UNIVERSITY OF CALIFORNIA, SAN DIEGO

Parking Function Polynomials and Their Relation to the Shuffle Conjecture

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by

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The dissertation of Angela Sue Hicks is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California, San Diego

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EPIGRAPH

A parking problem—the case of the capricious wives. *Let st. be a street with p parking places. A car occupied by a man and his dozing wife enters st. at the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space. Suppose st. to be initially empty and c cars arrive with independently capricious wives in each car. What is the probability that they all find parking places?*

—Alan G. Konheim and Benjamin Weiss in [KW66]

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On a more personal note, thanks to my family and friends, who were always encouraging, even when they couldn't figure out why I'd want to spend six years "learning how to park cars."

Professionally, I'd like to acknowledge the National Science Foundation, who supported this work through grant DMS 0800273.

A few of the figures, procedures, and definitions in Chapter 3 are common to several of my recent projects. As such, they are reproduced with permission from the paper "A Parking Function Bijection supporting the Haglund—Morse—Zabrocki Conjectures," that was advance access published in *Int. Math. Res. Notices* in December, 2012.

The interested reader will find a history of the shuffle conjecture in the appendix, written by Prof. Adriano Garsia and reproduced here with his permission.

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PUBLICATIONS

Two Parking Function Bijections: A Sharpening of the q,t-Catalan and Shroeder Theorems, *Int. Math. Res. Notices*, July 2011.

(with A. M. Garsia and A. Stout) The case $k = 2$ of the Shuffle Conjecture, *Journal of Combinatorics* Vol 2, 2011.

Connections between a family of recursive polynomials and parking function theory, *Proceedings of the 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC).* 2012.

(with Y. Kim) An explicit formula for ndinv, a new statistic for two-shuffle parking functions, *Journal of Combinatorial Theory, Series A* Vol 120, January 2013.

A Parking Function Bijection supporting the Haglund—Morse —Zabrocki Conjectures, *Int. Math. Res. Notices*, Advance Access published December 2012.

(with J.C. Aval, M. D'Adderio, M. Dukes, and Y. Le Borgne) Statistics on parallelogram polyominoes and a $q; t$-analogue of the Narayana numbers (preprint).

(with E. Leven) A refinement of the Shuffle Conjecture with cars of two sizes and $t = 1/q$ (preprint).

ABSTRACT OF THE DISSERTATION

Parking Function Polynomials and Their Relation to the Shuffle Conjecture

by

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The "Shuffle Conjecture" states that the bigraded Frobeneus characteristic of the space of diagonal harmonics (equal to ∇e_n) can be computed as the weighted sum of combinatorial objects called parking functions. In a 2010 paper Haglund, Morse, and Zabrocki studied the family of polynomials $\nabla C_{p_1} \ldots C_{p_k} 1$, where $p = (p_1, \ldots, p_k)$ is a composition and the C_a are certain rescaled Hall-Littlewood vertex operators. They conjecture that these polynomials enumerate a composition indexed family of parking functions weighted by the same statistics. This refinement of the nearly decade old "Shuffle Conjecture," when combined with properties of the Hall-Littlewood polynomials implies the existence of certain bijections between these families of parking functions. The existence of these bijections

then follows from some relatively simple properties of a certain recursively constructed family of polynomials. This work introduces those polynomials, explains their connection to the conjecture of Haglund, Morse, and Zabrocki, and explores some of their surprising properties, both proven and conjectured. The result is an intriguing new approach to the Shuffle Conjecture and a deeper understanding of some classical parking function statistics.

Chapter 1

Introduction

1.1 The Combinatorial Side—Parking Functions

We begin with a one way street with n cars and n parking places. Each driver has a preferred parking space, with the group of choices given by a vector π called a *preference function.*

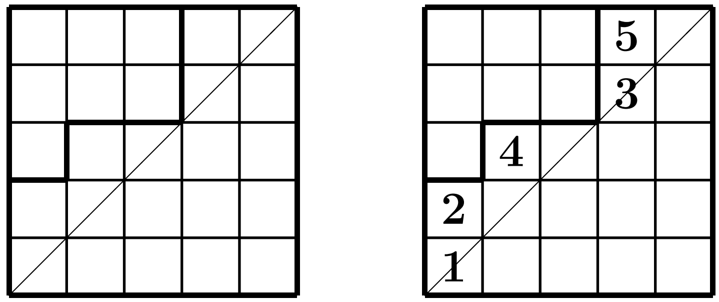
Example 1. $\pi = (1, 3, 3, 2)$ *is a preference function, representing that car 1 wants to park in space 1, car 2 wants to park in space 3 and so on.*

If we imagine that the cars park one by one, with the first car parking in space π_1 and the ith car parking in the first open space weakly after space π_i, it is easy to see that depending on the preference function, all the cars may or may not be able to park. If all the cars park successfully, π is called a *parking function.*

Example 2. $\pi = (1, 3, 3, 2)$ *is a parking function, since car 1 could park in space 1; car 2 could then park in space 3; car 3, upon seeing it's preferred space 3 was already filled would continue to space 4; and car 4 would then park in space 2.*

Example 3. *By contrast* $\pi = (1, 3, 3, 3)$ *is not a parking function, since car 4, upon proceeding to space 3, would find both it and space 4 occupied and be unable to park.*

Konheim and Weiss introduced parking functions in [KW66], but they have been reintroduced in various guises since then. A standard presentation defines



(a) (b)

Figure 1.1: A Dyck path $D\prime$ and a parking function $PF\prime$.

parking functions according to an easy to prove necessary and sufficient condition for a preference function to be a parking function:

Theorem 4. *Let $\alpha_1 \leq \cdots \leq \alpha_n$ be the rearrangement of $\pi \in [n]^n$ in weakly increasing order. Then π is a parking function exactly when $\alpha_i \leq i$ for all i. Equivalently, π is a parking function if and only if $\#\{i : \pi_i \leq j\} \geq j$.*

A graphical way of representing this latter condition is of principal importance to a number of results in parking function theory. In particular, one can represent a parking function as a particular kind of labeled Dyck path. First, start with an $n \times n$ grid with a "main diagonal" running from its southwest corner to its northeast corner. Then a Dyck path is a series of north and east steps beginning in the southwest and ending in the northeast, such that the path never crosses the main diagonal. Figure 1.1(a) shows a typical example. Figure 1.1(b) then gives a parking function, a labeled Dyck path. In particular, a parking function is represented by a Dyck path with its north steps labeled with the integers 1 to n such that integers in the same column (such as 3 and 5 in the example) increase from bottom to top. A third way of representing a parking function is closely related to the second, but easier to manipulate with a computer. From such a diagram, we can construct a two line array, by defining r_i as the integer in the i^{th} row (moving bottom to top) and by defining g_i as the number of full lattice cells between the Dyck path and the main diagonal in the i^{th} row. Thus Figure 1.1 (b) corresponds to the array

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Under this mapping, a two line array

$$PF = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ g_1 & g_2 & \cdots & g_n \end{bmatrix}$$

is a parking function exactly when:

1. (Dyck Path Condition.) For all i, g_i is a nonnegative integer with $g_1 = 0$ and for $i < n$, $0 \leq g_{i+1} \leq g_i + 1$.
2. (Increasing Column Condition.) $\begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$ is a permutation of 1 to n such that when $g_{i+1} = g_i + 1$, $r_{i+1} > r_i$.

To maintain the traditional parking function jargon we will hereafter refer to $r_1, r_2, \ldots, r_n$ as the "cars" and to $g_1, g_2, \ldots, g_n$ as their respective "diagonals". For instance we could say that "car r_i is in diagonal g_i". (Notice this means that cars along the main diagonal of a parking function are in diagonal 0.) The enumeration of parking functions has been known since their inception; here we reproduced the best known argument by Pollak, as reported in [Rio69].

Theorem 5. *There are exactly* $n + 1^{n-1}$ *parking functions of size* n.

Proof. Start with a vector $\pi \in [n+1]^n$ giving the preferences for n drivers on a circular one way street. Since the street is circular, each car will be able to park. See Figure 1.2. Once every car is parked, there will be exactly one parking space empty. Replacing preference i with preference $i+1 \bmod (n+1)$ for all i will result in a rotation of the final positions of the parked cars and the empty space. If the empty space is the $n+1$st space, then the original preference function must have

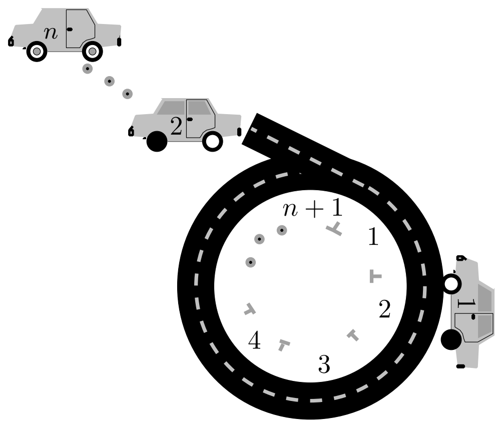


Figure 1.2: A parking function on a circular street.

been a parking function in the traditional (not circular) sense. Since exactly one of these $n+1$ rotations must then correspond to a parking function, we have exactly

$$\frac{(n+1)^n}{n+1}$$

parking functions as claimed. □

1.2 The Representation Theoretic Side—Diagonal Harmonics

The parking functions have a rich history connecting them to a simply defined space known as the diagonal harmonics.

Definition 6 (Diagonal Harmonics)**.** *A polynomial* $f[x_1, \ldots, x_n, y_1, \ldots, y_n]$ *is* diagonal harmonic *if and only if it is killed by all the differential operators:*

$$\Delta_{r,s} = \sum_{i=1}^{n} \partial_{x_i}^h \partial_{y_i}^k$$

That is:

$$DH_n = \left\{ f \in \mathbb{C}[X_n, Y_n] : \sum_{i=1}^{n} \partial_{x_i}^h \partial_{y_i}^k f[X, Y] = 0, \forall h + k > 0 \right\}$$

Example 7.

$$f[X,Y] = x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1 \in DH_2$$

The best known example of a diagonal harmonic is the Vandermonde determinant (either in X as given here or in Y):

Example 8.

$$\Delta_n(X) = \prod_{1\le i<j\le n} (x_j - x_i) \in DH_n$$

An"easy" fact about the diagonal harmonics that we include here without proof, first appeared, along with definition of the space, in [Hai94]. It highlights the quintessential importance of the Vandermonde determinant to the space.

Theorem 9. *For $P(x) \in \mathbb{Q}[x_1,\dots,x_n]$ let*

$$P(\partial_x) = P(\partial_{x_1},\dots,\partial_{x_n}).$$

For $a > 0$, let

$$R_a(\partial_x) = y_1\partial_{x_1}^a + \cdots + y_n\partial_{x_n}^a.$$

Then applying these two types of operators to the Vandermonde gives a spanning set for the diagonal harmonics:

$$DH_n = \operatorname{span}\left\{R_{a_1}(\partial_x)R_{a_2}(\partial_x)\dots R_{a_k}(\partial_x)P(\partial_x)\Delta_n(X) : P(x) \in \mathbb{Q}[X]\right\}_{a_1\le\cdots\le a_k\le n-1}$$

Directly applying this theorem and the definition of the diagonal harmonics, we get several other easily apparent facts about the space:

Theorem 10.

- *DH_n is a finite dimensional space, with polynomials of total degree at most $\binom{n}{2}$.*
- *If a polynomial $P[X,Y] \in DH_n$, then so are its derivatives.*
- *If a polynomial $P[X,Y] \in DH_n$, then so are its bi-homogeneous components.*

Example 11. *If we know that $(-x_2 + x_3 + 1)\,y_1 + (x_1 - x_3 - 1)\,y_2 + (x_2 - x_1)\,y_3 \in DH_3$, then we can conclude that $y_1 - y_2 \in DH_3$, as a bi-homogeneous component of the first.*

A much more difficult result, proved by Mark Haiman in [Hai01b] using deep theorems from algebraic geometry gives the dimension of the diagonal harmonics:

Theorem 12 (Haiman)**.**

$$\dim(DH_n) = (n+1)^{n-1}$$

The astute reader should recognize this enumeration from its previous mention, as giving the number of parking functions of size n; we will explore this connection in depth in an upcoming section. Let $\mathcal{H}_{r,s}(\mathrm{DH}_n)$ give diagonal harmonics of degree r in X and s in Y. Then we have a simple bi-grading of DH_n:

$$\mathrm{DH}_n = \bigoplus_s \bigoplus_r \underset{0 \leq r+s \leq \binom{n}{2}}{} \mathcal{H}_{r,s}(\mathrm{DH}_n)$$

and it makes sense to consider the "bi-variate Hilbert series" of DH_n:

$$F_{\mathrm{DH}_n}(q,t) = \sum \sum\nolimits_{0 \leq r+s \leq \binom{n}{2}} t^r q^s \dim(\mathcal{H}_{r,s}(\mathrm{DH}_n)).$$

Moreover, there is a natural *diagonal action* of S_n on DH_n:

$$\sigma P(x_1, \ldots, x_n; y_1, \ldots, y_n) = P(x_{\sigma_1}, \ldots, x_{\sigma_n}; y_{\sigma_1}, \ldots, y_{\sigma_n})$$

which does not change the bi-homogeneous degree of the polynomial.

Example 13. *Let* $\sigma = (1,3)$ *and*

$$f[X,Y] = (-x_2 + x_3 + 1)\, y_1 + (x_1 - x_3 - 1)\, y_2 + (x_2 - x_1)\, y_3.$$

Then

$$\begin{aligned} \sigma f[X,Y] &= (-x_2 + x_1 + 1)\, y_3 + (x_3 - x_1 - 1)\, y_2 + (x_2 - x_3)\, y_1 \\ &\in DH_3. \end{aligned}$$

This action induces a representation of S_n on DH_n. Thus, beyond studying the Hilbert series, one can consider the bi-graded Frobenius characteristic of the diagonal harmonics:

$$\mathrm{DH}_n[x;q,t] = \sum \sum\nolimits_{0 \leq r+s \leq \binom{n}{2}} t^r q^s \ \mathrm{F\ char}\ \mathcal{H}_{r,s}(\mathrm{DH}_n).$$

1.3 The Symmetric Function Side—Macdonald Polynomials and Nabla

The symmetric functions of degree n, $(\Lambda^{=n})$ have a number of well-studied bases, including the schur (s_λ), the power (p_λ), the homogeneous (h_λ), the monomial (m_λ), and the elementary (e_λ). A particularly useful concept when expressing symmetric function identities is plethystic notation, which is symbolically notated with brackets ([]). For any expression $E = E(t_1, t_2, \dots)$, we define:

$$p_k[E] := E(t_1^k, t_2^k, \dots)$$

Since any symmetric function F can then be expressed in terms of the power basis, $(F = Q_F(p_1, p_2, \ldots)$, we can then expand the definition to any symmetric function:

$$F[E] := Q_F(p_1, p_2, \dots)\Big|_{p_k \to p_k[E]}$$

A generalization of several of the symmetric function basis, introduced by Macdonald in [Mac95], which is particularly important to the topic at hand is the Macdonald polynomials $P_\lambda[X; q, t]$ and the closely related integral forms $J_\lambda[X; q, t]$. In fact, we are interested in a modification of the Macdonald polynomials originally introduced by Garsia and Procesi in [GH96b]:

$$\tilde{H}_\lambda[X; q, t] := t^{n(\lambda)} J_\lambda \left[\frac{X}{1 - \frac{1}{t}}; q, \frac{1}{t} \right]$$

where $n(\lambda) = \sum_{i=1}^{l(\lambda)} \lambda_i (i-1)$. Alternately, $\tilde{H}_\lambda$ is the is the unique symmetric function basis such that (using dominance partial order):

$$\tilde{H}_\lambda\Big|_{s_n} = 1,$$

$$\tilde{H}_\lambda = \sum_{\mu \le \lambda} s_\mu \left[\frac{X}{t-1} \right] c_{\mu\lambda}(q, t), \text{ and } \tilde{H}_\lambda = \sum_{\mu \ge \lambda} s_\mu \left[\frac{X}{1-q} \right] d_{\mu\lambda}(q, t).$$

A number of polynomial expressions involving the Macdonald Polynomials are calculated by summing or multiplying over statistics on the cells of a Ferrer's diagram of a given partition. (See Figure 1.3 for a pictorial definition of these statistics.) Using λ' to indicate the conjugate partition to λ:

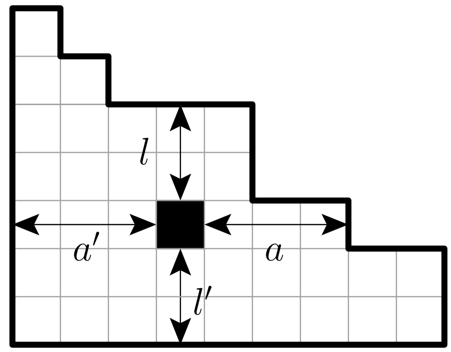


Figure 1.3: A standard tableau in French notation, with the leg ($l(c) = 2$), arm ($a(c) = 3$), coleg ($l'(c) = 3$), and coarm ($a'(c) = 2$) of a typical cell c indicated.

$$n(\lambda) = \sum_{c\in\lambda} l(c) = \sum_{i=1}^{l(\lambda)} \lambda_i(i-1) \qquad \tilde{h}_\lambda(q,t) = \prod_{c\in\lambda} q^{a(c)} - t^{l(c)+1}$$

$$T_\lambda = t^{n(\lambda)} q^{n(\lambda')} \qquad \tilde{h}_\lambda(q,t) = \prod_{c\in\lambda} q^{a(c)} - t^{a(c)+1}$$

$$B_\lambda(q,t) = \sum_{c\in\lambda} q^{a'(c)} t^{l'(c)} \qquad w_\lambda(q,t) = \tilde{h}_\lambda(q,t)\tilde{h}'_\lambda(q,t)$$

$$\Pi_\lambda(q,t) = \prod_{\substack{c\in\lambda \\ c\neq(0,0)}} \left(1 - q^{a'(c)} t^{l'(c)}\right)$$

Definition 14. *Then we can define a fundamental symmetric function operator* nabla*, introduced in [BG99], by:*

$$\nabla \tilde{H}_\mu[X;q,t] = T_\mu \tilde{H}_\mu[X;q,t].$$

1.4 A Connection—The Shuffle Conjecture

Combining results in a number of papers ([GH96a], [Hai01a], and [BG99]) and using the ∇ operator, we can express the Frobenius characteristic of the Diagonal Harmonics:

Theorem 15 (Haiman)**.**

$$\nabla e_n = DH_n[x;q,t] = \sum_{\mu\vdash n} \frac{(1-t)(1-q)T_\mu \tilde{H}_\mu \Pi_\mu B_\mu}{w_\mu}$$

Example 16.

$$\nabla e_3 = s_3 + \left(t + t^2 + tq + q + q^2\right) s_{2,1} \\ + \left(t^3 + t^2 q + tq + tq^2 + q^3\right) s_{1,1,1}$$

Intriguingly, representation theory tells us that this expression should always be a schur positive polynomial and not just a rational expression, as in the previous theorem. Recalling that the dimension of the space of diagonal harmonics on $2n$ variables is the same as the number of parking functions with n cars, several statistics on the parking functions have been introduced to explore these coefficients.

Definition 17. *The **area** of a parking function is the number of lattice squares between its Dyck path and the main diagonal.*

Note here that we do not add two half squares to get a whole square. Equivalently,

$$\text{area}(PF) = \sum g_i$$

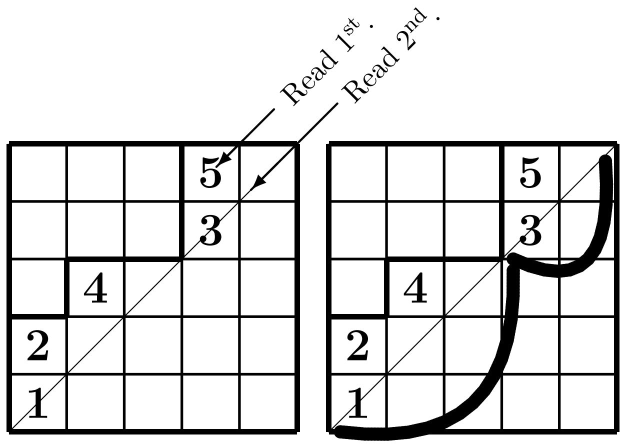
or, using the original preference function style expression, the area of a parking function is the sum of the additional distances that each car has to go beyond its preferred parking place in order to park.

Example 18. *For $PF\prime$ in Figure 1.1 (b),* $\text{area}(PF\prime) = 3$.

Definition 19. *Two cars in a parking function, are **primary attacking** if they are in the same diagonal. Two cars a and b are **secondary attacking** if a is to the left of b and b is in the diagonal just below that of a.*

Example 20. *The primary attacking pairs in $PF\prime$ are $\{1,3\}$, $\{2,5\}$, $\{2,4\}$, and $\{4,5\}$. The secondary attacking pairs are $\{2,3\}$ and $\{3,4\}$.*

Definition 21. *Two primary attacking pairs form a **primary diagonal inversion** (dinv) when the car on the left is smaller. Two secondary attacking pairs form a **secondary dinv** when the car on the left (called a above) is larger. The **dinv** of a parking function is the total number of pairs of cars forming either primary or secondary dinv.*



(a) (b)

Figure 1.4: The parking function $PF\prime$ has word $[5,4,2,3,1]$ and composition $[3,2]$.

Equivalently, using χ for the truth function,

$$\operatorname{dinv}(PF) = \sum_{i<j} \chi(g_i = g_j \text{ and } r_i < r_j) + \chi(g_i + 1 = g_j \text{ and } r_i > r_j).$$

Example 22. $\{1,3\}$, $\{2,5\}$, $\{2,4\}$ *and* $\{4,5\}$ *all form primary dinv in* $PF\prime$. $\{3,4\}$ *(but not* $\{2,3\}$*) forms secondary dinv. Thus* $\operatorname{dinv}(PF\prime) = 5$.

Definition 23 (reading word)**.** *The **reading word** (or simply word) of a parking function is formed by reading cars by diagonals, starting with the diagonal farthest from the main diagonal, reading cars in a diagonal from northeast to southwest. The **ides** of a parking function is the set of* r *occurring after* $r+1$ *in the word.*

Example 24. *As in Figure 1.4, reading the integers from the diagonal in* $PF\prime$ *containing* 4 *and then from the diagonal containing* 3*, following the arrow in 1.4 (a), we obtain the word* $[5,4,2,3,1]$. *The ides of* $PF\prime$ *is* $\{1,3,4\}$.

A fundamental basis for the quasi-symmetric functions, first introduced by Gessel in [Ges84], indexed here by subsets of $\{1,\cdots,n-1\}$, is Gessel's Fundamental basis with elements defined by

$$Q_S = \sum_{\substack{1 \le a_1 \le \cdots \le a_n \le n \\ i \in S \to a_i < a_{i+1}}} x_{a_1} x_{a_2} \dots x_{a_n}.$$

Then finally we combine the previous statistics to get a single quasi-symmetric weight from each parking function:

$$\mathrm{wt}(PF) = t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)}.$$

Now we can express the now decade old "shuffle" conjecture, first expressed by Haglund et al. in [HHL$^+$05b]:

Conjecture 25.

$$DH_n[x; q, t] = \sum_{PF \in PF_n} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}.$$

Chapter 2

The HMZ conjecture

To state several more recent results in this area, we need to introduce a certain family of modified Hall Littlewood Operators. Again using the brackets [ ] to indicate plethystic substitution, set for any symmetric function $F[X]$

$$C_a F[X] = \left(\frac{-1}{q}\right)^{a-1} \sum_{k \geq 0} F\left[X + \frac{1-q}{q} z\right]\Big|_{z^k} h_{a+k}[X].$$

Note that to simplify notation, for a composition $c = [c_1, \cdots, c_k]$, we use the convention

$$C_c F[X] = C_{c_1} \dots C_{c_k} F[X].$$

Frequently, we use C_c for $C_c 1$ when the meaning is otherwise clear. In fact, these particular polynomials are closely related to the original Hall- Littlewood polynomials, as shown in [HMZ12], by

$$Q'_\lambda[X; q] = (-q)^{l(\lambda) - |\lambda|} C_\lambda \left[X; \frac{1}{q}\right].$$

Next, we must introduce one additional parking function definition:

Definition 26 (composition)**.** *The **composition** of a parking function gives the number of lattice cells between successive intersections of the Dyck path and the main diagonal. (See Figure 1.4 (b).) Use $\mathcal{A}_c$ for the set of parking functions with composition c.*

In [HMZ12] Haglund, Morse and Zabrocki published the following conjecture, first formulated by themselves and N. Bergeron:

Conjecture 27. *For c a composition,*

$$\nabla C_c 1 = \sum_{PF \in \mathcal{A}_c} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}. \tag{2.1}$$

Note first that by summing over all compositions of n on both sides of (2.1), we get the original shuffle conjecture; thus Conjecture 27 is a sharpening. A classical approach, since both sides of the equation are symmetric functions, has been to verify the validity of (2.1) upon Hall scalar multiplication of both sides by every element of a symmetric function basis. An early combinatorial result of the author in [Hic12], combined with a symmetric function result in [GXZ12b] proves the following equality:

Theorem 28. *For any a and b,*

$$\langle \nabla C_p 1, e_a h_b \rangle = \left\langle \sum_{\text{comp}(PF)=p} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}, e_a h_b \right\rangle.$$

This result includes and sharpens the q,t-Catalan Theorem of [GH02] and the Schröder Theorem of [Hag04]. At the date of publication, the strongest such result, stated in [GXZ12a] gives equality when taking the inner product on each side of (2.1) with $e_a h_b h_c$. Although this approach has been very successful in several additional cases and inspired several surprising developments such as the the results in [DGZ] and [HK13], the increasing difficulty of the combinatorial part as well as the symmetric function part, as the complexity of the basis element increases, has prompted other approaches. The majority of this work is motivated by an alternate plan. There is a natural way to divide the proof of Conjecture 27 into two parts, namely:

1. (*Reduction.*) Reduce the HMZ conjecture to proving the partition case. That is, show that if (2.1) is true for all partitions p then it is true for all compositions.

2. (*Partition Problem.*) Prove that (2.1) is true when p is a partition.

The latter part relies primarily on the fact that the collection $\left\{C_{\mu_1} \cdots C_{\mu_n} 1\right\}_{\mu \vdash n}$ (where $\mu \vdash n$ indicates μ is a partition of n) is a symmetric function basis. The majority of this work focuses on the second part and is based on a commutativity law satisfied by the C_a operators.

2.1 The Partition Problem

The C operators have several useful properties. (See [GP92].) Among them:

1. Using $c \models n$ to indicate that c is a composition of n, we have:

$$e_n = \sum_{[c_1,\cdots,c_s]\models n} C_{c_1} C_{c_2} \dots C_{c_s} 1$$

2. The C operators obey the following commutativity law: For $a+1 \leq b$,

$$q(C_a C_b + C_{b-1} C_{a+1}) = C_b C_a + C_{a+1} C_{b-1}.$$

3. Using $\mu \vdash n$ to indicate that μ is a partition of n, $\left\{C_{\mu_1} \cdots C_{\mu_n} 1\right\}_{\mu \vdash n}$ is a basis for the homogeneous symmetric functions of degree n.

The first statement is exactly what allows us to conclude that the HMZ conjecture is a sharpening of the Shuffle Conjecture. The second, fundamental to the problem of reduction, we will study in depth in the remainder of this work. The third is key to a current approach to solving the Partition Problem. Inspired by the argument used in [HHL05a], proving Haglund's combinatorial formula for the Macdonald polynomials $\{\tilde{H}_\mu\}_\mu$, Garsia has made progress on the Partition Problem by means of the following elementary result in linear algebra:

Theorem 29. *Let V be a vector space with four bases:*

$$G = \langle G_1, \dots, G_n \rangle,\ H = \langle H_1, \dots, H_n \rangle,$$

$$\phi = \langle \phi_1, \dots, \phi_n \rangle,\ \textit{and}\ \psi = \langle \psi_1, \dots, \psi_n \rangle.$$

Say that G and H are both upper triangularly related to the ϕ basis and lower triangularly related to the ψ basis, that is that:

$$G_j = \sum_{i \leq j} \phi_i a_{i,j},\ G_j = \sum_{i \geq j} \psi_i b_{i,j},$$

$$H_j = \sum_{i \leq j} \phi_i c_{i,j},\ \textit{and}\ H_j = \sum_{i \geq j} \psi_i d_{i,j}.$$

Then there exist scalars s_j such that $G_j = s_j H_j$.

For sake of completeness, we include an elementary proof.

Proof. In particular, using matrix notation, say

$$\langle G \rangle = \langle \phi \rangle U_H = \langle \psi \rangle L_H \tag{2.2}$$

and

$$\langle H \rangle = \langle \phi \rangle U_H = \langle \psi \rangle L_H. \tag{2.3}$$

Then let M be defined by

$$M = \langle \phi \rangle^{-1} \langle \psi \rangle \tag{2.4}$$

Then

$$U_H L_H^{-1} = M = U_G L_G^{-1} \tag{2.5}$$

and thus

$$U_G^{-1} U_H = L_G^{-1} L_H = D. \tag{2.6}$$

Since this left hand side, as the product of two upper triangular matrices is also upper triangular and similarly the right hand side is lower triangular, we can conclude their product, call it D, is a diagonal matrix. Then

$$U_H = U_G D \tag{2.7}$$

and thus

$$\langle H \rangle = \langle \phi \rangle U_G D = \langle G \rangle D \tag{2.8}$$

as required. □

As noted previously, both $\{\nabla C_\mu\}_{\mu \vdash n}$ and

$$\left\{ \sum_{\mathrm{comp}(PF)=p} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)} \right\}_{\mu \vdash n}$$

are bases for the space of symmetric functions of degree n. It is tempting to hope that one might consider one of these basis as G and another as H and apply the previous theorem. Partial progress towards this goal is the following theorem:

Theorem 30 (Garsia, private communication)**.**

$$\{\nabla C_\mu\}_{\mu\vdash n} \text{ and } \left\{ \sum_{\mathrm{comp}(PF)=p} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)} \right\}_{\mu\vdash n}$$

are both upper triangularly related to the basis $\phi = \left\{ s_\mu \left[\frac{X}{q-1} \right] \right\}_{\mu\vdash n}$.

Moreover by Theorem 28, if a lower trangularity could be found, it would guarantee not just that the two basis are the same up to scalars, but that the two basis are truly identical. Thus to solve the Partition Problem, it remains to find a basis ψ which is lower triangularly related to the two bases.

Open Problem 31. *Find a basis* ψ *which is lower triangularly related to*

$$\{\nabla C_\mu\}_{\mu\vdash n} \text{ and } \left\{ \sum_{\mathrm{comp}(PF)=p} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)} \right\}_{\mu\vdash n}.$$

2.2 Reduction

Recall the commutativity relations on the C operators, in particular that for $a + 1 \leq b$,

$$q(C_a C_b + C_{b-1} C_{a+1}) = C_b C_a + C_{a+1} C_{b-1}.$$

In particular, reducing the HMZ conjecture then is equivalent to showing the existence of a number of parking function bijections implied by successive applications of these commutativity relations. That is, after setting

$$c = [c_1, \cdots, c_k] = [c', c_i, c_{i+1}, c''],$$

it is then easily seen that the above commutativity relations imply that for $c_i < c_{i+1} - 1$, we must have

$$q(C_{[c',c_i,c_{i+1},c'']} 1 + C_{[c',c_{i+1}-1,c_i+1,c"]} 1) = C_{[c',c_{i+1},c_i,c'']} 1 + C_{[c',c_i+1,c_{i+1}-1,c"]} 1.$$

Then setting

$$A_c = \sum_{PF\in\mathcal{A}_c} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)},$$

that is the weighted sum over all parking functions with composition c, we are forced to conjecture that:

$$q(A_{[c',c_i,c_{i+1},c'']} + A_{[c',c_{i+1}-1,c_i+1,c"]}) = A_{[c',c_{i+1},c_i,c'']} + A_{[c',c_i+1,c_{i+1}-1,c"]}. \tag{2.9}$$

Proving this equality for every composition and choice of i is exactly what is required to reduce the HMZ conjecture. Although this requires a number of bijections, we begin with the simplest, which uses two part compositions. We will refer to it as the *commutativity bijection* f:

Conjecture 32. *For $a \leq b-1$, there exists a bijection f*

$$f : \mathcal{A}_{(a,b)} \cup \mathcal{A}_{(b-1,a+1)} \leftrightarrow \mathcal{A}_{(b,a)} \cup \mathcal{A}_{(a+1,b-1)}$$

with the following properties:

1. *f increases the dinv statistic by exactly one*
2. *f preserves the area and the ides statistics*

Experience with the problem has led to the following additional conjecture, now verified experimentally through parking functions of size 14, that the map can be made to satisfy a "diagonal condition": f does not change the diagonal of any given car. Although f is defined only on parking functions with two parts, adding this diagonal condition gives the following remarkable theorem, perhaps the most important in this work, which allows us to concentrate on finding a single bijection:

Theorem 33. *If there exists a commutativity bijection f that satisfies the diagonal condition, then proving the HMZ conjecture is reduced to the partition case. Moreover, if f increases the dinv appropriately, it is enough to check that f satisfies the diagonal condition without verifying that f preserves the area and ides.*

Thus we need only search for a single set of bijections on two part parking functions to reduce the proof of the HMZ conjecture. We save the proof of this theorem for a later section.

Chapter 3

Parking Functions with a Given Diagonal Word

Theorem 33 suggests that to attempt our desired reduction, we must better understand the family of parking functions with the same sets of cars in the same diagonal. Imagine we start with a parking function

$$PF = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ g_1 & g_2 & \cdots & g_n \end{bmatrix}.$$

To find the other parking functions in the same family, we need to consider all the ways that we can arrange the columns of PF (referred to as “dominoes”) to get a legal parking function. A particularly nice way to think about doing this is to consider placing the cars by increasing diagonal, placing cars within a diagonal by decreasing car values, since we can then preserve the requirements for a parking function at every step. How can we place a domino

$$\begin{bmatrix} r_j \\ g_j \end{bmatrix}?$$

In particular, we can place it to the immediate right of any car in the same diagonal g_j or to the right of a smaller car on diagonal $g_j - 1$ without violating the Dyck Path Condition or the Increasing Column Condition. Thus we get a whole decision tree of choices as to where we can put any one car, as seen in Figure 3.1. Moreover, if we chose not to place a domino $\begin{bmatrix} r_j \\ g_j \end{bmatrix}$ beside a domino $\begin{bmatrix} r_i \\ g_i \end{bmatrix}$ where $g_j = g_i$ and r_i was placed before r_j (so that $r_i > r_j$) or otherwise $g_i = g_j - 1$ and $r_j > r_i$, then we are precisely in the situation that we have formed a diagonal inversion. Thus moving further left on our decision tree gives additional diagonal inversions. We can formalize this construction by introducing something called a diagonal word.

Definition 34. *The **diagonal word** of a parking function (*diagword(PF)*) is found by reading the diagonals, again (as with the original reading word) starting with the diagonal farthest from the main diagonal, but this time recording the cars within a diagonal in increasing order.*

Example 35. *The diagonal word of the parking function in Figure 1.4 is*

$$[2, 4, 5, 1, 3].$$

Notice that by splitting diagword(PF) at its descents, we get the contents of the diagonals of PF. (i.e. Notice that 2, 4, and 5 are on the first diagonal and 1 and 3 are in the main diagonal of $PF\prime$.) Thus two parking functions have the same diagonal word exactly when they contain the same cars on every diagonal. In [HL05], Haglund and Loehr were the first to describe a recursive operation for forming the parking functions with a given diagonal word $\tau = [\tau_1, \ldots, \tau_n]$. We give an equivalent formulation of this natural procedure here, expanding on our comments above, as we will use the procedure as a starting point in studying our bijections. For notational convenience, set

$$\overline{\tau} = [\overline{\tau}_0, \overline{\tau}_1, \ldots, \overline{\tau}_n] = [0, \tau_n, \ldots, \tau_1],$$

since the procedure produces parking functions by recursively adding cars, starting with τ_n and working forward. Using this notation, we reproduce the procedure here, using $\tau = [4, 2, 5, 1, 3]$ (and thus $\overline{\tau} = [0, 3, 1, 5, 2, 4]$) as an example.

Procedure 36. *1. Form dominoes from $\overline{\tau}$ by the following:*

(a) Split $\overline{\tau}$ at its assents to form v.

- *Ex.* $v = ([0], [3, 1], [5, 2], [4])$

(b) *Define* t_i *such that* $\overline{\tau}_i$ *is in* v_{t_i+2}.

Form the list $D = \left(\begin{bmatrix} \overline{\tau}_0 \\ t_0 \end{bmatrix} \begin{bmatrix} \overline{\tau}_1 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} \overline{\tau}_n \\ t_n \end{bmatrix} \right)$.

- *Ex.* $D = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right)$,

2. *Begin with* $V_0 = ([D_0])$. *(We will remove* D_0 *from our final parking functions. Here it is a convenient way to begin our recursion.)*

   - *Ex.* $V_0 = \left(\left[\begin{bmatrix} 0 \\ -1 \end{bmatrix} \right] \right)$

3. *Recursively, add* $D_i = \begin{bmatrix} \overline{\tau}_i \\ t_i \end{bmatrix}$ *to an element in* V_{i-1} *in all possible ways so that* D_i *is directly to the right of* $\begin{bmatrix} \overline{\tau}_j \\ t_j \end{bmatrix}$ *and either:*

   (a) $t_i = t_j$

   (b) $t_i = t_j + 1$ *and* $\overline{\tau}_i > \overline{\tau}_j$.

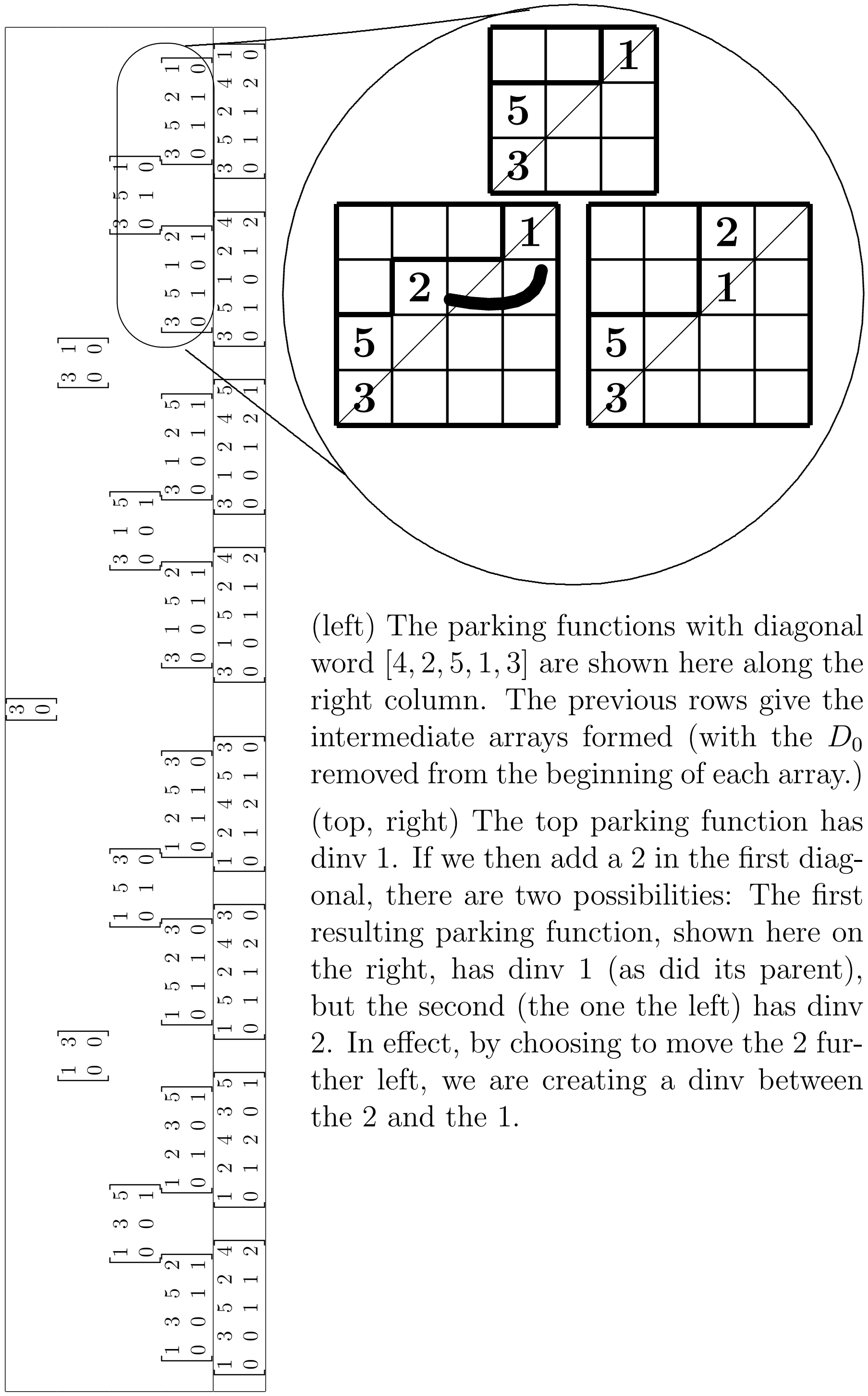
   Form V_i *by adding* D_i *in all possible ways to all the elements in* V_{i-1}.

   - *Ex. We may add* $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ *to* $\begin{bmatrix} 0 & 3 & 5 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ *and get* $\begin{bmatrix} 0 & 3 & 5 & 1 & 2 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 3 & 5 & 2 & 1 \\ -1 & 0 & 1 & 1 & 0 \end{bmatrix}$.

4. *Remove* $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ *from the beginning of every element in* V_n *to form all the parking functions with diagonal word* τ.

   - *Ex. See Figure 3.1 for the final family of parking functions with diagonal word* $[4, 2, 5, 1, 3]$.

Notice that although the actual positions to which we add D_i may vary depending on the particular element in V_{n-1}, the *number* of positions to which we



(left) The parking functions with diagonal word $[4, 2, 5, 1, 3]$ are shown here along the right column. The previous rows give the intermediate arrays formed (with the D_0 removed from the beginning of each array.)

(top, right) The top parking function has dinv 1. If we then add a 2 in the first diagonal, there are two possibilities: The first resulting parking function, shown here on the right, has dinv 1 (as did its parent), but the second (the one the left) has dinv 2. In effect, by choosing to move the 2 further left, we are creating a dinv between the 2 and the 1.

Figure 3.1: The parking functions with diagonal word $(4, 2, 5, 1, 3)$ formed recursively.

add D_i is constant across all the elements of V_{n-1}. We refer to this number as w_i and can calculate it directly as

$$w_i = \#\{\overline{\tau}_j : \overline{\tau}_j < \overline{\tau}_i \text{ and } t_j + 1 = t_i\} + \#\{\overline{\tau}_j : \overline{\tau}_j > \overline{\tau}_i \text{ and } t_j = t_i\},$$

These numbers (and the sets they count) have taken on a surprisingly important role in the remainder of this work. We next reproduce a theorem which hints at their importance.

Theorem 37 (Haglund & Loehr, [HL05])**.**

$$\sum_{\text{diagword}(PF)=\tau} t^{\text{area}(PF)} q^{\text{dinv}(PF)} = t^{\text{maj}(\tau)} \prod_{i=1}^{n} [w_i]_q$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$.

Proof. The proof of the theorem comes directly from the comments opening this section. First, notice that if $\text{diagword}(PF) = \tau$,

$$\text{area}(PF) = \sum_{i>0} t_i = \text{maj}(\tau),$$

since ascents (besides the first) in τ' correspond to descents in τ. Next, consider the cars in PF that are placed by Procedure 36 in PF before $\overline{\tau}_i$ and form a diagonal inversion with $\overline{\tau}_i$. These are exactly the sets

$$\{\overline{\tau}_j : \overline{\tau}_j > \overline{\tau}_i, t_i = t_j, \text{ and } \overline{\tau}_i \text{ to the left of } \overline{\tau}_j \text{ in } PF\} \text{ and}$$
$$\{\overline{\tau}_j : \overline{\tau}_j < \overline{\tau}_i, t_i = t_j + 1, \text{ and } \overline{\tau}_i \text{ to the left of } \overline{\tau}_j \text{ in } PF\}.$$

Notice that

$$\left\{ \begin{bmatrix} \overline{\tau}_j \\ t_j \end{bmatrix} : \overline{\tau}_j > \overline{\tau}_i, t_i = t_j \right\} \text{ and } \left\{ \begin{bmatrix} \overline{\tau}_j \\ t_j \end{bmatrix} : \overline{\tau}_j < \overline{\tau}_i, t_i = t_j + 1 \right\}$$

are exactly the elements we may place D_i beside in step (3) of Procedure 36. Consider adding D_i in all possible ways into $\pi \in V_{i-1}$. By definition, D_i can be placed in π in w_i distinct places, say to the right of dominoes $D_{k_1}, \cdots, D_{k_{w_i}}$ listed in the order they appear in π. Then the diagonal inversions formed when we place D_i in π are formed between D_i and the subset of $D_{k_1}, \cdots, D_{k_{w_i}}$ occurring to the right of D_i. Thus placing D_i directly to the right of $D_{k_{w_i}}$ (such that we see $D_{k_1}, \cdots, D_{k_{w_i}}, D_i$ occurring in this relative order in the result) will not create any new diagonal inversions in π, but each time we choose to place D_i further to the right, we create a new diagonal inversion, thus giving an increase in dinv of $0, 1, 2, \cdots, w_i - 1$ as required. (Again see Figure 3.1.) □

The set counted by w_i is important enough to warrant notation of its own. In general, given a parking function and a particular car τ_i, we'd like to know all the cars it could have been placed next to in Procedure 36 and the subset of these cars which were to its right in the parking function.

Definition 38 (degree set)**.** *Let the* ***degree set*** *of τ_i in PF be*

$$\mathrm{degset}(\tau_i, PF) = \{j : \overline{\tau}_j < \overline{\tau}_{n-i+1} \textit{ and } t_j + 1 = t_{n-i+1}\} \cup \{j : \overline{\tau}_j > \overline{\tau}_{n-i+1} \textit{ and } t_j = t_{n-i+1}\},$$

(Include 0 in the degree set of elements in the main diagonal.) Say that τ_i is of ***full degree*** *if τ_i as far to the left as possible when we construct PF, creating as many new diagonal inversions as possible.*

Definition 39 (dinv set)**.** *Let the* dinv set *of τ_k in a parking function PF be*

$$\mathrm{dinvset}(\tau_i, PF) = \{j : j \in \mathrm{degset}(\tau_i, PF) \textit{ and } \overline{\tau}_j \textit{ right of } \overline{\tau}_{n-i+1} \textit{ in } PF\}.$$

Next we state a quick lemma that will be useful in future proofs:

Lemma 40.

$$\mathrm{degset}(\overline{\tau}_i, PF) = \{i-1, \ldots, i-w_i\}$$

Proof. The observation is immediate, since in step 3.(a) of Procedure 36 we add car $\overline{\tau}_i$ beside cars in the same (descending) run that have already been added and in step 3.(b) we add the same car beside smaller cars in the next run in $\overline{\tau}$.) Since smaller cars in the next run occur first within the run, we are done. □

Finally, we observe that Procedure 36 suggests a new way to represent a parking function, in particular in a two line array we will refer to as the *dinv representation* of a parking function and notate with curly braces. On the top line, we give the diagonal word of a parking function; on the bottom, working from right to left, the number of cars to the right of τ_i that we could have placed τ_i to the right of in Procedure 36. That is, by

$$PF = \begin{Bmatrix} \tau_1 & \tau_2 & \dots & \tau_n \\ d_1 & d_2 & \dots & d_n \end{Bmatrix}$$

we mean the unique parking function with diagonal word τ and

$$d_i = \#\,\mathrm{dinvset}(\tau_i, PF)$$

for all i. In particular, $d_{n-i+1} < w_i$ for all i exactly when PF is a parking function. Of course it is easy to convert this form of a parking function to the original two line array

$$PF = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ g_1 & g_2 & \cdots & g_n \end{bmatrix}$$

by following Procedure 36. Note that conversely, if $\tau_k = c_i$,

$$d_k = \#\{j > i : (t_i = t_j \ \&\ c_j > c_i) \text{ or } (t_i = t_j + 1 \ \&\ c_j < c_i)\}.$$

Example 41.

$$\begin{Bmatrix} 2 & 3 & 5 & 1 & 4 \\ 2 & 1 & 0 & 1 & 0 \end{Bmatrix}$$

corresponds to parking function

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

3.1 Schedules

The w_i's mentioned above, which give the size of the degree set of τ_{n-i+1}'s become very important in the remainder of this work. Thus we give the following definition.

Definition 42. *Let the schedule of* τ *(*$\mathrm{sched}(\tau)$*) be the sequence* $W = (w_1, \ldots, w_n)$*, where each*

$$\begin{aligned} w_i &= \mathrm{degset}(\overline{\tau}_i, PF) \\ &= \#\{\overline{\tau}_j : \overline{\tau}_j < \overline{\tau}_i \text{ and } t_j + 1 = t_i\} + \#\{\overline{\tau}_j : \overline{\tau}_j > \overline{\tau}_i \text{ and } t_j = t_i\}. \end{aligned}$$

Next we analyze some properties of the resulting schedules.

Theorem 43. *For any schedule* W *corresponding to a diagonal word* τ*,* $w_1 = 1$ *and* $1 \leq w_i \leq w_{i-1} + 1$*. In particular, the degree set of* $\overline{\tau}_i$ *is always a subset of the union of* $\{i-1\}$ *and the degree set of* $\overline{\tau}_{i-1}$*.*

Proof. $w_1 = \#\{\tau_0\} = 1$.

Case 1 ($\overline{\tau}_i < \overline{\tau}_{i-1}$). If $\overline{\tau}_i < \overline{\tau}_{i-1}$, they are in the same diagonal, call it d, of our family of parking functions. Then for $\overline{\tau}_{i-1-j}$ and $j > 0$, if $\overline{\tau}_{i-1-j}$ is in diagonal d, i is in the degree set of both $\overline{\tau}_{i-1}$ and $\overline{\tau}_i$. Otherwise if $i-1-j$ is in the degree set of $\overline{\tau}_i$ then either $j = 0$ or $\overline{\tau}_i > \overline{\tau}_{i-1-j}$ and $\overline{\tau}_{i-1-j}$ is in diagonal $d-1$. In the latter case $\overline{\tau}_{i-1} > \overline{\tau}_i$ as well and $i-1-j$ is in the degree set of $\overline{\tau}_{i-1}$. Thus the degree set of $\overline{\tau}_i$ is a subset of the union of $\{i-1\}$ and the degree set of $\overline{\tau}_{i-1}$.

Case 2 ($\overline{\tau}_i > \overline{\tau}_{i-1}$). Then if $\overline{\tau}_{i-1}$ is in diagonal d, $\overline{\tau}_i$ is the first element entered into diagonal $d+1$. It is immediate from the definition of the degree set that every element entered in diagonal d before $\overline{\tau}_{i-1}$ is in the degree set of $\overline{\tau}_{i-1}$, since $\overline{\tau}_{i-1}$ must be the smallest element in diagonal d. Since $\overline{\tau}_i$ must be the largest element in diagonal $d+1$, the only elements in the diagonal set of $\overline{\tau}_i$ are a subset of those in diagonal d and thus again a subset of the union of $\{i-1\}$ and the degree set of $\overline{\tau}_{i-1}$.

□

Moreover, the set of all schedules is exactly characterized by the above restrictions. We give a recursive procedure for finding a reverse diagonal word $\overline{\tau}$ for a given schedule $W = (w_1, \ldots, w_n)$.

Definition 44. *Define the permutation* $\mathrm{append}(k, \sigma)$ *by replacing any element* i *in* σ *weakly greater than* k *by* $i+1$*, then appending* k *to the result.*

Example 45. $\text{append}(3, (1, 3, 5, 2, 4)) = (1, 4, 6, 2, 5, 3)$

Procedure 46. *Begin with* $\overline{\tau}^1 = (1)$. *For* i *from* 2 *to* n*:*

- *If* $i = w_i$*, let* $\overline{\tau}^i = \text{append}(1, \overline{\tau}^{i-1})$.
- *Otherwise, let* $\overline{\tau}^i = \text{append}(\overline{\tau}^{i-1}_{i-w_i} + 1, \overline{\tau}^{i-1})$.

Let $\overline{\tau} = \overline{\tau}^n$

Example 47. *If* $W = (1, 2, 2, 3, 2)$*, we get the following permutations in order, ending with one whose reverse has* W *as its schedule:*

$$\begin{gathered}(1)\\(2,1)\\(2,1,3)\\(2,1,4,3)\\(2,1,4,3,5)\end{gathered}$$

Theorem 48. *Every vector* W *such that* $w_1 = 1$ *and* $1 \leq w_i \leq w_{i-1} + 1$ *has a corresponding diagonal word* τ*, in particular the reverse of* $\overline{\tau}$ *from Procedure 46, such that* $\text{sched}(\tau) = W$.

Proof. Inductively, we assume that $\overline{\tau}^i_j$ has $\{j - w_j, \cdots j - 1\}$ in $\overline{\tau}^i$ as its degree set and that we have an $s_j \geq 0$ such that

$$\overline{\tau}^i_{j-s_j+1} > \cdots > \overline{\tau}^i_{j-1} > \overline{\tau}^i_j > \overline{\tau}^i_{j-w_j} > \cdots > \overline{\tau}^i_{j-s_j}$$

for $1 \leq j \leq i$ (where as usual we understand that where convenient we consider $\overline{\tau}^i_0 = 0$). Certainly this is trivially true for $i = 1$. Moreover, since the append operation does not change the relative size of elements, the inductive hypothesis holds for $\overline{\tau}^{i+1}_j$ where $j \leq i$. If $i+1 = w_{i+1}$, we have $W = (1, 2, \ldots, i+1, w_{i+2}, \ldots w_n)$ and $\overline{\tau}^{i+1} = (i+1, i, \ldots, 1)$ and again the hypothesis holds. If $i+1 \neq w_{i+1}$ we have two cases.

Case 1 ($s_i \geq w_{i+1}$). Then in particular $i+1-w_{i+1} \geq i-s+1$ and by construction

$$\overline{\tau}^{i+1} = \text{append}(\overline{\tau}^i_{i+1-w_{i+1}} + 1, \overline{\tau}^i).$$

Then we have:

$$\overline{\tau}^{i}_{i-s_i+1} > \cdots > \overline{\tau}^{i}_{i+2-w_{i+1}} > \overline{\tau}^{i+1}_{i+1} > \overline{\tau}^{i}_{i+1-w_{i+1}} > \cdots > \overline{\tau}^{i}_{i}$$

and thus

$$\overline{\tau}^{i+1}_{i+1} > \overline{\tau}^{i+1}_{i+1-w_{i+1}} > \cdots > \overline{\tau}^{i+1}_{i}$$

with all but the first element in this list (of course) giving the indices of the degree set of $\overline{\tau}^{i+1}_{i+1}$ in $\overline{\tau}^{i+1}$ as required.

Case 2 ($s_i < w_{i+1}$). Then

$$\overline{\tau}^{i}_{i-s_i+1} > \cdots > \overline{\tau}^{i}_{i-1} > \overline{\tau}^{i}_{i} > \overline{\tau}^{i}_{i-w_i} > \cdots > \overline{\tau}^{i}_{i-w_i+1} > \overline{\tau}^{i+1}_{i+1} > \overline{\tau}^{i}_{i+1-w_{i+1}} > \cdots > \overline{\tau}^{i}_{i-s_i}$$

and again we can replace the superscript i by $i+1$ in the equalities and determine that the degree set of $\overline{\tau}^{i+1}_{i+1}$ is

$$\{i - s_i + 1, \ldots, i-1, i, i+1-w_{i+1}, \ldots, i-s_i\}$$

as required.

□

Although this procedure gives a diagonal word for every schedule (and in fact a particularly nice one, as we will see soon) it does not give the only such diagonal word. In fact, there can be many diagonal words for every schedule; a schedule gives several possible linear orderings on $[n]$, with the total ordering giving all possible corresponding diagonal words. We will omit the details here because they are technical and not particularly important for the remainder of this work.

Example 49. *Both* $(1,4,5,2,3)$ *and* $(5,3,4,1,2)$ *have diagonal word* $(1,2,2,3,2)$.

We end with one additional observation that we will need later:

Theorem 50. *If* w_i *is the first non-maximal element of the schedule, that is*

$$W = (1, 2, \ldots, i-1, w_i, \ldots, w_n)$$

where $w_i < i$*, then* W *corresponds to parking functions with* $i-1$ *cars on the main diagonal.*

Proof. If a car $\overline{\tau}_i$ is not in the main diagonal, it can have only positive $j < i$ in its degree set and thus $w_i < i$. If $\overline{\tau}_i$ is in the main diagonal, conversely, it can be placed next to all the previously placed elements in the same diagonal or next to the (later removed) 0 car. Thus it has a degree set of exactly size i. □

In particular, later we will restrict ourselves to the study of parking functions with two parts, in which case we will also restrict our schedules to those that begin $(1, 2, 1, \dots)$ and $(1, 2, 2, \dots)$

Definition 51 (k part schedules)**.** *Call W a k* part schedule *if $w_i = i$ for $i \leq k$ and $w_{i+1} < i + 1$.*

In particular k part schedules correspond to parking functions with compositions of length k.

3.2 Schedule Trees

At this point, combining Haglund and Loehr's result (Theorem 37) with our new understanding of the possible schedules, we have a family of polynomials that we can describe directly (i.e. without resorting to first generating the parking functions) that gives us generating functions for the dinv- weighted parking functions working over all possible diagonal words. If we don't care about the multiplicity with which we get see these polynomials arise (since in fact, as we mentioned earlier, two diagonal words may correspond to the same schedule and thus the same polynomial), in particular we get

$$\{\prod_{i=1}^{n}[w_i]_q\}_{W=(w_1,\dots,w_n) \text{ a schedule of length } n}.$$

Moreover, in a very straightforward manner we may choose to associate these polynomials with a family of W-ary trees, where w_i gives the number of siblings in each family the ith generation and the end points (the nth generation), with input of a diagonal word with schedule W, gives the parking functions we wish to study. We already hint at this construction in Figure 3.1, but now we formally represent the polynomial corresponding to schedule $(1, 2, 2, 2, 1)$ with the tree in

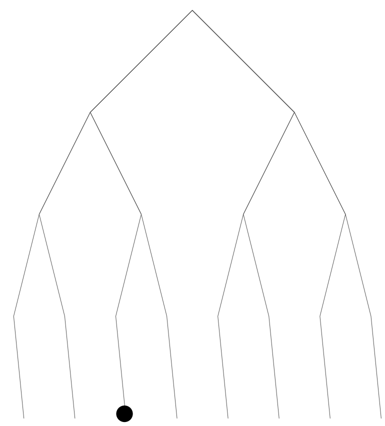


Figure 3.2: A tree corresponding to $W = (1, 2, 2, 2, 1)$.

Figure 3.2. We can determine the dinv of the parking function corresponding to any given endpoint by counting the number of siblings strictly to the right of every direct ancestor of the point (along with the siblings to the right of the point itself.) Thus in Figure 3.2 the point highlighted must correspond to a parking function with dinv $2 = 0 + 1 + 0 + 1 + 0$. Experimentally, choosing the associated diagonal word $\tau = (4, 2, 5, 1, 3)$, (where we leave it as an exercise that it has the appropriate schedule) we can follow the Procedure 36 to determine the corresponding parking function is

$$\begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

and independently verify it has the proper dinv. Note that the corresponding dinv representation of this parking function

$$\begin{Bmatrix} 4 & 2 & 5 & 1 & 3 \\ 0 & 1 & 0 & 1 & 0 \end{Bmatrix}$$

can be read directly from our choice of path down the tree to our point, where we interpret the last 0 in the bottom row as the start of our path, the 1 in the 4th column as the as the first leftward branch, the middle 0 as our next right branching and so on.

3.3 Statistics on the Trees

In addition to understanding the dinv and the area in the context of these generating functions and trees, we want to further analyze this construction to understand additional classical statistics not mentioned by Haglund and Loehr.

3.3.1 Composition

For a moment we put aside the ides calculations to focus on the composition. Initially, we focus on the case of two part compositions, motiving it only by the fact that we can index two part compositions in context by the size of their second part. In fact, this operation can be generalized, although we will show later why we conjecture it is enough to study just two parts.

Definition 52 (topset). *Let*

$$\text{topset}(PF) = \{i : \overline{\tau_i} \textit{ occurs in the second part of the parking function}\}$$

and $\text{top}(PF) = \#\,\text{topset}(PF)$.

Example 53.

$$\begin{bmatrix} 1 & 3 & 2 & 5 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

has $\overline{\tau} = (2, 1, 5, 4, 3)$, $\text{topset}(PF) = \{1, 3, 4\}$, *and* $\text{top}(PF) = 3$.

Next we must look more closely at our recursive construction for forming the parking functions, Procedure 36. Some observations:

1. If a domino $\begin{bmatrix} r \\ g \end{bmatrix}$ is placed directly to the right of another domino $\begin{bmatrix} r' \\ g' \end{bmatrix}$, it is in the same part as $\begin{bmatrix} r' \\ g' \end{bmatrix}$.

2. If $\begin{bmatrix} r \\ g \end{bmatrix}$ can be placed to the right of w elements, s of them in the topset, then the *first* s choices (from the right) of positions for $\begin{bmatrix} r \\ g \end{bmatrix}$ correspond to

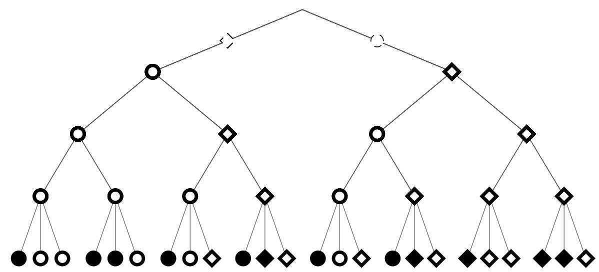


Figure 3.3: A tree corresponding to schedule $(1, 2, 2, 2, 3)$. Darkened elements give us elements of the i-descent set of the respective parking functions, while diamonds give elements in the top set.

increasing the size of the topset by one and the dinv by $0, \ldots, s-1$. The remaining $w - s$ choices leave the size of the topset fixed (since $\begin{bmatrix} r \\ g \end{bmatrix}$ is placed in the first part of the composition) and increase the dinv by $s, \ldots, w-1$.

3. As always, when we place a domino D, the w elements in its degree set are the last w elements placed in the parking functions by Procedure 36. Thus one can determine such an s as above by computing how many elements in the topset were added in the previous w steps.

See Figure 3.3 for an example of how we can use these observations to determine the size of the top set of a given parking function using a tree diagram. Recursively, we add two dotted vertices, a diamond just to the left of the root and a circle to the right. Then we recursively mark a set of k sibling vertices by looking at their k closest ancestors. Say m of the k ancestors have diamond vertices. Then decorate the m rightmost sibling vertices with diamonds and the remainder with circles. Diamonds represent adding elements to the top set and circles to the bottom set. Notice that since we cannot determine whether the first element is in the top or the bottom set until the second element is inserted, this single vertex in the tree is represented by the two top dashed vertices, depending on the ultimate choice. To determine the size of the top set of a given parking function, we count the number of diamonds along the path leading to its endpoint. Note that this operation does not depend at all on the corresponding diagonal word, just the schedule.

Example 54. *In Figure 3.3, any parking function corresponding to its right most leaf has top set size* 4 *and thus composition* $\{1, 4\}$.

Next we use the previous facts to define a recursive operator that in fact generates the set of polynomials in which we will ultimately be interested.

Definition 55. *Say*

$$B_{n,w} := \frac{1}{1-q}((z_n - q^w)P(z_1, \ldots, z_{n-1}, q) \tag{3.1}$$

$$+ (1 - z_n)P(z_1, \ldots, z_{n-w-1}, qz_{n-w}, \ldots, qz_{n-1}, q)) \tag{3.2}$$

$$P_{(1,2)}(Z, q) := qz_1 + z_2 \tag{3.3}$$

$$P_{(w_1,\ldots,w_n)}(Z, q) := B_{n,w_n} P_{w_1,\ldots,w_{n-1}}(Z_{n-1}, q) \tag{3.4}$$

$$R_W(z, q) := P_W(Z_n, q)\Big|_{z_1=z_2=\cdots=z_n=z} \tag{3.5}$$

Theorem 56. *If* $\operatorname{degseq}(\tau) = W$,

$$P_W(Z_n, q)\Big|_{\prod_{i\in S} z_i} = \sum_{\substack{\operatorname{diagword}(PF=\tau) \\ \operatorname{topset}(PF)=S}} q^{\operatorname{dinv}(PF)}. \tag{3.6}$$

In particular,

$$R_W(z, q) = \sum_{\operatorname{diagword}(PF)=\tau} q^{\operatorname{dinv}(PF)} z^{\operatorname{top}(PF)}. \tag{3.7}$$

Proof. The base case is easily checked by hand. Working inductively on the length of the schedule, assume the statement is true for the schedule $W' = (w_1, \ldots, w_{n-1})$ and the degree sequence formed by removing τ_1 from τ. (Although technically the result is not a permutation, it is easy to see we could apply the inductive hypothesis to the standard reduction of the sequence to a permutation, then return to our original numbering scheme without changing the statistics.) Let $W = (w_1, \ldots, w_{n-1}, w_n)$ be the degree sequence corresponding to τ. (Notice that it must agree with W' for all but its last element, which does not occur in W'.) When we add $\tau_1 = \overline{\tau_n}$ using Procedure 36, a typical monomial in $P_{W'}(Z_{n-1}; q)$, say $q^d \prod_{i\in S} z_i$, corresponds to a parking function with topset S. By our previous observations, using

$$a = |S \cap \{n - w_n, \ldots, n - 1\}|$$

its children will correspond to monomials

$$q^d \prod_{i\in S} z_i((1+q+\cdots+q^{a-1})z_n + (q^a+\cdots+q^{w_n-1})). \tag{3.8}$$

Adopting the notation that

$$a(S) = |S \cup \{n-w_n,\ldots,n-1\}|$$

and

$$c(S) = P_{W'}(Z_{n-1},q)\big|_{\prod_{i\in S} z_i},$$

$$P_W(Z_n,q) = B_{n,w_n} P_{W'}(Z_{n-1},q) \tag{3.9}$$

$$= \frac{1}{1-q}\left((z_n - q^w)\sum_{S\subset[n-1]} c(S)\prod_{i\in S} z_i\right. \tag{3.10}$$

$$\left. +(1-z_n)\sum_{S\subset[n-1]} c(S)\prod_{i\in S} z_i\Big|_{\substack{z_i\to qz_i\\ n-w_n\le i\le n-1}}\right) \tag{3.11}$$

$$= \frac{1}{1-q}\left((z_n - q^w)\sum_{S\subset[n-1]} c(S)\prod_{i\in S} z_i + (1-z_n)\sum_{S\subset[n-1]} c(S)q^{a(S)}\prod_{i\in S} z_i\right) \tag{3.12}$$

$$= \frac{1}{1-q}\sum_{S\subset[n-1]} c(S)\prod_{i\in S} z_i\left((1-q^{a(S)})z_n + q^{a(S)}(1-q^{w-a(S)})\right) \tag{3.13}$$

$$= \sum_{S\subset[n-1]} c(S)\prod_{i\in S} z_i((1+q+\cdots+q^{a(S)-1})z_n + (q^{a(S)}+\cdots+q^{w_n-1})) \tag{3.14}$$

The remainder of the theorem is an immediate consequence. □

3.3.2 Ides

To understand the ides term, we first need to distinguish between two different occurrences of ides. Let diagword$(PF) = \tau$. Notice that τ and the reading word of PF are similar, since τ gives the diagonals read from top to bottom *in increasing order* and word(PF) gives the diagonals read from top to bottom, *right to left*. Note that elements in the i-descent set of τ are exactly the set of j that occur in a later run (and thus a lower diagonal in PF) than $j+1$ in τ, since the runs are strictly increasing sequences. Thus ides$(\tau) \subset$ ides(PF).

Definition 57 (Forced ides)**.** *Let the forced i-descent set of a parking function PF with diagonal word τ be the i-descent set of τ. (*$\text{Fides}(PF) = \text{ides}(\tau)$*.)*

Definition 58 (Unforced ides)**.** *Let the unforced i-descent set of a parking function PF with diagonal word τ be the elements in i-descent set of PF that are not in* $\text{Fides}(PF)$*. (*$\text{Uides}(PF) = \text{ides}(PF) \setminus \text{Fides}(PF)$*.)*

Example 59. *When*

$$PF = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$

then $\text{diagword}(PF) = (2, 3, 5, 1, 4)$*,* $\text{Fides}(PF) = \{1, 4\}$*, and* $\text{Uides}(PF) = \{2\}$*.*

We have the following theorem that allows us to compute the unforced i-descent set directly from a parking function in diagonal notation.

Theorem 60. *If* $\text{diagword}(PF) = \tau$*,* $\tau_i \in \text{Uides}(PF)$ *if and only if* $\tau_i = \tau_{i+1} - 1$ *and* $d_i > d_{i+1}$*.*

Proof. From our argument above, we have that if $\tau_i \in \text{Uides}(PF)$ then τ_i and $\tau_i + 1$ must be in the same run and thus $\tau_i = \tau_{i+1} - 1$. Next, notice that in this case, the degree set of τ_i is exactly the degree set of τ_{i+1} union the index corresponding to τ_{i+1} itself $(n - i + 1)$, since any car except τ_{i+1} is larger (resp. smaller) than τ_i if and only if it is larger (smaller) than τ_{i+1}. Then for τ_{i+1} to occur before τ_i in the reading word of PF, it must occur to the right of τ_i. Thus any elements in the dinv set of τ_{i+1} are also in the dinv set of τ_i. Since the index corresponding to τ_{i+1} $(n - i)$ is also in the dinv set of τ_i, we have the desired $d_i > d_{i+1}$. Conversely, if $\tau_i = \tau_{i+1} - 1$ and $d_i \geq d_{i+1} + 1$, then using Procedure 36, we must first place τ_{i+1} directly to the left of the $d_{i+1} + 1$th object in its degree set, working from the right. Then we must place τ_i to the left of the $d_i + 1$th object in its degree set. Since again τ_i and τ_{i+1} have almost identical degree sets, we see that the objects to the right of τ_i must include the objects to the right of τ_{i+1} as well as τ_{i+1} itself. Thus we read τ_{i+1} before τ_i in the reading word of PF and $\tau_i \in \text{Uides}(PF)$. □

See Figure 3.3 for a method of recording these unforced i-descents in an example tree. When we take τ as the corresponding diagonal word, the blackened squares at level i correspond to an unforced i-descent at τ_{n-i+1}.

Example 61. *In Figure 3.3, if we let the diagonal word be* $(2,3,5,1,4)$*, the blackened squares represent unforced i-descents of* 2 *for certain parking functions. For example, we can conclude that the parking function corresponding to the third leaf from the right has i-descent set* $\{2\} \cup \text{ides}(\tau) = \{1,2,4\}$*.*

Corollary 62.

$$\text{ides}(\tau) = \bigcap_{\text{diag}(PF)=\tau} \text{ides}(PF)$$

Next, we explore what we can determine about the i-descent set of parking functions with a given schedule.

Definition 63 (Maximal set)**.** *Define the maximal set of a schedule* W *(*$\max(W)$*) to be the indices where the schedule increases as much as is allowed by the slow growth restriction:*

$$\max(W) = \{i : w_i = w_{i-1} + 1\}.$$

Definition 64 (Separating set)**.** *Let the separating set of a schedule be:*

$$\text{Sep}(W) = \{i - w_i\}.$$

Finally,

Definition 65 (Diagonal I-descent)**.** *Let the diagonal i-descent set of a permutation* τ *be the set*

$$\text{Diagides}(\tau) = \{i : \exists PF \in \text{diag}(\tau) \ \textit{s.t.}\ \overline{\tau}_i \in \text{ides}(PF) \setminus \text{ides}(\tau)\}.$$

That is, the set gives the indices of $\overline{\tau}$ *which are unforced i-descents for some parking function with diagonal word* τ*.*

Theorem 66. *If* $\text{sched}(\tau) = W$*, then*

$$\text{Diagides}(\tau) \subset \max(W) \setminus \text{Sep}(W).$$

Moreover, for a τ *constructed by Procedure 46, the containment is an equality.*

Proof. First, assume $i \in \text{Diagides}(\tau)$ for a parking function PF with diagonal word τ. Notice by the above arguments that if $\overline{\tau}_i \in \text{ides}(PF)$ but $\overline{\tau}_i \notin \text{ides}(\tau)$, it must be that $\overline{\tau}_i$ and $\overline{\tau}_i + 1$ are in the same diagonal and $\overline{\tau}_i + 1 = \overline{\tau}_{i-1}$. If $\overline{\tau}_i$ and $\overline{\tau}_i + 1$ are in the same diagonal then the degree set of $\overline{\tau}_i$ is exactly the degree set of $\overline{\tau}_i + 1$ along with $i-1$ itself, since any element distinct from $\overline{\tau}_i$ and $\overline{\tau}_i + 1$ has the same relative value to them both. Thus

$$w_i = w_{i-1} + 1.$$

Moreover, since $\overline{\tau}_i + 1 = \overline{\tau}_{i-1}$, for every $j > i$, $\overline{\tau}_j > \overline{\tau}_i$ ($\overline{\tau}_j < \overline{\tau}_i$) if and only if $\overline{\tau}_{i-1} > \overline{\tau}_j$ ($\overline{\tau}_{i-1} < \overline{\tau}_j$ respectively) so i should also be in the degree set of $\overline{\tau}_j$ if and only if $i-1$ is in the degree set of $\overline{\tau}_j$. But

$$\{j-1, j-2, \ldots, j-w_j\}$$

is the degree set of $\overline{\tau}_j$ by Lemma 40. Thus in particular $i \neq j - w_j$ and $i \notin \text{Sep}(W)$. For the second assertion, first notice that if i is in the maximal set of W, Procedure 46 appends $\overline{\tau}^{i-2}_{i-1-w_{i-1}} + 1$ and then $\overline{\tau}^{i-1}_{i-w_i} + 1 = \overline{\tau}^{i-1}_{i-(w_{i-1}+1)} + 1$—the same element twice—with the result that $\overline{\tau}^{i}_{i-1} = \overline{\tau}^{i}_{i} + 1$. Moreover, in the remaining steps $j > i$, $\overline{\tau}^{j}_{i}$ and $\overline{\tau}^{j}_{i-1}$ will only increase simultaneously, unless in some step we append $\overline{\tau}^{j-1}_{i} + 1$ to $\overline{\tau}^{j-1}$. But this happens exactly when $j - w_j = i$ and $i \in \text{Sep}(W)$. □

Definition 67 (top tau)**.** *Call the* top tau *of a schedule (*toptau(W)*) the τ resulting from applying Procedure 46 to W.*

Corollary 68. *Say that τ is the top tau of W and τ^1 is another diagonal word with the same schedule. Then if*

$$PF = \begin{Bmatrix} \tau_1 & \tau_2 & \ldots & \tau_n \\ d_1 & d_2 & \ldots & d_n \end{Bmatrix},$$

$$PF^1 = \begin{Bmatrix} \tau^1_1 & \tau^1_2 & \ldots & \tau^1_n \\ d_1 & d_2 & \ldots & d_n \end{Bmatrix},$$

and τ^1_i is an unforced i-descent of PF^1, τ_i is an unforced i-descent of PF.

Next we construct a list we call the consecutivities of τ.

Procedure 69.
- *First, define a set of sets by placing τ_1 in the first set, then repeatedly adding τ_i to the previous set if $\tau_i = \tau_{i-1} + 1$ and otherwise creating a new set containing τ_i.*
- *Second, rearrange the resulting sets such that the elements in the ith set are smaller than the elements in the jth set for $i < j$. Call the resulting list the consecutivities of τ ($\text{consec}(\tau)$).*

Example 70. $\text{consec}((3, 4, 5, 7, 6, 1, 2)) = (\{1, 2\}, \{3, 4, 5\}, \{6\}, \{7\})$

The consecutivities of τ (call them $(C_1, \ldots, C_k)$) correspond naturally to a Young subgroup $\text{Yconsec}(\tau) = S_{C_1} \times \cdots \times S_{C_k}$ which acts on a permutation in S_n by permuting the elements of C_j among themselves for each j. Call the elements of the subgroup the Young consecutivities of a permutation τ. In general, a well-studied action of S_n on the parking functions acts by permuting elements and then rearranging cars within columns so that they are strictly increasing. (In fact, such an action on the parking functions induces a representation that is isomorphic to the standard action on the diagonal harmonics tensored with the sign representation.) If we restrict ourselves to looking at the action by the subgroup of Young consecutivities, we have the following theorem:

Theorem 71. *Let $\text{diag}(PF) = \tau$ and $\sigma \in \text{Yconsec}(\tau)$. Then $\sigma(PF)$ acts strictly by permuting elements (with no rearranging columns necessary). Moreover, if $\text{ides}(PF) = \text{ides}(\tau)$, then:*

- $\text{comp}(\sigma(PF)) = \text{comp}(PF)$
- $\text{dinv}(\sigma(PF)) = \text{dinv}(PF) + \text{inv}(\sigma)$
- $\text{ides}(\sigma(PF)) = \text{ides}(PF) \cup \text{ides}(\sigma)$

Proof. The first observation is trivial, since if i and $i+1$ are in the same diagonal then i can sit in a column between two cars c_1 and c_2 if and only if $c_1 < i < c_2$ which happens exactly when $c_1 < i+1 < c_2$ and $i+1$ can sit in the same column. The composition is clearly unchanged by the operation. Similarly, say that i creates a dinv with some car $c \neq i+1$ in a parking function PF. Then c and i have the same relative value as c and $i+1$ and the parking function which results from interchanging i and $i+1$ has a dinv between c and $i+1$. Following this logic, if $\text{ides}(PF) = \text{ides}(\tau)$, every diagonal inversion in PF that is not between two elements in the same set of $\text{consec}(\tau)$ will remain a diagonal inversion in $\sigma(PF)$. In particular, if $\text{ides}(PF) = \text{ides}(\tau)$ then if $i < j$ are in the same set of $\text{consec}(\tau)$ and thus in the same diagonal, i must be read before j in the reading word of τ. Thus i is to the right of j as there is no diagonal inversion between the two. Moreover, if $i < j$ form an inversion in σ, by construction they must be in the same consecutive set. Interchanging i and j in PF thus places i to the left of j and creates a new diagonal inversion. The final conclusion is similar. □

Remark 72. Note that the previous theorem would also be true if we worked with just a subset of the connectivities, rather than all of $Yconsec(\tau)$.

In the following theorem, we introducing a set of variables z indexed by compositions of n to keep track of the composition of our parking functions. (We could also use z to index the size of the top set, as previously, but use the composition here, since the theorem applies in full generality to parking functions with any number of parts.)

Corollary 73. *If τ has corresponding schedule $W = (w_1, \ldots, w_n)$,*

$$\sum_{\text{diagword}(PF)=\tau} t^{\text{area}(PF)} q^{\text{dinv}(PF)} z_{\text{comp}(PF)} Q_{\text{ides}(PF)}$$

$$= \left(t^{\text{maj}(\tau)} \sum_{\sigma \in \text{Yconsec}(\tau)} q^{\text{inv}(\sigma)} Q_{\text{ides}(\sigma)\cup\text{ides}(\tau)} \right) \times \left(\sum_{\substack{\text{diagword}(PF)=\tau \\ \text{ides}(PF)=\emptyset}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} z_{\text{comp}(PF)} \right)$$

Note that in this case, our result depends not just on W but on τ. Also, notice that previous theorems could be applied together to generate the last term in the product directly, using our tree diagrams and selecting only those paths with no i-descent set. In fact, we also have the following theorem, which by a similar idea allows us to generate the sum on the left without resorting to this or to generating the parking functions, instead using the B_{n,w_n} operators. This is much more computationally effective.

Corollary 74. *If* τ *has corresponding schedule* W *and* $\tau_{n-2} > \tau_{n-1} < \tau_n$*,*

$$\left(\sum_{\mathrm{diagword}(PF)=\tau} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} z^{\mathrm{top}(PF)} Q_{\mathrm{ides}(PF)} \right) \left(\sum_{\sigma \in \mathrm{Yconsec}(\tau)} q^{\mathrm{inv}(\sigma)} \right) = R_W(z,q) \left(t^{\mathrm{maj}(\tau)} \sum_{\sigma \in \mathrm{Yconsec}(\tau)} q^{\mathrm{inv}(\sigma)} Q_{\mathrm{ides}(\sigma) \cup \mathrm{ides}(\tau)} \right)$$

Note that the conditions on τ are exactly what we need to ensure that the associated parking functions are two part parking functions.

Remark 75. Note that some of the early definitions, figures, and procedures in this chapter are also important to another work by the author, "A Parking Function Bijection supporting the Haglund—Morse —Zabrocki Conjectures," that was advance access published in *Int. Math. Res. Notices* in December, 2012 and as such have been reproduced here with permission. See the work for a related bijection defined in terms of small changes in the dinv representation of a parking function.

Chapter 4

Reduction

Next, we combine the proceeding theorems and begin applying them to simplifying the task at hand, that is reducing the HMZ conjecture. Recall that our goal for reduction, as stated previously, is to show that:

Conjecture 76 (Implicative Conjecture)**.** *The Partition case implies the HMZ conjecture. That is, if the HMZ conjecture is true for all partitions p then it is true for all compositions.*

Moreover, recall that when we use

$$A_c = \sum_{PF \in \mathcal{A}_c} t^{\text{area}(PF)} q^{\text{dinv}(PF)} Q_{\text{ides}(PF)},$$

proving the Implicative Conjecture is equivalent to the following:

Conjecture 77. *(Implicative Conjecture, v. 2) For all composition c:*

$$q(A_{[c',c_i,c_{i+1},c'']} + A_{[c',c_{i+1}-1,c_i+1,c"]}) = A_{[c',c_{i+1},c_i,c'']} + A_{[c',c_i+1,c_{i+1}-1,c"]}. \tag{4.1}$$

Notice that the commutativity conditions are a local property—a single application tells us something about interchanging two parts in the middle of a composition. Rather than defining each such bijection separately, we can define a single bijection on two parts—provided it satisfies the diagonal property —and naturally extend it to any number of parts. Thus we show that such a bijection g on two parts can be extended to a map $\tilde{g}$ on parking functions with any number of parts that only interchange elements within two parts. In fact, if we want the bijection which exchanges the i and $i+1$st part of our composition we claim that such a map $\tilde{g}^i$ is a natural expansion of g. In particular it leaves the elements of any parts except i and $i+1$ fixed. Then it applies g to the parking function formed by the cars in parts i and $i+1$ (suitably renumbered to give the cars 1 to n and then returned to their original numberings.)

Example 78. *If*

$$g\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 & 4 & 6 & 5 \\ 0 & 1 & 2 & 3 & 1 & 0 \end{bmatrix},$$

then

$$\tilde{g}^2\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \\ 0 & 1 & 0 & 1 & 2 & 3 & 1 & 0 \end{bmatrix}$$

Lemma 79. *If g satisfies the diagonal condition and is our desired bijection on two parts, then $\tilde{g}^i$ is our desired bijection for any i.*

Proof. First note that the renumbering required to apply g to the ith and $i+1$st part neither creates nor destroys dinv, so there is exactly one more dinv between elements in the ith and $i+1$st part than before we apply $\tilde{g}^i$. Moreover, for a car j in part i or $i+1$ and car k in a part other than i or $i+1$, since j and k don't change diagonal or relative position with each other there is a diagonal inversion between cars j and k in PF if and only if there is a diagonal inversion between cars j and k in $\tilde{g}^i(PF)$. Clearly area remains fixed. Finally, if j and k above happen to be consecutive, by the same argument we can see that j is read before k in the reading word of PF if and only if j is read before k in the reading word of $\tilde{g}^i(PF)$. If j and k are consecutive and in one or both of the parts i and $i+1$ in PF, then again j is read before k in PF if and only if j is read before k in the reading word of $\tilde{g}^i(PF)$ since j and k reduce to consecutive numbers in the smaller two part parking function and g does not change the i-descent set of this smaller parking function. □

Finally, we come to a previously promised key theorem in this work:

Theorem 80. *If there exists a bijective map that satisfies the diagonal condition, then it can be naturally expanded to a bijective map that works for every size partition. Moreover, it is equivalent to check the dinv and the composition alone or to only define the map on the parking functions without any unforced i-descent (or even only unforced i-descent for certain τ_j). Finally, it is equivalent to check any of these conditions on only those parking functions whose diagonal word is a top tau.*

Proof. Lemma 79 gives that the map can be extended. Clearly, as remarked previously, the diagonal condition forces the map to keep the area unchanged. Moreover, assume we have a weaker version of the Implicative Conjecture, in particular that: for any $k < n-k$, there exists a bijection f

$$f : \mathcal{A}_{\{k,n-k\}} \cup \mathcal{A}_{\{n-k-1,k+1\}} \leftrightarrow \mathcal{A}_{\{n-k,k\}} \cup \mathcal{A}_{\{k+1,n-k-1\}}$$

with the following properties:

1. f increases the *dinv* by exactly one

2. f satisfies the diagonal condition.

This happens exactly when for every τ and k

$$\sum_{\substack{\text{diagword}(PF)=\tau \\ \text{comp}(PF)=\{k,n-k\} \\ \text{or } \{n-k-1,k+1\}}} q^{\text{dinv}(PF)} = \sum_{\substack{\text{diagword}(PF)=\tau \\ \text{comp}(PF)=\{n-k,k\} \\ \text{or } \{k+1,n-k-1\}}} q^{\text{dinv}(PF)}. \tag{4.2}$$

By Theorem 71, this happens if and only if

$$\sum_{\substack{\text{diagword}(PF)=\tau \\ \text{comp}(PF)=\{k,n-k\} \\ \text{or } \{n-k-1,k+1\} \\ \text{ides}(PF)=\emptyset}} q^{\text{dinv}(PF)} = \sum_{\substack{\text{diagword}(PF)=\tau \\ \text{comp}(PF)=\{n-k,k\} \\ \text{or } \{k+1,n-k-1\} \\ \text{ides}(PF)=\emptyset}} q^{\text{dinv}(PF)} \tag{4.3}$$

since we can divide both sides of equation (4.2) by

$$\sum_{\sigma \in \text{Yconsec}(\tau)} q^{\text{dinv}(PF)}$$

to get (4.3). (Thus we can restrict our bijection to parking functions without unforced i-descents. Similarly, using the remark after Theorem 71, we can restrict our bijection to parking functions with only certain unforced i-descents.) Then multiplying by

$$\sum_{\sigma \in \text{Yconsec}(\tau)} q^{\text{dinv}(PF)} Q_{\text{ides}(\tau)\cup\text{ides}(\sigma)}$$

on both sides, the result (again by Theorem 71) is

$$\sum_{\substack{\text{diagword}(PF)=\tau \\ \text{comp}(PF)=\{k,n-k\} \\ \text{or } \{n-k-1,k+1\}}} q^{\text{dinv}(PF)} Q_{\text{ides}(PF)} = \sum_{\substack{\text{diagword}(PF)=\tau \\ \text{comp}(PF)=\{n-k,k\} \\ \text{or } \{k+1,n-k-1\}}} q^{\text{dinv}(PF)} Q_{\text{ides}(PF)}. \tag{4.4}$$

Since this happens for every k and τ iff and only if we have a bijection that respects the i-descent set, we have a (nonconstructive) proof that we need not check the i-descent. Finally, notice this final condition is similar for the set of all τ which correspond to the W, but most restrictive by Corollary 68, in the case that τ is a top tau of W. □

Since we need only study the composition and dinv, we formally define the polynomials:

$$S^\tau(z,q) = \begin{cases} \sum_{\text{diagword}(PF)=\tau} q^{\text{dinv}(PF)} z^{\text{top}(PF)} & \text{if } \tau_{n-2} > \tau_{n-1} < \tau_n \\ 0 & \text{otherwise} \end{cases}$$

and give a new stronger version of the Implicative Conjecture. (Note that $S^\tau(z,q)$ is by design nonzero exactly when we are looking at a family of 2 part parking functions.)

Conjecture 81 (Implicative Conjecture v. 3)**.** *For $k < n-k$*

$$q\left(S^\tau(z,q)\big|_{z^{n-k}+z^{k+1}}\right) = S^\tau(z,q)\big|_{z^k+z^{n-k-1}}.$$

Corollary 82. *The Implicative Conjecture v. 3 implies the Implicative Conjecture v. 2.*

If we study the set

$$\{S^\tau(z,q)\}_\tau$$

we find a plethora of repetitions. In fact, for parking functions of length 5 there are 40 distinct diagonal words, but only 14 distinct nonzero polynomials.

Example 83.

$$S^{(4,3,1,2)}(z,q) = z(q+1)(q+z^2) = S^{(1,4,2,3)}(z,q)$$

When we restrict ourselves to the top tau, we get exactly these polynomials, but in fact, using an earlier theorem, Theorem 56, we can now conclude that we may just as well study a set of polynomials we defined earlier using our recursive operator B_{n,w_n}.

Theorem 84. *The Implicative Conjecture v. 3 is true if and only if for all two part schedules W and $k < n-k$*

$$q\left(R^W(z,q)\big|_{z^{n-k}+z^{k+1}} \right) = R_W(z,q)\Big|_{z^k+z^{n-k-1}}.$$

4.1 Polynomial Properties

We begin this section with another way of generating $P_W(Z_n;q)$, in particular one which allows us to directly find the coefficient of any given monomial in the z_i's. Recall that $P_W(Z_n;q)$ is the original recursively defined polynomial which defines $R_W(z,q)$ by the equation:

$$R_W(z,q) := P_W(Z_n,q)\Big|_{z_1=z_2=\cdots=z_n=z}.$$

Theorem 85. *Let $W = (w_1, \ldots, w_n)$ and $S \subset [n]$ contain exactly one of 1 or 2. Let*

$$m_i = \begin{cases} 0 & \text{if } i = 1 \text{ and } 2 \in S \\ 1 & \text{if } i = 1 \text{ and } 1 \in S \text{ or } i = 2 \\ \#(S \cap \{i-1, i-2, \ldots, i-w_i\}) & \text{if } i > 2 \end{cases}.$$

Then the coefficient of $\prod_{i \in S} z_i$ in $P_W(Z_n;q)$ is nonzero if and only if

- *For all i in $S \backslash \{2\}$, $m_i \geq 1$*
- *For all i not in S, $w_i - m_i \geq 1$*

In this case, the coefficient of $\prod_{i\in S} z_i$ *in* $P_W(Z_n;q)$ *is exactly*

$$D_S^W(q) = \left(\prod_{i\in S}[m_i]_q\right)\left(\prod_{i\notin S} q^{m_i}[w_i - m_i]_q\right)$$

Proof. By construction this is the case for $W = (1,2)$. Working by induction, let $S \subset [n-1]$ and $W' = (w_1, \ldots, w_{n-1}$. We begin by applying B_{n,w_n} to a monomial $D_S^{W'}(q)\prod_{i\in S} z_i$. Assume that $m_n = \#(S \cap \{n-1, n-2, \ldots, n-w_n\})$.

$$B_{n,w_n}\left(D_S^{W'}(q)\prod_{i\in S} z_i\right) \tag{4.5}$$

$$= \frac{1}{1-q}\left((z_n - q^w)D_S^{W'}(q)\prod_{i\in S} z_i + (1-z_n)q^{m_n}D_S^{W'}(q)\prod_{i\in S} z_i\right) \tag{4.6}$$

$$= \left(z_n\left(\frac{1-q^{m_n}}{1-q}\right) + \left(\frac{q^{m_n} - q^{w_n}}{1-q}\right)\right) D_S^{W'}(q)\prod_{i\in S} z_i \tag{4.7}$$

$$= \left(z_n[m_n]_q + q^{m_n}[w_n - m_n]_q\right) D_S^{W'}(q)\prod_{i\in S} z_i \tag{4.8}$$

Assuming the statement holds for W', we may inductively replace $D_S^{W'}(q)$.

$$B_{n,w_n}\left(D_S^{W'}(q)\prod_{i\in S} z_i\right) = \left(z_n[m_n]_q + q^{m_n}[w_n - m_n]_q\right) \tag{4.9}$$

$$\times\left(\prod_{i\in S}[m_i]_q\right)\left(\prod_{i\notin S} q^{m_i}[w_i - m_i]_q\right)\prod_{i\in S} z_i \tag{4.10}$$

$$= \left(\prod_{i\in S\cup\{n\}}[m_i]_q\right)\left(\prod_{i\notin S\cup\{n\}} q^{m_i}[w_i - m_i]_q\right)\prod_{i\in S\cup\{n\}} z_i \tag{4.11}$$

$$+\left(\prod_{i\in S}[m_i]_q\right)\left(\prod_{i\notin S} q^{m_i}[w_i - m_i]_q\right)\prod_{i\in S} z_i \tag{4.12}$$

and thus we have proved the required equality for S and $S \cup \{n\}$ when $W = (w_1, \ldots, w_n)$. □

Notice this proof is extremely reminiscent to the proof of Theorem 56. In fact, we could have just as easily showed that D_S^W gives the correct coefficient

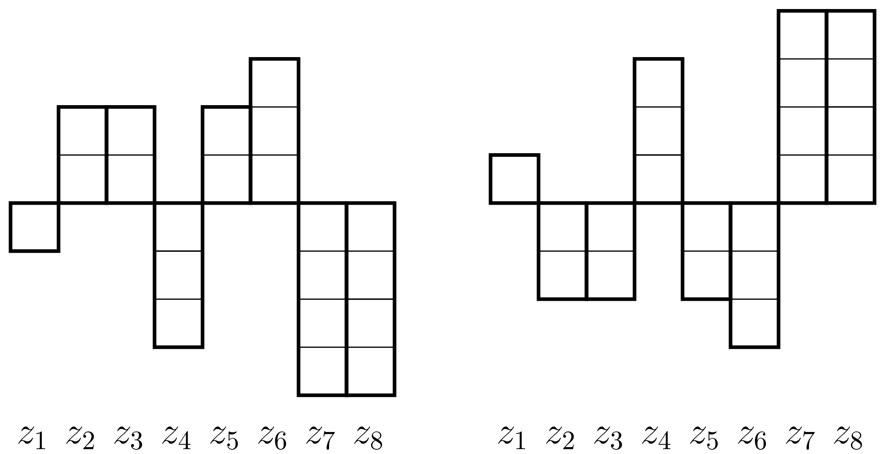


Figure 4.1: For $W = (1, 2, 2, 3, 2, 3, 4, 4)$ and $S = \{2, 3, 5, 6\}$ or $S = \{1, 4, 7, 8\}$, correct placement of bars in a diagram.

directly. Frequently, we find it easier to calculate D_S^W using an alternate diagram, which we will refer to as a *parking bar diagram*. To construct the diagram:

1. Place a "bar" of length w_i in each column, pointing upward if $i \in S$ or downward otherwise. See Figure 4.1.

2. Shade a single square closest to the main line in each of the first two columns. (This corresponds to the special conditions for m_i when $i \in \{1, 2\}$.) See Figure 4.2.

3. For $i > 2$, shade as many squares in column i as there are columns in the range $i-1, \ldots, i-w_i$ that are pointed the same direction as column i. See Figure 4.3. As an example, notice that the last column of the last diagram has two shaded squares, since two of the previous four bars are pointed upwards like the last bar.

We will refer to the diagram constructed this way as $\tilde{D}_S^W$. Finally, say that the weight of any bar in a diagram with squares $j, \ldots, j+k$ shaded, counting from the bottom of the bar, is

$$q^{j-1} + \cdots + q^{j+k-1}.$$

In particular, say the weight of a bar without shaded squares is 0. Say the weight

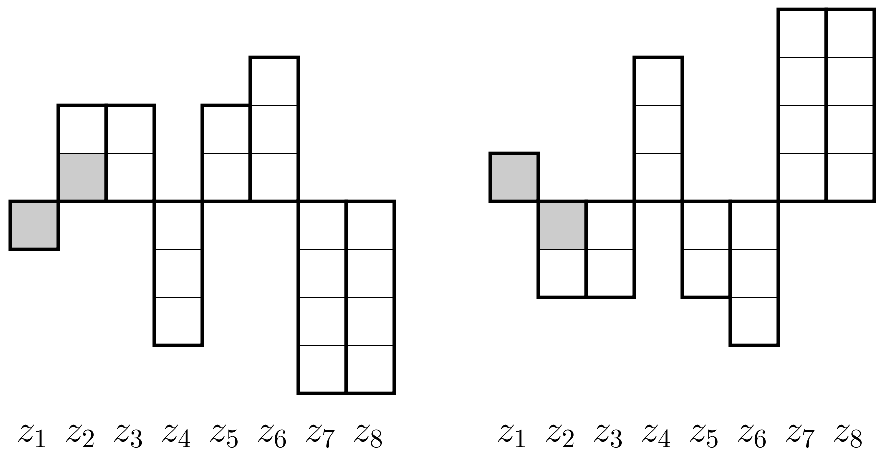


Figure 4.2: For $W = (1, 2, 2, 3, 2, 3, 4, 4)$ and $S = \{2, 3, 5, 6\}$ or $S = \{1, 4, 7, 8\}$, adding the initial shading.

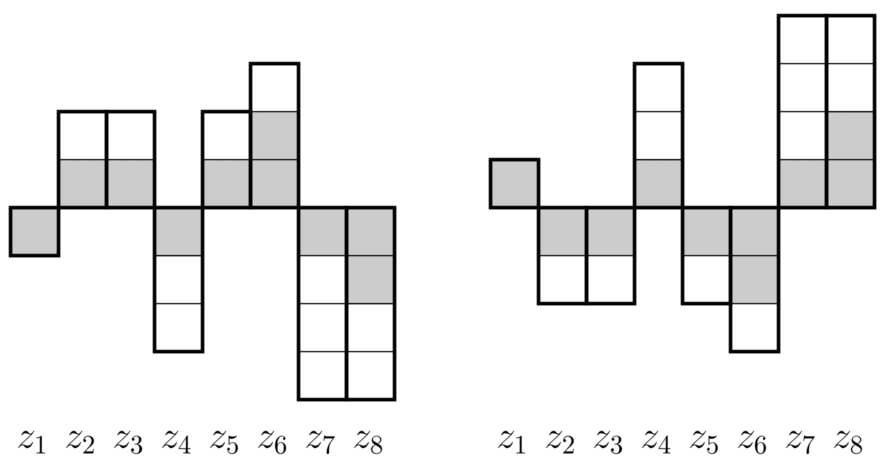


Figure 4.3: For $W = (1, 2, 2, 3, 2, 3, 4, 4)$ and $S = \{2, 3, 5, 6\}$ or $S = \{1, 4, 7, 8\}$, adding the remaining shading to get the appropriate diagram.

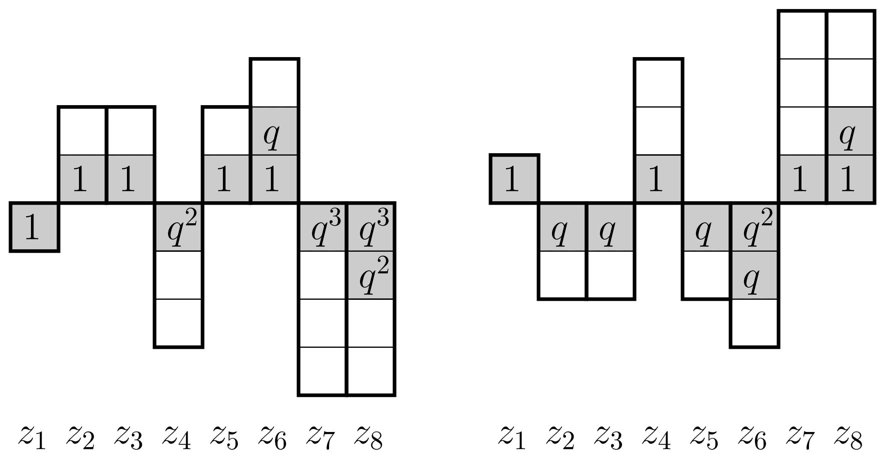


Figure 4.4: For $W = (1, 2, 2, 3, 2, 3, 4, 4)$ and $S = \{2, 3, 5, 6\}$ or $S = \{1, 4, 7, 8\}$, the respective diagrams with the weights labeled.

of a diagram (weight($\tilde{D}_S^W$)) is the product of the weights of its bars. See Figure 4.4.

Example 86. *Using Figure 4.4 we can conclude that for*

$$D_{\{2,3,5,6\}}^{(1,2,2,3,2,3,4,4)} = q^2(1+q)q^3(q^2+q^3).$$

Definition 87. *We say that i* acts *on $i-1, \ldots, i-w_i$. That is for $i > 2$, i acts on j exactly when we check column j to see if it is pointed in the same direction as i when we are shading the squares in the previous construction. Frequently we use the notation*

$$\mathrm{act}(i, W) = \{i-1, \ldots, i-w_i\}.$$

Note that this corresponds exactly to

$$\mathrm{degset}(\overline{\tau}_i, PF) = \{i-1, \ldots, i-w_i\},$$

although we choose to rename it here to emphasize that this is something that depends on our schedule, and not a particular choice of diagonal word. Formally, we give the following corollary.

Corollary 88. *The weight of a diagram corresponding to a schedule W and a set S containing exactly one of 1 or 2 is*

$$\mathrm{weight}(\tilde{D}_S^W) = D_S^W = P_W(Z_n, q)\big|_{\prod_{i \in S} z_i}.$$

We begin working with these diagrams with a simple observation:

Theorem 89. *If a diagram with nonzero weight has a single bar that is entirely shaded, the remaining bars to the right must be entirely shaded.*

Proof. If the jth column of a diagram $\tilde{D}_S^W$ is entirely shaded, that means in particular that $\mathrm{act}(j,W) \cup \{j\} \subset S$ or $\mathrm{act}(j,W) \cup \{j\} \subset S^c$. Since

$$\mathrm{act}(j+1,W) \subset \mathrm{act}(j,W) \cup \{j\},$$

bar $j+1$ must be in the same direction as j (since we assume the diagram has nonzero weight and thus that the $j+1$st column is not entirely unshaded) and similarly entirely shaded. □

Using the previous theorem, we can conclude the following about the relationship between $D_S^W(q)$ and $D_{S^c}^W(q)$:

Theorem 90. *Let $W = (w_1, \ldots, w_n)$ and $S \subset [n]$ contain exactly one of 1 or 2. Then*

$$D_{S^c}^W(1/q) = q^{n-(\sum w_i)} D_S^W(q).$$

Proof. Here, when we consider the set S^c, we use m_i^c in place of m_i for ease of notation. Notice that by definition, $m_i^c = w_i - m_i$. Furthermore, recall that

$$[n]_q\big|_{q\to 1/q} = \frac{[n]_q}{q^{n-1}}.$$

Then

$$\begin{aligned}
D_{S^c}^W(1/q) &= \left[\left(\prod_{i\in S^c}[m_i^c]_q\right)\left(\prod_{i\notin S^c} q^{m_i^c}[w_i - m_i^c]_q\right)\right]_{q\to 1/q} && (4.13)\\
&= \left[\left(\prod_{i\notin S}[w_i - m_i]_q\right)\left(\prod_{i\in S} q^{w_i-m_i}[m_i]_q\right)\right]_{q\to 1/q} && (4.14)\\
&= \left(\prod_{i\notin S} q^{m_i-w_i+1}[w_i - m_i]_q\right)\left(\prod_{i\in S} q^{1-w_i}[m_i]_q\right) && (4.15)\\
&= q^{n-(\sum w_i)}\left(\prod_{i\notin S} q^{m_i}[w_i - m_i]_q\right)\left(\prod_{i\in S}[m_i]_q\right) && (4.16)\\
&= q^{n-(\sum w_i)} D_S^W(q). && (4.17)
\end{aligned}$$

□

Although we have given the previous proof formally, using the m_i, an equally valid, if less formal proof, can be seen succinctly using our parking bar diagrams. We give the second proof here and hereafter will frequently return to this style of proof.

Proof. D_S^W and $D_{S^c}^W$ correspond to parking bar diagrams which have been flipped across the main line of the diagram. Thus for any particular column j, k_1 of the last w_j columns are pointed in the same direction as column j in $\tilde{D}_S^W$ if and only if k_1 of the last w_j columns are pointed in the same direction as column j in $\tilde{D}_{S^c}^W$. This means that in any given column, there are the same number of shaded squares in each diagram. See Figure 4.5. Since we take the k_1 highest powers of $[w_j]_q$ as the weight of the column if the column is pointed downward and the k_1 lowest powers if the column is pointed downwards, we have exactly the required relation in their weights. □

Corollary 91. *Let* $W = (w_1, \ldots, w_n)$*. Then*

$$P_W(Z_n; q) = q^{n-(\sum_i w_i)} \left(\prod_i z_i \right) P_W\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}; \frac{1}{q} \right).$$

For the next few proofs, to simplify notation, say that

$$R_W(z, q) = \sum_s B_s(q) z^s.$$

Corollary 92. *For all* $1 \leq s \leq n/2$,

$$B_s(q) + B_{n-s-1}(q) = q(B_{s+1}(q) + B_{n-s}(q)) \tag{4.18}$$

if and only if $R_s(q) = B_{s+1}(q) + B_{n-s}(q)$ *is palindromic, with*

$$R_s(q) = q^{n-(\sum w_i)-1} R_s(1/q). \tag{4.19}$$

Proof. By Corollary 91,

$$B_s(q) = q^{n-(\sum w_i)} B_s(1/q).$$

Then

$$B_s(q) + B_{n-s-1}(q) = q(B_{s+1}(q) + B_{n-s}(q)) \tag{4.20}$$

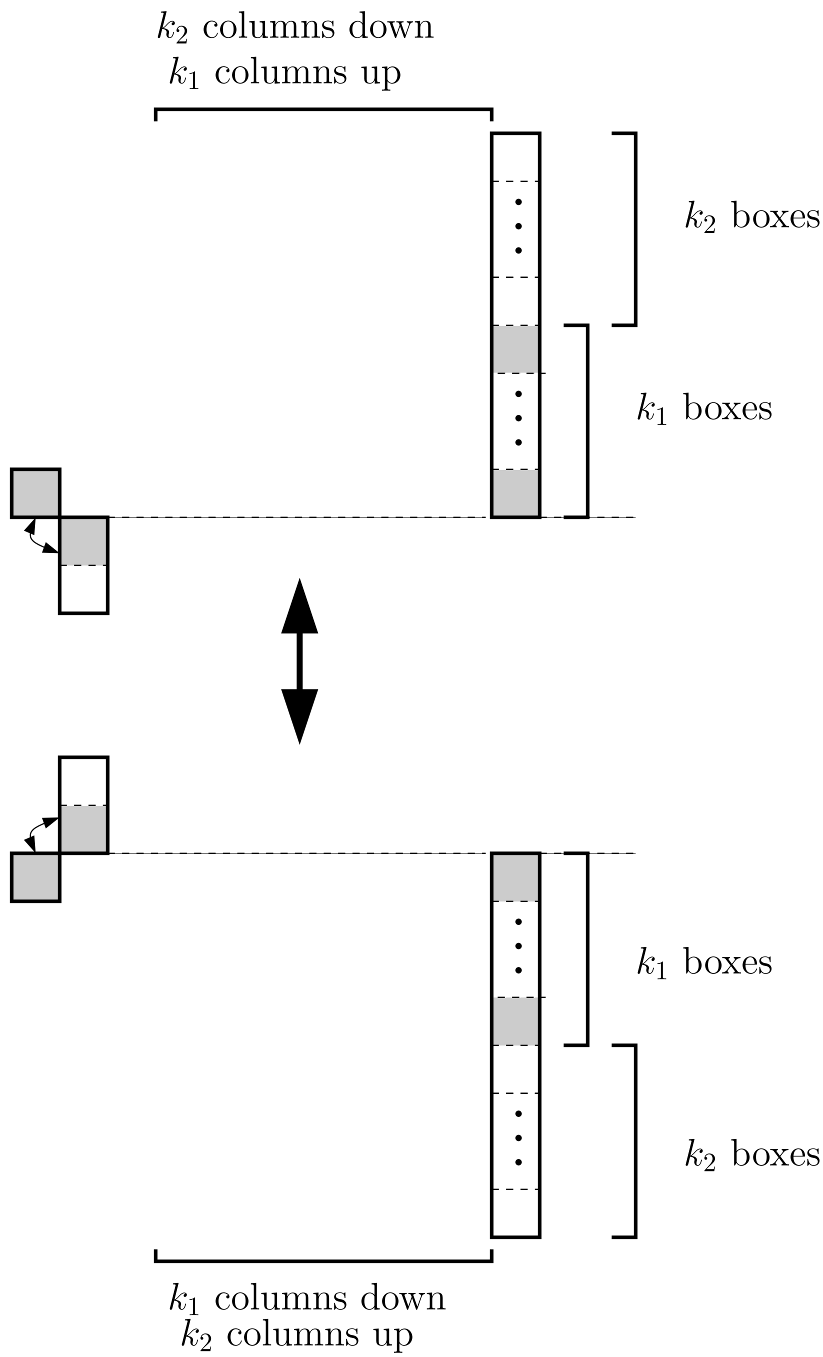


Figure 4.5: Two diagrams, flipped across the axis, correspond to complementary sets. Any given column has the same number of shaded boxes in each diagram. A bracket ending at a column j, above or below our diagram, gives the region that encompasses w_j squares. The curved arrows are meant to show that we would also like to consider the diagrams with the first bar down and the second up.

if and only if

$$q^{n-(\sum w_i)}(B_{s+1}(1/q)+B_{n-s}(1/q)) = q(B_{s+1}(q)+B_{n-s}(q)), \tag{4.21}$$

as required. □

Theorem 93.

$$R_W(1,q) = \prod_{i=1}^{n}[w_i]_q$$

Proof. Notice that in particular, $R_W(1,q)$ is the sum of the weights of all parking bar diagrams corresponding to W. Inductively, assume the result is true for smaller schedules, in particular when $W' = (w_1,\ldots,w_{n-1})$. (The base case is trivial.) Then any parking bar diagram we could form for W starts with a parking bar diagram for W'. Pick a typical such diagram $\tilde{D}$. From the diagram, in the last w_n columns there must be some k columns up and $w_n - k$ columns down. Then add a final bar to form a diagram for W as in Figure 4.6. This can be done in two ways, one which adds a weight $[k]_q$ and the second a weight $q^k[w_n-k]_q$. Thus the sum of the weights of the two new diagrams is

$$\text{weight}(\tilde{D})([k]_q + q^k[w_n-k]_q) = \text{weight}(\tilde{D})[w_n]_q.$$

Then summing over all diagrams for W' this way, we are done by induction. □

Theorem 94.

$$R_{(1,2,2,3,\ldots,k)}(z,q) = \sum_{s=1}^{k}(1+q)q^{k-s}[k-1]_q!z^s$$

Proof. First, notice that i acts on every $j<i$. The coefficient $R_{(1,2,2,3,\ldots,k)}(z,q)|_{z^s}$ comes from summing the weights of all diagrams with s bars up. First, notice that if $S_1 = \{1\}\cup S$ and $S_2 = \{2\}\cup S$, where $|S| = s-1$, then $D^{(1,2,2,3,\ldots,k)}_{S_1} = qD^{(1,2,2,3,\ldots,k)}_{S_2}$, since interchanging the first two bars does not change the weight of any bar except the first. Thus

$$R_{(1,2,2,3,\ldots,k)}(z,q)\Big|_{z^s} = (1+q)\sum_{\substack{S\subset\{3,\ldots,k+1\}\\ |S|=s-1}} D^{(1,2,2,3,\ldots,k)}_{S\cup\{2\}}. \tag{4.22}$$

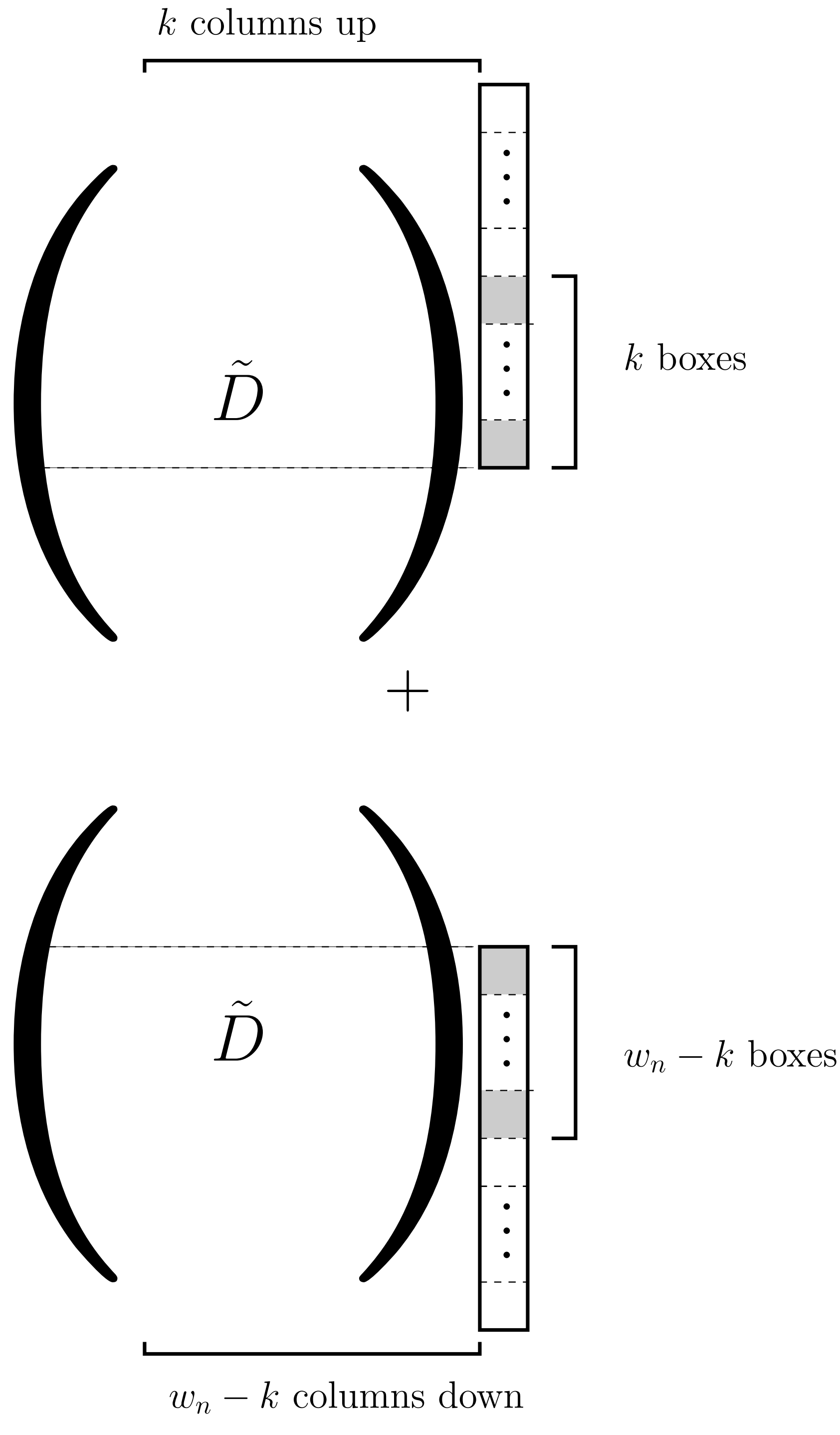


Figure 4.6: Two diagrams, formed from a smaller diagram $\tilde{D}$.

Rather than summing over subsets, let $r \in R(0^{k-s}, 1^{s-1})$ be a word with $k-s$ 0's and $s-1$ 1's. Use $D_r^{(1,2,2,3,\ldots,k)}$ for $D_S^{(1,2,2,3,\ldots,k)}$, where $s \in S$ if and only if $r_i = 1$. See Figure 4.7 for two such diagrams labeled by the corresponding r. As is obvious from the examples, the upward pointing columns beside the first will have in order $1, 2, 3, \ldots, s-1$ shaded boxes and thus correspond to weights

$$[1]_q, [2]_q, \ldots, [s-1]_q.$$

The downward pointing columns besides the first will have in order $1, 2, 3, \ldots, k-s$ boxes. Each will have at least one empty box corresponding to the second column, which points upwards; this gives a total q weight of q^{k-s}. Each additional empty box corresponds to some column (besides the second) which is to the left of a column i and pointed upwards. This is exactly the inversions in r. Thus we have:

$$R_{(1,2,2,3,\ldots,k)}(z,q)\big|_{z^s} = (1+q) \sum_{r \in R(0^{k-s},1^{s-1})} D_r^{(1,2,2,3,\ldots,k)} \tag{4.23}$$

$$(1+q)q^{k-s}[s-1]_q![k-s]_q! \sum_{r \in R(0^{k-s},1^{s-1})} q^{\mathrm{inv}(r)} \tag{4.24}$$

$$= (1+q)q^{k-s}[s-1]_q![k-s]_q! \begin{bmatrix} k-1 \\ s-1 \end{bmatrix}_q \tag{4.25}$$

$$= (1+q)q^{k-s}[k-1]_q \tag{4.26}$$

□

4.2 The Functional Equation

Rather than checking our bijection (or palindromicity) for every k, it turns out we can check a simple condition involving what we will hereafter refer to as the functional equation:

Conjecture 95 (Implicative Conjecture v. 4)**.**

$$(1-q/z)R_W(z,q) + z^{n-1}(1-qz)R_W(1/z,q) = (1+z^{n-1})(1-q)\prod_{i=1}^{n}[w_i]_q, \tag{4.27}$$

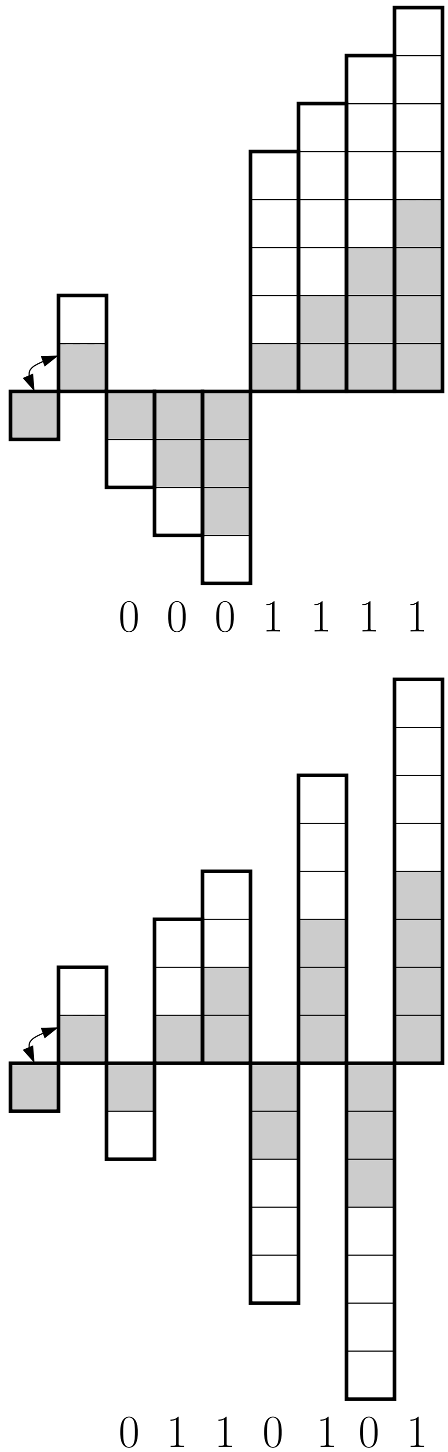


Figure 4.7: Diagrams corresponding to $D_r^{(1,2,2,3,\ldots,8)}$, with corresponding values of r labeled below each.

Definition 96. *We will hereafter refer to (4.27) as the* functional equation *and say that a two part schedule W* satisfies the functional equation *if we can verify Conjecture 95 for W.*

Then a key theorem in this work is the following:

Theorem 97. *If every two part schedule satisfies the functional equation, the Implicative conjecture is true. In particular, the Implicative Conjecture v. 3 is equivalent to the Implicative Conjecture v. 4.*

Proof. Rewriting the left hand side of our functional equation,

$$(1-q/z)R_W(z,q) + z^{n-1}(1-qz)R_W(1/z,q) \tag{4.28}$$

$$= \sum_{s=1}^{n-1} B_s(q)z^s - \sum_{s=1}^{n-1} qB_s(q)z^{s-1} + \sum_{s=1}^{n-1} B_s(q)z^{n-s-1} - \sum_{s=1}^{n-1} qB_s(q)z^{n-s}. \tag{4.29}$$

If a schedule satisfies the functional equation, then in (4.28) z^s must have vanishing coefficient when $1 \leq s \leq n-2$. This happens exactly when

$$B_s(q) - qB_{s+1}(q) + B_{n-s-1}(q) - qB_{n-s}(q) = 0 \tag{4.30}$$

as required. Moreover, note that the constant term in(4.29) is simply

$$-qB_1(q) + B_{n-1}(q)$$

Thus we have

$$(1-q/z)R_W(z,q) + z^{n-1}(1-qz)R_W(1/z,q) = (1+z^{n-1})(-qB_1(q) + B_{n-1}(q)) \tag{4.31}$$

For that remainder, we simply recall Theorem 93 that

$$R_W(1;q) = \prod_{i=1}^{n} [w_i]_q.$$

But then setting $z = 1$ in (4.31) gives

$$2(1-q)R_W(1,q) = R_W(1,q) = 2\left(B_{n-1}(q) - qB_1(q)\right)$$

□

Notice that we only define satisfying the functional equation with respect to two part schedules. In the remainder of this work, even when not mentioned explicitly, we assume that we work with two part schedules.

Remark 98. The proof gives us a similar version of the functional equation that we can express when working with all τ or even just top tau. In particular, the Implicative Conjecture is true if for every such τ, there exists a polynomial $T^\tau(q)$ such that

$$(1-q/z)S^\tau(z,q)+z^{n-1}(1-qz)S^\tau(1/z,q)=(1+z^{n-1})(1-q)T^\tau(q). \qquad (4.32)$$

Remark 99. A particularly elegant way to see that we need not check our bijections respect the i-descent set is observed from the fact that the "i-descent term"

$$\sum_{\sigma\in \mathrm{Yconsec}(\tau)} q^{\mathrm{inv}(\sigma)} Q_{\mathrm{ides}(\sigma)\cup\mathrm{ides}(\tau)},$$

will factor through this version of the functional equation, since it has no z. Thus we may either replace the Gessel quasisymmetric terms by 1 and get all parking functions or remove the "i-descent term" (or even the part of the term corresponding to a particular set of consecutivities, if we decide to sum over a subset of Yconsec(τ)) and check the functional equation. The result is the same in any case.

4.2.1 Families Satisfying the Functional Equation

Several families of schedules have been shown to satisfy the functional equation. First, recalling that schedules have a slow growth restriction, we return to the family of maximal two part schedules.

Theorem 100. $(1,2,2,3,\ldots,k)$ *satisfies the functional equation for any* k.

Proof. Recall from Theorem 94,

$$R_{(1,2,2,3,\ldots,k)}(z,q)=\sum_{s=1}^{k}(1+q)q^{k-s}[k-1]_q!z^s.$$

If $W = (1, 2, 2, \ldots, k)$,

$$(1 - q/z)R_W(z, q) + z^{n-1}(1 - qz)R_W(1/z, q) \tag{4.33}$$

$$= (1 - q/z)\left(\sum_{s=1}^{k}(1+q)q^{k-s}[k-1]_q!z^s\right) \tag{4.34}$$

$$+ z^k(1 - qz)\left(\sum_{s=1}^{k}(1+q)q^{k-s}[k-1]_q!z^{-s}\right) \tag{4.35}$$

$$= \left(\sum_{s=1}^{k}(1+q)q^{k-s}[k-1]_q!z^s\right) - \left(\sum_{s=0}^{k-1}(1+q)q^{k-s}[k-1]_q!z^s\right) \tag{4.36}$$

$$+ \left(\sum_{s=0}^{k-1}(1+q)q^{s}[k-1]_q!z^s\right) - \left(\sum_{s=1}^{k}(1+q)q^{s}[k-1]_q!z^s\right) \tag{4.37}$$

$$= \left((1+q)[k-1]_q!z^k\right) - \left((1+q)q^k[k-1]_q!\right) \tag{4.38}$$

$$+ ((1+q)[k-1]_q!) - ((1+q)q^k[k-1]_q!z^k) \tag{4.39}$$

$$= (1 + z^k)(1+q)(1-q^k)[k-1]_q! \tag{4.40}$$

$$= (1 + z^k)(1-q)[2]_q[k]_q! \tag{4.41}$$

□

Theorem 101. *If $W' = (w_1, \ldots, w_{n-1})$ and $W'' = (w_1, \ldots, w_{n-2})$ satisfy the functional equation, then so does $W = (w_1, \ldots, w_{n-1}, 1)$.*

Proof. Let $W''' = (w_1, \ldots, w_{n-3})$. Say

$$R_{W'}(z, q) = R^{n-2,n-1}(z, q) + R^{n-2}(z, q) + R^{n-1}(z, q) + R^{\emptyset}(z, q),$$

where

$$R^T = \sum_{S \cap \{n-2, n-1\} = T} D_S^{W'}.$$

See the second line of Figure 4.8. Then, as is evident by examining the four columns of Figure 4.8, we have the following equalities:

$$R_W(z,q) = zR^{n-2,n-1}(z,q) + R^{n-2}(z,q) + zR^{n-1}(z,q) + R^{\emptyset}(z,q) \quad (4.42)$$

$$R_{W'}(z,q) = R^{n-2,n-1}(z,q) + R^{n-2}(z,q) + R^{n-1}(z,q) + R^{\emptyset}(z,q) \quad (4.43)$$

$$[w_{n-1}]_q R_{W''}(z,q) = \frac{1}{z}R^{n-2,n-1}(z,q) + R^{n-2}(z,q) + \frac{1}{z}R^{n-1}(z,q) + R^{\emptyset}(z,q). \quad (4.44)$$

For example, if $\#(\text{act}(n-1,W') \cup S) = s$, $\#(\text{act}(n-2,W') \cup S) = r$ and $S = U \cup \{n-2,n-1\}$, then

$$D^{W}_{U\cup\{n-2,n-1,n\}} = [r]_q[s+1]_q D^{W'''}_U \quad (4.45)$$

$$D^{W'}_{U\cup\{n-2,n-1\}} = [r]_q[s+1]_q D^{W'''}_U \quad (4.46)$$

$$[s+1]_q D^{W''}_{U\cup\{n-2\}} = [r]_q[s+1]_q D^{W'''}_U \quad (4.47)$$

Noticing that the relative size of the subsets in the subscripts on the left increases by one as we read down the last lines, we can conclude that all the first terms beginning at (4.42) are equal. From the same lines, we can conclude that

$$R_W = zR^{n-2,n-1}(z,q) + R^{n-2}(z,q) + zR^{n-1}(z,q) + R^{\emptyset}(z,q) \quad (4.48)$$

$$= (1+z)R_{W'}(z,q) - z[w_{n-1}]_q R_{W''}(z,q). \quad (4.49)$$

Then

$$\begin{aligned}
&(1-q/z)R_W(z,q) + z^{n-1}(1-qz)R_W(1/z,q) \\
&\quad = \left(1-\frac{q}{z}\right)\left((1+z)R_{W'}(z,q) - z[w_{n-1}]_q R_{W''}(z,q)\right) \\
&\qquad + z^{n-1}(1-qz)\left(\left(1+\frac{1}{z}\right)R_{W'}\left(\frac{1}{z},q\right) - \frac{1}{z}[w_{n-1}]_q R_{W''}\left(\frac{1}{z},q\right)\right) \\
&\quad = (1+z)\big((1-q/z)R_{W'}(z,q) + z^{n-2}(1-qz)R_{W'}(1/z,q)\big) \\
&\qquad - z[w_{n-1}]_q\big((1-q/z)R_{W''}(z,q) + z^{n-3}(1-qz)R_{W''}(1/z,q)\big) \\
&\quad = \left((1+z)(1-q)\prod_{i=1}^{n-1}[w_i]_q(1+z^{n-2})\right) \\
&\qquad - \left(z[w_{n-1}]_q(1-q)\prod_{i=1}^{n-2}[w_i]_q(1+z^{n-3})\right) \\
&\quad = (1-q)\prod_{i=1}^{n}[w_i]_q(1+z^{n-1})
\end{aligned}$$

□

This type of theorem is so typical in form that we will use it to inspire a definition.

Definition 102. *Say W of length n* inductively *satisfies the functional equation if assuming that smaller schedules in length lexicographic order satisfy the functional equation allow us to conclude that W satisfies the functional equation.*

Corollary 103. *Any schedule of the form $W = (w_1, \ldots, w_{n-1}, 1)$ inductively satisfies the functional equation.*

This allows us to prove a seemingly weaker hypothesis:

Conjecture 104 (Implicative Conjecture v.5)**.** *Every two part schedule inductively satisfies the functional equation.*

By inducting on the length, we have that this conjecture is equivalent to the previous Implicative Conjecture. An additional theorem has such a similar proof to Theorem 101 that we will go ahead and state it here with a very brief proof. Throughout the remaining theorems, we will use the following notation: If $W = (w_1, \ldots, w_{n-1}, w_n)$,

$$W^1 = (w_1, \ldots, w_{n-1}, 1).$$

Moreover, let

$$R_W^{T,m} = \sum_{S \cap \{n-m+1,\ldots,n\} = T} D_S^W.$$

Lastly, define

$$S_W = R_W - [w_n]_q R_{W^1}.$$

Theorem 105. *If $W = (w_1, \ldots, w_{n-1}, w_n)$, then*

$$S_W = R_W - [w_n]_q R_{W^1} = (1-z)\left(R_W^{\{n-1\},2} - \frac{1}{z} R_W^{\{n\},2}\right).$$

Proof. This time, we have the following equalities, as seen in Figure 4.9:

$$R_W(z,q) = R_W^{\{n-1,n\},2}(z,q) + R_W^{\{n-1\},2}(z,q) + R_W^{\{n\},2}(z,q) + R_W^{\emptyset,2}(z,q) \quad (4.50)$$

$$[w_n]_q R_{W^1}(z,q) = R_W^{\{n-1,n\},2}(z,q) + zR_W^{\{n-1\},2}(z,q) + \frac{1}{z} R_W^{\{n\},2}(z,q) + R_W^{\emptyset,2}(z,q). \quad (4.51)$$

By subtraction, the result is immediate. □

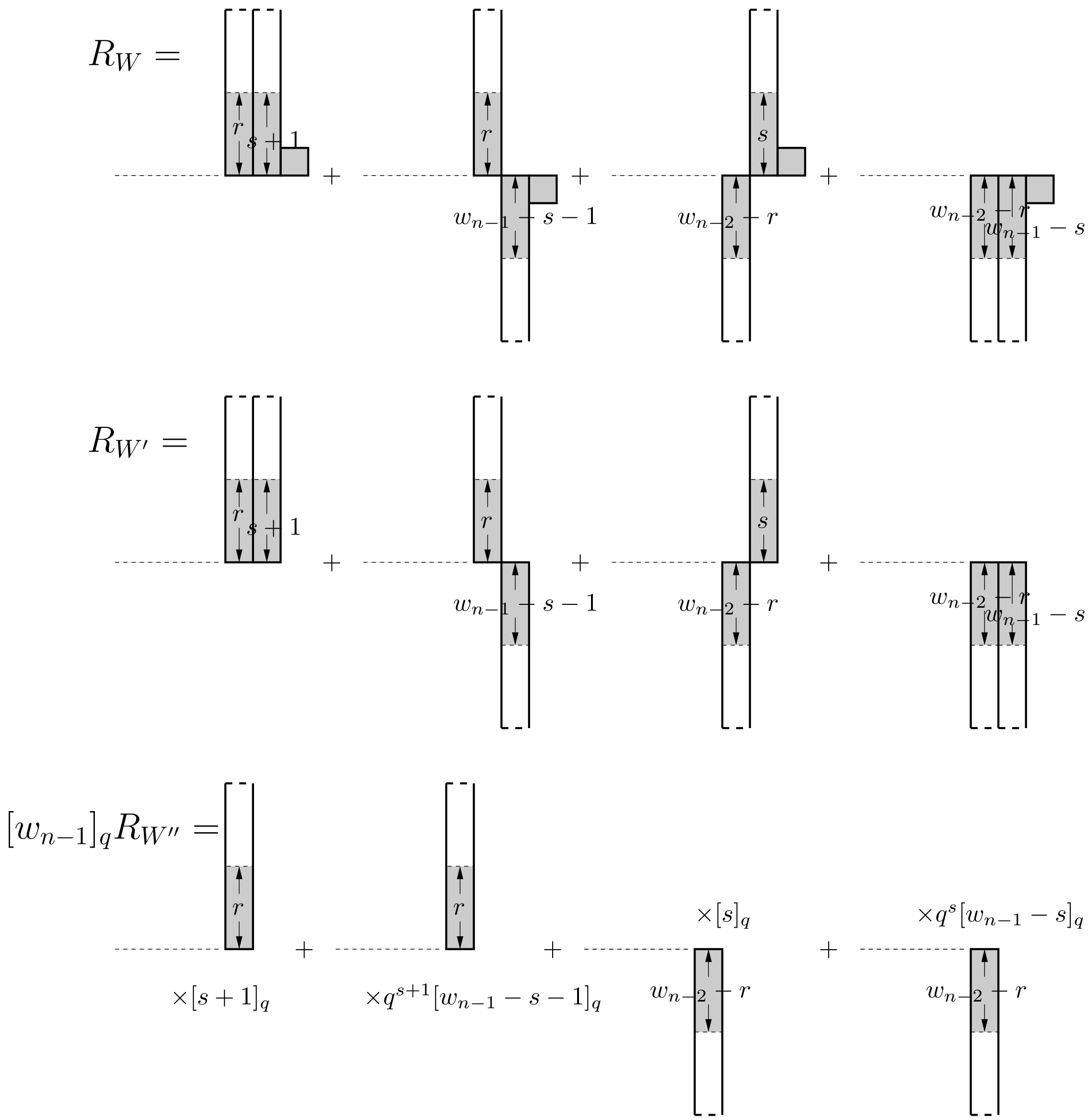


Figure 4.8: A way of splitting $R_{W'}$ and the remaining polynomials into four parts each. Notice that if the dotted lines are all replaced by a single diagram $\tilde{D}_U^{W'''}$, the q weight of elements within columns are identical.

This result is surprisingly useful, because of the next theorem.

Theorem 106. *Assume W^1 satisfies the functional equation. Then*

$$(1-q/z)S_W(z,q)+z^{n-1}(1-qz)S_W(1/z,q)=0, \tag{4.52}$$

if and only if W inductively satisfies the functional equation.

Proof. Assume W^1 satisfies the functional equation. Then

$$\begin{aligned}0 &= (1-q/z)S_W(z,q)+z^{n-1}(1-qz)S_W(1/z,q) && (4.53)\\ &= (1-q/z)(R_W(z,q)-[w_n]_qR_{W^1}(z,q)) && (4.54)\\ &\qquad + z^{n-1}(1-qz)\left(R_W\left(1/z,q\right)-[w_n]_qR_{W^1}\left(1/z,q\right)\right) && (4.55)\\ &= (1-q/z)R_W(z,q)+z^{n-1}(1-qz)R_W\left(1/z,q\right) && (4.56)\\ &\qquad -(1+z^{n-1})(1-q)\prod_{i=1}^{n}[w_i]_q. && (4.57)\end{aligned}$$

□

We now can give a strengthening of a previous result with an amazingly simple proof.

Theorem 107. *Assume $W=(w_1,\ldots,w_n)$, where for some $j>2$, $w_j=1$. Then W inductively satisfies the functional equation.*

Proof. If $j=n$, we are done. Otherwise, notice that by Theorem 89, since column j is completely shaded in a nonzero diagram D_S^W, so are the remaining columns. In particular this means that the last two columns in any nonzero diagram are either both up or both down. Thus in particular $R_W^{\{n-1\},2}$ and $R_W^{\{n\},2}$ are both identically zero and thus $S_W(z,q)=0$. By Theorem 106, W then inductively satisfies the functional equation. □

Theorem 108. *If $W=(w_1,\ldots,w_j,v,v+1,\ldots,v+a-1,a)$ then W inductively satisfies the functional equation and in particular $S_W=0$.*

Proof. Let $W'=(w_1,\ldots,w_j)$. Say $S\subset[j]$ such that $\#\{act(j+1,W)\cap S\}=s$. Then we begin by considering the weight of the diagram corresponding to the set

$$T_1=S\cup\{j+a-u+1,j+a-u+2,\ldots,j+a-1,j+a\}$$

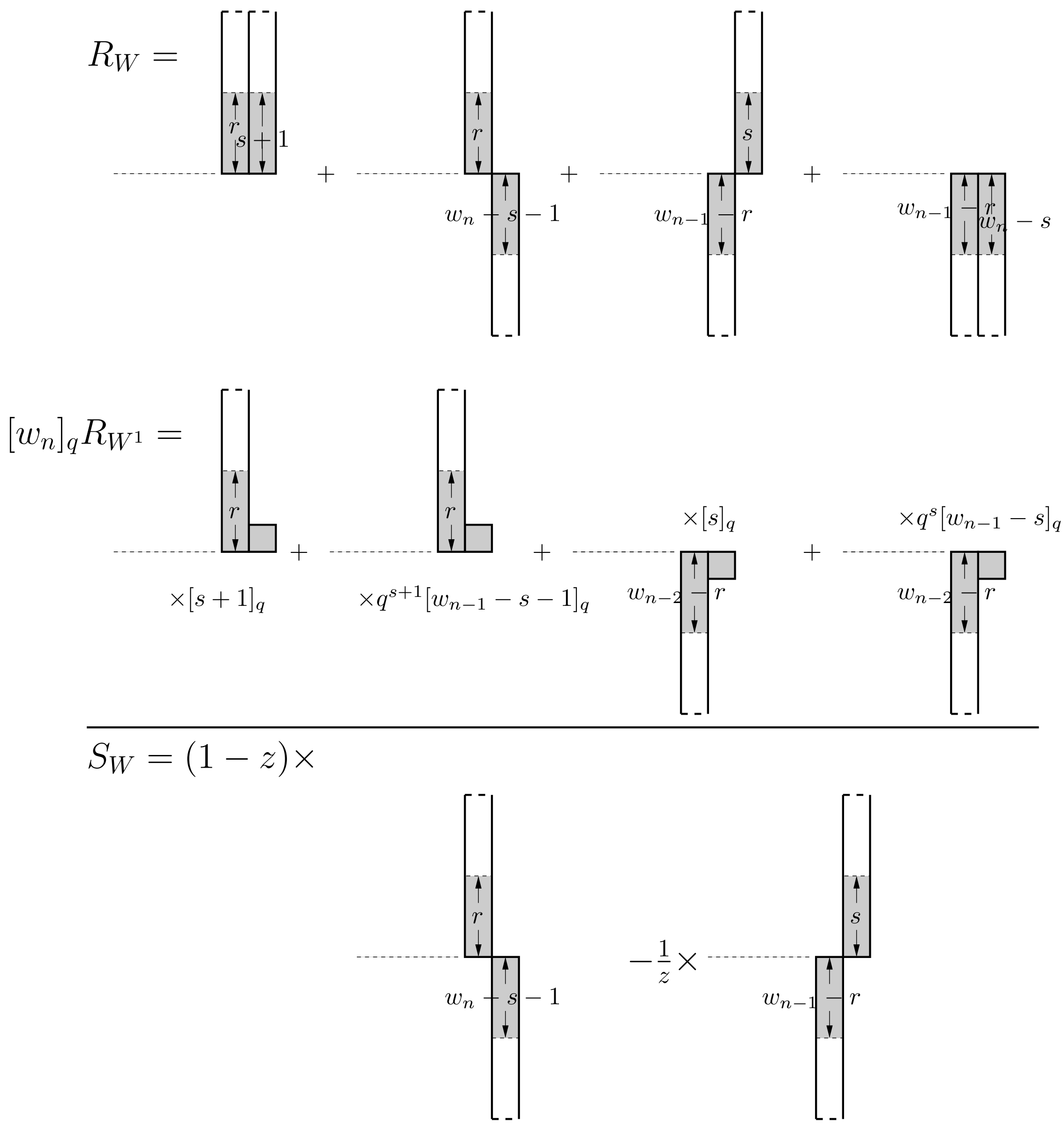


Figure 4.9: A way of splitting R_W and $[w_n]_q R_{W^1}$ each into four parts. Notice that again if the dotted lines are all replaced by a single diagram, the q weight of elements within columns are identical. The bottom line gives the difference of the first two.

and

$$T_2 = S \cup \{j+a-u, \ldots, j+a-1, j+a+1\},$$

that is the set that corresponds to the next $a-u$ down, u up, and then the last one (the one of length a) down and the set that corresponds to the next $a-u-1$ down, then u up, then one down, and the final bar up. See Figure 4.10 for an example. Notice that $\#\{T_1\} = \#\{T_2\} - 1$. By Theorem 105 they correspond to the same coefficient of S_W with opposite signs. Then the weight of the first j bars is $D_S^{W'}$. Starting with T_1, the weight of the next $a-u$ bars is exactly:

$$q^{s(a-u)} \prod_{i=0}^{a-u-1} [v-s+i]_q.$$

The weight of the next $u-1$ bars is

$$\prod_{i=0}^{u-2} [s+i]_q.$$

The weight of the final two bars is respectively:

$$[s+u-1]_q \text{ and } q^u[a-u]_q.$$

Thus

$$D_{T_1}^W = q^{s(a-u)} \prod_{i=0}^{a-u-1} [v-s+i]_q \prod_{i=0}^{u-2} [s+i]_q [s+u-1]_q q^u [a-u]_q.$$

A similar argument gives that

$$D_{T_2}^W = q^{s(a-u-1)} \prod_{i=0}^{a-u-2} [v-s+i]_q \prod_{i=0}^{u-1} [s+i]_q q^{s+u} [v-s+a-u-1]_q [u]_q.$$

To compute the weight of the remaining nonzero schedules counted by S_W that begin with $\tilde{D}_S^{W'}$ and correspond to the same coefficient of z, we need to vary which $u-1$ of the bars from the $j+1$st to the $j+a-2$nd columns are pointing upward in the diagram $\tilde{D}_{T_1}^W$ and which u of the bars are pointing upwards in the same columns of $\tilde{D}_{T_2}^W$. Like in the proof of Theorem 94, we can indicate this by attaching words in zeros and ones to these bars, where the zeros correspond to downward bars, the ones correspond to upward columns, and inversions in the word correspond to the relative increase in dinv from the dinv we calculated above. The additional weight then can again be seen as binomial coefficients, in this case

$$\begin{bmatrix} a-1 \\ u-1 \end{bmatrix}_q \text{ and } \begin{bmatrix} a-1 \\ u \end{bmatrix}_q$$

respectively. Since a quick calculation gives

$$\begin{bmatrix} a-1 \\ u-1 \end{bmatrix}_q D_{T_1}^W - \begin{bmatrix} a-1 \\ u \end{bmatrix}_q D_{T_2}^W = 0$$

summing over all sets S, we are done since then $S_W = 0$. □

Theorem 109. *Let $W' = (1, 2, w_3 \ldots, w_{n-2})$ and $(W')^1$ satisfy the functional equation. Then $W = (1, 2, v, w_3, \ldots, w_{n-2})$ also satisfies the functional equation for $v = 2, 2$ and $v = 2, 3$.*

Proof. The theorem is true if $w_3 = 1$ so let $w_3 = 2$. We consider nonzero diagrams corresponding to S_W. Notice that if 3 and 4 are both in S or both in S^c, then the bar of length 2 in column 5 of the diagram $\tilde{D}_S^W$ is entirely shaded. Then by Theorem 89 the remaining bars are entirely shaded and in particular pointing in the same direction as columns 3 and 4. Thus they do not correspond to diagrams counted by S_W. Thus we may assume either 3 or 4 is in S but not both. If $v = 2, 2$, just removing the third and fourth column will result in a nonzero diagram. In particular, the weights in the new third column onward will have the same weights as the original fifth column onward, since when we look left starting from these columns to determine the number of shaded squares, we either see identical (shifted) columns or (replacing the 3rd and 4th) the first and second column. Moreover, the result will have one less element in S and one less dinv, so $S_W = qzS_{W'}$. See Figure 4.11. Since W' and $(W')^1$ satisfy the functional equation, we have:

$$qz \times 0 = qz\left((1 - q/z)S_{W'}(z, q) + z^{n-3}(1 - qz)S_{W'}(1/z, q)\right) \tag{4.58}$$

$$= (1 - q/z)qzS_{W'}(z, q) + z^{n-1}(1 - qz)q1/zS_{W'}(1/z, q) \tag{4.59}$$

$$= (1 - q/z)S_W(z, q) + z^{n-1}(1 - qz)S_W(1/z, q) \tag{4.60}$$

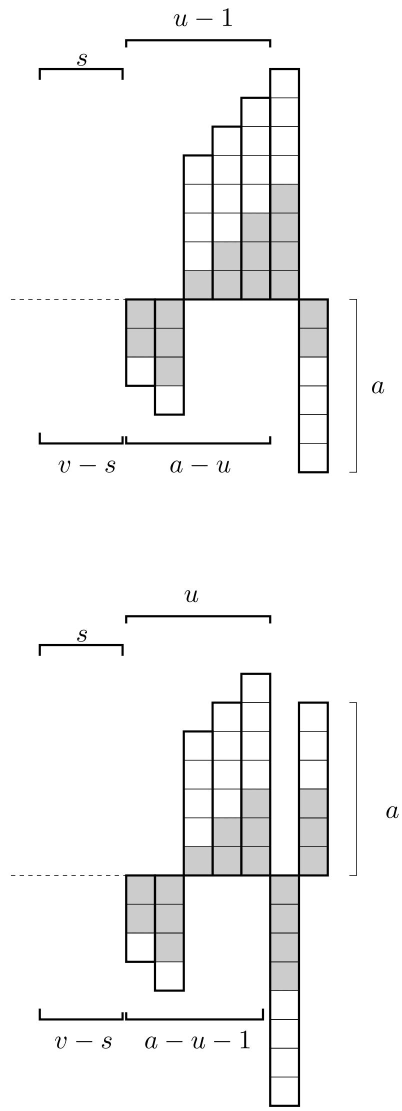


Figure 4.10: Some particular elements among the diagrams for $W = (w_1, \ldots, w_j, v, v+1, \ldots, v+a-1, a)$. The first corresponds to T_1 and the second to T_2. Replace the dotted line with a diagram for W' and take numbers above the diagrams to be the number of bars in that range pointing upward and the numbers below the diagrams to be the number in that range that are pointing downward.

The proof is similar when $v = 2, 3$, although this time we remove the third and fourth columns and flip the first and second to get the corresponding smaller diagram twice, each with one less element in S and with one or two less dinv. Again see Figure 4.11. Then we have $S_W = (q + q^2)zS_{W'}$. Again we have:

$$(q^2 + q)z \times 0 = (q^2 + q)z \left((1 - q/z)S_{W'}(z, q) + z^{n-3}(1 - qz)S_{W'}(1/z, q)\right) \tag{4.61}$$

$$= (1 - q/z)(q^2 + q)zS_{W'}(z, q) + z^{n-1}(1 - qz)(q^2 + q)1/zS_{W'}(1/z, q) \tag{4.62}$$

$$= (1 - q/z)S_W(z, q) + z^{n-1}(1 - qz)S_W(1/z, q) \tag{4.63}$$

□

This theorem gives us several infinite families that we may conclude satisfy the functional equation explicitly, including two easily described families:

Corollary 110. *Schedules of the form*

$$W = (1, 2, 2, 3, 2, 3, \ldots, 2, 3) \text{ and } W = (1, 2, 2, 2, \ldots, 2)$$

satisfy the functional equation.

Theorem 109 gives that schedules beginning $(1, 2, 2, 3, 2 \ldots)$ and $(1, 2, 2, 2, 2 \ldots)$ inductively satisfy the functional equation. Additionally we omit several similar removal proofs that give the following:

Theorem 111. *Schedules beginning with* $(1, 2, 2, 2, 3, 2 \ldots)$, $(1, 2, 2, 3, 3, 2 \ldots)$, $(1, 2, 2, 3, 4, 2 \ldots)$, $(1, 2, 2, 2, 3, 3, 2 \ldots)$, $(1, 2, 2, 2, 3, 4, 2 \ldots)$, *and* $(1, 2, 2, 3, 3, 3, 2 \ldots)$ *conditionally satisfy the functional equation.*

4.3 A Final Restatement of the Implicative Conjecture

We end this chapter with a final restatement of the Implicative Conjecture and some surrounding theorems. Unlike the previous incarnations of the conjecture, we do not directly use this restatement in further proofs. We give it here

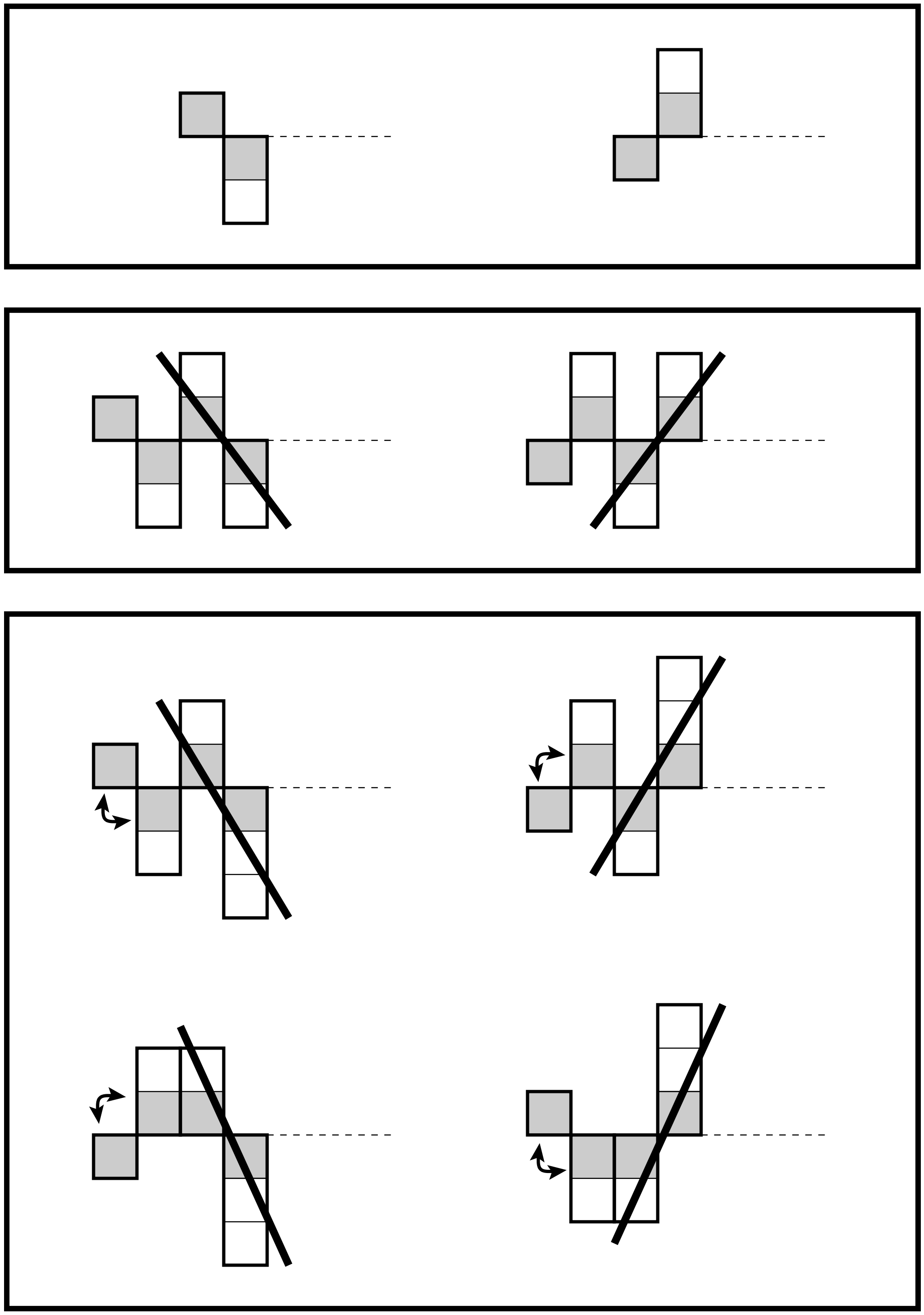


Figure 4.11: If we add to the diagrams above by placing additional bars, starting with one of length two, along each of the dotted line segments, new columns will have identical shading in every diagram. Thus we can express S_W in terms of $S_{W'}$.

merely as an interesting indication of some additional properties that the S_W must satisfy.

Conjecture 112 (Implicative Conjecture v.6)**.** *Let* $W = (w_1, \ldots, w_n)$*,* $W^1 = (w_1, \ldots, w_{n-1}, 1)$*, and*

$$S_W(z, q) = R_W(z, q) - [w_n]_q R_{W^1}(z, q).$$

Then

$$S_W(z, q) = (1 - z)(1 - qz)T_W(z, q),$$

where

$$T_W(z, q) = z^{n-2} T_W(1/z, q).$$

Theorem 113. *Implicative Conjecture v. 6 is equivalent to Implicative Conjecture v. 5.*

Proof. If W satisfies the inductively functional equation, then

$$0 = (1 - q/z)S_W(z, q) + z^{n-1}(1 - qz)S_W(1/z, q). \tag{4.64}$$

If we let $z = 1$ and $z = 1/q$ we respectively get:

$$0 = 2(1 - q)S(1, q)$$

and

$$0 = (1 - q^2)S_W(1/q, q)$$

and thus $S(1, q) = 0$ and $S(1/q, q) = 0$ as required for the factorization. If we assume the factorization, then we have

$$0 = (1 - q/z)(1 - z)(1 - qz)T_W(z, q) + z^{n-1}(1 - qz)(1 - 1/z)(1 - q/z)T_W(1/z, q) \tag{4.65}$$

$$= (1 - q/z)(1 - z)(1 - qz)T_W(z, q) + z^{n-2}(1 - qz)(z - 1)(1 - q/z)T_W(1/z, q) \tag{4.66}$$

if and only if

$$0 = T_W(z, q) - z^{n-2} T_W(1/z, q).$$

□

In fact, we have established the factorization for all two part schedules, although not the palindromicity.

Theorem 114. *For every W, $(1-z)$ divides $S_W(z,q)$.*

Proof. This follows directly from Theorem 105, once we notice that R_W has no constant term when viewed as a polynomial in z, so when we divide a subset of its monomials by z and subtract them from another subset, we are guaranteed to get a polynomial. □

The remaining factor is less immediate. We prove a stronger statement in the following theorems by a slightly circuitous path. We begin by splitting the weight of individual diagrams into two (sometimes possibly zero valued) pieces. In particular, when $n \notin S$, let

$$\text{weight}(\tilde{D}_S^W) = \text{weight}_1(\tilde{D}_S^W) + \text{weight}_2(\tilde{D}_S^W),$$

where $\text{weight}_1(\tilde{D}_S^W)$ is nonzero only when $n - w_n \notin S$, in which case it is the product of the weights of the columns except the nth times $q^{\text{act}(n,W)\cap S}$. That is, a nonzero value of $\text{weight}_1(\tilde{D}_S^W)$ corresponds to the weight of every column except the last times the lowest power of the weight of the nth column.

Theorem 115. *Let $W = (w_1, \dots, w_n)$ and $W' = (w_1, \dots, w_{n-1}, w_n - 1)$. Then if $n \notin S$, $\text{weight}_2(\tilde{D}_S^W) = q * \text{weight}(\tilde{D}_S^{W'})$*

Proof. Start with a diagram $\text{weight}(\tilde{D}_S^{W'})$. Imagine adding a square to the bottom of the right (downward facing) column. The result is the diagram $\tilde{D}_S^W$. If we look at the shading of $\text{weight}(\tilde{D}_S^W)$, we have to shade one additional square exactly when $n - w_n \notin S$. Besides that square, all the weights are identical to the weights of $\text{weight}(\tilde{D}_S^{W'})$, except that the squares in the last diagonal are now each weighted by an additional power of q. See the top two lines of Figure 4.12. □

Theorem 116. *Let $W = (w_1, \dots, w_n)$ and $W' = (w_1, \dots, w_{n-2}, w_n - 1)$. Let $S \subset [n-2] \backslash \{n - w_n\}$. Then*

$$q[w_{n-1}]_q \,\text{weight}_1(\tilde{D}_S^{W'}) = \text{weight}_1(\tilde{D}_S^W)q + \text{weight}_1(\tilde{D}_{S\cup\{n-1\}}^W)$$

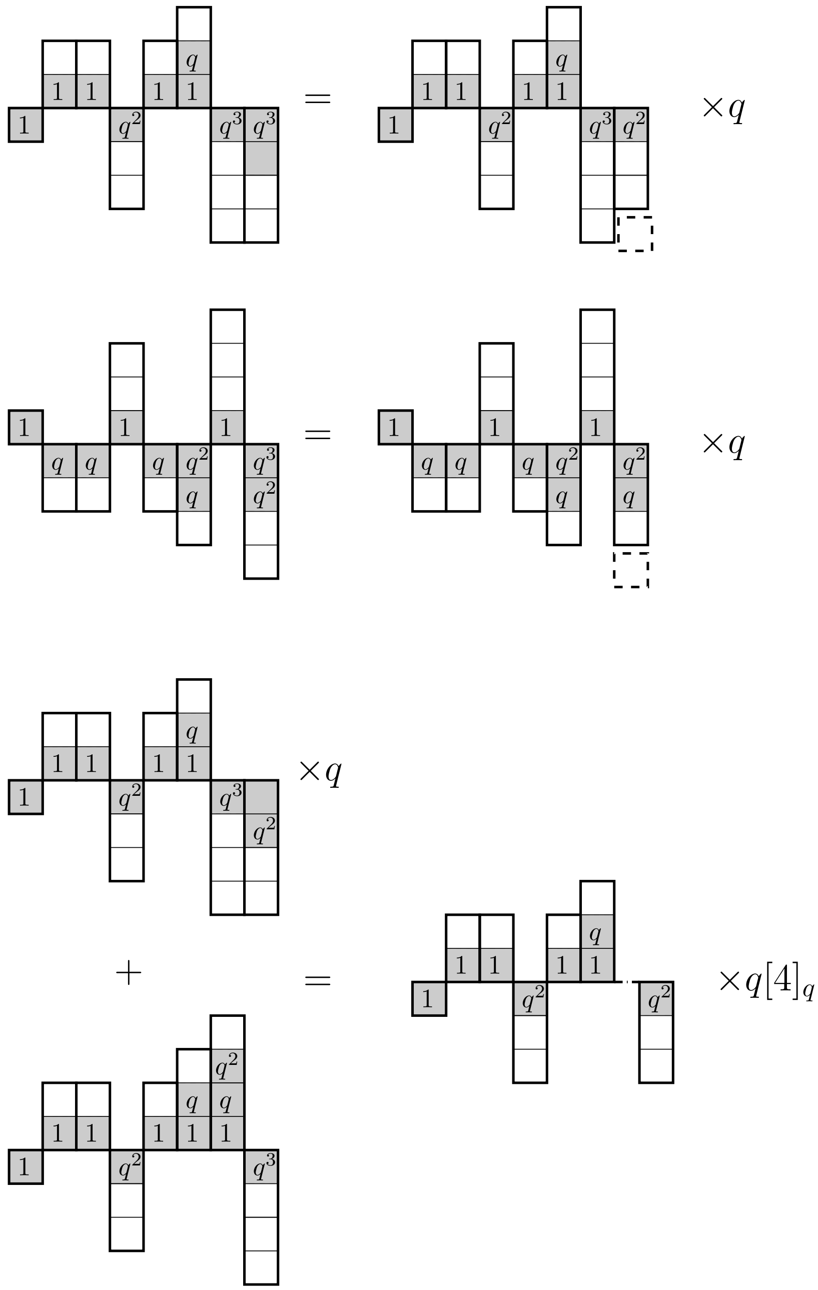


Figure 4.12: The top two diagrams represent the weight$_2(\tilde{D}_S^W)$ of two typical diagrams and the corresponding smaller diagrams used in our inductions. The bottom two diagrams correspond to weight$_1(\tilde{D}_S^W)$ for a typical smaller diagram.

Proof. Begin with the diagram $\tilde{D}_S^{W'}$. Imagine adding a downward facing bar between the last two columns and increasing the length of the last column by one. Then the last column will have exactly one new shaded square, so the least weight in the last column remains the same, despite the fact that we have increased the last column's length. We will have an additional new weight of $q^s[w_{n-1}-s]_q$ where $s = \text{act}(n-1, W) \cap S$ from the new (second to last) column. On the other hand, again begin with the diagram $\tilde{D}_S^{W'}$. Imagine adding an upward facing bar between the last two columns and increasing the length of the last column by one. Then the resulting last column will have one new unshaded square, thus increasing the weight by q. The weight of the new column will be exactly $[s]_q$. Again see Figure 4.12. □

Theorem 117. $W = (w_1, \ldots, w_n)$. *Then*

$$\sum_{S\subset[n-1]} (1/q)^{\#S} \,\text{weight}_1(\tilde{D}_S^W) = \prod_{i=3}^{n-1} [w_i]_q.$$

Proof. The proof is by induction and the base case is easily checked. Let $W' = (w_1, \ldots, w_{n-2}, w_n - 1)$.

$$\sum_{S\subset[n-1]} (1/q)^{\#S} \,\text{weight}_1(\tilde{D}_S^W) \tag{4.67}$$

$$= \sum_{S\subset[n-2]} (1/q)^{\#S} \,\text{weight}_1(\tilde{D}_S^W) + (1/q)^{\#S+1} \,\text{weight}_1(\tilde{D}_{S\cup\{n-1\}}^W) \tag{4.68}$$

$$= \sum_{S\subset[n-2]} [w_{n-1}]_q (1/q)^{\#S} \,\text{weight}_1(\tilde{D}_S^{W'}) \tag{4.69}$$

$$= [w_{n-1}]_q \prod_{i=3}^{n-2} [w_i]_q \tag{4.70}$$

where the last equality is by induction and the previous equality is a consequence of the previous theorem. □

Theorem 118. *Let* $W = (w_1, \ldots, w_n)$. *Then*

$$R_W^{\emptyset,1}(1/q, q) = \prod_{i=3}^{n} [w_i]_q$$

Proof. Again, the proof is by induction. Let $W' = (w_1, \ldots, w_{n-1}, w_n - 1)$.

$$R_W^{\emptyset,1}(1/q, q) = \sum_{S \subset [n-1]} (1/q)^{\#S} \operatorname{weight}(\tilde{D}_S^W) \tag{4.71}$$

$$= \sum_{S \subset [n-1]} (1/q)^{\#S} \operatorname{weight}_1(\tilde{D}_S^W) + \sum_{S \subset [n-1]} (1/q)^{\#S} \operatorname{weight}_2(\tilde{D}_S^W) \tag{4.72}$$

$$= \prod_{i=3}^{n-1} [w_i]_q + q \sum_{S \subset [n-1]} (1/q)^{\#S} \operatorname{weight}(\tilde{D}_S^{W'}) \tag{4.73}$$

$$= \prod_{i=3}^{n-1} [w_i]_q + q R_{W'}^{\emptyset,1}(1/q, q) \tag{4.74}$$

$$= \prod_{i=3}^{n-1} [w_i]_q + q[w_n - 1]_q \prod_{i=3}^{n-1} [w_i]_q \tag{4.75}$$

□

Theorem 119. *For every W, $(1 - qz)$ divides $S_W(z, q)$.*

Proof. First, by summing the equation in the statement of Theorem 90 over $n \in S$, notice that

$$z^n R_W^{\emptyset,1}(1/z, 1/q) = R^{\{n\},1}(z, q) q^{n - \sum_i w_i}.$$

Making the substitutions $q \to 1/q$ then $z \to q$ gives

$$q^n R_W^{\emptyset,1}(1/q, q) = R^{\{n\},1}(q, 1/q) q^{\sum_i w_i - n}. \tag{4.76}$$

Thus

$$R_W^{\{n\},1}(q, 1/q) = q^{2n - \sum_i w_i} \prod_{i=3}^{n} [w_i]_q. \tag{4.77}$$

Again substituting $q \to 1/q$ we have

$$R_W^{\{n\},1}(1/q, q) = q^{\sum_i w_i - 2n - (\sum_{i \geq 3} - n} \prod_{i=3}^{n} [w_i]_q \tag{4.78}$$

$$= q^{1-n} \prod_{i=3}^{n} [w_i]_q. \tag{4.79}$$

Then for every W,

$$R_W = (1 + q^{1-n}) \prod_{i=3}^{n} [w_i]_q.$$

Thus

$$\begin{aligned} S_W(1/q,q) &= R_W(1/q,q) - [w_n]_q R_{W^1}(1/q,q) \\ &= \left((1+q^{1-n}) \prod_{i=3}^{n} [w_i]_q \right) - [w_n]_q \left((1+q^{1-n}) \prod_{i=3}^{n-1} [w_i]_q \right) \\ &= 0 \end{aligned}$$

□

Chapter 5

A final theorem and a summary

We end with a final theorem that gives a broad range of families which inductively satisfy the functional equation. Recall a previous theorem, that: Maximal two part schedules, that is schedules of the form $(1, 2, 2, 3, \ldots, k)$, satisfy the functional equation for any k. If a schedule fails to be maximal, there must be a first entry that fails to be maximal.

Theorem 120. *Let $W = (w_1, \ldots, w_n)$. Let $j > 2$ be the first such that $w_j < j - 1$. Then if $w_j \neq w_{j-1}$, W inductively satisfies the functional equation.*

Proof. If $j < 5$, this theorem follows from Theorem 107. Assume $j \geq 5$ and thus $w_{j-1} \geq 3$. Let

$$W' = (w_1, \ldots, w_{j-2}, w_{j-1} - 1, w_j, \ldots, w_n)$$

and

$$W'' = (w_1, \ldots, w_{j-2}, w_{j-1} - 2, w_j, \ldots, w_n).$$

Notice the restriction on w_j means that these are both legal schedules. We claim that $2 \in \max(W) \setminus \operatorname{Sep}(W)$ and $2 \in \max(W'') \setminus \operatorname{Sep}(W'')$. First notice that $i - w_i \in \{0, 1\}$ if $i < j$ and

$$j - 1 - (w_{j-1} - 2) = j - 1 - (j - 2) = 1.$$

Next, notice that since $w_j \neq w_{j-1}$ and $w_j < j - 1$, $w_j < j - 2$. Then recalling the slow growth restriction, we know that for $i \geq j$, $w_i \leq w_j + j - i$. Thus

$$i - w_i \geq i - (w_j + i - j) = j - w_j > j - (j - 2) = 2.$$

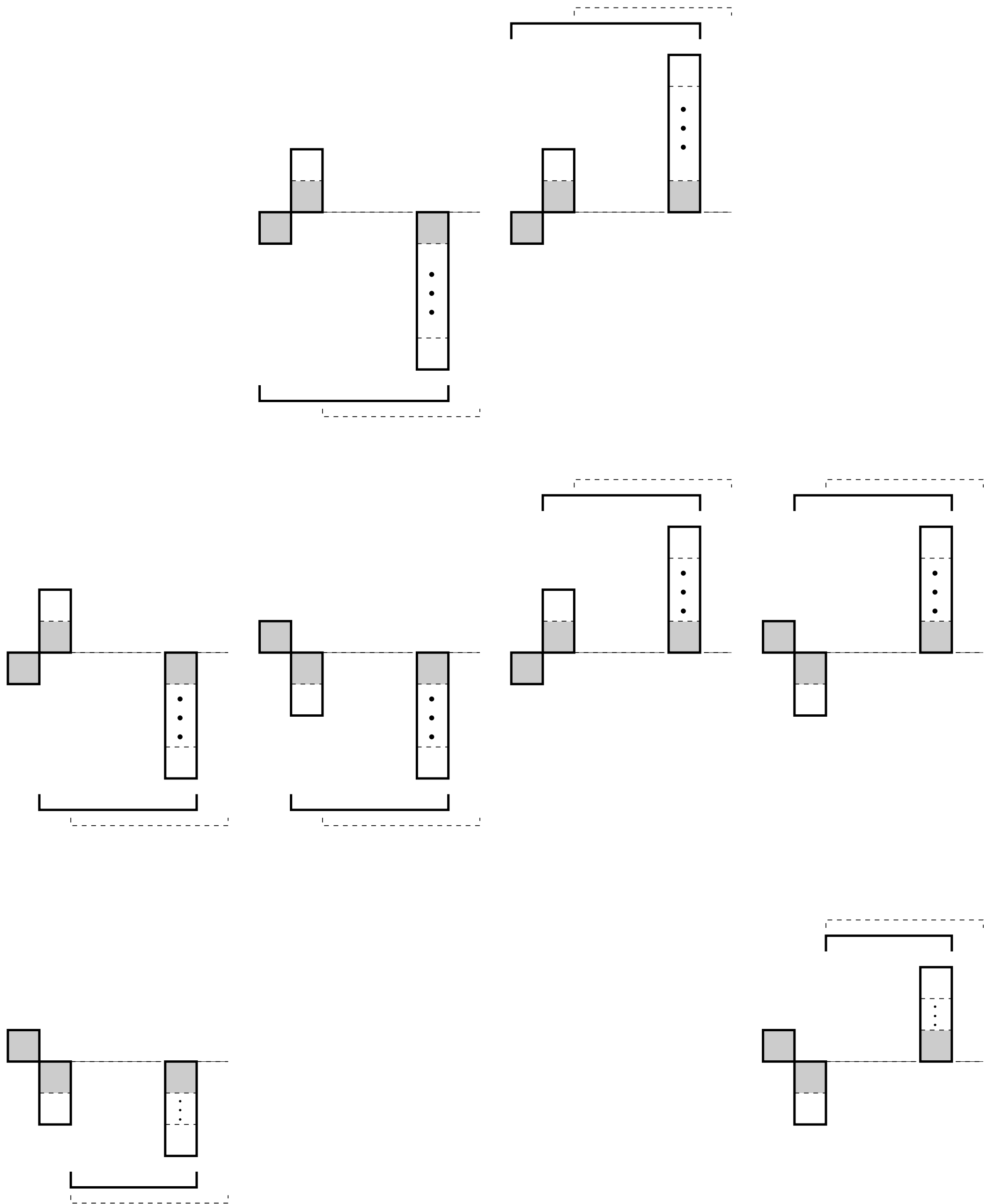


Figure 5.1: Diagrams in the same column have the same weight. Assume the missing middle diagrams are all of maximal length and are filled with bars in all possible combinations of up and down bars. The solid brackets mark the columns indexed by $\mathrm{act}(j-1, W)$ along with $j-1$ itself. The dotted brackets mark the most possible columns indexed by $\mathrm{act}(j, W)$ along with j itself. Notice that the length of the bar $j-1$ decreases by one moving down the columns, but the number of shaded squares stays fixed.

By Theorem 66, we may thus assume there is a top tau τ for W with an unforced i-descent at $\overline{\tau}_2$. By Remark 98, it is then enough to show that S^τ satisfies the functional equation. By Remark 99, it is enough to check the functional equation on just those parking functions without an unforced i-descent at $\overline{\tau}_2$. By Theorem 60, this corresponds to the weights of diagrams with $2 \in S$, since these are exactly those where $d_1 \leq d_2$. Let

$$R_W^2 = \sum_{2 \in S} z^{\#S} D_S^W.$$

Then by Remark 98, it is enough to show that when we plug R_W^2 into the functional equation, all but the constant and highest powers of z are 0. Assume W^1 satisfies the functional equation. By a similar argument, we can assume that when we plug $R_{W'}^2$ into the functional equation, all but the constant and highest powers of z are 0. By Theorem 71 and the following remark, notice that $R_W = R_W^2(1+q)$ (since $\sum_{\sigma \in S_2} q^{\mathrm{inv}(\sigma)} = (1+q)$ and $R_{W^1} = R_{W^1}^2(1+q)$. Then by Theorem 106 in order to show W inductively satisfies the functional equation, it is enough to show that when S_W is plugged into the functional equation, the result is zero. Notice that this happens if and only if the result is zero when

$$S_W^2 = R_W^2 - [w_n]_q R_W^2$$

is plugged into the functional equation. Similarly, let

$$R_{W''}^1 = \sum_{1 \in S} z^{\#S} D_S^{W''}.$$

By a similar argument, conclude that S_W^1 satisfies the functional equation. Next, assume that W' satisfies the functional equation. Finally, notice as can be seen by Figure 5.1, that

$$S_W^2 = S_{W'} - S_{W''}^1.$$

Then assuming that W' satisfies the functional equation, we have that

$$
\begin{aligned}
&(1-q/z)S_W(z,q) + z^{n-1}(1-qz)S_W(1/z,q) && (5.1)\\
&\qquad = (1-q/z)(S_{W'}(z,q) - S^1_{W''}(z,q)) && (5.2)\\
&\qquad\quad + z^{n-1}(1-qz)(S_{W'}(1/z,q) - S^1_{W''}(1/z,q)) && (5.3)\\
&\qquad = ((1-q/z)S_{W'}(z,q) + z^{n-1}(1-qz)S_{W'}(1/z,q)) && (5.4)\\
&\qquad\quad - ((1-q/z)S^1_{W''}(z,q) + z^{n-1}(1-qz)S^1_{W''}(1/z,q)) && (5.5)\\
&\qquad = 0 && (5.6)
\end{aligned}
$$

□

With this result, the majority of schedules are known to satisfy the functional equation.

5.1 Remaining Open Problems

Recall that our object in this work was to split the Haglund—Morse—Zabrocki Conjecture into two parts:

1. (*Reduction.*) Reduce the HMZ conjecture to proving the partition case. That is, show that if (2.1) is true for all partitions p then it is true for all compositions.

2. (*Partition Problem.*) The HMZ conjecture is true when p is a partition.

A remaining open question to which an affirmative answer would solve the Partition Problem is as follows:

Open Problem 121. *Is there a basis ψ which is lower triangularly related to*

$$\{\nabla C_\mu\}_{\mu\vdash n} \text{ and } \left\{ \sum_{\mathrm{comp}(PF)=p} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)} \right\}_{\mu\vdash n} ?$$

On the other hand, the remaining open question to which an affirmative answer would reduce the HMZ conjecture is as follows:

Open Problem 122. *For all two part schedules* $W = (w_1, \ldots, w_n)$ *where the first non-maximal entry* $j > 3$ *is not a repeat of* w_{j-1}*, does* W *satisfies the functional equation?*

Note that in particular, there is a plethora of experimental evidence in favor of an affirmative response. In particular, Rodemich has shown this using exhaustive search for schedules or length less than 15. Moreover, extending this result to all n would be a significant step towards proving the HMZ conjecture (and thus, of course, the original Shuffle Conjecture.)

Appendix A

An Historical Note

During a plane flight, soon before I began preparing this manuscript, my thesis advisor, Adriano Garsia, began repeating stories he's told me throughout my career, tying them together so that I could better understand how the Shuffle Conjecture had evolved. At one point, afraid that these stories would one day be lost to future generations, I asked him to help me write them down. I was pleasantly surprised when a few weeks later, he emailed me the following pages with citations. With his gracious permission, I'm reproducing them here with only minor formatting changes.

~ A. Hicks

A.1 An informal brief history of the Shuffle conjecture and related developments

by Adriano Garsia

Back in 1988 in a Paris Hotel Jan Macdonald showed me his new symmetric function basis $J_\mu[X;q,t]$ with some of its remarkable connections to combinatorics of standard tableaux. What excited me was that the coefficients $K_{\lambda,\mu}(q,t)$ that arose from the Schur function expansion

$$J_\mu[X;q,t] = \sum_{\lambda \vdash n} s_\lambda[(1-t)X]K_{\lambda,\mu}(q,t) \tag{A.1}$$

obtained in hand calculations by Macdonald (up to partitions of 6) turned out to be polynomials in $\mathbf{N}[q,t]$ with the additional property that

$$K_{\lambda,\mu}(1,1) = f_\lambda \qquad \text{(the number of standard tableaux of shape } \lambda). \tag{A.2}$$

This became the "*Macdonald q,t-Kostka conjecture*" [2]. Now (A.1) and (A.2) gives that the polynomial

$$H_\mu[X;q,t] = J_\mu[\frac{X}{1-t};q,t] = \sum_{\lambda\vdash n} s_\lambda[X]K_{\lambda,\mu}(q,t) \tag{A.3}$$

satisfies (for any $\mu \vdash n$)

$$H_\mu[X;1,1] = \sum_{\lambda\vdash n} s_\lambda[X] f_\lambda = e_1^n \tag{A.4}$$

which is the Frobenius characteristic of the left regular representation of S_n. This circumstance suggested me that an approach to prove these experimental findings in full generality was to construct a bigraded S_n module which had $H_\mu[X;q,t]$ as Frobenius characteristic. I quickly found out that there was a slight problem with this idea, for it could be easily shown that for all $\mu \vdash n$

$$K_{n,\mu}(q,t) = t^{n(\mu)} \text{ and} K_{1^n,\mu}(q,t) = q^{n(\mu'} \tag{A.5}$$

which forced the trivial to occur at a degree > 0 which seemed rather counterintuitive! So I decided that passing to

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} H_\mu[X;q,1/t] \tag{A.6}$$

would easily fix this problem. That is how the modified Macdonald polynomial $\tilde{H}_\mu[X;q,t]$ was born. So I started to translate all the properties that Macdonald had worked out for his polynomials into corresponding properties of the polynomials $\tilde{H}_\mu[X;q,t]$. The three stunning facts that emerged was the identities

$$\tilde{H}_{\mu'}[X;q,t] = \tilde{H}_\mu[X;t,q] \tag{A.7}$$

and that setting $T_\mu = t^{n(\mu)} q^{n(\mu')}$ we have

$$T_\mu \omega \tilde{H}_\mu[X;1/q,1/t] = \tilde{H}_\mu[X;q,t] \tag{A.8}$$

and last but not least

$$\tilde{H}_\mu[X;0,t] = \tilde{H}_\mu[X,t]$$

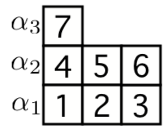
The latter being the Hall-Littlewood polynomial

$$\tilde{H}_\mu[X,t] = \sum_{\lambda \vdash n} s_\lambda[X] \tilde{K}_{\lambda,\mu}(q)$$

whose coefficients $\tilde{K}_{\lambda,\mu}(q)$ had been named "*Kotska-Foulkes*" and were the subject of the so called Kostka-Foulkes conjectures (solved in 1980 by Lascoux-Schutzenberger). Since it can be shown that

$$\tilde{H}_\mu[X,1] = h_\mu[X]$$

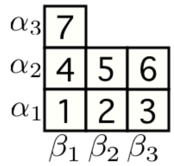
which is the Frobenius characteristic of the action of S_n on the left cosets of a Young subgroup of shape μ , I said to myself that a good test case of my approach to resolve the Macdonald q,t-Kostka conjecture was to construct a singly graded module with Frobenius characteristic $\tilde{H}_\mu[X,t]$. Since I was not inclined to reinvent a wheel I decided, before embarking on the project, to ask Claudio Procesi if this had already been done anywhere. His answer was stunning. Yes this had been done by the Algebraic Geometers which obtained $\tilde{H}_\mu[X,t]$ as the characteristic of the action of S_n on the schematic intersection of the diagonal matrices with the Nilpotent matrices of diagonal Jordan blocks of shape μ (!!!!). Unfortunately I quickly found out that all this was shown using machinery totally inaccessible to me. So I proposed to Procesi that we may work jointly to find a more elementary approach to prove the same result. Procesi then showed to me how an isomorphic S_n module could be obtained as a quotient of the ordinary polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by a well defined ideal I_μ. Since I have an in born distaste for quotients I decided that we should work with the orthogonal complement of I_μ which I called $\mathcal{H}_\mu$. We shortly noticed that $\mathcal{H}_\mu$ could be obtained by taking a point a in n-dimensional space with stabilizer the Young subgroup of shape μ, calling $[a]_{S_n}$ its S_n orbit and defining $I_{[a]_{S_n}}$ as the ideal of polynomials vanishing at the orbit $[a]_{S_n}$ then obtaining I_μ as $gr\, I_{[a]_{S_n}}$ (that is the ideal generated by the highest homogeneous components of the elements of $I_{[a]_{S_n}}$). This yielded that $\mathcal{H}_\mu$ was none other than a very natural



$$a = (\alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_2, \alpha_3)$$

Figure A.1: A diagram μ with labeled rows.

subspace of the Harmonics of S_n (the polynomials killed by all the S_n invariant differential operators with 0 constant term). After a few months of struggles I finally succeeded in proving that indeed $\tilde{H}_\mu[X,t]$ was the graded Frobenius characteristic of $\mathcal{H}_\mu$, [3]. Encouraged by this success I embarked on the more ambitious project of constructing the bigraded module that gave $\tilde{H}_\mu[X;q,t]$. Since Procesi was after his own pursuits and I dislike working alone I decided to lure Mark Haiman into my q,t-Kostka project. Mark Haiman was looking for a job at that time and found the idea of coming to La Jolla and working with me on this project very attractive. Surprisingly, it turned out to be a struggle to get him a job in La Jolla (as a pure Combinatorialist, which he was at that time). But I finally succeeded, but about a year after I first described to him my construction of the S_n module $\mathcal{H}_\mu$. Nevertheless we kept on working jointly on this project. To better understand these developments. I must give a more detailed description of the procedure I used in [3] to obtain the module $\mathcal{H}_\mu$. Given a partition $\mu \vdash n$ starting from the Ferrers' diagram of μ, I constructed a point $a = (a_1, a_2, \ldots, a_n)$ by the following procedure. I filled the cells of the diagram of μ with the numbers $1, 2, \ldots, n$ by rows as in Figure A.1. Then I placed the indeterminate α_j in position i if i was in row j as in the above example. Since the stabilizer of a under the permutation action of S_n is the Young subgroup $S_{1,2,3} \times S_{4,5,6} \times S_7$, the orbit $[a]_{S_n}$ consisted precisely of $n!/\mu!$ points all of which could be obtained by the same process. That is filling the diagram of μ in a row increasing manner then constructing the corresponding point by placing the indeterminate α_j in position i if i was in row j of the resulting tableau. This given, note that to obtain a polynomial in the ideal $I_{[a]_{S_n}}$ it suffices to concoct a way to kick any set of integers out of the diagram. For instance the



$$(a;b)= (\alpha_1,\alpha_1,\alpha_1,\alpha_2,\alpha_2,\alpha_2,\alpha_3\ ;\beta_1,\beta_2,\beta_3,\beta_1,\beta_2,\beta_3,\beta_1)$$

Figure A.2: A diagram μ with labeled rows and columns.

polynomial

$$(x_1-\alpha_1)(x_1-\alpha_2)(x_1-\alpha_3)$$

vanishes throughout the orbit $[a]_{S_n}$ and thus x_1^3 belongs to $grI_{[a]_{S_n}}$ and consequently the operator $\partial_{x_1}^3$ kills all the elements of the space $\mathcal{H}_{3,3,1} = grI_{[a]_{S_n}}^{\perp}$. Moreover, given an S_n invariant homogeneous polynomial $P(x)$ then $P(x)-P(a) \in I_{[a]_{S_n}}$ an thus $P(x) \in grI_{[a]_{S_n}}$ and likewise every element of $\mathcal{H}_\mu$ is killed by the differential operator $P(\partial_x)$. Using these tricks I was able to obtain that the Groebner basis of the ideal $grI_{[a]_{S_n}}$ was precisely the one that was needed to get the Hall-Littlewood polynomial to be the Frobenius characteristic of $\mathcal{H}_\mu$. Note that if I constructed a point a with trivial stabilizer then the resulting Frobenius characteristic would have to be that of a graded left regular representation. But it was a well known result that, for groups generated by reflections, the polynomials killed by the invariants (with 0 constant term) are a vector space of dimension the order of the group. Since for a regular orbit the corresponding harmonics have dimension the order of the group and they form a subspace of the Harmonics they necessarily end up filling the whole space of harmonics. Starting from this observation together with our need to get a bigraded space Mark Haiman suggested that we should work with two sets of variables $x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n$ and use the diagonal action, which is clearly not generated by reflection. To obtain a regular point Mark Haiman suggested that we should try extending my construction by labeling also the columns of the diagram as indicated in Figure A.2 Then for the second set of positions we place β_j in position i if i is in the j^{th} column of the tableau. This particular choice was essentially forced also for the reason that if we wanted $\tilde{H}_\mu[X;q,t]$ to be the bigraded Frobenius characteristic of the resulting module we would have to get the two Hall-Littlewoods $\tilde{H}_\mu[X;t]$ and $\tilde{H}_{\mu'}[S;q]$ as marginal Frobenius characteristics. The construction also looked promising since it beautifully explained the identity in (A.7). The next step came in trying to identify the unique (up to a scalar) element of the resulting module $\mathcal{H}_\mu[x;y]$ that afforded the alternating representation. This came out almost instantly by the kicking trick. For instance the polynomial

$$(x_1-\alpha_1)(x_1-\alpha_2)(y_1-\beta_1)$$

kicks 1 out of the diagram and therefore lies in $I_{[a;b]_{S_n}}$ and in particular all the elements of $\mathcal{H}_\mu[x;y]$ must be killed by the operator $\partial_{x_1}^2\partial_{y_1}$ In fact, by kicking 1 into each addable corner of the partition μ, we were left with only one choice for this alternant. The polynomial

$$\delta_\mu = \det \|x_i^{r_j} y_i^{s_j}\|_{i,j=1}^n$$

where the sequence $(r_1,s_1),(r_2,s_2),\ldots,(r_n,s_n)$ gave the coordinates of the cells of μ. That is

$$\sum_{j=1}^n t^{r_j} q^{s_j} = B_\mu(q,t)$$

This particular fact, made the whole construction even more promising, because by that time I knew that the $\tilde{H}_\mu[X;q,t]$ form of the Macdonald ∂_1 operator had precisely $B_\mu(q,t)$ as eigenvalue. We quickly verified that the linear span of the derivatives of Δ_μ filled the entire space $\mathcal{H}_\mu[x;y]$. We knew that we had to have the containment

$$\mathcal{L}[\partial_x^r\partial_y^s\Delta_\mu] \subseteq \mathcal{H}_\mu[x;y]$$

since orbit harmonics are derivative closed. This gave us the inequality

$$\dim \mathcal{L}[\partial_x^r\partial_y^s\Delta_\mu] \leq n! \tag{A.9}$$

So all we needed was the equality to give a natural bi-grading to the module $\mathcal{H}_\mu[x;y]$. Computer data widely confirmed the equality as well as the fact that my modification $\tilde{H}_\mu[X;q,t]$ of the Macdonald polynomial was indeed the bigraded Frobenius characteristic of $\mathcal{L}[\partial_x^r\partial_y^s\Delta_\mu]$. The equality in (A.9) also beautifully explained the identity in (A.8) since the map "*flip*" defined by setting

$$\textit{flip}\, P(x;y) = P(\partial_x;\partial_y)\Delta_\mu(x;y)$$

induces an automorphism of $\mathcal{H}_\mu[x;y]$ that complements the bi-degree of $\Delta_\mu(x;y)$ and twists each representation by the sign representation! The resulting "n!" conjecture was stated in [4]. Not too long after the discovery of $\Delta_\mu(x;y)$, Marc Haiman noticed that a very elementary argument showed that for any μ for which (A.9) was an equality the polynomial $\tilde{H}_\mu[X;q,t]$ had to be the bigraded Frobenius characteristic of $\mathcal{H}_\mu$. This was very much in contrast with the Hall-Littlewood case where the dimension was immediate but the identification of the Frobenius took most of the work. But another phenomenon had emerged quite early in this adventure. For me the natural space to be studied was the space $DH_n[x;y]$ of Harmonics of the diagonal action of S_n that contained all the $\mathcal{H}_\mu[x;y]$ spaces. Those are all the polynomials killed by the diagonally invariant operators with vanishing constant term. For Haiman that liked quotients and the user unfriendly "macaulay" (a Groebner bases software), the object of study was the diagonal co-invariants. Me in San Diego and Haiman in Boston began accumulating properties of this remarkable bi-graded module. A variety of truly mesmerizing conjectures started to emerge. The very first two was that the multiplicity of the alternating was the catalan number and the dimension of the space was $(n+1)^{n-1}$. Mark Haiman got a lot of help in Boston translating experimental findings into all kinds of conjectures. For instance it was Gessel that suggested the Parking Function space might provide a natural combinatorial setting for diagonal harmonics. In fact, its natural action of S_n only had to be twisted by the sign representation to give the same representation resulting from the action of S_n on $DH_n[x;y]$. It was Macdonald himself (that was in Boston at that time) that observed that the Frobenius of that action gave the coefficients of the compositional inverse of the formal series $F(z) = z/E(z)$ (with $E(z)$ the generating function of the elementaries). Most of the conjectures gathered in the 1990-91 period appeared in [5]. I discovered what came by the name "*$n!/2$ conjecture*" and an area we referred to as "*Science fiction*" gravid with conjectures that are still open. The $n!/2$ problem arises as follows. Say you have a partition $\mu \vdash n$ and two partitions $\alpha, \beta \vdash n-1$ obtained by removing any two corners of μ then computer data show that the intersection $\mathcal{H}_\alpha[x;y]$ with $\mathcal{H}_\beta[x;y]$ has dimension $n!/2$. That led me to my first (1990) false proof of the $n!$ conjecture. I was sure I could prove it using the "flip" map since Δ_α-flipping $\mathcal{H}_\alpha[x;y] \cap \mathcal{H}_\beta[x;y]$ sends this subspace into its complement in $\mathcal{H}_\alpha[x;y]$. Science fiction arises when we start studying the various intersections obtained by using more than two corners of μ. The Science Fiction conjectures are stated in [7]. Apparently there is something finer than the Hilbert scheme underlying Science Fiction since Mark was not able to explain this phenomenon. After several approaches to proving the $n!$ conjecture had run into granite walls, in 1992 I decided that we must consult Procesi for advice on how to attack this problem. Since I wanted Mark to be present I dragged him to Rome on the way to a 1992 Mittag Leffler meeting so that we could consult with the Grand Master of Italian Algebraic Combinatorics. In this historical encounter Mark Haiman asked Procesi, "What is true for two sets of variables and is false for three sets?" This was prompted from the fact that the $n!$ conjecture was blatantly false in three sets. Procesi's instantaneous answer was,"The Hilbert scheme. It is smooth for two and not for three," and at the end of our meeting he added, "Let me think of this, there may be a connection." The next day at our second and last meeting before departing for Sweden he totally MESMERIZED (!!!!) us by outlining a grand scheme for proving the $n!$ conjecture by a limiting process based on the algebraic geometry of the Hilbert scheme. Mark (nor I) knew any algebraic geometry at that time (I still don't). But he initiated his education into the subject that took him about 8 years before he could even think of realizing Procesi's grand scheme. He did eventually [13]. But in the meantime throughout the 1990's we made significant advances in the symmetric function part of (modified) Macdonald polynomial theory and the combinatorial part as well [6],[8],[9],[10], [11]]. The most significant step took place in (1995) by the discovery that the Attiah-Bott theorem gave an explicit expression for the bigraded Frobenius Characteristics the Diagonal Harmonics. That led to the discovery of the "*Nabla*" operator. Let me tell you how all that happened since it is kind of funny. Macdonald was visiting UCSD at that time and one day he asked Mark why he did not try to find out what the Attiah-Bott formula applied to the Hilbert Scheme gave concerning the module $DH_n[x;y]$. Mark replied that he could not prove that it was applicable.

Macdonald retorted, "Never mind use it anyway!" A day later Mark came with the formula! [6] It was only a rational expression and yet we knew it had to be a Schur positive symmetric polynomial. But it was beautiful! We quickly were able to derive from it many of the conjectures that had been formulated by the Combinatorial Mafia in 1990. But the great surprise was that I noted that the expansion of e_n in terms of the modified Macdonalds differed from the Attiah-Bott formula only by the coefficient $T_\mu = t^{n(\mu)}t^{n(\mu')}$ multiplying $\tilde{H}_\mu[X;q,t]$ (!!!!). Francois Bergeron suggested that the modified Macdonald eigen-operator that had those eigenvalues ought to be an interesting object of study since it apparently had such magic power of changing e_n into the bigraded Frobenius characteristic of the Diagonal Harmonics. That is how ∇ was born [7]. Nabla magic exploded! Francois quickly discovered a variety of Schur positivity conjectures and I noted that Nabla could translate many of my Science fiction discoveries into Nabla magics. A sea of discoveries and conjectures (Symmetric Function Theoretical, Representation Theoretical and Combinatorial) ensued most of which are still open [8],[9],[10]. One project I spent in an unbelievable amount of time was trying to prove the Catalan dimension conjecture for the Diagonal Harmonic alternants. We knew that combinatorially the Hilbert series of this submodule, which we denoted $C_n(q,t)$ and called the q,t-Catalan, had to be obtained by a q,t-enumeration of Dyck paths. That is there had to be two statistics $a(D)$, $b(D)$ giving

$$C_n(q,t) = \sum_{D\in\mathcal{D}_n} t^{a(D)}q^{b(D)} \tag{A.10}$$

and we also knew that both statistics had to have the distribution of the "area" statistic of Dyck paths. Here I must digress a bit about a manner of depicting Parking Functions which I created and in retrospect was perhaps the very best idea I had in this subject. Let me tell you how I got to use such a representation of Parking Functions in [6] . In the original paper of Konheim-Weiss [1], a "*Parking Function*" is defined as a *Preference Function that parks the cars.* To be more detailed here we have

(1) a one-way street with n parking spaces

(2) n cars arrive and enter the street in succession,

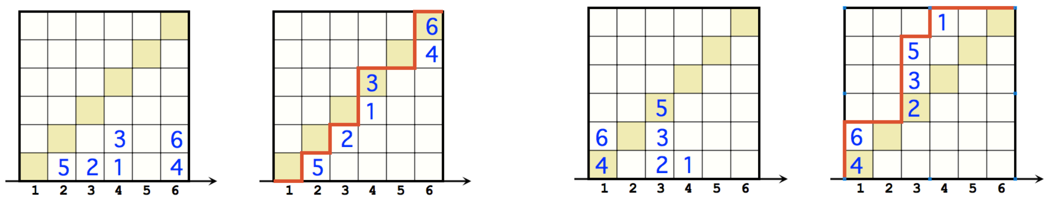


Figure A.3: Forming the parking functions.

(3) Each driver has a preferred parking space.

(4) When they reach the preferred place they park if it is free otherwise park in the first available space.

It is clear that a necessary condition that must be satisfied for all the cars to be able to park is that for any $1 \leq k \leq n$ the number of drivers that prefer to park in the first k places is at least k. *The miracle is that this condition is also sufficient.* Mathematically, we have a map

$$f = (f_1, f_2, \ldots, f_n) \, : \, [1, n] \to [1, n]$$

Denoting by $f'_1 \leq f'_2 \leq \cdots \leq f'_n$ the weakly increasing rearrangement of

$$f_1, f_2, \ldots, f_n.$$

The preference function parks the cars if and only if

$$f'_k \leq k \qquad \text{for all } k \in [1, n]$$

In most combinatorial texts this is how they define a parking function. But I am visual and I needed to "see" a parking function. So I depicted a preference function as increasing a piles of cars stacked on top of their preferred parking spaces as in the Figure A.3 to the left. Then to visualize whether it satisfied the necessary and sufficient condition I shifted the piles upwards so that the bottom each pile was at level of the top of pile that preceded it, as in the figure on its immediate right. to complete the picture we add a lattice (red) path surrounding the shifted piles. Now we see that the preference function parks the cars if and only if the path remains weakly above the (yellow) diagonal. This is easily identified with a "Dyck" path. Now we also see that the extra driving that the cars end up doing in addition to what they preferred to do is given by the number of lattice cells between the path and the (yellow) diagonal. From the start we chose the statistic $a(D)$ in (10) to be this "area". The search for the b statistic went on for nearly 6 years. Innumerable candidates were emailed to me periodically. Most of them worked up to $n = 4, 5$ but failed immediately after. When an April day of the year 2000 Jim Haglund showed up in my office with the bounce statistic. The miracle had happened! Computer data quickly confirmed the bounce up to $n = 9$ (that is a s far as we could compute $C_n(q,t)$ at that time). Mark Haiman was visiting Berkeley at that time so I sent him the news. By return mail we got another miracle (not to be out-done) he had concocted a competing statistic the "dinv" for " *diagonal inversions*" from his recently acquired Hilbert scheme intuition. We quickly verified (up to $n = 9$) that

$$C_n(q,t) = \sum_{D\in\mathcal{D}_n} t^{bounce(D)} q^{area(D)} = \sum_{D\in\mathcal{D}_n} t^{area(D)} q^{dinv(D)} \tag{A.11}$$

We quickly also geometrically established that the three statistics area, bounce and dinv had precisely the same distribution, but the symmetry

$$C_n(q,t) = C_n(t,q) \tag{A.12}$$

which is obvious from the diagonal harmonics point of view still remains a mystery to this date. It is at that time Remmel proposed his very promising PhD student Nick Loehr to find a combinatorial proof of (A.12). In retrospect this was not a bad idea since Loehr with the tenacity of of pit-bull created, in his attempts at proving (A.12), a vast machinery that was eventually conducive to his contributing substantial progress in the combinatorial part of this entire adventure. In the June of 2000 an intense daily joint effort (at the La Jolla Shores beach) with Haglund providing his exceptional combinatorial intuition and me providing my, by then quite vast apparatus of symmetric function identities accumulated in years of Macdonald polynomial manipulatorics) we succeeded in proving (A.11). By the end of this beach adventure Haglund acquired enough of a Macdonald polynomial tool kit to, very shortly after, in his truly outstanding "Shroeder" paper, totally out-do his teacher!!!. The dinv statistics was soon, by joint efforts of Loehr and Haglund extended to Parking Functions to conjecture the $DH_n(x;y)$ Hilbert series formula

$$F_{DH_n}(q;t) = \sum_{PF \in \mathcal{PF}_n} t^{area(PF)} q^{dinv(PF)} \tag{A.13}$$

and finally later by a multitude of efforts the "Shuffle conjecture" was finally brought to life. The word "shuffle" due to the original conjectured identity being

$$\langle \nabla e_n, h_{\mu_1} h_{\mu_2} \mathcal{D}ots h_{\mu_l} \rangle = \sum_{PF \in \mathcal{PF}} t^{area(PF)} q^{dinv(PF)} \mathcal{H}i(\sigma(PF) \in E_1 \sqcup\!\sqcup E_2 \sqcup\!\sqcup \mathcal{D}ots \sqcup\!\sqcup E_l) \tag{A.14}$$

where $\sigma(PF)$ is the permutation obtained by reading the cars from right to left by diagonals starting from the highest diagonal, the symbol "$\sqcup\!\sqcup$" denotes 'shuffling" and $E_1, E_2, \ldots, E_l$ are successive intervals of the word $123 \ldots n$ of lengths $\mu_1, \mu_2, \ldots, \mu_l$. I will stop here in recounting this saga since I rather prefer to make history that writing it. The Macdonald polynomial saga in fact continued to expand in the most fascinating way to this day and from the recent work of Gorsky-Negut it will undoubtedly continue for decades to follow.

A.1.1 Some suggestions about what to read

The Mark Haiman paper [5] should be the first to read, since although it appeared in 1994 it contains most of the results and conjectures that were derived in 1990-91. I prompted Mark to write all that up since it was full of very interesting conjectures. My contribution to that paper was the definition of the Diagonal Harmonics, and the so called "operator Conjecture" which states that the whole module of Diagonal Harmonics can be obtained by derivatives of the ordinary Vandermonde determinant followed by successive applications of the operators $D_r = \sum_{i=1}^{n} y_i \partial_{x_i}^r$. All my work on "Science Fiction" is not included there and appeared much later in the paper [7] with Francois. This is the paper that is usually quoted as the one where Nabla was created. This paper is full of conjectures that are still open to this date. Another paper that is full of conjectures that are still open is the "Lattice diagram polynomials" paper [10]. However the next paper to read is the one with Mark Haiman: *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, that contains the embryo of many results that were proved or conjectured later.

A.1.2 History of Tesler matrices

Tesler matrices were introduced but not directly published by Glenn Tesler to carry out on the computer experimentation with the $\tilde{H}_\mu[X;q,t]$ form of the higher index Macdonald Operators. We had done D_0 that has the eigenvalue $1-(1-t)(1-q)B_\mu(q,t)$ and I asked Glenn to carry that out for all the other Macdonald operators that appeared in the original paper. The resulting "Plethystic" form or "vertex Form" of these operators are stated and proved in [8] "Identities and positivity." Chapter 5. Theorem 5.1. Glenn did make public some statistic on his matrices by sending some enumerating sequences to the Sloane Encyclopedia. This was very fortunate since when Haglund searched in the Encyclopedia for a sequence he had obtained by computer experimentation while working on the research that led to his "... Hilbert series... " paper [19] he hit on Tesler item and thus brought Tesler matrices back to life! ' Thank you very much Neal Sloane for creating such a powerful research tool!!! One of the amazing coincidences of this story is that I noticed that the Shuffle Algebra that Andrei uses had its origin precisely from the same need: *giving a "vertex" form to the Macdonald operators!!* I included in the bibliography the work of Guoce Xin because of his powerful partial fraction tools for computing constant terms. These tools are precisely what permits to pass directly from constant term expressions to expansion in terms of standard tableaux.

A.1.3 Bibliography

(In chronological order, more or less)

[1] A. G. Konheim and B. Weiss, *An occupancy discipline and applications*, SIAM J. Applied Math. 14 (1966), 1266—1274.

[2] I. G. Macdonald, *A new Class of Symmetric Functions*, (1988) Publ. I.R.M.A. Strasbourg, Actes 20e Seminaire Lotharingien, 131—171.

[3] A. Garsia and Procesi, *On certain graded S_n-modules and the q-Kotska polynomials*, Adv. Math. **94** (1992), 82—138.

[4] A. Garsia and M. Haiman, *A graded representation module for Macdonald polynomials*, Proc. Nat. Acad. Sci. USA **90** (1993) 3607-3610. (Proofs of results announced in this paper appeared in the Foata Festschrift: Electronic J. Combin. 3 (1996), no. 2, Research Paper 24,)

[5] M. Haiman, *Conjectures on the quotient ring by diagonal invariants* J. Algebraic Combin. 3 (1994), no. 1, 17—76

[6] A.M. Garsia and M. Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combin. 5 (1996), no. 3, 191—244.

[7] F. Bergeron and A. M. Garsia, *Science fiction and Macdonald polynomials*, Algebraic methods and q-special functions (Montreal, QC, 1996), CRM Proc. Lecture Notes, vol. 22, Amer. Math. Soc., Providence, RI, 1999, pp. 1 —52.

[8] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler, *Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions*, Methods in Appl. Anal. 6 (1999), 363—420.

[9] A. Garsia, M. Haiman and G. Tesler, *Explicit Plethystic Formulas for the Macdonald q,t-Kostka Coefficients*, Séminaire Lotharingien de Combinatoire, B42m (1999), 45 pp.

[10] F. Bergeron, N. Bergeron, A. M. Garsia, M. Haiman and G. Tesler, *Lattice diagram polynomials and extended Pieri rules*, Advances in Math. 142 (1999), 244 —334.

[11] M. Zabrocki, UCSD Advancement to Candidacy Lecture Notes, Posted in http://www.math.ucsd.edu/~garsia/somepapers/.

[12] A. M. Garsia and J. Haglund, A proof of the q,t-Catalan positivity conjecture. Discrete Mathematics, 256 (2002), 677 —717.

[13] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14 (2001), 941 —1006.

[14] J. Haglund and Nick Loehr, *A conjectured combinatorial formula for the Hilbert series for diagonal harmonics*, Discrete Math. (Proceedings of FPSAC 2002 conference held in Melbourne, Australia.) 298 (2005), pp. 189 —204.

[15] J. Haglund, *A proof of the q,t-Schröder conjecture*, Internat. Math. Res. Notices 11 (2004), 525 —560.

[16] J. Haglund, M. Haiman, N. Loehr, J. Remmel and A. Ulyanov, *A combinatorial formula of the diagonal coinvariants*, Duke Math. J., **126** (2005), pp. 195 —232.

[17] G. Xin, *The ring of Malcev-Neumann series and the residue theorem*, PhD. Dissertation at Brandeis University (May 2004).

[18] G. Xin, *A fast algorithm for MacMahon's partition analysis*, Electron. J. Combin., 11 (2004), R58. (electronic)

[19] J. Haglund *A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants*, Adv. Math. 227 (2011), 2092 —2106.

[20], D. Armstrong, A. Garsia, J. Haglund, B. Rhoades, and B. Sagan, *Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics* , Journal of Combinatorics, 3 (2012), 451—494.

[21] A. M. Garsia, J. Haglund and G. Xin, *Constant term methods in the theory of Tesler matrices and Macdonald polynomial operators* , Annals of Combinatorics, to appear.

Bibliography

[BG99] F. Bergeron and A. M. Garsia. Science fiction and Macdonald's polynomials. In *Algebraic methods and q-special functions (Montréal, QC, 1996)*, volume 22 of *CRM Proc. Lecture Notes*, pages 1–52. Amer. Math. Soc., Providence, RI, 1999.

[DGZ] A. Duane, A. Garsia, and M. Zabrocki. A new dinv arising from the two part case of the shuffle conjecture. *Journal of Algebraic Combinatorics*, pages 1–33. http://dx.doi.org/10.1007/s10801-012-0382-0.

[Ges84] Ira M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[GH96a] A. M. Garsia and M. Haiman. A remarkable q,t-Catalan sequence and q-Lagrange inversion. *J. Algebraic Combin.*, 5(3):191–244, 1996. http://dx.doi.org/10.1023/A:1022476211638.

[GH96b] A. M. Garsia and M. Haiman. Some natural bigraded s_n-modules and q,t-kostka coefficients. In *electronic), The Foata Festschrift, http://www.combinatorics.org/Volume 3/volume3 2.html#R24*, page pp., 1996.

[GH02] A. M. Garsia and J. Haglund. A proof of the q,t-Catalan positivity conjecture. *Discrete Math.*, 256(3):677–717, 2002. http://dx.doi.org/10.1016/S0012-365X(02)00343-6.

[GP92] A. M. Garsia and C. Procesi. On certain graded S_n-modules and the q-Kostka polynomials. *Adv. Math.*, 94(1):82–138, 1992.

[GXZ12a] A. M. Garsia, G. Xin, and M. Zabrocki. A three shuffle case of the compositional parking function conjecture. *ArXiv e-prints*, August 2012.

[GXZ12b] A. M. Garsia, G. Xin, and M. Zabrocki. Hall-Littlewood operators in the theory of parking functions and diagonal harmonics. *Int. Math.*

Res. Not. IMRN, (6):1264–1299, 2012. http://imrn.oxfordjournals.org/content/early/2011/04/28/imrn.rnr060.abstract.

[Hag04] J. Haglund. A proof of the q,t-Schröder conjecture. *Int. Math. Res. Not.*, (11):525–560, 2004. http://dx.doi.org/10.1155/S1073792804132509.

[Hai94] Mark D. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3(1):17–76, 1994.

[Hai01a] Mark Haiman. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006 (electronic), 2001. http://dx.doi.org/10.1090/S0894-0347-01-00373-3.

[Hai01b] Mark Haiman. Vanishing theorems and character formulas for the hilbert scheme of points in the plane. *Invent. Math*, 149:371–407, 2001.

[HHL05a] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18(3):735–761 (electronic), 2005. http://dx.doi.org/10.1090/S0894-0347-05-00485-6.

[HHL+05b] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke Math. J.*, 126(2):195–232, 2005. http://dx.doi.org/10.1215/S0012-7094-04-12621-1.

[Hic12] Angela S. Hicks. Two parking function bijections: A sharpening of the q,t-Catalan and Shröder theorems. *International Mathematics Research Notices*, 2012(13):3064–3088, 2012. http://imrn.oxfordjournals.org/content/2012/13/3064.abstract.

[HK13] Angela Hicks and Yeonkyung Kim. An explicit formula for ndinv, a new statistic for two-shuffle parking functions. *Journal of Combinatorial Theory, Series A*, 120(1):64 – 76, 2013. http://www.sciencedirect.com/science/article/pii/S0097316512001306.

[HL05] J. Haglund and N. Loehr. A conjectured combinatorial formula for the Hilbert series for diagonal harmonics. *Discrete Math.*, 298(1-3):189–204, 2005. http://dx.doi.org/10.1016/j.disc.2004.01.022.

[HMZ12] J. Haglund, J. Morse, and M. Zabrocki. A Compositional Shuffle Conjecture Specifying Touch Points of the Dyck Path. *Canad. J. Math.*, 64(4):822–844, 2012. http://dx.doi.org/10.4153/CJM-2011-078-4.

[KW66] A. Konheim and B. Weiss. An occupancy discipline and applications. *SIAM Journal on Applied Mathematics*, 14(6):1266–1274, 1966.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Rio69] John Riordan. Ballots and trees. *Journal of Combinatorial Theory*, 6(4):408 – 411, 1969.