UC Santa Barbara
UC Santa Barbara Previously Published Works

Title

Two Brownian particles with rank-based characteristics and skew-elastic collisions

Permalink

Journal

Stochastic Processes and their Applications, 123(8)

ISSN

0304-4149

Authors

Fernholz, E Robert

Ichiba, Tomoyuki

Karatzas, Ioannis

Publication Date

2013-08-01

DOI

10.1016/j.spa.2013.03.019

Copyright Information

Peer reviewed

Two Brownian Particles with Rank-Based Characteristics and Skew-Elastic Collisions

E. Robert Fernholz[a], Tomoyuki Ichiba[b,*], Ioannis Karatzas[a,c]

[a] INTECH *Investment Management LLC, One Palmer Square, Suite 441, Princeton, NJ 08542*
[b] *Department of Statistics and Applied Probability, South Hall, University of California, Santa Barbara, CA 93106*
[c] *Department of Mathematics, Columbia University, MailCode 4438, New York, NY 10027*

Abstract

We construct a two-dimensional diffusion process with rank-dependent local drift and dispersion coëfficients, and with a full range of patterns of behavior upon collision that range from totally frictionless interaction, to elastic collision, to perfect reflection of one particle on the other. These interactions are governed by the left- and right-local times at the origin for the distance between the two particles. We realize this diffusion in terms of appropriate, apparently novel systems of stochastic differential equations involving local times, which we show are well posed. Questions of pathwise uniqueness and strength are also discussed for these systems.

  The analysis depends crucially on properties of a skew Brownian motion with two-valued drift of the bang-bang type, which we also study in some detail. These properties allow us to compute the transition probabilities of the original planar diffusion, and to study its behavior under time reversal.

Keywords: Diffusion, Local Time, Skew Brownian Motion, Time Reversal, Brownian Motion reflected on Brownian motion.
AMS 2000 Subject Classifications : Primary 60H10 · 60G44; secondary 60J55 · 60J60

1. Introduction

We construct a planar diffusion $(X_1(\cdot), X_2(\cdot))$ according to the following recipe: each of its component particles $X_1(\cdot)$ and $X_2(\cdot)$ behaves locally like Brownian motion. The characteristics of these random motions are assigned not by name, but by rank: the leader is assigned drift $-h \leq 0$ and dispersion $\rho \geq 0$, whereas the laggard is assigned drift $g \geq 0$ and dispersion $\sigma \geq 0$. One of the dispersions is allowed to vanish, but not both; similarly for the drifts. In the interest of concreteness and simplicity, we shall set

$$\lambda := g + h > 0, \qquad \rho^2 + \sigma^2 = 1. \qquad (1.1)$$

A bit more precisely, we shall construct a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ endowed with a filtration $\mathbf{F} = \{\mathfrak{F}(t)\}_{0 \leq t < \infty}$ that satisfies the "usual conditions" of right continuity and of augmentation by $\mathbb{P}$−negligible sets, and on it two pairs $(B_1(\cdot), B_2(\cdot))$ and $(X_1(\cdot), X_2(\cdot))$ of continuous, $\mathbf{F}$−adapted processes, such that $(B_1(\cdot), B_2(\cdot))$ is planar Brownian motion and $(X_1(\cdot), X_2(\cdot))$ a

---

continuous planar semimartingale that starts at some given site $(X_1(0), X_2(0)) = (x_1, x_2) \in \mathbb{R}^2$ on the plane and satisfies the dynamics

$$\mathrm{d}X_1(t) = \left(g\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}} - h\,\mathbf{1}_{\{X_1(t) > X_2(t)\}}\right)\mathrm{d}t + \left(\rho\,\mathbf{1}_{\{X_1(t) > X_2(t)\}} + \sigma\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}}\right)\mathrm{d}B_1(t)$$

$$+ \frac{1 - \zeta_1}{2}\,\mathrm{d}L^{X_1 - X_2}(t) + \frac{1 - \eta_1}{2}\,\mathrm{d}L^{X_2 - X_1}(t), \qquad (1.2)$$

$$\mathrm{d}X_2(t) = \left(g\,\mathbf{1}_{\{X_1(t) > X_2(t)\}} - h\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}}\right)\mathrm{d}t + \left(\rho\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}} + \sigma\,\mathbf{1}_{\{X_1(t) > X_2(t)\}}\right)\mathrm{d}B_2(t)$$

$$+ \frac{1 - \zeta_2}{2}\,\mathrm{d}L^{X_1 - X_2}(t) + \frac{1 - \eta_2}{2}\,\mathrm{d}L^{X_2 - X_1}(t). \qquad (1.3)$$

Here and in the sequel we denote by $L^X(\cdot) \equiv L^X(\cdot\,; 0)$ the right-continuous local time accumulated at the origin by a generic continuous semimartingale $X(\cdot)$, by $L_-^X(\cdot) \equiv L^{-X}(\cdot\,; 0)$ its left-continuous version, and by $\widehat{L}^X(\cdot) = (L^X(\cdot) + L_-^X(\cdot))/2$ its symmetric version; we collect in section 2 the necessary reminders from the theory of semimartingale local time.

Each time the particles collide, their trajectories are "dragged" by amounts proportional to the right local times, $L^{X_1 - X_2}(t)$ and $L^{X_2 - X_1}(t)$, respectively, that have been accumulated up to that instant t at the origin by the differences $X_1(\cdot) - X_2(\cdot)$ and $X_2(\cdot) - X_1(\cdot)$; this is the significance of the last two terms in each of (1.2), (1.3). With the notation

$$\zeta := 1 + \frac{\zeta_1 - \zeta_2}{2}, \qquad \eta := 1 - \frac{\eta_1 - \eta_2}{2}, \qquad (1.4)$$

the proportionality constants of these interactions, ζ_i and η_i for $X_i(\cdot)$ $(i = 1, 2)$, will be assumed to satisfy the conditions

$$\zeta + \eta \ne 0, \qquad 0 \le \alpha := \frac{\eta}{\eta + \zeta} \le 1. \qquad (1.5)$$

We shall discuss in detail the significance of these conditions for the system of equations (1.2), (1.3); in particular, the fact that they are not only sufficient but also necessary for the well-poseness of the above system of stochastic equations. For the time being, let us note that in the special case $\zeta_1 = \eta_1 = 1$, the trajectory $X_1(\cdot)$ of the first particle crosses the trajectory $X_2(\cdot)$ of the second particle without "feeling" it, that is, without being subjected to any local time drag; as we shall see in subsection 5.2, we obtain this same effect under the more general condition (5.5). Likewise, the second particle does not "feel" the first, when $\zeta_2 = \eta_2 = 1$ or, more generally, under the condition (5.6). When $\zeta_i = \eta_i = 1$ $(i = 1, 2)$, the local times vanish completely from (1.2), (1.3) and we are in the situation studied in detail by FERNHOLZ ET AL. [7]. In this case, the collisions of the particles are totaly frictionless.

At the other extreme $\zeta = 0 \ne \eta$ (respectively, $\eta = 0 \ne \zeta$) the trajectory $X_1(\cdot)$ of the first particle bounces off the trajectory $X_2(\cdot)$ of the second particle (resp., the other way round), as if the latter trajectory were a perfectly reflecting boundary; cf. subsection 5.1. Think of the second (resp., the first) particle as being "heavy", so that in collisions with the "light" first (resp., second) particle its motion is unaffected, while the light particle undergoes perfect reflection.

In between, for other values of the parameters, we have collisions that are neither totally frictionless (without local time drag), nor perfectly reflecting, but "elastic": The particles are subjected

2

in general to local time drag, and this kind of friction manifests itself in an asymmetric fashion –
due to the presence of both right- and left- local times at the origin $L^Y(\cdot) \equiv L^Y(\cdot\,;0+)$ and
$L^Y_-(\cdot) \equiv L^Y(\cdot\,;0-)$ in (1.2), (1.3) for the difference $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$. We call such colli-
sions "skew-elastic".

1.1. Preview

Under the conditions of (1.5), the system of equations (1.2), (1.3) will be shown in section 4 to
admit a weak solution, which is unique in the sense of the probability distribution; cf. Theorem
4.1. Using a common terminology: under the conditions of (1.5), the system of equations (1.2),
(1.3) is *well posed.*

A crucial rôle in establishing this result will be played by the properties of the difference
$Y(\cdot) = X_1(\cdot) - X_2(\cdot)$, for which we show that

$$W(t) := Y(t) - (x_1 - x_2) + \lambda \int_0^t \mathrm{sgn}\big(Y(s)\big)\,\mathrm{d}s - 2\,(2\,\alpha - 1)\,\widehat{L}^Y(t)\,, \qquad 0 \le t < \infty$$

is standard Brownian motion $W(\cdot)$ in the notation of (1.1), (1.5). To put a little differently: we
identify this difference $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ as a so-called *Skew Brownian Motion with Bang-
Bang drift*, a process that we study in detail in Section 6.

Similarly, recalling the notation of (1.4) and setting

$$\beta := \frac{\eta\,(\zeta_1 + \zeta_2) + \zeta(\eta_1 + \eta_2)}{2\,(\eta + \zeta)}\,,$$

we identify the process

$$V(t) := X_1(t) + X_2(t) - (x_1 + x_2) - (g - h)\,t - 2\,(1 - \beta)\,\widehat{L}^Y(t)\,, \qquad 0 \le t < \infty$$

as another standard Brownian motion, whose cross-variation with the Brownian motion $W(\cdot)$ is

$$\langle V, W \rangle(\cdot) = \langle V, Y \rangle(\cdot) = \gamma \int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\,\mathrm{d}t\,, \qquad \gamma := \rho^2 - \sigma^2\,.$$

These identifications allow us then to represent the motions $X_1(\cdot)$, $X_2(\cdot)$ of the individual parti-
cles in the form

$$X_1(t) = x_1 + \mu\,t + \rho^2\big(Y^+(t) - y^+\big) - \sigma^2\big(Y^-(t) - y^-\big) + \big(1 - \beta - \gamma\big)\,\widehat{L}^Y(t) + \rho\,\sigma\,Q(t)\,,$$

$$X_2(t) = x_2 + \mu\,t - \sigma^2\big(Y^+(t) - y^+\big) + \rho^2\big(Y^-(t) - y^-\big) + \big(1 - \beta - \gamma\big)\,\widehat{L}^Y(t) + \rho\,\sigma\,Q(t)\,;$$

here $Q(\cdot)$ is yet another standard Brownian motion, independent of the difference $Y(\cdot) = X_1(\cdot) -
X_2(\cdot)$, and

$$\mu = g\,\rho^2 - h\,\sigma^2\,.$$

This way we construct a weak solution to the system of equations (1.2), (1.3), and also show that
uniqueness in distribution holds for it.

Always under the conditions of (1.5), the system of equations (1.2), (1.3) is shown in section 4
actually to admit a *pathwise unique, strong solution;* cf. Theorem 4.2. Here we refine the LE GALL

3

[15] [16] methodology, that we used in the recent work FERNHOLZ ET AL. [7] to establish pathwise uniqueness for a generalization of the perturbed TANAKA equation of PROKAJ [26].

In fact, *the conditions in (1.5) turn out to be not just sufficient but also necessary for the well-posedness of the system (1.2), (1.3);* cf. Proposition 6.1. As we shall see in Remarks 3.1 and 3.2, this system admits no solution in the case $\eta = -\zeta \neq 0$; whereas it has lots of solutions, i.e., uniqueness in distribution fails for the system of equations (1.2) and (1.3), when $\eta = \zeta = 0$. Finally, if we do have $\eta + \zeta \neq 0$ yet (1.5) fails because $\alpha \notin [0, 1]$, it is seen in Remark 6.1 that the system (1.2), (1.3) once again fails to admit a solution.

Section 5 discusses some special configurations of the parameters η_i, ζ_i $(i = 1, 2)$ in (1.2), (1.3) that give rise to some rather interesting structure. We see, in particular in the non-degenerate case $\rho\,\sigma > 0$, that when $\beta = 0$ (respectively, $\beta = 2$), the trajectory $X_1(\cdot) \vee X_2(\cdot)$ of the "leader" (respectively, $X_1(\cdot) \wedge X_2(\cdot)$ of the "laggard") is Brownian motion with drift, with perfect reflection on the trajectory $X_1(\cdot) \wedge X_2(\cdot)$ of the "laggard" (respectively, $X_1(\cdot) \vee X_2(\cdot)$ of the "leader"), which is then another, independent Brownian motion with drift.

Section 6 develops the theory and properties of the Skew Brownian Motion with Bang-Bang drift. Finally, Section 7 uses these properties to compute the transition probabilities and the time-reversal of the planar diffusion $(X_1(\cdot), X_2(\cdot))$.

1.2. Extant Work and Open Questions

The study of multidimensional stochastic differential equations that involve a local time supported on a smooth hypersurface starts with the work of ANULOVA [1] and PORTENKO [22] [23] [24] [25]. To the best of our knowledge, systems of stochastic equations of the type

$$X_i(\cdot) = x_i + B_i(\cdot) + \sum_{j \neq i} q_{ij}\, \widehat{L}^{X_i - X_j}(\cdot)\,, \qquad i = 1, \cdots, n \tag{1.6}$$

for a suitable array of real constants $(q_{ij})_{1 \leq i, j \leq n}$, with $B_1(\cdot), \cdots, B_n(\cdot)$ independent standard Brownian motions, were studied first by SZNITMAN & VARADHAN [29]. In fact, these authors consider the more general model

$$\boldsymbol{X}(t) = \boldsymbol{x} + \boldsymbol{B}(t) + \sum_{k=1}^{N} \boldsymbol{q}_k\, \widehat{L}^{\boldsymbol{n}_k \cdot \boldsymbol{X}}(t)\,, \quad 0 \leq t < \infty\,, \tag{1.7}$$

where $\boldsymbol{X}(\cdot) := (X_1(\cdot), \ldots, X_n(\cdot))'$, $\boldsymbol{B}(\cdot) := (B_1(\cdot), \cdots, B_n(\cdot))'$ is Brownian motion in $\mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{R}^n$, the unit column vectors $\boldsymbol{n}_k$ generate pairwise distinct hyperplanes, and the column vectors $\boldsymbol{q}_k$ satisfy the orthogonality conditions $\boldsymbol{q}_k \cdot \boldsymbol{n}_k = 0$ for $k = 1, \ldots, N$. When $g = h = 0$ and $\sigma = \rho$, it can be verified – using the relationships (3.16) between the symmetric local time and the right local time – that the system (1.2)-(1.3) is equivalent to the model (1.7) with parameters $n = 2$, $N = 1$,

$$\boldsymbol{n}_1 := (1, -1)' / \sqrt{2}\,, \quad \text{and} \quad \boldsymbol{q}_1 := \big(\alpha(1-\zeta_1) + (1-\eta_1)(1-\alpha)\,, \ \alpha(1-\zeta_2) + (1-\eta_2)(1-\alpha)\big)'.$$

The orthogonality conditions amount then to $\eta = \zeta$. Thus, we can apply the results of SZNITMAN & VARADHAN [29], if $g = h = 0$, $\sigma = \rho$, $\eta = \zeta$ in our system (1.2)-(1.3).

4

There are rather obvious similarities, as well as differences, between the system (1.6) and that of (1.2), (1.3). In particular, it would be very interesting to extend the results of this paper to systems of stochastic differential equations of the type

$$X_i(\cdot) = x_i + \sum_{k=1}^{n} \int_0^{\cdot} \delta_k \, \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}} \, \mathrm{d}t + \sum_{k=1}^{n} \int_0^{\cdot} \sigma_k \, \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}} \, \mathrm{d}B_i(t)$$

$$+ \sum_{j \neq i} \left(q_{ij}^+ \, L^{X_i - X_j}(\cdot) + q_{ij}^- \, L^{X_j - X_i}(\cdot) \right), \qquad i = 1, \cdots, n \qquad (1.8)$$

for an arbitrary number $n \in \mathbb{N}$ of particles, with $x_1, \cdots, x_n$ and $\delta_1, \cdots, \delta_n$ given real constants, with $\sigma_1, \cdots, \sigma_n$ given positive constants, with suitable arrays $(q_{ij}^{\pm})_{1 \leq i,j \leq n}$ of real constants, the "descending order statistics" notation

$$\max_{1 \leq j \leq n} X_j(t) =: X_{(1)}(t) \geq X_{(2)}(t) \geq \cdots \geq X_{(n-1)}(t) \geq X_{(n)}(t) := \min_{1 \leq j \leq n} X_j(t),$$

and lexicographic breaking of ties. This system (1.8) exhibits both features of rank-dependent characteristics and skew-elastic collisions that are manifest in (1.2), (1.3), but involves several particles rather than just two.

2. On Semimartingale Local Time

Let us recall the notion of a continuous, real-valued semimartingale

$$X(\cdot) = X(0) + M(\cdot) + C(\cdot), \qquad (2.1)$$

where $M(\cdot)$ is a continuous local martingale and $C(\cdot)$ a continuous process of finite first variation such that $M(0) = C(0) = 0$. The *local time* $L^X(t; \xi)$ accumulated at a given "site" $\xi \in \mathbb{R}$ over the time-interval $[0, t]$ by this process, is

$$L^X(t; \xi) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{\xi \leq X(s) < \xi + \varepsilon\}} \, \mathrm{d}\langle X \rangle(s)$$

$$= \left(X(t) - \xi \right)^+ - \left(X(0) - \xi \right)^+ - \int_0^t \mathbf{1}_{\{X(s) > \xi\}} \, \mathrm{d}X(s), \qquad (2.2)$$

where $\langle X \rangle(\cdot) \equiv \langle M \rangle(\cdot)$. For every fixed $\xi \in \mathbb{R}$ this defines a nondecreasing, continuous and adapted process $L^X(\cdot; \xi)$ which is flat off the set $\{t \geq 0 : X(t) = \xi\}$, namely

$$\int_0^\infty \mathbf{1}_{\{X(t) \neq \xi\}} \, \mathrm{d}L^X(t; \xi) = 0; \quad \text{and we have also the property} \quad \int_0^\infty \mathbf{1}_{\{X(t) = \xi\}} \, \mathrm{d}\langle X \rangle(t) = 0.$$
$$(2.3)$$

On the other hand, for each fixed $T \in (0, \infty)$ the mapping $\xi \mapsto L^X(T; \xi)$ is almost surely RCLL (Right-Continuous on $[0, \infty)$, with Limits from the Left on $(0, \infty)$) paths, and has jumps of size

$$L^X(T; \xi) - L^X(T; \xi-) = \int_0^T \mathbf{1}_{\{X(t) = \xi\}} \, \mathrm{d}X(t) = \int_0^T \mathbf{1}_{\{X(t) = \xi\}} \, \mathrm{d}C(t). \qquad (2.4)$$

5

We shall employ also the notation

$$\widehat{L}^X(T;\xi) := \frac{1}{2}\left(L^X(T;\xi) + L^X(T;\xi-)\right) \tag{2.5}$$

for the so-called "symmetric local time" accumulated at the site ξ over the time interval $[0,T]$. For these local times we prefer to use the simpler notation

$$L^X(\cdot) \equiv L^X(\cdot\,;0), \qquad L_-^X(\cdot) \equiv L^X(\cdot\,;0-), \qquad \widehat{L}^X(\cdot) \equiv \widehat{L}^X(\cdot\,;0) \tag{2.6}$$

when we evaluate them at the origin $\xi = 0$, and note

$$L_-^X(\cdot) = L^{-X}(\cdot) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\,\varepsilon}\int_0^{\cdot}\mathbf{1}_{\{0\ge X(t)>-\varepsilon\}}\,\mathrm{d}\langle X\rangle(t)\,. \tag{2.7}$$

Finally, we recall the occupation time density formulae

$$\int_0^{\cdot}h(X(t))\,\mathrm{d}\langle X\rangle(t) = 2\int_{\mathbb{R}}L^X(\cdot\,;\xi)\,h(\xi)\,\mathrm{d}\xi = 2\int_{\mathbb{R}}\widehat{L}^X(\cdot\,;\xi)\,h(\xi)\,\mathrm{d}\xi\,, \tag{2.8}$$

valid for every Borel measurable function $h:\mathbb{R}\to[0,\infty)$, as well as the ITÔ-TANAKA formulae

$$f(X(\cdot)) = f(X(0)) + \int_0^{\cdot}D^-f(X(t))\,\mathrm{d}X(t) + \int_{\mathbb{R}}L^X(\cdot\,;\xi)\,f''(\mathrm{d}\xi)\,, \tag{2.9}$$

$$f(X(\cdot)) = f(X(0)) + \frac{1}{2}\int_0^{\cdot}\left(D^+f(X(t)) + D^-f(X(t))\right)\mathrm{d}X(t) + \int_{\mathbb{R}}\widehat{L}^X(\cdot\,;\xi)\,f''(\mathrm{d}\xi)\,. \tag{2.10}$$

Here $f:\mathbb{R}\to\mathbb{R}$ is the difference of two convex functions, $D^{\pm}f(\cdot)$ denote its derivatives from left and right, and $f''(\cdot)$ denotes its second derivative measure.

2.1. Tanaka Formulae

For a continuous, real-valued semimartingale $X(\cdot)$ as in (2.1), and with the conventions

$$\overline{\mathrm{sgn}}\,(x) := \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x)\,, \quad \mathrm{sgn}\,(x) := \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0]}(x)\,, \qquad x\in\mathbb{R} \tag{2.11}$$

for the symmetric and the left-continuous versions of the signum function, we obtain from (2.9), (2.10) the TANAKA formulae

$$|X(\cdot) - \xi| = |X(0) - \xi| + \int_0^{\cdot}\mathrm{sgn}\big(X(t) - \xi\big)\,\mathrm{d}X(t) + 2\,L^X(\cdot\,;\xi) \tag{2.12}$$

$$= |X(0) - \xi| + \int_0^{\cdot}\overline{\mathrm{sgn}}\big(X(t) - \xi\big)\,\mathrm{d}X(t) + 2\,\widehat{L}^X(\cdot\,;\xi)\,. \tag{2.13}$$

Applying (2.12) with $\xi = 0$ to the continuous, nonnegative semimartingale $|X(\cdot)|$, then comparing with the expression of (2.12) itself, we obtain the companion

$$2\,L^X(\cdot) - L^{|X|}(\cdot) = \int_0^{\cdot}\mathbf{1}_{\{X(t)=0\}}\,\mathrm{d}X(t) = \int_0^{\cdot}\mathbf{1}_{\{X(t)=0\}}\,\mathrm{d}C(t) \tag{2.14}$$

of the property (2.4). For the theory that undergirds these results we refer, for instance, to KARATZAS & SHREVE [13], section 3.7.

3. Analysis

Let us suppose that such a probability space as stipulated in section 1 has been constructed, and on it a pair $B_1(\cdot)$, $B_2(\cdot)$ of independent standard Brownian motions, as well as two continuous, nonnegative semimartingales $X_1(\cdot)$, $X_2(\cdot)$ such that the dynamics (1.2)-(1.3) are satisfied. We import the notation of FERNHOLZ ET AL. [7] : in addition to (1.1), we set

$$\nu = g - h\,, \quad y = x_1 - x_2\,, \quad z := x_1 + x_2 > 0\,, \quad r_1 = x_1 \vee x_2\,, \quad r_2 = x_1 \wedge x_2\,, \quad (3.1)$$

and introduce the difference and the sum of the two component processes, namely

$$Y(\cdot) := X_1(\cdot) - X_2(\cdot)\,, \qquad Z(\cdot) := X_1(\cdot) + X_2(\cdot)\,. \qquad (3.2)$$

3.1. Auxiliary Brownian Motions

We introduce also the two planar Brownian motions $\big(W_1(\cdot), W_2(\cdot)\big)$ and $\big(V_1(\cdot), V_2(\cdot)\big)$, given respectively by

$$W_1(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}B_1(t) - \int_0^\cdot \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}B_2(t)\,, \qquad (3.3)$$

$$W_2(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}B_1(t) - \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}B_2(t) \qquad (3.4)$$

and

$$V_1(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}B_1(t) + \int_0^\cdot \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}B_2(t)\,, \qquad (3.5)$$

$$V_2(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}B_1(t) + \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}B_2(t)\,. \qquad (3.6)$$

Finally, we construct the Brownian motions $W(\cdot)$, $V(\cdot)$, $Q(\cdot)$ and $W^\flat(\cdot)$, $V^\flat(\cdot)$, $U^\flat(\cdot)$ as

$$W(\cdot) := \rho\, W_1(\cdot) + \sigma\, W_2(\cdot)\,, \quad V(\cdot) := \rho\, V_1(\cdot) + \sigma\, V_2(\cdot)\,, \quad Q(\cdot) := \sigma\, V_1(\cdot) + \rho\, V_2(\cdot)\,, \quad (3.7)$$

$$W^\flat(\cdot) := \rho\, W_1(\cdot) - \sigma\, W_2(\cdot)\,, \quad V^\flat(\cdot) := \rho\, V_1(\cdot) - \sigma\, V_2(\cdot)\,, \quad U^\flat(\cdot) := \sigma\, W_1(\cdot) - \rho\, W_2(\cdot)\,; \quad (3.8)$$

we note the independence of $Q(\cdot)$ and $W(\cdot)$, the independence of $Q(\cdot)$ and $V^\flat(\cdot)$, and observe the intertwinements

$$V_j(\cdot) = (-1)^{j+1} \int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\mathrm{d}W_j(t) \quad (j=1,2)\,, \qquad V^\flat(\cdot) = \int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\mathrm{d}W(t) \quad (3.9)$$

and

$$V(\cdot) = \int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\mathrm{d}W^\flat(t)\,, \qquad Q(\cdot) = \int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\mathrm{d}U^\flat(t)\,. \qquad (3.10)$$

3.2. *The Difference and the Sum*

After this preparation, we observe that the difference $Y(\cdot)$ and the sum $Z(\cdot)$ from (3.2) satisfy, respectively, the stochastic integral equation

$$Y(\cdot) = y - \lambda \int_0^\cdot \operatorname{sgn}\big(Y(t)\big)\,\mathrm{d}t + (1 - \zeta)\, L^Y(\cdot) - (1 - \eta)\, L_-^Y(\cdot) + W(\cdot) \qquad (3.11)$$

which involves both the right- and the left- local time at the origin of its solution process $Y(\cdot)$, and the identity

$$Z(t) = z + \nu\, t + V(t) + \big(1 - \overline{\zeta}\,\big)\, L^Y(t) + \big(1 - \overline{\eta}\,\big)\, L_-^Y(t)\,, \qquad 0 \le t < \infty\,. \qquad (3.12)$$

We have used here the notation of (1.1), (3.1), (1.4), as well as

$$\overline{\zeta} := \frac{\zeta_1 + \zeta_2}{2}\,, \qquad \overline{\eta} := \frac{\eta_1 + \eta_2}{2}\,. \qquad (3.13)$$

We note from (2.3) and (3.11) that $Y(\cdot)$ is a continuous semimartingale with

$$\int_0^\infty \mathbf{1}_{\{Y(t)=0\}}\,\mathrm{d}t = \int_0^\infty \mathbf{1}_{\{Y(t)=0\}}\,\mathrm{d}\langle Y\rangle(t) = 0\,, \qquad \text{a.s.}, \qquad (3.14)$$

and that on the strength of (2.4), (2.3) we have

$$L^Y(\cdot\,;0) - L^Y(\cdot\,;0-) = \int_0^\cdot \mathbf{1}_{\{Y(t)=0\}}\big((1 - \zeta)\,\mathrm{d}L^Y(t;0) - (1 - \eta)\,\mathrm{d}L^Y(t;0-)\big)$$

$$= (1 - \zeta)\, L^Y(\cdot\,;0) - (1 - \eta)\, L^Y(\cdot\,;0-)$$

or equivalently

$$\zeta\, L^Y(\cdot) = \eta\, L_-^Y(\cdot)\,. \qquad (3.15)$$

From this relationship and (2.3)-(2.5), (2.14) we obtain

$$2\,\widehat{L}^Y(\cdot) = L^{|Y|}(\cdot) \qquad \text{and} \qquad L^Y(\cdot) = \alpha\, L^{|Y|}(\cdot)\,, \quad L_-^Y(\cdot) = (1 - \alpha)\, L^{|Y|}(\cdot)\,, \qquad (3.16)$$

where we introduce as in (1.5) the "skewness parameter"

$$\alpha := \frac{\eta}{\eta + \zeta}\,. \qquad (3.17)$$

- With this notation, and recalling (3.14) and (3.16), the equation (3.11) takes the form

$$Y(\cdot) = y - \lambda \int_0^\cdot \overline{\operatorname{sgn}}\big(Y(t)\big)\,\mathrm{d}t + 2\,\big(2\,\alpha - 1\big)\,\widehat{L}^Y(\cdot) + W(\cdot) \qquad (3.18)$$

of the equation for a *Skew Brownian Motion with Bang-Bang drift* (Skew Bang-Bang Brownian Motion, or SBBBM for short). This is a very close relative of the Skew Brownian motion, that was introduced by ITÔ & MCKEAN [10], [11] and was furhter studied by WALSH [30], HARRISON & SHEPP [9] ; see LEJAY [17] for a comprehensive survey.

The diffusion process $Y(\cdot)$ of (3.18) is studied in detail in section 6. It is a strong MARKOV and FELLER process, whose transition probabilities can be computed explicitly; see (6.13)-(6.15) below. In particular, it is shown in section 6 that, for $0 \le \alpha \le 1$, the stochastic equation (3.18) has a pathwise unique, strong solution and that the filtration identities

$$\mathfrak{F}^Y(t) = \mathfrak{F}^W(t), \qquad \forall \ t \in [0, \infty) \tag{3.19}$$

hold. Here and in what follows, given a process $\Xi : [0, \infty) \times \Omega \to \mathbb{R}^d$ with values in some Euclidean space and RCLL paths, we shall use the convention

$$\mathbf{F}^\Xi = \{\mathfrak{F}^\Xi(t)\}_{0 \le t < \infty}$$

for the smallest filtration to which $\Xi(\cdot)$ is adapted that satisfies the "usual conditions" of right continuity and augmentation by sets of measure zero.

• Similarly, and with the notation of (3.17), the expression (3.12) takes the form

$$X_1(t) + X_2(t) = Z(t) = z + \nu\, t + V(t) + 2\left(1 - \beta\right) \widehat{L}^Y(t), \qquad 0 \le t < \infty \tag{3.20}$$

where, as in subsection 1.1, we set

$$\beta := \frac{\eta\, \overline{\zeta} + \zeta\, \overline{\eta}}{\eta + \zeta}\,. \tag{3.21}$$

Remark 3.1. It is clear from (3.15) that the equation (3.11) can be written as

$$Y(\cdot) = y - \lambda \int_0^{\cdot} \operatorname{sgn}(Y(t))\, \mathrm{d}t + L^Y(\cdot) - L_-^Y(\cdot) + W(\cdot)\,. \tag{3.22}$$

To wit: skew Brownian motion with bang-bang drift solves the equation (3.22), for *any* value $\alpha \in [0, 1]$ of its skewness parameter. We conclude that *uniqueness in distribution fails for this equation (3.22), thus also for the equation (3.11) that governs the difference* $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ *when* $\eta = \zeta = 0\,.$

In particular, uniqueness in distribution cannot possibly hold for the system (1.2), (1.3) in this case $\eta = \zeta = 0\,.$

Remark 3.2. When $\eta = -\zeta \ne 0$ we get $L^Y(\cdot) + L_-^Y(\cdot) \equiv 0$ from (3.15), thus

$$L^Y(\cdot) \equiv L_-^Y(\cdot) \equiv 0\,, \tag{3.23}$$

and the equation (3.11) takes the form of Brownian motion with bang-bang drift

$$Y(\cdot) = y - \lambda \int_0^{\cdot} \operatorname{sgn}(Y(t))\, \mathrm{d}t + W(\cdot)\,. \tag{3.24}$$

This diffusion process was studied in detail by KARATZAS & SHREVE [12], who computed its transition probabilities and the joint distribution of the triple $\left(Y(t),\, L^Y(t),\, \int_0^t \mathbf{1}_{\{Y(s)>0\}}\, \mathrm{d}s \right)$. This diffusion *does* accumulate local time at the origin: indeed, on the strength of (2.4), (2.3), we have

$$L^Y(\cdot) - L_-^Y(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{Y(t)=0\}}\, \mathrm{d}Y(t) = \lambda \int_0^{\cdot} \mathbf{1}_{\{Y(t)=0\}}\, \mathrm{d}t = \lambda \int_0^{\cdot} \mathbf{1}_{\{Y(t)=0\}}\, \mathrm{d}\langle W\rangle(t) = 0$$

almost surely, but also $\mathbb{P}(L^Y(t) = L_-^Y(t) > 0) > 0$ for every $t \in (0, \infty)\,$; this contradicts (3.23). In fact, we have $\mathbb{P}(L^Y(t) = L_-^Y(t) > 0) = 1$ for $y = 0\,.$

We conclude that *the equation (3.11) for the difference* $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ *has no solution in the case* $\eta = -\zeta \ne 0\,.$ Thus, the system (1.2), (1.3) cannot possibly have a solution in this case.

3.3. Auxiliary Systems

From the equations of (3.11), (3.12) and using the notation in (3.2)-(3.7), we obtain a system of stochastic differential equations

$$\mathrm{d}X_1(t) \, = \, \Big(g \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} - h \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} \Big) \, \mathrm{d}t \, + \, \rho \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} \, \mathrm{d}W_1(t)$$

$$+ \, \sigma \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} \, \mathrm{d}W_2(t) \, + \, \frac{1-\zeta_1}{2} \, \mathrm{d}L^{X_1-X_2}(t) \, + \, \frac{1-\eta_1}{2} \, \mathrm{d}L^{X_2-X_1}(t) \,, \quad (3.25)$$

$$\mathrm{d}X_2(t) \, = \, \Big(g \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} - h \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} \Big) \, \mathrm{d}t \, - \, \rho \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} \, \mathrm{d}W_1(t)$$

$$- \, \sigma \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} \, \mathrm{d}W_2(t) \, + \, \frac{1-\zeta_2}{2} \, \mathrm{d}L^{X_1-X_2}(t) \, + \, \frac{1-\eta_2}{2} \, \mathrm{d}L^{X_2-X_1}(t) \,, \quad (3.26)$$

quite similar to that of (1.2), (1.3), but now driven by the planar Brownian motion $(W_1(\cdot), W_2(\cdot))$. In a totally analogous manner, we obtain also the system

$$\mathrm{d}X_1(t) \, = \, \Big(g \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} - h \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} \Big) \, \mathrm{d}t \, + \, \rho \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} \, \mathrm{d}V_1(t)$$

$$+ \, \sigma \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} \, \mathrm{d}V_2(t) \, + \, \frac{1-\zeta_1}{2} \, \mathrm{d}L^{X_1-X_2}(t) \, + \, \frac{1-\eta_1}{2} \, \mathrm{d}L^{X_2-X_1}(t) \,, \quad (3.27)$$

$$\mathrm{d}X_2(t) \, = \, \Big(g \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} - h \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} \Big) \, \mathrm{d}t \, + \, \rho \, \mathbf{1}_{\{X_1(t)\le X_2(t)\}} \, \mathrm{d}V_1(t)$$

$$+ \, \sigma \, \mathbf{1}_{\{X_1(t)>X_2(t)\}} \, \mathrm{d}V_2(t) \, + \, \frac{1-\zeta_2}{2} \, \mathrm{d}L^{X_1-X_2}(t) \, + \, \frac{1-\eta_2}{2} \, \mathrm{d}L^{X_2-X_1}(t) \,, \quad (3.28)$$

now driven by the planar Brownian motion $(V_1(\cdot), V_2(\cdot))$.

3.4. Skew Representations

In light of the TANAKA formula (2.13), of the equation (3.18) for the semimartingale $Y(\cdot)$, and of the last intertwinement in (3.9), we represent now the size of the "gap" between $X_1(t)$ and $X_2(t)$ as

$$\big|Y(t)\big| \, = \, |y| - \lambda \, t + V^\flat(t) + 2 \, \widehat{L}^Y(t) \qquad\qquad (3.29)$$

$$= \, |y| - \lambda \, t + V^\flat(t) + \widehat{L}^{|Y|}(t) \,, \qquad 0 \le t < \infty \,.$$

With the help of (3.9), (3.16), let us write the first Brownian motion in (3.8) as

$$W^\flat(\cdot) \, = \, \gamma \, W(\cdot) + \delta \, U^\flat(\cdot) \,, \qquad \text{where} \quad \gamma := \rho^2 - \sigma^2 \,, \quad \delta := \sqrt{1-\gamma^2} = 2 \, \rho \, \sigma \,.$$

With this notation, and with the help of (3.10), the Brownian motion $V(\cdot)$ in (3.7) takes the form

$$V(t) \, = \, \gamma \, V^\flat(t) + \delta \, Q(t) \, = \, \gamma \left(|Y(t)| - |y| + \lambda \, t - 2 \, \widehat{L}^Y(t) \right) + \delta \, Q(t) \,, \qquad 0 \le t < \infty \,.$$

We recall here from (3.10) the standard Brownian motion $Q(\cdot)$ which, being independent of $W(\cdot)$, is also independent of the process $Y(\cdot)$ in light of (3.19).

10

In conjunction with $X_1(t) - X_2(t) = Y(t)$ and the representation (3.20) for $X_1(t) + X_2(t)$, and with the notation

$$\mu := \frac{1}{2}\left(\nu + \lambda\,\gamma\right) = g\,\rho^2 - h\,\sigma^2\,,$$

we obtain from this expression the *skew representations* for the component processes themselves

$$X_1(t) = x_1 + \mu\,t + \rho^2\big(Y^+(t) - y^+\big) - \sigma^2\big(Y^-(t) - y^-\big) + \big(1 - \beta - \gamma\big)\,\widehat{L}^Y(t) + \rho\sigma Q(t) \quad (3.30)$$

$$X_2(t) = x_2 + \mu\,t - \sigma^2\big(Y^+(t) - y^+\big) + \rho^2\big(Y^-(t) - y^-\big) + \big(1 - \beta - \gamma\big)\,\widehat{L}^Y(t) + \rho\sigma Q(t) \quad (3.31)$$

in terms of the paths of the skew Brownian motion process $Y(\cdot)$ with bang-bang drift, and of the *independent* Brownian motion $Q(\cdot)$. In particular, this shows that *uniqueness in distribution* holds for the system of stochastic differential equations (1.2), (1.3).

Similar reasoning shows that uniqueness in distribution holds also for each of the systems (3.25), (3.26) and (3.27), (3.28).

Remark 3.3. It is clear from (3.29) that the absolute value of the skew Brownian motion with bang-bang drift in (3.18), for *any* value $\alpha \in [0,1]$ of the skewness parameter, is Brownian motion with drift $-\lambda$ and reflection at the origin. Arguing as in WALSH [30], Proposition 1, one can conclude that every diffusion process $Y(\cdot)$, for which $|Y(\cdot)|$ is Brownian motion with drift $-\lambda$ and reflected at the origin, is a skew Brownian motion with bang-bang drift.

4. Synthesis

We reverse now the steps of the analysis in section 3. Let us start with a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbf{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$ and with two independent, standard Brownian motion $W_1(\cdot)$, $W_2(\cdot)$ on it; we shall assume $\mathbf{F} \equiv \mathbf{F}^{(W_1, W_2)}$, i.e., that the $\mathbf{F}$ is the smallest filtration satisfying the usual conditions, to which the planar Brownian motion $(W_1(\cdot), W_2(\cdot))$ is adapted.

With given real constants ζ_1, ζ_2, η_1, η_2 and nonnegative constants g, h, ρ, σ that satisfy (1.1) and (1.5), with a given vector $(x_1, x_2) \in \mathbb{R}^2$, and with the notation of (3.1), we construct the pairs of independent Brownian motions

$$W(\cdot) := \rho\,W_1(\cdot) + \sigma\,W_2(\cdot)\,, \qquad U^\flat(\cdot) := \sigma\,W_1(\cdot) - \rho\,W_2(\cdot) \qquad (4.1)$$

and

$$U(\cdot) := \sigma\,W_1(\cdot) + \rho\,W_2(\cdot)\,, \qquad W^\flat(\cdot) := \rho\,W_1(\cdot) - \sigma\,W_2(\cdot) \qquad (4.2)$$

as in (3.8), (3.7). Clearly, $\mathbf{F}^{(W_1, W_2)} \equiv \mathbf{F}^{(W, U^\flat)} \equiv \mathbf{F}^{(U, W^\flat)}$.

We construct also the pathwise unique, strong solution $Y(\cdot)$ of the stochastic equation (3.18) driven by the Brownian motion $W(\cdot)$ introduced in (4.1). With the process $Y(\cdot)$ thus in place, we introduce by analogy with (3.9) the independent Brownian motions

$$V_1(\cdot) = \int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\,\mathrm{d}W_1(t)\,, \qquad V_2(t) = -\int_0^\cdot \mathrm{sgn}\big(Y(t)\big)\,\mathrm{d}W_2(t)\,, \qquad (4.3)$$

and by analogy with (3.7), (3.8) the two additional pairs of independent Brownian motions

$$V(\cdot) := \rho\, V_1(\cdot) + \sigma\, V_2(\cdot)\,, \qquad\qquad Q^{\flat}(\cdot) := \sigma\, V_1(\cdot) - \rho\, V_2(\cdot) \qquad (4.4)$$

and

$$Q(\cdot) := \sigma\, V_1(\cdot) + \rho\, V_2(\cdot)\,, \qquad\qquad V^{\flat}(\cdot) := \rho\, V_1(\cdot) - \sigma\, V_2(\cdot)\,. \qquad (4.5)$$

- We introduce also the continuous martingales

$$M_1(\cdot) := \int_0^{\cdot} \Big(\rho\, \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}W_1(t) + \sigma\, \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}W_2(t) \Big) \qquad (4.6)$$

$$= \int_0^{\cdot} \Big(\rho\, \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}V_1(t) + \sigma\, \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}V_2(t) \Big)\,,$$

$$M_2(\cdot) := -\int_0^{\cdot} \Big(\rho\, \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}W_1(t) + \sigma\, \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}W_2(t) \Big) \qquad (4.7)$$

$$= \int_0^{\cdot} \Big(\rho\, \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}V_1(t) + \sigma\, \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}V_2(t) \Big)\,,$$

with $\langle M_1, M_2 \rangle\,(\cdot) \equiv 0$ and quadratic variations

$$\langle M_1 \rangle(\cdot) = \int_0^{\cdot} \big(\rho^2\, \mathbf{1}_{\{Y(t)>0\}} + \sigma^2\, \mathbf{1}_{\{Y(t)\leq 0\}} \big)\mathrm{d}t\,, \qquad \langle M_2 \rangle(\cdot) = \int_0^{\cdot} \big(\rho^2\, \mathbf{1}_{\{Y(t)\leq 0\}} + \sigma^2\, \mathbf{1}_{\{Y(t)>0\}} \big)\mathrm{d}t\,.$$

There exist then independent Brownian motions $B_1(\cdot)\,,\ B_2(\cdot)$ on our filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})\,,\ \mathbf{F} = \{\mathfrak{F}(t)\}_{0\leq t<\infty}\,,$ so the continuous martingales of (4.6), (4.7) are cast in their DOOB representations as

$$M_1(\cdot) = \int_0^{\cdot} \Big(\rho\, \mathbf{1}_{\{Y(t)>0\}} + \sigma\, \mathbf{1}_{\{Y(t)\leq 0\}} \Big)\, \mathrm{d}B_1(t)\,, \quad M_2(\cdot) = \int_0^{\cdot} \Big(\rho\, \mathbf{1}_{\{Y(t)\leq 0\}} + \sigma\, \mathbf{1}_{\{Y(t)>0\}} \Big)\, \mathrm{d}B_2(t) \qquad (4.8)$$

in terms of independent Brownian motions $B_1(\cdot)\,,\ B_2(\cdot)$; for instance, by taking

$$B_1(\cdot) = \int_0^{\cdot} \Big(\mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}W_1(t) + \mathbf{1}_{\{Y(t)\leq 0\}}\, \mathrm{d}W_2(t) \Big)\,, \qquad (4.9)$$

$$B_2(\cdot) = -\int_0^{\cdot} \Big(\mathbf{1}_{\{Y(s)\leq 0\}}\, \mathrm{d}W_1(t) + \mathbf{1}_{\{Y(t)>0\}}\, \mathrm{d}W_2(t) \Big)\,. \qquad (4.10)$$

- Finally, we introduce the continuous, $\mathbf{F}-$adapted processes

$$X_1(\cdot) := x_1 + \int_0^{\cdot} \big(g\, \mathbf{1}_{\{Y(t)\leq 0\}} - h\, \mathbf{1}_{\{Y(t)>0\}} \big)\, \mathrm{d}t + M_1(\cdot) + \frac{1-\zeta_1}{2}\, L^Y(\cdot) + \frac{1-\eta_1}{2}\, L_-^Y(\cdot) \qquad (4.11)$$

and

$$X_2(\cdot) := x_2 + \int_0^{\cdot} \big(g\, \mathbf{1}_{\{Y(t)>0\}} - h\, \mathbf{1}_{\{Y(t)\leq 0\}} \big)\, \mathrm{d}t + M_2(\cdot) + \frac{1-\zeta_2}{2}\, L^Y(\cdot) + \frac{1-\eta_2}{2}\, L_-^Y(\cdot)\,. \qquad (4.12)$$

It is now easy to check $X_1(\cdot) - X_2(\cdot) = Y(\cdot)$; and from this, that the vector process $(X_1(\cdot),\, X_2(\cdot))$ solves the system (1.2)-(1.3), as well as the systems (3.25)-(3.26), (3.27)-(3.28), and

$$\mathrm{d}X_1(t) = \mathbf{1}_{\{X_1(t)\le X_2(t)\}}\big(g\,\mathrm{d}t + \sigma\,\mathrm{d}B_1(t)\big) + \mathbf{1}_{\{X_1(t)>X_2(t)\}}\big(-h\,\mathrm{d}t + \rho\,\mathrm{d}B_1(t)\big) + \kappa_1\,\mathrm{d}L^{|X_1-X_2|}(t)$$
(4.13)
$$\mathrm{d}X_2(t) = \mathbf{1}_{\{X_1(t)>X_2(t)\}}\big(g\,\mathrm{d}t + \sigma\,\mathrm{d}B_2(t)\big) + \mathbf{1}_{\{X_1(t)\le X_2(t)\}}\big(-h\,\mathrm{d}t + \rho\,\mathrm{d}B_2(t)\big) + \kappa_2\,\mathrm{d}L^{|X_1-X_2|}(t)$$
(4.14)

where $2\,\kappa_j := \alpha\,(1-\zeta_j) + (1-\alpha)\,(1-\eta_j)$, $j = 1,2$. It is also straightforward to verify the skew representations of (3.30), (3.31).

4.1. Ranks

Let us introduce explicitly the ranked versions (leader and laggard, respectively)

$$R_1(\cdot) := X_1(\cdot) \vee X_2(\cdot), \qquad R_2(\cdot) := X_1(\cdot) \wedge X_2(\cdot) \tag{4.15}$$

of the components of the vector process $(X_1(\cdot), X_2(\cdot))$ constructed in (4.11), (4.12). From (3.12) and (3.29), it is rather clear that we have

$$R_1(t) + R_2(t) = X_1(t) + X_2(t) = r_1 + r_2 + \nu\,t + V(t) + \big(1-\beta\big)\,L^{|Y|}(t), \quad 0 \le t < \infty$$

$$R_1(t) - R_2(t) = \big|X_1(t) - X_2(t)\big| = \big|Y(t)\big| = |y| - \lambda\,t + V^\flat(t) + L^{|Y|}(t), \tag{4.16}$$

and these representations lead to the expressions

$$R_1(t) = r_1 - h\,t + \rho\,V_1(t) + \big(1 - (\beta/2)\big)\,L^{R_1-R_2}(t), \qquad 0 \le t < \infty \tag{4.17}$$
$$R_2(t) = r_2 + g\,t + \sigma\,V_2(t) - (\beta/2)\,L^{R_1-R_2}(t), \qquad 0 \le t < \infty. \tag{4.18}$$

A few remarks are in order. The equations (4.17), (4.18) identify the processes $V_1(\cdot)$ and $V_2(\cdot)$ of (3.5), (3.6) as the independent Brownian motions associated with the diffusive motion of the *ranked particles*, the "leader" $R_1(\cdot)$ and the "laggard" $R_2(\cdot)$, respectively; whereas the independent Brownian motions $B_1(\cdot)$ in (1.2) and $B_2(\cdot)$ in (1.3) are associated with the specific "names" (indices, or identities) of the individual particles. On the other hand, with the help of the theory of the SKOROKHOD reflection problem (e.g., KARATZAS & SHREVE [13], page 210), we obtain from (4.16), (2.3) the identification of the "collision local time"

$$L^{R_1-R_2}(t) = L^{|Y|}(t) = \max_{0\le s\le t}\Big(-|y| + \lambda\,s - V^\flat(s)\Big)^+, \qquad 0 \le t < \infty. \tag{4.19}$$

Let us also observe that, in the non-degenerate case $\rho\,\sigma > 0$, the equations (4.16)-(4.19) and the second equation in (4.5) give the filtration comparisons

$$\mathfrak{F}^{(V_1,V_2)}(t) = \mathfrak{F}^{(R_1,R_2)}(t), \qquad 0 \le t < \infty, \tag{4.20}$$

$$\mathfrak{F}^{V^\flat}(t) = \mathfrak{F}^{|Y|}(t) \subsetneqq \mathfrak{F}^{Y}(t), \qquad 0 < t < \infty, \tag{4.21}$$

where the inclusion is strict, due to the fact that the process $Y(\cdot)$ changes its sign with positive probability during any time-interval $[0,t]$ with $t > 0$.

4.2. Filtration Comparisons, Weak and Strong Solutions

We have the following straightforward analogues of Propositions 4.1, 4.2 and of Theorems 4.1, 4.2 in FERNHOLZ ET AL. [7].

Proposition 4.1. *In the degenerate case* $\sigma = 0$*, thus* $\rho = 1$ *in light of (1.1), we have the relations*

$$\mathfrak{F}^{(R_1,R_2)}(t) = \mathfrak{F}^V(t) = \mathfrak{F}^{|X_1-X_2|}(t) \subsetneq \mathfrak{F}^{X_1-X_2}(t) = \mathfrak{F}^W(t) = \mathfrak{F}^{(X_1,X_2)}(t) \qquad (4.22)$$

for every $0 < t < \infty$*, where the inclusion is strict.*

In the special case $\beta = 1$ *of (5.2) we have in addition* $\boldsymbol{\sigma}(V(t)) = \boldsymbol{\sigma}(X_1(t) + X_2(t))$*, thus also* $\mathfrak{F}^V(t) = \mathfrak{F}^{X_1+X_2}(t)$*, for every* $0 \leq t < \infty$*.*

Proposition 4.2. *In the non-degenerate case* $\rho\sigma > 0$*, we have for every* $0 < t < \infty$ *the filtration relations*

$$\mathfrak{F}^{(V_1,V_2)}(t) = \mathfrak{F}^{(R_1,R_2)}(t) = \mathfrak{F}^{(|Y|,V)}(t) = \mathfrak{F}^{(|Y|,Q)}(t) \qquad (4.23)$$

$$\subsetneq \mathfrak{F}^{(Y,Q)}(t) = \mathfrak{F}^{(Y,V)}(t) = \mathfrak{F}^{(W_1,W_2)}(t) = \mathfrak{F}^{(X_1,X_2)}(t),$$

where the inclusion is strict.

Theorem 4.1. *The system of stochastic differential equations (1.2), (1.3) is well-posed, that is, has a weak solution which is unique in the sense of the probability distribution. The same is true for each of the systems of equations (3.25), (3.26) and (3.27), (3.28).*

On the other hand, the system of stochastic differential equations (3.25), (3.26) admits a strong solution, which is therefore pathwise unique; whereas the system (3.27), (3.28) admits no strong solution.

Theorem 4.2. *The system of stochastic differential equations (1.2), (1.3) admits a pathwise unique, strong solution; in particular, the filtration identity* $\mathfrak{F}^{(B_1,B_2)}(t) = \mathfrak{F}^{(X_1,X_2)}(t)$ *holds for all* $0 \leq t < \infty$*.*

Proof. Repeating almost verbatim the arguments in the proof of Theorem 5.1 in FERNHOLZ ET AL. [7], the question boils down to whether the filtration comparison

$$\mathfrak{F}^Y(t) \subseteq \mathfrak{F}^{(B_1,B_2)}(t), \qquad \forall \quad 0 \leq t < \infty \qquad (4.24)$$

holds. To decide this issue, we write the equation (3.18) as driven by the pair $(B_1(\cdot), B_2(\cdot))$; in other words, we use (3.7) and (3.3), (3.4) to express the skew Brownian motion $Y(\cdot)$ with bang-bang drift as solution of a stochastic differential equation driven by the planar Brownian motion $(B_1(\cdot), B_2(\cdot))$. Since this equation does admit a weak solution which is unique in distribution, the issue is whether this solution is also strong, that is, whether (4.24) holds.

This question is easy to settle in the isotropic case $\rho = \sigma = 1/\sqrt{2}$; then $W(\cdot) = (B_1(\cdot) - B_2(\cdot))/\sqrt{2}$, and the comparison (4.24) follows from the strong solvability of the equation (3.18) proved in section 6: $\mathfrak{F}^Y(t) = \mathfrak{F}^W(t) \subseteq \mathfrak{F}^{(B_1,B_2)}(t)$ holds for all $0 \leq t < \infty$, by virtue of (3.19).

In the non-isotropic case $\rho \neq \sigma$, we write (3.18) as the *extended skew Tanaka equation*

$$Y(t) = y + \frac{\rho - \sigma}{\sqrt{2}} \int_0^t \overline{\text{sgn}}(Y(s)) \, \mathrm{d}\boldsymbol{\beta}(s) - \frac{\rho + \sigma}{\sqrt{2}} \, \boldsymbol{\vartheta}(t) + 2(2\alpha - 1) \widehat{L}^Y(t), \quad 0 \leq t \leq T \qquad (4.25)$$

with $T \in (0, \infty)$ arbitrary but fixed. Here

$$\boldsymbol{\beta}(\cdot) := \frac{\boldsymbol{\beta}_1(\cdot) + \boldsymbol{\beta}_2(\cdot)}{\sqrt{2}}, \qquad \boldsymbol{\vartheta}(\cdot) := \frac{\boldsymbol{\beta}_1(\cdot) - \boldsymbol{\beta}_2(\cdot)}{\sqrt{2}}$$

are given standard, independent Brownian motions after an equivalent change of probability measure, and

$$\boldsymbol{\beta}_i(t) := B_i(t) - \frac{\lambda t}{\rho - \sigma}, \qquad 0 \le t \le T \qquad (i = 1, 2).$$

• Let us suppose that $Y_1(\cdot)$ and $Y_2(\cdot)$ are two solutions of the equation (4.25), defined on the same probability space and with respect to the same, independent standard Brownian motions $B_1(\cdot)$, $B_2(\cdot)$. Following LE GALL (1983), we shall show $L^{Y_1 - Y_2}(\cdot) \equiv 0$; we shall then argue that this implies also $Y_1(\cdot) \equiv Y_2(\cdot)$, a.s.

To this end, we consider the difference $D(\cdot) := Y_1(\cdot) - Y_2(\cdot)$, as well as the linear combinations

$$Z^{(u)}(\cdot) := (1 - u) Y_1(\cdot) + u Y_2(\cdot) \quad \text{for} \quad 0 \le u \le 1;$$

we introduce also a sequence $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{C}^1(\mathbb{R})$ of continuous and continuously differentiable functions that converge to $f_\infty(\cdot) := \overline{\text{sgn}}(\cdot)$ pointwise, and satisfy $\sup_{k \in \mathbb{N}} \|f_k\|_{TV} < \infty$. Since $\limsup_k \|f_k\|_{TV} \le \|f_\infty\|_{TV}$ obviously holds, this is only possible if $f_\infty(\cdot)$ is of bounded variation, and in this case an approximating sequence is easily obtained, e.g., by mollifiers. As in the proof of Theorem 8.1 of FERNHOLZ ET AL. [7], for every $\delta > 0$, $T > 0$, $k \ge 1$, we establish then

$$\mathbb{E}\Big[\int_0^T \frac{|f_k(Y_1(s)) - f_k(Y_2(s))|}{Y_1(s) - Y_2(s)} \mathbf{1}_{\{Y_1(s) - Y_2(s) > \delta\}} \, \mathrm{d}t \Big] \le c_1 \|f_k\|_{TV} \cdot \sup_{\xi, u} \mathbb{E}\big(2\widehat{L}^{(u)}(T, \xi)\big);$$

here $\widehat{L}^{(u)}(T, \xi)$ is the symmetric local time of $Z^{(u)}(\cdot)$ accumulated at the site $\xi \in \mathbb{R}$ over the time interval $[0, T]$, and c_1 is a constant chosen independently of k, u, δ. Letting $k \uparrow \infty$ and $\delta \downarrow 0$, we obtain

$$\mathbb{E}\Big[\int_0^T \frac{1}{D(t)} \mathbf{1}_{\{D(t) > 0\}} \, \mathrm{d}\langle D \rangle(t) \Big] \le 2 \mathbb{E}\Big[\int_0^T \frac{|f_\infty(Y_1(t)) - f_\infty(Y_2(t))|}{Y_1(t) - Y_2(t)} \mathbf{1}_{\{D(t) > 0\}} \, dt \Big]$$

$$\le 2 c_1 \|f_\infty\|_{TV} \cdot \sup_{\xi, u} \mathbb{E}\Big(2\widehat{L}^{(u)}(T, \xi)\Big).$$

Now the CAUCHY-SCHWARTZ inequality, the ITÔ isometry, and the TANAKA formula (2.13) applied to $Z^{(u)}(\cdot)$, allow us to estimate

$$\mathbb{E}\big(2\widehat{L}^{(u)}(T; \xi)\big) \le \mathbb{E}|Z^{(u)}(T) - Z^{(u)}(0)| + \big[\mathbb{E}(\langle Z^{(u)} \rangle(T))\big]^{1/2}$$

$$+ 2 (2\alpha - 1)\big(u \, \mathbb{E}\big(\widehat{L}^{Y_1}(T)\big) + (1 - u) \mathbb{E}\big(\widehat{L}^{Y_2}(T)\big)\big)$$

$$\le 2 \Big[\big[\mathbb{E}(\langle Z^{(u)} \rangle(T))\big]^{1/2} + 2 (2\alpha - 1)\Big(u \, \mathbb{E}\big(\widehat{L}^{Y_1}(T)\big) + (1 - u) \mathbb{E}\big(\widehat{L}^{Y_2}(T)\big)\Big)\Big].$$

The last term is bounded uniformly in (ξ, u), since $\langle Z^{(u)} \rangle(t) \le c_2 t$ and $\mathbb{E}\big(\widehat{L}^{Y_i}(T)\big) \le c_3$, for $i = 1, 2$ and for some constants c_2, c_3 that do not depend on (ξ, u). Thus, we obtain

$$\mathbb{E}\Big[\int_0^T \frac{1}{D(t)} \mathbf{1}_{\{D(t) > 0\}} \, \mathrm{d}\langle D \rangle(t) \Big] < \infty, \quad 0 < T < \infty. \tag{4.26}$$

Using Lemma 1.0 of LE GALL [15] (see also Exercise 3.7.12, pages 225-226 in KARATZAS & SHREVE [13]), we verify that (4.26) gives $L^D(\cdot) \equiv 0$. By exchanging the rôles of $Y_1(\cdot)$ and $Y_2(\cdot)$, we obtain also $L^{-D}(\cdot) = L^{Y_2-Y_1}(\cdot) \equiv 0$, as well as $\widehat{L}^D(\cdot) \equiv 0$. Furthermore, we note that on the strength of Corollary 2.6 of OUKNINE & RUTKOWSKI [19] this implies that the symmetric local time $\widehat{L}^M(\cdot)$ of the maximum

$$M(\cdot) := Y_1(\cdot) \vee Y_2(\cdot) = Y_1(\cdot) + \big(Y_2(\cdot) - Y_1(\cdot)\big)^+$$

is given as

$$\widehat{L}^M(\cdot) := \widehat{L}^{Y_1 \vee Y_2}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{Y_2(t) \leq 0\}} \, d\widehat{L}^{Y_1}(t) + \int_0^{\cdot} \mathbf{1}_{\{Y_1(t) < 0\}} \, d\widehat{L}^{Y_2}(t).$$

We combine now these results with the TANAKA formula, to obtain the dynamics of the maximum

$$M(\cdot) = y + \int_0^{\cdot} \mathbf{1}_{\{Y_1(t) \geq Y_2(t)\}} \, dY_1(t) + \int_0^{\cdot} \mathbf{1}_{\{Y_1(t) < Y_2(t)\}} \, dY_2(t) + L^{Y_2-Y_1}(\cdot)$$

$$= y + \frac{\rho - \sigma}{\sqrt{2}} \int_0^{\cdot} \overline{\mathrm{sgn}}(M(t)) \, d\boldsymbol{\beta}(t) - \frac{\rho + \sigma}{2} \, \boldsymbol{\vartheta}(\cdot) + 2\,(2\alpha - 1)\,\widehat{L}^M(\cdot),$$

and observe that these are the same as those of (4.25). But uniqueness in distribution holds for the equation (4.25), so the distribution of the process $M(\cdot)$ is the same as that of $Y_1(\cdot)$; and of course we have $M(\cdot) \geq Y_1(\cdot)$ a.s. This implies $M(\cdot) \equiv Y_1(\cdot)$, thus $Y_1(\cdot) \equiv Y_2(\cdot)$ a.s.

Therefore, the solution to (4.25) is pathwise unique, hence also strong by the theory of YA-MADA & WATANABE (e.g., KARATZAS & SHREVE [13], pages 308-311). $\qquad\square$

5. Some Special Cases

When $\alpha = 1/2$, that is, $\eta = \zeta \neq 0$ or equivalently

$$\eta_1 - \eta_2 = \zeta_2 - \zeta_1 \neq 2, \tag{5.1}$$

the equation (3.18) for the difference $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ becomes that of Brownian motion with bang-bang drift

$$Y(t) = y - \lambda \int_0^t \mathrm{sgn}\big(Y(s)\big) \, ds + W(t), \qquad 0 \leq t < \infty$$

as in (3.24). In this special case $\eta = \zeta \neq 0$ and with $\sigma = \rho$, the existence and uniqueness of (1.2)-(1.3) can be shown also by direct application of Theorem 3.5 of SZNITMAN & VARADHAN [29] and a GIRSANOV's change-of-measure, with the aid of the local time relationships (3.16).

On the other hand, when $\beta = 1$ or equivalently

$$\eta\,(1 - \overline{\zeta}) = \zeta\,(1 - \overline{\eta}), \tag{5.2}$$

we observe from (3.20) that the sum $X_1(\cdot) + X_2(\cdot)$ is just standard Brownian motion with drift $\nu = g - h$.

Let us single out now, and study, some more interesting special cases.

16

5.1. Perfect Reflection for Individual Particles Upon Collision

Suppose $\alpha = 1$, or equivalently $\zeta = 0$ and $\eta \neq 0$ from (1.5), that is

$$\zeta_2 - \zeta_1 \,=\, 2 \,\neq\, \eta_1 - \eta_2 \,, \tag{5.3}$$

and that $x_1 \geq x_2$. We see then from (3.16) that we have $L_-^Y(\cdot) \equiv L^{X_2-X_1}(\cdot) \equiv 0$, and that the equation (3.18) becomes

$$Y(t) \,=\, y - \lambda\,t + W(t) + L^Y(t)\,, \qquad 0 \leq t < \infty\,.$$

From the theory of the SKOROKHOD problem (e.g., KARATZAS & SHREVE [13], pages 209-210) we conclude

$$L^Y(t) \,=\, \max_{0 \leq s \leq t} \big(-y + \lambda\,s - W(s) \big)^+\,, \qquad 0 \leq t < \infty$$

thus $Y(\cdot) \geq 0$, and that strength and pathwise uniqueness hold; for more general results along these lines see CHITASHVILI & LAZRIEVA [5]. It is also clear from the last two displayed equations, that the filtration identity $\mathfrak{F}^Y(t) = \mathfrak{F}^W(t)$, $0 \leq t < \infty$ in (3.19) also holds.

In this case, then, when the particles collide, the trajectory $X_1(\cdot)$ of the first particle bounces off the trajectory $X_2(\cdot)$ of the second particle as if this latter were a perfectly reflecting lower boundary. We can visualize the situation by saying that, under the conditions of (1.5) and (5.3), the second particle is "heavy" (unaffected by collisions), whereas the first particle is "light" in that it bounces off (reflects perfectly) when colliding with the heavy particle.

- The "symmetric" situation obtains for $\alpha = 0$, that is $\zeta \neq 0$ and $\eta = 0$ or equivalently

$$\zeta_2 - \zeta_1 \,\neq\, 2 \,=\, \eta_1 - \eta_2 \,; \tag{5.4}$$

in this case and again with $x_1 \geq x_2$, when the two particles collide, the second particle bounces off the first as if this latter were a perfectly reflecting upper boundary; it is the first particle that is now "heavy", and the second that is "light".

5.2. Frictionless Collision

It follows also from (3.16) that the local times disappear entirely in (1.2) when we have the configuration of parameters $(1 - \zeta_1)\alpha + (1 - \eta_1)(1 - \alpha) = 0$, or equivalently

$$(1 - \zeta_1)\,\eta + (1 - \eta_1)\,\zeta \,=\, 0\,; \tag{5.5}$$

in this case the trajectory of the first particle crosses that of the second without "feeling it", that is, without being subjected to any local time drag.

Similarly, the second particle crosses the first in the same frictionless manner, that is, the local times disappear entirely in (1.3), if

$$(1 - \zeta_2)\,\eta + (1 - \eta_2)\,\zeta \,=\, 0\,. \tag{5.6}$$

- If both (5.5) and (5.6) hold, then all such crossings are *completely frictionless*. We note that (5.5) and (5.6) are both satisfied, if and only if

$$\eta_1 + \zeta_1 \,=\, \eta_2 + \zeta_2 \,=\, 2 \tag{5.7}$$

17

holds. This condition implies $\eta = \zeta$ (so when this common value is nonzero we are in the case $\alpha = 1/2$ mentioned at the start of the section), and is obviously satisfied in the special case $\eta_1 = \zeta_1 = \eta_2 = \zeta_2 = 1$ studied by FERNHOZ ET AL. [7]. However, (5.7) holds also for other configurations of parameters, for instance $\zeta_1 = \eta_2 = 1/2$, $\eta_1 = \zeta_2 = 3/2$.

The condition (5.7) gives the value $\beta = 1$ for the parameter of (3.21); back in (4.17), (4.18), this implies that the collision local time $L^{R_1 - R_2}(\cdot)$ "gets apportioned equally to the ranks".

5.3. Elastic Collisions

Beyond these two extremes of perfect reflection and frictionless collision – that is, for all other configurations of parameters – we have collisions that are "elastic": neither completely frictionless, nor perfectly reflecting.

5.4. Brownian motion reflected on an independent Brownian motion

Finally, let us consider the case $\beta = 0$ or equivalently $\eta\,\overline{\zeta} + \zeta\,\overline{\eta} = 0$, that is

$$2\left(\zeta_1 + \zeta_2 + \eta_1 + \eta_2\right) = \left(\zeta_1 + \zeta_2\right)\left(\eta_1 - \eta_2\right) - \left(\eta_1 + \eta_2\right)\left(\zeta_1 - \zeta_2\right) \qquad (5.8)$$

in light of (3.21) and (1.4), (3.13). This happens, for instance, when $\zeta_1 = 3/4$, $\zeta_2 = 9/4$, $\eta_1 = -4/3$, $\eta_2 = -8/3$; in this case we have $\alpha = 4/7$ and of course $\beta = 0$.

Under the condition (5.8), the laggard in (4.18) feels no pressure (local time drag) from the leader; it just evolves like Brownian motion with variance σ^2 and nonnegative drift. On the other hand, the leader in (4.17) evolves like an independent Brownian motion with variance ρ^2 and nonpositive drift, reflected off the trajectory of the laggard. Such a process has been studied by BURDZY & NUALART [4] (see also SOUCALIUC ET AL. [27], SOUCALIUC & WERNER [28]); here it arises as a special case of the ranked system (4.17), (4.18) for the particles whose motions are governed by the equations (1.2), (1.3).

We have in this case $\beta = 0$ an interesting fusion: the "perfect reflection" we saw in subsection 5.1, and the "frictionless motion" of subsection 5.2, are occurring here simultaneously – not for the motions of the individual particles, however, but rather for the motions of their ranked versions, the leader $R_1(\cdot)$ and the laggard $R_2(\cdot)$, respectively. To put it a little differently: starting with two particles that undergo skew-elastic collisions one is able, under the conditions of (1.5) and (5.8), to "simulate a heavy particle" (the laggard) and a "light" particle (the leader).

• The "reverse" situation obtains when $\beta = 2$ or equivalently $\eta\,\overline{\zeta} + \zeta\,\overline{\eta} = 2\left(\eta + \zeta\right)$, that is

$$2\left(\zeta_1 + \zeta_2 + \eta_1 + \eta_2\right) = 4\left(4 + \zeta_1 - \zeta_2 - \eta_1 + \eta_2\right) + \left(\zeta_1 + \zeta_2\right)\left(\eta_1 - \eta_2\right) - \left(\eta_1 + \eta_2\right)\left(\zeta_1 - \zeta_2\right); \qquad (5.9)$$

then it is the trajectory of the laggard (now the "light" particle) that gets reflected off that of the leader (now the "heavy" particle). This happens, for instance, when $\zeta_1 = 3/2$, $\zeta_2 = 3$, $\eta_1 = 7/3$, $\eta_2 = 1$; in this case we have $\alpha = 4/7$ and $\beta = 2$.

5.5. Some Simulations

The pictures (Figures 1-4) that follow present simulations of the processes $X_1(t)$ (in black) and $X_2(t)$ (in red) for $t \in [0,1]$, in black and red, respectively, with drifts $g = h = 1$ in the degenerate case $\rho = 0$.
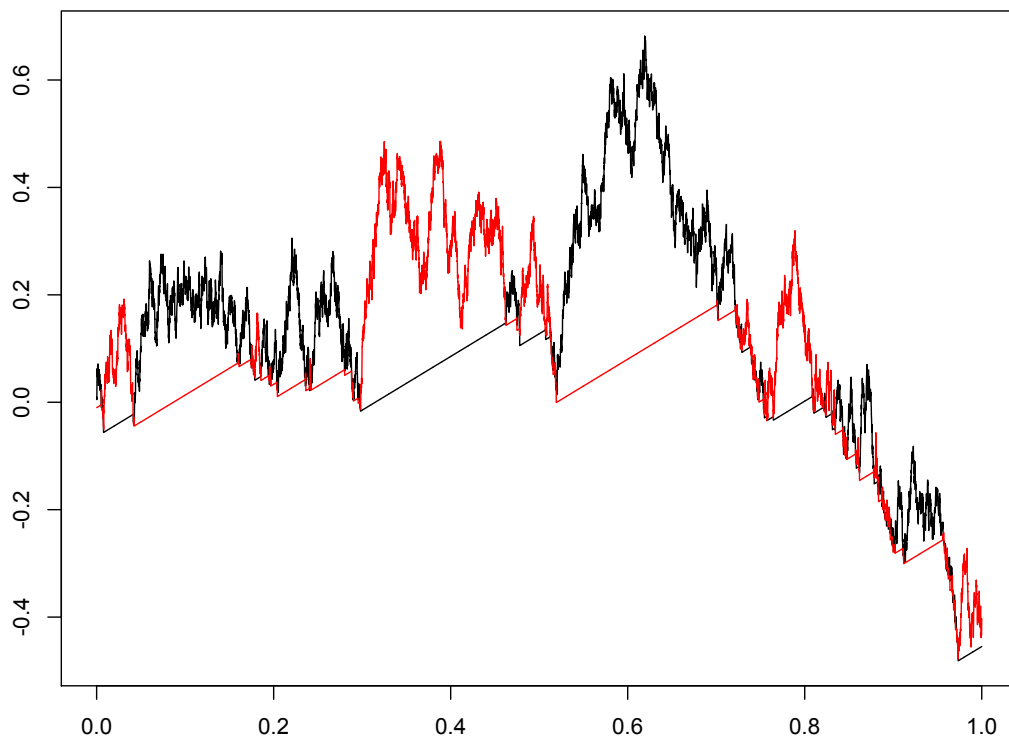
18

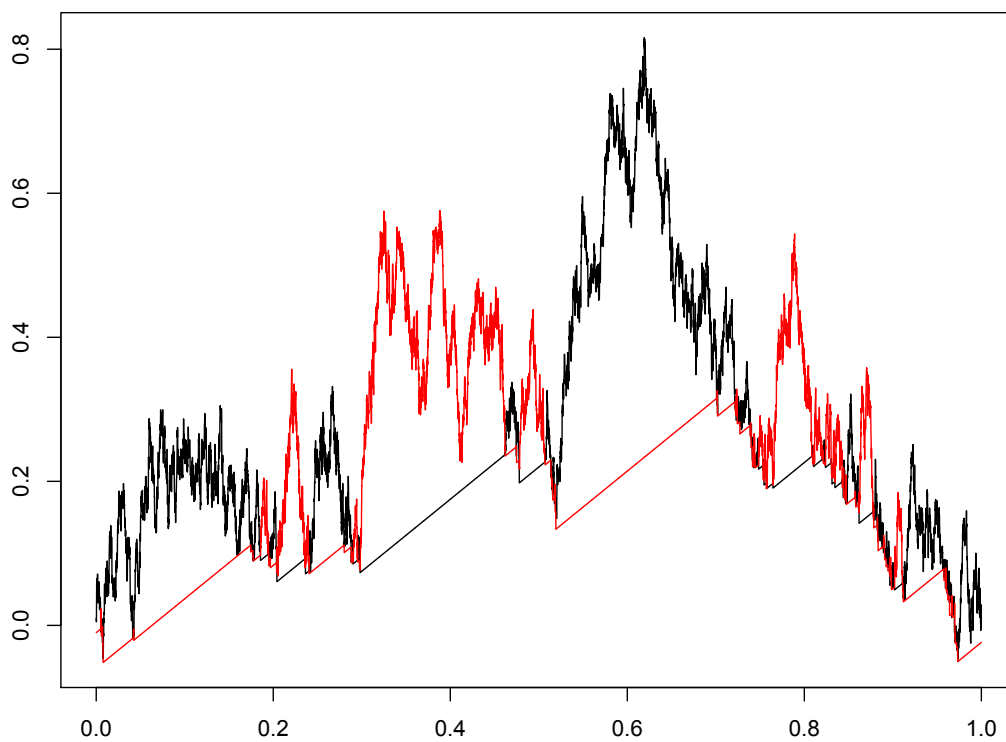Figure 1: $\zeta_1 = \zeta_2 = \eta_1 = \eta_2 = 1$; $\alpha = 1/2$, $\beta = 1$.

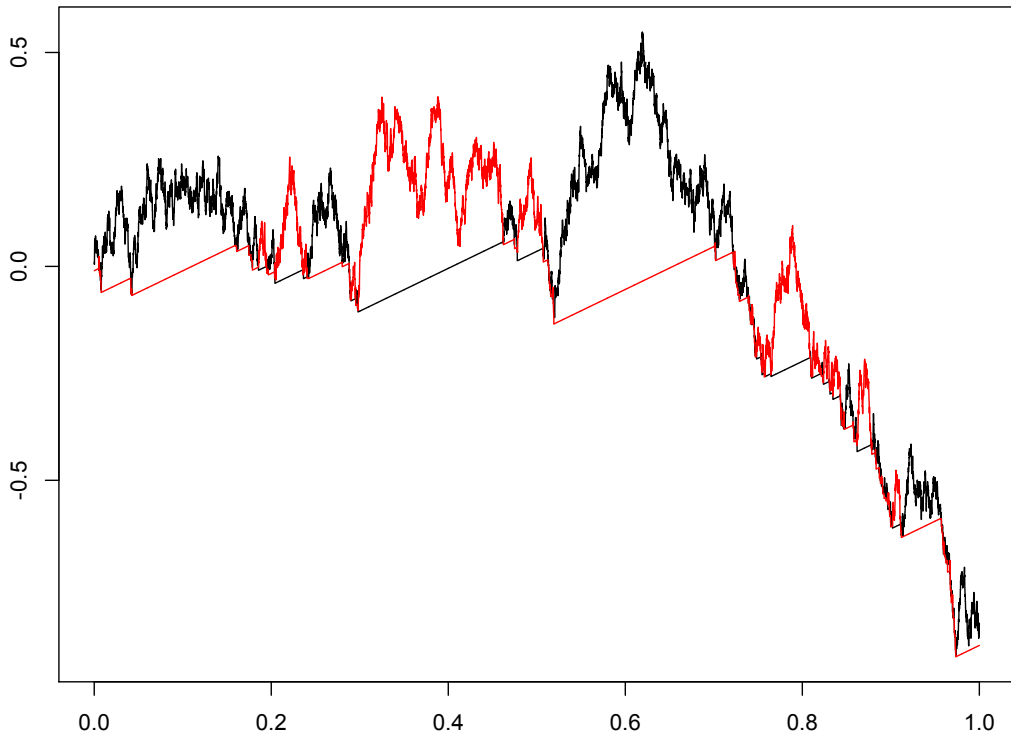Figure 2: $\zeta_1 = 0$, $\zeta_2 = \eta_1 = \eta_2 = 1$; $\alpha = 2/3$, $\beta = 2/3$.

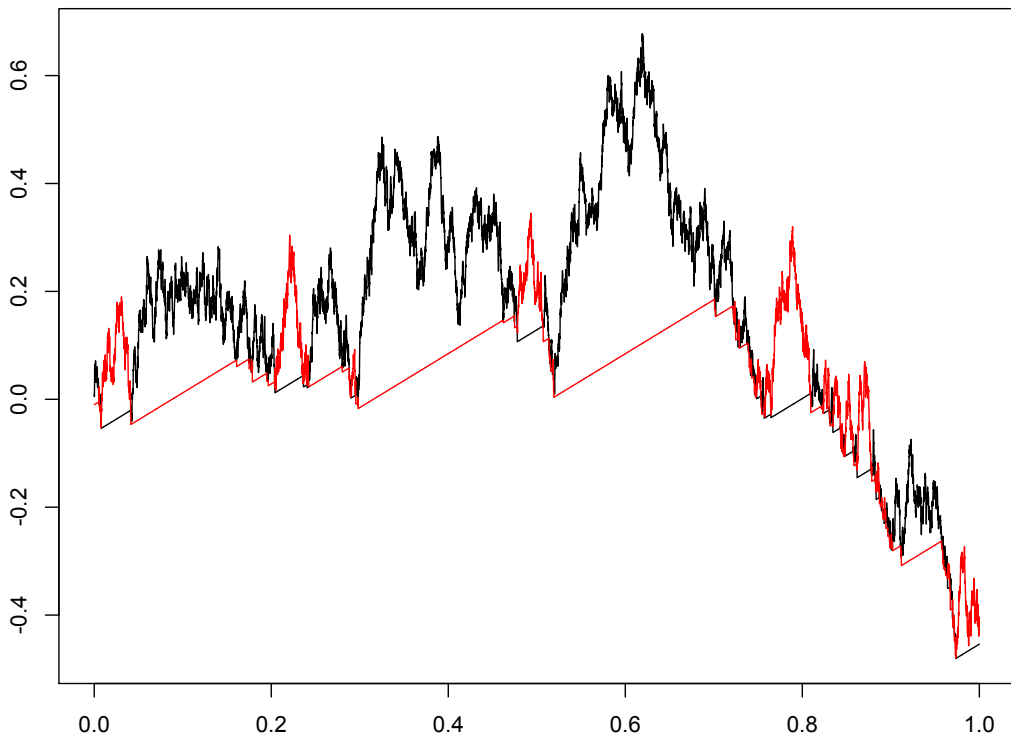Figure 3: $\zeta_2 = 2$, $\zeta_1 = \eta_1 = \eta_2 = 1$; $\alpha = 2/3$, $\beta = 4/3$.

Figure 4: $\zeta_1 = 0$, $\zeta_2 = 2$, $\eta_1 = \eta_2 = 1$; $\alpha = 1$, $\beta = 1$.

6. Skew Brownian Motion with Bang-Bang Drift

We study here the stochastic differential equation (3.18) for the skew Brownian motion with bang-bang drift

$$\mathfrak{b}(y) \ = \ -\lambda \, \mathrm{sgn}(y) \,, \qquad y \in \mathbb{R} \tag{6.1}$$

for some given constant $\lambda > 0$, with the notation of (2.11), and with skewness parameter $\alpha \in [0,1]$. The cases $\alpha = 0$ and $\alpha = 1$ have been discussed already in subsection 5.1, so we focus here on the range $0 < \alpha < 1$.

For this range of values of the skewness parameter, we choose to write the equation (3.18) in terms of the right-continuous local time of the unknown process at the origin, namely

$$Y(\cdot) \ = \ y_0 - \lambda \int_0^{\cdot} \mathrm{sgn}\big(Y(t)\big) \, \mathrm{d}t + W(\cdot) + \frac{2\,\alpha - 1}{\alpha} \, L^Y(\cdot) \,. \tag{6.2}$$

This equation is of the more general form

$$Y(\cdot) \ = \ y_0 + \int_0^{\cdot} \boldsymbol{\tau}(Y(t)) \, \mathrm{d}W(t) + \int_{\mathbb{R}} L^Y(\cdot, \xi) \, \boldsymbol{\nu}(\mathrm{d}\xi) \,, \tag{6.3}$$

with dispersion $\boldsymbol{\tau}(y) \equiv 1$ and measure

$$\nu(\mathrm{d}y) \ = \ 2\,\mathfrak{b}(y) \, \mathrm{d}y + \frac{2\,\alpha - 1}{\alpha} \, \boldsymbol{\delta}_0(\mathrm{d}y)$$

with $\mathfrak{b}(y) = -\lambda \, \mathrm{sgn}(y)$, $y \in \mathbb{R}$ as in (6.1) above, and with $\boldsymbol{\delta}_0(\cdot)$ the Dirac mass at the origin.

We shall deal with the equation (6.2) using a direct methodology that removes the parts of finite variation, that is, both the drift and the local time, and "reduces" (6.2) to a stochastic differential equation in natural scale

$$Z(\cdot) \ = \ \mathfrak{p}(y_0) + \int_0^{\cdot} \mathfrak{s}\big(Z(t)\big) \, \mathrm{d}W(t) \tag{6.4}$$

for appropriate functions $\mathfrak{p}(\cdot)$ and $\mathfrak{s}(\cdot)$. This approach was pioneered for the skew Brownian motion itself (i.e., with $\lambda = 0$) by HARRISON & SHEPP [9] , and for more general equations of the form (6.3) for suitable measurable functions $\boldsymbol{\tau}(\cdot)$ and measures $\boldsymbol{\nu}$ on $\mathcal{B}(\mathbb{R})$, by LE GALL [15] [16] and ENGELBERT & SCHMIDT [6]. The results in these works do not seem to cover the equation (6.2), but those in BASS & CHEN [3] do; we have preferred to detail a direct construction which is, in our opinion at least, quite simpler.

In this spirit, let us introduce the scale function

$$\mathfrak{p}(y) = \frac{1 - \alpha}{2\,\lambda} \left(e^{2\lambda y} - 1 \right), \ \ y > 0 \,; \qquad \mathfrak{p}(0) = 0 \,; \qquad \mathfrak{p}(y) = \frac{\alpha}{2\,\lambda} \left(1 - e^{-2\lambda y} \right), \ \ y < 0 \,.$$

This has left-continuous derivative

$$\mathfrak{p}'(y) \ = \ (1 - \alpha) \, e^{2\lambda y} \, \mathbf{1}_{(0,\infty)}(y) + \alpha \, e^{-2\lambda y} \, \mathbf{1}_{(-\infty,0]}(y) \,, \qquad y \in \mathbb{R}$$

which is bounded away from zero, and second derivative measure

$$\mathfrak{p}''(\mathrm{d}y) \ = \ -2\,\mathfrak{b}(y) \, \mathfrak{p}'(y) \, \mathrm{d}y + \big(1 - 2\,\alpha\big) \, \boldsymbol{\delta}_0(\mathrm{d}y) \,.$$

Likewise, we introduce the inverse

$$\mathfrak{q}(z) = \frac{1}{2\,\lambda}\log\left(1 + \frac{2\,\lambda\,z}{1-\alpha}\right), \quad z > 0\,; \quad \mathfrak{q}(0) = 0\,; \quad \mathfrak{q}(z) = \frac{-1}{2\,\lambda}\log\left(1 - \frac{2\,\lambda\,z}{\alpha}\right), \quad z < 0$$

of the function $\mathfrak{p}(\cdot)$, as well as its left-continuous derivative and its second-derivative measure

$$\mathfrak{q}'(z) = \left(1 - \alpha + 2\,\lambda\,z\right)^{-1}\mathbf{1}_{(0,\infty)}(z) + \left(\alpha - 2\,\lambda\,z\right)^{-1}\mathbf{1}_{(-\infty,0]}(z)\,,$$

$$\mathfrak{q}''(\mathrm{d}z) = 2\,\mathfrak{b}(z)\left(\mathfrak{q}'(z)\right)^2\mathrm{d}z + \frac{2\,\alpha - 1}{\alpha\,(1-\alpha)}\,\boldsymbol{\delta}_0(\mathrm{d}z)\,.$$

Analysis: Assume that a solution to (6.2) has been constructed; in particular, the process $Y(\cdot)$ is then a continuous semimartingale for which (3.14) holds a.s. We look at the process $Z(\cdot) := \mathfrak{p}(Y(\cdot))$ and apply the ITÔ-TANAKA rule

$$\mathrm{d}Z(t) = \mathfrak{p}'(Y(t))\,\mathrm{d}Y(t) + \mathrm{d}\int_{\mathbb{R}}L^Y(t,y)\,\mathfrak{p}''(\mathrm{d}y)$$

of (2.9), to obtain

$$\mathrm{d}Z(t) = \mathfrak{p}'(Y(t))\left[\mathfrak{b}(Y(t))\,\mathrm{d}t + \mathrm{d}W(t) + \frac{2\,\alpha - 1}{\alpha}\,\mathrm{d}L^Y(t)\right]$$

$$-\,\mathrm{d}\int_{\mathbb{R}}2\,\mathfrak{b}(y)\,\mathfrak{p}'(y)\,L^Y(t,y)\,\mathrm{d}y + (1 - 2\,\alpha)\,\mathrm{d}L^Y(t) = \mathfrak{p}'\big(\mathfrak{q}(Y(t))\big)\,\mathrm{d}W(t)\,.$$

We have used here the occupation-time-density formula (2.8), and the property $\mathfrak{p}'(0) = \alpha$. Now the piecewise-linear function

$$\mathfrak{s}(z) := \mathfrak{p}'\big(\mathfrak{q}(z)\big) = \frac{1}{\mathfrak{q}'(z)} = \left(1 - \alpha + 2\,\lambda\,z\right)\mathbf{1}_{(0,\infty)}(z) + \left(\alpha - 2\,\lambda\,z\right)\mathbf{1}_{(-\infty,0]}(z)\,, \quad z \in \mathbb{R} \quad (6.5)$$

is bounded away from the origin, so the process $Z(\cdot)$ is the pathwise unique, strong solution of the stochastic differential equation (6.4) for this new dispersion function (NAKAO [18]); and because $Y(\cdot)$ and $Z(\cdot)$ are bijections of each other, we have again the filtration identities

$$\mathfrak{F}^Z(t) = \mathfrak{F}^Y(t) = \mathfrak{F}^W(t)\,, \qquad 0 \le t < \infty\,. \tag{6.6}$$

Synthesis: Consider the strong solution $Z(\cdot)$ of the stochastic differential equation (6.4) with the new dispersion function of (6.5), and define the process $Y(\cdot) := \mathfrak{q}(Z(\cdot))$. Since $\mathrm{d}Z(t) = \mathfrak{s}(Z(t))\,\mathrm{d}W(t)$ and $\mathrm{d}\langle Z\rangle(t) = \mathfrak{s}^2(Z(t))\,\mathrm{d}t$, this process satisfies almost surely

$$\int_0^\infty \mathbf{1}_{\{Z(t)=0\}}\,\mathrm{d}t = \int_0^\infty \mathbf{1}_{\{Z(t)=0\}}\,\frac{\mathrm{d}\langle Z\rangle(t)}{\mathfrak{s}^2(Z(t))} \le \left(\min(\alpha,\,1-\alpha)\right)^{-2}\int_0^\infty \mathbf{1}_{\{Z(t)=0\}}\,\mathrm{d}\langle Z\rangle(t) = 0\,,$$

and we apply the ITÔ-TANAKA rule to obtain

$$\mathrm{d}Y(t) = \mathfrak{q}'(Z(t))\,\mathrm{d}Z(t) + \mathrm{d}\int_{\mathbb{R}}L^Z(t,z)\,\mathfrak{q}''(\mathrm{d}z)\,.$$

On the strength of the occupation-time-density formula and $\mathfrak{s}(\cdot)\,\mathfrak{q}'(\cdot)\equiv 1$, this gives

$$dY(t) \;=\; dW(t) + d\int_{\mathbb{R}} 2\,\mathfrak{b}(y)\,\big(\mathfrak{q}'(y)\big)^2 L^Y(t,y)\,dy \;+\; \frac{2\,\alpha-1}{\alpha\,(1-\alpha)}\,dL^Z(t)$$

$$=\; dW(t) \;+\; \mathfrak{b}(Y(t))\,\big(\mathfrak{q}'(Y(t))\big)^2\,\mathfrak{s}^2(Y(t))\,dt \;+\; \frac{2\,\alpha-1}{\alpha\,(1-\alpha)}\,dL^Z(t)$$

$$=\; dW(t) \;+\; \mathfrak{b}(Y(t))\,dt \;+\; \frac{2\,\alpha-1}{\alpha}\,dL^Y(t)\,,$$

that is, the equation (6.2) of the skew Brownian motion with bang-bang drift for the process $Y(\cdot)$. We have used here the comparison of the local times at the origin for these two processes:

$$L^Z(\cdot) \;=\; (1-\alpha)\,L^Y(\cdot)\,. \tag{6.7}$$

This last identity (6.7) can be justified as follows: We start by noting

$$L^Z(\cdot) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\,\varepsilon}\int_0^{\cdot}\mathbf{1}_{\{0<Z(t)<\varepsilon\}}\,d\langle Z\rangle(t) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\,\varepsilon}\int_0^{\cdot}\mathbf{1}_{\{0<Y(t)<\mathfrak{q}(\varepsilon)\}}\big(\mathfrak{p}'(Y(t))\big)^2\,dt$$

$$=\; (1-\alpha)\cdot\lim_{\varepsilon\downarrow 0}\frac{1-\alpha}{2\,\varepsilon}\int_0^{\cdot}\mathbf{1}_{\{0<Y(t)<\mathfrak{q}(\varepsilon)\}}\left(\frac{\mathfrak{p}'(Y(t))}{1-\alpha}\right)^2 dt\,.$$

On the event $\{0<Y(t)<\mathfrak{q}(\varepsilon)\}$ we have

$$1 \;\leq\; \frac{\mathfrak{p}'(Y(t))}{1-\alpha} \;\leq\; e^{\,2\lambda\,\mathfrak{q}(\varepsilon)}\,; \quad\text{and since}\quad \lim_{\varepsilon\downarrow 0}\left(\frac{\mathfrak{q}(\varepsilon)}{\varepsilon}\right) \;=\; \frac{1}{1-\alpha}\,,$$

we deduce the claimed identity of (6.7), namely

$$L^Z(\cdot) \;=\; (1-\alpha)\cdot\lim_{\varepsilon\downarrow 0}\frac{1}{2\,\mathfrak{q}(\varepsilon)}\int_0^{\cdot}\mathbf{1}_{\{0<Y(t)<\mathfrak{q}(\varepsilon)\}}\,dt \;=\; (1-\alpha)\,L^Y(\cdot)\,.$$

• Taken together, the Analysis and Synthesis parts of this argument establish the following result.

Theorem 6.1. *The equation (3.18) admits a pathwise unique, strong solution for all values of its "skewness parameter" $\alpha\in[0,1]$, and we have the filtration identity $\mathfrak{F}^Y(t)=\mathfrak{F}^W(t)$, $0\leq t<\infty$ in (3.19).*

Remark 6.1. As shown towards the end of section 3 in HARRISON & SHEPP [9], the stochastic equation (3.18) admits no solution for $\alpha\notin[0,1]$. Consequently, when $\eta+\zeta\neq 0$ holds but the condition (1.5) fails because $\alpha=\eta/(\eta+\zeta)\notin[0,1]$, the system of equations (1.2), (1.3) admits no solution.

Remark 6.2. We compute in the next subsection the transition probabilities of the diffusion $Y(\cdot)$. It follows from these computations, and in conjunction with the theory developed in PORTENKO [23] [24] [25], that this process has the strong MARKOV and FELLER properties.

From the Remarks 3.1, 3.2, 6.1 and in conjunction with Theorem 4.1, we obtain now the following result.

Proposition 6.1. *The conditions of (1.5) are not just sufficient but also necessary for the well-posedness of the system of equations (1.2), (1.3).*

6.1. Joint Distribution of SBBBM and its Local Time

Let us recall a construction of the skew Brownian motion (ITÔ & MC KEAN [10], [11], WALSH [30]). We take a Brownian motion starting from $y_0 \geq 0$, reflect it at the origin, and consider its excursions away from the origin. Then we change the sign of each excursion independently with probability $1 - \alpha \in (0, 1)$. The resulting process is positive with probability α, negative with probability $1 - \alpha$. This implies a *non-symmetric reflection principle* around the origin. We shall see that even in the presence of the bang-bang drifts

$$\mathfrak{b}(y) \; = \; -\lambda \operatorname{sgn}(y), \quad y \in \mathbb{R}$$

as in (6.1), this principle continues to hold for the skew Brownian motion. Thus, the joint distribution of SBBBM and its local time are derived here.

By GIRSANOV's theorem (e.g., KARATZAS & SHREVE [13] , section 3.5), we consider the "reference probability measure" $\mathbb{P}_\star$, under which the process $\int_0^\cdot \mathfrak{b}(Y(t)) \, \mathrm{d}t + W(\cdot)$ becomes standard Brownian motion. For every given $t \in [0, \infty)$ the RADON-NIKODÝM derivative on $\mathfrak{F}^Y(t)$ of the original measure with respect to the reference measure, is

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_\star}\bigg|_{\mathfrak{F}^Y(t)} \; = \; \exp\left\{ \int_0^t \mathfrak{b}(Y(s))\mathrm{d}W(s) + \frac{1}{2} \int_0^t \mathfrak{b}^2(Y(s))\mathrm{d}s \right\}$$

$$= \; \exp\left\{ \lambda\big(|y_0| - |Y(t)| + 2\widehat{L}^Y(t)\big) - \frac{\lambda^2}{2}t \right\};$$

we have used in this last equation the relationships (3.29), (3.9).

Under the reference probability measure $\mathbb{P}_\star$, the process $Y(\cdot)$ is skew Brownian motion which starting at y_0. As shown in WALSH [30] (see also LANG [14]), the transition probability density function $\mathfrak{p}_\star(t; y_0, \xi) = \mathbb{P}_\star(Y(t) \in \mathrm{d}\xi) / \mathrm{d}\xi$ for this process is given by

$$\mathfrak{p}_\star(t; y_0, \xi) \; = \; \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(y_0 - \xi)^2}{2t} \right\} + (2\alpha - 1) \cdot \operatorname{sgn}(\xi) \cdot \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(|y_0| + |\xi|)^2}{2t} \right\}$$

for $(\xi, y) \in \mathbb{R}^2$, $t > 0$. Moreover, by the method of elastic Brownian motion (e.g., KARATZAS & SHREVE [12], APPUHAMILLAGE ET AL. [2]) the joint distribution of the skew Brownian motion and its symmetric local time is computed as

$$\mathbb{P}_\star(Y(t) \in \mathrm{d}\xi, \, 2\widehat{L}^Y(t) \in \mathrm{d}b) \; =$$

$$= \; \big\{ 1 + (2\alpha - 1)\operatorname{sgn}(\xi) \big\} \cdot \frac{|\xi| + b + |y_0|}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(|\xi| + b + |y_0|)^2}{2t} \right\} \mathrm{d}\xi \, \mathrm{d}b; \quad b > 0$$

and

$$\mathbb{P}_\star\big(Y(t) \in \mathrm{d}\xi, \, 2\widehat{L}^Y(t) = 0\big) \; = \; \frac{1}{\sqrt{2\pi t}} \left[\exp\left\{ -\frac{(|y_0| - |\xi|)^2}{2t} \right\} - \exp\left\{ -\frac{(|y_0| + |\xi|)^2}{2t} \right\} \right] \mathrm{d}\xi$$

for $\xi \in \mathbb{R}$. Note that we have

$$1 + (2\alpha - 1)\operatorname{sgn}(\xi) \; = \; 2\alpha \quad \text{if} \; \xi > 0, \quad \text{and} \quad 1 + (2\alpha - 1)\operatorname{sgn}(\xi) \; = \; 2(1 - \alpha) \; \text{if} \; \xi \leq 0.$$

Thus, the non-symmetric reflection principle around the origin works intuitively even for the joint distribution. As expected, when there is no accumulation of local time at the origin, the skewness parameter α does not affect the transition probabilities.

• We bring the above formulae from the reference measure $\mathbb{P}_\star$ back to the original measure $\mathbb{P}$, by means of

$$\mathbb{P}\big(Y^\pm(t) \in A, Y^\mp(t) = 0, 2\widehat{L}^Y(t) \in B\big) = \tag{6.8}$$

$$= \exp\left\{\lambda|y_0| - \lambda^2 t/2\right\} \cdot \mathbb{E}^{\mathbb{P}_\star}\left[\exp\left\{2\widehat{L}^Y(t) - Y^\pm(t)\right\} \cdot \mathbf{1}_{\{Y^\pm(t)\in A, Y^\mp(t)=0, 2\widehat{L}^Y(t)\in B\}}\right]$$

for $(A, B) \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, $t > 0$. With $(\xi, b) \in [0, \infty) \times (0, \infty)$, the joint density functions are

$$\mathbb{P}(Y^+(t) \in \mathrm{d}\xi, Y^-(t) = 0, 2\widehat{L}^Y(t) \in \mathrm{d}b) = \tag{6.9}$$

$$= 2\alpha \cdot e^{-2\lambda\xi} \cdot \frac{\xi + b + |y_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\xi + b + |y_0| - \lambda t)^2}{2t}\right\} \mathrm{d}\xi\, \mathrm{d}b,$$

as well as

$$\mathbb{P}(Y^-(t) \in \mathrm{d}\xi, Y^+(t) = 0, 2\widehat{L}^Y(t) \in \mathrm{d}b) = \tag{6.10}$$

$$= 2(1 - \alpha) \cdot e^{-2\lambda\xi} \cdot \frac{\xi + b + |y_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\xi + b + |y_0| - \lambda t)^2}{2t}\right\} \mathrm{d}\xi\, \mathrm{d}b.$$

Whereas, when there is no accumulation of local time, we have

$$\mathbb{P}\left(Y^\pm(t) \in \mathrm{d}\xi, Y^\mp(t) = 0, 2\widehat{L}^Y(t) = 0\right) = \tag{6.11}$$

$$= \frac{1}{\sqrt{2\pi t}}\left(\exp\left\{-\frac{(\xi - |y_0| + \lambda t)^2}{2t}\right\} - e^{-2\lambda\xi} \cdot \exp\left\{-\frac{(\xi + |y_0| + \lambda t)^2}{2t}\right\}\right) \mathrm{d}\xi, \quad \xi > 0.$$

• The marginal density $\mathfrak{p}(t; y_0, \xi)\, \mathrm{d}\xi = \mathbb{P}(Y(t) \in \mathrm{d}\xi)$ of $Y(t)$ under the original probability measure $\mathbb{P}$ is obtained from

$$\mathbb{P}(Y(t) \in \mathrm{d}\xi) = \mathbb{P}(Y(t) \in \mathrm{d}\xi, 2\widehat{L}^Y(t) > 0) + \mathbb{P}(Y(t) \in \mathrm{d}\xi, 2\widehat{L}^Y(t) = 0) \cdot \mathbf{1}_{\{\xi\, y_0 > 0\}}, \tag{6.12}$$

where the second term takes care of the case when the local time is absent. If $\xi > 0$ and $y_0 > 0$, the marginal density becomes

$$\mathfrak{p}(t; y_0, \xi) = (2\alpha - 1)\, e^{-2\lambda\xi} \cdot \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(\xi + y_0 - \lambda t)^2}{2t}\right\} \tag{6.13}$$

$$+ \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(\xi - y_0 + \lambda t)^2}{2t}\right\} + (2\alpha) \cdot \frac{\lambda\, e^{-2\lambda\xi}}{\sqrt{2\pi t}} \int_{\xi + y_0}^\infty e^{-\frac{(u - \lambda t)^2}{2t}}\, \mathrm{d}u\,;$$

whereas, if $\xi < 0$ and $y_0 < 0$, this expression becomes

$$\mathfrak{p}(t; y_0, \xi) = (1 - 2\alpha)e^{2\lambda\xi} \cdot \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(-\xi - y_0 - \lambda t)^2}{2t}\right\} \tag{6.14}$$

$$+ \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(-\xi + y_0 + \lambda t)^2}{2t} \right\} + 2(1-\alpha) \cdot \frac{\lambda\, e^{2\lambda\xi}}{\sqrt{2\pi t}} \int_{-\xi - y_0}^{\infty} e^{-\frac{(u-\lambda t)^2}{2t}}\, \mathrm{d}u\,.$$

If $\xi\, y_0 \le 0$, then this expression becomes

$$\mathfrak{p}(t; y_0, \xi) = \left\{ 1 + (2\alpha - 1)\,\mathrm{sgn}(\xi) \right\} \cdot \frac{e^{-2\lambda|\xi|}}{\sqrt{2\pi t}} \cdot$$

$$\cdot \left[\exp\left\{ -\frac{(|\xi| + |y_0| - \lambda t)^2}{2t} \right\} + \lambda \int_{|\xi| + |y_0|}^{\infty} e^{-\frac{(u - \lambda t)^2}{2t}}\, \mathrm{d}u \right], \qquad (6.15)$$

where the term $\mathbb{P}(Y(t) \in \mathrm{d}\xi\,, 2\widehat{L}^Y(t) = 0)$ in (6.12) is now equal to zero.

Remark 6.3. Letting $t \to \infty$ in (6.13)-(6.15), we derive for the process $Y(\cdot)$ the stationary measure $\mathfrak{m}(\cdot) = \int \mathfrak{p}_\infty(\xi)\, \mathrm{d}\xi$ with the double-exponential probability density function

$$\mathfrak{p}_\infty(\xi) := \lim_{t \to \infty} \mathfrak{p}(t; y_0, \xi) = \alpha \cdot (2\lambda)\, e^{-2\lambda\xi} \cdot \mathbf{1}_{\{\xi > 0\}} + (1 - \alpha) \cdot (2\lambda)\, e^{2\lambda\xi} \cdot \mathbf{1}_{\{\xi \le 0\}}\,. \qquad (6.16)$$

It can be verified that (6.16) is the invariant distribution. Furthermore, it follows from the transition density (6.13)-(6.15) and the stationary distribution (6.16) that the following duality holds:

$$\int_{\mathbb{R}} g(y) \left(\int_{\mathbb{R}} f(\xi) \mathfrak{p}(t; y, \xi)\, \mathrm{d}\xi \right) \mathfrak{p}_\infty(y)\, \mathrm{d}y = \int_{\mathbb{R}} f(\xi) \left(\int_{\mathbb{R}} g(y) \mathfrak{p}(t; \xi, y)\, \mathrm{d}y \right) \mathfrak{p}_\infty(\xi)\, \mathrm{d}\xi,$$

for arbitrary bounded, measurable functions f, g, or equivalently

$$\int_{\mathbb{R}} g(y)\, \mathbb{E}_y\big[f(Y(t)) \big]\, \mathfrak{p}_\infty(y)\, \mathrm{d}y = \int_{\mathbb{R}} f(\xi)\, \mathbb{E}_\xi\big[g(Y(t)) \big]\, \mathfrak{p}_\infty(\xi)\, \mathrm{d}\xi\,; \quad t > 0\,. \qquad (6.17)$$

Here $\mathbb{E}_y$ stands for the expectation under the measure $\mathbb{P}_y$ induced by $Y(\cdot)$ which starts from $y \in \mathbb{R}$. Thus, under the probability measure

$$\mathbb{P}_\infty(\cdot) := \int_{\mathbb{R}} \mathbb{P}_y(\cdot)\, \mathfrak{m}(\mathrm{d}y)\,,$$

the process $Y(\cdot)$ is stationary; and moreover, given a fixed time $T \in (0, \infty)$, the time reversal

$$\widehat{Y}(t) := Y(T - t)\,, \qquad 0 \le t \le T \qquad (6.18)$$

satisfies

$$\mathbb{E}^{\mathbb{P}_\infty}\big[f_0(Y(t_0)) \cdots f_n(Y(t_n)) \big] = \mathbb{E}^{\mathbb{P}_\infty}\big[f_0(\widehat{Y}(t_n)) \cdots f_n(\widehat{Y}(t_0)) \big] \qquad (6.19)$$

for every integer $n \in \mathbb{N}$, collection of time points $0 = t_0 < t_1 < \cdots < t_n = T$, and bounded, measurable functions $f_0, \ldots, f_n$.

Remark 6.4. The infinitesimal generator of the process $Y(\cdot)$ may be defined formally by

$$[\mathcal{L}f](\xi) := \frac{1}{2}\, f''(\xi) - \lambda\, \mathrm{sgn}(\xi) f'(\xi) + 2(2\alpha - 1) f'(\xi)\, \delta_0(\xi)\,, \quad \xi \in \mathbb{R} \qquad (6.20)$$

26

for $f \in \mathcal{D} := C_0^\infty(\mathbb{R})$, where $\delta_0(\cdot)$ is the "Dirac delta function" at the origin. Here we use the parametrization for the symmetric local time $\widehat{L}^Y(\cdot)$. Let us denote formally the symmetric version of the density of $\mathfrak{m}$ by

$$\widehat{\mathfrak{p}}_\infty(\xi) = (2\alpha)\,\lambda\,e^{-\lambda\xi} \cdot \mathbf{1}_{\{\xi>0\}} + 2(1-\alpha)\,\lambda\,e^{\lambda\xi} \cdot \mathbf{1}_{\{\xi<0\}} + \lambda \cdot \mathbf{1}_{\{\xi=0\}}\,.$$

Then by direct calculation

$$\int_{\mathbb{R}} f(\xi)\,[\mathcal{L}g](\xi)\,\mathfrak{m}(\mathrm{d}\xi) = \int_{\mathbb{R}} g(\xi)\,[\mathcal{L}f](\xi)\,\mathfrak{m}(\mathrm{d}\xi)\,, \quad \int_{\mathbb{R}} [\mathcal{L}f]\,\mathfrak{m}(\mathrm{d}\xi) = 0\,; \quad f,\,g \in \mathcal{D}\,. \quad (6.21)$$

Applying Theorem 2.3 of FUKUSHIMA & STROOCK [8], we arrive at the same conclusion (6.19).

Remark 6.5. Let us define the time reversal of $Y(\cdot)$ as in (6.18). Following PARDOUX [20] and PETIT [21], we may show that the time reversal is a solution of the stochastic equation

$$\widehat{Y}(t) = \widehat{Y}(0) + W^\sharp(t) + 2\,(1-2\alpha)\,\widehat{L}^{\widehat{Y}}(t) +$$

$$+ \int_0^t \left(\lambda\,\mathrm{sgn}\big(\widehat{Y}(s)\big) + \frac{\partial}{\partial \xi}\log\mathfrak{p}\big(T-s; y_0, \widehat{Y}(s)\big) \right) \mathrm{d}s \quad (6.22)$$

for $0 \le t \le T$, where $W^\sharp(\cdot)$ is a standard Brownian motion with respect to the backwards filtration $\mathbf{F}^{\widehat{Y}}(\cdot)$ generated by the time-reversed process $\widehat{Y}(\cdot)$ of (6.18), and

$$\widehat{L}^{\widehat{Y}}(t) := \widehat{L}^Y(T) - \widehat{L}^Y(T-t)\,, \qquad 0 \le t \le T\,. \quad (6.23)$$

In the special case $y_0 = 0 = \widehat{Y}(T)$, the logarithmic derivative of the transition probability density function is

$$\frac{\partial}{\partial \xi}\log\mathfrak{p}\big(t; 0, \xi\big) = -2\,\lambda\,\mathrm{sgn}(\xi) - \frac{\xi}{t} \cdot \frac{\mathfrak{C}_1(t,\xi)}{\mathfrak{C}_1(t,\xi) + \mathfrak{C}_2(t,\xi)}\,,$$

where

$$\mathfrak{C}_1(t,\xi) := \exp\left(-\frac{(|\xi|+\lambda t)^2}{2t} \right)\,, \quad \mathfrak{C}_2(t,\xi) := \lambda\,e^{-2\lambda|\xi|} \int_{|\xi|}^\infty \exp\left(-\frac{(u-\lambda t)^2}{2t} \right) \mathrm{d}u\,.$$

Thus, the time reversal is a *skew Brownian bridge* with bang-bang drift

$$\widehat{Y}(\cdot) = \widehat{Y}(0) + W^\sharp(\cdot) + 2\,(1-2\,\alpha)\,\widehat{L}^{\widehat{Y}}(\cdot) -$$

$$- \int_0^\cdot \left[\lambda\,\mathrm{sgn}\big(\widehat{Y}(t)\big) + \frac{\widehat{Y}(t)}{T-t} \cdot \left(\frac{\mathfrak{C}_1}{\mathfrak{C}_1 + \mathfrak{C}_2} \right)\big(T-t, \widehat{Y}(t)\big) \right] \mathrm{d}t\,.$$

This result suggests that the time reversal of SBBBM in general looks like a skew Brownian bridge drifted towards the target point $\widehat{Y}(T) = y_0$.

7. Applications of the Skew Representations (3.30)-(3.31)

With the skew representations in section 3.4 and the joint distribution of $(Y(\cdot), \widehat{L}^Y(\cdot))$ in section 6.1 it is now straightforward to compute the transition density of the system (1.2)-(1.3) as well as its time reversal.

7.1. Transition Density

Let us discuss only some special cases, since the other cases are quite similar. For example, in the degenerate case with $\sigma = 0$, thus $\rho = 1$, $\gamma = 1$ and with $x_1 \geq x_2$, (3.30)-(3.31) become

$$X_1(t) = x_1 + g\,t + (Y^+(t) - y_0^+) - \beta\,\widehat{L}^Y(t)\,, \quad X_2(t) = x_2 + g\,t + (Y^-(t) - y_0^-) - \beta\,\widehat{L}^Y(t)\,,$$

for $0 \leq t < \infty$, where $y_0 = x_1 - x_2 \geq 0$, and hence the transition density of $(X_1(\cdot), X_2(\cdot))$ is

$$\mathbb{P}\big(X_1(t) \in \mathrm{d}\xi_1\,, \ X_2(t) \in \mathrm{d}\xi_2\big) = (2\alpha) \cdot \frac{2}{\beta} \cdot e^{-2\lambda(\xi_1 - \xi_2)} \cdot \frac{\mathfrak{c}_1}{\sqrt{2\,\pi\,t^3}} \exp\Big\{-\frac{(\mathfrak{c}_1 - \lambda\,t)^2}{2t}\Big\}\mathrm{d}\xi_1\mathrm{d}\xi_2\,,$$

$$\text{where} \quad \mathfrak{c}_1 := \xi_1 - \Big(\frac{2+\beta}{\beta}\Big)\xi_2 + x_1 + \Big(\frac{2-\beta}{\beta}\Big)x_2 + \frac{2}{\beta}\,g\,t\,,$$

if $\beta > 0$, $\xi_1 \geq \xi_2$ and $\xi_2 < x_2 + g\,t$. Similarly, by (skew) symmetry:

$$\mathbb{P}\big(X_1(t) \in \mathrm{d}\xi_1\,, \ X_2(t) \in \mathrm{d}\xi_2\big) = 2(1-\alpha) \cdot \frac{2}{\beta} \cdot e^{-2\lambda(\xi_2 - \xi_1)} \cdot \frac{\mathfrak{c}_2}{\sqrt{2\,\pi\,t^3}} \exp\Big\{-\frac{(\mathfrak{c}_2 - \lambda\,t)^2}{2t}\Big\}\mathrm{d}\xi_1\mathrm{d}\xi_2\,,$$

$$\text{where} \quad \mathfrak{c}_2 := \xi_2 - \Big(\frac{2+\beta}{\beta}\Big)\xi_1 + x_1 + \Big(\frac{2-\beta}{\beta}\Big)x_2 + \frac{2}{\beta}\,g\,t\,,$$

if $\beta > 0$, $\xi_2 \geq \xi_1$ and $\xi_1 < x_2 + g\,t$. If $\beta > 0$, $\xi_1 > \xi_2 = x_2 + g\,t$, then the local time $\widehat{L}^Y(\cdot)$ is absent, and the transition density is easily obtained from (6.11):

$$\mathbb{P}\big(X_1(t) \in \mathrm{d}\xi_1\,, \ X_2(t) = x_2 + g\,t\big) =$$

$$= \frac{1}{\sqrt{2\pi t}}\Big(\exp\Big\{-\frac{(a - x_1 + x_2 + \lambda t)^2}{2t}\Big\} - e^{-2\lambda\,a}\cdot\exp\Big\{-\frac{(a + x_1 - x_2 + \lambda t)^2}{2t}\Big\}\Big)\Big|_{a = \xi_1 - x_2 - g\,t}\,\mathrm{d}\xi_1\,.$$

• For another extreme example, in the degenerate case with $\sigma = 1$, thus $\rho = 0$, $\gamma = -1$ and with $x_1 \geq x_2$, (3.30)-(3.31) become

$$X_1(t) = x_1 - h\,t - (Y^-(t) - y_0^-) + (2-\beta)\,\widehat{L}^Y(t)\,, \quad X_2(t) = x_2 - h\,t - (Y^+(t) - y_0^+) + (2-\beta)\,\widehat{L}^Y(t)\,,$$

for $0 \leq t < \infty$. If $\beta < 2$, $\xi_1 \geq \xi_2$ and $\xi_1 > x_1 - h\,t$, then

$$\mathbb{P}\big(X_1(t) \in \mathrm{d}\xi_1\,, \ X_2(t) \in \mathrm{d}\xi_2\big) = (2\alpha) \cdot \frac{2}{2-\beta} \cdot e^{-2\lambda(\xi_1 - \xi_2)} \cdot \frac{\mathfrak{c}_3}{\sqrt{2\,\pi\,t^3}} \exp\Big\{-\frac{(\mathfrak{c}_3 - \lambda\,t)^2}{2t}\Big\}\mathrm{d}\xi_1\mathrm{d}\xi_2\,,$$

$$\text{where} \quad \mathfrak{c}_3 := \Big(\frac{4-\beta}{2-\beta}\Big)\xi_1 - \xi_2 - \Big(\frac{\beta}{2-\beta}\Big)x_1 - x_2 + \Big(\frac{4-\beta}{2-\beta}\Big)h\,t$$

If $\beta < 2$, $\xi_2 \geq \xi_1$ and $\xi_2 > x_1 - h\,t$, then

$$\mathbb{P}\big(X_1(t) \in \mathrm{d}\xi_1\,, \ X_2(t) \in \mathrm{d}\xi_2\big) = 2(1-\alpha) \cdot \frac{2\,e^{-2\lambda(\xi_2 - \xi_1)}}{2-\beta} \cdot \frac{\mathfrak{c}_4}{\sqrt{2\,\pi\,t^3}} \exp\Big\{-\frac{(\mathfrak{c}_4 - \lambda\,t)^2}{2t}\Big\}\mathrm{d}\xi_1\mathrm{d}\xi_2\,,$$

$$\text{where} \quad \mathfrak{c}_4 := \Big(\frac{4-\beta}{2-\beta}\Big)\xi_2 - \xi_1 - \Big(\frac{\beta}{2-\beta}\Big)x_1 - x_2 + \Big(\frac{4-\beta}{2-\beta}\Big)h\,t\,.$$

If $\beta < 2$, $\xi_1 = x_1 - h\,t > \xi_2$, then the local time $\widehat{L}^Y(\cdot)$ does not accumulate, that is, the transitions density is obtained from (6.11) :

$$\mathbb{P}(X_1(t) = x_1 - h\,t\,,\; X_2(t) \in \mathrm{d}\xi_2) =$$

$$= \frac{1}{\sqrt{2\pi t}}\left(\exp\left\{-\frac{(a - x_1 + x_2 + \lambda t)^2}{2t}\right\} - e^{-2\lambda a}\exp\left\{-\frac{(a + x_1 - x_2 + \lambda t)^2}{2t}\right\}\right)\Bigg|_{a = x_1 - \xi_2 - h\,t}\mathrm{d}\xi_1\,.$$

• For the isotropic variance case with $\rho = \sigma = 1/\sqrt{2}$, $\gamma = 0$ and $x_1 \geq x_2$ the difference and the sum of $X_1(\cdot)$ and $X_2(\cdot)$ are

$$X_1(\cdot) - X_2(\cdot) = Y(\cdot)\,,\quad X_1(\cdot) + X_2(\cdot) = x_1 + x_2 + \nu\,t + 2(2\alpha - 1)\widehat{L}^Y(\cdot) + Q(\cdot)\,,$$

where $(Y(\cdot), \widehat{L}^Y(\cdot))$ and $Q(\cdot)$ are independent. Thus the joint distribution of $(X_1(\cdot), X_2(\cdot))$ are obtained by integrating out the local time.

If $\alpha \in (0,1) \setminus \{1/2\}$, then the above transition densities are discontinuous on the diagonal line due to the skewness. If $\alpha = 1/2$ and $\beta = 1$, then these formulae are the same as those of the degenerate system studied in FERNHOLZ ET AL. [7]. The transition densities for all the other cases as well as the joint distribution of $(X_1(\cdot), X_2(\cdot), L^{X_1 - X_2}(\cdot))$ are computable from the skew representations (3.30)-(3.31) and the joint distribution (6.9)-(6.11) in a similar manner.

7.2. Time Reversal

We consider now the time-reversal

$$\widehat{X}_i(t) := X_i(T - t)\,,\quad \widetilde{X}_i(t) := X_i(T - t) - X_i(T)\,,\quad 0 \leq t \leq T\,,\; i = 1, 2 \qquad (7.1)$$

of the solution to the system (1.2)-(1.3) with the backwards filtration $\widetilde{\mathbf{F}} = \{\widetilde{\mathfrak{F}}(t)\}_{0 \leq t \leq T}$ generated by the random variable $Y(T)$ and by the time-reversal

$$\left(\widetilde{W}(t) := W(T - t) - W(T)\,,\; \widetilde{Q}(t) := Q(T - t) - Q(T)\right),\qquad 0 \leq t \leq T$$

of the planar Brownian motion :

$$\widetilde{\mathfrak{F}}(t) := \boldsymbol{\sigma}(Y(T)) \vee \mathfrak{F}^{(\widetilde{Q},\widetilde{W})}(t)\,,\quad \mathfrak{F}^{(\widetilde{Q},\widetilde{W})}(t) := \boldsymbol{\sigma}\left(\widetilde{Q}(\boldsymbol{\theta}), \widetilde{W}(\boldsymbol{\theta})\,; 0 \leq \boldsymbol{\theta} \leq t\right)\,,\quad 0 \leq t \leq T\,.$$

With some extra work in addition to the discussion of Remark 6.5 we may show that $\widehat{Y}(\cdot)$ is a diffusion (6.22) driven by the $\widehat{\mathbb{F}}$-Brownian motion $\widehat{W}^\sharp(\cdot)$ (c.f. PARDOUX [20] and section 3 of PETIT [21]). Combining the skew representations (3.30)-(3.31) with the time-reversals (6.22)-(6.23), we derive the time-reversed skew representation form

$$\widetilde{X}_1(t) = -\mu\,t + \rho^2\big(\widehat{Y}^+(t) - \widehat{Y}^+(0)\big) - \sigma^2\big(\widehat{Y}^-(t) - \widehat{Y}^-(0)\big) - (1 - \beta - \gamma)\widehat{L}^Y(t) + \rho\,\sigma\widehat{Q}(t)$$

$$= -\mu\,t + \int_0^t \big(\rho^2\mathbf{1}_{\{\widehat{Y}(s) > 0\}} + \sigma^2\mathbf{1}_{\{\widehat{Y}(s) \leq 0\}}\big)\mathrm{d}\widehat{Y}(s) - (1 - \beta - 2\gamma)\widehat{L}^Y(t) + \rho\,\sigma\widehat{Q}(t) \quad (7.2)$$

for $0 \leq t \leq T$, and

$$\widetilde{X}_2(t) = -\mu\,t - \sigma^2\big(\widehat{Y}^+(t) - \widehat{Y}^+(0)\big) + \rho^2\big(\widehat{Y}^-(t) - \widehat{Y}^-(0)\big) - (1 - \beta - \gamma)\widehat{L}^Y(t) + \rho\,\sigma\widehat{Q}(t)$$

$$= -\mu\,t - \int_0^t \big(\rho^2\mathbf{1}_{\{\widehat{Y}(s) \leq 0\}} + \sigma^2\mathbf{1}_{\{\widehat{Y}(s) > 0\}}\big)\mathrm{d}\widehat{Y}(s) - (1 - \beta - 2\gamma)\widehat{L}^Y(t) + \rho\,\sigma\widehat{Q}(t)\,. \quad (7.3)$$

Remark 7.1. By analogy with (7.1) we denote the time-reversal of ranks by $\widehat{R}_i(t) := R_i(T - t)$ for $0 \leq t \leq T$, $i = 1, 2$. Applying the TANAKA formula to (7.2)-(7.3), we may derive the time-reversed dynamics of $(R_1(\cdot), R_2(\cdot))$.

Remark 7.2. As we saw in Remarks 6.3-6.4, the process $Y(\cdot)$ is strictly time-reversible when started at its invariant distribution (6.16). Under this invariant distribution, the dynamics of the time-reversal of $(R_1(\cdot), R_2(\cdot))$ can be derived through the skew representations of (3.30)-(3.31).

References

[1] S.V. Anulova, Diffusion processes with singular characteristics. In "Stochastic Differential Systems, Filtering and Control": Proceedings of an I.F.I.P.-W.G. Conference, Vilnius, 1978, Lithuania. *Lecture Notes in Control and Information Systems* **25** (1980) 264-269. Springer-Verlag, New York.

[2] Th. Appuhamillage, B. Vrushali, E. Thomann, E. Waymire, B. Wood, Occupation and local times for skew Brownian motion with application to dispersion across an interface. *Ann. Appl. Probab.* **21** (2011) 183-214. (Correction: ibid., 2050-2051.)

[3] R. Bass, Z.Q. Chen, One-dimensional stochastic differential equations with singular and degenerate coëfficients. *Sankhyā* **67** (2005) 19-45.

[4] K. Burdzy, D. Nualart, Brownian motion reflected on Brownian motion. *Probab. Theory Relat. Fields* **122** (2002) 471-493.

[5] R.J. Chitashvili, N.L. Lazrieva, Strong solutions of stochastic differential equations with boundary conditions. *Stochastics* **5** (1981) 255-309.

[6] H.J. Engelbert, W. Schmidt, On one-dimensional stochastic differential equations with generalized drift. *Lecture Notes in Control and Information Systems* **69** (1985) 143-155. Springer-Verlag, NY.

[7] E.R. Fernholz, T. Ichiba, I. Karatzas, V. Prokaj, A planar diffusion with rank-based characteristics, and perturbed Tanaka equations. *Probab. Theory Relat. Fields,* to appear. Available at the site *http://arxiv.org/pdf/1108.3992* (posted August 19, 2011).

[8] M. Fukushima, D. Stroock, Reversibility of solutions to martingale problems. In "Probability, Statistical Mechanics, and Number Theory". *Adv. Math. Suppl. Stud.* **9** (1986) 107-123. Academic Press, Orlando, FL.

[9] J.M. Harrison, L.A. Shepp, On skew Brownian motion. *Ann. Probab.* **9** (1981) 309–313.

[10] K. Itô, H.P. Mc Kean Jr., Brownian Motion on a Half-Line. *Illinois J. Math.* **7** (1963) 181–231.

[11] K. Itô, H.P. Mc Kean Jr., *Diffusion Processes and Their Sample Paths.* Second Printing (Corrected), Springer Verlag, New York, 1974.

[12] I. Karatzas, S.E. Shreve, Trivariate density of Brownian motion, its local and occupation times, with application to stochastic control. *Ann. Probab.* **12** (1984) 819-828.

[13] I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus.* Second Edition, Springer Verlag, New York, 1991.

[14] R. Lang, Effective conductivity and skew Brownian motion. *J. Stat. Phys.* **80** (1995) 125-146.

[15] J.F. Le Gall, Applications des temps locaux aux équations différentielles stochastiques unidimensionelles. *Lecture Notes in Mathematics* **986** (1983) 15-31. Springer-Verlag, New York.

[16] J.F. Le Gall, One-dimensional stochastic differential equations involving the local times of the unknown process. *Lecture Notes in Mathematics* **1095** (1984) 51-82. Springer-Verlag, New York.

[17] A. Lejay, On the constructions of the skew Brownian motion. *Probability Surveys* **3** (2006) 413-466.

[18] S. Nakao, On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. *Osaka J. of Math.* **9** (1972) 513-518.

[19] Y. Ouknine, M. Rutkowski, Local times of functions of continuous semimartingales *Stochastic Anal. Appl.* **13** (1995) 211-231.

[20] E. Pardoux, Grossissement d'une filtration et retournement du temps. *Séminaire de Probabilités* XX, 1984/85, *Lecture Notes in Mathematics*, **1204** (1986) 48-55. Springer-Verlag, New York.

[21] F. Petit, Time reversal and reflected diffusion. *Stochastic Process. Appl.* **69** (1997) 25-53.

[22] N.I. Portenko, Generalized diffusion processes. *Lecture Notes in Mathematics* **550** (1976) 500-523. Springer-Verlag, New York.

[23] N.I. Portenko, Diffusion processes with generalized drift coëfficients. *Theory Probab. Appl.* **24** (1979) 62-78.

[24] N.I. Portenko, Stochastic differential equations with generalized drift vector. *Theory Probab. Appl.* **24** (1979) 338-353.

[25] N.I. Portenko, *Generalized Diffusion Processes*. Translations of Mathematical Monographs. American Mathematical Society, Providence, RI 1990.

[26] V. Prokaj, The solution of the perturbed Tanaka equation is pathwise unique. *Ann. Probab.*, to appear. (2011) arXiv:1104.0740.

[27] F. Soucaliuc, B. Tóth, W. Werner, Reflection and coalescence between independent one-dimensional Brownian motions. *Ann. Inst. Henri Poincaré, Sec. B* **36** (2000) 509-545.

[28] F. Soucaliuc, W. Werner, A note on reflecting Brownian motions. *Electron. Comm. Probab.* **7** (2002) 117-122.

[29] A.S. Sznitman, S.R.S. Varadhan, A multidimensional process involving local time. *Probab. Theory Relat. Fields* **71** (1986) 553-579.

[30] J.B. Walsh, A diffusion with discontinuous local time. In "Temps Locaux", *Astérisque* **52-53** (1978) 37-45.