Title
Essays on Nonparametric Identification and Estimation of All-Pay Auctions and Contests

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Publication Date
2019

UNIVERSITY OF CALIFORNIA

Los Angeles

Essays on Nonparametric Identification and Estimation of All-Pay Auctions and Contests

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Economics

by

Ksenia Shakhgildyan

2019

ABSTRACT OF THE DISSERTATION

Essays on Nonparametric Identification and Estimation
of All-Pay Auctions and Contests

by

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Doctor of Philosophy in Economics
University of California, Los Angeles, 2019
Professor Rosa Liliana Matzkin, Co-Chair
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My dissertation contributes to the structural nonparametric econometrics of auctions and contests with incomplete information. It consists of three chapters.

The first chapter investigates the identification and estimation of an all-pay auction where the object is allocated to the player with the highest bid, and every bidder pays his bid regardless of whether he wins or not. As a baseline model, I consider the setting, where one object is allocated among several risk-neutral participants with independent private values (IPV); however, I also show how the model can be extended to the multiunit case. Moreover, the model is not confined to the IPV paradigm, and I further consider the case where the bidders' private values are affiliated (APV). In both IPV and APV settings, I prove the identification and derive the consistent estimators of the distribution of the bidders' valuations using a structural approach similar to that of Guerre et al. (2000). Finally, I consider the model with risk-averse bidders. I prove that in general the model in this set-up is not identified even in the semi-parametric case where the utility function of the bidders is restricted to belong to the class of functions with constant absolute risk aversion (CARA).

The second chapter proves the identification and derives the asymptotically

normal estimator of a nonparametric contest of incomplete information with uncertainty. By uncertainty, I mean that the contest success function is not only determined by the bids of the players, but also by the variable, which I call uncertainty, with a nonparametric distribution, unknown to the researcher, but known to the bidders. This work is the first to consider the incomplete information contest with a nonparametric contest success function. The limiting case of the model when there is no uncertainty is an all-pay auction considered in the first chapter. The model with two asymmetric players is examined. First, I recover the distribution of uncertainty using the information on win outcomes and bids. Next, I adopt the structural approach of Guerre et al. (2000) to obtain the distribution of the bidders' valuations (or types). As an empirical application, I study the U.S. House of Representatives elections. The model provides a method to disentangle two sources of incumbency advantage: a better reputation, and better campaign financing. The former is characterized by the distribution of uncertainty and the latter by the difference in the distributions of candidates' types. Besides, two counterfactual analyses are performed: I show that the limiting expenditure dominates public campaign financing in terms of lowering total campaign spending as well as the incumbent's winning probability.

The third chapter is a semiparametric version of the second chapter. In the case when the data is sparse, some restrictions on the nonparametric structure need to be put. In this work, I prove the identification and derive the consistent estimator of a contest of incomplete information, in which an object is allocated according to the serial contest success function. As in previous chapters, I recover the distribution of the bidders' valuations from the data on observed bids using a structural approach similar to that of Guerre et al. (2000) and He and Huang (2018). As a baseline model, I consider the symmetric contest. Further, the model is extended to account for the bidders' asymmetry.

The dissertation of Ksenia Shakhgildyan is approved.

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2019

To my family

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ACKNOWLEDGEMENTS

I am indebted to my advisor, Rosa Liliana Matzkin. Without her guidance, support, encouragement, and patience, it would not be possible for me to write this thesis. Her door was always open for me every week, all the years I spent at UCLA. I am very grateful to my co-chair John Asker for his advice and always constructive and supportive critique. I would like to thank Shuyang Sheng for being always willing to listen and to talk, for being open to share her knowledge and experience with me. I thank Denis Chetverikov for very valuable discussions every time we had a conversation. I am grateful to Andres Santos for his feedback and suggestions, to Jinyong Hahn, Zhipeng Liao, and other members of Econometrics proseminar for their valuable comments. I thank Sushil Bikhchandani for serving on my committee. I thank Moshe Buchinsky for his constant support and belief in me. Special thank you to Gary Jacobson who has gratefully shared with me the data on Congressional Elections.

I also thank Jesper Soerensen, El Hadi Caoui, Renato Giroldo, Vitaly Titov and Konstantin Platonov for discussions and support; Alex Fon, Akina Ikudo and Byeonghyeon Jeong for making our office a place where you want to come every day; Diana Van Patten, Rustin Partow, Jon Gu for being great classmates; Ruslan Sverchkov for all the support when it was most needed; Daria Ostrovskaia and Dinara Akchurina for being by my side even being far away; Polina Abatcheva and Grigory Franguridi for being my closest friends in LA during my Ph.D. I thank Stefano Fiorin for being there for me every day.

Finally, this whole journey won't be possible without my family. Their support was unconditional during all my life and them I owe the most.

VITA

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CHAPTER 1

Nonparametric Identification and Estimation of All-Pay Auctions

1.1 Introduction

Nonparametric analysis of auction data is a widely discussed topic. For an extensive review see Athey and A. Haile (2007). The contribution of this paper is the identification and estimation of an all-pay auction. The object is allocated to the player with the highest bid, and every bidder pays his bid regardless of whether he wins or not. There are several reasons why the examining of this auction format is of interest. First, the underlying structural model is used to describe the players' behavior in many scenarios, in which the assumption that only the person who wins needs to pay his bid (as in case of the first-price auction) seems to be restrictive. For instance, the all-pay auction has been used to model elections, different kinds of contests and sports events, research and development as well as rent-seeking activity, such as lobbying (see Baye et al. (1993)). The other reason for considering this auction format is that as theoretical (see Krishna and Morgan (1997)) and experimental (see Noussair and Silver (2006)) results indicate, it raises greater revenue than the first-price auction and thus is good from the seller's perspective, although it is rarely used in real-world situations.

In this work, I prove the identification and derive the consistent estimator of an all-pay auction. I find the distribution of the bidders' valuations from the data on bids using a structural approach similar to that of Guerre et al. (2000). As a baseline model, I consider the setting where one object is allocated among several risk-neutral participants with independent private values (IPV). However, I also show how the model can be extended to the setting where $M > 1$ objects are distributed through the auction (see Barut et al. (2002) for the same set-up).

Moreover, the model is not confined to the IPV paradigm and can account for the case in which the bidders' private values are affiliated (APV). Laffont and Vuong (1996) show that "for any given level of competition any symmetric AV (affiliated values) model is observationally equivalent in terms of bids to some symmetric APV model". Thus, any CV (common values) model is equivalent observationally to some APV model while IPV setting is a particular case of the APV, making the APV setting the most general.

The estimation procedure (similar to that of Li et al. (2002)) makes it possible to obtain the joint distribution of private valuations from the observed bids. This distribution provides information to test the IPV assumption and to consider different policy interventions.

Finally, I consider the model with risk-averse bidders. As experimental studies of all-pay auctions show (see Fibich et al. (2006) and Barut et al. (2002)) the bidder's behavior is consistent with the risk-averse utility function. However, I prove that in general the model in this set-up is not identified even in the semi-parametric case where the utility function of the bidders is restricted to belong to the class of functions with constant absolute risk aversion (CARA).

The rest of the paper is organized as follows. In Section 2 I introduce notations and definitions used throughout the paper. Section 3 discusses the identification and estimation as well as the Monte Carlo simulations of the IPV model. Section 4 considers the APV setting. Similarly, identification analysis, estimation procedure as well as the Monte Carlo simulations are presented. Section 5 discusses the IPV setting with risk-averse bidders and proves the nonidentification result. Section 6 concludes.

1.2 Notations and Definitions

In this work, an all-pay auction model with N symmetric bidders is considered. Every bidder observes some private information with cumulative distribution function (CDF) $F \in \mathcal{F}$. Let $\mathcal{G}$ denote the set of all possible distributions of bids. Let us call the mapping from the private information to bids $\gamma \in \Gamma$, where $\gamma : \mathcal{F} \to \mathcal{G}$.

Nonparametric identification means that the econometrician can recover the distribution of private information from the observed bids. Formally,

Definition 1.1. *(Identification). A model* $(\mathcal{F}, \Gamma)$ *is identified if for every* $(F, F') \in \mathcal{F}^2$ *and* $(\gamma, \gamma') \in \Gamma^2$, $\gamma(F) = \gamma'(F') \Rightarrow (F, \gamma) = (F', \gamma')$.

There exist various specifications of the auctions models. The setting where each bidder observes his valuation of the good but not the values of the rest of the players is called the private value model. In contrast, when all bidders receive correlated signals about the value, the common values model is considered. Other dimensions are whether the bidders are symmetric or asymmetric and whether bidders' information is independent or affiliated (see Athey and Haile (2002)).

Let us denote by v_i the bidder i's private information (or type), $v = (v_1, ..., v_N)$. N is the number of (potential) bidders. The payoff of each bidder if he obtains one unit is represented by $U_i = u(v_i, V)$, where V is the common payoff component. It is further assumed that the utility function $u(\cdot)$ is continuous, non-negative, increasing in each argument, and common across bidders. Bidders might be risk-neutral or risk-averse. Here F denotes the joint cumulative distribution function of $(v_1, ..., v_N, V)$. This function is assumed to be symmetric in v_i (exchangeability). F, N and u are common knowledge. Thus bidders play the game of incomplete information.

Definition 1.2. *Bidders have* ***private values*** *if*

$$E[u(v_i, V)|(v_1, ..., v_N)] = E[u(v_i, V)|v_i] \quad \forall v_{-i}, U_i$$

In the private values setting we can distinguish between two cases, namely independent and affiliated values.

Definition 1.3. *The private values are **independent** if*

$$f_{v_i,v_j} = f_{v_i} f_{v_j},$$

where $f(\cdot)$ is the marginal distribution of the private signal.

Based on Milgrom and Weber (1982) the affiliation means the following:

Definition 1.4. *For variables with densities it is said that they are **affiliated** if for all v and $\hat{v}$*

$$f(v \vee \hat{v}) f(v \wedge \hat{v}) \geq f(v) f(\hat{v}),$$

where $\vee$ denotes the component-wise maximum and $\wedge$ denotes the component-wise minimum.

Affiliation means, that the bigger is the realization of one's value, the more likely it is that the other's value is also big.

1.3 IPV All-Pay Auction with Risk-Neutral Bidders

1.3.1 Model

I first focus on the IPV environment with N risk-neutral players and M identical goods. In this case:

Assumption 1.1. $u(v_i) = v_i$, $i = 1, ..., N$.

Assumption 1.2. *Each bidder draws a value v_i independently from a commonly known distribution $F(v)$ with support $[\underline{v}, \bar{v}]$.*

All the bidders are ex-ante symmetric. Here v_i is the private value for bidder i of possessing the good.

Assumption 1.3. *The bidders submit the bids b_i simultaneously knowing N, M, v_i and $F(v)$.*

Thus the distribution function $F(\cdot)$ is a common knowledge, while the valuations of other players are not observed, which makes the setting a game of incomplete information.

Assumption 1.4. *Each of N bidders pays b_i, regardless of whether or not he obtains a good.*

Assumption 1.5. *N bids are ordered from highest to lowest and all M highest bidders receive a good. If there is a tie for the M-th object, a lottery takes place and each of the bidders gets the object with equal probability.*

Therefore the bidder i's resulting payoff is $v_i - b_i$ if he obtains a good, and $-b_i$ otherwise. In expectation then the payoff to bidder i is:

$$E[U_i|v_i, v_{-i}] = v_i P[win|b_i, N, M, F(v)] - b_i$$

where $P[win|b_i, N, M, F(v)]$ is the probability that b_i is one of the M highest bids.

Following the literature, I consider the Bayesian equilibrium of this incomplete information game, which is strictly monotonic and symmetric (the existence can be proved as in Krishna and Morgan (1997)). For each valuation the corresponding bid is defined by the function $s(v) = b$. Since $s(v)$ is strictly monotonic it is invertible.

Given Assumptions 1.1-1.5, the win probability can be written as:

$$P[win|b_i, N, M, F(v)] = \sum_{j=N-M}^{N-1} \frac{(N-1)!}{(N-j-1)!j!} F(v_i)^j (1 - F(v_i))^{N-j-1}.$$

Given the winning probability I proceed to find the equation that characterizes the equilibrium.

Proposition 1.1. *Given Assumptions 1.1-1.5 and M=1, there exists a strictly increasing symmetric Bayesian equilibrium of the game described above:*

$$b_i = s(v_i, F, N) = (N-1)\int_{\underline{v}}^{v_i} v f(v) F(v)^{N-2} dv. \tag{1.1}$$

The first-order condition of this game can be written as:

$$v_i = \frac{s'(v_i)}{f(v_i)F(v_i)^{N-2}(N-1)}. \tag{1.2}$$

Proof: If the bid b corresponds to valuation v, $b = s(v)$, the winning probability is

$$P[win|b, N, M, F(v)] = F(v)^{N-1}$$

Therefore expected utility of a bidder whose valuation is v_i, but who bids as if his valuation was v is:

$$V(v_i, v) = v_i F(v)^{N-1} - s(v).$$

Using the First order condition (differentiating with respect to v and substituting $v = v_i$), we get:

$$0 = v_i(N-1)F(v_i)^{N-2} f(v_i) - s'(v_i).$$

From this differential equation we obtain the value v_i:

$$v_i = \frac{s'(v_i)}{f(v_i)F(v_i)^{N-2}(N-1)}.$$

It follows that the equilibrium strategy is

$$b_i = s(v_i, F, N) = (N-1)\int_{\underline{v}}^{v_i} v f(v)F(v)^{N-2}dv.$$

$\square$

The case M=1 was considered for simplicity of the presentation. It can be easily generalized for any M.[1]

Usually, valuations are unobserved for the econometrician, whereas bids are observed in the data. Let us denote by $G(\cdot)$ the distribution of bids. The next section discusses how to recover the distribution of v_i from the distribution of bids $G(\cdot)$ using equation (1.2).

1.3.2 Nonparametric Identification

In this section, I prove the nonparametric identification of the IPV model.

In structural estimation, the first main question is whether the parameters of the economic model are identified from the available data or not. The distribution $F(\cdot)$ of bidders' valuations is the only unknown element for the econometrician. The number N of bidders, and the bids b_i, $i = 1, ..., N$ are observed. Therefore the question is whether there exists a distribution F corresponding to the observables and whether this function is unique.

The bids distribution $G(\cdot)$ depends on $F(\cdot)$ not only through v_i but also through the equilibrium strategy $s(\cdot)$. Thus for the successful identification, both F, as well as the equilibrium strategy should be canceled out once the bids distribution and density are plugged in into the first-order condition (1.2).

[1]For instance, for $M = 2$: $Pr[win|b, N, M, F(v)] = F(v)^{N-2}(N-1-(N-2)F(v))$. It follows that

$$v_i = \frac{s'(v_i)}{f(v_i)F(v_i)^{N-3}(N-1)(N-2)(1-F(v_i))}.$$

Let $\mathbf{G}(\cdot)$ denote the joint distribution of $(b_1, ..., b_N)$. Then the following proposition, analogous to Theorem 1 in Guerre et al. (2000)[2], holds:

Proposition 1.2. *Let $N \geq M$. Let $\boldsymbol{G}(\cdot)$ belong to the set of absolutely continuous probability distributions with support $[\underline{b}, \bar{b}]^N$. There exists an absolutely continuous distribution of bidders' valuations $F(\cdot)$ such that $\boldsymbol{G}(\cdot)$ is the distribution of the equilibrium bids in the all-pay auction with independent private values if and only if:*

1. $\boldsymbol{G}(b_1, ..., b_N) = \prod_{i=1}^{N} G(b_i)$.
2. *The function $\xi(\cdot, N, G) \equiv \frac{1}{g(b_i)G(b_i)^{N-2}(N-1)}$ is strictly increasing on $[\underline{b}, \bar{b}]$ and its inverse is differentiable on $[\underline{v}, \bar{v}] = [\xi(\underline{b}, N, G), \xi(\bar{b}, N, G)]$.*

Moreover, when $F(\cdot)$ exists, it is unique with support $[\underline{v}, \bar{v}]$ and satisfies $F(v) = G(\xi^{-1}(v, N, G))$ for all $[\underline{v}, \bar{v}]$. In addition, $\xi(\cdot, N, G)$ is the quasi inverse of the equilibrium strategy in the sense that $\xi(b, N, G) = s^{-1}(b, N, F)$ for all $b \in [\underline{b}, \bar{b}]$.

Proof: For any $b \in [\underline{b}, \bar{b}] = [s(\underline{v}), s(\bar{v})]$ it holds that $G(b) = Pr(b_1 \leq b) = Pr(v_1 \leq s^{-1}(b)) = F(s^{-1}(b)) = F(v)$, where $b = s(v)$. Thus the bids distribution $G(\cdot)$ has support $[s(\underline{v}), s(\bar{v})]$ and its density is $g(b) = \frac{f(v)}{s'(v)}$, where $v = s^{-1}(b)$.

This allows us to rewrite the differential equation (1.2) above in terms of the distribution of bids, that is

$$v_i = \frac{1}{g(b_i)G(b_i)^{N-2}(N-1)}. \tag{1.3}$$

As a result, we obtain the expression for private value v_i as a function of the bids b_i, its distribution $G(\cdot)$, its density $g(\cdot)$, and the number of bidders N. The rest follows from the proof of Theorem 1 in Guerre et al. (2000).

□

[2] I use the same notation as in Guerre et al. (2000) for convenience.

The next section discusses the estimation procedure.

1.3.3 Nonparametric Estimation

In this section, the consistent plug-in estimation for the bidder's valuation is proposed.

Given equation (1.3), the plug-in estimator is constructed in the following way. The first step is to estimate the bids distribution $G(\cdot)$ and density $g(\cdot)$ as using them the econometrician would be able to find the corresponding valuations, which in turn can be used to estimate the density function $f(\cdot)$. More precisely, as $G(b)$ is the marginal distribution of equilibrium bids in N-bidder auctions and $g(b)$ is the associated density, they can be estimated using kernel function as follows.

Consider L_N - the number of N-bidders auctions. I index by l the l-th auction and use the observations $\{b_{il}, i = 1, ..., N, l = 1, ..., L_N\}$ to find the nonparametric estimates of $G(\cdot)$ and $g(\cdot)$. Thus,

$$\hat{G}(b) = \frac{1}{L_N} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}(b_{il} \leq b), \tag{1.4}$$

$$\hat{g}(b) = \frac{1}{L_N h_g} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K_g\Big(\frac{b - b_{il}}{h_g}\Big), \tag{1.5}$$

where h_g denotes the bandwidth and K denotes the kernel function.

As a result, we can estimate v_i by plugging in the estimates $\hat{G}$ and $\hat{g}$ into equation (1.3).

Assumption 1.6. *The data on* $\{b_i\}$ *is i.i.d.*

Assumption 1.7. *The density* $g(b)$ *has compact support, is continuously differentiable of order* $m \geq \delta + k$, $k \geq 2$, *with derivatives which are uniformly bounded.*

Assumption 1.8. *The kernel function is of order δ, it has compact support and is continuously differentiable on its support.*

Assumption 1.9. *As $L \to \infty$, $h_g \to 0$, $\sqrt{Lh_g} \to \infty$, $\sqrt{Lh_g^{1+2k}} \to 0$, where $L = L_N * N$.*

Proposition 1.3. *Given the Assumptions 1.1-1.5 about the model as well as Assumptions 1.6-1.9 are satisfied the following is the consistent estimator of the valuation of player i in auction l:*

$$\hat{v}_{il} \xrightarrow{p} v_{il},$$

where:

$$\hat{v}_{il} = \frac{1}{\hat{g}(b_{il})\hat{G}(b_{il})^{N-2}(N-1)}. \tag{1.6}$$

These are the pseudo values.

Proof: In the Appendix. □

In the next step to estimate the density $f(\cdot)$ I use the pseudo-sample $\{\hat{v}_{il}, i = 1, ..., N, l = 1, ..., L_N\}$ and the kernel function:

$$\hat{f}(v) = \frac{1}{L_N h_f} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K_f\Big(\frac{v - \hat{v}_{il}}{h_f}\Big). \tag{1.7}$$

Here h_f is the bandwidth and K_f - kernel function.

Note that the invertibility of the bid function is the key thing for identification as I relied on the assumption that the bidders use a strictly increasing bid function.[3]

[3]It is possible to account for the case when the number of bidders is not known to the participants, but they receive a signal with known distribution.

1.3.4 Monte Carlo Simulation

In this section, a Monte Carlo study is conducted. It is similar to the one in Guerre et al. (2000) and describes the estimation procedure in detail.

I assume that the data on $L = 500$ auctions with $N = 2$ players taking part in them is given. The number of auctions and players per se do not change the estimation procedure as the bidders are assumed to be ex-ante symmetric. What plays a role in the estimation is the total number of observations, which is given by $L*N$. In this study, the true distribution function F of valuation is log-normal with parameters zero and one, truncated at 0.055 and 2.5 that leads to leaving out 20% approximately of the original log-normal distribution. 1000 Monte Carlo replications are conducted. Next, each replication is described.

To start with, $L * N$ observations of valuations are drawn randomly and the corresponding equilibrium strategies b_{il}, $i = 1, 2$, $l = 1, ..., L$ defined in equation (1.1) are calculated. After that, given the bids, the CDF is estimated using (1.4) and the bids' density function is estimated using (1.5). Specifically, I use the triweight kernel:

$$K(u) = \frac{35}{32}(1 - u^2)^3 \mathbb{1}(|u| \leq 1).$$

This is a kernel of order 2. The important property is that it has compact support and the kernel function is continuously differentiable on its support. There are many other kernels satisfying the above properties. In its turn, $h_g = 1.06\hat{\sigma_b}(NL)^{-1/5}$, where $\hat{\sigma_b}$ is the estimated standard deviation of the bids. The order is $L^{-1/5}$ as according to the Theorem 3 in Guerre et al. (2000) when the valuations are not observed but should be estimated by choosing the bandwidths $h_g = c_g(logL/L)^{1/(2R+3)}$ and $h_f = c_f(logL/L)^{1/(2R+3)}$, where R is the number of bounded continuous derivatives of $f(\cdot)$, the optimal convergence rate can be reached. In our case $R = 1$. Constant 1.06 is the result of the so-called rule of thumb (see Hardle (1991)).

Knowing the estimated distribution and density of the bids we are ready to estimate the valuations. The issue here is that the estimator of density g is biased on the borders of the support. More precisely on $[\underline{b}, \underline{b} + \rho_g h_g/2)$ and on $(\bar{b} - \rho_g h_g/2, \bar{b}]$, where ρ_g is the length of the support of the kernel. In our case $\rho_g = 2$. If we consider b_{min} to be the minimum of the observed bids and b_{max} the maximum of the observed bids, then the $\hat{g}$ is unbiased on $[b_{min} + \rho_g h_g/2, b_{max} - \rho_g h_g/2]$. Thus I trim the estimated valuations specified in (1.6):

$$\hat{v}_{il} = \begin{cases} \frac{1}{\hat{g}(b_{il})\hat{G}(b_{il})^{N-2}(N-1)}, \\ \quad \text{if } b_{min} + \rho_g h_g/2 \leq b_{il} \leq b_{max} - \rho_g h_g/2 \\ \infty, otherwise \end{cases} \tag{1.8}$$

The final step is the estimation of the density function of valuations $\hat{f}(\cdot)$ using (1.7). Here $h_f = 1.06\hat{\sigma_v}(NL_T)^{-1/5}$, L_T is the number of auctions that are left after the trimming and $\hat{\sigma_b}$ is the estimated standard deviation of $\hat{v}_{il}$. In each replication I estimate $\hat{f}(\cdot)$ at 500 equally spaced points on $[0.055, 2.5]$.

Figure 1.1 presents the true density of the truncated log-normal distribution, and for each value of v in the support, the mean of the 1000 estimates $\hat{f}(v)$, together with the 5% quantile, and the 95% quantile.

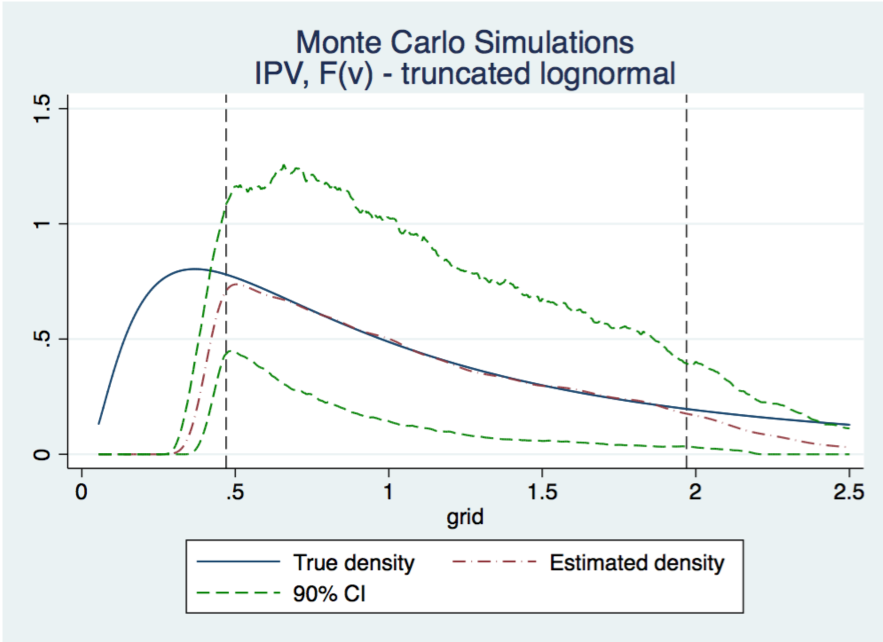


Figure 1.1: True and estimated densities of valuations in IPV model

The important result is that inside the interval marked by the vertical dashed lined defined by the average of $[s(b_{min} + h_g) + h_f, s(b_{max} - h_g) - h_f]$ the true density function is approximated by the mean of the 1000 Monte Carlo estimates almost perfectly. On the borders, the estimation is biased due to the bias of kernel estimators and trimming.

In addition in Figure 1.2 the true equilibrium strategy $b = s(v)$ is represented as well as for each $b \in [s(0.055), s(2.5)]$ the mean of the 1000 estimates $\hat{v}(b) = s^{-1}(b)$, together with the 5% quantile, and the 95% quantile.

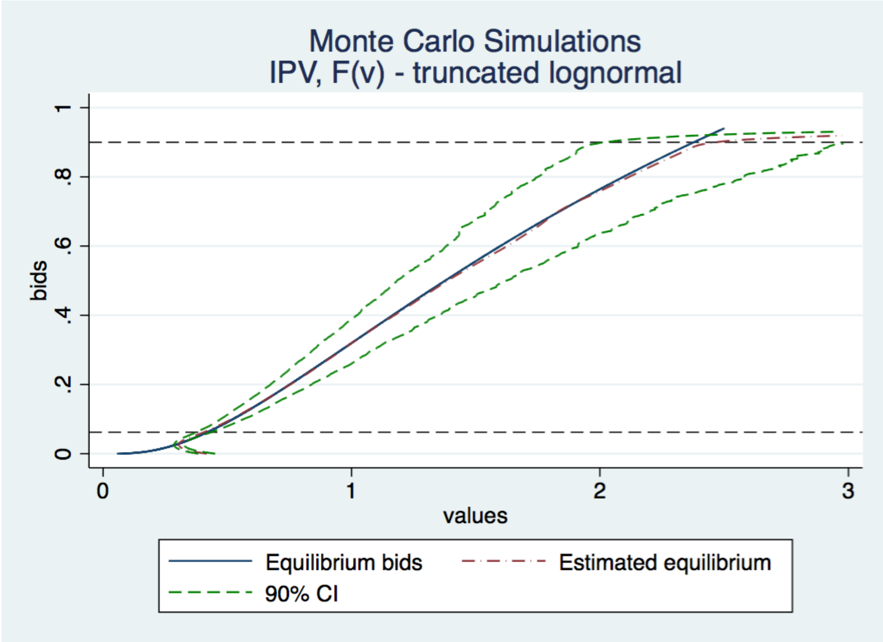


Figure 1.2: True and estimated equilibrium bids in IPV model

In this case inside the interval marked by the horizontal dashed lined defined by the average of $[b_{min} + h_g, b_{max} - h_g]$ the true equilibrium strategy is approximated by the mean of the 1000 Monte Carlo estimates almost perfectly. On the borders, the estimation is biased due to the bias of kernel estimators.

1.3.5 Observed Heterogeneity

The model can be extended to account for observed heterogeneity. Let N_l be the number of bidders in the l-th auction and X_l is the vector of observed characteristics. In this setting, the distribution of v_{il} for the l-th auction is the conditional distribution $F(\cdot|X_l, N_l)$ of valuations given (X_l, N_l). In its turn, the distribution of observed bids in the l-th auction is $G(\cdot|X_l, N_l)$. Thus

$$v_{il} = \frac{1}{g(b_{il}|X_l, N_l)G(b_{il}|X_l, N_l)^{N-2}(N-1)},$$

where

$$G(b|x,i) = \frac{G(b,x,i)}{f_l(x,i)}, \quad g(b|x,i) = \frac{g(b,x,i)}{f_l(x,i)}.$$

These ratios can be estimated using observations $\{(b_{il}, X_l, N_l, i = 1, ..., N_l, l = 1, ..., L\}$

$$\hat{G}(b,x,i) = \frac{1}{Lh_G^d} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbb{1}(b_{il} \leq b) K_G \left(\frac{x - X_l}{h_G}, \frac{i - N_l}{h_{GN}} \right),$$

$$\hat{g}(b,x,i) = \frac{1}{Lh_g^{d+1}} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} K_g \left(\frac{b - b_{il}}{h_g}, \frac{x - X_l}{h_g}, \frac{i - N_l}{h_{gN}} \right),$$

$$\hat{f}_l(x,i) = \frac{1}{Lh^d} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} K \left(\frac{x - X_l}{h}, \frac{i - N_l}{h} \right),$$

where h denotes the bandwidth and K denotes the kernel function.

As a result, we are able to estimate:

$$\hat{v}_{il} = \frac{1}{\hat{g}(b_{il}|X_l, N_l)\hat{G}(b_{il}|X_l, N_l)^{N-2}(N-1)}.$$

Next, using the pseudo sample $\{(\hat{v}_{il}, X_l), i = 1, ..., N_l, l = 1, ..., L\}$, we estimate nonparametrically the density $f(v|x)$ by $\hat{f}(v|x) = \frac{\hat{f}(v,x)}{\hat{f}(x)}$, where

$$\hat{f}(v,x) = \frac{1}{Lh_f^{d+1}} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} K_f \left(\frac{v - \hat{v}_{il}}{h_f}, \frac{x - X_l}{h_f} \right),$$

$$\hat{f}(x) = \frac{1}{Lh_x^d} \sum_{l=1}^{L} K_x \left(\frac{x - X_l}{h_x} \right),$$

where h denotes the bandwidth and K denotes the kernel function.

The procedure is very similar to the one before except for the fact that we condition of the observables and thus much more data is required.

1.4 APV All-Pay Auction with Risk-Neutral Bidders

1.4.1 Model

In this section, I consider the same set-up with the affiliated private values (APV).

Assumption 1.10. *Symmetric APV model is considered, thus all bidders are ex-ante identical. Each of the bidders knows the joint distribution of the valuations* $\boldsymbol{F}$.

The case of the first-price auction was considered in Li et al. (2002). The authors use the same idea as in Guerre et al. (2000) to make use of the kernel density estimators. As before I only consider Bayesian Nash equilibrium that is strictly increasing, differentiable and symmetric. At first, I consider just one unit of indivisible good for sale. Then the analysis will be extended to the case of M units in Section 4.5.

Each bidder i chooses a bid b_i to maximize his utility:

$$E[U_i|V_i = v_i, V_{-i} = v_{-i}] = v_i P[win|b_i, N, F(\cdot)] - b_i = v_i P[B_i \leq b_i|v_i] - b_i,$$

where $B_i = s(y_i)$, $y_i = \max_{j \neq i} v_j$, and $s(\cdot)$ is the equilibrium strategy.

Proposition 1.4. *Given Assumptions 1.1-1.5 and 1.10 are satisfied, as well as M=1, there exists the strictly increasing symmetric Bayesian equilibrium of the game described above:*

$$b_i = s(v_i) = \int_{\underline{v}}^{v_i} v \cdot f_{y_1|v_1}(v|v) dv. \tag{1.9}$$

The first-order condition of this game can be written as:

$$v_i = \frac{s'(v_i)}{f_{y_1|v_1}(v_i|v_i)}. \tag{1.10}$$

Proof: The expected utility of a bidder whose valuation is v_i, but who bids as if his valuation was v is:

$$V(v_i, v) = v_i \int_{\underline{v}}^{v} f_{y_1|v_1}(y|v)dy - s(v).$$

Using the First order condition (differentiating with respect to v and substituting $v = v_i$), we get:

$$0 = v_i \cdot f_{y_1|v_1}(v_i|v_i) - s'(v_i),$$

for all $v_i \in [\underline{v}, \bar{v}]$ such that $s(\underline{v}) = \underline{v}$. $f_{y_1|v_1}(\cdot|\cdot)$ is the notation for conditional density of y_1 given v_1. Here 1 is the index of any bidder as all of them are identical ex-ante. As a result, we get the following differential equation determining the bid function:

$$s'(v_i) = v_i \cdot f_{y_1|v_1}(v_i|v_i), \tag{1.11}$$

therefore

$$b_i = s(v_i) = \int_{\underline{v}}^{v_i} v \cdot f_{y_1|v_1}(v|v)dv.$$

From the differential equation (1.11) we obtain the value v_i:

$$v_i = \frac{s'(v_i)}{f_{y_1|v_1}(v_i|v_i)}.$$

This proves the proposition. □

Using the first-order condition we will be able to identify the model.

1.4.2 Nonparametric Identification

In this section, I prove the nonparametric identification of the APV model.

As in the case of the IPV model, the APV model is identified whenever the distribution function F can be found by the econometrician uniquely given the data on the bids.

Let $G_{B_1|b_1}(\cdot|\cdot)$ be the conditional distribution of B_1 given b_1 and $g_{B_1|b_1}(\cdot|\cdot)$ be the corresponding density.

Then the following proposition analogous to Proposition 1 in Li et al. (2002)[4] holds:

Proposition 1.5. *Let $N \geq 2$. Let $\boldsymbol{G}(\cdot)$ belong to the set of absolutely continuous probability distributions with support $[\underline{b}, \bar{b}]^N$. Then the symmetric APV model is identified. Moreover, distribution $\boldsymbol{G}(\cdot)$ with support $[\underline{b}, \bar{b}]^N$ can be rationalized by a symmetric APV model if and only if*

1. *$\boldsymbol{G}(\cdot)$ is symmetric and affiliated, and*
2. *the function $\xi(\cdot, N, \boldsymbol{G}) \equiv \frac{1}{g_{B_1|b_1}(b_i|b_i)}$ is strictly increasing on $[\underline{b}, \bar{b}]$.*

Proof: Analogously to Li et al. (2002):

$$G_{B_1|b_1}(B|b) = P(B_1 \leq B|b_1 = b) = P(y_1 \leq s^{-1}(B)|v_1 = s^{-1}(b)) =$$
$$= F_{y_1|v_1}(s^{-1}(B)|s^{-1}(b)).$$

Thus

$$g_{B_1|b_1}(B|b) = \frac{f_{y_1|v_1}(s^{-1}(B)|s^{-1}(b))}{s'(s^{-1}(B))}.$$

As a result, using the two equations above and condition $v = s^{-1}(b)$, the first-order condition (1.10) can be rewritten as:

$$v_i = \frac{1}{g_{B_1|b_1}(b_i|b_i)}. \tag{1.12}$$

[4] I use the same notation as in Li et al. (2002) for convenience.

The rest follows from the proof of Proposition 1 in Li et al. (2002).

□

The next section discusses the estimation procedure.

1.4.3 Nonparametric Estimation

In this section, the consistent plug-in estimation for the bidder's valuation is proposed.

Similar to the IPV case, the first step is the estimation of the conditional bid density $g_{B_1|b_1}(\cdot|\cdot)$ using the data on bids. In the next step, the pseudo valuations can be estimated using the equation (1.12). The last step is the estimation of the density of the valuations from the obtained pseudo values using kernel estimator.

Since

$$g_{B_1|b_1}(B|b) = \frac{g_{B_1,b_1}(B,b)}{g_{b_1}(b)}, \tag{1.13}$$

joint density should be estimated as well as the density of b_1.

Let L_N be the number N-bidders auctions. I index by l the l-th auction and use the observations $\{b_{il}, i = 1, ..., N, l = 1, ..., L_N\}$ to estimate nonparametrically $g_{B_1,b_1}(\cdot,\cdot)$ and $g_{b_1}(\cdot)$.

$$\hat{g}_{B_1,b_1}(B,b) = \frac{1}{L_N h_g^2} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K_g\Big(\frac{B - B_{il}}{h_g}, \frac{b - b_{il}}{h_g}\Big), \tag{1.14}$$

$$\hat{g}_{b_1}(b) = \frac{1}{L_N h} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K\Big(\frac{b - b_{il}}{h}\Big), \tag{1.15}$$

where h denotes the bandwidth and K denotes the kernel function.

Proposition 1.6. *Given the Assumptions 1.1-1.10 are satisfied and M=1 the following is the consistent estimator of the valuation of player i in auction l:*

$$\hat{v}_{il} \xrightarrow{p} v_{il},$$

where:

$$\hat{v}_{il} = \frac{1}{\hat{g}_{B_1|b_1}(b_{il}|b_{il})} = \frac{\hat{g}_{b_1}(b_{il})}{\hat{g}_{B_1,b_1}(b_{il},b_{il})}. \tag{1.16}$$

These are the pseudo values.

Proof: In the Appendix. □

To estimate the joint density $f(\cdot,...,\cdot)$ I use the pseudo-sample $\{\hat{v}_{il}, i = 1,...,N, l = 1,...,L_N\}$

$$\hat{f}(v_1,...,v_N) = \frac{1}{L_N h_f^N} \sum_{l=1}^{L_N} K_f\Big(\frac{v_1 - \hat{v}_{1l}}{h_f},...,\frac{v_N - \hat{v}_{Nl}}{h_f}\Big),$$

for any value $(v_1,...,v_N)$.

In its turn, to estimate the marginal density

$$\hat{f}(v) = \frac{1}{L_N h_f} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K_f\Big(\frac{v - \hat{v}_{il}}{h_f}\Big). \tag{1.17}$$

for any value $v \in [0,1]$.

Similarly to the IPV case, it is possible to account for the observed heterogeneity by conditioning on the unobservables.

1.4.4 Monte Carlo Simulation

In this section, the estimation is described step by step by conducting a Monte Carlo study.

I consider the scenario when the data on $L = 500$ auctions, each with $N = 2$ bidders is given. It could be easily generalized to account for the case when there is a different number of bidders. Following Li et al. (2002) I consider the simplest case of affiliated values distribution.

Private values are assumed to be the sum of the two uniform random variables $v_i = \gamma + u_i$, where γ is $U[0.25, 0.75]$, and the u_i's are independently drawn from $U[-0.25, 0.25]$ so that they are correlated through γ and $corr(v_i, v_j) = 0.5$. Then $f_\gamma(x) = 2$, $x \in [0.25, 0.75]$, $f_u(y) = 2$, $y \in [-0.25, 0.25]$, thus

$$f(v) = \int f_\gamma(v-y) f_u(y) dy = 2 \int_{-0.25}^{0.25} f_\gamma(v-y) dy = \begin{cases} \int_{-0.25}^{v-0.25} 4dy, & v \in [0, 0.5] \\ \int_{v-0.75}^{0.25} 4dy, & v \in [0.5, 1] \end{cases}$$

It follows that the marginal density of the valuations is triangular:

$$f(v) = \begin{cases} 4v, & v \in [0, 0.5] \\ 4 - 4v, & v \in [0.5, 1] \end{cases} \quad \text{and} \quad F(v) = \begin{cases} 2v^2, & v \in [0, 0.5] \\ 4v - 2v^2 - 1, & v \in [0.5, 1] \end{cases} \tag{1.18}$$

It can also be shown that

$$f_{y_1,v_1}(t,s) = \begin{cases} 8t, & t < s < 1/2, t > 0 \\ 8s, & s < t < 1/2, s > 0 \\ 8t - 8s + 4, & t < 1/2, 1/2 < s < t + 1/2 \\ 8s - 8t + 4, & s < 1/2, 1/2 < t < s + 1/2 \\ 8 - 8s, & s < 1, 1/2 < t < s \\ 8 - 8t, & t < 1, 1/2 < s < t \end{cases}$$

and

$$f_{y_1,v_1}(v,v) = \begin{cases} 8v, & v \in [0, 0.5] \\ 8 - 8v, & v \in [0.5, 1] \end{cases}$$

As a result,

$$f_{y_1|v_1}(v|v) = 2, \quad v \in [0, 1].$$

Similarly, in general case, for any N it can also be shown that

$$f_{y_1|v_1}(v|v) = \begin{cases} \frac{1-(2v-1)^{N-1}}{1-v}, & v \in [0, 0.5] \\ 2^{N-1}v^{N-2}, & v \in [0.5, 1]. \end{cases}$$

Thus we can find the corresponding bids. In case when $N = 2$ using (1.9)

$$b_i = \int_0^{v_i} 2v dv = v_i^2, \quad v_i \in [0, 1].$$

1000 Monte Carlo simulations are conducted. For each simulation 500 values of γ and 1000 values of u_i are drawn, and then v_i are calculated. Next, for each value draw the corresponding bid is calculated. Given the bids, the first step of estimation is conducted, using (1.14) and (1.15) the joint and marginal densities are estimated. In each estimation the triweight kernel is used:

$$K(u) = \frac{35}{32}(1-u^2)^3 \mathbb{1}(|u| \leq 1).$$

This is a kernel of order 2. It is continuously differentiable and has compact support, thus $rho_g = 2$. In its turn $h = 1.06\hat{\sigma_b}(NL)^{-1/5}$ and $h_g = 1.06\hat{\sigma_b}(NL)^{-1/6}$, where $\hat{\sigma_b}$ is the estimator of the standard deviation of the bids. Given the estimators of joint and marginal densities the pseudo values (1.16) are calculated and trimmed as in the case of the IPV model to account for the bias of kernel estimation:

$$\hat{v}_{il} = \begin{cases} \frac{1}{\hat{g}_{B_1|b_1}(b_{il}|b_{il})} = \frac{\hat{g}_{b_1}(b_{il})}{\hat{g}_{B_1,b_1}(b_{il},b_{il})}, \\ \quad \text{if } b_{min} + \rho_g h_g/2 \leq b_{il} \leq b_{max} - \rho_g h_g/2 \\ \infty, otherwise \end{cases} \tag{1.19}$$

The final step is the estimation of $\hat{f}(\cdot)$ using (1.17). Here $h_f = 1.06\hat{\sigma_v}(NL_T)^{-1/5}$, L_T is the number of auctions that are left after the trimming. In its turn, $\hat{\sigma_b}$ is the estimator of the standard deviation of the pseudo values. In each replication, I estimate $\hat{f}(\cdot)$ at 500 equally spaced points on $[0, 1]$. The Figure 1.3 presents the true triangular marginal density, and for each value of $v \in [0, 1]$, the mean of the 1000 estimates $\hat{f}(v)$, together with the 5% quantile, and the 95% quantile.

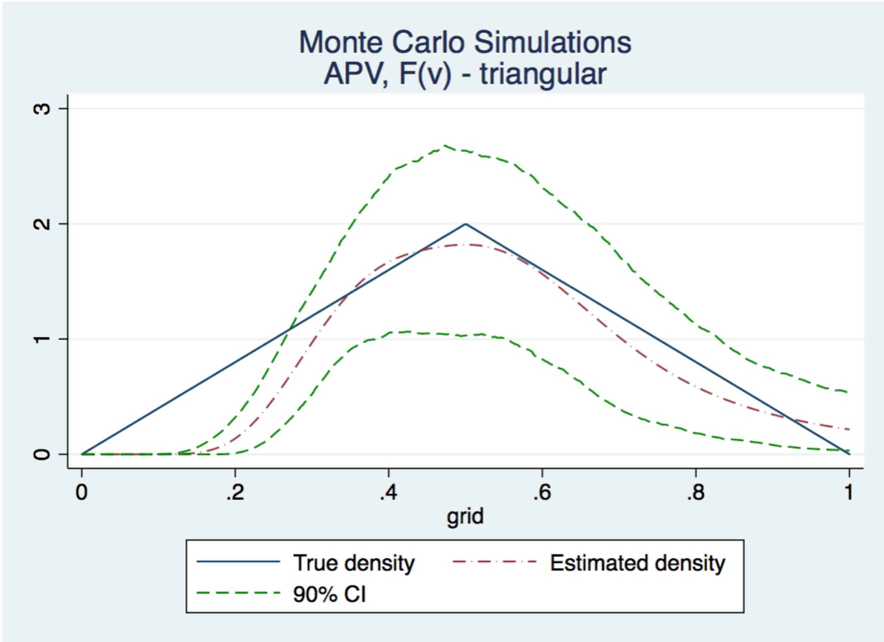


Figure 1.3: True and estimated densities of valuations in APV model

The Figure below describes the situation when the model is estimated as IPV, whereas in reality, it is an APV setting.

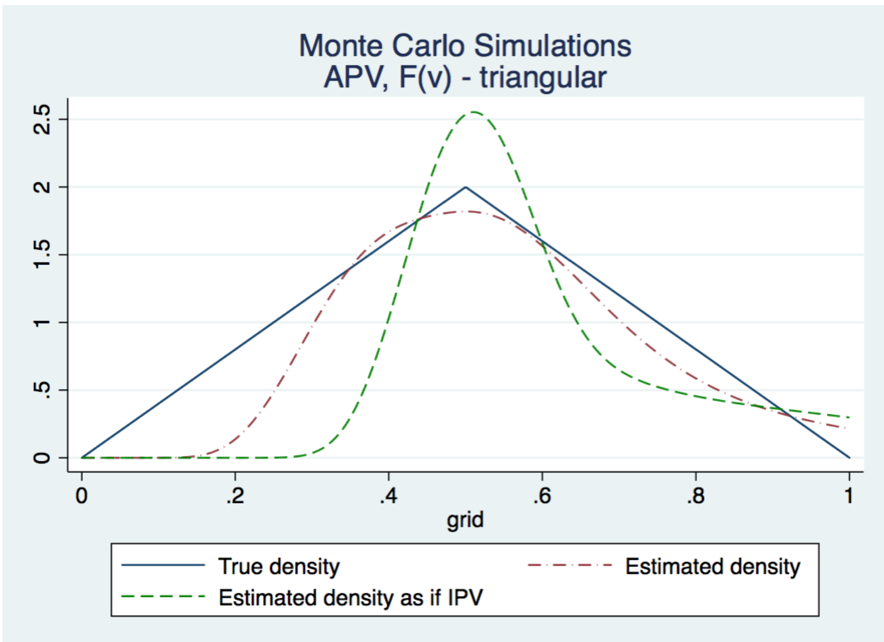


Figure 1.4: True and estimated densities of valuations: IPV and APV comparison

1.4.5 Model with Multiple Units

The model could be extended to the case with M units for sale. In this case, instead of B_1 I introduce $B_m = s(y_m)$, y_m is m-th largest bid among others' bids. In this case

$$s'(v_i) = v_i \cdot f_{y_m|v_1}(v_i|v_i).$$

And as a result we get:

$$v = \frac{1}{g_{B_m|b_1}(b|b)}.$$

I use the estimates:

$$\hat{g}_{B_m,b_1}(B,b) = \frac{1}{L_N h_g^2} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K_g\Big(\frac{B - B_{il}}{h_g}, \frac{b - b_{il}}{h_g}\Big),$$

$$\hat{g}_{b_1}(b) = \frac{1}{L_N h} \sum_{l=1}^{L_N} \frac{1}{N} \sum_{i=1}^{N} K\Big(\frac{b - b_{il}}{h}\Big),$$

where h denotes the bandwidth and K denotes the kernel function.

Thus:

$$\hat{v}_{il} = \frac{1}{\hat{g}_{B_m|b_1}(b_{il}|b_{il})} = \frac{\hat{g}_{b_1}(b_{il})}{\hat{g}_{B_m,b_1}(b_{il}|b_{il})}.$$

To estimate the joint density $f(\cdot, ..., \cdot)$ the pseudo-sample $\{\hat{v}_{il}, i = 1, ..., N, l = 1, ..., L_N\}$ is used as before.

1.5 IPV All-Pay Auction with Risk-Averse Bidders

1.5.1 Model

As in previous sections assume that there are N bidders, $i = 1, ..., N$. Each bidder draws a value v_i independently from a commonly known distribution $F(v)$ with support $[\underline{v}, \bar{v}]$. v_i is the private value that bidder i of possessing the good. In contrast to the previous set-up now each bidder is risk-averse, therefore he has utility function $U(\cdot)$, such that: $U(0) = 0$, $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$, which are standard assumptions. Without loss of generality let us normalize $U(1) = 1$.

Each bidder i knows the number of bidders N, his value v_i, as well as the distribution of the valuations of the other bidders $F(v)$ and utility function $U(\cdot)$. Again I consider the Bayesian equilibrium of this incomplete information game which is strictly monotonic and symmetric. The bid function is defined by $s(v) = b$. As $s(v)$ is strictly monotonic it is invertible, so $s^{-1}(b) = v$. In addition to these assumptions using the independence of the valuations, we can write the expected payoff to bidder i as:

$$E[U_i|V_i = v_i, V_{-i} = v_{-i}] = U(v_i - b_i)F^{N-1}(s^{-1}(b_i)) + U(-b_i)[1 - F^{N-1}(s^{-1}(b_i))].$$

Taking the first derivative with respect to the bid we get:

$$0 = -U'(v_i - b_i)F^{N-1}(s^{-1}(b_i)) + U(v_i - b_i)(N-1)F^{N-}(s^{-1}(b_i))f(s^{-1}(b_i))\frac{1}{s'(s^{-1}(b_i))} -$$

$$-U'(-b_i)[1 - F^{N-1}(s^{-1}(b_i))] - U(-b_i)(N-1)F^{N-}(s^{-1}(b_i))f(s^{-1}(b_i))\frac{1}{s'(s^{-1}(b_i))}.$$

Substituting $s^{-1}(b_i) = v_i$ and rearranging the terms we get the first-order differential equation which determines the bid function:

$$s'(v_i) = \frac{(N-1)f(v_i)F^{N-2}(v_i)[U(v_i - b_i) - U(-b_i)]}{U'(-b_i) + F^{N-1}(v_i)[U'(v_i - b_i) - U'(-b_i)]}. \tag{1.20}$$

Using this equation I prove that the model is not identified. Analogous result for the case of first-price auction is derived in Campo et al. (2011).

1.5.2 Nonidentification Result

I have shown in section 3.1 that $F(v_i) = G(b_i)$ and $\frac{f(v_i)}{s'(v_i)} = g(b_i)$, thus we can rewrite the equation 1.20 as

$$\frac{(N-1)g(v_i)G^{N-2}(v_i)[U(v_i - b_i) - U(-b_i)]}{U'(-b_i) + G^{N-1}(b_i)[U'(v_i - b_i) - U'(-b_i)]} = 1.$$

Following Guerre et al. (2009), let us call the model a set of structures $[U, F]$. A structure $[U, F]$ is non-identified if there exists another structure $[U', F']$ within the model that leads to the same equilibrium bid distribution. If no such structure $[U', F']$ exists for any $[U, F]$, the model is (globally) identified.

Proposition 1.7. *In general the IPV model with risk-averse bidders is not identified. Moreover, any structure $[U, F]$ in $U^{CARA} \times \mathcal{F}$ is not identified. Formally, consider $N = 2$, $U(x) = \frac{1-\exp(-ax)}{1-\exp(-a)}$, $a > 0$. Then any structure $[U, F]$ with $F(v) = \frac{2-a-2\exp(-av)}{(1-\exp(-av))(2-a)}$, $v \in [-\frac{1}{a}\ln\left(\frac{2-a}{2}\right), +\infty)$, where $a \in [1, 2)$, leads to the exponential distribution $G(b) = 1 - \exp^{-2b}$ on $[0, +\infty)$.*

Proof: Let's consider CARA utility function such that :

$$U(x) = \frac{1 - \exp(-ax)}{1 - \exp(-a)}, \quad a > 0.$$

Thus $U'(x) = \frac{a\exp(-ax)}{1-\exp(-a)} > 0$ and $U''(x) = \frac{-a^2\exp(-ax)}{1-\exp(-a)} < 0$.

Let's also fix N=2. In this case the differential equation becomes (I omit index i for simplicity):

$$1 = \frac{g(b)[1 - \exp(-av + ab) - 1 + \exp(ab)]}{a\exp(ab) + G(b)[a\exp(-av + ab) - a\exp(ab)]}$$

which (after deviding both numerator and denominator by $\exp(ab)$) is equivalent to

$$1 = \frac{g(b)[1 - \exp(-av)]}{a - aG(b)[1 - \exp(-av)]}.$$

Let's find v from this equation. $1 - \exp(-av) = \frac{a}{g(b)+aG(b)} \Rightarrow$

$$v = -\frac{1}{a} \ln \left(1 - \frac{a}{g(b) + aG(b)} \right).$$

Now let's consider exponential family of distributions $G(b) = 1 - \exp^{-\lambda b},\ b \geq 0$, $g(b) = \lambda \exp^{-\lambda b}$. Then $g(b) + aG(b) = \lambda \exp^{-\lambda b} + a(1 - \exp^{-\lambda b})$ and $\frac{\partial[g(b)+aG(b)]}{\partial b} = -\lambda^2 \exp^{-\lambda b} + a\lambda \exp^{-\lambda b} = \lambda \exp^{-\lambda b}(a - \lambda) < 0$ when $\lambda > a$. Thus in this case v is an increasing function of b.

Let's find the bid function: $g(b) + aG(b) = \frac{a}{1-\exp(-av)} \Rightarrow (\lambda - a)\exp^{-\lambda b} = \frac{a}{1-\exp(-av)} \Rightarrow$

$$b = s(v) = -\frac{1}{\lambda} \ln \Big(\frac{a \exp(-av)}{(1 - \exp(-av))(\lambda - a)} \Big).$$

In its turn

$$F(v) = G(b(v)) = 1 - \exp^{-\lambda b} = 1 - \frac{a \exp(-av)}{(1 - \exp(-av))(\lambda - a)} = \frac{\lambda - a - \lambda \exp(-av)}{(1 - \exp(-av))(\lambda - a)}.$$

$b \in [0; +\infty)$, therefore v is well-defined, but has the moving support since $v = -\frac{1}{a} \ln \left(\frac{\lambda - a}{\lambda} \right)$ if $b = 0$ and $v = +\infty$ if $b = +\infty$.

In particular, this is true if λ=2. This proves the proposition. $\square$

As a result, it was shown that this model is not identified even in the semi-parametric case where the utility function of the bidders is restricted to belong to the class of functions with constant absolute risk aversion (CARA).

1.6 Conclusion

In this work, I have proved the identification and derived the consistent estimator of an all-pay auction. I have adopted the structural approach of Guerre et al. (2000) and have proved that the distribution function of bidders' valuations is identified nonparametrically from the data in both IPV and APV settings. The important property of the estimation is that the determination of the equilibrium strategy is avoided since I only use the first-order condition. This allows the estimation of the distribution of valuations even in the case when the closed-form solution cannot be explicitly found. Finally, I considered the model with risk aversion. I show that this model is not identified even in the semi-parametric case where the utility function of the bidders is restricted to belong to the class of functions with constant absolute risk aversion (CARA).

1.7 Appendix

1.7.1 Proof of Proposition 1.3

Proof: By Theorem 1.1 from Li and Racine (2006):

$$\hat{g}(b_{il}) - g(b_{il}) = O_p(h_g^2 + (Lh_g)^{-1/2}) = o_p(1).$$

In its turn by the string law of large numbers empirical CDF converges almost surely to the true CDF, thus also converges in probability.

Thus, by the properties of convergence in probability and continuous mapping theorem,

$$\frac{1}{\hat{g}(b_{il})\hat{G}(b_{il})^{N-2}(N-1)} \xrightarrow{p} \frac{1}{g(b_{il})G(b_{il})^{N-2}(N-1)}.$$

$\square$

1.7.2 Proof of Proposition 1.6

Proof: By Theorem 1.1 from Li and Racine (2006):

$$\hat{g_{b_1}}(b_{il}) - g_{b_1}(b_{il}) = o_p(1).$$

Moreover, by Theorem 1.3 from Li and Racine (2006):

$$\hat{g}_{B_1,b_1}(b_{il}, b_{il}) - g_{B_1,b_1}(b_{il}, b_{il}) = o_p(1).$$

Thus, by the properties of convergence in probability and continuous mapping theorem,

$$\frac{\hat{g}_{b_1}(b_{il})}{\hat{g}_{B_1,b_1}(b_{il}, b_{il})} \xrightarrow{p} \frac{g_{b_1}(b_{il})}{g_{B_1,b_1}(b_{il}, b_{il})}.$$

$\square$

CHAPTER 2

Nonparametric Identification and Estimation of Contests with Uncertainty and an Application to U.S. House Elections

2.1 Introduction

In this work, I prove the identification and derive the asymptotically normal estimator of a nonparametric contest of incomplete information with uncertainty. This is the first paper to consider the identification and estimation of a model with a nonparametric contest success function which determines the winning probability. Similar to Guerre et al. (2000) and He and Huang (2018) I propose a method to estimate the distribution of bidders' private valuations (or types) from observed bids as well as the winning outcomes, which does not require any parametric assumptions or the Bayesian Nash equilibrium computation.

The contest is a natural model of costly competition as it describes situations when all players exert costly effort to achieve some goal (win the contest). This is a sunk cost as no matter whether a player wins or loses the bidder always pays the bid. Such interactions include a wide range of scenarios. The electoral competition was modeled using contest theory since the 1990: see, for example, Snyder (1989), Baron (1994) or Skaperdas and Grofman (1995). Moreover, it is used to model marketing and advertising by firms (Bell et al. (1975)); litigation (Farmer and Pecorino (1999), Bernardo et al. (2000), Hirshleifer and Osborne (2001), Baye et al. (2005)); research and development, patent race, procurement of innovative good, research contests (Taylor (1995), Che and Gale (2003)); sport events, arms race and rent-seeking activity, such as lobbying (Tullock (1980), Krueger (1974), Baye et al. (1993)).

The contest is defined by the contest success function that maps efforts (bids) into probabilities of winning for participating players (bidders).[1] In this work, I consider a contest with uncertainty. By uncertainty I mean that the contest success function is not only determined by the bids of the players, but also by a variable, which I call uncertainty, with a nonparametric distribution, known to the bidders, but unknown to the researcher. The model is described in detail in the next section.

Moreover, I consider the incomplete information contest in contrast to most of the theoretical papers on contests and auctions that consider games with complete information. In reality, it is more plausible to think that the bidders do not observe the private information of the other bidders. Fey (2008), Ryvkin (2010) and Ewerhart (2014) are a few of the papers providing the existence of equilibrium results in the context of incomplete information contests. The literature on nonparametric identification and estimation of incomplete information auctions and contests is very sparse as well. Only the first-price auctions were considered in detail in the block of papers originated from Guerre et al. (2000). In the previous chapter, I considered all-pay auctions (an extreme case of a contest). And the only paper that considers a contest as a game with incomplete information is the one by the He and Huang (2018). In that paper, the authors assume that the contest success function has the Tullock's form. I will show that the Tullock contest is a particular case of a contest with uncertainty considered in this work,

[1] All-pay auction is an extreme case of the contest when the bidder with the highest bid wins; thus, the winning probability is one if and only if the bidder has the highest bid. In reality, in the scenarios described above, it is common that the contestant with the highest bid can still lose, thus it is important to consider contests for empirical applications.

in a case when the distribution of uncertainty is known to be exponential.

I examine the model with two asymmetric players in this work. Every bidder pays his bid regardless of whether he wins or not. Each bidder has a valuation of the good, which is his private information, and knowing his own valuation, the number of bidders and the distribution of the other bidders' valuations, submits a bid in order to obtain an object. The model is a game of incomplete information in the sense that the bidders do not observe the other bidders' valuations, but the distributions of the valuations are common knowledge. As a result of the identification and estimation of the model, I recover the distributions of bidders' valuations. The novelty of the paper is that as the first step, I estimate the nonparametric distribution of uncertainty using the information on win outcomes and bids.

Importantly for the empirical application, the model can be reformulated in terms of types instead of valuations, meaning that each player instead has a different type, which is just the inverse of the valuation. The type characterizes how costly it is for the player to raise a bid, whereas the valuations are normalized to be 1. I show that this model is equivalent to the one with the different valuations' distributions.

As an empirical application, I consider the U.S. House of Representatives elections, which were also studied by He and Huang (2018) as an application of the Tullock contest. Bidders in this setting are considered to have different abilities to raise money (types described above), whereas the valuations are normalized to be 1. Using the model, I disentangle and estimate two potential advantages of the Incumbent. The first source of advantage is due to the fact that the Incumbent often has a better reputation and is more experienced than the Challenger. The other source of advantage is the Incumbent's better campaign financing. Only the latter can be regulated by the authority; thus, it is important that this source of advantage can be quantified separately from the reputation effect. A large body of empirical work studies the effect of campaign spending on the vote share in the context of Congressional elections starting from the pioneering work of Jacobson (1978). My work contributes to the literature by providing a method of recovering the incumbency advantage in campaign financing (characterized by the difference in type distributions between the candidates), as well as the advantage of the Incumbent due to the reputation (characterized by the uncertainty distribution). This is done using the information on the observed spendings as well as winning outcomes, and the nonparametric structural contest model.[2] Results of the model suggest that the Incumbent's advantage was prevalent throughout the sample period 1972-2016. Incumbents won in 93.8% of contests. Moreover, on average Incumbents spent 2.5 times as much as their Challengers. Using the structural model, I estimate that if the Incumbents were to spend as much as the Challengers they would win only 85% of the times. The knowledge of the type distributions allows policymakers to quantify the effect of different policy changes. In this work, I consider two different policy counterfactuals aimed at limiting the incumbency advantage: a public campaign financing of Challengers and a limit on Incumbents' expenditure. I show that the latter is more effective in terms of lowering both the Incumbents' winning probability as well as the total campaign spending. This is in accordance with He and Huang (2018) conclusions, but in contrast to the prevailing opinion that: "the problem is not equalizing spending between candidates but rather simply getting more money to Challengers so that they can mount competitive races," stated by Jacobson (1978).

The rest of the paper is organized as follows. In Section 2, I introduce the contest model with uncertainty. Section 3 discusses the nonparametric identification of the model. Section 4 considers the nonparametric estimation as well as the Monte Carlo simulations. The application to the U.S. House Elections is presented in Section 5. Section 6 concludes.

[2]This is in contrast to the Tullock contest considered in He and Huang (2018) that imposes the parametric structure on the winning probability.

2.2 Contest Model with Uncertainty

2.2.1 Notations and Definitions

In this work, a contest with $N = 2$ asymmetric risk-neutral bidders is considered. This is motivated by the nature of the application in which two candidates are competing for a seat in U.S. House of Representatives: one is the Incumbent and the other is the Challenger. The model can be easily extended to account for the arbitrary number of bidders.

Assumption 2.1. *Each bidder has a valuation of the good v_i, $i = 1, 2$, which is his private information. Each bidder draws this valuation v_i independently from a commonly known distribution $F_i(v)$ with support $[\underline{v_i}, \bar{v}_i]$, density f_i and quantile function $q_i = F_i^{-1}$, $i = 1, 2$.*

Assumption 2.2. *The bidders submit the bids b_i simultaneously.*

Assumption 2.3. *Each of N bidders pays b_i, regardless of whether or not he obtains a good.*

Moreover, the impact of the campaign spending on the winning probability is uncertain.

Assumption 2.4. *The real impact is $x_i = g(b_i, \epsilon_i)$, where ϵ_i is assumed to be independent of b_i.*

Assumption 2.5. *At the time of bidding, each bidder i knows the number of bidders, his own valuation v_i as well as $F_j(\cdot)$ and the distributions of uncertainties ϵ_i, $i = 1, 2$.*

The reason for such an Assumption 4 is that some of the voters have a preference for the Incumbent versus the Challenger due to the Incumbent's reputation, and this is no matter what would be the campaign spending and advertising. On the other hand, for other voters, the campaign expenditure determines their preference. The goal is to disentangle and estimate these two potential advantages of the Incumbent. The first source of advantage is due to the fact that the Incumbent often has a better reputation and is more experienced than the Challenger. The other source of advantage is the Incumbent's better skills in raising money for the campaign. In this work, I consider the case when the higher expenditures have a multiplicative effect on the political impact: as in Hillman and Riley (1989), where the model was introduced.

Assumption 2.6. *$x_i = b_i \cdot \epsilon_i$, $i = 1, 2$, where $H_\xi(\cdot)$ is the CDF of $\epsilon_2/\epsilon_1 := \xi$, whereas by $h_\xi(\cdot)$ the corresponding density function.*

Only positive x_i can lead to victory; thus, ϵ_i have positive support.

Moreover, let $w_i = 1$ if bidder i wins and $w_i = 0$ otherwise. Then the probability of winning of the first player given the bids is:

$$P(w_1 = 1|\ b_1, b_2) = P(x_1 > x_2 \mid b_1, b_2) = P(b_1\epsilon_1 > b_2\epsilon_2 \mid b_1, b_2) \tag{2.1}$$

where ϵ_1 and ϵ_2 are preferences for bidder 1 and bidder 2 respectively.

The expected payoff to bidder i participating in the contest, is given by:

$$E[U_i|v_i, F_j, H_\xi] = v_i P[w_i = 1|v_i, F_j, H_\xi] - b_i = v_i P(b_i\epsilon_i > b_j\epsilon_j|v_i, F_j, H_\xi) - b_i, \tag{2.2}$$

where $i = 1, 2$, $j = -i$.[3] The final payoff to the bidder i is $v_i - b_i$ if he obtains a good, and $-b_i$ if he does not obtain a good.

It is worth noting that:

[3]This model can be extended to account for the observables by assuming:

$$P[w_i = 1|v_i, F_j, H_\xi] = P[(b_i + m(X_i))\epsilon_i \geq (b_j + m(X_j))\epsilon_j|v_i, F_j, H_\xi],$$

where both function m and distribution of ϵ_2/ϵ_1 can be identified in the first step.

Proposition 2.1. *In a specific case when both ϵ_i and ϵ_j have an exponential distribution with parameter $\lambda = 1$, the contest described above is equivalent to the Tullock contest.*

Proof: In the Appendix. □

2.2.2 Equilibrium Characterization

I consider the strictly monotonic Bayesian equilibrium in this incomplete information game. Using the results of Athey (2001), the existence of equilibrium can be proved.

Proposition 2.2. *Given Assumptions 2.1-2.6 are satisfied, there exists a pure strategy increasing BNE of the incomplete information game formulated above.*

Proof: In the Appendix. □

For each valuation, the corresponding bid is defined by the function $s_i(v_i) = b_i$, $i = 1, 2$ that is the equilibrium bid strategy which maximizes the bidder i's expected payoff. Since $s_i(v_i)$ is strictly monotonic it is invertible and $s_i^{-1}(b_i) = v_i$.

Proposition 2.3. *Given Assumptions 1-6 as well as the assumption of strict monotonicity of the bidding strategies the first-order conditions of this game can be written as:*

$$v_1 = \frac{1}{\int\limits_{\underline{v}_2}^{\bar{v}_2} f_2(v_2) \frac{1}{s_2(v_2)} h_\xi \left(\frac{s_1(v_1)}{s_2(v_2)} \right) dv_2} \tag{2.3}$$

$$v_2 = \frac{1}{\int\limits_{\underline{v}_1}^{\bar{v}_1} f_1(v_1) \frac{s_1(v_1)}{s_2^2(v_2)} h_\xi \left(\frac{s_1(v_1)}{s_2(v_2)} \right) dv_1} \tag{2.4}$$

Proof: Under the assumptions of strict monotonicity of the bidding strategies and independent valuations, we can write the expected payoff to bidder 1 when his true valuation is v_1 but he bids as if it was v as:

$$E[U_1|v_1, F_2, H_\xi] =$$

$$= v_1 P[w_1 = 1|b, F_2, H_\xi] - b = v_1 P(b\epsilon_1 > b_2\epsilon_2) - b = v_1 P(b_2\xi < b) - b =$$

$$= v_1 \int_{\underline{b}_2}^{\bar{b}_2} \left[\int_0^{b/b_2} h_\xi(y)dy \right] g_2(b_2)db_2 - b =$$

$$v_1 \int_{\underline{v}_2}^{\bar{v}_2} \left[\int_0^{s_1(v)/s_2(v_2)} h_\xi(y)dy \right] f_2(v_2)dv_2 - s_1(v).$$

Using the First order condition (differentiating with respect to v and substituting $v = v_i$ and equating it to zero) we get the following equation for the valuation of player 1:

$$v_1 \int_{\underline{v}_2}^{\bar{v}_2} \frac{s_1'(v)}{s_2(v_2)} h_\xi \left(\frac{s_1(v)}{s_2(v_2)} \right) f_2(v_2)dv_2 - s_1'(v) = 0 \text{ when } v = v_1 \Rightarrow$$

$$v_1 = \frac{1}{\int_{\underline{v}_2}^{\bar{v}_2} f_2(v_2) \frac{1}{s_2(v_2)} h_\xi \left(\frac{s_1(v_1)}{s_2(v_2)} \right) dv_2}$$

Similarly for player 2 we can write the expected payoff to bidder 2 when his true valuation is v_2 but he bids as if it was v as:

$$E[U_2|v_2, F_1, H_\xi] =$$

$$= v_2 P[w_2 = 1|b, F_1, H_\xi] - b = v_2 P(b\epsilon_2 > b_1\epsilon_1) - b = v_2 P(\xi > b_1/b) - b =$$

$$= v_2 \int_{\underline{b}_1}^{\bar{b}_1} \left[\int_{b_1/b}^{\infty} h_\xi(y)dy \right] g_1(b_1)db_1 - b =$$

$$v_2 \int_{\underline{v}_1}^{\bar{v}_1} \left[\int_{s_1(v_1)/s_2(v)}^{\infty} h_\xi(y)dy \right] f_1(v_1)dv_1 - s_2(v).$$

By taking derivative with respect to v and equating it to zero we get the following equation for the valuation of player 2:

$$v_2 \int_{\underline{v}_1}^{\bar{v}_1} \frac{s_2'(v)s_1(v_1)}{s_2^2(v)} h_\xi \left(\frac{s_1(v_1)}{s_2(v)} \right) f_1(v_1)dv_1 - s_2'(v) = 0 \text{ when } v = v_2 \Rightarrow$$

$$v_2 = \frac{1}{\int_{\underline{v}_1}^{\bar{v}_1} f_1(v_1) \frac{s_1(v_1)}{s_2^2(v_2)} h_\xi \left(\frac{s_1(v_1)}{s_2(v_2)} \right) dv_1}$$

This proves the proposition. □

In our case, given the data, private values are unobserved for the econometrician, whereas bids are observed. Thus the goal would be to rewrite right hand sides of the equations (2.3) and (2.4) in terms of distribution of bids. The method is described in detail in the Section on Identification.

2.2.2.1 Representation in Terms of Types

The problem can be easily reformulated in terms of the types (how costly is it to raise a bid for the player). Expected payoff to bidder i in this case is given by:

$$\mathbb{E}[U_i|c_i, F_j(c), H_\xi] = P[w_i = 1|c_i, F_j, H_\xi] - c_i b_i = P(b_i\epsilon_i > b_j\epsilon_j|c_i, F_j, H_\xi) - c_i b_i, \tag{2.5}$$

where $i = 1, 2$, $j = -i$, $c_i = \frac{1}{v_i}$ and F_i is the type distribution function whereas f_i is the corresponding density.

Thus equations (2.3) and (2.4) can be written in terms of types:

$$c_1 = \int\limits_{\underline{c}_2}^{\bar{c}_2} f_2(c_2)\frac{1}{s_2(c_2)} h_\xi\left(\frac{s_1(c_1)}{s_2(c_2)}\right) dc_2 \tag{2.6}$$

and

$$c_2 = \int\limits_{\underline{c}_1}^{\bar{c}_1} f_1(c_1)\frac{s_1(c_1)}{s_2^2(c_2)} h_\xi\left(\frac{s_1(c_1)}{s_2(c_2)}\right) dc_1 \tag{2.7}$$

2.3 Nonparametric Identification

In this section, I prove that the parameters of the model are nonparametrically identified from available data, which is the main question in structural estimation.

In the presented model there are two unknown structural elements for the econometrician - the distribution of valuations $F(\cdot)$ as well as the distribution $H_\xi(\cdot)$ of $\epsilon_2/\epsilon_1 := \xi$, whereas the number of bidders, the bids themselves b_i, $i = 1, 2$ as well as the win results, are observed. Therefore the identification problem reduces to whether the distributions F and H_ξ are uniquely determined from observed bids and win outcomes. Note that the distribution $G(\cdot)$ of b_i depends on the underlying distribution $F(\cdot)$ not only through v_i, but also through the equilibrium strategy $s(\cdot)$.

Formally, let $\mathcal{G}$ denote the set of all distributions over the space of permitted bids and let p denote the win probability of the Incumbent, $F \in \mathcal{F}$ and $H \in \mathcal{H}$. Let us call the mapping from the private information to bids $\gamma \in \Gamma$, where $\gamma : \mathcal{F} \times \mathcal{H} \to \mathcal{G} \times p$. Then,

Definition 2.1. *(Identification). A model $(\mathcal{F}, \mathcal{H}, \Gamma)$ is identified if for every (F, F'), (H, H') and (γ, γ'), $\gamma(F, H) = \gamma'(F', H') \Rightarrow (F, H, \gamma) = (F', H', \gamma')$.*

The identification argument can be conducted in two steps. First:

Proposition 2.4. *The distribution of the ration of uncertainties ϵ_1/ϵ_2 is identified from the data on bids and win outcomes.*

Proof:

$$P(w_1 = 1) = P(b_1\epsilon_1 > b_2\epsilon_2) = P\left(\frac{\epsilon_2}{\epsilon_1} < \frac{b_1}{b_2}\right) := P\left(\xi < \frac{b_1}{b_2}\right) = H_\xi\left(\frac{b_1}{b_2}\right), \quad (2.8)$$

where I do not condition on bids for simplicity.

Thus the distribution of $\frac{\epsilon_1}{\epsilon_2}$ can be identified from observed win outcomes on the positive support by varying b_1/b_2. $\square$

In the second step, the distribution of ξ is used to recover the value distribution.

Proposition 2.5. [4] *Suppose that functions*

$$\lambda_1(b_i, N, G, H) \equiv \frac{1}{\int\limits_{\underline{b}_2}^{\bar{b}_2} g_2(b_2)\frac{1}{b_2}h_\xi\left(\frac{b_1}{b_2}\right)db_2}$$

and

$$\lambda_2(b_i, N, G, H) \equiv \frac{1}{\int\limits_{\underline{b}_1}^{\bar{b}_1} g_1(b_1)\frac{b_1}{b_2^2}h_\xi\left(\frac{b_1}{b_2}\right)db_1}$$

[4]The formulation of the proposition is similar to Theorem 1 in Guerre et al. (2000).

are strictly increasing on the support of bids $[\underline{b_i}, \bar{b_i}]$ *and their inverses are differentiable on the supports of valuations* $[\underline{v_i}, \bar{v}_i]$*. If* $G_i(\cdot)$ *are absolutely continuous probability distributions with support* $[\underline{b_i}, \bar{b_i}]$*, then there exists an absolutely continuous distribution of bidders' valuations* $F_i(\cdot)$ *corresponding to the distribution of bids. When* $F_i(\cdot)$ *exists, it is unique with support* $[\underline{v_i}, \bar{v}_i]$ *and satisfies* $F_i(v_i) = G_i(\lambda_i^{-1}(b_i, N, G, H))$ *for all* $[\underline{v_i}, \bar{v}_i]$*. In addition,* $\lambda_i(b_i, N, G, H)$ *is the quasi inverse of the equilibrium strategy in the sense that* $\lambda_i^{-1}(b_i, N, G, H) = s_i^{-1}(b_i, N, F_i, H)$ *for all* $b \in [\underline{b_i}, \bar{b_i}]$*. Moreover, the identifying equations can be rewritten in terms of quantile functions:*

$$q_1(t_1) = \frac{1}{\int_0^1 \frac{1}{r_2(t_2)} h_\xi \left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_2} \tag{2.9}$$

and

$$q_2(t_2) = \frac{1}{\int_0^1 \frac{r_1(t_1)}{r_2^2(t_2)} h_\xi \left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_1}. \tag{2.10}$$

where $t_1,\ t_2 \in (0,1)$*.*

Proof: Similar to Guerre et al. (2000), the identification result is based on the property that together with the distribution $F_i(\cdot)$ and the density $f_i(\cdot)$, the derivative of the strategy $s_i'(\cdot)$ can be canceled out from the differential equation.

Because b_i is a function of v_i, which is random and distributed as $F_i(\cdot)$, b_i is also random. Let's denote its distribution $G_i(\cdot)$ and quantile function $r_i(\cdot) = G_i^{-1}(\cdot)$, $i = 1, 2$.

For every $b \in [\underline{b_i}, \bar{b_i}] = [s_i(\underline{v_i}), s_i(\bar{v}_i)]$, we have $G_i(b) = Pr(b_i \leq b) = Pr(v_i \leq s_i^{-1}(b)) = F_i(s_i^{-1}(b)) = F_i(v)$, where $b_i = s_i(v_i)$. Thus, the distribution $G_i(\cdot)$ is absolutely continuous, has support $[s_i(\underline{v_i}), s_i(\bar{v}_i)]$ and density $g_i(b_i) = \frac{f_i(v_i)}{s_i'(v_i)}$, where $v_i = s_i^{-1}(b_i)$.

This allows us to rewrite the differential equation (2.3) above in terms of the distribution of bids, that is for the first bidder:

$$v_1 = \frac{1}{\int\limits_{\underline{b}_2}^{\bar{b}_2} g_2(b_2)\frac{1}{b_2}h_\xi\left(\frac{b_1}{b_2}\right)db_2} \tag{2.11}$$

In its turn, the equation (2.4) for the second bidder can be rewritten as:

$$v_2 = \frac{1}{\int\limits_{\underline{b}_1}^{\bar{b}_1} g_1(b_1)\frac{b_1}{b_2^2}h_\xi\left(\frac{b_1}{b_2}\right)db_1} \tag{2.12}$$

Thus equations now express the individual private values v_i as functions of the individual's equilibrium bids b_i, their distributions $G_i(\cdot)$, their densities $g_i(\cdot)$, the density h_ξ of the ratio of tastes ξ and the number of bidders N.

Let us denote $t_i = F_i(v_i)$ and $t_j = F_j(v_j)$, equivalently $v_i = q_i(t_i)$ and $v_j = q_j(t_j)$, where $q_i(\cdot)$ and $q_j(\cdot)$ are quantile functions of the distribution of valuations. As a result of monotonicity of the strategies $G_i(s_i(v_i)) = F_i(v_i)$, applying $r_i^{-1}(\cdot)$ to both sides of equality, where $r_i(\cdot)$ is quantile function of the bid distribution we get: $s_i(v_i) = r_i(F_i(v_i)) = r_i(t_i)$ and $s_j(v_j) = r_j(F_j(v_j)) = r_j(t_j)$. Moreover, $F_j(s_j^{-1}(s_i(v_i))) = G_j(s_i(v_i)) = G_j(r_i(t_i))$, $F_j(\bar{v_j}) = 1$ and $F_j(\underline{v_j}) = 0$. Using these equalities and changing variables we can rewrite the equations (2.11) and (2.12) above as:

$$q_1(t_1) = \frac{1}{\int\limits_0^1 \frac{1}{r_2(t_2)}h_\xi\left(\frac{r_1(t_1)}{r_2(t_2)}\right)dt_2} \tag{2.13}$$

and

$$q_2(t_2) = \frac{1}{\int\limits_0^1 \frac{r_1(t_1)}{r_2^2(t_2)} h_\xi\left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_1}, \tag{2.14}$$

where $t_1,\, t_2 \in (0,1)$. This proves the proposition. $\square$

Moreover, Proposition 5 can be reformulated in terms of types.

Proposition 2.6. *Suppose that functions*

$$\lambda_1^c(b_i, N, G, H) \equiv \int\limits_{\underline{b}_2}^{\bar{b}_2} g_2(b_2)\frac{1}{b_2} h_\xi\left(\frac{b_1}{b_2}\right) db_2$$

and

$$\lambda_2^c(b_i, N, G, H) \equiv \int\limits_{\underline{b}_1}^{\bar{b}_1} g_1(b_1)\frac{b_1}{b_2^2} h_\xi\left(\frac{b_1}{b_2}\right) db_1$$

are strictly decreasing on the support of bids $[\underline{b_i}, \bar{b_i}]$ and their inverses are differentiable on the supports of types $[\underline{c_i}, \bar{c}_i]$. If $G_i(\cdot)$ are absolutely continuous probability distributions with support $[\underline{b_i}, \bar{b_i}]$, then there exists an absolutely continuous distribution of bidders' private types $F_i(\cdot)$ corresponding to the distribution of bids. When $F_i(\cdot)$ exists, it is unique with support $[\underline{c_i}, \bar{c}_i]$ and satisfies $F_i(c_i) = 1 - G_i((\lambda_i^c)^{-1}(b_i, N, G, H))$ for all $[\underline{v_i}, \bar{v}_i]$. In addition, $\lambda_i^c(b_i, N, G, H)$ is the quasi inverse of the equilibrium strategy in the sense that $(\lambda_i^c)^{-1}(b_i, N, G, H) = s_i^{-1}(b_i, N, F_i, H)$ for all $b \in [\underline{b_i}, \bar{b_i}]$. Moreover, the identifying equations can be rewritten in terms of quantile functions and given by equations:

$$q_1^c(1 - t_1) = \int\limits_0^1 \frac{1}{r_2(t_2)} h_\xi\left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_2 \tag{2.15}$$

and

$$q_2^c(1-t_2) = \int_0^1 \frac{r_1(t_1)}{r_2^2(t_2)} h_\xi \left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_1, \tag{2.16}$$

where $t_1,\ t_2 \in (0,1)$.

Proof: Apply Proposition 5 and note $c_i = \frac{1}{v_i}$, $i = 1, 2$. $\square$

2.4 Nonparametric Estimation

In this section, I propose the asymptotically normal estimators of the density h_ξ and bidders' types.

If we knew the quantile functions $r_i(\cdot)$ as well as the distribution of ξ $H_\xi(\cdot)$, then we could use that to recover quantile functions of the bidders' valuations $q_i(\cdot)$. Let L be the number of auctions, l is the the l-th auction, $\{b_{il}, i = 1, 2, l = 1, ..., L\}$ are the observations of the bids, $\{w_{il}, i = 1, 2, l = 1, ..., L\}$ are the observations of the winning outcomes.[5]

In the first step, I estimate the distribution of the $\frac{\epsilon_1}{\epsilon_2}$ using kernels from the observed bids and winning outcomes. Specifically, consider bidder 1 winning probability:

$$\hat{H}_\xi(b) = \hat{P}(w_1 = 1 | b_1/b_2 = b) = \frac{\sum\limits_{l=1}^{L} w_{1l} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right)}{\sum\limits_{l=1}^{L} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right)}, \tag{2.17}$$

where $K(\cdot)$ is the kernel function and h is the bandwidth.

By taking the derivative with respect to b, we can find the estimator for the

[5] I assume that in each auction the same two types of bidders take part. In case when there are some observable characteristics of the bidders and enough data, the analysis is similar, with the only difference that we can condition on the observables.

corresponding density function:

$$\hat{h}_\xi(b) = \hat{H}'_\xi(b) =$$

$$= \frac{\sum_{l=1}^{L} w_{1l} K\left(\frac{b_{1l}/b_{2l}-b}{h}\right) \cdot \sum_{l=1}^{L} K'\left(\frac{b_{1l}/b_{2l}-b}{h}\right) - \sum_{l=1}^{L} K\left(\frac{b_{1l}/b_{2l}-b}{h}\right) \cdot \sum_{l=1}^{L} w_{1l} K'\left(\frac{b_{1l}/b_{2l}-b}{h}\right)}{h\left[\sum_{l=1}^{L} K\left(\frac{b_{1l}/b_{2l}-b}{h}\right)\right]^2}. \tag{2.18}$$

I use Frchet derivatives to find the asymptotic distribution. In terms of the density of the observables:

$$H_\xi(b) = \frac{\int w f(w,b) dw}{f(b)},$$

where $f(w,b)$ is the density of the vector (w,b) and $b = b_1/b_2$. By taking the derivative with respect to b we get:

$$h_\xi(b) = \frac{f(b)\int w \frac{\partial f(w,b)}{\partial b} dw - \frac{\partial f(b)}{\partial b}\int w f(w,b) dw}{f(b)^2} = \frac{f(b)\int w f'(w,b) dw - f'(b)\int w f(w,b) dw}{f(b)^2}$$

Assumption 2.7. *The data on $\{b_i, w_i\}$ is i.i.d.*

Assumption 2.8. *The density $f(b)$ has compact support, is continuously differentiable of order $m \geq \delta + k$, $k \geq 2$, with derivatives which are uniformly bounded.*

Assumption 2.9. *The kernel function is of order δ, it has compact support and is continuously differentiable on its support.*

Assumption 2.10. *As $L \to \infty$, $h \to 0$, $\sqrt{Lh^3} \to \infty$, $\sqrt{Lh^{3+2k}} \to 0$.*

Then the following theorem holds:

Theorem 2.1. *Given the assumptions about the model as well as that Assump-*

tions 2.7-2.10 are satisfied:

$$\hat{h}_\xi(b) \xrightarrow{p} h_\xi(b), \ \textit{and}$$

$$\sqrt{Lh^3}(\hat{h}_\xi(b) - h_\xi(b)) \rightarrow N(0, V_\xi),$$

where

$$V_\xi = \left[\frac{P(w=1|b)(1-P(w=1|b)))}{f^2(b)}\right] \int \left(\frac{\partial K(u)}{\partial u}\right)^2 du$$

Proof: In the Appendix. □

In its turn, the bid density can be estimated using the kernel estimator as follows:

$$\hat{g}_i(b_i) = \frac{1}{Lh} \sum_{l=1}^{L} K\left(\frac{b_i - b_{il}}{h}\right), \tag{2.19}$$

Then the pseudo-values are estimated using the combination of $\hat{h}_\xi(b)$ and $\hat{g}_{b_i}(b)$:

$$\hat{v}_1 = \frac{1}{\int_{\underline{\hat{b}}_2}^{\bar{\hat{b}}_2} \hat{g}_2(b_2) \frac{1}{b_2} \hat{h}_\xi\left(\frac{b_1}{b_2}\right) db_2} \tag{2.20}$$

and

$$\hat{v}_2 = \frac{1}{\int_{\underline{\hat{b}}_1}^{\bar{\hat{b}}_1} \hat{g}_1(b_1) \frac{b_1}{b_2^2} \hat{h}_\xi\left(\frac{b_1}{b_2}\right) db_1} \tag{2.21}$$

and pseudo types then are:

$$\hat{c}_1 = \int_{\hat{\underline{b}}_2}^{\hat{\bar{b}}_2} \hat{g}_2(b_2) \frac{1}{b_2} \hat{h}_\xi \left(\frac{b_1}{b_2} \right) db_2 \tag{2.22}$$

and

$$\hat{c}_2 = \int_{\hat{\underline{b}}_1}^{\hat{\bar{b}}_1} \hat{g}_1(b_1) \frac{b_1}{b_2^2} \hat{h}_\xi \left(\frac{b_1}{b_2} \right) db_1 \tag{2.23}$$

Then the following theorem holds:

Theorem 2.2. *Given the assumptions about the model as well as Assumptions 2.71-2.10 are satisfied:*

$$\hat{c_1}(b_1) \xrightarrow{p} c_1(b_1), \text{ and}$$
$$\sqrt{Lh^3}(\hat{c_1}(b_1) - c_1(b_1)) \to N(0, V),$$

where

$$V = \int_{\underline{b}_2}^{\bar{b}_2} g_2^2(b_2) \frac{1}{b_2^2} \left[\frac{P(w=1|\frac{b_1}{b_2})(1 - P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Similarly:

$$\hat{c_2}(b_2) \xrightarrow{p} c_2(b_2), \text{ and}$$
$$\sqrt{Lh^3}(\hat{c_2}(b_2) - c_2(b_2)) \to N(0, V),$$

where

$$V = \int\limits_{\underline{b}_1}^{\bar{b}_1} g_1^2(b_1) \frac{b_1^2}{b_2^4} \left[\frac{P(w=1|\frac{b_1}{b_2})(1-P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Proof: In the Appendix. □

$r_i(\cdot)$ can be estimated from observed bids:

$$\hat{r}_i(t) = b_i^{(\lceil Lt \rceil : L)}, \tag{2.24}$$

where $b_i^{(\lceil s \rceil : L)}$ is the s-th lowest order statistic out of L i.i.d. bids observations; $\lceil \cdot \rceil$ is the ceiling function.

In the second step, the quantile functions of the bidder's valuations are estimated:

$$\hat{q}_1(t_1) = \frac{1}{\int\limits_0^1 \frac{1}{\hat{r}_2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_2} \tag{2.25}$$

and

$$\hat{q}_2(t_2) = \frac{1}{\int\limits_0^1 \frac{\hat{r}_1(t_1)}{\hat{r}_2^2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_1}, \tag{2.26}$$

where t_1, $t_2 \in (0,1)$.

Note that the invertibility of the bid function is the key for identification as we relied on the assumption that the bidders use a strictly increasing bid function.

Similarly, we can estimate the quantile functions of types:

$$\hat{q}_1^c(1-t_1) = \int\limits_0^1 \frac{1}{\hat{r}_2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_2 \tag{2.27}$$

and

$$\hat{q}_2^c(1-t_2) = \int_0^1 \frac{\hat{r}_1(t_1)}{\hat{r}_2^2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_1, \tag{2.28}$$

where t_1, $t_2 \in (0,1)$.

Proposition 2.7. *(Csorgo (1983)) Let G be a twice differentiable distribution function, having finite support. Assume $\inf_{0<t<1} g(G^{-1}(t)) > 0$ and $\sup_{0<t<1} |g'(G^{-1}(t))| < \infty$. Then $\sup_{0<t<1} |\hat{r}(t) - r(t)| \xrightarrow{a.s.} 0$.*

$\Rightarrow$

$$\sup_{0<t<1} |\hat{r}(t) - r(t)| = o_p(1).$$

It can be proved that:

Proposition 2.8. *Under the same assumptions as above:*

$$\hat{q}_i(t) - q_i(t) = o_p(1),$$

$i = 1, 2$.

2.5 Monte Carlo Simulations

Example 2.1. *If the true quantile functions q_i, $i = 1, 2$ of the bidder's valuations are*

$$q_1(t_1) = \frac{1-k}{k} c_2^k c_1^{1-k} t_1^{1-k}, \quad q_2(t_2) = \frac{1+k}{k} c_2^{k+1} c_1^{-k} t_2^{1+k} \quad t_1, t_2 \in (0,1),$$

and the distribution of ξ is Beta-distribution: $h_\xi(x) = kx^{k-1}$, $0 \le x \le 1$, $0 < k < 1$, then there exist unique equilibrium bid functions $s_1(v_1) = c_1 F_1(v_1)$ and $c_2(v) = k_2 F_2(v)$ for any c_1 and c_2.

Let's consider $L = 200$ auctions with 2 bidders and 100 Monte Carlo replications. Then the following Figure 2.1 below presents the true quantile function, the mean, the 5% quantile, and the 95% quantile of the 100 estimates $\hat{q}_1(t)$ and $\hat{q}_2(t)$ for $c_1 = 4$, $c_2 = 2$ and $k = 0.2$.

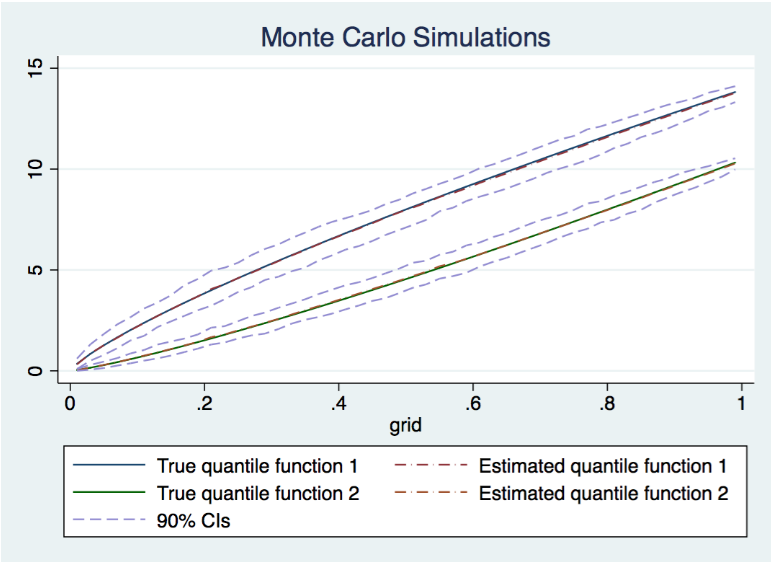


Figure 2.1: Results of Monte Carlo simulations

2.6 Application: U.S. House of Representatives

The theory described in the previous sections can be applied to quantify the incumbency advantage in the U.S. House of Representatives elections. Moreover, the model provides a method to separate the advantage into two parts. The first advantage is due to the better reputation of the Incumbent. It is characterized by the fact that even when both the Incumbent and the Challenger spend the same amount of money on their campaign, the probability that the Incumbent wins is estimated to be bigger than that of the Challenger. This probability is given by the $P(\xi < 1)$, which is determined by the distribution of uncertainty. In its turn, the second advantage is due to the difference in campaign financing, which is characterized by the difference in the quantile functions of candidates' types, where the type describes how costly is it for the candidate to raise money. I show that the Incumbent has a lower type and thus has better campaign financing. The important difference between these two advantages is that only the latter can be influenced by the policymakers, whereas the reputation can not. Thus it is crucial for policy implications to be able to distinguish and quantify them.

2.6.1 U.S. House Elections: Incumbents vs. Challengers

I use the data from U.S. House of Representatives elections.[6] These elections happen every two years. Currently, there are 435 voting seats; winners serve 2-year terms. To quantify the incumbency advantage, I use the data on 6578 Incumbent-Challenger elections during the 1972-2016 period. All the Incumbent's and the Challenger's expenditures are in $2016. The summary statistics is presented in Table 2.1 below. Incumbents won in 93.8% of contests. On average, Incumbents spend 2.5 times as much as the Challenger. Throughout the observed period, expenditures are increasing with only a slight decline starting in 2010. Please see Figure 2.2 below.

Table 2.1: Summary statistics of the Incumbent-Challenger elections

	Obs	Mean	Std. Dev.	Min	Max
Incumbent's Expenditures	6578	1057.67	1044.42	.198	26859.96
Challenger's Expenditures	6578	401.08	698.81	.002	10839.82
Vote share for Incumbent	6562	64.17	9.12	34.13	94.66
Incumbent winning dummy	6578	.938	.240	0	1

* Expenditures are in thousands of dollars

[6] I am very grateful to Gary Jacobson, Professor of Political Science at the University of California, San Diego, for providing me with his data.

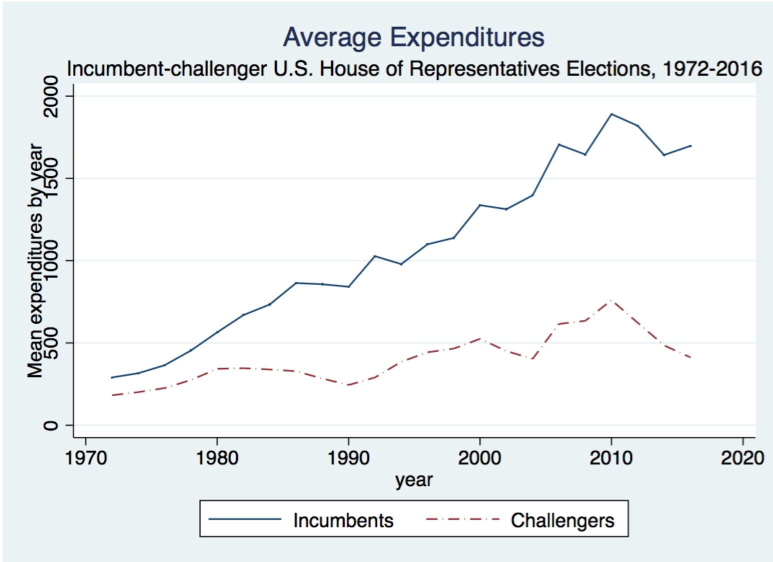


Figure 2.2: Average expenditures by election cycle

Thus we observe the data on 6578 auctions with two bidders each, and winning outcomes, where bidders are candidates and bids are expenditures.

The first step is the estimation of the distribution of uncertainty ξ using equations (2.17) and (2.18) above. The normal kernel and the optimal bandwidth are used. The results are shown below in Table 2.2.

Table 2.2: Cumulative distribution function $H_\xi(\cdot)$

b	1	2	3	4	5	7	7
$\hat{H}_\xi(b)$	0.85	0.88	0.91	0.94	0.97	0.98	1.00

Here b represents the ratio of the Incumbent's and the Challenger's bids. If $b = 1$, expenditures are equal, and $H_\xi(\cdot)$ represents the winning probability of the Incumbent in this case. Thus the first incumbency advantage is represented by 85% winning probability even in the case when the expenditures are the same.

The second step is the estimation of quantile functions of types. We start with the estimation of the quantile function of bids using equation (2.24) and then plug the estimates into equations (2.27) and (2.28) to find the quantile functions of types.

Figure 2.3 represents the results of the model estimation.[7] I divide all election cycles by decades. The result reflects the Incumbent's advantage in campaign financing as the Challenger's type first-order stochastically dominates the Incumbent's type distribution.

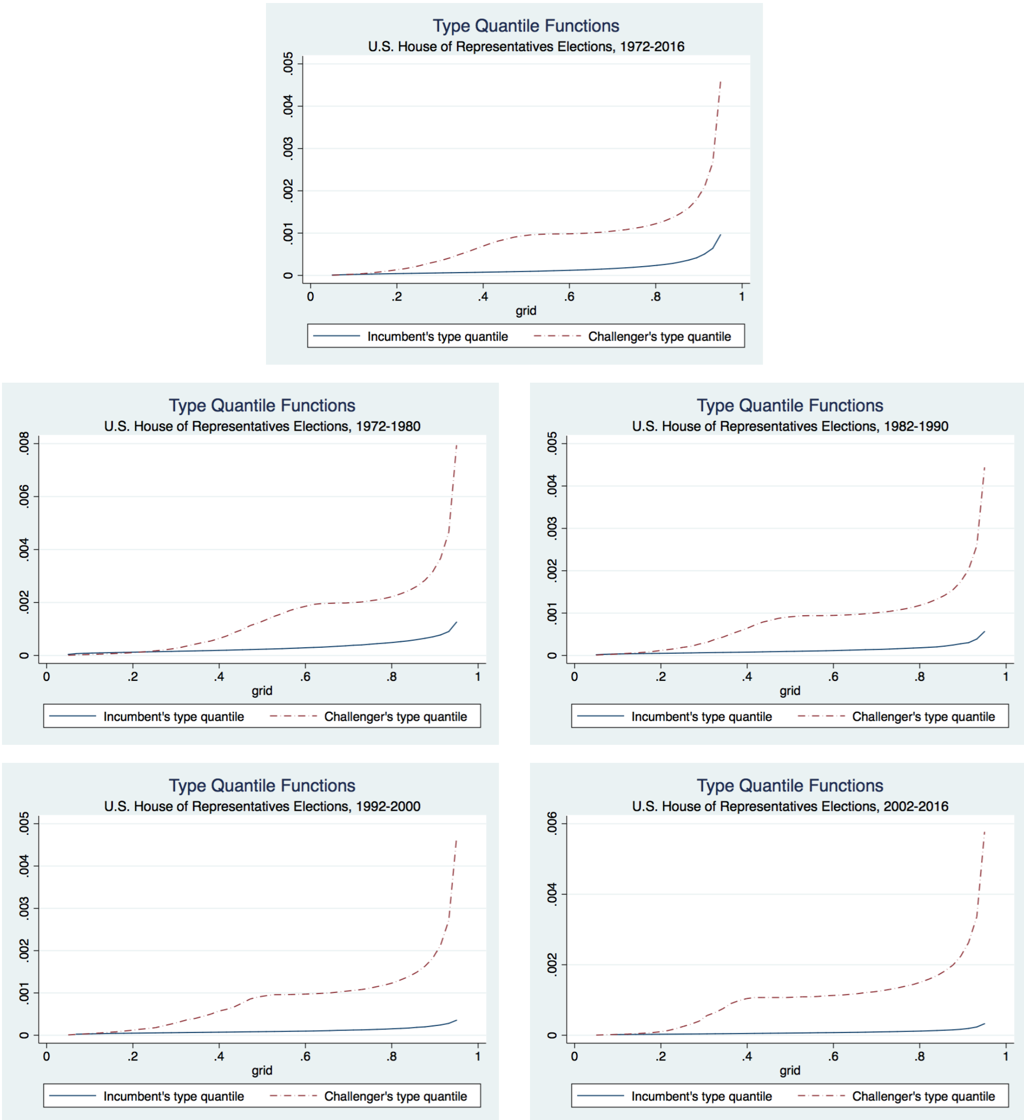


Figure 2.3: Estimated Quantile Functions of Types by Decade

[7]On the boundaries, the quantile functions were monotonized as the kernel estimators tend to be biased close to the boundaries.

I also present the change of the quantile functions over the decades in Figure 2.4 below.

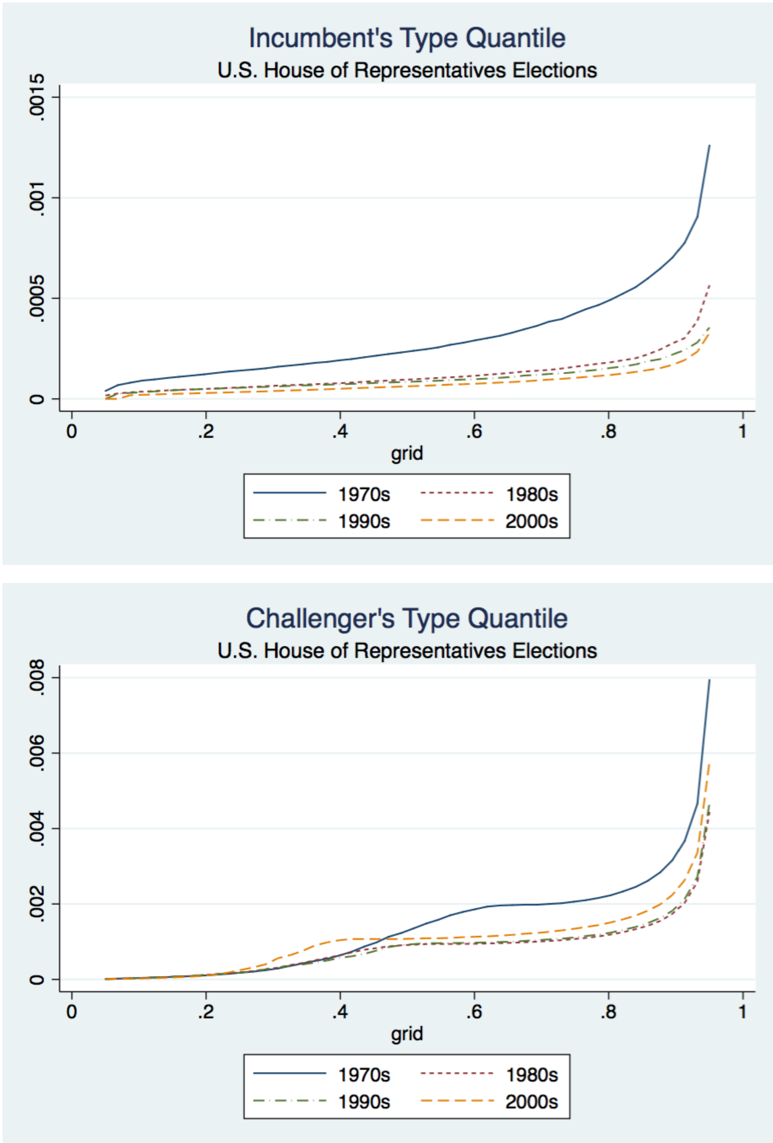


Figure 2.4: Estimated Quantile Functions of Types over Decades

We can also estimate the Incumbent's and the Challenger's valuation quantile functions instead. See Figure 2.5 below.

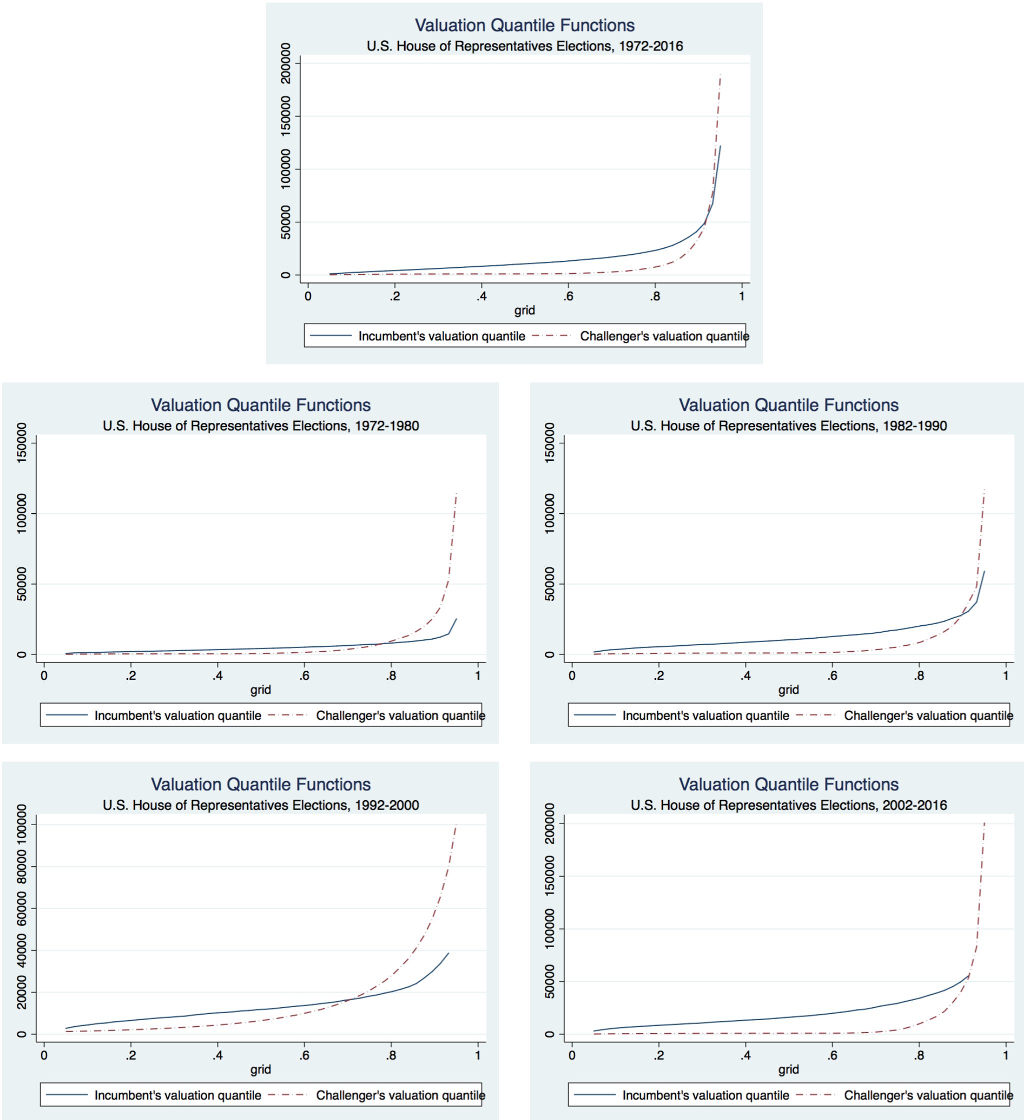


Figure 2.5: Estimated Quantile Functions of Valuations

2.6.2 Counterfactuals

Once the primitives of the model – such as the distribution of uncertainty and the type distributions – are estimated, researches have the capability to run the counterfactual simulations. Counterfactuals allow testing different ways to limit the incumbency advantage. Limiting the incumbency advantage is important for the following reasons. First, according to the prevalent opinion in political science, democracy is not possible without sufficient competition as well as the turnover of the seats in Congressional elections. Moreover, the increased total campaign spending is costly for society. Thus it would be useful to consider the policy that reduces the Incumbent's winning probability, as well as the total campaign spending.

Two well-known policies are the limit on expenditures and public campaign financing. According to Jacobson (1978): "Even though Incumbents raise money more easily from all sources, limits on contributions will not help Challengers because the problem is not equalizing spending between candidates but rather simply getting more money to Challengers so that they can mount competitive races." The reason behind that statement is that the marginal effect of the Challenger's expenditure on the probability to win is greater than that of the Incumbent. Although that is true, this logic doesn't take into account the underlying game between the Incumbent and the Challenger. In reality, as the Challenger increases expenditures, the low-type Incumbent also does so, and as a result, the effect on winning probability is uncertain.

Let us consider two policies, one by one and compare the conclusions.

2.6.2.1 Public Campaign Financing

First, I consider public campaign financing for the Challenger, which lowers his type's distribution. I quantify the effect of the limit case of the public financing of the Challengers such that the resulting type quantile function matches one of the Incumbents. This case eliminates the advantage due to the difference in types completely, since now the types are assumed to be the same.

I take the equal quantile functions of the Incumbent and the Challenger as given. The goal is to find the optimal strategies of the players using the equations (2.27) and (2.28) and solving the inverse problem of finding $\hat{r}_1(\cdot)$ and $\hat{r}_2(\cdot)$ from the $\hat{q}_1(\cdot)$ and $\hat{q}_2(\cdot)$. I do that by approximating the bid quantile function by the exponential distribution $-\frac{log(1-t)}{\lambda}$. After that, I calculate the Incumbent's winning probability knowing the bid strategies and the distribution of uncertainty. Results are presented in Figure 2.3 below.

Table 2.3: Public campaign financing: resulting winning probability

	Incumbent's probability of winning				
	All	72-80	82-90	92-2000	2002-2016
Original	0.938	0.929	0.953	0.941	0.935
With challenger's financing	0.896	0.907	0.917	0.895	0.847
Decrease	0.042	0.22	0.36	0.046	0.088

The Incumbent's winning probability decreases by 4.2% from 93.8% to 89.6%.

Moreover, the reform leads to the increase in expenditures of both candidates, see Table 2.4 below:

Table 2.4: Public campaign financing: resulting expenditures

	All	72-80	82-90	92-2000	2002-2016
	Mean of incumbent's expenditures				
Original	1057	394	792	1110	1650
With challenger's financing	1623	512	1092	1597	3381
Increase	566	118	300	487	1731
	Mean of challenger's expenditures				
Original	401	243	309	420	557
With challenger's financing	997	329	721	1073	1976
Increase	596	86	412	653	1419

* Expenditures are in thousands of $

2.6.2.2 Limit on Expenditure

The other popular policy is the limit on expenditure. I consider such a case that both candidates spend the same amount. Thus I do not allow the Incumbent to spend more than the Challenger. In this case, $b_1 = b_2$ and the Incumbent's winning probability becomes:

$$P(b_1\epsilon_1 > b_2\epsilon_2) = P(\epsilon_1 > \epsilon_2) = P(\epsilon_2/\epsilon_1 < 1) = H_\xi(1)$$

Using this formula and equation (2.17), I estimate the winning probability. Results are presented in Table 2.5 below.

Table 2.5: Limit on expenditure results

	Incumbent probability of winning				
	All	72-80	82-90	92-2000	2002-2016
Original	0.938	0.929	0.953	0.941	0.935
With the expenditure constraint	0.851	0.873	0.885	0.852	0.789
Decrease	0.087	0.056	0.068	0.089	0.146

It can be seen that the Incumbent's winning probability drops by 8.7% from 93.8% to 85.1%, a bigger change than with public campaign financing for the Challenger.

In conclusion, the Challenger's public financing is not as effective as the limit on expenditures in terms of both lowering the Incumbent's winning probability as well as on the total campaign spending. Thus by taking into account the game structure of the model, I have shown that the predictions change once the game-theoretical structure of the interactions between the candidates is taken into account.

2.7 Conclusion

In this work, I identified and estimated the incomplete information contest with nonparametric contest success function. As a result, I recovered the distribution of valuations or, alternatively, types from the bid distributions and win outcomes. Here types characterize how costly it is to raise the bid and is just the inverse of the valuation. This model provides the framework that can be applied to the variety of real-life scenarios such as marketing and advertising by firms, litigation, research and development, patent race, procurement of innovative good, research contests, sports events, arms race, rent-seeking activity, such as lobbying, as well as electoral competition. I apply the model to the U.S. House of Representatives elections, which were also studied by He and Huang (2018) in the case of Tullock contest. The model results show the incumbency advantage and can distinguish the two sources of it. Moreover, the decrease in how costly is it to raise money for the election over decades is observed. The knowledge of the types' distributions allows quantifying the effect of different policy changes such as limits on expenditures or funding for Challengers in order to eliminate incumbency advantage. By comparing these two policies, I found the former to be more effective.

2.8 Appendix

2.8.1 Proof of Proposition 2.1

Proof: In case when both ϵ_i and ϵ_j have exponential distribution with parameter $\lambda = 1$, $f_\epsilon(t) = e^{-t}$ and in its turn $F_\epsilon(t) = 1 - e^{-t}$. As a result:

$$P(w_1 = 1|\ b_1, b_2) = P(x_1 > x_2 \mid b_1, b_2) = P(b_1\epsilon_1 > b_2\epsilon_2 \mid b_1, b_2) =$$

$$= P(\epsilon_2 < \frac{b_1}{b_2}\epsilon_1 \mid b_1, b_2) = \int_0^{+\infty} F_\epsilon\left(\frac{b_1}{b_2}t\right) f_\epsilon(t)dt = \int_0^{+\infty} \left(1 - e^{-\frac{b_1}{b_2}t}\right) e^{-t}dt =$$

$$= 1 - \frac{1}{1 + \frac{b_1}{b_2}} = \frac{b_1}{b_1 + b_2}$$

which is the contest success function of the well-known Tullock contest. □

2.8.2 Proof of Proposition 2.2

Proof: Let us consider all assumptions required for the Theorem 6 in Athey (2001) to hold.

1. $f_i(\cdot)$ is density with respect to Lebesque measure, bounded and atomless.
2. $U_i = p_i(b_1, b_2)(v_i - b_i) + (1 - p_i(b_1, b_2))(-b_i)$ can be written in the general form considered in the paper.
3. Winner's payoff $v_i - b_i$ and loser's payoff $-b_i$ are continuous in (v_i, b) and bounded as v_i has a finite support $[\underline{v_i}, \bar{v_i}]$ and the bidders won't find it profitable to bid more than the valuation.
4. Expected utility $E[U_i] = \int p_i(b_i, s_j(v_j))f_j(v_j)dv_j - b_i$ is bounded and finite.

5. Single-crossing condition $\frac{\partial^2 U_i}{\partial v_i \partial b_i} \geq 0$ is satisfied as:

$$\frac{\partial^2 U_1}{\partial v_1 \partial b_1} = \frac{\partial P_1}{\partial b_1} = \frac{1}{b_1} h_\xi \left(\frac{b_1}{b_2} \right) > 0,$$

$$\frac{\partial^2 U_2}{\partial v_2 \partial b_2} = \frac{\partial P_2}{\partial b_2} = \frac{b_1}{b_2^2} h_\xi \left(\frac{b_1}{b_2} \right) > 0.$$

Thus all the assumptions of Theorem 6 in Athey (2001) are satisfied, hence there exists a pure-strategy Bayesian Nash Equilibrium in nondecreasing strategies. Since the single-crossing property holds with strict inequality, this equilibrium is actually in increasing strategies. □

2.8.3 Proof of Example 2.1

Proof: Given that the distribution of ξ is Beta-distribution: $h_\xi(x) = kx^{k-1}$, $0 \leq x \leq 1$, $0 < k < 1$, an equation (2.3) can be rewritten as

$$v_1 = \frac{1}{\int\limits_{\underline{v}_2}^{\bar{v}_2} \frac{1}{s_2(v_2)} k \frac{s_1(v_1)^{k-1}}{s_2(v_2)^{k-1}} dF_2(v_2)} = \frac{1}{k s_1(v_1)^{k-1} \int\limits_{\underline{v}_2}^{\bar{v}_2} s_2(v_2)^{-k} dF_2(v_2)}$$

In its turn, equation (2.4) becomes:

$$v_2 = \frac{1}{\int\limits_{\underline{v}_1}^{\bar{v}_1} \frac{s_1(v_1)}{s_2^2(v_2)} k \frac{s_1(v_1)^{k-1}}{s_2(v_2)^{k-1}} dF_1(v_1)} = \frac{1}{k s_2(v_2)^{-k-1} \int\limits_{\underline{v}_1}^{\bar{v}_1} s_1(v_1)^k dF_1(v_1)}$$

If we plug in $s_1(v_1) = c_1 F_1(v_1)$ and $s_2(v_2) = c_2 F_2(v_2)$ we get:

$$v_1 = \frac{1}{k(c_1 F_1(v_1))^{k-1} \int\limits_{\underline{v}_2}^{\bar{v}_2} (c_2 F_2(v_2))^{-k} dF_2(v_2)} = \frac{1}{\frac{k}{1-k} \frac{c_1^{k-1}}{c_2^k} (F_1(v_1))^{k-1}},$$

similarly

$$v_2 = \frac{1}{k(c_2 F_2(v_2))^{-k-1} \int\limits_{\underline{v}_1}^{\bar{v}_1} (c_1 F_1(v_1))^k dF_1(v_1)} = \frac{1}{\frac{k}{1+k} \frac{c_1^k}{c_2^{k+1}} (F_2(v_2))^{-k-1}}.$$

In terms of quantile functions this can be written as:

$$q_1(t_1) = \frac{1-k}{k} c_2^k c_1^{1-k} t_1^{1-k}, \quad q_2(t_2) = \frac{1+k}{k} c_2^{k+1} c_1^{-k} t_2^{1+k} \quad t_1, t_2 \in (0,1),$$

where $q_1 = F_1^{-1}$ and $q_2 = F_2^{-1}$. $\square$

2.8.4 Proof of Theorem 2.1

Proof: First, we would like to estimate $h_\xi(b)$ - the derivative of

$$H_\xi(b) = \frac{\int w f(w,b) dw}{f(b)}.$$

By taking the derivative with respect to b we get:

$$h_\xi(b) = \Phi(f) = \frac{f(b) \int w \frac{\partial f(w,b)}{\partial b} dw - \frac{\partial f(b)}{\partial b} \int w f(w,b) dw}{f(b)^2} =$$
$$\frac{f(b) \int w f'(w,b) dw - f'(b) \int w f(w,b) dw}{f(b)^2}$$

$$\Phi(f+h) =$$
$$= \frac{[f(b)+h(b)] \int w[f'(w,b)+h'(w,b)]dw - [f'(b)+h'(b)] \int w[f(w,b)+h(w,b)]dw}{[f(b)+h(b)]^2}$$

$$\Phi(f+h)-\Phi(f) =$$

$$\frac{f(b)^2[f(b)+h(b)]\int w[f'(w,b)+h'(w,b)]dw}{f(b)^2[f(b)+h(b)]^2} -$$
$$\frac{f(b)^2[f'(b)+h'(b)]\int w[f(w,b)+h(w,b)]dw}{f(b)^2[f(b)+h(b)]^2} +$$
$$\frac{-f(b)[f(b)+h(b)]^2\int wf'(w,b)dw}{f(b)^2[f(b)+h(b)]^2} +$$
$$\frac{f'(b)[f(b)+h(b)]^2\int wf(w,b)dw}{f(b)^2[f(b)+h(b)]^2}$$

$$\begin{aligned}
Num = f^3(b)\int wf'(w,b)dw + f^3(b)\int wh'(w,b)dw+\\
+f^2(b)h(b)\int wf'(w,b)dw + f^2(b)h(b)\int wh'(w,b)dw-\\
-f^2(b)f'(b)\int wf(w,b)dw - f^2(b)f'(b)\int wh(w,b)dw-\\
-f^2(b)h'(b)\int wf(w,b)dw - f^2(b)h'(b)\int wh(w,b)dw-\\
-f^3(b)\int wf'(w,b)dw - 2f^2(b)h(b)\int wf'(w,b)dw-\\
-f(b)h^2(b)\int wf'(w,b)dw + f'(b)f^2(b)\int wf(w,b)dw+\\
+2f'(b)f(b)h(b)\int wf(w,b)dw + f'(b)h^2(b)\int wf(w,b)dw =\\
f^3(b)\int wh'(w,b)dw - f^2(b)h(b)\int wf'(w,b)dw+\\
+f^2(b)h(b)\int wh'(w,b)dw - f^2(b)f'(b)\int wh(w,b)dw-\\
-f^2(b)h'(b)\int wf(w,b)dw - f^2(b)h'(b)\int wh(w,b)dw-\\
-f(b)h^2(b)\int wf'(w,b)dw + 2f'(b)f(b)h(b)\int wf(w,b)dw+\\
+f'(b)h^2(b)\int wf(w,b)dw = Q+P,
\end{aligned}$$

Where:

$$Q = f^3(b)\int wh'(w,b)dw - f^2(b)h(b)\int wf'(w,b)dw-$$
$$-f^2(b)f'(b)\int wh(w,b)dw - f^2(b)h'(b)\int wf(w,b)dw+$$
$$+2f'(b)f(b)h(b)\int wf(w,b)dw$$

$$P = f^2(b)h(b)\int wh'(w,b)dw - f^2(b)h'(b)\int wh(w,b)dw-$$
$$-f(b)h^2(b)\int wf'(w,b)dw + f'(b)h^2(b)\int wf(w,b)dw$$

Moreover:

$$\frac{1}{f^2(f+h)^2} = \frac{1}{f^4} + \frac{1}{f^2(f+h)^2} - \frac{1}{f^4} = \frac{1}{f^4} - \frac{2hf+h^2}{f^4(f+h)^2}$$

As a result:

$$\Phi(f+h) - \Phi(f) =$$
$$= \frac{Q}{f^4(b)} + \frac{P}{f^4(b)} - \frac{Q(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2} - \frac{P(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2} = D\Phi(f,h) + R\Phi(f,h),$$

Where:

$$D\Phi(f,h) = \frac{Q}{f^4(b)}$$

$$R\Phi(f,h) = \frac{P}{f^4(b)} - \frac{Q(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2} - \frac{P(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2}.$$

Thus, for some constant $A < \infty$:

$$|D\Phi(f,h)| < A||h|| \text{ and } \quad |R\Phi(f,h)| < A||h||^2.$$

Using Newey (1994), Lemma 5.3: $||h|| \xrightarrow{p} 0$. And thus,

$$sup|\Phi(f+h) - \Phi(f)| \leq A||h|| + a||h||^2 \xrightarrow{p} 0.$$

Now let us derive the asymptotic distribution.

$$\begin{aligned} Q = f^3(b) \int wh'(w,b)dw - f^2(b)h(b) \int wf'(w,b)dw- \\ -f^2(b)f'(b) \int wh(w,b)dw - f^2(b)h'(b) \int wf(w,b)dw+ \\ +2f'(b)f(b)h(b) \int wf(w,b)dw \end{aligned}$$

In its turn,

$$f(b) = \int f(w,b)dw.$$

Thus

$$\begin{aligned} Q = f^3(b) \int wh'(w,b)dw - f^2(b)h'(b) \int wf(w,b)dw + T = \\ = f^3(b) \int w(\hat{f}'(w,b) - f'(w,b))dw- \\ -f^2(b) \int ((\hat{f}'(w,b) - f'(w,b))dw) \int wf(w,b)dw + T = \\ = \int f^2(b) \left[wf(b) - \int wf(w,b)dw \right] (\hat{f}'(w,b) - f'(w,b))dw + T, \end{aligned}$$

where T depends only on h, but not h'. Thus the terms in T converge faster than the ones that depend on the derivative estimate.

As a result,

$$D = \int \left[\frac{wf(b) - \int wf(w,b)dw}{f^2(b)} \right] (\hat{f}'(w,b) - f'(w,b))dw + T.$$

Using Newey (1994), Lemma 5.3, we find that

$$\sqrt{Lh^3}(\Phi(f+h) - \Phi(f)) \to N\left(0, V_\xi\right),$$

where

$$V_\xi = \int \left[wf(b) - \int wf(w,b)dw \right]^2 \frac{f(w,b)}{f^4(b)} dw \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du.$$

As w can only take 2 values 0 and 1: $\int wf(w,b)dw = f(1,b)$ and

$$\begin{aligned} V_\xi = \left[\frac{f(1,b)^2 f(0,b) + f(0,b)^2 f(1,b)}{f^4(b)} \right] \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du = \\ = \left[\frac{f(1,b) f(0,b)}{f^3(b)} \right] \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du \end{aligned}$$

Moreover,

$$\begin{aligned} f(1,b) = f(b)P(w=1|b) \ and \\ f(0,b) = f(b)P(w=0|b) = f(b)(1 - P(w=1|b)), \end{aligned}$$

thus

$$V_\xi = \left[\frac{P(w=1|b)(1 - P(w=1|b)))}{f(b)} \right] \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

And as a result,

$$\hat{h}_\xi(b) \to h_\xi(b) \textit{ in probability, and}$$
$$\sqrt{Lh^3}(\hat{h}_\xi(b) - h_\xi(b)) \to N(0, V_\xi),$$

where

$$V_\xi = \left[\frac{P(w=1|b)(1-P(w=1|b)))}{f^2(b)}\right]\int\left(\frac{\partial K(u)}{\partial u}\right)^2 du$$

. $\square$

2.8.5 Proof of Theorem 2.2

Proof: Now let us denote by $\tilde{f}(w, b_1, b_2)$ the joint density of the vector (w, b_1, b_2) and consider:

$$c_1 = 1/v_1 = \int_{\underline{b}_2}^{\bar{b}_2} g_2(b_2)\frac{1}{b_2}h_\xi\left(\frac{b_1}{b_2}\right)db_2 := \tilde{\Phi}(b_1; f)$$

We also denote:

$$h_\xi\left(\frac{b_1}{b_2}\right) := \phi(b_1, b_2; \tilde{f})$$
$$g_2(b_2)\frac{1}{b_2} := \psi(b_2; \tilde{f})$$

Then

$$\tilde{\Phi}(b_1; \tilde{f}) = \int_{\underline{b}_2}^{\bar{b}_2} \psi(b_2; \tilde{f})\phi(b_1, b_2; \tilde{f})db_2$$

It follows that:

$$\tilde{\Phi}(b_1; \tilde{f} + \tilde{h}) - \tilde{\Phi}(b_1; \tilde{f}) =$$

$$\int_{\underline{b}_2}^{\bar{b}_2} \psi(b_2; \tilde{f} + \tilde{h}) \phi(b_1, b_2; \tilde{f} + \tilde{h}) db_2 - \int_{\underline{b}_2}^{\bar{b}_2} \psi(b_2; \tilde{f}) \phi(b_1, b_2; \tilde{f}) db_2$$

$$= \int_{\underline{b}_2}^{\bar{b}_2} \phi(b_1, b_2; \tilde{f}) D\psi(b_1, b_2; \tilde{f}) db_2 + \int_{\underline{b}_2}^{\bar{b}_2} D\psi(b_2; \tilde{f}) \phi(b_1, b_2; \tilde{f}) db_2 + the\ rest =$$

$$= \int_{\underline{b}_2}^{\bar{b}_2} \int_{w} g_2(b_2) \frac{1}{b_2} \left[\frac{wf(b) - \int wf(w,b)dw}{f^2(b)} \right] (\hat{f}'(w,b) - f'(w,b)) dw db_2 + the\ rest,$$

where $b = b_1/b_2$.

The rest converges faster as the rate of convergence of $\hat{f}'(w,b)$ is slower than that of $\hat{f}(w,b)$.

Thus:

$$\hat{c_1}(b_1) \to c_1(b_1)\ in\ probability,\ and$$
$$\sqrt{Lh^3}(\hat{c_1}(b_1) - c_1(b_1)) \to N(0, V),$$

where

$$V = \int_{\underline{b}_2}^{\bar{b}_2} g_2^2(b_2) \frac{1}{b_2^2} \left[\frac{P(w=1|\frac{b_1}{b_2})(1 - P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Similarly:

$$\hat{c_2}(b_2) \to c_2(b_2)\ in\ probability,\ and$$
$$\sqrt{Lh^3}(\hat{c_2}(b_2) - c_2(b_2)) \to N(0, V),$$

where

$$V = \int_{\underline{b}_1}^{\bar{b}_1} g_1^2(b_1) \frac{b_1^2}{b_2^4} \left[\frac{P(w=1|\frac{b_1}{b_2})(1-P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

$\square$

CHAPTER 3

Nonparametric Identification and Estimation of a Serial Contest

3.1 Introduction

In this work, I prove the identification and derive the consistent estimator of a contest model where an object is allocated according to the allocation rule determined by the serial contest success function (CSF). Every bidder pays his bid regardless of whether he wins or not. The model is a game of incomplete information in the sense that the bidders do not observe the other bidders' valuations, but the distribution of the valuations is common knowledge. Identification and estimation of the model primitives is a crucial part of any policy intervention. Similar to Guerre et al. (2000) and He and Huang (2018) I propose a method that allows the researcher to estimate the distribution of bidders' valuations using the data on their bids. This method does not require any parametric assumptions, nor does it require Bayesian Nash equilibrium strategy computation, which makes the method computationally attractive.

The contest is a natural model of costly competition as it describes situations where all players exert costly effort to achieve some goal (win the contest). This is sunk cost, in the sense that it is paid no matter whether a player wins or loses. Such interactions include a wide range of scenarios such as marketing and advertising by firms (Bell et al. (1975)), litigation (Hirshleifer and Osborne (2001), Baye et al. (2005)); research and development, patent race, procurement of innovative good, research contests (Taylor (1995), Che and Gale (2003)); sports events, arms race and rent-seeking activity, such as lobbying (Tullock (1980), Krueger (1974), Baye et al. (1993)). Electoral competition was also modeled using contest theory since the 1990s, see, for example, Baron (1994), Snyder (1989), or Skaperdas and Grofman (1995).

The contest is determined by the contest success function, which is the function that maps bidders' efforts to the win probability. See the survey by Konrad (2009) for an extensive discussion of different types of contests.[1] The probability of winning satisfies some standard assumptions which are described in detail in Section 2. Depending on the application, different contest success functions are more reasonable to use. In this work, the contest with serial CSF is considered. It was introduced by Alcalde and Dahm (2007), and the main characteristic of this contest is that the win probability depends on the percentage mark-up. The serial contest success function has several advantages over the other widely used CSF such as Tullock (1980) CSF, for which win probability depends on the ratio of the bids, and Hirshleifer (1989) CSF for which win probability depends on the difference of the bids. Compared to the Tullock's contest success function, serial CSF depends not only on the relative bids but also on the absolute differences of bids. In the case of election campaign spending, for instance, it is plausible to think that the difference in spendings matter. For instance, the difference between bids of 10 and 30 thousand dollars might be more impactful than the difference between the bids of 1000 and 3000. On the other hand, with respect to the Hirshleifer's difference-form contests, the serial contest weakens the absolute criterion in the mapping from bids to winning probability and, most importantly, it is homogeneous in bids. The last property is crucial for the applications in which the bid is the expenditure, as we would like to have the property that

[1]All-pay auction is an extreme case of the contest model when the bidder with the highest bid wins, thus the winning probability is one if and only if the bidder has the highest bid. In reality, in the scenarios described above it is common that the contestant with the highest bid can still lose, thus it is important to consider contests for empirical applications.

the win probability does not depend on the units of measurement (like dollars or thousands of dollars).

Most of the theoretical papers on contests and auctions consider games with complete information in the sense that players observe each other's valuations. In reality, it is more plausible to think that the bidders do not observe the private information of the other bidders. Fey (2008), Ryvkin (2010) and Ewerhart (2014) are a few of the papers providing the existence of equilibrium results in the context of the incomplete information contests. The literature on nonparametric identification and estimation of incomplete information auctions and contests is very sparse. There is a block of papers on the first-price auctions originated from Guerre et al. (2000). In my previous research project, I considered all-pay auctions (an extreme case of a contest). The only two papers that consider the identification and estimation of a contest as a game with incomplete information are He and Huang (2018) and my project on nonparametric identification and estimation of the contest model with uncertainty. He and Huang (2018) consider the case when the contest success function has the Tullock's form. In my project, which is presented in Chapter 2, I consider the nonparametric representation of the contest success function. But in the case when the data is sparse, we would need to put some restrictions on the nonparametric structure. Thus, one of the possibilities would be to assume the specific CSF and in this work, I discuss the case of the serial contest success function.

The rest of the paper is organized as follows. In Section 2, I introduce the symmetric contest model with serial CSF. The nonparametric identification and estimation, as well as the Monte Carlo simulations, are discussed in detail. Section 3 considers the asymmetric setting. Similarly, identification analysis, estimation procedure as well as the Monte Carlo simulations are presented. Section 4 concludes.

3.2 Symmetric Contest Model

3.2.1 Notations and Definitions

In this work, I consider a contest model with $N = 2$ risk-neutral bidders. The model can be easily extended to account for the arbitrary number of bidders.

Assumption 3.1. *Each bidder has a private valuation of the good v_i, $i = 1, 2$, which is his private information. He draws this valuation v_i from the distribution $F(v)$ with support $[\underline{v}, \bar{v}]$, density f and quantile function $q = F^{-1}$ independently from the other bidders.*

Assumption 3.2. *When bidders simultaneously submit their bids b_i they know the number of bidders, their own valuations v_i and $F(v)$.*

Thus this is a game with incomplete information.

Assumption 3.3. *Each of the bidders pays b_i, regardless of whether or not he obtains a good.*

The winner is determined according to the contest success function Ψ.

Definition 3.1. *(CSF) A contest success function is a mapping*

$$\Psi : \mathbb{R}_+^N \to \Delta^N,$$

such that for each $b = (b_1, ..., b_N) \in \mathbb{R}_+^N$, $\Psi(b)$ is in the $N-1$ dimensional simplex, i.e. $\Psi(b)$ is such that, for each i, $\Psi_i(b) \geq 0$, and $\sum_{i=1}^N \Psi_i(b) = 1$.

Ψ satisfies the following assumptions:

Monotonicity: $\Psi_i(b_i, b_{-i})$ is weakly increasing in b_i, for any $b_{-i} \in \mathbb{R}_+^{N-1}$, and any $i \in 1, ..., N$;

Zero bids: $\Psi_i(0, b_{-i}) = 0$ for any $b_{-i} \neq 0$, and any $i \in 1, ..., N$; moreover,

$\Psi_i(b_i, b_{-i}) > 0$ for any $b_i > 0$, and any $i \in 1, ..., N$;

Anonymity: $\Psi_i(b) = \Psi_i(b_{\phi(1)}, ..., b_{\phi(N)})$ for any permutation $\phi : 1, ..., N \to 1, ..., N$, any $b \in \mathbb{R}^N_+$, and any $i \in 1, ..., N$;

Smoothness: Ψ_i is continuous on $\mathbb{R}^N_+ \backslash \{0\}$, for any $i \in 1, ..., N$; moreover, the partial derivative $\frac{\partial \Psi_i(b_i, b_{-i})}{\partial b_i}$ exists and is continuous in b_{-i}, for any $b_i > 0$, and any $i \in 1, ..., N$.

Given the contest success function Ψ, the expected payoff to bidder i participating in the contest, is given by:

$$\mathbb{E}[u_i | v_i, N, F(v)] = v_i \mathbb{E}[\Psi_i(b_i, b_{-i}) | b_i, N, F(v)] - b_i.$$

The final payoff to the bidder i is $v_i - b_i$ if he wins, and $-b_i$ if he looses.

In this work, I consider the contest with serial CSF. It was introduced by Alcalde and Dahm (2007). The main characteristic of this type of contests is that the win probabilities depend on the percentage mark-up.

Definition 3.2. *(Serial CSF) If b is an ordered vector of bids such that $b_1 \geq b_2 \geq ... \geq b_n \geq 0$, then the serial CSF with economies of scale parameter $\alpha \geq 0$ assigns for all bidders i:*

$$\Psi_i(b) = \sum_{j=i}^{n} \frac{b_j^\alpha - b_{j+1}^\alpha}{j \cdot b_1^\alpha},$$

with $b_{n+1} = 0$. If b is degenerated, then fair lottery takes place.

The class of serial contest success functions can also be defined recursively as follows:

$$\Psi_i(b) = \Psi_{i+1}(b) + \frac{b_i^\alpha - b_{i+1}^\alpha}{i \cdot b_1^\alpha}, \quad \Psi_n(b) = \frac{b_n^\alpha}{n \cdot b_1^\alpha}.$$

As in the paper I consider 2 bidder case, the CSF takes the following form.

Assumption 3.4. *The winning probability is given by:*

$$\Psi_i(b_i, b_j) = \begin{cases} \frac{1}{2}\left(\frac{b_i}{b_j}\right)^{\alpha}, \textit{ if } b_i \leq b_j \\ 1 - \frac{1}{2}\left(\frac{b_j}{b_i}\right)^{\alpha}, \textit{ if } b_i \geq b_j, \end{cases}$$

where $i, j = 1, 2$.

In case when $\alpha = 1$:

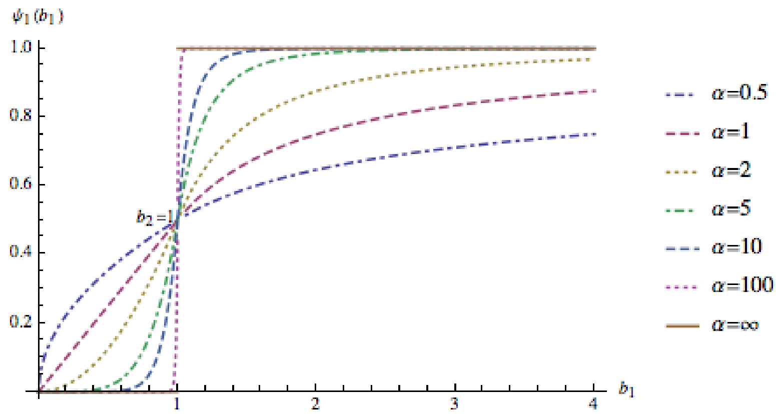
$$\Psi_i(b_i, b_j) = \begin{cases} \frac{1}{2} - \frac{b_j - b_i}{2b_j}, \text{ if } b_i \leq b_j \\ \frac{1}{2} + \frac{b_i - b_j}{2b_i}, \text{ if } b_i \geq b_j. \end{cases}$$

In its turn, the extreme case when $\alpha = \infty$ is equivalent to the all-pay auction:

$$\Psi_i(b_i, b_j) = \begin{cases} 0, \text{ if } b_i \leq b_j \\ 1, \text{ if } b_i \geq b_j. \end{cases}$$

Below is shown the contest success function (the win probability) of the first bidder once the bid of the second bidder b_2 is fixed at the value 1 for different values of the parameter α.

Figure 3.1: Serial contest success function of bidder 1



3.2.2 Equilibrium Characterization

Given Assumptions 3.1-3.4 the expected utility of player i can be written in the following way:

$$\mathbb{E}[u_i(v)|v_i, F(v)] = v_i \mathbb{E}[\Psi_i(b, b_{-i})|b, v_i, N, F(v)] - b_i =$$
$$v_i \left[\int_{b_i}^{\bar{b}} \frac{1}{2} \left(\frac{b_i}{b_j} \right)^{\alpha} dG(b_j) + \int_{\underline{b}}^{b_i} \left(1 - \frac{1}{2} \left(\frac{b_j}{b_i} \right)^{\alpha} \right) dG(b_j) \right] - b_i,$$

where $i = 1, 2,\ j = -i$. The final payoff to the bidder i is $v_i - b_i$ if he obtains a good, and $-b_i$ if he does not obtain a good.

I consider the Bayesian equilibrium in this incomplete information game which is symmetric and strictly monotonic. The existence can be proved using Athey (2001) Theorem 6.

Proposition 3.1. *Given Assumptions 3.1-3.4 there exists a pure strategy increasing BNE of the incomplete information game formulated above.*

Proof. The proof can be found in the Appendix. □

For each valuation, the corresponding bid is defined by the function $s(v) = b$ that is the equilibrium bid strategy which maximizes the bidder's expected payoff. $s(v)$ is invertible and $s^{-1}(b) = v$ given that it is strictly monotonic.

Proposition 3.2. *Given Assumptions 3.1-3.4 as well as the assumption of strict monotonicity of the bidding strategies the first-order conditions of this game can be written as:*

$$v_i = \frac{1}{\frac{\alpha}{2}[s(v_i)^{\alpha-1} \int_{v_i}^{\bar{v}} s(v_j)^{-\alpha} dF(v_j) + s(v_i)^{-\alpha-1} \int_{\underline{v}}^{v_i} s(v_j)^{\alpha} dF(v_j)]}, \tag{3.1}$$

where $i = 1, 2,\ j = -i$.

Proof: The expected payoff to bidder i when his true valuation is v_i but he bids as if it was v can be written as follows:

$$\begin{aligned}
\mathbb{E}[u_i(v)|v_i, F(v)] = v_i \mathbb{E}[\Psi_i(b, b_{-i})|b, v, N, F(v)] - b = \\
= v_i \left[\int_b^{\bar{b}} \frac{1}{2} \left(\frac{b}{b_j} \right)^{\alpha} dG(b_j) + \int_{\underline{b}}^{b} \left(1 - \frac{1}{2} \left(\frac{b_j}{b} \right)^{\alpha} \right) dG(b_j) \right] - b = \\
= v_i \left[\int_v^{\bar{v}} \frac{1}{2} \left(\frac{s(v)}{s(v_j)} \right)^{\alpha} dF(v_j) + \int_{\underline{v}}^{v} \left(1 - \frac{1}{2} \left(\frac{s(v_j)}{s(v)} \right)^{\alpha} \right) dF(v_j) \right] - s(v) = \\
= v_i \left[\frac{1}{2} s(v)^{\alpha} \int_v^{\bar{v}} s(v_j)^{-\alpha} dF(v_j) - \frac{1}{2} s(v)^{-\alpha} \int_{\underline{v}}^{v} s(v_j)^{\alpha} dF(v_j) + F(v) \right] - s(v),
\end{aligned}$$

since $b = s(v)$ and $G(b) = Pr(b_i \leq b) = Pr(v_i \leq s^{-1}(b)) = F(s^{-1}(b)) = F(v)$ as $s(v)$ is invertible.

Using the first order condition (FOC) (differentiating with respect to v and substituting $v = v_i$), we get:

$$\begin{aligned}
\frac{\partial \mathbb{E}[u_i]}{\partial v} = v_i [\frac{1}{2} \alpha s(v)^{\alpha-1} s'(v) \int_v^{\bar{v}} s(v_j)^{-\alpha} dF(v_j) - \frac{1}{2} s(v)^{\alpha} s(v)^{-\alpha} f(v) + \\
\frac{1}{2} \alpha s(v)^{-\alpha-1} s'(v) \int_{\underline{v}}^{v} s(v_j)^{\alpha} dF(v_j) - \frac{1}{2} s(v)^{-\alpha} s(v)^{\alpha} f(v) + f(v)] - s'(v) = 0,
\end{aligned}$$

when $v = v_i$.

From the differential equation above we obtain the following equation on the valuation and the strategy:

$$v_i = \frac{1}{\frac{\alpha}{2} [s(v_i)^{\alpha-1} \int_{v_i}^{\bar{v}} s(v_j)^{-\alpha} dF(v_j) + s(v_i)^{-\alpha-1} \int_{\underline{v}}^{v_i} s(v_j)^{\alpha} dF(v_j)]}.$$

□

In the model, the bids are observed from the data whereas the valuations are unknown for the econometrician. Thus, to be able to recover the valuations we should be able to eliminate both $F(\cdot)$ as well as $s(\cdot)$ from the write hand side of the equation (3.1) and represent it as a function of bids. The method is presented in the next section.

3.2.3 Nonparametric Identification

In this section, I prove that the parameters of the model are nonparametrically identified from available data.

The only unknown ingredient of the model is the distribution of valuations $F(\cdot)$, the number of bidders as well as the bids b_i, $i = 1, 2$, are observed. As a result, the question of identification boils down to the question of whether the distribution F can be uniquely recovered from observed bids.

Let's denote the distribution of b_i by $G(\cdot)$ and quantile function $r(\cdot) = G^{-1}(\cdot)$. Note that the distribution $G(\cdot)$ of b_i depends on the underlying distribution $F(\cdot)$ not only through v_i, but also through the equilibrium strategy $s(\cdot)$.

Formally, let $\mathcal{G}$ denote the set of all distributions over the space of permitted bids, $F \in \mathcal{F}$. Let's call the mapping from the private information to bids $\gamma \in \Gamma$, where $\gamma : \mathcal{F} \to \mathcal{G}$. Then,

Definition 3.3. *(Identification). A model $(\mathcal{F}, \Gamma)$ is identified if for every $(F, F') \in \mathcal{F}^2$ and $(\gamma, \gamma') \in \Gamma^2$, $\gamma(F) = \gamma'(F') \Rightarrow (F, \gamma) = (F', \gamma')$.*

Proposition 3.3. *Given Assumptions 3.1-3.4 are satisfied and $F(v)$ is continuous and strictly increasing on $[\underline{v}, \bar{v}]$, the quantile function of valuations is nonparametrically identified:*

$$q(t) = \frac{1}{\frac{\alpha}{2}[r(t)^{\alpha-1}\int\limits_t^1 r(t_j)^{-\alpha}dt_j + r(t)^{-\alpha-1}\int\limits_0^t r(t_j)^{\alpha}dt_j]}, \quad t \in (0, 1). \tag{3.2}$$

Proof: As it was shown above for every $b \in [\underline{b}, \bar{b}] = [s(\underline{v}), s(\bar{v})]$: $G(b) = F(v)$, where $b = s(v)$, thus $G(s(v)) = F(v)$. Let $r(t) = G^{-1}(t)$ and $q(t) = F^{-1}(t)$, where

$t \in (0,1)$. Then by changing variables (applying r to both sides of equation):

$$G(s(v)) = F(v) \Leftrightarrow s(v) = r(F(v)).$$

Moreover, $v = q(t)$, where $t = F(v)$.

Substituting the above expressions into the equation (3.1), we can rewrite the equation for the quantile function of valuations in terms of the quantile function of bids:

$$q(t) = \frac{1}{\frac{\alpha}{2}[r(t)^{\alpha-1}\int\limits_t^1 r(t_j)^{-\alpha}dt_j + r(t)^{-\alpha-1}\int\limits_0^t r(t_j)^{\alpha}dt_j]}, \quad t \in (0,1).$$

This proves the proposition. $\square$

3.2.4 Nonparametric Estimation

In this section, I propose the nonparametric estimators of the quantile function of bidders' valuations.

If we knew the quantile function $r(t)$, then we could use that to recover the quantile function of the bidders valuation $q(t)$. $r(t)$ is unknown, but can be estimated from observed bids:

$$\hat{r}(t) = b^{(\lceil nt \rceil : n)},$$

where $b^{(\lceil s \rceil : n)}$ is the s-th lowest order statistic out of n i.i.d. bids observations and $\lceil \cdot \rceil$ is the ceiling function.

Let L be the number of auctions, l is the l-th auction, $\{b_{il}, i = 1, 2, l = 1, ..., L\}$ are the observations. Then we can estimate the quantile function of valuations by plugging-in the estimators of quantile function of the bids into equation (3.2):

$$\hat{q}(t) = \frac{1}{\frac{\alpha}{2}[\hat{r}(t)^{\alpha-1}\int\limits_t^1 \hat{r}(t_j)^{-\alpha}dt_j + \hat{r}(t)^{-\alpha-1}\int\limits_0^t \hat{r}(t_j)^{\alpha}dt_j]}, \quad t \in (0,1). \tag{3.3}$$

Note that the invertibility of the bid function is the key for identification as we relied heavily on the assumption that the bidders use a strictly increasing bid function.

Proposition 3.4. *(Csorgo (1983)) Let G be a twice differentiable distribution function, having finite support. Assume $\inf\limits_{0<t<1} g(G^{-1}(t)) > 0$ and $\sup\limits_{0<t<1} |g'(G^{-1}(t))| < \infty$. Then $\sup\limits_{0<t<1} |\hat{r}(t) - r(t)| \xrightarrow{a.s.} 0$.*

$\Rightarrow$

$$\sup_{0<t<1} |\hat{r}(t) - r(t)| = o_p(1).$$

It can be proved that:

Proposition 3.5. *Under the same assumptions as above:*

$$\hat{q}(t) - q(t) = o_p(1).$$

Moreover, in case α is not known it can be estimated from the observed bids and win outcomes. As the CSF is:
$\Psi_i(b_i, b_j) = \begin{cases} \frac{1}{2}\left(\frac{b_i}{b_j}\right)^{\alpha}, \text{if } b_i \leq b_j \\ 1 - \frac{1}{2}\left(\frac{b_j}{b_i}\right)^{\alpha}, \text{if } b_i \geq b_j, \end{cases}$ where $i, j = 1, 2$, and it describes the probability of winning, thus α can be estimates using the Maximum Likelihood Estimator as the first step of the estimation procedure.

3.2.5 Monte Carlo Simulations

To analyze the performance of the estimator (3.3) I will run the Monte Carlo Simulations.

Example 3.1. *If the distribution of valuations is*

$$F(v) = \left(1 + \frac{\alpha-1}{\alpha+1} - \frac{\alpha-1}{\alpha}\frac{2k}{v}\right)^{\frac{1}{\alpha-1}}, \quad v \in [\underline{v}, \bar{v}], \; \alpha \neq 1,$$

or equivalently if quantile function q of the valuations is

$$q(t) = \frac{1}{\frac{\alpha}{2k(1-\alpha)}(t^{\alpha-1}-1) + \frac{\alpha}{2k(1+\alpha)}}, \quad t \in (0,1), \; \alpha \neq 1.$$

$q(0) = \frac{k(1+\alpha)(\alpha-1)}{\alpha^2} = \underline{v}$ *and* $q(1) = \frac{2k(\alpha+1)}{\alpha} = \bar{v}$. *Then there exists a unique equilibrium bid function $s(v) = kF(v)$, where $v \in [\underline{v}, \bar{v}]$ for any k.*

The proof can be found in the Appendix.

Let's consider $L = 200$ auctions with 2 bidders and 100 Monte Carlo replications. Then the following figure presents the true quantile function, the mean, the 5% quantile, and the 95% quantile of the 100 estimates $\hat{q}(t)$ for $k = 2$ and $\alpha = 3$.

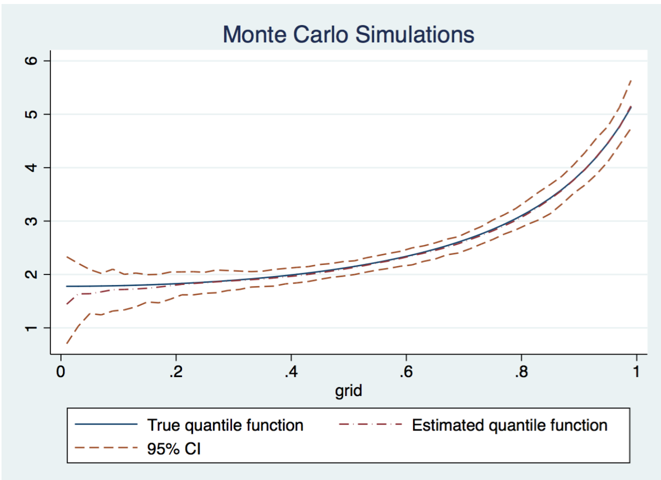


Figure 3.2: Monte Carlo results in case of symmetric bidders

As we see the estimator works very well in this case, the only issue might occur at the border since the quantile estimates are biased closed to the borders.

3.3 Asymmetric Contest Model

3.3.1 Nonparametric Identification

In this section, I prove the nonparametric identification of the model with asymmetric bidders.

I consider the case when bidders are asymmetric in a sense that they have different distributions of the valuations: $F_i(\cdot)$, $i = 1, 2$ with corresponding densities $f_i(\cdot)$. Denote by $s_i(v_i)$, $i = 1, 2$ strictly monotonic equilibrium strategies, thus they are invertible and $s_i^{-1}(b_i) = v_i$. The rest is the same as in the symmetric scenario.

Proposition 3.6. *If $F_i(v)$ are continuous and strictly increasing on $[\underline{v}, \bar{v}]$, the quantile functions of valuations are nonparametrically identified:*

$$q_i(t) = \frac{1}{\frac{\alpha}{2}[r_i(t)^{\alpha-1} \int\limits_{G_j(r_i(t))}^{1} r_j(t_j)^{-\alpha} dt_j + r_i(t)^{-\alpha-1} \int\limits_{0}^{G_j(r_i(t))} r_j(t_j)^{\alpha} dt_j]},$$

where $t \in (0, 1)$, $i = 1, 2$, $j = -i$.

Proof: Given that the strategies of both players are strictly monotonic and the valuations are independent, the expected payoff to bidder i when his true valuation is v_i but he bids as if it was v can be written as:

$$\mathbb{E}[u_i(v)|v_i, F_j(v)] = v_i \left[\int_b^{\bar{b_j}} \frac{1}{2}\left(\frac{b}{b_j}\right)^\alpha dG_j(b_j) + \int_{\underline{b_j}}^b \left(1 - \frac{1}{2}\left(\frac{b_j}{b}\right)^\alpha\right) dG_j(b_j)\right] - b =$$

$$= v_i \left[\int\limits_{s_j^{-1}(s_i(v))}^{\bar{v_j}} \frac{1}{2}\left(\frac{s_i(v)}{s_j(v_j)}\right)^\alpha dF_j(v_j) + \int\limits_{\underline{v_j}}^{s_j^{-1}(s_i(v))} \left(1 - \frac{1}{2}\left(\frac{s_j(v_j)}{s_i(v)}\right)^\alpha\right) dF_j(v_j)\right] - s_i(v) =$$

$$= v_i\frac{1}{2}s_i(v)^\alpha \int\limits_{s_j^{-1}(s_i(v))}^{\bar{v_j}} s_j(v_j)^{-\alpha} dF_j(v_j) - v_i\frac{1}{2}s_i(v)^{-\alpha} \int\limits_{\underline{v_j}}^{s_j^{-1}(s_i(v))} s_j(v_j)^\alpha dF_j(v_j) +$$

$$+ v_i F_j(s_j^{-1}(s_i(v))) - s_i(v),$$

since $b_i = s_i(v_i)$ and $G_i(b) = Pr(b_i \leq b) = Pr(v_i \leq s_i^{-1}(b)) = F_i(s_i^{-1}(b)) = F_i(v)$ as $s_i(v)$ is invertible.

Using the First order condition (differentiating with respect to v and substituting $v = v_i$), we get:

$$\frac{\partial \mathbb{E}[u_i]}{\partial v} = v_i[\frac{1}{2}\alpha s_i(v)^{\alpha-1}s_i'(v) \int\limits_{s_j^{-1}(s_i(v))}^{\bar{v_j}} s_j(v_j)^{-\alpha} dF_j(v_j) -$$

$$-\frac{1}{2}s_i(v)^\alpha s_j(s_j^{-1}(s_i(v)))^{-\alpha} f_j(s_j^{-1}(s_i(v)))\frac{\partial}{\partial v}(s_j^{-1}(s_i(v))) +$$

$$+\frac{1}{2}\alpha s_i(v)^{-\alpha-1}s_i'(v) \int\limits_{\underline{v_j}}^{s_j^{-1}(s_i(v))} s_j(v_j)^\alpha dF_j(v_j) -$$

$$-\frac{1}{2}s_i(v)^{-\alpha} s_j(s_j^{-1}(s_i(v)))^\alpha f_j(s_j^{-1}(s_i(v)))\frac{\partial}{\partial v}(s_j^{-1}(s_i(v))) +$$

$$+ f_j(s_j^{-1}(s_i(v)))\frac{\partial}{\partial v}(s_j^{-1}(s_i(v)))] - s_i'(v) = 0,$$

when $v = v_i$.

From the differential equation above we obtain the following equation on the valuation and the strategy:

$$v_i = \frac{1}{\frac{\alpha}{2}[s_i(v_i)^{\alpha-1}\int\limits_{s_j^{-1}(s_i(v_i))}^{\bar{v_j}} s_j(v_j)^{-\alpha}dF_j(v_j) + s_i(v_i)^{-\alpha-1}\int\limits_{\underline{v_j}}^{s_j^{-1}(s_i(v_i))} s_j(v_j)^{\alpha}dF_j(v_j)]}. \tag{3.4}$$

Let us now denote $t = F_i(v_i)$ and $t_j = F_j(v_j)$, equivalently $v_i = q_i(t)$ and $v_j = q_j(t_j)$, where $q_i(\cdot)$ and $q_j(\cdot)$ are quantile functions of the distribution of valuations. As a result of monotonicity of the strategies similar to the case with symmetric bidders $G_i(s_i(v_i)) = F_i(v_i)$, applying $r_i^{-1}(\cdot)$ to both sides of equality, where $r_i(\cdot)$ is quantile function of the bid distribution we get: $s_i(v_i) = r_i(F_i(v_i)) = r_i(t)$ and $s_j(v_j) = r_j(F_j(v_j)) = r_j(t_j)$. Moreover, $F_j(s_j^{-1}(s_i(v_i))) = G_j(s_i(v_i)) = G_j(r_i(t))$, $F_j(\bar{v_j}) = 1$ and $F_j(\underline{v_j}) = 0$. Using these equalities and changing variables we can rewrite the equation (3.4) above as:

$$q_i(t) = \frac{1}{\frac{\alpha}{2}[r_i(t)^{\alpha-1}\int\limits_{G_j(r_i(t))}^{1} r_j(t_j)^{-\alpha}dt_j + r_i(t)^{-\alpha-1}\int\limits_{0}^{G_j(r_i(t))} r_j(t_j)^{\alpha}dt_j]}, \tag{3.5}$$

where $t \in (0, 1)$, $i = 1, 2$, $j = -i$. This proves the proposition. □

3.3.2 Nonparametric Estimation

In this section, I propose nonparametric estimator of quantile functions of bidders' valuations.

If we knew the quantile functions $r_i(\cdot)$ as well as the distribution of bids $G_i(\cdot)$, then we could use that to recover quantile functions of the bidders' valuations $q_i(\cdot)$. Let L be the number of auctions, l is the l-th auction, $\{b_{il}, i = 1, 2, l = 1, ..., L\}$ are the observations. As in case of the symmetric valuations $r_i(\cdot)$ can be estimated from observed bids:

$$\hat{r}_i(t) = b_i^{(\lceil Lt \rceil : L)},$$

where $b_i^{(\lceil s \rceil : L)}$ is the s-th lowest order statistic out of L i.i.d. bids observations; $\lceil \cdot \rceil$ is the ceiling function. In its turn $G_i(\cdot)$ can be estimated as:

$$\hat{G}_i(b) = \frac{1}{L}\sum_{l=1}^{L} \mathbb{1}(b_{il} \leq b).$$

Thus, we can estimate the quantile function of valuations using the following plug-in estimator:

$$\hat{q}_i(t) = \frac{1}{\frac{\alpha}{2}[\hat{r}_i(t)^{\alpha-1} \int\limits_{\hat{G}_j(\hat{r}_i(t))}^{1} \hat{r}_j(t_j)^{-\alpha} dt_j + \hat{r}_i(t)^{-\alpha-1} \int\limits_{0}^{\hat{G}_j(\hat{r}_i(t))} \hat{r}_j(t_j)^{\alpha} dt_j]}, \tag{3.6}$$

where $t \in (0,1)$.

3.3.3 Monte Carlo Simulations

To analyze the performance of the estimator (3.6) I will run the Monte Carlo Simulations.

Example 3.2. *If the true quantile functions q_i, $i = 1, 2$ of the bidder's valuations are*

$$q_i(t) = \frac{1}{\frac{\alpha}{2}\left[\frac{k_i^{\alpha-1}k_j^{-\alpha}}{1-\alpha}t^{\alpha-1} + \frac{1}{k_j(1+\alpha)} + \frac{1}{k_j(\alpha-1)}\right]}, \quad t \in (0,1), \; i = 1, 2.$$

Then there exist unique equilibrium bid functions $s_1(v) = k_1 F_1(v)$ and $s_2(v) = k_2 F_2(v)$ for any k_1 and k_2.

The proof can be found in the Appendix.

Let's consider $L = 200$ auctions with 2 bidders and 100 Monte Carlo replications. Then the following figure presents the true quantile function, the mean, the 5% quantile, and the 95% quantile of the 100 estimates $\hat{q}_1(t)$ and $\hat{q}_2(t)$ for $k_1 = 5$, $k_2 = 6$ and $\alpha = 2$.

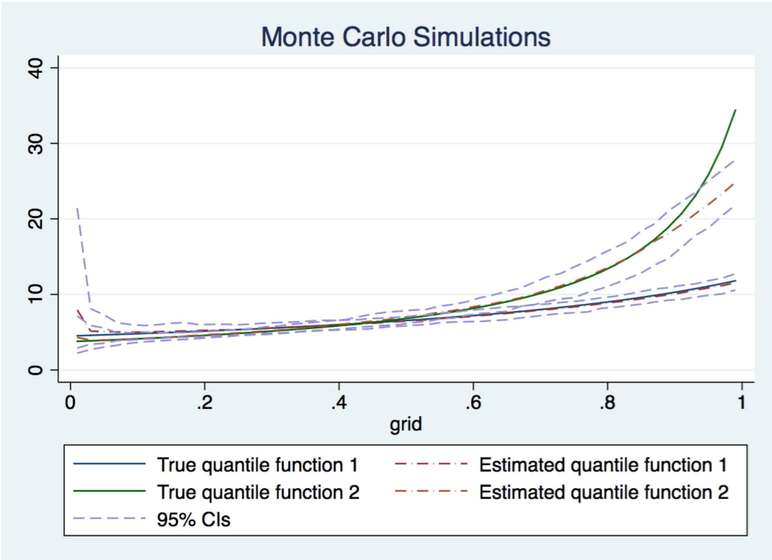


Figure 3.3: Monte Carlo results in case of asymmetric bidders

As we see the estimator works very well in this case, the only issue might occur at the border since the quantile estimates are biased closed to the borders.

3.3.4 Representation in Terms of Types

The problem can be easily reformulated in terms of the types, where the type characterizes how costly it is to raise a bid. Let c_i, $i = 1, 2$ be the type of bidder i. Expected payoff to bidder i in this case is given by:

$$\mathbb{E}[u_i|c_i, F_j(c)] = \mathbb{E}[\Psi_i(c_i, c_{-i})|b_i, c_i, N, F_j(c)] - c_i * b_i.$$

where $c_i = \frac{1}{v_i}$.

Then under the same conditions as before we can estimate the quantile functions of types:

$$\hat{q}_i(1-t) = \frac{\alpha}{2}[\hat{r}_i(t)^{\alpha-1} \int\limits_{\hat{G}_j(\hat{r}_i(t))}^{1} \hat{r}_j(t_j)^{-\alpha} dt_j + \hat{r}_i(t)^{-\alpha-1} \int\limits_{0}^{\hat{G}_j(\hat{r}_i(t))} \hat{r}_j(t_j)^{\alpha} dt_j],$$

where $t \in (0,1)$.

3.4 Conclusion

In this work, I identified and estimated the incomplete information contest model with serial contest success function with both symmetric and asymmetric bidders. As a result, I recover the distribution of valuations or, alternatively, types from the bid distribution. This model provides the framework that can be applied to the variety of real-life scenarios such as litigation, research and development, patent race, procurement of innovative good, research contests, sport, events, arms race, rent-seeking activity, such as lobbying, as well as electoral competition. The knowledge of the distribution of valuations or types allows the policymakers to quantify the effect of different policy changes. This is a semiparametric version of the model presented in Chapter 2, which can be applied in the case when the data is sparse and some restrictions need to be put on the nonparametric structure.

3.5 Appendix

3.5.1 Proof of Proposition 3.1

Proof: Let us consider the more general case when bidders are asymmetric in a sense that they have different distributions of the valuations: $F_i(\cdot)$, $i = 1, 2$ with corresponding densities $f_i(\cdot)$. Let us consider all assumptions required for the Theorem 6 in Athey (2001) to hold.

1. $f_i(\cdot)$ is density with respect to Lebesque measure, bounded and atomless.
2. $U_i = \Psi_i(b_1, b_2)(v_i - b_i) + (1 - \Psi_i(b_1, b_2))(-b_i)$ can be written in the general form considered in the paper.
3. Winner's payoff $v_i - b_i$ and loser's payoff $-b_i$ are continuous in (v_i, b) and bounded as v_i has a finite support $[\underline{v_i}, \bar{v}_i]$ and the bidders won't find it profitable to bid more that the valuation.
4. Expected utility $E[U_i] = \int \Psi_i(b_i, s_j(v_j)) f_j(v_j) dv_j - b_i$ is bounded and finite.
5. Single-crossing condition $\frac{\partial^2 U_i}{\partial v_i \partial b_i} \geq 0$ is satisfied as:

$$\frac{\partial^2 U_i}{\partial v_i \partial b_i} = \frac{\partial \Psi_i}{\partial b_i} = \begin{cases} \frac{1}{2b_j} > 0, \text{ if } b_i \leq b_j \\ \frac{b_j}{2b_i^2} > 0, \text{ if } b_i \geq b_j. \end{cases}$$

Thus all the assumptions of Theorem 6 in Athey (2001) are satisfied, hence there exists a pure-strategy Bayesian Nash Equilibrium in nondecreasing strategies. Since the single-crossing property holds with strict inequality, this equilibrium is actually in increasing strategies. □

3.5.2 Proof of Example 3.1

Proof: An equation (3.1) can be rewritten as

$$v_i \frac{\alpha}{2}[s(v_i)^{\alpha-1} \int_{v_i}^{\bar{v}} s(v_j)^{-\alpha} dF(v_j) + s(v_i)^{-\alpha-1} \int_{\underline{v}}^{v_i} s(v_j)^{\alpha} dF(v_j)] - 1 = 0.$$

If we consider $\alpha \neq 1$ and plug in $s(v) = kF(v)$ the equation above becomes:

$$\begin{aligned} v_i \frac{\alpha}{2}[(kF(v_i))^{\alpha-1} \int_{v_i}^{\bar{v}} (kF(v_j))^{-\alpha} dF(v_j) + \\ +(kF(v_i))^{-\alpha-1} \int_{\underline{v}}^{v_i} (kF(v_j))^{\alpha} dF(v_j)] - 1 = 0. \end{aligned}$$

Taking the integrals we get:

$$v_i \frac{\alpha}{2} \frac{1}{k} \left[F(v_i)^{\alpha-1} \left(\frac{1}{1-\alpha} - \frac{F(v_i)^{1-\alpha}}{1-\alpha} \right) + F(v_i)^{-\alpha-1} \frac{F(v_i)^{1+\alpha}}{1+\alpha} \right] - 1 = 0,$$

since $F(\bar{v}) = 1$.

Collecting the terms we get:

$$v_i \frac{\alpha}{2k} \left[F(v_i)^{\alpha-1} \frac{1}{1-\alpha} - \frac{1}{1-\alpha} + \frac{1}{1+\alpha} \right] = 1.$$

In the end from here:

$$v = \frac{1}{\frac{\alpha}{2k(1-\alpha)}(F(v)^{\alpha-1} - 1) + \frac{\alpha}{2k(1+\alpha)}},$$

or equivalently:

$$F(v) = \left(1 + \frac{\alpha-1}{\alpha+1} - \frac{\alpha-1}{\alpha} \frac{2k}{v} \right)^{\frac{1}{\alpha-1}}, \quad v \in [\underline{v}, \bar{v}], \ \alpha \neq 1.$$

This proves the statement. □

3.5.3 Proof of Example 3.2

Proof: An equation (3.4) can be rewritten as

$$v_i\frac{\alpha}{2}[s_i(v_i)^{\alpha-1}\int\limits_{s_j^{-1}(s_i(v_i))}^{\bar{v_j}} s_j(v_j)^{-\alpha}dF_j(v_j)+$$
$$+s_i(v_i)^{-\alpha-1}\int\limits_{\underline{v_j}}^{s_j^{-1}(s_i(v_i))} s_j(v_j)^{\alpha}dF_j(v_j)]-1=0.$$

If we consider $\alpha\neq 1$ and plug in $s_i(v)=k_iF_i(v_i)$ the equation above becomes:

$$v_i\frac{\alpha}{2}[k_i^{\alpha-1}F_i(v_i)^{\alpha-1}\int\limits_{s_j^{-1}(k_iF_i(v_i))}^{\bar{v_j}} k_j^{-\alpha}F_j(v_j)^{-\alpha}dF_j(v_j)+$$
$$+k_i^{-\alpha-1}F_i(v_i)^{-\alpha-1}\int\limits_{\underline{v_j}}^{s_j^{-1}(k_iF_i(v_i))} k_j^{\alpha}F_j(v_j)^{\alpha}dF_j(v_j)]-1=0.$$

Taking the integrals we get:

$$v_i\frac{\alpha}{2}\Bigg[\,k_i^{\alpha-1}k_j^{-\alpha}F_i(v_i)^{\alpha-1}\left(\frac{F_j(v_j)^{-\alpha+1}}{-\alpha+1}\right)\Bigg|_{s_j^{-1}(k_iF_i(v_i))}^{\bar{v_j}}+$$
$$+\ k_i^{-\alpha-1}k_j^{\alpha}F_i(v_i)^{-\alpha-1}\left(\frac{F_j(v_j)^{\alpha+1}}{\alpha+1}\right)\Bigg|_{\underline{v_j}}^{s_j^{-1}(k_iF_i(v_i))}\Bigg]-1=0.$$

Collecting the terms we get:

$$v_i\frac{\alpha}{2}\left[k_i^{\alpha-1}k_j^{-\alpha}F_i(v_i)^{\alpha-1}\frac{1}{-\alpha+1}+\frac{1}{k_2(\alpha-1)}+\frac{1}{k_2(\alpha+1)}\right]-1=0,$$

since $F_j(s_j^{-1}(k_iF_i(v_i)))=\frac{k_iF_i(v_i)}{k_j}$.

In the end from here:

$$v_i = \frac{1}{\frac{\alpha}{2}\left[\frac{k_i^{\alpha-1}k_j^{-\alpha}}{1-\alpha}F_i(v_i)^{\alpha-1} + \frac{1}{k_j(1+\alpha)} + \frac{1}{k_j(\alpha-1)}\right]}.$$

This proves the statement. □

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