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Thermodynamic scheme of inhomogeneous perfect fluid mixtures

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Abstract

We analyse the compatibility between the geometrodynamics and thermodynamics of a binary mixture of perfect fluids which describe inhomogeneous cosmological models. We generalize the thermodynamic scheme of general relativity to include the chemical potential of the fluid mixture with non-vanishing entropy production. This formalism is then applied to the case of Szekeres and Stephani families of cosmological models. The compatibility conditions turn out to impose symmetry conditions on the cosmological models in such a way that only the limiting case of the Friedmann–Robertson–Walker model remains compatible. This result is an additional indication of the incompatibility between thermodynamics and relativity.

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1. Introduction

When considering cosmological questions, it is conventional wisdom to work with the homogeneous and isotropic Friedmann–Robertson–Walker (FRW) models, since they reproduce many of the features of the observable universe. Nevertheless, the desire to describe details such as the process of structure formation, which leads to local inhomogeneities, requires the use of more general models. A natural framework to do this is found in the inhomogeneous cosmological models, which are exact solutions to Einstein's field equations, representing inhomogeneous universes and which contain the FRW models in a certain geometric limit, thus, generalizing them. Although the presence of inhomogeneities is crucial in the process of structure formation, especially in the context of galaxy formation, the inhomogeneous models to be investigated in this work would be insufficient to take into account all the details of the physics of structure formation (see, for instance, [1] for a recent review).

The most general classes of inhomogeneous models known are the Stephani [2] and Szekeres [3] solutions, which are irrotational perfect fluid exact solutions and have, in general, no isometries. Both families admit a barotropic equation of state only in the FRW limit, and this the question has arisen of how to find or generate non-trivial but physically meaningful equations of state. Attached to this question is the fact that an evolving universe is, from the thermodynamics point of view, a system out of equilibrium; thus, we must make sure that defining thermodynamic variables makes sense. In this direction, Coll and Ferrando [4] have shown that an exact solution admits a 'thermodynamic scheme', that is, it has well-defined thermodynamic variables compatible with the field equations, provided the integrability conditions of Gibbs equation are satisfied, which turns out to be equivalent to the existence of equations of state, not necessarily barotropic.

Following the thermodynamic scheme proposed in [4], Krasiński, Quevedo and Sussman [5–7] worked out the thermodynamic interpretation of the inhomogeneous cosmological models, and found that, in the general case, defining sound thermodynamics involves forcing isometries on the metrics. However, such analysis was made for a one-component perfect fluid, complying with particle conservation, which satisfies automatically $\dot{s} = 0$, i.e., for a flow with null entropy production.

In an attempt to understand such incompatibility between thermodynamics and perfect fluid inhomogeneous cosmological models, in this work, we study the simplest non-dissipative irreversible case in which the entropy production is in general non-vanishing: a binary mixture of perfect fluids. To this end, we generalize the thermodynamic scheme to the case of a binary mixture, and analyse both the integrability conditions of Gibbs equation and the condition of entropy production, due to the fluids mixing process. From the integrability conditions we are able to recognize the equations of state which describe the mixture, and which do not involve imposing isometries on the metrics. However, to satisfy the entropy production condition the metrics must reduce to those of the FRW cosmologies, a result that we interpret as a further indication to the incompatibility between thermodynamics and relativity.

Different aspects of binary mixtures of perfect fluids have been analysed in general relativity. Two-fluid models have been intensively applied to describe cosmological models in which one fluid represents radiation and the second one the matter content of the universe [8–16]. If the two fluids are comoving, then their energy–momentum tensor is effectively that of a single fluid. If the fluids are not comoving, then Letelier [17] has shown that the energy–momentum tensor can be transformed into a tensor explicitly exhibiting a preferred spatial direction which resembles the energy–momentum tensor of a viscous fluid. It was also shown in [17, 18] that an anisotropic fluid can be consistently described by two-perfect-fluid components. Krisch and Smalley [19] have investigated the propagation of discontinuities in the relativistic two-fluid system described by Letelier's tensor. Zimdahl, Pavon and Maartens [20, 21] have studied inflationary models as a mixture of two interacting and reacting fluids within the framework of irreversible thermodynamics. The authors consider an out-of-equilibrium dissipative mixture, with irreversible evolution in order to describe the reheating process in inflationary universe models. To this end, they assume the condition $\dot{s}_A = 0$ for each component A of the mixture. Finally, Cissoko [22] has analysed a mixture of two reacting and coupled perfect fluids flowing with distinct 4-velocities and has introduced the Lorentz factor, associated with the relative velocity of the fluids, as an additional thermodynamic variable. In all these works little attention has been paid to the study of the compatibility between the thermodynamic and geometric evolution of the models, especially when inhomogeneities are present.

This paper is structured as follows. In sections 2 and 3 we study the thermodynamics of a binary mixture of comoving perfect fluids, derive the expression for the entropy production due to the mixing process, and propose a thermodynamic scheme which explicitly includes the chemical potential. In section 4 we apply the conditions for the existence of a thermodynamic scheme to the Stephani and Szekeres inhomogeneous cosmological models. Finally, in section 5 we discuss our results and comment on additional problems for further research.

2. Thermodynamics of the source

Cosmology addresses, among other questions, one concerning the process of evolution of the universe as a whole. Such a process, according to general relativity, involves the evolution of both the source and the geometry, connected through Einstein's field equations. Besides satisfying the field equations, the source must be subject to the laws of thermodynamics, which formally involve quantities defined only in equilibrium. Thus, to describe an evolving system, from the thermodynamic viewpoint, we must extend the concepts and laws to the non-equilibrium case. The simplest way to do this, and the most invoked one in cosmology (implicitly or explicitly), is based on the local equilibrium hypothesis, which asserts that every point of a system not too far from equilibrium, has a neighbourhood which is in equilibrium, that is, in such a neighbourhood we can define uniquely all thermodynamic variables, with the same physical meanings as in equilibrium and satisfying all the relations the variables satisfy in equilibrium. Thus, the gradients and temporal changes of the variables in the system are connected through the relations they satisfy in equilibrium, i.e., we must treat them as scalar fields connected through the equilibrium relations. With this in mind, we now consider the source of the field in our models: a perfect fluid consisting of a binary mixture of perfect fluids. Its energy–momentum tensor is

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}, \tag{1}$$

where ρ and p are the total matter-energy density and the mixture's pressure, respectively, and u_α ($\alpha = 0, 1, 2, 3$) is the 4-velocity associated with the total matter flow. Such an energy–momentum tensor corresponds to a continuum in which energy transport occurs only due to matter transport. Thus, entropy production will be a consequence of the mixing process alone, as the composition of the mixture will be, in general, inhomogeneous. This tensor satisfies the conservation law $T^{\alpha\beta}{}_{;\beta} = 0$, which, in particular, implies the contracted Bianchi identity

$$\dot\rho + (\rho + p)\Theta = 0, \tag{2}$$

where $\Theta = u^\alpha{}_{;\alpha}$ is the scalar expansion. Besides, we restrict ourselves to the case in which particles of both components are conserved

$$(nu^\alpha)_{;\alpha} = 0, \tag{3}$$

where n is the total particle number density. The analogues of equations (2) and (3) for the case of a one-component perfect fluid, together with the corresponding Gibbs equation, imply automatically null entropy production, $\dot s = 0$. However, for the case of a binary mixture, Gibbs equation reads

$$T\,\mathrm{d}s = \mathrm{d}(\rho/n) + p\,\mathrm{d}(1/n) - \mu_1\,\mathrm{d}c - \mu_2\,\mathrm{d}c_2, \tag{4}$$

where s is the entropy per particle, T is the temperature, μ_1 and μ_2 are the chemical potentials of components 1 and 2, and c and c_2 are the fractional concentrations ($c + c_2 = 1$) of each kind of particle in the fluid element under consideration. It is clear that the fluid composition is determined by only one parameter, so we can write

$$T\,\mathrm{d}s = \mathrm{d}(\rho/n) + p\,\mathrm{d}(1/n) - \mu\,\mathrm{d}c, \tag{5}$$

where we have introduced the mixture's 'chemical potential', $\mu = \mu_1 - \mu_2$. Equation (5), together with (2) and (3), now implies

$$\dot{s} = -\frac{\mu}{T}\dot{c}. \tag{6}$$

This corresponds to the entropy production in a fluid in which the mixing process is the only irreversible process taking place, and it depends only on the change of the Gibbs free energy, due to change in composition, and on temperature. This represents the main difference with respect to the case of a one-component fluid, and it is our interest to exhibit its consequences.

3. Gibbs 1-form

As equation (6) states, to describe the thermodynamics of any fluid element of the mixture, we must introduce the conjugate pair of variables μ and c, in terms of which the entropy production is written. Thus, these variables play an important role in the description of thermodynamic evolution. On the other hand, the field equations determine ρ and p and their evolution (through the conservation law $T^{\alpha\beta}{}_{;\beta} = 0$), linking geometrical and physical evolution. What conditions are necessary and sufficient to guarantee compatibility between the field equations and the laws of thermodynamics? According to Coll and Ferrando [4], it is enough to demand the compliance of the integrability conditions of the Gibbs equation. Thus we consider the Gibbs 1-form

$$\Omega = Ts_{,\alpha}\,\mathrm{d}x^\alpha = [(\rho/n)_{,\alpha} + p(1/n)_{,\alpha} - \mu c_{,\alpha}]\,\mathrm{d}x^\alpha, \tag{7}$$

where a comma denotes partial differentiation and d is the exterior derivative. Its integrability conditions are

$$\mathrm{d}\Omega = 0 \tag{8}$$

$$\mathrm{d}\Omega \wedge \Omega = 0 \tag{9}$$

the first of which is a sufficient condition, of no physical significance, as it involves the existence of a 'heat function' which would imply unphysical properties of the fluid. The second condition, however, is a necessary and sufficient condition. It represents the differential relations that the equations of state (four in the case of a binary mixture in one phase) must satisfy in order to contain all the equilibrium information about the system, since not all of them are independent (only three in our case), as one of them can be deduced from the others via the Gibbs–Duhem equation. Its components are

$$\mathrm{d}\Omega \wedge \Omega = Z_{tij}\,\mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j + Z_{ijk}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{10}$$

$$Z_{tij} = -T\left(\frac{1}{n^2}p_{[,t}n_{,i} + \mu_{[,t}c_{,i}\right)s_{,j]} = 0 \tag{11}$$

$$Z_{ijk} = -T\left(\frac{1}{n^2}p_{[,i}n_{,j} + \mu_{[,i}c_{,j}\right)s_{,k]} = 0, \tag{12}$$

where square brackets denote antisymmetrization. Using the Gibbs 1-form, one can easily see that for the case $c = 1$, i.e., for a one-component fluid, one obtains conditions equivalent to those used by Krasiński *et al* [5–7].

Then, we say that a given cosmological model with a binary mixture of perfect fluids as source satisfies the thermodynamic scheme if the integrability conditions (11) and (12) are fulfilled and if it evolves in accordance with the entropy production law (6).

4. Equations of state and evolution

In this section we exploit the necessary and sufficient integrability condition, derived in the last section, in order to find the relations linking the new variables μ and c with the old ones, and establish the consequences of the entropy production condition. We do this for two general families of inhomogeneous solutions of Einstein's equations, namely the Szekeres and the Stephani solutions.

4.1. Szekeres solutions

The source in these models is an irrotational and geodesic perfect fluid, such that we can find local comoving coordinates in terms of which the metric has the form [23]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2\alpha}\,\mathrm{d}z^2 - \mathrm{e}^{2\beta}(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{13}$$

with $\alpha = \alpha(t, x, y, z)$, $\beta = \beta(t, x, y, z)$, functions to be determined from the field equations. In these coordinates the 4-velocity is $u^\mu = \delta^\mu_0$, which implies $\dot{u}^\mu = 0$ and so $p = p(t)$. For their study, we must consider separately the cases $\beta' = 0$ and $\beta \neq 0$, where $\beta' = \partial\beta/\partial z$.

In the case $\beta' = 0$, the solution is given by

$$\mathrm{e}^\beta = \frac{\Phi}{1 + \frac{1}{4}k(x^2 + y^2)} \tag{14}$$

$$\mathrm{e}^\alpha = \lambda + \Phi\Sigma, \tag{15}$$

where $\Phi = \Phi(t)$, k is an arbitrary constant, $\lambda = \lambda(t, z)$, Σ is determined by

$$\Sigma = \frac{\frac{1}{2}U(x^2 + y^2) + V_1 x + V_2 y + 2W}{1 + \frac{1}{4}k(x^2 + y^2)} \tag{16}$$

with $U = U(z)$, $V_1 = V_1(z)$, $V_2 = V_2(z)$, $W = W(z)$ arbitrary functions, Φ is given by

$$\frac{2\Phi_{,tt}}{\Phi} + \frac{\Phi^2_{,t}}{\Phi^2} + \kappa p + \frac{k}{\Phi^2} = 0 \tag{17}$$

and λ satisfies

$$\lambda_{,tt}\Phi + \lambda_{,t}\Phi_{,t} + \lambda\Phi_{,tt} + \lambda\Phi\kappa p = U + kW. \tag{18}$$

Both equations can be solved once the choice $p = p(t)$ has been made. The matter-energy density is given by

$$\kappa\rho = 2\left(\frac{\lambda\Phi_{,tt}}{\Phi} - \lambda_{,tt}\right)\mathrm{e}^{-\alpha} + \frac{3\Phi^2_{,t}}{\Phi^2} + \frac{3k}{\Phi^2}. \tag{19}$$

This family of spacetimes has in general no isometries, but when $(\lambda/\Phi)_{,t} = 0$ the solution reduces to a FRW model.

In the case $\beta' \neq 0$ we have

$$\mathrm{e}^\beta = \Phi\,\mathrm{e}^\nu \tag{20}$$

$$\mathrm{e}^\alpha = h\,\mathrm{e}^{-\nu}(\mathrm{e}^\beta)_{,z} \tag{21}$$

with $\Phi = \Phi(t, z)$, $\nu = \nu(x, y, z)$, $h = h(z)$ and

$$\mathrm{e}^{-\nu} = A(x^2 + y^2) + 2B_1 x + 2B_2 y + C \tag{22}$$

where $A = A(z)$, $B_1 = B_1(z)$, $B_2 = B_2(z)$, $C = C(z)$ and $h(z)$ are arbitrary functions, Φ is defined by

$$\frac{2\Phi_{,tt}}{\Phi} + \frac{\Phi_{,t}^2}{\Phi^2} + \kappa p \frac{k}{\Phi^2} = 0, \tag{23}$$

where $p = p(t)$ is an arbitrary function and $k = k(z)$ must satisfy

$$AC - B_1^2 - B_2^2 = \tfrac{1}{4}(h^{-2} + k). \tag{24}$$

This family has in general no isometries, but the FRW model results when $\Phi = zR(t)$ and $k = k_0 z^2$, where k_0 is a constant. We now analyse for both families the conditions under which the thermodynamic scheme is satisfied.

Let us consider condition $Z_{ijk} = 0$ for this solution, in which $p_{,i} = 0$, and so according to equation (12) it reduces to

$$\mu_{[,i} c_{,j} s_{,k]} = 0. \tag{25}$$

If we interpret this condition as an algebraic equation for the spatial gradients, ∇, of the variables entering it, the general solution of this equation can be written as $\nabla s = a\nabla\mu + b\nabla c$, with a and b arbitrary functions depending, in general, on all coordinates (t, x^i). On the other hand, the spatial components of the Gibbs 1-form (7) for a geodesic fluid imply the relation

$$\nabla s = \frac{1}{T}\nabla\left(\frac{\rho + p}{n}\right) - \frac{\mu}{T}\nabla c. \tag{26}$$

Hence we can identify $a = 1/T$ and $b = -\mu/T$ so that $\nabla\mu = \nabla[(\rho + p)/n]$. The integration of this last equation

$$\mu = \frac{\rho + p}{n} \tag{27}$$

yields our first equation of state where we are neglecting an additive function of time for two reasons: first, in a two-component system in one phase we have only three degrees of freedom, which, once fixed, determine all the remaining variables in such a way that if we consider ρ, p and n as independent, we cannot add to the variable μ an arbitrary function of time. Second, a chemical potential of this form can be interpreted in a natural way. In fact, considering the Gibbs 1-form as given in equation (7) and the chemical potential (27), we have

$$\mathrm{d}s = \frac{1}{nT}\,\mathrm{d}\rho - \frac{\mu}{nT}\,\mathrm{d}n - \frac{\mu}{T}\,\mathrm{d}c \tag{28}$$

i.e., the chemical potential of the fluid mixture must determine both the matter flow due to inhomogeneities in the total particle number density (with uniform composition) and the matter flow due to inhomogeneities in the composition (with uniform total particle number density). Now, if we substitute μ in the spatial components of the Gibbs 1-form, in terms of ρ and p, which are determined from the field equations, and of n (remember that particle conservation implies $n = f/\sqrt{\triangle}$, where $f(x^i)$ is an arbitrary function of spatial coordinates and $\triangle = \det(g_{ij})$), then we are in a position to recognize the rest of the equations of state of the mixture

$$c = \ln\frac{c_0}{n} \qquad T = \frac{T_0}{n} \qquad s = \frac{\mu}{T}, \tag{29}$$

where c_0 and T_0 are functions of time only. Thus, the system can accommodate, among others, the ideal gas equation of state, which, as simple as it is, is more realistic than the barotropic one.

Consider now the condition $Z_{tij} = 0$. For a geodesic fluid it can be written as the equation

$$\frac{\dot{p}}{n^2}\nabla s \times \nabla n + \dot{s}\nabla c \times \nabla\mu + \dot{\mu}\nabla s \times \nabla c + \dot{c}\nabla\mu \times \nabla s = 0 \tag{30}$$

which by using the components of the Gibbs 1-form (7) reduces to

$$\frac{\dot{p}}{n}\nabla s \times \left(\frac{1}{n}\nabla n + \nabla c\right) = 0. \tag{31}$$

This condition is trivially satisfied in view of the equation of state for c. Finally, we must demand that the set of thermodynamic variables satisfies the entropy production condition (6). By inserting the corresponding values given in equations (27) and (29), it can be shown that all the thermodynamic variables become functions of time only, and that the arbitrary functions which define the metric take the values corresponding to the FRW limit. Consequently, the Szekeres family of solutions does not satisfy the thermodynamic scheme in general, but only in the limiting FRW case.

4.2. Stephani solutions

These models represent the most general conformally flat solution with an irrotational perfect fluid as source, admit in general no isometries and generalize the FRW models. We can find local comoving coordinates in terms of which the metric has the form [23]

$$\mathrm{d}s^2 = D^2\,\mathrm{d}t^2 - V^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{32}$$

where

$$D = \frac{FV_{,t}}{V} \tag{33}$$

$$V = \frac{1}{R}\left\{1 + \frac{1}{4}k[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]\right\} \tag{34}$$

and $F(t)$, $R(t)$, $k(t)$, $x_0(t)$, $y_0(t)$ and $z_0(t)$ are arbitrary functions of time. The 4-velocity and state variables for this metric are given by

$$u^\alpha = D^{-1}\delta_0^\alpha \tag{35}$$

$$\rho = 3C^2 \tag{36}$$

$$p = -3C^2 + 2\frac{VCC_{,t}}{V_{,t}} \tag{37}$$

$$n = fV^3 \tag{38}$$

where $f(x^i)$ is an arbitrary function of spatial coordinates and $C(t)$ is defined by

$$k = R^2\left[C^2 - \frac{1}{F^2}\right]. \tag{39}$$

The FRW limit for this solution is obtained when k, x_0, y_0 and z_0 are all constants, or equivalently, when $V_{,t}/V$ is independent of the spatial coordinates $\{x, y, z\}$.

To study the thermodynamic behaviour of these solutions as a binary mixture of perfect fluids, we first analyse the condition $Z_{ijk} = 0$. Then, considering that in this case the thermodynamic system is the same as in the Szekeres models, we demand that the equations of state (cf equations (27) and (29))

$$\mu = \frac{\rho + p}{n} \qquad s = \frac{\mu}{T} \tag{40}$$

be satisfied. Moreover, introducing equations (40) into the Gibbs 1-form and considering that $\rho_{,i} = 0$, we find that

$$c = \ln\frac{c_0}{ns} \tag{41}$$

with c_0 again a function of time only. On the other hand, the condition $Z_{tij} = 0$ can be written as

$$\frac{\dot{\rho}}{n^2}\nabla p \times \left(\frac{1}{n}\nabla n + \nabla c\right) = 0, \tag{42}$$

where we have used equation (40) and the components of the Gibbs 1-form. Considering now the explicit form of the equations of state for the Stephani solutions, it can be shown that the integrability condition (42) is satisfied if

$$T = \frac{V}{V_{,t}}\frac{T_0}{n}, \tag{43}$$

where T_0 is a function of time only. Furthermore, we have to demand that the thermodynamic variables so defined satisfy the entropy production condition (6). This can be shown to be equivalent to demanding that the expression $V_{,t}/V$ depends on time only, a condition that reduces the Stephani metric to its FRW limit.

5. Concluding remarks

We have introduced the set of thermodynamic variables to describe the most general inhomogeneous cosmological models, considering that the source is a binary mixture of perfect fluids with, in general, non-vanishing entropy production. This process does not involve forcing isometries on the metrics. Nevertheless, demanding the compliance of the entropy production condition reduces the metrics to those of the FRW models. This condition must be regarded as a consequence of the local equilibrium hypothesis, so our results indicate certain incompatibility between thermodynamics and relativity.

We believe that in order to generalize thermodynamics to the relativistic case, one should first answer questions about how should one incorporate the second law of thermodynamics into the thermodynamic scheme and about the invariance of the thermodynamic variables. In particular, we note that when irreversible processes appear due to inhomogeneities in a system, their main effect is, precisely, the vanishing of the inhomogeneities, indicating the need to have a different thermodynamic framework, in which non-uniform equilibrium states should be possible.

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