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THE GENERALIZED-LEGENDRE ADDITION THEOREMS, SU(1,1), AND THE DIAGONALIZATION OF CONVOLUTION EQUATIONS †

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ABSTRACT

Several addition theorems involving the generalized Legendre functions of the first and second kind -- $P^j_{\mu\nu}(z)$ and $Q^j_{\mu\nu}(z)$ -- are (1) stated with convergence conditions; (2) interpreted in terms of the UIR's of $SU(1,1) \sim SO(2,1)$ in the discrete, continuous, and mixed bases; (3) proved both directly and indirectly by continuation from the SU(2) addition theorem. The relevant group theory is supplied in a set of appendices, along with detailed properties of the generalized Legendre functions. The problem of diagonalizing SU(1,1) convolution equations in the discrete and continuous bases is briefly considered; it is shown how the diagonalization of Abarbanel and Saunders arises as a special case of a more general result. Pertinent SU(1,1) expansion theorems are derived.

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I. Introduction

It is the aim of this paper to state, interpret, derive, and briefly apply the addition theorems associated with the "generalized Legendre functions" introduced by Azimov.[1]

These functions have appeared in the physics literature of partial wave analysis over the past 15 years in many guises, and their role as "harmonics" of $SU(1,1) \sim SO(2,1)$ is well understood, though not much has been said about the group theoretic status of the second-kind functions. The reason seems to be that, although the discrete-basis treatment of SU(1,1) was rather thoroughly handled by Bargmann[2] in 1947, the continuous-basis analysis was not effectively begun until 1967,[3] and it is with this continuous basis that the second-kind functions are associated.

In a non-group-theoretic context, the generalized Legendre functions of the first and second kind were defined and characterized by Azimov in 1966, two years after the work of Andrews and Gunson[4] of which Azimov was apparently unaware. Azimov's equations, allowing for arbitrary complex values of the helicity labels μ and ν, are more general than those of AG. Since it is through analytic continuation in these labels that the discrete and continuous bases of SU(1,1) are related, and since Azimov has provided such a complete set of formulas, we have adopted Azimov's notation for most of this paper: $P^j_{\mu\nu}(z)$ and $Q^j_{\mu\nu}(z)$.[5]

Most physicists are familiar with the first-kind addition theorem as it relates to the rotation group, e.g., spherical harmonics in electrostatics. The other addition theorems involving both first- and second-kind functions, or only second-kind functions, are much less well-known. What we have called the hybrid addition theorem was derived by Gunson[16] and later lectured upon by Hermann,[15] but the second-kind addition theorem seems to make an exclusive appearance in Azimov's paper. This formula reads [see (2.7)]

$$e^{-\mu\xi}\,\mathcal{Q}^j_{\mu\mu'}(z)\,e^{-\mu'\xi'} = \frac{1}{i\pi}\int_{-i\infty}^{i\infty} d\lambda\; e^{-\lambda\alpha}\,\mathcal{Q}^j_{\mu\lambda}(z_1)\,\mathcal{P}^j_{\lambda\mu'}(z_2). \tag{1.1}$$

Closely related to the notion of an addition theorem is a technique for simplifying an integral equation, known as diagonalization, in which some or all of the integrations are replaced with the sum appearing in an addition theorem. Often, the projected functions which appear in the diagonalized equation have some special significance which causes the diagonalized equation to be simpler and more comprehensible than the original equation. We can marvel at the simplicity of the elastic unitarity relation for spinless-particle scattering amplitudes expressed in partial waves,

$$\mathrm{Im}\,T_j = \tfrac{1}{2}|T_j|^2\,, \tag{1.2}$$

the standard example of a useful SU(2) diagonalization in particle physics. In Section VI of this paper we show, as an application of the addition theorems, how one might diagonalize certain SU(1,1) convolution equations in a similar manner.

The desire to clarify these diagonalizations has been our primary source of motivation for investigating the addition theorems in the first place. We have been interested in diagonalizing the various integral equations which arise in connection with the multiperipheral model for elementary-particle scattering amplitudes. The multiperipheral "bootstrap" idea is not new,[6] but has been recently infused with new life in the framework of the topological expansion of the S-matrix.[7] In particular, we hope that the second-kind diagonalization discussed in Section VI.2 can shed some light on the meaning of such topological entities as the twisted and untwisted reggeon propagators, or loops, which appear as kernels in the cylinder and planar bootstrap equations.[8]

Much of the material in this paper is standard SU(1,1) lore; we suggest that the value of the paper, if any, lies more in the interconnection of known facts that in the facts themselves. Nevertheless, to be reasonably self-contained, we have reproduced much of this SU(1,1) lore in the Appendices, where several topics are treated in somewhat non-standard fashion. Much use is made, for example, of the SU(1,1) Lie generators realized as regular-representation shift operators, and of the resultant Casimiric differential equations (Appendices C, D, F).

In Appendix D we give a quasi-derivation of the SU(1,1) matrix elements based on the Casimiric differential equations, but ultimately we rely on calculations in the literature. We suspect that the fact that the continuous-basis matrix elements are simply second-kind Legendre functions $Q^j_{\mu\mu'} \sim e^j_{\mu\mu'}$ has not been widely appreciated. In this vein we have slightly generalized the comments of Hermann[15] concerning the interpretation of the integral representations of the Legendre functions (Section V.5).

In Appendix G we derive from scratch the Peter-Weyl expansion theorem for SU(1,1), since this result is often quoted without proof in the literature. Our method of derivation, we feel, makes particularly clear the disposition of the "modified" expansion theorem for non-square-integrable functions, which is actually much simpler than the unmodified expansion theorem.

The complication inherent in the Peter-Weyl theorem in the discrete basis is exacerbated in the continuous basis by the appearance of the principal-series multiplicity index. Rather than interpret this extra index, we think we have made it go away in our S_o^+ semigroup expansion theorem (G.17), designed for use in conjunction with the second-kind addition theorem (Section VI.2). The projection part of this specialized expansion theorem, (G.17b), is reminiscent of the Froissart-Gribov projection of Regge theory, a fact we think will have a bearing on the definition of planar reggeon loops, as noted earlier.

Finally we comment on the derivations of the addition theorems. In Section III these theorems are in a sense derived, for special j values, because it is shown how the addition theorems reflect the Hilbert space completeness relations for the SU(1,1) UIR's in various bases. Somehow, we feel that this type of proof

lacks the punch of a direct non-group-theoretic derivation, a situation we have tried to remedy in Section IV, where we show how all the addition theorems follow from contortions of the SU(2) addition theorem which everybody believes. Unfortunately, these contortions may be found so discomforting that the reader is still not sure whether the addition theorems have in fact been proved. For this fastidious reader we provide Section V which contains our "best" and most interesting proof of the addition theorems. A byproduct of this proof is an understanding of the integration domain in the Legendre-function integral representations (Eqs. (5.16) and (5.17)).

The contents of this paper have been mostly described already. In Section II we state the addition theorems and related formulas with a minimum of comment. This section is independent of the rest of the paper, except that the Legendre functions appearing in the formulas are defined in Appendix H. This lengthy appendix contains the properties of the Legendre functions to which we constantly refer.

Throughout the paper we use the following terminology:

(1) representation: an explicit form of a Lie group.

(2) UIR: unitary irreducible representation.

(3) realization: an explicit form of a Lie algebra.

(4) differential generator: a realization of a Lie generator as a differential operator. We distinguish differential generators $\vec{G}_i$ from the generator matrices or abstract generators G_i by an over-arrow. (For 3-vectors we use the undertwiddle, $\underset{\sim}{x}$.)

(5) half-integer: m = "half-integer" if 2m = odd integer.

(6) integrality: that which distinguishes integers from half-integers ($\varepsilon = 0$ or $\frac{1}{2}$).

(7) Legendre function, Legendre equation: what Azimov calls a generalized Legendre function, and the generalized Legendre equation (see App. H).

II. Summary of the Addition and Multiplication Formulas

In this section we simply state the various addition theorems, their corresponding multiplication formulas, and certain special cases of both. The formulas are derived and interpreted in later sections of this paper. Nevertheless, in Section 7 we have tried to give at least one reference for each of the major formulas. Often, the conditions of validity stated in the literature are less general than those given here.

The variables z_1, z_2, z which appear in the following equations are always taken to lie on the principal sheet of the Legendre functions in which they appear. The cuts of the Legendre functions are shown in Figure 6. These cuts and the definition of the square roots $\sqrt{z^2 - 1} = \sqrt{z+1} \cdot \sqrt{z-1}$ are discussed in Appendix H.5.

More general versions of formulas (2.1), (2.5), and (2.14), with complex helicity labels, are given by Azimov.[1]

1. First-Kind Addition Theorem:

$$e^{-im\phi} P^j_{mm'}(z)\, e^{-im\phi'} = \sum_{n=-\infty}^{\infty} e^{-in\omega} P^j_{mn}(z_1)\, P^j_{nm'}(z_2). \tag{2.1}$$

In this formula, z_1, z_2, ω are independent complex variables in terms of which z, ϕ, ϕ' are given by

$$z = z_1 z_2 + \sqrt{z_1^2 - 1} \cdot \sqrt{z_2^2 - 1} \cos\omega \tag{2.2}$$

$$e^{-i\phi} = \frac{z_2\sqrt{z_1^2-1} + z_1\sqrt{z_2^2-1}\cos\omega - i\sqrt{z_2^2-1}\sin\omega}{\sqrt{z^2-1}}, \tag{2.3}$$

with $e^{-i\phi'}$ given by (2.3) with $z_1 \leftrightarrow z_2$. The label j is an arbitrary complex number, but the labels m, m' are either both integers (in which case the summation index n runs over the integers) or both half-integers (n runs over the half integers). In other words, m, m', n must have the same integrality.

The sum in (2.1) converges if z_1, z_2, ω respect the following condition:

$$\left|\frac{z_1+1}{z_1-1}\right| \cdot \left|\frac{z_2+1}{z_2-1}\right| > \exp\left(2|\mathrm{Im}(\omega)|\right). \tag{2.4}$$

If ω is real, (2.4) is satisfied by $\mathrm{Re}(z_1) > 0$, $\mathrm{Re}(z_2) > 0$ (but see Section IV.1 below). If ω is real and $z_i = \cos\theta_i$ with $|\theta_i| < \pi$, then (2.4) $\Rightarrow |\theta_1| + |\theta_2| < \pi$.

2. <u>Hybrid Addition Theorem:</u>

$$\begin{aligned} e^{-im\phi} Q^j_{mm'}(z)\, e^{-im'\phi'} &= \sum_{n=-\infty}^{\infty} e^{-in\omega} P^j_{mn}(z_1)\, Q^j_{nm'}(z_2) \\ &= \sum_{n=-\infty}^{\infty} e^{-in\omega} Q^j_{mn}(z_2)\, P^j_{nm'}(z_1). \end{aligned} \tag{2.5}$$

All the comments of Section 1 apply to (2.5) except those regarding convergence.

The convergence condition for (2.5) is

$$\left|\frac{z_1+1}{z_1-1}\right| \cdot \left|\frac{z_2-1}{z_2+1}\right|^{\sigma_2} > \exp(2|\mathrm{Im}(\omega)|), \tag{2.6}$$

where $\sigma_2 = \pm$ as $\mathrm{Re}(z_2) \gtrless 0$. If z_1, z_2, ω are real, (2.6) is satisfied by $z_2 > z_1 > 1$.

3. <u>Second-Kind Addition Theorem:</u>

$$e^{-\mu\xi} \not{Q}^j_{\mu\mu'}(z)\, e^{-\mu'\xi'} = \frac{1}{i\pi}\int_C d\lambda\, e^{-\lambda\alpha} \not{Q}^j_{\mu\lambda}(z_1)\, \not{Q}^j_{\lambda\mu'}(z_2). \tag{2.7}$$

In this formula (also valid with unslashed Q functions) z_1, z_2, α are independent complex variables in terms of which z, ξ, ξ' are given by

$$z = z_1 z_2 + \sqrt{z_1^2-1}\cdot\sqrt{z_2^2-1}\,\mathrm{ch}(\alpha) \tag{2.8}$$

$$e^{-\xi} = \left[z_2\sqrt{z_1^2-1} + z_1\sqrt{z_2^2-1}\,\mathrm{ch}\,\alpha - \sqrt{z_2^2-1}\,\mathrm{sh}\,\alpha\right] / \sqrt{z^2-1}, \tag{2.9}$$

with $e^{-\xi'}$ given by (2.9) with $z_1 \leftrightarrow z_2$. The labels j, μ, μ' are arbitrary complex numbers and the contour C is any contour running from $-i\infty$ to $+i\infty$ which separates the pole chains of the function $\Gamma(j+1+\lambda)\Gamma(j+1-\lambda)$, see Fig. 3. If $\mathrm{Re}(j) > -1$, C may be taken along the imaginary axis with no deformations.

The integration in (2.7) converges if z_1, z_2, α satisfy the condition

$$\frac{1}{2}\left|\arg\left(\frac{z_1+1}{z_1-1}\right)\right| + \frac{1}{2}\left|\arg\left(\frac{z_2+1}{z_2-1}\right)\right| + |\mathrm{Im}(\alpha)| < \pi. \tag{2.10}$$

For $z_1, z_2 > 1$, (2.10) requires only that $|\mathrm{Im}(\alpha)| < \pi$. For α

real, (2.10) is satisfied for all complex z_1, z_2, unless both these variables lie in the range $(-1,1)$.

4. Alternative Second-Kind Addition Theorem:

$$e^{-\mu\xi} \mathcal{Q}^{j}_{\mu\mu'}(z)\, e^{-\mu'\xi'} = -2\, \frac{\Gamma(j+1+\mu)}{\Gamma(j+1-\mu)} \sum_{m=j+1}^{\infty} (-1)^{m-j}\, e^{-m\alpha} \times \frac{\mathcal{Q}^{j}_{m\mu}(z_1)\mathcal{Q}^{j}_{m\mu'}(z_2)}{\Gamma(m-j)\Gamma(j+1+m)} . \tag{2.11}$$

The variables z, ξ, ξ' are given in terms of z_1, z_2, α exactly as in (2.8) and (2.9) above, and j, μ, μ' are again arbitrary complex numbers.

The convergence condition for (2.11) is

$$\left|\frac{z_1 - 1}{z_1 + 1}\right|^{\sigma_1} \cdot \left|\frac{z_2 - 1}{z_2 + 1}\right|^{\sigma_2} > \exp(-2\,\mathrm{Re}(\alpha)), \tag{2.12}$$

where $\sigma_1 = \pm$ as $\mathrm{Re}(z_1) \gtrless 0$ and $\sigma_2 = \pm$ as $\mathrm{Re}(z_2) \gtrless 0$. If z_1 and z_2 are imaginary, condition (2.12) is simply $\mathrm{Re}(\alpha) > 0$.

The alternative second-kind addition theorem is the analytic continuation of (2.7) obtained by closing the contour to the right, picking up the residues of $\Gamma(j+1-\lambda)$, and dropping the great circle.

5. Multiplication Formulas

The addition theorems (2.1), (2.5), and (2.7) are the fourier transforms of the following multiplication formulas:

$$P^{j}_{mn}(z_1)\, P^{j}_{nm'}(z_2) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\omega\, e^{+in\omega}\{e^{-im\phi}\, P^{j}_{mm'}(z)\, e^{-im'\phi'}\}, \tag{2.13}$$

$$P^{j}_{mn}(z_1)\, Q^{j}_{nm'}(z_2) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\omega\, e^{+in\omega}\{e^{-im\phi}\, Q^{j}_{mm'}(z)\, e^{-im'\phi'}\}, \tag{2.14}$$

$$\mathcal{Q}^{j}_{\mu\lambda}(z_1)\, \mathcal{Q}^{j}_{\lambda\mu'}(z_2) = \frac{1}{2} \int_{-\infty}^{\infty} d\alpha\, e^{+\lambda\alpha}\{e^{-\mu\xi}\, \mathcal{Q}^{j}_{\mu\mu'}(z)\, e^{-\mu'\xi'}\}. \tag{2.15}$$

These formulas are correct as stated provided that z_1, z_2 satisfy (2.4), (2.6), and (2.10), respectively, with ω and α real. For z_1, z_2 in violation of one of these conditions, the corresponding multiplication formula is still correct provided the integration contour is deformed around the branch point and attached cut which penetrates the nominal integration region. This branch point is the reflection via (2.2) or (2.8) of the $z = 1$ singularity of the Legendre functions into the plane of the integration variable ω or α.

6. Special Cases

When one of the helicity labels of $P^{j}_{\mu\nu}(z)$ or $Q^{j}_{\mu\nu}(z)$ vanishes, the resulting function is a regular associated Legendre function,

$$P^{j}_{\mu o}(z) = P^{\mu}_{j}(z) \qquad P^{j}_{o\mu}(z) = P^{-\mu}_{j}(z)$$

$$Q^{j}_{\mu o}(z) = Q^{\mu}_{j}(z) \qquad Q^{j}_{o\mu}(z) = Q^{-\mu}_{j}(z) .$$

We therefore obtain the following special cases of the addition theorems, with conditions as stated earlier:

$$P_j(z) = \sum_{n=-\infty}^{\infty} e^{-in\omega} P_j^{-n}(z_1)\, P_j^{n}(z_2) \tag{2.16}$$

$$Q_j(z) = \sum_{n=-\infty}^{\infty} e^{-in\omega} P_j^{-n}(z_1)\, Q_j^{n}(z_2) \tag{2.17}$$

$$Q_j(z) = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} d\lambda\, e^{-\alpha\lambda}\, \mathcal{Q}_j^{-\lambda}(z_1)\, \mathcal{Q}_j^{\lambda}(z_2) \tag{2.18}$$

$$Q_j(z) = -2 \sum_{m=j+1}^{\infty} (-1)^{m-j} e^{-m\alpha} \frac{\mathcal{Q}_j^{m}(z_1)\, \mathcal{Q}_j^{m}(z_2)}{\Gamma(m-j)\,\Gamma(j+1+m)} . \tag{2.19}$$

As special cases of the multiplication formulas we have

$$P_j^{-n}(z_1)\, P_j^{n}(z_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega\, e^{+in\omega} P_j(z) \tag{2.20}$$

$$P_j^{-n}(z_1)\, Q_j^{n}(z_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega\, e^{in\omega} Q_j(z) \tag{2.21}$$

$$\mathcal{Q}_j^{-\lambda}(z_1)\, \mathcal{Q}_j^{\lambda}(z_2) = \frac{1}{2} \int_{-\infty}^{\infty} d\alpha\, e^{\lambda\alpha} Q_j(z) \tag{2.22}$$

and specializing further,

$$P_j(z_1)\, P_j(z_2) = \frac{1}{\pi} \int_{0}^{\pi} d\omega\, P_j(z) \tag{2.23}$$

$$P_j(z_1)\, Q_j(z_2) = \frac{1}{\pi} \int_{0}^{\pi} d\omega\, Q_j(z) \tag{2.24}$$

$$Q_j(z_1)\, Q_j(z_2) = \int_{0}^{\infty} d\alpha\, Q_j(z) \tag{2,25}$$

with z still given by (2.2) or (2.8).

Using the following Jacobians (valid for z_1, z_2, z all real)

$$\frac{\partial\omega}{\partial z} = \int_{0}^{\pi} d\omega\, \delta\left(z - z_1 z_2 - \sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1}\cos\omega\right) = \frac{\theta(-k)}{\sqrt{-k}}$$

$$\frac{\partial\alpha}{\partial z} = \int_{0}^{\infty} d\alpha\, \delta\left(z - z_1 z_2 - \sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1}\operatorname{ch}\alpha\right) = \frac{\theta(z - z_+)}{\sqrt{+k}} ,$$

where

$$k = k(z_1, z_2, z) \equiv z_1^2 + z_2^2 + z^2 - 2 z_1 z_2 z - 1$$
$$= (z - z_+)(z - z_-)$$

and

$$z_\pm \equiv z_1 z_2 \pm \sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1} ,$$

equations (2.23) through (2.25) may be re-expressed as

$$P_j(z_1)\, P_j(z_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} dz\, P_j(z)\, \theta(-k)/\sqrt{-k} \tag{2.26}$$

$$P_j(z_1)\, Q_j(z_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} dz\, Q_j(z)\, \theta(-k)/\sqrt{-k} \tag{2.27}$$

$$Q_j(z_1)\,Q_j(z_2) = \frac{1}{2}\int_{-\infty}^{\infty} dz\; Q_j(z)\,\theta(z-z_+)/\sqrt{+k}\;. \tag{2.28}$$

7. References

(2.1)	Vilenkin	9	III.4.1(7); VI.4.1(6)
(2.5)	Gunson	16	(37)
(2.7)	Azimov	1	(42)
(2.13)	Vilenkin	9	III.4.3(1); VI.4.4(1)
(2.14)	Azimov	1	(43)
(2.15)	Azimov	1	(40)
(2.16)	Bateman	10	3.11(1) with $\psi = \omega + \pi$
(2.17)	Bateman	10	3.11(4) with $\psi = \omega + \pi$
(2.18)	CDM	27	(A.27), error × 2
(2.19)	Hobson	11	p. 384, error × 2 and phase
(2.19)	GR	12	8.795.3, error × 2 and phase
(2.19)	MO	13	p. 70, dropped in 3rd Ed.
(2.23)	ARR	14	(A.10)
(2.26)	ARR	14	(B-2.6)
(2.26)	AS	25	p.716 #1
(2.28)	ARR	14	(B.2-19) + (B.2-17)
(2.28)	AS	25	p. 716 #8, error in endpoint

III. Group-Theoretic Interpretation of the Addition Theorems

In general, an addition theorem is a consequence of the completeness of the set of vectors which spans the Hilbert space of a unitary irreducible group representation.

For example, the Hilbert space H^j of the UIR $D^j(g)$ of SU(2) is spanned by the complete set $\{|j,m\rangle\}$, where $2j = 0,1,2\ldots$ and $m = j,j-1\ldots-j$. The completeness relation is therefore

$$1^j = \sum_{m=-j}^{j} |jm\rangle\langle jm|, \tag{3.1}$$

where 1^j is the identity operator in H^j. Since the operators $D^j(g)$ which represent the elements of SU(2) in H^j have, by definition, the group property

$$D^j(g) = D^j(g_1)\,D^j(g_2), \tag{3.2}$$

where $g = g_1 g_2$, it follows from (3.1) that

$$\langle jm|D^j(g)|jm'\rangle = \sum_{n=-j}^{j} \langle jm|D^j(g_1)|jn\rangle\langle jn|D^j(g_2)|jm'\rangle. \tag{3.3}$$

In our parametrization of SU(2) we have $g = (\phi,\,\theta,\,\phi')$ and

$$\langle jm|D^j(g)|jm'\rangle = e^{-im\phi}\, d^j_{mm'}(\cos\theta)\, e^{-im'\phi'},$$

so that (3.3) becomes

$$e^{-im\phi}\, d^j_{mm'}(z)\, e^{-im'\phi'} = \sum_{n=-j}^{j} e^{-in\omega}\, d^j_{mn}(z_1)\, d^j_{nm'}(z_2)\,, \tag{3.4}$$

where $z_i = \cos\theta_i$, and expressions for ϕ, z, ϕ', ω are given in Appendix E.1. When (3.4) is converted to P functions via (H.13), we get a special case of the first-kind addition theorem (2.1).

In this section, we shall "interpret" the various addition theorems in terms of the UIR's of SU(1,1). We refer the reader at this point to Appendices A through E, whose contents are listed in the general Table of Contents at the beginning of this paper.

1. Unitarity of Matrix Elements

Unlike the d-functions, the $P^j_{mm'}$ are not actually unitary when thought of as matrix elements with indices m and m'. This fact, however, is not significant for the addition theorems (and their subsequent use in diagonalizing convolution equations) because the addition formulas are invariant under changes of the normalizing factor. By this we mean that if the unitary matrix elements $D^j_{mm'}(g)$ satisfy the addition theorem

$$D^j_{mm'}(g) = \sum_n D^j_{mn}(g_1)\, D^j_{nm'}(g_2),$$

then so do

$$\tilde{D}^j_{mm'}(g) \equiv \left[N^j_m/N^j_{m'}\right] D^j_{mm'}(g),$$

where N^j_m is an arbitrary function of m and j. According to (H.12) and (H.13), the functions P and d differ by just such a factor,

$$P^j_{mm'}(z) = \left[N^j_m/N^j_{m'}\right] d^j_{mm'}(z),$$

where $N^j_m = \sqrt{G^j_m}\,(\pm i)^m$, $\mathrm{Im}(z) \gtrless 0$.

2. First-Kind Addition Theorem

We have already shown how the first-kind addition theorem (2.1) may be understood in terms of the SU(2) UIR's when $2j = 0,1,2\ldots$ and $|m|,|m'| \leq j$. Alternatively, (2.1) may be construed as the group property of the SU(1,1) UIR matrix elements, taken in the discrete basis discussed in Appendix B. As j takes the special sets of values shown in Table B.1, the summation in the completeness relation (B.6) runs over the values shown in the right column of the Table. For each class of UIR we thereby obtain a special case of the general first-kind addition theorem. The fact that the summation is semi-infinite for the $D_k^\pm$ (and finite for SU(2)) is a consequence of the zeros of the P functions (see Appendix H.15 and Fig. 8 (g)).

The first-kind addition theorem for arbitrary complex j may be regarded as the analytic continuation of the C_q^0 and $C_q^{\frac{1}{2}}$ group properties away from $\mathrm{Re}(j) = -\frac{1}{2}$. In Section IV we will show that (2.1) is in fact the unique analytic continuation of the SU(2) addition theorem.

3. Second-Kind Addition Theorem

Given the completeness relation (B.12) for the Hilbert space associated with the D_k^+ UIR's in the continuous basis, and given the explicit continuous-basis D_k^+ matrix elements (D.8), we see at once that, when $2j = -1,0,1,\ldots,$ the second-kind addition theorem (2.7) is the D_k^+ group property in the continuous basis. The general result (2.7) is the unique analytic continuation of this group property in j away from the integers, and in μ, μ' away from the imaginary axes.

4. Alternative Second-Kind Addition Theorem

Equation (2.11) has -- when $2j$ = integer and z_1, z_2 are purely imaginary -- an interpretation similar to that discussed above. Using the discrete-basis completeness relation (B.6), but the mixed-basis D_k^+ matrix elements described in Appendix B.4 and given explicitly in (D.9), it is easy to show that the continuous-basis matrix elements of $U(g) = U(g_1)\,U(g_2)$ are

$$e^{-\mu\xi}\, \mathscr{E}^j_{\mu\mu'}\,(\mathrm{ch}\,\nu)\, e^{-\mu'\xi'} = 2\sum_{m=j+1}^{\infty} e^{i\pi m}\, e^{-im\omega}\, \frac{\mathscr{Q}^j_{\mu,-m}(i\,\mathrm{sh}\,\eta_1)\mathscr{Q}^j_{m\mu'}(i\,\mathrm{sh}\,\eta_2)}{\Gamma(m-j)\,\Gamma(m+j+1)}. \tag{3.5}$$

Equation (3.5) is a special case of (2.11) with $z_1 = -i\,\mathrm{sh}\,\eta_1$, $z_2 = i\,\mathrm{sh}\,\eta_2$, and $z = \mathrm{ch}\,\nu$. Alternatively, (2.11) is the analytic continuation of the mixed-basis addition theorem (3.5).

5. Hybrid Addition Theorem

For this theorem we give a different kind of group-theoretic interpretation taken from Hermann[15]. To conform with the notation of Hermann, and Gunson[16], we write (2.5) as

$$E^j_{mm'}(g) = \sum_{n=-\infty}^{\infty} D^j_{mn}(g_1)\, E^j_{nm'}(g_2)\,, \tag{3.6}$$

where

$$D^j_{mm'}(g) \equiv e^{-im\phi}\, d^j_{mm'}(z)\, e^{-im'\phi'}\,,$$

$$E^j_{mm'}(g) \equiv e^{-im\phi}\, e^j_{mm'}(z)\, e^{-im'\phi'}$$

Hermann's point of view is that, once one knows that (3.6) is true, one can write the E functions as matrix elements of an operator $E^j(g_c)$ which is related to the operator $D^j(g)$ of SU(2) by a certain Cauchy kernel transform. If we let $G = SU(2)$ and and $G_c = SL(2,C)$, then $E^j(g_c)$ is defined by

$$E^j(g_c) = \int_G dg\; C(g^{-1}g_c)\, D^j(g). \tag{3.7}$$

Here, $g_c \in G_c - G$, which includes SU(1,1), $2j = 0,1,2\ldots$, and D^j, E^j are operators in the Hilbert space H^j associated with the SU(2) UIR labelled by j. These operators possess the group multiplication property,

$$\begin{aligned} E^j(g_o g_c) &= \int_G dg\; C(g^{-1}[g_o g_c])\, D^j(g) \\ &= \int_G dg\; C([g_o^{-1}g]^{-1} g_c)\, D^j(g) \\ &= \int_G dg\; C(g^{-1}g_c)\, D^j(g_o g) \\ &= \int_G dg\; C(g^{-1}g_c)\, D^j(g_o)\, D^j(g) \\ &= D^j(g_o)\, E^j(g_c), \end{aligned} \tag{3.8}$$

where we have used the invariance of dg and the group property of the D^j. Taking matrix elements of (3.8) in H^j, we generate the hybrid addition theorem (3.6), which may then be continued to complex j.

From the completeness property of the SU(2) UIR's given in (G.19),

$$\frac{1}{2}\sum_{j=0}^{\infty}(2j+1)\,\mathrm{trace}\left[D^j(g_1^{-1})\,D^j(g_2)\right] = \delta(g_1-g_2),$$

we can solve (3.7) for the Cauchy kernel

$$\begin{aligned} C(g^{-1}g_c) &= \frac{1}{2}\sum_{j=0,1}^{\infty}(2j+1)\,\mathrm{trace}\left[D^j(g^{-1})\,E^j(g_c)\right] \\ &= \frac{1}{2}\sum_{j=0}^{\infty}(2j+1)\,\mathrm{trace}\,E^j(g^{-1}g_c), \end{aligned} \tag{3.9}$$

where trace means trace in H^j. An explicit expression for the Cauchy kernel is quoted in Gunson[16].

In passing, we point out that (3.9) and the matrix elements of (3.7),

$$E^j_{mm'}(g_c) = \int_G dg\; C(g^{-1}g_c)\,D^j_{mm'}(g)\,,$$

are the natural generalizations of the Heine and Neumann formulas,

$$\frac{1}{z_c - z} = \sum_{j=0}^{\infty}(2j+1)\,P_j(z)\,Q_j(z_c)$$

$$Q_j(z_c) = \frac{1}{2}\int_{-1}^{1} dz\cdot\frac{1}{z_c - z}\cdot P_j(z)\;.$$

IV. Derivation of the Addition Theorems from SU(2)

In this section -- without using any group theory -- we systematically derive from the SU(2) addition theorem all the addition theorems stated in Section II. The domains of convergence are emphasized. In Section V we shall present a direct and simultaneous proof of all the addition theorems using elementary group theoretic techniques.

In what follows, the variables z_1, z_2, and ω or α are treated as independent variables, while (ϕ, z, ϕ') or (ξ, z, ξ') are dependent and given by the set of equations $g = g_1 g_2$ in SL(2,C). We have relegated these details to Appendix E.

1. First-Kind Addition Theorem

As our starting point we take the SU(2) addition formula (3.4), which we assume is correct:

$$e^{-im\phi}\,d^j_{mm'}(z)\,e^{-im'\phi'} = \sum_{n=-j}^{j} e^{-in\omega}\,d^j_{mn}(z_1)\,d^j_{nm'}(z_2). \tag{4.1}$$

In (4.1), $z_i = \cos\theta_i$, $j = 0, \frac{1}{2}, 1 \ldots$, and (m,m') denotes a lattice point in region 5 of the helicity lattice diagram shown in Fig. 8(b). Figure 1 shows the specific helicity lattice for the second d-function in the summand of (4.1), and the line segment AA' represents the sum.

As a preliminary to the continuation of (4.1) in j, we replace the finite sum with an infinite sum to get

$$e^{-im\phi}\, d^{j}_{mm'}(z)\, e^{-im'\phi'} = \sum_{n=-\infty}^{\infty} e^{-in\omega}\, d^{j}_{mn}(z_1)\, d^{j}_{nm'}(z_2), \tag{4.2}$$

where n retains the integrality of j. When $j = 0,\frac{1}{2},1\ldots,$ Eqs. (4.1) and (4.2) are identical because we have extended the summation from segment AA' to segment BB' by adding segments AB and $A'B'$, both of which lie entirely within the "sense-nonsense" portion of the helicity lattice where $d^{j}_{nm'}(z_2)$ has square-root zeroes, as does $d^{j}_{mn}(z_1)$. For details on these zeros, see Appendix H.15.

We now consider the possibility of continuing Eq. (4.2) to complex j. From Table H.14 we observe that, when $-1 < z \leqslant 1$, $d^{j}_{mm'}(z)$ is Carlson in j. In fact, both sides of (4.2) are Carlson in j as long as the sum converges. Since (4.2) is true for $j = 0,1,2\ldots$ it follows from Carlson's Theorem that the equation is true for general complex j, with the unique Carlson continuation in j of $d^{j}_{mm'}$, provided by the hypergeometric function in (H.2) with (H.13).

It should be clear that, as $2j$ moves away from integral values, the portions of the sum in (4.2) represented in Fig. 1 by segments AB and $A'B'$ become "activated", and the question of convergence arises. If convergence is required, the analytic continuation of (4.2) in z_1, z_2, ω is to some extent restricted. This follows from the asymptotic behavior of the summand which is, according to (H.13), (H.32), (H.48) and (H.22),

$$\left|e^{-in\omega}\, d^{j}_{mn}(z_1)\, d^{j}_{nm'}(z_2)\right| \sim e^{|n|\cdot|\mathrm{Im}\,\omega|}\cdot |n|^{m+m'-1}\cdot e^{-\frac{1}{2}|n|\,\ln\left|\frac{z_1+1}{z_1-1}\right|}\cdot e^{-\frac{1}{2}|n|\,\ln\left|\frac{z_2+1}{z_2-1}\right|}$$

as $n \to \pm\infty$. The convergence condition is therefore

$$\left|\frac{z_1+1}{z_1-1}\right| \cdot \left|\frac{z_2+1}{z_2-1}\right| > \exp\left(2|\mathrm{Im}(\omega)|\right), \tag{4.3}$$

ignoring the possibility of power convergence. For ω real,

$$\left|\frac{z_1+1}{z_1-1}\right| \cdot \left|\frac{z_2+1}{z_2-1}\right| > 1. \tag{4.4}$$

Equation (4.4) is certainly satisfied by $\mathrm{Re}(z_1) > 0$, $\mathrm{Re}(z_2) > 0$ (as quoted in Bateman), but more generally, as a simple geometrical argument shows, (4.4) is satisfied and (4.2) converges if $\mathrm{Re}(z_2) > 0$ and z_1 lies anywhere outside a disc containing $z_1 = -1$ and lying entirely within the left-half z_1-plane, as illustrated in Fig. 2. Thus, a portion of the interval $(-1,1)$ near $z_1 = -1$ is necessarily excluded, a fact which reappears if we take $z_i = \cos\theta_i$, $|\theta_i| < \pi$, in which case (4.4) requires that $|\theta_1|+|\theta_2| < \pi$. (In order to show that the right-hand side of (4.2) is Carlson, we assume that θ_1 and θ_2 respect this condition prior to continuation.)

Converting (4.2) to P-functions via (H.13), we obtain the first-kind addition theorem given in (2.1),

$$e^{-im\phi}\, P^{j}_{mm'}(z)\, e^{-im'\phi'} = \sum_{n=-\infty}^{\infty} e^{-in\omega}\, P^{j}_{mn}(z_1)\, P^{j}_{nm'}(z_2), \tag{4.5}$$

with validity as described in Section II.1.

2. Hybrid Addition Theorem

Consider Eq. (4.5) above with $z_1, z_2 > 1$. We continue (4.5) onto the left hand cut in z_2 by taking $z_2 \to z_2 e^{\mp i\pi} = -z_2 \mp i\varepsilon$. Therefore, $\sqrt{z_2^2 - 1} \to e^{\mp i\pi}\sqrt{z_2^2 - 1}$. From (E.15), this implies that $z \to e^{\mp i\pi} z$ so $\sqrt{z^2 - 1} \to e^{\mp i\pi}\sqrt{z^2 - 1}$. Then (E.19) and its ϕ' counterpart tell us that $\phi \to \phi$ but $\phi' \to -\phi'$. The result of these changes is:

$$e^{-im\phi}\, P^j_{mm'}(-z \mp i\varepsilon)\, e^{+im'\phi'} = \sum_{n=-\infty}^{\infty} e^{-in\omega}\, P^j_{mn}(z_1)\, P^j_{nm'}(-z_2 \mp i\varepsilon) . \tag{4.6}$$

If in (4.6) we take $m' \to -m'$, multiply by $G^j_{-m'}$, and subtract the resultant equation from $e^{\mp i\pi j}$ times the original Eq. (4.5), we conclude with the help of identity (H.29) that

$$e^{-im\phi}\, Q^j_{mm'}(z)\, e^{-im'\phi'} = \sum_{n=-\infty}^{\infty} e^{-in\omega}\, P^j_{mn}(z_1)\, Q^j_{nm'}(z_2), \tag{4.7}$$

which is the hybrid addition theorem (2.5).

From the asymptotic behavior of the summand as $n \to \pm\infty$ given by (H.47) and (H.48),

$$\left| e^{-in\omega}\, P^j_{mn}(z_1)\, Q^j_{nm'}(z_2)\right| \sim e^{|n|\cdot|\mathrm{Im}\,\omega|}\cdot|n|^{m+|m'|-1}\cdot e^{-\frac{1}{2}|n|\cdot \ell n\left|\frac{z_1+1}{z_1-1}\right|}\; e^{+\frac{1}{2}|n|\cdot\left|\ell n\left|\frac{z_2+1}{z_2-1}\right|\right|} ,$$

we find that the convergence condition for (4.7) is

$$\frac{1}{2}\,\ell n\left|\frac{z_1+1}{z_1-1}\right| - \frac{1}{2}\left|\ell n\left|\frac{z_2+1}{z_2-1}\right|\right| > |\mathrm{Im}(\omega)| , \tag{4.8}$$

again ignoring the possibility of power convergence. Condition (4.8) is the same as that reported in (2.6). Again, if z_1, z_2, ω are real, (4.8) is satisfied by $z_2 > z_1 > 1$. More generally, a domain similar to that in Fig. 2 may be obtained.

The second form of the hybrid addition theorem shown in (2.5) follows trivially from (H.7), (H.23) and (H.32).

3. Second-Kind Addition Theorems

We apologize from the start for the apparent circuitousness of the present section, but remind the reader that a direct proof of the second-kind addition theorem may be found in the next section. There seems to be a certain amount of "analytic distance" between the addition theorems of the first and second kind.

In terms of the $\not{Q}$ functions, the hybrid addition theorem (4.7) is

$$e^{-im\phi}\, \not{Q}^j_{mm'}(z)\, e^{-im'\phi'} = \sum_{n=-\infty}^{\infty} (-1)^{m-n}\, e^{-in\omega}\, P^j_{mn}(z_1)\, \not{Q}^j_{nm'}(z_2) , \tag{4.9}$$

where for now we consider j, m, m', n to be integers. Starting with $z_1, z_2 > 1$, we continue (4.9) onto the left-hand cut in z_1 in a manner similar to that in which z_2 was treated in Section 2 above. This time $\phi \to -\phi$ and $\phi' \to +\phi'$ so we get

$$e^{+im\phi}\, \mathcal{Q}^j_{mm'}(-z\mp i\varepsilon)\, e^{-im'\phi'} = \sum_{n=-\infty}^{\infty} (-1)^{m-n}\, e^{-in\omega}\, P^j_{mn}(-z_1\mp i\varepsilon)\, \mathcal{Q}^j_{nm'}(z_2) \,. \tag{4.10}$$

If, in (4.10), we take $m \to -m$, multiply by $G^j_m(-1)^{j+1}$ and add the resultant equation to (4.9), we find, making use of identity (H.28) on the left and (H.29) on the right,

$$e^{-im\phi}\, \mathcal{Q}^j_{mm'}(z)\, e^{-im'\phi'} = -G^j_m \sum_{n=-\infty}^{\infty} (-1)^{j-n}\, e^{-in\omega}\, \frac{\mathcal{Q}^j_{nm}(z_1)\, \mathcal{Q}^j_{nm'}(z_2)}{\Gamma(n-j)\,\Gamma(j+n+1)} \,. \tag{4.11}$$

An examination of the gamma functions in (4.11) shows that the sum is really two distinct sums, one running from $-\infty$ to $-j-1$, and the other running from $j+1$ to $+\infty$; moreover, these two sums are the same, so the right side of (4.11) becomes

$$-2\, G^j_m \sum_{n=j+1}^{\infty} (-1)^{j-n}\, e^{-in\omega}\, \frac{\mathcal{Q}^j_{nm}(z_1)\, \mathcal{Q}^j_{nm'}(z_2)}{\Gamma(n-j)\,\Gamma(j+1+n)} \,. \tag{4.12}$$

Next, the sum in (4.12) may be Sommerfeld-Watson transformed to yield

$$\frac{1}{i\pi} \int_C dn\, e^{-in\omega}\, \mathcal{Q}^j_{mn}(z_1)\, \mathcal{Q}^j_{nm'}(z_2) \,,$$

where C is a clockwise contour containing $n = j+1,\ j+2,\ \ldots$. We now give ω a sufficiently large, negative imaginary part so that

ontour may be opened up. The new contour runs upward just to eft of $\mathrm{Re}(n) = j + 1$, but may be harmlessly shifted to $\mathrm{Re}(n) = 0$. ing variables $\xi = i\phi$, $\xi' = i\phi'$, $\alpha = i\omega$, $\lambda = n$ we find:

$$\xi\, \mathcal{Q}^j_{mm'}(z)\, e^{-m'\xi'} = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} d\lambda\, e^{-\lambda\alpha}\, \mathcal{Q}^j_{m\lambda}(z_1)\, \mathcal{Q}^j_{\lambda m'}(z_2). \tag{4.13}$$

So far, j,m,m' are still integers, but from Appendix H.14 an show that both sides of (4.13) are Carlson in j and m' hen, from (H.23), also in m. Taking $m \to \mu$ and $m' \to \mu'$ we n

$$\xi\, \mathcal{Q}^j_{\mu\mu'}(z)\, e^{-\mu'\xi'} = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} d\lambda\, e^{-\lambda\alpha}\, \mathcal{Q}^j_{\mu\lambda}(z_1)\, \mathcal{Q}^j_{\lambda\mu'}(z_2) \,, \tag{4.14}$$

now j,μ,μ' are all complex. Equation (4.14) is the second-addition theorem (2.7).

The convergence condition for (4.14) may be obtained from ehavior of the integrand as $\lambda \to \pm i\infty$. From (H.47) and (H.23) nd:

$$\mathcal{Q}^j_{\mu\lambda}(z_1)\, \mathcal{Q}^j_{\lambda\mu'}(z_2)| \sim e^{|\lambda|\cdot|\mathrm{Im}\,\alpha|} \cdot |\lambda|^{|\mathrm{Re}\,\mu| + |\mathrm{Re}\,\mu'| - 1} \cdot e^{-\pi|\lambda|}$$

$$\cdot\, e^{\frac{1}{2}|\lambda|\cdot\left|\arg\left(\frac{z_1+1}{z_1-1}\right)\right|} \cdot e^{\frac{1}{2}|\lambda|\cdot\left|\arg\left(\frac{z_2+1}{z_2-1}\right)\right|} \,.$$

Therefore the integration in (4.14) converges if

$$\frac{1}{2}\left|\arg\left(\frac{z_1+1}{z_2-1}\right)\right| + \frac{1}{2}\left|\arg\left(\frac{z_2+1}{z_2-1}\right)\right| + |\mathrm{Im}(\alpha)| < \pi . \qquad (4.15)$$

Since $\left|\arg\left(\frac{z_1+1}{z_1-1}\right)\right| = \pi$ only when $z_1 \in (-1,1)$, we see that when α is real, (4.14) converges for all z_1, z_2 except when both these variables lie in the range $(-1,1)$.

The contour in (4.14) plays the same role as the contour in the easily proven identity [GR 6.422 (3)]

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\lambda\, \Gamma(j+1+\lambda)\,\Gamma(j+1-\lambda) = 2^{-2j-2}\,\Gamma(2j+2) , \qquad (4.16)$$

which happens to be the double asymptotic limit of (4.14) as $z_1, z_2 \to \infty$. If $\mathrm{Re}(j) < -1$, the contour must be deformed as shown in Fig. 3 so as to continue to separate the pole chains of the integrand. As $2j \to$ a negative integer, the contour is pinched, generating the singularity appearing on the right side of (4.16).[17]

The alternative second-kind addition theorem (2.11) is obtained from (4.14) by running the Sommerfeld-Watson process in reverse, i.e., closing the contour to the right. Thus,

$$e^{-\mu\xi}\, \mathcal{Q}^{j}_{\mu\mu'}(z)\, e^{-\mu'\xi'} = -2\, G^{j}_{\mu} \sum_{m=j+1}^{\infty} (-1)^{m-j}\, e^{-m\alpha}\, \frac{\mathcal{Q}^{j}_{m\mu}(z_1)\,\mathcal{Q}^{j}_{m\mu'}(z_2)}{\Gamma(m-j)\,\Gamma(j+1+m)} ,$$

(4.17)

where j, μ, μ' are still complex. From (H.47) we have, as $\mathrm{Re}(m) \to +\infty$,

$$\left|\frac{e^{-m\alpha}\, \mathcal{Q}^{j}_{m\mu}(z_1)\,\mathcal{Q}^{j}_{m\mu'}(z_2)}{\Gamma(m-j)\,\Gamma(j+1+m)}\right| \sim e^{-\mathrm{Re}\, m \cdot \mathrm{Re}\,\alpha} \cdot |\mathrm{Re}\, m|^{|\mathrm{Re}\,\mu| + |\mathrm{Re}\,\mu'| - 1}$$

$$\cdot\, e^{\frac{1}{2}\mathrm{Re}\, m \cdot \left|\ell n\left|\frac{z_1+1}{z_1-1}\right|\right|} \cdot e^{\frac{1}{2}\mathrm{Re}\, m \cdot \left|\ell n\left|\frac{z_2+1}{z_2-1}\right|\right|} .$$

(4.18)

Thus, (4.17) converges when

$$\frac{1}{2}\left|\ell n\left|\frac{z_1+1}{z_1-1}\right|\right| + \frac{1}{2}\left|\ell n\left|\frac{z_2+1}{z_2-1}\right|\right| < \mathrm{Re}(\alpha) , \qquad (4.19)$$

which is the same as condition (2.12).

Interestingly, the mixed-basis addition theorem given in (3.5) just barely converges due to the $|\mathrm{Re}\, m|^{-1}$ shown in (4.18) and the rotating phase of the summand.

V. Group-Theoretic Proof of the Addition Theorems

So far we have "proven" the addition theorems in two different ways: first, the "proof by interpretation" given in Section III, and second, the "proof by continuation" (from the SU(2) addition theorem) given in Section IV. Whereas the first method relies on external calculations of UIR matrix elements, the second method depends on tedious manipulation, in particular, Carlson continuations.

In this section, we give a self-contained and direct proof of the multiplication formula corresponding to the second-kind addition theorem. This proof will automatically be valid for complex j,μ, and μ', and from the proof it will be obvious how to prove any addition theorem. The crucial facts turn out to be: (1) the Legendre functions are annihilated by the invariant Laplace operator of SU(1,1); (2) the integration appearing in the multiplication formula is the invariant integration of the subgroup K with respect to which SU(1,1) is reduced. For the first-kind multiplication formula, $K = SO(2)$, whereas for the second-kind, $K = SO(1,1)$.

The second-kind multiplication formula (2.15) reads

$$\mathscr{d}^{j}_{\mu\lambda}(g_1)\, \mathscr{d}^{j}_{\lambda\mu'}(z_2)\, e^{-\mu'\xi_2'} = \frac{1}{2}\int_{-\infty}^{\infty} d\xi_2\, e^{+\lambda\xi_2}\, \mathscr{d}^{j}_{\mu\mu'}(g) \quad , \tag{5.1}$$

where we have defined

$$\mathscr{d}^{j}_{\mu\mu'}(g) \equiv e^{-\mu\xi}\, \mathscr{d}^{j}_{\mu\mu'}(z)\, e^{-\mu'\xi'} \quad , \tag{5.2}$$

and we recall from Appendix E.3 the ordering of the parameters $g = g_1 g_2 \Longrightarrow (\xi,\nu,\xi') = (\xi_1,\nu_1,\xi_1')(\xi_2,\nu_2,\xi_2')$, and $\alpha = \xi_1' + \xi_2$.

The proof of (5.1), and thus of the second-kind addition theorem, conveniently divides into three parts. First, we show that both sides of (5.1) satisfy the same partial differential equation (the Laplace). Second, we show that both sides in fact solve the same ordinary differential equation (the Legendre). Third, we show that both sides are the same solution of this ordinary differential equation.

1. Part 1 of Proof

The Laplace operator of SU(1,1) is defined as

$$L^{(g)} \equiv \vec{J}^{2(g)} - j(j+1) \,, \tag{5.3}$$

where $\vec{J}^{2(g)}$ is the Casimir expressed in terms of the differential generators given in Appendix C. In particular, $\vec{J}^2$ was calculated for the continuous basis [SO(1,1) reduction] in Eq. (C.11).

If $f^{j}_{\mu\nu}(z)$ is a solution of the Legendre equation (H.1)

$$\mathscr{L}(j;\mu,\nu;z)\; f^{j}_{\mu\nu}(z) = 0 \,,$$

then the function $f^{j}_{\mu\nu}(g)$,

$$f^{j}_{\mu\nu}(g) \equiv e^{-\mu\xi} f^{j}_{\mu\nu}(z)\, e^{-\mu'\xi'} \quad ,$$

is a solution to the Laplace equation

$$L^{(g)}\, f^{j}_{\mu\nu}(g) = 0. \qquad (5.4)$$

The differential generators calculated in Appendix C.3 are the generators of the left-regular representation

$$T^{(g)}(g_1)\, f(g) = f(g_1^{-1} g), \qquad (5.5)$$

where

$$T^{(g)}(g_1) = \exp\left[-i\xi_1 \vec{K}_1^{(g)}\right] \exp\left[-i\nu_1 \vec{K}_2^{(g)}\right] \exp\left[-i\xi_1' \vec{K}_1^{(g)}\right] . \qquad (5.6)$$

Since $\left[\vec{J}^{2(g)}, \vec{K}_i^{(g)}\right] = 0$, it follows that $\left[L^{(g)}, T^{(g)}(g_1)\right] = 0$. This is why the Laplace operator is invariant[18]:

$$\begin{aligned} L^{(g)} f(g) &= L^{(g)} \left[T^{(g)}(g_1^{-1})\, f(g_1^{-1} g)\right] \\ &= T^{(g)}(g_1^{-1}) \left[L^{(g)}\, f(g_1^{-1} g)\right] \\ &= L^{(g_1 g)}\, f(g) . \end{aligned} \qquad (5.7)$$

That is, $L^{(g)} = L^{(g_1 g)}$ ~~acting~~ acting on $f(g)$ is left-invariant in the same sense that the Haar measure $d[g] = d[g_1 g]$ under $\int_G$ is left-invariant.

From the right-regular representation

$$T_R^{(g)}(g_1)\, f(g) = f(g\, g_1)$$

one may conclude that the Laplace operator is also right-invariant because, although the left-and right-shift differential generators are not the same, the Casimir and Laplace operators are the same whether expressed in terms of either left-or right-shift generators [see Appendix F] .

From the invariance of $L^{(g)}$, it is at once obvious that both sides of the multiplication formula (5.1) satisfy the Laplace equation $L^{(g_1)} f(g_1) = 0$:

$$\begin{aligned} L^{(g_1)}\, \mathcal{D}^{j}_{\mu\lambda}(g_1) &= 0 \\ L^{(g_1)}\, \mathcal{D}^{j}_{\mu\mu'}(g) &= L^{(g_1)}\, \mathcal{D}^{j}_{\mu\mu'}(g_1 g_2) = L^{(g_1 g_2)}\, \mathcal{D}^{j}_{\mu\mu'}(g_1 g_2) \\ &= 0. \end{aligned}$$

This completes part 1 of the proof.

2. Part 2 of Proof

We show here that both sides of (5.1) satisfy the Legendre equation $\mathcal{L}(j;\mu,\lambda;z_1)\, f(z_1) = 0$. This fact is obvious for the left-hand side of (5.1), and is almost obvious for the right-hand side. We have shown in part 1 above that the right-hand side of (5.1) $\equiv \mathrm{RHS}(g_1)$ satisfies the equation $L^{(g_1)} \mathrm{RHS}(g_1) = 0$. If we can show that

$$\mathrm{RHS}(g_1) = e^{-\mu\xi_1} \cdot e^{-\lambda\xi_1'} \widehat{\mathrm{RHS}}(z_1) \quad , \tag{5.8}$$

then it will follow that $\mathcal{L}(j;\mu,\lambda;z_1)\,\mathrm{RHS}(g_1) = 0$. Thus, part 2 of the proof is complete if we can demonstrate (5.8). This is where the $SO(1,1)$ invariant integration comes into play.

We begin by "exploding" $\mathcal{Q}(g_1 g_2)$ in (5.1) via the left-regular representation (5.5) and (5.6), so that

$$\mathrm{RHS}(g_1) = \frac{1}{2}\int_{-\infty}^{\infty} d\xi_2\, e^{\lambda\xi_2}\, T^{(g_2)}(g_1^{-1})\, \mathcal{Q}^j_{\mu\mu'}(g_2)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} d\xi_2\, e^{\lambda\xi_2} \exp(i\xi_1'\vec{K}_1^{(g_2)}) \exp(i\nu_1\vec{K}_2^{(g_2)})$$

$$\cdot \exp(i\xi_1\vec{K}_1^{(g_2)})\, \mathcal{Q}^j_{\mu\mu'}(g_2) \; . \tag{5.9}$$

From (C.11), $\vec{K}_1^{(g_2)} = -i\,\dfrac{\partial}{\partial\xi_2}$, so we may replace the rightmost exponential operator in (5.9) with $\exp(-\mu\xi_1)$, achieving half the goal of demonstrating (5.8). The leftmost operator cannot be taken through onto $\mathcal{Q}(g_2)$ because $\vec{K}_1$ and $\vec{K}_2$ do not commute. However, by considering the general form of the expression in (5.9), we may successfully expose the factor $e^{-\lambda\xi_1'}$ as follows:

$$\int_{-\infty}^{\infty} d\xi_2 e^{\lambda\xi_2} \exp\left(\xi_1' \frac{\partial}{\partial\xi_2}\right) F(\xi_2)$$

$$= \int_{-\infty}^{\infty} d\xi_2\, e^{\lambda\xi_2}\, F(\xi_2 + \xi_1') \tag{5.10}$$

(Eq. (5.10) continued on next page)

$$= e^{-\lambda\xi_1'} \int_{-\infty}^{\infty} d\xi_2\, e^{\lambda\xi_2}\, F(\xi_2) \; , \tag{5.10}$$

where we have in effect used the regular representation of $SO(1,1)$, the group multiplication property of $D(\xi_2) = e^{\lambda\xi_2}$, and the invariance of $\int_{SO(1,1)} d\left[\xi_2\right]$. Therefore,

$$\mathrm{RHS}(g_1) = e^{-\mu\xi_1}\, e^{-\lambda\xi_1'} \cdot \left\{\frac{1}{2}\int_{-\infty}^{\infty} d\xi_2\, e^{\lambda\xi_2} \exp(i\nu_1 \vec{K}_2^{(g_2)})\, \mathcal{Q}^j_{\mu\mu'}(g_2)\right\}, \tag{5.11}$$

which concludes part 2 of the proof.

3. Part 3 of Proof

We have shown that the Legendre equation

$$\mathcal{L}(j;\mu,\lambda;z_1)\, f(z_1) = 0 \tag{5.12}$$

is solved by both sides of the second-kind multiplication formula

$$\mathcal{Q}^j_{\mu\lambda}(z_1)\, \mathcal{Q}^j_{\lambda\mu'}(z_2) = \frac{1}{2}\int_{-\infty}^{\infty} d\alpha\, e^{\lambda\alpha}\left[e^{-\mu\xi}\mathcal{Q}^j_{\mu\mu'}(z)\, e^{-\mu'\xi'}\right], \tag{5.13}$$

and we now wish to show that both sides of (5.13) are in fact the <u>same</u> solution of (5.12). From (H.39) we see that, for complex j, the linear combination of $\mathcal{Q}^j_{\mu\lambda}(z_1)$ and $\mathcal{Q}^{-j-1}_{\mu\lambda}(z_1)$, which any solution of (5.12) must be, is completely determined by the asymptotic form as $z_1 \to \infty$. Thus, we shall prove that both sides of (5.13) are the same solution of (5.12), with the same coefficient[19], by showing that (5.13) is true as $z_1 \to \infty$. But, using the information given in (E.20) with $\omega = -i\alpha$ and (E.22), it is easily shown that the asymptotic limit of (5.13) as $z_1 \to \infty$ is a version of the integral representation for $Q^j_{\lambda\mu'}(z_2)$ given in (H.58). This concludes our proof.

4. Proofs of the Other Multiplication Formulas

The formulas (2.13) and (2.14) may be proven by the same procedure as above. Part 1 of the proof goes through intact, since it depends only on the general $g_1 \cdot g_2 = g$ structure of the multiplication formula. Part 2 goes through as above except $K = SO(2)$, so equations (5.10) are correspondingly different. It is here that the restriction that (m, m', n) be integers or half-integers arises. Part 3 is then shown by taking $z_1 \to \infty$ as above, then using (H.57) or (H.58).

Regarding the hybrid multiplication formula (2.14), we mention one detail which causes its proof to differ slightly from the others. In part 2, the proof that both sides of (2.14) solve the Legendre equation in z_1 fails when $z_1 = z_2$, because in this case $z = z_1 z_2 + \sqrt{z_1^2 - 1} \cdot \sqrt{z_2^2 - 1} \cos(\omega) \to 1$ as $\omega \to \pm\pi$, so the singularity of $Q^j_{mm'}(1)$ touches the endpoints of the integration in (2.14). This has the effect of causing a discontinuity in the right-hand side of (2.14), treated as a function of z_1, at $z_1 = z_2$, and for $z_1, z_2 > 1$ we find by the above procedure that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega\, e^{in\omega} \{ e^{-im\phi} Q^j_{mm'}(z)\, e^{-im'\phi'} \} \tag{5.14}$$

$$= \theta(z_2 - z_1)\, P^j_{mn}(z_1)\, Q^j_{nm'}(z_2) + \theta(z_1 - z_2)\, Q^j_{mn}(z_1)\, P^j_{nm'}(z_2) .$$

We have chosen the first term for the analytic continuation in z_1 and z_2 discussed in Section IV.2.

5. A Note on the Integral Representations for $\mathcal{Q}$ and P

From Eq. (5.11) derived above and Appendix C.3 it follows that

$$\mathcal{Q}^j_{\mu\lambda}(z_1)\, \mathcal{Q}^j_{\lambda\mu'}(z_2) = \frac{1}{2} \int_{-\infty}^{\infty} d\xi_2\, e^{\lambda\xi_2} \exp(i\nu_1 \vec{K}_2) \left[e^{-\mu\xi_2} \mathcal{Q}^j_{\mu\mu'}(z_2) \right] ,$$

where

$$\vec{K}_2 = i\,\mathrm{sh}\xi_2 \left(\mu' \,\mathrm{csch}\, \nu_2 + \mathrm{cth}\, \nu_2 \frac{\partial}{\partial \xi_2} \right) - i\,\mathrm{ch}\, \xi_2 \frac{\partial}{\partial \nu_2} .$$

Setting $\mu' = 0$ and taking $\nu_2 \to \infty$ on both sides we find, according to (H.39),

$$\mathcal{Q}^j_{\mu\lambda}(z_1) = \frac{\Gamma(j+1+\mu)}{\Gamma(j+1+\lambda)} \frac{1}{2} \int_{-\infty}^{\infty} d\xi_2\, e^{+\lambda\xi_2} \exp(+i\nu_1 \vec{k}_2)\, e^{-\mu\xi_2} , \tag{5.15}$$

where

$$\vec{k}_2 = i\,\mathrm{sh}\xi_2 \frac{\partial}{\partial \xi_2} + i(j+1)\,\mathrm{ch}\xi_2 .$$

Now, using the symmetry (H.22) and replacing $\mu \to -\mu'$, $\lambda \to -\mu$, $\xi_2 \to \alpha$, $z_1 \to z$ we get

$$\mathcal{Q}^j_{\mu\mu'}(\mathrm{ch}\,\nu) = \frac{\Gamma(j+1-\mu')}{\Gamma(j+1-\mu)} \frac{1}{2} \int_{-\infty}^{\infty} d\alpha\, e^{-\mu\alpha}\, e^{i\nu \vec{k}_2} \left[e^{\mu'\alpha}\right] \quad (5.16)$$

with

$$\vec{k}_2 = i \left[\mathrm{sh}\,\alpha\, \partial_\alpha + (j+1)\,\mathrm{ch}\,\alpha\right] .$$

A result similar to (5.16) follows from the equation corresponding to (5.11) in the proof of the first-kind multiplication formula. The answer may be quickly guessed by comparing (H.57) with (H.58) and using $\alpha = i\omega$:

$$P^j_{mm'}(\mathrm{ch}\,\nu) = \frac{\Gamma(-j+m)}{\Gamma(-j+m')} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega\, e^{-im\omega}\, e^{i\nu \vec{k}_2} \left[e^{im'\omega}\right] \quad (5.17)$$

where

$$\vec{k}_2 = i \left[\sin\omega\, \partial_\omega + (j+1)\cos\omega\right] .$$

Equations (5.16) and (5.17) allow us to interpret the Legendre integral representations (H.57) and (H.58) as "matrix elements" of the operator $\exp(i\nu\vec{k}_2)$, where $\vec{k}_2$ is a realization of the SU(1,1) Lie generator K_2 as a single-parameter differential operator. The full single-parameter Lie algebra appropriate to (5.16) is

$$\vec{k}_1 = i\partial_\alpha$$

$$\vec{k}_2 = i\left[\mathrm{sh}\,\alpha\cdot\partial_\alpha + (j+1)\,\mathrm{ch}\,\alpha\right]$$

$$\vec{J}_3 = -i\left[\mathrm{ch}\,\alpha\cdot\partial_\alpha + (j+1)\,\mathrm{sh}\,\alpha\right]$$

which is just the realization discussed by Mukunda[3], Eq. (4.15), and by Hermann[15], Eq. (5.3). Note that the SO(1,1) generator $\vec{k}_1$ is "diagonal".

In order to show <u>directly</u> that (5.16) and (H.58) are the same, one must compute the action of an exponentiated differential operator. A trick for doing this is given by Hermann, p. 104 (but sign error in Eq. (5.8)).

Considering the discussion of Appendix D, we are not surprised at this interpretation of the integral representations since, for special values of j and the helicity labels, the Legendre functions are precisely the SU(1,1) D^+_k UIR matrix elements, aside from inessential factors.

VI. Application: the Diagonalization of Convolution Equations

The group-theoretic addition theorems are particularly useful in diagonalizing integral equations of the convolution form. If A,B and C are functions defined on a Lie group G with invariant measure dg, consider the "integral equation"

$$A(g) = \left[B*C \right](g) = \int_G dg_1\, B(g_1)\, C(g_2) \qquad (6.1)$$

where $g_2 = g_1^{-1} g$. Let $D^\sigma_{k\,k'}(g)$ be the matrix elements of an irreducible representation of G labelled by the eigenvalues σ of the invariant operators of the Lie algebra (e.g., Casimir operators). Indices k and k' represent the eigenvalues of the simultaneously diagonalized generators in the basis $|\sigma,k\rangle$.
The functions $D^\sigma_{kk'}(g)$ satisfy an addition theorem

$$D^\sigma_{kk'}(g) = S_{k''}\; D^\sigma_{kk''}(g_1)\, D^\sigma_{k''k'}(g_2). \qquad (6.2)$$

Applying $\int_G dg\, D^\sigma_{k,k'}(g)$ to both sides of (6.1) we find

$$\int_G dg\, A(g)\, D^\sigma_{kk'}(g) = \int_G dg_1\, B(g_1) \int_G dg\, D^\sigma_{kk'}(g)\, C(g_2)$$
$$= \underset{k''}{S} \int_G dg_1\, B(g_1)\, D^\sigma_{kk''}(g_1) \cdot \int_G dg_2\, C(g_2) D^\sigma_{k''k'}(g_2),$$

where we have used the Haar invariance $\int_G dg = \int_G d\left[g_1 g_2\right] = \int_G dg_2,$

as well as the addition theorem (6.2). Defining the projections

$$f^\sigma_{kk'} \equiv \int_G dg\, f(g)\, D^\sigma_{kk'}(g), \qquad (6.3)$$

we arrive at the "diagonalized" equation

$$A^\sigma_{kk'} = S_{k''}\, B^\sigma_{kk''}\, C^\sigma_{k''k'}. \qquad (6.4)$$

1. Diagonalization in the Discrete Basis

Specifically, if A,B and C are defined on $G = SU(1,1)$, the equation[20]

$$A(\phi,\nu,\phi') = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \cdot \int_0^\infty d\nu_1 \cdot \mathrm{sh}\,\nu_1 \cdot \int_{-2\pi}^{2\pi} \frac{d\phi_1'}{4\pi}\, B(\phi_1,\nu_1,\phi_1')\, C(\phi_2,\nu_2,\phi_2') \qquad (6.5)$$

may be diagonalized by means of the first-kind Legendre addition theorem (2.1)

$$P^j_{mm'}(g) = \sum_{n=-\infty}^{\infty} P^j_{mn}(g_1)\, P^j_{nm'}(g_2), \qquad (6.6)$$

where

$$P^j_{mm'}(g) \equiv e^{-im\phi}\, P^j_{mm'}(\mathrm{ch}\,\nu)\, e^{-im'\phi'}. \qquad (6.7)$$

The resultant diagonalized equation is

$$A^j_{mm'} = \sum_{n=-\infty}^{\infty} B^j_{mn}\, C^j_{nm'} \qquad (6.8)$$

where

$$f^j_{mm'} \equiv \int_G dg\, f(g)\, P^j_{mm'}(g) \qquad (6.9)$$

with dg as shown in (6.5).

Once the diagonalized equation is "solved" for $A^j_{mm'}$ (e.g., if B is given and C is a function of A), the function $A(g)$ may be reconstructed from its projections according to (G.15a).

The diagonalization above was discussed in connection with the partial-wave analysis of particle scattering amplitudes by Serterio and Toller[21] (1964) and in further detail by Toller[22] (1965).

2. Diagonalization in the Continuous Basis

Assume now that the functions A,B,C are defined[23] only on the semigroup S_o^+ discussed in Appendix E.3. To diagonalize the equation

$$A(\xi,\nu,\xi') = \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \cdot \int_0^{\infty} d\nu_1 \cdot \text{sh}\, \nu_1 \int_{-\infty}^{\infty} \frac{d\xi_1'}{2\pi}\, B(\xi_1,\nu_1,\xi_1')\, C(\xi_2,\nu_2,\xi_2') \qquad (6.10)$$

we apply the second-kind Legendre addition theorem (2.7)

$$\mathcal{Q}^j_{\mu\mu'}(g) = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} d\lambda\, \mathcal{Q}^j_{\mu\lambda}(g_1)\, \mathcal{Q}^j_{\lambda\mu'}(g_2) \;, \qquad (6.11)$$

where

$$\mathcal{Q}^j_{\mu\mu'}(g) \equiv e^{-\mu\xi}\, \mathcal{Q}^j_{\mu\mu'}(\text{ch}\, \nu)\, e^{-\mu'\xi'} \;. \qquad (6.12)$$

The result is

$$A^j_{\mu\mu'} = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} d\lambda\, B^j_{\mu\lambda}\, C^j_{\lambda\mu'} \qquad (6.13)$$

where

$$f^j_{\mu\mu'} \equiv \int_{S_o^+} dg\, f(g)\, \mathcal{Q}^j_{\mu\mu'}(g) \qquad (6.14)$$

with dg as shown in (6.10). Again, if the diagonalized equation is solved for the $A^j_{\mu\mu'}$, the unprojected function $A(g)$ may be obtained from (G.17a).

3. The Diagonalization of Abarbanel and Saunders

Consider the following special case of (6.10),

$$A(-,\nu,-) = \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \int_0^{\infty} d\nu_1\, \text{sh}\, \nu_1\, B(-,\nu_1,-)\, C(\xi_2,\nu_2,-)$$

or

$$A(z) = \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \int_1^{\infty} dz_1\, B(z_1)\, C(\xi_2,z_2) \;, \qquad (6.15)$$

where a dash indicates an absence of functional dependence on the variable appearing in (6.10). We have removed the (non-compact) integration over ξ_1' and have set $\xi_1' = 0$.[24] It is easy to show that the diagonalization procedure is unaffected by the fact that the full invariant integration fails to appear in (6.15); only the projection of B is different.

From (6.13) we find, after cancelling delta functions, the following diagonalization of (6.15):

$$a^j_{oo} = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} d\lambda \; b^j_{o\lambda} \, c^j_{\lambda o} \tag{6.16}$$

where

$$a^j_{oo} = \int_1^\infty dz \; A(z) \, \mathcal{Q}^j_{oo}(z) \tag{6.17}$$

$$b^j_{o\lambda} = \int_1^\infty dz_1 \; B(z_1) \, \mathcal{Q}^j_{o\lambda}(z_1) \tag{6.18}$$

$$c^j_{\lambda o} = \int_{-\infty}^{\infty} \frac{d\xi_2}{2\pi} \int_1^\infty dz_2 \cdot C(\xi_2, z_2) \, e^{-\lambda\xi_2} \cdot \mathcal{Q}^j_{\lambda o}(z_2). \tag{6.19}$$

Specializing still further by removing the ξ_2-dependence from $C(\xi_2, z_2)$, we get $c^j_{\lambda o} = \delta(i\lambda)\, c^j_{oo}$, so the diagonalization of

$$A(z) = \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \int_1^\infty dz_1 \; B(z_1) \, C(z_2) \tag{6.20}$$

is

$$a_j = \frac{1}{\pi} \, b_j \, c_j \, , \tag{6.21}$$

with all projections of the form

$$a_j = \int_1^\infty dz \; A(z) \, Q_j(z) \, .$$

In terms of simplicity, (6.21) is comparable to (1.2).

The special case of the second-kind diagonalization given as (6.20) and (6.21) was discovered by Abarbanel and Saunders[25] (1970) and further analyzed by Cronström[26] (1974).

4. A Physics Comment

Briefly, the physical significance of the simplified convolution Eq. (6.15) may be understood in terms of Fig. 4 which shows, in schematic form, the typical multiperipheral integral equation (in a particular kinematic configuration, see CDM [27]). The x's mark "CDM frames"[27] and the variables shown are the boost parameters which link the frames in a manner similar to the usual Toller or BCP variables.[28] Of course these variables are also the SO(2,1) group variables we have been using all along in the continuous-basis S_o^+ semigroup parametrization, and $g = g_1 g_2$.

The multiperipheral integral equation symbolized by Fig. 4 is a statement of (s-channel) unitarity. This means that, roughly speaking, A,B, and C are the discontinuities of reggeon-reggeon scattering amplitudes, with "cluster masses" sensed by the variables ν, ν_1, and ν_2. We have included in C the "reggeon propagator" whose "energy" dependence is characterized by the variable ξ_2.

The loop integration in the multiperipheral equation is the Lorentz invariant d^4k, where k^μ is the 4-momentum of, say, the lower reggeon of the reggeon propagator. When this momentum is viewed from the leftmost CDM frame, one finds that:

$$d^4k = dT \cdot d\mathfrak{g}_1$$

where

$$\begin{aligned} dT &= k^2 dk dw \\ &= \sqrt{-\Delta(t,t_1,t_2)}\, dt_1 dt_2/8(-t) \\ &= \frac{1}{2} du'(-u')\, dz' \sqrt{1-z'^2} \end{aligned}$$

and

$$d\mathfrak{g}_1 = d\xi_1 \cdot d(\mathrm{ch}\, \nu_1) \ .$$

The variables k,w,t,t_1,t_2,u',z' are described in CDM and AS but are of no concern here. The point is that the "loop phase space" d^4k factorizes exactly into a residual "transverse integration" dT (which survives in the partially diagonalized equation), and the group phase space $d\mathfrak{g}_1$ which appears in (6.15).

In other words, the $t < 0$ multiperipheral equation is a convolution equation with respect to the S_o^+ semisubgroup of SO(2,1) and may therefore be exactly diagonalized by the second-kind addition theorem. This is in contrast to the approximate diagonalization obtained by use of the Mellin/Laplace/SO(1,1) transform which treats the integral equation as if it were a convolution with

respect to SO(1,1) rather than SO(2,1). A problem with this SO(1,1) or "rapidity" approximation is that certain potentially significant effects (such as threshold behavior) get washed out in the diagonalization process.

The reason that Abarbanel and Saunders were able to partially diagonalize the ASF equation using (6.20) and (6.21) is that the ASF equation has an energy-independent pion propagator in place of the more general reggeon propagator, i.e., $C(z_2)$ in place of $C(\xi_2,z_2)$. The diagonalization of a fully reggeized multiperipheral equation (such as the planar bootstrap) would look more like (6.16) or (6.13).[27,29] The variable λ is related to the analytic continuation of the helicity of the reggeon propagator in the same sense that the full projection given in (6.14) is the helicity continuation of the Froissart-Gribov projection with spin. We hope to clarify this comment in a future publication.

Acknowledgement

It is my pleasure to thank Prof. Geoff Chew for suggesting this line of research, and also Prof. Eyvind Wichmann and Jan Dash for some help along the way.

Note Added to Manuscript. After writing this report we have discovered, much to our embarrassment, the existence of the references listed below, in particular Ref. A. This pleasant paper (a follow-up to Ref. 25) contains on page 269 a statement of the second-kind addition theorem and multiplication formula, and makes the identification of the second-kind Legendre functions with the continuous-basis SU(1,1) matrix elements (albeit for the C_q rather than the D_k^+ series). We suspect that similar information is contained in Ref. C which we have been unable to locate. Moreover, Ref. A effects the diagonalization we have given in Section VI.2, though we might still claim to have done so with more generality and conciseness. Ref. B extends the work of Ref. A to the t=0 case. Ref. E describes the significance of the semigroup which we stumbled upon in our Appendix E.3. Refs. D and E discuss the possibility of projecting amplitudes onto (Banach) representations of the semigroups of SU(1,1) and SL(2,C) which support the multiperipheral integration in the $t < 0$ and t=0 cases. Finally, we note the criticism lodged by Ref. F against "improved" expansion theorems like our (G.17).

A) H.D.I. Abarbanel and L.M. Saunders, Ann.Phys.(N.Y.) 64,254 (1971).

B) H.D.I. Abarbanel and L.M. Saunders, Ann.Phys.(N.Y.) 69,583 (1972).

C) N.W. Macfadyen, Carnegie Mellon University Report, October 1969.

D) S. Ferrara et. al., Nucl. Phys. B53,366 (1973).

E) G. Soliani and M. Toller, Nuovo Cimento 15A,430 (1973).

F) N.W. Macfadyen, Commun. Math. Phys. 28,87 (1972).

Appendix A: Lie Generator Conventions

1. Lie Algebras and Weyl's Trick

The six abstract generators of SL(2,C) satisfy the Lie algebra

$$
\begin{aligned}
\left[J_i, J_j\right] &= i\,\varepsilon_{ijk}\, J_k \\
\left[J_i, K_j\right] &= i\,\varepsilon_{ijk}\, K_k \\
\left[K_i, K_j\right] &= -i\,\varepsilon_{ijk}\, J_k\,.
\end{aligned}
\qquad (A.1)
$$

From (A.1) and the Campbell-Hausdorff formula it follows that

$$
\begin{aligned}
e^{-i\phi J_i}\, J_j\, e^{i\phi J_i} &= \cos\phi\cdot J_j + \sin\phi\cdot\varepsilon_{ijk}\, J_k \\
e^{-i\phi J_i}\, K_j\, e^{i\phi J_i} &= \cos\phi\cdot K_j + \sin\phi\cdot\varepsilon_{ijk}\, K_k \\
e^{-i\nu K_i}\, J_j\, e^{i\nu K_i} &= \mathrm{ch}\,\nu\cdot J_j + \mathrm{sh}\,\nu\cdot\varepsilon_{ijk}\, K_k \\
e^{-i\nu K_i}\, K_j\, e^{i\nu K_i} &= \mathrm{ch}\,\nu\cdot K_j - \mathrm{sh}\,\nu\cdot\varepsilon_{ijk}\, J_k\,.
\end{aligned}
\qquad (A.2)
$$

The SU(2) subgroup of SL(2,C) is generated by

$$J_1,\ J_2,\ J_3$$

with the Lie algebra

$$\left[J_i, J_j\right] = i\,\varepsilon_{ijk}\, J_k$$

and Casimir

$$J^2 \equiv J_1^2 + J_2^2 + J_3^3\ .$$

For the SU(1,1) subgroup of SL(2,C) we choose the generators

$$K_1,\ K_2,\ J_3$$

with the Lie algebra

$$
\begin{aligned}
\left[J_3, K_1\right] &= i\,K_2 \qquad \left[J_3, K_2\right] = -i\,K_1 \\
\left[K_1, K_2\right] &= -i\,J_3
\end{aligned}
\qquad (A.3)
$$

and Casimir

$$J^2 = -K_1^2 - K_2^2 + J_3^2\ . \qquad (A.4)$$

The SU(1,1) Lie algebra may be obtained from that of SU(2) by the mapping

$$(J_1, J_2, J_3) \rightarrow (-i\,K_1, -i\,K_2, J_3)\ ,$$

a fact sometimes referred to as Weyl's Trick (see Appendix B.1). In the explicit realization of SL(2,C) given below, the above mapping is an identity.

There are several simple automorphisms of SU(2), two of which are the obvious cyclic permutations. Two more are

$$(J_1, J_2, J_3) \rightarrow (-J_1, -J_2, J_3)\ ,\ (J_2, -J_1, J_3)\ .$$

From these four, a list of 23 automorphisms may easily be constructed, allowing any generator to be mapped into (plus or minus) any other generator. Using Weyl's trick, the corresponding list of 23 automorphisms of SU(1,1) is at once found. Two of these are

$$(K_1,K_2,J_3) \rightarrow (-K_2,K_1,J_3)\ , \quad (iJ_3,K_2,\ i\ K_1). \qquad (A.5)$$

The first shows (see below) that our generators K_1,K_2 and J_3 are trivially automorphically connected to the "J_i" used by Mukunda[3],

$$"J_o" = J_3 = \frac{1}{2}\sigma_3$$

$$"J_1" = K_2 = \frac{1}{2}\, i\, \sigma_2$$

$$"J_2" = -K_1 = -\frac{1}{2}\, i\, \sigma_1.$$

The second automorphism in (A.5) is useful in interconnecting relations between the discrete and continuous-basis parametrizations of SU(1,1) (see Appendix C.3).

2. Explicit Realization of SL(2,C).

The Lie algebras given above have the following two-dimensional realization,

$$J_i = \frac{1}{2}\sigma_i \qquad\qquad K_i = \frac{1}{2}\, i\, \sigma_i \qquad (A.6)$$

where σ_i are the Pauli matrices. The matrices of the one-parameter subgroups may be found from

$$e^{\underset{\sim}{\alpha}\cdot\underset{\sim}{\sigma}} = \text{ch}\ \alpha + \alpha^{-1}\text{sh}\ \alpha(\underset{\sim}{\alpha}\cdot\underset{\sim}{\sigma})\ ,$$

where $\underset{\sim}{\alpha}$ = complex 3-vector and $\alpha = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{\frac{1}{2}}$. They are:

$$e^{-i\phi J_1} = \begin{pmatrix} C_{\phi/2} & -iS_{\phi/2} \\ -iS_{\phi/2} & C_{\phi/2} \end{pmatrix} \qquad e^{-i\nu K_1} = \begin{pmatrix} \text{ch}\ \frac{\nu}{2} & \text{sh}\ \frac{\nu}{2} \\ \text{sh}\ \frac{\nu}{2} & \text{ch}\ \frac{\nu}{2} \end{pmatrix}$$

$$e^{-i\phi J_2} = \begin{pmatrix} C_{\phi/2} & -S_{\phi/2} \\ S_{\phi/2} & C_{\phi/2} \end{pmatrix} \qquad e^{-i\nu K_2} = \begin{pmatrix} \text{ch}\ \frac{\nu}{2} & -i\text{sh}\ \frac{\nu}{2} \\ i\text{sh}\ \frac{\nu}{2} & \text{ch}\ \frac{\nu}{2} \end{pmatrix}$$

$$e^{-i\phi J_3} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \qquad e^{-i\nu K_3} = \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & e^{-\nu/2} \end{pmatrix}. \qquad (A.7)$$

Whereas the J_i are hermitian and the $e^{-i\phi J_i}$ are unitary, the K_i are anti-hermitian and the $e^{-i\nu K_i}$ are non-unitary.

3. Relation to the Lorentz Group

Throughout this paper we have avoided repeated mention of SO(3) with SU(2), and SO(2,1) with SU(1,1). Physical applications of the addition theorems (e.g., diagonalizations as in Sec. VI) usually involve these Lorentz subgroups rather than their SU counterparts. For this reason, we include here our convention for the connection between SL(2,C) and SO(3,1).

If we represent an arbitrary SL(2,C) group element by

$$g = e^{-i\left[\underset{\sim}{a}\cdot\underset{\sim}{J} + \underset{\sim}{b}\cdot\underset{\sim}{K}\right]}$$

the corresponding element of the (proper orthochronous) Lorentz group $SO(3,1)^+$ is given by

$$\Lambda^{\mu}_{\cdot\nu} = \frac{1}{2}\ \mathrm{trace}\left[\sigma_{\mu}\, g\, \sigma_{\nu}\, g^{\dagger}\right]$$

according to the usual homomorphic connection (see Rühl,[30] Eq.(1-6)),

$$X' = g\, X\, g^{+} \qquad\qquad X = \sigma_{\mu} x^{\mu} = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}$$

$$X' = \sigma_{\mu} x'^{\mu} \qquad\qquad x'^{\mu} = \Lambda^{\mu}_{\cdot\nu}\, x^{\nu} \ .$$

The 4-dimensional Lorentz generators defined by

$$\Lambda^{\mu}_{\cdot\nu} = \left(e^{-i\left[\underset{\sim}{a}\cdot\underset{\sim}{J} + \underset{\sim}{b}\cdot\underset{\sim}{K}\right]}\right)^{\mu}_{\cdot\nu}$$

are then given by

$$\underset{\sim}{a}\cdot\underset{\sim}{J} + \underset{\sim}{b}\cdot\underset{\sim}{K} = i \times \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix} .$$

According to this connection between SL(2,C) and $SO(3,1)^{+}$, the Lorentz transformations corresponding to (A.7) are of the active type, e.g.,

$$(e^{-i\nu K_1})^{\mu}_{\cdot\nu} = \begin{bmatrix} \mathrm{ch}\,\nu & \mathrm{sh}\,\nu & 0 & 0 \\ \mathrm{sh}\,\nu & \mathrm{ch}\,\nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (e^{-i\phi J_3})^{\mu}_{\cdot\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{\phi} & -S_{\phi} & 0 \\ 0 & S_{\phi} & C_{\phi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

(A.8)

Appendix B: Representations and Bases for SU(1,1)

In this appendix we derive the classes of UIR's for SU(1,1)[31] and define the meaning of discrete, continuous and mixed basis. In each basis the forms of the matrix elements are given, but the explicit functions are deferred to Appendix D.

1. The UIR's

The unitary irreducible representations (UIR's) of SU(1,1) are all of infinite dimension, since SU(1,1) is non-compact. In the discrete basis, to be described below, the basis vectors $|j,m\rangle$ which span the representation space of a UIR are eigenvectors of J^2 and J_3, just as in the usual SU(2) analysis. In fact, using Weyl's trick[32] as mentioned in Appendix A,

$$J_\pm \equiv J_1 \pm i\, J_2 \rightarrow -i\, K_\pm \qquad \text{where } K_\pm \equiv K_1 \pm i\, K_2 \ ,$$

and our knowledge of SU(2), we find for SU(1,1) that

$$K_\pm\, |j,m\rangle = \left[(m \pm \tfrac{1}{2})^2 - (j + \tfrac{1}{2})^2\right]^{\frac{1}{2}} |j,m\pm 1\rangle \qquad \text{(B.1)}$$

$$\langle j,m|\, K_- K_+ |j,m\rangle = ||K_+|j,m\rangle||^2 = (m + \tfrac{1}{2})^2 - (j + \tfrac{1}{2})^2 \ . \qquad \text{(B.2)}$$

To say that the UIR $\{|j,m\rangle\}$ is unitary is to say that the generators K_1, K_2, J_3 are hermitian with respect to the scalar product $\langle\,|\,\rangle$. Therefore, $\langle\,|\,\rangle$ had better be a scalar product. As (B.2) shows, this will only be true if $(m + \frac{1}{2})^2 > (j + \frac{1}{2})^2$

for all $|j,m\rangle$ in the representation. From this simple fact, and the truncation possibility implicit in (B.1), we immediately know all the UIR's. With the spectrum of J_3 restricted to integers and half-integers (for single-valued representations of SU(1,1)), the nontrivial UIR's are displayed in Table B.1.

TABLE B.1 The UIR's of SU(1,1)

name	range of j	J_3 spectrum
C_q^0	$j = -\frac{1}{2} + is$ (s real)	$m = 0, \pm 1, \pm 2, \ldots$
$C_q^{\frac{1}{2}}$	$j = -\frac{1}{2} + is$ (s real)	$m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$
E_q^0	$-\frac{1}{2} < j < 0$	$m = 0, \pm 1, \pm 2, \ldots$
D_k^+	$j = -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$	$m = j+1, j+2, \ldots$
D_k^-	$j = -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$	$m = -j-1, -j-2, \ldots$

The UIR's are called, respectively, the integral and half-integral continuous (or principal) series, the exceptional (or supplementary) series, and the positive and negative discrete series. The notation is that of Bargmann[1] who uses

$$q = -j(j+1) \qquad k = j+1 \qquad \text{(B.3)}$$

so that

$$q = k(1-k) = \frac{1}{4} + s^2 \tag{B.4}$$

$$k = \frac{1}{2} + is \qquad j = -\frac{1}{2} + \sqrt{\frac{1}{4} - q} .$$

2. Discrete Basis

The vectors $|j,m>$ which diagonalize J^2 and J_3,

$$J^2|j,m> = j(j+1)|j,m> \qquad J_3|j,m> = m|j,m> , \tag{B.5}$$

comprise the "discrete basis" of the UIR labelled by j, so-called because the spectrum of the compact generator J_3 is discrete. As indicated by (B.1), the points of the spectrum are separated by one unit and, since J_3 is hermitian, the spectrum lies on the real axis. The normalization and completeness of the $|j,m>$ are given by

$$<j,m|j,m'> = \delta_{m,m'} \qquad 1^j = \sum_m |j,m><j,m| , \tag{B.6}$$

where 1^j indicates the identity in the Hilbert Space H^j of the UIR, and the sum on m extends over the appropriate range as shown in Table B.1.

The abstract elements of the Lie group SU(1,1) are represented in H^j by operators $U(g)$, where g indicates some parametrization of the group elements. The UIR matrix elements in the discrete basis are then $<j,m|U(g)|j,m'>$.

The traditional parametrization of SU(1,1), and the one appropriate for taking discrete-basis matrix elements, is:[33]

$$U_1(\phi,\nu,\phi') = e^{-i\phi J_3} e^{-i\nu K_2} e^{-i\phi' J_3} . \tag{B.7}$$

If we restrict ϕ,ν,ϕ' to the regions

$$0 \leq \phi < 2\pi , \qquad -2\pi \leq \phi' < 2\pi , \qquad \nu \geq 0, \tag{B.8}$$

SU(1,1) is covered once, but SO(2,1) is covered twice. For SO(2,1) this double coverage can be removed by giving ϕ' the same range as ϕ.

The discrete-basis matrix elements then have the form

$$<j,m|U_1(\phi,\nu,\phi')|j,m'> = e^{-im\phi} e^{-im'\phi'} <j,m|e^{-i\nu K_2}|j,m'> . \tag{B.9}$$

3. Continuous Basis. If the non-compact generator K_1 is diagonalized instead of J_3,

$$J^2|j,p> = j(j+1)|j,p> \qquad K_1|j,p> = p|j,p> , \tag{B.10}$$

we have the "continuous basis" since the spectrum of K_1 is the continuous real line (again, K_1 is hermitian for UIR's). Actually, for the C_q UIR's, the spectrum of K_1 is the real line twice, and a bi-valued multiplicity index must be added in the kets, $|j,p,b\rangle$.[34] We shall be concerned only with the D_k^+ UIR's where there is no multiplicity index.

The normalization and completeness relation for the D_k^+ UIR's are

$$\langle j,p|j,p'\rangle = \delta(p-p') \qquad 1^j = \int_{-\infty}^{\infty} dp|j,p\rangle\langle j,p| , \tag{B.11}$$

where j takes the values shown in Table B.1. It turns out to be more convenient to use the purely imaginary variables $\mu = ip$ and $\mu' = ip'$ so that

$$K_1|j,\mu\rangle = -i\mu|j,\mu\rangle \qquad \langle j,\mu|j,\mu'\rangle = \delta(i\mu - i\mu')$$

$$1^j = (-i)\int_{-i\infty}^{i\infty} d\mu|j,\mu\rangle\langle j,\mu| . \tag{B.12}$$

This is the Mellin-Barnes contour which appears in the second-kind addition theorem (2.7).

An appropriate parametrization for the continuous basis is

$$U_2(\xi,\nu,\xi') = e^{-i\xi K_1} e^{-i\nu K_2} e^{-i\xi' K_1} . \tag{B.13}$$

The sector of the SU(1,1) group manifold which admits this parametrization with $\nu \geqslant 0$ forms a semigroup S_o^+ (see Appendix E.3),

so for $g \in S_o^+$, the continuous-basis matrix elements for the D_k^+ UIR's have the form

$$\langle j,\mu|U_2(\xi,\nu,\xi')|j,\mu'\rangle = e^{-\mu\xi} e^{-\mu'\xi'} \langle j,\mu|e^{-i\nu K_2}|j,\mu'\rangle . \tag{B.14}$$

4. Mixed Basis

Matrix elements in the "mixed basis" have J_3 diagonal on one side, and K_1 diagonal on the other. Any element of SU(1,1) can be parametrized in either of the forms

$$U_3(\phi,\eta,\xi') = e^{-i\phi J_3} e^{-i\eta K_2} e^{-i\xi' K_1} .$$
$$U_4(\xi,\eta,\phi') = e^{-i\xi K_1} e^{-i\eta K_2} e^{-i\phi' K_1} . \tag{B.15}$$

The mixed-basis matrix elements for D_k^+ then have the form

$$\langle j,m|U_3(\phi,\eta,\xi')|j,\mu'\rangle = e^{-im\phi} e^{-\mu'\xi'} \langle j,m|e^{-i\eta K_2}|j,\mu'\rangle$$
$$\langle j,\mu|U_4(\xi,\eta,\phi')|j,m'\rangle = e^{-\mu\xi} e^{-im'\phi'} \langle j,\mu|e^{-i\eta K_2}|j,m'\rangle . \tag{B.16}$$

Appendix C: The Lie Generators as Differential Operators on SU(1,1)

Following the approach of Bargmann[2] we show in general how Lie generators can be realized as differential operators on the group manifold itself. Then we explicitly calculate the operators for each of the SU(1,1) parametrizations. The main point of this effort is to obtain the second-order differential operators for the Casimir which are used in the following appendix to "compute" the explicit SU(1,1) matrix elements.

1. The Method

Consider an n-parameter Lie group associated with one of the classical matrix groups. Let the generators be G_i, the parameters p_i, and let $U(p)$ be a representation so that

$$U(p) = e^{-ip_1 G_{i_1}} e^{-ip_2 G_{i_2}} \dots e^{-iP_n G_{i_n}} \quad . \qquad (C.1)$$

In this chain of operators, some of the generators may appear more than once, others not at all.

The generators G_i can be realized as differential operators $\vec{G}_i$ on the manifold p (the parameter space) according to[35]

$$\vec{G}_i\, U(p) = -G_i\, U(p) \quad . \qquad (C.2)$$

Since the G_i satisfy the Lie algebra $[G_i,G_j] = c^k_{ij}\, G_k$, so do the $\vec{G}_i$:

$$\begin{aligned}
[\vec{G}_i,\vec{G}_j]\, U(p) &= (\vec{G}_i\vec{G}_j - \vec{G}_j\vec{G}_i)\, U(p) && (C.3)\\
&= (-\vec{G}_i G_j + \vec{G}_j G_i)\, U(p) \\
&= (-G_j\vec{G}_i + G_i\vec{G}_j)\, U(p)\ , \qquad [G_i,\vec{G}_j] = 0 \\
&= (G_j G_i - G_i G_j)\, U(p) \\
&= -[G_i,G_j]\, U(p) = -c^k_{ij}\, G_k\, U(p) \\
&= c^k_{ij}\, \vec{G}_k\, U(p) \quad . && (C.4)
\end{aligned}$$

If $U(p)$ is taken to be the "elementary" matrix representation, (C.2) is a simple system of equations which can be solved for the functions $X_{ij}(p)$ which characterize the operators $\vec{G}_i$,

$$\vec{G}_i = \sum_{j=1}^{n} X_{ij}(p)\, \frac{d}{dp_j} \quad . \qquad (C.5)$$

We find it more convenient to think of $U(p)$ as an abstract representation and, in effect, let the Campbell-Hausdorff identities do the work of solving these equations. This method is illustrated in the following sections.

2. Discrete Basis

We calculate the "differential generators" $\vec{G}_i$ for SU(1,1) using the standard Bargmann parametrization

$$U_1 = U_1(\phi,\nu,\phi') = e^{-i\phi J_3}\ e^{-i\nu K_2}\ e^{-i\phi' J_3} .$$

First,

$$\frac{\partial U_1}{\partial\phi} = -iJ_3U_1 \Rightarrow \vec{J}_3 = -i\,\partial_\phi . \qquad (C.6)$$

Next,

$$\frac{\partial U_1}{\partial\nu} = e^{-i\phi J_3}(-iK_2)\,e^{-i\nu K_2}\ e^{-i\phi' J_3}$$

$$= (-i)\ (e^{-i\phi J_3}\ K_2\, e^{i\phi J_3})\ U_1$$

$$= (-i)\,(\cos\phi\,K_2 - \sin\phi\,K_1)\,U_1 ,$$

where we have used one of the Campbell-Hausdorff identities (A.2). Therefore,

$$-i\,\partial_\nu = \cos\phi\ \vec{K}_2 - \sin\phi\ \vec{K}_1 . \qquad (C.7)$$

Finally,

$$\frac{\partial U_1}{d\phi'} = e^{-i\phi J_3}\ e^{-i\nu K_2}\ (-iJ_3)\ e^{-i\phi' J_3}$$

$$= (-i)\ \left[e^{-i\phi J_3}\ (e^{-i\nu K_2}\ J_3\ e^{i\nu K_2})\,e^{i\phi J_3}\right]\ U_1$$

(Equation continued on next page)

$$= (-i)\ \left[e^{-i\phi J_3}\,(\mathrm{ch}\,\nu\,J_3 + \mathrm{sh}\,\nu\,K_1)\,e^{i\phi J_3}\right]\ U_1$$

$$= (-i)\left[\ \mathrm{ch}\,\nu\,J_3\ +\ \mathrm{sh}\,\nu(\cos\phi\ K_1 + \sin\phi\ K_2)\right]U_1 .$$

Therefore,

$$-i\,\partial_{\phi}' = \mathrm{ch}\,\nu\,\vec{J}_3 + \mathrm{sh}\,\nu(\cos\phi\ \vec{K}_1 + \sin\phi\ \vec{K}_2) . \qquad (C.8)$$

Combining (C.6) through (C.8) we find

for $U_1(\phi,\nu,\phi') = e^{-i\phi J_3}\,e^{-i\nu K_2}\,e^{-i\phi' J_3}$:

$$\begin{aligned}
\vec{J}_3 &= -i\,\partial_\phi \\
\vec{K}_1 &= -i\cos\phi\,\Lambda + i\sin\phi\,\partial_\nu \\
\vec{K}_2 &= -i\sin\phi\,\Lambda - i\cos\phi\,\partial_\nu \qquad (C.9)\\
\vec{K}_\pm &\equiv \vec{K}_1 \pm i\,\vec{K}_2 = e^{\pm i\phi}\,(-i\Lambda \pm \partial_\nu),
\end{aligned}$$

where $\Lambda \equiv \frac{1}{\mathrm{sh}\,\nu}\,(\partial_\phi' - \mathrm{ch}\,\nu\ \partial_\phi)$.

The Casimir $J^2 \equiv -K_1^2 - K_2^2 + J_3^2$ is most easily computed from $J^2 = J_3^2 - \frac{1}{2}\left[K_+,K_-\right]_+$ with the result

$$\vec{J}^2 = \partial_\nu^2 + \text{cth}\,\nu\,\partial_\nu + (\Lambda^2 - \partial_\phi^2)$$

$$= \frac{1}{\text{sh}\,\nu}\,\partial_\nu\left[\text{sh}\,\nu\,\partial_\nu\right] + \frac{1}{\text{sh}^2\nu}\left[\partial_\phi^2 + \partial_{\phi'}^2 - 2\,\text{ch}\,\nu\,\partial_\phi\partial_{\phi'}\right]$$

$$= (z^2-1)\,\partial_z^2 + 2z\partial_z + \frac{1}{(z^2-1)}\left[\partial_\phi^2 + \partial_{\phi'}^2 - 2z\,\partial_\phi\partial_{\phi'}\right], \tag{C.10}$$

where $z = \text{ch}\,\nu$.

3. Continuous Basis

To get the corresponding expressions for the parametrization $U_2(\xi,\nu,\xi') = e^{-i\xi K_1}\,e^{-i\nu K_2}\,e^{-i\xi' K_1}$ we can repeat the above procedure. However, since this parametrization can be reached from the preceding parametrization via the automorphism $(K_1,K_2,J_3) \to (iJ_3,K_2,iK_1)$ and change of variables $\xi = i\phi$, $\xi' = i\phi'$, we can simply translate the above equations accordingly.[36] Therefore,

for $U_2(\xi,\nu,\xi') = e^{-i\xi K_1}\,e^{-i\nu K_2}\,e^{-i\xi' K_1}$:

$$\vec{K}_1 = -i\partial_\xi \qquad\qquad \Lambda = \frac{1}{\text{sh}\,\nu}\,(i\partial_{\xi'} - \text{ch}\,\nu\; i\partial_\xi)$$

$$\vec{J}_3 = -\text{ch}\,\xi\,\Lambda - i\,\text{sh}\,\xi\;\partial_\nu$$

$$\vec{K}_\pm = e^{\pm\xi}\,(-i\Lambda \pm \partial_\nu)$$

$$\vec{K}_2 = -\text{sh}\,\xi\,\Lambda - i\,\text{ch}\,\xi\;\partial_\nu$$

$$\vec{J}^2 = (z^2-1)\partial_z^2 + 2z\partial_z + \frac{1}{(z^2-1)}\left[(i\partial_\xi)^2 + (i\partial_{\xi'})^2 - 2z(i\partial_\xi)(i\partial_{\xi'})\right], \tag{C.11}$$

again with $z = \text{ch}\,\nu$.

4. Mixed Basis

By setting $u = i\cdot\frac{\pi}{2}$ in the third of equations (A.2) we find

$$e^{\frac{\pi}{2}K_2}\,J_3\,e^{-\frac{\pi}{2}K_2} = iK_1 \tag{C.12}$$

from which it easily follows that

$$U_3(\phi,\eta,\xi') = e^{-i\phi J_3}\;e^{-i\eta K_2}\;e^{-i\xi' K_1}$$

$$= \left[e^{-i\phi J_3}\;e^{-i\nu K_2}\;e^{-i\phi' J_3}\right]e^{-\frac{\pi}{2}K_2}, \tag{C.13}$$

where $\phi' = -i\xi'$ and $\nu = \eta + i\,\frac{\pi}{2}$. Now, let $\vec{G}_i(\phi,\nu,\phi')$ be one of the Bargmann-parametrization generators given in (C.9). According to (C.2) and (C.13),

$$\vec{G}_i(\phi,\nu,\phi')\,U_3 = -G_iU_3 = \vec{G}_i(\phi,\eta + i\,\tfrac{\pi}{2}, -i\xi')\,U_3\,. \tag{C.14}$$

The derivation of the first equality in (C.14) goes through exactly as in Section 2 above; it is unaffected by the presence of the factor $\exp(-\frac{\pi}{2}K_2)$ sitting on the right side of (C.13). The second equality in (C.14) indicates that the differential generators in the mixed-basis parametrization U_3 are the same as those in the U_1 parametrization with $\phi' \to -i\xi'$ and $\nu \to \eta + i\,\frac{\pi}{2}$. Therefore, for $U_3(\phi,\eta,\xi') = e^{-i\phi J_3} e^{-i\eta K_2} e^{-i\xi' K_1}$:

$$\vec{J}_3 = -i\partial_\phi \qquad \Lambda = \frac{1}{i\,\mathrm{ch}\,\eta}\left[(i\partial_{\xi'}) - (i\,\mathrm{sh}\,\eta)\,\partial_\phi\right]$$

$$\vec{K}_1 = -i\cos\phi\,\Lambda + i\sin\phi\,\partial_\eta$$

$$\vec{K}_\pm = e^{\pm i\phi}(-i\Lambda \pm \partial_\eta)$$

$$\vec{K}_2 = -i\sin\phi\,\Lambda - i\cos\phi\,\partial_\eta$$

$$\vec{J}^2 = \partial_\eta^2 + \mathrm{th}\,\eta\,\partial_\eta - \frac{1}{\mathrm{ch}^2\eta}\left[\partial_\phi^2 + (i\partial_{\xi'})^2 - 2(i\,\mathrm{sh}\,\eta)\,\partial_\phi(i\partial_{\xi'})\right]$$

$$= (z^2-1)\,\partial_z^2 + 2z\partial_z + \frac{1}{(z^2-1)}\left[\partial_\phi^2 + (i\partial_{\xi'})^2 - 2z\cdot\partial_\phi(i\partial_{\xi'})\right], \tag{C.15}$$

where now $z = i\mathrm{sh}\,\eta$.

Finally, for U_4 we apply the automorphism $(K_1,K_2,J_3) \to (iJ_3,K_2,iK_1)$ to the U_3 results (C.15) with the variable change $\phi \to -i\xi$, $\xi' \to -i\phi'$ to get

for $U_4(\xi,\eta,\phi') = e^{-i\xi K_1} e^{-i\eta K_2} e^{-i\phi' J_3}$:

$$\vec{K}_1 = -i\,\partial_\xi \qquad \Lambda = \frac{1}{i\,\mathrm{ch}\,\eta}\left[(-\partial_{\phi'}) - (i\,\mathrm{sh}\eta)(i\partial_\xi)\right]$$

$$\vec{J}_3 = -\mathrm{ch}\,\xi\,\Lambda - i\,\mathrm{sh}\,\xi\,\partial_\eta$$

$$\vec{K}_\pm = e^{\pm\xi}(-i\Lambda \pm \partial_\eta)$$

(Equation continued on next page)

$$\vec{J}^2 = (z^2-1)\,\partial_z^2 + 2z\partial_z + \frac{1}{(z^2-1)}\left[(i\partial_\xi)^2 + (\partial_{\phi'})^2 - 2z(i\partial_\xi)(-\partial_{\phi'})\right], \tag{C.16}$$

where again $z = i\,\mathrm{sh}\,\eta$.

Appendix D: The Casimiric Differential Equation and Explicit SU(1,1) Matrix Elements

In Appendix C we constructed realizations of the SU(1,1) Lie generators as differential operators on the group manifold according to

$$\vec{G}_i U(g) = -G_i U(g) \ ,$$

with G_i and $U(g)$ operators in a representation space, and $\vec{G}_i$ the differential generators in the parameters. In particular,

$$\vec{J}^2 U(g) = J^2 U(g) \ . \qquad \text{(D.1)}$$

Therefore in some basis $|j,a\rangle$ we have

$$\begin{aligned} \vec{J}^2 \langle j,a|U(g)|j,a'\rangle &= \langle j,a|\vec{J}^2 U(g)|j,a'\rangle \\ &= \langle j,a|J^2 U(g)|j,a'\rangle \\ &= \langle j,a|U(g)J^2|j,a'\rangle \ , \qquad \left[U(g),J^2\right] = 0 \\ &= j(j+1)\ \langle j,a|U(g)|j,a'\rangle \ . \qquad \text{(D.2)} \end{aligned}$$

so that the UIR matrix elements are eigenfunctions of the Casimiric differential operator. If we define $\mathcal{L}(j;\mu,\nu;z)$ as in (H.1), then application of $\vec{J}^2$ in the forms (C.10), (C.11), (C.15) and (C.16) to the matrix elements (B.9), (B.14) and (B.16) tells us, according

to (D.2), that

$$\mathcal{L}(j;m,m';\mathrm{ch}\,\nu)\ \langle j,m|e^{-i\nu K_2}|j,m'\rangle = 0, \qquad \text{(D.3)}$$

$$\mathcal{L}(j;\mu,\mu';\mathrm{ch}\,\nu)\ \langle j,\mu|e^{-i\nu K_2}|j,\mu'\rangle = 0, \qquad \text{(D.4)}$$

$$\mathcal{L}(j;m,\mu;i\,\mathrm{sh}\,\eta)\ \langle j,m|e^{-i\eta K_2}|j,\mu\rangle = 0, \qquad \text{(D.5)}$$

$$\mathcal{L}(j;\mu,-m;i\,\mathrm{sh}\,\eta)\ \langle j,\mu|e^{-i\eta K_2}|j,m\rangle = 0. \qquad \text{(D.6)}$$

Therefore, the matrix elements of all UIR's of SU(1,1) in the discrete, continuous and mixed bases are Legendre functions in the z-variable indicated. The only question that remains is: which Legendre functions, and what are the coefficients?

For the discrete-basis matrix elements we know that $m - m'$ = integer and $\langle j,m|j,m'\rangle = \delta_{m,m'}$. From (H.41) and (H.42),

$$\lim_{z\to 1} P^j_{mm'}(z) = \delta_{m,m'} \ , \qquad \lim_{z\to 1} Q^j_{mm'}(z) \sim (z-1)^{-|m-m'|/2} \ .$$

We regard this as evidence, if not proof, of the fact that all the discrete-basis UIR matrix-elements turn out to be, with a conventional phase choice,

$$\begin{aligned} \langle j,m|e^{-i\nu K_2}|j,m'\rangle &= (+i)^{m'-m}\sqrt{G^j_{m'm}}\ P^j_{mm'}(\mathrm{ch}\,\nu) \\ &= d^j_{mm'}(\mathrm{ch}\,\nu + i\varepsilon)\ , \quad \nu \geq 0. \qquad \text{(D.7)} \end{aligned}$$

As proof of this result, we take $K_2 = iJ_2$ in SL(2,C) and observe that (D.7) is exactly the analytic continuation of the SU(2) Wigner d-function onto the right-hand cut (see Fig. 6(c)); but see also Ref. 16 and references in footnote 31.

For the continuous-basis D_k^+ matrix elements, μ and μ' are both imaginary and $\langle j\mu|j\mu'\rangle = \delta(i\mu-i\mu')$. From (H.41) and (H.42),

$$\lim_{z\to 1} P^j_{\mu\mu'}(z) \sim (z-1)^{(\mu'-\mu)/2}, \quad \lim_{z\to 1} Q^j_{\mu\mu'}(z) = +\pi\delta(i\mu-i\mu') .$$

Again, this is suggestive of the result for the D_k^+ matrix element which is

$$\langle j,\mu|e^{-i\nu K_2}|j,\mu'\rangle = e^{i\frac{\pi}{2}(\mu-\mu')} \cdot \frac{1}{\pi} \cdot Q^j_{\mu\mu'}(\text{ch }\nu), \quad \nu \geqslant 0 . \tag{D.8}$$

This matrix element has been explicitly calculated by Pasupathy and Radhakrishnan[37] using the method of Mukunda and Radhakrishnan.[38]

From the work of Lindblad and Nagel,[39] it is possible to evaluate the basis transformation matrix $\langle j\mu|jm\rangle$ directly from the Lie algebra, and to conclude that the mixed-basis matrix element is a second-kind Legendre function. In a calculation based on Mukunda[3] and following the lines of footnote 34, we have found that, in a phase choice consistent with the continuous-basis matrix element, the D_k^+ mixed-basis matrix elements are:

$$\langle j,m|e^{-i\eta K_2}|j,\mu\rangle = A\cdot e^{+i\frac{\pi}{2}(m-\mu)}\, Q^j_{m,\mu}(+i\text{ sh }\eta),$$

$$\langle j,\mu|e^{-i\eta K_2}|j,m\rangle = A\cdot e^{+i\frac{\pi}{2}(m+\mu)}\, Q^j_{\mu,-m}(+i\text{ sh }\eta), \quad \eta \geqslant 0$$

where

$$A = \sqrt{\frac{2}{\pi}}\cdot\left[\Gamma(j+1+m)\Gamma(-j+m)\right]^{-\frac{1}{2}} . \tag{D.9}$$

These matrix elements may also be computed using the non-local (i.e., non-multiplier) construction of Mukunda and Radhakrishnan.[38] The mixed-basis matrix elements for the C_q series have been calculated by Kalnins;[40] see also CDM.[27]

We feel that all these matrix elements should be rigorously obtainable from the Casimiric differential equations and some boundary conditions without explicit construction of the representations, but we do not know how to do this.

Appendix E: Elaboration of $g = g_1 g_2$

Here we state in detail the relations implied by $g = g_1 g_2$ in $SU(2)$ and in the discrete, continuous, and mixed bases of $SU(1,1)$. In Section 5 the results are summarized and a relevant asymptotic limit taken. One may obtain equivalent parameter relations for $g = g_1 g_2$ in terms of half-angles by simply multiplying the $SL(2,C)$ matrices given in (A.7).

1. SU(2)

For each g in $g = g_1 g_2$ we use the parametrization and abbreviated notation,

$$g = e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\phi' J_3} \equiv \phi\,\theta\,\phi' .$$

Therefore,

$$\begin{aligned} g = g_1 g_2 \Rightarrow \phi\,\theta\,\phi' &= \phi_1\theta_1\phi_1' \cdot \phi_2\theta_2\phi_2' \\ \Rightarrow (\phi-\phi_1)\,\theta(\phi'-\phi_2') &= \theta_1(\phi_1'+\phi_2)\theta_2 \\ \Rightarrow \phi\,\theta\,\phi' &= \theta_1\omega\,\theta_2 . \end{aligned} \tag{E.1}$$

In the last line we have, without loss of generality, set $\phi_1 = \phi_2' = 0$ and defined $\omega = \phi_1' + \phi_2$. Applying (E.1) in $SO(3)$ to the z-like unit vector $(0,0,1)$ we find, using (A.8), the three equations

$$\begin{aligned} \sin\phi\,\sin\theta &= \sin\omega\,\sin\theta_2 \\ \cos\phi\,\sin\theta &= \cos\theta_1\,\sin\theta_2\cos\omega + \sin\theta_1\,\cos\theta_2 \\ \cos\theta &= \cos\theta_1\,\cos\theta_2 - \sin\theta_1\,\sin\theta_2\,\cos\omega. \end{aligned}$$

Six similar equations are obtained by substituting into these three the replacements suggested by

$$\begin{aligned} &\phi\,\theta\,\phi' = \theta_1\,\omega\,\theta_2 \\ \Rightarrow\ &\phi^{-1}\,\theta_1\,\omega = \theta\,\phi'\,\theta_2^{-1} \\ \Rightarrow\ &\omega\,\theta_2\,\phi'^{-1} = \theta_1^{-1}\phi\,\theta . \end{aligned}$$

The results are then summarized in an obvious notation,

$$\begin{aligned} C_\theta &= C_{\theta_1} C_{\theta_2} - S_{\theta_1} S_{\theta_2} C_\omega & S_\theta &= +\sqrt{1 - C_\theta^2} \\ C_\phi &= (C_{\theta_1} S_{\theta_2} C_\omega + S_{\theta_1} C_{\theta_2})/S_\theta & S_\phi &= S_{\theta_2} S_\omega / S_\theta . \end{aligned} \tag{E.2}$$

The equations for ϕ' are obtained from those for ϕ by $1 \leftrightarrow 2$.[41] By convention, we take $0 \leqslant \theta \leqslant \pi$ so that $\sin\theta \geqslant 0$.

2. SU(1,1): Discrete Basis

For each g in $g = g_1 g_2$ we use the parametrization

$$g = e^{-i\phi J_3} e^{-i\nu K_2} e^{-i\phi' J_3} \equiv \phi\,\nu\,\phi' .$$

Therefore,

$$g = g_1 \cdot g_2 \Rightarrow \phi\,\nu\,\phi' = \phi_1\nu_1\phi_1' \cdot \phi_2\nu_2\phi_2'$$

$$\Rightarrow (\phi - \phi_1)\nu(\phi' - \phi_2') = \nu_1(\phi_1' + \phi_2)\nu_2$$

$$\Rightarrow \phi\,\nu\,\phi' = \nu_1 \omega\, \nu_2 \quad , \qquad (E.3)$$

again removing redundant parameters. Since $K_2 = iJ_2$ in SL(2,C), the parameter relations are obtained from those of SU(2) given in (E.2) by the replacements

$$\theta \to i\nu \qquad \cos\theta \to \mathrm{ch}\,\nu \qquad \sin\theta \to i\,\mathrm{sh}\,\nu \quad ,$$

and the same for θ_1 and θ_2. Therefore we find,

$$\mathrm{ch}\,\nu = \mathrm{ch}\,\nu_1\ \mathrm{ch}\,\nu_2 + \mathrm{sh}\,\nu_1\ \mathrm{sh}\,\nu_2\ C_\omega . \qquad \mathrm{sh}\,\nu = +\sqrt{\mathrm{ch}^2\nu - 1}$$

$$C_\phi = (\mathrm{ch}\,\nu_1\ \mathrm{sh}\,\nu_2\ C_\omega + \mathrm{sh}\,\nu_1\ \mathrm{ch}\,\nu_2)/\mathrm{sh}\,\nu \qquad S_\phi = \mathrm{sh}\,\nu_2 S_\omega/\mathrm{sh}\nu .$$

$$(E.4)$$

with the expressions for ϕ' given again by $1 \leftrightarrow 2$. By convention, $\nu \geqslant 0$.

3. SU(1,1): Continuous Basis; the Semigroups $S_0^{\pm}$

For each g in $g = g_1 g_2$ we take

$$g = e^{-i\xi K_1} e^{-i\nu K_2} e^{-i\xi' K_1} \equiv \xi\,\nu\,\xi' \quad , \qquad (E.5)$$

so that

$$g = g_1 \cdot g_2 \Rightarrow \xi\,\nu\,\xi' = \xi_1\nu_1\xi_1' \cdot \xi_2\nu_2\xi_2'$$

$$\Rightarrow (\xi - \xi_1)\nu(\xi' - \xi_2') = \nu_1(\xi_1' + \xi_2)\nu_2$$

$$\Rightarrow \xi\,\nu\,\xi' = \nu_1\,\alpha\,\nu_2 \quad , \qquad (E.6)$$

where $\alpha = \xi_1' + \xi_2$, etc. From (A.2) we can turn K_1 into J_3 by

$$e^{-\frac{\pi}{2}K_2}\ K_1\ e^{\frac{\pi}{2}K_2} = -iJ_3 \qquad (E.7)$$

Therefore we rewrite (E.6) as

$$e^{-i\xi K_1} e^{-i\nu K_2} e^{-i\xi' K_1} = e^{-i\nu_1 K_2} e^{-i\alpha K_1} e^{-i\nu_2 K_2}$$

$$\Rightarrow e^{-i(-i\xi)J_3} e^{-i\nu K_2} e^{-i(-i\xi')J_3} = e^{-i\nu_1 K_2} e^{-i(-i\alpha)J_3} e^{-i\nu_2 K_2} .$$

But this is (E.3) with $\phi = -i\xi$, $\phi' = -i\xi'$, $\omega = -i\alpha$. Thus we translate (E.4) accordingly to get:

$$\operatorname{ch}\nu = \operatorname{ch}\nu_1 \operatorname{ch}\nu_2 + \operatorname{sh}\nu_1 \operatorname{sh}\nu_2 \operatorname{ch}\alpha \qquad (E.8)$$

$$\operatorname{ch}\xi = (\operatorname{ch}\nu_1 \operatorname{sh}\nu_2 \operatorname{ch}\alpha + \operatorname{sh}\nu_1 \operatorname{ch}\nu_2)/\operatorname{sh}\nu \qquad (E.9)$$

$$\operatorname{sh}\xi = \operatorname{sh}\nu_2 \operatorname{sh}\alpha/\operatorname{sh}\nu\ , \qquad (E.10)$$

with ξ' expressions given by $1 \to 2$.

An essential difference between the continuous-basis parametrization and those considered earlier is that not all of the SU(1,1) manifold is accessible to (E.5), e.g., the J_3 rotations are excluded. In a rough sense, only 1/5 of the SU(1,1) and 2/5 of the SO(2,1) manifold can be reached.[42] Therefore, if we define the sector of SU(1,1) accessible to (E.5) as S_o, it is not obvious that $g_1, g_2 \in S_o \Rightarrow g = g_1 g_2 \in S_o$. In fact, from (E.8) it is clear that if ν_1 and ν_2 have opposite sign, it is likely that $\operatorname{ch}\nu < 1$, $\Rightarrow \nu$ not real $\Rightarrow g \notin S_o$. In other words, S_o is not closed under group multiplication, although $g \in S_o \Rightarrow g^{-1} \in S_o$. (If S_o were closed it would be a non-trivial 3-parameter subgroup of SU(1,1), which is nonsense.)

On the other hand, if ν_1 and ν_2 have the same sign, $\operatorname{ch}\nu > 1$ and $g \in S_o$. Moreover, from (E.9) we see that ν has the same sign as ν_1 and ν_2. If we define S_o^+ as the half of S_o with $\nu > 0$, and S_o^- as the other half, then we have shown that S_o^+ is closed. However, S_o^+ is not a subgroup of SU(1,1) because, aside from the above remark, the inverses of the elements of S_o^+ all lie in S_o^-. An object such as S_o^+ is called a

semigroup, so S_o^+ and S_o^- are semisubgroups of SU(1,1).

4. SU(1,1): Mixed Basis

We take

$$g = e^{-i\xi K_1}\, e^{-i\nu K_2}\, e^{-i\xi' K_1} = \xi\,\nu\,\xi'$$

$$g_1 = e^{-i\xi_1 K_1}\, e^{-i\eta_1 K_2}\, e^{-i\phi_1' J_3} = \xi_1 \eta_1 \phi_1'$$

$$g_2 = e^{-i\phi_2 J_3}\, e^{-i\eta_2 K_2}\, e^{-i\xi_2' K_1} = \phi_2 \eta_2 \xi_2^1\ ,$$

so that g remains in the continuous-basis parametrization, but g_1 and g_2 are in mixed-basis form. Then,

$$g = g_1 g_2 \Rightarrow \xi\,\nu\,\xi' = \xi_1\eta_1\phi_1' \cdot \phi_2\eta_2\xi_2'$$

$$\Rightarrow (\xi - \xi_1)\nu(\xi' - \xi_2') = \eta_1(\phi_1' + \phi_2)\eta_2$$

$$\Rightarrow \xi\,\nu\,\xi' = \eta_1 \omega\, \eta_2\ ,$$

or

$$e^{-i\xi K_1}\, e^{-i\nu K_2}\, e^{-i\xi' K_1} = e^{-i\eta_1 K_2}\, e^{-i\omega J_3}\, e^{-i\eta_2 K_2}\ . \qquad (E.11)$$

Using (E.7), the right side of (E.11) becomes

$$e^{-i\left[\eta_1 - i\frac{\pi}{2}\right]K_2}\, e^{-i\left[i\omega\right]K_1}\, e^{-i\left[\eta_2 + i\frac{\pi}{2}\right]K_2}\ .$$

Therefore, (E.11) is the same as (E.6) with $\nu_1 = \eta_1 - i\,\frac{\pi}{2}$, $\nu_2 = \eta_2 + i\,\frac{\pi}{2}$, and $\alpha = i\omega$. Then we may convert (E.8)$\rightarrow$(E.10) according to

$$\begin{array}{lll} \mathrm{ch}\,\nu_1 \rightarrow -i\,\mathrm{sh}\,\eta_1 & \mathrm{ch}\,\nu_2 \rightarrow i\,\mathrm{sh}\,\eta_2 & \mathrm{sh}\,\alpha \rightarrow i\,S_\omega \\ \mathrm{sh}\,\nu_1 \rightarrow -i\,\mathrm{ch}\,\eta_1 & \mathrm{sh}\,\nu_2 \rightarrow i\,\mathrm{ch}\,\eta_2 & \mathrm{ch}\,\alpha \rightarrow C_\omega \end{array}$$

to find:

$$\mathrm{ch}\,\nu = \mathrm{sh}\,\eta_1 \mathrm{sh}\,\eta_2 + \mathrm{ch}\,\eta_1 \mathrm{ch}\,\eta_2\, C_\omega \tag{E.12}$$

$$\mathrm{sh}\,\xi = -(\mathrm{ch}\,\eta_2 S_\omega)/\mathrm{sh}\,\nu\,; \qquad \mathrm{sh}\,\xi' = +(\mathrm{ch}\,\eta_1 S_\omega)/\mathrm{sh}\,\nu \tag{E.13}$$

$$\mathrm{ch}\,\xi = (\mathrm{sh}\,\eta_1 \mathrm{ch}\,\eta_2 C_\omega + \mathrm{ch}\,\eta_1 \mathrm{sh}\,\eta_2)/\mathrm{sh}\,\nu\,; \quad \mathrm{ch}\,\xi' = (1 \leftrightarrow 2). \tag{E.14}$$

Although all of SU(1,1) is accessible to the mixed parametrizations g_1 and g_2, the product $g = g_1 g_2$ will not in general fall into the sector S_o defined above, in which case ξ, ν and ξ' are imaginary $\Rightarrow$ $g \in SU(2)$. For our purposes, we restrict to $\eta_1 \geqslant 0$, $\eta_2 \geqslant 0$ and $\cos\,\omega \geqslant 0$ in which case g ends up in S_o^+ as seen from (E.12) and (E.14) above.

5. Summary and Limit as $|z_1| \rightarrow \infty$.

The information described in the preceding sections can be summarized by the following redundant set of equations together with Table E.5:

$$z = z_1 z_2 + \sqrt{z_1^2 - 1}\,\sqrt{z_2^2 - 1}\,\cos\omega \tag{E.15}$$

$$z_2 = z\,z_1 - \sqrt{z^2 - 1}\,\sqrt{z_1^2 - 1}\,\cos\phi \tag{E.16}$$

$$\cos\phi = \left[z_2\sqrt{z_1^2 - 1} + z_1\sqrt{z_2^2 - 1}\,\cos\omega\right]/\sqrt{z^2 - 1} \tag{E.17}$$

$$\sin\phi = \sin\omega\,\sqrt{z_2^2 - 1}\,/\sqrt{z^2 - 1} \tag{E.18}$$

$$e^{\pm i\phi} = (z_2\sqrt{z_1^2 - 1} + z_1\sqrt{z_2^2 - 1}\,\mathrm{Cos}\,\omega \pm i\,\sin\omega\sqrt{z_2^2 - 1})/\sqrt{z^2 - 1}. \tag{E.19}$$

The expressions involving ϕ' are obtained from (E.16) through (E.19) by taking $\phi \rightarrow \phi'$ and $1 \leftrightarrow 2$.

Important asymptotic limits of (E.15) and (E.19) are:

$$|z_1| \rightarrow \infty : \quad z = z_1(z_2 + \sqrt{z_2^2 - 1}\,\cos\omega) \tag{E.20}$$

$$e^{\pm i\phi} = 1\,, \qquad e^{\pm i\phi'} = \left[\frac{\sqrt{z_2^2 - 1} + z_2\cos\omega \pm i\,\sin\omega}{z_2 + \sqrt{z_2^2 - 1}\,\cos\omega}\right] \tag{E.21}$$

$$e^{\pm\xi} = 1\,, \qquad e^{\pm\xi'} = \left[\frac{\sqrt{z_2^2 - 1} + z_2\mathrm{ch}\,\alpha \pm \mathrm{sh}\,\alpha}{z_2 + \sqrt{z_2^2 - 1}\,\mathrm{ch}\,\alpha}\right]. \tag{E.22}$$

TABLE E.5

	z_1	z_2	z	$\sqrt{z_1^2-1}$	$\sqrt{z_2^2-1}$	$\sqrt{z^2-1}$	ϕ	ϕ'	ω
1. SU(2)	C_{θ_1}	C_{θ_2}	C_θ	$i\,S_{\theta_1}$	$i\,S_{\theta_2}$	$i\,S_\theta$	ϕ	ϕ'	ω
2. discrete	$\mathrm{ch}\,\nu_1$	$\mathrm{ch}\,\nu_2$	$\mathrm{ch}\,\nu$	$\mathrm{sh}\,\nu_1$	$\mathrm{sh}\,\nu_2$	$\mathrm{sh}\,\nu$	ϕ	ϕ'	ω
3. continuous	$\mathrm{ch}\,\nu_1$	$\mathrm{ch}\,\nu_2$	$\mathrm{ch}\,\nu$	$\mathrm{sh}\,\nu_1$	$\mathrm{sh}\,\nu_2$	$\mathrm{sh}\,\nu$	$-i\xi$	$-i\,\xi'$	$-i\,\alpha$
4. mixed	$-i\mathrm{sh}\,\eta_1$	$i\mathrm{sh}\,\eta_2$	$\mathrm{ch}\,\nu$	$-i\mathrm{ch}\,\eta_1$	$i\mathrm{ch}\,\eta_2$	$\mathrm{sh}\,\nu$	$-i\xi$	$-i\xi'$	ω

Appendix F: The Regular Representations

The so-called regular representations are discussed in Chapter 1 of Vilenkin's excellent book;[9] we mention here only a few details relevant to Section V.

In a shift representation, the elements of a group G are represented by shift operators acting on a space L of functions which are in turn defined on a homogeneous space M. Thus,

$$T(g_1)\,f(x) \;=\; f(g_1^{-1}\,x), \tag{F.1}$$

where $g_1 \in G$, $f \in L$, $x \in M$. It is easy to show from (F.1) that

$$T(g_1)\,T(g_2) \;=\; T(g_1 g_2)\ .$$

For a Lie group, the operators $T(g_1)$ may be expressed in terms of the Lie generators, as e. g. in Eq. (5.6), and then these generators will be realized as differential operators in the variables of M.

It may be shown that any homogeneous space M is equivalent to G/H, the space of cosets of G with respect to some subgroup H. If we choose $H = \{1\}$, then $M = G$ and we have the "regular" representation,

$$T(g_1)\,f(g) \;=\; f(g_1^{-1}\,g)\ ; \tag{F.2}$$

where now the Lie generators are realized as differential operators of G itself, i.e., in the parameters of G. In fact, the generators of the regular representaion are exactly those generators constructed in Appendix C, as we now show.

First, in (F.2) we visualize $f(g)$ as a function of the matrix $U(p)$,

$$f(g(p)) = F\left[U(p)\right] ,$$

where, as in (C.1),

$$U(p) = e^{-ip_1 G_{i_1}} e^{-ip_2 G_{i_2}} \dots e^{-ip_n G_{i_n}} . \qquad (F.3)$$

We shall assume that (F.3) is symmetric in the sense that $G_{i_1} = G_{i_n}$, $G_{i_2} = G_{i_{n-1}}$, etc., and also that each of the generator matrices is either hermitian $G_i^\dagger = G_i$ or anti-hermitian $G_i^\dagger = -G_i$.

The notion of the derivative of a function of a matrix, which we need below, is easily shown to be

$$F\left[U(p) + \delta U\right] \approx F\left[U(p)\right] + \text{trace}\left[\delta U \cdot \nabla\right] F\left[U(p)\right] , \qquad (F.4)$$

where δU is a matrix of small parameters, and

$$\nabla_{ij} \equiv \frac{\partial}{\partial\, [U(p)]_{ji}} .$$

If we parametrize the operator $T(g(p))$ exactly as in (F.3) but with the operators $\vec{G}_i$ replacing the matrices G_i, we may compute the $\vec{G}_i$ by examining (F.2) near the identity using (F.4).

We find,

$$\vec{G}_i^{(p)} = -\text{trace}\left[G_i \cdot U(p) \cdot \nabla\right] . \qquad (F.5)$$

For example, in SU(1,1) this is

$$\vec{G}_i^{(\alpha,\beta)} = -\,\text{trace}\left[G_i \cdot \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \cdot \begin{pmatrix} \partial_\alpha & \partial_\beta \\ \partial_{\overline{\beta}} & \partial_{\overline{\alpha}} \end{pmatrix}\right] .$$

Applying (F.5) to the matrix $U(p)$ we find that

$$\vec{G}_i^{(p)}\, U(p) = -G_i U(p) ,$$

which shows that the $\vec{G}_i^{(p)}$ are the same as the generators constructed in Appendix C.

The representation (F.2) is the left-regular representation. One may also construct a right-regular representation on G according to

$${}_R T(g_1)\, f(g) = f(g\, g_1) ,$$

from which it may be shown that the right-shift generators ${}_R\vec{G}_i^{(p)}$ are given by

$${}_R\vec{G}_i^{(p)} = +\,\text{trace}\left[U(p) \cdot G_i \cdot \nabla\right]$$

$${}_R\vec{G}_i^{(p)} U(p) = +\, U(p) \cdot G_i .$$

The right-shift differential generators also satisfy the Lie algebra of G (see C.3). With the stipulations made above for the form of U(p), the left- and right-shift generators are related by

$$ {}_R\vec{G}_i(p) = \mp\left[\vec{G}_i(p^\dagger)\right]^* , \qquad (F.6)$$

with $\mp$ depending on whether $G_i^\dagger = \pm G_i$, and

$$p = (p_1, p_2, \ldots p_{n-1}, p_n)$$

$$p^\dagger \equiv (\mp p_n, \mp p_{n-1}, \ldots \mp p_2, \mp p_1) \qquad (F.7)$$

where the signs in (F.7) are $\mp$ depending on the hermiticity of the generator associated with each parameter in (F.3), $G_i^\dagger \equiv \pm G_i$.

From (F.6), left- and right-shift Casimir operators are related by

$${}_R\vec{\mathcal{C}}^{(p)} = \left[\vec{\mathcal{C}}^{(p^\dagger)}\right]^*$$

which, in our discrete basis parametrization of SU(1,1) becomes

$${}_R\vec{J}^2(\phi,\nu,\phi') = \left[\vec{J}^2(-\phi',\nu,-\phi)\right]^* .$$

From (C.10), the terms in $\vec{J}^2$ are all real, and $\vec{J}^2$ is symmetric under $\phi \leftrightarrow -\phi'$, so the Casimir (and Laplace operator of Section V) is the same in terms of left- or right-shift generators.

Appendix G: Expansion Theorems

In this section we derive the standard Peter-Weyl theorems for SU(1,1) and SU(2) using the Green's function method. In addition, we give a simplified expansion theorem for functions defined on $S_o^+ \subset SU(1,1)$.[43]

1. The Green's function method.

If L is a self-adjoint differential operator, then in the Hilbert space spanned by its eigenfunctions we have "Cauchy's formula,"[44]

$$-\frac{1}{2\pi i}\lim_{R\to\infty}\oint_{|\lambda|=R} d\lambda\,(L-\lambda)^{-1} = 1 . \qquad (G.1)$$

Defining the Green's function $g(x|y;\lambda)$ by

$$(L-\lambda)\,g(x|y;\lambda) = \delta(x-y) , \qquad (G.2)$$

application of the operator Eq. (G.1) to (G.2) shows that

$$\delta(x-y) = -\frac{1}{2\pi i}\oint_{|\lambda|=\infty} d\lambda\, g(x|y;\lambda). \qquad (G.3)$$

To be specific, we take

$$L = (1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{1}{(1-z^2)}\left[\mu^2+\nu^2-2\mu\nu z\right] , \qquad (G.4)$$

$$\lambda = -j(j+1) . \qquad (G.5)$$

Taking the solution of (G.5) ,

$$j = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda} = -\frac{1}{2} \mp i \sqrt{\lambda - \frac{1}{4}}, \quad \mathrm{Im}\,\lambda \gtrless 0,$$

it is easy to show that (G.3) becomes

$$\delta(x - y) = \frac{1}{2\pi i} \int_C dj(2j + 1)\, g(x|y;j) , \qquad (G.6)$$

where the contour C runs from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$, circumscribing the right-half j-plane at $|j| = \infty$.

The Green's function may be written as

$$g(x|y;j) = -u_1(x_<)u_2(x_>)/c(j)$$

with

$$c(j) = (z^2 - 1)\ W\{u_1,u_2\} ,$$

where u_1 and u_2 are solutions of the Legendre equation,

$$(L - \lambda)\, W(z) = \mathcal{L}(j;\mu,\nu;z)\ W(z) = 0,$$

with u_1 matching a boundary condition at the left end of an interval, u_2 at the right. For the interval $(1,\infty)$ we choose $P^j_{\mu\nu}$ for u_1 and the $z = \infty$ "limit point" solution $\mathcal{Q}^j_{\nu\mu}$ for u_2. From (H.11)

we have $c(j) = -1$ and

$$g(x|y;j) = + P^j_{\mu\nu}(x_<)\, \mathcal{Q}^j_{\nu\mu}(x_>) ,$$

so a completeness relation for functions on the interval $(1,\infty)$ is, from (G.6) ,

$$\delta(x - y) = \frac{1}{2\pi i} \int_C dj(2j + 1)\, P^j_{\mu\nu}(x)\, \mathcal{Q}^j_{\nu\mu}(y) \qquad (G.7)$$

with C as described above.

2. Discrete-Basis Expansion Theorem for SU(1,1)

As our starting point we take the above result,

$$\delta(z_1-z_2) = \frac{1}{2\pi i} \int_C dj(2j+1)\ P^j_{mm'}(z_1)\, \mathcal{Q}^j_{m'm}(z_2), \quad z_1, z_2 \geq 1. \qquad (G.8)$$

As the contour C is shifted left to $\mathrm{Re}(j) = -\frac{1}{2}$, it wraps a finite number of poles of the integrand so that (G.8) becomes

$$\delta(z_1-z_2) = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj(2j+1)\, P^j_{mm'}(z_1)\, \mathcal{Q}^j_{m'm}(z_2) \qquad (G.9)$$

$$+ \frac{1}{2} \sum_{j=\varepsilon}^{J} (2j+1)\,(-1)^{m-m'}\, P^j_{mm'}(z_1)\, P^j_{m'm}(z_2),$$

where $\varepsilon = 0$ or $\frac{1}{2}$ depending on the integrality of (m,m'), and

$$J \;=\; \max(|m|,|m'|) - |m-m'| - 1. \qquad (G.10)$$

The location of the above-mentioned poles is shown in Fig. 8(h), and the pole residues are given in (H.53). Note that there is no pole at $j = -\frac{1}{2}$ due to the factor $(2j+1)$. Since $P^j = P^{-j-1}$, the integration in (G.9) senses only the odd part of $\mathcal{Q}^j$ so we replace, via (H.33),

$$\mathcal{Q}^j_{m'm} \to \frac{1}{2}\left[\mathcal{Q}^j_{m'm} - \mathcal{Q}^{-j-1}_{m'm}\right] = \frac{\pi}{2}\cot\pi(j+\varepsilon)\,(-1)^{m-m'}\,P^j_{m'm}$$

to get

$$\delta(z_1-z_2) = \left\{\frac{1}{4i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dj}{\tan\pi(j+\varepsilon)} + \frac{1}{2}\sum_{j=\varepsilon}^{J}\right\}(2j+1)(-1)^{m-m'}$$

$$\cdot\, P^j_{mm'}(z_1)\,P^j_{m'm}(z_2)\;. \qquad (G.11)$$

Multiplying both sides by $e^{-im(\phi_1-\phi_2)}\; e^{-im'(\phi_1'-\phi_2')}$, summing on m and m', and using the order interchange suggested by Fig. 8(h),

$$\sum_{m,m'=-\infty}^{\infty}\;\sum_{j=\varepsilon}^{J} = \sum_{j=\varepsilon}^{\infty}\left[\sum_{m,m'=j+1}^{+\infty} + \sum_{m,m'=-j-1}^{-\infty}\right],$$

(G.11) may be rewritten as

$$\delta(g_1-g_2) = \frac{1}{4i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dj(2j+1)}{\tan\pi(j+\varepsilon)}\cdot\sum_{m,m'=-\infty}^{\infty}(-1)^{m-m'}\,P^j_{mm'}(g_1)P^j_{-m,-m'}(g_2)$$

(Equation continued on next page)

$$+\,\frac{1}{2}\sum_{j=\varepsilon}^{\infty}(2j+1)\left(\sum_{m,m'=+j+1}^{\infty} + \sum_{m,m'=-j-1}^{-\infty}\right)(-1)^{m-m'}\,P^j_{mm'}(g_1)\,P^j_{-m,-m'}(g_2),$$

$$(G.12)$$

where

$$P^j_{mm'}(g) \equiv e^{-im\phi}\,P^j_{mm'}(z = \mathrm{ch}\,\nu)\,e^{-im'\phi'}$$

and

$$\delta(g_1-g_2) \equiv 2\pi\delta(\phi_1-\phi_2)\cdot\delta(z_1-z_2)\cdot 4\pi\,\delta(\phi_1'-\phi_2')\;. \qquad (G.13)$$

Since $g_2 = (\phi_2, \nu_2, \phi_2')$, we have $g_2^{-1} = (\pi-\phi_2', \nu_2, \pi-\phi_2)$ and

$$P^j_{-m-m'}(g_2) = (-1)^{m-m'}\,P^j_{m'm}(g_2^{-1})\;,$$

from which we obtain the group-theoretic form of the completeness relation,

$$\delta(g_1-g_2) = \frac{1}{4i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dj(2j+1)}{\tan\pi(j+\varepsilon)}\;\mathrm{trace}^{C^{\varepsilon}_{q}}\left[P^j(g_1)P^j(g_2^{-1})\right]$$

(Equation continued on next page)

$$+ \frac{1}{2} \sum_{j=\varepsilon}^{\infty} (2j+1) \sum_{\sigma=\pm} \operatorname{trace}^{D_\kappa^\sigma} P^j(g_1) P^j(g_2^{-1}) ,$$

(G.14)

where $\varepsilon = 0$ or $\frac{1}{2}$ and the traces are in the Hilbert spaces labelled by the superscripts, see Table B.1.

Equation (G.14) is the Peter-Weyl theorem for SU(1,1).[45] Symbolically it reads

$$\delta(g_1-g_2) = S_j \operatorname{trace}^j \left[P^j(g_1) P^j(g_2^{-1})\right] ,$$

so the expansion theorem for functions square-integrable on SU(1,1) is

$$f(g_1) = S_j \operatorname{trace}^j \left[P^j(g_1) f^j\right] \qquad \text{(G.15a)}$$

$$f^j_{mm'} = \int_G dg_2\, f(g_2)\, P^j_{mm'}(g_2^{-1}) , \qquad \text{(G.15b)}$$

where dg is the invariant measure[46]

$$\int_G dg = \int_0^{2\pi} \frac{d\phi}{2\pi} \cdot \int_{-1}^{1} dz \cdot \int_{-2\pi}^{2\pi} \frac{d\phi'}{4\pi} .$$

Had we simply terminated the analysis back at Eq. (G.8) and let C be a vertical contour running up to the right of J given in (G.10), we would have obtained the expansion theorem[47]

$$f(z) = \frac{1}{2\pi i} \int_C dj(2j+1)\, P^j_{mm'}(z)\, f^j_{mm'}$$

$$f^j_{mm'} = \int_1^\infty dz\, f(z)\, \mathcal{Q}^j_{mm'}(z) ,$$

which is capable of handling functions $f(z)$ which are non-square-integrable in the usual sense, e.g., $f(z) = z^a$ with $\mathrm{Re}(a) > -\frac{1}{2}$ (see (H.39)). The above form cannot, however, be extended to a "full" expansion theorem on SU(1,1), like (G.15), without generating $D_k^\pm$ terms; but see (G.17) below.

3. Continuous-Basis Expansion Theorem for S_o^+

Again, we start with (G.7),

$$\delta(z_1-z_2) = \frac{1}{2\pi i} \int_C dj(2j+1)\, P^j_{\mu\mu'}(z_1)\, \mathcal{Q}^j_{\mu'\mu}(z_2).$$

Since μ and μ' are both imaginary (see Appendix B.3), the poles of the integrand lie entirely in the left half j-plane so that C may be taken to be any contour running up vertically to the right of $\mathrm{Re}(j) = -1$. Multiplying both sides by $e^{-\mu(\xi_1-\xi_2)}\, e^{-\mu'(\xi_1'-\xi_2')}$ and applying $(-i)^2 \int d\mu \cdot \int d\mu'$ we find

$$\delta(g_1-g_2) = (-i)^2 \int_{-i\infty}^{i\infty} d\mu \int_{-i\infty}^{i\infty} d\mu' \frac{1}{2\pi i} \int_C dj(2j+1)\, P^j_{-\mu-\mu'}(g_1)\, \mathcal{Q}^j_{\mu\mu'}(g_2) ,$$

(G.16)

where $P(g)$ and $\mathcal{Q}(g)$ are now functions defined on the semigroup S_o^+ discussed in Appendix E.3, e.g.,

$$\mathcal{D}^j_{\mu\mu'}(g_2) \equiv e^{-\mu\xi_2}\, \mathcal{D}^j_{\mu\mu'}(z_2)\, e^{-\mu'\xi_2'} \quad ,$$

and

$$\delta(g_1-g_2) = 2\pi\, \delta(\xi_1-\xi_2)\, \delta(z_1-z_2)\, 2\pi\, \delta(\xi_1'-\xi_2') \; .$$

Therefore, an expansion theorem for functions on S_o^+ is, from (G.16),

$$f(g_1) = \frac{1}{2\pi i} \int_C dj(2j+1)\,(-i)^2 \int_{-i\infty}^{i\infty} d\mu \int_{-i\infty}^{i\infty} d\mu'\, P^j_{-\mu-\mu'}(g_1)\, f^j_{\mu\mu'} \tag{G.17a}$$

$$f^j_{\mu\mu'} = \int_{S_o^+} dg_2\, f(g_2)\, \mathcal{D}^j_{\mu\mu'}(g_2) \tag{G.17b}$$

with[46]

$$\int_{S_o^+} dg = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \cdot \int_1^{\infty} dz \cdot \int_{-\infty}^{\infty} \frac{d\xi'}{2\pi} \; .$$

Once this expansion theorem has been established with imaginary helicity contours, the three contours appearing in (G.17a) may -- with care -- be shifted in their respective planes.

Although only useful for expanding functions defined on S_o^+, (G.17) is much simpler than the "full" continuous-basis expansion theorem obtained from (G.14) by replacing the helicity sums with helicity integrals, i.e., changing bases (see Mukunda[42], section 2; PR[37], section 3). Our expansion theorem has no discrete

series contributions, nor does it have the complications involving the bivalued multiplicity index associated with the continuous series UIR's in the continuous basis. In fact, one may show, by Sommerfeld-Watson-transforming the discrete series terms in PR equation (3.1) and by executing the multiplicity sums, that the full result reduces, for functions on S_o^+, to the expansion theorem (G.17) above.

4. Completeness Relation for SU(2)

On the interval (-1,1) we take $u_1 = P^j_{m'm}$ and $u_2 = \frac{1}{2}\left[Q^j_{mm'} + Q^{-j\mp 1}_{mm'}\right]$ so that (G.6) becomes

$$\delta(z_1-z_2) = -\frac{1}{2\pi i} \int_C dj(2j+1)\, \frac{1}{2}\left[Q^j_{mm'}(z_1) + Q^{-j\mp 1}_{mm'}(z_1)\right] P^j_{m'm}(z_2) \; .$$

Using (H.53) to evaluate the pole residues, and noting that the "background integral" at $\mathrm{Re}(j) = -\frac{1}{2}$ vanishes by the same symmetry noted above, we find

$$\delta(z_1-z_2) = \frac{1}{2} \sum_{j=\max(|m|,|m'|)}^{\infty} (2j+1)\, (-1)^{m-m'}\, P^j_{mm'}(z_1)\, P^j_{m'm}(z_2). \tag{G.18}$$

Again applying exponentials, summing on m and m', then changing order of summation, we obtain the usual SU(2) completeness relation

$$\delta(g_1-g_2) = \frac{1}{2} \sum_{j=\varepsilon}^{\infty} (2j+1)\, \mathrm{trace}^j \left[P^j(g_1)\, P^j(g_2^{-1})\right] \quad , \tag{G.19}$$

with $\delta(g_1-g_2)$ as given in (G.13), $z_i = \cos\theta_i$, and $\mathrm{tr}^j(A) = \sum_{m=-j}^{j} A_{mm}$.

Appendix H: Generalized Legendre Functions

In this appendix we give the definitions and selected properties of the generalized Legendre functions. The notation and nearly all the formulas below are due to Azimov,[1] though some are taken from Andrews and Gunson.[4] We have not included information on the recurrence relations or integrals (over z) of products of Legendre functions. In Eq. (H.59) we give the connection to the first-kind function used by Vilenkin.[9] Our standard reference for the hypergeometric functions is Bateman volume 1, referred to by the letter B.[10]

1. Differential Equation

The first- and second-kind (generalized) Legendre functions defined below are independent solutions of the differential equation

$$\mathcal{L}(j;\mu,\nu;z)\, w(z) = 0,$$

where

$$\mathcal{L}(j;\mu,\nu;z) \equiv (1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \left[j(j+1) - \frac{(\mu^2+\nu^2-2z\mu\nu)}{(1-z^2)}\right]. \tag{H.1}$$

If either $\nu = 0$ or $\mu = 0$, (H.1) is Legendre's differential equation B3.2 (1).

2. First-Kind Legendre Function P:

$$P^j_{\mu\nu}(z) \equiv \left(\frac{z-1}{2}\right)^{\frac{1}{2}(\nu-\mu)} \left(\frac{z+1}{2}\right)^{\frac{1}{2}(\nu+\mu)} F\left(j+1+\nu,-j+\nu;\nu-\mu+1;\frac{1-z}{2}\right)\Big/\Gamma(\nu-\mu+1). \tag{H.2}$$

$P^j_{\mu\nu}(z)$ is analytic in j,μ,ν and z, with zeros described in Section 15, and with cuts in z described in Section 5. From the linear shift formula B2.9 (4),

$$F(a,b;c;z) = (1-z)^{-b} F(c-a,b;c;\, z/(z-1)), \tag{H.3}$$

an alternative form for $P^j_{\mu\nu}(z)$ is found to be

$$P^j_{\mu\nu}(z) = \left(\frac{z+1}{2}\right)^j \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}(\nu-\mu)} F\left(-j-\mu,-j+\nu;\nu-\mu+1;\frac{z-1}{z+1}\right)\Big/\Gamma(\nu-\mu+1). \tag{H.4}$$

When $\nu = 0$, (H.2) reduces to entry (14) in Bateman's table B3.2:

$$P^j_{\mu 0}(z) = P^\mu_j(z) \qquad P^j_{00}(z) = P_j(z). \tag{H.5}$$

The most elementary properties of $P^j_{\mu\nu}$ are[48]

$$P^j_{-\nu,-\mu} = P^j_{\mu\nu} \qquad P^{-j-1}_{\mu\nu} = P^j_{\mu\nu}.$$

3. Second-Kind Legendre Function $\not{Q}$:

$$\not{Q}^j_{\mu\nu}(z) \equiv \frac{1}{2}\,\Gamma(j+1+\mu)\Gamma(j+1-\nu)\,\left(\frac{z-1}{2}\right)^{\frac{1}{2}(\mu-\nu)}\left(\frac{z+1}{2}\right)^{\frac{1}{2}(\mu+\nu)}\left(\frac{z-1}{2}\right)^{-j-1-\mu}$$

$$\times\; F(j+1+\mu, j+1+\nu; 2j+2; \frac{2}{1-z})\,/\Gamma(2j+2)\;. \qquad (H.6)$$

$\not{Q}^j_{\mu\nu}(z)$ is analytic in j,μ,ν and z except for the poles present in $\Gamma(j+1+\mu)\Gamma(j+1-\nu)$ and the cuts in z described in Section 5 below. The slash is introduced to avoid repetitious writing of the phase factor attached to the "true" Legendre functions,

$$\not{Q}^j_{\mu\nu} = e^{-i\pi(\mu-\nu)}\;Q^j_{\mu\nu}\;. \qquad (H.7)$$

When $\nu = 0$, (H.6) reduces to entry (37) in Bateman's table B 3.2:

$$\not{Q}^j_{\mu o}(z) = \not{Q}^\mu_j(z) \equiv e^{-i\pi\mu}\,Q^\mu_j(z)\;,$$

$$\not{Q}^j_{oo}(z) = Q^j_{oo}(z) = Q_j(z)\;. \qquad (H.8)$$

The elementary symmetry property is,

$$\not{Q}^j_{-\nu-\mu} = \not{Q}^j_{\mu\nu} \qquad \text{or} \qquad Q^j_{-\nu-\mu} = Q^j_{\mu\nu}\;.$$

4. Wronskians

From the asymptotic behaviors in z given below, one may quickly compute the following wronskians, $W(a,b) = ab' - ba'$:

$$(1-z^2)\,W(P^j_{\mu\nu}, P^j_{\nu\mu}) = \frac{2}{\pi}\,\sin\pi(\nu-\mu), \qquad (H.9)$$

$$(1-z^2)\,W(\not{Q}^j_{\mu\nu}, \not{Q}^{-j-1}_{\nu\mu}) = \frac{\pi}{2}\,S^j_{\nu\mu} \qquad (H.10)$$

$$(1-z^2)\,W(P^j_{\mu\nu}, \not{Q}^j_{\nu\mu}) = 1\;. \qquad (H.11)$$

This shows that P and $\not{Q}$ are always independent solutions of (H.1), whereas other pairs are not always so.

5. The z-plane Cut Structure

Throughout this paper we adhere to the convention that $f(z) = (z-1)^\alpha$ means a function cut from $z = 1$ to $z = -\infty$ with principal branch determined by $|\arg(z-1)| < \pi$, and $f(z) > 0$ when $z > 1$ and α real. In other words, $f(z) = \exp[\alpha\ln(z-1)]$ with $\ln(z-1)$ cut in the "usual" way. For z on the principal sheet, $\arg(1-z) = \arg(z-1) \mp i\pi$ for $\mathrm{Im}(z) \gtrless 0$, so that $(1-z)^\alpha = e^{\mp i\pi\alpha}(z-1)^\alpha$. It follows that $(1-z)^\alpha$ is a function cut from $z = 1$ to $z = +\infty$, but we continue to define the principal sheet by $|\arg(z-1)| < \pi$. These remarks are illustrated in Fig. 5.

With this in mind, we draw the cuts in z for $P^j_{\mu\nu}(z)$ and $\not{Q}^j_{\mu\nu}(z)$ as shown in Fig. 6(a) and (b), where we have slightly deformed the cuts for clarity. The peculiar way of cutting $\not{Q}^j_{\mu\nu}(z)$ from $z = 1$ is connected with the definition of $\tilde{\not{Q}}^j_{\mu\nu}(z)$ below and the resultant simplicity of the discontinuity formula (H.38).

6. The Functions $\tilde{P}$ and $\tilde{\mathcal{Q}}$

We define these functions by:

$$\left.\begin{aligned}\tilde{P}^j_{\mu\nu}(z) &\equiv P^j_{\mu\nu}(z)\cdot e^{\pm i\pi(\mu-\nu)/2} \\ \tilde{\mathcal{Q}}^j_{\mu\nu}(z) &\equiv \mathcal{Q}^j_{\mu\nu}(z)\cdot e^{\mp i\pi(\mu-\nu)/2}\end{aligned}\right\} \qquad \mathrm{Im}\, z \gtrless 0 \qquad (H.12)$$

$\tilde{P}$ and $\tilde{\mathcal{Q}}$ are simply new versions of P and $\mathcal{Q}$ with minus signs inserted into the first $(\frac{z-1}{2})$ factors appearing in (H.2) and (H.6), which is to say, the corresponding cuts are taken to the right instead of the left, as shown in Fig. 6 (c) and (d). For $\tilde{P}$, whis leaves the interval $(-1,1)$ uncut.

7. The Functions d and e

We define these in terms of the twiddled functions above:

$$d^j_{\mu\nu}(z) \equiv \sqrt{G^j_{\mu\nu}}\cdot \tilde{P}^j_{\nu\mu}(z) \ ,$$

$$e^j_{\mu\nu}(z) \equiv \sqrt{G^j_{\mu\nu}}\cdot \tilde{\mathcal{Q}}^j_{\nu\mu}(z) \ , \qquad (H.13)$$

where

$$G^j_{\mu\nu} = \frac{\Gamma(j+1+\mu)\Gamma(j+1-\nu)}{\Gamma(j+1-\mu)\Gamma(j+1+\nu)} \ .$$

These definitions coincide precisely with the functions used by Andrews and Gunson [4] for $(\mu,\nu) = (m,m')$ in all four of their regions (see (H.32) below and also Section 15). Clearly, d and e have the same z-plane structure as $\tilde{P}$ and $\tilde{\mathcal{Q}}$.

The advantages of the d and e functions are :

(1) when (m,m') are both integers or both half-integers, the "switch" symmetry relations are very simple (compare to (H.32) and (H.23)),

$$d^j_{m'm} = (-1)^{m'-m}\, d^j_{mm'} \qquad (= d^j_{-m,-m'})$$

$$e^j_{m'm} = (-1)^{m'-m}\, e^j_{mm'} \qquad (= e^j_{-m,-m'}); \qquad (H.14)$$

(2) the d functions <u>are</u> the SU(2) and SU(1,1) reduced matrix elements (see (D.7));

(3) The z-plane structure is that of $\tilde{P}$ and $\tilde{\mathcal{Q}}$ so that, from (H.38),

$$e^j_{mm'}(x+i\varepsilon) - e^j_{mm'}(x-i\varepsilon) = -i\pi\, d^j_{mm'}(x) \ , \quad -1 < x < 1; \qquad (H.15)$$

(4) the location of singularities in the helicity lattice is symmetric (see Section 15 below);

(5) workers in Regge theory are familiar with the d and e functions.

The principle disadvantage of the d and e functions is the price paid to get (H.14), namely, the appearance of square-roots of ratios of gamma functions. When μ and ν are arbitrary complex numbers, $(G^j_{\mu\nu})^{\frac{1}{2}}$ has a distinctly unpleasant cut structure in the j-plane, although it at least truncates when $(\mu,\nu) = (m,m')$, as shown in Fig. 2 of AG . We point out that square-roots of gamma functions do not appear in any of the relations involving P and $\mathcal{Q}$, and in general, since we are very interested in complex μ and ν, we shall avoid using the d and e functions, despite their advantages noted above.

8. Auxiliary Functions.

In deriving and simply stating the various properties of the Legendre functions which follow, much effort is saved by use of the following notation:

$$G^j_\mu \equiv \frac{\Gamma(j+1+\mu)}{\Gamma(j+1-\mu)} \tag{H.16}$$

$$G^j_{\mu\nu} \equiv \frac{\Gamma(j+1+\mu)\Gamma(j+1-\nu)}{\Gamma(j+1-\mu)\Gamma(j+1+\nu)} \tag{H.17}$$

$$S^j_{\mu\nu} \equiv \frac{\sin \pi(2j)}{\sin \pi(j-\mu)\sin \pi(j+\nu)} \tag{H.18}$$

$$L^j_{\mu\nu} \equiv \frac{\sin \pi(j+\mu)\sin \pi(j-\nu)}{\sin \pi(j-\mu)\sin \pi(j+\nu)} \tag{H.19}$$

These auxiliary functions have the following symmetries and inter-relations:

$$L^{-j-1}_{\mu\nu} = L^j_{\nu\mu} \qquad\qquad L^j_{\mu\nu}\, L^j_{\nu\mu} = 1$$

$$S^{-j-1}_{\mu\nu} = -S^j_{\nu\mu} \qquad\qquad S^j_{\mu\nu}\, L^j_{\nu\mu} = S^j_{\nu\mu}$$

$$G^{-j-1}_{\mu\nu} = L^j_{\mu\nu}\, G^j_{\mu\nu} \qquad\qquad S^j_{\mu\nu}\, G^j_{\mu\nu} = -S^{-j-1}_{\mu\nu}\, G^{-j-1}_{\mu\nu}$$

$$G^j_{\mu\nu}\, G^j_{\nu\mu} = 1 \qquad\qquad (L^j_{\mu\nu} - 1) = \sin \pi(\mu-\nu)\, S^j_{\mu\nu} .$$

When $(\mu,\nu) = (m,m')$ = both integers or both half-integers (j still general complex) we find

$$S^j_{mm'} = S^j_{m'm} = 2(-1)^{m-m'} \cot \pi(j+\varepsilon) = 2(-1)^{m+m'} \begin{cases} \cot \pi j, & \varepsilon = 0 \\ \tan \pi j, & \varepsilon = \frac{1}{2} \end{cases} \tag{H.20}$$

where ε = integrality of (m,m'). Moreover,

$$L^j_{mm'} = 1 \qquad\qquad G^{-j-1}_{mm'} = G^j_{mm'} .$$

9. Basic Properties of the Legendre Functions

From the definitions of P and $\mathcal{Q}$ and the linear shift B 2.9 (2),

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a,c-b;c;z), \tag{H.21}$$

we have the "switch-and-negate" relations

$$P^j_{-\nu-\mu} = P^j_{\mu\nu} \qquad\qquad \mathcal{Q}^j_{-\nu-\mu} = \mathcal{Q}^j_{\mu\nu}. \tag{H.22}$$

The "switch" relation for $\mathcal{Q}$, (H.23) below, is obvious from the definition of $\mathcal{Q}$. The corresponding relation for P derives from the famous connection formula relating $F(\ldots;z)$ to $F(\ldots;z^{-1})$, B 2.9 (34). Thus,

$$\mathcal{Q}^j_{\mu\nu} = G^j_{\mu\nu}\, \mathcal{Q}^j_{\nu\mu} \tag{H.23}$$

$$P^j_{\mu\nu} = G^j_{\mu\nu}\, P^j_{\nu\mu} + \frac{2}{\pi} \sin \pi(\mu-\nu)\, \mathcal{Q}^j_{\mu\nu} . \tag{H.24}$$

The symmetry under $j \to -j-1$ of P^j is apparent from (H.2). The corresponding relation for $\mathcal{Q}$ then follows from (H.24) and the relations given in Section 8:

$$P^j_{\mu\nu} = P^{-j-1}_{\mu\nu} \qquad \text{(H.25)}$$

$$\mathcal{Q}^j_{\mu\nu} = \mathcal{Q}^{-j-1}_{\mu\nu} + \frac{\pi}{2} S^j_{\mu\nu} G^j_{\mu\nu} P^j_{\nu\mu} . \qquad \text{(H.26)}$$

An alternative form of (H.26), explicitly displaying the symmetry of P in j, is

$$P^j_{\mu\nu} = \frac{2}{\pi} \left[\frac{\mathcal{Q}^j_{\mu\nu}}{S^j_{\nu\mu}} + \frac{\mathcal{Q}^{-j-1}_{\mu\nu}}{S^{-j-1}_{\nu\mu}} \right] . \qquad \text{(H.27)}$$

Next, from the linear shift (H.3) we find a simple relation between $Q(-z)$ and $Q(z)$. Combined with (H.26), this produces the second equation following:

$$\left.\begin{aligned} G^j_{-\nu} \mathcal{Q}^j_{\mu,-\nu}(-z) &= e^{\pm i\pi(j+1)} \mathcal{Q}^j_{\mu\nu}(z) = G^j_{\mu} \mathcal{Q}^j_{-\mu,\nu}(-z) && \text{(H.28)} \\ G^j_{-\nu} P^j_{\mu,-\nu}(-z) &= e^{\mp i\pi j} P^j_{\mu\nu}(z) - \frac{2}{\pi} e^{\pm i\pi\nu} \sin \pi(j+\mu) \\ &\qquad \times \mathcal{Q}^j_{\mu\nu}(z) && \text{(H.29)} \end{aligned}\right\} \operatorname{Im} z \gtrless 0 .$$

Converting (H.24) to $\tilde{P}$ and $\tilde{\mathcal{Q}}$ yields (H.30) below, which, when used in (H.29) to eliminate $\mathcal{Q}$, gives (H.31):

$$e^{\mp i\pi(\mu-\nu)} \tilde{P}^j_{\mu\nu}(z) = G^j_{\mu\nu} \tilde{P}^j_{\nu\mu}(z) + \frac{2}{\pi} \sin \pi(\mu-\nu) \tilde{\mathcal{Q}}^j_{\mu\nu}(z) \qquad \text{(H.30)}$$

$$\frac{1}{\pi} \sin \pi(\mu-\nu) \tilde{P}^j_{\mu,-\nu}(-z) = \left[\frac{\tilde{P}^j_{\mu\nu}(z)}{\Gamma(j+1-\nu)\Gamma(-j-\nu)} \right] - \left[\frac{\tilde{P}^j_{\nu\mu}(z)}{\Gamma(j+1-\mu)\Gamma(-j-\mu)} \right] . \qquad \text{(H.31)}$$

Obviously, these formulas can be combined and permuted <u>ad infinitum</u>.

When, with j complex, we let $(\mu,\nu) \to (m,m')$, both integers or both half-integers, many of the preceding formulas simplify. Most notably, (H.24) reduces to (H.32), and then (H.26) with (H.20) produces (H.33):

$$P^j_{mm'} = G^j_{mm'} P^j_{m'm} \qquad \text{(H.32)}$$

$$\mathcal{Q}^j_{mm'} = \mathcal{Q}^{-j\mp 1}_{mm'} + (-1)^{m-m'} \pi \cot \pi(j+\varepsilon) P^j_{mm'} . \qquad \text{(H.33)}$$

10. The Cut Discontinuities

The cuts of the various functions are shown in Fig. 6. It is implicit that the following formulas always give the <u>total</u> discontinuity across all cuts, which, as noted above, we take to be compressed onto the real axis.

For P we have, from (H.29) and (H.12),

$$\begin{aligned} P^j_{\mu\nu}(-x+i\varepsilon) - P^j_{\mu\nu}(-x-i\varepsilon) &= 2i\, G^j_{-\nu} \Big[\sin \pi j\, P^j_{\mu,-\nu}(x) \\ &\quad - \frac{2}{\pi} \sin \pi\nu \sin \pi(j+\mu)\, \mathcal{Q}^j_{\mu,-\nu}(x) \Big] , \quad x > 1 \end{aligned} \qquad \text{(H.34)}$$

$$P^j_{\mu\nu}(x+i\varepsilon) - P^j_{\mu\nu}(x-i\varepsilon) = -2i \sin \frac{\pi}{2}(\mu-\nu)\, \tilde{P}^j_{\mu\nu}(x), \quad -1 < x < 1 . \qquad \text{(H.35)}$$

For $\mathcal{Q}$ we have, from (H.28) and (H.30),

$$Q^j_{\mu\nu}(-x+i\varepsilon) - \mathcal{Q}^j_{\mu\nu}(-x-i\varepsilon) = 2i \sin \pi j \; G^j_{-\nu}\mathcal{Q}^j_{\mu,-\nu}(x) \,, \quad x > 1 \tag{H.36}$$

$$\mathcal{Q}^j_{\mu\nu}(x+i\varepsilon) - \mathcal{Q}^j_{\mu\nu}(x-i\varepsilon) = -\frac{i\pi}{2}\left[\tilde{P}^j_{\mu\nu}(x) + G^j_{\mu\nu}\tilde{P}^j_{\nu\mu}(x)\right] \sec \frac{\pi}{2}(\mu-\nu) \,, \quad -1 < x < 1, \tag{H.37}$$

but also from (H.30),

$$\tilde{\mathcal{Q}}^j_{\mu\nu}(x+i\varepsilon) - \tilde{\mathcal{Q}}^j_{\mu\nu}(x-i\varepsilon) = -i\pi \; \tilde{P}^j_{\mu\nu}(x), \qquad -1 < x < 1. \tag{H.38}$$

11. Asymptotic Behavior in z; Limits as $z \to 1$

The expression (H.6) for $\mathcal{Q}$ is an asymptotic expansion in z, i.e., $F \to 1$ as $|z| \to \infty$. Thus, for $|\arg(z)| < \pi$,

$$\lim_{|z|\to\infty} Q^j_{\mu\nu}(z) = 2^j\Gamma(j+1+\mu)\Gamma(j+1-\nu)\, z^{-j-1}/\Gamma(2j+2) \sim (z)^{-j-1} \tag{H.39}$$

From (H.27) it follows that

$$\lim_{|z|\to\infty} P^j_{\mu\nu}(z) = \frac{2^{j+1}\Gamma(j+1+\mu)\Gamma(j+1-\nu)z^{-j-1}}{\pi\, S^j_{\nu\mu}\,\Gamma(2j+2)} + (j \leftrightarrow -j-1) \sim (z)^j + (z)^{-j-1} \,. \tag{H.40}$$

We include here the limits of the Legendre functions as $z \to 1$. Defining $\varepsilon = \left(\frac{z-1}{2}\right)^{\frac{1}{2}}$ we find

$$\lim_{z\to 1} P^j_{\mu\nu}(z) = \varepsilon^{\nu-\mu}/\Gamma(\nu-\mu+1) \,, \quad \nu-\mu \neq -1,-2.. \tag{H.41}$$

$$= \varepsilon^{\mu-\nu}\, G^j_{\mu\nu}/\Gamma(\mu-\nu+1) \,, \quad \nu-\mu = -1,-2.. \,.$$

Inserting the above into (H.24) we get

$$\lim_{z\to 1} \mathcal{Q}^j_{\mu\nu}(z) = \tfrac{1}{2}\,\Gamma(\mu-\nu)\,\varepsilon^{\nu-\mu} \,, \qquad \mathrm{Re}(\mu-\nu) > 0 \,;$$

$$= \tfrac{1}{2}\,\Gamma(\nu-\mu)\cdot G^j_{\mu\nu}\cdot \varepsilon^{\mu-\nu}, \qquad \mathrm{Re}(\mu-\nu) < 0 \,;$$

$$= +\,\pi\,\delta(i\mu-i\nu) \,, \qquad \mathrm{Re}(\mu-\nu) = 0 \,. \tag{H.42}$$

The last result is a consequence of

$$\lim_{\varepsilon\to 0} \left(\frac{\varepsilon^{\pm ix}}{ix}\right) f(ix) = \mp\,\pi\,\delta(x)\, f(0) \,.$$

12. Asymptotic Behavior in j

For the regular associated Legendre functions the large-j behavior may be obtained from the quadratic hypergeometric transformations, e.g., B 3.2 (44), which puts j into the "c" position of $F(a,b;c;z)$. For the generalized Legendre functions this approach fails and we rely instead on Watson's application of the method of steepest descents to the standard hypergeometric integral representations. Watson's results[49] are, in part, reported in B 2.3 (16), from which we conclude that

$$\lim_{|j|\to\infty} \mathcal{Q}^{j}_{\mu\nu}(z) = \sqrt{\frac{\pi}{2}}\cdot(z^2-1)^{-\frac{1}{4}}\,(j)^{\mu-\nu-\frac{1}{2}}\; e^{-(j+\frac{1}{2})\xi}$$

$$\sim\; j^{\mu-\nu-\frac{1}{2}}\; e^{-j\xi}\quad, \qquad\qquad (H.43)$$

where $|\arg(j)| < \pi$ and $\xi = \ln(z+\sqrt{z^2-1}) = \mathrm{ch}^{-1}(z)$. The functions of z in (H.43) are cut in the usual way discussed in Section 5, e.g., $\xi(z) = \ln(z+\sqrt{z^2-1})$ is cut as shown in Fig. 7 (a), duplicating the cut structure shown in Fig. 6 (b). In Fig. 7 (b) we show the region of the ξ-plane which is the image of the principal sheet of the z-plane upon which the Legendre functions are defined. Watson's results are given in terms of the variable ξ.

The condition $|\arg(j)| < \pi$, which Watson gives for (H.43), keeps j away from the fictitious cut generated by $(j)^{\mu-\nu-\frac{1}{2}}$ and arising from the asymptotic limit of gamma functions. Recall that $\mathcal{Q}$ is actually meromorphic in j.

For P as $|j|\to\infty$ we use the above result for $\mathcal{Q}$ in (H.27), along with

$$\lim_{|j|\to\infty} S^{j}_{\nu\mu} = 2\,e^{\mp i\pi(\nu-\mu+\frac{1}{2})}\quad,\quad \mathrm{Im}\, j \gtrless 0,$$

to get

$$\lim_{|j|\to\infty} P^{j}_{\mu\nu}(z) = \frac{1}{\sqrt{2\pi}}\,(z^2-1)^{-\frac{1}{4}}\,(j)^{\mu-\nu-\frac{1}{2}}\; e^{+(j+\frac{1}{2})\xi} + (j \leftrightarrow -j-1)$$

$$\sim (j)^{\mu-\nu-\frac{1}{2}}\,\left[e^{j\xi} + e^{-j\xi}\right]\;. \qquad\qquad (H.44)$$

The identical result follows from Watson's formula B 2.3 (17). It seems to the present author that the above derivation indicates that (H.44) should be true for $|\arg(j)| < \pi$. However, Watson says [B 2.3 (17)] that (H.44) is true only for $|\arg(j)| \leqslant \frac{\pi}{2}$ plus a section of the left half j-plane,

$$-\frac{\pi}{2} - w_2 \;<\; \arg(j) \;<\; \frac{\pi}{2} + w_1$$

$$0 < w_i < \frac{\pi}{2} \qquad\qquad \text{for}\quad \mathrm{Re}(\xi) > 0.$$

We shall compromise by considering (H.44) to be true for $|\arg(j)| \leqslant \frac{\pi}{2}$.

13. Asymptotic Behavior in μ

Before giving these limits we draw attention to two errors in Bateman concerning the asymptotic limits of the hypergeometric function in the parameters. First, B 2.3 (10), which says that $\lim_{|c|\to\infty} F(a,b;c;z) = 1$ for $|\arg(c)| < \pi$, is only true for $|\arg(c)| \leqslant \frac{\pi}{2}$ plus a region in the left half c-plane, even when $|z| < 1$. Second, B 2.3 (13), (14), (15) are incorrect, as seen from $F(a,b;a;z) = (1-z)^{-b}$, and should be replaced by

$$\lim_{|b|\to\infty} F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(c-a)}\,(-bz)^{-a} + \frac{\Gamma(c)}{\Gamma(a)}\,(+bz)^{a-c}(1-z)^{c-a-b} \qquad (H.45)$$

for $|\arg(b)| < \pi$ and $|\arg(1-z)| < \pi$.

To get the large $|\mu|$ limit of $\mathcal{Q}$, we apply (H.45) to (H.6):

$$\lim_{|\mu|\to\infty} \mathcal{Q}^{j}_{\mu\nu}(z) = \tfrac{1}{2}\Gamma(j+1+\mu)\left[\left(\frac{z-1}{2}\right)^{\nu}(\mu)^{-j-1-\nu}\left(\frac{z+1}{z-1}\right)^{(\mu+\nu)/2} + G^{j}_{-\nu}\left(\frac{z-1}{2}\right)^{-\nu}(-\mu)^{-j-1+\nu}\left(\frac{z+1}{z-1}\right)^{-(\mu+\nu)/2}\right] \tag{H.46}$$

for $|\arg(\mu)| < \pi$ and $\left|\arg\left(\frac{z+1}{z-1}\right)\right| < \pi$, i.e., $z \notin (-1,1)$.

Schematically,

$$\lim_{|\mu|\to\infty} \mathcal{Q}^{j}_{\mu\nu}(z) \sim (\mu)^{-j-1\mp\nu}\, e^{\pm\frac{1}{2}\mu \ln\left(\frac{z+1}{z-1}\right)}\, \Gamma(j+1+\mu) \tag{H.47}$$

whichever choice of signs gives the worst case.

To get the large $|\mu|$ behavior of $P^{j}_{\mu\nu}$, it would appear that we could use the above $\mathcal{Q}$ result in (H.27) to get an answer valid for $|\arg(\mu)| < \pi$. However, the result so obtained is not correct due to a cancellation of leading terms between the two $\mathcal{Q}$ functions. Instead, we content ourselves with the large $|\mu|$ behavior of $P^{j}_{\nu\mu}(z) = P^{j}_{-\mu,-\nu}(z)$ which follows directly from (H.2), with $\arg(\mu)$ restricted as noted above:

$$\lim_{|\mu|\to\infty} P^{j}_{\nu\mu}(z) = \left(\frac{z^2-1}{4}\right)^{-\frac{\nu}{2}} \cdot \left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}\mu} / \Gamma(\mu+1-\nu) \tag{H.48}$$

with $|\arg(\mu)| \leqslant \frac{\pi}{2}$ and $|\arg(z+1)| < \pi$, i.e., $z \neq -1$.

The large $|\nu|$ behavior follows from the above results and the symmetry properties given in Section 9.

14. Carlson Conditions

A function $f(j)$ is said to be "Carlson" if $f(j)$ is analytic in $\mathrm{Re}(j) \geqslant 0$ and bounded so that $|f(j)| < M\, e^{k|j|}$ with $k < \pi$ as $|j| \to \infty$ on all rays in the right half plane including the imaginary rays, i.e., $|\arg(j)| \leqslant \frac{\pi}{2}$. For example, $\mathrm{sh}\,(\pi j)$ and $\sin(\pi j)$ are not Carlson.

From the asymptotic limit (H.43), it follows that $Q^{j}_{\mu\nu}(z)$ is Carlson in j if $|\mathrm{Im}(\xi)| < \pi$ and $\mathrm{Re}(\xi) > -\pi$. Since $|\mathrm{Im}(\xi)| = \pi$ corresponds to $z < -1$, and since $\mathrm{Re}(\xi) > -\pi$ includes $\mathrm{Re}(\xi) > 0$, we conclude that $Q^{j}_{\mu\nu}(z)$ is Carlson in j for all z on the principal sheet except for $z < -1$.

From (H.44), the corresponding conditions for $P^{j}_{\mu\nu}(z)$ are $|\mathrm{Im}(\xi)| < \pi$ and $-\pi < \mathrm{Re}(\xi) < \pi$. The portion of this domain on the principal sheet of z, $0 < \mathrm{Re}(\xi) < \pi$, is the interior of the ellipse

$$\left[\frac{\mathrm{Re}(z)}{\mathrm{ch}\,\pi}\right]^2 + \left[\frac{\mathrm{Im}(z)}{\mathrm{sh}\,\pi}\right]^2 = 1, \tag{H.49}$$

but cut from $z = -1$ to the left. [See ellipse A in Fig. 7 (a).]

In the variable μ, the limit (H.48) indicates that $P^{j}_{\nu\mu}(z)$ is Carlson provided that $\left|\arg\left(\frac{z+1}{z-1}\right)\right| < \pi$, i.e., $z \notin (-1,1)$. [Recall that as $\mu \to \pm i\infty$, $|\Gamma(\mu)| \sim \exp(-\frac{1}{2}\pi|\mu|)$.]

Finally, from (H.47) we see that $\mathcal{Q}^{j}_{\nu\mu}(z) = \mathcal{Q}^{j}_{-\mu,-\nu}(z)$ is Carlson in μ for all z on the principal sheet.

These results are summarized in Table H.14.

The significance of a function $f(j)$ being Carlson lies in Carlson's Theorem which states:[50] the set of numbers f_j, $j = 0,1,2\ldots$

may be interpolated by many analytic functions, but at most one such function can be Carlson.

TABLE H.14

Conditions for which the Legendre functions are Carlson

	$\mathcal{Q}^j_{\nu\mu}(z)$	$P^j_{\nu\mu}(z)$
j	$z \neq -1$	z interior of (H.49)
μ	all z	$z \notin (-1,1)$

15. Zeros and Poles of $P^j_{mm'}$ and $Q^j_{mm'}$

When the helicity labels μ and ν are both integers or both half-integers, we rename them m and m' and refer to the functions $P^j_{mm'}$ and $Q^j_{mm'}$ as being "on the helicity lattice". These functions are, as we have seen in Appendix D, associated with the SU(2) and SU(1,1) UIR matrix elements taken in the discrete basis. As $d^j_{mm'}$ and $e^j_{mm'}$, the helicity-lattice Legendre functions were studied in detail by Andrews and Gunson.[4] In this section, we discuss the singularities in j of these functions.

A convenient tool for displaying the j-plane singularities of a function $f^j_{mm'}$ is the helicity lattice diagram used by Andrews and Gunson. For example, Fig. 8 (a) shows the location of the poles, zeros, double poles, and double zeros of the function

$$G^j_{mm'} = \frac{\Gamma(j+1+m)\Gamma(j+1-m')}{\Gamma(j+1-m)\Gamma(j+1+m')} .$$

The meaning of the diagram is illustrated by this example: if (m,m') are the coordinates of lattice point P shown in Fig. 8 (a), and if $2j_o$ = integer is the length of the edge of the central square, then $G^j_{mm'}$ has a simple pole as $j \to j_o$.

In Fig. 8 (b) we show the same diagram with regions labelled 1 through 9. Region 5, including the points on the square, is associated with the SU(2) UIR's and is sometimes called the "sense-sense" region since both helicity labels m, m' are less, in magnitude, than the angular momentum label j. Regions 2, 4, 6, 8 are then "sense-nonsense" and regions 1, 3, 7, 9 are "nonsense-nonsense". As Table B.1 shows, regions 3 and 7 are associated with the SU(1,1) UIR's D_k^+ and D_k^-.

We now discuss the zeros of $P^j_{mm'}$. With j complex, as $(\mu,\nu) \to (m,m')$ we have, from (H.24),

$$P^j_{mm'}(z) = G^j_{mm'} P^j_{m'm}(z). \qquad (H.50)$$

For $m' \geqslant m$, the meaning of $P^j_{mm'}$ is clear from (H.2); for $m > m'$, we may regard (H.50) as the definition of $P^j_{mm'}$. This definition corresponds to the usual manner of treating $F(a,b;c;z)/\Gamma(c)$ when $c \to$ negative integer, see, e.g., B 2.8 (19). From (H.50) it then follows that $P^j_{mm'}$ has possible zeros or double zeros when $m > m'$ due to $G^j_{mm'}$. The locations of the zeros[51] of $P^j_{mm'}$ are shown in Fig. 8 (c).

In similar fashion, the poles[51] and double poles of $Q^j_{mm'}$ are indicated in Fig. 8 (d). These poles arise from the gamma functions in the numerator of (H.6).

In the remaining diagrams we have indicated the zeros and singularities of related functions. The notation $\sqrt{0}$ denotes a "square-root zero", i.e., a branch point $(j - j_o)^{\frac{1}{2}}$. Similarly, $\sqrt{x}$ denotes a "square-root pole", $(j-j_o)^{-\frac{1}{2}}$.

From relation (H.29),

$$\frac{2}{\pi} \sin \pi(j+m)\, e^{\pm i\pi m'} \mathcal{Q}^j_{mm'}(z) = e^{\mp i\pi j} P^j_{mm'}(z) - G^j_{-m'} P^j_{m,-m'}(-z), \qquad (H.51)$$

we may deduce two useful facts. First, for (m,m') in region 5, $\mathcal{Q}$ has no poles, so

$$e^{\mp i\pi j_o} P^{j_o}_{mm'}(z) = G^{j_o}_{-m'} P^{j_o}_{m,-m'}(-z) \;. \quad /5 \qquad (H.52)$$

Second, in regions 3 and 7 associated with the $D_k^{\pm}$, the residues of the poles in $\mathcal{Q}^j_{mm'}$ are given by the first term in (H.51), since the second term has zeros in these regions. Thus,

$$\frac{1}{2\pi i} \oint_{j_o} Q^j_{mm'}(z)\, dj = \frac{1}{2} P^{j_o}_{mm'}(z) \;. \quad /3,7 \qquad (H.53)$$

In terms of the d and e functions (see Section 7) these last two equations may be written as

$$d^{j_o}_{mm'}(z) = (-1)^{j_o-m}\, d^{j_o}_{m,-m'}(-z) \qquad /5$$

$$\frac{1}{2\pi i} \oint_{j_o} e^j_{mm'}(z)\, dj = \frac{1}{2} d^{j_o}_{mm'}(z) \;. \qquad /3,7$$

16. Integral Representations

The first- and second-kind Legendre functions defined in (H.2) and (H.6) may be expressed as single integrals of the same integrand[52]

$$P^j_{\mu\mu'}(\mathrm{ch}\,\nu) = \frac{\Gamma(-j+\mu)}{\Gamma(-j+\mu')} \frac{1}{2\pi i} \int_{-\mathrm{th}\frac{\nu}{2}}^{(0,+)} f(s)ds, \quad \mathrm{Re}(-j+\mu') > 0 \qquad (H.54)$$

$$\mathcal{Q}^j_{\mu\mu'}(\mathrm{ch}\,\nu) = \frac{\Gamma(j+1-\mu')}{\Gamma(j+1-\mu)} \cdot \frac{1}{2} \int_0^{\infty} f(s)ds \;, \quad \mathrm{Re}(j+1+\mu) > 0 \qquad (H.55)$$

where

$$f(s) = s^{j-\mu}\left(s\cdot \mathrm{ch}\,\frac{\nu}{2} + \mathrm{sh}\,\frac{\nu}{2}\right)^{-j-1+\mu'}\left(\mathrm{ch}\,\frac{\nu}{2} + s\cdot \mathrm{sh}\,\frac{\nu}{2}\right)^{-j-1-\mu'}$$

$$\propto s^{\mu'-\mu-1}\left(1+s^{-1}\mathrm{th}\,\frac{\nu}{2}\right)^{-j-1+\mu'}\left(s + \mathrm{cth}\,\frac{\nu}{2}\right)^{-j-1-\mu'} .$$

In Fig. 9 we sketch the cuts of the integrand and the two integration contours. When $\mu' - \mu$ = integer, one of the cuts vanishes allowing the contour for P to be simplified,

$$P^j_{mm'}(\mathrm{ch}\,\nu) = \frac{\Gamma(-j+m)}{\Gamma(-j+m')} \frac{1}{2\pi i} \oint_{|s|=1} ds\, s^{j-m}\left(s\cdot \mathrm{ch}\,\frac{\nu}{2} + \mathrm{sh}\,\frac{\nu}{2}\right)^{-j-1+m'}$$

$$\times \left(\mathrm{ch}\,\frac{\nu}{2} + s\cdot \mathrm{sh}\,\frac{\nu}{2}\right)^{-j-1-m'} . \qquad (H.56)$$

Equation (H.54) may be verified by making the substitution $s = -(\mathrm{th}\,\frac{\nu}{2})\,t$, then using a version of B 2.12 (3),

$$\frac{-1}{2\pi i}\int_1^{(0^+)} dt(-t)^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} = \frac{\Gamma(c-b)}{\Gamma(c)\Gamma(1-b)}F(a,b;c;z).$$

Equation (H.55) is proved with the substitution $s = +(\mathrm{cth}\,\frac{\nu}{2})\,t$ and subsequent application of B 2.12 (5).

In this section we are using $z = \mathrm{ch}\,\nu$ only for convenience; there is no implication that $z \geqslant 1$. In fact, all the integral representations given here are valid for complex z off the cuts shown in Fig. 6. For example, $\mathrm{sh}\,\frac{\nu}{2} = (\frac{z-1}{2})^{\frac{1}{2}}$, cut according to Section 5.

With the replacement $s = e^{\alpha} = e^{+i\omega}$ and use of the identities

$$(e^{\alpha}\mathrm{ch}\,\frac{\nu}{2} + \mathrm{sh}\,\frac{\nu}{2})\,(\mathrm{ch}\,\frac{\nu}{2} + e^{\alpha}\mathrm{sh}\,\frac{\nu}{2}) = e^{\alpha}(\mathrm{ch}\,\nu + \mathrm{sh}\,\nu\,\mathrm{ch}\,\alpha),$$

$$(e^{\alpha}\mathrm{ch}\,\frac{\nu}{2} + \mathrm{sh}\,\frac{\nu}{2})\,(\mathrm{ch}\,\frac{\nu}{2} + e^{\alpha}\mathrm{sh}\,\frac{\nu}{2})^{-1} = \left[\frac{\mathrm{ch}\,\nu + \mathrm{sh}\,\nu\,\mathrm{ch}\,\alpha}{\mathrm{sh}\,\nu + \mathrm{ch}\,\nu\,\mathrm{ch}\,\alpha - \mathrm{sh}\,\alpha}\right] \equiv I,$$

formulas (H.56) and (H.55) may be recast as

$$P^{j}_{mm'}(\mathrm{ch}\,\nu) = \frac{\Gamma(-j+m)}{\Gamma(-j+m')}\cdot\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} d\omega\, e^{-im\omega}(\mathrm{ch}\,\nu+\mathrm{sh}\,\nu\cos\omega)^{-j-1+m'}$$
$$\times\,(\mathrm{sh}\,\nu+\mathrm{ch}\,\nu\,\mathrm{ch}\,\alpha-i\sin\omega)^{-m'}, \tag{H.57}$$

$$\mathcal{Q}^{j}_{\mu\mu'}(\mathrm{ch}\,\nu) = \frac{\Gamma(j+1-\mu')}{\Gamma(j+1-\mu)}\cdot\frac{1}{2}\int_{-\infty}^{\infty} d\alpha\, e^{-\mu\alpha}(\mathrm{ch}\,\nu+\mathrm{sh}\,\nu\,\mathrm{ch}\,\alpha)^{-j-1+\mu'}$$
$$\times\,(\mathrm{sh}\,\nu+\mathrm{ch}\,\nu\,\mathrm{ch}\,\alpha-\mathrm{sh}\,\alpha)^{-\mu'}. \tag{H.58}$$

Endless variations of (H.57) and (H.58) arise from the symmetry relations given in Section 9 (e.g., $P^{j} = P^{-j-1}$), from taking $\alpha \to -\alpha$, $\omega \to -\omega$, and from further versions of the expression I defined above,

$$I = \left[\frac{\mathrm{sh}\,\nu+\mathrm{ch}\,\nu\,\mathrm{ch}\,\alpha+\mathrm{sh}\,\alpha}{\mathrm{ch}\,\nu+\mathrm{sh}\,\nu\,\mathrm{ch}\,\alpha}\right]$$
$$= \left[\frac{\mathrm{sh}\,\nu+\mathrm{ch}\,\nu\,\mathrm{ch}\,\alpha+\mathrm{sh}\,\alpha}{\mathrm{sh}\,\nu+\mathrm{ch}\,\nu\,\mathrm{ch}\,\alpha-\mathrm{sh}\,\alpha}\right]^{\frac{1}{2}}$$
$$= \left[\frac{e^{\pm\alpha} + \mathrm{th}\,\frac{\nu}{2}}{1 + e^{\pm\alpha}\mathrm{th}\,\frac{\nu}{2}}\right]^{\pm 1}.$$

For example,

$$\mathcal{Q}^{j}_{\mu\mu'}(\mathrm{ch}\,\nu) = \pi\,\frac{\Gamma(j+1-\mu')}{\Gamma(j+1-\mu)}\cdot\Big\{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{-\mu\alpha}(\mathrm{ch}\,\nu+\mathrm{sh}\,\nu\,\mathrm{ch}\,\alpha)^{-j-1}$$
$$\times\left(\frac{e^{\alpha} + \mathrm{th}\,\frac{\nu}{2}}{1 + e^{\alpha}\mathrm{th}\,\frac{\nu}{2}}\right)^{\mu'}\Big\}.$$

This version appears in CDM[27] as (A.8) in their calculation of the + + C_q-class UIR matrix element, which they call $d^{j}_{\mu+,\mu'+}(\nu^{-1})$.

Chapters 3 and 6 of Vilenkin's book[9] provide an imposing quantity of information on the functions $P^{j}_{mm'}(z)$, including further integral representations. The connection to Vilenkin's function $\mathfrak{P}^{j}_{mm'}(z)$ is found by comparing (H.56) with Vilenkin VI 3.3 (1):

$$\mathfrak{P}^{j}_{mm'}(z) = \frac{\Gamma(j+1+m')}{\Gamma(j+1+m)} P^{j}_{mm'}(z) \qquad \text{(H.59)}$$

The integral representations (H.57) and (H.58), which are central to Part 3 of our addition theorem proof of Section V, are given a group-theoretic interpretation in Section V.5.

FOOTNOTES AND REFERENCES

† This report was done with support from the United States Energy Research and Development Administration.

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17. This pinch is the source of Regge cuts in the diagonalized multiperipheral equation (see Eq. (6.13)), unless the "kinematic" poles in Fig. 3 are somehow cancelled in the projection (6.14).

18. The classical Laplace operator $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is an invariant operator of the Euclidean group E(3).

19. The addition theorem (2.7) is clearly true as $z_1 \to 1$ since $\lim_{z_1\to 1} \mathscr{Q}^j_{\mu\lambda}(z_1) = \pi\delta(i\mu - i\lambda)$. However, this does not prove (2.7) because the coefficient is not determined by this limit.

20. It should be emphasized that the parameters $g_2 = (\phi_2, \nu_2, \phi_2')$ are dependent variables given by $g_2 = g_1^{-1}g$ as in Appendix E.

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23. To visualize the diagonalization it is helpful to extend the definitions of A, B, and C to the entire group manifold via

$$A(g) = \theta(g)A(g) \quad \text{where} \quad \theta(g) = \begin{cases} 1 & g \in S_o^+ \\ 0 & g \notin S_o^+ \end{cases}$$

24. In the SO(3) analog of going from (6.10) to (6.15), one would take $B(\phi_1,\theta_1,\phi_1') \to B(-,\theta_1,-)$ and then $\int_0^{2\pi} d\phi_1'/2\pi = 1$. In particle physics applications of these equations, usually the product $B(g_1)\,C(g_2)$ depends only on the sum $\omega = \phi_1' + \phi_2$ (the Toller angle) or its continuation $\alpha = \xi_1' + \xi_2$, in which

case ϕ_1' or ξ_1' may be regarded as a redundant variable and the "Toller" dependence taken into the object $C(g_2)$. See, e.g., Fig. 4.

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32. J. G. Kuriyan, N. Mukunda and E. C. G. Sudarshan, J. Math. Phys. <u>9</u>, 2100 (1968).

33. Bargmann used $e^{-i2\mu J_3}\ e^{-i2\zeta K_1}\ e^{-i2\nu J_3}$.

34. Mukunda's concrete interpretation of this fact[3] is that the change of variable which takes one from Bargmann's "circle" multiplier representation, where $J_3 = i\,\partial/\partial\phi$, to a space where $K_1 = i\,\partial/\partial_q$, maps Bargmann's circle into two real lines in the complex q-plane. On the other hand, the D_k^+ representation is associated with functions analytic inside Bargmann's circle, hence analytic in the strip between the two lines in the q-plane, so for D_k^+ the two lines are not "independent" and there is no need for a multiplicity index.

35. Bargmann uses $G_i = M_r$ or L_r, $\vec{G}_i = -\chi_r$, and $p_i = a_i$. See, e.g., Bargmann's equations (1.26), (1.37), (4.7), (4.17) to (4.20), also (10.5). For an understanding of Bargmann's "preliminary remarks", e.g., equations (1.1) to (1.4), see L. O'Raifeartaigh, Matscience Report 25 (Inst. of Math. Sciences, Madras, 1964). In Appendix F we show that the $\vec{G}_i$ generate the left-regular representation.

36. The point is that if $G_i \to G_i'$ is an automorphism of the Lie algebra, the Campbell-Hausdorff identities (A.2) will be the same in G_i' as they are in G_i, since they are derived directly from the Lie algebra. Because our derivation of the differential generators $\vec{G}_i$ uses only the C-H identites, the new operators $\vec{G}_i'$ will be given by the same expressions as the $\vec{G}_i$.

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45. For other statements of this theorem see §13 of Ref. 2, Eq. (14.5) of Ref. 4, or Section VI 5.3 of Ref. 9.

46. $dg \propto \Pi\, dp_i/|\det X|$ with X defined in (C.5) and implicitly given in (C.9), (C.11), (C.15) and (C.16).

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is given as Theorem 11.3.3 of Einar Hille, Analytic Function Theory (Ginn, Boston, 1962), Vol. II, p. 64.

51. More generally, as follows from (H.24) when $\mu - \nu = 1,2,3\ldots$, $P^j_{\mu\nu}$ has two finite chains of zeros; $\nu \leqslant j \leqslant \mu-1$ and $-\mu \leqslant j \leqslant -\nu - 1$. For any μ and ν, $Q^j_{\mu\nu}$ has two semi-infinite chains of poles, $j \leqslant -\mu - 1$ and $j \leqslant \nu - 1$.

52. This fact is of course no coincidence; see B 2.1 (12) and nearby discussion. The contour notation is explained in B 1.6.

FIGURE CAPTIONS

Fig. 1 The helicity lattice for d^j_{nm}' and the summation segments for Eqs. (4.1) and (4.2).

Fig. 2 Cross hatch shows convergence domain of (4.2) in z_1 for a typical value of z_2 with $\mathrm{Re}(z_2) > 0$.

Fig. 3 Integration contour for (4.16), (4.14) or (2.7), when $\mathrm{Re}(j) < -1$.

Fig. 4 Kinematic structure of a typical multiperipheral equation.

Fig. 5 Principal sheet for $(z-1)^{\alpha}$. With $|\arg(z-1)| < \pi$, $(1-z) = (z-1)\, e^{\mp i\pi}$, $\mathrm{Im}(z) \gtrless 0$.

Fig. 6 Cuts of Legendre functions. All cuts, deformed for clarity, are taken to lie on the real axis. $\tilde{P}$ and $\tilde{Q}$ have the same cuts as P and Q except that one cut has been swung around from left to right. F indicates the hypergeometric cut in each case.

Fig. 7 (a) Principal sheet of $\xi(z) = \mathrm{ch}^{-1}(z) = \ln\left[z + \sqrt{z^2 - 1}\right]$ showing square-root and logarithmic cuts.

(b) Region of ξ-plane corresponding to the z-sheet shown in (a). Level curves are drawn to indicate the nature of the mapping; ellipses are not drawn to scale.

Fig. 8 Helicity lattice diagrams.

Fig. 9 Squiggles show cut choice for integrand of (H.54) and (H.55). Solid lines are integration contours.

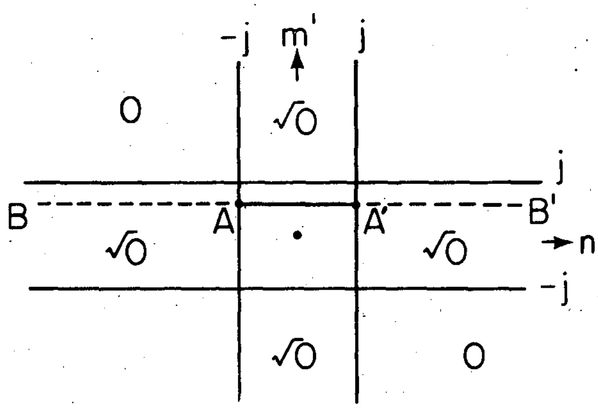


Fig. 1

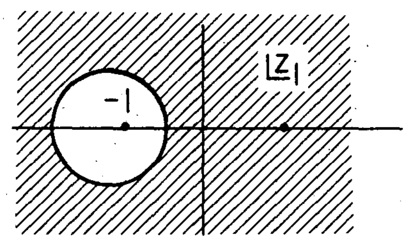
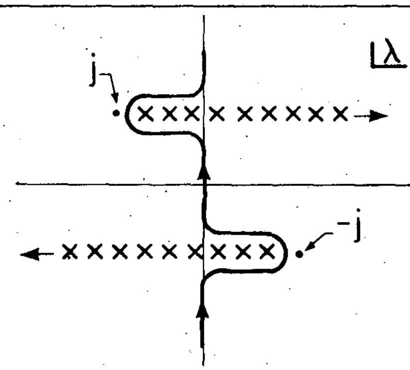


Fig. 2



XBL 7610 4297

Fig. 3

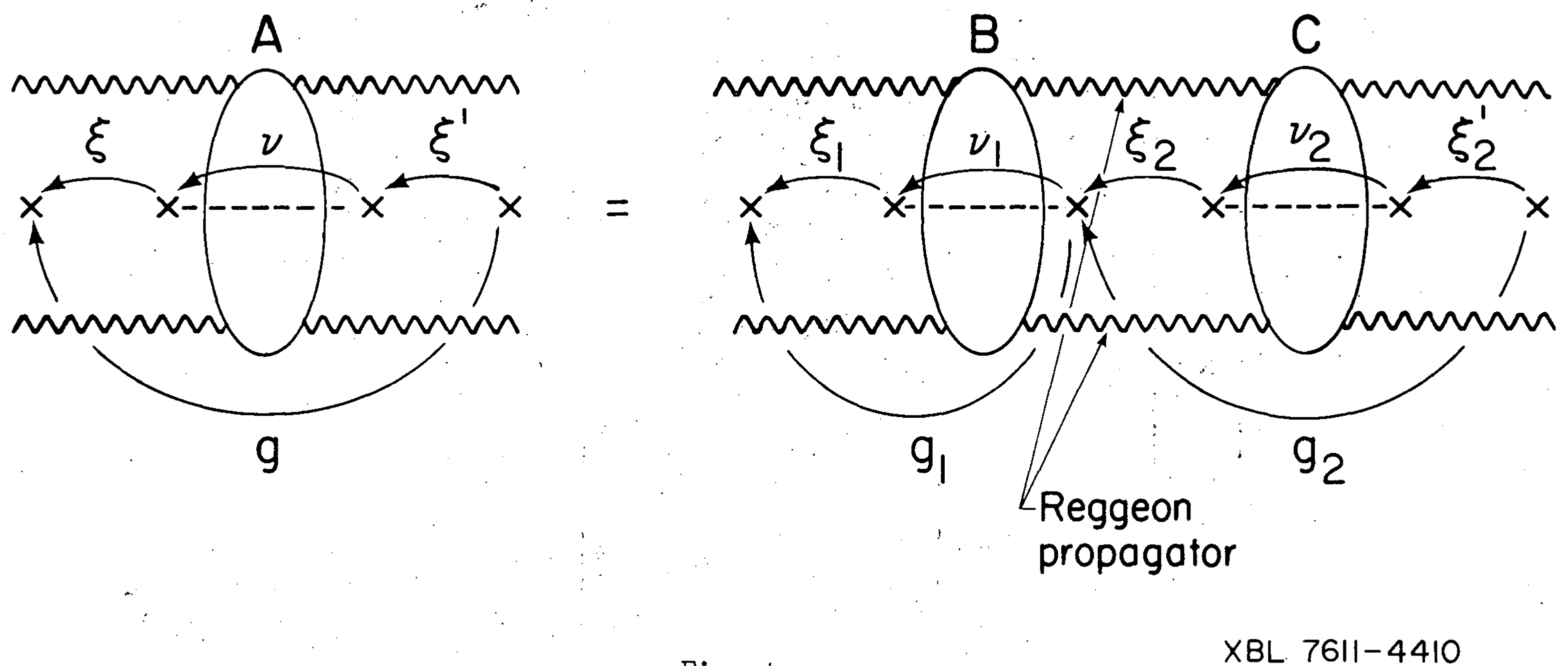
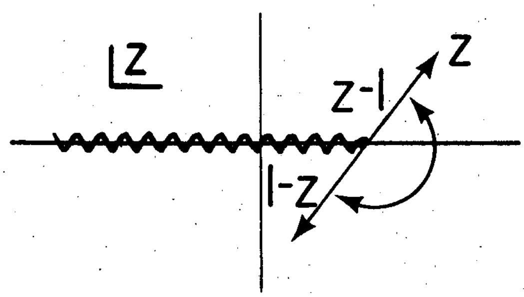
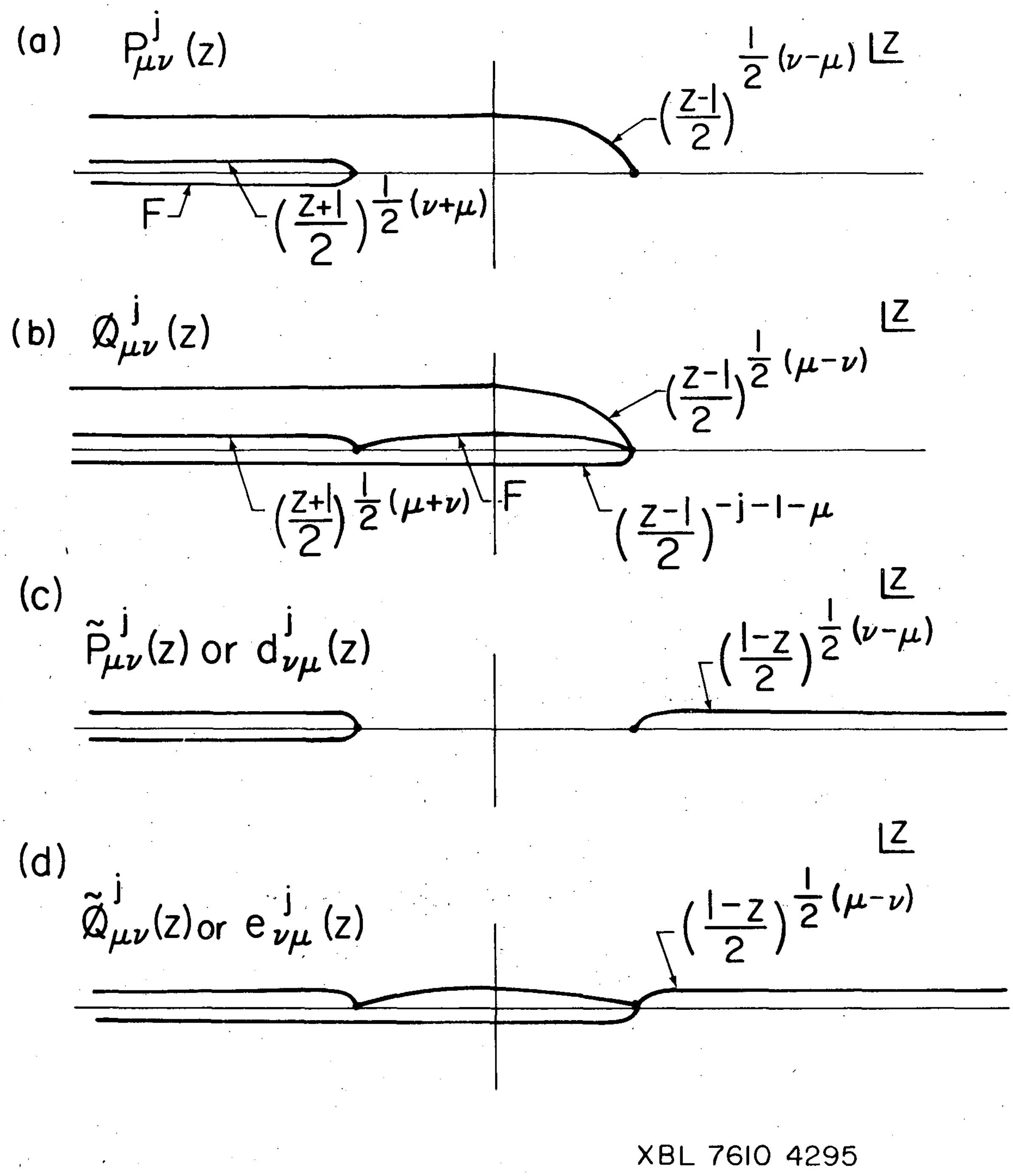

A
ξ
ν
ξ'
g
=
B
C
ξ_1
ν_1
ξ_2
ν_2
ξ_2'
g_1
g_2
Reggeon propagator
XBL 7611-4410

Fig. 4



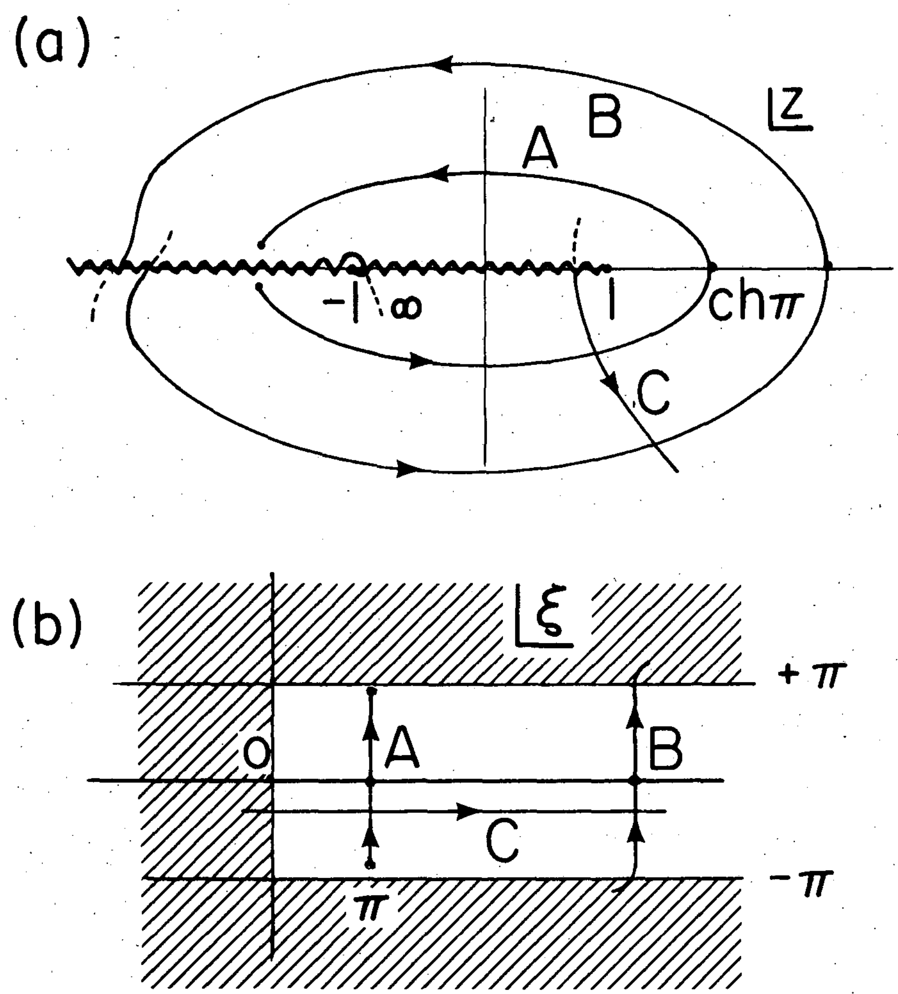
XBL 7610 4296

Fig. 5



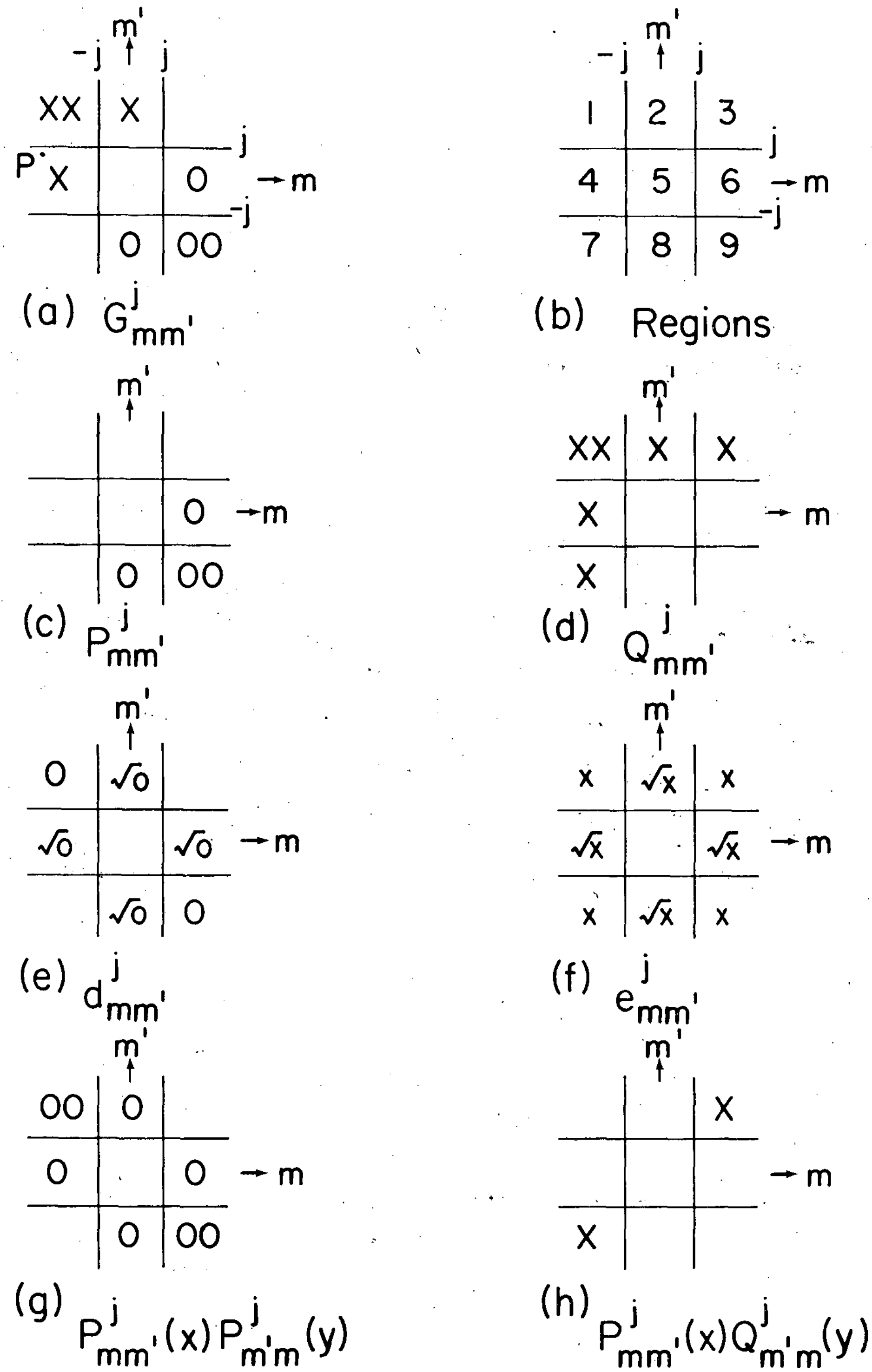
(a) $P^j_{\mu\nu}(z)$
$\left(\frac{z-1}{2}\right)^{\frac{1}{2}(\nu-\mu)}$
z
$\left(\frac{z+1}{2}\right)^{\frac{1}{2}(\nu+\mu)}$
F
(b) $Q^j_{\mu\nu}(z)$
z
$\left(\frac{z-1}{2}\right)^{\frac{1}{2}(\mu-\nu)}$
$\left(\frac{z+1}{2}\right)^{\frac{1}{2}(\mu+\nu)}$
F
$\left(\frac{z-1}{2}\right)^{-j-1-\mu}$
(c) $\tilde{P}^j_{\mu\nu}(z)$ or $d^j_{\nu\mu}(z)$
z
$\left(\frac{1-z}{2}\right)^{\frac{1}{2}(\nu-\mu)}$
(d) $\tilde{Q}^j_{\mu\nu}(z)$ or $e^j_{\nu\mu}(z)$
z
$\left(\frac{1-z}{2}\right)^{\frac{1}{2}(\mu-\nu)}$
XBL 7610 4295

Fig. 6



XBL 7610 4294

Fig. 7



XBL 7610 4293

Fig. 8

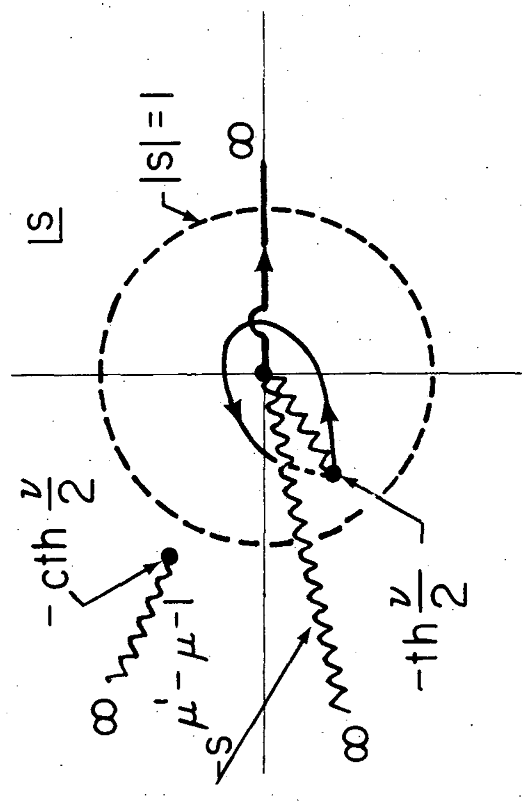


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Fig. 9

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