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THE SHARP CONSTANT IN THE WEAK (1,1) INEQUALITY FOR THE SQUARE FUNCTION: A NEW PROOF

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ABSTRACT. In this note we give a new proof of the sharp constant $C = e^{-1/2} + \int_0^1 e^{-x^2/2}\,dx$ in the weak $(1,1)$ inequality for the dyadic square function. The proof makes use of two Bellman functions $\mathbb{L}$ and $\mathbb{M}$ related to the problem, and relies on certain relationships between $\mathbb{L}$ and $\mathbb{M}$, as well as the boundary values of these functions, which we find explicitly. Moreover, these Bellman functions exhibit an interesting behavior: the boundary solution for $\mathbb{M}$ yields the optimal obstacle condition for $\mathbb{L}$, and vice versa.

1. INTRODUCTION

In this paper we consider weak inequalities for the dyadic square function:

$$S\varphi(x) := \left(\sum_{I \in \mathcal{D}} (\varphi, h_I)^2 \frac{\mathbb{1}_I(x)}{|I|} \right)^{1/2},$$

where $(\cdot, \cdot)$ denotes the usual inner product in $L^2(\mathbb{R})$, $\mathcal{D}$ is the standard collection of dyadic intervals on the real line, and $\{h_I\}_{I \in \mathcal{D}}$ are the (L^2–normalized) Haar functions:

$$h_I(x) := \frac{1}{\sqrt{|I|}}(\mathbb{1}_{I_-}(x) - \mathbb{1}_{I_+}(x)),$$

where I_- and I_+ denote the left and right halves of I, respectively. In particular, we look at localized versions of S, applied to compactly supported functions; for a dyadic interval $J \in \mathcal{D}$, let

$$S_J^2\varphi := \sum_{I \subset D,\, I \subseteq J} (\varphi, h_I)^2 \frac{\mathbb{1}_I}{|I|} = \sum_{I \subset J} |\Delta_I \varphi|^2 \mathbb{1}_I,$$

where $\Delta_I\varphi$ denotes the martingale difference

$$\Delta_I \varphi := \frac{1}{2}(\langle\varphi\rangle_{I_+} - \langle\varphi\rangle_{I_-})(\mathbb{1}_{I_+} - \mathbb{1}_{I_-}) = (\varphi, h_I)h_I.$$

Note that $S_J^2\varphi = S[(\varphi - \langle\varphi\rangle_J)\mathbb{1}_J]$, where $\langle\varphi\rangle_J := \frac{1}{|J|}\int_J \varphi\,dx$, so we may always assume that $\operatorname{supp}(\varphi) \subset J$.

We are looking for the sharp constant C in the inequality

$$|\{x \in J : S_J^2\varphi(x) \geq \lambda\}| \leq C\frac{1}{\sqrt{\lambda}} \int_J |\varphi|,$$

for all $\varphi \in L^1(J)$ and $J \in \mathcal{D}$. It was conjectured by Bollobas in [2], and it was later proved by Osekowski in [4], that this constant is

$$(1.1) \qquad C = \Psi(1), \text{ where } \Psi(\tau) = \tau\Phi(\tau) + e^{-\tau^2/2} \text{ and } \Phi(\tau) = \int_0^\tau e^{-x^2/2}\,dx.$$

In this paper we give a new proof of this fact, using Bellman functions.

B. Bollobas published [2] in 1982 but apparently he initiated this problem in mid-70's, as it is said in [2] that he invented the problem to entertain professor Littlewood. In [2] a certain constant and a certain special function (the Bellman function of an underlying problem) were invented. But the fact that the constant and the function of Bollobas are precisely the best constant and the Bellman function correspondingly were proved only in 2008 by A. Osekowski. in [4]. We give here a different proof of this fact, and we list also some extra properties of the function found by Bollobas in [2].

We begin by defining the standard Bellman function for the above listed problem:

Definition 1. *Given $f \in \mathbb{R}$, $F \geq |f|$, and $\lambda > 0$, define:*

$$\mathbb{M}(f, F, \lambda) := \sup \frac{1}{|J|} |\{x \in J : S_J^2 \varphi(x) \geq \lambda\}|,$$

where the supremum is over all functions φ, supported in $J \in \mathcal{D}$, such that $\langle \varphi \rangle_J = f$ and $\langle |\varphi| \rangle_J = F$. We say that any such φ is an admissible function for $\mathbb{M}(f, F, \lambda)$.

As shown in Proposition 2.1, this function has the expected properties, such as a main inequality and an obstacle condition. Also as expected, we show in Theorem 2.3 that $\mathbb{M}$ is the so-called "least supersolution" for its main inequality.

Next, we define another Bellman function, also associated to this problem:

Definition 2. *Given $f \in \mathbb{R}$, $0 \leq p \leq 1$, and $\lambda > 0$, define:*

$$\mathbb{L}(f, p, \lambda) := \inf \langle |\varphi| \rangle_J,$$

where the infimum is over all functions φ, supported in $J \in \mathcal{D}$, such that

$$\langle \varphi \rangle_J = f \quad and \quad \frac{1}{|J|} |\{x \in J : S_J^2 \varphi(x) \geq \lambda\}| = p.$$

We say that any such φ is an admissible function for $\mathbb{L}(f, p, \lambda)$.

This definition is inspired by Bollobas [2] – see Remark 2.7 for details of the connection to Bollobas's definition. Being defined as an infimum, this function will have most of the mirrored properties of $\mathbb{M}$ – replace concavity with convexity for example. These are detailed in Proposition 2.2. Also mirroring $\mathbb{M}$, we show in Theorem 2.5 that $\mathbb{L}$ is the so-called "greatest subsolution" for its main inequality.

Using the standard methods, we obtain so-called "obstacle conditions" for $\mathbb{M}$ and $\mathbb{L}$, namely

$$\mathbb{M}(f, F, \lambda) = 1, \forall F \geq \sqrt{\lambda} \quad \text{and} \quad \mathbb{L}(f, p, \lambda) = |f|, \forall |f| \geq \sqrt{\lambda}.$$

While these suffice, as expected, to prove the least supersolution and greatest subsolution results, there is no reason to believe they are optimal. That is, $\mathbb{M}$ could very well be equal to 1 for some points where $F < \sqrt{\lambda}$, for instance. As it turns out, we may obtain the optimal obstacle condition for $\mathbb{M}$ from information about $\mathbb{L}$, and vice versa.

In Section 3 we explore the connections between $\mathbb{M}$ and $\mathbb{L}$. We show in Theorem 3.1 that $\mathbb{L}(f, p, \lambda)$ is the smallest value of F for which $\mathbb{M}(f, F, \lambda) \geq p$, and $\mathbb{M}(f, F, \lambda)$ is the largest value of p such that $\mathbb{L}(f, p, \lambda) \leq F$:

$$\mathbb{L}(f, p, \lambda) = \inf\{F \geq |f| : \mathbb{M}(f, F, \lambda) \geq p\} \quad \text{and} \quad \mathbb{M}(f, F, \lambda) = \sup\{p \in [0, 1] : \mathbb{L}(f, p, \lambda) \leq F\}.$$

These relationships are further improved in Proposition 3.4, where we show that in certain domains (ultimately the really "interesting" parts of the domains), we have in fact that

$$\mathbb{M}(f, \mathbb{L}(f, p, \lambda), \lambda) = p \quad \text{and} \quad \mathbb{L}(f, \mathbb{M}(f, F, \lambda), \lambda) = F.$$

Then the value of $\mathbb{M}$ along the boundary $F = |f|$:

$$\mathbb{M}_b(f, \lambda) := \mathbb{M}(f, |f|, \lambda) = \sup\{p \in [0, 1] : \mathbb{L}(f, p, \lambda) = |f|\},$$

yields the optimal obstacle condition for $\mathbb{L}$, and the value of $\mathbb{L}$ along the boundary $p = 1$:

$$\mathbb{L}_b(f, \lambda) := \mathbb{L}(f, 1, \lambda) = \inf\{F \geq |f| : \ \mathbb{M}(f, F, \lambda) = 1\},$$

yields the optimal obstacle condition for $\mathbb{M}$. We find $\mathbb{M}_b$ and $\mathbb{L}_b$ explicitly in Section 5. See Section 3.1 and Figures 1 and 2 for a description of the optimal obstacle conditions for $\mathbb{M}$ and $\mathbb{L}$ obtained from these boundary values.

In Section 4 we give the new proof of the sharp constant in (1.1). The inequality

$$\mathbb{M}(0, F, \lambda) \leq \frac{F}{\mathbb{L}(0, 1, \lambda)} = \frac{F}{\mathbb{L}_b(0, \lambda)},$$

originally proved by Bollobas [2] is given a more detailed proof in Theorem 4.1. This, combined with the relationship

$$\mathbb{L}(f, \mathbb{M}(f, F, \lambda), \lambda) = F,$$

and the expression of $\mathbb{L}_b$ obtained in Theorem 3.3, then yields the desired sharp constant C, as detailed in Corollary 4.3. The proof is significantly simplified once we find, in Proposition 4.2, the values of $\mathbb{M}$ and $\mathbb{L}$ at $f = 0$.

2. Properties of the Bellman Functions $\mathbb{M}$ and $\mathbb{L}$

2.1. **Basic Properties.** In this section we prove the basic properties of $\mathbb{M}$ and $\mathbb{L}$, such as the main inequalities, convexity, monotonicity, and obstacle conditions.

Proposition 2.1. *The Bellman function $\mathbb{M}(f, F, \lambda)$ in Definition 1 has the following properties:*

    1). $\mathbb{M}$ *is independent of the choice of interval $J \in \mathcal{D}$ in its definition.*

    2). *Domain and Range:* $\mathbb{M}$ *has convex domain $\Omega_{\mathbb{M}} := \{(f, F, \lambda) : |f| \leq F; \lambda > 0\}$, and $0 \leq \mathbb{M} \leq 1$.*

    3). $\mathbb{M}$ *is decreasing in λ.*

    4). $\mathbb{M}$ *is even in f.*

    5). Homogeneity:

$$\tag{2.1} \mathbb{M}(f, F, \lambda) = \mathbb{M}(tf, |t|F, t^2\lambda), \ \forall t \neq 0.$$

    6). Obstacle Condition:

$$\tag{2.2} \mathbb{M}(f, F, \lambda) = 1, \ \forall \lambda \leq F^2.$$

    7). Main Inequality: *For all triplets (f, F, λ), $(f_{\pm}, F_{\pm}, \lambda_{\pm})$ in the domain with $f = \frac{1}{2}(f_- + f_+)$, $F = \frac{1}{2}(F_- + F_+)$, and $\lambda = \min(\lambda_-, \lambda_+)$, there holds:*

$$\tag{2.3} \mathbb{M}\left(f, F, \lambda + \left(\frac{f_+ - f_-}{2}\right)^2\right) \geq \frac{1}{2}\left(\mathbb{M}(f_+, F_+, \lambda_+) + \mathbb{M}(f_-, F_-, \lambda_-)\right).$$

    8). $\mathbb{M}$ *is concave and continuous in variables f and F.*

    9). $\mathbb{M}$ *is maximal at $f = 0$:*

$$\tag{2.4} \mathbb{M}(f, F, \lambda) \leq \mathbb{M}(f, F, \lambda - f^2) \leq \mathbb{M}(0, F, \lambda).$$

    10). $\mathbb{M}$ *is non-decreasing in F; $\mathbb{M}$ is non-increasing in f for $f \geq 0$ (and non-decreasing in f for $f \leq 0$).*

Proof. **1).** follows by the standard considerations. Properties **2).** and **3).** are obvious. Property **4).** follows since φ is admissible for $\mathbb{M}(f, F, \lambda)$ if and only if $-\varphi$ is admissible for $\mathbb{M}(-f, F, \lambda)$, and in this case $S_J^2\varphi = S_J^2(-\varphi)$. To see homogeneity, **5).**, note that φ is admissible for $\mathbb{M}(f, F, \lambda)$ if and only if $t\varphi$ is admissible for $\mathbb{M}(tf, |t|F, t^2\lambda)$, and in this case $S_J^2(t\varphi) = t^2 S_J^2\varphi$.

Next, we prove the obstacle condition, **6).** Given a point (f, F, λ) in the domain, consider the function $\varphi = f\mathbb{1}_J + F\sqrt{|J|}h_J$. Then $\langle\varphi\rangle_J = f$, $S_J^2\varphi = F^2\mathbb{1}_J$, and $\langle|\varphi|\rangle_J = F$. So φ is admissible for $\mathbb{M}(f, F, \lambda)$, and if $\lambda \leq F^2$, $\{x \in J : S_J^2\varphi(x) \geq \lambda\} = J$, so $\mathbb{M}(f, F, \lambda) = 1$.

To prove the main inequality **7).**, let $J \in \mathcal{D}$ be a dyadic interval, and let $\varphi_\pm$ be functions supported on $J_\pm$, admissible for $\mathbb{M}(f_\pm, F_\pm, \lambda_\pm)$, and which give the supremum up to some $\epsilon > 0$:

$$\text{supp}(\varphi_\pm) \subset J_\pm; \ \langle \varphi_\pm \rangle_{J_\pm} = f_\pm; \ \langle |\varphi_\pm| \rangle_{J_\pm} = F_\pm,$$

and

$$\frac{1}{|J_\pm|} \left| \{ x \in J_\pm : S^2_{J_\pm} \varphi_\pm \geq \lambda_\pm \} \right| > \mathbb{M}(f_\pm, F_\pm, \lambda_\pm) - \epsilon.$$

Now define φ on J by concatenation: $\varphi := \varphi_- \mathbb{1}_{J_-} + \varphi_+ \mathbb{1}_{J_+}$. Then $\langle \varphi \rangle_J = f$ and $\langle |\varphi| \rangle_J = F$, so φ is admissible for $\mathbb{M}(f, F, \lambda)$. Moreover:

$$S^2_J \varphi = |\Delta_J \varphi|^2 \mathbb{1}_J + \sum_{I \subset J_-} |\Delta_I \varphi_-|^2 \mathbb{1}_I + \sum_{I \subset J_+} |\Delta_I \varphi_+|^2 \mathbb{1}_I$$

$$= \frac{1}{4}(f_+ - f_-)^2 \mathbb{1}_J + S^2_{J_-} \varphi_- + S^2_{J_+} \varphi_+.$$

Then:

$$\mathbb{M}(f, F, \lambda + \frac{1}{4}(f_+ - f_-)^2) \geq \frac{1}{|J|} \left| \{ x \in J : S^2_{J_-} \varphi_-(x) + S^2_{J_+} \varphi_+(x) \geq \lambda \} \right|$$

$$\geq \frac{1}{2|J_-|} \left| \{ x \in J_- : S^2_{J_-} \varphi_-(x) \geq \lambda_- \} \right| + \frac{1}{2|J_+|} \left| \{ x \in J_+ : S^2_{J_+} \varphi_+(x) \geq \lambda_+ \} \right|$$

$$> \frac{1}{2} \left(\mathbb{M}(f_-, F_-, \lambda_-) + \mathbb{M}(f_+, F_+, \lambda_+) \right) - \epsilon.$$

Since this holds for all $\epsilon > 0$, the main inequality (2.3) is proved.

To prove **8).**, rewrite the main inequality in a more convenient form:

$$(2.5) \qquad \frac{1}{2} \left(\mathbb{M}(f + a, F + b, \lambda) + \mathbb{M}(f - a, F - b, \lambda) \right) \leq \mathbb{M}(f, F, \lambda + a^2) \leq \mathbb{M}(f, F, \lambda),$$

for all $a \in \mathbb{R}$ and $|b| \leq F$. Letting $a = 0$ and $b = 0$ above, we obtain that $\mathbb{M}$ is *midpoint concave* in variables F and f, respectively. Since $\mathbb{M}$ is measurable, this is enough to show that $\mathbb{M}$ is continuous and concave in F and f (see page 60 in [3], and the references therein [1, 5]).

For **9).**, take $f = 0$ and $b = 0$ in (2.5):

$$\mathbb{M}(0, F, \lambda + a^2) \geq \frac{1}{2} \left(\mathbb{M}(a, F, \lambda) + \mathbb{M}(-a, F, \lambda) \right) = \mathbb{M}(a, F, \lambda),$$

where the last equality follows because $\mathbb{M}$ is even in the first variable.

Finally, to see **10).** note that by the obstacle condition (2.2), $\mathbb{M}(f, \cdot, \lambda)$ is concave and has a maximum at $F = \sqrt{\lambda}$, and is constant for $F \geq \sqrt{\lambda}$. Similarly, $\mathbb{M}(\cdot, F, \lambda)$ is even, concave, and by (2.4) has a maximum at $f = 0$. $\qquad \square$

Note that if $F = 0$, the only admissible function is $\varphi = 0$ a.e. so

$$(2.6) \qquad\qquad\qquad \mathbb{M}(0, 0, \lambda) = 0, \ \forall \lambda > 0.$$

Proposition 2.2. *The Bellman function $\mathbb{L}(f, p, \lambda)$ in Definition 2 has the following properties:*

    1). $\mathbb{L}$ *is independent of the choice of interval $J \in \mathcal{D}$ in its definition.*

    2). Domain and Range: $\mathbb{L}$ *has convex domain $\Omega_{\mathbb{L}} := \{ (f, p, \lambda) : f \in \mathbb{R}; p \in [0, 1], \lambda > 0 \}$. As for the range:*

$$(2.7) \qquad\qquad\qquad |f| \leq \mathbb{L}(f, p, \lambda) \leq (1 - p)|f| + p \max(|f|, \sqrt{\lambda}).$$

    3). $\mathbb{L}$ *is increasing in λ.*

    4). $\mathbb{L}$ *is even in f.*

    5). Homogeneity:

$$(2.8) \qquad\qquad\qquad \mathbb{L}(tf, p, t^2 \lambda) = |t| \mathbb{L}(f, p, \lambda), \ \forall t \neq 0.$$

6). Obstacle Condition:

$$\mathbb{L}(f, p, \lambda) = |f|, \ \forall |f| \geq \sqrt{\lambda}. \tag{2.9}$$

7). Main Inequality: *For all triplets* (f, p, λ), $(f_\pm, p_\pm, \lambda_\pm)$ *in the domain with* $f = \frac{1}{2}(f_- + f_+)$, $p = \frac{1}{2}(p_- + p_+)$, *and* $\lambda = \min(\lambda_-, \lambda_+)$, *there holds:*

$$\mathbb{L}\left(f, p, \lambda + \left(\frac{f_+ - f_-}{2}\right)^2\right) \leq \frac{1}{2}\left(\mathbb{M}(f_+, p_+, \lambda_+) + \mathbb{M}(f_-, p_-, \lambda_-)\right). \tag{2.10}$$

8). $\mathbb{L}$ *is convex and continuous in variables* f *and* p.

9). $\mathbb{L}$ *is minimal at* $f = 0$:

$$\mathbb{L}(0, p, \lambda) \leq \mathbb{L}(0, p, \lambda + f^2) \leq \mathbb{L}(f, p, \lambda). \tag{2.11}$$

10). $\mathbb{L}$ *is non-decreasing in* p; $\mathbb{L}$ *is non-decreasing in* f *for* $f \geq 0$ *(and non-increasing in* f *for* $f \leq 0$).

Proof. The proofs of properties **1).**, **4).** and **5).** are similar to those for $\mathbb{M}$. It is also straightforward to prove

$$\mathbb{L}(f, p, \lambda + a^2) \leq \frac{1}{2}\left(\mathbb{L}(f + a, p + b, \lambda) + \mathbb{L}(f - a, p - b, \lambda)\right), \tag{2.12}$$

a weaker form of (2.10) – note that we don't know yet that $\mathbb{L}$ is increasing in λ, a property that is not so obvious in this case. We may see now however that $\mathbb{L}$ is convex and continuous in p, by letting $a = 0$ in (2.12).

Next, we prove the range condition **2).** (2.7), and note that in this case the obstacle condition **6).** (2.9) follows directly from the range condition, since

$$|f| \leq \mathbb{L}(f, p, \lambda) \leq (1 - p)|f| + p \max(|f|, \sqrt{\lambda}) \leq \max(|f|, \sqrt{\lambda}).$$

The first inequality is obvious, as any function φ admissible for $\mathbb{L}(f, p, \lambda)$ satisfies $\langle |\varphi| \rangle_J \geq |\langle \varphi \rangle_J| = |f|$. We now prove the second inequality, and begin with some simple examples. When $p = 1$, consider the function $\varphi = f \mathbb{1}_J + \sqrt{\lambda}\sqrt{|J|}h_J$. Then $S_J^2 \varphi = \lambda \mathbb{1}_J$, so φ is admissible for $\mathbb{L}(f, 1, \lambda)$, and then:

$$\mathbb{L}(f, 1, \lambda) \leq \langle |\varphi| \rangle_J = \frac{1}{2}|f + \sqrt{\lambda}| + \frac{1}{2}|f - \sqrt{\lambda}| = \max\{|f|, \sqrt{\lambda}\}.$$

If $p = \frac{1}{2}$, then consider for example the function $\varphi = f\mathbb{1}_J + \sqrt{\lambda}\sqrt{|J_-|}h_{J_-}$. Then $S_J^2 \varphi = \lambda\mathbb{1}_{J_-}$, so φ is admissible for $\mathbb{L}(f, 1/2, \lambda)$, and then:

$$\mathbb{L}(f, 1/2, \lambda) \leq \langle |\varphi| \rangle_J = \frac{1}{4}|f + \sqrt{\lambda}| + \frac{1}{4}|f - \sqrt{\lambda}| + \frac{1}{2}|f| = \frac{1}{2}|f| + \frac{1}{2}\max\{|f|, \sqrt{\lambda}\}.$$

Now suppose that $p \in (0, 1)$ is a *dyadic rational*, that is $p = \frac{k}{2^N}$ for some integers $N \geq 1$ and $1 \leq k \leq 2^N - 1$. On some dyadic interval J, let $\mathcal{I}$ denote any collection of k subintervals in the Nth generation $J_{(N)}$ of dyadic descendants of J, and let

$$\varphi = f\mathbb{1}_J + \sqrt{\lambda}\sum_{I \in \mathcal{I}}\sqrt{|I|}h_I.$$

Then $S_J^2 \varphi = \lambda\mathbb{1}_{\{\cup I : I \in \mathcal{I}\}}$, so

$$\frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) \geq \lambda\}| = \frac{|\bigcup_{I \in \mathcal{I}} I|}{|J|} = \frac{k}{2^N} = p,$$

and $\mathbb{L}(f, p, \lambda) \leq \langle |\varphi| \rangle_J$. Now for every $I \in \mathcal{I}$, on $I_\pm$, $\varphi = f \pm \sqrt{\lambda}$, and $\varphi = f$ off $\cup_{I \in \mathcal{I}}I$. So

$$\langle |\varphi| \rangle_J = \frac{1}{|J|}\left(\max\{|f|, \sqrt{\lambda}\}\sum_{I \in \mathcal{I}}|I| + |f||J \setminus \cup_{I \in \mathcal{I}}I|\right) = (1 - p)|f| + p\max\{|f|, \sqrt{\lambda}\}.$$

Therefore the second inequality in (2.7) holds for all dyadic rationals $p \in (0,1)$, and by continuity of $\mathbb{L}$ in p and density of the dyadic rationals in $[0,1]$, the result follows.

Also note that, taking $p = 0$ in (2.7), we see that

$$(2.13) \qquad\qquad\qquad \mathbb{L}(f, 0, \lambda) = |f|.$$

Thus $\mathbb{L}(f, \cdot, \lambda)$ is convex in $p \in [0,1]$ and has a minimum at $p = 0$, so $\mathbb{L}$ is non-decreasing in p. In turn, this allows us to prove property **3).**, that $\mathbb{L}$ is non-decreasing in λ: suppose $\lambda_1 \leq \lambda_2$ and let φ be admissible for $\mathbb{L}(f, p, \lambda_2)$. Then $\langle \varphi \rangle_J = f$ and

$$p = \frac{1}{|J|}|\{x \in J : S_J^2 \varphi(x) \geq \lambda_2\}| \leq \frac{1}{|J|}|\{x \in J : S_J^2 \varphi(x) \geq \lambda_1\}| =: q.$$

So φ is also admissible for $\mathbb{L}(f, q, \lambda_1)$, where $q \geq p$, which means

$$\langle |\varphi| \rangle_J \geq \mathbb{L}(f, q, \lambda_1) \geq \mathbb{L}(f, p, \lambda_1).$$

Since this holds for all φ admissible for $\mathbb{L}(f, p, \lambda_2)$, we have $\mathbb{L}(f, p, \lambda_2) \geq \mathbb{L}(f, p, \lambda_1)$.

Having the desired monotonicity in λ then gives the full form of the main inequality **7).** (2.10), as well as $\mathbb{L}(f, p, \lambda) \leq \mathbb{L}(f, p, \lambda + a^2)$. So letting $b = 0$ in (2.12) with we obtain convexity (and continuity) in f – so property **8).** is also proved. Let $f = 0$ and $b = 0$ in (2.12) and we obtain **9).**, minimality of $\mathbb{L}$ at $f = 0$. Finally, we may then finish proving **10).**: since $\mathbb{L}(\cdot, p, \lambda)$ is even, convex, and minimal at $f = 0$, the claimed monotonicity in f follows.

$\square$

2.2. $\mathbb{M}$ is the Least Supersolution. Consider the main inequality for $\mathbb{M}$ in more generality:

$$(2.14) \qquad\qquad m(f, F, \lambda + a^2) \geq \frac{1}{2}\Big(m(f + a, F + b, \lambda) + m(f - a, F - b, \lambda)\Big).$$

Definition 3. *We say that a function $m(f, F, \lambda)$ defined on $\Omega_{\mathbb{M}}$ is a supersolution of the main inequality* (2.14) *provided that m is non-negative, continuous, and satisfies*

    1). The main inequality (2.14);
    2). The obstacle condition $m(f, F, \lambda) = 1$, whenever $\lambda \leq F^2$.

Theorem 2.3. *If m is any supersolution as defined above, then $\mathbb{M} \leq m$.*

Proof. Obviously, it suffices to show that if m is a supersolution, then

$$(2.15) \qquad\qquad |J|m(f, F, \lambda) \geq |\{x \in J : S_J^2 \varphi(x) \geq \lambda\}|,$$

for any function φ supported in $J \in \mathcal{D}$ with $\langle \varphi \rangle_J = f$ and $\langle |\varphi| \rangle_J = F$. The first key observation is that it suffices to prove (2.15) for functions φ with *finite Haar expansion*.

Remark 2.4. *Some caution is needed when working in L^1, so we recall here the classical Haar system on $[0,1)$. Consider $J = [0,1)$ and arrange its dyadic subintervals (and hence also their corresponding Haar functions) in lexicographical order:*

$$J_n := \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right), \quad \forall n = 2^k + j - 1, \ k \geq 0, \ 1 \leq j \leq 2^k.$$

So

$$J_1 = J; \ \ J_2 = J_-, J_3 = J_+; \ \ J_4 = J_{-_-}, J_5 = J_{-_+}, \ldots$$

The classical result of Haar states that for every $\varphi \in L^p[0,1)$, $1 \leq p < \infty$, the Haar series

$$\varphi_N(x) := \mathbb{1}_{[0,1)}(x)\int_0^1 \varphi + \sum_{k=1}^N (\varphi, h_{J_k})h_{J_k}(x)$$

converges to φ in $L^p[0,1)$ and almost everywhere. The reason for caution in our problem is that, while for $p > 1$ the Haar functions form an unconditional basis for $L^p[0,1)$, the most we can say for

$p = 1$ is that $\{\mathbb{1}_{[0,1)}\} \cup \{h_{J_k}\}_{k \geq 1}$ is a Schauder basis. That is, we may rearrange the Haar series in such a way that it becomes divergent.

This result transfers in an obvious way to any dyadic interval $J \in \mathcal{D}$, and we use the notation

$$\{h_{J_k}\}_{k \geq 1}$$

whenever we must keep track of the ordering of the subintervals of J. We say this is the Haar system adapted to J.

Returning to our problem, suppose (2.15) holds for functions with finite Haar expansion, and let φ with $\text{supp}(\varphi) \subset J$, $\langle \varphi \rangle_J = f$ and $\langle |\varphi| \rangle_J = F$. Then the Haar series

$$\varphi_N := f\mathbb{1}_J + \sum_{k=1}^{N} (\varphi, h_{J_k}) h_{J_k}$$

converges to φ in $L^1(J)$ and almost everywhere. Moreover, $\langle \varphi_N \rangle_J = f$ and $F_N := \langle |\varphi_N| \rangle_J \to F$ as $N \to \infty$. Then

$$|J|m(f, F_N, \lambda) \geq |\{x \in J : S_J^2 \varphi_N(x) \geq \lambda\}|$$

for all N. Since m is continuous[1], taking limit as $N \to \infty$ on both sides above yields exactly the desired conclusion.

So now suppose $\text{supp}(\varphi) \subset J$, $\langle \varphi \rangle_J = f$, $\langle |\varphi| \rangle_J = F$. The goal is to show that if m is any supersolution,

$$|J|m(f, F, \lambda) \geq |E|, \text{ where } E := \{x \in J : S_J^2 \varphi(x) \geq \lambda\}.$$

Suppose further that there is some dyadic level $N > 0$ such that

$$\varphi = f\mathbb{1}_J + \sum_{\substack{I \subset J \\ |I| \geq |J|2^{-N}}} (\varphi, h_I) h_I.$$

Remark that $S_J^2 \varphi$ is constant on each $I \in J_{(N)}$, so E is then a disjoint union of intervals $I \in J_{(N)}$ (unless E is empty, in which case we are done). For every $I \subset J$, let

$$f_I := \langle \varphi \rangle_I; \ \ F_I := \langle |\varphi| \rangle_J; \ \ \lambda_I := \lambda - \sum_{K : I \subsetneq K \subset J} \Delta_K^2 \varphi.$$

Then note that

$$f = f_J = \frac{1}{2}(f_{J_+} + f_{J_-}); \ \ F = F_J = \frac{1}{2}(F_{J_+} + F_{J_-}); \ \ \lambda = \lambda_J; \ \ \Delta_J^2 \varphi = \frac{1}{4}(f_{J_+} - f_{J_-})^2 \leq F_J^2.$$

Now, we describe the iteration procedure:

- If $\lambda \leq \Delta_J^2 \varphi$, then the obstacle condition gives that $|J|m(f, F, \lambda) = |J| \geq |E|$, and we are done.
- Otherwise, we have $\lambda_{J_+} = \lambda_{J_-} = \lambda - \Delta_J^2 \varphi > 0$, so then we apply the main inequality for m to obtain:

$$|J|m(f, F, \lambda) \geq |J_-|m(f_{J_-}, F_{J_-}, \lambda_{J_-}) + |J_+|m(f_{J_+}, F_{J_+}, \lambda_{J_+}).$$

  - If $\lambda_{J_+} \leq \Delta_{J_+}^2 \varphi \leq F_{J_+}^2$, then this becomes

$$|J|m(f, F, \lambda) \geq |J_-|m(f_{J_-}, F_{J_-}, \lambda_{J_-}) + |J_+|,$$

    and if we iterate further, we only do so on J_-. Also note that, in this case, $\lambda_I \leq 0$ for any $I \in J_{(N)}$ with $I \subsetneq J_+$.
  - Otherwise, iterate the J_+ term further, with $\lambda_{J_{+-}} = \lambda_{J_{++}} = \lambda - \Delta_J^2 \varphi - \Delta_{J_+}^2 \varphi > 0$.

---

[1]Note from the proof of Theorem 2.3 that it suffices to require m be continuous in the variable F.

Continuing this process down to the last dyadic level N, we have

$$(2.16) \qquad |J|m(f, F, \lambda) \geq \sum_{I \in J_{(N)}: \lambda_I > 0} |I|m(f_I, F_I, \lambda_I) + \sum_{I \in J_{(N)}: \lambda_I \leq 0} |I|.$$

Finally, it is easy to see that for any $I \in J_{(N)}$, we have $I \subset E$ if and only if $\lambda_I \leq \Delta_I^2 \varphi$, and again by the obstacle condition, if $I \subset E$ and $\lambda_I > 0$, then $m(f_I, F_I, \lambda_I) = 1$. So (2.16) gives us the desired conclusion:

$$|J|m(f, F, \lambda) \geq \sum_{I \in J_{(N)}: I \subset E} |I| = |E|.$$

$$\square$$

2.3. $\mathbb{L}$ **is the Greatest Subsolution.** Let us also consider the main inequality for $\mathbb{L}$ in more generality:

$$(2.17) \qquad \ell(f, p, \lambda + a^2) \leq \frac{1}{2}\left(\ell(f + a, p + b, \lambda) + \ell(f - a, p - b, \lambda)\right).$$

Definition 4. *We say that a function $\ell(f, p, \lambda)$ defined on $\Omega_{\mathbb{L}}$ is a subsolution for the main inequality* (2.17) *provided that ℓ is non-negative, continuous, and satisfies*

1). *The main inequality* (2.17);
2). *Range/Obstacle Condition:* $|f| \leq \ell(f, p, \lambda) \leq \max\{|f|, \sqrt{\lambda}\}$;
3). *Boundary Condition:* $\ell(f, 0, \lambda) = |f|$.

Theorem 2.5. *If ℓ is any subsolution as defined above, then $\ell \leq \mathbb{L}$.*

Proof. We must prove that $\ell(f, p, \lambda) \leq \langle|\varphi|\rangle_J$ for any function φ on J with $\langle\varphi\rangle_J = f$ and $\frac{1}{|J|}|E| = p$, where $E = \{x \in J : S_J^2 \varphi(x) \geq \lambda\}$. As before, we may assume that there is some dyadic level $N \geq 0$ below which the Haar coefficients of φ are zero, and assume that p is a dyadic rational[2].

If $\lambda \leq \Delta_J^2 \varphi$, then by condition 2):

$$\ell(f, p, \lambda) \leq \max\{|f|, \sqrt{\lambda}\} \leq \max\{|f|, |\Delta_J \varphi|\} \leq \langle|\varphi|\rangle_J,$$

and we are done. Otherwise, put $\lambda_{J_{\pm}} = \lambda - \Delta_J^2 \varphi > 0$, $f_{J_{\pm}} = \langle\varphi\rangle_{J_{\pm}}$, and

$$p_{J_{\pm}} = \frac{1}{|J_{\pm}|}|\{x \in J_{\pm} : S_{J_{\pm}}^2 \varphi(x) \geq \lambda_{J_{\pm}}\}|.$$

Then by the Main Inequality:

$$|J|\ell(f, p, \lambda) \leq |J_-|\ell(f_{J_-}, p_{J_-}, \lambda_{J_-}) + |J_+|\ell(f_{J_+}, p_{J_+}, \lambda_{J_+}).$$

If $\lambda_{J_{\pm}} \leq \Delta_{J_{\pm}}^2 \varphi$, it follows as before that $|J_{\pm}|\ell(f_{J_{\pm}}, p_{J_{\pm}}, \lambda_{J_{\pm}}) \leq \int_{J_{\pm}} |\varphi|$, and otherwise we iterate further on $J_{\pm}$.

Continuing in this way down to the last level N and putting $\lambda_I := \lambda - \Delta_{I^{(1)}}^2 \varphi - \ldots - \Delta_J^2 \varphi$ for every $I \in J_{(N)}$, the previous iterations have covered all cases where $\lambda_I \leq 0$, and we have

$$(2.18) \qquad |J|\ell(f, p, \lambda) \leq \sum_{I \in J_{(N)}: \lambda_I \leq 0} \int_I |\varphi| + \sum_{I \in J_{(N)}: \lambda_I > 0} |I|\ell(f_I, p_I, \lambda_I).$$

Now note that for $I \in J_{(N)}$:

$$p_I = \frac{1}{|I|}|\{I : S_I^2 \varphi \geq \lambda_I\}| = \frac{1}{|I|}|\{I : \Delta_I^2 \varphi \geq \lambda_I\}| = \begin{cases} 0 & , \text{if } I \not\subset E \\ 1 & , \text{if } I \subset E. \end{cases}$$

So, if $I \not\subset E$, then we use the boundary condition 3):

$$\ell(f_I, p_I, \lambda_I) = \ell(f_I, 0, \lambda_I) = |f_I| \leq \langle|\varphi|\rangle_I,$$

---

[2]Note that only continuity of ℓ in the variable p is used in the proof of Theorem 2.5.

and if $I \subset E$, or $\lambda_I \leq \Delta_I^2 \varphi$, we use condition 2) as before to obtain $\ell(f_I, p_I, \lambda_I) \leq \max\{|f_I|, |\Delta_I \varphi|\} \leq \langle|\varphi|\rangle_I$. Finally, (2.18) becomes:

$$|J|\ell(f, p, \lambda) \leq \sum_{I \in J_{(N)}} \int_I |\varphi| = \int_J |\varphi|.$$

$\square$

Remark 2.6. *Later in Section 5, we will look at subsolutions for the particular case $\mathbb{L}(f, 1, \lambda)$. We note that the boundary condition 3). above will no longer be needed there: when $p = 1$, we are looking only at functions φ with $S_J^2 \varphi \geq \lambda$ almost everywhere on J, so at the end of the proof, there will be no intervals left outside E, and there will be no terms of the form $\ell(f_I, 0, \lambda_I)$.*

Remark 2.7. *Our definition of the Bellman function $\mathbb{L}$ was inspired by Bollobas [2], who worked with*

$$L_B(s, h) := \inf\left\{\int_0^1 |\varphi|\, dx : supp(\varphi) \subset [0, 1];\ \int_0^1 \varphi\, dx = h;\ S\varphi \equiv s\ on\ [0, 1]\right\}.$$

We claim that $L_B(s, h) = \mathbb{L}(h, 1, s^2)$. In fact, we may define $\mathbb{L}(f, p, \lambda)$ in general by replacing "$\geq \lambda$" with "$= \lambda$." To see this, let

$$\mathbb{L}'(f, p, \lambda) := \inf\{\langle|\varphi|\rangle_J : supp(\varphi) \subset J;\ \langle\varphi\rangle_J = f;\ \frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) = \lambda\}| = p\}.$$

We claim that $\mathbb{L}' = \mathbb{L}$. Suppose φ is admissible for $\mathbb{L}'(f, p, \lambda)$. Then

$$q := \frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) \geq \lambda\}| \geq \frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) = \lambda\}| = p,$$

so φ is also admissible for $\mathbb{L}(f, q, \lambda)$ with $q \geq p$. Then, since $\mathbb{L}$ is non-decreasing in the second variable, $\langle|\varphi|\rangle_J \geq \mathbb{L}(f, q, \lambda) \geq \mathbb{L}(f, p, \lambda)$. This shows that $\mathbb{L}' \geq \mathbb{L}$. To see the converse, we note that $\mathbb{L}'$ is a subsolution for the main inequality (2.17), as in Definition 4. It is easy to show in the usual way that $\mathbb{L}'$ satisfies (2.17). Moreover, $\mathbb{L}'$ satisfies the same range condition (2.7) as $\mathbb{L}$: $|f| \leq \mathbb{L}'(f, p, \lambda) \leq (p - 1)|f| + p \max(|f|, \sqrt{\lambda})$. The proof of this inequality for $\mathbb{L}$ goes through identically for $\mathbb{L}'$, since the test functions φ we constructed for each dyadic rational p really satisfied $\{x \in J : S_J^2\varphi(x) \geq \lambda\} = \{x \in J : S_J^2\varphi(x) = \lambda\}$. Then by Theorem 2.5 it follows that $\mathbb{L}' \leq \mathbb{L}$.

3. Relationships between $\mathbb{M}$ and $\mathbb{L}$

Theorem 3.1. *$\mathbb{L}(f, p, \lambda)$ is the smallest value of F for which $\mathbb{M}(f, F, \lambda) \geq p$:*

$$(3.1) \qquad\qquad \mathbb{L}(f, p, \lambda) = \inf\{F \geq |f| : \mathbb{M}(f, F, \lambda) \geq p\}.$$

Moreover, $\mathbb{M}(f, F, \lambda)$ is the largest value of p such that $\mathbb{L}(f, p, \lambda) \leq F$:

$$(3.2) \qquad\qquad \mathbb{M}(f, F, \lambda) = \sup\{p \in [0, 1] : \mathbb{L}(f, p, \lambda) \leq F\}.$$

Proof. Suppose $\mathbb{M}(f, F, \lambda) \geq p$ and let $\epsilon > 0$. Then there is a function φ on $J \in \mathcal{D}$ such that:

$$\langle\varphi\rangle_J = f,\ \ \langle|\varphi|\rangle_J = F,\ \ q := \frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) \geq \lambda\}| > p - \epsilon.$$

Then φ is admissible for $\mathbb{L}(f, q, \lambda)$, and since $\mathbb{L}$ is non-decreasing in the second variable,

$$\mathbb{L}(f, p - \epsilon, \lambda) \leq \mathbb{L}(f, q, \lambda) \leq \langle|\varphi|\rangle_J = F.$$

Since this holds for all $\epsilon > 0$, $\mathbb{L}(f, p, \lambda) \leq F$ for all F such that $\mathbb{M}(f, F, \lambda) \geq p$. Further, for every $\epsilon > 0$ there is a function φ on $J \in \mathcal{D}$ such that

$$\langle\varphi\rangle_J = f,\ \ \frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) \geq \lambda\}| = p,\ \ F := \langle|\varphi|\rangle_J < \mathbb{L}(f, p, \lambda) + \epsilon.$$

But φ is admissible for $\mathbb{M}(f, F, \lambda)$, and then clearly $\mathbb{M}(f, F, \lambda) \geq p$. This proves (3.1). The other equation (3.2) follows similarly. $\square$

3.1. **Optimal Obstacle Conditions for $\mathbb{M}$ and $\mathbb{L}$.** Looking back at the obstacle condition (2.2) for $\mathbb{M}$, namely $\mathbb{M}(f, F, \lambda) = 1$ whenever $F \geq \sqrt{\lambda}$, there is no reason to think this condition is optimal. That is, there well could be values of F strictly smaller than $\sqrt{\lambda}$ where $\mathbb{M}$ is 1. As it turns out, the optimal obstacle condition for $\mathbb{M}$ can be obtained from information about $\mathbb{L}$. Since $\mathbb{M} \leq 1$, taking $p = 1$ in (3.1), we obtain exactly this:

$$(3.3) \qquad \mathbb{L}(f, 1, \lambda) = \inf\{F \geq 0 : \mathbb{M}(f, F, \lambda) = 1\},$$

On the other hand, the obstacle condition for $\mathbb{L}$ really comes from its range, $|f| \leq \mathbb{L}(f, p, \lambda) \leq \max\{|f|, \sqrt{\lambda}\}$, which clearly shows that $\mathbb{L} = |f|$ whenever $|f| \geq \lambda$. However, this says nothing about p, and we do know that, for example, $\mathbb{L}(f, 0, \lambda) = |f|$ regardless of the behavior of f and λ. What other values of p could this hold for? This is again obtained precisely from information about $\mathbb{M}$, by letting $F = |f|$ in (3.2):

$$(3.4) \qquad \mathbb{M}(f, |f|, \lambda) = \sup\{p \in [0, 1] : \mathbb{L}(f, p, \lambda) = |f|\}.$$

So, if we find the expressions for $\mathbb{L}$ and $\mathbb{M}$ along these boundaries of their domains, we also obtain the optimal obstacle conditions for $\mathbb{M}$ and $\mathbb{L}$, respectively.

We denote these boundary values of $\mathbb{M}$ and $\mathbb{L}$ by $\mathbb{M}_b$ and $\mathbb{L}_b$, respectively, defined as follows. For $f \geq 0$ and $\lambda > 0$,

$$(3.5) \qquad \mathbb{M}_b(f, \lambda) := \mathbb{M}(f, |f|, \lambda) = \sup \frac{1}{|J|} |\{x \in J : S_J^2 \varphi(x) \geq \lambda\}|,$$

where the supremum is over all functions φ on J with $\varphi \geq 0$ a.e. and $\langle \varphi \rangle_J = f$. Note that since $\mathbb{M}$ is even in f, it suffices to consider $\mathbb{M}_b$ for $f \geq 0$. Moreover, the only admissible functions for $\mathbb{M}(f, |f|, \lambda)$ are those with $\varphi \geq 0$ a.e. (for $f \geq 0$) or $\varphi \leq 0$ a.e. (for $f \leq 0$). Similarly,

$$(3.6) \qquad \mathbb{L}_b(f, \lambda) := \mathbb{L}(f, 1, \lambda) = \inf\{\langle |\varphi| \rangle_J : \mathrm{supp}(\varphi) \subset J;\ \langle \varphi \rangle_J = f;\ S_J^2 \varphi \geq \lambda \text{ a. e. on } J\}.$$

We find these functions in Section 5, where we prove the following results.

Theorem 3.2. *The function $\mathbb{M}_b$ is given by*

$$(3.7) \qquad \mathbb{M}_b(|f|, \lambda) = \mathbb{M}(f, |f|, \lambda) = \begin{cases} \frac{\Phi\left(\frac{|f|}{\sqrt{\lambda}}\right)}{\Phi(1)}, & |f| < \sqrt{\lambda} \\ 1, & |f| \geq \sqrt{\lambda}. \end{cases} = \min\left(\frac{\Phi(|f|/\sqrt{\lambda})}{\Phi(1)}, 1\right),$$

where

$$\Phi(\tau) := \int_0^\tau e^{-x^2/2}\, dx,$$

for all $\tau \geq 0$.

Theorem 3.3. *The function $\mathbb{L}_b$ is given by*

$$(3.8) \qquad \mathbb{L}_b(f, \lambda) = \mathbb{L}(f, 1, \lambda) = \begin{cases} \frac{\sqrt{\lambda}\Psi\left(\frac{|f|}{\sqrt{\lambda}}\right)}{\Psi(1)}, & 0 \leq |f| < \sqrt{\lambda} \\ |f|, & |f| \geq \sqrt{\lambda}. \end{cases} = \sqrt{\lambda} \max\left(\frac{\Psi(|f|/\sqrt{\lambda})}{\Psi(1)}, \frac{|f|}{\sqrt{\lambda}}\right),$$

where

$$\Psi(\tau) = \tau\Phi(\tau) + e^{-\tau^2/2},$$

for all $\tau \geq 0$.

3.2. **The functions θ and η.** To visualize the optimal obstacle conditions induced by $\mathbb{M}_b$ and $\mathbb{L}_b$ for $\mathbb{L}$ and $\mathbb{M}$, respectively, we use homogeneity of $\mathbb{M}$ and $\mathbb{L}$ to reduce the discussion to functions of two variables. Specifically, from (2.1) and (2.8), we write

$$(3.9) \qquad \mathbb{M}(f, F, \lambda) = \mathbb{M}(f/\sqrt{\lambda}, F/\sqrt{\lambda}, 1) =: \theta(\tau, \gamma) \quad \text{and} \quad \mathbb{L}(f, p, \lambda) =: \sqrt{\lambda}\eta(\tau, p),$$

where $\tau = f/\sqrt{\lambda}$ and $\gamma = F/\sqrt{\lambda}$. Thus θ is defined on $\Omega_\theta := \{0 \leq |\tau| \leq \gamma\}$ with values in $[0, 1]$, and η is defined on $\Omega_\eta := \{0 \leq p \leq 1; \tau \in \mathbb{R}\}$ with values in $|\tau| \leq \eta(\tau, p) \leq (1 - p)|\tau| + \max(|\tau|, 1)$. It is also clear that θ and η are even in τ, so we often restrict our attention to the domains Ω_θ^+ and Ω_η^+ where $\tau \geq 0$. Other properties that θ and η inherit from $\mathbb{M}$ and $\mathbb{L}$ are easy to check:

- $\theta(0, 0) = 0$ and $\eta(\tau, 0) = \tau$.
- θ is maximal at $\tau = 0$, and η is minimal at $\tau = 0$:

$$\theta(|\tau|, \gamma) \leq \theta(0, \gamma); \quad \eta(0, p) \leq \eta(|\tau|, p).$$

- θ is decreasing in τ for $\tau \geq 0$, and is increasing in γ. η is increasing in both $\tau \geq 0$ and p.
- θ is concave in both τ and γ, and η is convex in both τ and p.
- The original obstacle conditions (2.2) and (2.9) for $\mathbb{M}$ and $\mathbb{L}$ translate to

$$\theta(\tau, \gamma) = 1, \ \forall \gamma \geq 1 \quad \text{and} \quad \eta(\tau, p) = |\tau|, \ \forall |\tau| \geq 1.$$

Moreover, (3.3) and (3.4) become

$$\eta(\tau, 1) = \inf\{\gamma \geq |\tau| : \theta(\tau, \gamma) = 1\} \quad \text{and} \quad \theta(\tau, |\tau|) = \sup\{p : \eta(|\tau|, p) = |\tau|\}.$$
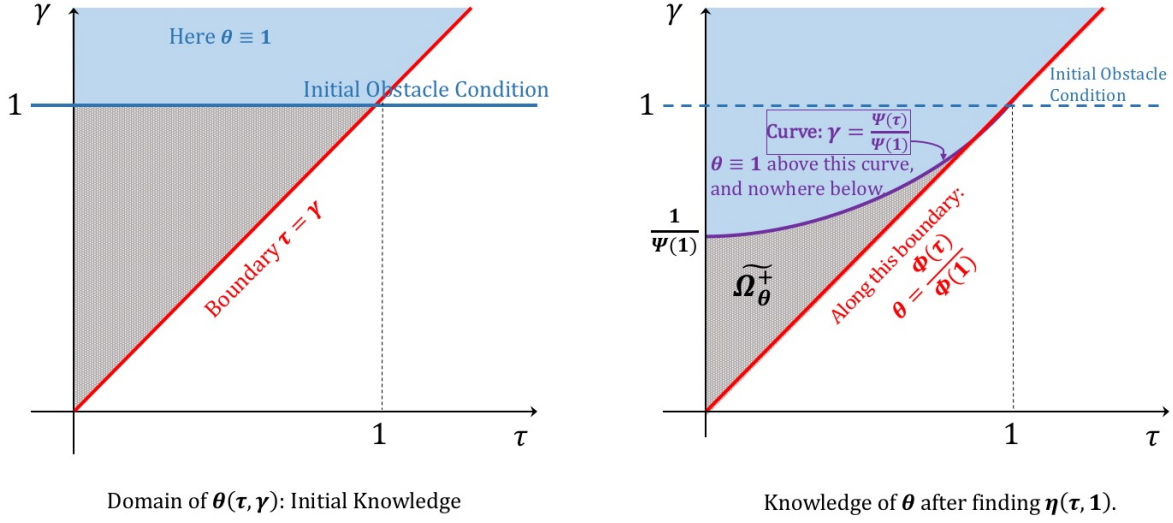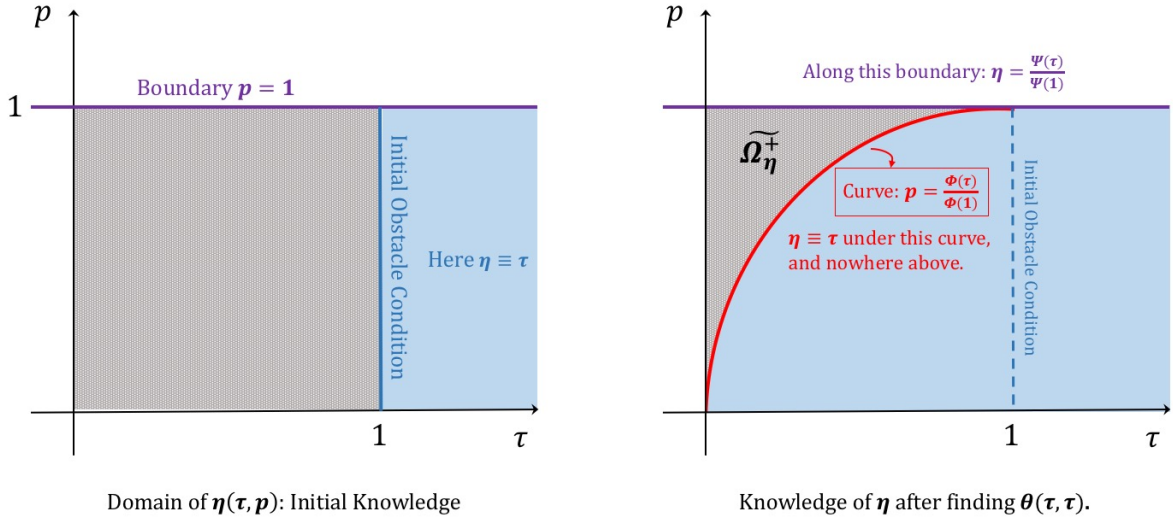
The expression for $\mathbb{L}_b$ gives that

$$\eta(\tau, 1) = \begin{cases} \frac{\Psi(|\tau|)}{\Psi(1)}, & 0 \leq |\tau| < 1 \\ |\tau|, & |\tau| \geq 1. \end{cases},$$

which yields the optimal obstacle condition for θ (see Figure 1). Similarly, $\mathbb{M}_b$ gives that

$$\theta(\tau, \tau) = \begin{cases} \frac{\Phi(|\tau|)}{\Phi(1)}, & 0 \leq |\tau| \leq 1 \\ 1, & \tau \geq 1. \end{cases},$$

which yields the optimal obstacle condition for η (see Figure 2).

FIGURE 1. Initial and optimal Obstacle Conditions for θ.

FIGURE 2. Initial and optimal Obstacle Conditions for η.

Let us give some special names to the "interesting" parts of the domains of θ and η, where they are unknown. We denote by $\widetilde{\Omega_\theta}$ the part of the domain of θ that lies underneath the obstacle condition curve $\gamma = \eta(\tau, 1)$:

$$\widetilde{\Omega_\theta} := \{(\tau, \gamma) : 0 \le |\tau| \le 1;\ |\tau| \le \gamma \le \frac{\Psi(|\tau|)}{\Psi(1)} = \eta(\tau, 1)\},$$

and by $\widetilde{\Omega_\eta}$ the part of the domain of η that lies above the obstacle condition curve $p = \theta(\tau, |\tau|)$:

$$\widetilde{\Omega_\eta} := \{(\tau, p) : 0 \le |\tau| \le 1;\ p \ge \frac{\Phi(|\tau|)}{\Phi(1)} = \theta(\tau, |\tau|)\}.$$

As the next proposition shows, in these domains we can improve the results of Theorem 3.1.

Proposition 3.4. *The functions $\mathbb{M}$ and $\mathbb{L}$ satisfy*

(3.10) $$\mathbb{M}\big(f, \mathbb{L}(f, p, \lambda), \lambda\big) = p, \text{ for all } (\tau = f/\sqrt{\lambda}, \, p) \in \widetilde{\Omega}_\eta,$$

that is, for all

$$0 \le |f| \le \sqrt{\lambda} \text{ and } p \ge \frac{\Phi(|f|/\sqrt{\lambda})}{\Phi(1)}.$$

Similarly,

(3.11) $$\mathbb{L}\big(f, \mathbb{M}(f, F, \lambda), \lambda\big) = F, \text{ for all } (\tau = f/\sqrt{\lambda}, \, \gamma = F/\sqrt{\lambda}) \in \widetilde{\Omega}_\theta,$$

that is, for all

$$0 \le |f| \le \sqrt{\lambda} \text{ and } |f| \le F \le \frac{\sqrt{\lambda}\Psi(|f|/\sqrt{\lambda})}{\Psi(1)}.$$

.

Proof. The relationships between $\mathbb{M}$ and $\mathbb{L}$ in Theorem 3.1 translate in θ–η language to

(3.12) $$\eta(\tau, p) = \inf\{\gamma \ge \tau : \theta(\tau, \gamma) \ge p\} \quad \text{and} \quad \theta(\tau, \gamma) = \sup\{0 \le p \le 1 : \eta(\tau, p) \le \gamma\}.$$

Now fix some $0 \le \tau \le 1$. If $p < \theta(\tau, \tau)$ (below the obstacle condition curve for η), then $\eta(\tau, p) = \tau$ and $\theta(\tau, \gamma) \ge \theta(\tau, \tau) > p$ for all $\gamma \ge \tau$, so indeed $\gamma = \tau$ is the smallest possible value of γ where $\theta(\tau, \gamma) \ge p$. If, on the other hand, $1 \ge p \ge \theta(\tau, \tau)$, or $(\tau, p) \in \widetilde{\Omega}_\eta$, then there exists a $\gamma \ge \tau$ such that $\theta(\tau, \gamma) = p$. So, in this case, we may rewrite the first equation in (3.12) as

$$\eta(\tau, p) = \inf\{\gamma \ge \tau : \theta(\tau, \gamma) = p\},$$

and then obviously

(3.13) $$\theta(\tau, \eta(\tau, p)) = p, \text{ for all } (\tau, p) \in \widetilde{\Omega}_\eta.$$

This is exactly (3.10). Similarly, we have that

(3.14) $$\eta(\tau, \theta(\tau, \gamma)) \text{ for all } (\tau, \gamma) \in \widetilde{\Omega}_\theta.$$

$\square$

4. THE SHARP INEQUALITY FOR THE SQUARE FUNCTION

The following result is an adaptation of Lemma 2 in Bollobas [2].

Theorem 4.1. *The functions $\mathbb{M}$ and $\mathbb{L}$ satisfy:*

(4.1) $$\mathbb{M}(0, F, \lambda) \le \frac{F}{\mathbb{L}(0, 1, \lambda)} = \frac{F}{\mathbb{L}_b(0, \lambda)},$$

for all $F \ge 0$ and $\lambda > 0$.

Proof. Let φ be a function on $J \in \mathcal{D}$ with $\int_J \varphi = 0$ and finite Haar expansion (up to some dyadic level $N \ge 0$):

$$\varphi = \sum_{I \subset J} (\varphi, h_I) h_I = \sum_{k=1}^{2^{N+1}-1} a_k h_{J_k},$$

where in the last term we are keeping track of the ordering in the Haar system adapted to J, as in Remark 2.4. Fix some $\lambda > 0$ and let

$$p := \frac{1}{|J|} |\{J : S_J^2 \varphi \ge \lambda\}| =: c\langle |\varphi| \rangle_J,$$

and suppose that $0 < p < 1$. Put the intervals in the last generation $J_{(N)}$ into two ("good" and "bad") categories:

$$J_{(N)} = \mathcal{I}_g \cup \mathcal{I}_b,$$

where $\mathcal{I}_g$ is the collection of intervals $I \in J_{(N)}$ with $S_J^2\varphi \geq \lambda$ on I, and $\mathcal{I}_b$ are the remaining ones where $S_J^2\varphi < \lambda$. Then clearly

$$\left|\cup_{I\in\mathcal{I}_g}I\right| = p|J| \text{ and } \left|\cup_{I\in\mathcal{I}_b}I\right| = (1-p)|J|.$$

Now, for each $I \in \mathcal{I}_b$, let the function:

$$\psi_I := \sum_{k=1}^{2^{N+1}-1} \frac{1}{\sqrt{2^{N+1}}}a_k h_{I_k^-} + \sum_{k=1}^{2^{N+1}-1} \frac{1}{\sqrt{2^{N+1}}}a_k h_{I_k^+},$$

where each $\{h_{I_k^-}\}$ and $\{h_{I_k^+}\}$ denote the (ordered) Haar systems adapted to I_- and I_+, respectively. Essentially, this amounts to

$$\psi_I = \mathbb{1}_{I_-}\psi_{I_-} + \mathbb{1}_{I_+}\psi_{I_+},$$

where each $\psi_{I_\pm}$ is a *copy of φ adapted to $I_\pm$*, so

$$\langle|\psi_{I_\pm}|\rangle_{I_\pm} = \langle|\psi_I|\rangle_I = \langle|\varphi|\rangle_J.$$

Now, let

$$\varphi_1 := \varphi + \sum_{I\in\mathcal{I}_b} \psi_I.$$

Then $\int_J \varphi_1 = 0$, and

$$\langle|\varphi_1|\rangle_J \leq \langle|\varphi|\rangle_J\big(1 + (1-p)\big).$$

The square function $S_J^2\varphi_1$ equals $S_J^2\varphi$ on $\cup_{I\in\mathcal{I}_g}I$, while on any $I \in \mathcal{I}_b$:

$$|\{I : S_J^2\varphi_1 \geq \lambda\}| \geq |I_-|p + |I_+|p = |I|p.$$

So φ_1 satisfies

$$\frac{1}{|J|}|\{J : S_J^2\varphi_1 \geq \lambda\}| \geq p\big(1 + (1-p)\big).$$

Continuing this process, we obtain a sequence of functions $\{\varphi_n\}_n$, supported on J, each with $\int_J \varphi_n = 0$ and

$$\frac{1}{|J|}|\{J : S_J^2\varphi_n \geq \lambda\}| \geq p\big(1 + (1-p) + \ldots + (1-p)^n\big) \xrightarrow[n\to\infty]{} 1,$$

and

$$\langle|\varphi_n|\rangle_J \leq \langle|\varphi|\rangle_J\big(1 + (1-p) + \ldots + (1-p)^n\big) \xrightarrow[n\to\infty]{} \frac{1}{p}\langle|\varphi|\rangle_J.$$

Letting $\widetilde{\varphi} = \lim \varphi_n$, we have

$$\langle\widetilde{\varphi}\rangle_J = 0, \ \ \langle|\widetilde{\varphi}|\rangle_J \leq \frac{1}{p}\langle|\varphi|\rangle_J, \ \ S_J^2\widetilde{\varphi} \geq \lambda \text{ a.e. on } J.$$

Therefore $\widetilde{\varphi}$ is admissible for $\mathbb{L}(0,1,\lambda)$, so

$$\mathbb{L}(0,1,\lambda) \leq \langle|\widetilde{\varphi}|\rangle_J \leq \frac{1}{p}\langle|\varphi|\rangle_J = \frac{1}{c}.$$

We then have that

$$\frac{1}{|J|}|\{J : S_J^2\varphi \geq \lambda\}| \leq \frac{\langle|\varphi|\rangle_J}{\mathbb{L}(0,1,\lambda)},$$

for all φ on J with mean zero, and all $\lambda > 0$, which yields exactly (4.1). $\qquad\square$

Next, we find the values of $\mathbb{M}$ and $\mathbb{L}$ for $f = 0$.

Proposition 4.2. *If $f = 0$, the functions $\mathbb{M}$ and $\mathbb{L}$ are given by:*

(4.2)
$$\mathbb{M}(0, F, \lambda) = \begin{cases} \frac{F}{\mathbb{L}(0,1,\lambda)} = \frac{F}{\sqrt{\lambda}}\Psi(1), & \text{if } F \le \frac{\sqrt{\lambda}}{\Psi(1)} \\ 1, & \text{if } F > \frac{\sqrt{\lambda}}{\Psi(1)}. \end{cases}$$

and

(4.3)
$$\mathbb{L}(0, p, \lambda) = p\,\mathbb{L}(0, 1, \lambda) = \frac{p\sqrt{\lambda}}{\Psi(1)}.$$

Proof. Consider $\gamma \mapsto \theta(0, \gamma)$. We know that $\theta(0,0) = 0$ and $\theta(0, \gamma) = 1$ for all $\gamma \ge \frac{1}{\Psi(1)}$ (see Figure 1). But θ is *concave* in γ, so $\theta(0, \cdot)$ lies above its secant line between $(0, 0, 0)$ and $(0, \frac{1}{\Psi(1)}, 1)$. This line has equation $y(\gamma) = \Psi(1)\gamma$, so

$$\theta(0, \gamma) \ge \Psi(1)\gamma, \text{ for all } 0 \le \gamma \le \frac{1}{\Psi(1)}.$$

But Theorem 4.1 says that $\theta(0, \gamma) \le \Psi(1)\gamma$, so then

$$\theta(0, \gamma) = \begin{cases} \Psi(1)\gamma, & \text{if } 0 \le \gamma \le \frac{1}{\Psi(1)} \\ 1, & \text{otherwise.} \end{cases}$$

Now let $p \in [0, 1]$. Then $p = \theta(0, \gamma)$ for $\gamma = \frac{p}{\Psi(1)}$. Then by (3.14),

$$\eta(0, p) = \eta(0, \theta(0, \gamma)) = \gamma = \frac{p}{\Psi(1)},$$

proving that

$$\eta(0, p) = \frac{p}{\Psi(1)}.$$

$\square$

Corollary 4.3. *The sharp constant C in the inequality*

$$\frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) \ge \lambda\}| \le C\frac{1}{\sqrt{\lambda}}\langle|\varphi|\rangle_J, \text{ for all } \varphi \in L^1(J),\ J \in \mathcal{D},$$

is given by $C = \Psi(1)$.

Proof. Obviously

(4.4)
$$C = \sup_{f, F, \lambda} \frac{\mathbb{M}(f, F, \lambda)\sqrt{\lambda}}{F} = \sup_{F, \lambda} \frac{\mathbb{M}(0, F, \lambda)\sqrt{\lambda}}{F} = \Psi(1),$$

where the second equality follows since $\mathbb{M}(f, F, \lambda) \le \mathbb{M}(0, F, \lambda)$, and the last equality follows from (4.2). $\square$

5. Proofs of the Boundary Values $\mathbb{M}_b$ and $\mathbb{L}_b$ of $\mathbb{M}$ and $\mathbb{L}$

In this section we prove Theorems 3.2 and 3.3.

5.1. **The boundary case $\mathbb{M}_b(f, \lambda)$.** Recall that

$$\mathbb{M}_b(f, \lambda) := \sup \frac{1}{|J|}|\{x \in J : S_J^2\varphi(x) \ge \lambda\}|, \forall f \ge 0, \lambda > 0,$$

where the supremum is over all functions φ on J with $\varphi \ge 0$ a.e. and $\langle\varphi\rangle_J = f$. Then $\mathbb{M}_b$ has the obvious properties:

- Domain: $\Omega_{\mathbb{M}_b}^+ = \{f \ge 0, \lambda > 0\}$; Range: $0 \le \mathbb{M}_b \le 1$;
- $\mathbb{M}_b$ is decreasing in λ;
- Homogeneity: $\mathbb{M}_b(f, \lambda) = \mathbb{M}_b(tf, t^2\lambda)$, for all $t > 0$;
- Obstacle Condition: $\mathbb{M}_b(f, \lambda) = 1$, for all $f \ge \sqrt{\lambda}$;

- Boundary Condition: $\mathbb{M}_b(0, \lambda) = 0$, for all $\lambda > 0$;
- Main Inequality: For any pairs in the domain with $f = \frac{1}{2}(f_+ + f_-)$, $\lambda = \min\{\lambda_\pm\}$:

$$(5.1) \qquad \mathbb{M}_b\left(f, \lambda + \left(\frac{f_+ - f_-}{2}\right)^2\right) \geq \frac{1}{2}\big(\mathbb{M}_b(f_+, \lambda_+) + \mathbb{M}_b(f_-, \lambda_-)\big);$$

- $\mathbb{M}_b$ is continuous;
- $\mathbb{M}_b$ is concave and non-decreasing in f;
- Least Supersolution: If $m(f, \lambda)$ is a continuous non-negative function on $\Omega_{\mathbb{M}_b}^+$ which satisfies (5.1) and the obstacle condition, then $\mathbb{M}_b \leq m$.

Rewriting the Main Inequality (5.1) in a more convenient form:

$$\mathbb{M}_b(f, \lambda) \geq \frac{1}{2}\big(\mathbb{M}_b(f - a, \lambda - a^2) + \mathbb{M}_b(f + a, \lambda - a^2)\big), \forall f \geq a \geq 0, \lambda > a^2,$$

it is easy to use Taylor's formula and obtain the infinitesimal version of (5.1):

$$(5.2) \qquad (\mathbb{M}_b)_{ff} - 2(\mathbb{M}_b)_\lambda \leq 0.$$

Using homogeneity of $\mathbb{M}_b$, we put:

$$\mathbb{M}_b(f, \lambda) = \mathbb{M}_b(f/\sqrt{\lambda}, 1) =: \alpha(\tau), \text{ where } \tau = \frac{f}{\sqrt{\lambda}}.$$

Then from (2.6),

$$\alpha : [0, \infty) \to [0, 1] \text{ with } \alpha(0) = 0 \text{ and } \alpha(\tau) = 1, \forall \tau \geq 1.$$

The inequality (5.2) becomes:

$$(5.3) \qquad \tau\alpha'(\tau) + \alpha''(\tau) \leq 0.$$

Let us look at the differential equation $\tau y'(\tau) + y''(\tau) = 0$ for $\tau \geq 0$. The general solution is:

$$y(\tau) = C\Phi(\tau) + D, \text{ where } \Phi(\tau) := \int_0^\tau e^{-x^2/2} \, dx.$$

Imposing $y(0) = 0$ and $y(1) = 1$, we obtain an obvious candidate for our function α:

$$(5.4) \qquad y(\tau) = \begin{cases} \frac{\Phi(\tau)}{\Phi(1)}, & 0 \leq \tau \leq 1 \\ 1, & \tau \geq 1. \end{cases}$$

The first thing we should check is that this function satisfies the (discrete) main inequality (5.1). This is the content of the following lemma, which we prove shortly:

Lemma 5.1. *The function $m(f, \lambda) = y(\tau)$, where $\tau = f/\sqrt{\lambda}$ and y is the function in (5.4), is a supersolution for (5.1).*

Obviously, this gives us that $M(f, \lambda) \leq m(f, \lambda)$. To see that we have, in fact, equality, we consider a new variable:

$$S := \Phi(\tau),$$

and observe that for a function g:

$$(5.5) \qquad \big(\tau g'(\tau) + g''(\tau)\big)e^{\tau^2} = \frac{d^2 g}{dS^2} = g_{SS}.$$

So (5.3) is equivalent to $\alpha_{SS} \leq 0$, or α being concave in the variable S. It is easy to see that:

If $g(S)$ is a concave non-negative function for $S \geq 0$, then the ratio $\frac{g(S)}{S}$ is non-increasing.

Thus, if we put $\alpha(\tau) := g(S)$, we have that for all $0 \leq \tau \leq 1$:

$$\frac{g(S)}{S} = \frac{\alpha(\tau)}{\Phi(\tau)} \geq \frac{g(\Phi(1))}{\Phi(1)} = \frac{\alpha(1)}{\Phi(1)} = \frac{1}{\Phi(1)},$$

which gives exactly that $\mathbb{M}_b(f, \lambda) \geq m(f, \lambda)$. Therefore

$$(5.6) \qquad \mathbb{M}_b(|f|, \lambda) = \mathbb{M}(f, |f|, \lambda) = \begin{cases} \frac{\Phi\left(\frac{|f|}{\sqrt{\lambda}}\right)}{\Phi(1)}, & |f| < \sqrt{\lambda} \\ 1, & |f| \geq \sqrt{\lambda}. \end{cases}$$

Proof of Lemma 5.1. We define the quantities:

$$(5.7) \qquad X^+_{\tau,x} := \frac{\tau + x}{\sqrt{1 - x^2}} \text{ and } X^-_{\tau,x} := \frac{\tau - x}{\sqrt{1 - x^2}},$$

for all $\tau \geq 0$, and $0 \leq x < 1$, $x \leq \tau$. We claim that, for all $0 \leq x \leq \tau < 1$, the function Φ satisfies

$$(5.8) \qquad 2\Phi(\tau) \geq \Phi(X^+_{\tau,x}) + \Phi(X^-_{\tau,x}).$$

In what follows, suppose $\tau \in [0, 1)$ is fixed, and we wish to show that

$$2\Phi(\tau) \geq g(x), \ \forall \, 0 \leq x \leq \tau, \text{ where } g(x) := \Phi(X^+_{\tau,x}) + \Phi(X^-_{\tau,x}).$$

Since $g(0) = 2\Phi(\tau)$, it suffices to show that g is non-increasing. We have

$$\frac{d}{dx} X^+_{\tau,x} = \frac{1 + \tau x}{(1 - x^2)^{3/2}} \text{ and } \frac{d}{dx} X^-_{\tau,x} = -\frac{1 - \tau x}{(1 - x^2)^{3/2}},$$

and then

$$g'(x) \leq 0 \Leftrightarrow \frac{1 + \tau x}{1 - \tau x} \leq e^{\frac{2\tau x}{1 - x^2}}$$

$$\Leftrightarrow 0 \leq G(x), \text{ where } G(x) = \frac{2\tau x}{1 - x^2} - \log\left(\frac{1 + \tau x}{1 - \tau x}\right).$$

Since $G(0) = 0$, it suffices to show that G is non-decreasing. A simple computation shows that

$$G'(x) = 2\tau \left(\frac{1 + x^2}{(1 - x^2)^2} - \frac{1}{1 - x^2\tau^2}\right) \geq 0, \ \forall \, 0 \leq x \leq \tau < 1.$$

This completes the proof for (5.8).

Returning to Lemma 5.1, recall that we wish to show that

$$2m(f, \lambda) \geq m(f + a, \lambda - a^2) + m(f - a, \lambda - a^2), \ \forall \, f \geq a \geq 0, \ \lambda > a^2,$$

where $m(f, \lambda) = y(\tau)$, and $y(\tau) = \min(\Phi(\tau)/\Phi(1), 1)$, for $\tau = \frac{f}{\sqrt{\lambda}} \geq 0$. Using the homogeneity of m, we can rewrite this in terms of y. Moreover, letting $x := \frac{a}{\sqrt{\lambda}}$, we have that $0 \leq x < 1$ and also $x \leq \tau$, so we may use exactly the quantities $X^+_{\tau,x}$ and $X^-_{\tau,x}$ defined in (5.7) to rewrite the inequality we have to prove:

$$(5.9) \qquad 2y(\tau) \geq y(X^+_{\tau,x}) + y(X^-_{\tau,x}), \ \forall \, \tau \geq 0, 0 \leq x < 1, x \leq \tau.$$

If $\tau < 1$, then it is easy to see that $X^-_{\tau,x} \leq \tau < 1$, so (5.9) becomes

$$2\Phi(\tau) \geq \Phi(X^-_{\tau,x}) + \Phi(1)y(X^+_{\tau,x}).$$

If $X^+_{\tau,x} < 1$, this becomes exactly (5.8). If $X^+_{\tau,x} \geq 1$, the inequality follows again by (5.8) and monotonicity of Φ:

$$\Phi(X^-_{\tau,x}) + \Phi(1) \leq \Phi(X^-_{\tau,x}) + \Phi(X^+_{\tau,x}) \leq 2\Phi(\tau).$$

Finally, when $\tau \geq 1$, $y(\tau) = 1$, and since $y \leq 1$ always, $2 = 2y(\tau) \geq y(X^+_{\tau,x}) + y(X^-_{\tau,x})$.

$\square$

5.2. **The boundary case** $\mathbb{L}(f, 1, \lambda)$. Define

$$\mathbb{L}_b(f, \lambda) := \mathbb{L}(f, 1, \lambda) = \inf\{\langle|\varphi|\rangle_J : \operatorname{supp}(\varphi) \subset J; \langle\varphi\rangle_J = f; S_J^2\varphi \geq \lambda \text{ a. e. on } J\}.$$

Some of the obvious properties $\mathbb{L}_b$ inherits from $\mathbb{L}$ are:

- Domain: $\Omega_{\mathbb{L}_b} := \{(f, \lambda) : f \in \mathbb{R}; \lambda > 0\}$;
- $\mathbb{L}_b$ is increasing in λ and even in f;
- Homogeneity: $\mathbb{L}_b(tf, t^2\lambda) = |t|\mathbb{L}_b(f, \lambda)$;
- Range/Obstacle Condition: $|f| \leq \mathbb{L}_b(f, \lambda) \leq \max\{|f|, \sqrt{\lambda}\}$;
- Main Inequality:

(5.10) $$2\mathbb{L}_b(f, \lambda) \leq \mathbb{L}_b(f - a, \lambda - a^2) + \mathbb{L}_b(f + a, \lambda - a^2), \ \forall\, |a| < \sqrt{\lambda}.$$

- $\mathbb{L}_b$ is convex in f, and recall from (2.11) that $\mathbb{L}_b$ is minimal at $f = 0$:

(5.11) $$\mathbb{L}_b(0, \lambda) \leq \mathbb{L}_b(f, \lambda), \ \forall\, f,$$

  therefore $\mathbb{L}_b$ is non-decreasing in f for $f \geq 0$, and non-increasing in f for $f \leq 0$;
- Greatest Subsolution: If $\ell(f, \lambda)$ is any continuous non-negative function on $\Omega_{\mathbb{L}_b}$ which satisfies the main inequality

(5.12) $$2\ell(f, \lambda) \leq \ell(f + a, \lambda - a^2) + \ell(f - a, \lambda - a^2)$$

  and the range condition $\ell(f, \sqrt{\lambda}) \leq \max\{|f|, \sqrt{\lambda}\}$, then $\ell \leq \mathbb{L}_b$. See Remark 2.6.

Using homogeneity, we write

$$\mathbb{L}_b(f, \lambda) = \sqrt{\lambda}\mathbb{L}_b\left(\frac{f}{\sqrt{\lambda}}, 1\right) =: \sqrt{\lambda}b(\tau), \text{ where } \tau := \frac{f}{\sqrt{\lambda}}.$$

Then $b : \mathbb{R} \to [0, \infty)$, b is even in τ, and from (5.11):

(5.13) $$b(0) \leq b(\tau), \ \forall\, \tau.$$

Moreover, b satisfies

(5.14) $$b(\tau) = |\tau|, \ \forall\, |\tau| \geq 1.$$

Using again Taylor's formula, the infinitesimal version of (5.10) is

$$(\mathbb{L}_b)_{ff} - 2(\mathbb{L}_b)_\lambda \geq 0.$$

In terms of b, this becomes

(5.15) $$b''(\tau) + \tau b'(\tau) - b(\tau) \geq 0.$$

Since b is even, we focus next only on $\tau \geq 0$.

The general solution to the differential equation $z''(\tau) + \tau z'(\tau) - z(\tau) = 0$ for $\tau \geq 0$ is

$$z(\tau) = C\Psi(\tau) + D\tau, \text{ where } \Psi(\tau) = \tau\Phi(\tau) + e^{-\tau^2/2}, \ \forall\, \tau \geq 0.$$

Note that

(5.16) $$\Psi'(x) = \Phi(x), \ \Psi''(x) = e^{-x^2/2}.$$

Given our condition that $b(\tau) = \tau$ for all $\tau \geq 1$, a reasonable candidate for our function b is one already proposed by Bollobas [2]:

(5.17) $$z(\tau) := \begin{cases} \frac{\Psi(\tau)}{\Psi(1)}, & 0 \leq \tau < 1 \\ \tau, & \tau \geq 1. \end{cases}$$

In other words, a candidate for $\mathbb{L}_b$ is

(5.18) $$L(f, \lambda) = \begin{cases} \sqrt{\lambda}\frac{\Psi\left(\frac{|f|}{\sqrt{\lambda}}\right)}{\Psi(1)}, & \sqrt{\lambda} \geq |f|, \\ |f|, & \sqrt{\lambda} \leq |f|. \end{cases}$$

Our first goal will be to prove:

Lemma 5.2. *The function L defined in* (5.18) *satisfies* (5.12).

Since it is easy to verify that L satisfies the range condition $L(f, \lambda) \leq \max\{|f|, \sqrt{\lambda}\}$, we have then that L is a subsolution of (5.12), and so

$$L \leq \mathbb{L}_b.$$

Now we want to prove the opposite inequality

(5.19) $$\mathbb{L}_b \leq L.$$

Recall that we write $\mathbb{L}_b(f, \lambda) = \sqrt{\lambda} b(\tau)$, where $\tau = \frac{f}{\sqrt{\lambda}}$. We look only at $\tau \geq 0$. Consider again a new variable

(5.20) $$T := \frac{\tau}{\Psi(\tau)}, \quad \tau \geq 0.$$

Then

$$\frac{dT}{d\tau} = \frac{e^{-\tau^2/2}}{\Psi^2(\tau)},$$

which shows that T is strictly increasing in τ. Moreover, it is easy to check that for a function g, we have

(5.21) $$\frac{d^2}{dT^2}\left(\frac{g(\tau)}{\Psi(\tau)}\right) = \Psi^3(\tau) e^{\tau^2}(g'' + \tau g' - g).$$

So, if we circle back to our function b, and denote

$$\beta(T) := \frac{b(\tau)}{\Psi(\tau)},$$

the infinitesimal main inequality (5.15) for b is equivalent to $\beta_{TT} \geq 0$, or β being convex in the variable T. Now note that

$$\beta'(T) = \left(b'(\tau)\Psi(\tau) - b(\tau)\Phi(\tau)\right)e^{\tau^2/2}.$$

Since $T = 0$ only at $\tau = 0$, we have

$$\beta'(T)|_{T \to 0_+} = b'(0_+) \geq 0,$$

where $b'(0_+)$ denotes the right derivative of b at 0. This is non-negative because b is a convex, even function. So now we have that $\beta(T)$ is convex and $\beta'(0_+) \geq 0$, showing that β is *non-decreasing* for $T \geq 0$. Finally, we have then that for any $0 \leq \tau < 1$:

$$\frac{b(\tau)}{\Psi(\tau)} \leq \frac{b(1)}{\Psi(1)} = \frac{1}{\Psi(1)},$$

therefore

$$b(\tau) \leq \frac{\Psi(\tau)}{\Psi(1)}, \quad \forall \tau \in [0, 1],$$

which is exactly $\mathbb{L}_b \leq L$. So Theorem 3.3 is proved, provided we have Lemma 5.2, which we prove next.

Proof of Lemma 5.2. In fact, the proof is given in [2]. It is slightly sketchy and leaves some cases to the reader, so here we follow the proof of [2] in more details. By symmetry we can think that $x \geq 0$. Case 1) will be when both points $x \pm t, \lambda - t^2)$ lie in Π (that is over parabola $\lambda = x^2$).

Notice that $L(x, \lambda) = \max(\sqrt{\lambda}\frac{\Psi\left(\frac{|x|}{\sqrt{\lambda}}\right)}{\Psi(1)}, |x|)$ if we are in Π. Thus convexity of $|x|$ and this remark finish this case.

Case 1). We follow [2]. Put

$$(5.22) \qquad X(x,\tau) := \frac{x+\tau}{(1-\tau^2)^{1/2}}, \quad \tau \in [0,x], \; x \in [0,1] \,.$$

Then (5.12) in our case can be rewritten as (e. g. $\tau := a/\sqrt{\lambda}, x = f/\sqrt{\lambda}$):

$$(5.23) \qquad 2\Psi(x) \le \Psi(X(x,\tau)) + \Psi(X(x,-\tau)),$$

which is correct for $\tau = 0$. Let us check that

$$(5.24) \qquad \frac{\partial}{\partial \tau}(\Psi(X(x,\tau)) + \Psi(X(x,-\tau))) \ge 0 \,.$$

Using (5.16), we get the equality

$$\frac{\partial}{\partial \tau}(\Psi(X(x,\tau)) + \Psi(X(x,-\tau))) = \frac{1}{1-\tau^2}(\Phi(X(x,\tau)) - \Phi(X(x,-\tau)))$$

$$\frac{x\tau}{1-\tau^2}(\Phi(X(x,\tau)) + \Phi(X(x,-\tau))) - \frac{\tau}{(1-\tau^2)^{1/2}}(X(x,\tau))\Phi(X(x,\tau)) + X(x,-\tau))\Phi(X(x,-\tau)))$$

$$- \frac{\tau}{(1-\tau^2)^{1/2}}(e^{-X(x,\tau)^2/2} + e^{-X(x,-\tau)^2/2}) \,.$$

After plugging (5.22) this simplifies to

$$\frac{\partial}{\partial \tau}(\Psi(X(x,\tau)) + \Psi(X(x,-\tau))) = (\Phi(X(x,\tau)) - \Phi(X(x,-\tau)))$$

$$- \frac{\tau}{(1-\tau^2)^{1/2}}(e^{-X(x,\tau)^2/2} + e^{-X(x,-\tau)^2/2}) \,.$$

But $\frac{\tau}{(1-\tau^2)^{1/2}} = \frac{1}{2}(X(x,\tau) - X(x,-\tau))$, so to prove (5.24) one needs to check the following inequality:

$$(5.25) \qquad \frac{1}{X(x,\tau) - X(x,-\tau)} \int_{X(x,-\tau)}^{X(x,\tau)} e^{-s^2/2}ds \ge \frac{1}{2}(e^{-X(x,\tau)^2/2} + e^{-X(x,-\tau)^2/2}) \,.$$

This inequality holds because in our case 1) we have $X(x,-\tau) \in [-1,1], X(x,\tau) \in [-1,1]$, and function $s \to e^{-s^2/2}$ is concave on the interval $[-1,1]$. (It is easy that for every concave function on an interval, its average over the interval is at least its average over the ends of the interval.)

Case 2). Now suppose that the left point $(x-t, \lambda - t^2)$ lies on parabola. By homogeneity we can always think that $\lambda = 1$. We continue to consider by symmetry $x \ge 0$ only. If $(x-t, 1-t^2)$ is such that $(x-t)^2 = 1 - t^2$ then we need to show that

$$(5.26) \qquad 2\frac{\Psi(x)}{\Psi(1)} \le 2t$$

Clearly $0 \le t \le 1$, $0 \le x \le t$. From $(x-t)^2 = 1 - t^2$ we obtain that $t - \sqrt{1-t^2} =: x(t) \ge 0$, so $t \ge \frac{1}{\sqrt{2}}$, and the inequality (5.26) simplifies to

$$x(t) \le \Psi^{-1}(\Psi(1)t), \quad 1 \ge t \ge \frac{1}{\sqrt{2}}$$

The left hand side is convex and the right hand side is concave. Since at $t = 1$ and $t = \frac{1}{\sqrt{2}}$ the inequality holds then it holds on the whole interval $[1/\sqrt{2}, 1]$.

So we proved that if the left point already left Π (and then automatically the right point also already left it), the desired inequality holds.

Case 3). It remains to show that if the right point already left Π but the left point is in Π, then (5.12) still holds. Again by homogeneity we can always think that $\lambda = 1$. Then the required

inequality amounts to

$$2\Psi(x) \le \sqrt{1-t^2}\Psi\left(\frac{t-x}{\sqrt{1-t^2}}\right) + \Psi(1)(x+t)$$

where either $\sqrt{1-t^2} - t \le x \le t \le 1$ or $\sqrt{1-t^2} - t \le t \le x \le 1$. It is the same as to show

(5.27)
$$\Psi\left(\frac{t-x}{\sqrt{1-t^2}}\right) + \Psi(1)\left(\frac{t - (\frac{2\Psi(x)}{\Psi(1)} - x)}{\sqrt{1-t^2}}\right) \ge 0$$

for all $0 \le x \le 1$ if $\frac{\sqrt{2-x^2}-x}{2} < t < \frac{x+\sqrt{2-x^2}}{2}$. Te left inequality says that the right point already crossed parabola $\partial\Pi$ and the right inequality says that the left point is still inside Π.

Let as show that the derivative in t of the left hand side of (5.27) is nonnegative. If this is the case then we are done. Ψ is increasing (see (5.16)), and since $xt \le 1$ therefore $t \mapsto \Psi\left(\frac{t-x}{\sqrt{1-t^2}}\right)$ is increasing as a composition of two increasing functions. By the same logic, to check the monotonicity of the map $t \mapsto \frac{t-(\frac{2\Psi(x)}{\Psi(1)}-x)}{\sqrt{1-t^2}}$ it is enough to verify that $t(\frac{2\Psi(x)}{\Psi(1)} - x) \le 1$. The latter inequality follows from the following two simple inequalities

(5.28)
$$\Psi(x) \ge \Psi(1)x, \quad 0 \le x \le 1$$

(5.29)
$$\left(\frac{x+\sqrt{2-x^2}}{2}\right)\left(\frac{2\Psi(x)}{\Psi(1)} - x\right) \le 1, \quad 0 \le x \le 1$$

Indeed, to verify (5.28) notice that $\frac{d}{dx}\frac{\Psi(x)}{x} = \frac{x\Phi(x)-\Psi(x)}{x^2} = -\frac{e^{-\frac{x^2}{2}}}{x^2} < 0$, therefore $\frac{\Psi(x)}{x} \ge \Psi(1) \ge \frac{\Psi(1)}{2}$.

To verify (5.29) it is enough to show that

$$\frac{\Psi(x)}{\Psi(1)x} \le \frac{1}{x^2 + x\sqrt{2-x^2}} + \frac{1}{2}$$

If $x = 1$ we have equality. Taking derivative of the mapping $x \to \frac{\Psi(x)}{\Psi(1)x} - \frac{1}{x^2+x\sqrt{2-x^2}} - \frac{1}{2}$ in x we obtain

$$\frac{2}{x^2}\left(-\frac{e^{-\frac{x^2}{2}}}{2\Psi(1)} + \frac{x+\frac{1-x^2}{\sqrt{2-x^2}}}{(x+\sqrt{2-x^2})^2}\right) \ge 0$$

To prove the last inequality it is the same as to show that $\frac{\sqrt{2-x^2}+x(2-x^2)}{x\sqrt{2-x^2}+1-x^2} \le \Psi(1)e^{\frac{x^2}{2}}$. For the exponential function we use the estimate $e^{\frac{x^2}{2}} \ge 1 + \frac{x^2}{2}$. We estimate $\sqrt{2-x^2}$ from above in the numerator by $\sqrt{2}(1-\frac{x^2}{4})$, and we estimate $\sqrt{2-x^2}$ from below in the denominator by $(1-\sqrt{2})(x-1)+1$ (as $x \to \sqrt{2-x^2}$ is concave). Thus it would be enough to prove that

$$\frac{\sqrt{2}(1-\frac{x^2}{4}) + x(2-x^2)}{\sqrt{2}x(1-x)+1} \le \Psi(1)\left(1+\frac{x^2}{2}\right), \quad 0 \le x \le 1$$

If we further use the estimates $\Psi(1) \ge \frac{29}{28}$, and $\frac{41}{29} \le \sqrt{2} \le \frac{17}{12}$ (for denominator and numerator correspondingly), then the last inequality would follow from

$$\frac{29}{240} \cdot \frac{246x^4 - 486x^3 + 233x^2 - 12x - 8}{29 + 41x - 41x^2} \le 0$$

The denominator has the positive sign. The negativity of $246x^4 - 486x^3 + 233x^2 - 12x - 8 \le 0$ for $0 \le x \le 1$ follows from the Sturm's algorithm, which shows that the polynomial does not have roots on $[0,1]$. Since at point $x = 0$ it is negative therefore it is negative on the whole interval. $\qquad \square$

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