda - \delta) \cdot y. \end{aligned} \tag{2.18}$$

Thus, the eigenvalues of $\mathcal{D}P(y_o)$ are of the form $(\lambda - \delta)$ for all eigenvalues λ of $\mathcal{D}Q(y_o)$, and $\mathcal{D}P(y_o)$ is singular when δ is equal to one of the eigenvalues of $\mathcal{D}Q(y_o)$.

We use this fact to estimate the norm of $\Gamma_o = \mathcal{D}P(y_o)^{-1}$. We know (see Halmos, Ref. 5, page 182) that this norm does not exceed the largest eigenvalue of Γ_o, which is the reciprocal of the smallest eigenvalue of $\mathcal{D}P(y_o)$. As in the statement of Theorem 1, assume that δ is closer to λ than to any other eigenvalue of $\mathcal{D}Q(y_o)$. Then

$$\|\Gamma_o\| \le 1/\ |\delta - \lambda| \tag{2.19}$$

Lemma 6: If y is an element of $\mathcal{F}$, and R is an operator on $\mathcal{F}$ or on $(\mathcal{F} \to \mathcal{F})$, then

$$\begin{aligned} \|\Gamma_o\, y\| &\le \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) \cdot \|\mathcal{D}Q(y_o)^{-1}\, y\|; \\ \|\Gamma_o \circ R\| &\le \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) \cdot \|\mathcal{D}Q(y_o)^{-1} \circ R\|. \end{aligned} \tag{2.20}$$

Proof:

$$(2.21) \quad \|\Gamma_o\, y\| \le \|\Gamma_o \circ DQ(y_o)\| \cdot \|DQ(y_o)^{-1}\, y\|;$$

$$\|\Gamma_o \circ R\| \le \|\Gamma_o \circ DQ(y_o)\| \cdot \|DQ(y_o)^{-1} \circ R\|.$$

$$\|\Gamma_o \circ DQ(y_o)\| = \|\Gamma_o \circ (DP(y_o) + \delta \cdot I)\|$$

but Γ_o is a linear operator:

$$(2.22) \quad \|\Gamma_o \circ DQ(y_o)\| = \|\Gamma_o \circ DP(y_o) + \delta \cdot \Gamma_o\|$$

$$= \|I + \delta \cdot \Gamma_o\|$$

$$\le \|I\| + |\delta| \cdot \|\Gamma_o\|.$$

$$\le 1 + |\delta|/|\delta - \lambda|.$$

Lemma 7: If $Qy_o = 0$, then

$$(2.23) \quad \|\Gamma_o \circ Py_o\| \le \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) \cdot |\delta| \cdot \|DQ(y_o)^{-1}\, y_o\|$$

Proof: We apply Lemma 6 to $y = Py_o$:

$$(2.24) \quad \|\Gamma_o \circ Py_o\| \le \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) \cdot \|DQ(y_o)^{-1} \circ Py_o\|.$$

But $Py_o = Qy_o - \delta \cdot y_o$ and $Qy_o = 0$ so $Py_o = -\delta \cdot y_o$:

$$(2.25) \quad \|DQ(y_o)^{-1} \circ Py_o\| = |\delta| \cdot \|DQ(y_o)^{-1}\, y_o\|.$$

3. Completion of Theorem 1

We see that the hypothesis of Theorem 1 satisfies the requirements of Lemma 5, with

$$(2.26) \quad A = \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) \cdot A_o;$$

$$\eta_o = \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) \cdot |\delta| \cdot \|DQ(y_o)^{-1}\, y_o\|.$$

The first of these statements follows from Lemma 6 with $R = D^2Q(\tilde{y})$; however, it must be realized that we have used two names for the same object. According to Lemma 3, the second-order derivative is independent of δ:

$$D^2P(\tilde{y}) = D(DQ(\tilde{y}) - \lambda \cdot I) = D^2Q(\tilde{y}). \tag{2.27}$$

The second statement is the same as Lemma 7, since $\eta_0 = \|\Gamma_0 \circ Py_0\|$. The last hypothesis of Theorem 1 assures us that $A \cdot \eta_0 \leq \frac{1}{4}$.

A variant of Theorem 1 is useful in case y_0 is an approximate, rather than an exact solution for $\delta = 0$. Assume that $\|Qy_0\| \leq \epsilon$. The effect of this modification appears only in Lemma 7. There, and in the last hypothesis of Theorem 1, the factor

$$|\delta| \cdot \|DQ(y_0)^{-1} y_0\| \tag{2.28}$$

may be changed to

$$(|\delta| \cdot \|DQ(y_0)^{-1} y_0\| + \epsilon \cdot \|DQ(y_0)^{-1}\|) \tag{2.29}$$

which is an estimate for $\|DQ(y_0)^{-1} \circ Py_0\|$ when $\|Qy_0\| \leq \epsilon$.

We note that the <u>uniqueness</u> of the solution (if it exists) depends only upon the Lipschitz condition: if

$$\|\Gamma_0 \circ D^2P(\tilde{y})\| \leq A \tag{2.30}$$

for all $\tilde{y} \in S(y_0, r)$, with $A = 1/r$, we may take r' slightly smaller than r without violating (2.30) in $S(y_0, r')$; then Lemma 2 is satisfied with $L = Ar' < Ar = 1$. The last part of the proof of the Fixed Point Theorem (see Appendix) then shows that there is at most one solution in $S(y_0, r')$.

III. THE ODD-HARMONIC SOLUTIONS OF THE DUFFING PROBLEM

A. Operator Form of the Duffing Problem

For Duffing's differential equation:

$$(3.1) \qquad \gamma^2 D^2 y(\theta) + y(\theta) + \beta \cdot y^3(\theta) = \cos\theta,$$

with boundary conditions $y(\pi/2) = 0$; $Dy(0) = 0$, we seek harmonic solutions; i.e., solutions of the form

$$(3.2) \qquad y(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta.$$

Because of the odd symmetry of Eq. (3.1), the sine coefficients b_k are all zero and the second boundary condition is automatically satisifed. The first boundary condition further implies that the even-numbered cosine coefficients are also zero, so that Eq. (3.2) reduces to the odd-cosine form

$$(3.3) \qquad y(\theta) = \sum_{k=1}^{\infty} a_{2k-1} \cos(2k-1)\cdot\theta.$$

The following integral equation is equivalent to Eq. (3.1) with the given boundary conditions:

$$(3.4) \qquad -\gamma^2 \cdot y(\theta) + \int_{\theta}^{\pi/2} dt_2 \int_0^{t_2} [y(t_1) + \beta \cdot y^3(t_1) - \cos t_1]\, dt_1 = 0.$$

This equation may be rewritten in operator form. Let us first define the special operators J and G:

$$(3.5a) \qquad J\, y(\theta) = \int_{\theta}^{\pi/2} dt_2 \int_0^{t_1} y(t_1)\, dt_1;$$

$$(3.5b) \qquad G\, y(\theta) = (y(\theta))^3.$$

The use of these operators allows us to write Eq. (3.4) in the form $Py = 0$, where

$$(3.6) \qquad Py = -\gamma^2 \cdot y + J\,(y + \beta \cdot Gy - \cos).$$

Now let the "constant operator" $\underset{\sim}{\cos}$ be defined by: $\underset{\sim}{\cos}\, y = \cos$. In other words, $\underset{\sim}{\cos}$ is an operator which maps any function y into the cosine function. Also, let I be the identity operator. Now Eq. (3.6) becomes:

$$(3.7) \qquad P = -\gamma^2 \cdot I + J \circ (I + \beta \cdot G - \underset{\sim}{\cos}).$$

It should be noted that the operator P contains two parameters, γ and β, which have not yet been specified. For given values of γ and β, P is a (nonlinear) mapping of the space $\mathcal{L}^2\,[0, \pi/2]$ into itself.

It is obvious that the operator P is of the form studied in Chapter II, with $\delta = \gamma^2 - \gamma_o^{\,2}$ and

$$(3.8) \qquad Q = -\gamma_o^{\,2} \cdot I + J \circ (I + \beta \cdot G - \underset{\sim}{\cos}).$$

Also, we have the Frechet derivative

$$(3.9) \qquad \mathcal{D}Q(y_o) = -\gamma_o^{\,2} \cdot I + J \circ (I + \beta \cdot \mathcal{D}G(y_o))$$

$$\mathcal{D}G(y_o)\, y(\theta) = 3 \cdot (y_o(\theta))^2 \cdot y(\theta),$$

and the second-order derivative

$$(3.9a) \qquad \mathcal{D}^2Q(y_o) = \beta \cdot J \circ \mathcal{D}^2G(y_o)$$

$$\mathcal{D}^2G(y_o)\, y_1\, y_2 = 6\, y_o \cdot y_1 \cdot y_2$$

B. High-Order Resonance

We consider first the situation in case $\gamma_o = 0$. Then the problem $Qy = 0$ corresponds to the "reduced equation"

$$(3.10) \qquad y_o(\theta) + \beta \cdot y_o^3(\theta) = \cos\theta$$

$$y_o(\pi/2) = 0, \quad Dy(0) = 0.$$

A solution of this problem for $\beta = 1$ is shown in Table 4.2 (Chapter IV).

The standard "singular perturbation" theory for equations of this type, in particular for problems in which the order of the differential equation changes by 2 for $\gamma = 0$, has been elucidated by Wasow (Ref. 6; see also Cesari, Ref. 7, pp 195 ff). Except in a certain case which he calls "parametrically irregular", Wasow concludes that the solution of the differential equation converges to y_o on an interval containing $\gamma = 0$. Checking Wasow's hypotheses as applied to the Duffing problem, we find that Eq. (3.1) is "parametrically irregular", so that we are not surprised to find, as a result of computation studies, indications that there may be arbitrarily small values of γ for which there is no solution close to y_o. These are values of γ for which $\mathcal{D}P(y_o)$ is nearly singular and the theory of the previous chapter cannot be used to assert the existence of solutions.

Thus, for the Duffing problem a different approach must be adopted. In spite of the fact that as $\gamma \to 0$ there are infinitely many regions of "resonance", we find that these regions are separated by "valleys" which may be characterized, for instance, by the fact that $\|\mathcal{D}P(y_o)^{-1} \circ Py_o\|$ is small. We shall apply Theorem 1 to show that a sequence of values of γ can be chosen from these "valleys", in such a way that as $\gamma \to 0$ via the chosen sequence, the corresponding sequence of solutions tends to y_o.

We introduce the auxiliary function $q'(\theta)$ which has the property (compare Eq. (3.9)) that

$$(3.11) \qquad q'(\theta) \cdot y(\theta) = \mathcal{D}Q(y_o)\, y(\theta).$$

Theorem 2: For the problem of Eq. (3.1) let β be fixed and let y_o be defined by Eq. (3.10). Assume that β is such that the following hypotheses hold, where

$$q'(\theta) = 1 + 3 \cdot \beta \cdot y_o^2(\theta). \tag{3.12}$$

(i). There exist r and $A = {}^1/2r$ such that for all $\tilde{y} \in S(y_o, r)$

$$\|6 \cdot \beta \cdot \tilde{y} / q'\| \le A. \tag{3.13}$$

(ii). There exists constants a_1 and a_2 such that for $0 \le \theta \le \pi/2$

$$0 < a_1 < q'(\theta) < a_2. \tag{3.14}$$

(iii). The constants A, a_1, a_2, η_o satisfy

$$A \cdot (a_2^4 / a_1^3) \cdot \eta_o < 1 \tag{3.15}$$

where $\eta_o = \|y_o / q'\|$.

Then there is an infinite sequence of values $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ tending to zero, and a corresponding sequence of solutions $\{y_1, y_2, y_3, \ldots\}$ where y_ν is the solution of Eq. (3.1) for $\gamma = \gamma_\nu$, such that y_ν tends to y_o. Furthermore, y_ν is unique in that there is no other solution of Eq. (3.1) in $S(y_o, r)$ for $\gamma = \gamma_\nu$.

Proof: We only need to show the existence of a sequence $\{\gamma_\nu\}$ such that for $\delta = \gamma_\nu^2$ $(\nu = 1, 2, 3, \ldots)$

$$\left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right)^2 \cdot |\delta| \le (a_2^4 / 4 \cdot a_1^3) \tag{3.17}$$

for some a_1 and a_2 which satisfy assumptions (ii) and (iii). The remaining hypotheses of Theorem 2 imply those of Theorem 1, since

$$\|6 \cdot \beta \cdot \tilde{y} / q'\| = \|DQ(y_o)^{-1} \circ D^2Q(\tilde{y})\| \tag{3.18}$$

$$\|y_o / q'\| = \|DQ(y_o)^{-1} \circ y_o\|,$$

as may be verified from Eq. (3.9) or by direct use of the definition of the Frechet derivative (Appendix 1 or Liusternik, Ref. 3, page 183) as applied to the definition of Q, Eq. (3.8).

The crucial point in Eq. (3.17) is clearly the choice of δ such that $|\delta - \lambda|$ is not too small. Here λ is defined, as in Theorem 1, as the eigenvalue of $\mathcal{D}Q(y_o)$ which is closest to δ, so that in Lemma 6 we can use the estimate of Eq. (2.19):

$$\|\Gamma_o\| \leq 1/\ |\delta - \lambda|.$$

We therefore turn to an investigation of the eigenvalues of $\mathcal{D}Q(y_o)$.

1. <u>Eigenvalues of the Variational Differential Equation.</u>

We seek those values of λ for which there exists a nontrivial solution y of

$$\mathcal{D}Q(y_o)\ y = \lambda \cdot y \tag{3.19}$$

(compare Eq. (2.18)). The same equation may be written, in view of Eq's. (3.5, 3.9) since $\gamma_o = 0$, as follows:

$$\int_\theta^{\pi/2} dt_2 \int_0^{t_2} (1 + 3 \cdot \beta \cdot y_o^2(t_2)) \cdot y(t_2)\ dt_2 = \lambda \cdot y(\theta) \tag{3.20}$$

and so the problem is equivalent to finding eigenvalues of the "variational" differential equation

$$\lambda\ D^2y(\theta) + (1 + 3 \cdot \beta \cdot y_o^2(\theta)) \cdot y(\theta) = 0, \tag{3.21}$$
$$y(\pi/2) = 0, \quad Dy(0) = 0.$$

To this problem we may apply "Sturm's first comparison theorem" (Ince, Ref. 8, p. 228). As before, let

$$q'(\theta) = 1 + 3 \cdot \beta \cdot y_o^2(\theta) \tag{3.22}$$

and assume that $0 < a_1 < q'(\theta) < a_2$ for $0 \leq \theta \leq \pi/2$. Let $t = \theta/\lambda^{\frac{1}{2}}$ and let $f(t) = q'(\lambda^{\frac{1}{2}} \cdot t)$ for $0 \leq t \leq \pi/(2 \cdot \lambda^{\frac{1}{2}})$, and consider the following problem:

$$(3.23) \qquad D^2y(t) + f(t) \cdot y(t) = 0, \quad y(0) = 1, \quad Dy(0) = 0.$$

We see that $y(\pi/(2 \cdot \lambda^{\frac{1}{2}})) = 0$ if and only if λ is an eigenvalue of Eq. (3.21). The Sturm-Lionville theory (Ince, Ref. 8, page 232) assures us that there are an infinite number of eigenvalues λ_j of Eq. (3.21) which may be labeled so that there are exactly j zeroes of the solution of Eq. (3.21) in the interval $0 \leq \theta \leq \pi/2$.

We apply the Comparison Theorem to Eq. (3.23), replacing f(t) in by each of the constants a_1 and a_2, and conclude that λ_j can be bounded as follows:

$$(3.24) \qquad \frac{a_1}{(2j-1)^2} < \lambda_j < \frac{a_2}{(2j-1)^2}.$$

We apply the same theorem in a different way to obtain a lower bound for $(\lambda_j - \lambda_{j+1})$. Comparing Eq. (3.23) with the following problem:

$$(3.25) \qquad D^2y(t) + a_2 \cdot y(t) = 0, \quad y(\pi/(2 \cdot \lambda_j^{\frac{1}{2}})) = 0, \quad Dy(0) = 0,$$

we conclude that the next zero of Eq. (3.25) to the right of $(\pi/(2 \cdot \lambda_j^{\frac{1}{2}}))$, is to the left of the next zero of Eq. (3.23):

$$(3.26) \qquad (\pi/2\, \lambda_j^{\frac{1}{2}}) + (\pi/a_2^{\frac{1}{2}}) < (\pi/2\, \lambda_{j+1}^{\frac{1}{2}}).$$

A little algebra applied to Eq. (3.26) gives the conclusion

$$(3.27) \qquad \lambda_j - \lambda_{j+1} > a_1^{3/2} / 2 \cdot a_2^{\frac{1}{2}} \cdot j^3.$$

The estimates of Eq.'s (3.24) and (3.26) hold for each j : ($j = 1, 2, 3, \ldots$). By Eq. (3.24), $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$. For any fixed j, we may take

$$\delta = (\lambda_j + \lambda_{j+1})/2 \tag{3.28}$$

so that we may take $\lambda = \lambda_j$ and

$$\begin{aligned} &\delta < \lambda_j \\ &|\delta - \lambda| = \tfrac{1}{2}(\lambda_j - \lambda_{j+1}) \end{aligned} \tag{3.29}$$

and we obtain with the aid of Eq.'s (3.24) and (3.27) the following estimate:

$$\left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right)^2 \cdot |\delta| < \left(1 + j \cdot \left(\frac{a_2}{a_1}\right)^{3/2} \cdot \frac{4j^2}{(2j-1)^2}\right)^2 \cdot \frac{a_2}{(2j-1)^2} . \tag{3.30}$$

This is a strict inequality; hence, if j is large enough we may drop the first "1" on the right and replace $(2j-1)^2$ by $4j^2$, leaving

$$\left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right)^2 \cdot |\delta| < a_2^{\,4} / 4 \cdot a_1^{\,3} \tag{3.31}$$

for all sufficiently large j, if δ is chosen in accordance with Eq. (3.28). We may avoid all smaller values of j and begin with

$$\begin{aligned} \gamma_1^{\,2} &= \delta_1 = \tfrac{1}{2}(\lambda_j + \lambda_{j+1}), \\ \gamma_2^{\,2} &= \delta_2 = \tfrac{1}{2}(\lambda_{j+1} + \lambda_{j+2}), \\ &\ldots, \end{aligned} \tag{3.32}$$

for sufficiently large j. The smaller values of j satisfy a similar theorem in which assumption (iii) is replaced by the more complicated expression obtained from the right-hand side of Eq. (3.30).

2. Effect of the Parameter β Upon the High-Order Resonances

The hypotheses of Theorem 2 depend only upon β and upon y_o. But y_o in turn depends only upon β, since it is the solution of the algebraic equation

$$(3.33) \qquad y(\theta) + \beta \cdot y^3(\theta) = \cos\theta,$$

and for any given value of θ the solution can be found by solving this cubic equation.

For $\beta \geq 0$, we can obtain more information a priori. In this case, we can clearly find a_1 such that $0 < a_1 < q'(\theta)$ (in fact $a_1 \geq 1$ for $\beta > 0$) and assumption (ii) can always be satisfied. The following steps show that in this case we can satisfy assumption (i) as well by taking

$$(3.34) \qquad r < -\tfrac{1}{2}\,\|y_o\| + \tfrac{1}{2}\sqrt{(\|y_o\|^2 + \ a_1/3\,\beta)}.$$

This implies the following:

$$(3.35) \qquad (r + \tfrac{1}{2}\,\|y_o\|)^2 < \tfrac{1}{4}(\|y_o\|^2 + a_1/3\,\beta)$$

$$r^2 + r\|y_o\| \ < \ a_1/12\,\beta$$

$$\frac{6\beta}{a_1}(\|y_o\| + r) < \frac{1}{2\,r} = A$$

$$\|6\,\beta\,(y_o + r)/q'\| < A.$$

If β is near zero, we may take r quite large in Eq. (3.34) and hence A is quite small; also q' stays close to 1 so a_2 and a_1 are nearly equal; thus, $(a_2{}^4/a_1{}^3) \cong 1$ and $\eta_o \cong \|y_o\|$. Accordingly, hypothesis (iii) of Theorem 2 is easily satisfied. In Chapter IV, some computations are made to determine the largest value of β for which assumption (iii) holds.

C. Low-Order Resonance

Theorem 1, besides forming a basis for the proof of Theorem 2, is also useful in quantitative studies of the solutions of the Duffing problem for particular values of the parameters.

If we let γ_o be some parameter value (other than 0) for which the solution is known, and let Q be defined (by Eq. (3.8)) as the basic operator which we have been considering, with $\gamma = \gamma_o$, then Theorem 1 asserts (under certain conditions) the existence of a certain interval of values of γ for which one and only one solution exists in a neighborhood of the solution for γ_o. It may be possible to extend this interval by choosing a new value of γ_o for which we have (by Theorem 1) existence and uniqueness of a solution, and obtaining a new interval for this new γ_o. Very little can be said *a priori* about such a procedure, partly because the operator Q, and, hence, also the eigenvalue λ, depends upon γ_o in a way which is difficult to predict. The numerical studies which are reported in Chapter IV are devoted largely to a quantitative study of this procedure, including the choice of "starting values" for γ_o and the corresponding "known" solutions.

IV. NUMERICAL STUDIES OF NONLINEAR RESONANCE

A. Discretization of the Duffing Problem

For numerical work, it is necessary to replace the Duffing problem by a related problem whose solution can be obtained by a finite procedure. A fairly common practice is to restrict attention to a finite number of points in the interval of interest and to replace all derivatives by finite-difference approximations. But when periodic solutions are being studied, it is often advantageous to transform the problem by assuming a solution in the form of a Fourier series, and to obtain the finite related ("discretized") problem by truncating the series. This technique, of course, extends to the case in which solutions are assumed to be representable in any given series form; the discretization is easier when the basic functions are orthogonal, because the "best" representation in a given number of terms is then obtained by simple truncation of the series.

It is necessary, when this technique is used, to determine the effect of the transformation upon all parts of the problem -- in particular, upon derivatives and integrals and upon nonlinear operations. In many cases - including the case of sine-cosine series representations - differentiation and integration have a very simple form consisting of easy operations upon the individual coefficients of the series. Some of the very simplest nonlinear operations can also be performed directly upon the coefficients: for example, the coefficients of the series, obtained by multiplying two given Fourier series, can be expressed in terms of the coefficients of the two given series. For more complicated nonlinear operations, however, it may be necessary to transform the problem back into the original space by a Fourier synthesis transformation, σ, then to perform the indicated nonlinear operation G, and finally to perform a Fourier analysis $\sigma^\dagger$ upon the result. Thus, the nonlinear operation G is replaced by a mapping G* of the space of Fourier coefficients into itself according to the formula $G^* = \sigma^\dagger \circ G \circ \sigma$. As usual, the composition of operators is from right to left.

We shall now consider in detail the process of transforming the Duffing problem from a space of functions defined on the real interval $[0, \pi/2]$ into a space whose elements consist of sequences of numbers which are the coefficients of the series obtained by representing the functions of the original space as a Fourier series. As we have seen in Eq. (3.3), the boundary conditions in our case allow us to ignore all of the sine terms and all of the even-numbered cosine terms.

1. The Odd-Cosine Hilbert Space

The following symbols conform to a system of notation developed by Professor R. J. DeVogelaere (Ref. 9). The closed real interval $[0, \pi/2]$ is called E, and the set of all positive integers is called ω. Equation (3.3) is rewritten as follows:

$$(4.1) \qquad y_E(\theta) = \Sigma_{k\varepsilon\omega}\, y_\omega[k] \cdot \cos(2k-1)\,\theta. \qquad (\theta \,\varepsilon\, E)$$

If $\Sigma_{k\varepsilon\omega}(y_\omega[k])^2 < \infty$, then the sequence of coefficients $\{y_\omega[k]: k \,\varepsilon\, \omega\}$ is an element of a Hilbert space $\mathcal{F}_\omega$, and Eq. (4.1) defines a complete synthesis operation τ which is a mapping from $\mathcal{F}_\omega$ to a function space $\mathcal{F}_E$ which contains y_E and which is a subset of $\mathcal{L}^2[0, \pi/2]$. Specifically, we define the space $\mathcal{F}_E$ as the image under τ of $\mathcal{F}_\omega$. Thus, each function $y_E \,\varepsilon\, \mathcal{F}_E$ has a representation in the form given by Eq. (4.1). The analysis operation τ^{-1} can be written, according to the theory of general Fourier expansions (see Rudin, Ref. 10, page 154) as follows:

(4.2) $$y_\omega [k] = \frac{\pi}{4} \int_E y_E(\theta) \cdot \cos (2k - 1)\, \theta \cdot d\theta. \qquad (k \varepsilon \omega)$$

a. The Duffing Problem in Hilbert space. Using this notation, we have in the function space $\mathcal{F}_E$ the special operators J_E and G_E as in Eq. (3.5):

(4.3) $$J_E\, y_E(\theta) = \int_\theta^{\pi/2} dt_2 \int_0^{t_2} y_E(t_1)\, dt_1;$$

$$G_E\, y_E(\theta) = (y_E(\theta))^3. \qquad (\theta \varepsilon E)$$

Equation (3.6) becomes

(4.4) $$P_E\, y_E = -\gamma^2 \cdot y_E + J_E \circ (y_E + \beta \cdot G_E\, y_E - \cos)$$

and, using $\underset{\sim\sim\sim}{\cos}_E\, y_E = \cos$:

(4.5) $$P_E = -\gamma^2 \cdot I_E + J_E \circ (I_E + \beta \cdot G_E - \underset{\sim\sim\sim}{\cos}_E).$$

The Frechet derivative at $\bar{y}_E$ is also written:

(4.6) $$\mathcal{D}P_E(\bar{y}_E) = -\gamma^2 \cdot I_E + J_E \circ (I_E + \beta \cdot \mathcal{D}G_E(\bar{y}_E));$$

where $\mathcal{D}G_E(\bar{y}_E)\, y_E = 3\, (\bar{y}_E)^2 \cdot y_E$.

We now define the corresponding operators in the Hilbert space $\mathcal{F}_\omega$, by means of the mappings τ and τ^{-1}. The operator P_ω is defined as follows:

(4.7) $$P_\omega = \tau^{-1} \circ P_E \circ \tau$$

so that, if $y_E = \tau\, y_\omega$, we have

(4.8) $$P_\omega\, y_\omega = \tau^{-1} \circ P_E \circ \tau\, y_\omega = \tau^{-1} \circ P_E\, y_E.$$

We also define I_ω as the identity on $\mathcal{F}_\omega$, and $J_\omega = \tau^{-1} \circ J_E \circ \tau$,

$G_\omega = \tau^{-1} \circ G_E \circ \tau$, and $\underset{\sim}{\cos}_\omega = \tau^{-1} \circ \underset{\sim}{\cos}_E \circ \tau$ so that $\underset{\sim}{\cos}_\omega\, y_\omega = \cos_\omega$, where $\cos_\omega$ is the sequence $\{1, 0, 0, 0, \ldots\}$. Thus,

$$(4.9) \qquad P_\omega = \tau^{-1} \circ [-\gamma^2 \cdot I_E + J_E \quad (I_E + \beta \cdot G_E - \underset{\sim}{\cos}_E)] \circ \tau$$

$$= -\gamma^2 \cdot I_\omega + J_\omega \circ (I_\omega + \beta \cdot G_\omega - \underset{\sim}{\cos}_\omega).$$

Of particular importance is the form assumed by J_ω:

$$(4.10) \qquad J_\omega\, y_\omega = \tau^{-1} \circ J_E \circ \tau\, y_\omega = \tau^{-1} \circ J_E\, y_E;$$

$$J_E\, y_E(\theta) = \int_\theta^{\pi/2} dt_2 \int_0^{t_2} [\Sigma_{k\varepsilon\omega}\, y_\omega\, [k] \cdot \cos\,(2k - 1) \cdot t_1]\, dt_1$$

$$= \Sigma_{k\varepsilon\omega}\, y_\omega\, [k] \int_\theta^{\pi/2} dt_2 \int_0^{t_2} \cos\,(2k - 1) \cdot t_1\, dt_1$$

$$= \Sigma_{k\varepsilon\omega} \frac{1}{(2k-1)^2} \cdot y_\omega\, [k] \cos\,(2k - 1) \cdot \theta; \qquad (\theta \;\varepsilon\; E)$$

$$(J_\omega\, y_\omega)\, [k] = \frac{1}{(2k-1)^2}\, y_\omega\, [k]. \qquad (k \;\varepsilon\; \omega)$$

The Frechet derivative at $\bar{y}_\omega$ has the following form:

$$(4.11) \qquad \mathcal{D}P_\omega(\bar{y}_\omega) = -\gamma^2 \cdot I_\omega + J_\omega \circ (I_\omega + \beta \cdot \mathcal{D}G_\omega(\bar{y}_\omega)).$$

b. Discretization in the Hilbert Space: Let m be a fixed integer larger than 1. We define the following two finite sets of integers:

$$(4.12) \qquad N = \{0, 1, \ldots, m - 1\}$$

$$K = \{1, 2, \ldots, m\}$$

and the sample function H_N from N into E:

$$(4.13) \qquad H_N[n] = n\, \pi/2\, m. \qquad (n \;\varepsilon\; N)$$

Let the sampled function space $\mathcal{F}_N$ consist of functions y_N defined on the set of points $\{H_N\,[n] : n \;\varepsilon\; N\}$, which is the same as the set of points

$\{0, \pi/2\,m, \ldots, (m-1)\,\pi/2\,m\}$. Let the finite-dimensional vector space $\mathcal{F}_K$ consist of sequences of the form $\{y_K[1], \ldots, y_K[m]\}$.

Figure 4.1 shows the relations between the two "infinite" spaces $\mathcal{F}_E$ and $\mathcal{F}_\omega$, and the two "finite" spaces $\mathcal{F}_N$ and $\mathcal{F}_K$. The arrows indicate operators which map one of the spaces into another. The definitions of these operators are summarized in Table 4.1. A close inspection of this table shows some redundancy; in particular, $\psi = \tau \circ \phi_K$, $\psi^\dagger = \rho_K \circ \tau^{-1}$, and $\sigma = \rho \circ \psi$. Also, because $\psi^\dagger \circ \psi = I_K$, it follows (from the definition $\phi = \psi \circ \sigma^\dagger$) that $\psi^\dagger \circ \phi = \sigma^\dagger$.

The vector space $\mathcal{F}_K$ is particularly well suited for computational work. In this space we have the operator

$$(4.14)\qquad P_K = \rho_K \circ P_\omega \circ \phi_K$$

$$= \rho_K \circ \tau^{-1} \circ P_E \circ \tau \circ \phi_K$$

$$= \psi^\dagger \circ P_E \circ \psi.$$

In particular, for the Duffing problem,

$$(4.15)\qquad P_K = -\gamma^2 \cdot I_K + J_K \circ (I_K + \beta \cdot G_K - \underset{\sim}{\cos}_K),$$

where I_K is the $(m \times m)$ identity matrix; J_K is the constant diagonal matrix (cf. Eq. (3.18))

$$(4.16)\qquad J_K[k1, k2] = 0 \qquad \text{for } k1 \neq k2,$$

$$= \frac{1}{(2 \cdot k1 - 1)^2} \qquad \text{for } k1 = k2; \quad (k1, k2 \in K)$$

and for any y_K, $\underset{\sim}{\cos}_K\, y_K = \cos_K = \{1, 0, 0, \ldots, 0\}$. The nonlinear operator G_K is interpreted via the relation

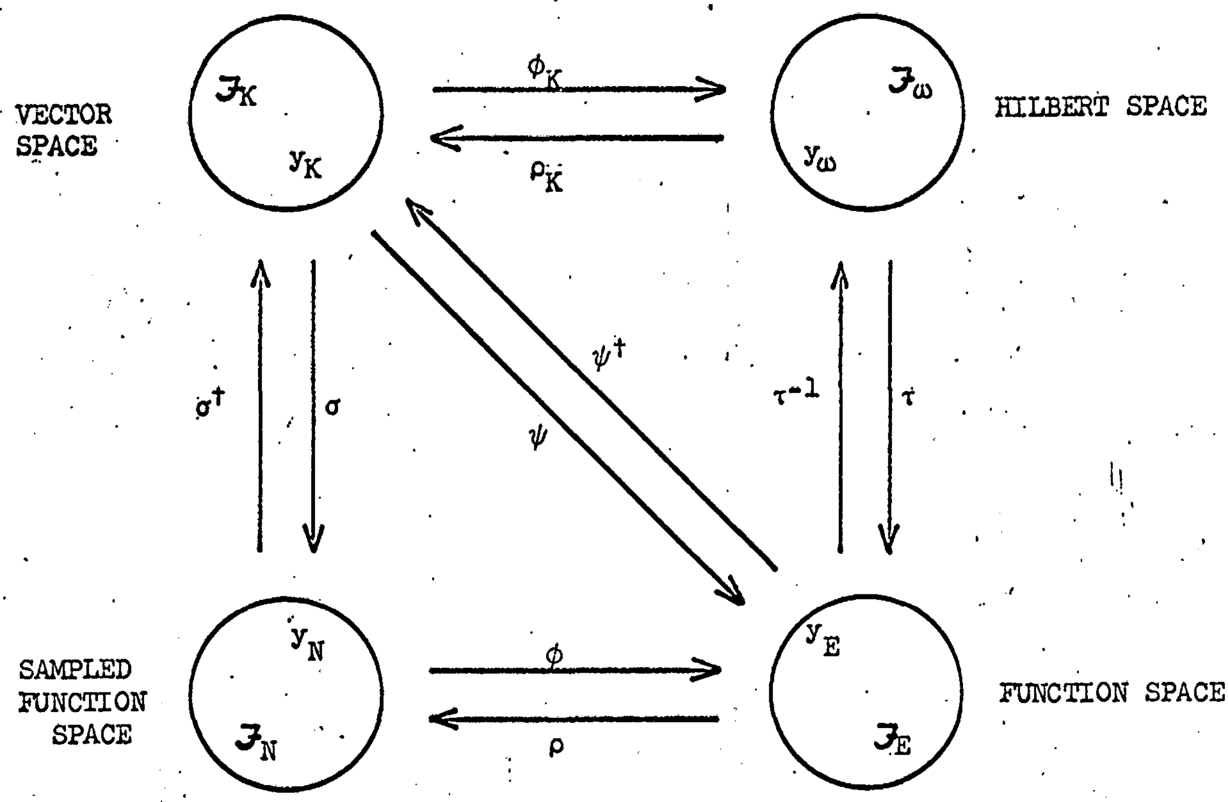

FIGURE 4.1

RELATIONS BETWEEN FINITE AND INFINITE SPACES

Table 4.1 Operators Involved in Discretization

Symbol	Name	Definition	
τ	Complete Synthesis	$y_E(\theta) = \Sigma_{k \varepsilon \omega} y_\omega [k] \cdot \cos (2k - 1) \theta$	$(\theta \varepsilon E)$
τ^{-1}	Complete Analysis	$y_\omega [k] = \frac{\pi}{4} \int_E y_E(\theta) \cdot \cos (2k - 1) \theta \, d\theta$	$(k \varepsilon \omega)$
σ	Finite Synthesis	$y_N [n] = \Sigma_{k \varepsilon K} y_K [k] \cdot \cos (2k - 1) H_N [n]$	$(n \varepsilon N)$
$\sigma^\dagger$	Finite Analysis*	$y_K [k] = \Sigma_{n \varepsilon N} w_N [n] \cdot y_N [n] \cdot \cos (2k - 1) H_N [n]$	$(k \varepsilon K)$
ψ	Mixed Synthesis	$y_E(\theta) = \Sigma_{k \varepsilon K} y_K [k] \cdot \cos (2k - 1) \theta$	$(\theta \varepsilon E)$
$\psi^\dagger$	Mixed Analysis	$y_K(k) = \frac{\pi}{4} \int_E y_E(\theta) \cdot \cos (2k - 1) \theta \, d\theta$	$(k \varepsilon K)$
ρ_K	Truncation	$y_K(k) = y_\omega(k)$	$(k \varepsilon K)$
ϕ_K	Elongation	$y_\omega(k) = y_K(k)$ for $k \varepsilon K$; $y_\omega(k) = 0$ for $k \varepsilon (\omega - K)$	
ρ	Restriction	$y_N [n] = y_E(H_N [n])$	$(n \varepsilon N)$
ϕ	Interpolation	$\phi \, y_N = \psi \circ \sigma^\dagger y_N$	$(y_N \varepsilon \mathcal{F}_N)$

*Note: $w_N [0] = {}^1/m$; $w_N [n] = {}^2/m$ for $n \neq 0$.

$$(4.17)\qquad \begin{aligned} G_K &= \psi^\dagger \circ G_E \circ \psi \\ &= \psi^\dagger \circ \phi \circ G_N \circ \rho \circ \psi \\ &= \sigma^\dagger \circ G_N \circ \sigma \end{aligned}$$

That is, in order to form $G_K\, y_K$, the "finite synthesis" operator $y_N = \sigma\, y_K$ is performed, then $G_N\, y_N$ is calculated at all of the points $H_N\,[n] = n\, \pi/2\, m$ $(n \in N)$, and finally $G_K\, y_K$ is obtained by "finite analysis" $\sigma^\dagger$ from $G_N\, y_N$.

The Frechet derivative is given by

$$(4.18)\qquad \mathcal{D}P_K(y_K) = -\gamma^2 \cdot I_K + J_K \circ (I_K + \beta \cdot \mathcal{D}G_K(y_K)).$$

Being a linear transformation on a finite-dimensional vector space, clearly $\mathcal{D}P_K(y_K)$ must have a matrix representation. We have seen how to represent I_K and J_K, but the representation of $\mathcal{D}G_K(y_K)$ as a matrix is not obvious. In Appendix II, we show that

$$(4.19)\qquad \mathcal{D}G_K(y_K)\,[k1,\, k2] = \tfrac{1}{2}\, g'_{K'}\,[\mu(k1 - k2)] + \tfrac{1}{2}\, g'_{K'}\,[\mu(k1 + k2 - 1)] \qquad (k1,\, k2 \in K)$$

where $K' = \{0, 1, 2, \ldots, m\}$, $\mu(j) = \min\,(|j|,\, 2\,m - |j|)$, (see Appendix II), and

$$(4.20)\qquad g'_{K'}\,[k] = \Sigma_{n \in N}\, w_N\,[n] \cdot g'_N\,[n] \cdot \cos\,(2k)\, H_N\,[n]. \qquad (k \in K')$$

Here $w_N\,[n]$ is defined the same way as in Table 4.1, and

$$(4.21)\qquad g'_N\,[n] = \partial\, g_N\,[n]/\partial\, y_N\,[n], \qquad g_N\,[n] = (G_N\, y_N)\,[n].$$

By analogy to $\sigma^\dagger$, we may define $\sigma'^\dagger$ by Eq. (4.20) and write $g'_{K'} = \sigma'^\dagger g'_N$.

2. <u>The Computational Procedure</u>.

Solutions to the problem $P_K\, y_K = 0$ are obtained by the "(ordinary) Newton iteration method", as follows:

$$(4.22) \qquad y_K^{(\nu+1)} = T\, y_K^{(\nu)} = -\mathcal{D}P_K(y_K^{(\nu)})^{-1} \circ P_K\, y_K^{(\nu)} + y_K^{(\nu)}$$

$$(\nu = 0, 1, 2, \ldots)$$

where P_K and $\mathcal{D}P_K(y_K)$ are defined by Eq's. (4.15, 4.18). Starting, then, from $y_K^{(0)} = \psi^\dagger\, y_E$ or $y_K^{(0)} = \sigma^\dagger\, y_N$, the following computational steps are performed at each iteration on the current value of the vector y_K:

(4.23)

(a) $y_N := \sigma\, y_K$;

(b) $g_N := G_N\, y_N$;

$g'_N := D\, G_N\, y_N$;

(c) $g_K := \sigma^\dagger\, g_N = G_K\, y_K$;

$g'_{K'} := \sigma'^\dagger\, g'_N$;

(d) $\mathcal{D}G_K(y_K)\,[k1, k2] := \frac{1}{2}\, g'_{K'}\,[\mu(k1 - k2)]$

$+ \frac{1}{2}\, g'_{K'}\,[\mu(k1 + k2 - 1)]$; $\qquad (k1, k2 \in K)$

(e) $P_K\, y_K := -\gamma^2 \cdot y_K + J_K \circ (y_K + \beta \cdot g_K - \cos_K)$;

(f) $\mathcal{D}P_K(y_K) := -\gamma^2 \cdot I_K + J_K \circ (I_K + \beta \cdot \mathcal{D}G_K(y_K))$;

(g) Solve: $\mathcal{D}P_K(y_K)\ \eta_K = -P_K\, y_K$;

(h) $y_K := y_K + \eta_K$.

These steps are repeated iteratively until $\|\eta_K\|$ is smaller than some pre-assigned tolerance.

The matrix formed at step (f) is also used in eigenvalue calculations.

B. Numerical Results

The purpose of this section is to present detailed quantitative information concerning solutions of the Duffing problem (Eq. 1.1) for various values of the parameters β and γ (particularly in the region $0 < \beta \leq 1$, $0 \leq \gamma < 1$) as well as to describe the procedure by which the numbers were calculated. Data is given to substantiate the theoretical results of the previous chapters [11] in particular (a) to illustrate the assertion of Theorem 2, and (b) to demonstrate rigorously the behavior of the "principal" branch of the solution in the neighborhood of the resonances of low order.

Theorem 2 asserts that under certain conditions on β a sequence of values of γ exists on which the solution to the Duffing problem tends to the solution of the "reduced" problem for $\gamma = 0$. In the first part of this section, we examine the validity of the hypotheses of Theorem 2 for various values of β, and then we show how the quantity $A \cdot \eta_o$ of Lemma 5 (which must be less than $\frac{1}{4}$ to insure the existence of a solution) depends upon γ and the role, in this dependence, of the eigenvalues of the variational differential equation.

In the second part of this section, we start from certain non-resonance values of $\gamma = \gamma_o$ for which we can assert the uniqueness of the solution in a certain neighborhood of the "reduced" solution. Having found a solution for such a value of γ_o in the designated neighborhood, we find from Theorem 1 an interval of values of γ on which the solution has a unique connected branch. We define a new γ_o in this interval and continue the calculation, extending the unique connected branch until we approach a region of resonance. Starting from a different non-resonant value of γ_o, we approach the same resonance region from the other side. The results of such calculations for the resonances of order 5, 7, 9, and 11 are reported, and we find in each of these cases that it is possible to extend the separate branches, corresponding to values of γ which approach the region of resonance from above and from below, until a common value of γ is reached for which two different solutions are obtained, one being connected to each branch.

1. High-Order Resonance

Here we are dealing with the situation in which a solution is "known" for $\gamma = 0$, and we want to make certain assertions about the amount by which the solution changes when γ is changed to a different value. The first computational task, then, is to obtain this "known" solution.

a. The "reduced" solution: $\gamma = 0$. Since the machinery (described in Section A, paragraph 2) for solving the Duffing problem in the general case must be available, we can obtain the solution of the reduced equation by simply putting in $\gamma = 0$. As a starting value, we use the solution of the trivial problem

(4.24) $$y(\theta) = \cos \theta$$

which is obtained by setting $\gamma = 0$ and $\beta = 0$. The corresponding starting vector is given by

(4.25) $$y_K[1] = 1.0$$

$$y_K[k] = 0.0 \quad \text{for } k \neq 1, \quad k \in K.$$

We perform the iteration procedure of Eq. (4.23) with $\gamma = 0$, $\beta = 1$ until $\|\eta_K\| < 10^{-14}$. The first 34 components are larger than 10^{-14}. These are listed in Table 4.2.

TABLE 4.2. SOLUTION OF THE REDUCED EQUATION FOR BETA=1.0

K	2K-1	K'TH COMPONENT OF Y
1	1	0.72898856416250
2	3	-0.05618862750920
3	5	0.01214496257033
4	7	-0.00344531646982
5	9	0.00111301116134
6	11	-0.00038820329000
7	13	0.00014243895049
8	15	-0.00005418104009
9	17	0.00002117238511
10	19	-0.00000844848240
11	21	0.00000342805537
12	23	-0.00000141010665
13	25	0.00000058667667
14	27	-0.00000024644842
15	29	0.00000010438430
16	31	-0.00000004452950
17	33	0.00000001911505
18	35	-0.00000000825084
19	37	0.00000000357890
20	39	-0.00000000155920
21	41	0.00000000068199
22	43	-0.00000000029936
23	45	0.00000000013183
24	47	-0.00000000005823
25	49	0.00000000002579
26	51	-0.00000000001145
27	53	0.00000000000510
28	55	-0.00000000000227
29	57	0.00000000000102
30	59	-0.00000000000045
31	61	0.00000000000020
32	63	-0.00000000000009
33	65	0.00000000000004
34	67	-0.00000000000002

THE REDUCED EQUATION IS OBTAINED BY SETTING GAMMA=0 IN DUFFING'S DIFFERENTIAL EQUATION. THE K'TH COMPONENT OF Y, LISTED HERE, IS THE COEFFICIENT OF COS(2K-1) IN THE ODD-COSINE FOURIER EXPANSION OF THE SOLUTION.

b. The effect of the parameter β. As our first task, we set out to find values of β in the range $0 < \beta \leq 1$ for which Theorem 2 holds. Since $3\beta \cdot y(\theta) \geq 0$, it is obvious that $q'(\theta) \geq 1$ for all $\theta \in E = [0, \pi/2]$. Therefore, we can estimate $\|1/q'\| \leq 1$. Hence, assumption (i) will be satisfied if $6\beta \cdot (\|y_0\| + r) \leq 1/2 \cdot r$, or if

$$(4.26) \qquad r^2 + \|y_0\| \cdot r - \frac{1}{12 \cdot \beta} \leq 0.$$

This condition obviously holds for r sufficiently small; the largest value for which it is true is obtained by replacing the inequality sign by equality: the resulting quadratic equation has one positive real root,

$$(4.27) \qquad r = \tfrac{1}{2}\left(-\|y_0\| + \sqrt{\left(\|y_0\|^2 + \frac{1}{3 \cdot \beta}\right)}\right).$$

The values of $\|y_0\|$ and of r are shown in the first columns of Table 4.3 Assumption (ii) is satisfied with $a_1 = 1$, $a_2 = 1 + 3 \cdot \beta \cdot \|y\|^2$. For Assumption (iii), we estimate η_0 by $\eta_0 \leq \|y_0\|$ since $\|1/q'\| \leq 1$, and we use the fact that $A = 1/2\ r$. The results are given in the remaining columns of Table 4.3. It is clear that the value $\beta = 0.1$ is small enough, but these estimates (which are undeniably quite conservative) do not justify the value $\beta = 0.2$.

The assumptions of Theorem 2 are used to prove a priori (that is, from an examination of the "reduced" problem only) the existence of a sequence of values of γ on which the solution of the Duffing problem tends to the solution of the "reduced" problem for $\gamma = 0$. On the other hand, when we actually calculate the solution with $\beta = 1.0$, we find a set of values of γ for which this convergence appears to be taking place. Of course, we cannot demonstrate the existence of an infinite sequence of points tending to a certain value, merely by exhibiting a finite ordered set of points, each one being closer than the previous one to the desired value.

Table 4.3. Effect of the Parameter β

β	$\|y_o\|$	r	A	$A\cdot\eta_o$
0.02	0.986	1.607	0.311	0.307
0.05	0.966	0.895	0.559	0.540
0.10	0.939	0.557	0.898	0.843
0.20	0.896	0.338	1.479	1.325
0.40	0.835	0.201	2.488	2.078
1.00	0.733	0.100	5.000	3.665

Here $\|y_o\|$ is the norm of the solution of the reduced equation for the corresponding value of β, and r is obtained from Eq. (4.27). Also $A = 1/2\,r$ and we take $\eta_o = \|y_o\|$.

Hence (until closer estimates are found), the assertion of the existence of the desired infinite sequence for values of β as large as 1.0 must be regarded as a conjecture. The calculations (with $\beta = 1.0$) for values of γ down to about 0.067 are described in the following paragraphs.

c. Matrix form of the eigenvalue problem. We recall (compare Eq. (3.21), (3.28)) that the sequence of values of γ (described in Theorem 2) on which the solution of the Duffing problem tends to the "reduced" solution is to be chosen by taking $\gamma^2 = \bar{\lambda}$, for values of $\bar{\lambda}$ which are halfway between the eigenvalues of the variational differential equation.

Comparing Chapter I, Section B, paragraph 1 with Appendix B, we see that the matrix which we form at step (f) of the procedure of Eq. (4.23) is suitable for the calculation of eigenvalues of the variational problem. We take as y_K the solution which is listed in Table 4.2 and perform steps (a) to (d) and step (f) of the procedure of Eq. (4.23), with $\gamma = 0$. The resulting matrix is symmetric (as we see from Eq. (4.19), since $\mu\ (k1 - k2) = \mu\ (k2 - k1)$), and so we can find the eigenvalues by Givens' method (see White, Ref. 11, pp 398 ff.).

Table 4.4 lists the ten largest eigenvalues, along with the corresponding values of $\lambda_j \cdot (2 \cdot j - 1)^2$. Further values (not listed) indicate that the sequence $\{\lambda_j \cdot (2 \cdot j - 1)^2\}$ approaches a limit which is close to 1.76746. Corresponding resonant values of $\gamma = \lambda^{\frac{1}{2}}$ are given by $\gamma \cdot (2 \cdot j - 1) \approx 1.32946$.

d. Convergence conditions for Newton's Method. We check the conditions of Lemma 5, Chapter II, for various values of γ in order to see whether we can assert, for some r, the existence and uniqueness of a solution of the Duffing problem in the neighborhood $S(y_o, r)$ of the solution y_o of the reduced equation. We choose about 20 values of γ between each pair

Table 4.4

Largest Eigenvalues of the Variational Problem

j	$2j-1$	λ_j	$\gamma_j = \sqrt{\lambda_j}$	$\lambda_j \cdot (2j-1)^2$
1	1	2.14292	1.46387	2.14292
2	3	0.20073	0.44803	1.80655
3	5	0.07101	0.26648	1.77529
4	7	0.03610	0.19001	1.76905
5	9	0.02182	0.14772	1.76760
6	11	0.01461	0.12085	1.76729
7	13	0.01046	0.10226	1.76726
8	15	0.00785	0.08863	1.76729
9	17	0.00612	0.07820	1.76732
10	19	0.00490	0.06997	1.76735

The column headed 2j-1 gives the order of harmonic resonance corresponding to the eigenvalue λ_j. The last column shows that λ_j approaches a constant multiple of $(2j-1)^{-2}$.

of eigenvalues: specifically, we take $\gamma_j = 13.2946/j$ for $j = 1, 2, \ldots, 199$. The resonances are near $j = 10, 30, 50, \ldots, 190$. We may think of $j/10$ as the "order" of the resonance, since it corresponds to the frequency of the resonating harmonic.

Figures 4.2 through 4.6 show the main steps in this computation. Noting (compare Eq. (3.9)) that

$$(4.28) \qquad D^2P(\tilde{y})\, y_1\, y_2\, (\theta) = 6 \cdot \beta \cdot J \circ \tilde{y}\,(\theta) \cdot y_1(\theta) \cdot y_2(\theta)$$

we see that

$$(4.29) \qquad \|\Gamma_o \circ D^2P(\tilde{y})\| \leq 6 \cdot |\beta| \cdot \|\Gamma_o \circ J\| \cdot \|\tilde{y}\|$$

$$\leq 6 \cdot |\beta| \cdot \|\Gamma_o \circ J\| \cdot (\|y_o\| + r).$$

Recalling that $\Gamma_o = DP(y_o)^{-1}$, we see that the norm $\|\Gamma_o \circ J\|$ is large when the smallest eigenvalue of $J^{-1} \circ DP(y_o)$ is near zero. Figure 4.2 shows the 3 or 4 smallest eigenvalues of $J^{-1} \circ DP(y_o)$ for the chosen values of γ. Figure 4.3 shows the corresponding <u>largest</u> eigenvalues of $(\Gamma_o \circ J)$, which are the reciprocals of the values shown in Fig. 4.2 In Fig. 4.4, the norm $\|\Gamma_o \circ J\|$ is shown, which is obtained by taking the absolute value of the eigenvalues of Fig. 4.3. The special diamond-shaped points in Figs. 4.2 - 4.4 indicate the values of γ for which the two largest eigenvalues of $\Gamma_o \circ J$ are equal in magnitude: these are the points at which $\|\Gamma_o \circ J\|$ is locally smallest.

Looking no farther than Fig. 4.4, one might conjecture the impossibility of finding a sequence of values of γ tending to zero (moving indefinitely to the left) on which the hypotheses of Lemma 5 are valid; however, it is η_o which saves the day. Figure 4.5 shows $\|\eta_o\|$, and Fig. 4.6 shows the product $\|\Gamma_o \circ J\| \cdot \|\eta_o\|$ from the two previous graphs,

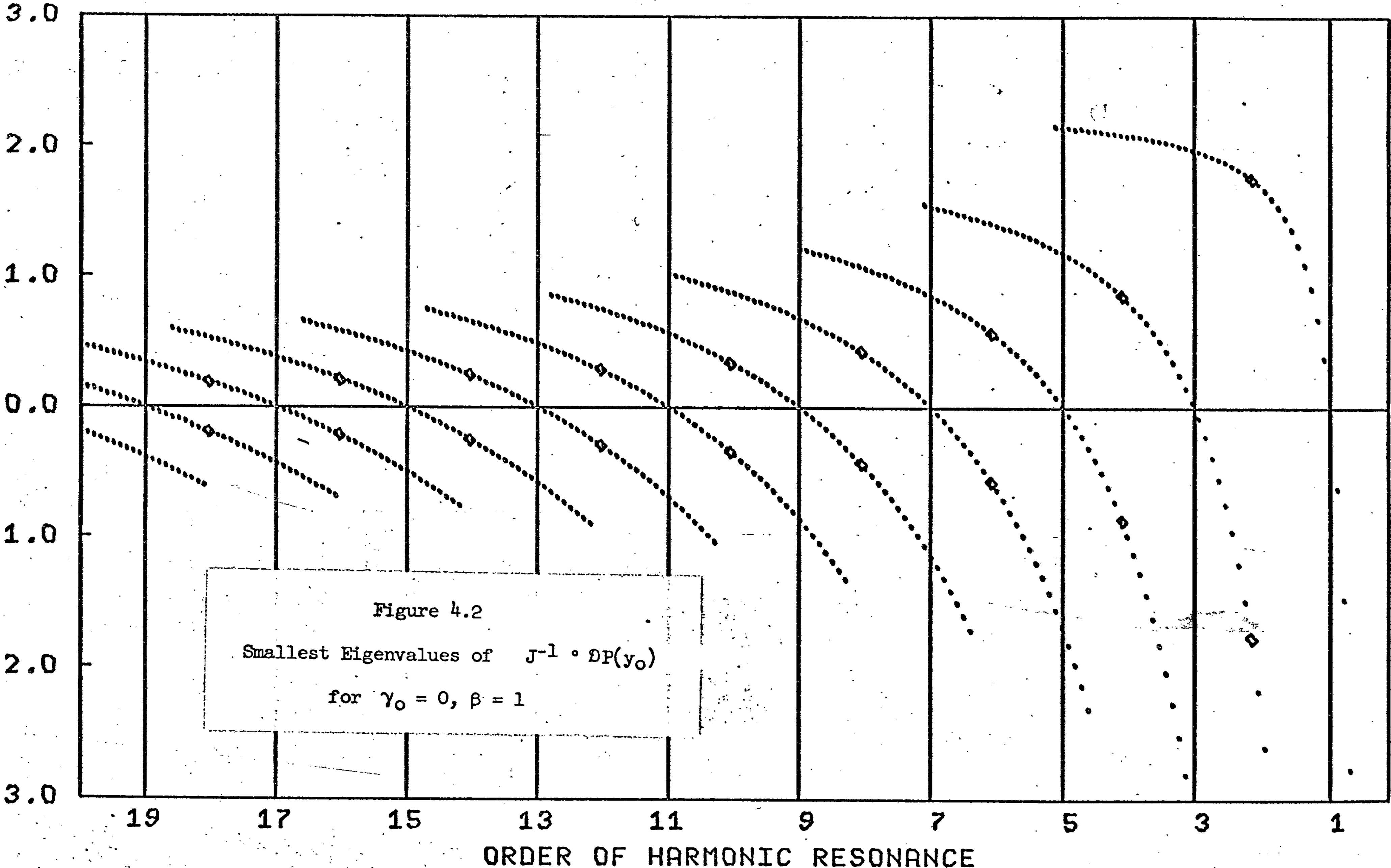

Figure 4.2

Smallest Eigenvalues of $J^{-1} \circ \mathcal{D}P(y_o)$

for $\gamma_o = 0$, $\beta = 1$

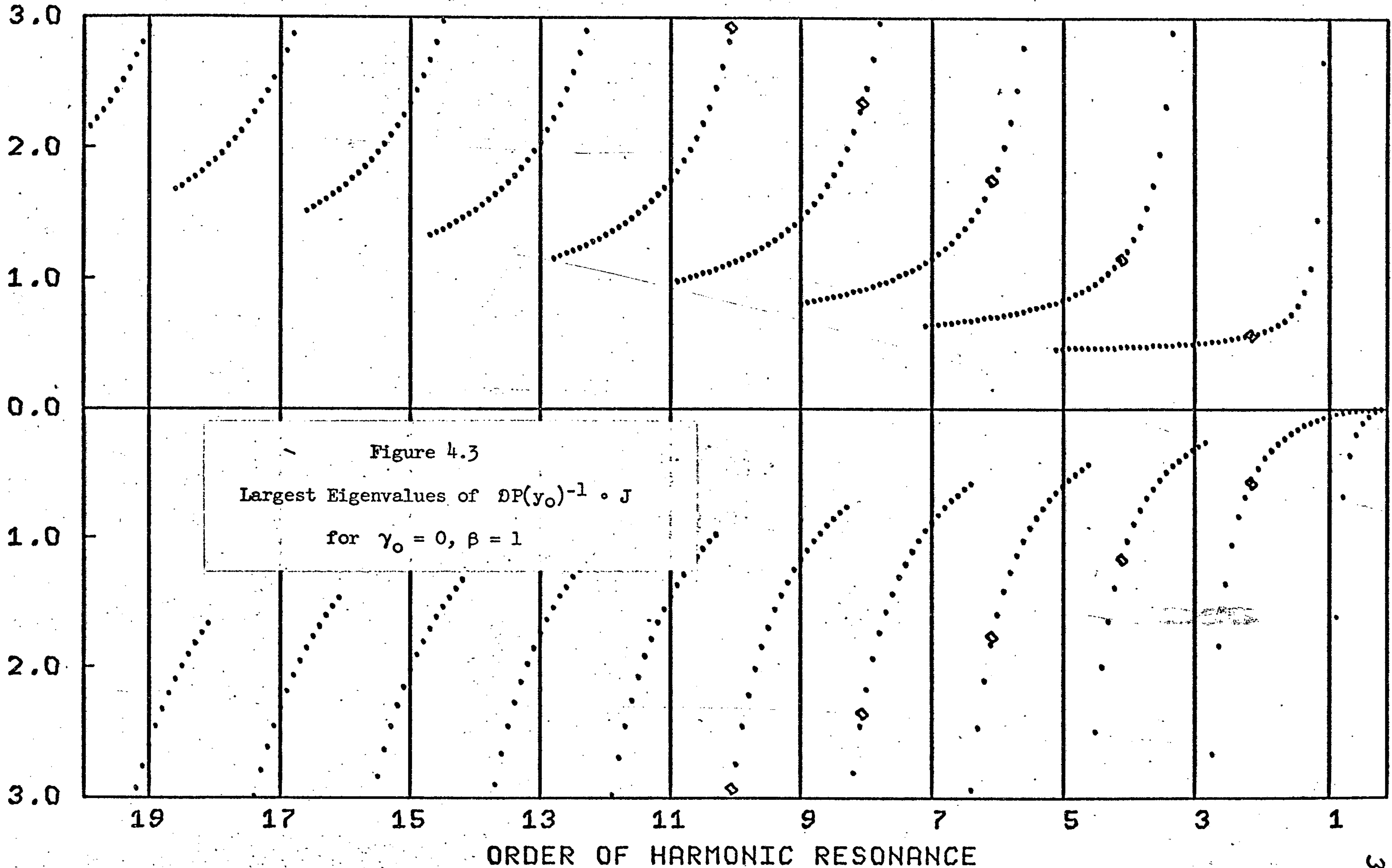

Figure 4.3

Largest Eigenvalues of $DP(y_o)^{-1} \circ J$

for $\gamma_o = 0$, $\beta = 1$

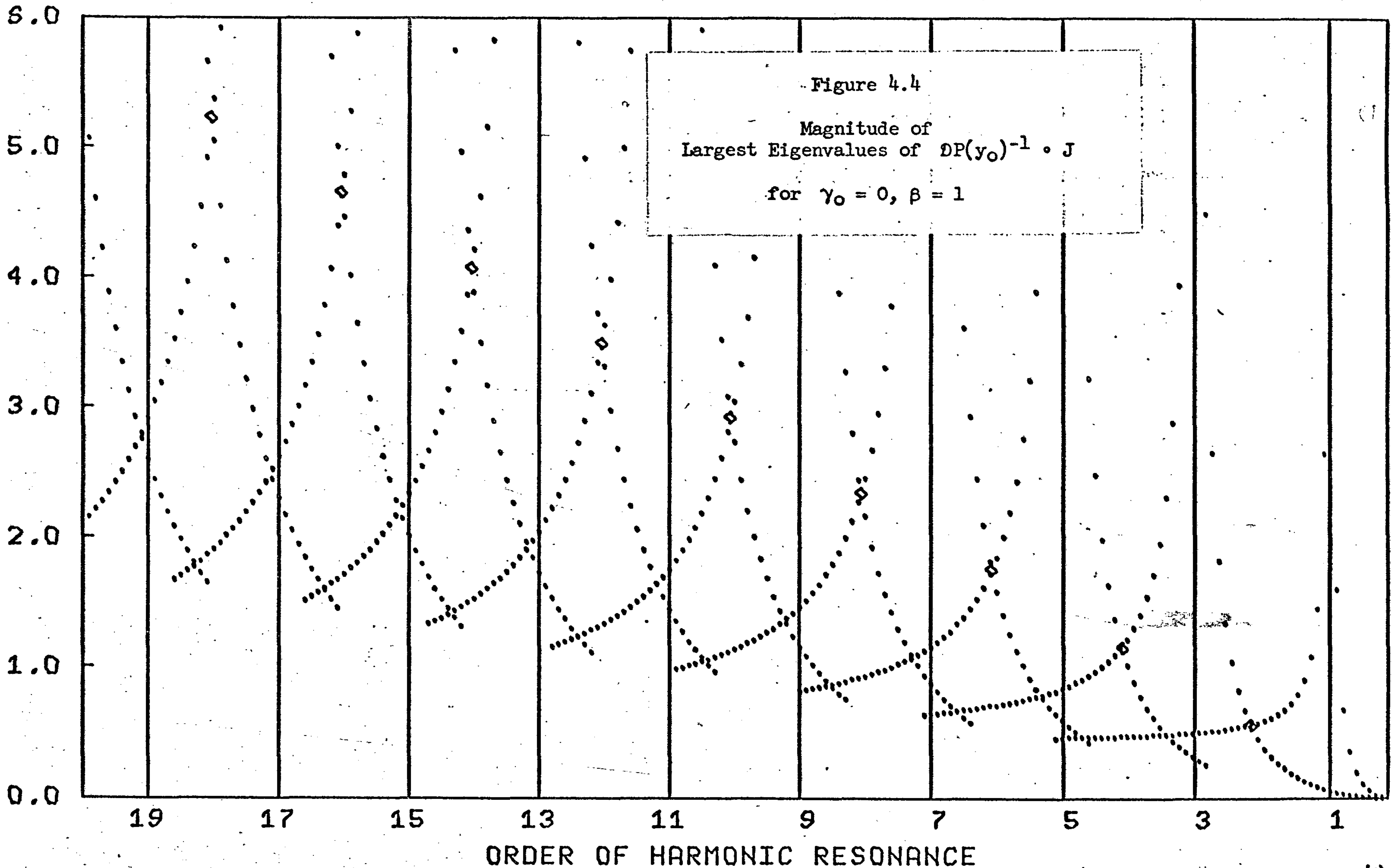

Figure 4.4

Magnitude of Largest Eigenvalues of $\mathcal{D}P(y_o)^{-1} \circ J$

for $\gamma_o = 0$, $\beta = 1$

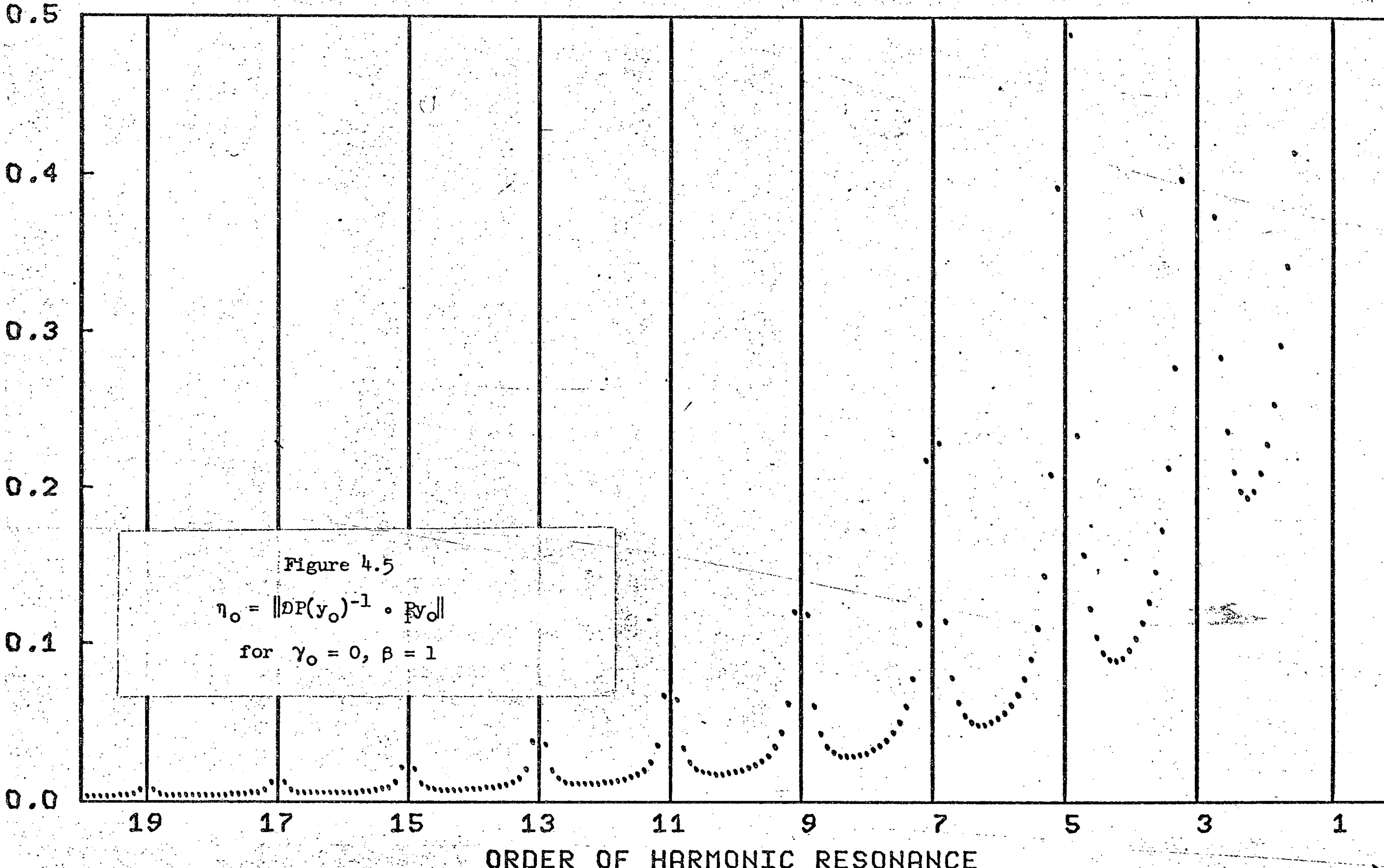

Figure 4.5

$\eta_o = \|DP(y_o)^{-1} \circ Py_o\|$

for $\gamma_o = 0$, $\beta = 1$

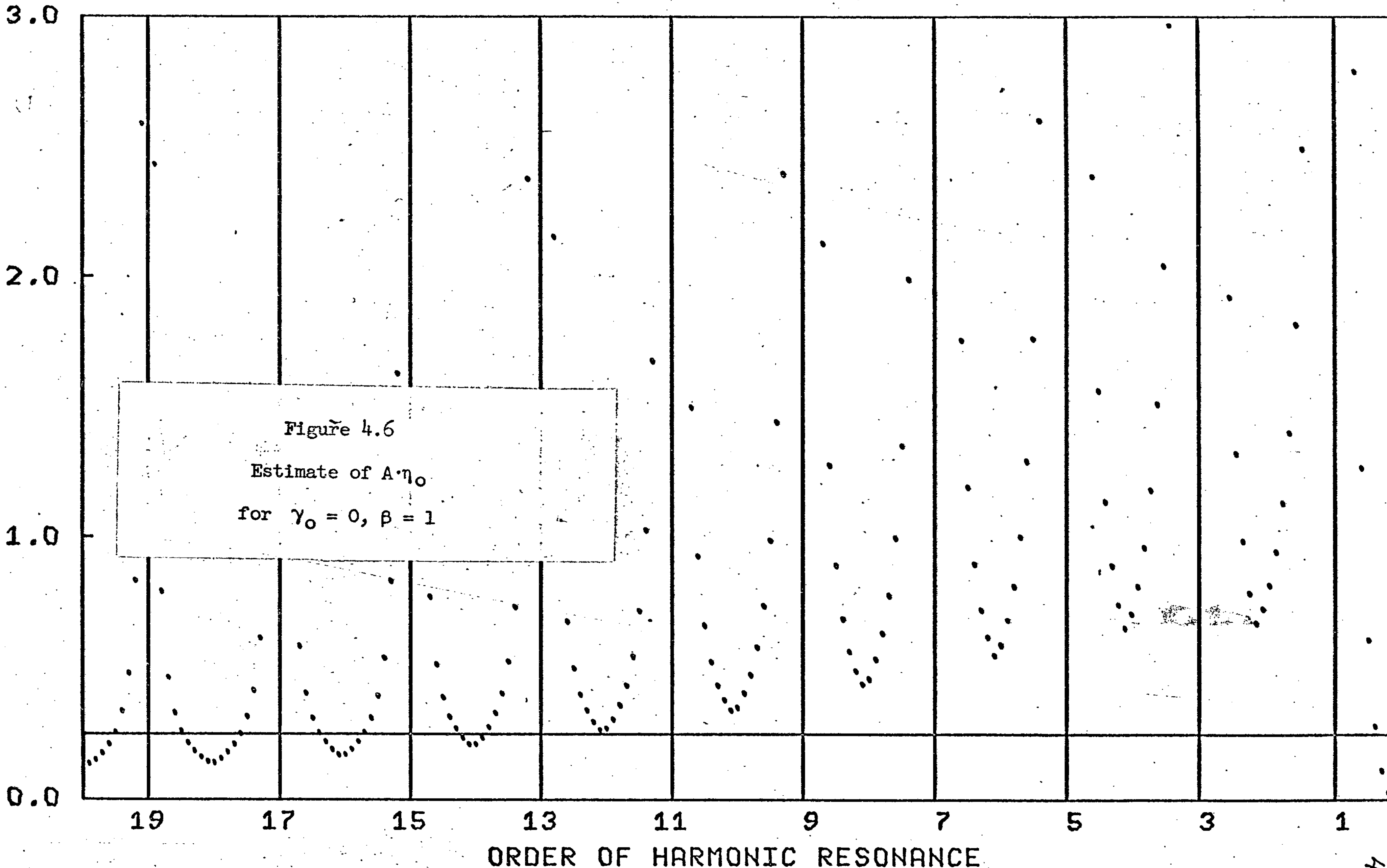

Figure 4.6

Estimate of $A \cdot \eta_o$

for $\gamma_o = 0$, $\beta = 1$

multiplied also by the value (independent of γ) $6 \cdot |\beta| \cdot (\|y_o\| + r)$ as required by Eq. (4.29). Those values of γ for which the points in Fig. 4.6 lie below the horizontal line at $\frac{1}{4}$ are the ones for which all of the hypotheses of Lemma 5 are satisfied, so that we can guarantee the existence of a unique solution in a neighborhood of y_o.

The points shown in Fig. 4.6 are also listed in Table 4.7 which is discussed in the last part of this section.

e. Remark. The matrix $\mathcal{D}G_K(y_K)$ is close to a diagonal matrix. We recall (Eq. (4.19)) that the [k1, k2] element of this matrix is $\frac{1}{2}$ of the sum of two components of the vector $g'_{K'}$. But only the first few of these components are significant, as shown in Table 4.5, and the composition of the matrix, shown in Table 4.6, is such that these few relatively large components occur only near the diagonal and in the upper corner. A very rough approximation to this matrix is obtained by ignoring all components of the vector except the first; then, the matrix is a diagonal matrix with all diagonal elements equal to 0.901.

For $\gamma = 0$, $\mathcal{D}P_K(y_K) = J_K \circ (I_K + \beta \cdot \mathcal{D}G_K(y_K))$; and our crude approximation would give a diagonal matrix whose j'th diagonal element equals $(1.901)/(2 \cdot j - 1)^2$. Thus, $\mathcal{D}P_K(y_K) - \gamma^2 \cdot I_K$ has a zero on the diagonal, according to our crude approximation, when $\gamma^2 = (1.901)/(2 \cdot j - 1)^2$. This approximation may be compared to the actual values $\gamma^2 = \lambda_j \simeq 1.767/(2 \cdot j - 1)^2$. The discrepancy is, of course, due to the not-insignificant values of the off-diagonal elements.

When γ is near one of these critical values, there is a small value in some position on the diagonal. Our crude approximation leads us to expect that $\Gamma_o = \mathcal{D}P_K(y_K)^{-1}$ will have a large number in the corresponding position on the diagonal, and this is indeed the case. Thus, $\|\Gamma_o\|$ is large. However, $\|\Gamma_o \circ J\|$ is much smaller, especially if the critical

Table 4.5 Components of the Vector $g'_{K'}$

k	$g'_{K'}[k]$
0	1.80211
1	0.33603
2	-0.04785
3	0.01778
4	-0.00354
5	0.00118
6	-0.00042
7	0.00016
8	-0.00006
9	0.00002
10	-0.00001
11	0.00000

Table 4.6 Composition of the Matrix $DG_K(y_K)$

	0 + 1	1 + 2	2 + 3	3 + 4	4 + 5	...
	1 + 2	0 + 2	1 + 4	2 + 5	3 + 6	...
(a)	2 + 3	1 + 4	0 + 5	1 + 6	2 + 7	...
	3 + 4	2 + 5	1 + 6	0 + 7	1 + 8	...
	4 + 5	3 + 6	2 + 7	1 + 8	0 + 9	...
	...	...	...	...	...	...
	0 + 1	1 + 2	2 + 3	3	-	...
	1 + 2	0 + 3	1	2	3	...
(b)	2 + 3	1	0	1	2	...
	3	2	1	0	1	...
	-	3	2	1	0	...
	...	...	...	...	...	...

In (a), the indices of all components of $g'_{K'}$ are listed as they appear in the matrix. The effect of ignoring all components beyond the third is shown in (b).

diagonal element is not too near the top, since J multiples the $j^{\underline{th}}$ diagonal element by $1/(2j-1)^2$. In this case, we cannot estimate $\|\Gamma_o \circ J\|$ at all well by $\|\Gamma_o\| \cdot \|J\|$ since $\|J\| = 1$.

This difficulty of making close estimates is, of course, one of the annoyances of numerical analysis. We see it in one of its simpler forms whenever we are multiplying two objects without being able to prove in advance that they are nearly orthogonal. We will encounter another example in the next section, when we try to estimate $\sup\{\|\Gamma_o \circ J\,\tilde{y}\| : \tilde{y} \in S(y_o, r)\}$ by computing $\|\Gamma_o \circ J\| \cdot (\|y_o\| + r)$, since we find it very difficult to take advantage of the fact that $\tilde{y}$ has a small "tail" which further reduces the effect of the large element of Γ_o.

2. Low-Order Resonance

We would like to define the "principal branch" of the solution, as γ varies, in a natural way. Accordingly, we begin by calculating the size of the neighborhood about the "reduced" solution within which the actual solution is unique. If there are any values of γ such that a solution can be shown to exist in this neighborhood, then that solution (which is unique) for such a value of γ will be called the principal solution. If we can then show the existence of a connected branch of the solution as γ varies, that branch will be called a "principal branch".

a. Starting values. We recall the remark made at the end of Chapter II, that the uniqueness of a solution (whose existence is known or can be otherwise assumed) depends only upon the Lipschitz condition. Specifically, if $\Gamma_o = \mathcal{D}P(y_o)^{-1}$ exists, and if for all $\tilde{y} \in S(y_o, r)$ it is true that

(4.30) $$\|\Gamma_o \circ \mathcal{D}^2 P(\tilde{y})\| \leq A,$$

where $A < {}^1/r$ (compare Lemma 5 of Chapter II), then we conclude that $\|D^2T(\tilde{y})\| = \|\Gamma_o \circ D^2P(\tilde{y})\| \leq A$ and we may take $L = A \cdot r < 1$ in Lemma 2. Here T is defined as in Eq. (2.10):

$$(4.31) \qquad Ty = DP(y_o)^{-1} \circ Py + y$$

As in Chapter II, we conclude that T satisfies a Lipschitz condition with $L < 1$, so that T has at most one fixed point and therefore P has at most one root in $S(y_o, r)$.

We now wish to apply this result using as y_o the solution of the reduced problem ($\gamma = 0$). As in the high-order resonance case, we shall use the estimate (compare Eq. (4.29)).

$$(4.32) \qquad \|\Gamma_o \circ D^2P(\tilde{y})\| \leq 6 \cdot |\beta| \cdot \|\Gamma_o \circ J\| \cdot (\|y_o\| + r),$$

where $\|\Gamma_o \circ J\|$ is the reciprocal of the smallest eigenvalue of $(\Gamma_o \circ J)^{-1} = J^{-1} \circ DP(y_o)$, and is pictured in Fig. 4.4 for $\beta = 1$. We can satisfy Eq. (4.30) if we can find a value of r such that

$$(4.33) \qquad 6 \cdot |\beta| \cdot \|\Gamma_o \circ J\| \cdot (\|y_o\| + r) < \frac{1}{r}.$$

We choose $k < 1$ (say $k^2 = 0.98$) and solve for r in the following equation:

$$(4.34) \qquad 6 \cdot x \cdot (y + \frac{r}{k^2}) = \frac{1}{r}$$

with $x = |\beta| \cdot \|\Gamma_o \circ J\|$ and $y = \|y_o\|$. If we write

$$(4.35) \qquad F(r) = r^2 + k^2 \cdot y \cdot r - k^2/6x$$

we see that Eqs. (4.34) and (4.33) are satisfied when $F(r) = 0$. It is clear (since x, y, and k^2 are non-negative) that F(r) has a zero between 0 and $k/\sqrt{6x}$. This value of r is listed in Table 4.7, for the same 199 values of γ discussed earlier in this section: i.e., $\gamma_j = 13.2946/j$ for $j = 1, 2, \ldots, 199$, and with $\beta = 1$.

TABLE 4.7. EXISTENCE OF PRINCIPAL SOLUTION (PAGE 1 OF 5)

J	GAMMA(J)	Y-Y(0)		R(0)	R	A*ETA(0)
1	13.29460	0.73693	P	4.99398	3.78601	0.36169
2	6.64730	0.75435	P	2.28662	1.86667	0.37433
3	4.43153	0.78476	P	1.36996	1.20744	0.40193
4	3.32365	0.83060	P	0.90005	0.85732	0.45521
5	2.65892	0.89615		0.60792	0.62421	0.55783
6	2.21577	0.98978		0.40379	0.44143	0.77548
7	1.89923	1.13276		0.24826	0.27403	1.38654
8	1.66182	1.47410		0.12096	0.05251	11.16711
9	1.47718	0.81152		0.00888	0.21229	3.32832
10	1.32946	0.66136		0.07746	0.20051	2.60326
11	1.20860	0.54596		0.13176	0.19311	2.05584
12	1.10788	0.45656		0.16940	0.18875	1.64335
13	1.02266	0.38684		0.19699	0.18653	1.33200
14	0.94961	0.33219		0.21800	0.18577	1.09636
15	0.88631	0.28928		0.23444	0.18599	0.91757
16	0.83091	0.25571		0.24764	0.18686	0.78173
17	0.78204	0.22978	P	0.25842	0.18822	0.67884
18	0.73859	0.21035	P	0.26733	0.18993	0.60198
19	0.69972	0.19675	P	0.27481	0.18131	0.57281
20	0.66473	0.18863	P	0.28120	0.15851	0.60307
21	0.63308	0.18605	P	0.28665	0.13771	0.65893
22	0.60430	0.18937	P	0.24993	0.11881	0.74993
23	0.57803	0.19945	P	0.21119	0.10176	0.89139
24	0.55394	0.21774		0.17489	0.08649	1.10855
25	0.53178	0.24664		0.14066	0.07303	1.44224
26	0.51133	0.29001		0.10821	0.06129	1.96339
27	0.49239	0.35417		0.07725	0.05086	2.81627
28	0.47481	0.45140		0.04754	0.04010	4.46917
29	0.45843	0.63106		0.01886	0.02069	12.20294
29	0.45843	0.93195		0.01887	0.02485	17.53804
30	0.44315	0.36578		0.00880	0.03248	5.13713
31	0.42886	0.27804		0.03298	0.04110	3.02597
32	0.41546	0.22246		0.05372	0.04838	2.03234
33	0.40287	0.18448		0.07176	0.05488	1.47440
34	0.39102	0.15725		0.08761	0.06085	1.12781
35	0.37985	0.13706		0.10165	0.06637	0.89833
36	0.36929	0.12177		0.11419	0.07149	0.73934
37	0.35931	0.11009	P	0.12546	0.07625	0.62590
38	0.34986	0.10125	P	0.13565	0.08064	0.54374
39	0.34089	0.09479	P	0.14490	0.08470	0.48440

TABLE 4.7. EXISTENCE OF PRINCIPAL SOLUTION (PAGE 2 OF 5)

J	GAMMA(J)	Y-Y(0)		R(0)	R	A*ETA(0)
40	0.33236	0.09054	P	0.15334	0.08506	0.45874
41	0.32426	0.08852	P	0.15788	0.07461	0.50298
42	0.31654	0.08899	P	0.13807	0.06467	0.57422
43	0.30918	0.09253	P	0.11890	0.05527	0.68803
44	0.30215	0.10020	P	0.10032	0.04646	0.87358
45	0.29544	0.11390		0.08226	0.03830	1.18733
46	0.28901	0.13706		0.06468	0.03097	1.74329
47	0.28286	0.17623		0.04753	0.02471	2.77458
48	0.27697	0.24373		0.03078	0.01963	4.78308
49	0.27132	0.36735		0.01436	0.01414	10.00623
50	0.26589	0.26561		0.00176	0.01348	7.94256
51	0.26068	0.17717		0.01681	0.01982	3.57109
52	0.25567	0.13272		0.03045	0.02530	2.09095
53	0.25084	0.10631		0.04290	0.03049	1.39127
54	0.24620	0.08906		0.05431	0.03544	1.00456
55	0.24172	0.07706		0.06482	0.04017	0.76890
56	0.23740	0.06832	P	0.07452	0.04466	0.61499
57	0.23324	0.06179	P	0.08352	0.04891	0.50935
58	0.22922	0.05686	P	0.09189	0.05293	0.43441
59	0.22533	0.05319	P	0.09970	0.05673	0.38030
60	0.22158	0.05062	P	0.10699	0.05922	0.34716
61	0.21794	0.04909	P	0.10893	0.05217	0.37837
62	0.21443	0.04874	P	0.09588	0.04536	0.42784
63	0.21103	0.04983	P	0.08309	0.03879	0.50675
64	0.20773	0.05293	P	0.07055	0.03246	0.63742
65	0.20453	0.05916		0.05825	0.02641	0.86776
66	0.20143	0.07076		0.04616	0.02071	1.31211
67	0.19843	0.09281		0.03427	0.01552	2.27711
68	0.19551	0.13814		0.02256	0.01124	4.64716
69	0.19268	0.23987		0.01103	0.00820	11.05415
70	0.18992	0.20343		0.00035	0.00650	12.02436
71	0.18725	0.11492		0.01124	0.01142	3.85163
72	0.18465	0.08065		0.02137	0.01584	1.95165
73	0.18212	0.06286		0.03082	0.02014	1.20019
74	0.17966	0.05215		0.03968	0.02431	0.82814
75	0.17726	0.04506	P	0.04799	0.02831	0.61677
76	0.17493	0.04007	P	0.05582	0.03215	0.48480
77	0.17266	0.03640	P	0.06320	0.03582	0.39680
78	0.17044	0.03365	P	0.07017	0.03933	0.33530
79	0.16829	0.03158	P	0.07676	0.04268	0.29100

TABLE 4.7. EXISTENCE OF PRINCIPAL SOLUTION (PAGE 3 OF 5)

J	GAMMA(J)	Y-Y(0)		R(0)	R	A*ETA(0)
80	0.16618	0.03008	P	0.08301	0.04564	0.25995
81	0.16413	0.02909	P	0.08325	0.04036	0.28231
82	0.16213	0.02867	P	0.07349	0.03521	0.31667
83	0.16018	0.02893	P	0.06387	0.03019	0.37021
84	0.15827	0.03017	P	0.05440	0.02529	0.45760
85	0.15641	0.03295	P	0.04504	0.02054	0.61152
86	0.15459	0.03853		0.03582	0.01594	0.91505
87	0.15281	0.04995		0.02670	0.01157	1.62461
88	0.15107	0.07661		0.01769	0.00762	3.76404
89	0.14938	0.15550		0.00879	0.00482	12.04866
90	0.14772	0.15655		0.00003	0.00347	16.97923
91	0.14609	0.07307		0.00855	0.00747	3.67476
92	0.14451	0.04867		0.01661	0.01120	1.63730
93	0.14295	0.03740		0.02423	0.01485	0.95234
94	0.14143	0.03103	P	0.03146	0.01839	0.64043
95	0.13994	0.02697	P	0.03834	0.02181	0.47126
96	0.13849	0.02418	P	0.04488	0.02511	0.36846
97	0.13706	0.02216	P	0.05111	0.02828	0.30090
98	0.13566	0.02065	P	0.05706	0.03133	0.25395
99	0.13429	0.01951	P	0.06275	0.03427	0.22005
100	0.13295	0.01865	P	0.06818	0.03710	0.19501
101	0.13163	0.01806	P	0.06739	0.03296	0.21137
102	0.13034	0.01773	P	0.05958	0.02884	0.23596
103	0.12907	0.01774	P	0.05186	0.02479	0.27304
104	0.12783	0.01821	P	0.04424	0.02083	0.33184
105	0.12662	0.01944	P	0.03669	0.01695	0.43302
106	0.12542	0.02208	P	0.02922	0.01316	0.63017
107	0.12425	0.02777		0.02183	0.00947	1.09564
108	0.12310	0.04205		0.01450	0.00596	2.62412
109	0.12197	0.09505		0.00724	0.00306	11.51495
110	0.12086	0.11779		0.00004	0.00203	21.60094
111	0.11977	0.04532		0.00697	0.00540	3.12018
112	0.11870	0.02926		0.01365	0.00862	1.26703
113	0.11765	0.02249		0.02004	0.01176	0.71631
114	0.11662	0.01884	P	0.02615	0.01482	0.47836
115	0.11561	0.01659	P	0.03201	0.01777	0.35244
116	0.11461	0.01507	P	0.03763	0.02063	0.27671
117	0.11363	0.01398	P	0.04302	0.02340	0.22706
118	0.11267	0.01316	P	0.04820	0.02608	0.19246
119	0.11172	0.01254	P	0.05319	0.02867	0.16728

TABLE 4.7. EXISTENCE OF PRINCIPAL SOLUTION (PAGE 4 OF 5)

J	GAMMA(J)	Y-Y(0)		R(0)	R	A*ETA(0)
120	0.11079	0.01206	P	0.05799	0.03118	0.14843
121	0.10987	0.01171	P	0.05660	0.02785	0.16071
122	0.10897	0.01150	P	0.05010	0.02442	0.17904
123	0.10809	0.01144	P	0.04366	0.02105	0.20569
124	0.10721	0.01160	P	0.03727	0.01774	0.24645
125	0.10636	0.01212	P	0.03094	0.01447	0.31425
126	0.10551	0.01334	P	0.02467	0.01127	0.44234
127	0.10468	0.01612	P	0.01844	0.00812	0.73859
128	0.10386	0.02346		0.01226	0.00506	1.71863
129	0.10306	0.05438		0.00613	0.00224	8.94238
130	0.10227	0.08529		0.00005	0.00130	24.17850
131	0.10149	0.02766		0.00590	0.00422	2.42779
132	0.10072	0.01771		0.01162	0.00703	0.93616
133	0.09996	0.01381	P	0.01711	0.00977	0.52716
134	0.09921	0.01180	P	0.02241	0.01243	0.35502
135	0.09848	0.01059	P	0.02751	0.01502	0.26447
136	0.09775	0.00978	P	0.03243	0.01754	0.20988
137	0.09704	0.00920	P	0.03719	0.01998	0.17383
138	0.09634	0.00876	P	0.04178	0.02236	0.14844
139	0.09564	0.00843	P	0.04621	0.02467	0.12974
140	0.09496	0.00816	P	0.05050	0.02691	0.11553
141	0.09429	0.00796	P	0.04880	0.02410	0.12534
142	0.09362	0.00782	P	0.04322	0.02118	0.13963
143	0.09297	0.00775	P	0.03769	0.01829	0.15966
144	0.09232	0.00779	P	0.03220	0.01544	0.18923
145	0.09169	0.00800	P	0.02675	0.01264	0.23649
146	0.09106	0.00854	P	0.02133	0.00987	0.32209
147	0.09044	0.00986	P	0.01596	0.00714	0.51212
148	0.08983	0.01354		0.01062	0.00446	1.12244
149	0.08923	0.03015		0.00531	0.00188	5.90571
150	0.08863	0.05925		0.00004	0.00091	23.89183
151	0.08804	0.01684		0.00513	0.00347	1.79162
152	0.08746	0.01094		0.01012	0.00595	0.68135
153	0.08689	0.00879	P	0.01495	0.00837	0.39007
154	0.08633	0.00771	P	0.01961	0.01073	0.26794
155	0.08577	0.00708	P	0.02414	0.01302	0.20311
156	0.08522	0.00666	P	0.02851	0.01526	0.16347
157	0.08468	0.00636	P	0.03276	0.01745	0.13689
158	0.08414	0.00612	P	0.03688	0.01958	0.11789
159	0.08361	0.00594	P	0.04087	0.02165	0.10368

TABLE 4.7. EXISTENCE OF PRINCIPAL SOLUTION (PAGE 5 OF 5)

J	GAMMA(J)	Y-Y(0)		R(0)	R	A*ETA(0)
160	0.08309	0.00579	P	0.04475	0.02368	0.09270
161	0.08258	0.00568	P	0.04289	0.02124	0.10092
162	0.08207	0.00559	P	0.03800	0.01869	0.11252
163	0.08156	0.00553	P	0.03316	0.01616	0.12834
164	0.08106	0.00552	P	0.02834	0.01367	0.15094
165	0.08057	0.00559	P	0.02355	0.01121	0.18571
166	0.08009	0.00581	P	0.01879	0.00878	0.24576
167	0.07961	0.00641	P	0.01406	0.00638	0.37190
168	0.07913	0.00820	P	0.00936	0.00401	0.75504
169	0.07867	0.01681		0.00468	0.00169	3.66571
170	0.07820	0.03979		0.00003	0.00068	21.36454
171	0.07775	0.01037		0.00454	0.00295	1.29280
172	0.07729	0.00701	P	0.00897	0.00517	0.50163
173	0.07685	0.00587	P	0.01327	0.00733	0.29666
174	0.07641	0.00531	P	0.01744	0.00944	0.20921
175	0.07597	0.00499	P	0.02150	0.01150	0.16168
176	0.07554	0.00478	P	0.02544	0.01352	0.13196
177	0.07511	0.00462	P	0.02928	0.01549	0.11164
178	0.07469	0.00449	P	0.03301	0.01741	0.09687
179	0.07427	0.00439	P	0.03665	0.01930	0.08566
180	0.07386	0.00431	P	0.04018	0.02114	0.07688
181	0.07345	0.00424	P	0.03826	0.01898	0.08398
182	0.07305	0.00418	P	0.03391	0.01672	0.09376
183	0.07265	0.00414	P	0.02960	0.01448	0.10681
184	0.07225	0.00411	P	0.02530	0.01227	0.12502
185	0.07186	0.00412	P	0.02104	0.01008	0.15215
186	0.07148	0.00420	P	0.01679	0.00791	0.19701
187	0.07109	0.00446	P	0.01257	0.00577	0.28564
188	0.07072	0.00529	P	0.00837	0.00365	0.53445
189	0.07034	0.00964		0.00419	0.00155	2.27937
190	0.06997	0.02612		0.00003	0.00054	17.82197
191	0.06961	0.00656		0.00407	0.00258	0.93633
192	0.06924	0.00473	P	0.00805	0.00457	0.38193
193	0.06888	0.00415	P	0.01193	0.00652	0.23530
194	0.06853	0.00387	P	0.01571	0.00843	0.17037
195	0.06818	0.00371	P	0.01939	0.01030	0.13393
196	0.06783	0.00360	P	0.02297	0.01213	0.11056
197	0.06749	0.00352	P	0.02647	0.01393	0.09428
198	0.06714	0.00345	P	0.02988	0.01568	0.08226
199	0.06681	0.00339	P	0.03321	0.01741	0.07303

Table 4.7 also lists, for each of these values of γ, the distance $\|y - y_o\|$, where y is a solution of the Duffing problem for the given value of γ, and y_o is the "reduced" solution for $\gamma = 0$. For those values of γ such that $\|y - y_o\| \leq r$, y is the "principal" solution.

The last two columns in Table 4.7 list two other quantities. One of these is the radius of the neighborhood about the actual solution within which existence and uniqueness can be asserted according to Lemma 5. The assumption that $A \cdot \eta_o \leq \frac{1}{4}$ is obviously satisfied since $\eta_o = 0$. The radius is found exactly as in Eq. (4.33) except that on the right-hand side $A = {}^1/2\ r$; hence, F(r) is re-defined as follows:

$$(4.36) \qquad F(r) = r^2 + k^2 \cdot y \cdot r - k^2/12\ x.$$

Now F(r) has a zero between 0 and $k/\sqrt{12\ x}$. The quantity in the last column of Table 4.7 is the value of $A \cdot \eta_o$ for the reduced solution. Whenever this quantity is less than $\frac{1}{4}$ we know that a "principal" solution exists. As we noticed earlier, we are unable to estimate this quantity precisely. In fact, we find a "principal" solution in cases where the estimated value of $A \cdot \eta_o$ is as large as 0.89.

b. Extending a principal branch toward a resonance. Now we wish to apply Theorem 1, using as γ_o a value for which a principal solution (i.e., a unique solution in a neighborhood of the "reduced" solution) can be found. We now use y_o to designate the solution corresponding to γ_o instead of the "reduced" solution (as before when $\gamma_o = 0$ was assumed). We form the matrix $DQ(y_o)$ as described in Eqs. (3.56 and (3.57), and steps (a) to (f) of Eq. (3.61) with $\gamma = \gamma_o$. Let λ be the smallest (in magnitude) eigenvalue of this matrix. We shall restrict our attention to values of δ between 0 and λ, so that λ is the eigenvalue closest to δ: thus, we consider the case in which γ is moving from γ_o toward a resonance.

(Recall that $\delta = \gamma^2 - \gamma_o^2$: compare Eq. (3.8)). When δ is between 0 and λ, we find that $\delta/(\delta - \lambda)$ is negative, so that

$$(4.37) \qquad \left(1 + \left|\frac{\delta}{\delta - \lambda}\right|\right) = \left(1 - \frac{\delta}{\delta - \lambda}\right) = \left(\frac{\lambda}{\lambda - \delta}\right).$$

Thus, the main condition of Theorem 1 is the following:

$$(4.38) \qquad A_o \cdot \left(\frac{\lambda}{\lambda - \delta}\right)^2 \cdot |\delta| \cdot \|\mathcal{D}Q(y_o)^{-1}\, y_o\| \le \tfrac{1}{4},$$

where $A_o = 1/2\, r$. The value of r is calculated (as described in the preceding paragraph) as "the radius of the neighborhood about the actual solution within which existence and uniqueness can be asserted", and is listed in the penultimate column of Table 4.7. The vector $z = \mathcal{D}Q(y_o)^{-1}\, y_o$ is calculated by solving the linear system: $\mathcal{D}Q(y_o)\; z = y_o$.

The next step is to find the largest value of δ such that Theorem 1 is satisfied for all $\bar{\delta}$ such that $0 \le \bar{\delta} \le \delta$. This value of δ is a root of the following equation:

$$(4.39) \qquad A_o \cdot \left(\frac{\lambda}{\lambda - \delta}\right)^2 \cdot |\delta| \cdot \|z\| = \tfrac{1}{4}$$

$$4 \cdot (\lambda^2 \cdot \|z\| \cdot A_o) \cdot |\delta| = (\lambda - \delta)^2$$

$$2 \cdot s \cdot \delta = (\lambda - \delta)^2$$

where we let $s = 2 \cdot (\lambda^2 \cdot \|z\| \cdot A_o) \cdot \text{sgn}\,(\delta)$, and we note that s has the same sign as δ since A_o is non-negative. Expanding the right-hand side, we have

$$(4.40) \qquad 2 \cdot s \cdot \delta = \lambda^2 - 2\,\delta\,\lambda + \delta^2$$

$$\delta^2 - 2\,(\lambda + s) \cdot \delta + \lambda^2 = 0$$

The discriminant of this quadratic equation is $(\lambda^2 + 2 \cdot \lambda \cdot s + s^2 - \lambda^2)$

or $(2 \cdot \lambda \cdot s + s^2)$. This is always positive, since we have assumed that δ has the same sign as λ, so that the sign of s is the same as that of λ. Therefore, there are two real roots:

$$(4.41) \qquad \delta = (\lambda + s) \pm \sqrt{(2 \cdot \lambda \cdot s + s^2)}.$$

We must choose the negative sign of the radical in order to have δ between 0 and λ. The values of λ, along with the values of δ obtained from Eq. (4.41), are listed in Table 4.8.

We conclude that there is one and only one solution of the Duffing problem for all values of δ up to this maximum, and we easily see that as γ varies (with $\gamma^2 = \gamma_o^2 + \delta$ for these values of δ) the solution varies continuously and for each value of γ isolated within some neighborhood. We say that a "connected branch" of the solution exists on the interval between γ_o and any value of γ such that $\gamma^2 - \gamma_o^2$ is less than the maximum value of δ given by Eq. (4.41).

We can re-set γ_o to some value near the end of this interval, and make another step. Designating now by y_o the solution (which has been shown to exist) for the new value of γ_o, we again form the matrix $\mathcal{D}Q(y_o)$ and find its smallest eigenvalue, which we call λ. We also re-evaluate A_o and z, and solve as before in Eqs. (4.39) to (4.41) to obtain a new value of δ, and thus we extend the connected branch over the new interval.

We cannot say a priori how far the branch can be extended in this way. For $\beta = 1$, the procedure described above has actually been carried out for starting values on each side of each of the resonances of order 5, 7, 9, and 11. It was found in each case that the connected branches can be continued from both sides until a common value of γ is reached. For this common value, two different solutions are obtained, one on a branch connected to a starting value on one side of the resonance and the

TABLE 4.8. EXTENSION OF PRINCIPAL BRANCH (PAGE 1 OF 5)

ORDER	GAMMA	EIGENVALUE	INTERVAL
0.10	13.29460	-1.7575E 02	-1.6637E 02
0.20	6.64730	-4.3185E 01	-3.6898E 01
0.30	4.43153	-1.8632E 01	-1.3844E 01
0.40	3.32365	-1.0024E 01	-6.2184E 00
0.50	2.65892	-6.0083E 00	-2.9463E 00
0.60	2.21577	-3.7582E 00	-1.3331E 00
0.70	1.89923	-2.2412E 00	-4.7446E-01
0.80	1.66182	-5.0091E-01	-1.6566E-02
0.90	1.47718	-1.7029E 00	-2.2498E-01
1.00	1.32946	-1.3577E 00	-1.9599E-01
1.10	1.20860	-1.0997E 00	-1.7593E-01
1.20	1.10788	-9.0147E-01	-1.6113E-01
1.30	1.02266	-7.4581E-01	-1.4937E-01
1.40	0.94961	-6.2131E-01	-1.3919E-01
1.50	0.88631	-5.2030E-01	-1.2969E-01
1.60	0.83091	-4.3722E-01	-1.2032E-01
1.70	0.78204	-3.6816E-01	-1.1074E-01
1.80	0.73859	-3.1024E-01	-1.0070E-01
1.90	0.69972	-2.6126E-01	-8.7575E-02
2.00	0.66473	-2.1968E-01	-7.1341E-02
2.10	0.63308	-1.8418E-01	-5.6187E-02
2.20	0.60430	-1.5375E-01	-4.2357E-02
2.30	0.57803	-1.2787E-01	-3.0352E-02
2.40	0.55394	-1.0607E-01	-2.0588E-02
2.50	0.53178	-8.7732E-02	-1.3199E-02
2.60	0.51133	-7.2604E-02	-8.0075E-03
2.70	0.49239	-6.0026E-02	-4.5603E-03
2.80	0.47481	-4.8020E-02	-2.2632E-03
2.90	0.45843	-2.6355E-02	-4.5823E-04
2.90	0.45843	3.8379E-02	5.3142E-04
3.00	0.44315	4.2125E-02	1.6092E-03
3.10	0.42886	5.0130E-02	2.9506E-03
3.20	0.41546	5.7141E-02	4.5428E-03
3.30	0.40287	6.3662E-02	6.3351E-03
3.40	0.39102	6.9828E-02	8.2650E-03
3.44	0.38647	-7.1210E-02	-9.0286E-03
3.50	0.37985	-6.6648E-02	-9.9319E-03
3.60	0.36929	-5.9494E-02	-1.1158E-02
3.70	0.35931	-5.2869E-02	-1.1958E-02
3.80	0.34986	-4.6736E-02	-1.2279E-02
3.90	0.34089	-4.1057E-02	-1.2100E-02

TABLE 4.8. EXTENSION OF PRINCIPAL BRANCH (PAGE 2 OF 5)

ORDER	GAMMA	EIGENVALUE	INTERVAL
4.00	0.33236	-3.5801E-02	-1.1207E-02
4.10	0.32426	-3.0940E-02	-9.2887E-03
4.20	0.31654	-2.6452E-02	-7.2914E-03
4.30	0.30918	-2.2324E-02	-5.3639E-03
4.40	0.30215	-1.8555E-02	-3.6574E-03
4.50	0.29544	-1.5163E-02	-2.2829E-03
4.60	0.28901	-1.2197E-02	-1.2903E-03
4.70	0.28286	-9.7465E-03	-6.6095E-04
4.80	0.27697	-7.8655E-03	-3.1139E-04
4.90	0.27132	-5.9329E-03	-1.1251E-04
4.94	0.26912	-4.0158E-03	-3.9713E-05
4.94	0.26912	3.6442E-03	3.8568E-05
5.00	0.26589	5.4288E-03	1.2965E-04
5.10	0.26068	7.7034E-03	3.8119E-04
5.20	0.25567	9.7516E-03	7.6262E-04
5.30	0.25084	1.1742E-02	1.2714E-03
5.40	0.24620	1.3680E-02	1.8858E-03
5.50	0.24172	1.5557E-02	2.5741E-03
5.60	0.23740	1.7361E-02	3.3004E-03
5.64	0.23572	1.8062E-02	3.5930E-03
5.70	0.23324	-1.7037E-02	-3.8645E-03
5.80	0.22922	-1.5268E-02	-4.1618E-03
5.90	0.22533	-1.3585E-02	-4.2601E-03
6.00	0.22158	-1.1984E-02	-4.1197E-03
6.10	0.21794	-1.0462E-02	-3.5153E-03
6.20	0.21443	-9.0167E-03	-2.8410E-03
6.30	0.21103	-7.6454E-03	-2.1484E-03
6.40	0.20773	-6.3486E-03	-1.4966E-03
6.50	0.20453	-5.1300E-03	-9.3838E-04
6.60	0.20143	-4.0014E-03	-5.1140E-04
6.70	0.19843	-2.9936E-03	-2.3146E-04
6.80	0.19551	-2.1830E-03	-8.5411E-05
6.90	0.19268	-1.6457E-03	-2.7392E-05
6.96	0.19101	-1.0133E-03	-6.2911E-06
6.96	0.19101	7.3733E-04	4.3069E-06
7.00	0.18992	1.2648E-03	1.9662E-05
7.10	0.18725	2.1730E-03	9.9186E-05
7.20	0.18465	3.0158E-03	2.5199E-04
7.30	0.18212	3.8501E-03	4.8078E-04
7.40	0.17966	4.6689E-03	7.7334E-04
7.50	0.17726	5.4662E-03	1.1118E-03
7.60	0.17493	6.2389E-03	1.4774E-03
7.70	0.17266	6.9859E-03	1.8511E-03
7.74	0.17176	-7.2715E-03	-1.9977E-03
7.80	0.17044	-6.8314E-03	-2.0935E-03
7.90	0.16829	-6.1193E-03	-2.1680E-03

TABLE 4.8. EXTENSION OF PRINCIPAL BRANCH (PAGE 3 OF 5)

ORDER	GAMMA	EIGENVALUE	INTERVAL
8.00	0.16618	-5.4333E-03	-2.1327E-03
8.10	0.16413	-4.7725E-03	-1.8535E-03
8.20	0.16213	-4.1362E-03	-1.5320E-03
8.30	0.16018	-3.5236E-03	-1.1902E-03
8.40	0.15827	-2.9345E-03	-8.5487E-04
8.50	0.15641	-2.3692E-03	-5.5281E-04
8.60	0.15459	-1.8298E-03	-3.0718E-04
8.70	0.15281	-1.3227E-03	-1.3485E-04
8.80	0.15107	-8.7071E-04	-4.0800E-05
8.90	0.14938	-5.6096E-04	-8.4743E-06
8.98	0.14805	-2.0524E-04	-4.3823E-07
8.98	0.14805	2.7013E-04	1.4928E-06
9.00	0.14772	3.9741E-04	4.3396E-06
9.10	0.14609	8.4576E-04	4.0169E-05
9.20	0.14451	1.2728E-03	1.2474E-04
9.30	0.14295	1.6961E-03	2.5942E-04
9.40	0.14143	2.1113E-03	4.3431E-04
9.50	0.13994	2.5163E-03	6.3698E-04
9.60	0.13849	2.9104E-03	8.5576E-04
9.70	0.13706	3.2934E-03	1.0800E-03
9.80	0.13566	-3.6327E-03	-1.2930E-03
9.90	0.13429	-3.2681E-03	-1.3346E-03
10.00	0.13295	-2.9142E-03	-1.3138E-03
10.10	0.13163	-2.5709E-03	-1.1554E-03
10.20	0.13034	-2.2376E-03	-9.7087E-04
10.30	0.12907	-1.9141E-03	-7.7174E-04
10.40	0.12783	-1.6002E-03	-5.7129E-04
10.50	0.12662	-1.2959E-03	-3.8368E-04
10.60	0.12542	-1.0013E-03	-2.2268E-04
10.70	0.12425	-7.1780E-04	-1.0132E-04
10.80	0.12310	-4.5038E-04	-2.9197E-05
10.90	0.12197	-2.3269E-04	-3.6525E-06
10.98	0.12108	-1.4312E-04	-4.2652E-07
10.98	0.12108	7.1532E-05	1.7666E-07
11.00	0.12086	1.5342E-04	1.3095E-06
11.10	0.11977	4.0674E-04	2.2479E-05
11.20	0.11870	6.5160E-04	8.0031E-05
11.30	0.11765	8.9333E-04	1.7287E-04
11.40	0.11662	1.1302E-03	2.9176E-04
11.50	0.11561	1.3618E-03	4.2727E-04
11.60	0.11461	1.5876E-03	5.7176E-04
11.70	0.11363	1.8080E-03	7.1893E-04
11.80	0.11267	2.0230E-03	8.6295E-04
11.84	0.11229	-2.0722E-03	-9.0885E-04
11.90	0.11172	-1.9465E-03	-9.1484E-04

TABLE 4.8. EXTENSION OF PRINCIPAL BRANCH (PAGE 4 OF 5)

ORDER	GAMMA	EIGENVALUE	INTERVAL
12.00	0.11079	-1.7411E-03	-8.9378E-04
12.10	0.10987	-1.5407E-03	-7.9204E-04
12.20	0.10897	-1.3453E-03	-6.7366E-04
12.30	0.10809	-1.1547E-03	-5.4556E-04
12.40	0.10721	-9.6868E-04	-4.1471E-04
12.50	0.10636	-7.8729E-04	-2.8880E-04
12.60	0.10551	-6.1045E-04	-1.7592E-04
12.70	0.10468	-4.3836E-04	-8.5036E-05
12.80	0.10386	-2.8101E-04	-2.5871E-05
12.90	0.10306	-1.2078E-04	-2.4180E-06
13.00	0.10227	7.0035E-05	5.3195E-07
13.10	0.10149	2.2647E-04	1.5787E-05
13.20	0.10072	3.7877E-04	6.0215E-05
13.30	0.09996	5.2866E-04	1.3014E-04
13.40	0.09921	6.7560E-04	2.1686E-04
13.50	0.09848	8.1951E-04	3.1331E-04
13.60	0.09775	9.6034E-04	4.1443E-04
13.70	0.09704	1.0982E-03	5.1652E-04
13.80	0.09634	1.2331E-03	6.1627E-04
13.86	0.09592	-1.3039E-03	-6.7050E-04
13.90	0.09564	-1.2513E-03	-6.6869E-04
14.00	0.09496	-1.1218E-03	-6.4679E-04
14.10	0.09429	-9.9507E-04	-5.7554E-04
14.20	0.09362	-8.7099E-04	-4.9394E-04
14.30	0.09297	-7.4951E-04	-4.0603E-04
14.40	0.09232	-6.3057E-04	-3.1561E-04
14.50	0.09169	-5.1412E-04	-2.2694E-04
14.60	0.09106	-4.0011E-04	-1.4470E-04
14.70	0.09044	-2.8855E-04	-7.4667E-05
14.80	0.08983	-1.7966E-04	-2.4643E-05
14.90	0.08923	-7.5614E-05	-2.2578E-06
15.00	0.08863	3.6688E-05	2.8109E-07
15.10	0.08804	1.3958E-04	1.2850E-05
15.20	0.08746	2.4015E-04	4.9665E-05
15.30	0.08689	3.3906E-04	1.0482E-04
15.40	0.08633	4.3617E-04	1.7055E-04
15.50	0.08577	5.3145E-04	2.4172E-04
15.60	0.08522	6.2493E-04	3.1514E-04
15.70	0.08468	7.1665E-04	3.8860E-04
15.80	0.08414	8.0665E-04	4.6024E-04
15.88	0.08372	-8.6888E-04	-5.1121E-04
15.90	0.08361	-8.5133E-04	-5.0877E-04

TABLE 4.8. EXTENSION OF PRINCIPAL BRANCH (PAGE 5 OF 5)

ORDER	GAMMA	EIGENVALUE	INTERVAL
16.00	0.08309	-7.6456E-04	-4.8703E-04
16.10	0.08258	-6.7940E-04	-4.3437E-04
16.20	0.08207	-5.9581E-04	-3.7531E-04
16.30	0.08156	-5.1378E-04	-3.1225E-04
16.40	0.08106	-4.3320E-04	-2.4724E-04
16.50	0.08057	-3.5412E-04	-1.8267E-04
16.60	0.08009	-2.7648E-04	-1.2127E-04
16.70	0.07961	-2.0025E-04	-6.6571E-05
16.80	0.07913	-1.2620E-04	-2.4258E-05
16.90	0.07867	-5.2670E-05	-2.4837E-06
17.00	0.07820	2.1351E-05	1.8247E-07
17.10	0.07775	9.2390E-05	1.1434E-05
17.20	0.07729	1.6205E-04	4.3038E-05
17.30	0.07685	2.3059E-04	8.7490E-05
17.40	0.07641	2.9797E-04	1.3834E-04
17.50	0.07597	3.6422E-04	1.9208E-04
17.60	0.07554	4.2936E-04	2.4675E-04
17.70	0.07511	4.9354E-04	3.0110E-04
17.80	0.07469	5.5635E-04	3.5391E-04
17.88	0.07435	6.0599E-04	3.9446E-04
17.90	0.07427	-6.0512E-04	-3.9743E-04
18.00	0.07386	-5.4418E-04	-3.7677E-04
18.10	0.07345	-4.8425E-04	-3.3646E-04
18.20	0.07305	-4.2531E-04	-2.9224E-04
18.30	0.07265	-3.6735E-04	-2.4550E-04
18.40	0.07225	-3.1062E-04	-1.9751E-04
18.50	0.07186	-2.5419E-04	-1.4917E-04
18.60	0.07148	-1.9896E-04	-1.0249E-04
18.70	0.07109	-1.4470E-04	-5.9456E-05
18.80	0.07072	-9.1227E-05	-2.3781E-05
18.90	0.07034	-3.8802E-05	-2.8770E-06
19.00	0.06997	1.3413E-05	1.3712E-07
19.10	0.06961	6.4420E-05	1.0695E-05
19.20	0.06924	1.1455E-04	3.8168E-05
19.30	0.06888	1.6395E-04	7.4308E-05
19.40	0.06853	2.1257E-04	1.1415E-04
19.50	0.06818	2.6047E-04	1.5540E-04
19.60	0.06783	3.0766E-04	1.9690E-04
19.70	0.06749	3.5410E-04	2.3787E-04
19.80	0.06714	3.9985E-04	2.7775E-04
19.90	0.06681	4.4491E-04	3.1591E-04

other one connected to a starting value on the other side. Thus, we have a rigorously defined "principal solution" for values of γ as small as about 0.11.

In Table 4.8 we show, for several values of γ, the smallest eigenvalue λ, and the magnitude of δ. For $\gamma = 0.26911$, 0.19101, 0.14804, and 0.12107 two solutions are given, one of which is on each of the two branches.

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V. APPENDIX

A. Calculus of Operators in Banach Space

A Banach space is a complete normed linear space. The set of all linear mappings of a Banach space $\mathcal{J}_1$ into a Banach space $\mathcal{J}_2$ is in turn a Banach space, and is designated by $(\mathcal{J}_1 \to \mathcal{J}_2)$.

1. The Frechet Derivative.

This paragraph is based on Liusternik (Ref. 3) and Kantorovich (Ref. 4). Let P be an operator (in general nonlinear) which maps $\mathcal{J}_1$ into $\mathcal{J}_2$. If, for a given element $y \in \mathcal{J}_1$, there is a linear operation $H \in (\mathcal{J}_1 \to \mathcal{J}_2)$ such that

$$\|P(y + h) - P(y) - H(h)\| \leq \|h\| \cdot \epsilon\,(\|h\|) \tag{5.1}$$

where $\epsilon\,(\|h\|) \to 0$ as $\|h\| \to 0$, then P is said to be Frechet-differentiable at y, and H is called the Frechet derivative* of P at y. We write:

$$H = \mathcal{D}P(y). \tag{5.2}$$

Thus, $\mathcal{D}P(y)$ is an element of the space $(\mathcal{J}_1 \to \mathcal{J}_2)$. On the other hand, $\mathcal{D}P$ is a (possibly nonlinear) mapping from $\mathcal{J}_1$ into $(\mathcal{J}_1 \to \mathcal{J}_2)$. If $\mathcal{D}P$ is Frechet-differentiable at z, we call $\mathcal{D}(\mathcal{D}P)(z)$ the second-order Frechet derivative of P at z and write:

$$\mathcal{D}(\mathcal{D}P)(z) = \mathcal{D}^2P(z). \tag{5.3}$$

Thus, $\mathcal{D}^2P(z) \in (\mathcal{J}_1 \to (\mathcal{J}_1 \to \mathcal{J}_2))$.

The statements made in Lemma 3, Chapter II are verified by substituting the proposed $\mathcal{D}P(y)$ for H in the defining equation, Eq. (5.1).

In particular, the real numbers (suitably normed) form a Banach

* Note: H(h) is called the Frechet differential of P at y.

space, and a differentiable function of one real variable is Frechet-differentiable. We verify from Eq. (5.1) that if p is differentiable at a, then

$$Dp(a) = \mathcal{D}p(a) \tag{5.4}$$

where Dp(a) denotes the ordinary derivative of p at a.

2. The Mean Value Theorem for Banach Space.

This paragraph is based on Kantorovich (Ref. 4, p. 161). The Mean Value Theorem for Banach space states that

$$\|P(y + h) - P(y)\| \leq \|h\| \cdot \sup \{\|\mathcal{D}P(\bar{y})\| : \bar{y} \in C(y, y + h)\}, \tag{5.5}$$

where $C(y, y + h)$ denotes the set $\{y + c\, h: 0 \leq c \leq 1\}$. To prove this theorem, we let $x = P(y + h) - P(y)$. As a corollary of the Hahn-Banach theorem, it can be shown (see Liusternik, Ref. 3, page 98) that there exists a linear functional T defined on $\mathfrak{F}$ such that $\|T\| = 1$ and $Tx = \|x\|$. We select such a functional T and define

$$F(c) = T(P(y + c \cdot h)). \tag{5.6}$$

We find that $\mathcal{D}F(c) = h \cdot T \circ \mathcal{D}P(y + c\, h)$. Also it is clear that

$$Tx = T(P(y + h) - P(y)) = F(1) - F(0). \tag{5.7}$$

Applying the ordinary Mean Value Theorem to F, which is a continuous function of a real variable if P is Frechet -differentiable, we have:

$$F(1) - F(0) = DF(c) = \mathcal{D}F(c) = h \cdot T \circ \mathcal{D}P(y + c \,.\, h) \tag{5.8}$$

$$= h \cdot T \circ \mathcal{D}P(\bar{y})$$

where y is some element of $C(y, y + h)$. But:

$$(5.9)\qquad \|P(y+h)-P(y)\| = \|x\| = Tx = h\cdot T\circ DP(\bar{y})$$

$$\le \|h\|\cdot\|T\|\cdot\|DP(\bar{y})\| = |h|\cdot\|DP(\bar{y})\|$$

$$\le \|h\|\cdot \sup\{\|DP(\bar{y})\| : \bar{y}\in C(y, y+h)\}.$$

3. <u>A Fixed-Point Theorem</u>

This paragraph has been compiled from various sources including Liusternik (Ref. 3) and DeVogelaere (Ref. 9).

Let $\mathcal{F}$ be a complete metric space, with metric denoted by $\rho(x, y)$. Let $\mathcal{E}$ be a subset of $\mathcal{F}$, and T a mapping of $\mathcal{E}$ into $\mathcal{F}$. Given $y_o \varepsilon \mathcal{E}$, write $y_o = T^\nu y_o$ if $T^{\nu-1}y_o \varepsilon \mathcal{E}$.

If there exists r such that $S(y_o, r)\subset \mathcal{E}$, and if T satisfies Lipschitz' condition with $K < 1$ on $S(y_o, r)$, and if $\rho(y_o, Ty_o) \le (1-K)\cdot r$, then there is one and only one fixed point y_∞ of T in $S(y_o, r)$, and $\lim\{y_\nu : \nu\to\infty\} = y_\infty$.

<u>Proof</u>; One nice proof of this theorem begins with the following lemma:

<u>Lemma</u>: If $y_\nu \varepsilon S(y_o\ r)$ for all $\nu < q$, and if $p < q$, then $\rho(y_p, y_q) \le K^p\cdot r$.

<u>Proof</u>:

$$(5.10)\qquad \rho(y_\nu, y_{\nu+1}) \le K\cdot\rho(y_{\nu-1}, y_\nu) \le \dots \le K^\nu\cdot\rho(y_o, y_1).$$

$$\rho(y_p, y_q) \le \rho(y_p, y_{p+1}) + \dots + \rho(y_{q-1}, y_q)$$

$$= \Sigma_{\nu=p}^{q-1}\rho(y_\nu, y_{\nu+1}) \le \Sigma_{\nu=p}^{q-1} K^\nu\,\rho(y_o, y_1)$$

$$\le \rho(y_o, y_1)\,\Sigma_{\nu=p}^{\infty} K^\nu = \rho(y_o, y_1)\cdot K^p/(1-K)$$

$$\le K^p\cdot r$$

The proof of the Fixed Point Theorem is now easy:

(1). For all ν, $y_\nu \varepsilon S(y_0, r)$. Using induction on ν, we see by the lemma that if $y_\nu \varepsilon S(y_0, r)$ for $\nu < q$ then $\rho(y_0, y_q) \leqq r$, hence, $y_\nu \varepsilon S(y_0, r)$ for $\nu = q$.

(2). For sufficiently large p, and $q > p$, we find that $\rho(y_p, y_q)$ can be made arbitrarily small since $K^p \to 0$. Thus, we have a Cauchy sequence which in a complete metric space has a limit. We may call this limit y_∞.

(3). The limit point y_∞ is a fixed point of T, since

$$\rho(T\, y_\infty, y_\infty) \leqq \rho(T\, y_\infty, y_{\nu+1}) + \rho(y_{\nu+1}, y_\infty)$$

$$\leqq \rho(y_\infty, y_\nu) + \rho(y_\infty, y_{\nu+1})$$

which is arbitrarily small for sufficiently large ν.

(4). The fixed point is unique: suppose that $y_\infty = T\, y_\infty$ and $\overline{y}_\infty = T\, \overline{y}_\infty$; then

$$\rho(y_\infty, \overline{y}_\infty) = \rho(T\, y_\infty, T\, \overline{y}_\infty) \leqq K \cdot \rho(y_\infty, \overline{y}_\infty),$$

$$(1 - K) \cdot \rho(y_\infty, \overline{y}_\infty) \leqq 0$$

But $(1 - K) > 0$ and $\rho(y_\infty, \overline{y}_\infty) \geqq 0$ so $\rho(y_\infty, \overline{y}_\infty) = 0$.

B. Matrix Representation of $\mathcal{D}G(y)$ in the Space $\mathcal{J}_K$.

This information is based on Professor DeVogelaere's lectures (Ref. 9).

We abbreviate the notation by writing G instead of G_K and $\overline{G}$ instead of G_N, also $y = y_K$ and $\overline{y} = y_N$, etc. (compare Eqs. (4.17, 4.21)). Thus, we begin with

(5.11)
$$\overline{g} = \overline{G}\,\overline{y}$$
$$g = G\,y$$

where $G = \sigma^{\dagger} \circ \overline{G} \circ \sigma$. Since $\overline{y} = \sigma\, y$, it follows that $g = \sigma^{\dagger} \circ \overline{G}\,\overline{y} = \sigma^{\dagger}\overline{g}$. We recall the definitions of σ and $\sigma^{\dagger}$ from Table 4.1, and see that

(5.12)
$$\overline{y}\,[n] = \Sigma_K\, y\,[k] \cdot \cos\,((2 \cdot k - 1) \cdot H\,[n]), \qquad (n \in N)$$
$$g\,[k] = \Sigma_n\, \overline{w}\,[n] \cdot \overline{g}\,[n] \cdot \cos\,((2 \cdot k - 1) \cdot H\,[n]) \qquad (k \in K)$$

To further shorten the notation, we write

(5.13)
$$C\,[k, n] = \cos\,((2 \cdot k - 1) \cdot H\,[n] \qquad (k \in K, n \in N)$$

and re-write Eq. (5.12):

(5.14)
$$\overline{y}\,[n] = \Sigma_K\, y\,[k] \cdot C\,[k, n], \qquad (n \in N)$$
$$g\,[k] = \Sigma_n\, \overline{w}\,[n] \cdot \overline{g}\,[n] \cdot C\,[k, n]. \qquad (k \in K)$$

Now we assert that the operator $\mathcal{D}G(y)$ can be represented by the matrix G' defined by

(5.15)
$$G'\,[k1, k2] = \partial g\,[k1]/\partial y\,[k2]. \qquad (k1, k2 \in K)$$

To see this, we must show (compare Eq. (5.1)) that

$$(5.16) \qquad \|G(y+\eta) - G(y) - G' \cdot \eta\| \le \|\eta\| \cdot \epsilon\,(\|\eta\|).$$

Looking at the individual elements on the left, we have

$$(5.17) \qquad (G(y+\eta))\,[k1] - (G(y))\,[k1] - \Sigma_{k2}\, G'\,[k1, k2] \cdot \eta\,[k2]. \qquad (k1 \in K)$$

Here $(G(y))\,[k1]$ is a function of all of the variables $\{y\,[k2] : k2 \in K\}$ and we see that if we substitute Eq. (5.15) into Eq. (5.17) we obtain an expression of the familiar form

$$(5.18) \qquad F(y_1 + \eta_1, \ldots) - F(y_1, \ldots) - \Sigma_k \frac{\partial F}{\partial y_k} \cdot \eta_k$$

so that the condition expressed by Eq. (5.16) is clearly valid.

Now from Eq. (5.14) we see that

$$(5.19) \qquad \partial \bar{y}\,[n] \,/\, \partial y\,[k] = C\,[k, n]$$

$$\partial g\,[k] \,/\, \partial \bar{g}\,[n] = \bar{w}\,[n] \cdot C\,[k, n] \qquad (k \in K,\ n \in N)$$

Now, by the Chain Rule we have:

$$(5.20) \qquad \frac{\partial g\,[k1]}{\partial y\,[k2]} = \Sigma_n \left(\frac{\partial g\,[k1]}{\partial \bar{g}\,[n]}\right) \cdot \left(\frac{\partial \bar{g}\,[n]}{\partial \bar{y}\,[n]}\right) \cdot \left(\frac{\partial \bar{y}\,[n]}{\partial y\,[k2]}\right)$$

$$= \Sigma_n\, (\bar{w}\,[n] \cdot C\,[k1, n]) \cdot \left(\frac{\partial \bar{g}\,[n]}{\partial \bar{y}\,[n]}\right) \cdot (C\,[k2, n]).$$

$$(k1, k2 \in K)$$

Let us write $\bar{g}'\,[n] = \partial \bar{g}\,[n] \,/\, \partial \bar{y}\,[n]$; then

$$(5.21) \qquad \partial g\,[k1] \,/\, \partial y\,[k2] = \Sigma_n\, \bar{w}\,[n] \cdot \bar{g}'\,[n] \cdot C\,[k1, n] \cdot C\,[k2, n].$$

$$(k1, k2 \in K)$$

Returning to Eq. (5.13), we see

$$(5.22) \qquad C\,[k1, n] \cdot C\,[k2, n] = \cos\,((2 \cdot k1 - 1) \cdot H\,[n]) \cdot \cos\,((2 \cdot k2 - 1) \cdot H\,[n]),$$

$$(k1, k2 \in K,\ n \in N)$$

which, by means of the identity

$$\text{(5.23)} \qquad \cos a \cdot \cos b = \tfrac{1}{2} \cos (a + b) + \tfrac{1}{2} \cos (a - b),$$

may be reduced to the following form:

$$\text{(5.24)} \qquad C\,[k1, n] \cdot C\,[k2, n] = \tfrac{1}{2} \cos (2 \cdot (k1 - k2) \cdot H\,[n]) + \tfrac{1}{2} \cos (2 \cdot (k1 + k2 - 1) \cdot H\,[n]). \qquad (k1, k2 \in K;\ n \in N)$$

If we let

$$\text{(5.25)} \qquad CE\,[k, n] = \cos (2 \cdot k \cdot H\,[n]) \qquad (n \in N)$$

then Eq. (5.21) becomes

$$\text{(5.26)} \qquad \partial g\,[k1] / \partial y\,[k2] = \tfrac{1}{2} \Sigma_n \overline{w}\,[n] \cdot \overline{g}'\,[n] \cdot (CE\,[k1 - k2, n] + CE\,[k1 + k2 - 1, n]),$$

$$= \tfrac{1}{2} \Sigma_n \overline{w}\,[n] \cdot \overline{g}'\,[n] \cdot CE\,[k1 - k2, n]$$

$$+ \tfrac{1}{2} \Sigma_n \overline{w}\,[n] \cdot g'\,[n] \cdot CE\,[k1 + k2 - 1, n]. \qquad (k1, k2 \in K;\ n \in N)$$

We now define an <u>even-cosine</u> analysis operation as follows:

$$\text{(5.27)} \qquad g'\,[k] = \Sigma_n \overline{w}\,[n]\, \overline{g}'\,[n]\, CE\,[k, n]. \qquad (k \in K')$$

where K′ = {0, 1, 2, ..., m}. The only remaining difficulty is that if k1 and k2 are elements of K = {1, 2, ..., m} then (k1 - k2) and (k1 + k2 - 1) do not always lie within K′. It would, of course, be possible to define g′ [k] by Eq. (5.27) for all k on the larger set K′′ = {-2m, ..., 0, ..., 2m}. It turns out, however, that the desired result can be accomplished even if k is restricted to the smaller set, and the amount of computation is thereby considerably reduced. To this end, we introduce a mapping μ from K'' to K' which has the property that

(5.28) $$CE\,[\mu(k),\, n] = CE\,[k,\, n]$$

for all k between -2m and 2m. Let

(5.29) $$\mu(k) = \min\,(|k|,\, 2m - |k|). \qquad (k \in K'')$$

It may be verified that μ has the property required by Eq. (5.28) and that $\mu(k) \in K'$ when $k \in K''$. We may thus write

(5.30) $$\partial g\,[k1] \,/\, \partial y\,[k2] = \tfrac{1}{2}\, g'\,[\mu(k1 - k2)] + \tfrac{1}{2}\, g'\,[\mu(k1 + k2 - 1)]. \qquad (k1,\, k2 \in K)$$

Thus, the assertion of Eq. (4.19) as to the form of $\mathcal{D}G(y)$ is verified. We note that this matrix is symmetric, since $\mu(k1 - k2) = \mu(k2 - k1)$.

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