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MEKLER'S CONSTRUCTION AND GENERALIZED STABILITY

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Abstract. Mekler's construction gives an interpretation of any structure in a finite relational language in a pure group (nilpotent of class 2 and exponent $p > 2$, but not finitely generated in general). Even though this construction is not a bi-interpretation, it is known to preserve some model-theoretic tameness properties of the original structure including stability and simplicity. We demonstrate further that k-dependence of the theory is preserved, for all $k \in \mathbb{N}$, and that NTP_2 is preserved. We apply this result to obtain first examples of strictly k-dependent pure groups.

1. Introduction

Mekler's construction [11] provides a general method to interpret any structure in a finite relational language in a pure 2-nilpotent group of finite exponent (the resulting group is typically not finitely generated). This is not a bi-interpretation, however it tends to preserve various model-theoretic tameness properties. First Mekler proved that for any cardinal κ the constructed group is κ-stable if and only if the initial structure was [11]. Afterwards, it was shown by Baudisch and Pentzel that simplicity of the theory is preserved, and by Baudisch that, assuming stability, CM-triviality is also preserved [2]. See [10, Section A.3] for a detailed exposition of Mekler's construction.

The aim of this paper is to investigate further preservation of various generalized stability-theoretic properties from Shelah's classification program [14]. We concentrate on the classes of k-dependent and NTP_2 theories.

The classes of k-dependent theories (see Definition 4.1), for each $k \in \mathbb{N}$, were defined by Shelah in [16], and give a generalization of the class of NIP theories (which corresponds to the case $k = 1$). See [7, 9, 15] for some further results about k-dependent groups and fields and connections to combinatorics. In Theorem 4.7 we show that Mekler's construction preserves k-dependence. Our initial motivation was to obtain algebraic examples that witness the strictness of the k-dependence hierarchy. For $k \geq 2$, we will say that a theory is *strictly k-dependent* if it is k-dependent, but not $(k-1)$-dependent. The usual combinatorial example of a strictly k-dependent theory is given by the random k-hypergraph. The first example of a strictly 2-dependent group was given in [9] (it was also considered in [17, Example 4.1.14]):

Example 1. Let G be $\oplus_\omega \mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field with p elements. Consider the structure $(G, \mathbb{F}_p, 0, +, \cdot)$, where 0 is the neutral element, $+$ is addition in G, and $\cdot$ is the bilinear form $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ from G to $\mathbb{F}_p$. This group is not NIP, but is 2-dependent. In the case $p = 2$, this structure is mutually interpretable with an

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extra-special p-group (see e.g. the appendix in [12]), hence providing an example of a strictly 2-dependent pure group.

In Corollary 4.8 we use Mekler's construction to show that for every k, there is a strictly k-dependent pure group.

The class of NTP_2 theories was defined in [13] (see Definition 5.1). It gives a common generalization of simple and NIP theories (along with containing many new important examples), and more recently it was studied in e.g. [3–6]. In Theorem 5.6 we show that Mekler's construction preserves NTP_2.

The paper is organized as follows. In Section 2 we review Mekler's construction and record some auxiliary lemmas, including the key lemma about type-definability of partial transversals and related objects (Proposition 2.14). In Section 3 we prove that NIP is preserved. In Section 4 we discuss indiscernible witnesses for k-dependence and give a proof that Mekler's construction preserves k-dependence. As an application, for each $k \geq 2$ we construct a strictly k-dependent pure group and discuss some related open problems. Finally, in Section 5 we prove that Mekler's construction preserves NTP_2.

2. Preliminaries on Mekler's construction

We review Mekler's construction from [11], following the exposition and notation in [10, Section A.3] (to which we refer the reader for further details).

Definition 2.1. A graph (binary, symmetric relation without self-loops) is called *nice* if it satisfies the following two properties:

(1) there are at least two vertices, and for any two distinct vertices a and b there is some vertex c different from a and b such that c is joined to a but not to b;
(2) there are no triangles or squares in the graph.

For any graph C and an odd prime p, we define a 2-nilpotent group of order p denoted by $G(C)$ which is generated freely by the vertices of C by imposing that two generators commute if and only if they are connected by an edge in C.

Now, let C be a nice graph and consider the group $G(C)$. Let G be any model of $\mathrm{Th}(G(C))$. We consider the following $\emptyset$-definable equivalence relations on the elements of G.

Definition 2.2. Let g and h be elements of G, then

- $g \sim h$, if $C_G(g) = C_G(h)$.
- $g \approx h$ if there is some natural number r and c in $Z(G)$ such that $g = h^r \cdot c$.
- $g \equiv_Z h$ if $g \cdot Z(G) = h \cdot Z(G)$.

Note that $g \equiv_Z h$ implies $g \approx h$, which implies $g \sim h$.

Definition 2.3. Let g be an element of G. We say that g is *of type* q if there are q-many different $\approx$-equivalence classes in the $\sim$-class $[g]_\sim$ of g. Moreover, we say that g is *isolated* if $[g]_\approx = [g]_{\equiv_Z}$.

All elements of G can be partitioned into four different $\emptyset$-definable classes (see [10, Section A.3] for the details):

(1) elements of type 1 which are not isolated, also referred to as *elements of type* 1^ν (in $G(C)$ this class includes the elements given by the vertices of C),
(2) elements of type 1 which are isolated, also referred to as *elements of type* 1^ι,

(3) elements of type p, and
(4) elements of type $p-1$.

The elements of the latter two types are always non-isolated.

Definition 2.4. For every element $g \in G$ of type p, the elements of G which commute with g are precisely the elements $\sim$-equivalent to g, and an element b of type 1^ν together with the elements $\sim$-equivalent to b. Such an element b is called a *handle of* g, and is definable from g up to $\sim$-equivalence.

Note here, that the center of G as well as the quotient $G/Z(G)$ are elementary abelian p-groups. Hence they can be viewed as $\mathbb{F}_p$-vector spaces. In the latter, being independent over some supergroup of $Z(G)$ refers to linear independence in terms of the corresponding $\mathbb{F}_p$-vector space.

Definition 2.5. Let G be a model of $\mathrm{Th}(G(C))$. We define the following:

- A 1^ν*-transversal* of G is a set X^ν consisting of one representative for each $\sim$-class of elements of type 1^ν in G.
- An element is *proper* if it is not a product of any elements of type 1^ν in G.
- A p*-transversal* of G is a set X^p of representatives of $\sim$-classes of proper elements of type p in G which is maximal with the property that if Y is a finite subset of X_p and all elements of Y have the same handle, then Y is independent modulo the subgroup generated by all elements of type 1^ν in G and $Z(G)$.
- A 1^ι*-transversal* of G is a set X^ι of representatives of $\sim$-classes of proper elements of type 1^ι in G which is maximal independent modulo the subgroup generated by all elements of types 1^ν and p in G, together with $Z(G)$.
- A set $X \subseteq G$ is a *transversal of* G if $X = X^\nu \sqcup X^p \sqcup X^\iota$, where X^ν, X^p and X^ι are some transversals of the corresponding types.

Notation 1. For a given (partial) transversal X, we denote by X^ν, X^p, and X^ι the elements in X of the corresponding types.

Lemma 2.6. *Let $G \models \mathrm{Th}(G(C))$. Given a small tuple of variables $\bar{x} = \bar{x}^\nu{}^\frown\bar{x}^p{}^\frown\bar{x}^\iota$, there is a partial type $\Phi(\bar{x})$ such that for any tuples $\bar{a}^\nu, \bar{a}^p$ and $\bar{a}^\iota$ in G, we have that $G \models \Phi(\bar{a}^\nu, \bar{a}^p, \bar{a}^\iota)$ if and only if every element in $\bar{a}^\nu, \bar{a}^p$ and $\bar{a}^\iota$ is of type $1^\nu, p$ and 1^ι, respectively, and $\bar{a} = \bar{a}^\nu{}^\frown\bar{a}^p{}^\frown\bar{a}^\iota$ can be extended to a transversal of G.*

Proof. By inspecting Definition 2.5. □

Fact 2.7. [10, Corollary A.3.11] *Let C be a nice graph. There is an interpretation Γ such that for any model G of $\mathrm{Th}(G(C))$, we have that $\Gamma(G)$ is a model of $\mathrm{Th}(C)$. More specifically, the graph $\Gamma(G) = (V, R)$ is given by the set of vertices $V = \{g \in G : g \text{ is of type } 1^\nu, g \notin Z(G)\}/\approx$ and the (well-defined) edge relation $([g]_\approx, [h]_\approx) \in R \iff g, h$ commute.*

The full set of transversal gives another important graph, a so called cover of a nice graph, which we define below.

Definition 2.8. (1) Let C be an infinite nice graph. A graph C^+ containing C as a subgraph is called a *cover* of C if for every vertex $b \in C^+ \setminus C$, either there is a unique vertex a in C that is joined to b and this vertex a has infinitely many adjacent vertices in C, or b is joined to no vertex in C^+.

(2) A cover C^+ of C is a λ*-cover* if

- for every vertex a in C the number of vertices in $C^+ \setminus C$ joined to a is λ if a is joined to infinitely many vertices in C, and zero otherwise;
- the number of new vertices in $C^+ \setminus C$ which are not joined to any other vertex in C^+ is λ.

Observe that a cover of a nice graph is generally not a nice graph.

Remark 2.9. Given a 1^ν-transversal X_ν of G, we identify the elements of X_ν with the set of vertices of $\Gamma(G)$ by mapping $x \in X_\nu$ to its class $[x]_\approx$. Then a set of transversals X can be viewed as a cover of the nice graph given by the elements of type 1^ν in X, with the edge relation given by commutation.

Fact 2.10. [10, Theorem A.3.14, Corollary A.3.15] *Let G be a model of* $\mathrm{Th}(G(C))$ *and let X be a transversal of G.*

(1) There is a subgroup of $Z(G)$ which we denote by H_X such that $G = \langle X\rangle \times H_X$.
(2) The group H_X is an elementary abelian p-group, in particular $\mathrm{Th}(H_X)$ *is stable and eliminates quantifiers.*
(3) If G is saturated, then both the graph $\Gamma(G)$ and the group H_X are also saturated (as $|H_X| = |G|$ and $\mathrm{Th}(H_X)$ *is uncountably categorical).*
(4) If G is a saturated model of $\mathrm{Th}(G(C))$*, then every automorphism of $\Gamma(G)$ can be lifted to an automorphism of G (equivalently, one could work with a special model instead of a saturated one to avoid any set-theoretic issues).*
(5) $\langle X\rangle \cong G(X)$ via an isomorphism which is the identity on the elements in X (where X is viewed as a graph as in Remark 2.9).

The following lemma is a refinement of Fact 2.10(4) and [2, Lemma 4.12].

Lemma 2.11. *Let G be a saturated model of* $\mathrm{Th}(G(C))$*, X be a transversal, and $H_X \leq Z(G)$ be such that $G = \langle X\rangle \times H_X$. Let Y and Z be two small subsets of X and let $\bar h_1, \bar h_2$ be two tuples in H_X. Suppose that*

- *there is a bijection f between Y and Z which respects the 1^ν-, p-, and 1^ι-parts, the handles, and* $\mathrm{tp}_\Gamma(Y^\nu) = \mathrm{tp}_\Gamma(f(Y^\nu))$,
- $\mathrm{tp}_{H_X}(\bar h_1) = \mathrm{tp}_{H_X}(\bar h_2)$.

Then there is an automorphism of G coinciding with f on Y and sending $\bar h$ to $\bar k$.

Proof. By Remark 2.9, we identify $\Gamma(G)$ with X^ν. By saturation of $\Gamma(G)$, $f \restriction Y_\nu$ extends to an automorphism σ of the graph X^ν. As X is a $|G|$-cover of X^ν by saturation of G and f respects the 1^ν-, p-, and 1^ι-parts and the handles, σ extends to an automorphism τ of the graph X agreeing with f. By Fact 2.10(5), we have that $\langle X\rangle \cong G(X)$ and τ lifts to an automorphism of the group $G(X)$, hence to an automorphism $\tilde\tau$ of $\langle X\rangle$ extending f by construction. As H_X is saturated by Fact 2.10(3), there is an automorphism ρ of H_X which maps $\bar h_1$ to $\bar h_2$. We can now take the cartesian product of $\tilde\tau \times \rho$ to obtain an automorphism of G which extends f and maps $\bar h_1$ to $\bar h_2$. □

Next, we observe that in Fact 2.10 the choice of a transversal and an elementary abelian subgroup of the center in the decomposition of G can be made entirely independently of each other.

Lemma 2.12. *Let G be any model of* $\mathrm{Th}(G(C))$*, let X be a transversal of G. Then we have $G' = \langle X\rangle'$.*

Proof. Let H be a subgroup of $Z(G)$ as in Fact 2.10, such that $G = \langle X\rangle \times H$. It is enough to show that for all $g, g' \in G$, we have that $[g, g']$ is in $\langle X\rangle'$. We choose $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ and $h, k \in H$ such that $g = \prod_{i=1}^n x_i \cdot h$ and $g' = \prod_{j=1}^m y_j \cdot k$. Then, using that G is 2-nilpotent, we have

$$[g, g'] = [\prod_{i=1}^n x_i \cdot h, \prod_{j=1}^m y_j \cdot k] = \prod_{i=1}^n \prod_{j=1}^m [x_i, y_j],$$

which is in $\langle X\rangle'$. □

Remark 2.13. This implies that in fact $\langle X\rangle' \cap Z(G)$ is the same for any transversal X of G, as it coincides with $G' \cap Z(G)$.

Proposition 2.14. *Let G be a model of* $\mathrm{Th}(G(C))$*, and let $\bar{x} = \bar{x}^\nu{}^\frown\bar{x}^p{}^\frown\bar{x}^\iota$ and $\bar{y}$ be two small tuples of variables. Then there is a partial type $\pi(\bar{x}, \bar{y})$ such that for any tuples of pairwise distinct elements $\bar{a} = \bar{a}^\nu{}^\frown\bar{a}^p{}^\frown\bar{a}^\iota$ and $\bar{b}$ from G we have that $G \models \pi(\bar{a}, \bar{b})$ if and only if we can extend $\bar{a}$ to a transversal X of G and find a subset $H \subseteq Z(G)$ containing $\bar{b}$ which is linearly independent over G', so that $G = \langle X\rangle \times \langle H\rangle$.*

Proof. Let $\Psi(v_i : i \in \kappa)$ be the partial type consisting of the formulas

$$\forall g_0, \ldots, g_{2m} \left(\bigwedge_{\alpha_0,\ldots,\alpha_n \in p^{\times n+1} \setminus \{0,\ldots,0\}} \left(v_{i_0}^{\alpha_0} \cdot \ldots \cdot v_{i_n}^{\alpha_n} \neq \prod_{j=1}^m [g_{2j}, g_{2j+1}] \right)\right)$$

for all $m, n \in \omega$ and $i_0, \ldots, i_n \in \kappa$. An easy inspection yields that for any tuple $\bar{b}$ in $Z(G)$, $\bar{b} \models \Psi(\bar{y})$ if and only if $\bar{b}$ is linearly independent in the elementary abelian p-group $Z(G)$ over G' (seen as an $\mathbb{F}_p$-vector space). Combining this with Remark 2.13, there is a subgroup H of G containing $\bar{b}$ such that for any transversal X of G, we have that $G = \langle X\rangle \times H$. Combining this with Lemma 2.6, we can conclude. □

3. NIP

We begin with the simplest case demonstrating that NIP is preserved. Recall the following basic characterization of NIP.

Fact 3.1. *(see e.g. [1]) Let T be a complete first-order theory and let $\mathbb{M} \models T$ be a monster model. Let κ be the regular cardinal $|T|^+$. Then the following are equivalent.*

(1) T is NIP.

(2) For every indiscernible sequence $I = (\bar{a}_i : i \in \kappa)$ of finite tuples and a finite tuple $\bar{b}$ in $\mathbb{M}$, there is some $\alpha < \kappa$ such that $\mathrm{tp}(\bar{b}\bar{a}_i) = \mathrm{tp}(\bar{b}\bar{a}_j)$ *for all $i, j > \alpha$.*

As in Section 2, let C be a nice graph and let $G(C)$ be the 2-nilpotent group of order p which is freely generated by the vertices of C by imposing that two generators commute if and only if they are connected by an edge in C.

Theorem 3.2. $\mathrm{Th}(C)$ *is NIP if and only if* $\mathrm{Th}(G(C))$ *is NIP.*

Proof. If $\mathrm{Th}(G(C))$ is NIP, then $\mathrm{Th}(C)$ is also NIP as C is interpretable in $G(C)$.

Now, we want to prove the converse. Let $G \models \mathrm{Th}(G(C))$ be a saturated model, and assume that $\mathrm{Th}(G(C))$ has IP but $\mathrm{Th}(C)$ is NIP. Fix κ to be $(\aleph_0)^+$. Then there is some formula $\phi(\bar{x}, \bar{y}) \in L_G$, and a sequence $I = (\bar{a}_i : i \in \kappa)$ in G shattered by

$\phi(\bar{x}, \bar{y})$, i.e. such that for every $S \subseteq \kappa$, there is some $\bar{b}_S$ in G satisfying $G \models \phi(\bar{b}_S, \bar{a}_i)$ if and only if $i \in S$.

Let X be a transversal for G and $H \subseteq Z(G)$ a set of elements linearly independent over G' and such that $G = \langle X \rangle \times \langle H \rangle$. Then for each $i \in \kappa$ we have, slightly abusing notation, $\bar{a}_i = t_i(\bar{x}_i, \bar{h}_i)$ for some L_G-term t_i and some finite tuples $\bar{x}_i = \bar{x}_i^{\nu\frown}\bar{x}_i^{p\frown}\bar{x}_i^{\iota}$ from X where $\bar{x}_i^{\nu}, \bar{x}_i^{p}, \bar{x}_i^{\iota}$ list all of the elements of type $1^{\nu}, p, 1^{\iota}$ in $\bar{x}_i$, respectively, and $\bar{h}_i$ from H. After adding some elements of type 1^{ν} to the beginning of the tuple and changing the term t_i accordingly, we may assume that for each $i \in \kappa$ and $j < |x_i^p|$, the handle of the j-*th* element of $\bar{x}_i^p$ is the j-th element of $\bar{x}_i^{\nu}$ (there might be some repetitions of elements of type 1^{ν} as different elements of type p might have the same handle). As $\kappa > |L_G| + \aleph_0$, passing to a cofinal subsequence and reordering the tuples if necessary, we may assume that:

(1) $t_i = t \in L_G$ and $|\bar{x}_i|$ and $|\bar{h}_i|$ are constant for all $i \in \kappa$,
(2) $|\bar{x}_i^{\nu}|, |\bar{x}_i^{p}|, |\bar{x}_i^{\iota}|$ are constant for all $i \in \kappa$.

Consider the L_G-formula $\phi'(\bar{x}, \bar{y}') = \phi(\bar{x}; t(\bar{y}_1, \bar{y}_2))$ with $\bar{y}' := \bar{y}_1^\frown \bar{y}_2$ and $|\bar{y}_1| = |\bar{x}_i|$ and $|\bar{y}_2| = |\bar{h}_i|$. Let $\bar{a}'_i := \bar{x}_i^\frown \bar{h}_i$. Then the sequence $I' := (\bar{a}'_i : i \in \kappa)$ is shattered by $\phi'(\bar{x}, \bar{y}')$. Note however that I' is generally not indiscernible.

To fix this, let $J = (\bar{x}'_i, \bar{h}'_i : i \in \kappa)$ be an L_G-indiscernible sequence of tuples in G with the same EM-type as I'. Then we have:

(1) J is still shattered by $\phi'(\bar{x}, \bar{y}')$,
(2) for each $i \in \kappa$ and $j < |x_i^p|$, we have that the handle of the j-th element of $(\bar{x}'_i)^p$ is the j-th element of $(\bar{x}'_i)^{\nu}$ (since being a handle is a definable condition, see Definition 2.4, and the corresponding property was true on all elements in I').
(3) The set of all elements of G appearing in the sequence $(\bar{x}'_i : i \in \kappa)$ still can be extended to some transversal X' of G.
(4) The set of all elements of G appearing in the sequence $(\bar{h}'_i : i \in \kappa)$ can be extended to some set $H' \subseteq Z(G)$ linearly independent over G' and such that $G = \langle X' \rangle \times \langle H' \rangle$.

The last two conditions hold as the sets of all elements appearing in the sequences $(\bar{x}_i : i \in \kappa)$ and $(\bar{h}_i : i \in \kappa)$ satisfied the respective conditions, these conditions are type-definable by Proposition 2.14 and J has the same EM-type as I'.

Now let $\bar{b} \in G$ be such that both sets $\{i \in \kappa : G \models \phi'(\bar{b}, \bar{a}'_i)\}, \{i \in \kappa : G \models \neg\phi'(\bar{b}, \bar{a}'_i)\}$ are cofinal in κ. Then $\bar{b} = s(\bar{z}, \bar{k})$ for some term $s \in L_G$ and some finite tuples $\bar{z}$ in X' and $\bar{k}$ in H'. Write $\bar{z} = \bar{z}^{\nu\frown}\bar{z}^{p\frown}\bar{z}^{\iota}$, with $\bar{z}^{\nu}, \bar{z}^{p}, \bar{z}^{\iota}$ listing the elements of the corresponding types in $\bar{z}$. In the same way as extending $\bar{x}_i$, we may add elements to the tuple $\bar{z}$ and assume that the handle of the j-th element of $\bar{z}^p$ is the j-th element of $\bar{z}^{\nu}$.

Consider all of the elements in $\bar{z}^{\nu}$ and $((\bar{x}'_i)^{\nu} : i \in \kappa)$ as elements in $\Gamma(G)$ — a saturated model of $\mathrm{Th}(C)$, and note that as $\Gamma(G)$ is interpretable in G we have that the sequence $((\bar{x}'_i)^{\nu} : i \in \kappa)$ is also indiscernible in $\Gamma(G)$. As $\mathrm{Th}(\Gamma(G))$ is NIP, by Fact 3.1 there is some $\alpha < \kappa$ such that $\mathrm{tp}_\Gamma(\bar{z}^{\nu}(\bar{x}'_i)^{\nu}) = \mathrm{tp}_\Gamma(\bar{z}^{\nu}(\bar{x}'_j)^{\nu})$ for all $i, j > \alpha$. Moreover, using indiscernibility of the sequence $(\bar{x}'_i)$ and possibly throwing away finitely many elements from the sequence, we have that

$$(\bar{x}'_i)^p \cap \bar{z}^p = (\bar{x}'_j)^p \cap \bar{z}^p, (\bar{x}'_i)^\iota \cap \bar{z}^\iota = (\bar{x}'_j)^\iota \cap \bar{z}^\iota \text{ (as tuples)}$$

and $\bar{x}'_i \cap \bar{x}'_j$ is constant, for all $i, j \in \kappa$. Thus, for any $i, j > \alpha$ mapping $\bar{x}'_i\bar{z}$ to $\bar{x}'_j\bar{z}$ preserving the order of the elements defines a bijection $\sigma_{i,j}$ such that:

(1) $\sigma_{i,j}$ is equal to $\sigma_{i,j}$ on $(\bar{x}'_i)^{\nu}\bar{z}^{\nu}$, hence $\mathrm{tp}_\Gamma((\bar{x}'_i)^{\nu}\bar{z}^{\nu}) = \mathrm{tp}_\Gamma(\sigma_{i,j}((\bar{x}'_i)^{\nu}\bar{z}^{\nu}))$,

(2) the map $\sigma_{i,j}$ fixes $\bar{z}$,
(3) the map $\sigma_{i,j}$ respects the 1^{ν}-, p- and 1^{ι}-parts and the handles (since the handle of the j-*th* element of $\bar{x}_i^p$ is the j-th element of $\bar{x}_i^{\nu}$).

Now consider $\bar{k}$ and $(\bar{h}_i : i \in \kappa)$ as tuples of elements in $\langle H' \rangle$, which is a model of the stable theory $\mathrm{Th}(\langle H' \rangle)$. Moreover, as $(h_i : i \in \kappa)$ is L_G-indiscernible and $\mathrm{Th}(\langle H' \rangle)$ eliminates quantifiers, $(h_i : i \in \kappa)$ is also indiscernible in the sense of $\mathrm{Th}(\langle H' \rangle)$. Hence, by stability, there is some $\beta \in \kappa$ such that $\mathrm{tp}_{\langle H' \rangle}(\bar{k}\bar{h}_i) = \mathrm{tp}_{\langle H' \rangle}(\bar{k}\bar{h}_j)$ for all $i, j > \beta$.

Now, Lemma 2.11 gives us an automorphism of G sending $\bar{x}_i\bar{h}_i\bar{z}\bar{k}$ to $\bar{x}_j\bar{h}_j\bar{z}\bar{k}$, so $\mathrm{tp}_G(\bar{x}_i\bar{h}_i/\bar{z}\bar{k}) = \mathrm{tp}_G(\bar{x}_j\bar{h}_j/\bar{z}\bar{k})$ for all $i, j > \max\{\alpha, \beta\}$. This contradicts the choice of $\bar{b} = s(\bar{z}, \bar{k})$. ☐

An alternative argument for NIP. An alternative proof can be provided relying on the previous work of Mekler and set-theoretic absoluteness.

Recall that the *stability spectrum* of a complete theory T is defined as the function

$$f_T(\kappa) := \sup\{|S_1(M)| : M \models T, |M| = \kappa\}$$

for all infinite cardinals κ.

For the following two facts see e.g. [8] and references there.

Fact 3.3. *(Shelah) Let T be a theory in a countable language.*
(1) It T is NIP, then $f_T(\kappa) \leq (\operatorname{ded}\kappa)^{\aleph_0}$ for all infinite cardinals κ.
(2) If T has IP, then $f_T(\kappa) = 2^{\kappa}$ for all infinite cardinals κ.

It is possible that in a model of ZFC, $\operatorname{ded}\kappa = 2^{\kappa}$ for all infinite cardinals κ (e.g. in a model of the Generalized Continuum Hypothesis). However, there are models of ZFC in which these two functions are different.

Fact 3.4. *(Mitchell) For every cardinal κ of uncountable cofinality, there is a cardinal preserving Cohen extension such that $(\operatorname{ded}\kappa)^{\aleph_0} < 2^{\kappa}$.*

In the original paper of Mekler [11] it is demonstrated that if C is a nice graph and $\mathrm{Th}(C)$ is stable, then $\mathrm{Th}(G(C))$ is stable. More precisely, the following result is established (in ZFC).

Fact 3.5. *Let C be a nice graph. Then $f_{\mathrm{Th}(G(C))}(\kappa) = f_{\mathrm{Th}(C)}(\kappa) + \aleph_0$ for all infinite cardinals κ.*

Finally, note that the property "T is NIP" is a finitary formula-by-formula statement, hence set-theoretically absolute. Thus in order to prove Theorem 3.2, it is enough to prove it in *some* model of ZFC. Working in Mitchell's model for some κ of uncountable cofinality (hence $(\operatorname{ded}\kappa)^{\aleph_0} + \aleph_0 < 2^{\kappa}$), it follows immediately by combining Facts 3.3 and 3.5.

4. Preservation of k-dependence

We are following the notation from [7], and begin by recalling some of the facts there.

Definition 4.1. A formula $\varphi(x; y_0, \ldots, y_{k-1})$ has the *k-independence property* (with respect to a theory T), if in some model there is a sequence $(a_{0,i}, \ldots, a_{k-1,i})_{i \in \omega}$ such that for every $s \subseteq \omega^k$ there is b_s such that

$$\models \phi\left(b_s; a_{0,i_0}, \ldots, a_{k-1,i_{k-1}}\right) \Leftrightarrow (i_0, \ldots, i_{k-1}) \in s.$$

Here $x, y_0, \ldots, y_{k-1}$ are tuples of variables. Otherwise we say that $\varphi(x, y_0, \ldots, y_{k-1})$ is *k-dependent*. A theory is *k-dependent* if it implies that every formula is k-dependent.

To characterize k-dependence in a formula-free way, we have to work with a more complicated form of indiscernibility.

Definition 4.2. Fix a language $L^k_{\mathrm{opg}} = \{R(x_0, \ldots, x_{k-1}), <, P_0(x), \ldots, P_{k-1}(x)\}$. An *ordered k-partite hypergraph* is an L^k_{opg}-structure $\mathcal{A} = (A; <, R, P_0, \ldots, P_{k-1})$ such that:

(1) A is the (pairwise disjoint) union $P_0^{\mathcal{A}} \sqcup \ldots \sqcup P_{k-1}^{\mathcal{A}}$,
(2) $R^{\mathcal{A}}$ is a symmetric relation so that if $(a_0, \ldots, a_{k-1}) \in R^{\mathcal{A}}$ then $P_i \cap \{a_0 \ldots a_{k-1}\}$ is a singleton for every $i < k$,
(3) $<^{\mathcal{A}}$ is a linear ordering on A with $P_0(A) < \ldots < P_{k-1}(A)$.

Fact 4.3. *Let $\mathcal{K}$ be the class of all finite ordered k-partite hypergraphs, and let $\mathcal{K}^* = \{A : A \subseteq B \in K\}$ be the hereditary closure of $\mathcal{K}$. Then $\mathcal{K}^*$ is a Fraïssé class, and its limit is called the* ordered k-partite random hypergraph*, which we will denote by $G_{k,p}$. An ordered k-partite hypergraph $\mathcal{A}$ is a model of* $\mathrm{Th}(G_{k,p})$ *if and only if:*

- *$(P_i(A), <)$ is a model of DLO for each $i < k$,*
- *for every $j < k$, any finite disjoint sets $A_0, A_1 \subset \prod_{i<k, i \neq j} P_i(A)$ and $b_0 < b_1 \in P_j(A)$, there is $b_0 < b < b_1$ such that: $R(b, \bar{a})$ holds for every $\bar{a} \in A_0$ and $\neg R(b, \bar{a})$ holds for every $\bar{a} \in A_1$.*

We denote by $O_{k,p}$ the reduct of $G_{k,p}$ to the language $L^k_{\mathrm{op}} = \{<, P_0(x), \ldots, P_{k-1}(x)\}$.

Definition 4.4. Let T be a theory in the language L, and let $\mathbb{M}$ be a monster model of T.

(1) Let I be a structure in the language L_0. We say that $\bar{a} = (a_i)_{i \in I}$, with a_i a tuple in $\mathbb{M}$, is *I-indiscernible* over a set of parameters $C \subseteq \mathbb{M}$ if for all $n \in \omega$ and all $i_0, \ldots, i_n$ and $j_0, \ldots, j_n$ from I we have:
$$\mathrm{qftp}_{L_0}(i_0, \ldots, i_n) = \mathrm{qftp}_{L_0}(j_0, \ldots, j_n) \Rightarrow$$
$$\mathrm{tp}_L(a_{i_0}, \ldots, a_{i_n}/C) = \mathrm{tp}_L(a_{j_0}, \ldots, a_{j_n}/C).$$
For any $L_1 \subseteq L_0$, $(a_i)_{i \in I}$ is said to be L_1-indiscernible if it is $(I \restriction L_1)$-indiscernible.
(2) For L_0-structures I and J, we say that $(b_i)_{i \in J}$ is *based on* $(a_i)_{i \in I}$ over a set of parameters $C \subseteq \mathbb{M}$ if for any finite set Δ of $L(C)$-formulas, and for any finite tuple $(j_0, \ldots, j_n)$ from J there is a tuple $(i_0, \ldots, i_n)$ from I such that:
   - $\mathrm{qftp}_{L_0}(j_0, \ldots, j_n) = \mathrm{qftp}_{L_0}(i_0, \ldots, i_n)$ and
   - $\mathrm{tp}_\Delta(b_{j_0}, \ldots, b_{j_n}) = \mathrm{tp}_\Delta(a_{i_0}, \ldots, a_{i_n})$.

The following fact gives a method for finding $G_{k,p}$-indiscernibles using structural Ramsey theory.

Fact 4.5. [7, Corollary 4.8] *Let $C \subseteq \mathbb{M}$ be a small set of parameters.*

(1) For any $\bar{a} = (a_g)_{g \in O_{k,p}}$, there is some $(b_g)_{g \in O_{k,p}}$ which is $O_{k,p}$-indiscernible over C and is based on $\bar{a}$ over C.
(2) For any $\bar{a} = (a_g)_{g \in G_{k,p}}$, there is some $(b_g)_{g \in G_{k,p}}$ which is $G_{k,p}$-indiscernible over C and is based on $\bar{a}$ over C.

Fact 4.6. [7, Proposition 6.3] *Let T be a complete theory and let $\mathbb{M} \models T$ be a monster model. For any $k \in \mathbb{N}$, the following are equivalent:*

(1) T is k-dependent.
(2) For any $(a_g)_{g\in G_{k,p}}$ and b with a_g, b finite tuples in $\mathbb{M}$, if $(a_g)_{g\in G_{n,p}}$ is $G_{n,p}$-indiscernible over b and L^k_{op}-indiscernible (over $\emptyset$), then it is L^k_{op}-indiscernible over b.

We are ready to prove the main theorem of the section.

Theorem 4.7. *For any $k \in \mathbb{N}$ and a nice graph C, $\mathrm{Th}(C)$ is k-dependent if and only if $\mathrm{Th}(G(C))$ is k-dependent.*

Proof. Let $G \models \mathrm{Th}(G(C))$ be a saturated model, let X be a transversal, and let H be a set in $Z(G)$ which is linearly independent over G' such that $G = \langle X\rangle \times \langle H\rangle$. Moreover, fix κ to be $\aleph_0^+$.

As in the NIP case, if $\mathrm{Th}(G(C))$ is k-dependent, then $\mathrm{Th}(C)$ is also k-dependent as C is interpretable in $G(C)$.

Now suppose that $\mathrm{Th}(C)$ is k-dependent but $\mathrm{Th}(G(C))$ has the k-independence property witnessed by the formula $\varphi(x; y_0, \dots, y_{k-1}) \in L_G$. By compactness we can find a sequence $(a_{0,\alpha}, \dots, a_{k-1,\alpha})_{\alpha\in\kappa}$ such that for any $s \subseteq \kappa^k$ there is some b_s such that

$$\models \phi\left(b_s; a_{0,\alpha_0}, \dots, a_{k-1,\alpha_{k-1}}\right) \Leftrightarrow (\alpha_0, \dots, \alpha_{k-1}) \in s.$$

By the choice of X and H, for each $i < k$ and $\alpha \in \kappa$, there is some term $t_{i,\alpha} \in L_G$ and some finite tuples $\bar{x}_{i,\alpha}$ from X and $\bar{h}_{i,\alpha}$ from H such that $a_{i,\alpha} = t_{i,\alpha}(\bar{x}_{i,\alpha}, \bar{h}_{i,\alpha})$. As $\kappa > |L_G| + \aleph_0$, passing to a subsequence of length κ for each $i < k$ we may assume that $t_{i,\alpha} = t_i$ and $\bar{x}_{i,\alpha} = \bar{x}^{\nu}_{i,\alpha}{}^\frown\bar{x}^{p}_{i,\alpha}{}^\frown\bar{x}^{\iota}_{i,\alpha}$ with $\bar{x}^{\nu}_{i,\alpha}, \bar{x}^{p}_{i,\alpha}, \bar{x}^{\iota}_{i,\alpha}$ listing all elements of the corresponding type in $\bar{x}_{i,\alpha}$ and $|\bar{x}^{\nu}_{i,\alpha}|, |\bar{x}^{p}_{i,\alpha}|, |\bar{x}^{\iota}_{i,\alpha}|$ constant for all $i < j$ and $\alpha \in \kappa$. Moreover, in the same way as in the NIP case, we add the handles of the elements in the tuple $\bar{x}^{p}_{i,\alpha}$ to the beginning of $\bar{x}^{\nu}_{i,\alpha}$. Taking $\psi(x; y'_0, \dots, y'_{k-1}) := \phi(x; t_0(y'_0), \dots, t_{k-1}(y'_{k-1}))$, we see that the sequence $(\bar{x}_{0,\alpha}, \bar{h}_{0,\alpha}, \dots, \bar{x}_{k-1,\alpha}, \bar{h}_{k-1,\alpha} : \alpha \in \kappa)$ is shattered by ψ, i. e. for each $A \subset \kappa^k$ there is some $\bar{b}$ such that $G \models \psi(\bar{b}; \bar{x}_{i_0}{}^\frown\bar{h}_{i_0}, \dots, \bar{x}_{i_{k-1}}{}^\frown\bar{h}_{i_{k-1}})$ if and only if $(i_0, \dots, i_{k-1}) \in A$. We define an L_{op}-structure on κ by interpreting each of the $P_i, i < k$ as some countable disjoint subsets of κ, and choosing any ordering isomorphic to $(\mathbb{Q}, <)$ on each of the P_i's. We pass to the corresponding subsequences of $(\bar{x}_{i,\alpha} : \alpha \in \kappa)$, namely for each $i \in k$, we consider the sequence given by $(\bar{x}_{i,\alpha} : \alpha \in P_i)$. Taking these k different sequences together we obtain the sequence $(\bar{x}_g{}^\frown\bar{h}_g : g \in O_{k,p})$ indexed by $O_{k,p}$. This sequence is shattered in the following sense: for each $A \subset P_0 \times \cdots \times P_{k-1}$ there is some $\bar{b} \in G$ such that $G \models \psi(\bar{b}; \bar{x}_{g_0}{}^\frown\bar{h}_{g_0}, \dots, \bar{x}_{g_{k-1}}{}^\frown\bar{h}_{g_{k-1}})$ if and only if $(g_0, \dots, g_{k-1}) \in A$.

By Fact 4.5(1), let $(\bar{y}_g{}^\frown\bar{m}_g : g \in O_{k,p})$ be an $O_{k,p}$-indiscernible in G based on $(\bar{x}_g{}^\frown\bar{h}_g : g \in O_{k,p})$. Observe that, using Proposition 2.14 as in the proof of Theorem 3.2, we still have:

(1) $(\bar{y}_g{}^\frown\bar{m}_g : g \in O_{k,p})$ is shattered by ψ,
(2) the handle for each jth element in the tuple $\bar{y}^{p}_g$ is the jth element of the tuple $\bar{y}^{\nu}_g$,
(3) the set of all elements of G appearing in $(\bar{y}_g : g \in O_{k,p})$ is a partial transversal, hence can be extended to a transversal Y of G,

(4) the set of all elements of G appearing in $(\bar{m}_g : g \in O_{k,p})$ is still a set of elements in $Z(G)$ linearly independent over G', hence can be extended to a linearly independent set M such that $G = \langle Y\rangle \times \langle M\rangle$.

We can expand $O_{k,p}$ to $G_{k,p}$ (see Fact 4.3). As $(\bar{y}_g{}^\frown \bar{m}_g : g \in O_{k,p})$ is shattered by ψ, we can find an element $b \in G$ such that $G \models \psi(b; \bar{y}_{g_0}, \bar{k}_{g_0}, \ldots, \bar{y}_{g_{k-1}}, \bar{m}_{g_{k-1}}) \iff G_{k,p} \models R(g_0, \ldots, g_{k-1})$, for all $g_i \in P_i$. We can write $b = s(\bar{z}, \bar{\ell})$ for some term $s \in L_G$ and some finite tuples $\bar{z} = \bar{z}^\nu{}^\frown \bar{z}^p{}^\frown \bar{z}^\iota$ in Y and $\bar{\ell}$ in K. As usual, extending $\bar{z}^\nu$ if necessary, we may assume that $\bar{z}$ is closed under handles. Taking $\theta(x'; y'_0, \ldots, y'_{k-1}) := \psi(s(x'); y'_0, \ldots, y'_{k-1})$, we still have that

$$G \models \theta(\bar{z}\bar{\ell}; \bar{y}_{g_0}, \bar{m}_{g_0}, \ldots, \bar{y}_{g_{k-1}}, \bar{m}_{g_{k-1}}) \iff G_{k,p} \models R(g_0, \ldots, g_{k-1})$$

for all $g_i \in P_i$.

By Fact 4.5(2), we can find $(\bar{z}_g{}^\frown \bar{\ell}_g : g \in G_{k,p})$ which is $G_{k,p}$-indiscernible over $\bar{z}{}^\frown \bar{\ell}$ and is based on $(\bar{y}_g{}^\frown \bar{m}_g : g \in G_{k,p})$ over $\bar{z}{}^\frown \bar{\ell}$. Then we have:

(1) $G \models \theta(\bar{z}\bar{\ell}; \bar{z}_{g_0}, \bar{\ell}_{g_0}, \ldots, \bar{z}_{g_{k-1}}, \bar{\ell}_{g_{k-1}}) \iff G_{k,p} \models R(g_0, \ldots, g_{k-1})$, for all $g_i \in P_i$;
(2) for $\bar{z}_g = \bar{z}_g^\nu{}^\frown \bar{z}_g^p{}^\frown \bar{z}_g^\iota$ we have that:
 - all of these tuples are of fixed length and list elements of the corresponding type,
 - the handle of the j-th element of $\bar{z}_g^p$ is the j-th element of $\bar{z}_g^\nu$;
(3) the set of all elements of G appearing in $\bar{z}$ and $(\bar{z}_g : g \in G_{k,p})$ is a partial transversal, hence can be extended to some transversal Z of G;
(4) the set of all elements of G appearing in $\bar{\ell}$ and $(\bar{\ell}_g : g \in G_{k,p})$ is still a set of elements in $Z(G)$ linearly independent over G', hence can be extended to a linearly independent set L such that $G = \langle Z\rangle \times \langle L\rangle$;
(5) $(\bar{z}_g{}^\frown \bar{\ell}_g : g \in G_{k,p})$ is L^k_{op}-indiscernible over $\emptyset$ (follows since $(\bar{z}_g{}^\frown \bar{\ell}_g : g \in G_{k,p})$ is based on $(\bar{y}_g{}^\frown \bar{k}_g : g \in G_{k,p})$, which was L^k_{op}-indiscernible, as in the proof of [7, Lemma 6.2]).

Consider now all of the elements in $\bar{z}^\nu$ and $(\bar{z}_g^\nu : g \in G_{k,p})$ as elements in $\Gamma(G)$, a saturated model of $\mathrm{Th}(C)$, and note that as $\Gamma(G)$ is interpretable in G, we have that the sequence $(\bar{z}_g^\nu : g \in G_{k,p})$ is also $G_{k,p}$-indiscernible over $\bar{z}^\nu$ and is L^k_{op}-indiscernible over $\emptyset$, both in $\Gamma(G)$. As $\mathrm{Th}(C)$ is k-dependent, it follows by Fact 4.6 that $(\bar{z}_g^\nu : g \in G_{k,p})$ is L^k_{op}-indiscernible over $\bar{z}^\nu$ in $\Gamma(G)$. Hence for any finite tuples $g_0, \ldots, g_n, q_0, \ldots, q_n \in G_{k,p}$ such that $\mathrm{tp}_{L^k_{\mathrm{op}}}(\bar{g}) = \mathrm{tp}_{L^k_{\mathrm{op}}}(\bar{q})$, we have that $\mathrm{tp}_\Gamma(\bar{z}_{g_0}^\nu, \ldots, \bar{z}_{g_n}^\nu / \bar{z}^\nu)$ is equal to $\mathrm{tp}_\Gamma(\bar{z}_{q_0}^\nu, \ldots, \bar{z}_{q_n}^\nu / \bar{z}^\nu)$. Now, using L^k_{op}-indiscernibility and that $\bar{z}$ is finite, for each $i < k$ there is some finite $\lambda_i \subseteq P_i$ such that for all $g \neq q \in P_i$ both greater than λ_i we have

$$\bar{z}_g^p \cap \bar{z}^p = \bar{z}_q^p \cap \bar{z}^p, \bar{z}_g^\iota \cap \bar{z}^\iota = \bar{z}_q^\iota \cap \bar{z}^\iota \text{ (as tuples)}$$

and $\bar{z}_g \cap \bar{z}_q$ is constant. Thus, for any $g_0, \ldots, g_{k-1}, q_0, \ldots, q_{k-1}$ such that $g_i, q_i > \lambda_i$ and $g_i, q_i \in P_i$, we get that mapping $\bar{z}_{g_0}, \ldots, \bar{z}_{g_{k-1}}, \bar{z}$ to $\bar{z}_{q_0}, \ldots, \bar{z}_{q_{k-1}}, \bar{z}$ preserving the positions of the elements in the tuples defines a bijection $\sigma_{\bar{g},\bar{q}}$ such that:

(1) $\mathrm{tp}_\Gamma(\bar{z}_{g_0}^\nu, \ldots, \bar{z}_{g_{k-1}}^\nu, \bar{z}^\nu) = \mathrm{tp}_\Gamma(\sigma_{\bar{g},\bar{q}}(\bar{z}_{g_0}^\nu, \ldots, \bar{z}_{g_{k-1}}^\nu, \bar{z}^\nu))$,
(2) the map $\sigma_{\bar{g},\bar{q}}$ fixes $\bar{z}$,
(3) the map $\sigma_{\bar{g},\bar{q}}$ respects the 1^ν-, p- and 1^ι-parts and the handles.

Next we consider all of the elements in $\bar{\ell}$ and $(\bar{\ell}_g : g \in G_{k,p})$ as elements in $\langle L\rangle$, a saturated model of the stable theory $\mathrm{Th}(\langle L\rangle)$. By quantifier elimination, we still have that $(\bar{\ell}_g : g \in G_{k,p})$ is both L^k_{op}-indiscernible and $G_{k,p}$-indiscernible over $\bar{\ell}$ in $\langle L \rangle$. As $\langle L \rangle$ is stable, so in particular k-dependent, by Fact 4.6, $(\bar{\ell}_g : g \in G_{k,p})$ is L^k_{op}-indiscernible over $\bar{\ell}$.

Now let $\bar{g}, \bar{q} \in G_{k,p}$ be such that $g_i, q_i > \lambda_i$ and $g_i, q_i \in P_i$ for all $i < k$, and such that $G_{k,p} \models R(g_0, \dots, g_{k-1}) \wedge \neg R(q_0, \dots, q_{k-1})$ holds. Then by the choice of $\bar{z}^\frown \bar{\ell}$ we have that $G \models \theta(\bar{z}\bar{\ell}; \bar{z}_{g_0}, \bar{\ell}_{g_0}, \dots, \bar{z}_{g_{k-1}}, \bar{\ell}_{g_{k-1}}) \wedge \neg\theta(\bar{z}\bar{\ell}; \bar{z}_{q_0}, \bar{\ell}_{q_0}, \dots, \bar{z}_{q_{k-1}}, \bar{\ell}_{q_{k-1}})$. On the other hand, combining the last two paragraphs and using Lemma 2.11, we find an automorphism of G sending $(\bar{z}_{g_0}, \bar{\ell}_{g_0}, \dots, \bar{z}_{g_{k-1}}, \bar{\ell}_{g_{k-1}})$ to $(\bar{z}_{q_0}, \bar{\ell}_{q_0}, \dots, \bar{z}_{q_{k-1}}, \bar{\ell}_{q_{k-1}})$ and fixing $\bar{z}\bar{\ell}$ — a contradiction.

□

Corollary 4.8. *For every $k \geq 2$, there is a strictly k-dependent pure group G. Moreover, we can find such a G with a simple theory.*

Proof. For each $k \geq 2$, let A_k be the random k-hypergraph. It is well-known that $\mathrm{Th}(A_k)$ is simple. Moreover, A_k is clearly not $(k-1)$-dependent, as witnessed by the edge relation, and it is easy to verify that A_k is k-dependent (as it eliminates quantifiers and all relation symbols are at most k-ary, see e.g. [7, Proposition 6.5]).

Now A_k, as well as any other structure in a finite relational language, is bi-interpretable with some nice graph C_k by [10, Theorem 5.5.1 + Exercise 5.5.9], so C_k also has all of the aforementioned properties. Then Mekler's construction produces a group $G(C_k)$ with all of the desired properties, by Theorem 4.7 and preservation of simplicity from [2]. □

This corollary gives first examples of strictly k-dependent groups, however many other questions about the existence of strictly k-dependent algebraic structures remain.

Problem 4.9. (1) Are there pseudofinite strictly k-dependent groups, for $k > 2$? The strictly 2-dependent group in Example 1 is pseudofinite.
(2) Are there $\aleph_0$-categorical strictly k-dependent groups? We note that Mekler's construction doesn't preserve $\aleph_0$-categoricity in general.
(3) Are there strictly k-dependent fields, for any $k \geq 2$? We conjecture that there aren't any with a simple theory. It is proved in [9] that any k-dependent PAC field is separably closed, and there are no known examples of fields with a simple theory which are not PAC.

5. Preservation of NTP_2

We recall the definition of NTP_2 (and refer to [4] for further details).

Definition 5.1. (1) A formula $\phi(x, y)$, with x, y tuples of variables, has TP_2 if there is an array $(a_{i,j} : i, j \in \omega)$ of tuples in $\mathbb{M} \models T$ and some $k \in \omega$ such that:
  (a) for all $i \in \omega$, the set $\{\phi(x, a_{i,j}) : j \in \omega\}$ is k-inconsistent.
  (b) for all $f : \omega \to \omega$, the set $\{\phi(x, a_{i,f(i)}) : i \in \omega\}$ is consistent.
(2) A theory T is NTP_2 if no formula has TP_2 relatively to it.

Remark 5.2. If T is not NTP_2, one can find a formula as in Definition 5.1(1) with $k = 2$.

We will use the following formula-free characterization of NTP_2 from [4, Section 1].

Fact 5.3. *Let T be a theory and $\mathbb{M} \models T$ a monster model. Let $\kappa := |T|^+$. The following are equivalent:*

(1) T is NTP_2.
(2) For any array $(a_{i,j} : i \in \kappa, j \in \omega)$ of finite tuples with mutually indiscernible rows *(i.e. for each $i \in \kappa$, the sequence $\bar{a}_i := (a_{i,j} : j \in \omega)$ is indiscernible over $\{a_{i',j} : i' \in \kappa \setminus \{i\}, j \in \omega\}$) and a finite tuple b, there is some $\alpha \in \kappa$ satisfying the following: for any $i > \alpha$ there is some b' such that* $\mathrm{tp}(b/a_{i,0}) = \mathrm{tp}(b'/a_{i,0})$ *and $\bar{a}_i$ is indiscernible over b'.*

The following can be proved using finitary Ramsey theorem and compactness, see [4, Section 1] for the details.

Fact 5.4. *Let $(a_{\alpha,i} : \alpha, i \in \kappa)$ be an array of tuples from $\mathbb{M} \models T$. Then there is an array $(b_{\alpha,i} : \alpha, i \in \kappa)$ with mutually indiscernible rows* based on $(a_{\alpha,i} : \alpha, i \in \kappa)$, *i.e. such that for every finite set of formulas Δ, any $\alpha_0, \ldots, \alpha_{n-1} \in \kappa$ and any strictly increasing finite tuples $\bar{j}_0, \ldots, \bar{j}_{n-1}$ from κ, there are some strictly increasing tuples $\bar{i}_0, \ldots, \bar{i}_{n-1}$ from κ such that*

$$\models \Delta((b_{\alpha_0,i} : i \in \bar{j}_0), \ldots, (b_{\alpha_{n-1},i} : i \in \bar{j}_{n-1})) \iff$$
$$\models \Delta((a_{\alpha_0,i} : i \in \bar{i}_0), \ldots, (a_{\alpha_{n-1},i} : i \in \bar{i}_{n-1})).$$

Remark 5.5. If $\phi(x,y)$ and $(a_{\alpha,i} : \alpha, i \in \kappa)$ satisfy the condition in Definition 5.1(1) and $(b_{\alpha,i} : \alpha, i \in \kappa)$ is based on it, then $\phi(x,y)$ and $(b_{\alpha,i} : \alpha, i \in \kappa)$ also satisfy the condition in Definition 5.1(1).

Theorem 5.6. *For any nice graph C, we have that* $\mathrm{Th}(G(C))$ *is* NTP_2 *if and only if* $\mathrm{Th}(C)$ *is* NTP_2.

Proof. As before, let $G \models \mathrm{Th}(G(C))$ be a monster model, let X be a transversal, and let H be a set in $Z(G)$ which is linearly independent over G' such that $G = \langle X \rangle \times \langle H \rangle$. Moreover, fix κ to be $\aleph_0^+$. If $\mathrm{Th}(G(C))$ is NTP_2 then $\mathrm{Th}(C)$ is also NTP_2 as C is interpretable in $G(C)$.

Now suppose that $\mathrm{Th}(C)$ is NTP_2, but $\mathrm{Th}(G(C))$ has TP_2. By compactness and Remark 5.2 we can find some formula $\phi(x,y)$ and an array $(\bar{a}_{i,j} : i,j \in \kappa)$ of tuples in G witnessing TP_2 as in Definition 5.1(1). Then for all $i,j \in \kappa$ we have $\bar{a}_{i,j} = t_{i,j}(\bar{x}_{i,j}, \bar{h}_{i,j})$ for some terms $t_{i,j} \in L_G$ and some finite tuples $\bar{x}_{i,j}$ from X and $\bar{h}_{i,j}$ from H.

As $\kappa > |L_G| + \aleph_0$, passing to a subsequence of each row, and then to a subsequence of the rows, we may assume that $t_{i,j} = t \in L_G$ and $\bar{x}_{i,j} = \bar{x}^{\nu}_{i,j} {}^\frown \bar{x}^{p}_{i,j} {}^\frown \bar{x}^{\iota}_{i,j}$ with $|\bar{x}^{\nu}_{i,j}|, |\bar{x}^{p}_{i,j}|, |\bar{x}^{\iota}_{i,j}|, |\bar{h}_{i,j}|$ constant for all $i,j \in \kappa$. Again as in the NIP case, we add the handles of the elements in the tuple $\bar{x}^{p}_{i,\alpha}$ to the beginning of $\bar{x}^{\nu}_{i,\alpha}$ for all $i,j \in \kappa$. Taking $\psi(x,y') := \phi(x, t(y'))$ with $|y'| = |\bar{x}_{i,j} {}^\frown \bar{h}_{i,j}|$ and $\bar{b}_{i,j} := \bar{x}_{i,j} {}^\frown \bar{h}_{i,j}$, we have that $\psi(x,y') \in L_G$ and the array $(\bar{b}_{i,j} : i,j \in \kappa)$ still satisfy the condition in Definition 5.1(1).

By Fact 5.4, let $(\bar{c}_{i,j} : i,j \in \kappa)$ with $\bar{c}_{i,j} = \bar{y}_{i,j} {}^\frown \bar{m}_{i,j}$ be an array with mutually indiscernible rows based on $(\bar{b}_{i,j} : i,j \in \kappa)$. Then, arguing as in the proofs of Theorems 3.2 and 4.7 using type-definability of the relevant properties from Proposition 2.14 and Remark 5.5, we have:

(1) $\psi(x,y')$ and the array $(\bar{c}_{i,j} : i,j \in \kappa)$ satisfy the condition in Definition 5.1(1);
(2) For $\bar{y}_{i,j} = \bar{y}^{\nu}_{i,j} {}^\frown \bar{y}^{p}_{i,j} {}^\frown \bar{y}^{\iota}_{i,j}$ we have that:

- all of these tuples are of fixed length and list elements of the corresponding type,
- the handle of the n-th element of $\bar{y}^p_{i,j}$ is the n-th element of $\bar{y}^\nu_{i,j}$;

(3) the set of all elements of G appearing in $(\bar{y}_{i,j} : i, j \in \kappa)$ is a partial transversal of G and can be extended to a transversal Y of G;

(4) the set of all elements of G appearing in $(\bar{m}_{i,j} : i, j \in \kappa)$ is a set of elements in $Z(G)$ linearly independent over G', hence can be extended to a set of generators M such that $G = \langle Y \rangle \times \langle M \rangle$.

Let now $\bar{b}$ be a tuple in G such that $G \models \{\psi(\bar{b}, \bar{c}_{i,0}) : i \in \kappa\}$. We have that $\bar{b} = s(\bar{y}, \bar{m})$ for some term $s \in L_G$ and some finite tuples $\bar{y}$ in Y and $\bar{m}$ in M. Let $\bar{y} = \bar{y}^\nu{}^\frown\bar{y}^p{}^\frown\bar{y}^\iota$, each listing the elements of the corresponding type. In the same way as for each of the $\bar{y}_{i,j}$'s, we add the handles of the elements in the tuple $\bar{y}^p$ to the beginning of $\bar{y}^\nu$ so that the handle of the n-th element of $\bar{y}^p$ is the n-th element of $\bar{y}^\nu$. Taking $\theta(x', y') := \psi(s(x'), y')$, we still have that $\bar{y}^\frown\bar{m} \models \{\theta(x', \bar{c}_{i,0}) : i \in \kappa\}$ and the set of formulas $\{\theta(x', \bar{c}_{i,j}) : j \in \kappa\}$ is 2-inconsistent for each $i \in \kappa$. Moreover, after possibly throwing away finitely many rows, we may assume that the rows are mutually indiscernible over $\bar{y}^\frown\bar{m} \cap \bigcup\{\bar{c}_{i,0} : i \in \kappa\}$ (if an element of $\bar{y}^\frown\bar{m}$ appears in $\bar{c}_{i,0}$, then the rows of the array $(\bar{c}_{i',j} : i' \in \kappa, i' \neq i, j \in \kappa)$ are mutually indiscernible over it). This implies that if $z \in \bar{y} \cap \bar{y}_{i,0}$ for some i and z is the n-th element in the tuple $\bar{y}_{i,0}$, then it is the n-th element in any tuple $\bar{y}_{j,0}$ with $j \in \kappa$.

Consider all of the elements in $\bar{y}^\nu$ and $(\bar{y}^\nu_{i,j} : i, j \in \kappa)$ as elements in $\Gamma(G)$, a saturated model of $\mathrm{Th}(C)$, and note that as $\Gamma(G)$ is interpretable in G we have that the array $(\bar{y}^\nu_{i,j} : i, j \in \kappa)$ has mutually indiscernible rows in $\Gamma(G)$. As $\mathrm{Th}(\Gamma(G))$ is NTP_2, it follows by Fact 5.3 that there is some $\alpha \in \kappa$ such that for each $i > \alpha$ there is some tuple $\bar{y}'^\nu$ such that $\mathrm{tp}_\Gamma(\bar{y}^\nu/\bar{y}^\nu_{i,0}) = \mathrm{tp}_\Gamma(\bar{y}'^\nu/\bar{y}^\nu_{i,0})$ and the sequence $(\bar{y}^\nu_{i,j} : j \in \kappa)$ is L_Γ-indiscernible over $\bar{y}'^\nu$, i. e. $\mathrm{tp}_\Gamma(\bar{y}^\nu, \bar{y}^\nu_{i,0}) = \mathrm{tp}_\Gamma(\bar{y}'^\nu, \bar{y}^\nu_{i,0})$. Let σ_0 be the bijection which maps $\bar{y}^\nu{}^\frown\bar{y}_{i,0}$ to $\bar{y}'^\nu{}^\frown\bar{y}_{i,0}$. Now we want to extend this bijection to $\bar{y}^\frown\bar{y}_{i,0}$ in a type and handle preserving way. To do so, we have to choose an image for each element in $\bar{y}^p{}^\frown\bar{y}^\iota$. Let z be the n-th element of $\bar{y}^p$ and let u be the n-th element of $\bar{y}^\nu$ (i. e. the handle of z).

- If $z \notin \bar{y}^p_{i,0}$, then choose z' to be any element in Y^p which has handle $\sigma_1(u)$ and is not contained in $\bar{y}^p_{i,0}$.
- If $z \in \bar{y}^p_{i,0}$, then we have that σ_1 fixes z as well as the handle u of z. In this case let z' be equal to z.

Now, we extend σ_0 to σ by mapping z to z' and fixing each element of $\bar{y}^\iota$. Let $\bar{y}' = \bar{y}'^\nu{}^\frown\sigma(\bar{y}^p)^\frown\bar{y}^\iota$. Then we have that for all $y \in \bar{y}^\frown\bar{y}_{i,0}$:

(1) σ is well defined;
(2) σ fixes all elements in $\bar{y}_{i,0}$;
(3) σ respects types and handles by construction;
(4) $\mathrm{tp}_\Gamma(\bar{y}^\nu, \bar{y}^\nu_{i,0}) = \mathrm{tp}_\Gamma(\sigma(\bar{y}^\nu, \bar{y}^\nu_{i,0}))$ as $\sigma(y) = \sigma_0(y)$ for all $y \in \bar{y}^\nu{}^\frown\bar{y}^\nu_{i,0}$.

Now consider $\bar{m}$ and $(\bar{m}_{i,j} : i, j \in \kappa)$ as tuples of elements in $\langle M \rangle$, which is a model of the stable theory $\mathrm{Th}(\langle M \rangle)$. Moreover, as $(\bar{m}_{i,j} : i, j \in \kappa)$ has L_G-mutually indiscernible rows and $\mathrm{Th}(\langle M \rangle)$ eliminates quantifiers, $(\bar{m}_{i,j} : i, j \in \kappa)$ has mutually indiscernible rows in the sense of $\mathrm{Th}(\langle M \rangle)$. Hence, by Fact 5.3 again, there is some $\beta \in \kappa$ such that for each $i > \beta$ there is some $\tau \in \mathrm{Aut}(\langle M \rangle)$ fixing $\bar{m}_{i,0}$ and such that $(\bar{m}_{i,j} : j \in \kappa)$ is indiscernible over $\bar{m}' := \tau(\bar{m})$.

Fix some $i > \max\{\alpha, \beta\}$ and let $\bar{y}'$ and $\bar{m}'$ be chosen as above. Then by Lemma 2.11 we find an automorphism of G which maps $\bar{y}\bar{m}^\frown\bar{y}_{i,0}\bar{m}_{i,0}$ to $\bar{y}'(\bar{m}')^\frown\bar{y}_{i,0}\bar{m}_{i,0}$, hence

$$\mathrm{tp}_G(\bar{y}'\bar{m}'/\bar{y}_{i,0}\bar{k}_{i,0}) = \mathrm{tp}_G(\bar{y}\bar{m}/\bar{y}_{i,0}\bar{m}_{i,0}).$$

In particular, $G \models \theta(\bar{y}'\bar{m}', \bar{y}_{i,0}\bar{m}_{i,0})$. We will show that

$$\mathrm{tp}_G(\bar{y}_{i,0}\bar{m}_{i,0}/\bar{y}'\bar{m}') = \mathrm{tp}_G(\bar{y}_{i,1}\bar{m}_{i,1}/\bar{y}'\bar{m}'),$$

which would then contradict the assumption that $\{\theta(x', \bar{y}_{i,j}\bar{h}_{i,j}) : j \in \kappa\}$ is 2-inconsistent.

We show that sending $\bar{y}'\bar{y}_{i,0}$ to $\bar{y}'\bar{y}_{i,1}$ is a well-defined bijection f_0. The only thing to check is that if the nth element z of $\bar{y}_{i,0}$ is an element of $\bar{y}'$, then the nth element of $\bar{y}_{i,1}$ is equal to z. This is true as by construction we have that the sequence $(\bar{y}_{i,j} : j \in \kappa)$ is indiscernible over $(\bar{y}' \cap \bigcup_{i\in\kappa} \bar{y}_{i,0})$. Moreover, we have the following properties for f_0:

(1) f_0 fixes all elements in $\bar{y}'$ (by construction);
(2) f_0 respects types and handles (by construction);
(3) $\mathrm{tp}_\Gamma(\bar{y}'^\nu, \bar{y}^\nu_{i,0}) = \mathrm{tp}_\Gamma(f_0(\bar{y}'^\nu, \bar{y}^\nu_{i,0}))$ (since by the choice of $\bar{y}'^\nu$ above, we have that $(\bar{y}^\nu_{i,j} : j \in \kappa)$ is indiscernible over $\bar{y}'^\nu$ in $\Gamma(G)$).

Similarly, by the choice of $\bar{m}'$ above, the sequence $(\bar{m}_{i,j} : j \in \kappa)$ is indiscernible over $\bar{m}'$, so $\mathrm{tp}_{\langle M\rangle}(\bar{m}_{i,0}, \bar{m}') = \mathrm{tp}_{\langle M\rangle}(\bar{m}_{i,1}, \bar{m}')$

Again, Lemma 2.11 gives us an automorphism of G sending $\bar{y}_{i,0}\bar{m}_{i,0}$ to $\bar{y}_{i,1}\bar{m}_{i,1}$ and fixing $\bar{y}'\bar{m}'$, as wanted.

□

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