Title
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Publication Date
1967-03-20

UCRL-17429

University of California

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Berkeley, California

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Submitted to Physical Review

UCRL-17429
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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ABSTRACT

The analytic structure of helicity amplitudes is derived from basic analyticity properties. Previous derivations relied on crossing properties and extra assumptions.

I. INTRODUCTION

The problem of expressing scattering amplitudes in terms of functions of scalar invariants without introducing extra singularities has been solved by Hepp[1] and Williams.[2] Their solution has a form that is not convenient, however, for many practical purposes. This is in part because it involves a reduction of the amplitude to its irreducible components. Though such a reduction is in principle straightforward, it is in practice cumbersome. Moreover, the irreducible components, though the natural mathematical quantities, are not nice physically. For example, the irreducible components mix different parity eigenstates. This means that the condition of invariance under space reflection does not lead to any simple reduction in the number of irreducible components. It leads rather to complicated relations between different irreducible components. For this reason, among others, the elegant results of Hepp and Williams have had little or no practical application.

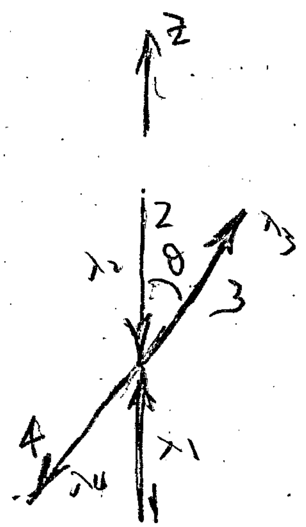
For many purposes the most convenient form of the scattering amplitude is in terms of helicity amplitudes. The helicity amplitudes, like any others, become functions of scalar invariants when evaluated in the center-of-mass frame. This is because the components of the momentum vectors become functions of scalar invariants. However, the functions that express these components in terms of the invariants have numerous kinematic singularities, which the amplitude itself is expected to inherit. Also, the various rotations and boosts needed to define the helicity amplitudes have kinematic singularities. Thus the analytic structure of the helicity amplitudes, considered as functions of the scalar invariants, would be expected to be very complicated. It turns out, however, that most of the singularities cancel, leaving the helicity amplitudes with reasonably simple analyticity properties. The purpose of the present paper is to show this.

The result is not new, having been obtained already by Hara[3] and Wang.[4] Their method is, however, circuitous. Rather than starting directly from the basic momentum-space analyticity properties, or equivalently from the analyticity properties deduced by Hepp and Williams, they base their conclusions on consistency with well-known crossing relations for helicity amplitudes. Since the crossing relations are themselves derived from the basis momentum-space analyticity properties, their procedure is evidently permissible. But it is roundabout. One would expect it to be simpler to work directly with the basic properties, and this is indeed the case.

There is a second reason for reconsidering the question. The method of Wang makes essential use of an extra assumption. This assumption is that if certain singular kinematic functions with zeros are divided out of the helicity amplitude, then the resulting function has no kinematic singularities in certain variables. Any such singularities necessarily arise from a failure of a generalized Legendre expansion to converge, and it is asserted that this is a dynamical question. While this seems reasonable, it is not absolutely convincing, since we do not yet fully understand the dynamics of elementary-particle systems. Thus it is not absolutely inconceivable that a kinematic singularity could cause the series to diverge. In any case the question arises whether this assumption is a dynamical assumption that goes beyond the basic analyticity properties used by Hepp and Williams. We find that this extra assumption is not really needed.

An assumption essentially equivalent to the extra assumption of Wang is made also by Hara, who relies heavily on perturbation theory.

As in the work of Hara and Wang, only four-particle reactions are considered. It is further assumed that the two initial particles have unequal masses, and that the two final particles have unequal masses. The passage to equal mass limits has been discussed by Wang.



II. SINGULARITIES AT $s = 0$

The helicity amplitude is given by

$$H = RS , \tag{2.1}$$

where S is the S matrix and R is a product of rotation operators R_j , one for each final particle. The center of mass frame is used and the z-axis is taken to lie along the direction of one of the incoming particles. The helicity λ_1 of this particle is just the z-component of its spin. The other incoming particle has helicity λ_2 , which is minus the z-component of its spin. The two final particles move in the x-z plane, the first moving in the direction θ, the second in the opposite direction.

The two rotations R_j act on the spin spaces of the two final particles, and each gives a rotation through angle θ. Specifically, for either final particle j, one has

$$R_j = \exp(i\theta J_{jy}) , \tag{2.2}$$

where J_{jy} is the y component of the spin vector J_j that acts in the spin space of the final particle j.

The S matrix is related to the M function by[5]

$$S = BM \,, \tag{2.3}$$

where B is a product of boosts, one for each particle. We work in the representation where all indices are either lower dotted or lower undotted. For a particle associated with a lower undotted index the boosts for particle j is expressed in terms of its covariant velocity $v_j = p_j/m_j$ by $v_j^2 \cdot v_j = 1$

$$B_j = D^{J_j}[(v_j \cdot \tilde{\sigma})^{\frac{1}{2}}] \,, \tag{2.4}$$

which acts on M by multiplication from the left. For a particle associated with a lower dotted index the boost is given by the same function of its velocity acting on M by multiplication from the right. For any unitary 2-by-2 matrix A the matrix $D^J[A]$ is just the $(2J + 1)$-by-$(2J + 1)$ matrix that represents the rotation specified by A in the $(2J + 1)$-dimensional irreducible representation of the rotation group.[6] The matrix elements of the $D^J[A]$ are homogeneous polynomials in the matrix elements of A, and $D^J[A]$ for general A is defined by analytic continuation.

Consider first a system consisting of one spin-$\frac{1}{2}$ particle and one spin-zero particle. Then the boosts B_j take the form

$$B_j \;=\; D^{\frac{1}{2}}[(v_j \cdot \tilde{\sigma})^{\frac{1}{2}}]$$

$$\;=\; (v_j \cdot \tilde{\sigma})^{\frac{1}{2}}$$

$$\;=\; (v_j{}^{o} + 1 - \underset{\sim}{v}_j \cdot \underset{\sim}{\sigma})/(2v_j{}^{o} + 2)^{\frac{1}{2}}$$

$$\;=\; \frac{1}{\sqrt{2}}\left[(v_j{}^{o} + 1)^{\frac{1}{2}} - (v_j{}^{o} - 1)^{\frac{1}{2}}\, \hat{\underset{\sim}{v}}_j \cdot \underset{\sim}{\sigma}\right]. \qquad (2.5)$$

The rotation matrix R_f has elements

$$R_{\lambda\mu} \;=\; \pm\left(\cos\frac{\theta}{2}\right)^{|\lambda+\mu|}\left(\sin\frac{\theta}{2}\right)^{|\lambda-\mu|}, \qquad (2.6)$$

where the sign is minus for $\lambda - \mu = -1$ and plus otherwise.

The basic analyticity assumption is that the M functions are analytic functions of the components of the momentum vectors, except at dynamical singularities.[5] It follows from this, and Lorentz invariance, that M can be written in the form[7,8]

$$M \;=\; a\, v_f \cdot \sigma \;+\; b\, v_i \cdot \sigma \;+\; c\, w \cdot \sigma$$

$$\;+\; d\, v_f \cdot \sigma\, w \cdot \tilde{\sigma}\, v_i \cdot \sigma, \qquad (2.7)$$

where the coefficients a, b, c, and d are meromorphic functions of the scalar invariants with, at most, simple poles at $\phi = 0$. Here ϕ is given by[7,4]

$$\phi = (stu - sa^2 - tb^2 - uc^2 + 2\,abc) \;, \qquad (2.8)$$

where s, t, and u are the Mandelstam variables and

$$a = (m_1^2 + m_2^2 - m_3^2 - m_4^2)/2$$

$$b = (m_1^2 + m_3^2 - m_2^2 - m_4^2)/2$$

$$c = (m_1^2 + m_4^2 - m_2^2 - m_3^2)/2 \;.$$

The surface $\phi = 0$, which is the set of points where the rank of the gram determinant is less than three, includes the boundaries of the physical regions. The possibility of poles at $\phi = 0$ is the problem considered by Hepp and Williams: the M function itself has no such poles, but the linear dependence of the various terms of Eq. (2.7) at $\phi = 0$ allows the individual terms to have them.

The vector w in Eq. (2.7) is the total energy vector and v_i and v_f are the covariant velocities of the initial and final fermions. The expansion Eq. (2.7) is obtained by first writing M in the form[7,8]

$$M = \sum_{i=1}^{3} c_i\, p_i \cdot \sigma + c_4 [p_1\, p_2\, p_3] \cdot \sigma , \tag{2.9}$$

where the vector $[p_1\ p_2\ p_3]$ is the box product defined using the alternating symbol $\epsilon_{\mu\nu\rho\lambda}$. Evaluation of the constants c_i by means of trace formulas[7,8] shows that they are holomorphic except for dynamical singularities and for possible poles at $\phi = 0$. A rearrangement of terms then gives the meromorphy of the coefficients in Eq. (2.7).

The nondynamical singularities of H fall into three categories. First, there are the possible singularities at $\phi = 0$. Second, there are singularities where some $v_j^{\,0} = \pm 1$. And finally, there are possible singularities where the components of the vectors v_i, v_f, and w, when expressed as functions of the scalar invariants, have singularities. Evaluating the energy-momentum vectors in the center of mass frame one has[3,4]

$$p_a^{\,0} = (s + m_a^{\,2} - m_b^{\,2})/2s^{\frac{1}{2}} , \tag{2.10a}$$

$$p^2 = [s - (m_a + m_b)^2]\,[s - (m_a - m_b)^2]/4s , \tag{2.10b}$$

$$\cos\theta = [2st + s^2 - s \sum_i m_i^{\,2} + (m_a^{\,2} - m_b^{\,2})(m_c^{\,2} - m_d^{\,2})] \times (4spp')^{-1} , \tag{2.10c}$$

and

$$\sin\theta = 2[s\,\phi(s,t)]^{\frac{1}{2}}/4spp' . \tag{2.10d}$$

Thus for $\phi \neq 0$ the singularities of the components occur only at $s \equiv W^2 = 0$, where their behavior is as follows:

$$p_j^{\,o} \sim \frac{1}{W},$$

$$p_j \equiv |\underset{\sim}{p}_j| \sim \frac{1}{W},$$

$$\cos\theta \sim 1,$$

and

$$\sin\theta \sim W.$$

Accordingly M itself has terms in W^{-1} and $W^{o} = 1$. Each boost has terms like $1/\sqrt{W}$ and $\sqrt{W}$, and $R_{\lambda\mu}$ goes like $W^{|\lambda-\mu|}$. Thus H appears to have a nasty behavior at $W = 0$.

Using the relations[9]

$$(v_f\cdot\tilde{\sigma})^{\frac{1}{2}}\, v_f\cdot\sigma = (v_f\cdot\sigma)^{\frac{1}{2}} \tag{2.11a}$$

and

$$v_i\cdot\sigma\,(v_i\cdot\tilde{\sigma})^{\frac{1}{2}} = (v_i\cdot\sigma)^{\frac{1}{2}}, \tag{2.11b}$$

one obtains, in the center of mass frame,

$$S = BM = a(v_f \cdot \sigma)^{\frac{1}{2}} (v_i \cdot \tilde{\sigma})^{\frac{1}{2}} + b(v_f \cdot \tilde{\sigma})^{\frac{1}{2}} (v_i \cdot \sigma)^{\frac{1}{2}}$$

$$+ \; c\, W(v_f \cdot \tilde{\sigma})^{\frac{1}{2}} (v_i \cdot \tilde{\sigma})^{\frac{1}{2}} + d\, W(v_f \cdot \sigma)^{\frac{1}{2}} (v_i \cdot \sigma)^{\frac{1}{2}} \,. \tag{2.12}$$

Let the fermion be the initial particle that moves along the z axis. Then Eq. (2.5) gives

$$(v_i \cdot \tilde{\sigma})^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \left[(v_i^{\,0} + 1)^{\frac{1}{2}} - \hat{\lambda}_i (v_i^{\,0} - 1)^{\frac{1}{2}} \right] \tag{2.13a}$$

and

$$(v_i \cdot \sigma)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \left[(v_i^{\,0} + 1)^{\frac{1}{2}} + \hat{\lambda}_i (v_i^{\,0} - 1)^{\frac{1}{2}} \right], \tag{2.13b}$$

where $\hat{\lambda}_i = 2\lambda_i = \pm 1$.

Noting that $R_f\, B_f = R_f\, B(v_f) = B(v_f')\, R_f$, with $\underset{\sim}{v}_f' = |\underset{\sim}{v}_f|\, \hat{e}_z$, one obtains

$$H_{\lambda_f \lambda_i} = R_{\lambda_f \lambda_i} F_{\lambda_f \lambda_i} \,, \tag{2.14}$$

where

$$\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} V_f^{\circ} \quad -\begin{pmatrix}1&0\\0&-1\end{pmatrix} \lambda_i (V_i^{\circ}-1)\right]$$

$$\begin{aligned} 2F_{\lambda_f \lambda_i} = {} & a\left[(v_f{}^{\circ}+1)^{\frac{1}{2}} + \hat{\lambda}_f (v_f{}^{\circ}-1)^{\frac{1}{2}}\right]\left[(v_i{}^{\circ}+1)^{\frac{1}{2}} - \hat{\lambda}_i (v_i{}^{\circ}-1)^{\frac{1}{2}}\right] \\ & + b\left[(v_f{}^{\circ}+1)^{\frac{1}{2}} - \hat{\lambda}_f (v_f{}^{\circ}-1)^{\frac{1}{2}}\right]\left[(v_i{}^{\circ}+1)^{\frac{1}{2}} + \hat{\lambda}_i (v_i{}^{\circ}-1)^{\frac{1}{2}}\right] \\ & + c\, W\left[(v_f{}^{\circ}+1)^{\frac{1}{2}} - \hat{\lambda}_f (v_f{}^{\circ}-1)^{\frac{1}{2}}\right]\left[(v_i{}^{\circ}+1)^{\frac{1}{2}} - \hat{\lambda}_i (v_i{}^{\circ}-1)^{\frac{1}{2}}\right] \\ & + d\, W\left[(v_f{}^{\circ}+1)^{\frac{1}{2}} + \hat{\lambda}_f (v_f{}^{\circ}-1)^{\frac{1}{2}}\right]\left[(v_i{}^{\circ}+1)^{\frac{1}{2}} + \hat{\lambda}_i (v_i{}^{\circ}-1)^{\frac{1}{2}}\right]. \end{aligned} \tag{2.15}$$

Here $\hat{\lambda}_j = 2\lambda_j = \pm 1$. Observe that

$$(v^{\circ}+1)^{\frac{1}{2}} \pm (v^{\circ}-1)^{\frac{1}{2}} = W^{\mp\frac{1}{2}} f^{\pm}(W^2) = W^{\mp\frac{1}{2}} f^{\pm}(s) \tag{2.16}$$

and that

$$R_{\lambda_f \lambda_i} \sim \begin{cases} 1 & \text{for} \quad \lambda_i = \lambda_f \\ W & \text{for} \quad \lambda_i \neq \lambda_f \end{cases} . \tag{2.17}$$

Combining Eqs. (2.14) through (2.17) we see that H is analytic in $s = W^2$ at $W = 0$, $\phi \neq 0$, except at dynamical singularities.

This proof is only for the simplest case of a spin-$\frac{1}{2}$ particle scattering on a spin zero particle. Yet it allows us to immediately conclude that the result holds also in general: If the two initial masses are different and the two final masses are different then the four-particle helicity amplitude is analytic in s and t at s = 0, except perhaps on $\phi = 0$ and at dynamical singularities. To get the general result, one merely observes that at $\phi \neq 0$ the higher-spin particles are kinematically equivalent to sets of spin-$\frac{1}{2}$ particles combined by Clebsch-Gordan coefficients. That is, as long as the physical vectors provide a non-singular set of basic vectors, one may pass freely between the two forms by using Clebsch-Gordan coefficients. The fact that the spin-$\frac{1}{2}$ particle parts are analytic at $s = 0$ then implies that the entire function is analytic at $s = 0$. The technicalities are given in the appendix.

III. SINGULARITIES AT $v_j^{\,o} = \pm 1$

Let the reaction be $a + b \rightarrow c + d$. Then in the center of mass frame one has

$$(v_a^{\,o} + 1) = (W + m_a + m_b)(W + m_a - m_b)/2Wm_a\,, \qquad (3.1a^+)$$

$$(v_a^{\,o} - 1) = (W - m_a - m_b)(W - m_a + m_b)/2Wm_a\,, \qquad (3.1a^-)$$

$$(v_b^{\,o} + 1) = (W + m_a + m_b)(W + m_b - m_a)/2Wm_b\,, \qquad (3.1b^+)$$

$$(v_b^{\,o} - 1) = (W - m_a - m_b)(W - m_b + m_a)/2Wm_b\,, \qquad (3.1b^-)$$

and four similar equations involving c and d. Thus the zeros of the functions $v_j^{\,o} \pm 1$ lie at $W = \pm(m_j \pm m_k)$.

As in the preceding section the physical particles of spins J_a, J_b, J_c, and J_d are considered to be combinations of N_a, N_b, N_c, and N_d spin-$\frac{1}{2}$ particles respectively. The N_j particles that combine to give particle j all have velocity v_j.

Applying to each spin-$\frac{1}{2}$ particle the manipulation that led to Eq. (2.14) one obtains

$$\begin{aligned} H_{\Lambda_f \Lambda_i} &= R_{\Lambda_f \Lambda_i} F_{\Lambda_f \Lambda_i} \\ &\equiv R(\theta)_{\Lambda_f \Lambda_i} F\left[(v_\alpha^{\,o} + 1)^{\frac{1}{2}},\ \lambda_\alpha(v_\alpha^{\,o} - 1)^{\frac{1}{2}};\ s,\ t\right]. \end{aligned} \qquad (3.2)$$

Here $R(\theta)_{\Lambda_f \Lambda_i}$ is the matrix element of the rotation operator $R(\theta)$ between the spin states specified by Λ_f and Λ_i. If the spins of the initial and final particles are different, then some Clebsch-Gordan coefficients will occur. The exact expression is given in the appendix. The function F is a combination of the F parts of Eq. (2.14). It contains one factor of either $(v_\alpha{}^0 + 1)^{\frac{1}{2}}$ or $\lambda_\alpha(v_\alpha{}^0 - 1)^{\frac{1}{2}}$ for each initial spin-$\frac{1}{2}$ particle, where λ_α is the helicity and v_α is the velocity of this particle. There is a similar factor for each final spin-$\frac{1}{2}$ particle.

The helicities λ_α satisfy

$$\sum_{\alpha\epsilon j} \lambda_\alpha = \lambda_j , \tag{3.3}$$

where λ_j is the helicity of particle j, and the j on the left represents the set of indices referring to the spin-$\frac{1}{2}$ particles that form particle j. The vital property of F is that each factor $(v_\alpha{}^0 - 1)^{\frac{1}{2}}$ appears multiplied by the corresponding helicity λ_α, and conversely, whereas the factors $(v_\alpha{}^0 + 1)^{\frac{1}{2}}$ have no such factors.

For each of the N_j particles that make up particle j there is, in each term of F, one factor $(v_j{}^0 + 1)^{\frac{1}{2}}$ or one factor $(v_j{}^0 - 1)^{\frac{1}{2}}$. Let $N_j{}^+$ be the number of factors $(v_j{}^0 + 1)$ in a given term, and let $N_j{}^-$ be the number of factors $(v_j^{\,o} - 1)$. Then one has

$$N_j^+ + N_j^- = N_j. \tag{3.4}$$

Define also

$$N_a^\pm + N_b^\pm = N_i^\pm \tag{3.5a}$$

and

$$N_c^\pm + N_d^\pm = N_f^\pm . \tag{3.5b}$$

Invariance under space inversion is assumed. This implies[10] that

$$F_{\Lambda_f \Lambda_i} = \eta \, F_{-\Lambda_f, -\Lambda_i} , \tag{3.6}$$

where η is either +1 or -1. For simplicity we assume it is +1, since the argument in the other case is essentially the same. Equation (3.6) is obtained from Eqs.(43), (31) and (A1) of Jacob and Wick,[10] by noticing that the identity

$$R_{\lambda_f \lambda_i}(\theta) = (-1)^{(\lambda_f - \lambda_i)} R_{-\lambda_f, -\lambda_i}(\theta) \tag{3.7}$$

for the individual spin-$\frac{1}{2}$ particles implies

$$R_{\Lambda_f \Lambda_i}(\theta) = \pm(-1)^{\Lambda_f - \Lambda_i} R_{-\Lambda_f, -\Lambda_i}(\theta) , \qquad (3.8)$$

since the sum of magnetic quantum numbers is preserved under Clebsch-Gordan composition. The $\pm$ sign in Eq. (3.8) is shown in the appendix to be $(-1)^{J_a + J_b - J_c - J_d}$.

Parity-conserving amplitudes are defined by

$$\begin{aligned} F^{\pm}_{\Lambda_f \Lambda_i} &\equiv F_{\Lambda_f \Lambda_i} \pm F_{-\Lambda_f \Lambda_i} \\ &= \tfrac{1}{2}\left[F_{\Lambda_f \Lambda_i} + F_{-\Lambda_f, -\Lambda_i} \pm F_{-\Lambda_f, \Lambda_i} \pm F_{\Lambda_f, -\Lambda_i}\right] \\ &= \tfrac{1}{2}\left[1 + (-1)^{N_f^- + N_i^-} \pm (-1)^{N_f^-} \pm (-1)^{N_i^-}\right] F_{\Lambda_f \Lambda_i} \\ &= \tfrac{1}{2}\left[1 \pm (-1)^{N_f^-}\right]\left[1 \pm (-1)^{N_i^-}\right] F_{\Lambda_f \Lambda_i} , \end{aligned} \qquad (3.9)$$

where the numbers N_f^- and N_i^- defined in Eq. (3.5) are regarded as operators in Eq. (3.9). The next to last line in Eq. (3.9) follows from the fact that a term of F having N_j^- factors $(v_j^{\,o} - 1)^{\frac{1}{2}}$ has also N_j^- factors λ_α, as mentioned earlier. The sign of Λ_j is reversed by reversing the sign of all of these, as shown in the appendix. From Eq. (3.9) we see that N_f^- and N_i^- are both even for terms contributing to F^+, and both odd for terms contributing to F^-.

Consider first the singularities at sums and differences of the initial particle mass. These are contained in the factor

$$
\begin{aligned}
&(v_a{}^o+1)^{N_a{}^+/2}(v_a{}^o-1)^{N_a{}^-/2}(v_b{}^o+1)^{N_b{}^+/2}(v_b{}^o-1)^{N_b{}^-/2}(2Wm_a)^{N_a/2}(2Wm_a)^{N_b/2} \\
&= (W+m_a+m_b)^{N_i{}^+/2}\,(W-m_a-m_b)^{N_i{}^-/2} \\
&\quad \times (W+m_a-m_b)^{(N_a{}^+ + N_b{}^-)/2}\,(W-m_a+m_b)^{(N_a{}^- + N_b{}^+)/2} \\
&= (W+m_a+m_b)^{(N_a + N_b - N_i{}^-)/2}\,(W-m_a-m_b)^{N_i{}^-/2} \\
&\quad \times (W+m_a-m_b)^{(N_a+N_i{}^- - 2N_a{}^-)/2}(W-m_a+m_b)^{(N_b+N_i{}^- - 2N_b{}^-)/2} .
\end{aligned}
\tag{3.10}
$$

The cases BB, FF, and FB, for which the two initial particles are bosons, fermions, and one of each, are considered separately. Bearing in mind that $N_i{}^-$ is even for F^+ but odd for F^-, one immediately sees that the square-root singularities for the various cases are as follows:

$$F^{+}\ (BB) \sim 1$$

$$F^{-}\ (BB) \sim [S - (m_a - m_b)^2]^{\frac{1}{2}}\ [S - (m_a + m_b)^2]^{\frac{1}{2}}$$

$$F^{+}\ (FF) \sim [S - (m_a - m_b)^2]^{\frac{1}{2}}$$

$$F^{-}\ (FF) \sim [S - (m_a + m_b)^2]^{\frac{1}{2}}$$

$$F^{+}\ (FB) \sim [(W + m_a + m_b)\ (W + m_a - m_b)]^{\frac{1}{2}}$$

$$F^{-}\ (FB) \sim [(W - m_a - m_b)\ (W - m_a + m_b)]^{\frac{1}{2}}\ , \qquad (3.11)$$

where in the FB case the fermion is particle a. If the factor indicated in Eq. (3.11) is divided out, along with the analogous factor coming from the final particles, then the resulting functions are analytic at $v_j^{\,o} = \pm 1$. For cases BB and FF the factors divided out are analytic in s at s = 0 and hence the helicity amplitude remains analytic in s. In the FB case one retains only analyticity in W at W = 0.

IV. SINGULARITIES AT $\phi = 0$

The singularities at $\phi = 0$ were the only ones that occurred in the expansions considered by Hepp and Williams; these were the cause of all the difficulty. In the present approach singularities are introduced in W at zero, and at sums and differences of masses. The compensation is simplicity at $\phi = 0$.

The behavior of M at points of $\phi = 0$ not lying on dynamical singularities is given by

$$M_{\alpha\beta} \sim (\sin\theta)^{|\alpha-\beta|} . \qquad (4.1)$$

Here $\sim$ means equal to within a factor holomorphic in s and t, except at $s = 0$, at $v_j^{\,o} = \pm 1$, ~~or~~ and at dynamical singularities. One way to verify Eq. (4.1) is to consider the power-series expansion[11]

$$M = \Sigma\, c^{+}_{\ell m}\, (p_3 - \bar{p}_3)^{\ell}\, (p_x + ip_y)^m + \Sigma\, c^{-}_{\ell m}\, (p_3 - \bar{p}_3)^{\ell}\, (p_x - ip_y)^m , \qquad (4.2)$$

where the $c^{\pm}_{\ell m}$'s are matrices in spin space. The Lorentz-invariance property of M,

$$\Lambda_s\, M(\Lambda^{-1}K) \;=\; M(K), \qquad (4.3)$$

specialized to rotations about the z axis and expressed in differential form, gives[12]

$$(J_{fz} - J_{iz})\, M(K) + (L_{fz} - L_{iz})\, M(K) = 0, \tag{4.4}$$

where J and L are the spin and orbital-angular-momentum operators respectively. Application of (4.4) to (4.2) gives

$$(C^{\pm}_{\ell m})_{\alpha\beta} = C^{\pm}_{\ell}\, \delta_{\pm m, \beta-\alpha}. \tag{4.5}$$

That is, every term in the power-series expansion of $M_{\alpha\beta}$ about $p_x = p_y = 0$ contains the components p_x and p_y in the precise form $(p_x \pm ip_y)^{|\alpha-\beta|}$, where the $\pm$ is the sign of $\beta - \alpha$. Equation (4.1) then follows from Eq. (2.10). The boost has the same singularity structure at $\sin\theta = 0$ that M has. One observes that a product of two matrices having this structure also has this structure. Here one uses the fact that $\sin^2\theta$ is regular in s and t, which follows from (2.10d) and (2.8). Thus the product BM = S has the same structure at $\phi = 0$ that M has:

$$S_{\alpha\beta} \sim (\sin\theta)^{|\alpha-\beta|}. \tag{4.6}$$

The singularities of $R(\theta)$ at $\phi = 0$ are given by[4]

$$R(\theta)_{\lambda\mu} \sim (\sin\frac{\theta}{2})^{|\lambda-\mu|} (\cos\frac{\theta}{2})^{|\lambda+\mu|} .$$

The right side is essentially the same as that of Eq. (4.1). The apparent differences comes from the fact that λ is minus the z component of spin at $\cos\frac{\theta}{2} = 0$.

Using again the fact that any product of matrices having the singularity structure of Eq. (4.1) also has this singularity structure, we see that, apart from dynamical singularities, the function

$$H_{\Lambda_f \Lambda_i} \Big/ (\sin\frac{\theta}{2})^{|\Lambda_f-\Lambda_i|} \cos\frac{\theta}{2}^{|\Lambda_f+\Lambda_i|}$$

is regular in s and t at points of $\phi = 0$ where W is not equal to zero or to sums or differences of masses.

V. COMBINED RESULT

The function $\sin\frac{\theta}{2}$ behaves like a multiple of $W = s^{\frac{1}{2}}$ at $s = 0$, $\phi \neq 0$, whereas $\cos\frac{\theta}{2}$ is regular. Thus, apart from dynamical singularities, the function

$$\bar{H}_{\Lambda_f \Lambda_i} \equiv \frac{H_{\Lambda_f \Lambda_i}}{(\sin\frac{\theta}{2})^{|\Lambda_f-\Lambda_i|} \cos\frac{\theta}{2}^{|\Lambda_f+\Lambda_i|}} \tag{5.1}$$

is, when multiplied by $W^{|\Lambda_f-\Lambda_i|}$, analytic in t and s at $s = 0$ for $\phi \neq 0$ and $|v_j^{\,o}| \neq 1$. It is also analytic in t and s at $\phi = 0$ for $s \neq 0$ and $|v_j^{\,o}| \neq 1$. Thus by virtue of the theorem on isolated singularities[13] it is analytic in s and t except at $|v_j^{\,o}| = 1$, apart from dynamical singularities. Thus if m is the maximum of $|\Lambda_f-\Lambda_i|$ and $|\Lambda_f+\Lambda_i|$ and $G^{\pm}$ are the functions given by (3.11), then the function

$$\hat{H}^{\pm}_{\Lambda_f \Lambda_i} = (\bar{H}_{\Lambda_f,\Lambda_i} \pm \bar{H}_{-\Lambda_f,\Lambda_i})\, W^m/G^{\pm} \tag{5.2}$$

is analytic in t and s (W for the FB case) except at dynamical singularities.

APPENDIX A. REDUCTION TO SPIN-$\frac{1}{2}$ PARTICLES

What must be shown is that the set of M functions $M(ab \to cd)$ representing the physical process $ab \to cd$ can be represented in terms of spin-$\frac{1}{2}$ particles in a way such that the scalar coefficients of the expansion are analytic at $\phi \neq 0$, except at the dynamical singularities of M itself.

A system of n spin-$\frac{1}{2}$ particles is described by 4^n matrix elements. At a point $\phi \neq 0$ the set of matrices $b_1 = v_f \cdot \sigma$, $b_2 = v_i \cdot \sigma$, $b_3 = w \cdot \sigma$, and $b_4 = v_f \cdot \sigma \; w \cdot \tilde{\sigma} \; v_i \cdot \sigma$ span a single-particle spin space. Thus one can express the 4^n matrix elements $M_{\alpha_1 \dot{\beta}_1 \alpha_2 \dot{\beta}_2 \cdots \alpha_n \dot{\beta}_n}$ in terms of the 4^n coefficients $C_{\lambda_1 \lambda_2 \cdots \lambda_n}$ by

$$M_{\alpha_1 \dot{\beta}_1 \cdots \alpha_n \dot{\beta}_n} = \sum_{\lambda_i = 1}^{4} C_{\lambda_1 \lambda_2 \cdots \lambda_n} \, b_{\lambda_1 \alpha_1 \dot{\beta}_1} \, b_{\lambda_2 \alpha_2 \dot{\beta}_2} \cdots b_{\lambda_n \alpha_n \dot{\beta}_n} \quad . \tag{A.1}$$

The transformation between the C's and the $M_{\alpha_1 \cdots \dot{\beta}_n}$'s is nonsingular at $\phi \neq 0$, and hence for $\phi \neq 0$ the coefficients $C_{\lambda_1 \cdots \lambda_n}$ are analytic functions of the scalar invariants wherever the $M_{\alpha_1 \cdots \dot{\beta}_n}$'s are analytic functions of the momenta.[7,1,2] The problem is therefore solved if one can construct functions $M_{\alpha_1 \cdots \dot{\beta}_n}$ that, when contracted with appropriate Clebsch-Gordan coefficients, give the physical functions $M(ab \to cd)$, and moreover are analytic in the momenta wherever the functions $M(ab \to cd)$ are. A set of such functions $M_{\alpha_1 \cdots \dot{\beta}_n}$ can be constructed by imposing extra conditions on the $M_{\alpha_1 \cdots \dot{\beta}_n}$ in such a way that these functions become unique analytic functions of the functions $M(ab \to cd)$.

The extra conditions we impose can be regarded as the conditions that would arise if the $M_{\alpha_1 \cdots \dot{\beta}_n}$ were made to describe also certain additional fictitious reactions. Assume for the moment that $J_a + J_b = J_c + J_d$. Then the physical particles a, b, c, and d are formed by Clebsch-Gordan composition on the sets of spin-$\frac{1}{2}$ particles Γ_a, Γ_b, Γ_c, and Γ_d, respectively, where the number of particles in Γ_j is $N_j = 2J_j$. The N_j particles of Γ_j can be combined in only one way to give a particle of spin $J_j = N_j/2$. However, they can be combined in a variety of ways to give particles of lesser spin. To be specific, let the particles of Γ_j first be numbered in some arbitrary way. Then the first and second particles of Γ_j can be combined to give systems of spins $J^+ = \frac{1}{2} + \frac{1}{2}$ and $J^- = \frac{1}{2} - \frac{1}{2}$. The third particle can be added in various possible ways to give three possible systems, having spins $J^{++} = J^+ + \frac{1}{2}$, $J^{+-} = J^+ - \frac{1}{2}$, and $J^{-+} = \frac{1}{2}$, respectively. Continuing, one finally obtains a single system of spin J_j, $N_j - 1$ systems of spin $J_j - 1$, $\frac{1}{2}(N_j - 1)(N_j - 2) - 1$ systems of spin $J_j - 2$, and so on. The total number of spin states of all these systems is

$$(2J_j+1) + (N_j-1)(2J_j-1) + [\tfrac{1}{2}(N_j-1)(N_j-2)-1]\,(2J_j-3)\ldots = 2^{N_j} . \tag{A.2}$$

It can be shown[2] that there is an orthogonal transformation relating these 2^{N_j} states to the 2^{N_j} basic states of the N_j particles of Γ_j. Thus one can consider that the functions $M_{\alpha_1 \cdots \dot{\beta}_n}$ describe a whole set of reactions. And because the relevant transformation is orthogonal, the $M_{\alpha_1 \cdots \dot{\beta}_n}$ are analytic functions of the functions describing these various processes. Therefore if one takes all the auxilliary processes to vanish, then the $M_{\alpha_1 \cdots \dot{\beta}_n}$'s will be uniquely defined analytic functions of physical scattering amplitudes $M(ab \to cd)$, which is want we wanted.

The Clebsch-Gordan composition that takes $2J$ spin-$\frac{1}{2}$ particles into one spin-J particle is given by forming the completely symmetrized sum of the states of the proper z component of angular momentum. Thus reversal of the elementary helicities simply reverses the total helicity.

The above discussion covered the case $J_a + J_b = J_c + J_d$. If this condition is not satisfied, then for certain j we have $N_j - 2J_j \equiv e_j > 0$. The labels a, b, c, and d can be chosen so that the e_j are even for all j. Then a physical particle j with $e_j > 0$ is formed by contracting each of the first $e_j/2$ pairs to spin zero. Now the helicity Λ_j can be reversed by reversing the helicities of merely the last $N_j - e_j$ particles of Γ_j. The numbers N_j^+, N_j^-, and N_j used in Section III refer only to these last $N_j - e_j = 2J_j$ particles. Because $N_j - 2J_j$ is even, it can be ignored in (3.10).

The functions $M(ab \to cd)$ are constructed from the $M_{\alpha_1 \cdots \dot{\beta}_n}$ by Clebsch-Gordan composition. Let this relation be represented by

$$M(ab \to cd) = \mathcal{C}_a \, \mathcal{C}_b \, M_{\alpha_1 \cdots \dot{\beta}_n} \, \mathcal{C}_c \, \mathcal{C}_d \, . \tag{A.3}$$

The rotation operator $R_{\Lambda_f \Lambda_i}(\theta)$ is given by the identical composition:

$$R_{\Lambda_f \Lambda_i}(\theta) = \mathcal{C}_a \, \mathcal{C}_b \left[\prod_{\rho=1}^{n} R_{\lambda_{f\rho} \lambda_{i\rho}}(\theta) \right] \mathcal{C}_c \, \mathcal{C}_d \, . \tag{A.4}$$

In cases where $J_a + J_b = J_c + J_d$, Eq. (A.4) shows that (3.7) implies (3.8). In the other cases, however, each pair of particles that is contracted to the singlet state, which is antisymmetric, gives an extra minus sign. The sign (3.8) is therefore $(-1)^{J_a + J_b - J_c - J_d}$. This sign must be included in the η of (3.6), which becomes, therefore, simply the product $\eta_c \eta_d / \eta_a \eta_b$.

The dynamical singularities lie on surfaces defined by invariants.[5] The domain of regularity is therefore I_+-saturated, in the terminology of Hepp.[1] This means that any regular point is the image of some point in momentum space for which the rank of the gram determinant $|k_i \cdot k_j|$ is equal to the number of linearly independent vectors. The power-series expansion (4.2) can be considered an expansion about such a point. The (p_x, p_y, p_z) are the relative momenta in a frame where the total energy-momentum vector P^μ is a pure time-like vector. The existence of such a frame is assured by Lemma 2 of Hall and Wightman.[14] The component P^o is fixed by the mass constraints and is an analytic function of the (p_x, p_y, p_z) at $p_x = p_y = 0$ except when $p_3^2 = m_j^2$ for some j. Thus except possibly at these points, and at dynamical singularities, the M function is analytic in the (p_x, p_y, p_z) at $p_x = p_y = 0$, and the power-series expansion (4.2) is valid. Accordingly the relation (4.1) is valid except possibly when $p^2 = - m_j^2$, or equivalently, as one sees from (2.10a) when

s^2 is $(m_a{}^2 - m_b{}^2)^2$ or $(m_c{}^2 - m_d{}^2)^2$. But M considered as a function of invariants is not singular on these surfaces, as one also sees from (2.10a). Thus (4.1) holds everywhere, except at dynamical singularities.

FOOTNOTES AND REFERENCES

* This work was done under the auspices of the U. S. Atomic Energy Commission.

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