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C-Graded Vertex Algebras and their Representations

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ℂ-GRADED VERTEX ALGEBRAS AND THEIR REPRESENTATIONS

A dissertation submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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Table of Contents

Abstract

$\mathbb{C}$-Graded Vertex Algebras and their Representations

by

Robert A. Laber

In this thesis we consider two related classes of vertex algebras. The first class we consider consists of objects called $\mathbb{C}$-graded vertex algebras. These are vertex algebras with additional structure that allows for the construction of a Zhu algebra with a sufficiently well-behaved representation theory. This additional structure is minimal in the sense that it is necessary for the construction of the Zhu algebra. Given a $\mathbb{C}$-graded vertex algebra, we provide a construction of the Zhu algebra and a pair of functors which are inverse bijections between the appropriate module categories.

The second class we consider arises from considering conformal deformations of vertex operator algebras. These structures are called pseudo vertex operator algebras, and their main distinguishing feature is that the operator $L(0)$ is not assumed to be semi-simple and is permitted to have complex eigenvalues. Similar theories have been studied: In the context of logarithmic conformal field theory, for example, $L(0)$ is not required to be semi-simple on modules. Here, we extend that notion to allow $L(0)$ to be non semi-simple on V itself. We show how to construct a family of pseudo vertex operator algebras from a given vertex operator algebra, and we prove that all such pseudo vertex operator algebras are $\mathbb{C}$-graded vertex algebras. We then prove that every pseudo vertex operator algebra obtained via conformal deformation of a lattice vertex operator algebra is regular, which means that the category of admissible modules is semi-simple.

To my parents,
for their unending love and support.

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Part I

$\mathbb{C}$-Graded Vertex Algebras

Chapter 1

Introduction

1.1 Overview

A common theme in the study of algebraic objects is the study of the representations of that object. This study is often conducted by examining the category of modules for a given object. Understanding the modules for an object can illuminate certain features that may not be accessible from a structural study of the object itself. We utilize this general heuristic in the study of vertex algebras and vertex operator algebras.

There are three classes of modules for a vertex operator algebra V, each emphasizing different features. A vertex operator algebra can be roughly summarized as a structure that is:

- A vertex algebra,
- A graded algebra,
- A sum of eigenspaces for the distinguished operator $L(0)$.

For each of these characteristics, we have a class of V-module that reflects that characteristic. A *weak* V-module M is a module for V as a vertex algebra. In particular, no requirements are made regarding a grading on M. An *admissible* V-module M is a module for V as a graded algebra. In this case, we require a grading on M that is compatible with the grading on V. In particular, the grading on M is compatible with the *modes* of homogeneous elements in V. Lastly, an *ordinary* V-module M is a module with a grading that is induced by $L(0)$-eigenvalues.

It is easily seen that there is a forgetful functor from the category of vertex operator algebras to the category of vertex algebras, and this functor induces a forgetful functor from the category of admissible V-modules to the category of weak V-modules. Otherwise stated, the category of admissible V-modules is a full subcategory of the category of weak V-modules. Moreover, it follows immediately from the definitions that an ordinary V-module is an admissible V-module. We then have the following inclusions:

$$\{ \text{ ordinary } V\text{-modules } \} \subseteq \{ \text{ admissible } V\text{-modules } \} \subseteq \{ \text{ weak } V\text{-modules } \}$$

It is demonstrated in several places ([DLM2], [ABD]), as well as in Section 6.1 of this work, that a simple admissible V-module is a simple ordinary V-module. Therefore, the categories of admissible V-modules and ordinary V-modules share the same simple objects. For this reason, we concern ourselves primarily with the category of admissible V-modules.

One of the most important tools for studying the admissible modules of a vertex operator algebra V is the *Zhu algebra* $A(V)$. This is an associative algebra which arises by considering a product on a certain quotient of V. The importance of $A(V)$ comes from the relationship between the representation theories of V and $A(V)$: Under suitable conditions, there is a natural bijection between the simple objects in the category of $A(V)$-modules and the simple objects in the category of admissible V-modules. This effectively reduces the study of admissible V-modules to the study of modules for the associative (and often finite-dimensional) algebra $A(V)$.

1.2 Motivation

In Part I of this work, we seek to extend the idea of the Zhu algebra to a broader class of vertex algebras.

The primary motivation for the generalization of the Zhu algebra comes from the notion of a conformal deformation of a vertex operator algebra, which we study in detail in Part II of this thesis. A conformal deformation yields a structure which is not in general a vertex operator algebra, yet nevertheless admits a suitable Zhu algebra. This leads one to ask whether there is a broad class of vertex algebras that support the construction of an associative algebra which preserves some aspects of the corresponding representation theory. In particular, we seek a minimal set of conditions that allow for the construction of a Zhu algebra.

A natural place to start would be to try to define an associative algebra $A(V)$ from a vertex algebra V. The lack of grading on V, however, is problematic, since the modes of elements (i.e., operators $a(n)$ for $a \in V$ and $n \in \mathbb{Z}$) do not act as weighted operators on V. Thus, there is no notion of a *zero mode* (see Section 3.2), nor any notion of a *lowest weight space* of V. Any subsequent notion of an associative algebra associated to V would therefore be fundamentally distinct from the Zhu algebra we seek to generalize.

This leads to the notion of a $\mathbb{C}$-graded vertex algebra, which is a vertex algebra equipped with a grading that is compatible with the modes of elements of V, and which is generated by its lowest weight vectors. We prove that these two additional features are sufficient for the construction of an associative algebra $A(V)$ with the desired representation theory (see Theorem 5).

Chapter 2

Vertex Algebras

2.1 Basic Definitions and Notation

We let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers and complex numbers, respectively. Given a complex number a, we let $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ denote the real and imaginary parts of a, respectively. For a vector space V, we let $V[[x]]$ denote the ring of formal power series in x with coefficients in V, and we let $V[[x, x^{-1}]]$ denote the ring of formal Laurent series in x with coefficients in V. That is,

$$V[[x]] = \left\{ \sum_{n=0}^{\infty} a_n \, x^n \quad | \quad a_n \in V \right\}$$

and

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} a_n \, x^n \quad | \quad a_n \in V \right\}.$$

We denote by $\mathrm{End}(V)$ the ring of linear endomorphisms of a vector space V. All vector spaces are assumed to be defined over $\mathbb{C}$.

2.1.1 Formal Calculus

In this section we let V denote an arbitrary vector space. We introduce two important formal sums. First, we have the exponential

$$e^{Tx} = \sum_{j=0}^{\infty} \frac{T^j}{j!} \, x^j \ \in (\mathrm{End}(V))[[x]],$$

where T is any element of $\mathrm{End}(V)$ and x is any indeterminate. We also have the δ-function, which is defined as the formal Laurent series

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n \ \in \mathbb{C}[[x, x^{-1}]].$$

This sum has the property that, for any formal Laurent series $f(x) \in V[[x, x^{-1}]]$ such that the product $\delta(x)f(x)$ exists, one has

$$\delta\left(\frac{x}{y}\right) f(x) = \delta\left(\frac{x}{y}\right) f(y).$$

This δ-function should not be confused with the *Dirac delta*

$$\delta_{a,b} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

which also appears in this work.

We have the linear operators

$$\begin{aligned} \frac{\partial}{\partial x} : V[[x, x^{-1}]] &\to V[[x, x^{-1}]] \\ \sum_{n\in\mathbb{Z}} a_n x^n &\mapsto \sum_{n\in\mathbb{Z}} n a_n x^{n-1} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Res}_x : V[[x, x^{-1}]] &\to V \\ \sum_{n\in\mathbb{Z}} a_n x^n &\mapsto a_{-1}. \end{aligned}$$

If $f(x)$ and $g(x)$ are elements of $V[[x, x^{-1}]]$ such that the product $f(x)g(x)$ is defined, then the operator $\frac{\partial}{\partial x}$ acts as a derivation:

$$\frac{\partial}{\partial x}\left(f(x)g(x)\right) = \left(\frac{\partial}{\partial x} f(x)\right) g(x) + f(x)\left(\frac{\partial}{\partial x} g(x)\right). \tag{2.1}$$

It is clear that

$$\mathrm{Res}_x \left(\frac{\partial}{\partial x} f(x)\right) = 0$$

for any $f(x) \in V[[x, x^{-1}]]$, so using (2.1) we obtain the formula

$$\mathrm{Res}_x \left(\left(\frac{\partial}{\partial x} f(x)\right) g(x)\right) = -\mathrm{Res}_x \left(f(x)\left(\frac{\partial}{\partial x} g(x)\right)\right).$$

We also have the change of variables formula for $w = f(x) \in V[[x, x^{-1}]]$:

$$\mathrm{Res}_w\, g(w) = \mathrm{Res}_x \left(g(f(x))\frac{\partial}{\partial x} f(x)\right). \tag{2.2}$$

2.1.2 Binomials

Definition 2.1.1 (Binomial Coefficients)**.** *Let $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$. We define the binomial coefficient $\binom{\alpha}{k}$ as*

$$\begin{aligned}\binom{\alpha}{k} &:= \frac{1}{k!}\prod_{i=0}^{k-1}(\alpha - i) \\ &= \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}.\end{aligned} \tag{2.3}$$

Definition 2.1.2 (Binomial Expansion)**.** *Let $\alpha \in \mathbb{C}$. We define the binomial expansion of $(x+y)^\alpha$ as*

$$(x+y)^\alpha := \sum_{i=0}^{\infty} \binom{\alpha}{i} x^{\alpha-i} y^i. \tag{2.4}$$

In other words, any binomial is to be expanded as a power series in the second variable. One observes that if $\alpha \in \mathbb{N}$, then the previous two definitions coincide with the usual definitions. It is important to note that, using our binomial expansion convention (2.4), the expressions $\frac{1}{x-y}$ and $-\frac{1}{y-x}$ are not equal as formal power series. Rather, we treat $\frac{1}{x-y}$ as defined in the region $x > y$ and $-\frac{1}{y-x}$ as defined in the region $y > x$, so we have

$$\frac{1}{x-y} + \frac{1}{y-x} = x^{-1}\delta\left(\frac{y}{x}\right).$$

2.2 Vertex Algebras

We now come to the primary object of study in Part I of this thesis. The reader should be aware that there are various formalisms for the following axiomatic foundations. For alternative treatments, see [B], [FHL], or [FLM].

Definition 2.2.1 (Vertex Algebra)**.** *A* vertex algebra *is a triple $(V, Y, \mathbb{1})$ consisting of a vector space V, a distinguished vector $\mathbb{1} \in V$, and a linear map*

$$\begin{aligned} Y : V &\to (\mathrm{End}(V))\,[[z, z^{-1}]] \\ a &\mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a(n)\, z^{-n-1}\end{aligned}$$

such that the following hold:

(i) (Truncation) For any $a, b \in V$, we have $a(n)b = 0$ for all sufficiently large n,

(ii) (Vacuum) $Y(\mathbb{1}, z) = \mathrm{Id}_V$,

(iii) (Creativity) For any $a \in V$, $a(n)\mathbb{1} = 0$ if $n \geq 0$ and $a(-1)\mathbb{1} = a$,

(iv) (Translation Covariance) There is a linear operator $T \in \mathrm{End}(V)$ with the property that

$$[T, Y(a,z)] = Y(Ta, z) = \frac{\partial}{\partial z} Y(a,z),$$

(v) (Jacobi Identity) For any $a, b \in V$, one has

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2) \quad &- \quad z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y(b, z_2)Y(a, z_1) \\ &= \quad z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2). \end{aligned}$$

Remark. From the creativity property, we see that the map $Y(\cdot, z)$ is injective. It then follows that $T\mathbb{1} = 0$, since

$$Y(T\mathbb{1}, z) = \frac{\partial}{\partial z} Y(\mathbb{1}, z) = 0.$$

Remark. The translation covariance condition yields the following useful formula: For any $a \in V$ and any $n \in \mathbb{Z}$, one has

$$(Ta)(n) = -na(n-1). \tag{2.5}$$

Remark. It should be noted that the Jacobi identity above resembles the familiar Jacobi identity from Lie algebras, which can be expressed as

$$\mathrm{ad}(u)\,\mathrm{ad}(v) - \mathrm{ad}(v)\,\mathrm{ad}(u) = \mathrm{ad}(\mathrm{ad}(u)v).$$

The resemblance is clear when one substitutes $Y(\cdot, z)$ for $\mathrm{ad}(\cdot)$.

Taking various residues of the Jacobi identity yields three important formulas which will be used repeatedly in the sequel. We have the *commutator formula*,

$$[a(n), b(m)] = \sum_{i=0}^{\infty} \binom{n}{i} (a(i)b)(n+m-i), \tag{2.6}$$

the *associator formula*,

$$(a(n)b)(m) = \sum_{i=0}^{\infty}(-1)^i \binom{n}{i} \Big(a(n-i)b(n+i) - (-1)^n b(n+m-i)a(i)\Big), \quad (2.7)$$

and the *skew-symmetry formula*,

$$a(n)b = \sum_{i=0}^{\infty}(-1)^{i+n+1}\frac{T^i}{i!}(b(n+i)a). \quad (2.8)$$

It should be noted that the sums above are all well defined. The sums appearing in the commutator and skew-symmetry formulas are finite sums due to the truncation axiom. The associator formula is an equality of operators, and the truncation axiom again ensures that the right hand side yields a finite sum when applied to an element of V.

2.3 $\mathbb{C}$-Graded Vertex Algebras

Definition 2.3.1 ($\mathbb{C}$-Graded Vertex Algebra)**.** *A vertex algebra V is said to be* $\mathbb{C}$-graded *if it has the following two properties:*

(i) V is a direct sum

$$V = \bigoplus_{\mu\in\mathbb{C}} V_\mu,$$

such that for any element $a \in V_\lambda$, one has

$$a(n)V_\mu \subseteq V_{\mu+\lambda-n-1}, \quad (2.9)$$

(ii) V is generated by a set of lowest weight vectors*, where a lowest weight vector is a vector $v \in V_\mu$ for some $\mu \in \mathbb{C}$ that satisfies the following: For any $a \in V_\lambda$, if $a(n)v \neq 0$, then either $n = \lambda - 1$ or $n < \mathrm{Re}(\lambda) - 1$.*

An element $a \in V_\mu$ is said to be *homogeneous of weight* μ, and we denote this by $|a| = \mu$. We can then define the operator $L \in \mathrm{End}(V)$ as the linear extension of the map

$$\begin{aligned} V_\mu &\rightarrow V_\mu \\ a &\mapsto \mu a = |a|a. \end{aligned}$$

We find it convenient to introduce some notation here. To simplify property (*ii*) above, when given two complex numbers a and b, we write

$$a \prec b$$

if either $a = b$ or $\mathrm{Re}(a) < \mathrm{Re}(b)$. Using this notation and formula (2.9), we have an alternative characterization of lowest weight vectors. A homogeneous vector $v \in V_\mu$ is a lowest weight vector if the following implication holds: For any homogeneous $a \in V$, if $a(n)v \neq 0$, then

$$|v| \prec |a(n)v|.$$

Proposition 2.3.1. *Let $V = \bigoplus_{\mu\in\mathbb{C}} V_\mu$ be a $\mathbb{C}$-graded vertex algebra. Then*

(i) $\mathbb{1} \in V_0$,

(ii) If $a \in V_\mu$ then $Ta \in V_{\mu+1}$.

Proof. From the vacuum axiom, we know that $\mathbb{1}(-1)$ is the identity operator on V. Therefore, if $a \in V$ is homogeneous of weight λ, then $\mathbb{1}(-1)a = a$ must also be homogeneous of weight λ. In view of (2.9), we deduce that $\mathbb{1}$ cannot be a sum of homogeneous vectors of different weights, and therefore, $\mathbb{1}$ itself is a homogeneous vector. Now use (2.9) again to see that

$$\lambda = |a| = |\mathbb{1}(-1)a| = \lambda + |\mathbb{1}| - (-1) - 1,$$

whence we easily obtain $|\mathbb{1}| = 0$.

Now suppose $a \in V$ is homogeneous vector. From (2.5) we have

$$(Ta)(n) = -na(n-1)$$

for all $n \in \mathbb{Z}$. From this one sees that Ta must be homogeneous, since its modes preserve the homogeneous subspaces of V. Then we use (2.9) to calculate

$$|Ta| = |(Ta)(-1)\mathbb{1}| = |a(-2)\mathbb{1}| = |\mathbb{1}| + |a| + 2 - 1 = |a| + 1,$$

which proves (*ii*). □

2.4 Modules

2.4.1 Weak Modules

Definition 2.4.1 (Weak V-Module)**.** *Let V be a vertex algebra. A* weak V-module *is a pair (M, Y_M) consisting of a vector space M and a linear map*

$$\begin{aligned} Y_M : V &\to (\mathrm{End}(M))\,[[z, z^{-1}]] \\ a &\mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a_M(n)\; z^{-n-1} \end{aligned}$$

such that the following hold:

(i) (Truncation) For any $a \in V$ and any $w \in M$, we have $a_M(n)w = 0$ for all sufficiently large n,

(ii) (Vacuum) $Y_M(\mathbb{1}, z) = \mathrm{Id}_M$,

(iii) (Translation Covariance) For any $a \in V$,

$$[T, Y_M(a, z)] = Y_M(Ta, z) = \frac{\partial}{\partial z} Y_M(a, z),$$

(iv) (Jacobi Identity) For any $a, b \in V$ and any $w \in M$, one has

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2)w \;&-\; z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y_M(b, z_2)Y_M(a, z_1)w \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2)w. \end{aligned}$$

When there is no risk of confusion, we may simply denote by M the weak V-module (M, Y_M), and we may also suppress the subscript M from the expressions $a_M(n)$ and $Y_M(a, z)$.

Remark. We note that V is a weak module over itself. Indeed, the axioms for a weak V-module are similar to those for a vertex algebra, the only difference being that weak V-modules have no creativity property.

Definition 2.4.2 (Homomorphism of V-Modules)**.** *Let M and N be weak V-modules. A linear map $\phi : M \to N$ is said to be a* homomorphism of V-modules *if*

$$\phi(a_M(k)u) = a_N(k)(\phi(u)) \tag{2.10}$$

for all $a \in V$, $k \in \mathbb{Z}$ and $u \in M$.

2.4.2 Admissible Modules

We now define an important class of module for $\mathbb{C}$-graded vertex algebras. These modules play a central role in the construction of the Zhu algebra.

Remark. The reader may be familiar with the notion of admissible module over a vertex operator algebra. Our choice of terminology here is deliberate: If a $\mathbb{C}$-graded vertex algebra V happens to be a vertex operator algebra, then the two notions of admissible V-module coincide.

Definition 2.4.3 (Admissible V-Module)**.** *Let V be a $\mathbb{C}$-graded vertex algebra. An* admissible V-module *is a weak V-module M with a grading of the form*

$$M = \bigoplus_{0 \prec \mu} M(\mu)$$

such that $M(0) \neq 0$ and for any homogeneous $a \in V_\lambda$, one has

$$a_M(n)M(\mu) \subseteq M(\mu + \lambda - n - 1). \tag{2.11}$$

If $M = \bigoplus_{0 \prec \mu} M(\mu)$ is an admissible V module, then we say that an element $u \in M(\mu)$ is *homogeneous of weight* μ. A homogeneous element $u \in M$ is called a *lowest weight vector* if the following holds: For any homogeneous $a \in V$, if $a_M(n)u \neq 0$, then $|u| \prec |a_M(n)u|$.

We see that any $\mathbb{C}$-graded vertex algebra V is an admissible module over itself as follows: Since V is generated by its lowest weight vectors, every element in V is a sum of elements of the form

$$a^r(n_r)a^{r-1}(n_{r-1})...a^1(n_1)w,$$

where a^i is homogeneous element in V and $w \in V$ is a lowest weight vector. We define $V(0)$ as the space of lowest weight vectors, and we define

$$\deg\left(a^r(n_r)a^{r-1}(n_{r-1})...a^1(n_1)w\right) := \sum_{i=1}^{r}\left(|a^i| - n_i - 1\right).$$

One can use the commutator formula (2.6) and the fact that w is a lowest weight vector to show that either

$$\mathrm{Re}\left(\sum_{i=1}^{r}\left(|a^i| - n_i - 1\right)\right) > 0$$

or

$$\left(\sum_{i=1}^{r} \left(|a^i| - n_i - 1 \right) \right) = 0.$$

Therefore, we define $V(\lambda)$ to be the space of all $v \in V$ with $\deg(v) = \lambda$, and this endows V with the structure of an admissible V-module.

Chapter 3

The Zhu Algebra

Throughout this section we let $V = (V, Y, \mathbb{1})$ be a $\mathbb{C}$-graded vertex algebra with grading $V = \bigoplus_{\mu \in \mathbb{C}} V_\mu$. We now fix some notation. For $a \in V_\mu$, we let $\overline{|a|}$ denote the ceiling of the real part of μ, i.e.,

$$\overline{|a|} := \min\{\, n \in \mathbb{Z} \mid n \geq \mathrm{Re}(\mu) \,\}.$$

Let V^r be the set of all elements $a \in V$ with $r = |a| - \overline{|a|}$. One has that

$$V = \bigoplus_{r \in \mathbb{C}} V^r.$$

Of course, $a \in V^0$ if and only if $|a| = \overline{|a|}$, or equivalently, $La = \overline{|a|}$. In light of formula (2.9), we can then characterize V^0 as the vertex subalgebra of V consisting of all integrally graded vectors.

3.1 Construction of $A(V)$

Here we construct the Zhu algebra $A(V)$ associated to a $\mathbb{C}$-graded vertex algebra.

Remark. The construction of $A(V)$ in this section resembles the original construction of $A(V)$ as described in [Z]. In that paper, V is assumed to be a vertex operator algebra (see Section 5.1), however, one notices that only the integral grading on V is used to construct $A(V)$. This means that if V is a $\mathbb{C}$-graded vertex algebra that carries an integral grading, i.e., if $V = V^0$, then we are essentially in the case considered in [Z], and we may therefore appeal to the results proved in that work.

Definition 3.1.1. *Let $a \in V^r$ be homogeneous. Define the products "$\circ$" and "$\star$" on V as the linear extensions of the following:*

$$a \circ b := \operatorname{Res}_z \frac{(1+z)^{\overline{|a|}+\delta_{r,0}-1}}{z^{1+\delta_{r,0}}} Y(a,z)b$$

and

$$a \star b := \delta_{r,0} \operatorname{Res}_z \frac{(1+z)^{|a|}}{z} Y(a,z)b.$$

We define $O(V)$ to be the linear span of all elements of the form $a \circ b$ for $a, b \in V$.

Lemma 3.1.1. *If $r \neq 0$, then $V^r \subseteq O(V)$.*

Proof. Let $a \in V^r$ be homogeneous. Then

$$a = \operatorname{Res}_z \frac{(1+z)^{\overline{|a|}-1}}{z} Y(a,z)\mathbb{1} = a \circ \mathbb{1} \in O(V).$$

□

Lemma 3.1.2. *For any homogeneous $a \in V$, one has*

$$(T+L)a \equiv 0 \pmod{O(V)}. \tag{3.1}$$

In particular, $Ta \equiv -La \pmod{O(V)}$.

Proof. Suppose that $a \in V^r$ for some r. Clearly $La \in V^r$, and $Ta \in V^r$ by Proposition 2.3.1. If $r \neq 0$, the result follows because $(T+L)a \in V^r \subseteq O(V)$. Otherwise, $a \in V^0$, and we have

$$(T+L)a = Ta + |a|a = \operatorname{Res}_z \frac{(1+z)^{|a|}}{z^2} Y(a,z)\mathbb{1} \in O(V),$$

where we use the fact that $Ta = a(-2)\mathbb{1}$. □

Lemma 3.1.3. *For any homogeneous $a \in V^r$ and any $m \geq n \geq 0$, we have*

$$\operatorname{Res}_z \frac{(1+z)^{\overline{|a|}+\delta_{r,0}-1+n}}{z^{1+\delta_{r,0}+m}} Y(a,z)b \ \in O(V). \tag{3.2}$$

Proof. The proof is the same as [Z] Lemma 2.1.2 if one replaces $L(-1)$ by T. $\square$

Lemma 3.1.4. *Let $a, b \in V$ be homogeneous elements. Then*

$$Y(a,z)b \equiv (1+z)^{-|a|-|b|} Y\left(b, \frac{-z}{1+z}\right) a \pmod{O(V)}. \tag{3.3}$$

Proof. This result was proved in [Z] during the proof of Lemma 2.1.3, but we provide additional details here. Using skew symmetry (2.8), we have

$$\begin{aligned} Y(a,z)b &= e^{zT} Y(b,-z)a \\ &= \sum_{i\in\mathbb{Z}} e^{zT} b(i)a\,(-z)^{-i-1} \\ &= \sum_{i\in\mathbb{Z}} \sum_{j\geq 0} \frac{z^j T^j}{j!} b(i)a\,(-z)^{-i-1}. \end{aligned}$$

Recall that $La + Ta \in O(V)$ for any $a \in V$, so we have the following congruence (mod $O(V)$):

$$\begin{aligned} \frac{z^j T^j}{j!} b(i)a &= \frac{z^j}{j!}(-L)T^{j-1} b(i)a \\ &\equiv (-1)\frac{z^j}{j!}(|a|+|b|-i-1+j-1)T^{j-1}b(i)a \\ &\equiv (-1)\frac{z^j}{j!}(|a|+|b|-i-1+j-1)(-L)T^{j-2}b(i)a \\ &\equiv (-1)^j\frac{z^j}{j!}(|a|+|b|-i-1+j-1)\dots(|a|+|b|-i-1)b(i)a \\ &= (-1)^j\frac{z^j}{j!}b(i)a\prod_{k=0}^{j-1}(|a|+|b|-i-1+k) \\ &= \binom{-|a|-|b|+i+1}{j} b(i)a\; z^j, \end{aligned}$$

where in the last equality we used the binomial convention (2.3). Returning to the previous calculation, we have the congruence (mod $O(V)$)

$$\begin{aligned} Y(a,z)b &\equiv \sum_{i\in\mathbb{Z}}\sum_{j\geq 0} \binom{-|a|-|b|+i+1}{j} b(i)a\; z^j(-z)^{-i-1} \\ &= \sum_{i\in\mathbb{Z}} (-1)^{i+1} b(i)a\,(1+z)^{-|a|-|b|+i+1} z^{-i-1} \\ &= (1+z)^{-|a|-|b|} Y\left(b, \frac{-z}{1+z}\right) a. \end{aligned}$$

$\square$

The following lemma and its proof are analogs of Lemma 2.1.3 of [Z].

Lemma 3.1.5. *If a and b are homogeneous elements in V^0, then we have the identities*

$$a \star b \equiv \mathrm{Res}_z \frac{(1+z)^{|b|-1}}{z} Y(b,z)a \pmod{O(V)} \tag{3.4}$$

and

$$a \star b - b \star a \equiv \mathrm{Res}_z\, (1+z)^{|a|-1} Y(a,z)b \pmod{O(V)}. \tag{3.5}$$

Proof. Recall that $|a| = \overline{|a|}$ and $|b| = \overline{|b|}$ since $a, b \in V^0$. Then use (3.3) to calculate (mod $O(V)$):

$$\begin{aligned} a \star b &= \mathrm{Res}_z\; Y(a,z)b \frac{(1+z)^{|a|}}{z} \\ &\equiv \mathrm{Res}_z\; Y\left(b, \frac{-z}{1+z}\right) a\; \frac{(1+z)^{|a|}}{z} (1+z)^{-|a|-|b|} \\ &= \mathrm{Res}_z\; Y\left(b, \frac{-z}{1+z}\right) a\; \frac{(1+z)}{z} (1+z)^{-|b|-1} \\ &= \mathrm{Res}_w\; Y(b,w)a\; \frac{(1+w)^{|b|-1}}{w}, \end{aligned}$$

where in the last equality, we set $w = \frac{-z}{1+z}$ and used the change of variables formula (2.2). This proves (3.4). Formula (3.5) follows directly from (3.4) and the definition of "$\star$". $\square$

Proposition 3.1.6. *$O(V)$ is a left ideal of V with respect to the "$\star$" product.*

Proof. We must show that $a \star (b \circ c) \in O(V)$ for any $a, b, c \in V$. Note that if $a \in V^r$ and $r \neq 0$, then $a \star (b \circ c) = 0 \in O(V)$. Therefore, we may assume that $a \in V^0$. If $b \in V^0$, then the result is true by Theorem 2.1.1 of [Z]. The only remaining case is when $a \in V^0$ and $b \in V^r$ for some $r \neq 0$. We calculate

$$\begin{aligned} a \star (b \circ c) - b \circ (a \star c) &= \mathrm{Res}_z \mathrm{Res}_w\; Y(a,z)Y(b,w)c \frac{(1+z)^{|a|}}{z} \frac{(1+w)^{\overline{|b|}-1}}{w} \\ &\quad - \mathrm{Res}_w \mathrm{Res}_z\; Y(b,w)Y(a,z)c \frac{(1+z)^{|a|}}{z} \frac{(1+w)^{\overline{|b|}-1}}{w} \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Res}_w \operatorname{Res}_{z-w} \ Y(Y(a,z-w)b,w)c\frac{(1+z)^{|a|}}{z}\frac{(1+w)^{\overline{|b|}-1}}{w} \\
&= \operatorname{Res}_w \operatorname{Res}_{z-w} \sum_{i,j=0}^{\infty} \binom{|a|}{i}(-1)^j(z-w)^{i+j}\frac{(1+w)^{\overline{|b|}+|a|-1-i}}{w^{2+j}} \\
&\qquad \cdot Y(Y(a,z-w)b,w)c \\
&= \sum_{i,j=0}^{\infty} \binom{|a|}{i}(-1)^j \operatorname{Res}_w \frac{(1+w)^{\overline{|b|}+|a|-1-i}}{w^{2+j}} Y(a(i+j)b,w)c \\
&= \sum_{i,j=0}^{\infty} \binom{|a|}{i}(-1)^j \operatorname{Res}_w \frac{(1+w)^{\overline{|a(i+j)b|}+j}}{w^{2+j}} Y(a(i+j)b,w)c,
\end{aligned}
$$

which is a sum of terms in $O(V)$ by (3.2). Since $b \circ (a \star c)$ is clearly in $O(V)$, and $a \star (b \circ c) - b \circ (a \star c) \in O(V)$ by the previous calculation, we see that $a \star (b \circ c) \in O(V)$, which proves that $O(V)$ is a left ideal of V with respect to the star product. □

Proposition 3.1.7. *$O(V)$ is a right ideal of V.*

Proof. We must show that any element of the form $(a \circ b) \star c$ is in $O(V)$. It suffices to prove this for homogeneous elements a, b, and c. By the definition of the star product, we know that if $a \circ b \in V^r$ for some nonzero r, then $(a \circ b) \star c = 0 \in O(V)$ for any $c \in V$. Therefore, we assume $a \circ b \in V^0$. It is clear that $V^0 \star V^r \subseteq V^r$ for any r, so if $c \in V^r$ for some $r \neq 0$, then $(a \circ b) \star c \in V^r \subseteq O(V)$. We are then reduced to the case where $(a \circ b) \in V^0$ and $c \in V^0$. If $a, b \in V^0$, then the result follows from [Z], so the only remaining case is where $a \in V^r$ for some nonzero r, $a \circ b \in V^0$, and $c \in V^0$. By (3.4) it suffices to show that

$$x = \operatorname{Res}_z \frac{(1+z)^{|c|-1}}{z} Y(c,z)(a \circ b) \in O(V).$$

We calculate

$$
\begin{aligned}
x &= \operatorname{Res}_z \operatorname{Res}_w \frac{(1+z)^{|c|-1}(1+w)^{\overline{|a|}-1}}{zw} Y(c,z)Y(a,w)b \\
&= \operatorname{Res}_w \operatorname{Res}_{z-w} Y(Y(c,z-w)a,w)b \ \frac{(1+z)^{|c|-1}(1+w)^{\overline{|a|}-1}}{zw} \\
&\qquad + \operatorname{Res}_w \operatorname{Res}_z Y(a,w)Y(c,z)b \ \frac{(1+z)^{|c|-1}(1+w)^{\overline{|a|}-1}}{zw}.
\end{aligned}
$$

Since $a \notin V^0$, we see that the second term in the above sum is in $O(V)$. Therefore we have (mod $O(V)$)

$$
\begin{aligned}
x &\equiv \operatorname{Res}_w \operatorname{Res}_{z-w} Y(Y(c, z-w)a, w)b \, \frac{(1+z)^{|c|-1}(1+w)^{\overline{|a|}-1}}{zw} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{|c|-1}{i} \operatorname{Res}_w \operatorname{Res}_{z-w} Y(Y(c, z-w)a, w)b \\
&\qquad \cdot \frac{(z-w)^{i+j}(1+w)^{\overline{|a|}+|c|-2-i}}{w^{2+j}} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{|c|-1}{i} \operatorname{Res}_w Y(c(i+j)a, w)b \, \frac{(1+w)^{\overline{|a|}+|c|-2-i}}{w^{2+j}} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{|c|-1}{i} \operatorname{Res}_w Y(c(i+j)a, w)b \, \frac{(1+w)^{\overline{|c(i+j)a|}-1+j}}{w^{2+j}}.
\end{aligned}
$$

Again, since $c \in V^0$ and $a \notin V^0$, we see that $c(i+j)a \notin V^0$, and we use (3.2) to see that the last line above is a sum of terms in $O(V)$. We conclude that $x \equiv 0$ (mod $O(V)$), and the result follows. □

Theorem 1. *Define $A(V) := V/O(V)$. Then $A(V)$ is an associative algebra with respect to the "$\star$" product.*

Proof. We may assume that $a, b, c \in V^0$, since otherwise $a\star(b\star c) = (a\star b)\star c = 0$ in $A(V)$. This result was proved in [Z]. □

3.2 The functor Ω

The goal of this section is to construct a functor Ω from the category of admissible V-modules to the category of $A(V)$-modules. Throughout, we let $M = \bigoplus_{0\prec\mu} M(\mu)$ be an admissible V-module.

Definition 3.2.1 (Vacuum Space)**.** *The* vacuum space $\Omega(M)$ *of M is the set of all lowest weight vectors of M.*

Proposition 3.2.1. *Let M be a simple admissible V-module. Then $\Omega(M) = M(0)$.*

Proof. First, observe that $\Omega(M)$ is a graded subspace of M, since it consists of lowest weight vectors. In particular,

$$\Omega(M)(\mu) = \Omega(M) \cap M(\mu).$$

It is clear that $M(0) \subseteq \Omega(M)$. If $\Omega(M)(\mu) \neq 0$ for some $\mu \neq 0$, then M contains a lowest weight vector w of weight μ with $\mathrm{Re}(\mu) > 0$. Since w is a lowest weight vector, it generates a submodule W of the form

$$W = \bigoplus_{\mu \prec \lambda} W(\lambda).$$

Note that any homogeneous vector $u \in W$ satisfies $\mathrm{Re}(|u|) > 0$. This shows that $M(0) \cap W = 0$, but this contradicts the fact M must equal W, as M is simple. Thus, $\Omega(M)(\mu) = 0$ if $\mu \neq 0$. □

Remark. Recall from (2.11) that for any admissible V-module M, and any homogeneous element $a \in V$, one has

$$a(n)M(\mu) \subseteq M(|a| + \mu - n - 1).$$

From this, it is evident that not every element $a \in V$ has a mode that acts on the graded subspaces of M. Indeed, if

$$|a| + \mu - n - 1 = \mu,$$

we see that $|a| = n + 1$. In particular, $|a| \in \mathbb{Z}$, which is only true for those elements $a \in V^0$.

Definition 3.2.2 (Zero Mode)**.** *Let $a \in V$ be a homogeneous element. We define the* zero mode $o(a)$ *of a as*

$$o(a) = a(\overline{|a|} - 1).$$

We extend this definition to all of V by linearity.

Note that if $a \in V^r$ for some nonzero r, then $\overline{|a|} - 1 > \mathrm{Re}(|a|) - 1$, and so $o(a) = a(\overline{|a|} - 1)$ annihilates any lowest weight vector. On the other hand, if $a \in V^0$, then $\overline{|a|} = |a|$, so this definition of $o(a)$ reduces to the definition given in [Z].

Theorem 2. *Let M be a simple admissible V-module. Then there is an action of the associative algebra $A(V)$ on $\Omega(M)$, with $a \in A(V)$ acting via its zero mode $o(a)$.*

Proof. Proposition 3.2.1 shows that $\Omega(M) = M(0)$, so we must show that $O(V)$ annihilates $M(0)$, and that $o(a \star b) = o(a)o(b)$ on $M(0)$ for $a, b \in A(V)$. First, we show that $O(V)$ annihilates $M(0)$. By the previous discussion, we see that V^r annihilates $M(0)$ whenever $r \neq 0$. Therefore, we must show that $V^0 \cap O(V)$ annihilates $M(0)$. For this it suffices to show that

$$o(a \circ b)M(0) = 0$$

for $a \circ b \in V^0$. If $a, b \in V^0$, this result was proved in [Z]. Thus, we assume that $a \in V^r$ for some nonzero r, and $a \circ b \in V^0$. Note that, in this case, $b \notin V^0$, and we have

$$\overline{|a(n)b|} = \overline{|a|} + \overline{|b|} - n - 2$$

for any $n \in \mathbb{Z}$. Now let $w \in M(0)$. Following [DLM2], we use a property of the δ function to rewrite the Jacobi identity as

$$z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)Y(a,z_1)Y(b,z_2)w - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(b,z_2)Y(a,z_1)w$$
$$= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)Y(Y(a,z_0)b,z_2)w.$$

Since $a, b \notin V^0$, we know that $a(\overline{|a|}-1)w = b(\overline{|b|}-1)w = 0$. Thus, $\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{\overline{|a|}-1} z_2^{\overline{|b|}-1}$ of the left hand side is equal to 0. Then we have

$$\begin{aligned}
0 &= \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{\overline{|a|}-1} z_2^{\overline{|b|}-1} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)Y(Y(a,z_0)b,z_2)w \\
&= \mathrm{Res}_{z_2} z_2^{\overline{|b|}-1}(z_2+z_0)^{\overline{|a|}-1}Y(Y(a,z_0)b,z_2)w \\
&= \mathrm{Res}_{z_2}\sum_{i=0}^{\infty}\binom{\overline{|a|}-1}{i} z_0^i z_2^{\overline{|a|}+\overline{|b|}-2-i}Y(Y(a,z_0)b,z_2)w \\
&= \mathrm{Res}_{z_0} z_0^{-1}\mathrm{Res}_{z_2}\sum_{i=0}^{\infty}\binom{\overline{|a|}-1}{i} z_0^i z_2^{\overline{|a|}+\overline{|b|}-2-i}Y(Y(a,z_0)b,z_2)w \\
&= \sum_{i=0}^{\infty}\binom{\overline{|a|}-1}{i}(a(i-1)b)(\overline{|a|}+\overline{|b|}-2-i)w \\
&= o(a \circ b)w.
\end{aligned}$$

Altogether, this shows that $O(V)$ annihilates $M(0)$. To see that $o(a \star b) = o(a)o(b)$ on $M(0)$, observe that if $a \notin V^0$, then $o(a)$ annihilates $M(0)$, and $a \star b = 0$, so the result holds. If $a \in V^0$ and $b \notin V^0$, then $a \star b \in O(V)$, and therefore $o(a \star b) = 0 = o(b) = o(a)o(b)$ on $M(0)$. The only remaining case is when $a, b \in V^0$, and this case was proved in [Z]. □

The following corollary is the main result of this section.

Corollary 3.2.2. *Let M be any admissible V-module. Then there is an action of $A(V)$ on $\Omega(M)$. In particular, Ω defines a covariant functor from the category of admissible V-modules to the category of $A(V)$-modules.*

Proof. The action of $A(V)$ on $\Omega(M)$ follows from the fact that $\Omega(M)$ is a sum of lowest weight spaces of M, hence a sum of modules for $A(V)$. The remaining functorial properties of Ω follow from (2.10), together with the fact that an element $a \in A(V)$ acts on $\Omega(M)$ via its zero mode $o(a)$. □

3.3 The functor Λ

In this section we provide a natural construction of an admissible V-module from an $A(V)$-module, thereby giving a functor Λ from the category of $A(V)$-modules to the category of admissible V-modules. This construction requires several steps. We first construct a Lie algebra V_{Lie}, whose elements are the modes of elements of V. We endow this Lie algebra with a suitable grading, and then construct an induced module for V_{Lie} from a simple $A(V)$-module. Finally, we take a quotient of this induced module, and show that it is an admissible V-module with a unique simple quotient.

3.3.1 The Lie Algebra V_{Lie}

In [DLM2] we encounter a Lie algebra which plays a role in their construction of an admissible V-module from a module for the twisted Zhu algebra $A_g(V)$. In this section, we construct a Lie algebra V_{Lie} which plays an analogous role. The construction of V_{Lie} is similar to the construction found in [DLM2], but with some notable differences. These differences primarily stem from the fact that our construction of V_{Lie} does not yield a Lie algebra with a suitable triangular decomposition, since certain subspaces of V_{Lie} do not close on a Lie algebra.

Let t be an indeterminate, and set $\mathfrak{L}(V) = \mathbb{C}[t, t^{-1}] \otimes V$. There is a natural vertex algebra structure on $\mathbb{C}[t, t^{-1}]$, and so $\mathfrak{L}(V)$ is a vertex algebra, as it is a tensor product of vertex algebras (see [K], [LL]). The translation covariance operator of $\mathfrak{L}(V)$ is given by

$$D = \frac{d}{dt} \otimes 1 + 1 \otimes T.$$

A standard calculation shows that the space

$$\mathfrak{L}(V)/D\mathfrak{L}(V)$$

carries the structure of a Lie algebra with Lie bracket given by

$$[a + D\mathfrak{L}(V), b + D\mathfrak{L}(V)] = a(0)b + D\mathfrak{L}(V).$$

Therefore, we define

$$V_{Lie} := \mathfrak{L}(V)/D\mathfrak{L}(V).$$

We let a_n denote the image of $t^n \otimes a$ in V_{Lie}. Then note that

$$[a_n, b_m] = \sum_{i=0}^{\infty} \binom{n}{i} (a(i)b)_{m+n-i}. \tag{3.6}$$

We give a complex grading to V_{Lie} by declaring, for homogeneous $a \in V$,

$$\deg(a_n) = |a| - n - 1.$$

We then have the Lie subalgebras

$$(V_{Lie})^+ = \langle\ a_n \mid \mathrm{Re}(\deg(a_n)) > 0\ \rangle$$

and

$$(V_{Lie})^0 = \langle\ a_n \mid \deg(a_n) = 0\ \rangle.$$

The space

$$(V_{Lie})^- = \langle a_n \mid \mathrm{Re}(\deg(a_n)) < 0 \quad \text{or} \quad \mathrm{Re}(\deg(a_n)) = 0 \text{ and } \mathrm{Im}(\deg(a_n)) \neq 0\ \rangle$$

is not a Lie subalgebra of V_{Lie}. Indeed, if $|a| = n + i\lambda$ for some $n \in \mathbb{Z}$ and nonzero $\lambda \in \mathbb{R}$, and $|b| = n - i\lambda$, then $a_{\overline{|a|}-1} = a_{n-1}$ and $b_{\overline{|b|}-1} = b_{n-1}$ are in $(V_{Lie})^-$, but

$$[a_{n-1}, b_{n-1}] = \sum_{i=0}^{\infty} \binom{n-1}{i} (a(i)b)_{2n-2-i} \in (V_{Lie})^0.$$

Therefore, we have that

$$(V_{Lie})^{\leq 0} = (V_{Lie})^0 \oplus (V_{Lie})^-$$

is a Lie subalgebra of V_{Lie}.

Of course, it is easy to see that $(V_{Lie})^0$ is spanned by elements of the form $a_{|a|-1}$ for homogenous $a \in V^0$. This gives a surjection

$$V^0 \twoheadrightarrow (V_{Lie})^0.$$

Proposition 3.3.1. *The kernel of the map*

$$\begin{aligned} \phi : V^0 &\rightarrow (V_{Lie})^0 \\ a &\mapsto a_{|a|-1} \end{aligned}$$

is $(T+L)V^0$.

Proof. The zero element in the quotient space $(V_{Lie})^0$ is equal to the space

$$D \langle a_n \mid \deg(a_n) = -1 \rangle .$$

A typical element in this space is a sum

$$\sum_j D(b^j_j) = \sum_j j(b^j_{j-1}) + (Tb^j)_j$$

for some homogeneous $b^j \in V_j$. One notes that $j(b^j_{j-1}) = \phi(Lb^j)$ and $(Tb^j)_j = \phi(Tb^j)$. The result then follows from the fact that

$$\sum_j D(b^j_j) = \sum_j \phi(Lb^j) + \phi(Tb^j) = \phi\left((T+L)\sum_j b^j\right).$$

For further discussion, see Section 4 of [DLM2]. □

The next lemma was proved in [DLM2]. Recall that any associative algebra has a natural Lie structure given by the commutator bracket.

Lemma 3.3.2. *The map*

$$\begin{array}{rcl} (V_{Lie})^0 & \to & A(V)_{Lie} \\ a_{|a|-1} & \mapsto & a + O(V) \end{array}$$

is a Lie algebra epimorphism, where $A(V)_{Lie}$ is the natural Lie algebra on $A(V)$.

Proof. First observe that we have the inclusions

$$(T+L)V^0 \subseteq O(V^0) \subseteq O(V) \cap V^0,$$

which give rise to the corresponding epimorphisms

$$(V_{Lie})^0 \cong V^0/\left((T+L)V^0\right) \twoheadrightarrow A(V^0) \cong V^0/O(V^0) \twoheadrightarrow A(V) \cong V^0/(O(V) \cap V^0),$$

which in turn induce the desired linear epimorphism. We use (3.5) and (3.6) to see that $a_{|a|-1} \mapsto a + O(V)$ is a Lie algebra morphism. □

3.3.2 The Space $M(U)$

Throughout this section, we let U be a module for the associative algebra $A(V)$.

Since U is a module for $A(V)$, and hence for $A(V)_{Lie}$, we can lift U to a module for $(V_{Lie})^0$ via

$$\begin{array}{ccccc} (V_{Lie})^0 & \twoheadrightarrow & A(V)_{Lie} & \to & \mathrm{End}(U) \\ a_{|a|-1} & \mapsto & a + O(V) & \mapsto & o(a). \end{array}$$

Thus, $a_{|a|-1} \in (V_{Lie})^0$ acts as $o(a)$ on U. We want to extend this action to

$$(V_{Lie})^{\leq 0} = (V_{Lie})^0 \oplus (V_{Lie})^-,$$

and we do so by letting $(V_{Lie})^-$ annihilate U. It is not immediately clear that this definition does in fact yield an action of $(V_{Lie})^{\leq 0}$, however, the next result shows that this is indeed the case.

Proposition 3.3.3. *The linear extension of the map*

$$\begin{array}{rcll} \phi : (V_{Lie})^{\leq 0} & \to & A(V)_{Lie} & \\ a_{|a|-1} & \mapsto & a + O(V) & \quad (a_{|a|-1} \in (V_{Lie})^0) \\ a_n & \mapsto & O(V) & \quad (a_n \in (V_{Lie})^-) \end{array}$$

is a morphism of Lie algebras.

Proof. We have already seen that ϕ restricted to $(V_{Lie})^0$ is a morphism of Lie algebras. Therefore, it suffices to show that the image of the space

$$(V_{Lie})^0 \cap [(V_{Lie})^-, (V_{Lie})^-]$$

under ϕ lies in $O(V)$.

Assume that $a_n, b_m \in (V_{Lie})^-$, and $[a_n, b_m] \in (V_{Lie})^0$. Then it follows that $\overline{|a|} = \overline{|b|} = \mathrm{Re}(|a|) = \mathrm{Re}(|b|)$, and $|a|+|b| \in \mathbb{Z}$. We also have $n = m = \overline{|a|}-1$. We calculate

$$\begin{aligned} \phi\left([a_n, b_m]\right) &= \phi\left(\left[a_{\overline{|a|}-1}, b_{\overline{|b|}-1}\right]\right) \\ &= \phi\left(\sum_{i\geq 0} \binom{\overline{|a|}-1}{i} (a(i)b)_{\overline{|a|}+\overline{|b|}-2-i}\right) \\ &= \phi\left(\sum_{i\geq 0} \binom{\overline{|a|}-1}{i} (a(i)b)_{\overline{|a(i)b|}-1}\right) \\ &= \sum_{i\geq 0} \binom{\overline{|a|}-1}{i} (a(i)b) \\ &= \mathrm{Res}_z \left(Y(a,z)b\,(1+z)^{\overline{|a|}-1}\right). \end{aligned}$$

Note that an element of the form $b \circ a - a \circ b$ is in $O(V)$ by definition of $O(V)$. Therefore, the lemma follows from the claim that

$$\mathrm{Res}_z \left(Y(a,z)b\,(1+z)^{\overline{|a|}-1}\right) \equiv b \circ a - a \circ b \pmod{O(V)}. \tag{3.7}$$

To see this congruence, we use (3.3) and the fact that $|a| + |b| = \overline{|a|} + \overline{|b|}$ to calculate (mod $O(V)$)

$$\begin{aligned}
a \circ b &= \operatorname{Res}_z \left(Y(a,z)b \, \frac{(1+z)^{\overline{|a|}-1}}{z} \right) \\
&\equiv \operatorname{Res}_z \left(Y(b, \frac{-z}{1+z})a \, \frac{(1+z)^{\overline{|a|}-1}}{z} (1+z)^{-\overline{|a|}-\overline{|b|}} \right) \\
&= \operatorname{Res}_z \left(Y(b, \frac{-z}{1+z})a \, \frac{(1+z)^{-\overline{|b|}+1}}{-z} \frac{\partial}{\partial z} \left(\frac{-z}{1+z} \right) \right) \\
&= \operatorname{Res}_w \left(Y(b,w)a \, \frac{(1+w)^{\overline{|b|}}}{w} \right),
\end{aligned}$$

where in the last equality we use formula (2.2). Then we have (mod $O(V)$)

$$\begin{aligned}
a \circ b - b \circ a &\equiv \operatorname{Res}_z \left(Y(a,z)b \, \frac{(1+z)^{\overline{|a|}-1}}{z} \right) - \operatorname{Res}_z \left(Y(a,z)b \, \frac{(1+z)^{\overline{|a|}}}{z} \right) \\
&= \operatorname{Res}_z \left(Y(a,z)b \, (1+z)^{\overline{|a|}-1} \right) \left(\frac{1}{z} - \frac{1+z}{z} \right) \\
&= -\operatorname{Res}_z \left(Y(a,z)b \, (1+z)^{\overline{|a|}-1} \right),
\end{aligned}$$

which proves (3.7), and hence the result. □

To summarize, we have an action of the Lie subalgebra $(V_{Lie})^{\leq 0}$ on U, where $a_{|a|-1}$ acts as $o(a)$, and $(V_{Lie})^{-}$ annihilates. We now consider the induced module

$$M(U) = \operatorname{Ind}_{\mathcal{U}((V_{Lie})^{\leq 0})}^{\mathcal{U}(V_{Lie})} U \cong \mathcal{U}(V_{Lie}) \otimes_{\mathcal{U}((V_{Lie})^{\leq 0})} U \cong S((V_{Lie})^{+}) \otimes_{\mathbb{C}} U, \quad (3.8)$$

where $S(V)$ denotes the symmetric algebra on V and $\mathcal{U}(\cdot)$ denotes the universal enveloping algebra.

Note that $M(U)$ inherits a $\mathbb{C}$-grading from $S((V_{Lie})^{+})$ if we assert that the subspace U has degree 0. More precisely, we can write

$$M(U) = \bigoplus_{0 \prec \mu} M(U)(\mu) \quad (3.9)$$

where $M(U)(0) = U$.

3.3.3 Action of V on $M(U)$

We define an action of V on $M(U)$ via

$$Y_{M(U)}(v,z) = \sum_{n\in\mathbb{Z}} v_n\, z^{-n-1}, \tag{3.10}$$

so that $v(n) = v_n$ for all $v \in V$ and $n \in \mathbb{Z}$. Our goal is to show that a certain quotient of $M(U)$ is an admissible V-module. Note that $(M(U), Y_{M(U)})$ satisfies the requirements to be an admissible V-module, except for the Jacobi identity.

Establishing the Jacobi identity directly is often difficult. We therefore make use of the following result (see [LL] Proposition 3.4.3, or [FHL] for a systematic treatment):

Proposition 3.3.4. *Let $u \in M(U)$ and $a, b \in V$. Assume that the formal sums $Y(a,z)b$, $Y_{M(U)}(a,z)u$ and $Y_{M(U)}(b,z)u$ are truncated as in Definitions 2.2.1 and 2.4.1. Then the Jacobi identity as applied to the elements a, b and u is equivalent to the existence of integers k and l, depending only on a and u, that satisfy*

$$(z_1 - z_2)^k [Y_{M(U)}(a,z_1), Y_{M(U)}(b,z_2)]u = 0 \tag{3.11}$$

and

$$(z_0 + z_2)^l \Big(Y_{M(U)}(Y(a,z_0)b, z_2)u - Y_{M(U)}(a, z_0+z_2) Y_{M(U)}(b,z_2)u \Big) = 0. \tag{3.12}$$

Remark. Conditions (3.11) and (3.12) above are often called *weak commutativity* and *weak associativity*, respectively.

To see that the operators $Y_{M(U)}(a,z)$ satisfy (3.11), we make use of Theorem 2.3 (iv) of [K], which states that (3.11) is equivalent to the commutator formula

$$[a(n), b(m)]\, u = \sum_{i=0}^{\infty} \binom{n}{i} (a(i)b)(m+n-i)\, u. \tag{3.13}$$

The definition of the action (3.10) and commutator formula (3.6) imply that (3.13) holds, so (3.11) holds as well.

It is not in general true that the operators $Y_{M(U)}(a,z)$ satisfy (3.12). Therefore, we must work with a suitable quotient of $M(U)$.

In particular, let W be the subspace of $M(U)$ spanned by the coefficients of $z_0^i z_2^j$ in the expressions

$$(z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}} Y(a, z_0+z_2) Y(b, z_2) u - (z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}} Y(Y(a, z_0) b, z_2) u \tag{3.14}$$

for homogeneous $a \in V^r$, $b \in V$, and $u \in U$. Now define

$$\bar{M}(U) = M(U)/\mathcal{U}(V_{Lie})W.$$

Note that $\mathcal{U}(V_{Lie})W$ is a graded V_{Lie}-submodule of $M(U)$, and so $\bar{M}(U)$ inherits the $\mathbb{C}$-grading from $M(U)$. It is not yet clear that $\bar{M}(U)(0) \neq 0$, but this will be established in the course of proving Theorems 3 and 4.

Proposition 3.3.5. *Let M be a module for V_{Lie} with the property that there is a subspace X of M such that X generates M as a module for V_{Lie} and, for any $a \in V^r$ and any $u \in X$, there is an integer k such that*

$$(z_0+z_2)^k Y(a, z_0+z_2) Y(b, z_2) u = (z_2+z_0)^k Y(Y(a, z_0) b, z_2) u \tag{3.15}$$

for any $b \in V$. Then M is a weak V-module.

Proof. See [DLM2] Proposition 6.1, and take $r = s = 0$. □

Theorem 3. *The space $\bar{M}(U)$ is an admissible V-module. Moreover, $\bar{M}(U)$ is a universal object in the sense that given any admissible V-module N and any morphism of $A(V)$-modules $\phi : U \to \Omega(N)$, there is a unique morphism of admissible V-modules $\bar{M}(U) \to N$ which extends ϕ. In particular, if ϕ is surjective, then N is a quotient of $\bar{M}(U)$.*

Proof. Since U generates $M(U)$ as a V_{Lie}-module, it follows that $U + \mathcal{U}(V_{Lie})W$ generates $\bar{M}(U)$ as a V_{Lie}-module. Clearly, $U + \mathcal{U}(V_{Lie})W$ satisfies property (3.15), so Proposition 3.3.5 implies that $\bar{M}(U)$ is a weak V-module. Moreover, by previous remarks, we know that $\bar{M}(U)$ satisfies the grading requirements, except for possibly the requirement that $\bar{M}(U)(0) \neq 0$, but this is demonstrated in the proof of Theorem 4.

In fact, in proving Theorem 4 we will show that $\bar{M}(U)(0) = U$. The universal property that $\bar{M}(U)$ satisfies will then be a simple consequence of the construction of $\bar{M}(U)$ and Definition 2.4.2. □

3.3.4 Construction of $\Lambda(U)$

We now describe the construction of the Λ functor. Note that $M(U)$ has a unique maximal graded V_{Lie}-submodule J subject to $J \cap U = 0$. Indeed, one may take J to be the sum of all submodules N satisfying $N \cap U = 0$. The main result of this section is the following:

Theorem 4. *The space $\Lambda(U) = M(U)/J$ is an admissible V-module with the property that the lowest weight space of $\Lambda(U)$ is U.*

The main goal in the proof of this theorem is to show that $\mathcal{U}(V_{Lie})W \subseteq J$, and most of this section is devoted to this goal. Once we establish this fact, it will be clear that $\Lambda(U)$ is an admissible V-module, as it is a quotient of $\bar{M}(U)$.

First, we note that we can give an alternative characterization of J as follows: Let U^* denote the set of linear functionals on U. Given $u' \in U^*$, we can extend u' to all of $M(U)$ be declaring that u' annihilates all graded subspaces of $M(U)$ which are not equal to U. One then sees that the V_{Lie}-submodule J can be described alternatively as

$$J = \{v \in M(U) \mid \left\langle u', xv \right\rangle = 0 \text{ for all } u' \in U^*,\ x \in \mathcal{U}(V_{Lie})\ \}, \tag{3.16}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing $U^* \otimes M(U) \to \mathbb{C}$.

We use this characterization of J in what follows. The next lemma and its proof are modifications of Lemma 6.7 of [DLM2]. Since there is little risk of confusion, we omit the subscripts $M(U)$ from the operators $Y_{M(U)}(a, z)$.

Lemma 3.3.6. *Let $a \in V$ be homogeneous with $a \in V^r$ for some nonzero r, let $u' \in U^*$, and let $u \in U$. Then, for any $i, j \geq 0$, we have*

$$\begin{aligned} &\mathrm{Res}_{z_0}\, z_0^{-1+i}(z_0 + z_2)^{\overline{|a|}-1+j} \left\langle u', Y(a, z_0 + z_2)Y(b, z_2)u \right\rangle \\ &\quad = \mathrm{Res}_{z_0}\, z_0^{-1+i}(z_2 + z_0)^{\overline{|a|}-1+j} \left\langle u', Y(Y(a, z_0)b, z_2)u \right\rangle \end{aligned}$$

for any $b \in V$.

Proof. Since $j \geq 0$ and $|a| \neq \overline{|a|}$, we know that $a_{\overline{|a|}-1+j}$ is an element of $(V_{Lie})^-$, and therefore $a(\overline{|a|} - 1 + j)$ annihilates u. Then, for any $i \geq 0$, we have

$$\mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\overline{|a|}-1+j} Y(b, z_2)Y(a, z_1)u = 0.$$

Note that we also have the commutator formula

$$[Y(a,z_1),Y(b,z_2)] = \mathrm{Res}_{z_0}\, z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(a,z_0)b,z_2),$$

which is simply a reformulation of (3.11). Then

$$\begin{aligned}
&\mathrm{Res}_{z_0}\, z_0^i(z_0+z_2)^{\overline{|a|}-1+j}Y(a,z_0+z_2)Y(b,z_2)u\\
&\quad= \mathrm{Res}_{z_1}(z_1-z_2)^i z_1^{\overline{|a|}-1+j}Y(a,z_1)Y(b,z_2)u\\
&\quad= \mathrm{Res}_{z_1}(z_1-z_2)^i z_1^{\overline{|a|}-1+j}[Y(a,z_1),Y(b,z_2)]u\\
&\quad= \mathrm{Res}_{z_0}\, Res_{z_1}(z_1-z_2)^i z_1^{\overline{|a|}-1+j}z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(a,z_0)b,z_2)u\\
&\quad= \mathrm{Res}_{z_0}\, Res_{z_1} z_0^i z_1^{\overline{|a|}-1+j}z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)Y(Y(a,z_0)b,z_2)u\\
&\quad= \mathrm{Res}_{z_0}\, z_0^i(z_2+z_0)^{\overline{|a|}-1+j}Y(Y(a,z_0)b,z_2)u,
\end{aligned}$$

which proves the lemma for $i \geq 1$. Now assume that $i = 0$. Then

$$\begin{aligned}
&\mathrm{Res}_{z_0}\, z_0^{-1}(z_0+z_2)^{\overline{|a|}-1+j}\left\langle u', Y(a,z_0+z_2)Y(b,z_2)u\right\rangle\\
&\quad= \sum_{k=0}^{\infty}\binom{j}{k}\mathrm{Res}_{z_0}(z_0+z_2)^{\overline{|a|}-1}z_0^{k-1}z_2^{j-k}\left\langle u', Y(a,z_0+z_2)Y(b,z_2)u\right\rangle\\
&\quad= \sum_{k=1}^{\infty}\binom{j}{k}\mathrm{Res}_{z_0}(z_2+z_0)^{\overline{|a|}-1}z_0^{k-1}z_2^{j-k}\left\langle u', Y(Y(a,z_0)b,z_2)u\right\rangle\\
&\qquad+\mathrm{Res}_{z_0}(z_0+z_2)^{\overline{|a|}-1}z_0^{-1}z_2^{j}\left\langle u', Y(a,z_0+z_2)Y(b,z_2)u\right\rangle
\end{aligned}$$

The lemma then follows from the claims that

$$\mathrm{Res}_{z_0}\, z_0^{-1}(z_0+z_2)^{\overline{|a|}-1}\left\langle u', Y(a,z_0+z_2)Y(b,z_2)u\right\rangle = 0 \tag{3.17}$$

and

$$\mathrm{Res}_{z_0}\, z_0^{-1}(z_2+z_0)^{\overline{|a|}-1}\left\langle u', Y(Y(a,z_0)b,z_2)u\right\rangle = 0. \tag{3.18}$$

To see (3.17), recall that $b(\overline{|b|}-1-n)u = 0$ since $b_{\overline{|b|}-1+n} \in (V_{Lie})^-$ for $n \leq 1$. Then we have

$$\begin{aligned}
&\left\langle u', Y(a,z_0+z_2)Y(b,z_2)u\right\rangle\\
&\quad= \sum_{n\geq 0}\left\langle u', a(\overline{|a|}-1+n)b(\overline{|b|}-1-n)u\right\rangle(z_0+z_2)^{-\overline{|a|}-n}z_2^{-\overline{|b|}+n}.
\end{aligned}$$

This gives

$$\begin{aligned}
&\mathrm{Res}_{z_0}\, z_0^{-1}(z_0+z_2)^{\overline{|a|}-1}\left\langle u', Y(a, z_0+z_2)Y(b,z_2)u\right\rangle \\
&\quad = \mathrm{Res}_{z_0} \sum_{n\geq 0} \left\langle u', a(\overline{|a|}-1+n)b(\overline{|b|}-1-n)u\right\rangle (z_0+z_2)^{-n-1} z_2^{-\overline{|b|}+n-1} \\
&\quad = \mathrm{Res}_{z_0} \sum_{n\geq 0}\sum_{k\geq 0} \binom{-n-1}{k} \left\langle u', a(\overline{|a|}-1+n)b(\overline{|b|}-1-n)u\right\rangle \\
&\qquad\quad \cdot z_2^{-\overline{|b|}+n-1+k} z_0^{-n-1-k} \\
&\quad = \left\langle u', a(\overline{|a|}-1)b(\overline{|b|}-1)u\right\rangle z_2^{-\overline{|b|}-1} \\
&\quad = 0,
\end{aligned}$$

where the last equality holds because $a \in V^r$, so $a(\overline{|a|}-1)$ annihilates $b(\overline{|b|}-1)u$. This proves (3.17). An analogous calculation shows that

$$\begin{aligned}
&\mathrm{Res}_{z_0}\, z_0^{-1}(z_2+z_0)^{\overline{|a|}-1}\left\langle u', Y(Y(a,z_0)b, z_2)u\right\rangle \\
&\quad = \sum_{k\geq 0} \binom{\overline{|a|}-1}{k} \left\langle u', (a(k-1)b)(|a(k-1)b|-1)u\right\rangle z_2^{-|a|-|b|+\overline{|a|}-1},
\end{aligned}$$

where we make the assumption that $\overline{|a(j)b|} = |a(j)b|$ for $j \in \mathbb{Z}$. This assumption is justfied because otherwise the element $a(j)b$ would have a zero mode that does not preserve the graded subspace U, which would then imply that

$$\left\langle u', (a(j)b)(k)u\right\rangle = 0$$

for any $k \in \mathbb{Z}$. Then (3.18) follows from the fact that

$$\sum_{k\geq 0} \binom{\overline{|a|}-1}{k} (a(k-1)b)(|a(k-1)b|-1)u = o(a\circ b)u = 0,$$

which holds since $O(V)$ annihilates U. □

Lemma 3.3.7. *Let a, u, and u' be as in the previous lemma. Let $j \geq 0$ and $i \in \mathbb{Z}$. Then we have*

$$\begin{aligned}
&\mathrm{Res}_{z_0}\, z_0^{i}(z_0+z_2)^{\overline{|a|}-1+j}\left\langle u', Y(a, z_0+z_2)Y(b,z_2)u\right\rangle \\
&\quad = \mathrm{Res}_{z_0}\, z_0^{i}(z_2+z_0)^{\overline{|a|}-1+j}\left\langle u', Y(Y(a,z_0)b, z_2)u\right\rangle
\end{aligned}$$

for any $b \in V$.

Proof. The result holds for $i \geq -1$ by the previous lemma. The proof then follows from the T derivative property together with induction on i. A complete proof is given in [DLM2] Lemma 6.8. $\square$

Proposition 3.3.8. *The following holds for any homogeneous $a \in V^r$, $b \in V$, $u' \in U^*$, $u \in U$ and $j \geq 0$:*

$$\left\langle u', (z_0 + z_2)^{\overline{|a|}-1+\delta_{r,0}+j} Y(a, z_0 + z_2) Y(b, z_2) u \right\rangle = \left\langle u', (z_2 + z_0)^{\overline{|a|}-1+\delta_{r,0}+j} Y(Y(a, z_0) b, z_2) u \right\rangle . \tag{3.19}$$

In particular, U^* annihilates W.

Proof. Assume that $b \in V^s$ for some s. If $r = 0$ and $s \neq 0$, then both sides of (3.19) are equal to 0. This follows from the fact that the modes of a act on V^0, but the modes of b do not, i.e., for any $i \in \mathbb{Z}$,

$$b(i)V^0 \cap V^0 = 0.$$

Now assume that $r = s = 0$. Then this is essentially the case where $V = V^0$, so we appeal to [DLM2] Proposition 6.5. The only remaining case is where $r \neq 0$, and this case follows from the previous two lemmas. $\square$

Proposition 3.3.9. *For any $u' \in U^*$, one has*

$$\left\langle u', \mathcal{U}(V_{Lie}) W \right\rangle = 0.$$

In particular, $\mathcal{U}(V_{Lie}) W \subseteq J$.

Proof. Let X be the subset of $\mathcal{U}(V_{Lie})$ that satisfies

$$\left\langle u', X W \right\rangle = 0.$$

To prove the proposition, we must show that $X = \mathcal{U}(V_{Lie})$. Note that (3.9) and Proposition 3.3.8 imply that $\mathcal{U}((V_{Lie})^+) \subseteq X$ and $\mathcal{U}((V_{Lie})^0) \subseteq X$. Therefore, it suffices to show that $\mathcal{U}((V_{Lie})^-) \subseteq X$. Since $\mathcal{U}((V_{Lie})^-)$ is generated by elements of the form c_k with c homogeneous and $k \geq \overline{|c|} - 1$, and since $1 \in X$, it suffices to show that $Xc_k \subseteq X$. Let $x \in X$, $a \in V^r$, $b \in V$, $u \in U$, and $u' \in U^*$. We will show that

$$\begin{aligned} &\left\langle u', (z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xc_k \cdot Y(a, z_0+z_2)Y(b, z_2)u \right\rangle \\ &\quad = \left\langle u', (z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xc_k \cdot Y(Y(a, z_0)b, z_2)u \right\rangle \end{aligned} \tag{3.20}$$

Note that

$$\overline{|c(i)a|} = \overline{|c|} + \overline{|a|} - i - 1 - \epsilon,$$

where ϵ is equal to 0 or 1. We may further assume that $c(i)a \in V^s$ for some s, so that we have

$$\overline{|a|} - 1 + \delta_{r,0} + k - i = \overline{|c(i)a|} - 1 + \delta_{s,0} + \alpha,$$

where $\alpha = k - \overline{|c|} + 1 - \delta_{s,0} + \delta_{r,0} + \epsilon$. Note that $\alpha \geq 0$ since $k \geq \overline{|c|} - 1$, and if $r \neq 0$ and $s = 0$, then ϵ must equal 1.

We also have the following formula, which is equivalent to (3.6) :

$$[c_k, Y(a, z_0)] = \sum_{i=0}^{\infty} \binom{k}{i} z_0^{k-i} Y(c(i)a, z_0).$$

For notational convenience, we set

$$\beta = \left\langle u', (z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xc_k \cdot Y(a, z_0+z_2)Y(b, z_2)u \right\rangle,$$

and then calculate:

$$
\begin{aligned}
\beta &= \left\langle u', (z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(b,z_2)c_k u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}(z_0+z_2)^{k-i+\overline{|a|}-1+\delta_{r,0}}\ xY(c(i)a,z_0+z_2)Y(b,z_2)u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}z_2^{k-i}(z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(c(i)b,z_2)u\right\rangle \\
&= \left\langle u', (z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(b,z_2)c_k u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}(z_0+z_2)^{\overline{|c(i)a|}-1+\delta_{s,0}+\alpha}\ xY(c(i)a,z_0+z_2)Y(b,z_2)u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}z_2^{k-i}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(Y(a,z_0)c(i)b,z_2)u\right\rangle \\
&= \left\langle u', (z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(b,z_2)c_k u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}(z_2+z_0)^{\overline{|c(i)a|}-1+\delta_{r,0}+\alpha}\ xY(Y(c(i)a,z_0)b,z_2)u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}z_2^{k-i}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(Y(a,z_0)c(i)b,z_2)u\right\rangle \\
&= \left\langle u', (z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(b,z_2)c_k u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}(z_2+z_0)^{k-i+\overline{|a|}-1+\delta_{r,0}}\ xY(Y(c(i)a,z_0)b,z_2)u\right\rangle \\
&\quad + \left\langle u', \sum_{i=0}^{\infty}\binom{k}{i}z_2^{k-i}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(c(i)Y(a,z_0)b,z_2)u\right\rangle \\
&\quad - \left\langle u', \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{k}{i}\binom{i}{j}z_2^{k-i}z_0^{i-j}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(Y(c(j)a,z_0)b,z_2)u\right\rangle
\end{aligned}
$$

For $i, j \geq 0$, we have the identity

$$\binom{k}{i}\binom{i}{j} = \binom{k}{j}\binom{k-j}{i-j},$$

which we use to see that the second and fourth terms in the last expression cancel:

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{k}{i}\binom{i}{j}z_2^{k-i}z_0^{i-j}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(Y(c(j)a,z_0)b,z_2)u\\
&\quad=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i}\binom{i}{j}z_2^{k-i}z_0^{i-j}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(Y(c(j)a,z_0)b,z_2)u\\
&\quad=\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\binom{k}{j}\binom{k-j}{i-j}z_2^{k-i}z_0^{i-j}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(Y(c(j)a,z_0)b,z_2)u\\
&\quad=\sum_{j=0}^{\infty}\binom{k}{j}(z_2+z_0)^{k-j+\overline{|a|}-1+\delta_{r,0}}\ xY(Y(c(j)a,z_0)b,z_2)u.
\end{aligned}$$

Returning to the previous calculation, we have

$$\begin{aligned}
\beta &= \left\langle u',(z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(b,z_2)c_ku\right\rangle\\
&\qquad+\left\langle u',\sum_{i=0}^{\infty}\binom{k}{i}z_2^{k-i}(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xY(c(i)Y(a,z_0)b,z_2)u\right\rangle\\
&= \left\langle u',(z_0+z_2)^{\overline{|a|}-1+\delta_{r,0}}\ xY(a,z_0+z_2)Y(b,z_2)c_ku\right\rangle\\
&\qquad+\left\langle u',(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ x\left[c_k,Y(Y(a,z_0)b,z_2)\right]u\right\rangle\\
&= \left\langle u',(z_2+z_0)^{\overline{|a|}-1+\delta_{r,0}}\ xc_k\cdot Y(Y(a,z_0)b,z_2)u\right\rangle,
\end{aligned}$$

where in the last step we used the fact that if $c_ku\neq 0$, then $c_ku\in U$, since $k\geq\overline{|c|}-1$. This proves (3.20), and thus we have shown that $X=\mathcal{U}(V_{Lie})$, as desired. □

We now return to the proof of Theorem 4. Recall that our main goal was to show that $\mathcal{U}(V_{Lie})W\subseteq J$, which we accomplished in Proposition 3.3.9. Since $U+J$ generates $\Lambda(U)$ as a V_{Lie}-module, and since $W=0$ in $\Lambda(U)$, we can use Proposition 3.3.5 to see that $\Lambda(U)$ is a weak V module, and it is a quotient of $\bar{M}(U)$. Moreover, J is a graded submodule of $M(U)$, so $\Lambda(U)$ inherits the $\mathbb{C}$-grading from $M(U)$ and is therefore an admissible V-module. Since $J\cap U=0$, we further have that $\Omega(\Lambda(U))=\Lambda(U)(0)=U$. This proves Theorem 4 and completes the proof of Theorem 3.

Proposition 3.3.10. *Let U be a simple $A(V)$-module. Then $\Lambda(U)$ is a simple admissible V-module.*

Proof. The assertion that $\Lambda(U)$ is simple is equivalent to the assertion that J is a maximal $\mathcal{U}(V_{Lie})$-submodule of $M(U)$. Let X be a graded $\mathcal{U}(V_{Lie})$-submodule of $M(U)$ which properly contains J. Then X necessarily has nonempty intersection with $M(U)(0) = U$, and so the space $X \cap U$ is a nonzero $A(V)$-submodule of U. Since U is a simple $A(V)$-module, it must be that $X \cap U = U$, and since U generates $M(U)$ as a $\mathcal{U}(V_{Lie})$-module, it follows that $X = M(U)$. This shows that J is a maximal graded submodule of $M(U)$, and hence $\Lambda(U)$ is a simple V-module. □

3.4 A Categorical Bijection

In this section we establish the main result of this chapter.

Proposition 3.4.1. *Let M be a simple admissible V-module. Then $\Omega(M)$ is a simple $A(V)$-module.*

Proof. Recall from Proposition 3.2.1 that $\Omega(M) = M(0)$. Assume that $M(0)$ contains a nonzero $A(V)$-submodule U, and let N denote the admissible V-module generated by U. We use Theorem 3 to see that N is a quotient of $\bar{M}(U)$ and $N(0) = U$. On the other hand, the simplicity of M implies that $N = M$, hence $N(0) = M(0) = U$. □

Theorem 5. *The functors Λ and Ω induce mutually inverse bijections on the isomorphism classes of the categories of simple $A(V)$-modules and simple admissible V-modules.*

Proof. First, Theorem 4 implies that $\Omega(\Lambda(U)) = U$ for any $A(V)$-module U so that $\Omega \circ \Lambda = Id$ on the full category of $A(V)$-modules. Now assume that M is a simple V-module. Then Propositions 3.4.1 and 3.3.10 show that $\Omega(M)$ is a simple $A(V)$-module and $\Lambda(\Omega(M))$ is a simple V-module. Then M and $\Lambda(\Omega(M))$ are both simple quotients of the universal object $\bar{M}(\Omega(M))$. Since $\bar{M}(\Omega(M))$ has a unique maximal ideal J subject to $J \cap \Omega(M) = 0$, it follows that $M \cong \bar{M}(\Omega(M))/J \cong \Lambda(\Omega(M))$. □

Remark. The failure of Λ and Ω to be inverse bijections on the full categories of $A(V)$-modules and admissible V-modules follows from the existence of admissible V-modules which are not completely reducible. For example, suppose that M is an indecomposable admissible V-module that contains a proper admissible submodule N with $\Omega(N) \subseteq M(\mu)$ for some nonzero μ. Then as an $A(V)$-module, $\Omega(M)$ has a direct sum decomposition

$$\Omega(M) = \Omega(N) \oplus P,$$

where P is an $A(V)$-submodule which contains the nonzero subspace $M(0)$. The V-module $\Lambda(\Omega(M))$ then has the corresponding direct sum decomposition

$$\Lambda(\Omega(M)) = \Lambda(\Omega(N)) \oplus \Lambda(P),$$

which is clearly not isomorphic to the indecomposable module M.

Part II

Pseudo Vertex Operator Algebras

Chapter 4

Introduction

4.1 Overview

A vertex operator algebra V is a $\mathbb{Z}$-graded vertex algebra with a distinguished element $\omega \in V$, called the *conformal vector* (see Section 5.1.1). In general, V contains many conformal vectors, and each of these vectors induces an alternative vertex operator algebra structure on V (see [DM], [Lian], [MN]). Interestingly, this "shifted" vertex operator algebra structure is not in general isomorphic to the original vertex algebra structure, and this idea has been explored in various contexts to obtain new examples of vertex operator algebras. Noteworthy examples include the triplet algebra $\mathcal{W}(p)$ (see [AM2]) and the work of Dong and Mason in constructing examples of "exotic" vertex operator algebras (see [DM]).

The triplet algebra is one example of a class of vertex operator algebra called *logarithmic* vertex operator algebras. These logarithmic theories are characterized by the fact that $\omega(1)$ does not necessarily act semisimply on admissible modules. Such theories have been studied extensively, often under the name *logarithmic conformal field theory* (see [AM1], [AM2], [AM3], [F1], [F2], [FFHST], [G], [GK]). These logarithmic theories are interesting in part because they give rise to novel features in the representation theory of the Virasoro algebra, including examples of modules for the Virasoro algebra which are not completely reducible.

4.2 Motivation

The motivation for the results of Part II of this thesis comes from the two related ideas mentioned above. The first concerns the notion of a *conformal shift* of a vertex operator algebra $V = (V, Y, \mathbb{1}, \omega)$. As discussed above, one can shift the conformal vector ω to obtain a new conformal vector ω' with the property that $V' = (V, Y \mathbb{1}, \omega')$ is another vertex operator algebra. We may view ω and ω' as the endpoints of a continuous path $\Gamma \subset \mathbb{C}^k$ (see Section 5.1.2). The intermediate points $\omega'' \in \Gamma$ are conformal vectors in the sense that they generate the Virasoro algebra, but it is not in general true that $V'' = (V, Y, \mathbb{1}, \omega'')$ is a vertex operator algebra. One then asks what kind of structure does V'' possess? We show that, for a class of vertex operator algebra called *strongly regular*, each intermediate ω'' gives rise to a structure which we call a *pseudo vertex operator algebra*. Thus, any strongly regular vertex operator algebra generates a space of pseudo vertex operator algebras, and we view Γ as a path in this space.

The second motivation comes from the study of logarithmic conformal field theories. Given the importance of these objects, it is natural to relax the assumptions on $\omega(1)$ to include the case that $\omega(1)$ does not have integral spectrum, nor does it act semisimply on any V-module, including V itself. We prove that these relaxed assumptions describe the intermediate structures V'' discussed above (see Theorem 9). Therefore, using the framework of pseudo vertex operator algebras, we seek to understand the nature of the relationship between conformal shifts of vertex operator algebras and the corresponding representation theory.

Chapter 5

Vertex Operator Algebras

5.1 Vertex Operator Algebras

A vertex operator algebra is a particular type of vertex algebra. We introduce some background.

5.1.1 Definitions

Definition 5.1.1 (Virasoro Lie Algebra)**.** *The* Virasoro Lie algebra *is the vector space*

$$Vir_c = \mathbb{C}k \oplus \bigoplus_{n \in \mathbb{Z}} L(n),$$

with Lie algebra structure given by

$$\begin{aligned} [k, L(n)] &= 0, \\ [L(n), L(m)] &= (n-m)L(n+m) - c\frac{m^3 - m}{12}\delta_{m+n,0}k, \end{aligned} \tag{5.1}$$

where c is a complex number called the central charge.

Definition 5.1.2 (Vertex Operator Algebra)**.** *Let c be a complex number. A* vertex operator algebra *of central charge c is a quadruple $(V, Y, \mathbb{1}, \omega)$ consisting of a vertex algebra $(V, Y, \mathbb{1})$, and a distinguished vector $\omega \in V$ called the conformal vector (or Virasoro vector). If we set $L(n) = \omega(n+1)$ for each n, so that*

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)\ z^{-n-2},$$

then we require that the operators $\{L(n)\}_{n\in\mathbb{Z}}$ generate the Virasoro Lie algebra (with k acting as Id_V). We also require the following:

(i) $L(-1)$ *is the translation covariance operator* T *for the vertex algebra* $(V, Y, \mathbb{1})$,

(ii) V *carries a truncated* $\mathbb{Z}$*-grading*

$$V = \bigoplus_{n=N}^{\infty} V_n$$

for some $N \in \mathbb{Z}$*, such that each graded subspace* V_n *is of finite dimension,*

(iii) $L(0)$ *is semisimple on* V*, with* $L(0)$ *acting as the scalar* n *on each* V_n*.*

When there is no danger of confusion, we often denote by V *the vertex operator algebra* $(V, Y, \mathbb{1}, \omega)$*.*

Definition 5.1.3 (CFT Type)**.** *A vertex operator algebra* $V = \bigoplus_{n\in\mathbb{Z}} V_n$ *is said to be of* CFT type *if* $V_n = 0$ *for any* $n < 0$ *and* $V_0 = \mathbb{C}\mathbb{1}$*.*

Remark. The letters "CFT" in the previous definition are an abbreviation of "conformal field theory". Most of the first examples of vertex operator algebras were of CFT type (see [FLM], [D]), and little work had been done in exploring the existence of non-CFT type vertex operator algebras. Dong and Mason were among the first to systematically produce new families of non-CFT type vertex operator algebras, demonstrating that such structures exist in abundance (see [DM]).

Proposition 5.1.1. *Let* V *be a CFT type vertex operator algebra. Then* V_1 *carries the structure of a Lie algebra with bracket given by* $[a, b] = a(0)b$*.*

Proof. First, we recall that $L(-1) = T$ annihilates $V_0 = \mathbb{C}\mathbb{1}$ and $a(n)b = 0$ for $n \geq 2$. Then the skew-symmetry (2.8) immediately gives

$$[a, b] = a(0)b = -b(0)a = -[b, a] .$$

Similarly, the Jacobi identity for Lie algebras is an immediate consequence of the associator formula (2.7) with $m = n = 0$. $\square$

Definition 5.1.4 (C_2-Cofinite)**.** *A vertex operator algebra V is called C_2-*cofinite *if the space*

$$C_2(V) = \{a(-2)b \mid a, b, \in V\}$$

has finite codimension in V.

Remark. The C_2-cofiniteness condition is an internal condition that is related to certain finiteness conditions in the representation theory of a given vertex operator algebra V. A routine argument shows that the codimension of $C_2(V)$ in V is an upper bound on the dimension of $A(V)$. C_2-cofiniteness therefore implies that a vertex operator algebra V has a finite dimensional Zhu algebra $A(V)$, which, together with mild additional assumptions, implies that there are only finitely many isomorphism classes of simple admissible V-modules (see [ABD], [DLM1], [L2], [Z]).

5.1.2 Conformal Shifts

In this section we assume that $V = \bigoplus_{n \geq 0} V_n$ is a CFT type vertex operator algebra of central charge c. We note that if $h \in V_1$, then $h(1)h$ and $L(1)h$ are both elements in V_0, and are therefore equal to scalar multiples of $\mathbb{1}$.

Proposition 5.1.2. *Let $h \in V_1$ and let α and β be the complex numbers defined by $h(1)h = \alpha\mathbb{1}$ and $L(1)h = \beta\mathbb{1}$. Then $\omega_h = \omega + L(-1)h$ satisfies the Virasoro relations with central charge $c_h = c + 12(\beta - \alpha)$.*

Proof. We let $Y(\omega_h, z) = \sum_{n \in \mathbb{Z}} L_h(n) z^{-n-2}$, and use (2.5) to obtain

$$L_h(n) = L(n) - (n + 1)h(n). \tag{5.2}$$

We then calculate:

$$\begin{aligned} [L_h(m), L_h(n)] &= [L(m), L(n)] + (m + 1)(n + 1)[h(m), h(n)] \\ &\quad -(n + 1)[L(m), h(n)] - (m + 1)[h(m), L(n)] \end{aligned}$$

$$
\begin{aligned}
= \quad & [L(m), L(n)] + (m+1)(n+1)[h(m), h(n)] \\
& -(n+1)\sum_{i=0}^{\infty}\binom{m+1}{i}(\omega(i)h)(m+n+1-i) \\
& +(m+1)\sum_{i=0}^{\infty}\binom{n+1}{i}(\omega(i)h)(m+n+1-i) \\
= \quad & [L(m), L(n)] + (m+1)(n+1)[h(m), h(n)] \\
& -(n+1)\sum_{i=0}^{\infty}\binom{m+1}{i}(L(i-1)h)(m+n+1-i) \\
& +(m+1)\sum_{i=0}^{\infty}\binom{n+1}{i}(L(i-1)h)(m+n+1-i) \\
= \quad & [L(m), L(n)] + (m+1)(n+1)[h(m), h(n)] \\
& -(n+1)\Big((L(-1)h)(m+n+1) + (m+1)(L(0)h)(m+n)\Big) \\
& -(n+1)\left(\frac{m(m+1)}{2}(L(1)h)(m+n-1)\right) \\
& +(m+1)\Big((L(-1)h)(m+n+1) + (n+1)(L(0)h)(m+n)\Big) \\
& +(m+1)\left(\frac{n(n+1)}{2}(L(1)h)(m+n-1)\right) \\
= \quad & (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} + (n+1)(m+1)[h(m), h(n)] \\
& +(m-n)(L(-1)h)(m+n+1) \\
& +(m+1)(n+1)\frac{(n-m)}{2}(L(1)h)(m+n-1) \\
= \quad & (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} - (m-n)(m+n+1)(h)(m+n) \\
& +(n+1)(m+1)[h(m), h(n)] \\
& +(m+1)(n+1)\frac{(n-m)}{2}(L(1)h)(m+n-1) \\
= \quad & (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} \\
& +(n+1)(m+1)\left([h(m), h(n)] + \frac{(n-m)}{2}(L(1)h)(m+n-1)\right) \\
= \quad & (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} \\
& +(n+1)(m+1)\left(\sum_{i=0}^{\infty}\binom{m}{i}(h(i)h)(m+n-i)\right) \\
& +(n+1)(m+1)\left(\frac{(n-m)}{2}(\beta\mathbb{1})(m+n-1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} \\
&\quad +(n+1)(m+1)\Big((h(0)h)(m+n) + m(h(1)h)(m+n-1)\Big) \\
&\quad +(n+1)(m+1)\frac{(n-m)}{2}\beta\delta_{m+n,0}\,\mathrm{Id} \\
&= (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} \\
&\quad +(n+1)(m+1)\left(m(\alpha\mathbb{1})(m+n-1) + \frac{(n-m)}{2}\beta\delta_{m+n,0}\,\mathrm{Id}\right) \\
&= (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} \\
&\quad +(n+1)(m+1)\left(m\alpha\delta_{m+n,0}\,\mathrm{Id} + \frac{(n-m)}{2}\beta\delta_{m+n,0}\,\mathrm{Id}\right) \\
&= (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c\,\mathrm{Id} + \frac{m^3-m}{12}12(\beta-\alpha)\delta_{m+n,0}\,\mathrm{Id} \\
&= (m-n)L_h(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}(c+12(\beta-\alpha))\,\mathrm{Id}
\end{aligned}
$$

□

Remark. In what follows, we will concern ourselves primarily with the case where h is a *primary* vector, i.e., $L(1)h = 0$, so that $\beta = 0$ in the previous proposition.

Definition 5.1.5. *The quadruple* $(V, Y, \mathbb{1}, \omega_h)$ *is called a* shifted vertex operator algebra*, and is denoted* V^h*.*

Remark. We emphasize here that a shifted vertex operator algebra is not in general a vertex operator algebra, the choice of terminology being historical ([DM]). The shifted vertex operator algebra V^h is, however, a vertex algebra, since shifting the conformal vector does not change the underlying vertex algebra structure. Indeed, V^h satisfies all of the requirements to be a vertex operator algebra except for possibly the semisimplicity of $L_h(0)$ and the grading requirements on V. From (5.2) we see that

$$L_h(0) = L(0) - h(0),$$

so $L_h(0)$ is semisimple if and only if $h(0)$ is a semisimple operator. Moreover, the integrality and finite dimensionality of the $L_h(0)$-eigenspaces are determined by the $h(0)$-eigenspace decomposition of V. Even if $h(0)$ is semisimple on V with integral spectrum, it is not clear if the $L_h(0)$-eigenspaces are finite dimensional.

Remark. Since V and V^h share the same underlying vertex algebra, it follows that they have the same set of weak modules. Thus, the categories of weak V-modules and weak V^h-modules are equivalent.

Chapter 6

Pseudo Vertex Operator Algebras

6.1 Pseudo Vertex Operator Algebras

Definition 6.1.1 (Pseudo Vertex Operator Algebra)**.** *A* pseudo vertex operator algebra *is a $\mathbb{C}$-graded vertex algebra $V = \bigoplus_{\mu \in \mathbb{C}} V_\mu$ with the following additional properties:*

(i) There is a vector $\omega \in V_2$, called the conformal vector*, such that the operators $\{L(n)\}$ defined by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)\, z^{-n-2}$ generate the Virasoro Lie algebra,*

(ii) $L(-1) = T$,

(iii) For any $v \in V_\mu$, there is some $n \in \mathbb{N}$ such that $(L(0) - \mu)^n v = 0$, and $\dim(V_\mu) < \infty$ for all $\mu \in \mathbb{C}$,

(iv) $\mathrm{Re}(\mu) \geq |\,\mathrm{Im}(\mu)|$ for all but finitely many $\mu \in \mathrm{Spec}_V(L(0))$.

Remark. A pseudo vertex operator algebra satisfies the axioms of a vertex operator algebra with the exception of the semisimplicity of $L(0)$ and the integrality of the spectrum of $L(0)$. Moreover, since $\omega \in V_2$, it follows from (2.9) that any pseudo vertex operator algebra contains the vertex operator algebra generated by ω.

We let V denote a pseudo vertex operator algebra. The conformal structure of V allows us to define a new class of V-module.

Definition 6.1.2 (Ordinary V-Module)**.** *A weak V-module M is called* ordinary *if M has a $\mathbb{C}$-grading induced by $L_M(0)$-eigenvalues*

$$M = \bigoplus_{\mu \in \mathrm{Spec}_M(L_M(0))} M(\mu),$$

such that each graded subspace $M(\mu)$ is finite dimensional and satisfies

$$(L_M(0) - \mu)^n M(\mu) = 0$$

for some $n \in \mathbb{N}$. We also require that $\mathrm{Re}(\mu) > 0$ for all but finitely many $\mu \in \mathrm{Spec}_M(L_M(0))$.

Remark. An ordinary V-module is an admissible module for V as a $\mathbb{C}$-graded vertex algebra. We also note that if V is a vertex operator algebra, the definition of ordinary V-module agrees with the usual definition of ordinary V-module as found in the literature (see [DLM1], [L2]).

Remark. If V is a pseudo vertex operator algebra, then a simple admissible V-module M has a very particular grading. First, the simplicity of M implies that $\Omega(M) = M(\lambda)$ for some $\lambda \in \mathbb{C}$. Then we have

$$M = \bigoplus_{\mu \in \mathrm{Spec}_V(L(0))} M(\lambda + \mu). \tag{6.1}$$

From this, it is evident that each graded subspace $M(\lambda+\mu)$ is of finite dimension, and therefore we see that a simple admissible V-module is a simple ordinary V-module.

Definition 6.1.3 (Conformal Weight)**.** *Let M be a simple admissible V-module with grading as in* (6.1)*. The complex number λ is called the* conformal weight *of M.*

6.2 Pseudo Vertex Operator Algebras via Conformal Shift

The main goal of this section is to construct a family of pseudo vertex operator algebras via conformal shifts of a vertex operator algebra V. In order to ensure that the shifted vertex operator algebra V^h is a pseudo vertex operator algebra, we restrict our attention to a class of vertex operator algebras called *strongly regular*. This is a reasonably general class of vertex operator algebra.

6.2.1 Strongly Regular Vertex Operator Algebras

Definition 6.2.1. *A vertex operator algebra V is called* strongly regular *if V is C_2-cofinite, CFT type, $L(1)V_1 = 0$, and the category of admissible V-modules is semisimple.*

Remark. The property that the category of admissible V-modules is semisimple is often called *rationality*.

Remark. It is shown in [L2] and [DLM2] that if V is a strongly regular vertex operator algebra, then a simple admissible V-module is a simple ordinary V-module.

The following two theorems are proved in [M]. Theorem 6 is a strengthening of Proposition 5.1.1.

Theorem 6. *Let V be a strongly regular vertex operator algebra. Then the Lie algebra V_1 is reductive.*

This theorem allows us to work with a Cartan subalgebra of V_1. Indeed, a reductive Lie algebra $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{a},$$

where $\mathfrak{g}_{ss}$ is a semisimple Lie algebra and $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. In this case, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{ss}$, then $H = \mathfrak{h} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$.

Theorem 7. *Let V be a strongly regular vertex operator algebra. Then any Cartan subalgebra $H \subseteq V_1$ has a basis $\{h_1, \ldots, h_r\}$ such that each h_i satisfies the following two properties:*

(i) $h_i(0)$ is a semisimple operator with integral eigenvalues,
(ii) $h_i(1)h_i \in 2\mathbb{Z}\mathbb{1}$.

Now we consider a strongly regular simple vertex operator algebra V. In this case, it is known that V has finitely many inequivalent simple admissible modules, which we denote by $\{M_1 = V, ..., M_k\}$ (see [ABD], [DLM2] or [L2]). Let h be an element in V_1 with the property that $h(0)$ acts semisimply on every M_i with integral spectrum. Then for each $i \in \{1, ...k\}$, we define

$$J^i(\tau, z) = \mathrm{Tr}_{M_i}\, q^{L(0)-c/24}\zeta^{h(0)},$$

where $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi iz}$. Here we have omitted the subscripts from the expressions $L_{M_i}(0)$ and $h_{M_i}(0)$, since the spaces on which these operators act should be clear from context.

We note that $J^i(\tau, z)$ is, up to an overall shift, a power series in q:

$$J^i(\tau, z) = q^{s_i}\sum_{n=0}^{\infty}\sum_{r\in\mathbb{Z}} c^i(n, r)q^n\zeta^r. \tag{6.2}$$

where $s_i = -c/24 + \lambda_i$ and λ_i is the conformal weight of M_i. We have the following result ([KM]):

Theorem 8. *Let V be a strongly regular simple vertex operator algebra. Let $H \subseteq V_1$ be a Cartan subalgebra of V_1, and let $h \in H$ be such that $h(0)$ is semisimple on every M_i with integral spectrum, and $h(1)h = \beta\mathbb{1}$ with $\beta \in 2\mathbb{Z}$. Then the functions $J^i(\tau, z)$ are holomorphic in $\mathbb{H} \times \mathbb{C}$, and the following functional equations hold for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $(u, v) \in \mathbb{Z}^2$:*

(i) There are scalars $a_{ij}(\gamma)$ depending only on γ such that

$$J^i\left(\gamma\tau, \frac{z}{c\tau + d}\right) = e^{\pi icz^2\beta/(c\tau+d)}\sum_{j=1}^{r} a_{ij}(\gamma)J^j(\tau, z),$$

(ii) There is a permutation $i \mapsto i'$ of $\{1, ..., k\}$ such that

$$J^i(\tau, z + u\tau + v) = e^{-\pi i\beta(u^2\tau+2uz)}J^{i'}(\tau, z). \tag{6.3}$$

Our goal is to use the transformation property (6.3) to deduce information about the coefficients $c^i(n, r)$. For each $i \in \{1, ..., k\}$, set

$$d_i = \max_{j\in\{1,...,k\}} |s_i - s_j|,$$

and set $m = \beta/2 \in \mathbb{Z}$, with β defined as in the statement of Theorem 8. We then have the following proposition:

Proposition 6.2.1. *With the previous notation, one has* $c^i(n,r) = 0$ *if*

$$r^2 > m^2 + 4m(n + d_i).$$

Proof. The transformation property (6.3) implies that

$$c^{i'}(n + ru + mu^2 + s_i - s_{i'}, r + 2um) = c^i(n,r).$$

Then using (6.2), we see that $c^i(n,r) = 0$ if $n + ru + mu^2 + s_i - s_{i'} < 0$. From this we can see that $c^i(n,r) = 0$ whenever there is an integer u such that $(n + d_i) + ru + mu^2 < 0$.

The condition $(n+d_i)+ru+mu^2 < 0$ is equivalent to the condition that the quadratic polynomial $f(x) = (n + d_i) + rx + mx^2$ has a negative value when evaluated at some integer u. Since m is a positive number, elementary algebra tells us that this condition is satisfied if the roots of f are more than 1 unit apart, i.e., if

$$\frac{\sqrt{r^2 - 4m(n + d_i)}}{2m} > \frac{1}{2},$$

which is equivalent to $r^2 > m^2 + 4m(n + d_i)$. □

Note that the coefficient $c^i(n,r)$ is the dimension of the space

$$M_i(n,r) = \{\ u \in M_i \mid L(0)u = nu \ \text{ and } \ h(0)u = ru\ \}.$$

In particular, one has

$$c^1(n,r) = \dim\{\ v \in V_n \mid h(0)v = rv\ \}.$$

Now define $h^n \in \mathbb{Z}_{\geq 0}$ to be the largest element of the set

$$\{\quad |m| \quad \mid \quad m \text{ is an eigenvalue of } h(0) \text{ on } V_n\ \},$$

where $|m|$ denotes the usual absolute value of m. In other words, $c^1(n, h^n) \neq 0$ and $c^1(n,r) = 0$ if $|r| > h^n$.

Remark. In the next few results, we make use of the so-called "big O" notation. To be precise, we say that $f(n)$ is *big O* of $g(n)$ as $n \to \infty$ if there is a nonnegative real number c such that

$$\lim_{n\to\infty} \left| \frac{f(n)}{g(n)} \right| \le c,$$

and we denote this by $f(n) \sim O(g(n))$.

Lemma 6.2.2. *Let V and h be as above. Then the following inequality holds for every n:*

$$(h^n)^2 \le n(4m) + m^2 + 4md_1.$$

In other words, $h^n \sim O(\sqrt{n})$ as $n \to \infty$.

Proof. By Proposition 6.2.1, it follows that $(h^n)^2 \le m^2 + 4m(n + d_1)$ for every n. □

Of course, one obtains a similar statement if we replace h by any complex scalar multiple λh of h. In this case, the previous proposition says that $|\lambda h^n| \sim O(\sqrt{n})$. This leads us to the following extension of Lemma 6.2.2:

Lemma 6.2.3. *Let V be a simple strongly regular vertex operator algebra and let $h \in V_1$ be such that $h(0)$ is semisimple. Then $|h^n| \sim O(\sqrt{n})$ as $n \to \infty$.*

Proof. Since h is a semisimple element of V_1, we know that h is an element of some Cartan subalgebra H of V_1. Now using a basis $\{h_1, ..., h_k\}$ of H as in Theorem 7, we write

$$h = \sum_{i=1}^{k} \lambda_i h_i$$

for some $\lambda_i \in \mathbb{C}$, so that

$$h(0) = \sum_{i=1}^{k} \lambda_i h_i(0). \tag{6.4}$$

Since $\{h_i(0)\}$ is a set of commuting, semisimple operators, it follows that

$$0 \le |h^n| \le \sum_{i=1}^{k} |\lambda_i h_i^n|.$$

Our prior remarks show that each of the above summands satisfies $|\lambda_i h_i^n| \sim O(\sqrt{n})$, and so it follows that $|h^n| \sim O(\sqrt{n})$. □

Lemma 6.2.4. *Let V and h be as in Proposition 6.2.3, and consider $L_h(0) = L(0) - h(0)$. Then $\mathrm{Re}(\mu) \geq |\,\mathrm{Im}(\mu)|$ for all but finitely many $\mu \in \mathrm{Spec}_V(L_h(0))$.*

Proof. Making use of the decomposition (6.4), one sees that

$$\mathrm{Spec}_V(L_h(0)) \;\subseteq\; \mathbb{Z} - \sum_{i=1}^{k} \lambda_i \mathbb{Z} \;\subset\; \mathbb{C}.$$

Any $\mu \in \mathrm{Spec}_V(L_h(0))$ is of the form

$$\mu = n - \sum_{i=1}^{k} \lambda_i a_i,$$

where $a_i \in \mathrm{Spec}_{V_n}(h_i(0)) \subset \mathbb{Z}$.

If $h(0)$ does not act as 0 on V_n, then $h_i^n > 0$ for some i. Without loss of generality we may assume $h_1^n > 0$. Now choose $m \in \mathbb{R}$ such that

$$mh_1^n \;>\; k \cdot \max_i |\,\mathrm{Re}(\lambda_i h_i^n)| + k \cdot \max_i |\,\mathrm{Im}(\lambda_i h_i^n)|.$$

Using Lemma 6.2.3, we see that that $n - mh_1^n > 0$ for all but finitely many n. Therefore, since $h_1^n > 0$, we have

$$n - k \cdot \max_i |\,\mathrm{Re}(\lambda_i h_i^n)| - k \cdot \max_i |\,\mathrm{Im}(\lambda_i h_i^n)| \;>\; n - mh_1^n \;>\; 0$$

for all but finitely many n. In particular,

$$n - k \cdot \max_i |\,\mathrm{Re}(\lambda_i h_i^n)| \;>\; k \cdot \max_i |\,\mathrm{Im}(\lambda_i h_i^n)|.$$

for all but finitely many n.

Then we have

$$
\begin{aligned}
\operatorname{Re}\left(n - \sum_{i=1}^{k} \lambda_i a_i\right) &= n - \operatorname{Re}\left(\sum_{i=1}^{k} \lambda_i a_i\right) \\
&\geq n - \left|\operatorname{Re}\left(\sum_{i=1}^{k} \lambda_i a_i\right)\right| \\
&\geq n - k \cdot \max_i |\operatorname{Re}(\lambda_i h_i^n)| \\
&> k \cdot \max_i |\operatorname{Im}(\lambda_i h_i^n)| \\
&\geq \sum_{i=1}^{k} |\operatorname{Im}(\lambda_i h_i^n)| \\
&\geq \sum_{i=1}^{k} |\operatorname{Im}(\lambda_i a_i)| \\
&\geq \left|\operatorname{Im}\left(\sum_{i=1}^{r} \lambda_i a_i\right)\right|
\end{aligned}
$$

for all but finitely many n. Since $h(0)$ has only finitely many eigenvalues on each V_n, it follows that the above inequality holds for all but finitely many $L_h(0)$-eigenvalues μ. □

Lemma 6.2.5. *Let V and h be as in Proposition 6.2.3. Then the $L_h(0)$ eigenspaces are all finite dimensional.*

Proof. Assume to the contrary that μ is an eigenvalue corresponding to an infinite dimensional $L_h(0)$-eigenspace. Since each V_n is of finite dimension, it follows that there must be an infinite number of $k \in \mathbb{N}$ for which

$$k - \mu \in \operatorname{Spec}_{V_k}(h(0)),$$

which is impossible due to Lemma 6.2.3. □

Theorem 9. *Let V be a simple strongly regular vertex operator algebra . Then V^h is a simple pseudo vertex operator algebra for any $h \in V_1$.*

Proof. We first remark that simplicity of V^h is equivalent to simplicity of V since they are both the same underlying vertex algebra. The C_2-cofiniteness of

V implies that V is finitely generated, that is,

$$V = \left\langle \bigoplus_{n=0}^{N} V_n \right\rangle.$$

Then the action of $h(0)$ on V is completely determined by the action of $h(0)$ on $\bigoplus_{n=0}^{N} V_n$, which is a finite dimensional V_1-module. Therefore, we consider the abstract Jordan decomposition $h(0) = h^{ss}(0) + h^{n}(0)$, with $h^{ss}(0)$ a semisimple operator and $h^{n}(0)$ a nilpotent operator such that $h^{ss}(0)$ and $h^{n}(0)$ commute. The theory of Lie algebras then ensures that the operators $h^{ss}(0)$ and $h^{n}(0)$ are modes of elements h^{ss}, $h^{n} \in V_1$. Thus any $h \in V_1$ decomposes as $h = h^{ss} + h^{n}$, where $h^{ss}(0)$ is a semisimple operator on V and $h^{n}(0)$ is a nilpotent operator on V that commutes with $h^{ss}(0)$.

First assume that $h^{n} = 0$. Then $L_h(0) = L(0) - h^{ss}(0)$ is a semisimple operator. In this case V^h has a grading

$$V^h = \bigoplus_{\mu \in \mathrm{Spec}_{V^h}(L(0))} V_\mu,$$

which satisfies (2.9). Moreover, we know that $\mathbb{1} \in V_0$, so $V_0 \neq 0$. This fact, together with Lemma 6.2.4, implies that V^h has a lowest weight space V^h_λ. Since V^h is simple as a vertex algebra, it is generated by V^h_λ. This shows that V^h is a $\mathbb{C}$-graded vertex algebra. It is clear from (5.2) that $L(-1) = L_h(-1)$. Therefore, recalling Lemmas 6.2.5 and 6.2.4, we see that V^h is a pseudo vertex operator algebra.

If h^{n} is not zero, then $h^{n}(0)$ simply acts on each $L_h^{ss}(0)$-eigenspace since $[h^{n}(0), L_h^{ss}(0)] = 0$, and so we see that the generalized $L_h(0)$-eigenspaces are precisely the $L_h^{ss}(0)$-eigenspaces. The finite dimensionality of each $L_h^{ss}(0)$-eigenspace V^h_λ implies that $(L_h(0) - \lambda)$ is a nilpotent operator on V^h_λ. The result follows. □

6.3 Regularity of Lattice Pseudo Vertex Operator Algebras

Let V_L be the vertex operator algebra associated to a rank k positive definite even lattice L (see [FLM], [K], or [LL]). Dong [D] proved that V_L is a rational vertex operator algebra, and that the irreducible ordinary modules for V_L correspond to cosets of L in its dual lattice L°. In particular, the irreducible ordinary V_L-modules are of the form

$$V_{L-\lambda} = M(1) \otimes \mathbb{C}[L-\lambda],$$

where $\lambda \in L^\circ$ and $\mathbb{C}[L-\lambda]$ is the corresponding module for the twisted group algebra $\mathbb{C}\{L\}$.

We recall Theorem 3.16 of [DLM1]:

Theorem 10. *Let L be any positive definite even lattice. Then any weak V_L-module is completely reducible, and any simple weak V_L-module is isomorphic to $V_{L-\lambda}$ for some λ in L°. In other words, V_L is regular.*

Since V_L is a simple, strongly regular vertex operator algebra, we know that for any $h \in V_1$, one obtains a pseudo vertex operator algebra by shifting the conformal structure on V_L by the element h. Specifically, we have the pseudo vertex operator algebra $V_L^h = (V_L, Y, \mathbb{1}, \omega_h)$, where $\omega_h = \omega - L(-1)h$.

Theorem 11. *The pseudo vertex operator algebra V_L^h is regular. In other words, any weak V_L^h-module is a direct sum of simple ordinary V_L^h-modules.*

Before beginning the proof of Theorem 11, we recall some details about partition functions. For any coset $L-h$ of L in H, we define the formal sum

$$\theta_{L-h}(q) = \sum_{\alpha\in L-h} q^{\langle\alpha,\alpha\rangle/2} = q^{\langle h,h\rangle/2} \sum_{\alpha\in L} q^{\langle\alpha,\alpha\rangle/2-\langle h,\alpha\rangle}.$$

We define the partition function of a V_L^h-module M as

$$Z_{M,V_L^h}(q) = \mathrm{Tr}_M\, q^{L_h(0)-c_h/24}.$$

This expression is to be treated only as a formal sum, since it may contain complex powers of q. Since we are primarily interested in the spectrum of $L_h(0)$ we only need to consider the operator $L_h^{ss}(0) = L(0) - h^{ss}(0)$. Therefore, we assume that $h^n = 0$ (see the proof of Theorem 9). For $\lambda \in L^\circ$, we consider the partition function of the V_L^h-module $V[L-\lambda]$. If $u \in M(1)(n)$ and $\alpha \in L-\lambda$, we have

$$(L(0) - h(0))(u \otimes e^\alpha) = \left(n + \frac{1}{2}\langle \alpha, \alpha \rangle - \langle h, \alpha \rangle\right)(u \otimes e^\alpha). \tag{6.5}$$

As a formal sum, we have that

$$\mathrm{Tr}_{V[L-\lambda]}\, q^{L_h(0)} = \sum_{\mu \in \mathbb{C}} \dim(V[L-\lambda](\mu))\, q^\mu,$$

where $V[L-\lambda](\mu)$ is the $L_h(0)$-eigenspace with eigenvalue μ. Then we calculate:

$$\begin{aligned}
Z_{V[L-\lambda],V_L^h} &= \mathrm{Tr}_{V[L-\lambda]}\, q^{L_h(0)-c_h/24} \\
&= \mathrm{Tr}_{V[L-\lambda]}\, q^{L(0)-h(0)-k/24+\langle h,h\rangle/2} \\
&= Z_{M(1),V_L}(q) \cdot \mathrm{Tr}_{V[L-\lambda]}\, q^{L(0)-h(0)+\langle h,h\rangle/2} \\
&= Z_{M(1),V_L}(q) \cdot q^{\langle h,h\rangle/2} \sum_{\alpha \in \mathbb{C}[L-\lambda]} q^{\langle\alpha,\alpha\rangle/2-\langle h,\alpha\rangle} \\
&= Z_{M(1),V_L}(q) \cdot \theta_{L-h-\lambda}(q).
\end{aligned}$$

Proof of Theorem 11. Let M be any weak V_L^h-module. Since V_L^h has the same modes as V_L, it follows that M is a weak module for V_L. Due to the regularity of V_L, M must decompose as a direct sum of simple ordinary V_L-modules. By [D] we know that each simple ordinary V_L-module must be of the form $V_{L-\lambda}$ for some $\lambda \in L^\circ$. Thus, M decomposes as a sum of simple weak V_L^h-modules, each of which is of the form $V_{L-\lambda}$. Moreover, we calculated above that the partition function of $V_{L-\lambda}$ as a V_L^h-module satisfies

$$Z_{V[L-\lambda],V_L^h} = Z_{M(1),V_L}(q) \cdot \theta_{L-h-\lambda}(q),$$

and this is sufficient to show that $V_{L-\lambda}$ satisfies the grading requirements for ordinary V_L^h-modules. Thus, M is a sum of simple ordinary V_L^h-modules. □

Corollary 6.3.1. *$A(V_L^h)$ is a finite dimensional semisimple algebra.*

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