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Design of Coupled Harmonic Oscillators for Synchronization and Coordination

Xinmin Liu and Tetsuya Iwasaki

***Abstract*—Synchronization and coordination of coupled oscillators are fundamental behaviors in complex dynamical systems. This paper considers the design of coupled harmonic oscillators to generate an orbitally stable limit cycle of prescribed oscillation profile. Based on the Floquet theory and averaging techniques, necessary and sufficient conditions are obtained for nonlinear coupling functions to achieve local exponential convergence to a desired orbit. Unlike globally convergent methods based on contraction analysis, the result applies to oscillators without flow invariance properties. Insights into coordination mechanisms are gained through interpretation of the coupling structure as a directed graph. The theory is illustrated by simple tutorial examples.**

I. Introduction

Coupled oscillators are found in various contexts in nature; e.g., synchronization of flashing fireflies [1], formation flight of migrating birds with coordinated flapping motions [2], pathological synchronization of neural oscillations in Parkinson's disease [3], and coordinated oscillations of the central pattern generator (CPG) in animal locomotion [4]. Feedback control theories to achieve/break synchronization and coordination of coupled oscillators would be useful for designing bio-inspired engineering systems and for developing treatments and assistive devices for certain medical conditions, including walking rehabilitation [5], dental training, and speech therapy [6].

Synchronization problems have been formulated and solved in the literature. Given a set of interconnected n subsystems (or agents) with outputs $y_1, \ldots, y_n$, the agents are said to be synchronized if all the outputs converge to a single (nontrivial) trajectory, i.e., there exists $y_o(t)$ such that

$$|y_i(t) - y_o(t)| \to 0 \quad \text{as} \quad t \to \infty, \quad \forall\, i = 1, \ldots, n.$$

The notion of coordination is an extension of synchronization. With prescribed constants $\gamma_i, \varphi_i \in \mathbb{R}$, the agents are said to be coordinated if there exists $y_o(t)$ such that

$$|y_i(t) - \gamma_i y_o(t + \varphi_i)| \to 0 \quad \text{as} \quad t \to \infty, \quad \forall\, i = 1, \ldots, n.$$

For coupled oscillator problems, the steady state output y_o is a periodic signal, while constant y_o has also been considered for synchronization of arbitrary (not necessarily oscillatory) subsystems, and the property is often termed consensus. The problem is to determine under what condition the synchronization or coordination occurs.

Within the linear system framework, synchronization (or consensus) of coupled harmonic oscillators [7], integrator agents [8], and general subsystems [9] have been considered, where the agents are homogeneous and share common dynamics. It was shown that, essentially, synchronization occurs when each agent is marginally stable and the directed graph underlying the inter-agent coupling has positive weights and contains a spanning tree. These results have been extended to deal with heterogeneous agents [10–12]. In all these references, the magnitude of the synchronized output y_o has to depend on the initial state since the system is linear autonomous. For certain coupled oscillator applications [13], however, it is important to have convergence to a prescribed amplitude with orbital stability, which requires nonlinear dynamics.

Weakly coupled nonlinear oscillators have been analyzed using perturbation and averaging techniques [14]. The orbit of each oscillator remains close to its intrinsic limit cycle after weak coupling with other oscillators. Hence, stability of the amplitude is not an issue, and the behavior of each oscillator can be described by a scalar phase variable after a state coordinate transformation [15–17]. The resulting model, called the phase coupled oscillators, provides a deep insight into phase coordination mechanisms, but the method does not seem directly useful for the design of coupled oscillators since it is difficult to convert the coupling function of the phase variables back to that of the original variables.

The framework of the master stability equation [18, 19] is directly useful for both analysis and design of coupling functions with a specific (e.g. diffusive) structure for synchronization with orbital stability [20]. The method is further extended to guarantee robust synchronization for intervals of diffusive coupling gains through averaging under time-scale separation by weak or strong coupling [21]. While the general framework is very powerful, it applies only to synchronization of homogeneous subsystems since the decomposition leading to the master stability equation requires such setting. The method does not work if the subsystems are heterogeneous or if the analysis/design of coordinated oscillations is considered.

The method of harmonic balance has been applied for analyzing coordinated oscillations of CPGs — neuronal circuits for controlling rhythmic body movements. In particular, the explicit relationship is revealed between the neuronal coupling structure and the resulting oscillation profile (frequency, amplitude, phase) [22, 23]. The result is simple and useful not only for the analysis but also for the design of synthetic CPGs. However, stability of the oscillation is not rigorously

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guaranteed due to the harmonic approximations, except for an extension to the case of rational phases [13].

The contraction analysis [24] provides a rigorous method for analyzing and synthesizing global convergence to a limit cycle orbit. Convergence properties of specific coupled oscillators are proven for synchronization by partial contraction analysis [25] and for coordination by flow invariance [26]. While the method is very powerful for certain systems, coordination of coupled oscillators is difficult and has not been solved when the flow invariance property is not available.

In this paper, we address the open problem of analyzing/designing coupled nonlinear oscillators for synchronization and coordination in the absence of the flow invariance property. The coupled oscillator problem is motivated by feedback control of mechanical systems, for which the flow invariance property crucial in the contraction analysis is absent. We show how the nonlinear coupling function can be constructed to achieve a periodic solution that is sinusoidal with a prescribed oscillation profile. Moreover, orbital stability of the limit cycle is theoretically proven and is guaranteed in the design of the coupling function.

Section II formulates and motivates a coupled oscillator problem, and describes an outline of our approach. Section III provides analysis results to characterize orbital stability. We first linearize the nonlinear system around the periodic solution, then decompose the resulting linear periodic system to remove the one-dimensional dynamics associated with the phase shift on the orbit. The exponential stability of the resulting reduced order linear periodic system is then equivalent to orbital stability of the solution (Theorem 1). When the oscillators are weakly coupled, we show through an averaging theorem that the stability condition can be given by Hurwitz properties of coupling matrices (Theorem 2). Based on these analysis results, Section IV proposes a procedure for designing coupling functions, where the graph theory is used to impose structural constraints (Theorem 3), and coupling optimization for fast convergence is discussed. Finally, the proposed method is illustrated by numerical examples in Section V.

Notation: Define the set $\mathbb{Z}_n := \{1, 2, \dots, n\}$. The column vector x and diagonal matrix X with entries $x_1, \dots, x_n$ are denoted by $x = \mathrm{col}\,(x_1, \dots, x_n)$ and $X = \mathrm{diag}(x_1, \dots, x_n) = \mathrm{diag}(x)$. Trigonometric functions act on a matrix elementwise. Define $C_x := \mathrm{diag}(\cos(x))$, $S_x := \mathrm{diag}(\sin(x))$, and $\mathbb{1} := \mathrm{col}\,(1, \dots, 1)$. For matrices A and B of the same dimensions, the element-wise product is denoted by $A \cdot B$.

II. Problem Statement and Approach

A. Coupled oscillator problem

Consider a set of coupled harmonic oscillators

$$\ddot{q} + \omega^2 q = h(q, \dot{q}), \quad q(t) \in \mathbb{R}^n \tag{1}$$

where the left hand side defines a set of n harmonic oscillators with frequency ω, and the right hand side defines their coupling dynamics through a continuously differentiable function h. Without the coupling term h, the system (1) has the general solution $q(t) = \xi(t)$, where

$$\xi_i(t) := \gamma_i \sin(\omega t + \varphi_i), \tag{2}$$

for $i \in \mathbb{Z}_n$ with arbitrary amplitude $\gamma_i \in \mathbb{R}$ and phase $\varphi_i \in \mathbb{R}$ that depend on the initial condition. Roughly speaking, our objective is to design the coupling function h so that the oscillation ξ_i occurs in the steady state, with prescribed values of the amplitudes γ_i and phases φ_i. Clearly, the coupling term must vanish on the periodic orbit, i.e.,

$$h(\xi, \dot{\xi}) \equiv 0 \tag{3}$$

is required for h when ξ is a solution of (1). In addition, we require a stability property, that is, the oscillation ξ should attract nearby trajectories. However, we do not require convergence of $q(t)$ to $\xi(t)$. Instead, we consider the design to be successful if convergence to $\xi(t + c)$ is achieved for some $c \in \mathbb{R}$. This property corresponds to formation of an oscillation pattern disregarding the time shift, and is known as the notion of orbital stability as defined below.

Let us express system (1) in the state space as

$$\dot{x} = f(x), \quad f(x) := \begin{bmatrix} x_2 \\ h(x_1, x_2) - \omega^2 x_1 \end{bmatrix}, \tag{4}$$

where $x := \mathrm{col}\,(q, \dot{q})$. The system (4) has a periodic solution $x = \chi$ where $\chi := \mathrm{col}\,(\xi, \dot{\xi})$ when (3) is satisfied. The periodic orbit is a closed curve in the state space parametrized by $\chi(t)$ with $t \in \mathbb{R}$. The notion of *orbital stability* means that any trajectory that comes sufficiently close to the orbit will eventually converge to the orbit. More precisely, the periodic solution χ is said to be *orbitally stable* if there exists $\varepsilon > 0$ with the following property: Whenever $\|x(t_1) - \chi(t_2)\| < \varepsilon$ for some $t_1, t_2 \in \mathbb{R}$, there exists a constant c such that

$$\lim_{t \to \infty} \|x(t) - \chi(t + c)\| = 0. \tag{5}$$

In this case, we also say ξ is orbitally stable, with a slight abuse of language. If the convergence is exponential, a periodic solution is said to be orbitally exponentially stable.

Now, the problem addressed in this paper can be stated as follows: *Find a necessary and sufficient condition on the coupling function h such that ξ is a periodic solution to (1) and is orbitally exponentially stable, and propose a systematic procedure for designing the coupling function.* This is a problem of achieving *coordinated oscillations* of $q_i(t)$ with prescribed amplitudes γ_i and specific timing to each other (i.e. relative phase $\varphi_i - \varphi_j$). In the special case where $\gamma_i = \gamma_j$ and $\varphi_i = \varphi_j$ for all $i, j \in \mathbb{Z}_n$, all the variables $q_i(t)$ should converge to a single orbit, which we refer to as *synchronized oscillations*.

For simplicity, we will impose the following.

Assumption 1: The target oscillation $\xi_i(t)$ has normalized amplitudes $\gamma_i = 1$ and frequency $\omega = 1$ for all $i \in \mathbb{Z}_n$.

This assumption can be made without loss of generality. To generate harmonic oscillations with arbitrary frequency ω and amplitudes γ_i, we can simply scale the oscillators as follows. Let function $h(q, \dot{q})$ be designed for the target oscillation $q_i(t) = \sin(t + \varphi_i)$. Then the scaled coupling function $\hbar(q, \dot{q})$ given by

$$\hbar(q, \dot{q}) := \omega^2 \Gamma\, h(\Gamma^{-1} q, \Gamma^{-1} \dot{q}/\omega), \tag{6}$$

achieves orbitally stable solution $q_i(t) = \gamma_i \sin(\omega t + \varphi_i)$ for (1) with h replaced by $\hbar$, where Γ is the diagonal matrix with the (i,i) entry being γ_i.

The coupled oscillator problem can be motivated as coordination of multiple agents with the double integrator dynamics arising from first principles of physics. For instance, movements of a general class of m degrees-of-freedom mechanical systems can be described by the Euler-Lagrange equation, leading to a differential equation of the form

$$\mathsf{J}(p)\ddot{p} + \mathsf{g}(p,\dot{p}) = \mathsf{B}(p)u \tag{7}$$

where $p(t) \in \mathbb{R}^m$ are the generalized coordinates, $u(t) \in \mathbb{R}^n$ are the force inputs, $\mathsf{J}(p)$ is the moment of inertia matrix, and $\mathsf{g}(p,\dot{p})$ is a nonlinear function representing the Coriolis and centrifugal forces, damping, and stiffness. Suppose $m > n$ and the n actuators independently drive the first n coordinates

$$q := \mathcal{H}p, \quad \mathcal{H} := [\ I_n \ \ 0\ ].$$

Multiplying $H(p) := \mathcal{H}\mathsf{J}(p)^{-1}$ from left, (7) gives

$$\ddot{q} + g(p,\dot{p}) = B(p)u, \quad g(p,\dot{p}) := H(p)\mathsf{g}(p,\dot{p}), \quad B(p) := H(p)\mathsf{B}(p)$$

A motion coordination problem is to find a feedback controller to generate inputs u such that the generalized coordinates q_i converge to the prescribed harmonic oscillations ξ_i in (2) for $i \in \mathbb{Z}_n$. Such controller can be found using a solution to the coupled oscillator problem, i.e., by selecting u as the state feedback

$$u = B(p)^{-1}(h(q,\dot{q}) - \omega^2 q + g(p,\dot{p}))$$

such that the resulting closed-loop system is of the form (1).

We aim to achieve the desired oscillation as part of a stable limit cycle of the autonomous closed-loop system rather than as a forced response to harmonic reference inputs. This is because the former method is expected to provide fast and smooth convergence back to the desired orbit after perturbations due to disturbances since the notion of orbital stability allows for arbitrary time shift without insisting on a preprogrammed reference signal as in the latter method. Also, autonomous oscillations emerged from network interactions may potentially be more robust against failures than forced oscillations commanded by a centralized reference generator if the design is made modular by distributed feedback.

B. Outline of approach

This section outlines our approach to the oscillator analysis in a general setting. We will integrate a series of classical results into a condition for orbital stability of a periodic solution. The general framework will be applied to the coupled oscillators (1) in the next section.

Consider an autonomous system $\dot{x} = f(x)$, where $x(t) \in \mathbb{R}^{\mathfrak{n}}$ is the state vector, and $f : \mathbb{R}^{\mathfrak{n}} \to \mathbb{R}^{\mathfrak{n}}$ is a locally Lipschitz function. Suppose the system has a T-periodic solution $x = \chi$. The linearization of the system around χ is given by

$$\dot{z} = A(t)z, \quad A(t) := \frac{\partial f}{\partial x}(\chi(t)), \quad z := x - \chi. \tag{8}$$

Note that $A(t) = A(t+T)$ holds for all $t \in \mathbb{R}$ due to the periodicity of χ. The orbital stability of χ is related to stability properties of the linear periodic system as explained below.

Let us first recall a basic notion. The Floquet multipliers for linear T-periodic system $\dot{z} = Az$ are defined as the eigenvalues of the fundamental matrix $\Theta(T)$, where

$$\dot{\Theta} = A\Theta, \quad \Theta(0) = I. \tag{9}$$

When the system is obtained from the linearization around a periodic orbit as in (8), one of the Floquet multipliers is always on the unit circle because $\dot{z} = Az$ admits a periodic solution $z = \dot{\chi}$. The remaining $\mathfrak{n}-1$ Floquet multipliers determine the orbital stability of χ.

Lemma 1: Consider the system $\dot{x} = f(x)$ and suppose it admits a T-periodic solution $\chi(t) \in \mathbb{R}^{\mathfrak{n}}$. The solution χ is orbitally exponentially stable if and only if $\mathfrak{n}-1$ Floquet multipliers of the linearized system (8) lie strictly inside the unit circle on the complex plane.

Proof: See Theorem 1.1 of [27]. ■

The condition for orbital stability in Lemma 1 is numerically verifiable, but is not directly useful for gaining insights into the coordination mechanisms or for choosing the coupling function to achieve desired oscillations. To proceed further, one may examine a subsystem describing the dynamics in the invariant subspace associated with the $\mathfrak{n}-1$ Floquet multipliers not on the unit circle. For this purpose, we use the Lyapunov transformation [28].

Let $L(t)$ be a matrix-valued function such that $L(t)$ is nonsingular and continuously differentiable, and $L(t)$, $L^{-1}(t)$, and $\dot{L}(t)$ are bounded for $t > 0$. Then $z = L\mathsf{x}$ is called a Lyapunov transformation, converting the system (8) into

$$\dot{\mathsf{x}} = \mathcal{A}(t)\mathsf{x}, \quad \mathcal{A}(t) = L^{-1}AL - L^{-1}\dot{L}. \tag{10}$$

If $L(t)$ is T-periodic, the Floquet multipliers are preserved and hence the two systems (8) and (10) share the same stability properties. A careful choice of the Lyapunov transformation can lead to isolation of the dynamics associated with the Floquet multiplier on the unit circle. For instance, L may be chosen so that the last column of $\mathcal{A}$ is zero, and removal of the last row of the state equation gives a reduced-order system

$$\dot{\mathsf{e}} = \mathcal{A}(t)\mathsf{e}, \tag{11}$$

where $\mathsf{e} \in \mathbb{R}^{\mathfrak{n}-1}$ is the first $\mathfrak{n}-1$ entries of $\mathsf{x} \in \mathbb{R}^{\mathfrak{n}}$. The orbital stability of the periodic solution χ is then guaranteed by exponential stability of this system.

Lemma 2: (Floquet theorem) The linear periodic system $\dot{\mathsf{e}} = \mathcal{A}(t)\mathsf{e}$ is exponentially stable if and only if all Floquet multipliers lie inside the unit circle on the complex plane.

Proof: See Theorem 4.2.3 of [28]. ■

Finally, the stability property follows if the average of $\mathcal{A}(t)$ over one period is Hurwitz, provided its norm is sufficiently small, or the rate of change of the state $\|\dot{\mathsf{e}}\|$ is sufficiently slow in comparison with the time scale of the cycle period T. A precise statement is given below, where $\mathcal{A}(t) = \varepsilon A(t)$ with $\varepsilon > 0$ capturing the size of $\dot{\mathsf{e}}$.

Lemma 3: Consider the linear periodic system

$$\dot{\mathsf{e}} = \varepsilon \mathcal{A}(t)\mathsf{e}, \quad \mathcal{A}(t+T) = \mathcal{A}(t). \tag{12}$$

Define the average dynamics as

$$\bar{\mathcal{A}} := \int_0^T \mathcal{A}(t)dt.$$

Then there exists $\varepsilon_o > 0$ such that the system is exponentially stable for all $0 < \varepsilon < \varepsilon_o$ if and only if $\bar{\mathcal{A}}$ is Hurwitz.

Proof: See Theorem 3.3 of [29]. ∎

The Hurwitz property of $\bar{\mathcal{A}}$ is often easier to deal with than the Floquet multipliers since $\bar{\mathcal{A}}$ depends linearly on the entries of $\mathcal{A}(t)$ while the dependence of $\Theta(T)$ is more complex. This simplicity is gained at the expense of restricting our attention to the case where ε is sufficiently small, which turns out to correspond to weak coupling of oscillators as shown in the next section.

III. Coupling analysis for orbital stability

A. Basic theory

In this section, we will take the approach outlined in the previous section and derive a condition on the coupling function h for orbital exponential stability of the periodic solution ξ in (2) for the coupled oscillators (1), or equivalently, the periodic solution χ for the state space system (4). As mentioned earlier, we impose Assumption 1 without loss of generality.

Orbital stability of χ is guaranteed, through Lemma 1, by a condition on the Floquet multipliers of the linearized system (8) with

$$A(t) = \begin{bmatrix} 0 & I \\ -I + \mathsf{K} & \mathsf{D} \end{bmatrix}, \tag{13}$$

$$\mathsf{K} := \frac{\partial h}{\partial q}(\xi, \dot{\xi}), \quad \mathsf{D} := \frac{\partial h}{\partial \dot{q}}(\xi, \dot{\xi}). \tag{14}$$

While such condition is useful for analysis, it is not directly useful for design of the coupling function h. For design, we need to search for a Lyapunov transformation to isolate the dynamics associated with the Floquet multiplier on the unit circle. To this end, we consider the original nonlinear system (1) and transform it into the polar coordinates. The idea is that the Floquet mode on the unit circle can be identified as the one that governs the absolute (as opposed to relative) evolution of the phase variables. Below, we formally state the result and then provide a detailed proof.

Theorem 1: Consider the coupled oscillators described by (1). Let the target oscillation $\xi(t) \in \mathbb{R}^n$ be given by (2) with desired frequency ω, amplitude γ_i, and phase φ_i for $i \in \mathbb{Z}_n$, where Assumption 1 is imposed. Define K and D by (14), and $\eta(t) \in \mathbb{R}^n$ by $\eta_i(t) := t + \varphi_i$ for $i \in \mathbb{Z}_n$, and let P and Q be arbitrary $n \times (n-1)$ matrices satisfying

$$P^{\mathsf{T}} \begin{bmatrix} Q & \mathbb{1} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{15}$$

The signal $q = \xi$ is a solution of (1) if and only if (3) is satisfied. In this case, the following statements are equivalent:

(i) The solution $q = \xi$ is orbitally exponentially stable.

(ii) All the Floquet multipliers of $\dot{w} = \mathsf{A}(t)w$ with

$$\mathsf{A}(t) := \begin{bmatrix} C_\eta \\ -S_\eta \end{bmatrix} \begin{bmatrix} \mathsf{D} & \mathsf{K} \end{bmatrix} \begin{bmatrix} C_\eta & -S_\eta \\ S_\eta & C_\eta \end{bmatrix}, \tag{16}$$

except for one at 1, are inside the unit circle.

(iii) All the Floquet multipliers of $\dot{\mathsf{e}} = \mathcal{A}(t)\mathsf{e}$ with

$$\mathcal{A}(t) := \mathsf{P}^{\mathsf{T}}\mathsf{A}(t)\mathsf{Q}, \qquad \begin{matrix} \mathsf{P} := \mathrm{diag}(I, P), \\ \mathsf{Q} := \mathrm{diag}(I, Q), \end{matrix} \tag{17}$$

are inside the unit circle.

Proof: Consider the coordinate transformation $(q, \dot{q}) \leftrightarrow (r, \theta)$ defined by

$$q_i = r_i \sin\theta_i, \quad \dot{q}_i = r_i \cos\theta_i, \quad i \in \mathbb{Z}_n.$$

In the new coordinates, the limit cycle (2) is described as

$$r_i(t) = 1, \quad \theta_i(t) = \eta_i(t), \quad \eta_i(t) := t + \varphi_i, \tag{18}$$

and the system (1) can be transformed into

$$\begin{bmatrix} \dot{r} \\ R\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} + \begin{bmatrix} C_\theta \\ -S_\theta \end{bmatrix} h(S_\theta r, C_\theta r) \tag{19}$$

where $R := \mathrm{diag}(r)$. Linearizing the system around the orbit and defining A as in (16), we obtain

$$\dot{\mathsf{w}} = \mathsf{A}(t)\mathsf{w}, \quad \mathsf{w} := \begin{bmatrix} \rho \\ \vartheta \end{bmatrix}, \quad \begin{matrix} \rho_i := r_i - 1, \\ \vartheta_i := \theta_i - \eta_i, \end{matrix} \tag{20}$$

It can be verified that the two linear systems, defined by $A(t)$ and $\mathsf{A}(t)$ in (13) and (16), are related by the Lyapunov transformation

$$\mathsf{A} = \mathcal{L}^{-1}A\mathcal{L} - \mathcal{L}^{-1}\dot{\mathcal{L}}, \quad \mathcal{L} := \begin{bmatrix} S_\eta & C_\eta \\ C_\eta & -S_\eta \end{bmatrix}$$

and hence share the same set of Floquet multipliers, proving (i) ⇔ (ii) through Lemma 1.

We now isolate the Floquet multiplier on the unit circle. The trajectory $\mathsf{w}(t)$ is on the target orbit if and only if $\rho_i(t) \equiv 0$ and $\vartheta_i(t) \equiv c$ for some constant c, in which case, we have $q(t) = \xi(t + c)$. This motivates us to consider the transformation $\vartheta \leftrightarrow (\phi, \psi)$, where $\phi \in \mathbb{R}^{n-1}$ are the relative phases and $\psi \in \mathbb{R}$ is the absolute phase. For instance, a choice is given by $\phi_i := \vartheta_i - \vartheta_n$ for $i \in \mathbb{Z}_{n-1}$ and $\psi = \vartheta_n$. More generally, ϕ and ψ can be defined by

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} := \begin{bmatrix} P^{\mathsf{T}} \\ p^{\mathsf{T}} \end{bmatrix} \vartheta, \quad \begin{bmatrix} P^{\mathsf{T}} \\ p^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} Q & \mathbb{1} \end{bmatrix} = I.$$

System (20) can then be described as

$$\dot{\mathsf{x}} = \mathcal{A}(t)\mathsf{x}, \quad \mathsf{x} := \begin{bmatrix} \mathsf{e} \\ \psi \end{bmatrix}, \quad \mathsf{e} := \begin{bmatrix} \rho \\ \phi \end{bmatrix}, \tag{21}$$

for an appropriately defined $\mathcal{A}$, and the target orbit is described by $\mathsf{e}(t) \equiv 0$ and $\psi(t) \equiv c$. The coupling condition (3) ensures that the orbit $q(t) = \xi(t+c)$ is a solution of (1), which implies that the constant trajectory $(\mathsf{e}, \psi) = (0, c)$ is a solution to (21). This in turn indicates that the last column of $\mathcal{A}(t)$ is zero, and the dynamics of ψ are characterized by the eigenvalue of $\mathcal{A}(t)$ at the origin, giving rise to the Floquet multiplier at one [30]. The other $2n-1$ Floquet multipliers are associated with the dynamics of e which are decoupled from ψ and given by (11), where $\mathcal{A}(t)$ is the matrix in (17) obtained by removing the last row and last column from $\mathcal{A}(t)$. Thus, from Lemma 1, orbital exponential stability of χ is equivalent to exponential stability of (11), which can be characterized as (ii) by Lemma 2. ∎

B. Weak parametric coupling

Through Theorem 1, the coupled-oscillator design problem is reduced to the choice of the coupling function h in (1) such that the vanishing condition (3) is satisfied and the linear periodic system (11) is stable. An explicit parametrization of all such h would be desired for systematic design, but is difficult to obtain. Here we instead focus on a specific coupling structure motivated by dynamics of mechanical systems, and develop characterizations of feasible coupling functions that are directly useful for the design.

We consider the following class of coupling functions

$$h(q,\dot q) := -\varepsilon(M\mathscr{E}(q,\dot q)+D)\dot q - \varepsilon(N\mathscr{E}(q,\dot q)+K)q, \quad (22)$$

where D, K, M, and N are constant $n\times n$ matrices, $\varepsilon\in\mathbb{R}$ is the coupling strength parameter, and $\mathscr{E}(q,\dot q)$ is a matrix-valued function. With this choice, system (1) resembles a mechanical system and is described by

$$\ddot q + \varepsilon(M\mathscr{E}(q,\dot q)+D)\dot q + (I+\varepsilon N\mathscr{E}(q,\dot q)+\varepsilon K)q = 0. \quad (23)$$

The idea is to specify the mode shape (or the relative amplitude and phase) by the linear stiffness and damping terms $D\dot q + Kq$, and stabilize the amplitude by the nonlinear damping and stiffness terms $M\mathscr{E}(q,\dot q)\dot q + N\mathscr{E}(q,\dot q)q$. The function $\mathscr{E}(q,\dot q)$ is chosen to vanish on the limit cycle ξ, i.e., $\mathscr{E}(\xi,\dot\xi)\equiv 0$, and to provide the amplitude error in the neighborhood of ξ. In particular, we propose matrix-valued function $\mathscr{E}(q,\dot q)$ that is diagonal with the (i,i) entry given by

$$\mathscr{E}_{ii}(q,\dot q) := \frac{1}{2}(r_i^2-1) = \frac{1}{2}\left(q_i^2+\dot q_i^2-1\right). \quad (24)$$

This function is motivated by classical amplitude-dependent damping terms used in the Andronov-Hopf oscillator [26, 31] and energy balance method [32]. If M is diagonal with positive entries, the i^{th} diagonal entry of damping term $\varepsilon M\mathscr{E}$ has a positive/negative sign when the amplitude r_i is larger/smaller than one. Similarly, the term $\varepsilon N\mathscr{E}$ adjusts the stiffness. Thus, these terms can provide nonlinear effects that stabilize the amplitudes at $r_i = 1$. If $M = N = 0$, then the coupled system is linear and orbital stability can never be achieved.

Assuming weak coupling and exploiting the averaging result in Lemma 3, a fairly simple characterization useful for design can be obtained for the coupling parameters (D,K,M,N) that guarantee orbital stability of ξ. The following theorem is the main result of this section.

Theorem 2: Consider the system (1) with the coupling function (22), where $\mathscr{E}(q,\dot q)$ is the diagonal function specified by (24). Let the target oscillation $\xi(t)\in\mathbb{R}^n$ be given by (2) with desired frequency ω, amplitude γ_i, and phase φ_i for $i\in\mathbb{Z}_n$, where Assumption 1 is imposed. Define

$$\begin{aligned}\Delta &:= \mathsf{C}_\varphi\cdot D - \mathsf{S}_\varphi\cdot K, \quad \Phi := \mathsf{C}_\varphi\cdot M - \mathsf{S}_\varphi\cdot N,\\ \nabla &:= \mathsf{S}_\varphi\cdot D + \mathsf{C}_\varphi\cdot K, \quad \Psi := \mathsf{S}_\varphi\cdot M + \mathsf{C}_\varphi\cdot N,\end{aligned} \quad (25)$$

where S_φ and C_φ are, respectively, the matrices with (i,j) entries $\sin(\varphi_i-\varphi_j)$ and $\cos(\varphi_i-\varphi_j)$. Then the periodic signal $q=\xi$ is a solution of the system if and only if

$$\Delta\mathbb{1} = \nabla\mathbb{1} = 0 \quad (26)$$

holds. In this case, the following statements are equivalent:

(i) The periodic solution $q=\xi$ is orbitally exponentially stable for sufficiently small $\varepsilon>0$.

(ii) All the eigenvalues of

$$\bar{\mathsf{A}} := \varepsilon\pi\begin{bmatrix} -(\Delta+\Phi) & -\nabla \\ \nabla+\Psi & -\Delta\end{bmatrix} \quad (27)$$

except for one at the origin have negative real parts.

(iii) All the eigenvalues of

$$\overline{\mathcal{A}} := \mathsf{P}^{\mathsf{T}}\bar{\mathsf{A}}\mathsf{Q} \quad (28)$$

have negative real parts, with P and Q in (17) and (15).

Proof: Assuming that the vanishing coupling condition (3) is satisfied, the linearization of the coupling term around the limit cycle ξ is given by

$$\begin{aligned} h(q,\dot q) &\cong \mathsf{K}(q-\xi)+\mathsf{D}(\dot q-\dot\xi),\\ \mathsf{K} &:= \frac{\partial h}{\partial q}(\xi,\dot\xi) = -\varepsilon K - \varepsilon(MC_\eta+NS_\eta)S_\eta,\\ \mathsf{D} &:= \frac{\partial h}{\partial \dot q}(\xi,\dot\xi) = -\varepsilon D - \varepsilon(MC_\eta+NS_\eta)C_\eta,\end{aligned} \quad (29)$$

where we noted that

$$\begin{aligned}\mathscr{E}_{ii}\dot q_i &\cong (\cos\eta_i\sin\eta_i)(q_i-\xi_i)+(\cos^2\eta_i)(\dot q_i-\dot\xi_i),\\ \mathscr{E}_{ii}q_i &\cong (\sin^2\eta_i)(q_i-\xi_i)+(\sin\eta_i\cos\eta_i)(\dot q_i-\dot\xi_i).\end{aligned}$$

Define the new design parameters (∇,Δ) and (Φ,Ψ) by

$$\begin{bmatrix}\Delta & \nabla\\ -\nabla & \Delta\end{bmatrix} := \Omega_\varphi\begin{bmatrix}D & K\\ -K & D\end{bmatrix}\Omega_\varphi^{\mathsf{T}},$$

$$\begin{bmatrix}\Phi & \Psi\\ -\Psi & \Phi\end{bmatrix} := \Omega_\varphi\begin{bmatrix}M & N\\ -N & M\end{bmatrix}\Omega_\varphi^{\mathsf{T}},$$

where

$$\Omega_\varphi := \begin{bmatrix}C_\varphi & S_\varphi\\ -S_\varphi & C_\varphi\end{bmatrix}.$$

These equations define bijective mappings $(D,K)\leftrightarrow(\Delta,\nabla)$ and $(M,N)\leftrightarrow(\Phi,\Psi)$. In particular, using the identities

$$\begin{aligned}\mathsf{C}_\varphi\cdot X &= C_\varphi X C_\varphi + S_\varphi X S_\varphi,\\ \mathsf{S}_\varphi\cdot X &= S_\varphi X C_\varphi - C_\varphi X S_\varphi,\end{aligned}$$

the equations are equivalent to (25).

Using the new parameters and substituting (29) into (16), matrix $\mathsf{A}(t)$ in (16) can be expressed as

$$\mathsf{A}(t) = \frac{1}{\pi}\mho(t)\bar{\mathsf{A}}, \quad \mho(t) := \begin{bmatrix}C_\eta\\ -S_\eta\end{bmatrix}\begin{bmatrix}C_\eta\\ -S_\eta\end{bmatrix}^{\mathsf{T}} \quad (30)$$

with $\bar{\mathsf{A}}$ defined in (27). Now, by Theorem 1, the periodic solution ξ is orbitally exponentially stable if and only if the linear periodic system $\dot e = \mathcal{A}(t)e$ is exponentially stable, where $\mathcal{A}(t) := \mathsf{P}^{\mathsf{T}}\mathsf{A}(t)\mathsf{Q}$ is of order ε. When the harmonic oscillators are weakly coupled, i.e., when $\varepsilon>0$ is sufficiently small, the stability of the linear periodic system is equivalent to stability of its averaged dynamics due to Lemma 3. Noting that the integral of $\mho(t)$ over one cycle period $T=2\pi$ is given by πI, we see that $\bar{\mathsf{A}}$ is the average of $\mathsf{A}(t)$. Hence, it follows from (17) that $\overline{\mathcal{A}}$ in (28) is the average of $\mathcal{A}(t)$, proving (i) $\Leftrightarrow$ (iii). The equivalence of (ii) and (iii) under (26) can be shown through a simple similarity transformation involving P and Q.

Finally, we show that (26) is equivalent to condition (3) for the coupling to vanish on the target orbit. Note that

$$\begin{aligned} h(\xi,\dot\xi)\equiv 0 \quad &\Leftrightarrow \quad D\dot\xi + K\xi \equiv 0 \\ &\Leftrightarrow \quad (DC_\varphi + KS_\varphi)\mathbb{1}\cos t \\ &\qquad +(KC_\varphi - DS_\varphi)\mathbb{1}\sin t \equiv 0. \end{aligned}$$

It is straightforward to verify that the coefficients of the sinusoids are zero if and only if (26) holds. ∎

For the synchronization case $\varphi_i = 0$ for all $i \in \mathbb{Z}_n$, all the entries of C_φ are one and $\mathsf{S}_\varphi = 0$, and hence the transformation in (25) reduces to

$$D = \Delta, \quad K = \nabla, \quad M = \Phi, \quad N = \Psi.$$

Thus, the parameters $(\Delta, \nabla, \Phi, \Psi)$ may be considered as a version of the stiffness and damping parameters (D, K, M, N), properly transformed in accordance with the coordination requirement specified by φ_i for $i \in \mathbb{Z}_n$. Note from (25) that $M = \Phi$ and $N = \Psi$ are true for any phase specification φ_i when Φ and Ψ are restricted to be diagonal.

C. Basic mechanisms underlying stable coordination

The purpose of this section is to uncover basic mechanisms for orbital stabilization and coordination of the coupled oscillators in Theorem 2. For brevity, some special cases are considered. We will also compare our result with one of the leading results [26] and point out a nontrivial progress made in our paper relative to the state of the art.

Recall from Theorem 2 that the average of the linearized dynamics (11) for the amplitudes ρ and relative phases ϕ are given by $\dot{\mathsf{e}} = \overline{\mathcal{A}}\mathsf{e}$ with $\mathsf{e} := \mathrm{col}\,(\rho, \phi)$ where $\overline{\mathcal{A}}$ is defined by (27) and (28). If we choose $\nabla = \Psi = 0$, then $\bar{\mathsf{A}}$ in (27) is block diagonal and decouples the dynamics as

$$\dot\rho = -\varepsilon\pi(\Delta + \Phi)\rho, \quad \dot\phi = -\varepsilon\pi P^{\mathsf{T}}\Delta Q\phi.$$

Thus, the phase ϕ is stabilized by the coupling through Δ, and the amplitude ρ is stabilized with the aid of the additional nonlinear damping provided through Φ. The core of the coupled oscillators design is the search for matrices Δ and Φ to stabilize these dynamics. We will consider this design problem in Section IV-B exploiting the graph theory.

To gain a deeper understanding of the coordination mechanism, we will express the coupled oscillators in Theorem 2 in terms of the state $y_i := \mathrm{col}\,(q_i, \dot q_i)$ of each oscillator. First note that an alternative form of (1) with $\omega = 1$ is given by

$$\dot y_i = Sy_i + Fv_i, \quad S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{aligned} v_i &:= \mathrm{col}\,(*, h_i(y)), \\ F &:= \mathrm{diag}(0,1). \end{aligned}$$

A major challenge stems from the fact that the each oscillator can be influenced only through the channel specified by F. But this structure of F is important for control of physical systems such as (7) since the force input can directly affect the acceleration, but not the velocity. If $F = I$, arbitrary coupling of oscillators $\dot y_i = Sy_i$ is possible, and the coordination problem is much easier. To make this point clear, let us consider the simple case where $\Phi = 2dI$ and $\nabla = \Psi = 0$ in Theorem 2, and state the condition for coordinated oscillations in comparison with the one from [26] which assumes $F = I$.

Corollary 1: Let $\varepsilon, d \in \mathbb{R}$, $\varphi \in \mathbb{R}^n$, $\Delta \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{2\times 2}$ be given, where ε and d are positive and $\Delta\mathbb{1} = 0$. Consider the coupled oscillators

$$\dot y_i = \mathfrak{f}_o(y_i) + \mathfrak{h}_i(y), \quad i \in \mathbb{Z}_n \tag{31}$$

with the oscillator dynamics $\mathfrak{f}_o$ of $y_i(t) \in \mathbb{R}^2$ and the coupling function $\mathfrak{h}_i$ of $y(t) := \mathrm{col}\,(y_1, \dots, y_n)$ given by

$$\mathfrak{f}_o(y_i) := Sy_i + \varepsilon d(1 - \|y_i\|^2)Fy_i, \tag{32}$$

$$\mathfrak{h}_i(y) := \varepsilon \sum_{j=1}^{n} \delta_{ij} F\big(y_i - \Omega_{ij} y_j\big), \tag{33}$$

where δ_{ij} is the (i,j) entry of Δ, φ_i is the i^{th} entry of φ, and

$$\Omega_{ij} := \Omega_i \Omega_j^{\mathsf{T}}, \quad \Omega_i := \begin{bmatrix} \cos\varphi_i & \sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{bmatrix}.$$

The periodic signal $y = \mathfrak{x}$ with

$$\mathfrak{x} := \mathrm{col}\,(\mathfrak{x}_1, \dots, \mathfrak{x}_n), \quad \mathfrak{x}_i(t) := \begin{bmatrix} \sin(t + \varphi_i) \\ \cos(t + \varphi_i) \end{bmatrix}$$

is a solution with the following properties:

(i) when $F = \mathrm{diag}(0,1)$, the solution $\mathfrak{x}$ is orbitally stable if all the eigenvalues of $P^{\mathsf{T}}\Delta Q$ have positive real parts, and ε is sufficiently small;

(ii) when $F = I$, the solution $\mathfrak{x}$ is globally orbitally stable[1] if the symmetric part of $P^{\mathsf{T}}\Delta Q - dI$ is positive definite;

where P and Q are the matrices satisfying (15) and $P = Q$.

Proof: Statement (i) is a special case of Theorem 2, where

$$y_i := \mathrm{col}\,(q_i, \dot q_i), \quad \Phi = 2dI, \quad \Psi = 0, \quad \nabla = 0.$$

In particular, orbital stability of the periodic solution ξ is equivalent to the Hurwitz property of $\overline{\mathcal{A}}$ in (28). With the above choices of Ψ and ∇, matrix $\overline{\mathcal{A}}$ becomes block diagonal and its eigenvalues are those of $-\varepsilon\pi(\Delta+\Phi)$ and $-\varepsilon\pi P^{\mathsf{T}}\Delta Q$. The matrix Δ has an eigenvalue at 0 since $\Delta\mathbb{1} = 0$, in which case the matrix $-P^{\mathsf{T}}\Delta Q$ is Hurwitz if and only if the remaining $n-1$ eigenvalues of Δ have positive real parts. This can be seen from the similarity transformation

$$\begin{bmatrix} Q & \mathbb{1} \end{bmatrix}^{-1} \Delta \begin{bmatrix} Q & \mathbb{1} \end{bmatrix} = \begin{bmatrix} P^{\mathsf{T}}\Delta Q & 0 \\ p^{\mathsf{T}}\Delta Q & 0 \end{bmatrix}. \tag{34}$$

Thus, if Φ in $-(\Delta + \Phi)$ moves the zero eigenvalue of $-\Delta$ to the left half complex plane without making the other eigenvalues migrate to the right half plane, then $-(\Delta + \Phi)$ would be Hurwitz. An obvious choice of such Φ is $\Phi = 2dI$.

Statement (ii) essentially follows from the synchronization condition for coupled subsystems based on the contraction theory (eqn (10) in [26]), and global orbital stability of $\mathfrak{x}_i$ for each subsystem $\dot y_i = \mathfrak{f}_o(y_i)$ (Example 7.2 in [33]). ∎

Clearly, (31) can be seen as a set of subsystems $\dot y_i = \mathfrak{f}_o(y_i)$ coupled through $\mathfrak{h}_i(y)$. Each subsystem is a nonlinear oscillator having a periodic solution $\mathfrak{x}_i$ with an arbitrary phase $\varphi_i \in \mathbb{R}$. The nonlinear term in $\mathfrak{f}_o(y_i)$ adds an amplitude-dependent corrective effect to the linear oscillator $\dot y_i = Sy_i$

[1] Here we mean that, with an arbitrary initial state $y(0)$ not on the stable manifold of the unstable equilibrium at the origin, the solution $y_i(t)$ converges to $\mathfrak{x}_i(t + c)$ for some $c \in \mathbb{R}$ that depends on the initial state $y(0)$.

to regulate the amplitude at $\|y_i\| = 1$. The coupling $\mathfrak{h}_i(y)$ is diffusive and vanishes on the orbit $y = \mathfrak{x}$ since $\mathfrak{x}_i = \Omega_{ij}\mathfrak{x}_j$, but generates corrective inputs to each oscillator when the inter-oscillator coordination has an error. Thus the similar stabilization mechanisms work for cases of (i) and (ii), but the proven stability property is only local for (i) but is global for (ii) although our numerical experience indicates that $\mathfrak{x}$ in (i) may also be globally orbitally stable.

The difficulty associated with the singular structure of F in (i) may be best illustrated by showing why the problem simplifies when $F = I$. Note that (31) can be written as

$$\dot{y} = \mathfrak{f}(y) + \mathfrak{h}(y), \tag{35}$$

where $\mathfrak{f}(y)$ and $\mathfrak{h}(y)$ are defined by stacking $\mathfrak{f}_o(y_i)$ and $\mathfrak{h}_i(y)$ for $i \in \mathbb{Z}_n$ in columns, respectively. Coordinated oscillations $y_i(t) = \mathfrak{x}_i(t + c)$ are achieved with some $c \in \mathbb{R}$ when $\Omega_i^{\mathsf{T}} y_i = \Omega_j^{\mathsf{T}} y_j$ for all $i, j \in \mathbb{Z}_n$, or equivalently, when $y \in \mathbb{M}$ where $\mathbb{M}$ is the subspace spanned by columns of $M := \mathrm{col}\,(\Omega_1, \ldots, \Omega_n)$. The special property of (35) with $F = I$ is that $\mathbb{M}$ is a flow-invariant subspace[2] of $\mathfrak{f}$, which can be verified by noting that $\Omega_i \mathfrak{f}_o(y_o) = \mathfrak{f}_o(\Omega_i y_o)$ holds for all $y_o \in \mathbb{R}^2$. Since the coupling vanishes on $\mathbb{M}$, i.e., $\mathfrak{h}(y) = 0$ for $y \in \mathbb{M}$, we see that $\mathbb{M}$ is a flow-invariant subspace of $\mathfrak{f} + \mathfrak{h}$. Now, it was shown in [26] that global convergence to $\mathbb{M}$ is guaranteed by contraction[3] of $\mathfrak{f} + \mathfrak{h}$ in the orthogonal subspace of $\mathbb{M}$, which in turn is implied by the diffusive coupling through Δ, sufficiently strong relative to the amplitude-dependent damping d, as stated in Corollary 1.

On the other hand, the contraction method does not work when $F = \mathrm{diag}(0, 1)$ due to the lack of the flow-invariance property; $\mathbb{M}$ is no longer a flow-invariant subspace of $\mathfrak{f}$ since $\Omega_i \mathfrak{f}_o(y_o) \neq \mathfrak{f}_o(\Omega_i y_o)$ in general. To see this difficulty more closely, note that the coordinated oscillations of y_i are equivalent to synchronization of $\mathfrak{y}_i := \Omega_i^{\mathsf{T}} y_i$. The dynamics of $\mathfrak{y}_i$, when uncoupled, are described by $\dot{\mathfrak{y}}_i = \mathfrak{f}_i(\mathfrak{y}_i) := \Omega_i^{\mathsf{T}} \mathfrak{f}_o(\Omega_i \mathfrak{y}_i)$. If $F = I$, then $\mathfrak{f}_i = \mathfrak{f}_o$ for all $i \in \mathbb{Z}_n$, and the coordination problem is equivalent to the synchronization of *identical* oscillators $\dot{\mathfrak{y}}_i = \mathfrak{f}_o(\mathfrak{y}_i)$. In this case, $\mathfrak{f}$ has a flow-invariant subspace $\mathbb{M}_o$ spanned by $\mathrm{col}\,(I, \ldots, I)$, which corresponds to the synchronized state. If $F = \mathrm{diag}(0, 1)$, however, we have $\mathfrak{f}_i \neq \mathfrak{f}_j$ for $i \neq j$, and coordination of identical oscillators $\dot{y}_i = \mathfrak{f}_o(y_i)$ is equivalent to synchronization of *heterogeneous* oscillators $\dot{\mathfrak{y}}_i = \mathfrak{f}_i(\mathfrak{y}_i)$, which is more difficult than synchronization of identical oscillators $\dot{\mathfrak{y}}_i = \mathfrak{f}_o(\mathfrak{y}_i)$ since $\mathbb{M}_o$ is not a flow-invariant subspace of $\mathfrak{f}(\mathfrak{y}) := \mathrm{col}\,(\mathfrak{f}_1(\mathfrak{y}_1), \ldots, \mathfrak{f}_n(\mathfrak{y}_n))$.

IV. Design for synchronization and coordination

A. General procedure

Based on the analysis results in the previous section, we now propose a method for designing the coupling function h in (1) to achieve a desired oscillation $\xi(t)$ in (2) as a stable limit cycle. In particular, we use the coupling function of the form (22) with (24), parametrized by $\varepsilon \in \mathbb{R}$ and (D, K, M, N). We search for the coupling parameter (D, K, M, N) satisfying the conditions in Theorem 2. Orbital stability can then be guaranteed for sufficiently small $\varepsilon > 0$. The process yields a coupling function $h(q, \dot{q})$ for the normalized target oscillations with $\omega = \gamma_i = 1$ for $i \in \mathbb{Z}_n$. To achieve desired oscillations with arbitrary frequency and amplitudes, the coupling function can be modified as in (6). More specifically, we propose the following.

Design Procedure

0. Set the frequency ω, amplitudes γ_i, and phases φ_i for the desired oscillation $\xi_i(t)$, $i \in \mathbb{Z}_n$, in (2).
1. Find $n \times n$ matrices $(\Delta, \nabla, \Phi, \Psi)$ such that condition (26) is satisfied and $\overline{\mathcal{A}}$ defined by (27) and (28) is Hurwitz, where ε is an arbitrary positive number.
2. Find $\varepsilon > 0$ such that the maximum magnitude of the Floquet multipliers of (11) with $\mathcal{A}(t)$ defined by (17) and (30), and $\eta_i := t + \varphi_i$, is less than one.
3. Set the coupling function $h(q, \dot{q})$ as in (22) where $\mathscr{E}(q, \dot{q})$ is defined by (24), and (D, K, M, N) are determined from $(\Delta, \nabla, \Phi, \Psi)$ by solving (25) as

$$\begin{aligned} D &= \mathsf{S}_\varphi \cdot \nabla + \mathsf{C}_\varphi \cdot \Delta, \quad K = \mathsf{C}_\varphi \cdot \nabla - \mathsf{S}_\varphi \cdot \Delta, \\ M &= \mathsf{S}_\varphi \cdot \Psi + \mathsf{C}_\varphi \cdot \Phi, \quad N = \mathsf{C}_\varphi \cdot \Psi - \mathsf{S}_\varphi \cdot \Phi. \end{aligned} \tag{36}$$

   Determine the scaled coupling function $\hbar(q, \dot{q})$ by (6) using ω and γ_i for $i \in \mathbb{Z}_n$. Then $q = \xi$ with (2) is an orbitally exponentially stable periodic solution of the coupled oscillators (1) with h replaced by $\hbar$.

In the design, Step 1 can be executed as follows. First note that condition (26) is equivalent to

$$\bar{\mathsf{A}}\, \mathbb{1}_o = 0, \quad \mathbb{1}_o := \mathrm{col}\,(0, \mathbb{1}).$$

It can be shown using standard results in linear algebra that the set of all matrices $\bar{\mathsf{A}}$ satisfying the linear equations

$$\bar{\mathsf{A}}\, \mathbb{1}_o = 0, \quad \overline{\mathcal{A}} = \mathsf{P}^{\mathsf{T}} \bar{\mathsf{A}} \mathsf{Q}$$

is parametrized by

$$\bar{\mathsf{A}} = (\mathsf{Q}\overline{\mathcal{A}} + \mathbb{1}_o a^{\mathsf{T}})\mathsf{P}^{\mathsf{T}},$$

where $a \in \mathbb{R}^{2n-1}$ is an arbitrary vector. Hence, $(\Delta, \nabla, \Phi, \Psi)$ can be found by choosing arbitrary Hurwitz matrix $\overline{\mathcal{A}}$ and vector a, calculating $\bar{\mathsf{A}}$ as above, and solving (27) for $(\Delta, \nabla, \Phi, \Psi)$. In Step 2, Theorem 2 guarantees that a sufficiently small $\varepsilon > 0$ satisfies the Floquet multiplier condition. To verify that a chosen value of ε is sufficiently small, one can calculate the fundamental matrix $\Theta(t)$ defined by (9) with $A := \mathcal{A}$ through numerical integration, and make sure that

$$\tau(\varepsilon) < 1, \quad \tau(\varepsilon) := \max_i |\lambda_i[\Theta(2\pi)]|, \tag{37}$$

holds, where the maximum is taken over all $2n-1$ eigenvalues λ_i of $\Theta(2\pi)$. Finally, Step 3 is a straightforward computation.

The design steps proposed above progressively use the desired oscillation profile $(\omega, \varphi, \gamma)$. In particular, $(\Delta, \nabla, \Phi, \Psi)$ in Step 1 is completely independent of the design specifications for ξ, and therefore, once a matrix quadruple $(\Delta, \nabla, \Phi, \Psi)$ with a desired property (e.g. convergence speed of the average dynamics $\overline{\mathcal{A}}$) is found, it can be used for different oscillation

[2] A subspace $\mathbb{M} \subset \mathbb{R}^n$ is said to be a *flow-invariant subspace* of function $f : \mathbb{R}^n \to \mathbb{R}^n$ if $f(x) \in \mathbb{M}$ for all $x \in \mathbb{M}$.

[3] A function $f(x)$ is said to be *contracting* if the symmetric part of $\frac{\partial f}{\partial x}(x)$ is uniformly negative definite for $x \in \mathbb{R}^n$.

profiles. In Step 2, the value of ε is determined with the knowledge of the phase parameters φ_i only. The values of the desired amplitudes γ_i are not used until Step 3. Hence, the parameter ε should be revised only when the desired phase values are changed, and the parameters ε and $(\Delta, \nabla, \Phi, \Psi)$ fixed in the first two steps can be used for various frequency and amplitude specifications. The formula (36) in Step 3 shows that a change of φ_i modifies the i^{th} row/column of (D, K, M, N), and thus requires adjustments only in the coupling gains between the i^{th} oscillator and those connected to it.

B. Structured coupling matrix

Design Procedure given in the previous section allows us to find a coupling function $h(q, \dot{q})$ to achieve desired oscillations ξ_i with orbital stability. In general, the procedure leads to a design with all-to-all coupling, i.e., each oscillator may receive inputs from all the other oscillators. However, it may be desired in certain applications that the coupling between oscillators is constrained to have a specific structure (e.g. oscillator 1 is allowed to receive input signals from oscillators 2 and 3, but not from oscillator 4). This section shows how to incorporate such structural constraints into the coupling function design.

The interconnection structure of the coupled oscillators is dictated by the structure of the coupling matrices (D, K, M, N); the i^{th} oscillator sends its state information $(q_i, \dot{q}_i)$ directly to the j^{th} oscillator when one of these matrices has a nonzero (j, i) entry. The structure is preserved under the change of variables between (D, K, M, N) and $(\Delta, \nabla, \Phi, \Psi)$ as seen in (25) and (36). Therefore, a constraint on the interconnection structure can be imposed by requiring each of $(\Delta, \nabla, \Phi, \Psi)$ to belong to $\mathbb{W}$ in Step 1 of Design Procedure, where $\mathbb{W}$ is a set of structured $n \times n$ real matrices with (j, i) entry being zero when no direct connection is allowed from the i^{th} oscillator to the j^{th} oscillator.

In what follows, we use the graph theory to show how structured coupling matrices $\Delta, \nabla, \Phi, \Psi \in \mathbb{W}$ can be found to satisfy the conditions in Step 1 of Design Procedure. To this end, let us introduce the following simplifying assumption.

Assumption 2: The coupling matrix ∇ or $\nabla + \Psi$ is zero, and Φ is diagonal. Moreover, off-diagonal entries of Δ are nonpositive, and diagonal entries of Φ are nonnegative.

Under this assumption, $\overline{\mathcal{A}}$ in (28) is block triangular, and orbital stability of the target oscillations is guaranteed by the choice of Δ and Φ, independently of the choice of ∇ and Ψ. In particular, the orbital stability condition is given by $-(\Delta+\Phi)$ and $-P^{\mathsf{T}}\Delta Q$ being Hurwitz, as in the special case $\nabla = \Psi = 0$ discussed in Section III-C. We will thus focus on the design of $\Delta, \Phi \in \mathbb{W}$ to satisfy the Hurwitz properties.

To exploit the graph theory, let us introduce some basic concepts. For a given coupling matrix Δ satisfying (26), let G be the directed graph (digraph) having Δ as its Kirchhoff matrix. That is, each oscillator is considered as a vertex of G, and there is an edge e_{ij} directed from the ith vertex to the jth vertex when $i \neq j$ and $\delta_{ji} \neq 0$, i.e., the (j, i) entry of Δ is nonzero. The digraph G is said to contain a *spanning tree* if removal of some (or no) edges from G defines a digraph such that one of the vertices (called the *root* of the tree) receives no inputs (no edge is directed toward it) and every other vertex receives one input (exactly one edge is directed toward it). The *weight of edge* e_{ij} is $-\delta_{ji}$ for $i \neq j$, and the *weight of a tree* is the product of the weights of all the edges in the tree.

The following result gives necessary and sufficient conditions for stability of the average system in (28).

Lemma 4: Let $\Delta, \Phi \in \mathbb{R}^{n\times n}$ be given. Suppose Φ is diagonal with nonnegative entries, and Δ has nonpositive off-diagonal entries and satisfies $\Delta \mathbb{1} = 0$. Let G be the digraph having Δ as its Kirchhoff matrix, and P and Q be matrices satisfying (15). The following statements are equivalent.

(i) $-P^{\mathsf{T}}\Delta Q$ is Hurwitz.
(ii) The digraph G contains a spanning tree.

If these statements hold, the following two are equivalent.

(iii) $-(\Delta + \Phi)$ is Hurwitz.
(iv) There exists an index $j \in \mathbb{Z}_n$ such that vertex j is a root of a spanning tree contained in G, and the (j, j) entry of Φ is positive.

Proof: See the appendix. ■

The result in Lemma 4 shows that the average dynamics (28) are stable when all the oscillators are connected by an embedded tree structure (statement (ii)) and the oscillator at the root of the tree is properly regulated by a nonlinear damping (statement (iv)). In general, existence of a spanning tree is essential for coordination (and for synchronization as a special case) which requires sharing of information among all agents. The additional condition at the root ensures convergence of amplitudes to prescribed values γ_i. The main result in this section can now be stated as follows.

Theorem 3: Consider the coupled oscillators described by (1), (22), (24), and (36). Suppose Assumptions 1 and 2 and (26) are satisfied. Let G be the digraph with Kirchhoff matrix Δ. The periodic solution (2) is orbitally exponentially stable for sufficiently small $\varepsilon > 0$ if and only if G contains a spanning tree with root $j \in \mathbb{Z}_n$ and the (j, j) entry of Φ is positive.

Proof: By Theorem 2, the periodic solution is orbitally stable for sufficiently small $\varepsilon > 0$ if and only if both $-P^{\mathsf{T}}\Delta Q$ and $-(\Delta + \Phi)$ are Hurwitz. The result then follows from Lemma 4. ■

The following result shows how to check existence of a spanning tree and identify its root vertex.

Lemma 5: Consider a digraph G with the associated Kirchhoff matrix $\mathcal{K}$. If vertex i is the root of a spanning tree of G, the (i, i)-cofactor $\det(\mathcal{K}_i^-)$ equals the sum of the weights of all spanning trees rooted at i, where $\mathcal{K}_i^-$ denotes the matrix obtained by deleting the i^{th} column and row from $\mathcal{K}$. In particular, if the weights of all edges are set to 1, then $\det(\mathcal{K}_i^-)$ is the number of spanning trees rooted at i. If there is no spanning tree rooted at vertex i, then $\det(\mathcal{K}_i^-) = 0$.

Proof: This is a dual version of the weighted matrix-tree theorem for spanning (converging) trees (see Theorem 2.1 of [34]) and can be proven by reversing the direction of every edge while keeping the same weight. ■

To design a coupling function with a desired structure $\mathbb{W}$, we need to verify that the graph structure imposed by $\mathbb{W}$

contains a spanning tree because otherwise orbital stability cannot be achieved as stated in Theorem 3. For this purpose, Lemma 5 can be used. Let $\mathcal{K} \in \mathbb{W}$ be the matrix such that each off-diagonal entry is equal to -1 whenever a nonzero entry is allowed by $\mathbb{W}$, and diagonal entries are set to satisfy $\mathcal{K}\mathbb{1} = 0$. Let G be the digraph having $\mathcal{K}$ as its Kirchhoff matrix. For each $i \in \mathbb{Z}_n$, if $\det(\mathcal{K}_i^-) \neq 0$, then G contains a spanning tree with its root at vertex i.

Once existence of a spanning tree is verified for a desired coupling structure $\mathbb{W}$, Step 1 of Design Procedure in Section IV can be executed with structural constraints $(\Delta, \nabla, \Phi, \Psi) \in \mathbb{W}^4$ using Theorem 3. First, let $\Delta \in \mathbb{W}$ be an arbitrary matrix such that each off-diagonal entry is negative whenever a nonzero entry is allowed and diagonal entries are set to satisfy $\Delta\mathbb{1} = 0$. Let Φ be a diagonal matrix such that the (j, j) entry is positive and the other diagonal entries are nonnegative, where $j \in \mathbb{Z}_n$ is the index of a root vertex. Finally, let $\nabla, \Psi \in \mathbb{W}$ be arbitrary matrices such that $\nabla\mathbb{1} = 0$ holds and at least one of ∇ and $\nabla + \Psi$ is zero.

When each of Φ and Ψ is diagonal with a nonzero value only at the (j, j) entry, the j^{th} oscillator can act as the leader to regulate the amplitudes at γ uniformly over the oscillators by modifying $\mathscr{E}$ as $\mathscr{E}_{jj}(q, \dot{q}) = ((r_j/\gamma)^2 - 1)/2$. This is because the scaling by $\Gamma = \gamma I$ in (6) only affects $\mathscr{E}$ as indicated above.

C. Optimal design for fast convergence

The design process described in the previous sections leaves large freedom in the choice of design parameters ε and $(\Delta, \nabla, \Phi, \Psi) \in \mathbb{W}$. The design theory is based on weak coupling of oscillators, and therefore corrective inputs from other oscillators, when perturbed from the target orbit, are small, resulting in slow convergence. This section discusses how to optimize the design parameters to speed up the convergence to the target oscillation.

Under weak coupling (small $\varepsilon > 0$), the maximum Floquet multiplier of the linearized dynamics (11) with $\mathcal{A}(t)$ in (17) is close to the unit circle, and hence convergence to the target orbit is slow. To speed up, we can optimize ε instead of arbitrarily choosing ε to satisfy (37) in Step 2 of Design Procedure. In particular, we can strengthen the coupling by minimizing the maximum magnitude of the Floquet multipliers:

$$\tau_o = \min_{\varepsilon > 0} \tau(\varepsilon), \tag{38}$$

A practical way for solving the optimization is to grid the ε parameter space, plot the function $\tau(\varepsilon)$ and look for the minimum. The value of $\tau(\varepsilon)$ for a given ε can be obtained through numerical integration of (9) with $A := \mathcal{A}$.

The optimal value τ_o is guaranteed to be less than one as required for orbital stability because Theorem 2 ensures $\tau(\varepsilon) < 1$ for sufficiently small $\varepsilon > 0$. In fact, it can be shown using the Peano-Baker series [35] that $\tau(0) = 1$ and $\tau'(0) = \sigma < 0$ hold, where σ is the largest real part of the eigenvalues of $\overline{\mathcal{A}}$. Hence, it is a reasonable option to minimize σ during the search for $(\Delta, \nabla, \Phi, \Psi) \in \mathbb{W}^4$ in Step 1. Let us explore this idea for the special case $\nabla = 0$, which was considered in Section IV-B under Assumption 2. In this case, the matrix $\overline{\mathcal{A}} := \mathsf{P}\bar{\mathsf{A}}\mathsf{Q}$ is block triangular with $-\varepsilon\pi(\Delta + \Phi)$ and $-\varepsilon\pi P^{\mathsf{T}}\Delta Q$ on the diagonal, and hence the convergence rate $|\sigma|$ is determined by these two matrices. Since the effect of Φ is to add nonlinear damping to speed up the convergence, we choose to focus on $-P^{\mathsf{T}}\Delta Q$ and attempt to minimize the largest real part of its eigenvalues over Δ.

Recall from the standard Lyapunov theory that the largest real part of the eigenvalues of $-P^{\mathsf{T}}\Delta Q$ is less than $\gamma/2$ if and only if there exists X such that

$$P^{\mathsf{T}}\Delta Q X + X(P^{\mathsf{T}}\Delta Q)^{\mathsf{T}} + \gamma X > 0, \quad X = X^{\mathsf{T}} > I. \tag{39}$$

Thus we may minimize γ over X and $\Delta \in \mathbb{W}$ subject to this condition and $\Delta\mathbb{1} = 0$, where a magnitude normalization, e.g., $\|\Delta\| < 1$, should be added to the constraint to make γ bounded below. This is a structured stabilization problem which is difficult to solve with theoretical guarantee for optimality or even feasibility in general. However, the graph theoretic result in Lemma 4 shows exactly when it is feasible — when the digraph underlying $\mathbb{W}$ has a spanning tree. Moreover, heuristic numerical methods [36–38] appear to work reasonably well for (39) due to relatively large design freedom in $\Delta \in \mathbb{W}$. We will illustrate this by numerical examples in the next section.

The maximum Floquet multiplier τ_o indicates the rate of convergence only in the neighborhood of the desired orbit. The convergence speed for an initial state away from the orbit can be evaluated, through simulations after the design, in terms of the instantaneous amplitude and phase errors defined by

$$e_A(t) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2}, \quad e_P(t) = \sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2}, \tag{40}$$

where

$$a_i = |p_i|/\gamma_i - 1, \quad b_{ij} = (\angle p_i - \angle p_j) - (\varphi_i - \varphi_j),$$
$$p_i := (\dot{q}_i/\omega) + \sqrt{-1}\, q_i$$

with $\angle(\cdot)$ being the angle of a complex number defined so that $|b_{ij}| \leq \pi$.

V. Illustrative examples

Consider the coupled harmonic oscillators in (1) with $n = 5$. We design structured coupling functions $h(q, \dot{q})$ following Design Procedure, where the five oscillators are allowed to couple as shown in the digraph in Fig. 1. The coupling structure $\mathbb{W}$ is defined by

$$\mathbb{W} := \{\, R \cdot \mathcal{K} :\ R \in \mathbb{R}^{5\times 5} \,\},$$
$$\mathcal{K} := \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}, \tag{41}$$

where $\mathcal{K}$ is the Kirchhoff matrix of the digraph, with all weights set to 1. Note that $\mathbb{W}$ is the set of 5×5 matrices that share the location of zero entries with $\mathcal{K}$. Simple calculations show that $\det(\mathcal{K}_i^-) = 11$ for $i = 1, 2$ and $\det(\mathcal{K}_i^-) = 0$ for $i = 3, 4, 5$. Hence, the digraph contains 22 spanning trees; half of them are rooted at vertex 1, and the other half at vertex 2. Therefore, from Theorem 3, orbital stability is achieved with

weak coupling under Assumption 2 if Φ has a positive value at the $(1,1)$ or $(2,2)$ entry, and $\Delta \in \mathbb{W}$ satisfies $\Delta \mathbb{1} = 0$ and is given by $\Delta = R \cdot \mathcal{K}$ for some $R \in \mathbb{R}^{5\times 5}$ with positive entries.

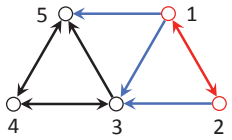

Fig. 1. Digraph for oscillator architecture. Each circle is an oscillator (vertex), and arrows indicate allowable signal flow which corresponds to $\mathbb{W}$.

We consider the following cases for the choice of $(\Delta, \nabla, \Phi, \Psi)$ in Step 1 of Design Procedure:

(a) $\Delta = \mathcal{K}$, $\Phi = \mathrm{diag}(0,0,1,1,1)$, $\nabla = \Psi = 0$, where $\Phi_{ii} = 0$ at all root vertices $i = 1$ and 2.

(b) $\Delta = \mathcal{K}$, $\Phi = \mathrm{diag}(1,0,0,1,0)$, $\nabla = \Psi = 0$, where $\Phi_{ii} > 0$ at a root vertex $i = 1$.

(c) $\Delta = R \cdot \mathcal{K}$ and R is optimized by the D-K iteration [36] to minimize γ subject to $\Delta \mathbb{1} = 0$, $\|\Delta\| < 1$, and (39). Same Φ as (b), and $\nabla = \Psi = 0$.

(d) Same (Δ, Φ) as (c), with $\nabla = \Delta/5$ and $\Psi = I/5$.

Orbital stability is not achieved for case (a) since, for any spanning tree with its root at $j \in \mathbb{Z}_n$, the (j,j) entry of Φ is zero. In particular, the amplitudes may not converge to desired values due to the lack of damping, although the relative phases will converge since the digraph contains a spanning tree with positive weights. Stability of $\overline{\mathcal{A}}$ is guaranteed by Lemma 4 for cases (b) and (c), but not for case (d) due to violation of Assumption 2. For case (d), stability was verified after the design with additional terms (∇, Ψ). The coupled oscillators for cases (a)–(c) can be seen as the system in (31)-(33) with $F = \mathrm{diag}(0,1)$ and d in (32) replaced by Φ_{ii}. For comparison, we consider the same system with $F = I$ and $d = 1/2$, resulting in the coupled Andronov-Hopf oscillators [26]. We label this case as (e):

(e) System (31)-(33) with $d = 1/2$ (corresponding to $\Phi = I$ and $\nabla = \Psi = 0$), the same Δ as (c), and $F = I$.

For the above five cases, we proceed to Step 2 and optimize ε as in (38) where the maximum Floquet multiplier $\tau(\varepsilon)$ defined in (37) is minimized for faster convergence to each of the synchronized oscillations with $\varphi = 0$, and coordinated oscillations with $\varphi = \mathrm{col}\,(0, 20, 50, 90, 140)$ degrees. The $\tau(\varepsilon)$ curves are shown in Fig. 2. For case (a), $\tau(\varepsilon) \geq 1$ for all $\varepsilon \geq 0$ and the orbit is not stable due to the lack of damping at root vertices, consistently with Theorem 3. For all the other cases, the curves start at $\tau(0) = 1$ with negative slopes to achieve orbital stability $\tau(\varepsilon) < 1$ for sufficiently small $\varepsilon > 0$ as guaranteed by Theorem 2. Case (c) is similar to case (b) but the minimum of $\tau(\varepsilon)$ is smaller than that of (b) as a result of the Δ optimization. The minimum of $\tau(\varepsilon)$ in case (d) is even smaller than that of (c) due to the additional coupling ∇ and nonlinear stiffness Ψ. For case (e), we have $\tau(\varepsilon) \to 0$ as $\varepsilon \to \infty$ and thus the stronger the coupling, the faster the convergence. We do not see this property for cases (b)–(d) because[4] the orbital

[4]The reasoning was justified by observing the absence of this property for case (c) with modified $\Phi = I$, which corresponds to case (e) with $F = \mathrm{diag}(0,1)$ (result not shown).

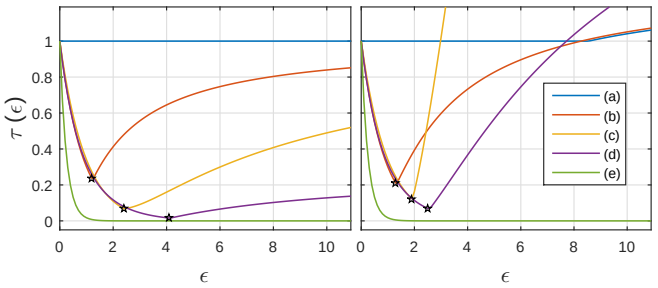

Fig. 2. Maximum magnitude of Floquet multipliers, $\tau(\varepsilon)$, as a function of ε. Left: synchronization. The optimizers are $\varepsilon_o = 1.2$, 2.4, 4.1 for cases (b), (c), (d), respectively. Right: coordination. The optimizers are $\varepsilon_o = 1.3$, 1.9, 2.5 for cases (b), (c), (d), respectively.

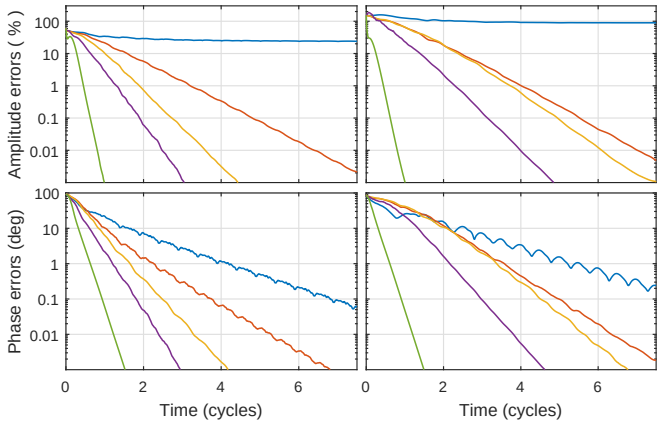

Fig. 3. Instantaneous amplitude and phase errors averaged over 100 samples of random initial conditions. Left column: synchronization. Right column: coordination. The line colors are as indicated in the legend of Fig. 2.

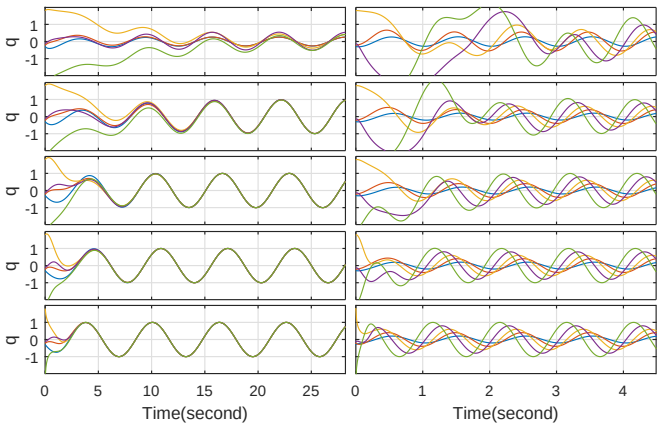

Fig. 4. Simulated time courses of $q(t)$ for the ten cases with a same initial condition. Left column: synchronization. Right column: coordination. Top to bottom: cases (a)-(e).

stabilization effect in (32) and (33) is through $F = \mathrm{diag}(0,1)$ and is not uniformly applied to the two entries of $y_i \in \mathbb{R}^2$. In fact, strong coupling (i.e. large $\varepsilon > 0$) destabilizes the orbit ($\tau(\varepsilon) > 1$) when coordination ($\varphi \neq 0$) is considered. The result illustrates difficulty of achieving coordination in the absence of the flow-invariance property.

We now proceed to Step 3 to determine the coupling function $h(q, \dot{q})$. The amplitudes are uniform $\gamma = \mathbb{1}$ for synchronization cases with $\omega = 1$, and nonuniform $\gamma = \mathrm{col}\,(0.2, 0.4, 0.6, 0.8, 1)$ for coordination cases with $\omega = 2\pi$.

The optimal values $\varepsilon := \varepsilon_o$ are used for all the cases except for (a) and (e) where ε is chosen as $\varepsilon = 2$. The nonlinear oscillator thus designed for each case is simulated with random initial conditions $q_i(0)$ and $\dot{q}_i(0)$, sampled from the normal distribution with mean zero and standard deviation 1. The system is simulated for 100 samples of the initial condition, and the amplitude and phase errors in (40) are calculated and averaged over the 100 samples at each time instant The time courses of the geometric average errors are shown in Fig. 3 and the time courses of $q(t)$ are shown in Fig. 4 for a typical sample case. We see that the orbit in case (a) is not stable, without convergence to the specified unity amplitude. The linear design of [7] assumes $\nabla = \Phi = \Psi = 0$ and would give an unstable synchronization result similar to case (a). The other cases give stable orbits with the convergence rate somewhat correlated with the linear analysis based on $\tau(\varepsilon)$.

VI. Conclusion

Synchronization and coordination problems are considered for a set of linear harmonic oscillators interconnected through nonlinear diffusive coupling. With a general coupling function, a necessary and sufficient condition is presented for orbital stability of a given harmonic oscillation, in terms of Floquet multipliers. The general result is specialized to the case of weak parametric coupling, where the coupling structure is physically motivated by nonlinear stiffness and damping for mechanical systems. Due to the weakness of the coupling, the orbital stability condition reduces through averaging to an eigenvalue condition for a constant coupling matrix. We have also provided graph theoretic interpretations of the orbital stability conditions under diffusive coupling. In particular, it is shown that the phase convergence is guaranteed if the digraph contains a spanning tree, in which case, the amplitude convergence is guaranteed if an oscillator at the root of a spanning tree has a nonlinear damping term. Based on the analysis results, a systematic procedure is proposed for the design of coupling functions to achieve amplitude regulation and phase coordination. It is shown how the coupling structure can be constrained and how the coupling parameters can be optimized for faster convergence. The theoretical results are illustrated by numerical examples. Finally, we remark that it may be possible to extend our results for a general class of weakly coupled Hamiltonian systems within the formal framework of the Birkhoff normal form theory [41, 42].

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Appendix

Here we provide a proof of Lemma 4. Let us first introduces some notation. For each $k \in \mathbb{Z}_n$, define $\mathbb{Z}_n^k$ to be the set of all combinations of k indices from $\mathbb{Z}_n$, that is,

$$\mathbb{Z}_n^k := \{(i_1, \ldots, i_k) : 1 \le i_1 < \ldots < i_k \le n \}.$$

For a matrix $F \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{Z}_n^k$, the principal submatrix obtained by deleting k rows and k columns specified by $z := (i_1, \ldots, i_k)$, is denoted by $F_z^- \in \mathbb{C}^{(n-k)\times(n-k)}$. Similarly, the submatrix obtained by retaining only those rows and columns is denoted by $F_z^+ \in \mathbb{C}^{k\times k}$.

We now prove Lemma 4. In view of the proof of Corollary 1, $-P^{\mathsf{T}}\Delta Q$ is Hurwitz if and only if the eigenvalues of Δ are all in the open right half plane except for one at the origin. This condition is then equivalent to (ii) due to Lemma 3.3 of [39].

Next we show (iii) $\Leftrightarrow$ (iv) assuming that (ii) holds. By Gershgorin circle theorem, every nonzero eigenvalue of $-(\Delta+\Phi)$ has a strictly negative real part. Thus, $-(\Delta+\Phi)$ is Hurwitz if and only if $-(\Delta+\Phi)$ has no eigenvalue at the origin, *i.e.*, $\det(\Delta+\Phi) \neq 0$. By Theorem 2.3 of [40],

$$\det(\Delta+\Phi) = \det(\Delta) + \det(\Phi) + \sum_{k=1}^{n-1} \sum_{z\in\mathbb{Z}_n^k} \det(\Phi_z^+)\det(\Delta_z^-). \tag{42}$$

Since every diagonally dominant real matrix with nonnegative diagonal entries has a nonnegative determinant, we have $\det(\Delta_z^-) \ge 0$ for all $z \in \mathbb{Z}_n^k$ and $k \in \mathbb{Z}_n$, and hence every term on the right hand side is nonnegative.

Suppose statement (iv) holds. Then, by Lemma 5, the principal submatrix Δ_j^- has a positive determinant, $\det(\Delta_j^-) > 0$, since all the edges of the graph are assumed to have positive weights. By assumption, Φ_j^+, the j^{th} diagonal entry of Φ, is positive. Therefore, the term $\det(\Phi_j^+)\det(\Delta_j^-)$ appearing on the right hand side of (42) is positive, and thus $\det(\Delta+\Phi) > 0$. This completes the proof of (iv) $\Rightarrow$ (iii).

Suppose statement (iv) does not hold. By reordering the vertices of the digraph, we may assume, without loss of generality, the following structure for Δ:

$$\Delta = \begin{bmatrix} \Delta_{11} & 0 \\ \Delta_{21} & \Delta_{22} \end{bmatrix},$$

where $\Delta_{11} \in \mathbb{R}^{m\times m}$, vertices $i \in \mathbb{Z}_m$ are the roots of spanning diverging trees, and none of the other vertices is

TABLE I
Variables and parameters

$q(t)$	$\mathbb{R}^n$	original oscillator variables
$\xi(t)$	$\mathbb{R}^n$	target oscillation for q
ω	$\mathbb{R}$	frequency of ξ
γ_i	$\mathbb{R}$	amplitude of ξ_i
φ_i	$\mathbb{R}$	phase of ξ_i
i	$\mathbb{Z}_n$	index for oscillators
$x(t)$	$\mathbb{R}^{2n}$	col $(q, \dot{q})$
$\chi(t)$	$\mathbb{R}^{2n}$	col $(\xi, \dot{\xi})$
$z(t)$	$\mathbb{R}^{2n}$	perturbation $x - \chi$
(r, θ)	$\mathbb{R}^n \times \mathbb{R}^n$	polar coordinates for $(q, \dot{q})$
$\eta(t)$	$\mathbb{R}^n$	target for $\theta(t)$, $\eta_i = t + \varphi_i$
(ρ, ϑ)	$\mathbb{R}^n \times \mathbb{R}^n$	perturbation $(r - \mathbb{1}, \theta - \eta)$
$\mathcal{w}(t)$	$\mathbb{R}^{2n}$	col (ρ, ϑ)
$\phi(t)$	$\mathbb{R}^{n-1}$	relative phases $\phi_i = \vartheta_i - \vartheta_n$
$\psi(t)$	$\mathbb{R}$	absolute phase $\psi = \vartheta_n$
$\mathfrak{e}(t)$	$\mathbb{R}^{2n-1}$	col (ρ, ϕ)
$\mathfrak{x}(t)$	$\mathbb{R}^{2n}$	col $(\mathfrak{e}, \psi)$
$y_i(t)$	$\mathbb{R}^2$	col $(q_i, \dot{q}_i)$
$A(t)$	$\mathbb{R}^{2n\times 2n}$	linearized dynamics for z
$\mathsf{A}(t)$	$\mathbb{R}^{2n\times 2n}$	linearized dynamics for $\mathcal{w}$
$\mathcal{A}(t)$	$\mathbb{R}^{2n\times 2n}$	linearized dynamics for $\mathfrak{x}$
$\mathscr{A}(t)$	$\mathbb{R}^{(2n-1)\times(2n-1)}$	linearized dynamics for $\mathfrak{e}$
ε	$\mathbb{R}$	coupling strength
(D, K)	$(\mathbb{R}^{n\times n})^2$	linear damping and stiffness
(M, N)	$(\mathbb{R}^{n\times n})^2$	nonlinear damping and stiffness
$(\Delta, \nabla, \Phi, \Psi)$	$(\mathbb{R}^{n\times n})^4$	φ-transformed (D, K, M, N)

the root of any spanning diverging tree. The $(1,2)$ block is zero by definition of a root because otherwise there is a root $i \notin \mathbb{Z}_m$ (see Fig. 1 and (41) for an example with $\Delta := \mathcal{K}$ and $m = 2$). Let us redefine Φ accordingly. Then Φ is still diagonal, and the first m diagonal entries of Φ are zero since (iv) is violated. Thus we have $\Phi = \mathrm{diag}(0, \Phi_{22})$, which implies $\det(\Delta + \Phi) = \det(\Delta_{11})\det(\Delta_{22} + \Phi_{22}) = 0$ since $\det(\Delta_{11}) = 0$ due to $\Delta_{11}\mathbb{1} = 0$. Thus $-(\Delta + \Phi)$ cannot be Hurwitz, proving (iii) $\Rightarrow$ (iv).

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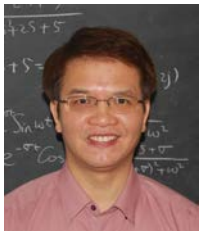

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