Title
Mathematical Study of The Nonlinear Singular Integral Magnetic Field Equation

Author
Friedman, Mark J

Publication Date
1979-06-01

LBL-9096
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA, BERKELEY, CA

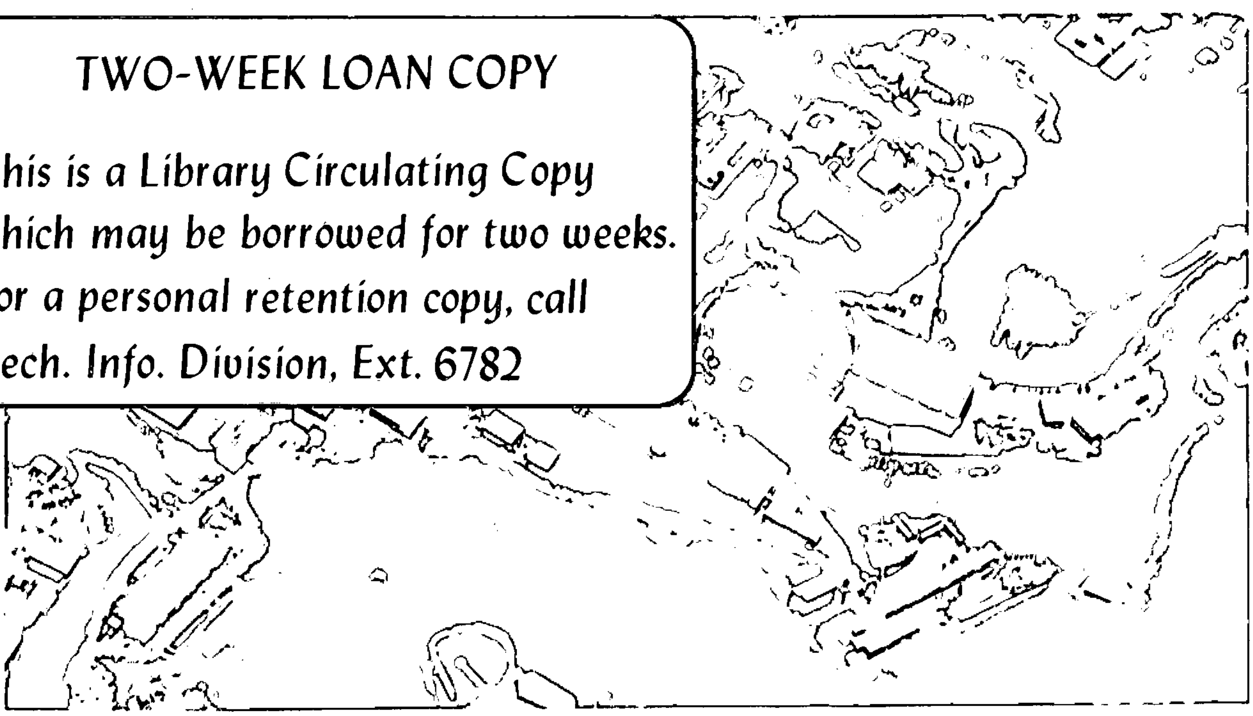
Physics, Computer Science & Mathematics Division

Submitted to Society for Industrial and Applied Mathematics, Numerical Analysis

MATHEMATICAL STUDY OF THE NONLINEAR SINGULAR INTEGRAL MAGNETIC FIELD EQUATION

Mark J. Friedman

June 1979



Prepared for the U. S. Department of Energy
under Contract W-7405-ENG-48

DISCLAIMER

MATHEMATICAL STUDY OF THE NONLINEAR SINGULAR INTEGRAL MAGNETIC FIELD EQUATION

Mark J. Friedman

Computer Science and Applied Mathematics
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Part 1

Abstract

We consider the nonlinear singular integral magnetic field equation $R\bar{M} = h\bar{M} + A\bar{M} = \bar{H}a$, in the Hilbert space of vector-functions $\bar{L}^2(\Omega)$, where $\bar{M}$ is the magnetization vector, $(h\bar{M})(x) = \bar{g}(\bar{M}(x),x)$ is the total field, and $(A\bar{M})(x) = -\frac{1}{4\pi}$ grad div $\int_\Omega \frac{\bar{M}(y)}{r}\,dy$. We prove that: i) A is bounded, with $\|A\| = 1$; ii) A is self-adjoint; iii) A is positive semi-definite, with $(A\bar{M},\bar{M}) \geq 0$.

Uniqueness is proved in case h is strictly monotone; existence of R^{-1} and its continuity are proved in case h is strongly monotone, continuous and bounded. In this case the Galerkin method (and, if the magnetic material is also isotropic, the Ritz method) is shown to yield a numerical solution of the equation.

0. INTRODUCTION

Magnetic field calculations have applications to various engineering devices [3]. Generally, the magnetostatic problem is that of determining the magnetization or magnetic field in a highly permeable three-dimensional body of complex geometric configuration. There are but a few mathematical investigations of the problem in the nonlinear case known to this author. In [15] a uniqueness theorem and in [7] a uniqueness and existence theorem have been proven.

Currently, magnetic field problems can be formulated using either a differential equation approach or an integral equation one. In the Russian electromechanic literature, the latter approach has been pioneered by I. I. Pekker [10,11] and in the Western literature by A. H. Halacsy [4]. Now it is widely used (see for example, [1-6,8,12,14]. But the author is unaware of any publications containing a rigorous mathematical analysis of the integral equation approach. In his opinion, such analysis might clarify both the possibilities of this approach and the restrictions on it, and might also suggest more effective techniques for solving the equation.

1. FORMULATION OF THE PROBLEM. PRINCIPLE RESULTS

The general problem is to solve the nonlinear singular integral equation [10,11]

$$\bar{g}(\bar{M}(x),x) - \frac{1}{4\pi}\,\text{grad div}\int_{\Omega}\frac{\bar{M}(y)}{r}\,dy = \bar{H}_a(x)\,, \qquad (x = (x_1,x_2,x_3)\in\Omega) \qquad (1.1)$$

for the magnetization $\bar{M}(x) = (M_1(x), M_2(x), M_3(x))$ given an "applied field" $\bar{H}_a(x)$. Here Ω is a bounded open region in R_3 (=Euclidean 3-space) with a sufficiently smooth boundary S, which we imagine to be filled with a ferromagnetic material; $\bar{r}(y,x) = x-y$, $r = |x-y|$, $\bar{H}(x) = \bar{g}(\bar{M}(x),x)$ is the net field in Ω considered as a function of $\bar{M}$; $\bar{g}$ is, as a rule, an experimental function representing the magnetic permeability of the ferromagnetic material which varies with the magnetization. This relationship is usually given by a single valued magnetization curve, called the $\bar{M}-\bar{H}$ characteristic of the magnetic material; obtained by neglecting the hysteresis effects. The integral represents the demagnetization field due to spatial distribution of magnetization.

In this paper we first study the singlar integral operator. Then we choose the appropriate functional space, $\bar{L}^2(\Omega)$, for studying the equation. Making use of the monotone operator method we prove the existence and uniqueness of the solution of (1.1), and the continuity of the inverse operator. (The latter result implies the so-called "correctness" property, i.e., that the solution depends continuously on the right side of the equation). We then justify the application of the Galerkin method for solving the equation in the general case and the Ritz method in the isotropic case. Note that the conditions (3.4) and (3.5) imposed on g are valid for isotropic as well as nonlinear anisotropic media in the limiting case of no hysteresis or very strong hysteresis (permanent magnets).

In later papers we plan to establish the Tucker stability of the Galerkin procedure, obtain perturbation estimates, consider some applications of these estimates, and give a more detailed analysis of the spectrum of the singular integral operator.

2. ANALYSIS OF THE SINGULAR INTEGRAL OPERATOR

Let us set

$$\bar{\Psi}(x) = -\frac{1}{4\pi}\int_{\Omega} \frac{\bar{M}(y)}{r}\, dy \quad . \qquad (2.0)$$

We also use the following notation

$$(\bar{u},\bar{v})_G = \int_G \bar{u}(x)\bar{v}(x)dx \;, \quad \|\bar{u}\|_G^2 = \int_G \bar{u}^2(x)\,dx \;, \quad (\bar{u},\bar{v})_\Omega \equiv (\bar{u},\bar{v}) \;, \quad \|\bar{u}\|_\Omega \equiv \|\bar{u}\| \;,$$

where G is a region in R_3.

Lemma 2.1. Let $\bar{M}(x)$ be smooth in $\bar{\Omega} = \Omega \cup S$ and $\bar{M}(x) = 0$ in $R_3 \setminus \Omega$. Then the following identities hold:

$$\bar{M} = \bar{M}_1 + \bar{M}_2 \;, \qquad (2.1)$$

where

$$\bar{M}_1(x) = \text{grad div}\,\bar{\Psi}(x) = \frac{1}{4\pi}\,\text{grad}\left(\int_S \frac{\bar{M}(y)\cdot\bar{n}(y)}{r}\, dS_y - \int_\Omega \frac{\text{div}\,\bar{M}(y)}{r}\, dy\right),$$

$$\bar{M}_2(x) = -\text{rot rot}\,\bar{\Psi}(x) = \frac{1}{4\pi}\,\text{rot}\left(-\int_S \frac{\bar{n}(y)\times\bar{M}(y)}{r}\, dS_y + \int_\Omega \frac{\text{rot}\,\bar{M}(y)}{r}\, dy\right)$$

$$\|\text{grad div}\,\bar{\Psi}\|_\Omega^2 = \|\bar{M}\|_\Omega^2 - \|\text{grad div}\,\bar{\Psi}\|_{R_3\setminus\Omega}^2 - \|\text{rot rot}\,\bar{\Psi}\|_{R_3}^2 \;, \qquad (2.2)$$

$$(\bar{M},\, \text{grad div}\,\bar{\Psi})_\Omega = \|\text{grad div}\,\bar{\Psi}\|_\Omega^2 + \tfrac{1}{2}\|\text{grad div}\,\bar{\Psi}\|_{R_3\setminus\Omega}^2 + \tfrac{1}{2}\|\text{rot rot}\,\bar{\Psi}\|_{R_3\setminus\Omega}^2 \qquad (2.3)$$

Proof. The identity (2.1) follows from the identities

$$\Delta\bar{\Psi} = -\text{rot rot}\,\bar{\Psi} + \text{grad div}\,\bar{\Psi} \;, \qquad (2.4)$$

$$\Delta\bar{\Psi}(x) = \bar{M}(x) \;, \qquad (x \in \Omega) \;, \qquad (2.5)$$

$$\operatorname{div} \bar{\Psi}(x) = -\frac{1}{4\pi}\int_\Omega \bar{M}(y)\cdot \operatorname{grad}_x \frac{1}{r}\, dy = \frac{1}{4\pi}\int_\Omega \bar{M}(y)\cdot \operatorname{grad}_y \frac{1}{r}\, dy$$

$$= \frac{1}{4\pi}\left(\int_S \frac{\bar{M}(y)\cdot\bar{n}(y)}{r}\, dS_y - \int_\Omega \frac{\operatorname{div} \bar{M}(y)}{r}\, dy\right) , \qquad (2.6)$$

$$-\operatorname{rot} \bar{\Psi}(x) = -\frac{1}{4\pi}\int_\Omega \bar{M}(y)\times \operatorname{grad}_x \frac{1}{r}\, dy = \frac{1}{4\pi}\int_\Omega \bar{M}(y)\times \operatorname{grad}_y \frac{1}{r}\, dy$$

$$= \frac{1}{4\pi}\left(-\int_\Omega \operatorname{rot}_y \frac{\bar{M}(y)}{r}\, dy + \int_\Omega \frac{\operatorname{rot} \bar{M}(y)}{r}\, dy\right)$$

$$= \frac{1}{4\pi}\left(-\int_S \frac{\bar{n}(y)\times\bar{M}(y)}{r}\, dS_y + \int_\Omega \frac{\operatorname{rot} \bar{M}(y)}{r}\, dy\right) . \qquad (2.7)$$

We proceed now to prove (2.2).

$$(\operatorname{rot}\operatorname{rot}\bar{\Psi}, \operatorname{grad}\operatorname{div}\bar{\Psi})_\Omega = \int_\Omega \operatorname{rot}\operatorname{rot}\bar{\Psi}\cdot\operatorname{grad}\operatorname{div}\bar{\Psi}\, dx$$

$$= \int_\Omega \operatorname{div}(\operatorname{div}\bar{\Psi} \quad \operatorname{rot}\operatorname{rot}\bar{\Psi})\, dx$$

$$= \int_S (\operatorname{div}\bar{\Psi})(\operatorname{rot}\operatorname{rot}\bar{\Psi}\cdot\bar{n}^+)\, dS .$$

Since $\operatorname{div}\bar{\Psi}$ and $\operatorname{rot}\operatorname{rot}\bar{\Psi}\cdot\bar{n}$ are continuous when crossing S [13] we have

$$\int_S (\operatorname{div}\bar{\Psi})(\operatorname{rot}\operatorname{rot}\bar{\Psi}\cdot\bar{n}^+)dS = -\int_S (\operatorname{div}\bar{\Psi})(\operatorname{rot}\operatorname{rot}\bar{\Psi}\cdot\bar{n}^-)dS$$

$$= \int_{R_3\setminus\Omega} \operatorname{rot}\operatorname{rot}\bar{\Psi}\cdot\operatorname{grad}\operatorname{div}\bar{\Psi}\, dx ,$$

where $\bar{n}=\bar{n}^+$ is the outer normal and $\bar{n}^-$ is the inner normal to S.

As a result

$$(\operatorname{rot}\operatorname{rot}\bar{\Psi}, \operatorname{grad}\operatorname{div}\bar{\Psi})_\Omega = -(\operatorname{rot}\operatorname{rot}\bar{\Psi}, \operatorname{grad}\operatorname{div}\bar{\Psi})_{R_3\setminus\Omega} . \qquad (2.8)$$

Further, (2.4), (2.5) and (2.8) give

$$\|\bar{M}\|_\Omega^2 = \|\text{rot rot } \bar{\Psi}\|_\Omega^2 - 2(\text{rot rot } \bar{\Psi}, \text{ grad div } \bar{\Psi})_\Omega + \|\text{grad div } \bar{\Psi}\|_\Omega^2$$

$$= \|\text{rot rot } \bar{\Psi}\|_\Omega^2 + 2(\text{rot rot } \bar{\Psi}, \text{ grad div } \bar{\Psi})_{R_3\setminus\Omega} + \|\text{grad div } \bar{\Psi}\|_\Omega^2 . \tag{2.9}$$

From (2.4) and the identity $\Delta\bar{\Psi} = 0$ for x in $R_3\setminus\Omega$ it follows that

$$0 = \|\text{rot rot } \bar{\Psi}\|_{R_3\setminus\Omega}^2 - 2(\text{rot rot } \bar{\Psi}, \text{ grad div } \bar{\Psi})_{R_3\setminus\Omega} + \|\text{grad div } \bar{\Psi}\|_{R_3\setminus\Omega}^2 . \tag{2.10}$$

Adding (2.9) and (2.10) gives (2.2); (2.3) follows from (2.4), (2.7) and (2.10) and the fact that

$$(\text{grad div } \bar{\Psi}, \bar{M})_\Omega = \|\text{grad div } \bar{\Psi}\|_\Omega^2 - (\text{rot rot } \bar{\Psi}, \text{ grad div } \bar{\Psi})_\Omega$$

$$= \|\text{grad div } \bar{\Psi}\|_\Omega^2 + (\text{grad div } \bar{\Psi}, \text{ rot rot } \bar{\Psi})_{R_3\setminus\Omega}$$

$$= \|\text{grad div } \bar{\Psi}\|_\Omega^2 + \tfrac{1}{2}\|\text{grad div } \bar{\Psi}\|_{R_3\setminus\Omega}^2 + \tfrac{1}{2}\|\text{rot rot } \bar{\Psi}\|_{R_3\setminus\Omega}^2$$

This ends the proof.

Let $\bar{L}^2(\Omega)$ be the Hilbert space of real square-summable in Ω vector functions $\bar{M}(x)$ with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. We define in $\bar{L}^2(\Omega)$ the operator

$$(A\bar{M})(x) = \text{grad div } \bar{\Psi}(x) = -\frac{1}{4\pi} \text{ grad div} \int_\Omega \frac{\bar{M}(y)}{r}\, dy . \tag{2.11}$$

We take the functions $\bar{M}(x)$ satisfying the conditions of Lemma 2.1 for the domain of definition D(A) of A.

Theorem 2.1. Operator A is

i) bounded, with $\|A\| = 1$,

ii) self-adjoint,

iii) positive semi-definite, with $\inf\limits_{\|\bar{M}\| = 1}(A\bar{M},\bar{M}) = 0$.

Proof. From Eq. (2.2) it follows that $\|A\|_{D(A)} \leq 1$. If $M \in D(A)$ and $\mathrm{rot}\,\bar{M}=0$, $\bar{M}\times\bar{n}|_S = 0$, then (2.1) gives $A\bar{M} = \bar{M}$. Thus $\|A\|_{D(A)} = 1$. Extension of A by continuity onto $\bar{L}^2(\Omega)$ results in $\|A\| = 1$. Using (2.6) and the theorem on differentiation of integrals with weak singularities [9, p.139] we obtain (compare with [5, pp.12,22-27] and [7, p.379]

$$(A\bar{M})(x) = -\frac{1}{4\pi}\,\mathrm{grad}\,\mathrm{div}\int_\Omega \frac{\bar{M}(y)}{r}\,dy = \frac{1}{4\pi}\int_\Omega \bar{M}(y)\cdot \mathrm{grad}_y \frac{1}{r}\,dy$$

$$= -\frac{1}{4\pi}\int_\Omega \left(\bar{M}(y)\cdot\mathrm{grad}_y\right)\mathrm{grad}_y \frac{1}{r}\,dy - \frac{1}{3}\bar{M}(x)$$

$$= \frac{1}{4\pi}\int_\Omega \left[-\frac{\bar{M}(y)}{r^3} + \frac{\bar{M}(y)\cdot\bar{r}(y,x)}{r^5}\,r(x,y)\right]dy - \frac{1}{3}\bar{M}(x) \quad , \tag{2.12}$$

where the latter integral is understood in the Cauchy sense, (i.e.,

$$\int_\Omega f(x,y)dy = \lim_{r\to 0}\int_{\Omega\setminus B(x,r)} f(x,y)\,dy,$$

where $B(x,r)$ is a ball of radius r with the center at x). Since the kernel of the latter integral is even, according to [9, p.162] A is self-adjoint.

To prove the third statement we note that due to Eq. (2.3), $(A\bar{M},\bar{M}) \geq 0$, and due to Eq. (2.1), $(A\bar{M},\bar{M}) = 0$ if $\mathrm{div}\,\bar{M} = 0, \bar{M}\cdot\bar{n}|_S = 0$.

The properties of A established above suggest the choice of $\bar{L}^2(\Omega)$ as the natural space in which to study (1.1) and attempt a numerical solution.

Remark.

The established properties of A have a physical interpretation which is as follows: let us denote by $\bar{H}_i(x)$ the field $\bar{H}(x) - \bar{H}_a(x)$ induced by $\bar{M}(x)$. Then Eq. (1,1) is written as

$$-\bar{H}_i = A\bar{M} . \tag{2.13}$$

The application of (iii) gives $-(\bar{H}_i,\bar{M}) = (A\bar{M},\bar{M}) \geq 0$, and therefore

$$(\bar{H}_i,\bar{M}) \leq 0 . \tag{2.14}$$

Now by (2.13) and (i), $-(\bar{H}_i,\bar{M}) = (A\bar{M},\bar{M}) \leq \|\bar{M}\|^2$, and using (2.14), $(\bar{H}_i,\bar{M}) \leq \|\bar{M}\|^2$, or

$$\left|\left(\bar{H}_i, \frac{\bar{M}}{\|\bar{M}\|}\right)\right| \leq \|\bar{M}\| . \tag{2.15}$$

The inequality (2.14) indicates that the mean angle between the induced field $\bar{H}_i$ and the magnetization $\bar{M}$ in Ω is not less than $\pi/2$; (2.15) indicates that the mean value of the projection of $\bar{H}_i$ onto $\bar{M}$ is not more than the mean value of $\bar{M}$ in Ω. Thus, (2.14) and (2.15) give a rigorous mathematical interpretation to the well known maxims among electrical engineers that, "the induced field is directed opposite the net field, or magnetization," and "the induced field is less than the magnetization."

3. NUMERICAL SOLUTION BY THE GALERKIN AND RITZ METHODS

We now study the general equation (1.1) in $\bar{L}^2(\Omega) \equiv \bar{L}^2$. We make the assumption that $\bar{H}_a \in \bar{L}^2$, and $\bar{g}\big(\bar{M}(x),x\big) \in \bar{L}^2$ for all $\bar{M} \in \bar{L}^2$ and define in $\bar{L}^2$ the operator

$$(h\bar{M})(x) = \bar{g}\big(\bar{M}(x),x\big) , \qquad (\bar{M} \in \bar{L}^2) ; \tag{3.1}$$

(1.1) is rewritten in operator form as

$$R\bar{M} = h\bar{M} + A\bar{M} = \bar{H}_a , \qquad (\bar{H}_a \in \bar{L}^2) . \tag{3.2}$$

Definition. An operator $R:\bar{L}^2 \to \bar{L}^2$ is called *monotone* if for any $\bar{M}_1, \bar{M}_2 \in \bar{L}^2$ we have $(R\bar{M}_1 - R\bar{M}_2,\ \bar{M}_1 - \bar{M}_2) \geq 0$. If $(\geq)$ is replaced by $(>)$, it is *strictly* monotone, and if 0 is replaced by $c\|\bar{M}_1 - \bar{M}_2\|^2$, where $c > 0$ does not depend on $\bar{M}$, it is *strongly* monotone.

For h the monotonicity condition is written as

$$\int_\Omega \Big(\bar{g}(\bar{M}_1(x),x) - \bar{g}(\bar{M}_2(x),x)\Big) \cdot (\bar{M}_1(x) - \bar{M}_2(x))dx \geq 0 \quad . \tag{3.3}$$

Since A is positive semi-definite, it is monotone. It follows that R is monotone if h is monotone. Due to [16, p.194] this implies

Theorem 3.1.

If h satisfies (3.3) with $(\geq)$ replaced by $(>)$, then (3.2) [or (1.1)] cannot have more than one solution.

To prove the existence of the inverse operator R^{-1} and its continuity we subject h to stronger conditions. Let $\bar{g}(\bar{M},x)$ be continuous with respect to $\bar{M}$ for almost all $x \in \Omega$, measurable in Ω with respect to x for all $\bar{M}$, and satisfy the inequalities

$$(\bar{g}(\bar{M}_1,x) - \bar{g}(\bar{M}_2,x)) \cdot (\bar{M}_1 - \bar{M}_2) \geq c(\bar{M}_1 - \bar{M}_2)^2 \tag{3.4}$$

for almost all $x \in \Omega$, where $\bar{M}_1, \bar{M}_2 \in R_3$ are arbitrary and $c > 0$,

$$|g_i(\bar{M},x)| \leq a_i(x) + b\sum_{k=1}^{3} |M_k| \quad , \qquad (i = 1,2,3) \, , \tag{3.5}$$

where $a(x) \in L^2(\Omega)$, and $b > 0$. It is easy to see that (3.4) amounts to the condition of the strong monotonicity for h. By [16, p.62] (3.5) implies that h is bounded and continuous in $\bar{L}^2$. By [16, p.273] there holds

Theorem 3.2.

If $\bar{g}$ satisfies (3.4) and (3.5), then (3.2) [or (1.1)] has the unique solution $\bar{M}_o$ in $\bar{L}^2$ for each $\bar{H}_a$ in $\bar{L}^2$, and the inverse operator R^{-1} is continuous.

The condition (3.4) means that the angle between the increments of $\bar{H}$ and $\bar{M}$ is less than $\pi/2$. It is valid for the magnetic materials mentioned in Section 1; the condition (3.5) is valid for all known materials.

For the numerical solution, (3.2) must be replaced by a sequence of finite-dimensional equations. This can be done by a Galerkin method. Let $\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_n, \ldots$ be a coordinate system in $\bar{L}^2$, i.e., $\bar{\phi}_n \in \bar{L}^2$, $\bar{\phi}_n$ are linearly independent, and every $\bar{M}$ in $\bar{L}^2$ can be represented as

$$\bar{M} = \sum_{k=1}^{\infty} \alpha_k(\bar{M})\, \bar{\phi}_k ,$$

where α_k are functionals. We seek the solution in the form

$$M_n = \sum_{k=1}^{n} \alpha_k\, \bar{\phi}_k . \tag{3.6}$$

The coefficients α_k are obtained from the system of equations

$$\left(R\left(\sum_{k=1}^{n} \alpha_k\, \bar{\phi}_k\right), \bar{\phi}_i\right) = (\bar{H}_a, \bar{\phi}_i) , \quad (i = 1, 2, \ldots, n) \tag{3.7}$$

which is called the Galerkin system.

Theorem 3.3.

If $h : \bar{L}^2 \to \bar{L}^2$ satisfies (3.4) and (3.5), then the Galerkin approximations $\bar{M}_n$ exist for each n and $\bar{M}_n$ converges to $\bar{M}_o$ in $\bar{L}^2$. Moreover,

$$\|\bar{M}_n - \bar{M}_o\| \leq \frac{1}{\sqrt{c}} (R\bar{M}_n - \bar{H}_a, \bar{M}_o) \quad . \tag{3.8}$$

Proof. For $\bar{M} \in \bar{L}^2$ with sufficiently large $\|\bar{M}\|$ we have $\|\bar{M}\| > 1/c \, \|R\bar{O} - \bar{H}_a\| > 0$, which gives, using (3.4),

$$(R\bar{M} - \bar{H}_a, \bar{M}) = (R\bar{M} - R\bar{O}, \bar{M}) + (R\bar{O} - \bar{H}_a, \bar{M}) \geq c\|\bar{M}\|^2 - |(R\bar{O} - \bar{H}_a, \bar{M})|$$

$$\geq \|\bar{M}\| \left(c\|\bar{M}\| - \|R\bar{O} - \bar{H}_a\|\right) > 0 \quad .$$

Now R satisfies the conditions of Theorem 23.3 [16], which gives the conclusions asserted above.

Consider now the particular case of the isotropic magnetic material when h has the form

$$(h\bar{M})(x) = H(M(x), x) \frac{\bar{M}(x)}{M(x)} \quad , \tag{3.9}$$

where $M = |\bar{M}|$, and satisfies (3.4) and (3.5).

Definition. An operator $R : \bar{L}^2 \to \bar{L}^2$ is called *potential* if there exists a function $F(\bar{M})$ on $\bar{L}^2$ such that $R\bar{M} = \operatorname{grad} F(\bar{M})$.

Lemma 3.1.

If $h : \bar{L}^2 \to \bar{L}^2$ satisfies (3.9), then $R : \bar{L}^2 \to \bar{L}^2$ is potential

Proof. Consider on $\bar{L}^2$ the functional

$$F(\bar{M}) = \int_\Omega dx \int_0^{M(x)} H(M, x) \, dM + \tfrac{1}{2} (A\bar{M}, \bar{M}) - (\bar{H}_a, \bar{M}) \quad . \tag{3.10}$$

Let us calculate $\operatorname{grad} F(\bar{M})$. For arbitrary $\bar{L} \in \bar{L}^2$, we have

$$(\text{grad } F(\bar{M}),\bar{L}) = \frac{d}{dt}\left(F(\bar{M}+t\bar{L})\right)_{t=0}$$

$$= \frac{d}{dt}\left[\int_0^{|\bar{M}(x)+t\bar{L}(x)|} H(M,x)\,dM + \tfrac{1}{2}(A(\bar{M}+t\bar{L}),\bar{M}+t\bar{L}) - (\bar{H}_a,\bar{M}+t\bar{L})\right]\Bigg|_{t=0}$$

$$= \int_\Omega dx\; H(M(x),x)\,\frac{\bar{M}(x)\cdot\bar{L}(x)}{M(x)} + (A\bar{M},\bar{L}) - (\bar{H}_a,\bar{L}) \quad .$$

It follows that $\text{grad } F(\bar{M}) = R\bar{M} - \bar{H}_a$.

From (3.4), (3.5), and Lemma 3.1, by [16, p.113], it follows that the problem of minimizing $F(\bar{M})$ is equivalent to that of solving (3.2). We remark that functional (3.10) is the energy of the corresponding magnetic system and Eq. (3.2) is the condition that minimizes it.

For the numerical minimization of (3.10), the Ritz method can be used. In this case we seek the approximate solution $\bar{M}_n$ in the form (3.6), and the problem of minimizing $F(\bar{M})$ is replaced by the problem of minimizing each of the functions

$$\phi(\alpha_1,\alpha_2,\ldots,\alpha_n) = F(\bar{M}_n) = F\left(\sum_{k=1}^{n}\alpha_k\bar{\phi}_k\right) \quad . \tag{3.11}$$

of n variables $(n\to\infty)$. It is easy to see [16, p.168] that this problem is equivalent to that of solving (3.7), which is also called the Ritz system in this case.

Acknowledgments

The author wishes to express gratitude to Prof. Daniel Gallin for editing the manuscript. This work was done under the auspices of the U.S. Department of Energy under Contract W-7405-ENG-48.

REFERENCES

[1] H.N.Bertram, "On the convergence of iterative solutions of the integral magnetic field equation," IEEE Trans. on Mag. 11, No. 3, (1975) 928-933.

[2] C.J.Collie, N.J.Diserence, M.J.Newman, C.W.Trowbridge, Proceedings of 4th Conference on Magnetic Fields, Electrical Engineering Dept., College of Engineering, University of Nevada, Reno, Nevada (1973).

[3] J.S.Colonias, "Calculation of magnetic fields for engineering devices," IEEE Trans. on Mag. 12, No. 6, (1976) 1030-1035.

[4] A.A.Halacsy, Proceedings of 2nd Conference on Analysis of Magnetic Fields, Electrical Engineering Dept., College of Engineering, University of Nevada, Reno, Nevada (1969).

[5] A.A.Halacsy, Three-Dimensional Magnetic Fields, Nevada (1973).

[6] A.A.Halacsy, Proceedings of 4th Conference on Magnetic Fields, Electrical Engineering Dept., College of Engineering, University of Nevada, Reno, Nevada (1973).

[7] F.I.Hăntila, "A method of solving stationary magnetic fields in nonlinear media," Rev. Roumaine Sci. Tech. Ser. Electrotech. Energet 20, No. 3, (1975) 397-407.

[8] H.C.Karmaker, S.D.Robertson, "Integral equation formulation for the solution of magnetic field problems," IEEE Trans. PAS 92, No. 2, (1973) 824-835.

[9] S.G.Mikhlin, Multidimensional Singular Integrals and Integral Equations, (Phiz. Mat. Izdat., Moscow, 1962), in Russian.

[10] I.I.Pekker, "Calculation of magnetic systems by integration over field sources," Isv. VUZ Electromekh. 9, (1964) 1047-1051, in Russian.

[11] I.I.Pekker, "On calculation of magnetic systems by integration over field sources," Isv. VUZ Electromekh. 9, (1968) 940-943, in Russian.

[12] I.I.Pekker, M.I.Pekker, A.G.Kirsanov, "Calculation of magnetic torus with air gap by integration over field sources," Isv. VUZ Electromekh. 1, (1975) 11-15.

[13] M.I.Pekker, I.I.Pekker, "On satisfaction of boundary conditions when calculating magnetic systems by integration over field sources," Isv. VUZ Electromekh. 8, (1973) 904-909.

[14] R.J.Potter, "Digital magnetic recording theory," IEEE Trans. Mag. 10, (1974) 502-508.

[15] C.W.Steele, "Uniqueness theorem for static electric and magnetic field in saturable media," J. Appl. Phys. 44, (1973) 3790-3791.

[16] M.M.Veinberg, Variational and Monotone Operators Method, (Nauka, Moscow, 1972), in Russian.

MATHEMATICAL STUDY OF THE NONLINEAR SINGULAR INTEGRAL MAGNETIC FIELD EQUATION*

Mark J. Friedman

Computer Science and Applied Mathematics
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Part 2

Abstract

For the numerical treatment of the nonlinear singular integral magnetic field equation $R\bar{M} = h\bar{M} + A\bar{M} = \bar{H}_a$, which has been considered by the author in Part 1, Tucker stability is established in the case where $(h\bar{M})(x) = \bar{g}(\bar{M}(x),x)$ is a bounded, continuous, and strongly monotone operator in $\bar{L}^2$. In the special cases $\bar{g}(\bar{M},x) = c\bar{M}$ and $\bar{g}(\bar{M},x) = g(M,x)\,\bar{M}/M$, which are important for engineering applications explicit perturbation estimates are obtained.

---

*This work was performed under the auspices of the U.S. Department of Energy under Contract W-7405-ENG-48.

4. PRELIMINARIES

In this paper we continue the numeration of Part 1 [5]. This series of papers deals with theoretical analysis of the nonlinear integral equation

$$\bar{g}(\bar{M}(x),x) - \frac{1}{4\pi} \operatorname{grad} \operatorname{div} \int_{\Omega} \frac{\bar{M}(y)}{|x-y|} dy = \bar{H}_a(x) \quad , \qquad x \in \Omega \quad (1.1)$$

and the Galerkin method for its solution, $x = (x_1, x_2, x_3)$, Ω is a region in R^3, the bars refer to 3-vectors, $\bar{M}(x)$ is the magnetization, and $\bar{H}(x) = \bar{g}(\bar{M}(x),x)$ is the net field. (The key assumption is that $\bar{g}$ is monotone increasing in $\bar{M}$.) In the isotropic case, $\bar{H} = \frac{1}{\mu-1}\bar{M}$, $\bar{B} = \frac{\mu}{\mu-1}\bar{M}$, where $\mu = \mu(M,x)$ is the permeability, $M = |\bar{M}|$. $\bar{H}_a$ is a known "applied field" (usually produced by currents). Equation (1.1) is considered as the operator equation

$$R\bar{M} = h\bar{M} + A\bar{M} = \bar{H}_a \qquad (3.2)$$

in the Hilbert space of vector-functions $\bar{L}^2 = \bar{L}^2(\Omega)$, where $h = \bar{g}$ and A is the differential integral operator in the second term on the left of (1.1). By $\| \ \|$ and $(\ , \ )$ we denote norm and scalar product.

The validity and applicability of (1.1) has been discussed in Ref. [5]. The main results of [5] are given by the following two theorems.

Theorem 2.1.

Operator A is

i) bounded, with $\|A\| = 1$;

ii) self-adjoint ;

iii) positive semi-definite, with $\inf_{\|\bar{M}\| = 1} (A\bar{M},\bar{M}) = 0$.

Theorem 3.3.

Let $\bar{g}(\bar{M},x)$ be continuous with respect to $\bar{M}$ for almost all $x \in \Omega$, measurable in Ω with respect to x for all $\bar{M}$ and satisfy the inequalities

$$(\bar{g}(\bar{M}_1,x) - \bar{g}(\bar{M}_2,x)) \cdot (\bar{M}_1 - \bar{M}_2) \geqslant c(\bar{M}_1 - \bar{M}_2)^2 \quad , \tag{3.4}$$

for almost all $x \in \Omega$, where $\bar{M}_1, \bar{M}_2 \in R^3$ are arbitrary, $c > 0$;

$$|g_i(\bar{M},x)| \leqslant a_i(x) + b\sum_{k=1}^{3} |M_k| \quad , \qquad (i = 1,2,3) \quad , \tag{3.5}$$

where $a(x) \in L^2(\Omega)$, $b > 0$. Then the Galerkin approximations $\bar{M}_n = \sum_{k=1}^{n} \alpha_k \bar{\phi}_k$ exist for each n, and $\bar{M}_n$ converges to the unique solution $\bar{M}_o$ of (3.2).

The Galerkin process (3.7) for (3.2) can be written in the form

$$R_n\alpha^{(n)} = (G_n + A_n)\alpha^{(n)} = H^{(n)} \quad , \tag{4.1}$$

where $\alpha^{(n)} = (\alpha_1, \ldots, \alpha_n)^T$, $H^{(n)} = ((\bar{H}_a,\bar{\phi}_1), \ldots, (\bar{H}_a,\bar{\phi}_n))^T$,
$G_n\alpha^{(n)} = \left(\bar{g}(\sum_{k=1}^{n} \alpha_k \bar{\phi}_k),\bar{\phi}_1\right), \ldots, \left(\bar{g}(\sum_{k=1}^{n} \alpha_k\bar{\phi}_k)\bar{\phi}_n\right)^T$; $A_n = (A\bar{\phi}_k,\bar{\phi}_\ell)_{k,\ell=1}^{n}$.

Let $\tilde{\gamma}_{k\ell}$ denote the errors arising in the computation of $G_n + A_n$, $\tilde{\Gamma}_n = (\tilde{\gamma}_{k\ell})$ be the error matrix, and $\delta^{(n)}$ be the corresponding error in $H^{(n)}$. (The discrepancy arising in the approximate solution of Eq. (4.1) can also be included in $\delta^{(n)}$.) Then instead of the exact Galerkin process (4.1), we solve the "nonexact" one

$$\tilde{R}\,\tilde{\alpha}^{(n)} = (G_n + A_n + \tilde{\Gamma}_n)\tilde{\alpha}^{(n)} = H^{(n)} + \delta^{(n)} \tag{4.2}$$

and obtain the nonexact solution

$$\tilde{\bar{M}}_n(x) = \sum_{k=1}^{n} \tilde{\alpha}_k \bar{\phi}_k(x) \quad . \tag{4.3}$$

Let us denote by $\bar{L}_n^2$ the finite-dimensional subspace of $\bar{L}^2$ spanned by the functions $\bar{\phi}_1(x), \dots, \bar{\phi}_n(x)$, and let P_n be the orthogonal projection of $\bar{L}^2$ onto $\bar{L}_n^2$ for each $n \geq 1$. Now the Galerkin process (4.1) can be written in $\bar{L}_n^2$ as

$$P_n R\bar{M}_n = P_n h\bar{M}_n + P_n A\bar{M}_n = P_n \bar{H}_a \quad . \tag{4.4}$$

To write the process (4.4) in $\bar{L}_n^2$ we use the approach of Ref. [2, p.260]. Let $\bar{S}_n : \bar{L}^2 \to R^n$ be the linear operator defined by $\bar{S}_n\bar{M} = ((\bar{M},\bar{\phi}_1), \dots, (\bar{M},\bar{\phi}_n))^T$ for each $\bar{M} \in \bar{L}^2$ and let S_n be the restriction of $\bar{S}_n$ to $\bar{L}_n^2$. It follows that $\| S_n \| = \| \bar{S}_n \|$, S_n^{-1} exists and $S_n^{-1}\bar{S}_n = P_n$. Moreover, if $\bar{M}_n = \sum_{i=1}^{n} \beta_i \bar{\phi}_i$ is any element in $\bar{L}_n^2$, then $S_n\bar{M}_n = K_n \beta^{(n)}$, where K_n is the Gramm matrix given by $K_n = ((\phi_k,\phi_\ell))_{k,\ell=1}^n$, for each $n \geq 1$. Since $\beta^{(n)} = K_n^{-1} S_n \bar{M}_n$, the above discussion implies that the perturbed process (4.2) can be written in $\bar{L}_n^2$ as

$$(P_n R + \Delta R_n)\tilde{\bar{M}}_n = P_n h\tilde{\bar{M}}_n + P_n A\tilde{\bar{M}}_n + \Delta R_n \tilde{\bar{M}}_n = \bar{P}_n \bar{H}_a + \bar{h}_n \quad , \tag{4.5}$$

where $\quad \bar{h}_n = S_n^{-1}\delta^{(n)} \in \bar{L}_n^2 \quad$, and $\quad \Delta R_n = S_n^{-1}\tilde{\Gamma}_n K_n^{-1} S_n : \bar{L}_n^2 \to \bar{L}_n^2$.

In order to investigate the stability of a general nonlinear process one can apply a theorem of Tucker [8]. His definition of stability is based on the one of Mikhlin [3]. We would also like to refer to a paper by Omodei and Anderssen [4], where Tucker stability have been investigated for certain classes of nonlinear elliptic boundary value problems; and a paper by Hertling and Schiop [1], where the Tucker stability has been established for some classes of Hammerstein equations.

In section 5 we establish Tucker stability for the Galerkin process (4.4) in the general case wherein the operator h is bounded, continuous, and strongly monotone. The Tucker stability provides the stability of the numerical process. However, for various applications, the explicit perturbation estimates can be useful. Following basically the approach of Mikhlin [3] and Krasnoselskii and others [2], we obtain in Section 6 such perturbation estimates for important special cases. In the case where $\bar{g}(\bar{M},x) \equiv c\bar{M}$, $c = \text{const} > 0$, we obtain detailed perturbation estimates and illustrate by an example their usefulness for engineering applications. Then we obtain some estimates in the case $\bar{g}(\bar{M},x) \equiv g(M,x)\,\bar{M}/M$, certain natural restrictions being imposed on g.

Remark 4.1. We consider coordinate functions $\{\bar{\phi}_k\}_{k=1}^{\infty}$, where the k^{th} coordinate function $\bar{\phi}_k$ is independent of the dimension of the subspace in which the approximate solution is sought. However, if piece-wise polynomial coordinate functions are used, then the accuracy of the Ritz-Galerkin approximations is improved by refining the mesh, and this leads to a completely new set of coordinate functions. It is easy to show that all our results are valid in the latter case as well.

5. TUCKER STABILITY

<u>Definition 5.1</u> [7]. An operator $\tilde{R}_n$ is said to lie in an $\Omega_n \equiv (\alpha^{(n)}, r_n, b_n)$ neighborhood of R_n if $\tilde{R}_n - R_n = b_n U_n$, where U_n is nonexpansive in $K_n(\alpha^{(n)}, r_n) = \{\beta^{(n)} : \|\beta^{(n)} - \alpha^{(n)}\|_n \leq r_n\}$ (where $\| \ \|_n$ denotes the Euclidean norm) and $\|U_n \alpha^{(n)}\|_n \leq \|\alpha^{(n)}\|_n$ independently of n.

Then the stability definition is as follows:

<u>Definition 5.2</u> [7]. The numerical process (4.1) is stable at $\{\alpha^{(n)}\}_{n=1}^{\infty}$ if for each r_n there exist neighborhoods $V_n(0,\eta_n)$, numbers p_n and constants s,t such that, if $\tilde{R}_n$ is in an $\Omega_n \equiv (\alpha^{(n)}, r_n, b_n)$ neighborhood of R_n with $b_n \leq p_n$ and $\delta^{(n)} \in V_n$, then the perturbed numerical process (4.2) is solvable and

$$\|\tilde{\alpha}^{(n)} - \alpha^{(n)}\|_n \leq s\, b_n + t\|\delta^{(n)}\|_n \quad ,$$

where s and t aredindependent of n but may depend on the sequence $\{\alpha^{(n)}\}_{n=1}^{\infty}$.

<u>Theorem 5.1</u>. If the coordinate system $\{\bar{\phi}_k\}_{k=1}^{n}$ is strongly minimal in $\bar{L}^2$ in the sense of Mikhlin [3], and the operator h satisfies conditions (3.4) and (3.5), then the Galerkin process (4.1) for Eq. (3.2) is Tucker stable.

<u>Proof</u>. Equation (3.5) gives the continuity of h. Together with $\|A\| = 1$ by Theorem 2.1, this gives the continuity of R_n.

By Theorem 3.3, $\bar{M}_n$ converges to $\bar{M}_o$ in $\bar{L}^2$, and hence there exists a constant λ_o independent of i, such that

$$0 < \lambda_o \leq \frac{\|\bar{M}_n\|^2}{\|\alpha^{(n)}\|_n^2} \quad .$$

It follows that $\|\alpha^{(n)}\|$ are bounded above independently of n.

By (3.4), (3.7) and (4.1) the strong minimality of the coordinate system, for all $\alpha^{(n)}, \beta^{(n)} \in R^n$ we have the following chain of inequalities:

$$\|R_n \alpha^{(n)} - R_n \beta^{(n)}\|_n = \left\|\bar{g}\left(\sum_{k=1}^{n} \alpha_k \bar{\phi}_k\right) - \bar{g}\left(\sum_{k=1}^{n} \beta_k \bar{\phi}_k\right) + \sum_{k=1}^{n} A\bar{\phi}_k(\alpha_k - \beta_k)\right\|$$

$$\geq c \left\| \sum_{k=1}^{n} (\alpha_k - \beta_k)\bar{\phi}_k \right\| \geq \lambda_0^{\frac{1}{2}} c \|\alpha^{(n)} - \beta^{(n)}\|_n \quad . \tag{5.1}$$

We also used here the inequality

$$\|\bar{g}(\bar{M}) - \bar{g}(\bar{L})\| \ \|\bar{M} - \bar{L}\| \geq (\bar{g}(\bar{M}) - \bar{g}(\bar{L}), \bar{M} - \bar{L}) \quad .$$

Tucker has proven [7] that the continuity of R_n, the uniform boundedness of $\{\|\alpha^{(n)}\|_n\}$, together with (5.1), ensure that the numerical process (4.1) is stable.

Remark 5.1. The Tucker stability of the numerical process (4.4) can be shown similarly.

6. THE PERTURBATION ESTIMATES

Let $\bar{H} = \bar{g}(\bar{M},x) \equiv \frac{1}{\mu-1}\bar{M}$ (or $\bar{B} = \mu\bar{H}$), μ being known as the permeability of the magnetic material. For simplicity we suppose here that the coordinate systems $\{\bar{\phi}_k\}_{k=1}^n$ are orthonormal which implies $K_n \equiv I_n$, where I_n is an identity matrix; and that for any $\bar{M} \in \bar{L}_n^2$, $0 < \lambda_0 \leq (P_n A\bar{M},\bar{M}) \leq \Lambda_0 < 1$, where λ_0,Λ_0 may depend on coordinate systems. The latter assumption is justified as follows. By Theorem 2.1, $0 \leq (A\bar{M},\bar{M}) \leq (\bar{M},\bar{M})$

for any $\bar{M} \in \bar{L}_n^2$. It can be shown (we shall do it in Part 3), that $\text{Ker } A = \{\bar{u} : \bar{u} = \text{rot } \bar{v},\ \ \bar{v} \in \bar{w}_2^1(\Omega),\ \ \text{div } \bar{v} = 0,\ \ \bar{v} \times \bar{n}|_s = 0\}$, $\text{Ker}(I - A) = \{\bar{u} : \bar{u} = \text{grad } \phi,\ \ \phi \in w_2^1(\Omega),\ \phi|_s = 0\}$. It is easy to choose $\bar{\phi}_k$ so that $\bar{\phi}_k \notin \text{Ker } A \cup \text{Ker}(I-A)$ as, for example, in the case when $\bar{\phi}_k$ are the characteristic functions of parallelpipeds.

1) $\underline{\mu = \text{const} > 1}$

Equation (3.2), the exact Ritz process (4.4), and the perturbed one (4.5) are rewritten, respectively, as

$$R\bar{M} \equiv \frac{1}{\mu - 1}\bar{M} + A\bar{M} = \bar{H}_a \qquad (\bar{M}, \bar{H}_a \in \bar{L}^2) \quad , \tag{6.1}$$

$$P_n R\bar{M} = \frac{1}{\mu - 1}\bar{M}_n + P_n A\bar{M} = P_n \bar{H}_a \quad , \tag{6.2}$$

$$(P_n R + \Delta R_n)\tilde{\bar{M}}_n \equiv \frac{1}{\mu + \Delta\mu - 1}\tilde{\bar{M}}_n + P_n A\tilde{\bar{M}}_n + F_n\tilde{\bar{M}}_n = P_n\bar{H}_a + \bar{h}_n. \tag{6.3}$$

Here $\Delta\mu$ is the maximum by modulus error in μ (the error in μ is usually known since μ is usually obtained from experiment). From physical considerations, $\mu + \Delta\mu - 1 > 0$; $\Gamma_n = (\gamma_{k\ell})$, where $\gamma_{k\ell}$ denotes the errors arising in the computation of $(A\bar{\phi}_k, \bar{\phi}_\ell)$. We shall suppose that Γ_n is self-adjoint; $F_n = S_n^{-1}\Gamma_n S_n$, $\|F_n\| = \|\Gamma_n\|$. We denote by $\bar{M}_o$, $\bar{M}_n = P_n\bar{M}_o$, $\tilde{\bar{M}}_n$ the solutions of (6.1), (6.2), and (6.3), respectively. Let us also set $\Delta = 1 + (\lambda_o - \|\Gamma_n\|)(\mu + \Delta\mu - 1)$.

Theorem 6.1. Let $\Delta > 0$ for all $n > n_o$. Then Eq. (6.3) is solvable and

$$\frac{\|\tilde{\bar{M}}_n - \bar{M}_n\|}{\|\bar{M}_n\|} \leqslant \Delta^{-1}\left[\frac{|\Delta\mu|}{\mu-1} + (\mu+\Delta\mu-1)\,\|\Gamma_n\| + [1+\Lambda_o(\mu-1)]\left(1+\frac{\Delta\mu}{\mu-1}\right)\frac{\|\delta^{(n)}\|}{\|P_n\bar{H}_a\|}\right], \tag{6.4}$$

$$\frac{\|\tilde{\bar{M}}_n - \bar{M}_o\|}{\|\bar{M}_o\|} \leqslant \Delta^{-1}\left[\frac{|\Delta\mu|}{\mu-1} + (\mu+\Delta\mu-1)\,\|\Gamma_n\| + \left(1+\frac{\Delta\mu}{\mu-1}\right)\mu\,\frac{\|\delta^{(n)}\|}{\|P_n\bar{H}_a\|} + \frac{\|(I-P_n)\bar{M}_o\|}{\|\bar{M}_o\|}\right], \tag{6.5}$$

$$\frac{\mu-1}{\mu}\,\|R\tilde{\bar{M}}_n - \bar{H}_a\| \leqslant \|\tilde{\bar{M}}_n - \bar{M}_o\| \leqslant (\mu-1)\,\|R\tilde{\bar{M}}_n - \bar{H}_a\| \quad . \tag{6.6}$$

Proof. The spectral theory for self-adjoint operators [6] implies that

$$\left\|\left(P_nA + \frac{1}{\mu+\Delta\mu-1}\,I\right)^{-1}\right\|^{-1} = \min_{\lambda\in\{\lambda_o,\ldots,\Lambda_o\}}\left|\lambda + \frac{1}{\mu+\Delta\mu-1}\right| = \lambda_o + \frac{1}{\mu+\Delta\mu-1}\,, \tag{6.7}$$

in addition $\Delta > 0$ implies $\|\Gamma_n\| < \lambda_o + 1/(\mu+\Delta\mu-1)$. Together with (6.7), this gives $\|\Gamma_n\|\left\|\left(P_nA + \frac{1}{\mu+\Delta\mu-1}\,I\right)^{-1}\right\| < 1$, and thus by a well known theorem the process (6.3) is solvable. Transforming (6.2),

$$(P_nR + \Delta R_n)\bar{M}_n = \Delta R_n\bar{M}_n + P_n\bar{H}_a$$

and subtracting the result from (6.3) gives

$$(P_nR + \Delta R_n)(\tilde{\bar{M}}_n - \bar{M}_n) = -\Delta R\bar{M}_n + \bar{h}_n \tag{6.8}$$

or

$$\tilde{\bar{M}}_n - \bar{M}_n = (P_nR + \Delta R_n)^{-1}\,(-\Delta R_n\bar{M}_n + \bar{h}_n) \tag{6.9}$$

and

$$\tilde{\bar{M}}_n - \bar{M}_o = (P_nR + \Delta R_n)^{-1}(-\Delta R_n\bar{M}_n + \bar{h}_n) - (I - P_n)\bar{M}_o \quad . \tag{6.10}$$

Equation (6.9) gives

$$\frac{\|\tilde{\bar{M}}_n - \bar{M}_n\|}{\|\bar{M}_n\|} \leqslant \left\|(P_nR + \Delta R_n)^{-1}\Delta R_n\right\| + \left\|(P_nR+\Delta R_n)^{-1}\right\| \frac{\|\delta^{(n)}\|}{\|\bar{M}_n\|} . \qquad (6.11)$$

From (6.7) and (6.3), by the perturbation theory for symmetric matrices [8],

$$\left\|(P_nR+\Delta R_n)^{-1}\right\| \leqslant \left(\frac{1}{\mu+\Delta\mu-1} + \lambda_o - \|\Gamma_n\|\right)^{-1} = \frac{\mu+\Delta\mu-1}{\Delta} . \qquad (6.12)$$

Further,

$$\Delta R_n = -\frac{\Delta\mu}{(\mu+\Delta\mu-1)(\mu-1)} I + F_n ,$$

$$\|\Delta R_n\| \leqslant \frac{|\Delta\mu|}{(\mu+\Delta\mu-1)(\mu-1)} + \|\Gamma_n\| . \qquad (6.13)$$

From Eq. (6.2),

$$\|P_nR\| \, \|\bar{M}_n\| \geqslant \|P_n\bar{H}_a\| , \qquad \|P_nR\| = \frac{1}{\mu-1} + \Lambda_o .$$

It follows that

$$\|\bar{M}_n\| \geqslant \frac{\mu-1}{1+\Lambda_o(\mu-1)} \|P_n\bar{H}_a\| . \qquad (6.14)$$

Now (6.4) follows from (6.11) to (6.14).

Taking into account

$$\|\bar{M}_o\| \geqslant \frac{\mu-1}{\mu}\|\bar{H}_a\| , \qquad (6.15)$$

we obtain (6.5) from (6.10), (6.12) and (6.13). Further, (6.1) implies

$$R(\tilde{\bar{M}}_n - \bar{M}_o) = R\tilde{\bar{M}}_n - \bar{H}_a , \qquad \tilde{\bar{M}}_n - \bar{M}_o = R^{-1}(R\tilde{\bar{M}}_n - \bar{H}_a) ,$$

which gives (6.6), and the theorem is proven.

Remark 6.1. The above analysis may give some other useful perturbation estimates. For example, (6.8) implies the estimate of $\|\tilde{\bar{M}}_n - \bar{M}_n\|$ from below in the case that Γ_n is not necessarily symmetric.

Remark 6.2. The inequality (6.5) implies the condition for consistency of the errors of various types:

$$\frac{|\Delta\mu|}{\mu - 1} \leqslant (\mu + \Delta\mu - 1)\,\|\Gamma_n\| = \mu\left(1 + \frac{\Delta\mu}{\mu - 1}\right)\frac{\|\delta^{(n)}\|}{\|\bar{H}_a\|} = \Delta\,\frac{\|(I - P_n)\bar{M}_o\|}{\|\bar{M}_o\|}$$

$$< \; 1 + \lambda_o(\mu + \Delta\mu - 1) \quad .$$

Remark 6.3. It is easy to show that the estimates (6.4) to (6.6) cannot be improved.

Remark 6.4. The remarks analogous to the ones 6.1 to 6.3 are also valid for the following theorems.

Formulating (6.1) - (6.3) in terms of $\bar{B}$ gives

$$S\bar{B} \;\equiv\; \frac{1}{\mu}\,\bar{B} + \frac{\mu - 1}{\mu}\,A\bar{B} + \bar{H}_a \quad , \tag{6.16}$$

$$P_n S\bar{B}_n \;=\; \frac{1}{\mu}\,\bar{B}_n + \frac{\mu - 1}{\mu}\,P_n A\bar{B}_n \;=\; P_n\bar{H}_a \quad , \tag{6.17}$$

$$(P_n S + \Delta S_n)\tilde{\bar{B}}_n \;\equiv\; \frac{1}{\mu + \Delta\mu}\,\tilde{\bar{B}}_n + \frac{\mu + \Delta\mu - 1}{\mu + \Delta\mu}\,(P_n A\tilde{\bar{B}}_n + F_n\tilde{\bar{B}}_n) \;=\; P_n\bar{H}_a + \bar{h}_n \;. \tag{6.18}$$

We denote by $\bar{B}_o$, $\bar{B}_n = P_n\bar{B}_o$, $\tilde{\bar{B}}_n$ the solutions of (6.16) to (6.18), respectively.

Theorem 6.2. Let $\Delta > 0$ for all $n > n_o$. Then Eq. (6.18) is solvable and

$$\frac{\|\tilde{\bar{B}}_n - \bar{B}_n\|}{\|\bar{B}_n\|} \leqslant \Delta^{-1}\left[(1-\lambda_o)\,\frac{|\Delta\mu|}{\mu} + (\mu+\Delta\mu-1)\,\|\Gamma_n\| + [1+\Lambda_o(\mu-1)]\left(1+\frac{\Delta\mu}{\mu}\right)\frac{\|\delta^{(n)}\|}{\|P_n\bar{H}_a\|}\right], \tag{6.19}$$

$$\frac{\|\tilde{\bar{B}}_n - \bar{B}_o\|}{\|\bar{B}_n\|} \leqslant \Delta^{-1}\left[(1-\lambda_o)\,\frac{|\Delta\mu|}{\mu} + (\mu+\Delta\mu-1)\,\|\Gamma_n\| + (\mu-\Delta\mu)\,\frac{\|\delta^{(n)}\|}{\|\bar{H}_a\|}\right] + \frac{\|(I-P_n)\bar{B}_o\|}{\|\bar{B}_o\|} \quad , \tag{6.20}$$

$$\|S\tilde{\bar{B}}_n - \bar{H}_a\| \leqslant \|\tilde{\bar{B}}_n - \bar{B}_o\| \leqslant \mu\,\|S\tilde{\bar{B}}_n - \bar{H}_a\| \quad . \tag{6.21}$$

The proof is completely analogous to that of Theorem 6.1.

Formulating (6.1) to (6.3) in terms of $\bar{H}$ gives

$$T\bar{H} \equiv \bar{H} + (\mu-1)A\bar{H} = \bar{H}_a \quad , \tag{6.22}$$

$$P_n T\bar{H}_n = \bar{H}_n + (\mu-1)P_n AH_n = P_n\bar{H}_a \quad , \tag{6.23}$$

$$(P_nT + \Delta T_n)\tilde{\bar{H}}_n \equiv \tilde{\bar{H}}_n + (\mu+\Delta\mu-1)(P_nA\tilde{\bar{H}}_n + F_n\bar{H}_n) = P_n\bar{H}_a + \bar{h}_n \quad . \tag{6.24}$$

We denote by $\bar{H}_o$, $\bar{H}_n = P_n\bar{H}_o$, $\tilde{\bar{H}}_n$ the solutions of (6.22) to (6.24), respectively.

Theorem 6.3. Let $\Delta > 0$ for all $n < n_o$. Then Eq. (6.24) is solvable and

$$\frac{\|\tilde{\bar{H}}_n - \bar{H}_n\|}{\|\bar{H}_n\|} < \frac{\Lambda_o|\Delta\mu|}{1 + c(\Delta - 1)} + \Delta^{-1}\left[(\mu + \Delta\mu - 1)\ \|\Gamma_n\| + [1 + \Lambda_o(\mu - 1)]\frac{\delta^{(n)}}{P_n\bar{H}_a}\right], \tag{6.25}$$

$$\frac{\|\tilde{\bar{H}}_n - \bar{H}_o\|}{\|\bar{H}_o\|} < \frac{\Lambda_o|\Delta\mu|}{1 + c(\Delta - 1)} + \Delta^{-1}\left[(\mu + \Delta\mu - 1)\ \|\Gamma_n\| + \mu\frac{\|\delta^{(n)}\|}{\|P_n\bar{H}_a\|}\right] + \frac{\|(I - P_n)\bar{H}_o\|}{\|\bar{H}_o\|}, \tag{6.26}$$

where
$$c = \begin{cases} 1\,, & \text{if } \lambda_o - \|\Gamma_n\| \leqslant 0\,, \\ \dfrac{\Lambda_o}{\lambda_o}\,, & \text{if } \lambda_o - \|\Gamma_n\| > 0\,, \end{cases}$$

$$\frac{1}{\mu}\|T\tilde{\bar{H}}_n - \bar{H}_a\| \leqslant \|\tilde{\bar{H}}_n - \bar{H}_o\| \leqslant \|T\tilde{\bar{H}}_n - \bar{H}_a\|\ . \tag{6.27}$$

The proof is analogous to that of Theorem 6.1. The distinction is that $\|(P_nT + \Delta T_n)^{-1}P_nA\|$ can be estimated in two ways: either by using the same method as in the proof of Theorem 6.1:

$$\|(P_nT + \Delta T_n)^{-1}P_nA\| \leqslant \|(P_nT + \Delta T_n)^{-1}\|\ \|P_nA\| \leqslant \frac{\Lambda_o}{\Delta}\,,$$

or by using

$$\|[(P_nA)^{-1} + (\mu + \Delta\mu - 1)I]^{-1}\| = \left(\frac{1}{\Lambda_o} + \mu + \Delta\mu - 1\right)^{-1},$$

as

$$\|(P_nT+\Delta T_n)^{-1}P_nA\| = \left\|[(P_nA)^{-1} + (\mu+\Delta\mu-1)(I-(P_nA)^{-1}F_n)^{-1}]\right\| \leq \frac{\Lambda_o}{1+\frac{\Lambda_o}{\lambda_o}(\Delta-1)} .$$

To illustrate the engineering applications of the above theorems we consider the following

Example 6.1. It is often required to estimate both the magnetization and induction, i.e., the pair $(\bar{M},\bar{B})$ inside magnetic material. Since it might be too expensive to solve both Eqs. (6.3) and (6.18), as a rule one proceeds in either of two ways: (i) one solves (6.3) and obtains the pair $\left(\tilde{\bar{M}}_n, \tilde{\tilde{\bar{B}}}_n \equiv \frac{\mu+\Delta\mu}{\mu+\Delta\mu-1}\tilde{\bar{M}}_n\right)$, or (ii) one solves (6.18) and obtains the pair $\left(\tilde{\tilde{\bar{M}}}_n \equiv \frac{\mu+\Delta\mu-1}{\mu+\Delta\mu}\tilde{\bar{B}}_n, \tilde{\bar{B}}_n\right)$. The question is: which way is preferable, i.e., which of the latter two pairs is closer to the pair $(\bar{M}_n,\bar{B}_n)$?

If we solve (6.3), then from the identity

$$\tilde{\tilde{\bar{B}}}_n - \bar{B}_n = \frac{\mu+\Delta\mu}{\mu+\Delta\mu-1}\left(-\frac{\mu+\Delta\mu-1}{\mu+\Delta\mu}\bar{B}_n + \tilde{\bar{M}}_n\right) = \frac{\mu+\Delta\mu}{\mu+\Delta\mu-1}\left(\frac{\mu-1}{\mu}\bar{B}_n - \frac{\mu+\Delta\mu-1}{\mu+\Delta\mu}\bar{B}_n + \tilde{\bar{M}}_n - \bar{M}_n\right)$$

$$= \frac{\mu+\Delta\mu}{\mu+\Delta\mu-1}\left[\frac{-\Delta\mu}{\mu(\mu+\Delta\mu)}\bar{B}_n + \frac{\mu-1}{\mu}\left(\frac{\mu-1}{\mu}\right)^{-1}(\tilde{\bar{M}}_n - \bar{M}_n)\right]$$

$$= \frac{-\Delta\mu}{(\mu+\Delta\mu-1)}\bar{B}_n + \frac{(\mu+\Delta\mu)(\mu-1)}{(\mu+\Delta\mu-1)\mu}\left(\frac{\mu-1}{\mu}\right)^{-1}(\tilde{\bar{M}}_n - \bar{M}_n)$$

we obtain the error estimate for $\tilde{\tilde{\bar{B}}}_n$

$$\frac{\|\tilde{\tilde{\bar{B}}}_n - \bar{B}_n\|}{\|\bar{B}_n\|} \leq \frac{(\mu+\Delta\mu)(\mu-1)}{(\mu+\Delta\mu-1)\mu}\frac{\|\tilde{\bar{M}}_n - \bar{M}_n\|}{\|\bar{M}_n\|} + \frac{|\Delta\mu|}{\mu(\mu+\Delta\mu-1)} . \tag{6.28}$$

If we solve (6.18), then from the identity

$$\bar{\tilde{\tilde{M}}}_n - \bar{M}_n = \frac{-\Delta\mu}{(\mu+\Delta\mu)(\mu-1)} \bar{M}_n + \frac{(\mu+\Delta\mu-1)\mu}{(\mu+\Delta\mu)(\mu-1)} \left(\frac{\mu}{\mu-1}\right)^{-1} (\bar{\tilde{B}}_n - \bar{B}_n)$$

we obtain the error estimate for $\bar{\tilde{\tilde{M}}}_n$

$$\frac{\|\bar{\tilde{\tilde{M}}}_n - \bar{M}_n\|}{\|\bar{M}_n\|} \leqslant \frac{(\mu+\Delta\mu-1)\mu}{(\mu+\Delta\mu)(\mu-1)} \frac{\|\bar{\tilde{B}}_n - \bar{B}_n\|}{\|\bar{B}_n\|} + \frac{|\Delta\mu|}{(\mu+\Delta\mu)(\mu-1)} . \tag{6.29}$$

From (6.28) and (6.4),

$$\frac{\|\bar{\tilde{B}}_n - \bar{B}_n\|}{\|\bar{B}_n\|} \leqslant \left\{ \frac{1}{1-\frac{1}{\mu+\Delta\mu}} \frac{|\Delta\mu|}{\mu} + \frac{1+\frac{\Delta\mu}{\mu}}{1+\frac{\Delta\mu}{\mu-1}} (\mu+\Delta\mu-1) \|\Gamma_n\| \right.$$

$$\left. + [1+\Lambda_o(\mu-1)] \left(1+\frac{\Delta\mu}{\mu}\right) \frac{\|\delta^{(n)}\|}{\|P_n\bar{H}_a\|} + \frac{|\Delta\mu|}{(\mu+\Delta\mu)(\mu-1)} \right\} \Delta^{-1} . \tag{6.30}$$

From (6.29) and (6.19),

$$\frac{\|\bar{\tilde{\tilde{M}}}_n - \bar{M}_n\|}{\|\bar{M}_n\|} \leqslant \left\{ \left(1-\frac{1}{\mu+\Delta\mu}\right)(1-\lambda_o) \frac{|\Delta\mu|}{\mu-1} + \frac{1+\frac{\Delta\mu}{\mu-1}}{1+\frac{\Delta\mu}{\mu}} (\mu+\Delta\mu-1) \|\Gamma_n\| \right.$$

$$\left. + [1+\Lambda_o(\mu-1)] \left(1+\frac{\Delta\mu}{\mu-1}\right) \frac{\|\delta^{(n)}\|}{\|P_n\bar{H}_a\|} + \frac{|\Delta\mu|}{(\mu+\Delta\mu)(\mu-1)} \right\} \Delta^{-1} . \tag{6.31}$$

Now, comparing (6.30) with (6.19) and (6.31) with (6.4), we can answer the primary question in various situations. If, for example, $\mu+\Delta\mu$ is close to 1, then the perturbation error (6.30) can become much bigger than the one (6.19); on the other hand, the estimates (6.31) and

(6.4) can be close. In this situation the pair $(\tilde{\tilde{\bar{M}}}_n, \tilde{\bar{B}}_n)$ seems to be closer to the pair $(\bar{M}_n, \bar{B}_n)$ than the one $(\tilde{\bar{M}}_n, \tilde{\tilde{\bar{B}}}_n)$; i.e., procedure (ii), solving (6.18), is preferable.

2) $\underline{\mu = \mu(H,x)}$

Let $\mu(H,x)$ be a function such that $\frac{\partial}{\partial H}(\mu H)$ exists and is piecewise continuous with respect to H for all $x \in \Omega$ and measurable in Ω with respect to x for every fixed $H \in [0,\infty)$. Let us set $\mu_d = \partial B/\partial H$. We now require

$$1 < \mu_{min} \leqslant \mu, \mu_d \leqslant \mu_{max} < +\infty \ , \ |\Delta\mu_d| \leqslant |\Delta\mu| \quad . \tag{6.32}$$

The error estimates for the perturbed Ritz process can be obtained on the basis of the following

Lemma 6.1. [2, p.293] Let C be an operator in a Banach space F which is Frèchet-differentiable for $\|x - x_*\| \leqslant \delta_*$ where x_* is a fixed point of F, $\delta_* > 0$. Assume that the linear operator $C'(x_*)$ is continuously invertible in F, and for some δ_o and q $(0 < \delta_o \leqslant \delta_*, \ 0 \leqslant q < 1)$,

$$\sup_{\|x - x_*\| \leqslant \delta_o} \| [C'(x_*)]^{-1} [C'(x) - C'(x_*)] \| \leqslant q \ , \tag{6.33}$$

$$\alpha = \| [C'(x_*)]^{-1} Cx_* \| \leqslant \delta_o(1-q) \quad . \tag{6.34}$$

Then the equation $Cx = 0$ has the unique solution x_o in the ball $\|x - x_*\| \leqslant \delta_o$, and x_o satisfies the estimate

$$\frac{\alpha}{1+q} \leqslant \|x_o - x_*\| \leqslant \frac{\alpha}{1-q} \quad . \tag{6.35}$$

To show how Lemma 6.1 works, we obtain (without detailed proof) some perturbation estimates, useful for the example 6.1, and indicate how to obtain others.

Calculating $(P_n R + \Delta R_n)'(\bar{M}_n)$ gives

$$(P_n R + \Delta R_n)'(\bar{M}_n)\bar{L} = \frac{d}{dt}(P_n R + \Delta R_n)(\bar{M} + t\bar{L})|_{t=0}$$

$$= P_n\left[(\alpha_d(M_n) - \alpha(M_n))\frac{(\bar{M}_n\cdot\bar{L})^2}{M_n^2}\bar{M}_n + \alpha(M_n)\bar{L} + A\bar{L}\right]$$

$$+ F_n\bar{L} \quad (\bar{L} \in \bar{L}_n^2) \quad . \tag{6.36}$$

where $\alpha_d(M_n) = [\mu_d(M_n) + \Delta\mu_d(M_n) - 1]^{-1}$, $\alpha(M_n) = [\mu(M_n) + \Delta\mu(M_n) - 1]^{-1}$. The operator (6.36) is self-adjoint. Let [3] $\Omega = \Omega^+ U \Omega^-$, where $\alpha_d(M_n) - \alpha(M_n) \geqslant 0$ for $x \in \Omega^+$ and $\alpha_d(M_n) - \alpha(M_n) < 0$ for $x \in \Omega$. Taking into account $(\bar{M}_n\cdot\bar{L})^2 \leqslant M_n^2 L^2$ gives

$$\int_{\Omega^+} P_n\left[(\alpha_d(M_n) - \alpha(M_n))\frac{(\bar{M}_n\cdot\bar{L})^2}{\bar{M}_n^2} + \alpha(M_n)\bar{L}^2\right]dx \geqslant \int_{\Omega^+} P_n\,\alpha(M_n)L^2\,dx$$

$$\geqslant (\mu_{max} + |\Delta\mu| - 1)^{-1}\,\|\bar{L}\|^2 \quad ,$$

$$\int_{\Omega^-} P_n\left[(\alpha_d(M_n) - \alpha(M_n))\frac{(\bar{M}_n\cdot\bar{L})^2}{M_n^2} + \alpha(M_n)L^2\right]dx \geqslant \int_{\Omega^-} P_n\alpha_d(M_n)L^2\,dx$$

$$\geqslant (\mu_{max} + |\Delta\mu| - 1)^{-1}\,\|\bar{L}\|^2 \quad ,$$

which leads to the estimate

$$\|[(P_n R + \Delta R_n)'(\bar{M}_n)]^{-1}\| \leqslant (\mu_{max} + |\Delta\mu| - 1)\,[1 + (\lambda_o - \|\Gamma_n\|)(\mu_{max} + \Delta\mu - 1)]^{-1} \quad . \tag{6.37}$$

The proof of the following estimates

$$\|\Delta R_n\| \leqslant (\mu_{min} + |\Delta\mu| - 1)^{-1} (\mu_{min} - 1)^{-1} |\Delta\mu| + \|\Gamma_n\| \quad , \tag{6.38}$$

$$\|\bar{M}_n\| \geqslant [1 + \Lambda_o(\mu_{min} - 1)]^{-1} (\mu_{min} - 1) \|P_n \bar{H}_a\| \quad , \tag{6.39}$$

is analogous to that of (6.13) and (6.14). The application of Lemma 6.1 with $F = \bar{L}_n^2$, $C = P_n R + \Delta R_n$, $x_* = \bar{M}_n$, $x_o = \tilde{\bar{M}}_n$, $\alpha \leqslant \alpha_{\bar{M}}$, $q = q_{\bar{M}}$ gives the estimate

$$\frac{\|\tilde{\bar{M}}_n - \bar{M}_n\|}{\|\bar{M}_n\|} \leqslant \frac{\alpha_{\bar{M}}}{1 - q_{\bar{M}}} \quad , \tag{6.40}$$

where the estimate $\alpha_{\bar{M}}$ for

$$\frac{\alpha_{\bar{M}}}{\|\bar{M}_n\|} = \frac{\mu_{max} + |\Delta\mu| - 1}{1 + (\lambda_o - \|\Gamma_n\|)(\mu_{max} + |\Delta\mu| - 1)}$$

$$\times \left[\frac{|\Delta\mu|}{(\mu_{min} + \Delta\mu - 1)(\mu_{min} - 1)} + \|\Gamma_n\| + \frac{1 + \Lambda_o(\mu_{min} - 1)}{\mu_{min} - 1} \frac{\|\delta^{(n)}\|}{\|P_n \bar{H}_a\|} \right] \tag{6.41}$$

is obtained from (6.37)-(6.39). The following estimates, (6.42) and (6.43), are obtained in the same way as those of (6.40) and (6.41), by applying Lemma 6.1 with $F = \bar{L}_n^2$, $C = P_n S + \Delta S_n$, $x_* = \bar{B}_n$, $x_o = \tilde{\bar{B}}_n$, $\alpha \leqslant \alpha_{\bar{B}}$, $q = q_{\bar{B}}$.

$$\frac{\alpha_{\bar{B}}}{\|\bar{B}_n\|} = \frac{\mu_{max} + |\Delta\mu|}{1 - (\lambda_o - \|\Gamma_n\|)(\mu_{max} + |\Delta\mu| - 1)} \left[\frac{(1-\lambda_o)|\Delta\mu|}{\mu_{min}(\mu_{min} + |\Delta\mu| - 1)} \right.$$

$$\left. + \frac{\mu_{max} + \Delta\mu - 1}{\mu_{max} + \Delta\mu} \|\Gamma\| + \frac{1 + \Lambda_o(\mu_{min} - 1)}{\mu_{min}} \frac{\|\delta^{(n)}\|}{\|P_n \bar{H}_a\|} \right] , \tag{6.42}$$

$$\frac{\|\tilde{\bar{B}}_n - \bar{B}_n\|}{\|\bar{B}_n\|} \leqslant \frac{\alpha_{\bar{B}}}{1 - q_{\bar{B}}} \quad . \tag{6.43}$$

Further, the estimate for $q_{\bar{M}}$ can be obtained from (6.36) and (6.37); the estimate for $q_{\bar{B}}$ can be obtained in the same way. To obtain the estimates similar to (6.28) and (6.29) we must use Lemma 6.1.

Acknowledgment

I wish to express gratitude to S. G. Mikhlin for bringing my attention to the Tucker stability and for his interest in my work.

REFERENCES

[1] J.Hertling, A.I.Schiop, "The stability of the Ritz-Galerkin method for Hammerstein equations," Math. Comp. 29 (1975) 486-488.

[2] M.A.Krasnoselskii, G.M.Vainikko, P.P.Zabreiko, J.B.Rututchii, and V.Ja.Stecenko, Approximate Solution of Operator Equations, (Walters-Noordhoff, Groningen, 1972).

[3] S.G.Mikhlin, The Numerical Performance of Variation Methods, (Walters-Noordhoff, Groningen, 1971).

[4] B.J.Omodei and R.S.Anderssen, "Stability of the Galerkin-Ritz procedure for nonlinear two point boundary value problems," Numer. Math. 24 (1975) 27-38.

[5] M.J.Friedman, "Mathematical study of the nonlinear singular integral magnetic field equation, Part I," (to appear in SIAM J. of Appl. Math.).

[6] W.Rudin, Functional Analysis (McGraw-Hill Book Co., New York, 1973).

[7] T.S.Tucker, "Stability of nonlinear computing schemes," SIAM J. Numer. Anal. 6 (1969) 72-81.

[8] J.M.Wilkinson, The Algebraic Eigenvalue Problem, (Clarendon Press, Oxford, 1975).

MATHEMATICAL STUDY OF THE NONLINEAR SINGULAR INTEGRAL MAGNETIC FIELD EQUATION*

Mark J. Friedman

Computer Science and Applied Mathematics
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Part 3

Abstract

We extend the results of Part 1 on the spectrum of the singular integral operator

$$(A\bar{M})(x) = -\frac{1}{4\pi}\,\text{grad div}\int_{\Omega}\frac{\bar{M}(y)}{r}\,dy \quad .$$

As an application we obtain an estimate of the lower bound of the spectrum of the magnetic field operator $R\bar{M} = h\bar{M} + A\bar{M}$ from $\bar{L}^2(\Omega)$ into the subspace J of generalized solenoidal vector-functions from $\bar{L}^2$. Here $\bar{M}$ is the magnetization vector, $h\bar{M} = \dfrac{\bar{M}}{\mu(M,x) - 1}$ is the total field, $A\bar{M}$ is the induced field, and Ω is a domain in R_3.

*This work was performed under the auspices of the U.S. Department of Energy under Contract W-7405-ENG-48.

7. INTRODUCTION

In this paper we keep the notation and the enumeration of Part 1 [5] and Part 2 [6].

In Part 1 [5] we began the investigation of the spectrum in $\bar{L}^2$ of the singular integral operator

$$(A\bar{M})(x) = \text{grad div } \bar{\psi}(x) \equiv -\frac{1}{4\pi} \text{grad div} \int_\Omega \frac{\bar{M}(y)}{r} dy , \quad x \in \Omega . \tag{2.11}$$

We showed there that, i) A is bounded, with $\|A\| = 1$; ii) A is self-adjoint; and iii) A is positive semi-definite, with $(A\bar{M},\bar{M}) \geq 0$. The present paper extends the results of Part 1 [5]. The principle result is given by Theorem 8.1, and follows from the classical potential theory, elementary properties of pseudodifferential operators on a compact manifold without edge, and the decomposition (8.19) of $\bar{L}^2$ into a direct sum [1].

Theorem 8.1 can have various applications to the investigation of the magnetic field equation

$$R\bar{M} = h\bar{M} + A\bar{M} = \bar{H}_A \tag{6.1}$$

and numerical methods for its solution. As an example we obtain an estimate of the lower bound of the spectrum of $R:\bar{L}^2 \to J$, where J is a subspace of generalized solenoidal vector-functions from $\bar{L}^2$. This choice is natural, since in applications we always have $\bar{H}_A \in J$. For simplicity, we consider the isotropic case. We denote by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,S}$ the norms in $H^s = H^s(\Omega)$, $H^s(S)$ (see [3] for the definition of these spaces. We shall also use the notation $\|\cdot\|_{0,\Omega} \equiv \|\cdot\|$, $H^0 = L^2$.

8. THE SPECTRUM OF A AND R

Let us introduce some subspaces of $\bar{L}^2$ [1].

$$\mathring{J} = \{\bar{M}:\bar{M} = \text{rot } \bar{F} , \ \bar{F} \in \bar{H}^1 , \ \text{div}\bar{F} = 0 , \ \bar{F} \times \bar{n}|_S = 0\} \tag{8.1}$$

$$\mathring{G} = \{\bar{M}:\bar{M} = \text{grad } \psi, \ \psi \in \bar{H}^1 , \ \psi|_S = 0 \} \tag{8.2}$$

$$U = \{\bar{M} : \bar{M} = \text{grad}\ \psi,\ \psi \in H^1,\ \Delta\psi = 0\} \tag{8.3}$$

Here $\bar{H}^1 = \bar{H}^1(\Omega)$ is the space of vector-functions $\bar{F} = (F_1, F_2, F_3)$, $F_i \in H^1$, $i = 1,2,3$.

Theorem 8.1.

Let the boundary S of Ω be twice continuously differentiable. Then

1. $\text{Ker}\ A = \overset{\circ}{J}$
2. $\text{Ker}(A - I) = \overset{\circ}{G}$
3. Within the interval (0,1) the spectrum of A is not more than countable: $\lambda = \frac{1}{2}$ is the unique limit point; each value $\lambda \neq \frac{1}{2}$ is regular or has a finite multiplicity; U is an invariant subspace of A, and the eigenfunctions of A in U form a complete orthogonal system in U.

We divide the proof into Lemmas 8.1 - 8.4:

Lemma 8.1. For any eigenvalue λ of A, the smooth eigenfunctions of A form a set which is dense in the set of all eigenfunctions of A in $\bar{L}^2$, corresponding to λ.

Proof — Let $\lambda, \bar{M}$ satisfy

$$A\bar{M} - \lambda\bar{M} = 0 \tag{8.4}$$

Setting $\bar{M}(x) \equiv 0$ for any $x \in R_3 - \Omega$, (8.4) is rewritten as a convolution in R_3:

$$-\frac{1}{4\pi}\ \text{grad div}\left(\bar{M} * \frac{1}{|y|}\right)(x) - \lambda\bar{M}(x) = 0\ , \qquad x \in \Omega \tag{8.5}$$

Now let $\rho(x) \in C^\infty(R_3)$ be such a function that $\rho(x)$ has a compact support in R_3, $\int_{R_3} \rho(x)dx = 1$. Setting $\rho_\epsilon(x) = \epsilon^{-3}\rho(x/\epsilon)$ for $\epsilon > 0$, we have from the properties of convolution

$$\lambda\rho_\epsilon * \bar{M} = \rho_\epsilon * \frac{1}{4\pi}\left(\text{grad div}\left(\bar{M} * \frac{1}{|y|}\right)\right) = \frac{1}{4\pi}\ \text{grad div}\left((\rho_\epsilon * \bar{M}) * \frac{1}{|y|}\right),$$

i.e., the smooth function $\bar{M}^{\epsilon} = \rho_{\epsilon} * \bar{M}$ satisfies (8.4). To end the proof we note that $\bar{M}^{\epsilon}$ converges to $\bar{M}$ in $\bar{L}^2$ as $\epsilon \to 0$.

Lemma 8.2. Ker $A = \mathring{J}$

Proof — Consider A on the linear set $D(A) = \bar{C}^1(\Omega) \cap \bar{C}(\bar{\Omega})$ of vector-functions with the components from $C^1(\Omega) \cap C(\bar{\Omega})$. From the proof of Lemma 2.1 [5] we have the identities

$$\bar{M}(x) = -\text{rot rot}\bar{\psi}(x) + \text{grad div } \bar{\psi}(x) \quad , \qquad x \in \Omega \tag{8.6}$$

$$\text{div}\int_{\Omega} \frac{\bar{M}(y)}{r}\, dy = -\int_{S} \frac{\bar{M}(y)\cdot\bar{n}_y}{r}\, dS + \int_{\Omega} \frac{\text{div }\bar{M}(y)}{r}\, dy \ , \qquad x \in \Omega \ . \tag{8.7}$$

Using the identity (8.6), Eq. (8.4) for $\lambda = 0$ is written as

$$\bar{M}(x) - \frac{1}{4\pi} \text{rot rot} \int_{\Omega} \frac{\bar{M}(y)}{r}\, dy = 0 \quad , \qquad x \in \Omega \ .$$

This implies

$$\text{div } \bar{M}(x) = 0 \quad , \qquad x \in \Omega \ . \tag{8.8}$$

Using the Green's formula, it follows

$$(\bar{M}\cdot\bar{n}, 1)_{0,S} = 0 \ . \tag{8.9}$$

From (8.7) and (8.8) it follows that for $\lambda = 0$, (8.4) can be written as

$$\text{grad } v(x) = \frac{1}{4\pi} \text{grad} \int_{S} \frac{\sigma(y)}{r}\, dS_y = 0 \ , \qquad x \in \Omega \ , \tag{8.10}$$

where we used the notation $\sigma = \bar{M}\cdot\bar{n}$. Taking into account the jump conditions on S for the derivatives of the single layer potential when x approaches S from the interior

$$\left(\frac{\partial v}{\partial x_i}\right)\Big|_S = \frac{\sigma(x)}{2} \cos(\bar{n}, x_i) + \frac{1}{4\pi} \int_{S} \sigma(y) \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) dS_y \ , \qquad i = 1,2,3$$

we obtain from (8.10)

$$\frac{\sigma}{2} + T\sigma \equiv \frac{\sigma}{2} + \frac{1}{4\pi}\int_S \sigma(y) \frac{\partial}{\partial n_x}\left(\frac{1}{r}\right) dS_y = 0 , \qquad x \in S . \quad (8.11)$$

From (8.9) we see that $(\sigma,1)_{o,S} = 0$. We therefore consider Eq. (8.11) in the space $\tilde{C}(S) = \{\sigma \in C(S), (\sigma,1)_{o,S} = 0\}$. It is the homogeneous equation for the interior Neumann problem. The conjugate equation is the equation for the exterior Dirichlet problem. It is known to have only the trivial solution in $\tilde{C}(S)$ (see, for example, [7]). By the standard Fredholm theory (8.11) also has only a trivial solution in $\tilde{C}(S)$. It follows that

$$\mathrm{Ker}A \cap D(A) \subset \{\bar{M} \in D(A) : \mathrm{div}\,\bar{M} = 0 , \quad \bar{M}\cdot\bar{n}|_S = 0\} .$$

The inverse inclusion follows immediately from the identity (8.7). By Lemma 8.1 the closure of $\mathrm{Ker}A \cap D(A)$ in $\bar{L}^2$ is Ker A, and by [1, Theorem 3.2] this closure coincides with $\mathring{J}$, and thus the lemma is proved.

Lemma 8.3. $\mathrm{Ker}(A-1) = \mathring{G}$.

Proof — Consider again A on $D(A) = \bar{C}^1(\Omega) \cap \bar{C}(\bar{\Omega})$. For $\lambda = 1$, (8.4) is written as

$$\bar{M}(x) + \frac{1}{4\pi}\,\mathrm{grad}\,\mathrm{div}\int_\Omega \frac{\bar{M}(y)}{r}\,dy = 0 , \qquad x \in \Omega . \quad (8.12)$$

This implies $\mathrm{rot}\,\bar{M} = 0$. Let us set

$$\phi(x) = -\frac{1}{4\pi}\,\mathrm{div}\int_\Omega \frac{\bar{M}(y)}{r}\,dy , \qquad x \in R_3 . \quad (8.13)$$

Then $\bar{M} = \mathrm{grad}\,\phi$ in Ω. It is easy to verify that ϕ satisfies the boundary value problem

$$\Delta\phi = \mathrm{div}\,\bar{M} , \qquad x \in \Omega , \quad (8.14)$$

$$\phi^+ = \phi^- , \qquad x \in S , \quad (8.15)$$

$$\Delta\phi = 0 \quad , \qquad x \in R_3 - \bar{\Omega} \tag{8.16}$$

$$\frac{\partial\phi^-}{\partial n} = 0 \quad , \qquad x \in S \tag{8.17}$$

$$\lim \phi(x) = 0 \quad , \qquad \text{for } |x| \to \infty \tag{8.18}$$

where (+) and (-) denote, respectively, the inner and outer limits on S. Here (8.14),(8.15),(8.16) and (8.18) follow immediately from (8.7) and the properties of the space potential and the single layer one. From (8.12) $\bar{M}^+ \cdot \bar{n} - \frac{\partial\phi^+}{\partial n} = 0$, together with $\frac{\partial\phi^+}{\partial n} - \frac{\partial\phi^-}{\partial n} = \bar{M}^+ \cdot \bar{n}$ this gives (8.17). The problem of (8.16)-(8.18) has only the trivial solution, and therefore by (8.15), $\phi^+ = 0$. It follows

$$\mathrm{Ker}(A - I) \cap D(A) \subseteq \{\bar{M} \in D(A) : \bar{M} = \mathrm{grad}\,\phi \ , \quad \phi^+ = 0\} \ .$$

On the other hand, $\bar{M} = \mathrm{grad}\,\phi$, $\phi^+ = 0$ implies $\mathrm{rot}\,\bar{M} = 0$, $\bar{M}^+ \times \bar{n} = 0$ for $x \in \Omega$. Together with the identities (8.6),

$$\mathrm{rot}\,\bar{\psi} = \frac{1}{4\pi} \int_S \frac{\bar{n}(y) \times \bar{M}(y)}{r}\, dS_y - \int_\Omega \frac{\mathrm{rot}\,\bar{M}(y)}{r}\, dy \quad , \tag{2.7}$$

this gives the inverse inclusion. From Lemma 8.1, the closure of $\mathrm{Ker}(A - I) \cap D(A)$ is $\mathrm{Ker}(A - I)$; and from the results in Ref. [1] the former is $\mathring{G}$. Thus the lemma is proved.

Let us denote by λ_o the minimum eigenvalue of T in $\tilde{C}(S)$ and by Λ_o the maximum one.

<u>Lemma 8.4</u>. In the interval (0,1) the spectrum of A in $\bar{L}^2$ is not more than countable:

1) λ_o is the lower bound, Λ_o is the upper bound, $0 < \lambda_o \leq \Lambda_o < 1$; $\lambda = \frac{1}{2}$ is the unique limit point; each value $\lambda \neq \frac{1}{2}$ is regular or has a finite multiplicity.

2) U is an invariant subspace of A, and the eigenfunctions of A in U form a complete orthogonal system in U.

Proof — By [1]

$$\bar{L}^2 = \mathring{J} \oplus U \oplus \mathring{G} \tag{8.19}$$

and therefore from Lemmas 8.2 and 8.3 it follows that the spectrum of A on (0,1) is a subset of U. We now reduce the problem of the investigation of the spectrum of A on U to that of the investigation of the spectrum of T on $H_o^{-\frac{1}{2}}(S) = \{\sigma \in H^{-\frac{1}{2}}; \langle\sigma,1\rangle_{H^{-\frac{1}{2}}(S)\times H^{\frac{1}{2}}(S)} = 0\}$.

For $\bar{M}\in U$ let us define a potential u by setting $\operatorname{grad} u = \bar{M}$, $\int_S u\,dS = 0$. The norm in $H^1(\Omega)$ can be defined as

$$\|u\|^2_{1,\Omega} = \int_\Omega (\operatorname{grad} u)^2\, dx + \left(\int_S u\,dS\right)^2 \tag{8.20}$$

For u defined above, we therefore have:

$$\|u\|^2_{1,\Omega} = \int_\Omega (\operatorname{grad} u)^2 dx = \int_\Omega (\bar{M})^2 dx = \|\bar{M}\|^2 \tag{8.21}$$

Now let λ be a point of the spectrum of A in U. Then $\exists$ a sequence $(\bar{M}_n)\subset U$ such that $\bar{M}_n = \operatorname{grad} u_n$, $\Delta u_n = 0$, $\int_S u\,dS = 0$, $\|\bar{M}_n\| = 1$, $\|A\bar{M}_n - \lambda\bar{M}_n\| \to 0$ as $n\to\infty$. It follows that

$$\left\|\operatorname{div}\int_\Omega \frac{\bar{M}_n(y)}{r}\,dy - \lambda u_n\right\|_{1,\Omega} \to 0$$

as $n\to\infty$. By [3] for a harmonic function u, the mapping $u \to \partial u^+/\partial n$ is continuous from $H^1(\Omega)$ to $H^{-\frac{1}{2}}(S)$. From the proof of Lemma 8.2 for $\sigma_n = \bar{M}_n^+\cdot\bar{n} \equiv \partial u_n^+/\partial n$ it follows that $\|T\sigma_n - (\frac{1}{2}-\lambda)\sigma_n\|_{-\frac{1}{2},S} \to 0$ as $n\to\infty$. The operator T can be considered as a pseudodifferential operator in $H^{-\frac{1}{2}}(S)$.

Its principle symbol [2; p.197],[4, p.211] is $T^{\circ}(x,\xi_x) = i\cos(\bar{n}_x,\xi_x) \equiv 0$, where (x,ξ_x) is an element of the cotangent bundle $T^*(S)$. By [2, p.198, Lemma 21.5] T is compact in $H^s(S)$ for all real s, in our case $s = -\frac{1}{2}$. The first statement of the Lemma now follows from the standard properties of a compact operator and the fact that the spectrum of A on (0,1) is a subset of that of T. The second statement follows from (8.7) and (8.8) and the fact that A is self-adjoint in $\bar{L}^2$ [5].

Let us denote by P_1, P_2, P_3 the orthogonal projectors of $\bar{L}^2$ onto $\mathring{J}$, U, $\mathring{G}$, respectively. By [1]

$$J \equiv \mathring{J} \oplus U = \{\bar{M} : \bar{M} = \text{rot}\,\bar{F},\quad \bar{F} \in \bar{H}_2^1\} \quad .$$

Theorem 8.2.

Let $R : \bar{L}^2 \to J$ be such an operator that

$$(R\bar{M})(x) = \frac{\bar{M}(x)}{\mu(M,x) - 1} + (A\bar{M})(x) \quad , \qquad 1 < \mu_{min} \leqslant \mu \leqslant \mu_{max} < \infty \quad .$$

Then there holds

$$(R\bar{M},\bar{M}) \geqslant \left(\frac{1}{\mu_{max} - 1} + \frac{\mu_{min} - 1}{\mu_{max} - 1}\;\frac{1}{\mu_{min} - \lambda_o(\mu_{min} - 1)}\,\lambda_o\right)\|\bar{M}\|^2 \tag{8.22}$$

Proof — Let us define $\lambda = \text{const} \geqslant 0$ by

$$\frac{1}{\mu_{max} - 1} + \lambda = \inf_{\substack{\|\bar{M}\| = 1 \\ R\bar{M} \in J}} (R\bar{M},\bar{M}) \quad , \tag{8.23}$$

By Theorem 8.1 we have the following chain of equalities and inequalities:

$$\frac{1}{\mu_{max}-1} + \lambda_o(\|P_2\bar{M}\|^2 + \|P_3\bar{M}\|^2) \leq \left(\frac{\bar{M}}{\mu-1}, \bar{M}\right) + (A\,P_2\bar{M}, P_2\bar{M}) + \|P_3\bar{M}\|^2$$

$$= (R\bar{M},\bar{M}) = \left(\frac{\bar{M}}{\mu-1}, (P_2+P_3)\bar{M}\right) + \left(A(P_2+P_3)\bar{M}, (P_2+P_3)\bar{M}\right)$$

$$\leq \left(\frac{1}{\mu_{min}-1} + 1\right) \|(P_2+P_3)\bar{M}\| \quad . \qquad (8.24)$$

Now (8.22) follows from (8.23),(8.24), and the following inequalities

$$\lambda_o \inf_{\substack{\|\bar{M}\|=1 \\ R\bar{M}\in J}} \left(\|P_2\bar{M}\|^2 + \|P_3\bar{M}\|^2\right) \leq \lambda \quad ,$$

$$\frac{1}{\mu_{max}-1} + \lambda \leq \left(\frac{1}{\mu_{min}-1} + 1\right) \inf_{\substack{\|\bar{M}\|=1 \\ R\bar{M}\in J}} \left(\|P_2\bar{M}\|^2 + \|P_3\bar{M}\|^2\right) \quad .$$

References

[1] E.B.Bykhovsky, N.V.Smirnov, "On the orthogonal decompositions of the space of vector-functions which are squarely summable over the domain," TrMIAN-SSSR-59 (1960) 6-36.

[2] G.I.Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations (Nauka, Moscow, 1973).

[3] J.L.Lions, E.Magenes, Nonhomogeneous Boundary Value Problems and Applications, Vol. 1 (Springer, Berlin, 1972).

[4] S.G.Mikhlin, Multidimensional Singular Integrals and Integral Equations, (Phiz. Mat. Izdat., Moscow, 1962).

[5] M.J.Friedman, "Mathematical study of the nonlinear singular integral magnetic field equation, Part 1," to appear in SIAM J. Appl. Math.

[6] M.J.Friedman, "Mathematical study of the nonlinear singular integral magnetic field equation, Part 2," To appear in SIAM J. on Numer. Anal.

[7] V.S.Vladimirov, Equations of Mathematical Physics, (Nauka, Moscow, 1971).

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720