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Rational minimal-twist motions on curves with rotation-minimizing Euler-Rodrigues frames

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Abstract

A *minimal twist frame* $(\mathbf{f}_1(\xi), \mathbf{f}_2(\xi), \mathbf{f}_3(\xi))$ on a polynomial space curve $\mathbf{r}(\xi)$, $\xi \in [0, 1]$ is an orthonormal frame, where $\mathbf{f}_1(\xi)$ is the tangent and the normal-plane vectors $\mathbf{f}_2(\xi), \mathbf{f}_3(\xi)$ have the least variation between given initial and final instances $\mathbf{f}_2(0), \mathbf{f}_3(0)$ and $\mathbf{f}_2(1), \mathbf{f}_3(1)$. Namely, if $\boldsymbol{\omega} = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3$ is the frame angular velocity, the component ω_1 does not change sign, and its arc length integral has the smallest value consistent with the boundary conditions. We consider construction of curves with *rational* minimal twist frames, based on the Pythagorean-hodograph curves of degree 7 that have rational rotation-minimizing *Euler-Rodrigues frames* $(\mathbf{e}_1(\xi), \mathbf{e}_2(\xi), \mathbf{e}_3(\xi))$ — i.e., the normal-plane vectors $\mathbf{e}_2(\xi), \mathbf{e}_3(\xi)$ have no rotation about the tangent $\mathbf{e}_1(\xi)$. A set of equations that govern the construction of such curves with prescribed initial/final points and tangents, and total arc length, is derived. For the resulting curves $\mathbf{f}_2(\xi), \mathbf{f}_3(\xi)$ are then obtained from $\mathbf{e}_2(\xi), \mathbf{e}_3(\xi)$ by a monotone rational normal-plane rotation, subject to the boundary conditions. A selection of computed examples is included to illustrate the construction, and it is shown that the desirable feature of a uniform rotation rate (i.e., $\omega_1 = \text{constant}$) can be accurately approximated.

Keywords: rigid body motions; minimal twist frame; rotation-minimizing frame; Euler-Rodrigues frame; spatial motion planning; Pythagorean-hodograph curves.

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1 Introduction

An *adapted* orthonormal frame $(\mathbf{f}_1(\xi), \mathbf{f}_2(\xi), \mathbf{f}_3(\xi))$ on a given space curve $\mathbf{r}(\xi)$ comprises the unit tangent $\mathbf{f}_1(\xi) = \mathbf{r}'(\xi)/|\mathbf{r}'(\xi)|$, and orthogonal unit vectors $\mathbf{f}_2(\xi), \mathbf{f}_3(\xi)$ that span the normal plane. The frame derivatives with respect to arc length s determine its angular velocity $\boldsymbol{\omega}$ through the relations

$$\frac{d\mathbf{f}_1}{ds} = \boldsymbol{\omega} \times \mathbf{f}_1, \quad \frac{d\mathbf{f}_2}{ds} = \boldsymbol{\omega} \times \mathbf{f}_2, \quad \frac{d\mathbf{f}_3}{ds} = \boldsymbol{\omega} \times \mathbf{f}_3. \quad (1)$$

A well-known instance is the *Frenet frame* comprising the tangent $\mathbf{t}$, principal normal $\mathbf{n}$, and binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, with angular velocity defined in terms of the curvature κ and torsion τ of $\mathbf{r}(\xi)$ by the *Darboux vector*, $\kappa \mathbf{b} + \tau \mathbf{t}$ [28]. The component $\tau \mathbf{t}$ specifies the rate of rotation of the normal plane vectors $\mathbf{n}$ and $\mathbf{b}$ about the tangent $\mathbf{t}$, and is determined by the requirement that the principal normal $\mathbf{n}$ always points toward the center of curvature.

If, instead of $\mathbf{n}$ and $\mathbf{b}$, normal-plane vectors $\mathbf{u}$ and $\mathbf{v}$ are specified so as to exhibit no instantaneous rotation about the tangent $\mathbf{t}$ (i.e., the frame angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$), then $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ is known as a *rotation-minimizing frame* (RMF), or a *Bishop frame* [3]. Such frames have diverse applications in computer animation, motion planning, robotics, swept surface constructions, and many schemes for their approximation have recently been developed — see [17, 23, 24, 25, 27, 29, 30, 32, 33, 34]. Several recent studies develop the theory to characterize [8, 13, 14, 20, 21] and algorithms to construct [1, 15, 18] space curves with *rational* rotation-minimizing frames: these curves must be Pythagorean-hodograph (PH) curves [7], since only PH curves admit rational unit tangents. A review of these results may be found in [9].

The construction of an RMF on a pre-defined curve $\mathbf{r}(\xi)$ is an *initial value problem* — i.e., specifying the orientation of the normal-plane vectors at any curve point determines their orientation at every other point. Consequently, it is not possible to construct rotation-minimizing rigid-body motions along a pre-determined path with prescribed initial *and* final orientations. Thus, in previous studies [15, 18] of rational rotation-minimizing motions with given initial/final positions and orientations, the curve $\mathbf{r}(\xi)$ was not determined *a priori*, but was rather an outcome of the construction procedure.

However, it is often desirable to *independently* specify a curve $\mathbf{r}(\xi)$ and a rational adapted frame $(\mathbf{f}_1(\xi), \mathbf{f}_2(\xi), \mathbf{f}_3(\xi))$ along it, compatible with given end points and frame orientations. To accommodate this requirement a novel type of adapted frame, the *minimal twist frame* (MTF) associated with a pre-defined curve and initial/final orientations, was introduced in [19]. Consistent

with the boundary conditions, the distinctive properties of an MTF are:

- the angular velocity component $\boldsymbol{\omega} \cdot \mathbf{f}_1$ in the tangent direction does not change sign, i.e., $\mathbf{f}_2, \mathbf{f}_3$ maintain a constant sense of rotation about $\mathbf{f}_1$;
- the arc-length integral of $\boldsymbol{\omega} \cdot \mathbf{f}_1$ along $\mathbf{r}(\xi)$ has the least possible absolute value, i.e., the total amount of rotation of $\mathbf{f}_2, \mathbf{f}_3$ about $\mathbf{f}_1$ is minimized.

A further desirable feature is the maintenance of a fixed rotation rate of the normal-plane vectors $\mathbf{f}_2, \mathbf{f}_3$ about the tangent $\mathbf{f}_1$ — i.e., $\boldsymbol{\omega} \cdot \mathbf{f}_1 = \text{constant}$ — although for a *rational* MTF this can only be approximately achieved.

The construction of rational MTFs on spatial PH quintics was considered in [19], by imposing a rational quartic rotation on the normal-plane vectors of the *Euler–Rodrigues frame* (ERF), a rational adapted frame defined on any PH curve [5]. This rotation must (i) cancel out the tangent component of the ERF angular velocity, and (ii) match the given initial/final MTF instances. The method permits *a priori* specification of the path geometry, independent of the MTF construction, and performs well if the ERF angular velocity has only mild variation of its tangent component, but subdivision is necessary to ensure good results in the presence of strong ERF variation.

As an alternative to the method in [19], we consider here the construction of MTFs on pre-defined PH curves with given initial/final points and frames, and a specified total arc length,¹ that admit rotation-minimizing ERFs. The simplest examples are certain degree 7 PH curves characterized by quadratic coefficient constraints [5, 18]: they have more shape freedoms than the RRMF quintics, and also lower degree rational RMFs. Their primary advantage, in the present context, is a significantly reduced burden on the quartic normal-plane rotation generating the MTF from the ERF: since the ERF is rotation-minimizing, it has a zero angular velocity component in the tangent direction, and requirement (i) on the normal-plane rotation is eliminated — its degrees of freedom can be exploited to satisfy requirement (ii) with $\boldsymbol{\omega} \cdot \mathbf{f}_1$ more nearly constant — an important consideration in, for example, the construction of “tubular” surfaces with asymmetric cross-sectional profiles.

We emphasize again the important conceptual distinction between RMFs and MTFs. An RMF can not, in general, satisfy prescribed initial and final frame orientations along a pre-defined space curve. In order to match initial and final RMF instances — as in [15, 18] — the curve is not specified *a priori*, but is rather an *a posteriori* outcome of the construction process. Thus when

¹The ability to impose a prescribed arc length on a polynomial free-form curve segment is unique to the PH curves — for example, see [10, 11, 26].

different initial or final RMF normal plane vectors are specified, the methods in [15, 18] generate *different curves* (see Example 5 below). This is evidently unacceptable in contexts where one needs to construct several frames, with different initial/final orientations, along a fixed curve.

An MTF, on the other hand, specifies a smooth adapted frame variation along a *pre-defined curve*, on which one can freely specify the initial and final MTF normal-plane vectors (see Figure 6 below). An MTF on a pre-defined curve cannot (in general) be an RMF, since prescribed initial and final frames imply a minimum (non-zero) “twist” of the normal-plane vectors along the curve, that is incompatible with the zero-twist property of an RMF. Hence, the methods developed herein offer an alternative to those of [19], but address a problem conceptually distinct from that treated in [15, 18].

The present study focuses on construction of MTFs on the class of degree 7 PH curves that have rotation-minimizing ERFs, rather than general quintic PH curves as in [19]. The main advantages of this approach are as follows.

- (1) The twist of the ERF on these degree 7 PH curves is precisely zero. For a general PH quintic, however, it is non-zero and it must be computed *a priori* by a factorization of the parametric speed and a rational function integration using a partial fraction decomposition [19].
- (2) The rational quartic rotation generating the MTF from the ERF need not compensate for the variation of the ERF angular velocity, which can often be quite severe: see Example 6 and Figure 8 below. This typically yields a near-constant tangent component of the MTF angular velocity (see Figure 4 below), and avoids the need to invoke subdivision methods at the inflections or extrema of the ERF angular velocity.

Note, however, that the MTFs constructed herein are of slightly higher degree (10 versus 8) than in [19], and the *a priori* construction of degree 7 PH curves that have rotation-minimizing ERFs is more involved. One may expect that simpler and more efficient construction procedures will emerge from deeper investigation, but that is beyond the scope of the present study.

The remainder of this paper is organized as follows. After reviewing key properties of PH curves in Section 2, we summarize the characterization of the degree 7 PH curves that possess rotation-minimizing ERFs in Section 3. This is used in Sections 4 and 5 to develop and solve a system of four quadratic equations in four real variables that identify such a curve with prescribed end points and tangents, and total arc length. These equations also incorporate two free parameters, which may be exploited to optimize the curve shape and

ensure the existence of solutions. Based upon the resulting curves, Section 6 describes the MTF construction by means of a rational normal-plane rotation of the ERF vectors to satisfy the specified boundary conditions and achieve a nearly-constant tangent component of the MTF angular velocity. Finally, Section 7 summarizes the principal results of this study, and suggests further possible developments of the methodology.

2 Spatial PH curves

The distinctive property of a polynomial *Pythagorean-hodograph* (PH) curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$ is that the derivative $\mathbf{r}'(\xi) = (x'(\xi), y'(\xi), z'(\xi))$ has components satisfying the Pythagorean condition

$$x'^2(\xi) + y'^2(\xi) + z'^2(\xi) = \sigma^2(\xi) \quad (2)$$

for some polynomial $\sigma(\xi)$. The *quaternion* and *Hopf map* forms [6, 12] are two alternative models for the construction of PH curves — the former generates a Pythagorean hodograph $\mathbf{r}'(\xi)$ from a quaternion² polynomial

$$\mathcal{A}(\xi) = u(\xi) + v(\xi)\mathbf{i} + p(\xi)\mathbf{j} + q(\xi)\mathbf{k} \quad (3)$$

and its conjugate $\mathcal{A}^*(\xi) = u(\xi) - v(\xi)\mathbf{i} - p(\xi)\mathbf{j} - q(\xi)\mathbf{k}$ through the product³

$$\begin{aligned} \mathbf{r}'(\xi) = \mathcal{A}(\xi)\mathbf{i}\mathcal{A}^*(\xi) &= [u^2(\xi) + v^2(\xi) - p^2(\xi) - q^2(\xi)]\mathbf{i} \\ &+ 2[u(\xi)q(\xi) + v(\xi)p(\xi)]\mathbf{j} + 2[v(\xi)q(\xi) - u(\xi)p(\xi)]\mathbf{k}, \end{aligned} \quad (4)$$

and the latter generates a Pythagorean hodograph from complex polynomials

$$\boldsymbol{\alpha}(\xi) = u(\xi) + \mathbf{i}v(\xi), \quad \boldsymbol{\beta}(\xi) = q(\xi) + \mathbf{i}p(\xi) \quad (5)$$

through the expression

$$\mathbf{r}'(\xi) = (|\boldsymbol{\alpha}(\xi)|^2 - |\boldsymbol{\beta}(\xi)|^2, 2\operatorname{Re}(\boldsymbol{\alpha}(\xi)\overline{\boldsymbol{\beta}}(\xi)), 2\operatorname{Im}(\boldsymbol{\alpha}(\xi)\overline{\boldsymbol{\beta}}(\xi))). \quad (6)$$

The parametric speed (i.e., the derivative of arc length s with respect to the curve parameter ξ) is defined in these two representations by

$$\sigma(\xi) = |\mathbf{r}'(\xi)| = |\mathcal{A}(\xi)|^2 = |\boldsymbol{\alpha}(\xi)|^2 + |\boldsymbol{\beta}(\xi)|^2, \quad (7)$$

²Calligraphic characters such as $\mathcal{A}$ are used to denote quaternions, their scalar (real) and vector (imaginary) parts being indicated by $\operatorname{scal}(\mathcal{A})$ and $\operatorname{vect}(\mathcal{A})$. Bold symbols denote either complex numbers or vectors in $\mathbb{R}^3$ — the meaning should be clear from the context.

³Note that products of the form $\mathcal{A}(\xi)\mathbf{i}\mathcal{A}^*(\xi)$ always generate pure vector quaternions.

and the polynomial arc length function $s(\xi)$ is the integral of $\sigma(\xi)$.

The equivalence of (4) and (6) is seen by taking $\mathcal{A}(\xi) = \boldsymbol{\alpha}(\xi) + \mathbf{k}\boldsymbol{\beta}(\xi)$, where the imaginary unit $\mathbf{i}$ is identified with the quaternion basis element $\mathbf{i}$. It will be advantageous to simultaneously employ both representations.

The quaternion polynomial (3) and complex polynomials (5) are defined in terms of the Bernstein basis

$$b_i^m(\xi) = \binom{m}{i} (1-\xi)^{m-i} \xi^i, \quad i = 0, \dots, m$$

of degree m on $\xi \in [0, 1]$ as

$$\mathcal{A}(\xi) = \sum_{i=0}^m \mathcal{A}_i b_i^m(\xi), \quad \boldsymbol{\alpha}(\xi) = \sum_{i=0}^m \boldsymbol{\alpha}_i b_i^m(\xi), \quad \boldsymbol{\beta}(\xi) = \sum_{i=0}^m \boldsymbol{\beta}_i b_i^m(\xi),$$

where the coefficients are related by $\mathcal{A}_i = \boldsymbol{\alpha}_i + \mathbf{k}\boldsymbol{\beta}_i$ for $i = 0, \dots, m$.

The *Euler–Rodrigues frame* (ERF) is a rational adapted frame defined on spatial PH curves [5], that serves as a “reference frame” for the identification of other rational adapted frames on PH curves. In terms of the quaternion form (4), it is specified by

$$\mathbf{e}_1(\xi) = \frac{\mathcal{A}(\xi) \mathbf{i} \mathcal{A}^*(\xi)}{|\mathcal{A}(\xi)|^2}, \quad \mathbf{e}_2(\xi) = \frac{\mathcal{A}(\xi) \mathbf{j} \mathcal{A}^*(\xi)}{|\mathcal{A}(\xi)|^2}, \quad \mathbf{e}_3(\xi) = \frac{\mathcal{A}(\xi) \mathbf{k} \mathcal{A}^*(\xi)}{|\mathcal{A}(\xi)|^2}, \quad (8)$$

where $\mathbf{e}_1 = \mathbf{r}'/|\mathbf{r}'|$ is the curve tangent, and $\mathbf{e}_2, \mathbf{e}_3$ span the normal plane.

3 Rotation-minimizing ERFs

Using the Hopf map form (5)–(6), RRMF curves can be characterized [8, 22] by the existence of a complex polynomial $\mathbf{w}(\xi) = a(\xi) + \mathbf{i}b(\xi)$ such that

$$\frac{\text{Im}(\overline{\boldsymbol{\alpha}}(\xi)\boldsymbol{\alpha}'(\xi) + \overline{\boldsymbol{\beta}}(\xi)\boldsymbol{\beta}'(\xi))}{|\boldsymbol{\alpha}(\xi)|^2 + |\boldsymbol{\beta}(\xi)|^2} = \frac{\text{Im}(\overline{\mathbf{w}}(\xi)\mathbf{w}'(\xi))}{|\mathbf{w}(\xi)|^2}. \quad (9)$$

The left-hand side in (9) is the component of the ERF angular velocity in the direction of the curve tangent $\mathbf{e}_1 = \mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$. When the condition (9) is satisfied, a rational RMF $(\mathbf{f}_1(\xi), \mathbf{f}_2(\xi), \mathbf{f}_3(\xi))$ can be generated from the ERF by a rational normal-plane rotation. Namely, we set $\mathbf{f}_1(\xi) = \mathbf{e}_1(\xi)$ and

$$\begin{bmatrix} \mathbf{f}_2(\xi) \\ \mathbf{f}_3(\xi) \end{bmatrix} = \frac{1}{a^2(\xi) + b^2(\xi)} \begin{bmatrix} a^2(\xi) - b^2(\xi) & -2a(\xi)b(\xi) \\ 2a(\xi)b(\xi) & a^2(\xi) - b^2(\xi) \end{bmatrix} \begin{bmatrix} \mathbf{e}_2(\xi) \\ \mathbf{e}_3(\xi) \end{bmatrix}. \quad (10)$$

For curves that satisfy (9) with $\text{Im}(\overline{\alpha}(\xi)\alpha'(\xi) + \overline{\beta}(\xi)\beta'(\xi)) \equiv 0$, the ERF $(\mathbf{e}_1(\xi), \mathbf{e}_2(\xi), \mathbf{e}_3(\xi))$ is itself rotation-minimizing, and the rotation (10) is not required to obtain a rational RMF. It was shown in [5] that the simplest true space curves with this property are PH curves of degree 7.

To define a PH curve of degree 7, we use cubic complex polynomials

$$\begin{aligned}\alpha(\xi) &= \alpha_0 b_0^3(\xi) + \alpha_1 b_1^3(\xi) + \alpha_2 b_2^3(\xi) + \alpha_3 b_3^3(\xi), \\ \beta(\xi) &= \beta_0 b_0^3(\xi) + \beta_1 b_1^3(\xi) + \beta_2 b_2^3(\xi) + \beta_3 b_3^3(\xi),\end{aligned}\quad (11)$$

in (6). For these polynomials, we have

$$\begin{aligned}|\alpha(\xi)|^2 \pm |\beta(\xi)|^2 &= (|\alpha_0|^2 \pm |\beta_0|^2) b_0^6(\xi) + \text{Re}(\overline{\alpha}_0 \alpha_1 \pm \overline{\beta}_0 \beta_1) b_1^6(\xi) \\ &+ \left[\frac{2}{5} \text{Re}(\overline{\alpha}_0 \alpha_2 \pm \overline{\beta}_0 \beta_2) + \frac{3}{5} (|\alpha_1|^2 \pm |\beta_1|^2) \right] b_2^6(\xi) \\ &+ \left[\frac{1}{10} \text{Re}(\overline{\alpha}_0 \alpha_3 \pm \overline{\beta}_0 \beta_3) + \frac{9}{10} \text{Re}(\overline{\alpha}_1 \alpha_2 \pm \overline{\beta}_1 \beta_2) \right] b_3^6(\xi) \\ &+ \left[\frac{2}{5} \text{Re}(\overline{\alpha}_1 \alpha_3 \pm \overline{\beta}_1 \beta_3) + \frac{3}{5} (|\alpha_2|^2 \pm |\beta_2|^2) \right] b_4^6(\xi) \\ &+ \text{Re}(\overline{\alpha}_2 \alpha_3 \pm \overline{\beta}_2 \beta_3) b_5^6(\xi) + (|\alpha_3|^2 \pm |\beta_3|^2) b_6^6(\xi),\end{aligned}\quad (12)$$

$$\begin{aligned}2\alpha(\xi)\overline{\beta}(\xi) &= 2\alpha_0\overline{\beta}_0 b_0^6(\xi) + (\alpha_0\overline{\beta}_1 + \alpha_1\overline{\beta}_0) b_1^6(\xi) \\ &+ \frac{2}{5}(\alpha_0\overline{\beta}_2 + \alpha_2\overline{\beta}_0 + 3\alpha_1\overline{\beta}_1) b_2^6(\xi) \\ &+ \frac{1}{10}[\alpha_0\overline{\beta}_3 + \alpha_3\overline{\beta}_0 + 9(\alpha_1\overline{\beta}_2 + \alpha_2\overline{\beta}_1)] b_3^6(\xi) \\ &+ \frac{2}{5}(\alpha_1\overline{\beta}_3 + \alpha_3\overline{\beta}_1 + 3\alpha_2\overline{\beta}_2) b_4^6(\xi) \\ &+ (\alpha_2\overline{\beta}_3 + \alpha_3\overline{\beta}_2) b_5^6(\xi) + 2\alpha_3\overline{\beta}_3 b_6^6(\xi),\end{aligned}\quad (13)$$

$$\begin{aligned}\overline{\alpha}(\xi)\alpha'(\xi) + \overline{\beta}(\xi)\beta'(\xi) &= 3(\overline{\alpha}_0\alpha_1 + \overline{\beta}_0\beta_1) b_0^4(\xi) \\ &+ \frac{3}{2}(\overline{\alpha}_0\alpha_2 + \overline{\beta}_0\beta_2) b_1^4(\xi) \\ &+ \frac{1}{2}[3(\overline{\alpha}_1\alpha_2 + \overline{\beta}_1\beta_2) + \overline{\alpha}_0\alpha_3 + \overline{\beta}_0\beta_3] b_2^4(\xi) \\ &+ \frac{3}{2}(\overline{\alpha}_1\alpha_3 + \overline{\beta}_1\beta_3) b_3^4(\xi) \\ &+ 3(\overline{\alpha}_2\alpha_3 + \overline{\beta}_2\beta_3) b_4^4(\xi).\end{aligned}\quad (14)$$

If the curve is to have a rotation-minimizing ERF, the imaginary part of the polynomial (14) must vanish identically, i.e., we must have

$$\begin{aligned}\text{Im}(\overline{\alpha}_0\alpha_1 + \overline{\beta}_0\beta_1) &= \text{Im}(\overline{\alpha}_0\alpha_2 + \overline{\beta}_0\beta_2) = 0, \\ 3\text{Im}(\overline{\alpha}_1\alpha_2 + \overline{\beta}_1\beta_2) + \text{Im}(\overline{\alpha}_0\alpha_3 + \overline{\beta}_0\beta_3) &= 0, \\ \text{Im}(\overline{\alpha}_1\alpha_3 + \overline{\beta}_1\beta_3) &= \text{Im}(\overline{\alpha}_2\alpha_3 + \overline{\beta}_2\beta_3) = 0.\end{aligned}\quad (15)$$

These equations impose 5 real constraints on the 16 degrees of freedom in the complex coefficients (α_i, β_i) for $i = 0, \dots, 3$.

Definition 1. The PH curve specified by (5) and (6) is in *canonical form* if $(\alpha_0, \beta_0) = (w, 0)$ for real $w \neq 0$, so that $(\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0)) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$.

A regular curve, with $\mathbf{r}'(0) \neq \mathbf{0}$, can always be mapped to canonical form through a spatial rotation. Since the canonical form assumption amounts to adoption of a particular coordinate system, any results deduced for canonical form PH curves apply to PH curves in general position. For a canonical-form curve, the rotation-minimizing ERF conditions (15) simplify [18] as follows.

Lemma 1. *In canonical form with $(\alpha_0, \beta_0) = (w, 0)$ the degree 7 PH curves that have rotation-minimizing ERFs are characterized by the conditions*

$$\alpha_1 = \frac{\text{Im}(\bar{\beta}_3 \beta_1)}{\text{Im}(\alpha_3)}, \quad \alpha_2 = \frac{\text{Im}(\bar{\beta}_3 \beta_2)}{\text{Im}(\alpha_3)}, \quad \text{Im}(w\alpha_3 + 3\bar{\beta}_1 \beta_2) = 0. \quad (16)$$

The expressions (16) assume that $\text{Im}(\alpha_3) \neq 0$. This assumption is in fact necessary to obtain a true space curve — see Remark 1 in [18].

4 Curve construction scheme

Given initial/final points $\mathbf{p}_i, \mathbf{p}_f$ and frames $(\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i), (\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f)$ we desire a spatial PH curve $\mathbf{r}(\xi)$ of prescribed total arc length $L > |\mathbf{p}_f - \mathbf{p}_i|$, equipped with a rational MTF $(\mathbf{f}_1(\xi), \mathbf{f}_2(\xi), \mathbf{f}_3(\xi))$ such that

$$\mathbf{r}(0) = \mathbf{p}_i, \quad (\mathbf{f}_1(0), \mathbf{f}_2(0), \mathbf{f}_3(0)) = (\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i),$$

$$\mathbf{r}(1) = \mathbf{p}_f, \quad (\mathbf{f}_1(1), \mathbf{f}_2(1), \mathbf{f}_3(1)) = (\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f).$$

The first step is to construct a degree 7 PH curve with a rotation-minimizing ERF, that matches the specified initial/final points, tangents, and arc length. The rational MTF will then be constructed by applying a rational rotation to the ERF normal-plane vectors.

Henceforth, we assume $\mathbf{p}_i = (0, 0, 0)$ and that the curve $\mathbf{r}(\xi)$ is in canonical form with $(\alpha_0, \beta_0) = (w, 0)$ and $(\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i) = (\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0)) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$. Since integrating the hodograph (6) introduces the free integration constant $\mathbf{p}_i = \mathbf{r}(0)$, the former choice incurs no loss of generality. The latter is achieved through a spatial rotation R that transforms⁴ the curve to canonical form — R must also be applied to the vectors $(\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f)$ and $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$. Once the

⁴This is most easily realized as a sequence of two rotations — the first being a rotation that maps $\mathbf{t}_i$ onto $\mathbf{i}$, and the second being a rotation about $\mathbf{i}$ that maps $\mathbf{u}_i, \mathbf{v}_i$ onto $\mathbf{j}, \mathbf{k}$.

interpolation problem is solved for this standard configuration, the inverse rotation R^{-1} may be applied to restore the curve to general position.

The curve construction requirements in canonical form are as follows:

- (a) match $\mathbf{e}_1(1)$ to the specified end-tangent $\mathbf{t}_f$;
- (b) match $\mathbf{r}(1) - \mathbf{r}(0)$ to the displacement $\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$;
- (c) match the total arc length to the prescribed value L ;
- (d) satisfy the rotation-minimizing ERF conditions (16).

The interpolation condition (a) corresponds to the requirement that

$$\mathbf{e}_1(1) = \frac{(|\boldsymbol{\alpha}_3|^2 - |\boldsymbol{\beta}_3|^2, 2 \operatorname{Re}(\boldsymbol{\alpha}_3 \overline{\boldsymbol{\beta}_3}), 2 \operatorname{Im}(\boldsymbol{\alpha}_3 \overline{\boldsymbol{\beta}_3}))}{|\boldsymbol{\alpha}_3|^2 + |\boldsymbol{\beta}_3|^2} = \mathbf{t}_f. \quad (17)$$

With $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ we set $\mathbf{t}_f = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$, and condition (17) is then equivalent to

$$\frac{|\boldsymbol{\alpha}_3|^2 - |\boldsymbol{\beta}_3|^2}{|\boldsymbol{\alpha}_3|^2 + |\boldsymbol{\beta}_3|^2} = \cos \theta, \quad \frac{2 \boldsymbol{\alpha}_3 \overline{\boldsymbol{\beta}_3}}{|\boldsymbol{\alpha}_3|^2 + |\boldsymbol{\beta}_3|^2} = \sin \theta \exp(i\phi). \quad (18)$$

For the satisfaction of condition (a), these equations yield

$$\boldsymbol{\alpha}_3 = \rho w \cos \frac{1}{2}\theta \exp(i(\phi + \psi)), \quad \boldsymbol{\beta}_3 = \rho w \sin \frac{1}{2}\theta \exp(i\psi), \quad (19)$$

where ρ and ψ are free parameters. The parameter ψ controls the orientation of the ERF vectors $\mathbf{e}_2(1)$, $\mathbf{e}_3(1)$ about the tangent $\mathbf{e}_1(1)$, and it also influences the curve shape. The parameter ρ controls the relative magnitudes of $\mathbf{r}'(1)$ and $\mathbf{r}'(0)$, namely $|\mathbf{r}'(1)| = \rho^2 |\mathbf{r}'(0)|$. Choosing $\rho = 1$ ensures that symmetric data will yield symmetric interpolants, but in practice this is found to be too restrictive (since solutions then exist for only restricted ranges of ψ).

Setting $\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = (\Delta x, \Delta y, \Delta z)$ and using the Hopf map form (6), the satisfaction of condition (b) yields the real equation

$$\int_0^1 |\boldsymbol{\alpha}(\xi)|^2 - |\boldsymbol{\beta}(\xi)|^2 d\xi = \Delta x, \quad (20)$$

together with the complex equation

$$\int_0^1 2 \boldsymbol{\alpha}(\xi) \overline{\boldsymbol{\beta}}(\xi) d\xi = \Delta y + i \Delta z, \quad (21)$$

while condition (c) is equivalent to the real equation

$$\int_0^1 |\boldsymbol{\alpha}(\xi)|^2 + |\boldsymbol{\beta}(\xi)|^2 d\xi = L. \quad (22)$$

Equations (20) and (22) may be replaced by the simpler conditions

$$\int_0^1 |\boldsymbol{\alpha}(\xi)|^2 d\xi = \frac{1}{2}(L + \Delta x) \quad \text{and} \quad \int_0^1 |\boldsymbol{\beta}(\xi)|^2 d\xi = \frac{1}{2}(L - \Delta x). \quad (23)$$

Finally, in standard configuration, the rotation-minimizing ERF conditions are achieved by expressing $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ in terms of $\boldsymbol{\alpha}_3, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$ as in (16), while $w, \boldsymbol{\alpha}_3, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ must also satisfy the third condition in (16).

We now take stock of the available free variables and imposed conditions. In canonical form $\boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}_0$ depend only on the free parameter w , while the interpolation of $\mathbf{t}_f$ implies that $\boldsymbol{\alpha}_3$ and $\boldsymbol{\beta}_3$ are specified by (19), where ρ and ψ are further free parameters. By Lemma 1, the coefficients of the cubics (11) must satisfy the conditions (16) if the curve is to have a rotation-minimizing ERF. The first two conditions specify $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ in terms of $\boldsymbol{\alpha}_3, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$. Thus, only $w, \rho, \psi, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ remain as free variables, embodying seven scalar degrees of freedom, and they must satisfy five scalar constraints — namely, the third condition in (16) and the single complex equation (21) and two real equations (23) that yield the prescribed displacement $\Delta \mathbf{p}$ and arc length L . Hence, the curve construction problem incorporates two residual freedoms.

For brevity, we henceforth write

$$c = \cos \frac{1}{2}\theta, \quad s = \sin \frac{1}{2}\theta, \quad t = \tan \frac{1}{2}\theta, \quad \lambda = \cos(\phi + \psi), \quad \mu = \sin(\phi + \psi). \quad (24)$$

Setting $\boldsymbol{\beta}_1 = w \mathbf{z}_1 \exp(i\psi)$, $\boldsymbol{\beta}_2 = w \mathbf{z}_2 \exp(i\psi)$ with $\mathbf{z}_1 = x_1 + i y_1$, $\mathbf{z}_2 = x_2 + i y_2$ and using (16) and (19) the coefficients of the cubics (11) then become

$$(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = w (1, t y_1 / \mu, t y_2 / \mu, \rho c (\lambda + i \mu)), \quad (25)$$

$$(\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3) = w (0, \mathbf{z}_1 \exp(i\psi), \mathbf{z}_2 \exp(i\psi), \rho s \exp(i\psi)), \quad (26)$$

and they must also satisfy the third condition in (16). Recall that $\text{Im}(\boldsymbol{\alpha}_3) \neq 0$ for a space curve. Since $w \neq 0$ for a regular curve, we must have $c \mu \neq 0$.

Now the definite integral of a Bernstein-form polynomial of degree n is just the sum of its coefficients divided by $n + 1$. Hence, using (12) and (13), the two real equations (23) become

$$\begin{aligned} & 10 |\boldsymbol{\alpha}_0|^2 + 6 |\boldsymbol{\alpha}_1|^2 + 6 |\boldsymbol{\alpha}_2|^2 + 10 |\boldsymbol{\alpha}_3|^2 \\ & + 10 \text{Re}(\overline{\boldsymbol{\alpha}}_0 \boldsymbol{\alpha}_1 + \overline{\boldsymbol{\alpha}}_2 \boldsymbol{\alpha}_3) + 4 \text{Re}(\overline{\boldsymbol{\alpha}}_0 \boldsymbol{\alpha}_2 + \overline{\boldsymbol{\alpha}}_1 \boldsymbol{\alpha}_3) \\ & + \text{Re}(\overline{\boldsymbol{\alpha}}_0 \boldsymbol{\alpha}_3) + 9 \text{Re}(\overline{\boldsymbol{\alpha}}_1 \boldsymbol{\alpha}_2) = 35 (L + \Delta x), \end{aligned} \quad (27)$$

$$\begin{aligned}
& 10 |\beta_0|^2 + 6 |\beta_1|^2 + 6 |\beta_2|^2 + 10 |\beta_3|^2 \\
& + 10 \operatorname{Re}(\bar{\beta}_0 \beta_1 + \bar{\beta}_2 \beta_3) + 4 \operatorname{Re}(\bar{\beta}_0 \beta_2 + \bar{\beta}_1 \beta_3) \\
& + \operatorname{Re}(\bar{\beta}_0 \beta_3) + 9 \operatorname{Re}(\bar{\beta}_1 \beta_2) = 35 (L - \Delta x), \tag{28}
\end{aligned}$$

while the complex equation (21) becomes

$$\begin{aligned}
& 20 \alpha_0 \bar{\beta}_0 + 12 \alpha_1 \bar{\beta}_1 + 12 \alpha_2 \bar{\beta}_2 + 20 \alpha_3 \bar{\beta}_3 \\
& + 10(\alpha_0 \bar{\beta}_1 + \alpha_1 \bar{\beta}_0 + \alpha_2 \bar{\beta}_3 + \alpha_3 \bar{\beta}_2) + 4(\alpha_0 \bar{\beta}_2 + \alpha_2 \bar{\beta}_0 + \alpha_1 \bar{\beta}_3 + \alpha_3 \bar{\beta}_1) \\
& + \alpha_0 \bar{\beta}_3 + \alpha_3 \bar{\beta}_0 + 9(\alpha_1 \bar{\beta}_2 + \alpha_2 \bar{\beta}_1) = 70 (\Delta y + i \Delta z). \tag{29}
\end{aligned}$$

We substitute (25)–(26) into the third of equations (16), and into equations (27)–(28). For equation (29), we multiply both sides by $\exp(i\psi)$, substitute (25)–(26), and separate real and imaginary parts. On clearing denominators, this yields a system of five real equations of the form

$$w^2 f_0(x_1, y_1, x_2, y_2) = k_i, \quad i = 0, \dots, 4, \tag{30}$$

where

$$f_0(x_1, y_1, x_2, y_2) := c \mu \rho + 3(x_1 y_2 - x_2 y_1),$$

$$\begin{aligned}
f_1(x_1, y_1, x_2, y_2) &:= \mu^2 (10 + 10 c^2 \rho^2 + c \lambda \rho) + t \mu (10 y_1 + 4 y_2) \\
&+ s \lambda \mu \rho (4 y_1 + 10 y_2) + 3 t^2 (2 y_1^2 + 3 y_1 y_2 + 2 y_2^2),
\end{aligned}$$

$$\begin{aligned}
f_2(x_1, y_1, x_2, y_2) &:= 10 s^2 \rho^2 + s \rho (4 x_1 + 10 x_2) \\
&+ 6 (x_1^2 + y_1^2 + x_2^2 + y_2^2) + 9 (x_1 x_2 + y_1 y_2),
\end{aligned}$$

$$\begin{aligned}
f_3(x_1, y_1, x_2, y_2) &:= s \mu \rho (20 c \lambda \rho + 1) + \mu (10 x_1 + 4 x_2) \\
&+ s t \rho (4 y_1 + 10 y_2) + 4 c \mu \rho (\lambda x_1 + \mu y_1) + 10 c \mu \rho (\lambda x_2 + \mu y_2) \\
&+ 12 t (x_1 y_1 + x_2 y_2) + 9 t (x_1 y_2 + x_2 y_1),
\end{aligned}$$

$$\begin{aligned}
f_4(x_1, y_1, x_2, y_2) &:= 20 c s \mu^2 \rho^2 - \mu (10 y_1 + 4 y_2) + 4 c \mu \rho (\mu x_1 - \lambda y_1) \\
&+ 10 c \mu \rho (\mu x_2 - \lambda y_2) - 6 t (2 y_1^2 + 3 y_1 y_2 + 2 y_2^2).
\end{aligned}$$

and, for given values of ρ and ψ , we define the constants

$$k_0 := 0, \quad k_1 := 35 (L + \Delta x) \mu^2, \quad k_2 := 35 (L - \Delta x),$$

$$k_3 := 70 (\cos \psi \Delta y - \sin \psi \Delta z) \mu, \quad k_4 := 70 (\sin \psi \Delta y + \cos \psi \Delta z) \mu.$$

Since $w \neq 0$ for a regular curve, instance $i = 0$ of equations (30) implies that

$$g_1(x_1, y_1, x_2, y_2) := c \mu \rho + 3(x_1 y_2 - x_2 y_1) = 0, \quad (31)$$

while for $i = 2, 3, 4$ we observe that

$$k_1 [w^2 f_i(x_1, y_1, x_2, y_2) - k_i] - k_i [w^2 f_1(x_1, y_1, x_2, y_2) - k_1] = 0.$$

Hence, since $w \neq 0$, for $i = 2, 3, 4$ we must have

$$g_i(x_1, y_1, x_2, y_2) := k_1 f_i(x_1, y_1, x_2, y_2) - k_i f_1(x_1, y_1, x_2, y_2) = 0. \quad (32)$$

Equations (31) and (32) define a system $g_i(x_1, y_1, x_2, y_2) = 0$, $i = 1, \dots, 4$ of quadratic equations in the four variables x_1, y_1, x_2, y_2 from which w has been eliminated ($c, s, \lambda, \mu, \Delta x, \Delta y, \Delta z, L$ being known constants for fixed values of the parameters ψ and ρ). This system defines four quadric hypersurfaces in $\mathbb{R}^4$ which, in general, possess at most 16 distinct real intersection points.

For each real solution of the system $g_i(x_1, y_1, x_2, y_2) = 0$, $i = 1, \dots, 4$ the corresponding value of w^2 can be determined from any of the expressions

$$w^2 = \frac{k_i}{f_i(x_1, y_1, x_2, y_2)}, \quad i = 1, \dots, 4. \quad (33)$$

Since (w, x_1, y_1, x_2, y_2) and $(-w, -x_1, -y_1, -x_2, -y_2)$ define the same curve we may, without loss of generality, choose the positive w defined by (33).

Now note that from the equation $g_1(x_1, y_1, x_2, y_2) = 0$ we may write

$$y_1 = \frac{3x_1 y_2 + c \mu \rho}{3x_2}, \quad (34)$$

and on substituting this into $g_2(x_1, y_1, x_2, y_2) = 0$ and clearing denominators we obtain a quadratic equation

$$a_2 y_2^2 + a_1 y_2 + a_0 = 0 \quad (35)$$

in y_2 , with coefficients dependent on x_1, x_2 — namely,

$$\begin{aligned} a_2 &= 9(k_1 - k_2 t^2)(2x_1^2 + 3x_1 x_2 + 2x_2^2), \\ a_1 &= 3(k_1 - k_2 t^2) c \mu \rho (4x_1 + 3x_2) \\ &\quad - 6k_2 \mu x_2 (5t x_1 + 2s \lambda \rho x_1 + 2t x_2 + 5s \lambda \rho x_2), \\ a_0 &= [30s \rho x_2^2 (s \rho + x_2) + 18x_2^2 (x_1^2 + x_2^2) + 3x_1 x_2^2 (4s \rho + 9x_2) + 2c^2 \mu^2 \rho^2] k_1 \\ &\quad - [3(10 + 10c^2 \rho^2 + c \lambda \rho) x_2^2 + 2s^2 \rho^2 + 10s \rho x_2 + 4c s \lambda \rho^2 x_2] \mu^2 k_2. \end{aligned}$$

For given real x_1, x_2 values, the discriminant $\delta = a_1^2 - 4 a_2 a_0$ of equation (35) must be non-negative to obtain a real solution y_2 , with the corresponding real value of y_1 being obtained from (34). The values thus identified automatically satisfy $g_1(x_1, y_1, x_2, y_2) = g_2(x_1, y_1, x_2, y_2) = 0$.

In principle, it is possible to eliminate three of the variables from the four quadratic equations $g_i(x_1, y_1, x_2, y_2) = 0$ for $i = 1, \dots, 4$ to obtain a univariate polynomial of degree 16 in the remaining variable. Since this polynomial is of even degree, there is no *a priori* guarantee that it has a real root — which is necessary for the existence a solution. The complicated dependence of the equations (31)–(32) on the input quantities $L, \Delta x, \Delta y, \Delta z, \theta, \phi$ precludes an elementary analysis of the existence of solutions for all possible combinations of their values. However, the presence of the free parameters ρ, ψ increases the likelihood that real solutions can always be found through their adjustment. This is borne out by numerical tests on a large variety of computed examples, in which no cases of non-existent solutions were observed.

5 Solution procedure

The solution scheme adopted herein is to employ a grid of initial x_1, x_2 values of magnitude ~ 1 , to initiate Newton–Raphson iterations to solve the system (31)–(32). For each pair x_1, x_2 the discriminant δ of the quadratic (35) is first computed: when $\delta < 0$ the x_1, x_2 pair is rejected; but when $\delta > 0$, two pairs of y_2 values are obtained as roots of (35) with corresponding y_1 values defined by (34), and these two x_1, y_1, x_2, y_2 sets serve as initial values for the Newton–Raphson iteration. This approach increases the likelihood that the iterations will rapidly converge to a real solution of the system (31)–(32).

A tolerance of 10^{-12} on the L_2 norm of the system (31)–(32) was imposed for convergence of the Newton–Raphson iterations. Although (for given ρ, ψ values) not all initial x_1, x_2 choices for which $\delta > 0$ resulted in convergence, in most cases a solution of high accuracy was obtained in 4–8 iterations. The Jacobian matrix for the system (31)–(32) was computed symbolically in the **Maple** computer algebra system. The resulting expression for its determinant is very complicated, but by factorization it is found to be identically zero only in the degenerate case $\mu = 0$ (which corresponds to a planar curve). Using a PC with a modest CPU, the computation time for fixed values of ρ, ψ and initial x_1, x_2 values on a 10×10 grid is ~ 0.008 seconds.

For each real solution w, x_1, y_1, x_2, y_2 of the system defined by (31)–(33), the complex coefficients (25)–(26) may be determined with $\mathbf{z}_1 = x_1 + i y_1$ and

$\mathbf{z}_2 = x_2 + \mathbf{i} y_2$. Substituting $\mathcal{A}(\xi)$ with quaternion coefficients $\mathcal{A}_i = \boldsymbol{\alpha}_i + \mathbf{k} \boldsymbol{\beta}_i$, $i = 0, \dots, 3$ into (4) and integrating then yields the Bézier form

$$\mathbf{r}(\xi) = \sum_{i=0}^7 \mathbf{p}_i b_i^7(\xi),$$

with control points $\mathbf{p}_0, \dots, \mathbf{p}_7$ given by

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{p}_0 + \mathcal{A}_0 \mathbf{i} \mathcal{A}_0^* / 7, \\ \mathbf{p}_2 &= \mathbf{p}_1 + (\mathcal{A}_0 \mathbf{i} \mathcal{A}_1^* + \mathcal{A}_1 \mathbf{i} \mathcal{A}_0^*) / 14, \\ \mathbf{p}_3 &= \mathbf{p}_2 + (\mathcal{A}_0 \mathbf{i} \mathcal{A}_2^* + 3 \mathcal{A}_1 \mathbf{i} \mathcal{A}_1^* + \mathcal{A}_2 \mathbf{i} \mathcal{A}_0^*) / 35, \\ \mathbf{p}_4 &= \mathbf{p}_3 + (\mathcal{A}_0 \mathbf{i} \mathcal{A}_3^* + \mathcal{A}_3 \mathbf{i} \mathcal{A}_0^* + 9 \mathcal{A}_1 \mathbf{i} \mathcal{A}_2^* + 9 \mathcal{A}_2 \mathbf{i} \mathcal{A}_1^*) / 140, \\ \mathbf{p}_5 &= \mathbf{p}_4 + (\mathcal{A}_1 \mathbf{i} \mathcal{A}_3^* + 3 \mathcal{A}_2 \mathbf{i} \mathcal{A}_2^* + \mathcal{A}_3 \mathbf{i} \mathcal{A}_1^*) / 35, \\ \mathbf{p}_6 &= \mathbf{p}_5 + (\mathcal{A}_2 \mathbf{i} \mathcal{A}_3^* + \mathcal{A}_3 \mathbf{i} \mathcal{A}_2^*) / 14, \\ \mathbf{p}_7 &= \mathbf{p}_6 + \mathcal{A}_3 \mathbf{i} \mathcal{A}_3^* / 7, \end{aligned} \tag{36}$$

where we set $\mathbf{p}_0 = \mathbf{p}_i$. By construction, this curve has a rotation-minimizing ERF, with initial instance $(\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0)) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ and the final tangent $\mathbf{e}_1(1) = \mathbf{t}_f$. Also, $\mathbf{p}_7 - \mathbf{p}_0$ coincides with the specified displacement $\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$, and the total arc length is equal to the prescribed value L .

The shape of the interpolant $\mathbf{r}(\xi)$ is sensitive to the parameters ρ and ψ , and it is desirable to identify values that yield “optimum” shape consistent with the specified end points and tangents, total arc length, and satisfaction of the rotation-minimizing ERF conditions (16). Optimal shape is typically identified by minimization of the elastic bending energy, defined [28] as

$$E = \int_0^L \kappa^2 ds, \tag{37}$$

where s and κ denote the arc length and curvature along $\mathbf{r}(\xi)$, while L is the total arc length. The curvature is given by

$$\kappa(\xi) = \frac{|\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)|}{\sigma^3(\xi)},$$

where the parametric speed (7) is the derivative $ds/d\xi$ of arc length s with respect to the parameter ξ . Now every PH curve satisfies [16] the relation

$$|\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)|^2 = 4\sigma^2(\xi) |\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}'(\xi) - \boldsymbol{\alpha}'(\xi)\boldsymbol{\beta}(\xi)|^2,$$

and since $ds = \sigma(\xi) d\xi$, the bending energy integral (37) reduces to

$$E = 4 \int_0^1 \frac{|\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}'(\xi) - \boldsymbol{\alpha}'(\xi)\boldsymbol{\beta}(\xi)|^2}{\sigma^3(\xi)} d\xi.$$

For a spatial PH curve of degree 7, the integrand has a numerator of degree 8, and the cube of the parametric speed (7) as denominator. Since $\sigma(\xi)$ is of degree 6, it does not admit a closed-form solution for its roots, and an exact partial-fraction expansion of the integrand is not possible. However, an adaptive numerical quadrature — e.g., the Simpson rule [31] — gives rapid convergence to machine precision, since the integrand is non-negative.

Note that curves minimizing E subject to given boundary conditions, but without constraining the total arc length L , need not be finite [2] since loops of length $L \sim 2\pi r$ and curvature $\kappa \sim 1/r$ satisfy $E \rightarrow 0$ as $r \rightarrow \infty$. However, this is not of concern in the present context, since the arc length L of $\mathbf{r}(\xi)$ is fixed *a priori* as an element of the curve construction process.

To optimize the curve shape with respect to the free parameters ρ, ψ a set of values $(\rho, \psi) \in [\rho_{\min}, \rho_{\max}] \times [0, 2\pi)$ is used. Since the curve end derivatives have magnitudes $|\mathbf{r}'(0)| = w^2$ and $|\mathbf{r}'(1)| = \rho^2 w^2$, and large disparities in them are incompatible with favorable shape properties, we choose $\rho_{\min} = 1/\sqrt{2}$ and $\rho_{\max} = \sqrt{2}$, so that $|\mathbf{r}'(0)|$ and $|\mathbf{r}'(1)|$ differ by a factor no greater than 2. For each distinct solution obtained, the value of the energy integral (37) is computed by numerical quadrature, and compared with that for solutions corresponding to other ρ, ψ values, to identify the minimum-energy solution. In the following examples, the parameters were sampled with increments $\delta\rho = 0.1/\sqrt{2}$ and $\delta\psi = \pi/90$ (a hierarchical refinement of the sampled values may be invoked if identification of the optimal solution to a higher accuracy is desired). Cycling over the different ρ, ψ values to optimize the curve shape required a computation time of ~ 5.0 seconds.

Example 1. For the data $\Delta\mathbf{p} = (1.25, -0.25, 0.75)$, $L = 1.25 |\Delta\mathbf{p}|$, $\theta = \pi/3$, $\phi = \pi$, the minimum energy solution with $E \approx 1.1563$ occurs for parameter values $\rho \approx 1.32$ and $\psi \approx 1.16\pi$, and is identified by

$$\begin{aligned} w &= 1.531804, \\ (x_1, y_1) &= (-0.304813, 0.508329), \\ (x_2, y_2) &= (0.114495, 0.395953). \end{aligned}$$

The corresponding complex coefficients (25)–(26) are

$$\begin{aligned}\alpha_0 &= 1.531804 + 0.000000i, & \beta_0 &= 0.000000 + 0.000000i, \\ \alpha_1 &= 0.957587 + 0.000000i, & \beta_1 &= 0.777819 - 0.468314i, \\ \alpha_2 &= 0.745894 + 0.000000i, & \beta_2 &= 0.129891 - 0.617865i, \\ \alpha_3 &= 1.546118 + 0.822085i, & \beta_3 &= -0.892652 - 0.474631i,\end{aligned}$$

and from the quaternion coefficients $\mathcal{A}_i = \alpha_i + \mathbf{k}\beta_i$ the curve control points are determined using (36) as

$$\begin{aligned}\mathbf{p}_0 &= (0.000000, 0.000000, 0.000000), \\ \mathbf{p}_1 &= (0.335203, 0.000000, 0.000000), \\ \mathbf{p}_2 &= (0.544751, 0.170210, 0.102481), \\ \mathbf{p}_3 &= (0.617982, 0.309264, 0.233441), \\ \mathbf{p}_4 &= (0.693456, 0.380316, 0.364809), \\ \mathbf{p}_5 &= (0.818552, 0.394800, 0.547700), \\ \mathbf{p}_6 &= (0.957971, 0.255809, 0.750000), \\ \mathbf{p}_7 &= (1.250000, -0.250000, 0.750000).\end{aligned}$$

Figure 1 illustrates the constructed degree 7 PH curve, with a rotation-minimizing ERF, satisfying the specified boundary conditions.

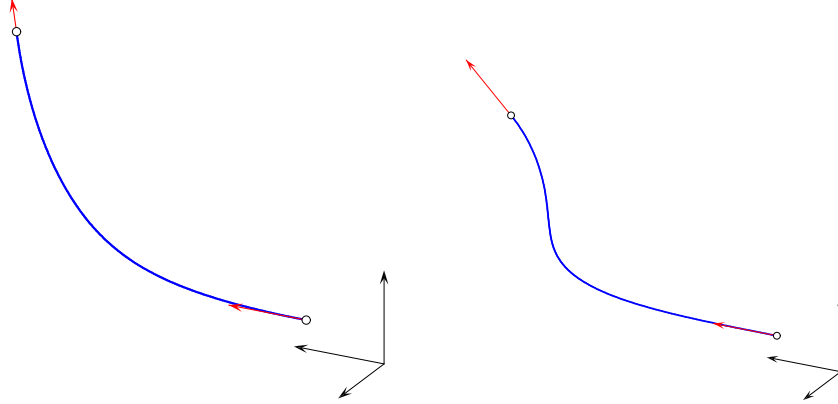


Figure 1: The degree 7 PH curves with rational RMFs, matching given end points, tangents, and arc lengths, in Example 1 (left) and Example 2 (right). The red arrows indicate the specified initial and final tangent vectors.

Example 2. For the data $\Delta \mathbf{p} = (1.0, 0.5, 1.0)$, $L = 1.25 |\Delta \mathbf{p}|$, $\theta = \pi/4$, $\phi = \pi/2$, the minimum energy solution with $E \approx 1.4770$ occurs for parameter values $\rho \approx 0.60$ and $\psi \approx 1.59 \pi$, and is identified by

$$\begin{aligned} w &= 2.315250, \\ (x_1, y_1) &= (0.124714, 0.119303), \\ (x_2, y_2) &= (0.630238, 0.169719). \end{aligned}$$

In this case, the complex coefficients (25)–(26) are

$$\begin{aligned} \alpha_0 &= 2.315250 + 0.000000i, & \beta_0 &= 0.000000 + 0.000000i, \\ \alpha_1 &= 0.391327 + 0.000000i, & \beta_1 &= 0.348568 - 0.195369i, \\ \alpha_2 &= 0.556694 + 0.000000i, & \beta_2 &= 0.802388 - 1.280515i, \\ \alpha_3 &= 1.227328 + 0.375232i, & \beta_3 &= 0.155426 - 0.508376i, \end{aligned}$$

and from the quaternion coefficients $\mathcal{A}_i = \alpha_i + \mathbf{k}\beta_i$ we obtain the curve control points

$$\begin{aligned} \mathbf{p}_0 &= (0.000000, 0.000000, 0.000000), \\ \mathbf{p}_1 &= (0.765769, 0.000000, 0.000000), \\ \mathbf{p}_2 &= (0.895200, 0.115289, 0.064618), \\ \mathbf{p}_3 &= (0.968291, 0.244828, 0.247137), \\ \mathbf{p}_4 &= (0.968769, 0.315289, 0.342362), \\ \mathbf{p}_5 &= (0.818274, 0.415596, 0.497109), \\ \mathbf{p}_6 &= (0.805067, 0.500000, 0.805067), \\ \mathbf{p}_7 &= (1.000000, 0.500000, 1.000000). \end{aligned}$$

Figure 1 illustrates the constructed degree 7 PH curve with a rotation–minimizing ERF satisfying the specified boundary conditions.

In both the above examples, the end point displacements, final tangents, and total arc lengths agree with the prescribed data to an accuracy of 10^{-12} or better. Likewise, evaluating the RRMF conditions (15) using the computed coefficients yields residuals of 10^{-12} or better.

6 Rational MTF construction

For an adapted frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ with angular velocity $\boldsymbol{\omega} = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3$ along a space curve $\mathbf{r}(\xi)$, $\xi \in [0, 1]$ of arc length L , the twist T is defined by

$$T = \int_0^L \omega_1 \, ds = \int_0^1 \omega_1 \sigma \, d\xi,$$

$\sigma(\xi) = |\mathbf{r}'(\xi)|$ being the parametric speed of $\mathbf{r}(\xi)$. By construction, the degree 7 PH curves considered in Section 4 have ERFs satisfying $\omega_1 \equiv 0$, and thus $T = 0$. To construct a rational MTF $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ subject to specified boundary conditions from the ERF $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ along such curves, we invoke the rational normal-plane rotation

$$\begin{bmatrix} \mathbf{f}_2(\xi) \\ \mathbf{f}_3(\xi) \end{bmatrix} = \frac{1}{a^2(\xi) + b^2(\xi)} \begin{bmatrix} a^2(\xi) - b^2(\xi) & 2a(\xi)b(\xi) \\ -2a(\xi)b(\xi) & a^2(\xi) - b^2(\xi) \end{bmatrix} \begin{bmatrix} \mathbf{e}_2(\xi) \\ \mathbf{e}_3(\xi) \end{bmatrix}, \quad (38)$$

for relatively prime polynomials $a(\xi)$, $b(\xi)$. This amounts to orienting $\mathbf{f}_2, \mathbf{f}_3$ relative to $\mathbf{e}_2, \mathbf{e}_3$ through the angular function

$$\theta(\xi) = 2 \tan^{-1} \frac{b(\xi)}{a(\xi)}. \quad (39)$$

The curve is assumed to be in canonical form, with $(\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0)) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$. Let $\mathbf{f}_2(0), \mathbf{f}_3(0)$ and $\mathbf{f}_2(1), \mathbf{f}_3(1)$ be given initial and final MTF normal-plane vectors. Their orientations θ_i, θ_f relative to the ERF are defined by

$$(\cos \theta_i, \sin \theta_i) = (\mathbf{f}_2(0) \cdot \mathbf{e}_2(0), \mathbf{f}_2(0) \cdot \mathbf{e}_3(0)),$$

$$(\cos \theta_f, \sin \theta_f) = (\mathbf{f}_2(1) \cdot \mathbf{e}_2(1), \mathbf{f}_2(1) \cdot \mathbf{e}_3(1)).$$

Since an MTF can always satisfy initial/final orientations with a total twist not less than $-\pi$ or greater than $+\pi$, the *reduced twist* $T_{\min}$ is employed in lieu of the nominal value $T = \theta_f - \theta_i$. When $T \in (-\pi, +\pi]$ we take $T_{\min} = T$. Otherwise, we define $T_{\min}$ by adding to or subtracting from T the integer multiple of 2π that yields a value in the domain $(-\pi, +\pi]$.

If $\boldsymbol{\omega} = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3$ is the MTF angular velocity, the *average* value of its tangent component is

$$\bar{\omega}_1 = \frac{T_{\min}}{L}.$$

As previously noted, ω_1 must not change sign for an MTF, i.e., $\mathbf{f}_2$ and $\mathbf{f}_3$ must exhibit a consistent *sense* of rotation about $\mathbf{f}_1$. A further desirable feature is that $\omega_1 \equiv \text{constant}$ ($= \bar{\omega}_1$), i.e., a fixed *rate* of rotation of $\mathbf{f}_2$ and $\mathbf{f}_3$ about $\mathbf{f}_1$. However, this cannot be achieved exactly by the rational MTF (38) since it corresponds to a rotation angle $\theta(\xi) = \bar{\omega}_1 s(\xi)$, and it is impossible to define polynomials $a(\xi)$, $b(\xi)$ such that

$$\frac{a^2(\xi) - b^2(\xi)}{a^2(\xi) + b^2(\xi)} = \cos(\bar{\omega}_1 s(\xi)), \quad \frac{2a(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)} = \sin(\bar{\omega}_1 s(\xi)).$$

Hence, the goal is to minimize the variation of ω_1 about the mean value $\bar{\omega}_1$.

The rotation (38)–(39) may be written in terms of a complex polynomial $\mathbf{w}(\xi) = a(\xi) + i b(\xi)$, such that

$$\exp(i\theta(\xi)) = \frac{\mathbf{w}^2(\xi)}{|\mathbf{w}(\xi)|^2} = \frac{a^2(\xi) - b^2(\xi) + i 2a(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)}, \quad (40)$$

from which we obtain

$$\theta'(\xi) = 2 \frac{\text{Im}(\bar{\mathbf{w}}(\xi)\mathbf{w}'(\xi))}{|\mathbf{w}(\xi)|^2} = 2 \frac{a(\xi)b'(\xi) - a'(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)}. \quad (41)$$

Thus, the normal-plane rotation (38) yields an angular velocity component

$$\omega_1(\xi) = \frac{d\theta}{ds} = \frac{d\xi}{ds} \frac{d\theta}{d\xi} = \frac{\theta'(\xi)}{\sigma(\xi)} \quad (42)$$

for the frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ in the tangent direction $\mathbf{f}_1$. To obtain an MTF, $\mathbf{w}(\xi)$ must be specified such that ω_1 does not change sign, and $\mathbf{f}_2, \mathbf{f}_3$ match given initial and final instances with minimum total variation between them.

As remarked in [19], the polynomial $\mathbf{w}(\xi)$ must be at least quadratic if the MTF is to accurately approximate the condition $\omega_1 = \text{constant}$. For the case of spatial PH quintics in [19], the angular velocity component (42) induced by $\mathbf{w}(\xi)$ was required to cancel out the ERF tangential angular velocity, and also superpose a near-constant rotation rate $\bar{\omega}_1$ consistent with the specified boundary conditions. For cases in which the ERF has strong angular velocity variations, it was necessary to subdivide the domain $[0, 1]$ of $\mathbf{r}(\xi)$ to ensure that $\omega_1(\xi)$ does not change sign. The degree 7 PH curves employed here have rotation-minimizing ERFs (with zero tangential angular velocity) that allow a simpler and more accurate identification of the polynomial that generates an MTF from the ERF — $\mathbf{w}(\xi)$ is not required to cancel the ERF tangential angular velocity: it is only required to impose a rotation rate of $\mathbf{f}_2, \mathbf{f}_3$ relative to $\mathbf{e}_2, \mathbf{e}_3$ that approximates the mean value $\bar{\omega}_1$.

For the complex quadratic Bernstein-form polynomial

$$\mathbf{w}(\xi) = \mathbf{w}_0(1 - \xi)^2 + \mathbf{w}_1 2(1 - \xi)\xi + \mathbf{w}_2 \xi^2$$

one can verify [19] that, with γ a free parameter, the coefficients

$$\begin{aligned} \mathbf{w}_0 &= \exp(i\frac{1}{2}\theta_i), \quad \mathbf{w}_2 = \gamma \exp(i\frac{1}{2}\theta_f), \\ \mathbf{w}_1 &= \bar{\omega}_1 \frac{\exp(i\frac{1}{2}\theta_f) \sigma(0) + \gamma \exp(i\frac{1}{2}\theta_i) \sigma(1)}{4 \sin \frac{1}{2}\Delta\theta} \end{aligned}$$

yield $\theta(0) = \theta_i$, $\theta(1) = \theta_f$, and $\omega_1(0) = \omega_1(1) = \bar{\omega}_1$. The parameter γ can be used to minimize the root-mean-square deviation of ω_1 about $\bar{\omega}_1$, defined by

$$\delta\omega_1 = \left[\frac{1}{L} \int_0^1 \left(\frac{\omega_1(\xi)}{\bar{\omega}_1} - 1 \right)^2 \sigma(\xi) d\xi \right]^{1/2}. \quad (43)$$

In general, the integral does not permit a closed-form reduction, but it can be efficiently computed to high accuracy by an adaptive quadrature such as the Simpson rule. A simple scheme for minimization without derivatives [4] can then be used to find the γ value that achieves its minimum value.

Example 3. For the curve constructed in Example 1, the orientations of the initial and final MTF normal plane vectors relative to the ERF are specified by the angles $\theta_i = 0$ and $\theta_f = -\frac{1}{2}\pi$. The parameter value $\gamma \approx 0.58$ is found to minimize the deviation (43) of ω_1 about $\bar{\omega}_1$, yielding $\delta\omega_1 \approx 0.033$ — see Figure 2. For this γ value, Figure 3 illustrates the curve with the prescribed initial/final MTF normal plane vectors, the ERF variation along the curve, and the constructed MTF satisfying the specified boundary conditions. Note that the ERF matches the initial but not final normal plane vectors, whereas the MTF matches both. Figure 4 plots the ratio $\omega_1/\bar{\omega}_1$ of the MTF angular velocity component ω_1 to the mean value $\bar{\omega}_1$. It is evident that ω_1 accurately approximates the desired constant angular velocity condition.

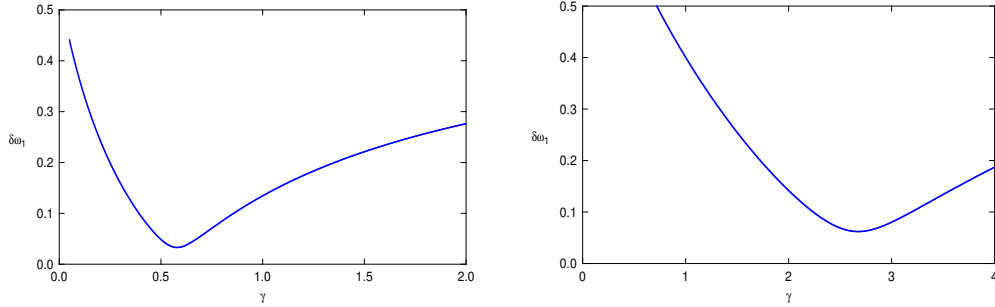


Figure 2: Dependence of the rms deviation (43) of the MTF angular velocity component ω_1 on the parameter γ in Example 1 (left) and Example 2 (right).

Example 4. For the curve in Example 2, we impose angles $\theta_i = 0$ and $\theta_f = \pi$. As seen in Figure 2, the parameter $\gamma \approx 2.67$ is found to identify the minimum value $\delta\omega_1 \approx 0.062$ of (43). Figure 5 illustrates the curve together with the

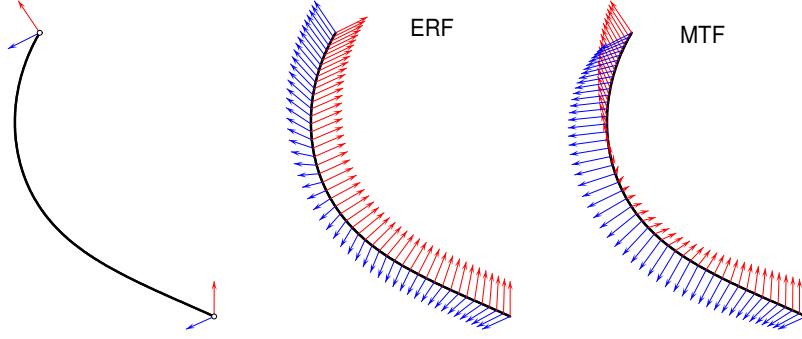


Figure 3: Left: the degree 7 PH curve with rational rotation-minimizing ERF in Example 1, showing desired initial and final MTF normal plane vectors. Center: the rotation-minimizing ERF matches the initial, but not the final, MTF vectors. Right: the rational MTF matching the boundary conditions, obtained by a rational rotation of the ERF normal plane vectors.

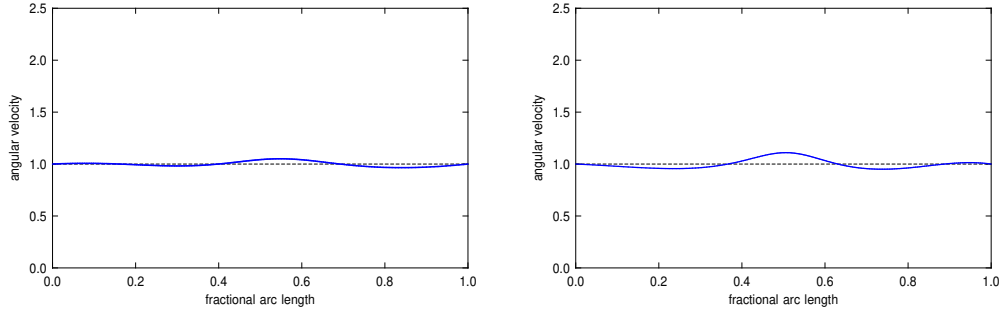


Figure 4: Variation of the angular velocity ratio $\omega_1/\bar{\omega}_1$ for the MTF satisfying given boundary conditions on the degree 7 PH curves with rational rotation-minimizing ERFs constructed in Example 1 (left) and Example 2 (right).

specified initial/final normal plane vectors, the ERF, and the constructed rational MTF, and Figure 4 plots the ratio $\omega_1/\bar{\omega}_1$ of the MTF angular velocity component ω_1 to the mean value $\bar{\omega}_1$.

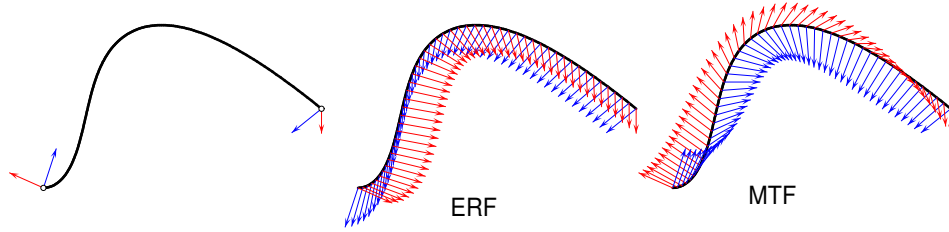


Figure 5: Left: the degree 7 PH curve with rational rotation-minimizing ERF in Example 2, showing desired initial and final MTF normal plane vectors. Center: the rotation-minimizing ERF matches the initial, but not the final, MTF vectors. Right: the rational MTF satisfying the boundary conditions, obtained by a rational rotation of the ERF normal plane vectors.

It is evident in Figure 4 that the rotation-minimizing ERF property of the degree 7 PH curves employed herein allows the MTF angular velocity ω_1 to closely approximate the constant value $\bar{\omega}_1$ without the need for subdivision, as was necessary for general PH quintics with strong ERF variations in [19], although the MTFs are now of somewhat higher degree (10 versus 8).

We use Example 4 to illustrate the ability of the MTF to match arbitrary initial/final orientations of the normal plane vectors on a unique curve. Since the ERF is rotation-minimizing, the nominal data $\theta_i = 0$ and $\theta_f = \pi$ yields a twist of maximum magnitude $T = T_{\min} = \pi$, shown on the left in Figure 6. In the center and on the right in Figure 6, we show the MTF when θ_f is increased by $2\pi/3$ and $4\pi/3$, respectively (note that these two cases correspond to the reduced twist values $T_{\min} = -\pi/3$ and $T_{\min} = \pi/3$).

Example 5. To highlight the distinction between the spatial motions defined by rational MTFs, considered herein and in [19], and those specified by the rational RMFs constructed in [15, 18], we re-visit Example 1 from [15]. This involves the construction of a quintic RRMF curve matching end points $\mathbf{p}_i =$

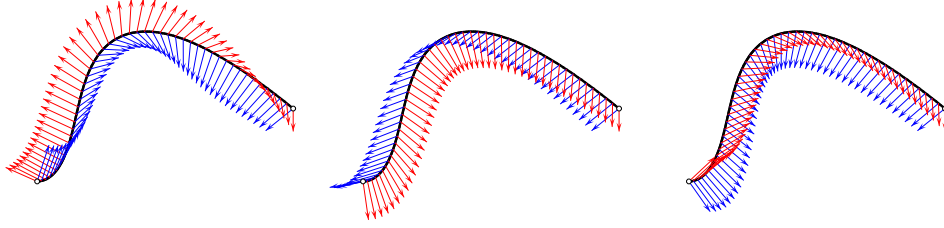


Figure 6: Three different MTFs on the *same* degree 7 PH curve, as specified by fixed initial normal-plane vectors (right curve point) and rotations of the final normal-plane vectors (left curve point) by angles $\pi, 2\pi/3, 4\pi/3$ relative to the orientation stated in Example 4. The pre-determined PH curve admits *arbitrary* orientations of the initial and final MTF normal-plane vectors.

$(0, 0, 0)$ and $\mathbf{p}_f = (1, 0, 0)$, and end frames

$$\begin{aligned} \mathbf{t}_i &= (0.707107, 0.707107, 0.000000), \\ \mathbf{u}_i &= (0.000000, 0.000000, -1.000000), \\ \mathbf{v}_i &= (-0.707107, 0.707107, 0.000000), \\ \mathbf{t}_f &= (0.804738, -0.310617, 0.505879), \\ \mathbf{u}_f &= (0.310617, -0.505879, -0.804738), \\ \mathbf{v}_f &= (0.505879, 0.804738, -0.310617). \end{aligned}$$

Two solutions exist, and the interpolant corresponding to the smaller λ value is shown on the left in Figure 7. The final frame is then modified by rotating $\mathbf{u}_f$ and $\mathbf{v}_f$ through angles $2\pi/3$ and $4\pi/3$ about $\mathbf{t}_f$. Each instance again yields two solutions, and the interpolants corresponding to the smaller λ are shown in the middle and on the right in Figure 7 — they differ markedly from the initial interpolant, illustrating the fact that RMFs are not compatible⁵ with different adapted end frame orientations on a fixed curve.

Example 6. To illustrate the superior ability of the degree 7 PH curves with rotation-minimizing ERFs to generate MTFs with nearly constant tangential angular velocity components, compared to general quintic PH curves, we revisit Example 3 in [19]. Because of the strong ERF variation, it was necessary to invoke subdivision to ensure a proper MTF, defined over sub-domains of

⁵Since this property is intrinsic to RMFs, it also applies to the degree 7 RRMF curves considered in [18].

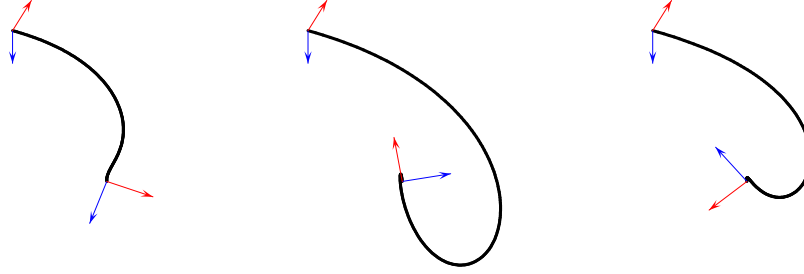


Figure 7: Solutions to the RRMF quintic Hermite interpolation problem [15] corresponding to fixed end points $\mathbf{p}_i$, $\mathbf{p}_f$ and initial frame vectors $\mathbf{u}_i$, $\mathbf{v}_i$ but different final vectors $\mathbf{u}_f$, $\mathbf{v}_f$. The shape of the resulting quintic RRMF curves is dependent on the orientation of the end frames, precluding the ability to define solutions with different end frames on a fixed curve (as with the MTF).

the $[0, 1]$ parameter interval, in that example. However, using a degree 7 PH rotation-minimizing ERF curve that has the same end points and tangents, and total arc length, a proper MTF with a nearly constant tangential angular velocity component $\omega_1 = \theta'/\sigma$ is easily achieved using a single rational quartic rotation. Figure 8 compares the angular velocity behavior in these two cases.

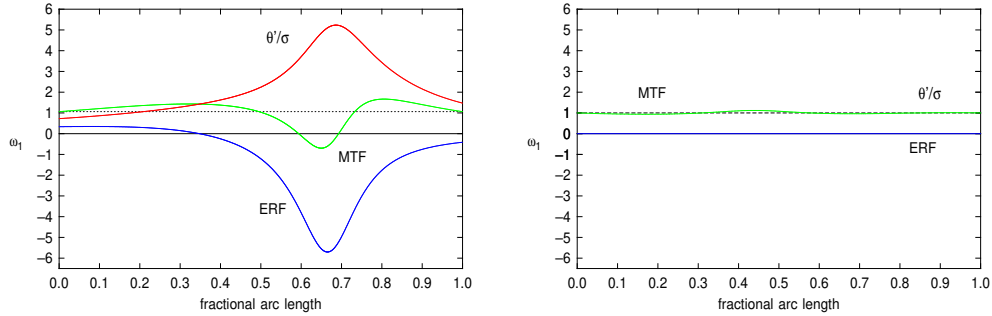


Figure 8: Left: angular velocity plots for the quintic PH curve in Example 3 of [19]: a single rational quartic rotation does not achieve a proper MTF over the entire curve (the angular velocity changes sign), and subdivision must be invoked. Right: a degree 7 PH curve with a rotation-minimizing ERF easily achieves a proper MTF with near-constant angular velocity.

7 Closure

The specification of normal–plane orientation along a spatial curve, subject to prescribed boundary conditions, is a fundamental problem of interest in a variety of applications, including path planning for robot end effectors, swept surface constructions, and elucidating the geometry of DNA molecules.

Among all possible adapted orthonormal frames, those of “minimal twist” (i.e., that exhibit the least total rotation of the normal–plane vectors about the tangent, consistent with the boundary conditions) are of special interest. The focus of this paper has been to formulate exact (i.e., rational) solutions to the problem of constructing such a *minimal twist frame* (MTF), based on degree 7 spatial Pythagorean–hodograph (PH) curves that admit a rotation–minimizing Euler–Rodrigues frame (ERF). The system of equations defining such curves with prescribed end points, end tangents, and total arc length was developed, and it was shown that solutions of this system can be obtained to machine precision by iterative numerical methods. The equations incorporate two free parameters, that can be exploited for shape optimization.

The rotation–minimizing nature of the ERF renders the construction of a rational MTF much easier than with an earlier approach [19] based upon spatial PH quintics, in which the rational rotation generating the MTF from the ERF must cancel out the tangent component of the ERF angular velocity. Consequently, it is possible to more closely approximate the ideal of an MTF with constant tangential angular velocity (which is not exactly achievable by a rational MTF), without appealing to subdivision of the curve.

The construction of degree 7 PH curves with rational rotation–minimizing frames, based upon determining the solutions to a system of four quadratic equations in four real variables, with coefficients dependent on several input parameters, is the most challenging aspect of the present approach. Further investigations may help to improve the efficiency of the numerical solution procedure, and address open questions concerning the existence of solutions for all possible sets of input data, and the numbers of distinct solutions.

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