Title

Theory and Applications of Pull-Back Operator Methods in Dynamical Systems

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Publication Date

2020

Peer reviewed|Thesis/dissertation

University of California
Santa Barbara

Theory and Applications of Pull-Back Operator Methods in Dynamical Systems

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mechanical Engineering

by

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December 2020

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December 2020

Theory and Applications of Pull-Back Operator Methods in Dynamical Systems

I dedicate this dissertation to my aunt Blanca Hernandez.

Acknowledgements

First and foremost, I would like to acknowledge my heavenly Father God, my Lord and Saviour Jesus Christ, and the Holy Spirit. Their love, blessings, and protection have abounded in my life; without them, I would not be where I am today, all glory be to them.

I would also like to thank my parents Otto Manrique Avila and Miriam Del Carmen Avila, for their love and support throughout my life. You sacrificed everything by leaving your country in search of a better life for your children. I can not thank you both enough. I am who I am today because of your love and nurture. Thank you.

I am also grateful for the support, love, and beautiful memories that my brothers Edwin Giovanni Avila, Otto Alexander Avila, and my sister Amy Mary Avila have given me throughout my life. I pray the best for them in their future endeavors.

I am also thankful to my wife, Abigail Avila, for her love, support, and constant belief in my capabilities. You are my best friend, a blessing to have, and you complete me. I look forward to a happy and long life together.

Lastly, I want to thank my advisor Dr. Igor Mezić for his support and mentorship throughout the years. He has been a great advisor and a friend to me. I thank my committee members for their patience and guidance. I am grateful to Dr. Ryan Mohr, Dr. Milan Korda, Dr. Hassan Arbabi, Dr. Alexadre Mauroy, Dr. Matthew Kvalheim and Dr. Saber Jafarpour; they have all been supportive friends and academic elders. Lastly, I thank my research group Cory Brown, Gowtham San Seenivasaharagavan, Poorva Shukla, Mathias Wanner, and Michael Banks, for their support and friendship over the years.

Curriculum Vitæ

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Avila, A.M., Mezić, I. Data-driven analysis and forecasting of highway traffic dynamics. Nat Commun 11, 2090 (2020). https://doi.org/10.1038/s41467-020-15582-5

Abstract

Theory and Applications of Pull-Back Operator Methods in Dynamical Systems

by

Allan Manrique Avila

During the 1930s, researchers realized that an abstract dynamical system induces a group of linear operators acting on the space of square-integrable functions. For measure-preserving systems, the induced operator is unitary and self-adjoint. As such, its spectrum is restricted to the unit circle and has been shown to encode many important statistical and geometric properties of the dynamical system. Since then, the induced Koopman group of operators' spectral properties have drawn an immense amount of research interest over the last decade. Due to the rise of computing capabilities and data availability, there has been an explosive amount of research into developing data-driven algorithms that can compute the spectrum numerically from data.

In the first part of this dissertation, we demonstrate how Koopman operator methods can offer a model-free, data-driven approach to analyze and forecast highway traffic dynamics. By obtaining a decomposition of data sets collected by the Federal Highway Administration and the California Department of Transportation, we can reconstruct observed data, distinguish any growing or decaying patterns, and obtain a hierarchy of previously identified and never before identified spatiotemporal patterns. Furthermore, it is demonstrated how this methodology can be utilized to forecast highway network conditions. The developed forecasting scheme readily generalizes to the much-needed scenario of multi-lane highway networks without any loss to its performance or efficiency. Also, we do not rely on large historical training data nor parameter tuning or selection. Thereby providing a completely efficient and accurate method of analyzing and forecasting

traffic patterns at the levels required by modern intelligent transportation systems.

In the second part of this dissertation, we consider the equivalent induced linear operators acting on the space of sections of the tangent, cotangent, and tensor bundles of the state space. We begin by first demonstrating how these operators are indeed natural generalizations of Koopman operators acting on functions. The fundamental insight lies in understanding the connection between the differential geometric concept of pulling back objects (functions, vector fields, covector fields, tensor fields) under a diffeomorphism and how their pull-back relates to the Lie derivative of that object. We then draw connections between the various operators' spectrum and characterize the algebraic and differential topological properties of their spectrum. We describe these operators' discrete spectrum for linear dynamical systems and derive spectral type expansions for linear vector fields. The expansions derived resemble the familiar spectral expansion of functions under the Koopman operator. We define the notion of an "eigendistribution" and provide conditions for when an eigendistribution is integrable. We then demonstrate how to recover the foliations arising from their integral manifolds via the level sets of Koopman eigenfunctions. Many of the results presented in the second part of this dissertation stem from well-known differential geometric concepts. Prior work on such generalized operators on vector fields exists but has remained mostly unnoticed by the growing Koopman operator community.

We conclude with an application to differential geometry where the well-known fact that the flows of commuting vector fields commute is generalized. Specifically, we show that the flows of two vector fields commute, subject to an appropriate rescaling of the flow time, if and only if one vector field is an eigensection of the other vector field. The eigenvalue prescribes the required time scaling, and we recover the original statement that the flows of commuting vector fields commute as a particular case of our result. We also apply our results to the study of a hyperbolic toral automorphism known as Arnold's Cat map. We demonstrate that the Lyapunov exponents are contained within the spectrum of the induced operator on vector fields, and we recover the stable and unstable foliations via the level sets of the joint eigenfunctions of the stable and unstable eigendistributions.

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Chapter 1

Introduction

Historically, the field of dynamical systems arose from studying the behavior of physical systems. The study of a specific system required identifying and solving the appropriate differential equations that model the system of interest. The result is a solution of the dynamical system's behavior as a function of time. Despite several procedures for solving differential equations, many differential equations do not admit closed-form analytic solutions. Thus, despite yielding a precise and complete description of the system's behavior, this technique is very limited to particular types of "solvable" systems. Furthermore, the class of solvable differential equations is so small it is seldom the case that a system of interest is explicitly solvable.

The advent of the "qualitative theory" of dynamical systems enabled insight into the possible types of behaviors a system can exhibit without explicitly solving the differential equations. Poincare's work accomplished this by reformulating the underlying differential equations in terms of a vector field defined over the space of states that the system can exhibit. The attention is now shifted away from an explicit, value-versus-time, solution of a differential equation and instead placed on quantifying the geometry of the state space. One now places special attention to dynamically invariant geometrical objects such as fixed points, periodic orbits, limit cycles, invariant manifolds, and attractors. Nevertheless, one frequently encounters complicated systems for which computing the geometrical invariants of interest is a difficult, if not intractable, task. In the presence of nonlinearity, one often resorts to linearization techniques around fixed points or special orbits. As a result, the system's behavioral properties, such as stability, can only be guaranteed locally, in a neighborhood of the linearization. Yet another obstacle can arise from the high dimensionality of the system itself. In such cases, a geometric analysis of the state space is complicated, and one resorts to reducing the system's dimensionality. The dimensionality reduction is typically accomplished by exploiting symmetries, obtaining invariants of motion, or finding a topological conjugacy to a simpler system. Lastly and perhaps the most obstructive situation is when one lacks knowledge of the dynamical equations, in which case, the geometric analysis can not even proceed.

To address the previously described obstacles, one could attempt to characterizing the dynamics of a system directly from data or measurements. Mathematically, one would interpret a measurement of the system as a function defined over the state space. From this perspective would then seek to characterize the dynamics by studying the evolution of functions under the system's dynamics. This point of view arose during the 1930s when it was realized that an abstract dynamical system induces a group of linear operators acting on the space of square-integrable functions. The induced group of operators are linear, and it has been shown that their spectrum encodes many important statistical and geometric properties of the dynamical system. The method of studying a dynamical system via the spectral properties of its induced operator was pioneered by Bernard Koopman, whose name has been attributed to the induced group of operators.

1.1 Contributions of this Dissertation

Chapter 2:

Chapter 2 contains no results by the author and is intended to provide the setting of this dissertation and motivate the results to come.

Chapter 3:

In chapter 3 we demonstrate how the Koopman mode decomposition (KMD) can offer a model-free, data-driven approach for analyzing and forecasting traffic dynamics. By obtaining a KMD of data sets collected by the US Federal Highway Administration and the California Department of Transportation, we are able to reconstruct observed data, distinguish any growing or decaying patterns, and obtain a hierarchy of previously identified and never before identified spatiotemporal patterns. Furthermore, it is demonstrated how the KMD can be utilized to forecast highway network conditions.

Chapter 4:

In chapter 4 we consider "Koopman-type" operators acting on the space of vector fields, covector fields and arbitrary tensor fields induced by a dynamical system. We first demonstrate how these operators are indeed natural generalizations of the standard Koopman operators and then study how the spectrum of the operators on vector fields and covector fields relates to the underlying dynamics.

1.2 Guide For the Reader

1. For ease of reference, the dissertation provides the relevant chapter citations at the end of each chapter.

2. Mathematical concepts, definitions and results are collected and weaved into the relevant sections. For the reader who is familiar with the topics of a certain section

these portions of the dissertation can be skipped over.

3. For the proofs of results that already exist we simply reference an appropriate citation, not necessarily the original citation.

1.3 Permissions

The entire contents of chapter 3, section 2.6 and section 5.1.1 are from:

Avila, A.M., Mezić, I. Data-driven analysis and forecasting of highway traffic dynamics. Nat Commun 11, 2090 (2020). https://doi.org/10.1038/s41467-020-15582-5

Chapter 2

Koopman Operator Methods in Dynamical Systems

The Koopman family of operators of a dynamical system is a group of infinite-dimensional, linear operators that describe the time evolution of an observable (measurable quantity) under the dynamics of the system[Koo31; KN32]. From this viewpoint, one can utilize measurements and data to interpret a complex system's underlying dynamics via the associated Koopman operator's spectral properties. The spectrum of the Koopman operator leads to a "triple decomposition" of a nonlinear and non-stationary dynamical system into its mean, periodic, and fluctuating components[MB04; Mez05]. Specifically, this linear operator's discrete spectrum (eigenvalues and eigenfunctions) accurately describes the mean (period zero) and periodic components of a nonlinear dynamical system. The continuous spectrum (spectral measure) of this linear operator captures the system's stochastic or chaotic dynamics.

2.1 The Koopman Group of a Dynamical System

In this section, we formally define the Koopman group of operators induced by a dynamical system. The mathematical setting will be in the context of ergodic measure-preserving dynamical systems, which is the setting in which the operators were originally conceived. However, we will also have an eye towards laying the foundation for chapter 4 in which the equivalent induced operators on sections of the tangent and cotangent bundles will be considered.

2.1.1 Relevant Definitions and Results

We begin by establishing some notation and recalling some basic definitions and results.

Dynamical Systems

Definition 2.1.1. *An* ***abstract dynamical system*** *is a triplet* $(\mathcal{M}, \mu, S^t)$ *where* $(\mathcal{M}, \mu)$ *is a measure space together with a one-parameter group of automorphisms* S^t*. If* $t \in \mathbb{R}$ *the dynamical system is said to be a* ***continuous-time dynamical system****. If* $t \in \mathbb{Z}$ *the dynamical system is said to be a* ***discrete-time dynamical system****.*

To avoid any possible confusion, S^t with $t \in \mathbb{R}$ will always denote a continuous-time dynamical system and S^n with $n \in \mathbb{Z}$ will always denote a discrete-time dynamical system. Both S^t and S^n will also be assumed to be diffeomorphisms for every t and n. Lastly, the space $\mathcal{M}$ will be assumed to be a finite measure space with $\mu(\mathcal{M}) = 1$

Definition 2.1.2. *The space* $\mathcal{M}$ *is referred to as the* ***state space*** *of the dynamical system.*

Definition 2.1.3. *Let* $x \in \mathcal{M}$*,* $t \in \mathbb{R}$ *and* $n \in \mathbb{Z}$*. The map*

1. *$S : \mathcal{M} \times \mathbb{Z} \to \mathcal{M}$ defined as $S(x, n) = S^n(x)$ is called the **flow map** or simply the **map of a discrete-time dynamical system**.*

2. *$S : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ defined as $S(x, t) = S^t(x)$ is called the **flow of a continuous-time dynamical system**.*

In the continuous-time case, we also assume the state space $\mathcal{M}$ has the structure of a smooth manifold, and we denote its tangent bundle (see 4.1.14) by $T\mathcal{M}$, and denote by $\mathfrak{X}(T\mathcal{M})$ a vector space of vector fields on $\mathcal{M}$.

Definition 2.1.4. *The **generator** of a*

1. *discrete-time dynamical system S^n, denoted by S, is defined to be the first iterate map $S = S(x, 1) = S^1(x)$.*

2. *continuous-time dynamical system is usually a vector field $\mathbf{F} : \mathcal{M} \to T\mathcal{M}$ such that $\frac{d}{dt} S^t = \mathbf{F} \circ S^t$ for all $t \in \mathbb{R}$.*

In light of definitions 2.1.3-2.1.4 we will interchangeably denote a discrete-time dynamical system by the discrete group S^n or its generator S. Also, since we will only consider a continuous-time dynamical system generated by a vector field, we will interchangeably denote a continuous-time dynamical system by the continuous group S^t or its vector field generator $\mathbf{F}$.

Definition 2.1.5. *An abstract dynamical system is said to be **measure preserving** if for all measureable subsets $A \subset \mathcal{M}$*

1.
$$\mu(S^{-t}(A)) = \mu(A) \tag{2.1}$$

for all $t \in \mathbb{R}$ in the continuous-time case

2.

$$\mu(S^{-1}(A)) = \mu(A) \tag{2.2}$$

in the discrete-time case.

Definition 2.1.6. *An abstract dynamical system is said to be* ***ergodic*** *if*

1. *in the continuous-time case, the only measureable subsets* $A \subset \mathcal{M}$ *such that* $S^{-t}(A) \subset A$*, for all* $t \in \mathbb{R}^+$*, have either full* $\mu(A) = 1$ *or zero* $\mu(A) = 0$ *measure.*

2. *in the discrete-time case, the only measureable subsets* $A \subset \mathcal{M}$ *such that* $S^{-1}(A) \subset A$ *have either full* $\mu(A) = 1$ *or zero* $\mu(A) = 0$ *measure.*

Definition 2.1.7. *An abstract dynamical system is said to be* ***weakly mixing*** *if*

1. *in the continuous-time case, for any pair of measureable subsets* $A,B \subset \mathcal{M}$ *we have that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\mu(S^{-t}(A) \cap B) - \mu(A)\mu(B)|dt = 0 \tag{2.3}$$

2. *in the discrete-time case, for any pair of measureable subsets* $A,B \subset \mathcal{M}$ *we have that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |\mu(S^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0 \tag{2.4}$$

Definition 2.1.8. *An abstract dynamical system is said to be* ***strongly mixing****, or simply* ***mixing****, if*

1. *in the continuous-time case, for any pair of measureable subsets* $A,B \subset \mathcal{M}$ *we have that*

$$\lim_{t\to\infty} \mu(S^{-t}A \cap B) = \mu(A)\mu(B) \tag{2.5}$$

2. *in the discrete-time case, for any pair of measureable subsets* A,$B \subset \mathcal{M}$ *we have that*

$$\lim_{n\to\infty} \mu(S^{-n}A \cap B) = \mu(A)\mu(B) \tag{2.6}$$

Semigroups

Definition 2.1.9. *Let* $\mathcal{M}$ *be a Banach space, and suppose that to every* $t \in [0,\infty)$ *is associated an operator* $Q(t)$*, such that*

1. $Q(0) = I$
2. $Q(s+t) = U(s)U(t)$ *for all* $s, t \geq 0$
3. $\lim_{t\to 0} \|Q(t)x - x\| = 0$ *for all* $x \in \mathcal{M}$

then $\{Q(t)\}$ *is called a* ***strongly continuous semigroup*** *of operators denoted* C_0*- semi-group.*

Definition 2.1.10. *For* $x \in \mathcal{M}$ *and* $\epsilon > 0$ *we can associate with* $\{Q(t)\}$ *the operators* $\mathbf{A}_\epsilon$*, by*

$$\mathbf{A}_\epsilon x = \frac{1}{\epsilon}[Q(\epsilon)x - x] \tag{2.7}$$

the operator $\mathbf{A}$ *defined by*

$$\mathbf{A}x = \lim_{\epsilon\to 0} \mathbf{A}_\epsilon x \tag{2.8}$$

is said to be the ***infinitesimal generator*** *of the* C_0*-semigroup* $\{Q(t)\}$ *for all* x *in the domain of* $\mathbf{A}$.

Spectrum of An Operator

Definition 2.1.11. *The* ***spectrum*** *of a linear operator* Q*, denoted* $\sigma(Q)$ *is the set of scalars* $\{\lambda \in \mathbb{C} | Q - \lambda I$ *is not invertible*$\}$. *Furthermore,*

1. *a scalar λ is said to be an* ***eigenvalue*** *of Q if $Q - \lambda I$ is not injective. The set of all eigenvalues of Q, denoted $\sigma_p(Q)$, is said to be the* ***point*** *or* ***discrete spectrum*** *of Q.*

2. *the set of all scalars λ such that $Q - \lambda I$ is injective, has dense range but is not surjective, denoted $\sigma(Q)_c$, is said to be the* ***continuous spectrum*** *of Q.*

3. *the set of all scalars λ such that $Q - \lambda I$ is injective but does not have dense range, denoted $\sigma(Q)_r$, is said to be the* ***residual spectrum*** *of Q.*

2.1.2 The Induced Koopman Operators

Throughout this dissertation, we will denote by $C(\mathcal{M}, \mathbb{F})$ a vector space of functions from the state space $\mathcal{M}$ to a field of scalars $\mathbb{F}$. Of course, this prescription is rather vague, and we do so to avoid an over commitment to any particular choice of function space. Thus, we allow the reader to infer from the context what additional structure the space $C(\mathcal{M}, \mathbf{F})$ should or could have. For example, when we speak of the Koopman group generator, which is a differential operator, we assume the functions in $C(\mathcal{M}, \mathbb{F})$ are at least C^1-differentiable. On the other hand, when speaking of the group itself, the functions in $C(\mathcal{M}, \mathbf{F})$ need not be differentiable nor even continuous. However, in many cases, we will consider the Hilbert space of complex-valued, square-integrable functions on $\mathcal{M}$, denoted by $L_2(\mathcal{M}, \mu)$. The inner product is of course, the usual L_2 inner product given by $\langle f, g \rangle = \int_{\mathcal{M}} f \cdot \bar{g} d\mu$, where $\bar{g}$ denotes the complex conjugate of g.

Definition 2.1.12. *Let $C(\mathcal{M}, \mathbb{F})$ denote a vector space of functions from $\mathcal{M}$ to a field of scalars $\mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The space $C(\mathcal{M}, \mathbb{F})$ is called the* ***space of observables*** *and an element $f \in C(\mathcal{M}, \mathbb{F})$ is called an* ***observable****.*

Definition 2.1.13. *For a function $f \in C(\mathcal{M}, \mu)$ the* ***Koopman group of operators*** *induced by a*

1. *continuous-time dynamical system S^t is defined as*

$$U^t f(x) = f \circ S^t(x) \tag{2.9}$$

2. *discrete-time dynamical system S is defined as*

$$U^n f(x) = f \circ S^n(x) \tag{2.10}$$

For the Koopman group of operators induced by a continuous-time dynamical system S^t, it is straightforward to check that the induced Koopman group is a C_0-semigroup and in fact a C_0-group since we assumed that S^t is an automorphism for every $t \in \mathbb{R}$. The fact that condition (3) in definition 2.1.9 holds for the Koopman group implies the existence of a generator which can be shown to be $\frac{d}{dt}|_{t=0}Q(t)$ [Rud73]. Specifically, the infinitesimal generator of the Koopman C_0-group is the first-order differential operator, which assigns to a function f its directional derivative in the direction of the vector field $\mathbf{F} : \mathcal{M} \to T\mathcal{M}$ generating S^t.

Proposition 2.1.1. *The Koopman group of operators induced by a continuous time dynamical system S^t is a C_0-group and its infinitesimal generator, denoted by L, is given by*

$$Lf(x) = \nabla f(x) \cdot \mathbf{F}(x) \tag{2.11}$$

Proof: See [Mez20a]

Since the induced Koopman group is a group of linear operators there is an associated spectrum.

Definition 2.1.14. *A scalar* $\lambda \in \mathbb{C}$ *and a function* $\psi(x) \in C(\mathcal{M}, \mathbb{F})$ *such that for a discrete-time or continuous-time dynamical system satisfy*

$$U^n \psi(x) = \lambda^n \psi(x),\ n \in \mathbb{Z} \tag{2.12}$$

in the discrete-time case or

$$U^t \psi(x) = e^{\lambda t} \psi(x),\ t \in \mathbb{R} \tag{2.13}$$

in the continuous-time case are said to be a ***Koopman eigenvalue*** *and* ***Koopman eigenfunction****, respectively.*

Also, for a continuous-time dynamical system if

$$\mathcal{L}^n \psi(x) = \lambda^n \psi(x),\ n \in \mathbb{Z} \tag{2.14}$$

then λ *and* $\psi(x)$ *are said to be an* ***eigenvalue of the Koopman generator*** *and* ***eigenfunction of the Koopman generator****, respectively.*

2.1.3 The Induced Koopman Operators Revisited

The introduction of the Koopman group of a dynamical system has thus far been purely analytical namely, in terms of composition operators. To lay the foundation for chapter 4 it worth noting, what could be said to be, an equivalent differential geometric formulation of the Koopman group of operators. To this end, let us, for the moment, assume the absence of a dynamical system and let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds. Again, we denote by $C(\mathcal{M}, \mathbb{F})$ a vector space of functions and also assume the existence of a mapping $S : \mathcal{M} \to \mathcal{N}$. Let us discuss under what conditions on S and by which means can one take a function f defined on $\mathcal{N}$ or $\mathcal{M}$ and create from it a function on the other space. We will explicitly state the regularity of S or f when required. When f is a function defined on $\mathcal{N}$, this can be accomplished for a continuous, not necessarily differentiable, mapping S via the differential geometric concept of pulling-back a scalar function f by a mapping S. Similarly, if one instead considers a function f defined on $\mathcal{M}$ and assumes the mapping S is a homeomorphism, one can then define a function on $\mathcal{N}$ via the push-forward. Specifically, we have the following definitions.

Definition 2.1.15. *Let $S : \mathcal{M} \to \mathcal{N}$ be a continuous mapping and $f \in C(\mathcal{N}, \mathbb{F})$. The* ***pull-back*** *of f by S is defined as*

$$S^* f(x) = f \circ S(x) \in C(\mathcal{M}, \mathbb{F})$$

If S is a homeomorphism then the ***push-forward*** *of $f \in C(\mathcal{M}, \mathbb{F})$ by S is defined as*

$$S_* f(x) = f \circ S^{-1}(x) \in C(\mathcal{N}, \mathbb{F})$$

It should be clear to see that if the mapping $S : \mathcal{M} \to \mathcal{M}$ is a dynamical system, the pull-back of f by S is precisely the action of the Koopman operator on f. Thus, the induced Koopman group of operators of a dynamical system could equivalently be called the induced pull-back group of operators on functions. In the presence of a vector field, $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$ one can also define the differential geometric concept of a Lie derivative operator on functions.

Definition 2.1.16. *Let $f \in C(\mathcal{M}, \mathbb{F})$ be a differentiable function and $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$. The* ***Lie derivative*** *of f, denoted $\mathcal{L}_{\mathbf{F}}$, said to be* ***along*** *or* ***in the direction of*** $\mathbf{F}$ *is defined as*

$$\mathcal{L}_{\mathbf{F}} f(x) = \nabla f(x) \cdot \mathbf{F}(x) \tag{2.15}$$

Proposition 2.1.2. *The collection of operators $\mathcal{L}_{(\cdot)}$ forms a vector space, a $C(\mathcal{M}, \mathbb{F})$-module and is isomorphic to $\mathfrak{X}(T\mathcal{M})$ as both.*

Proof: see [AMR83] page 216.

Proposition 2.1.2 demonstrates that every vector field $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$ induces an operator $\mathcal{L}_{\mathbf{F}}$. The relation between the Lie derivative of vector field and the flow it generates is a standard result in differential geometry.

Theorem 2.1.1. *Let $f \in C(\mathcal{M}, \mathbb{F})$ be a differentiable function, $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$ and let S^t be the flow generated by $\mathbf{F}$. Then*

$$\frac{d}{dt}\bigg|_{t=0} S^{t*} f(x) = \mathcal{L}_{\mathbf{F}} f(x) \tag{2.16}$$

Proof: See [AMR83] page 213.

In the language of semigroup theory theorem 2.16 is stating that the infinitesimal generator of the C_0-group of pull-back operators is the Lie derivative operator. It is also clear that the Lie derivative and the generator of the Koopman are the same operators. The fact that the Koopman group of operators, a functional analytic concept, can also be formulated in terms of purely differential geometric concepts will be the fundamental motivation for the contents of chapter 4. We ask the reader to keep the observations mentioned above in mind.

Lastly, note that the quantity $\mathcal{L}_{\mathbf{F}} f(x) = \nabla f(x) \cdot \mathbf{F}(x)$ could equivalently be expressed as $\mathcal{L}_{\mathbf{F}} f(x) = \mathbf{F}^{\intercal}(x) \cdot (\nabla f(x))^{\intercal}$ where $(\nabla(\cdot))^{\intercal}$ denotes the transpose of the gradient row vector. Throughout the rest of this dissertation we adopt the latter expression since it better demonstrates that $\mathcal{L}_{\mathbf{F}} = \mathbf{F}^{\intercal}(x) \cdot \nabla(\cdot)$ namely, that it is an operator.

2.2 Background & History

2.2.1 The spectral Invariants of Dynamical System

Koopman showed that for a measure-preserving system, the induced group of linear operators are unitary over L_2 functions [Koo31]. Hence, by the spectral theorem for unitary operators, the spectrum of the Koopman group of operators induced by a measure-preserving system is restricted to the unit circle [Rud73].

Definition 2.2.1. *The* ***spectrum of a dynamical system*** *is the spectrum of the Koopman group of operators induced by the dynamical system.*

Definition 2.2.2. *A dynamical system is said to have:*

1. *A* ***pure discrete spectrum*** *or* ***pure point spectrum*** *if the eigenfunctions of the induced operator form a basis for* $L_2(\mathcal{M}, \mu)$*.*

2. *A* ***pure continuous spectrum*** *if the only eigenfunctions of the induced operator are the constant functions.*

3. *A* ***mixed spectrum*** *if the dynamical system has neither a pure point spectrum nor a pure continuous spectrum.*

4. *A* ***Lebesgue Spectrum*** L^I *if there exists an orthonormal basis of* $L_2(\mathcal{M}, \mu)$ *formed by the function* $g(x) = 1$ *and some set of functions* $f_{i,j}(x)$ $i \in I$, $j \in \mathbb{Z}$ *such that*

$$U f_{i,j} = f_{i,j+1} \tag{2.17}$$

The cardinality of I *is said to be the* ***multiplicity of the Lebesgue spectrum****. If* I *is countably infinite then the dynamical system is said to have* ***countably infinite***

Lebesgue spectrum *and if I has only one element the dynamical system is said to have* ***simple Lebesgue spectrum****.*

It was quickly realized that the spectrum of a dynamical system encodes many important statistical and geometric properties of the dynamics [Koo31; KN32; Hal49]. For example, many ergodic properties of a dynamical system are reflected in the spectrum.

Theorem 2.2.1. *Let $(\mathcal{M}, \mu, S)$ be a discrete-time, measure-preserving, ergodic, dynamical system and U^n the induced Koopman group of operators.*
The dynamical system is ergodic

1. *if and only if the only eigenfunctions of U^n at eigenvalue $\lambda = 1$ are μ-almost everywhere constant.*
2. *if and only if 1 is a simple eigenvalue of U.*
3. *the absolute value of every eigenfunction is constant μ-almost everywhere.*

The dynamical system is weakly mixing

1. *if and only if*
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U^n f, g\rangle - \langle f, 1\rangle \cdot \langle 1, g\rangle| = 0 \tag{2.18}$$
for every $f, g \in L_2(\mathcal{M}, \mu)$
2. *if and only if U^n has a continuous spectrum.*

The dynamical system is mixing

1. *if and only if*
$$\lim_{n\to\infty} \langle U^n f, g\rangle = \langle f, 1\rangle \cdot \langle 1, g\rangle \tag{2.19}$$
for every $f, g \in L_2(\mathcal{M}, \mu)$

2. *if and only if the only eigenvalue of* U *is* 1*.*

3. *if it has a Lebesgue spectrum.*

Proof:

1. For the ergodic properties see [Hal49; Hal56; AA68].

2. For the weakly mixing properties see [Hal49; Hal56].

3. For the mixing properties see [Hal49; Hal56; AA68].

Definition 2.2.3. *Two dynamical systems* $(\mathcal{M}, \mu, S)$ *and* $(\mathcal{N}, \nu, T)$ *are said to be* ***spectrally isomorphic dynamical systems*** *if there induced operators* U_S *and* U_T *are unitarily equivalent. In other words, there exists a unitary operator* $V : L_2(\mathcal{M}, \mu) \to L_2(\mathcal{N}, \nu)$ *such that*

$$U_S = V^* U_T V \tag{2.20}$$

Definition 2.2.4. *The* ***spectral invariants*** *of a dynamical system are properties which are the same for spectrally isomorphic dynamical systems.*

2.2.2 The Isomorphism Problem in Dynamical Systems

At the time of its inception, the Koopman group of operators were generally perceived as a possible framework for solving one of the most famous and elusive problems in ergodic dynamical systems theory known as the isomorphism problem. The isomorphism problem was set forth by John Von Neumann [Von32b; RW12; Wei72] and is concerned with classifying dynamical systems up to an isomorphism. This call for classification is in the spirit of similar classification problems in other branches of mathematics, such as topology (classification of manifolds) or group theory (classification of groups).

Definition 2.2.5. *Two discrete-time, measure-preserving, dynamical systems* $(\mathcal{M}, \mu, S)$ *and* $(\mathcal{N}, \nu, T)$ *are said to be* ***isomorphic dynamical systems*** *if there is a bijection* $h : \mathcal{M} \to \mathcal{N}$ *such that*

1. h *and* h^{-1} *are measureable.*

2. $\nu(h(A)) = \mu(A)$ *for all measureable subsets* $A \subset \mathcal{M}$*.*

3. $h(S(x)) = T(h(x))$ *for all* $x \in \mathcal{M}$*.*

Initially, the isomorphism problem was considered for discrete-time systems only but the problem in the continuous-time case has an equivalent definition. One of the main approaches to tackling the isomorphism problem [RW12] is concerned with with finding a sufficiently large number of invariants of a dynamical system which collectively provide sufficient conditions for two dynamical systems to be isomorphic. One example is the spectral invariants of a dynamical system. Since equivalent operators share a set of eigenvalues the spectrum itself is clearly a spectral invariant. Also, if two dynamical systems $(\mathcal{M}, \mu, S)$ and $(\mathcal{N}, \nu, T)$ are isomorphic then they are spectrally isomorphic [RW12]. For example, if h is the isomorphism then the operator V defined as

$$Vf(x) = f(h^{-1}(y)) \tag{2.21}$$

for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$ provides the spectral isomorphism. The pertinent question now is whether the spectrally isomorphic also implies isomorphic.

Systems With Pure Discrete Spectrum

The fact that many dynamical properties are reflected in the induced operators' spectrum, lead Von Neumann and others to conjecture that the spectral invariants should be strong enough to classify dynamical systems. Von Neumann gave the first breakthrough in 1932, which gave a partial resolution to the isomorphism problem.

Theorem 2.2.2 (Von Neumann's discrete spectrum theorem)**.** *Two dynamical systems* $(\mathcal{M}, \mu, S)$ *and* $(\mathcal{N}, \nu, T)$ *with pure point spectrum are isomorphic if and only if they are spectrally isomorphic.*

Proof: See [Von32b].

The following result by Paul Halmos and Von Nuemann complemented the discrete spectrum theorem.

Theorem 2.2.3.

1. *Every countable subgroup of the circle group* $\mathbb{T} = \{z | z \in \mathbb{C}, |z| = 1\}$ *is the point spectrum of an ergodic dynamical system with discrete spectrum.*

2. *Every ergodic dynamical system with discrete spectrum is isomorphic to a translation on a compact abelian group.*

Proof: See [HN42]

The previous theorems give a complete classification of ergodic dynamical systems with pure discrete spectrum and Von Neumann conjectured that theorem 2.2.2 should also hold for systems with pure continuous spectrum [Von32a]. Unfortunately, further research showed that the conjecture does not hold for general systems with a mixed [RW12; Hal49; Hal56] or continuous spectrum [Kol85; Kol59; Sin59b; Sin59a].

Systems With Mixed Spectrum

In an unpublished letter written in 1941 by Von Neumann to Stanislaw Ulam, Von Neumann constructed a counterexample to his conjecture by providing an example showing that two spectrally isomorphic dynamical systems with a mixed spectrum need not be isomorphic. To see how Von Neumann accomplished this, let G be a family of functions whose constant absolute value is equal to 1. Furthermore, let $G^{'}$ be the family of all functions whose constant absolute value is equal to 1 and for which there exists a function $g \in G$ such that $Uf(x) = f \circ S(x) = g(x)f(x)$, $f \in G^{'}$. Denote by G_1 the set of all constant functions and define G_n inductively as $G_{n+1} = G_n^{'}$ for $n \in \mathbb{Z}^+$. Now, since for an ergodic dynamical system the only eigenfunctions $\psi(x)$ at eigenvalue $\lambda = 1$ are the constant functions, $G_2 = G_1^{'}$ is the set of all constant functions of absolute value 1 and $G_3 = G_1^{''}$ is the set of all eigenfunctions of absolute value 1.

Definition 2.2.6. *The functions in $G_3, G_4, G_5, \ldots$ are said to be **generalized eigenfunctions** with **generalized eigenvalues** in $G_2, G_3, G_4, \ldots$*

We remark that there is an unfortunate clash of terminology here with the generalized eigenfunctions in the sense of those corresponding to a linear dynamical system with degenerate spectrum and with a generalized eigenfunction in the distributional sense.

Now, if the two dynamical systems are isomorphic then their induced operators are related via the operator V given by equation 2.21. As such, $U_S f(x) = g(x)f(x)$ if and only if $U_T V f(x) = V g(x) f(x)$, and hence f is a generalized eigenfunction of U_S if and only if $V f(x)$ is a generalized eigenfunction of U_T. Von Nuemann then proceeds to construct two dynamical systems $(\mathcal{M}, \mu, S)$ and $(\mathcal{M}, \nu, T)$ with mixed spectrum which are spectrally isomorphic but shows that $G_4 \neq G_3$ for S while on the other hand $G_4 = G_3$ for T and thus S and T are not isomorphic.

It is useful to note that if $H \subset G$ then $H^{'} \subset G^{'}$ from which one can see that the sequence $\{G_n\}$ is increasing namely, $(G_1 \subset G_2 \subset G_3 \subset \ldots)$. Additionally, if it ever happens that $G_n = G_{n+1}$ then $G_n = G_{n+k}$ for all k. Denote by $n(S)$ the least positive integer for which this happens, without excluding the possibility that $n(S)$ be infinite. The function $n(S)$ is an invariant of S [Hal56] and thus, if S and T are isomorphic then $n(S) = n(T)$. A counterexample similar to Von Neumann's is constructed by Paul Halmos [Hal49; Hal56] in which he constructs two dynamical systems that are spectrally isomorphic but for which $n(S) \neq n(T)$.

Systems With Pure Continuous Spectrum

The Von Neumann and Halmos counterexamples thus showed that spectrally isomorphic systems with a mixed spectrum need not be isomorphic. However, since for a dynamical system with a pure continuous spectrum, the only eigenfunctions are constant eigenfunctions: $G_1 = G_2 = G_3 = \cdots$. Thus, the same construction does not apply to two spectrally isomorphic dynamical systems with a pure continuous spectrum. This gave hope that the spectral invariants may still be strong enough to characterize systems pure continuous spectrum. Unfortunately, Kolmogorov and Sinai's works soon realized that two spectrally isomorphic dynamical systems with pure continuous spectrum need not be isomorphic.

Specifically, in 1958 Kolmogorov [Kol85; Kol59] introduced the notion of entropy of a dynamical system, and Sinai improved the concept in 1959 [Sin59b]. The next ingredient in Kolmogorov's work dealt with a specific dynamical system known as the Bernoulli shift on n symbols. It was shown that the Kolmogorov-Sinai entropy of a Bernoulli shift on n symbols is equal to $\sum_{i=1}^{n} p_i \log(p_i)$ where p_i is the probability of the n'th symbol. From this, one can see that given an arbitrary nonnegative $a \in \mathbb{R}$, there exists a Bernoulli shift whose Kolmogorov-Sinai entropy is equal to a. Lastly, it was shown that the Kolmogorov-Sinai entropy is an invariant of the dynamical system [Kol85; Kol59]. Thus, two dynamical systems with different Kolmogorov-Sinai entropy can not be isomorphic. As a result, two Bernoulli shifts with different entropy can not be isomorphic. However, in terms of the spectral characterization of the Bernoulli shifts, we have the following result.

Proposition 2.2.1. *The Bernoulli shifts have a countable Lebesgue spectrum and are spectrally isomorphic to each other.*

Proof: See [AA68] pages 30-31.

Combining this result with the entropic properties of the Bernoulli shift mentioned, we have the following:

Theorem 2.2.4.

1. *There are dynamical systems with a pure continuous spectrum that are spectrally isomorphic but are not isomorphic.*

2. *There is an uncountable number of dynamical systems that are spectrally isomorphic but not isomorphic.*

In summary, a dynamical system's spectral invariants are not strong enough to characterize systems with continuous or mixed spectrum. To make matters worst, Halmos showed that the set of mixing, measure-preserving dynamical systems is a comeagre set in the strong neighborhood topology [Hal44]. Specifically, this shows that systems with a continuous spectrum are very generic.

Over time there have been several results involving the spectral invariants of ergodic and non-ergodic dynamical systems and how they relate to the isomorphism problem [Lin75; Ede17; Cho63; Bro72; Zim76; Mac64; Kwi81; NW72].

2.3 The Spectral Decomposition of A Dynamical System

Since the induced operator was mostly utilized as a method for tackling the isomorphism problem, a detailed spectral decomposition of the operator was not actively pursued until 2004-2005 in the works of Mezić [MB04; Mez05]. In this section, we summarize those works.

We begin by recalling the some definitions and the various ergodic theorems.

Definition 2.3.1. *The* ***time average*** *of a function $f \in C(\mathcal{M}, \mathbb{C})$, denoted by f^*, if it exists, is defined*

1. *for a continuous-time dynamical system as*

$$f^*(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f \circ S^t(x) dt \tag{2.22}$$

2. *for a discrete-time dynamical system as*

$$f^*(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ S^n(x) \tag{2.23}$$

Definition 2.3.2. *The* ***space average*** *of a function $f \in C(\mathcal{M}, \mathbb{C})$, denoted by $\bar{f}$, if it exists, is defined for a continuous or discrete-time dynamical system as*

$$\bar{f}(x) = \int_{\mathcal{M}} f(x) d\mu \tag{2.24}$$

Theorem 2.3.1 (Birkhoff's point-wise ergodic theorem)**.** *Let S be a measure preserving transformation w.r.t to μ of a finite measure space (X, μ). For any $f \in L_1(X, \mu)$*

$$f^* = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ S^n(x) \tag{2.25}$$

exists and

$$f^*(x) = \bar{f}(x) \tag{2.26}$$

μ-almost everywhere.

Theorem 2.3.2 (Von Neumann's mean ergodic theorem)**.** *Let H be a Hilbert space and $U : H \to H$ be a unitary operator. Let P be the orthogonal projection onto the eigenspace $\mathcal{E}_{\lambda=1} = \{\psi \in H | U\psi = \psi\}$, then for any $f \in H$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} U^n f = Pf \tag{2.27}$$

Theorem 2.3.2 shows that the eigenfunctions of the Koopman group of operators at eigenvalue 1 can be obtained by infinite-time averages and theorem 2.3.1 guarantees that these averages exist.

Definition 2.3.3. *The operator $P_S f(x) = f^*(x)$ or $P_{S^t} f(x) = f^*(x)$ corresponding to a discrete-time or continuous-time dynamical system is called the* ***time-averaging operator****.*

The time-averaging operator can be considered as a member of a family of averaging-type operators.

Definition 2.3.4. *The operators P_S^ω defined by*

$$P_S^\omega f(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} e^{2\pi i k\omega} f \circ S^k(x) \tag{2.28}$$

for a scalar $\omega \in \mathbb{C}$, are called the ***harmonic time-averaging operators****.*

Definition 2.3.5. *The* ***harmonic time average*** *of a function $f \in C(\mathcal{M}, \mathbb{C})$, if it exists, is defined as*

1. *for a continuous-time dynamical system as*

$$f_\omega^*(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\omega t} f \circ S^t(x) dt \tag{2.29}$$

2. for a discrete-time dynamical system as

$$f^*_\omega(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-2\pi i\omega k} f \circ S^k(x) \tag{2.30}$$

As one might expect the harmonic averages also play a role in the spectrum of the induced Koopman operators.

Theorem 2.3.3. *The harmonic time-averages f^*_ω of a function $f \in C(\mathcal{M}, \mathbb{C})$ are eigenfunctions of the Koopman group of operators at eigenvalue $e^{i\omega t}$ for a continuous-time dynamical system and $e^{2\pi i\omega}$ for a discrete-time dynamical system*

Proof: See [Pet83; Mez05]

The existence of harmonic averages is guaranteed by the following ergodic theorem.

Theorem 2.3.4 (Wiener & Wintner's ergodic theorem)**.** *Let S be a measure preserving transformation w.r.t to μ of a finite measure space (X, μ). For any $f \in L^1(X, \mu)$*

$$f^*_\omega = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} f(S^k x) e^{-k\omega} \tag{2.31}$$

exists μ-almost everywhere and for every $-\infty < \omega < \infty$.

Much like the time-averaging operator, it can also be shown that the harmonic time-averaging operators are orthogonal projectors onto the eigenspace $\mathcal{E}_{\lambda=e^{2\pi i\omega}}$ [Yos95]. Furthermore, the harmonic time-averaging operators are nonzero only on, at most, a countable set of ω's [WW41] and when it is non zero the works of Mezić have shown that they provide substantial amount of information regarding the dynamics of the system [MW99; LM10; MB04].

Again, since the induced operators of a measure-preserving system are unitary, the operators admit a unique decomposition $U = U_s + U_r$ into a singular and regular part

[Ple69]. Moreover, the operator U_s has a pure discrete spectrum, and the operator U_r has a pure continuous spectrum. Mezić leveraged this fact to give the following spectral decomposition and a spectral expansion for the evolution of an L_2 observable.

Theorem 2.3.5. *Let U^n or U^t be the Koopman group of operators induced by an ergodic, measure-preserving, discrete-time or continuous-time dynamical system. Let $dE(\lambda)$ denote the spectral measure associated with a resolution of the identity, denoted E, on subsets of the spectrum of U^n or U^t. The following spectral decomposition holds*

1. *for the continuous-time case*

$$U^t = U_s^t + U_s^t = P_{S^t} + \sum_k e^{2\pi i \omega_k t} P_{S^t}^{\omega_k} + \int_{-\infty}^{\infty} e^{2\pi i \lambda t} dE(\lambda) \tag{2.32}$$

2. *for the discrete-time case*

$$U^n = U_s^n + U_r^n = P_S + \sum_k e^{2\pi i \omega_k n} P_S^{\omega_k} + \int_{-1}^{1} e^{2\pi i \lambda} dE(\lambda) \tag{2.33}$$

Theorem 2.3.6. *Let U^n or U^t be the Koopman group of operators induced by an ergodic, measure-preserving, discrete-time or continuous-time dynamical system and let $f \in L_2(\mathcal{M}, \mu)$ be an arbitrary square-integrable observable. Denote by $P_{\lambda_k}(f(x))$ the projection of $f(x)$ onto the k'th eigenspace $\mathcal{E}_{\lambda_k} = span\{\psi_k(x)\}$, where $\psi_k(x)$ is an eigenfunction of U^n or U^t at eigenvalue λ_k. Also, denote by $\hat{f}(\beta)$ the spectral density of f. The following spectral expansion for the time evolution of f holds*

1. *for the continuous-time case*

$$U^t f(x) = f^*(x) + \sum_k e^{2\pi i \omega_k t} P_{\lambda_k}(f(x))\psi_k(x) + \int_{-\infty}^{\infty} e^{2\pi i \beta t} \hat{f}(\beta)\phi(x,\beta) d\beta \tag{2.34}$$

2. for the discrete-time case

$$U^n f(x) = f^*(x) + \sum_k e^{2\pi i \omega_k n} P_{\lambda_k}(f(x))\psi_k(x) + \int_{-1}^{1} e^{2\pi i \beta n} \hat{f}(\beta)\phi(x,\beta)d\beta \qquad (2.35)$$

We conclude this section with the following definitions.

Definition 2.3.6. *The* ***spectral decomposition of a dynamical system****, if it exists, is the spectral decomposition of its induced Koopman group of operators given by theorem 2.3.5.*

Definition 2.3.7. *Let $(\lambda, \psi(x))$ be an eigen-pair of U^n or U^t the* ***Koopman mode*** *of a scalar observable $f(x)$ at eigenvalue λ, if it exists, is the projection $P_\lambda(f(x))$ of $f(x)$ onto the eigenfunction $\psi(x)$. The definition also holds for a vector valued observable or a field of observables.*

Definition 2.3.8. *The* ***Koopman mode decomposition*** *of an observable $f(x)$, if it exists, is the spectral expansion given by theorem 2.3.6.*

2.4 Algebraic and Topological Properties of Eigenfunctions

In this section, we summarize some of the algebraic and topological properties that eigenfunctions enjoy. The purpose of this summary is to motivate the equivalent algebraic and topological properties that will hold for the "Koopman-type" operators studied in chapter 4.

2.4.1 Algebraic Properties of Eigenfunctions

Theorem 2.4.1. *Let U^n or U^t be the Koopman group of operators induced by a discrete-time or continuous-time dynamical system. Denote by $\mathcal{A}$ a subset of $C(\mathcal{M}, \mathbb{C})$ that is a commutative algebra of functions containing the constant function equal to 1. Also, denote by $\mathcal{A}_U$ the eigenfunctions of U^n or U^t that also belong to $\mathcal{A}$. Then $\mathcal{A}_U$ forms a commutative monoid under pointwise products. Specifically,*

1. *in the continuous-time case, if $\psi_1(x), \psi_2(x) \in \mathcal{A}_U$ are eigenfunctions at eigenvalue $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ then $\psi_1(x) \cdot \psi_2(x)$ is an eigenfunction at eigenvalue $e^{(\lambda_1+\lambda_2)t}$.*

2. *in the discrete-time case, if $\psi_1(x), \psi_2(x) \in \mathcal{A}_U$ are eigenfunctions at eigenvalue λ_1 and λ_2 then $\psi_1(x) \cdot \psi_2(x)$ is an eigenfunction at eigenvalue $\lambda_1 \cdot \lambda_2$.*

Proof: See [Koo31; Mez20b].

It is worth noting that it is possible for a chosen space of observables to not a priori be an algebra of functions. For example, the product of two square-integrable functions is not guaranteed to be square-integrable. Thus, it may be possible for the product of two L_2 eigenfunctions to not be in L_2. To avoid this issue, theorem 2.4.1 relies on a subspace $\mathcal{A}$ which forms an algebra (thus closed under products) and considers only the eigenfunctions belonging to the algebra $\mathcal{A}$.

2.4.2 Topological Properties of Eigenfunctions

We now summarize the behavior of eigenfunctions in the presence of a topological conjugacy. Consider two dynamical systems, one defined on the space $\mathcal{M}$ generated by a vector field $\mathbf{F}(x) \in \mathfrak{X}(T\mathcal{M})$ or a mapping $S(x)$ and another on the space $\mathcal{N}$ generated by the vector field $\mathbf{G}(y) \in \mathfrak{X}(T\mathcal{N})$ or a mapping $T(y)$. Furthermore, denote by $S^t_{\mathbf{F}}(x)$ and $S^t_{\mathbf{G}}(y)$ the flows generated by $\mathbf{F}(x)$ and $\mathbf{G}(y)$. Lastly, denote the associated Koopman group of operators by $U^t_{\mathbf{F}}, U^t_{\mathbf{G}}$, or U_S, U_T.

Definition 2.4.1. *The two dynamical systems* $(S^t_{\mathbf{F}}, S^t_{\mathbf{G}})$ *or* (S, T) *are said to be* ***topologically conjugate*** *if there exists a homeomorphism* $h : \mathcal{M} \to \mathcal{N}$ *such that*

$$h \circ S^t_{\mathbf{F}}(x) = S^t_{\mathbf{G}} \circ h(x) \\ \text{or} \\ h \circ S(x) = T \circ h(x) \tag{2.36}$$

If h *is a diffeomorphism then* h *is said to be a* ***diffeomorphic conjugacy****.*

Theorem 2.4.2. *Let* $\psi_k(x)$ *be an eigenfunction of* $U^t_{\mathbf{G}}$ *or* U_T *at eigenvalue* $e^{\lambda_k t}$ *or* λ_k*. Then the function* $\psi_k \circ h(x)$ *is also an eigenfunction of* $U^t_{\mathbf{F}}$ *or* U_S *at eigenvalue* $e^{\lambda_k t}$ *or* λ_k*.*

Proof: See [Mez05].

Theorem 2.4.2 shows that the spectral properties of the induced Koopman operators transform nicely under conjugacy. In the language of definition 2.1.15 one would say that eigenfunctions pull-back to eigenfunctions under a conjugacy.

2.5 Spectral Expansions for Linear Systems With Simple Spectrum

In this section, we summarize the spectral properties of the Koopman group induced by a linear dynamical system. Again, this summary's sole purpose is to motivate and foreshadow some of the results that will appear in chapter 4.

In what follows, we will consider a linear continuous-time dynamical system generated by the vector field $\mathbf{F}(x) = \mathbf{A}x$ or a linear discrete-time dynamical system generated by the transformation $S(x) = \mathbf{A}x$ with $\mathbf{A} \in \mathbb{R}^{n\times n}$. We will assume that the matrix $\mathbf{A}$ has distinct eigenvalues λ_i and a full set of linearly independent right and left eigenvectors. Trivially, every constant function is an eigenfunction of the Koopman group. The next question is: What are the linear eigenfunctions corresponding to a linear dynamical system?

Theorem 2.5.1. *Let λ_i, w_i denote the the eigenvalues and left eigenvectors of the matrix $\mathbf{A}$ corresponding to the linear continuous-time or discrete-time dynamical system.*

1. *for the continuous-time case, the linear functions $\psi_i(x) = \langle x, w_i \rangle$ are linear eigenfunctions of U^t at eigenvalue $e^{\lambda_i t}$.*
2. *for the discrete-time case, the the linear functions $\psi_i(x) = \langle x, w_i \rangle$ are linear eigenfunctions of U at eigenvalue λ_i.*

Proof: See [Mez20b]

Since the matrix $\mathbf{A}$ was assumed to be diagonalizable, the eigenvectors form a basis for $\mathbb{R}^n$. Motivated by this we can expand the state x in the eigenvector basis as follows

$$x = \sum_{i=1}^{n} \langle x, w_i \rangle v_i = \sum_{i=1}^{n} \psi_i(x) v_i \tag{2.37}$$

Theorem 2.5.2. *The time evolution of an arbitrary linear observable $f(x) = \mathbf{C}x$, $\mathbf{C} \in \mathbb{R}^{n \times n}$ under the dynamics $\mathbf{F}(x) = \mathbf{A}x$, is given by the following spectral expansion.*

$$U^t f(x) = \sum_{i=1}^{n} e^{\lambda_i t} \psi_i(x) \mathbf{C} v_i \tag{2.38}$$

Proof: See [Mez20b].

The expansion above provides the evolution of a linear observable in terms of the eigenvalues, eigenfunctions and the Koopman modes, which in this case are $\mathbf{C}v_i$. However, it is worth noting that if the linear observable was instead $g(x) = \mathbf{D}x$ then only the modes would change to $\mathbf{D}v_i$, indicating that the eigenvalues and eigenfunctions are intrinsic to the dynamics while the Koopman modes depend on the modality of observation. In a similar fashion it is also possible to describe the time evolution of a nonlinear real analytic observable under linear or nonlinear dynamics. For the details of such an expansion we refer the reader to [Mez20b].

2.6 Numerical Computation of the Spectrum

As previously mentioned, in the context of modern-day engineering systems, one usually deals with nonlinear and typically high dimensional dynamical systems for which explicit formulas that describe the dynamics are not available. The difficulty in deriving such equations and the ever-growing capabilities of computers, algorithms, and data collection have dramatically shifted researchers' attention towards data-driven techniques. This paradigm shift towards data-driven techniques coupled with algorithms for computing spectral properties of the Koopman operator from data has fueled an explosive revival in research interest into Koopman operator techniques.

The ability to compute the spectrum of the Koopman group of operators directly from real-world data is indeed an appealing feature. With the Koopman mode decomposition in hand, one can decompose the observed quantity into a hierarchy of simpler, yet dynamically important, periodic sub-patterns that describe a complex system's behavior. Specifically, the Koopman modes describe the shape of dynamically important spatiotemporal patterns found within the data, and the eigenvalues describe how these modes evolve (grow or decay) in time. Specifically, the real part of a Koopman eigenvalue yields the growth or decay rate of a mode and measures how long the pattern persists within the data. Similarly, the imaginary part of a Koopman eigenvalue produces the period of oscillation of the mode and measures how frequently the pattern repeats within the data. In systems with a pure discrete spectrum, the Koopman modes allow one to reconstruct and forecast the observed data accurately. At this point, several algorithms exist for computing the spectrum of a dynamical system directly from data. A large portion of these methods belong to the class of algorithms known as dynamic mode decomposition (DMD)[Sch10; AM17; Row+09; WKR15; Tu+14; Kut+16; Li+17; JSN14] and the applications range over a wide span of systems [PRR20].

2.6.1 The Hankel Dynamic Mode Decomposition Algorithm

In this dissertation, we have made no contributions to developing or improving any of the numerical algorithms that exist. For this reason, a complete account of all the various types of algorithms is not the focus of this chapter. However, in the next chapter, we will utilize the Hankel dynamic mode decomposition algorithm (Hankel-DMD)[AM17] to approximate the Koopman mode decomposition of highway traffic data. For this reason, we believe it necessary to summarize at least the specific procedure we have utilized.

Formally, we assume to have a time-ordered data matrix $\mathbf{X}$, which contains a total of m data vectors. Typically, the velocity or density profile along the highway, at an instant in time i, constitutes a single data vector labeled $\mathbf{x}_i$ and corresponds to the i'th column of the data matrix $\mathbf{X}$. The number of rows in $\mathbf{X}$, labeled by k, is dictated by the number of locations along the highway at which the velocity or density is measured (number of sensors). Hence, the data matrix has the following form shown below.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_m \end{bmatrix} \tag{2.39}$$

Again, in equation (2.39) m is the number of data snapshots acquired, $\mathbf{x}_i \in \mathbb{R}^k$, $i \in \{1, \dots, m\}$ is a single data vector at time i and $\mathbf{X}$ is a $k \times m$ matrix.

In our work we begin by computing the mean subtracted data matrix $\hat{\mathbf{X}}$ according to equation (2.40) shown below.

$$\hat{\mathbf{X}} = \begin{bmatrix} \mathbf{x}_1 - \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i & \mathbf{x}_2 - \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i & \dots & \mathbf{x}_m - \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i \end{bmatrix} \tag{2.40}$$

It is clear to see that $\hat{\mathbf{X}}$ is obtained by computing the time averages of the original data and subtracting the computed average from the data. This is motivated by the fact that time averages of the system are in the spectrum of the Koopman group and so we pre-compute this quantity. The mean-subtracted data matrix is then fed to the Hankel-DMD algorithm[AM17] which is a combination of a time-delay (Hankel matrix) embedding that is followed by an exact dynamic mode decomposition algorithm [Tu+14] (Exact-DMD). The method of delay embedding is a state-space reconstruction technique that has been shown to recover the attractor of the original dynamical system generating the data [Tak81; Rob08; DS11].

A delay embedding $d \in \mathcal{N}$ is then chosen and the mean subtracted data is embedded as shown below in equation (2.41).

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_m \end{bmatrix} \longrightarrow \mathbf{H} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_{m-d} \\ \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \cdots & \mathbf{x}_{m-d+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_d & \mathbf{x}_{d+1} & \mathbf{x}_{d+2} & \cdots & \mathbf{x}_m \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1 & \cdots & \mathbf{h}_l \end{bmatrix} \tag{2.41}$$

Now, what the exact Exact-DMD algorithm seeks to approximate is a finite dimensional representation of the Koopman operator which must satisfy the following relation shown below in (2.42).

$$\mathbf{H}_2 = \mathbf{K}\mathbf{H}_1 + \mathbf{r} \tag{2.42}$$

Where $\mathbf{K}$ is a finite matrix representation of the Koopman operator and $\mathbf{r}$ is a residual error term due to the fact that we only have a finite-dimensional approximation of a possibly infinite expansion. $\mathbf{H}_1$ and $\mathbf{H}_2$ are the time-shifted matrices as shown below.

$$\mathbf{H}_1 = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{l-1} \end{bmatrix} \tag{2.43a}$$

$$\mathbf{H}_2 = \begin{bmatrix} \mathbf{h}_2 & \mathbf{h}_3 & \dots & \mathbf{h}_{l} \end{bmatrix} \tag{2.43b}$$

The Exact-DMD obtains this approximation by minimizing the residual term in a least squares sense. By utilizing the singular value decomposition (SVD) of $\mathbf{H}_1 = \mathbf{U\Sigma W}^*$ we can rewrite (2.42) as shown below.

$$\mathbf{H}_2 = \mathbf{K}\mathbf{H}_1 = \mathbf{KU\Sigma W}^* + \mathbf{r} \tag{2.44}$$

Multiplying both sides of (2.44) with $\mathbf{U}^*$, and recalling that minimizing the residual term requires $\mathbf{r}$ to be orthogonal to $\mathbf{U}$ we obtain the following expression.

$$\mathbf{U}^*\mathbf{H}_2 = \mathbf{U}^*\mathbf{KU\Sigma W}^* + \mathbf{U}^*\mathbf{r} = \mathbf{U}^*\mathbf{KU\Sigma W}^* \tag{2.45}$$

By rearranging equation (2.45) we obtain a matrix $\mathbf{S}$ that is related to $\mathbf{K}$ via a similarity transformation.

$$\mathbf{U}^*\mathbf{H}_2\mathbf{W\Sigma}^{-1} = \mathbf{U}^*\mathbf{KU} \equiv \mathbf{S} \tag{2.46}$$

Since $\mathbf{K}$ and $\mathbf{S}$ are related they share common eigenvalues and the eigenvectors are the same up to the similarity transformation. Hence, if $(\lambda_i, \mathbf{w}_i)$ are an eigen-pair of $\mathbf{S}$ then $(\lambda_i, \mathbf{v} = \mathbf{U}\mathbf{w}_i)$ is an eigen-pair of $\mathbf{K}$.

Furthermore, since the sampled data produced a discrete-time description of an originally continuous-time process, the eigenvalues $\{\lambda_i\}$ we obtained lie on the unit circle. Therefore, the continuous-time eigenvalues can be recovered by $\omega_i = \frac{ln(\lambda_i)}{T}$, where T is the sampling rate at which the data was collected. Finally, we can use the numerically approximated KMD to obtain a description of the observed data points $\mathbf{x}_i$ via the following expression.

$$\mathbf{x}_{kmd}(t) = \sum_{i=1}^{l} b_0 v_i e^{\omega_i t} = \mathbf{V}\mathbf{e}^{\omega t}\mathbf{b}_0 \tag{2.47}$$

Where $\mathbf{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ is a matrix whose columns are the eigenvectors $\mathbf{v}_i$ and $\mathbf{b}_0$ is an initial amplitude coefficient associated with the initial data snapshot $\mathbf{x}_1$, specifically $\mathbf{b}_0 = \mathbf{V}^\dagger \mathbf{x}_1$. Where $\dagger$ represents the Moore-Penrose pseudoinverse of a matrix and $\mathbf{e}^{\omega t}$ represents a diagonal matrix whose elements are $e^{\omega_i t}$.

A detailed discussion on how we select the number of delays can be referenced in the online methods section. Pseudocode of the Hankel-DMD algorithm can be referenced in supplemental algorithm 1 and the corresponding source code is made available according to the code availability statement.

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Chapter 3

Applications of Koopman Operator Methods to Highway Traffic Dynamics

Highway traffic congestion in 2013 cost Americans $124 billion in direct and indirect losses [TC14]. This number is higher for some European countries [TC14] and is only expected to rise without the development of intelligent transportation systems (ITS) and the accurate forecasting of traffic conditions that ITS rely on to mitigate traffic. Traditionally, the analyzing and forecasting of highway traffic was performed via simulations of mathematical models [VKG14; Dag95]. However, the combination of data availability, modern processing capabilities, and development of machine learning (ML) algorithms has enabled an enormous amount of research into empirical data-driven algorithms [ZZH14; Zhu+18; WHL04; OS84; VDW96; Sun+03; IA02; Wu+18; Cla03; Tak+]. The first class of data-driven algorithms were primarily parametric models that rely on the user's ability to accurately estimating the model's parameters. Historically, this point of view led to the use of linear and nonlinear regression techniques [WHL04; Sun+03], Kalman filtering [OS84] and time-series models [VDW96; IA02]. Recently there has been a large and growing interest in non-parametric models that rely on historical training data to estimate their own parameters. Among the most popular of these methods include the neural and deep neural network models (NN,DNN) [SHG12; Wu+18; Ngu+18; Gir+10], K-nearest neighbors [Cla03; Tak+], Bayesian networks [Zhu+16] and (parametric/non-parametric) hybrid models [ZZH14]. Nevertheless, describing and forecasting the time evolution of traffic systems remains a challenging problem [VKG14; BAR15; NN05; SH07; Oh+18; BS16; JS18; Com+13].

A majority of the published literature in this field has focused on testing and validating a particular method's ability to describe real-world traffic data accurately. However, the issue discussed much less than accuracy is that of a model or algorithm's capability of generalizing to a real-world implementation [VKG14; Oh+18]. Many of the traditional mathematical models are known to be unfeasible for real-time implementation, tedious to solve numerically, and depend on parameter accuracy [VKG14]. The modern ML-based methods also rely heavily on accurate parameters. They typically require large amounts of training data [ZZH14; Zhu+18; Wu+18; Oh+18], which is usually limited and costly to collect [NN05; Oh+18]. Therefore, even if the state-of-the-art traffic models were accurate, typically, they would require an unrealistic amount of data collection and parameter tuning to function across differing highways [ZBT08; Zhu+18; Oh+18]. The first attempts at empirically characterizing highway traffic's country-specific differences can be found in [ZBT08], where traffic data from the United States, United Kingdom, and Germany were empirically analyzed and compared. This international comparison was motivated because different countries have different infrastructure, vehicle class mix, driving rules, and even different driver behavior. Indeed, the works of [ZBT08] confirm key differences in the periods of oscillation and speeds of propagation of traffic jams between the three countries. It is further stated how this country-specific dynamics of traffic will require the re-calibration of current models or the development of more general models. The findings of [ZBT08] validate our view, in that some of the shortcomings of previous research approaches are not primarily their lack of accuracy but more so their heavy dependence on parameters and large amounts of training data. This renders many state-of-the-art techniques developed today unfeasible for a large scale global implementation across differing highways [Oh+18].

In addition to traffic data's stochastic features, wave-like patterns have also been identified within traffic data [NN05; Dag95; SH07; Ahn05; AC07; CM01; ALC10; TBO12]. The exact cause of such traffic waves is still an open topic, although several mechanisms have been proposed [Dag02; Wil08; SH07]. A common theme across many of these proposed mechanisms is the effect that lane changing can have on a highway system. The empirical works of Ahn [AC07; ALC10] and Laval [LD06] provide evidence showing that lane changing maneuvers are critical in the development of traffic waves. Unfortunately, research into multi-lane traffic dynamics has proven to be an even more challenging task [MR13; AC07; Dag02; Wil08]. The complex lane changing dynamics and human interaction within a multi-lane highway have restricted many state-of-the-art techniques to analyze and forecast traffic at the highway corridor scale. Furthermore, generalizing these techniques to the multi-lane scenario is typically difficult [MR13; Dag02; VKG14; Wil08]. Ultimately, traffic management is generally applied at the network level [Oh+18]. However, an accurate and efficient method for the analysis and forecasting of multi-lane highway network conditions is perhaps the most challenging and strongly lacking component of modern ITS [VKG14; SHG12; BS16; JS18; Com+13]. Furthermore, many state-of-the-art techniques frequently require extensive parameter tuning and the proper pre-processing of raw data to perform adequately [Oh+18]. This has lead to the common practice of removing previously computed seasonal averages, aggregating, and smoothing raw data [SH07; TH03; VKG14; Che+01]. Additionally, the differing dynamics between weekday, weekend, holiday, and adverse weather conditioned traffic has led to the common practice of utilizing case-specific training data to forecast only case-specific data [Che+12; Oh+18;

ZZH14; VK12]. Lastly, the challenge in forecasting multiple detector data typically results in verifying methods over only a single or possibly few detectors [Wu+18; Che+12; ZZH14; TBO12]. Therefore, many state-of-the-art benchmarks have been obtained at the highway corridor (single lane) level, over a limited number of sensor locations, and incapable of generalizing to handle the multi-lane network scenario without extensive re-training.

Overall, a systematic and accurate method for identifying, analyzing, and forecasting spatiotemporal traffic features from data is still an open and challenging issue [VKG14; BAR15; NN05; SH07]. In this work, we demonstrate how the Koopman operator's spectral properties, specifically the Koopman mode decomposition (KMD), can offer a model-free, parameter-free, data-driven approach for accurately identifying, analyzing, and forecasting spatiotemporal traffic patterns. The methods we develop allow one to distinguish any growing or decaying phenomena and obtain a hierarchy of coherent spatiotemporal patterns hidden within the data. Furthermore, the forecasting scheme we propose readily generalizes to the much-needed scenario of multi-lane highway networks without any loss to its performance or efficiency. We do not rely on large historical training data, nor do we distinguish between weekday, weekend, holiday, or adverse weather conditions. Our method's performance does not rely on parameter tuning or selection. Thereby providing a completely efficient and accurate method of analyzing and forecasting traffic patterns at the levels required by modern ITS.

3.1 Koopman Mode Analysis of Spatiotemporal Highway Traffic Data

We begin by studying the Next Generation Simulation (NGSIM) data set collected by the US Federal Highway Administration. The NGSIM data set provides the precise location of every vehicle, its lane position, and location relative to other vehicles for every one-tenth of a second on 2100ft and 1640ft segments of the southbound US-101 and eastbound I-80 highways, respectively. Overall, the NGSIM data provides a microscopic description of traffic in that it is the individual vehicles that are tracked and not the velocity or density of the bulk, macroscopic flow. However, in this work, we are interested in identifying macroscopic spatiotemporal patterns. Therefore, the we convert the NGSIM trajectory data into spatiotemporal data via the binning method developed by [Edi63] and utilized by [Jin10; Bel+15]. This procedure allows the construction of macroscopic velocity and density profiles from vehicle trajectory data. The resulting spatiotemporal data is a matrix whose columns correspond to time. Its rows correspond to a position along the highway, and the entries contain the velocity, density, or flow at that location and time. The resulting spatiotemporal data for the US-101 highway is shown below in figure 3.1, and the I-80 highway data can be referenced in supplementary figures 1-2. A more detailed discussion on the binning method and formulas can be referenced in the online methods section, and access to the spatiotemporal data is also made available.

In this work, we categorize traffic patterns according to [SH07]. In addition to the well known free-flowing and congested traffic states, some of the various patterns identified by [SH07] are the pinned localized cluster (PLC), moving localized cluster (MLC), stop and go waves (SGW), and oscillating congested traffic (OCT). PLC type traffic oscillations do not propagate along the highway but are instead pinned or localized at a specific spatial location. On the other hand, MLC type phenomena, also called traffic jams, propagate backward along the highway, affect the entire highways, and their amplitudes are not perturbed by on or off-ramps. The SGW and the OCT, according to [SH07] are almost indistinguishable without the proper data filtering technique, and thus, in this work, we refer to both as SGW or traffic waves. The presence of such patterns for the US-101 highway data can be seen below in figure 3.1a.

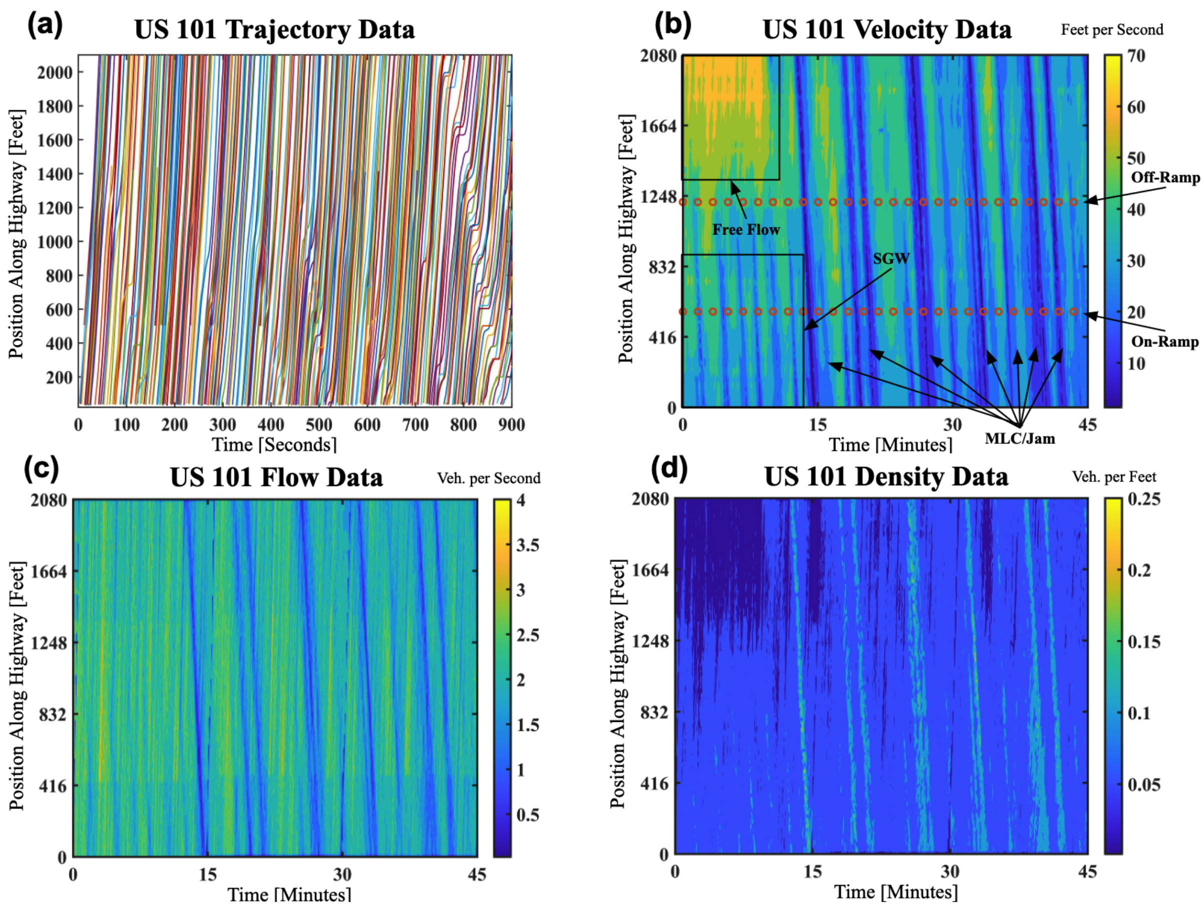

Figure 3.1: (3.1a) Plot of the trajectory data for the first fifteen minutes. The data was collected between the hours of 7:50 am-8:35 am, during the onset of congestion. The section of the highway studied consists of five main lanes, a single on and off-ramp, and an auxiliary lane between the on and off-ramp. Every colored line corresponds to a unique vehicle. (3.1b) Spatiotemporal velocity data. The locations of the ramps have been labeled with dark-orange dotted lines. During the first twelve minutes, the post-off-ramp section of the highway experiences a period of free-flowing traffic. However, during this same time, the highway's mid and pre-on-ramp sections are experiencing stop and go wave traffic, labeled as SGW. During the last thirty minutes of the study, the highway experiences a series of moving localized clusters labeled ML, which correspond to traffic jams. (3.1c) Spatiotemporal flow data. The flow data is obtained as the product of the velocity and density data sets. (3.1d) As expected, the density appears to be the inverse of the velocity profile. Specifically, one can observe that periods corresponding to free-flowing traffic have smaller density and periods corresponding to traffic jams resulting from high density. The source data underlying figures 3.1b-d are provided in the Source Data file.

By applying a KMD to the velocity data in figure 3.1, we seek to uncover traffic patterns that may be hidden within the data. Patterns revealed by the KMD for the US-101 highway density and flow and the I-80 highway can be referenced in supplementary figures 4-12 at `https://www.nature.com/articles/s41467-020-15582-5`. In our works, we sort the resulting Koopman modes according to their period of oscillation. Therefore, the slowest evolving pattern is what we refer to as the "first" mode and so on. A table listing the exact periods of oscillation of the modes discussed can be referenced in supplementary table 1.

By plotting some of the leading Koopman modes in figure 3.2-3.3, we find, that the first three modes (figure 3.2a-3.2c), modes five (figure 3.2e), ten (figure 3.3d), eleven, eighteen, nineteen (supplementary figure 3a,3d,3e), and thirteen (figure 3.2e) all share the common structure of a PLC. Specifically, their amplitude is entirely localized around the post-off-ramp (1280ft-2100ft) section of the highway. However, mode five is also spatially localized about the mid-ramp section of the highway. The double-peaked structure of mode five strongly resembles a sort of spatial harmonic feature of modes one through three. Interestingly, mode five along with modes eight (figure 3.3b), nine (figure 3.3c), and thirteen (figure 3.3e) display a standing wave node at precisely the on and off-ramp locations, which have been labeled with dark orange dotted lines. Modes sixteen (figure 3.3f), twenty, twenty-one, twenty-five, and twenty-eight (supplementary figures 3f,4a and supplementary figures 4d,4f) differ from the other PLC waves in that their amplitudes appear to grow or decay in time. The ability to uncover such growing and decaying patterns is a strongly distinguishing feature between the KMD and a Fourier analysis. It is also clear to see how the amplitudes of mode seven (figure 3.3a) and twenty-six (supplementary figure 4e) are unperturbed as they travel along the highway, indicating that they correspond to traffic jams that affect the entire highway as they propagate by. Several modes within figures 3.2-3.3 are harmonics of the first mode, which corresponds well with the known fact that harmonics of eigenvalues are also eigenvalues of the Koopman operator [MB04; Mez05; Mez20]. A complete list containing the periods of oscillation of the modes we discussed can be referenced in supplementary table 1 .

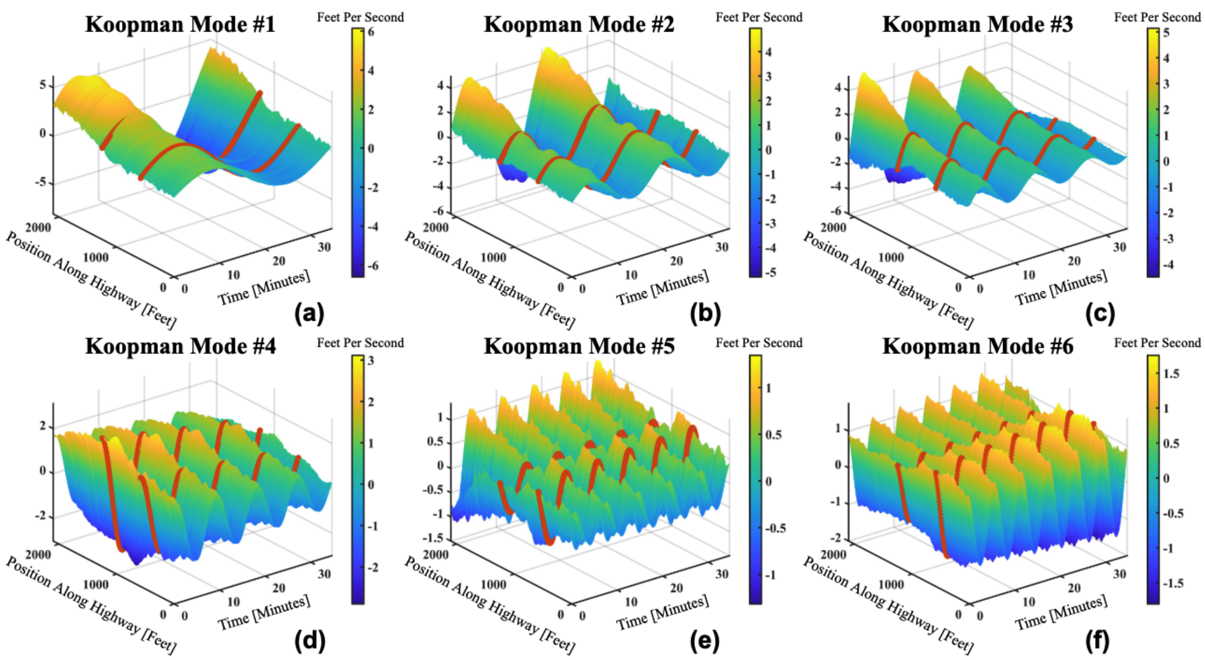

Figure 3.2: The on/off-ramp locations have been labeled with dark orange dotted lines. Modes one to three have a very localized structure near the post-off-ramp section, indicating that they correspond to pinned localized clusters (PLC). Furthermore, the first three modes capture the general transition from high to low velocities during the onset of traffic. Mode 5 seems to be a spatial harmonic of the first three modes in that it has another peaked structure in the mid-ramp section of the highway. Modes four and six provide clear evidence for the pumping effect, where an apparent increase in amplitude followed by a decrease can be seen in these modes as they propagate past the off and on-ramps, respectively. Overall, the Koopman modes uncover complex spatiotemporal wave structures that are hidden within traffic data. Furthermore, according to the imaginary part of its corresponding eigenvalue, every mode oscillates with a single known frequency. This can be contrasted to a Fourier analysis that would yield modes and frequencies specific to the highway positions. A complete list containing the periods of oscillation of the modes we discussed can be referenced in supplementary table 1.

We now demonstrate how the patterns we identify relate to previous research efforts and offer new insight. The empirical findings of Ahn [Ahn05] demonstrate how the amplitude of a traffic wave decreases when it propagates upstream past an on-ramp. Similarly, it was postulated that the amplitude should increase when propagating past an off-ramp. However, no validation for the off-ramp scenario is found by [Ahn05; ZBT08]. This phenomenon was referred to as the "pumping effect" [ZBT08; Ahn05; AC07]. Evidence of this effect is clearly displayed by figures 3.2d, 3.2f and 3.3c. However, figure 3.3b seems to display a decrease in amplitude followed by another decrease when propagating past the off and on-ramp, respectively. This phenomenon, to the author's knowledge, has not been reported by other empirical studies. Furthermore, analyzing time-series data from multiple sensors across large distances is regarded as a more challenging problem than a single or local group of sensor data [Che+12]. The works of [Che+12] confirm that similar frequencies are usually detected across nearby sensors, and remote sensors usually detect differing frequencies. It is believed differences in frequencies across distant sensors is due to the effect that on and off-ramps have on the volume of cars that flow by a specific group of detectors [ZZH14]. We emphasize that the Koopman modes we obtain disprove this notion by uncovering patterns defined across all detector locations yet oscillate with a single frequency regardless of the presence of on and off-ramps.

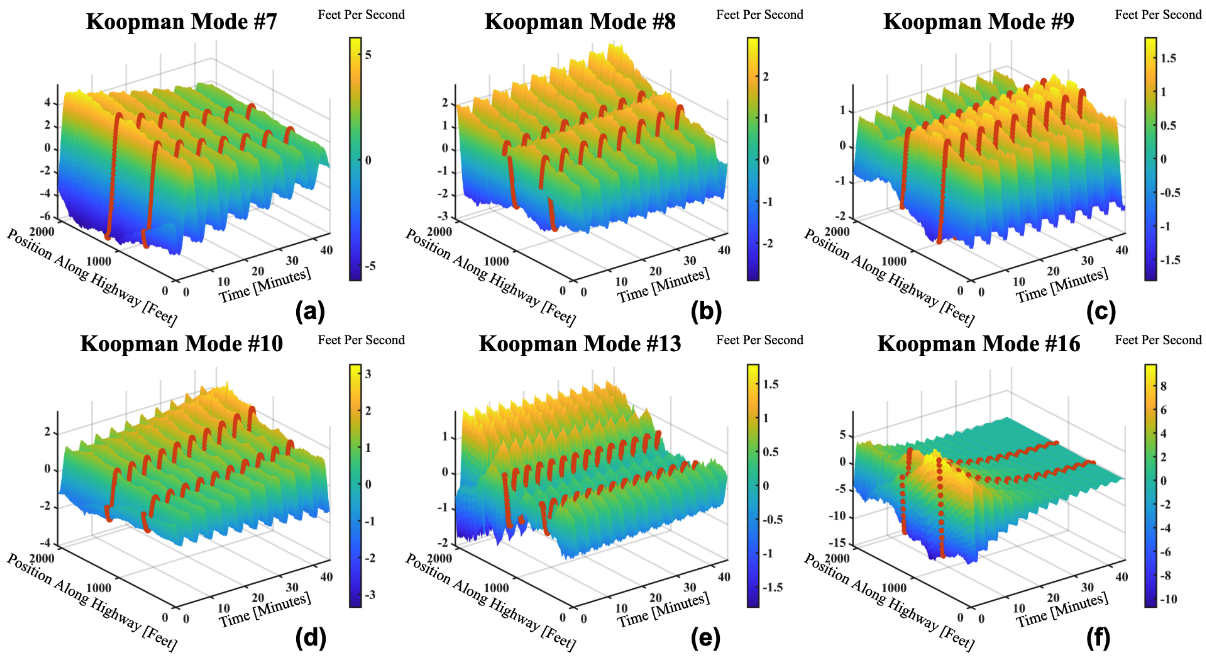

Figure 3.3: The on/off-ramp locations have been labeled with dark orange dotted lines. Mode seven propagates across the entire highway without disturbances to its amplitude and thus corresponds to a highway wide traffic jam, also known as a moving localized cluster (MLC). Mode nine provides further evidence for the pumping effect, where an apparent increase in amplitude followed by a decrease can be seen as the mode propagate past the off and on-ramps, respectively. However, mode eight seems to decrease in amplitude followed by another decrease when propagating past the off and on-ramp, respectively. This phenomenon, to the author's knowledge, has not been reported by other empirical studies. Modes ten and sixteen demonstrate our method's ability to uncover growing or decaying patterns. Specifically, mode sixteen appears to contain its amplitude almost entirely during the first ten to fifteen minutes and is concentrated in the highway's pre-on-ramp location. This indicates that mode sixteen corresponds to the stop and go waves (SGW) present during the exact same region of the spatiotemporal data in figure 3.1a. The exact growth or decay rate of the mode is dictated by the real part of its corresponding eigenvalue. This again is a distinguishing feature of our methodology from a Fourier analysis in that Fourier modes do not capture growing or decaying features. Lastly, mode thirteen also demonstrates a double-peaked structure resembling a spatial harmonic feature of figures 3.2a-c. A complete list containing the periods of oscillation of the modes we discussed can be referenced in supplementary table 1.

The works of Kim [TM04] postulate that high-frequency oscillations are more likely to decay in time. Evidence for such a phenomenon can be found by plotting the eigenvalues over the unit circle in the complex plane (figure 3.4a). Eigenvalues whose magnitudes are within a very narrow threshold (.001) of the unit circle are labeled neutral and correspond to persistent sub-patterns that neither decay nor grow in time. Likewise, eigenvalues with a magnitude larger (or smaller) than this threshold fall outside (or inside) the unit circle and have been labeled as unstable (or stable). What can be seen from figure 3.4a is a cluster of neutral eigenvalues on the far right side (within the dark-grey box) of the unit circle corresponding to the slowest frequencies. This confirms the works of [TM04] in that the slowest evolving patterns persist in time. Furthermore, the works of Gartner [NMR01] find that larger amplitudes typically accompany patterns associated with longer periods of oscillation. We find evidence for this trend by plotting each Koopman mode's average amplitude against its period (figure 3.4b). Indeed, one can clearly see a drop in amplitude for decreasing periods of oscillation.

In addition to the daily and weekly cycles, the works of Dendrinos [Den10] demonstrate the existence of intra-day (less than 24 hours) patterns [Den10]. Plotting the oscillation periods for the first fifteen modes (figure 3.4c) clearly verifies this phenomenon. Further evidence of the existence of intra-day as well as intra-week patterns can be referenced in supplementary figures 13-15. Furthermore, the periods of oscillation we identify are stable across various observation choices such as velocity, density, flow, or a concatenation. Lastly, we demonstrate that the modes we recover are indeed physically relevant to the dynamics by plotting the modes corresponding to the stable, unstable, and neutral eigenvalues separately (figures 3.4d-f). We then superimpose them (figures 3.4g-h) along with the previously removed average (figure 3.4i) to reconstruct the original data. This demonstrates that the modes we have uncovered are dynamically important sub-patterns and reconstruct the data when superimposed together.

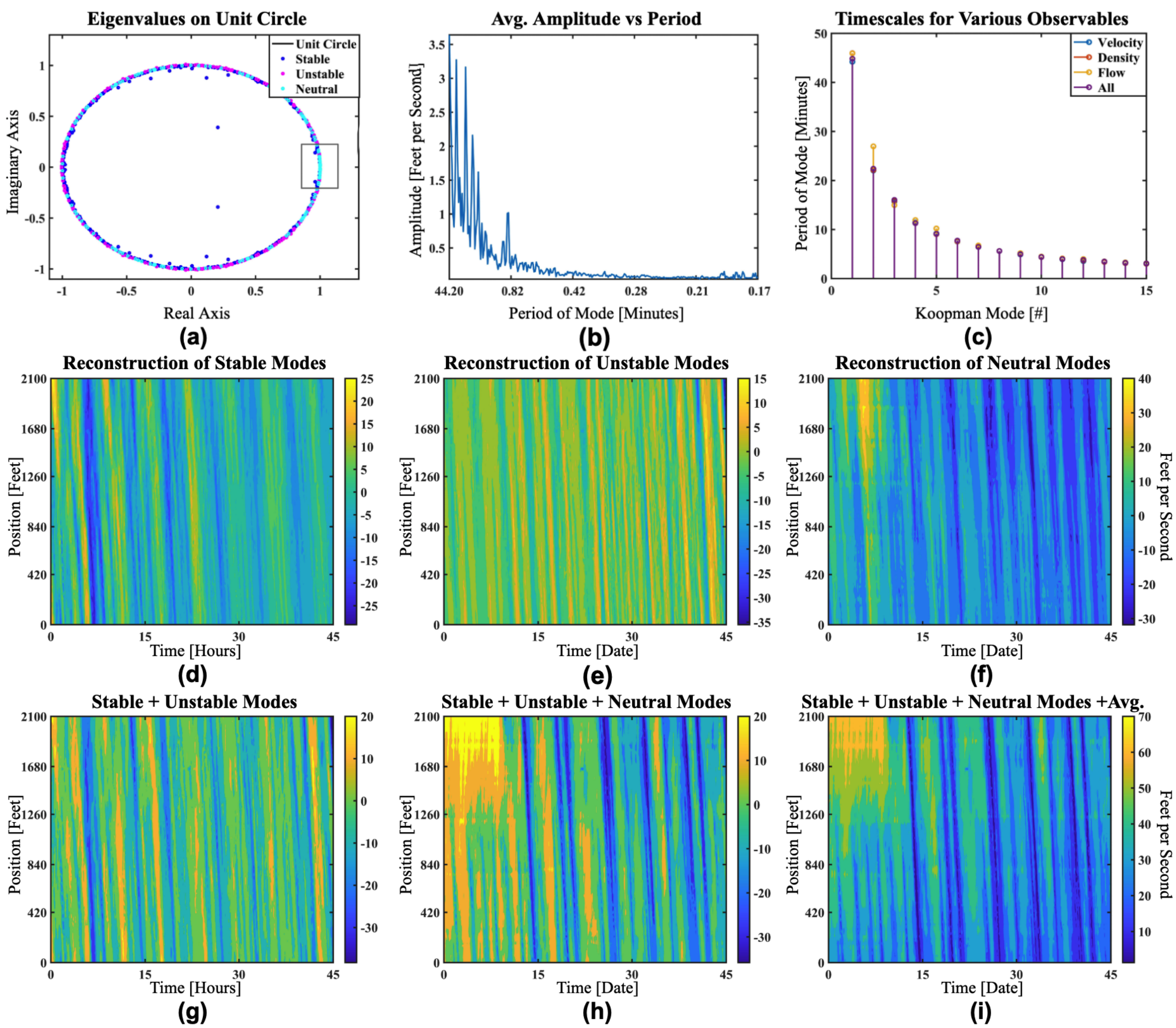

Figure 3.4: (3.4a) Plotting the color-coded eigenvalues on the unit circle confirms the findings of Kim [TM04] in that slowly evolving patterns persist in time. (3.4b) Plotting the average amplitude of the modes against their period confirms the findings of Gartner [NMR01] in that slower evolving patterns carry larger amplitudes. (3.4c) Plotting the timescales identified for various data choices demonstrates the KMD to be a robust methodology for extracting fundamental frequencies regardless of the modality of observation and verify the existence of intra-day patterns. (3.4d-3.4i) Plots of the corresponding stable, unstable, and neutral modes and various superpositions of them. These figures verify that the individual modes correspond to coherent and dynamically important sub-patterns and, when superimposed, reconstruct the data.

3.2 Koopman Mode Analysis of Multi-Lane Highway Traffic Data

We demonstrate how the KMD readily generalizes to the multi-lane scenario by analyzing multi-lane spatiotemporal density data for the US-101 highway. The multi-lane data was generated by binning the individual lanes of the NGSIM data. With the addition of an extra spatial coordinate (Lane #), the Koopman modes are now two-dimensional spatiotemporal patterns and best visualized as a video, which can be referenced in the online supplementary videos 1-14. However, we have also provided figures containing snapshots of the videos over an entire cycle in supplementary figures 16-22.

The first multi-lane mode (supplementary figure 16) is a spatially localized PLC about the post-off-ramp section of the highway similar to the first mode of the corridor-wide (single lane) analysis. The second and the third mode (supplementary figures 17-18) are again harmonics of the first. Interestingly, modes four, five, and ten (supplementary figures 19-21) display a dynamic lane-changing (zig-zag) motion within the mid-ramp section where lane-changing maneuvers are highest due to merging/diverging vehicles. The seventh mode is plotted below in figure 3.5. From top-left (figure 3.5a) to bottom-right (figure 3.5f) one can observe how the seventh mode corresponds to an MLC that affects the entire highway. However, the MLC's travel is out of phase across the different lanes, giving the jam an apparent top-left to bottom-right travel. We encourage the reader to reference the supplementary videos 1-14 online to properly visualize the multi-lane modes.

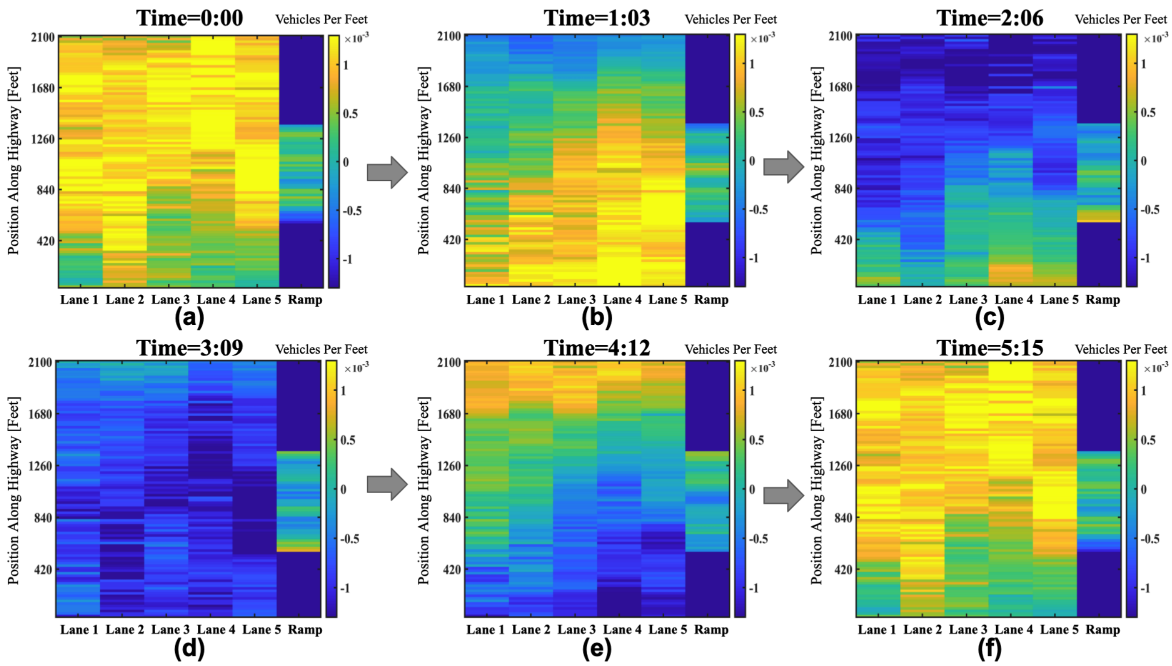

Figure 3.5: The mode has a period of approximately six minutes, and the time for each figure is given in minutes-seconds. Mode seven captures the dynamics of a highway wide traffic jam. It is interesting to note how the moving localized cluster's (MLC) travel can be out of phase across differing lanes. This results in the apparent top-left to a bottom-right direction of travel and is a feature impossible to recover from a single-lane analysis. Lastly, one can observe that the vehicles' on-ramp density is not merging at the time of peak congestion. Specifically, the on-ramp is most heavily congested during figures 3.5c-d at which point the MLC has already propagated by. However, mode fourteen, a harmonic of this mode, clearly displays the opposite effect. This demonstrates how our multi-lane analysis can be utilized to verify the successful timing of static ramp metering algorithms and identify the correct timescales for dynamic ramp metering algorithms. Mode fourteen can be referenced in supplementary figure 22.

Lastly, one can see how the merging on-ramp densities (multiplied by five for visual purposes) and highway densities are out of phase. This indicates the ramp metering's successful timing by verifying that incoming vehicles are not allowed to merge while the highway is jammed. The proper implementation of ramp metering algorithms has been shown to improve traffic congestion significantly, reduce travel times, and reduce accidents between merging and flowing traffic [CMR05; PR95; LHO06]. However, the proper tuning of the control algorithm's parameters and identifying congestion patterns is critical for a successful implementation. Figure 3.5 and supplementary figures 6-12 demonstrate how our analysis can be utilized to identify multi-lane and on-ramp congestion patterns along with their associated timescales. This information can, in turn, be used to verify the proper timing of static ramp meters and incorporated into the development of dynamic ramp metering algorithms.

3.3 Forecasting the California Performance Measurement System

The California Department of Transportation (Caltrans) Performance Measurement System (PeMs) data set is a real-time monitoring system for hundreds of highways across California. This measurement system processes 2GB of real-time data per day and provides access to years of historical data. The historical and real-time nature of the PeMs repository has led to its widespread use for implementing and verifying forecasting methodologies [Che+01; Zhu+18]. In this work, we have implemented a moving horizon Hankel dynamic mode decomposition (MH-HDMD). The algorithm utilizes a subset of s data vectors (sampling window) to forecast the next f data vectors (forecast window) updated every h time steps (horizon window). The specific highways we study are as follows. First, we forecast a week and month's worth of data for one-hundred and three-hundred-mile sections of the eastbound Interstate 10 highway (I-10) and northbound Interstate 5 highway (I-5), respectively. Next, a network of the ten largest highways connecting Los Angeles to the greater southern California area is forecasted for the week of Christmas 12/21-12/26 of 2016. Additionally, we forecast one-hundred miles of the northbound US-101 highway during the deadly [Por17] southern California rainstorm that occurred February 17, 2017. The data for this example was collected from February 16-17, 2017. In all cases, we utilize the last fifteen minutes to forecast for the next fifteen minutes, updating our forecasts every fifteen minutes. The original and forecasted data for the weekly and Christmas holiday network data sets are shown below in figure 3.6. The original and forecasted data for the I-5 and US-101 data sets can be referenced in supplementary figures 23-24.

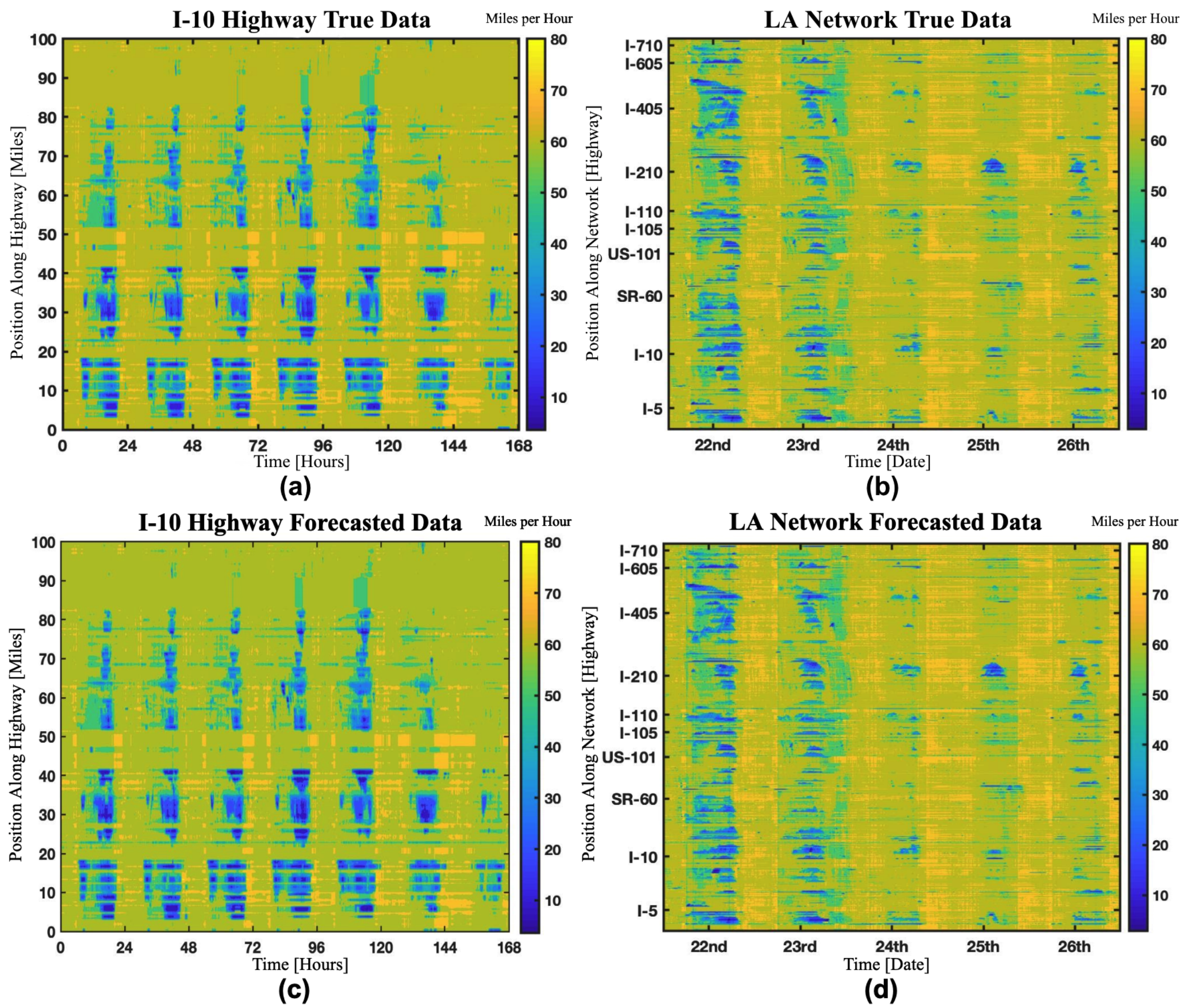

Figure 3.6: (3.6a-b) Raw data collected from PeMs for the I-10E and southern California network. The data was collected for a week and five days, respectively. The I-10 highway seems to be mostly congested throughout the week until Sunday. As expected, all highways within the network seem to be congested on the days leading up to Christmas eve (12/21-12/22). Interestingly, there appears to be drastic relief in congestion during and after the actual holiday dates of 12/24-12/26. (3.6c-d) Forecasted data generated by the MH-HDMD utilizing the last fifteen minutes of data to forecast the next fifteen minutes. The network was forecasted by concatenating data obtained for individual highways and applying the MH-HDMD algorithm to the concatenated data. By visual inspection alone, the raw and forecasted data sets are indistinguishable, indicating an accurate forecast. Nevertheless, there is an error present within our forecasts, which can be referenced in figure 3.7. The source data underlying figures 3.7a-b are provided in the Source Data file.

We quantify the performance of our method by computing the mean absolute error (MAE), mean relative error (MRE), root mean squared error (RMSE), the spatial and temporal averages of the mean absolute error (SMAE, TMAE), and the spatial and temporal correlations (SCorr, TCorr) according to formulas (3.1)-(3.5) described below.

$$\text{MAE} = \frac{1}{N}\sum_{i,j=1}^{n,m} |\mathbf{T}_{i,j} - \mathbf{F}_{i,j}| \tag{3.1}$$

$$\text{MRE} = \frac{1}{N}\sum_{i,j=1}^{n,m} \frac{|\mathbf{T}_{i,j} - \mathbf{F}_{i,j}|}{|\mathbf{T}_{i,j}|} \tag{3.2}$$

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i,j=1}^{n,m} |\mathbf{T}_{i,j} - \mathbf{F}_{i,j}|^2} \tag{3.3}$$

In the above formulas $\mathbf{T}$ is the true data matrix, $\mathbf{F}$ the forecasted data matrix, n is the number of rows in $\mathbf{T}$ (number of sensor locations), m the number of columns in $\mathbf{T}$ (number of time points) and $N = n \cdot m$ is the total number of elements in $\mathbf{T}$. The spatial and temporal averages of the absolute error (SMAE) and (TMAE) as well as the spatial and temporal correlations (SCorr) and (TCorr) are computed according to formulas 3.4-3.5 shown below. Where $\mathbf{E} = |\mathbf{T} - \mathbf{F}|$ is the absolute error matrix. The average value of correlation coefficients across different detectors (rows of $\mathbf{T}$) is used to compute what we refer to as the spatial correlation. The temporal correlation is computed by reshaping (vectorizing) $\mathbf{T}$ and $\mathbf{F}$ into a single vector time series and computing their corresponding correlations.

$$\text{TMAE} = \frac{1}{m}\sum_{j=1}^{m} \mathbf{E}_{i,j}, \;\; \text{SMAE} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}_{i,j} \tag{3.4}$$

$$\text{SCorr} = \frac{1}{n}\sum_{j=1}^{m} Corr(\mathbf{T}_{i,j}, \mathbf{F}_{i,j}), \;\; \text{TCorr} = Corr(\mathbf{T}_{i,j}, \mathbf{F}_{i,j}) \tag{3.5}$$

The SMAE and TMAE for the 1-week I-10E, 1-month I-5N, and holiday network data sets are plotted below in figure 3.7a-3.7f. Our method obtains an average MAE between one to two miles per hour across all detectors, for weeks/months data, across differing highways and network of highways. Furthermore, since highway traffic's unpredictability renders an absolutely perfect forecast impossible, we plot, in figures 3.7g-i, the probability distributions of the original and forecasted velocities. It is clear to see the near-perfect matches between the raw traffic data statistics and our forecasted data statistics. Furthermore, we can match the distribution of higher velocities with much more accuracy than lower velocities. Nevertheless, the subplots of the lower velocity distributions demonstrate that although states of congestion are inherently more unpredictable [Oh+18], our forecasts match the statistics of the data. This indicates that despite the higher variability believed to exist within congested traffic, wave-like patterns account for most of the system dynamics. A similar error analysis for the US-101 highway data set can be referenced in supplementary figure 25. A complete summary of our error analysis for all highways studied can be referenced in supplementary table 1.

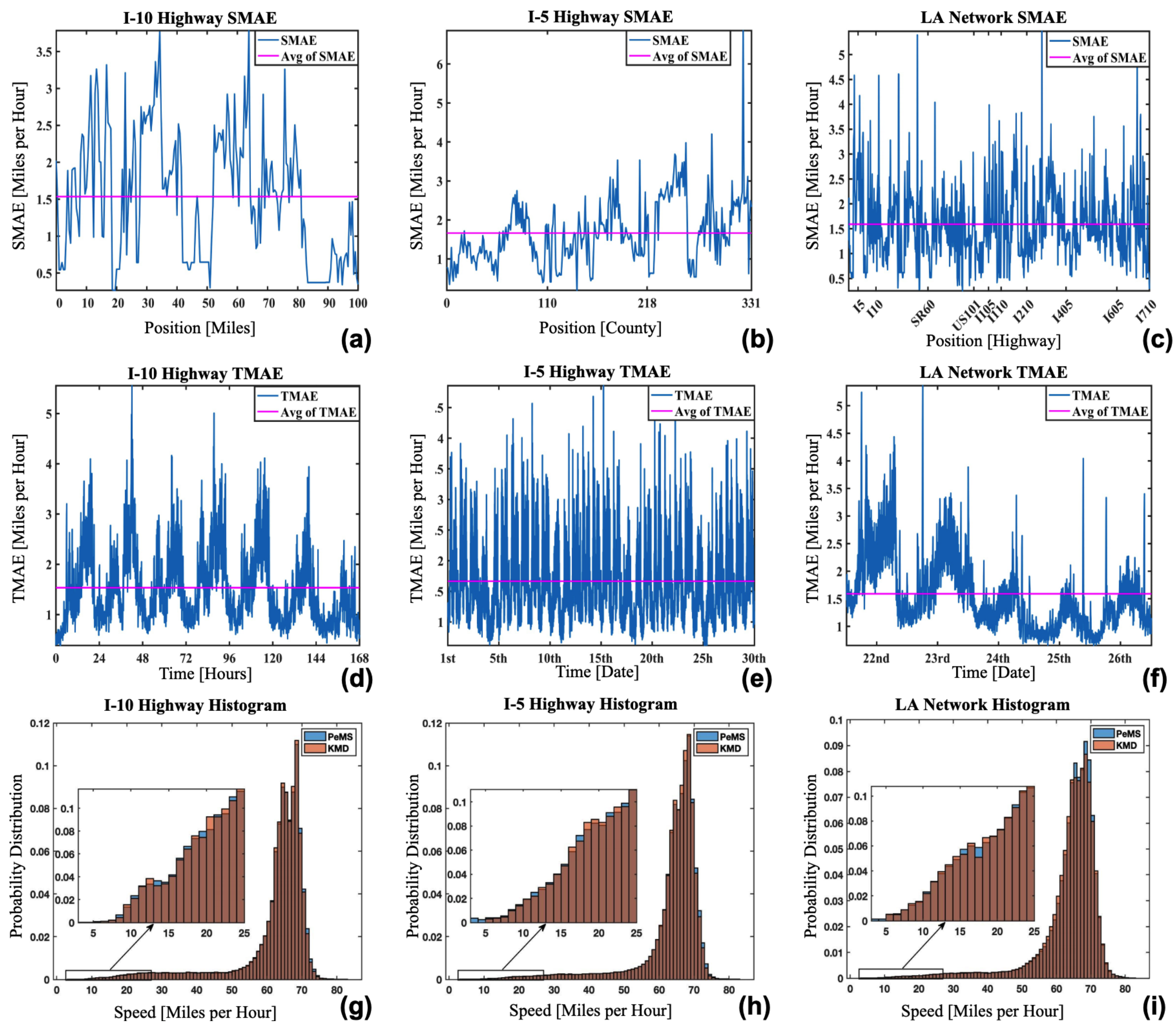

Figure 3.7: (3.7a-3.7c) Spatial MAE across detector locations and its average across all detectors for the I-10, I-5, and Los Angeles network. The SMAE is stable across different highways and networks of highways and is on average between one to two miles per hour. (3.7d-3.7f) Temporal MAE and its time average. Again, it is evident that our method is stable across differing highways and spatial scales and stable across a wide range of time scales (days, week, month). (3.7g-3.7i) Normalized histograms of highway velocities for both true and forecasted data sets. The near-perfect matches for low and high speeds indicate that despite the unpredictability present in a highway system, our forecasts' statistics are nearly identical to the real physical system statistics. A complete summary of the error analysis for all highways can be referenced in supplementary table 2.

We can further investigate how the mean absolute error varies for different choices of (s, f), by simulating for values of sampling and forecasting windows that are multiples of fifteen minutes. For every choice of (s, f, f) we record the MAE and generate a colormap for the I-10 highway (figure 3.8a) and the US-101 (figure 3.8b). Intuitively, our error should increase with longer forecast windows for a fixed sampling rate, a phenomenon previously observed by others [Oh+18]. The counter-intuitive aspect of figures 3.8a-d is that for a fixed forecasting window, increasing the sampling window's size hinders our forecasts. This is best seen by looking across rows and observing how the error increases. The results in figure 3.8 suggest that the accurate forecasting of traffic is dependant on the most current traffic conditions and not necessarily on the historical past. This indicates that costly training over extensive amounts of historical data is unnecessary and may hinder the ability to forecast. Although this is directly contrary to what many researchers believe[Oh+18], it is, in fact, beneficial as it indicates that accurate forecasts can be obtained efficiently with limited data.

3.4 Forecasting and Analysis of Multi-Lane Network Highway Traffic Data

This section demonstrates how the KMD can be utilized to analyze and forecast highway traffic at the multi-lane network scale. We do so by applying the KMD to highway occupancy data for a network of highways within Los Angeles. The highway occupancy is a normalization of the highway density by the maximum density of the highway. We plot below in figure (3.9a) a map, taken from Map data ©2019 Google, of Los Angeles. The highways studied are highlighted, and a plot of the twenty-four-hour Koopman modes (figure 3.9b) along with the average phase and magnitude (figure

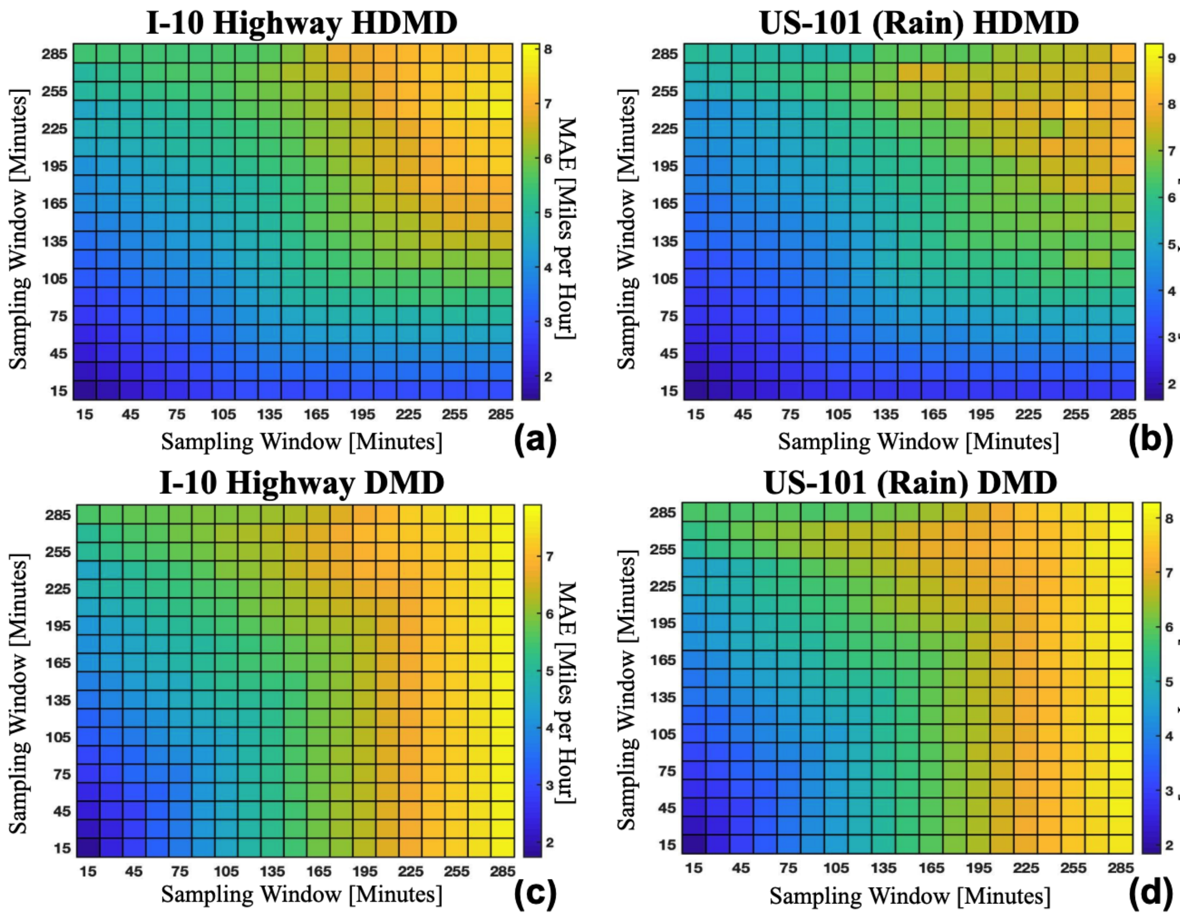

Figure 3.8: (3.8a-b) The top row was generated utilizing the MH-DMD algorithm. (3.8c-d) The bottom row was generated utilizing the DMD algorithm. It is evident that the Hankel-DMD can provide greater accuracy than a standard DMD analysis. This is best seen by observing the increase of blue-colored squares in figures 3.8a-b versus figures 3.8c-d. In both cases, the results of figure 3.8 suggest that traffic forecasting is a temporally local task and that current and future conditions depend very little on the historical past. Furthermore, increasing the amount of historical data utilized can hinder the accuracy of forecasts. This demonstrates the need for intelligent transportation systems to have sufficient analytic capabilities of detecting dynamic traffic patterns in real-time and that training over extensive amounts of historical data is not necessary.

3.9c,3.9d) is also shown. The magenta horizontal lines in figure 3.9d serve to divide the data corresponding to differing highways.

From the twenty-four-hour mode itself, we can immediately observe that the onset of congestion occurs in a specific order within the network. Specifically, the I-105W, I-10W, and I-405N are congested first, then the I-710N followed by the I-110N. One can conclude this by observing that the (green) areas of congestion for every highway are staggered; their staggering order reveals the order in which they are congested. The order of congestion indicated corresponds with the well-known fact that morning traffic travels in the direction of Los Angeles from San Bernardino County (east to west) and Orange County (southeast to northwest). This is further verified by observing the near equal phases for the west and northbound directions coming into LA (figure 3.9c) and the nearly identical, but opposite, phases for the east and southbound directions leaving LA (figure 3.9i).

Furthermore, figure 3.9d displays the highest level of magnitude within the I-10E and I-105E highways, indicating that, on a twenty-four-hour basis, traffic congestion is heaviest along these highways than all others within the network. This is also in line with the fact that the I-10E and I-105E highways connect LA to Orange and San Bernardino County. Interestingly, it is the east directions corresponding to the afternoon rushes that are most heavily congested. This suggests that more vehicles are traveling outside of LA in the afternoon than the original number of morning commuters entering LA.

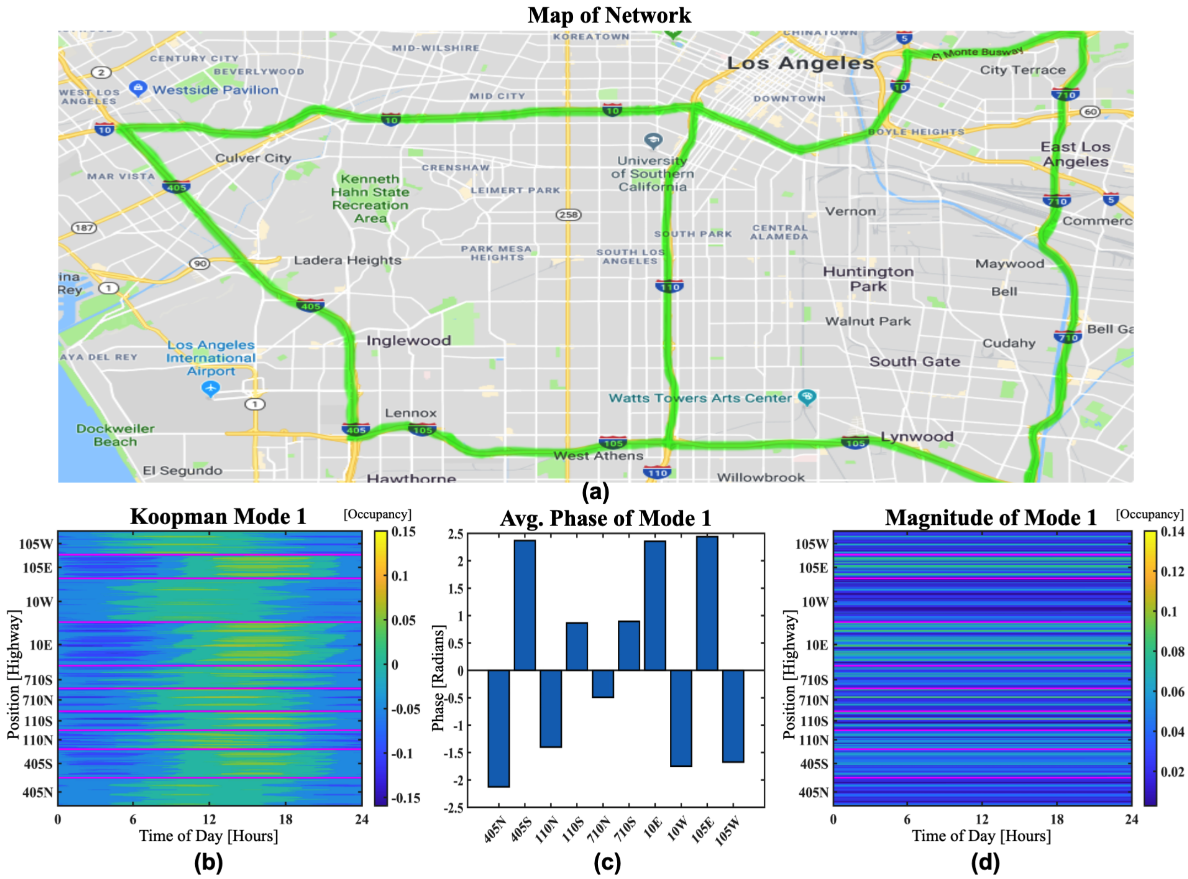

Figure 3.9: (3.9a) Map of the multi-lane network obtained from Map data ©2019 Google. The highways studied are highlighted, and data for three lanes of both north-south and east-west directions are collected for the entire day of December 20th, 2018. (3.9b) The twenty-four-hour Koopman mode reveals the order of congestion within the network. Specifically, by observing the staggering of the amplitude, one can see that during the morning rush, the westbound I-105 and I-10 along with the northbound I-405 and I-110 jam first. In the afternoon, traffic switches directions, and it is the eastbound and southbound directions of the previously mentioned highways which are jammed. This corresponds with the well-known fact the morning commuters generally travel from San Bernardino (east to west) and Orange County (southeast to northwest) into Los Angeles. (3.9c) A plot of the average phase of the mode sorted by highway confirming the previously mentioned synchrony of congestion. Specifically, the north and west directions seem to be in phase with each other and likewise for the south and east directions. (3.9d) The magnitude of the twenty-four-hour mode along with magenta-colored lines used to divide the differing highways. The magnitude of the mode reveals that the eastbound I-10 and I-105 highways are more occupied than the other highways.

Lastly, we forecast the highway occupancy data for the above-mentioned multi-lane network of highways within Los Angeles. Again, we utilize the last fifteen minutes to forecast the next fifteen minutes and visualize our results as a video, which can be referenced in the supplementary video 15. We plot below snapshots from the multi-lane network forecast video for the morning (figure 3.10a-b) and afternoon (figure 3.10c-d) rush hours. The forecasted and real traffic conditions at 5:45 am, and 6:00 pm are shown. For every plot within figure 3.10, the top and bottom horizontal highways correspond to the (I-10E, I-10W), and (I-105E, I-105W), respectively. The Left, center, and right vertical highways correspond to the (I-405S, I-405N), (US-110S, US-110N), and the (I-710S, I-710N), all of these highways are highlighted within figure 3.9a. Both morning and afternoon forecasts demonstrate a high level of similarity with the real conditions. However, the forecasts were available between five to fifteen minutes before the actual conditions occurring. The corresponding error analysis, shown in figures 3.10e-3.10g, validates that our forecasts remain accurate over the entire day. We encourage the reader to view the whole video of the forecasting results for the Los Angeles multi-lane network, which is available online in supplementary video 15.

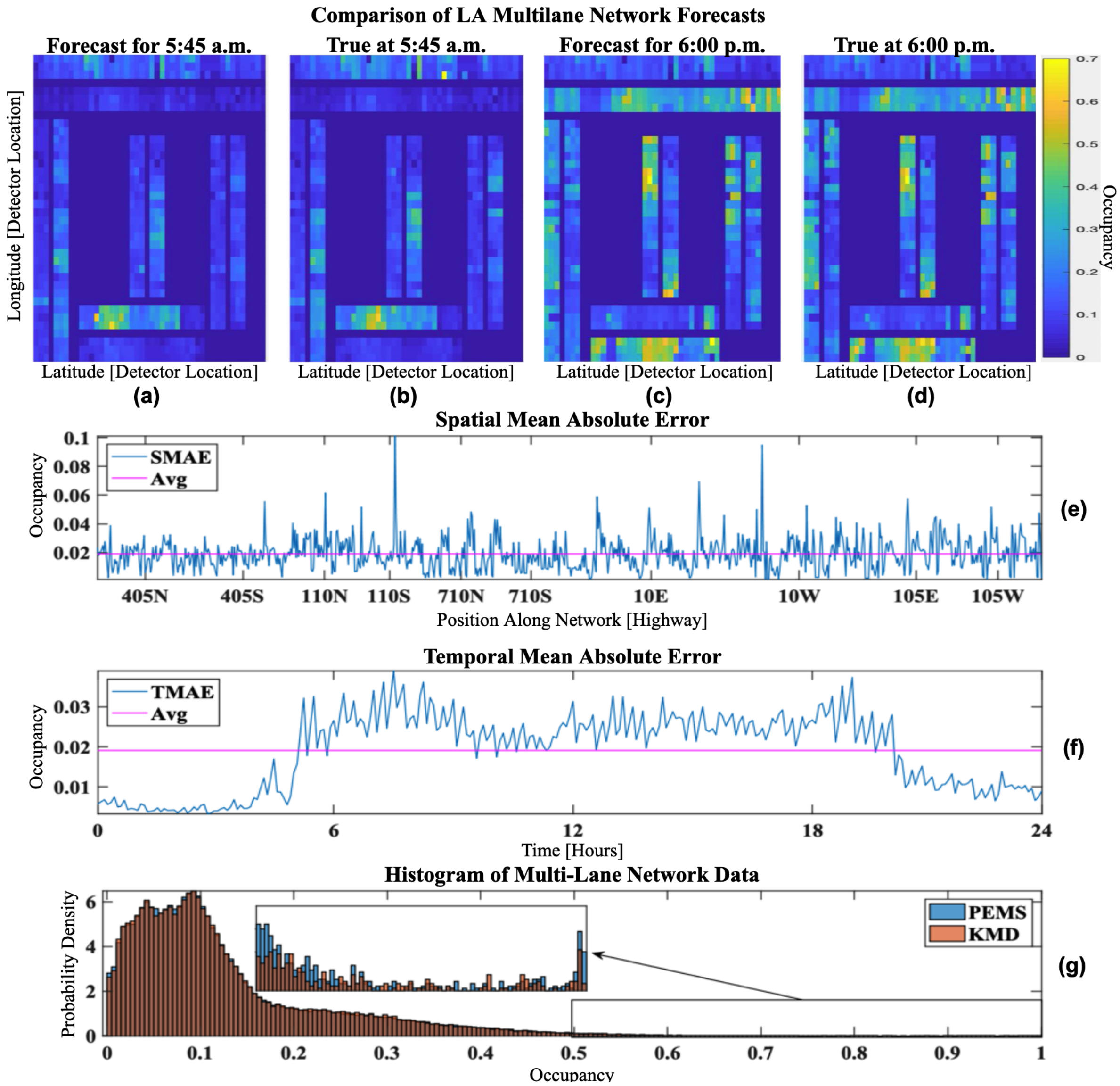

Figure 3.10: For every plot within figure 3.10a-3.10d the top and bottom horizontal highways correspond to the I-10, and I-105, respectively. The Left, center, and right vertical highways correspond to the I-405, I-110, and I-710. (3.10e and 3.10f) Plots of the spatial and temporal mean absolute errors indicating that, on average, we obtain an error of two percent occupancy. The highway occupancy is a measure corresponding to the highway density's normalization by the maximum density of the highway. (3.10g) Histogram of the raw and forecasted data sets indicating that our forecasts' probability density functions match the statistics of the system. We encourage the reader to reference the complete video of the forecasts available online in supplementary video 15.

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Chapter 4

Pull-Back Operator Methods in Dynamical Systems

In this chapter, we consider the induced group of linear operators acting on the space of sections of the tangent, kth cotangent, and general tensor bundle of the state space. To the author's knowledge, prior results exist primarily for the operators that act on sections of the tangent bundle. In some instances, these results can be related to the equivalent operators acting on differential one-forms by duality. However, the operators acting on higher-order forms or arbitrary tensor fields, to our knowledge, have not been previously investigated. The one exception to this is the Ruelle-Perron-Frobenius operator acting on densities, which is equivalent to the space of generalized top-degree differential forms.

4.1 The Pull-Back Groups of a Dynamical System

In this section, we will convince the reader that the induced group of operators on sections we consider are natural generalizations of the standard Koopman group of operators on functions. The fundamental insight lies in understanding the connection between the differential geometric concepts of pulling back objects (functions, vector fields, covector fields, tensor fields) under a diffeomorphism and how their pull-back relates to the Lie derivative of that object.

4.1.1 The General Setting

We first provide the setting in a general form to illustrate the idea. We begin by collecting some relevant definitions and results.

Vector Bundles

Definition 4.1.1. *Let E and F be vector spaces with U an open subset of E. The Cartesian product $U \times F$ is said to be a* ***local vector bundle****. The space U is said to be the* ***base space****, which can be identified with $U \times \{0\}$ called the* ***zero section****. Furthermore, for $u \in U$, $\{u\} \times F$ is said to be the* ***fiber*** *of u, which inherits the vector space structure of F. Lastly, the map $\pi : U \times F \to U$ given by $\pi(u, f) = u$ is said to be the* ***projection map*** *of $U \times F$.*

Definition 4.1.2. *Let S be a set. A* ***local bundle chart*** *of S is a pair (W, α) where $W \subset S$ and $\alpha : W \to U \times F$ is a bijection onto a local bundle. A* ***vector bundle atlas on*** *S is a family $\mathcal{B} = \{(W_i, \alpha_i)\}$ of local bundle charts satisfying*

1. *$S = \bigcup\{U_i\}$*
2. *$\mathcal{B}$ covers S*
3. *For any two local bundle charts (W_i, α_i) and (W_j, α_j) in $\mathcal{B}$ with $W_i \bigcap W_j \neq \emptyset$, $\alpha_i(W_i \bigcap W_j)$ is a local vector bundle map, and the transition map $\gamma_{ji} = \alpha_j \circ \alpha_i^{-1}$ restricted to $\alpha_i(W_i \bigcap W_j)$ is a C^∞ local vector bundle isomorphism.*

Definition 4.1.3. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two vector bundle atlases on S, they are said to be* ***equivalent*** *if $\mathcal{B}_1 \bigcup \mathcal{B}_2$ is also a vector bundle atlas.*

Definition 4.1.4. *An equivalence class of vector bundle atlases on a set S is said to be a* ***vector bundle structure****.*

Definition 4.1.5. *Let S be a set and $\mathcal{V}$ a vector bundle structure on S. A* ***vector bundle****, denoted by E, is a pair $(S, \mathcal{V})$.*

Definition 4.1.6. *For a vector bundle $E = (S, \mathcal{V})$ the* ***zero section*** *or* ***base*** *is defined by*

$$B = \{e \in E |\ \textit{there exists } (W, \alpha) \in \mathcal{V} \textit{ and } u \in U \textit{ with } e = \alpha^{-1}(u, 0)\}$$

Essentially, B is the union of all the zero sections of the local vector bundles.

Theorem 4.1.1. *Let E be a vector bundle. The base B of E is a submanifold of E there is a surjective and C^∞ map $\pi : E \to B$ said to be the* ***projection map****. For each $b \in B$, $\pi^{-1}(b)$, is said to be the* ***fiber over*** *b, and has a vector space structure.*

Proof: See [AMR83] page 139.

Due to the properties mentioned in theorem 4.1.1 one typically denotes a vector bundle by $\pi : E \to B$ in place of $E = (S, \mathcal{V})$ and if the base and projection map are understood then the vector bundle is simply denoted by E. We will use this convention throughout and also denote the fiber over b as E_b.

Definition 4.1.7. *Let $U \times E$ and $U' \times E'$ be local vector bundles. A map $g : U \times E \to U' \times E'$ is said to be a C^r* ***local vector bundle map*** *if it has the form $g(u, e) = (g_1(u), g_2(u) \cdot e)$ where $g_1 : U \to U'$ and $g_2 : U \to L(E, E')$ are C^r. Where $L(E, E')$ represents the space of linear transformations from E to E'. If g is a bijection then g is said to be a* ***local vector bundle isomorphism****.*

Definition 4.1.8. *Let E and E' be two vector bundles. A map $g : E \to E'$ is said to be a C^r* ***vector bundle mapping*** *if for each $e \in E$ and each local bundle chart (V, ψ) of E' for which $g(e) \in V$ there is a local bundle chart (W, ϕ) with $g(W) \subset V$ such that the $\psi \circ g \circ \phi^{-1}$ is a C^r local vector bundle mapping. If $\psi \circ g \circ \phi$ is a C^r local vector bundle isomorphism then g is said to be a* ***vector bundle isomorphism****.*

Proposition 4.1.1. *Let E and E' be two vector bundles and let $f : E \to E'$ be a vector bundle map. Then*

1. *f preserves the zero section: $f(B) \subset B$.*
2. *f induces a unique mapping $f_B : b \to B'$ such that $\pi' \circ f = f_B \circ \pi$.*
3. *A smooth map $g : E \to E'$ is a vector bundle map if and only if there is a smooth map $g_B : B \to B'$ such that $\pi' \circ g = g_B \circ \pi$ and g restricted to each fiber is a linear continuous map into a fiber.*

Proof: See [AMR83] page 142.

Definition 4.1.9. *A vector bundle map $g : E \to E$ such that $\pi' \circ g = g_B \circ \pi$ is said to* ***cover*** *g_B.*

Definition 4.1.10. *Let E be a vector bundle and $t \in \mathbb{R}$ or $\mathbb{Z}$. A* ***vector bundle flow*** *is a pair one-parameter group $(S^t, \boldsymbol{S}^t)$ for which $\boldsymbol{S}^t$ is a vector bundle map which covers a flow S^t on B for every t.*

Definition 4.1.11. *Let E be a vector bundle and $U \subset B$ an open set. A map $f : U \to E$ such that for each $b \in U$, $\pi \circ f(b) = b$ is said to be a* ***local section*** *of E. If $U = B$ then f is called a* ***global section*** *or simply a* ***section*** *of E.*

Definition 4.1.12. *The vector space of sections, denoted $\mathbf{\Gamma}(E)$ on a vector bundle E will be called the* ***space of sections of*** *E.*

The Group of Operators Induced By a Vector Bundle Flow

Let $(\mathcal{M}, E, \pi)$ denote a continuous complex vector bundle with linear fibers (each fiber is a linear vector space) defined over a compact metric space $\mathcal{M}$. Let $t \in \mathbb{R}$ or $\mathbb{Z}$ and $(S^t, \boldsymbol{S}^t)$ be a vector bundle flow of automorphisms of E. By defining a Finsler metric $|\cdot|$ on E one induces a norm $\|\cdot\|$ on each fiber E_x. In this setting, the space of continuous complex sections $\boldsymbol{\Gamma}(E)$ forms a Banach space with norm $\|\mathbf{F}\| = \sup_{x\in\mathcal{M}}|\mathbf{F}(x)|$ for $\mathbf{F} \in \boldsymbol{\Gamma}(E)$. By compactness of $\mathcal{M}$, the topology of $\boldsymbol{\Gamma}(E)$ is independent of the choice of a specific Finsler metric. The vector bundle flow $(S^t, \boldsymbol{S}^t)$ naturally induces a C_0-group, denoted $\boldsymbol{S}^{t*}$, of bounded linear operators acting on $\boldsymbol{\Gamma}(E)$ defined for $\mathbf{G}(x) \in \boldsymbol{\Gamma}(E)$, as follows:

$$\boldsymbol{S}^{t*}\mathbf{G}(x) = \boldsymbol{S}^{-t} \circ \mathbf{G} \circ S^t(x) \tag{4.1}$$

with $t \in \mathbb{R}$ or $\mathbb{Z}$. When $t \in \mathbb{R}$, the infinitesimal generator of the group, again denoted by $\mathcal{L}$, is given by:

$$\mathcal{L}\mathbf{F} = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{S}^{t*}\mathbf{F} \tag{4.2}$$

In this work we consider the case when the base of E is either a smooth manifold $\mathcal{M}$ or a smooth Riemannian manifold $(\mathcal{M}, g)$ and in particular is the state space of a discrete-time or continuous-time dynamical system. In the case of a Riemannian manifold, the metric $g(\cdot, \cdot)$ provides the fiber-wise norm; otherwise, we rely on a properly defined Finsler metric as previously described. As in chapter 2, in the continuous-time case, we will only consider a dynamical system $S^t_{\mathbf{F}}$, which is generated by a complete C^1 vector field $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$. In this case, the vector bundle flow $(S^t, \boldsymbol{S}^t)$ is provided by the tangent flow generated by $S^t_{\mathbf{F}}$ namely, $(S^t_{\mathbf{F}}, \boldsymbol{S}^t_{\mathbf{F}}) = (S^t_{\mathbf{F}}, \nabla S^t_{\mathbf{F}})$. The specific choice of fibers (tangent spaces, cotangent spaces, tensor spaces) of E will lead to the various induced operators we describe in the following sections. We will first consider the induced operators on sections of the tangent bundle.

4.1.2 The Induced Operators on Sections of the Tangent Bundle

The Tangent Bundle of $\mathcal{M}$

Definition 4.1.13. *Let $\mathcal{M}$ be a manifold and $m \in \mathcal{M}$. A **curve** at m is a C^1 map $c : I \to \mathcal{M}$ from an interval $I \subset \mathbb{R}$ into $\mathcal{M}$ with $0 \in I$ and $c(0) = m$. Let c_1 and c_2 be two curves at m and (U, α) a chart with $m \in U$. Then it is said that c_1 and c_2 **are tangent at** m **with respect to** α if and only if $(\alpha \circ c_1)'(0) = (\phi \circ c_2)'(0)$.*

Definition 4.1.14. *The **tangent space to $\mathcal{M}$ at** m is the set of equivalence classes of curves at m:*

$$T_m\mathcal{M} = \{[c]_m|\ c \text{ is a curve at } m\}$$

Furthermore, the tangent bundle of $\mathcal{M}$, denoted $T\mathcal{M}$ is defined as $T\mathcal{M} = \bigsqcup_{m\in\mathcal{M}} T_m\mathcal{M}$. The mapping $\pi_m : T\mathcal{M} \to \mathcal{M}$ defined by $\pi_{\mathcal{M}}([c]_m) = m$ is called the tangent bundle projection map of $\mathcal{M}$.

Definition 4.1.15. *Let $\mathcal{M}$ be a manifold, and $T\mathcal{M}$ the tangent bundle. A **vector field** is a section of $T\mathcal{M}$. The space of sections of the tangent bundle of a manifold will be denoted $\mathfrak{X}(T\mathcal{M})$*

The Induced Group of Operators

Let the vector bundle E be the tangent bundle $T\mathcal{M}$ of the manifold $\mathcal{M}$. In this setting the operator $\boldsymbol{S}^{t*} : \mathfrak{X}(T\mathcal{M}) \to \mathfrak{X}(T\mathcal{M})$ induced by a continuous or discrete-time dynamical system has the following explicit form.

$$\boldsymbol{S}_{\mathbf{F}}^{t*}\mathbf{G}(x) = \nabla S_{\mathbf{F}}^{-t}|_{S^t(x)} \cdot \mathbf{G} \circ S_{\mathbf{F}}^{t}(x) \tag{4.3}$$

with $t \in \mathbb{R}$ or $\mathbb{Z}$. The fact that we have labeled the operator with a superscript $*$ is indicative of the fact that the operator corresponds to the differential geometric concept of a pull-back operation.

Definition 4.1.16. *Let $f : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism and $\mathbf{G} \in \mathfrak{X}(T\mathcal{N})$. The **pull-back**, denoted by f^*, of $\mathbf{G}$ by f is defined as*

$$f^*\mathbf{G}(x) = (\nabla f)^{-1} \cdot \mathbf{G} \circ f(x) \in \mathbf{G} \in \mathfrak{X}(T\mathcal{M}) \tag{4.4}$$

*and the **push-forward** is defined as*

$$f_*\mathbf{G}(x) = \nabla f \cdot \mathbf{G} \circ f^{-1}(x) \in \mathbf{G} \in \mathfrak{X}(T\mathcal{N}) \tag{4.5}$$

It should be clear to see that a similar story to that of chapter 2 is unfolding. Namely, if the mapping f is the flow of a dynamical system, the pull-back of $\mathbf{G}(x)$ by f is precisely the action of the induced group of operators on sections of $T\mathcal{M}$. Thus, the induced group of operators $\boldsymbol{S}^{t*}$ of a dynamical system can equivalently be called the induced pull-back group of operators on sections of the tangent bundle. Similarly, one can also define the differential geometric concept of a Lie derivative operator on vector fields.

Definition 4.1.17. *Let $\mathbf{G}, \mathbf{F} \in \mathfrak{X}(T\mathcal{M})$ and $\mathcal{L}_{\mathbf{F}}, \mathcal{L}_{\mathbf{G}} : C(\mathcal{M}, \mathbb{F}) \to C(\mathcal{M}, \mathbb{F})$ be the associated Lie derivative operators acting on functions. The **Lie derivative**, denoted*

$\mathcal{L}_{\mathbf{F}}\mathbf{G}$, *of* $\mathbf{G}$ *said to be* ***along*** *or* ***in the direction of*** $\mathbf{F}$ *is defined as*

$$\mathcal{L}_{\mathbf{F}}\mathbf{G} = \mathcal{L}_{\mathbf{F}}\mathcal{L}_{\mathbf{G}} - \mathcal{L}_{\mathbf{G}}\mathcal{L}_{\mathbf{F}} \tag{4.6}$$

Theorem 4.1.2. *Let* $\mathbf{G}, \mathbf{F} \in \mathfrak{X}(T\mathcal{M})$*. The lie derivative of* $\mathbf{G}$ *in the direction of* $\mathbf{F}$ *can be computed as*

$$\mathcal{L}_{\mathbf{F}}\mathbf{G}(x) = \nabla\mathbf{G}(x) \cdot \mathbf{F}(x) - \nabla\mathbf{F}(x) \cdot \mathbf{G}(x) \tag{4.7}$$

Proof: See [AMR83] pages 223-224.

As expected the the C_0-group of pull-back operators $\boldsymbol{S}^{t*}$ are related to the Lie derivative operators.

Theorem 4.1.3. *Let* $\mathbf{G}, \mathbf{F} \in \mathfrak{X}^(T\mathcal{M})$ *and denote by* $S^t_{\mathbf{F}}$ *the flow generated by* $\mathbf{F}$*. Then*

$$\frac{d}{dt}\Big|_{t=0} \boldsymbol{S}^{t*}_{\mathbf{F}}\mathbf{G}(x) = \mathcal{L}_{\mathbf{F}}\mathbf{G}(x) \tag{4.8}$$

Proof: See [AMR83] page 220.

4.1.3 The Induced Operators on Sections of the the kth Cotangent Bundle

As mentioned, we will also consider the induced operators on sections of the kth cotangent bundle. These spaces correspond to differential forms which are themselves covariant tensor fields with certain properties. Since we will also propose the study of the induced operators on sections of tensor bundles, we will set the stage in the framework of tensor fields and recover the differential forms case along the way.

Tensor Bundles

Throughout the next set of definitions, E and F will be vector spaces, and E' F' will denote their dual space. The space of all linear mappings from E to a space F will be denoted $L(E;F)$.

Definition 4.1.18. *For a vector space E let $T^r_s(E) = L^{r+s}(\overbrace{E^*,\cdots,E^*}^{r},\overbrace{E,\cdots,E}^{s};\mathbb{R})$. The elements of $T^r_s(E)$ are called **tensors on** E of **contravariant order** r and **covariant order** s; or simply of **type** $\binom{r}{s}$.*

Definition 4.1.19. *Given $\boldsymbol{t}_1 \in T^{r_1}_{s_1}(E)$ and $\boldsymbol{t}_2 \in T^{r_2}_{s_2}(E)$ their **tensor product** is the tensor $\boldsymbol{t}_1 \otimes \boldsymbol{t}_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ defined as*

$$\begin{aligned}&\boldsymbol{t}_1 \otimes \boldsymbol{t}_2(\beta^1,\cdots,\beta^{r_1},\gamma^1,\cdots,\gamma^{r_2},f_1,\cdots,f_{s_1},g_1,\cdots,g_{s_2}) = \\ &\boldsymbol{t}_1(\beta^1,\cdots,\beta^{r_1},f_1,\cdots,f_{s_1})\boldsymbol{t}_2(\gamma^1,\cdots,\gamma^{r_2},g_1,\cdots,g_{s_2})\end{aligned} \tag{4.9}$$

Definition 4.1.20. *The **interior product**, denoted ι, of a vector $v \in E$ or a linear functional $\beta \in E'$ with a tensor $\boldsymbol{t} \in T^r_s(E)$ is a $\binom{r}{s-1}$ or $\binom{r-1}{s}$ type tensor, respectively, defined by*

$$\begin{aligned}&\iota_v\boldsymbol{t}(\beta^1,\cdots,\beta^r,v_1,\cdots,v_{s-1}) = \boldsymbol{t}(\beta^1,\cdots,\beta^r,v,v_1,\cdots,v_{s-1}) \\ &\iota^\beta\boldsymbol{t}(\beta^1,\cdots,\beta^{r-1},v_1,\cdots,v_s) = \boldsymbol{t}(\beta,\beta^1,\cdots,\beta^{r-1},v_1,\cdots,v_s)\end{aligned} \tag{4.10}$$

Definition 4.1.21. *If $f : E \to F$ is a linear mapping the **pull-back mapping**, denoted $f^{**} \in L(T^0_s(E), T^0_s(F))$, of f is defined as*

$$f^{**}\boldsymbol{t}(e_1,\cdots,e_s) = \boldsymbol{t}(f(e_1),\cdots,f(e_s)) \tag{4.11}$$

*where $\boldsymbol{t} \in T^0_s(F)$. When f is an isomorphism the **push-forward mapping**, denoted*

$f_{**} \in L(T^r_s(F), T^r_s(E))$, *of* f *is defined as*

$$f_{**}\boldsymbol{t}(\beta^1, \cdots, \beta^r, e_1, \cdots, e_s) = \boldsymbol{t}(\tilde{f}(\beta^1), \cdots, \tilde{f}(\beta^r), f^{-1}(e_1), \cdots, f^{-1}(e_s)) \tag{4.12}$$

for $\boldsymbol{t} \in T^r_s(F)$. *Where* $\tilde{f} : F' \to E'$ *is the adjoint of* f.

Definition 4.1.22. *If* $f : U \times F \to U' \times F'$ *is a local vector bundle mapping such that for each* $u \in U$, f_u *is an isomorphism, then let* $f^{loc}_* : U \times T^r_s(F) \to U' \times T^r_s(F')$ *be defined as*

$$f^{loc}_*(u, t) = (f_0(u), (f_0)_{**}\boldsymbol{t}) \tag{4.13}$$

where $\boldsymbol{t} \in T^r_s(F)$. *We call the mapping* f^{loc}_* *a* ***local tensor bundle mapping***

Definition 4.1.23. *Let* (B, E, π) *be a vector bundle and* $E_b = \pi^{-1}(b)$ *the fiber over* $b \in B$. *Define the* ***bundle of type*** $\binom{r}{s}$ ***tensors of*** E *as* $T^r_s(E) = \cup_{b\in B} T^r_s(E_b)$

Definition 4.1.24. *If* (B, E, π) *and* (B', E', π') *are two vector bundles and* $(f, f_0) : E \to E'$ *is a vector bundle isomorphism for every* $b \in B$ *then define the* ***tensor bundle isomorphism*** $f^{glo}_* : T^r_s(E) \to T^r_s(E')$, *as* $f^{glo}_* | T^r_s(E_b) = (f_b)^{loc}_*$

The Tensor Bundle of $\mathcal{M}$

Definition 4.1.25. *Let* $\mathcal{M}$ *be a manifold. The tensor bundle* $T^r_s(\mathcal{M})$ *is called the* ***tensor bundle of contravariant order*** r ***and covariant order*** s; *or simply* ***of type*** $\binom{r}{s}$.

Definition 4.1.26. *A* ***tensor field of type*** $\binom{r}{s}$ *on* $\mathcal{M}$ *is a smooth section of the tensor bundle* $T^r_s(\mathcal{M})$. *The space of smooth sections of* $T^r_s(\mathcal{M})$ *is denoted* $\mathcal{T}^r_s(\mathcal{M})$

Definition 4.1.27. *The* ***algebra of tensor fields*** *on* $\mathcal{M}$, *denoted* $\mathcal{T}(M)$, *is direct sum*

$$\mathcal{T}^0_0(\mathcal{M}) \oplus \mathcal{T}^0_1(\mathcal{M}) \oplus \mathcal{T}^1_0(\mathcal{M}) \oplus \cdots \tag{4.14}$$

together with the vector space structure of each $\mathcal{T}_i^j(\mathcal{M})$ and the tensor product $\otimes$.

Definition 4.1.28. *If $f : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism and $\boldsymbol{t} \in \mathcal{T}_s^r$. The **pull-back**, denoted $f^*\boldsymbol{t}$, of $\boldsymbol{t} \in \mathcal{T}_s^r(\mathcal{M})$ by f is defined as*

$$f^*\boldsymbol{t} = (\nabla f^{-1})_*^{glo} \circ \boldsymbol{t} \circ f(x) \tag{4.15}$$

*the **push-forward**, denoted $f_*\boldsymbol{t}$, of $\boldsymbol{t} \in T_s^r(F)$ by f is defined as*

$$f_*\boldsymbol{t} = (\nabla f)_*^{glo} \circ \boldsymbol{t} \circ f^{-1}(x) \tag{4.16}$$

Exterior Forms

Definition 4.1.29. *Let $\sigma \in S_k$ where S_k denotes the permutation group on k elements. A covariant tensor $\boldsymbol{t} \in T_k^0$ is called **skew symmetric** if*

$$\boldsymbol{t}(e_1, \cdots, e_k) = sign(\sigma)\boldsymbol{t}(e_\sigma(1), \cdots, e_\sigma(k)) \tag{4.17}$$

Definition 4.1.30. *The subspace of skew symmetric elements of $T_k^0(E)$ is denoted $\Lambda_k(E)$ and the elements are called **exterior k-forms**.*

Definition 4.1.31. *The **Alternation mapping Alt**: $T_k^0(E) \to T_k^0(E)$ is defined as*

$$Alt\ \boldsymbol{t}(e_1, \cdots, e_k) = \frac{1}{k!}\sum_{\sigma \in S_k} sign(\sigma)\boldsymbol{t}(e_\sigma(1), \cdots, e_\sigma(k)) \tag{4.18}$$

Definition 4.1.32. *If $\boldsymbol{\alpha} \in T_k^0(E)$ and $\boldsymbol{\beta} \in T_l^0(E)$ their **wedge product** denoted $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in \Lambda_{k+l}(E)$ is defined as*

$$\boldsymbol{\alpha}(x) \wedge \boldsymbol{\beta}(x) = \frac{(k+l)!}{k!l!} Alt(\boldsymbol{\alpha}(x) \otimes \boldsymbol{\beta}(x)) \tag{4.19}$$

The kth Cotangent Bundle of $\mathcal{M}$

Definition 4.1.33. *The tensor bundle $T_s^0(\mathcal{M})$ is called the* ***cotangent bundle of*** $\mathcal{M}$ *and is denoted $T^*\mathcal{M}$.*

Definition 4.1.34. *A* ***covector field*** *or* ***differential one-form*** *or* ***cosection*** *is an element of $\mathcal{T}_1^0(\mathcal{M})$.*

Definition 4.1.35. *The vector bundle of exterior k-forms on the tangent spaces of $\mathcal{M}$ is called the* ***kth cotangent bundle of*** $\mathcal{M}$.

Definition 4.1.36. *A* ***differential k-form*** *is a section of the kth cotangent bundle of $\mathcal{M}$. The space of smooth sections of the kth cotangent bundle will be denoted $\Omega_k(T^*\mathcal{M})$.*

The Induced Group of Operators

As before, we consider the case of of a discrete-time or continuous-time dynamical system generating a vector bundle flow $(S^t, \boldsymbol{S}^t) = (S^t, \nabla S^t)$, $t \in \mathbb{R}$ or $\mathbb{Z}$ on a smooth manifold $\mathcal{M}$ or Riemannian manifold $(\mathcal{M}, g)$. Let $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s$ be a collection of s tangent vectors at $x \in \mathcal{M}$. The induced group of operators on the space of differential k-forms $\boldsymbol{S}^{t*} : \Omega_k(T^*\mathcal{M}) \to \Omega_k(T^*\mathcal{M})$ have the following form

$$\boldsymbol{S}^{t*}\boldsymbol{\alpha}(x)(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k) = \boldsymbol{\alpha} \circ S^t(x)(\nabla S^t|_x \cdot \boldsymbol{v}_1, \cdots, \nabla S^t|_x \cdot \boldsymbol{v}_s) \tag{4.20}$$

As in the case of functions and vector fields these operators also correspond to a pull-back operation.

Definition 4.1.37. *If $f : \mathcal{M} \to \mathcal{N}$ is a mapping, $\boldsymbol{t} \in \mathcal{T}_s^0$ and $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s\}$ a collection of s tangent vectors at the point x. The* ***pull-back****, denoted $f^*\boldsymbol{t}$, of $\boldsymbol{t} \in \mathcal{T}_s^0(F)$ by f is defined as*

$$f^*\boldsymbol{t}(x)(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s) = \boldsymbol{t} \circ f(x)(\nabla f|_x \boldsymbol{v}_1, \cdots, \nabla f|_x \boldsymbol{v}_s) \tag{4.21}$$

The notion of a Lie derivative can also be defined for a one-form and from there used to build the definition for a higher order form or arbitrary tensor for that matter.

Definition 4.1.38. *Let* $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$ *and* $\boldsymbol{\alpha}(x) \in \Omega_1(T^*\mathcal{M})$. *The* ***Lie Derivative****, denoted* $\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}$ *said to be* ***along*** *or* ***in the direction of*** $\mathbf{F}$ *is defined as follows*

$$\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}(x) = \mathbf{F}^{\intercal}(x) \cdot (\nabla \boldsymbol{\alpha}^{\intercal}(x))^{\intercal} + \boldsymbol{\alpha}(x) \cdot \nabla \mathbf{F}(x) \tag{4.22}$$

Definition 4.1.39. *Let* $\mathbf{F}, \mathbf{G}_1, \cdots, \mathbf{G}_s \in \mathfrak{X}(T\mathcal{M})$, $\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^r \in \Omega_1(T^*\mathcal{M})$ *and* $\boldsymbol{t} \in \mathcal{T}^r_s(\mathcal{M})$. *The* ***Lie Derivative****, denoted* $\mathcal{L}_{\mathbf{F}}\boldsymbol{t}$ *said to be* ***along*** *or* ***in the direction of*** $\mathbf{F}$ *can be shown to have the following form*

$$\begin{aligned}
&(\mathcal{L}_{\mathbf{F}}\boldsymbol{t})(x)(\boldsymbol{\alpha}^1(x), \cdots, \boldsymbol{\alpha}^r(x), \mathbf{G}_1(x), \cdots, \mathbf{G}_s(x)) \\
&= \mathcal{L}_{\mathbf{F}}\boldsymbol{t}(x)(\boldsymbol{\alpha}^1(x), \cdots, \boldsymbol{\alpha}^r(x), \mathbf{G}_1(x), \cdots, \mathbf{G}_s(x)) \\
&+ \sum_{i=1}^{r} \boldsymbol{t}(x)(\boldsymbol{\alpha}^1(x), \cdots, \mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_i(x), \cdots, \boldsymbol{\alpha}^r(x), \mathbf{G}_1(x), \cdots, \mathbf{G}_s(x)) \\
&+ \sum_{i=1}^{s} \boldsymbol{t}(x)(\boldsymbol{\alpha}^1(x), \cdots, \boldsymbol{\alpha}^r(x), \mathbf{G}_1(x), \cdots, \mathcal{L}_{\mathbf{F}}\mathbf{G}_i(x), \cdots, \mathbf{G}_s(x))
\end{aligned} \tag{4.23}$$

The relationship between the Lie derivative of functions and vector fields to their respective pull-backs can be shown to also hold for an arbitrary tensor field.

Theorem 4.1.4. *Let* $\mathbf{F} \in \mathfrak{X}(T\mathcal{M})$ *and* $\boldsymbol{t} \in \mathcal{T}^r_s(\mathcal{M})$ *and denote by* $S^t_{\mathbf{F}}$ *the flow generated by* $\mathbf{F}$. *Then*

$$\frac{d}{dt}\bigg|_{t=0} \boldsymbol{S}^{t*}_{F}\boldsymbol{t}(x) = \mathcal{L}_{\mathbf{F}}\boldsymbol{t}(x) \tag{4.24}$$

Proof: See [AMR83] pages 304-306.

4.2 Background & History

We now summarize prior works on the operator on sections of the tangent bundle. Again, to our knowledge, besides the Ruelle-Perron-Frobenius operator, only the operator on sections of tangent bundle has been previously studied. By no means is this section an exhaustive literature review, nor is it our intention to be one. The history we provide, we hope, will convince the reader of the fruitfulness of studying the spectrum of the induced operators beyond the well known Koopman operator.

4.2.1 A New Spectral invariant

The earliest results, to our knowledge, can be traced back to the works of John Mather [Mat68] who utilized the spectrum of the induced group of operators on the space of continuous sections of the tangent bundle to characterize the hyperbolicity of a discrete-time diffeomorphism.

Definition 4.2.1. *A discrete-time dynamical system $(\mathcal{M}, S)$ is said to be **Anosov** or **hyperbolic** if there exists two subbundles E^S and E^u such that*

1. *The tangent bundle of $\mathcal{M}$ splits as a continuous Whitney sum $T\mathcal{M} = E^s \oplus E^u$.*
2. *For every positive integer n and for some Riemannian metric g:*

$$\begin{aligned} \|\nabla S^n \boldsymbol{v}\|_g &\geq a \cdot e^{\lambda n} \|\boldsymbol{v}\|_g, \quad \|\nabla S^{-n} \boldsymbol{v}\|_g \leq b \cdot e^{-\lambda n} \|\boldsymbol{v}\|_g, \text{ if } \quad \boldsymbol{v} \in E^u \\ \|\nabla S^n \boldsymbol{v}\|_g &\leq b \cdot e^{-\lambda n} \|\boldsymbol{v}\|_g, \quad \|\nabla S^{-n} \boldsymbol{v}\|_g \geq a \cdot e^{\lambda n} \|\boldsymbol{v}\|_g, \text{ if } \quad \boldsymbol{v} \in E^s \end{aligned} \tag{4.25}$$

*The constants a, b, λ are positive and independent of n and $\boldsymbol{v}$ but a and b depend on the metric g. The subbundle E^u is said to be **the unstable subbundle** and the subbundle E^s is said to be **the stable subbundle**.*

Definition 4.2.2. *An operator Q is said to be* ***hyperbolic*** *if it contains no eigenvalues of modulus* 1*.*

Specifically, Mather showed that a discrete-time dynamical system $S : \mathcal{M} \to \mathcal{M}$ is Anosov if and only if $1 - \boldsymbol{S}_* : \mathfrak{X}(T\mathcal{M}) \to \mathfrak{X}(T\mathcal{M})$ is an automorphism or, equivalently, the spectrum of $\boldsymbol{S}_*$ contains no eigenvalues of modulus 1. Mather's result provides a characterization of a diffeomorphism's hyperbolicity entirely in terms of the "hyperbolicity" of $\boldsymbol{S}_*$.

To fully appreciate Mather's result, recall that the Koopman operator's spectrum on functions encoded many dynamic properties of a dynamical system (ergodicity, weak mixing, mixing) referred to as the spectral invariants. Furthermore, recall that a measure-preserving system induces a group of unitary operators on the space $L^2(\mathcal{M}, \mu)$. As such, the spectrum of the induced operator in functions is restricted to the unit circle. What Mather's result shows is that by considering the induced operator on the space of continuous sections of the tangent bundle, one arrives at a new spectral invariant which is not a spectral invariant of the induced operator on functions, namely hyperbolicity. Mather's result has since been extended in several directions. For a measure-preserving system one can define an inner product on $\mathfrak{X}(T\mathcal{M})$ as

$$\langle \mathbf{F}(x), \mathbf{G}(x) \rangle = \int_{\mathcal{M}} g(\mathbf{F}(x), \mathbf{G}(x)) d\mu. \tag{4.26}$$

Let $\mathfrak{X}(T\mathcal{M})$ be the space of continuous sections of the tangent bundle. The completion of $\mathfrak{X}(T\mathcal{M})$ under the the induced norm is the space of square-integrable sections denoted $\mathfrak{X}_2(T\mathcal{M})$. Carmen Chicone and Richard Swanson (C&S) have shown the following

Theorem 4.2.1.

1. *For a measure-preserving, discrete-time dynamical system*

$$\sigma(\boldsymbol{S}^*, \mathfrak{X}(T\mathcal{M})) = \sigma(\boldsymbol{S}^*, \mathfrak{X}_p(T\mathcal{M})) \tag{4.27}$$

2. *For a measure-preserving, continuous-time dynamical system , if the non-periodic points are dense then*

$$e^{\sigma(\mathcal{L}, \mathfrak{X}_2(T\mathcal{M}))t} = \sigma(\boldsymbol{S}^{t*}, \mathfrak{X}_2(T\mathcal{M})) \tag{4.28}$$

3. *For a continuous-time dynamical system*

$$e^{\sigma(\mathcal{L}, \mathfrak{X}(T\mathcal{M}))t} = \sigma(\boldsymbol{S}^{t*}, \mathfrak{X}(T\mathcal{M})) \tag{4.29}$$

Proof: See [CS80]

Definition 4.2.3. *The operator $\mathcal{L}$ is said to be **infinitesimally hyperbolic** if it is invertible equivalently, if it has no eigenvalues with zero real part.*

By well known results in semigroup theory the following spectral inclusion $e^{(\sigma(\mathcal{L}_{\mathbf{F}})t)} \subseteq \sigma(S_{\mathbf{F}}^{t*})$ holds for an arbitrary C_0-semigroup [HPS08]. From this one can see that hyperbolicity implies infinitesimal hyperbolicity. A corollary of part 1 of theorem 4.2.1 is that the spectral invariant of hyperbolicity carries over when working in the space of square-integrable sections. A corollary, of parts 2 and 3 of Theorem 4.2.1 is that the flow $S_{\mathbf{F}}^t$ is hyperbolic if and only if the generator $\mathcal{L}_{\mathbf{F}}$ is infinitesimally hyperbolic on $\mathfrak{X}(T\mathcal{M}/[\mathbf{F}])$ or $\mathfrak{X}_2(T\mathcal{M}/[\mathbf{F}])$. C&S strengthen the result in [CS81b] where they prove the following,

Theorem 4.2.2 (Annular Hull Theorem)**.** *The **annular hull** denoted $\mathcal{A}$, of a set of complex numbers, consists of the disjoint union of all circles centered at the origin which* *intersect the set.*

$$e^{(\sigma(\mathcal{L}_{\mathbf{F}})t)} \subseteq \sigma(S^{t*}_{\mathbf{F}}) \subseteq \mathcal{A}(e^{(\sigma(\mathcal{L}_{\mathbf{F}})t)}) \tag{4.30}$$

The significance of the result is that while the spectrum of $\mathcal{L}_{\mathbf{F}}$ may not always exponentiate to the spectrum of $\boldsymbol{S}^{t*}$ the infinitesimal hyperbolicity of $\mathcal{L}_{\mathbf{F}}$ does exponentiate. Furthermore, since one does not need to know the flow $S^t_{\mathbf{F}}$ to compute the spectrum of $\mathcal{L}_{\mathbf{F}}$, the infinitesimal hyperbolicity condition provides an unintegrated condition for showing a flow is hyperbolic. C&S exploit this to show that normal hyperbolicity, in the sense of Hirsch, Pugh, and Shub [HPS77], can be determined from spectral properties of $\mathcal{L}_{\mathbf{F}}$.

Definition 4.2.4. *Let $H^1(T\mathcal{M})$ denote the Sobolev space of sections of the tangent bundle with one square-integrable weak derivative. A discrete-time dynamical system S is said to be* ***infinitesimally ergodic*** *if the operator $S_* - 1 : H^1(T\mathcal{M}) \to H^1(T\mathcal{M})$ has a dense range.*

The concept of infinitesimal ergodicity was introduced for discrete-time dynamical systems by J. Robbin [Rob72] who proves that infinitesimally ergodic implies ergodic. C&S have extended this result to continuous-time dynamical systems. They utilize this to show that the geodesic flow on a compact Riemannian manifold $\mathcal{M}$ of constant negative curvature is hyperbolic and ergodic by showing that $\mathcal{L}_{\mathbf{G}}$ is invertible and infinitesimally ergodic [CS81c], where $\mathbf{G} \in \mathfrak{X}(T^2\mathcal{M})$ is the geodesic spray. Although it was well-known [AA68] that the geodesic flow on a Riemannian manifold of negative curvature was Anosov and ergodic the original proofs were somewhat involved. The fact that these results followed from spectral properties of the induced operators, we hope, illustrates their value. For other results on these operators, we refer the reader to [CS81d; CS81a; Mañ77; Kre19; Sei20; Yan91; Guc72]

4.2.2 Notation & Terminology

We first fix the notation and terminology for this chapter.

1. The word **section** will be reserved for sections of the tangent bundle and will thus be synonymous with the word vector field. The two words will be used interchangeably in a few cases, but section will be the preferred choice.

2. The word **cosection** will be reserved for differential one-forms, and all higher-order forms will be referred to as differential k-forms. If a statement is made for k-forms without explicitly specifying k, it also regards the $k = 1$ case.

3. As before, The flow of a continuous-time dynamical system will be denoted $S^t_{\mathbf{F}}$ and will be generated by a vector field $\mathbf{F}(x)$. The transformation generating a discrete-time dynamical system will be denoted by the mapping S and its iterates S^n.

4. When speaking of eigenvalues/eigenfunctions/eigensections/eigenforms of a dynamical system that is not specified we mean the the spectrum of the dynamics generated by $\mathbf{F}(x)$ or $S(x)$.

5. The Koopman group of operators on functions will be denoted by its usual symbol $U^t_{\mathbf{F}}, t \in \mathbb{R}$ or $U^n_S, n \in \mathbb{Z}$. The pullback group of operators on sections will be denoted by $V^t_{\mathbf{F}}, t \in \mathbb{R}$ or $V^n_S, n \in \mathbb{Z}$. The pullback group of operators on differential forms (of all degrees) will be denoted $W^t_{\mathbf{F}}, t \in \mathbb{R}$ or $W^n_S, n \in \mathbb{Z}$. The subscript denotes the vector field or transformation that is inducing the operator.

6. In terms of the infinitesimal generators, they will all be referred to as the **Lie derivative operator**, sometimes **Lie bracket** in the case of sections and sometimes **Koopman generator** in the case of functions. They will all be denoted as $\mathcal{L}$ and

the specific type of Lie derivative operator will be clear from the object it is acting on.

7. The symbol $\psi(x) \in C(\mathcal{M}, \mathbb{F})$ will be reserved for eigenfunctions of $U^t_{\mathbf{F}}$. The symbol $\mathbf{\Phi}(x) \in \mathfrak{X}(T\mathcal{M})$, will be reserved for eigensections of $V^t_{\mathbf{F}}$. The symbols $\boldsymbol{\omega}(x) \in \Omega^1(T^*\mathcal{M})$ will be reserved for eigencosections and $\boldsymbol{\alpha}(x) \in \Omega^k(T^*\mathcal{M})$ will be reserved for eigen k-forms of $W^t_{\mathbf{F}}$ respectively.

8. The symbol λ will always denote the eigenvalue of an operator. If the eigenvalues of several operators appear in a single statement the symbols λ^{ψ}, $\lambda^{\mathbf{\Phi}}$ and $\lambda^{\boldsymbol{\omega}}$ (or $\lambda^{\boldsymbol{\alpha}}$) will be used to distinguish between the eigenvalues of $U^t_{\mathbf{F}}$, $V^t_{\mathbf{F}}$ and $W^t_{\mathbf{F}}$, respectively.

9. As in chapter 2, we allow the reader to infer from the context what additional structure the spaces $C(M, F), \mathfrak{X}(T\mathcal{M}), \Omega(T^*\mathcal{M})$ should or could have. For example, when we speak of the infinitesimal generators, which are first-order differential operators, we assume the functions/sections/forms are at least C^1-differentiable. On the other hand, when speaking of the group itself, the functions/sections/forms need not be differentiable nor even continuous.

4.3 Connections Amongst the Induced Operators

Perhaps the most immediate curiosity that might come to one's mind is determining how the spectrum of U^t, V^t, and W^t relate to each other. For example, can knowledge of one operator's spectrum help determine or generate more spectral quantities of another? This section is devoted to answering such questions and finding the interplay between the various operators' spectrum. Many of the results will follow naturally from well-known differential geometric concepts.

4.3.1 Lie Derivatives of Eigenfunctions & Eigenforms

One might ask if the eigenfunctions of $U_{\mathbf{F}}^t$ are preserved under Lie differentiation in the direction of an eigensection $\mathbf{\Phi}(x)$ of $V_{\mathbf{F}}^t$ and indeed this turns out to be true as the first part of proposition 4.3.1 shows.

Proposition 4.3.1. *For a continuous-time or discrete-time dynamical system, let $\psi_i(x)$ be an eigenfunction, $\mathbf{\Phi}_j(x)$ an eigensection. Then*

1. *in the continuous-time case, $\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)$ is also an eigenfunction of the Koopman generator at eigenvalue $\lambda_i^{\psi} + \lambda_j^{\mathbf{\Phi}}$ equivalently, at eigenvalue $e^{(\lambda_i^{\psi}+\lambda_j^{\mathbf{\Phi}})t}$ for $U_{\mathbf{F}}^t$.*

2. *in the discrete-time case, $\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)$ is also an eigenfunction of U_S at eigenvalue $\lambda_i^{\psi} + \lambda_j^{\mathbf{\Phi}}$.*

3. *If the eigenspace corresponding to $\psi(x)$ is one-dimensional and $\lambda^{\mathbf{\Phi}} = 0$ then $\psi(x)$ is also an eigenfunction of $\mathcal{L}_{\mathbf{\Phi}_j}$ and $U_{\mathbf{\Phi}_j}^t$.*

Proof: The following formula will be useful when showing the above result holds for the groups.

$$\nabla S_{\mathbf{F}}^{-t}|_{S_{\mathbf{F}}^t(x)}(\mathbf{\Phi}_j \circ S_{\mathbf{F}}^t(x)) = e^{\lambda_j^{\mathbf{\Phi}}t}\mathbf{\Phi}_j(x) \iff (\mathbf{\Phi}_j \circ S_{\mathbf{F}}^t(x)) = e^{\lambda_j^{\mathbf{\Phi}}t}\nabla S_{\mathbf{F}}^t|_x\mathbf{\Phi}_j(x) \qquad (4.31)$$

and

$$\nabla S^{-1}|_{S(x)}(\mathbf{\Phi}_j \circ S(x)) = \lambda_j^{\mathbf{\Phi}}\mathbf{\Phi}_j(x) \iff (\mathbf{\Phi}_j \circ S(x)) = \lambda_j^{\mathbf{\Phi}}\nabla S|_x\mathbf{\Phi}_j(x) \qquad (4.32)$$

For the first claim, in the continuous-time case, we have

$$
\begin{aligned}
\mathcal{L}_{\mathbf{F}}\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x) &= \mathcal{L}_{[\mathbf{F},\mathbf{\Phi}_j]}\psi_i(x) + \mathcal{L}_{\mathbf{\Phi}_j}\mathcal{L}_{\mathbf{F}}\psi_i(x) \\
&= \mathcal{L}_{\lambda_j^{\mathbf{\Phi}}\mathbf{\Phi}_j}\psi_i(x) + \lambda_i^{\psi}\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x) \\
&= (\lambda_j^{\mathbf{\Phi}} + \lambda_i^{\psi})\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)
\end{aligned}
\tag{4.33}
$$

for the generator and for the group we have

$$
\begin{aligned}
U_{\mathbf{F}}^t\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x) &= (\mathbf{\Phi}_j^{\intercal}(x)\cdot(\nabla\psi_i(x))^{\intercal})\circ S_{\mathbf{F}}^t(x) \\
&= \mathbf{\Phi}_j^{\intercal}(S_{\mathbf{F}}^t(x))\cdot(\nabla\psi_i(S_{\mathbf{F}}^t(x)))^{\intercal} \\
&= e^{\lambda_j^{\mathbf{\Phi}}t}\mathbf{\Phi}_j^{\intercal}(x)(\nabla S_{\mathbf{F}}^t|_x)^{\intercal}\cdot(\nabla\psi_i(S_{\mathbf{F}}^t(x)))^{\intercal} \\
&= e^{\lambda_j^{\mathbf{\Phi}}t}\mathbf{\Phi}_j^{\intercal}(x)\cdot(\frac{\partial}{\partial x}(\psi\circ S_{\mathbf{F}}^t(x)))^{\intercal} \\
&= e^{\lambda_j^{\mathbf{\Phi}}t}\mathbf{\Phi}_j^{\intercal}(x)\cdot e^{\lambda_i^{\psi}t}(\frac{\partial}{\partial x}\psi(x))^{\intercal} \\
&= e^{\lambda_j^{\mathbf{\Phi}}t}e^{\lambda_i^{\psi}t}\mathbf{\Phi}_j^{\intercal}(x)\cdot(\nabla\psi(x))^{\intercal} \\
&= e^{(\lambda_i^{\psi}+\lambda_j^{\mathbf{\Phi}})t}\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)
\end{aligned}
\tag{4.34}
$$

In the discrete-time case we have

$$
\begin{aligned}
U_S\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x) &= (\mathbf{\Phi}_j^{\intercal}(x)\cdot(\nabla\psi_i(x))^{\intercal})\circ S(x) \\
&= \mathbf{\Phi}_j^{\intercal}(S(x))\cdot(\nabla\psi_i(S(x)))^{\intercal} \\
&= \lambda_j^{\mathbf{\Phi}}\mathbf{\Phi}_j^{\intercal}(x)(\nabla S|_x)^{\intercal}\cdot(\nabla\psi_i(S(x)))^{\intercal} \\
&= \lambda_j^{\mathbf{\Phi}}\mathbf{\Phi}_j^{\intercal}(x)\cdot(\frac{\partial}{\partial x}(\psi\circ S(x)))^{\intercal} \\
&= \lambda_j^{\mathbf{\Phi}}\mathbf{\Phi}_j^{\intercal}(x)\cdot\lambda_i^{\psi}(\frac{\partial}{\partial x}\psi(x))^{\intercal} \\
&= \lambda_j^{\mathbf{\Phi}}\lambda_i^{\psi}\mathbf{\Phi}_j^{\intercal}(x)\cdot(\nabla\psi(x))^{\intercal} \\
&= (\lambda_i^{\psi}\lambda_j^{\mathbf{\Phi}})\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)
\end{aligned}
\tag{4.35}
$$

For the proof of the second claim observe that since $\lambda^{\mathbf{\Phi}} = 0$, then

$$\mathcal{L}_{\mathbf{F}}\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x) = \lambda_i^{\psi}\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)$$

This means that $\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)$ is an eigenfunction of $\mathcal{L}_{\mathbf{F}}$ at λ_i^{ψ}. However, since the eigenspace corresponding to $(\lambda_i^{\psi}, \psi_i(x))$ is one-dimensional $\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x)$ must be a scalar multiple of $\psi_i(x)$. In other words, $\mathcal{L}_{\mathbf{\Phi}_j}\psi_i(x) = \tilde{\lambda}\psi_i(x)$ for some $\tilde{\lambda} \in \mathbb{C}$. The same argument can be repeated to show the claim also holds for $U_{\mathbf{\Phi}}^t$.

Now, since functions are technically forms of degree zero it is plausible to expect that the contents of proposition 4.3.1 (or most results involving eigenfunctions) holds for differential k-forms.

Proposition 4.3.2. *For a continuous-time or discrete-time dynamical system, let $\mathbf{\Phi}_j(x)$ be an eigensection and $\boldsymbol{\alpha}_k(x)$ be an eigen k-form. Then,*

1. *in the continuous-time case, $\mathcal{L}_{\mathbf{\Phi}_j}\boldsymbol{\alpha}_k(x)$ is also an eigen k-form of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $\lambda_j^{\mathbf{\Phi}} + \lambda_k^{\boldsymbol{\alpha}}$ equivalently, at eigenvalue $e^{(\lambda_j^{\mathbf{\Phi}} + \lambda_k^{\boldsymbol{\alpha}})t}$ for $W_{\mathbf{F}}^t$.*

2. *in the discrete-time case, $\mathcal{L}_{\mathbf{\Phi}_j}\boldsymbol{\alpha}_k(x)$ is also an eigen k-form of U_S at eigenvalue $\lambda_j^{\mathbf{\Phi}}\lambda_k^{\boldsymbol{\alpha}}$.*

3. *If the eigenspace corresponding to $\boldsymbol{\alpha}_k(x)$ is one dimensional and $\lambda^{\mathbf{\Phi}} = 0$ then $\boldsymbol{\alpha}_k(x)$ is also an eigen k-form of $\mathcal{L}_{\mathbf{\Phi}_j}$ and $W_{\mathbf{\Phi}_j}^t$.*

Proof: For the first claim, we make us of the fact that

$$\mathcal{L}_{\mathbf{F}}\mathcal{L}_{\mathbf{\Phi}_j}\boldsymbol{\alpha}_k(x) = \mathcal{L}_{[\mathbf{F},\mathbf{\Phi}_j]}\boldsymbol{\alpha}_k(x) + \mathcal{L}_{\mathbf{\Phi}_j}\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_k(x) \tag{4.36}$$

also holds for a differential k-form, see [AMR83] page 366. The rest of the calculation is the same as in 4.3.1

For the group we will make use of the fact that

$$g^*\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_k(x) = \mathcal{L}_{g^*\mathbf{F}}g^*\boldsymbol{\alpha}_k(x) \tag{4.37}$$

holds for a diffeomorphism $g : \mathcal{M} \to \mathcal{M}$, see [AMR83] page 372. Thus,

$$\begin{aligned} W^t_{\mathbf{F}}\mathcal{L}_{\mathbf{\Phi}_j}\boldsymbol{\alpha}_k(x) &= \boldsymbol{S}^{t*}_{\mathbf{F}}\mathcal{L}_{\mathbf{\Phi}_j}\boldsymbol{\alpha}_k(x) \\ &= \mathcal{L}_{\boldsymbol{S}^{t*}_{\mathbf{F}}\mathbf{\Phi}_j}\boldsymbol{S}^{t*}_{\mathbf{F}}\boldsymbol{\alpha}_k(x) \\ &= \mathcal{L}_{e^{\lambda^{\mathbf{\Phi}}_j t}\mathbf{\Phi}_j} e^{\lambda^{\boldsymbol{\alpha}}_k t}\boldsymbol{\alpha}_k(x) \\ &= e^{(\lambda^{\mathbf{\Phi}}_j + \lambda^{\boldsymbol{\alpha}}_k)t}\mathcal{L}_{\mathbf{\Phi}_j}\boldsymbol{\alpha}_k(x) \end{aligned} \tag{4.38}$$

The argument for the second claim is the same as in proposition 2.1.2.

4.3.2 Interior Products of Eigenforms and Eigensections

We now consider how an eigenform $\boldsymbol{\omega}(x)$, in the presence of an eigensection $\mathbf{\Phi}(x)$, can lead to a new eigenform. Specifically, we consider $\iota_{\mathbf{\Phi}_j}\boldsymbol{\omega}(x)$ where $\iota_{\mathbf{G}} : \Omega^k(T^*\mathcal{M}) \to \Omega^{k-1}(T^*\mathcal{M})$ for $\mathbf{G} \in \mathfrak{X}(T\mathcal{M})$ is the interior product.

Proposition 4.3.3. *For a continuous-time or discrete-time dynamical system, let $\mathbf{\Phi}_i(x)$ be an eigensection and $\boldsymbol{\alpha}_j(x)$ be an eigen k-form. Then*

1. *in the continuous-time case, $\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x)$ is an eigen $(k-1)$-form of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $\lambda^{\mathbf{\Phi}}_i + \lambda^{\boldsymbol{\alpha}}_j$ equivalently, at eigenvalue $e^{(\lambda^{\mathbf{\Phi}}_i + \lambda^{\boldsymbol{\alpha}}_j)t}$ of $W^t_{\mathbf{F}}$.*
2. *in the discrete-time case, $\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x)$ is an eigen $(k-1)$-form of at eigenvalue $\lambda^{\mathbf{\Phi}}_i\lambda^{\boldsymbol{\alpha}}_j$ of W_S.*

Proof: In the continuous-time case, we make use of the fact that

$$\mathcal{L}_{\mathbf{F}}\iota_{\mathbf{G}} - \iota_{\mathbf{F}}\mathcal{L}_{\mathbf{G}} = \iota_{\mathcal{L}_{\mathbf{F}}\mathbf{G}} \tag{4.39}$$

holds for $\mathbf{F}, \mathbf{G} \in \mathfrak{X}(T\mathcal{M})$, see [AMR83] page 366. Thus, for the generator, we have that

$$\begin{aligned}
\mathcal{L}_{\mathbf{F}}\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) &= \iota_{\mathcal{L}_{\mathbf{F}}\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) + \iota_{\mathbf{\Phi}_i}\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_j(x) \\
&= \lambda_i^{\mathbf{\Phi}}\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) + \lambda_j^{\boldsymbol{\alpha}_j}\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) \\
&= (\lambda_i^{\mathbf{\Phi}} + \lambda_j^{\boldsymbol{\alpha}_j})\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x).
\end{aligned} \tag{4.40}$$

For the group we make use use of the fact that

$$g^*\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) = \iota_{g^*\mathbf{\Phi}_i}g^*\boldsymbol{\alpha}_j(x) \tag{4.41}$$

holds for a diffeomorphism $g : \mathcal{M} \to \mathcal{M}$, see [AMR83] page 364. Thus, we have that

$$\begin{aligned}
W_{\mathbf{F}}^t\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) &= \boldsymbol{S}_{\mathbf{F}}^{t*}\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) \\
&= \iota_{\boldsymbol{S}_{\mathbf{F}}^{t*}\mathbf{\Phi}_i}\boldsymbol{S}_{\mathbf{F}}^{t*}\boldsymbol{\alpha}_j(x) \\
&= \iota_{e^{\lambda_i^{\mathbf{\Phi}}t}\mathbf{\Phi}_i}e^{\lambda_j^{\boldsymbol{\alpha}}t}\boldsymbol{\alpha}_j(x) \\
&= e^{(\lambda_i^{\mathbf{\Phi}}+\lambda_j^{\boldsymbol{\alpha}})t}\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x).
\end{aligned} \tag{4.42}$$

In the discrete-time case we have

$$\begin{aligned}
W_S\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) &= \boldsymbol{S}^*\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x) \\
&= \iota_{S_{\mathbf{F}}^*\mathbf{\Phi}_i}S_{\mathbf{F}}^*\boldsymbol{\alpha}_j(x) \\
&= \iota_{\lambda_i^{\mathbf{\Phi}}\mathbf{\Phi}_i}\lambda_j^{\boldsymbol{\alpha}}\boldsymbol{\alpha}_j(x) \\
&= \lambda_i^{\mathbf{\Phi}}\lambda_j^{\boldsymbol{\alpha}}\iota_{\mathbf{\Phi}_i}\boldsymbol{\alpha}_j(x).
\end{aligned} \tag{4.43}$$

Proposition 4.3.3 demonstrates how an eigensection and an eigen k-form can be used to generate and eigen $(k-1)$-form. In the specific case of an eigencosection we get the following corollary regarding eigenfunctions.

Corollary 4.3.0.1. *For a continuous-time or discrete-time dynamical system, let* $\mathbf{\Phi}_i(x)$ *be an eigensection and* $\boldsymbol{\omega}_j(x)$ *be an eigencosection. Then*

1. *in the continuous-time case,* $\iota_{\mathbf{\Phi}_i}\boldsymbol{\omega}_j(x)$ *is an eigenfunction of the Koopman generator at eigenvalue* $\lambda_i^{\mathbf{\Phi}} + \lambda_j^{\boldsymbol{\omega}}$ *equivalently, at eigenvalue* $e^{(\lambda_i^{\mathbf{\Phi}}+\lambda_j^{\boldsymbol{\omega}})t}$ *for* $U_{\mathbf{F}}^t$.
2. *in the discrete-time case* $\iota_{\mathbf{\Phi}_i}\boldsymbol{\omega}_j(x)$ *is an eigen eigenfunction of* U_S *at eigenvalue* $\lambda_i^{\mathbf{\Phi}}\lambda_j^{\boldsymbol{\omega}}$.

4.4 Algebraic and Differential Topological Properties of Eigensections and Eigenforms

In this section, we develop the algebraic and differential topological properties that eigensections and eigenforms possess.

4.4.1 Algebraic Properties of Eigensections

We begin by recalling that the induced operators' eigenfunctions enjoy algebraic properties that allow one to create new eigenfunctions out of old ones. Specifically, theorem 2.4.1 stated that the pointwise product of two eigenfunctions is again an eigenfunction and, under certain conditions, a set of Koopman eigenfunctions forms an abelian monoid. Thus, it is reasonable to expect that a similar situation should hold for the other induced operators. In any case, the insight lies in understanding the appropriate algebraic structure of the objects one is considering.

Algebraic Properties of Eigensections

Definition 4.4.1. *Let $\mathcal{N}$ be a vector space over a field $\mathbb{F}$. The vector space $\mathcal{N}$ together with an operation, denoted $[\cdot,\cdot] : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$, called the **Lie product** or **Lie bracket** such that*

1. *$[\cdot,\cdot]$ is bilinear over $\mathbb{F}$.*
2. *$[n,n] = 0$ for all $n \in \mathcal{N}$.*
3. *$[n_1,n_2] = -[n_2,n_1]$ for all $n_1, n_2 \in \mathcal{N}$.*
4. *$[n_1,[n_2,n_3]] + [n_2,[n_3,n_1]] + [n_3,[n_1,n_2]] = 0$.*

*is said to be a **Lie Algebra**. The last property is called the **Jacobi identity**.*

Proposition 4.4.1. *Let $\mathfrak{X}(T\mathcal{M})$ denote the space of smooth vector fields. The Lie bracket $[\mathbf{F}(x), \mathbf{G}(x)] = \mathcal{L}_{\mathbf{F}}\mathbf{G}(x)$ on $\mathfrak{X}(T\mathcal{M})$, along with the vector space structure of $\mathfrak{X}(T\mathcal{M})$, form a Lie algebra.*

Proof: See [AMR83] page 222.

Proposition 4.4.2. *For a continuous-time or discrete-time dynamical system, the set of all smooth eigensections, form a Lie sub-algebra of $\mathfrak{X}(T\mathcal{M})$. Specifically, if $\mathbf{\Phi}_i$,$\mathbf{\Phi}_j$ are eigensections, then*

1. *in the continuous-time case, $[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$ is also an eigensection of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $\lambda_i + \lambda_j$ equivalently at eigenvalue $e^{(\lambda_i+\lambda_j)t}$ for $V^t_{\mathbf{F}}$.*
2. *in the discrete-time case, $[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$ is also an eigensection of V_S at eigenvalue $\lambda_i\lambda_j$.*

Proof: For the first claims we have

$$
\begin{aligned}
[\mathbf{F},[\mathbf{\Phi}_i,\mathbf{\Phi}_j]] &= -[\mathbf{\Phi}_j,[\mathbf{F},\mathbf{\Phi}_i]] - [\mathbf{\Phi}_i,[\mathbf{\Phi}_j,\mathbf{F}]] \\
&= [[\mathbf{F},\mathbf{\Phi}_i],\mathbf{\Phi}_j] + [\mathbf{\Phi}_i,[\mathbf{F},\mathbf{\Phi}_j]] \\
&= [\lambda_i\mathbf{\Phi}_i,\mathbf{\Phi}_j] + [\mathbf{\Phi}_i,\lambda_j\mathbf{\Phi}_j] \\
&= (\lambda_i+\lambda_j)[\mathbf{\Phi}_i,\mathbf{\Phi}_j]
\end{aligned} \tag{4.44}
$$

for the generator. For the group, we make use of the fact that

$$
g^*[\mathbf{F}(x),\mathbf{G}(x)] = [g^*\mathbf{F}(x), g^*\mathbf{G}(x)] \tag{4.45}
$$

holds for $\mathbf{F},\mathbf{G} \in \mathfrak{X}(t\mathcal{M})$ and a diffeomorphism $g:\mathcal{M}\to\mathcal{M}$, see [AMR83] pages 222-223 and 369. Thus, we have that

$$
\begin{aligned}
V^t_{\mathbf{F}}[\mathbf{\Phi}_i,\mathbf{\Phi}_j] &= \boldsymbol{S}^{t*}_{\mathbf{F}}[\mathbf{\Phi}_i,\mathbf{\Phi}_j] \\
&= [\boldsymbol{S}^{t*}_{\mathbf{F}}\mathbf{\Phi}_i, \boldsymbol{S}^{t*}_{\mathbf{F}}\mathbf{\Phi}_j] \\
&= [e^{\lambda_i t}\mathbf{\Phi}_i, e^{\lambda_j t}\mathbf{\Phi}_j] \\
&= e^{(\lambda_i+\lambda_j)t}[\mathbf{\Phi}_i,\mathbf{\Phi}_j].
\end{aligned} \tag{4.46}
$$

(2) In the discrete-time case we have

$$
\begin{aligned}
V_S[\mathbf{\Phi}_i,\mathbf{\Phi}_j] &= \boldsymbol{S}^{t*}[\mathbf{\Phi}_i,\mathbf{\Phi}_j] \\
&= [\boldsymbol{S}^{t*}\mathbf{\Phi}_i, \boldsymbol{S}^{t*}\mathbf{\Phi}_j] \\
&= [\lambda_i\mathbf{\Phi}_i, \lambda_j\mathbf{\Phi}_j] \\
&= \lambda_i\lambda_j[\mathbf{\Phi}_i,\mathbf{\Phi}_j].
\end{aligned} \tag{4.47}
$$

Of course, there is nothing that prevents us from considering higher order Lie products.

To that end, denote by $\mathcal{L}_{\mathbf{\Phi}_i}^{n_r}\mathcal{L}_{\mathbf{\Phi}_j}^{m_s}\cdots\mathcal{L}_{\mathbf{\Phi}_i}^{n_1}\mathcal{L}_{\mathbf{\Phi}_j}^{m_1}$, the following nested brackets of eigensections.

$$\mathcal{L}_{\mathbf{\Phi}_i}^{n_r}\mathcal{L}_{\mathbf{\Phi}_j}^{m_s}\cdots\mathcal{L}_{\mathbf{\Phi}_i}^{n_1}\mathcal{L}_{\mathbf{\Phi}_j}^{m_1} := \overbrace{[\mathbf{\Phi}_i,\cdots[\mathbf{\Phi}_i,}^{n_r}\overbrace{[\mathbf{\Phi}_j,\cdots[\mathbf{\Phi}_j,}^{m_s}\ldots\overbrace{[\mathbf{\Phi}_i,\cdots[\mathbf{\Phi}_i,}^{n_1}\overbrace{[\mathbf{\Phi}_j\cdots,\mathbf{\Phi}_j]}^{m_1}]\ldots]] \quad (4.48)$$

.

Proposition 4.4.3. *Let $t\in\{1,2,\ldots r\}$, $u\in\{1,2,\ldots s\}$ and $n_t,m_u\in\{1,2,\cdots\}$. If $m_1=1$ and $n_t,m_u>0$ for $t\geq 1$ and $u\geq 2$ then*

1. *in the continuous-time case, $\mathcal{L}_{\mathbf{\Phi}_i}^{n_r}\mathcal{L}_{\mathbf{\Phi}_j}^{m_s}\cdots\mathcal{L}_{\mathbf{\Phi}_i}^{n_1}\mathcal{L}_{\mathbf{\Phi}_j}^{m_1}$ is also an eigensection of $\mathcal{L}_{\mathbf{F}}$ with eigenvalue $(\sum_{t=1}^{r}n_t\lambda_i+\sum_{u=1}^{s}m_u\lambda_j)$ and at eigenvalue $e^{(\sum_{t=1}^{r}n_t\lambda_i+\sum_{u=1}^{s}m_u\lambda_j)t}$ for $V_{\mathbf{F}}^t$.*

2. *in the discrete-time case, $\mathcal{L}_{\mathbf{\Phi}_i}^{n_r}\mathcal{L}_{\mathbf{\Phi}_j}^{m_s}\cdots\mathcal{L}_{\mathbf{\Phi}_i}^{n_1}\mathcal{L}_{\mathbf{\Phi}_j}^{m_1}$ is also an eigensection of V_S with eigenvalue eigenvalue $(\Pi_{t=1}^{r}n_t\lambda_i\Pi_{u=1}^{s}m_u\lambda_j)$.*

Proof: We begin by noting that, since $[\mathbf{G},\mathbf{G}]=0$ for any $\mathbf{G}\in\mathfrak{X}(T\mathcal{M})$, we have that

1. if $m_1>1$ then the inner most term is $\overbrace{[\mathbf{\Phi}_j\cdots,\mathbf{\Phi}_j]}^{m_1}=0$ and thus $\mathcal{L}_{\mathbf{\Phi}_i}^{n_r}\mathcal{L}_{\mathbf{\Phi}_j}^{m_s}\cdots\mathcal{L}_{\mathbf{\Phi}_i}^{n_1}\mathcal{L}_{\mathbf{\Phi}_j}^{m_1}=0$ which is never a valid eigensection.

2. if $m_1=0$ and $n_1>0$ then a similar situation arises, in that the inner most term is now $\overbrace{[\mathbf{\Phi}_i\cdots,\mathbf{\Phi}_i]}^{n_1}=0$ and thus $\mathcal{L}_{\mathbf{\Phi}_i}^{n_r}\mathcal{L}_{\mathbf{\Phi}_j}^{m_s}\cdots\mathcal{L}_{\mathbf{\Phi}_i}^{n_1}\mathcal{L}_{\mathbf{\Phi}_j}^{m_1}=0$ which is never a valid eigensection.

In light of this, we assume $m_1=1$ and $n_t,m_u>0$ for $t\geq 1$ and $u\geq 2$ as the assumption states. The proof is by induction. The first step would be to take the base case of t, u, n_t, and show the base case on m_u holds.

We first show that

$$[\mathbf{F}, \mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j}] = [\mathbf{F}, \mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] = (n_1\lambda_i + \lambda_j)[\mathbf{\Phi}_i, \mathbf{\Phi}_j] = (n_1\lambda_i + m_1\lambda_j)[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$$

This is done by induction on the integer n_1, the base case of which is

$$[\mathbf{F}, \mathcal{L}^{1}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] = [\mathbf{F}, [\mathbf{\Phi}_i, \mathbf{\Phi}_j]] = (\lambda_i + \lambda_j)[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$$

and corresponds exactly with proposition 4.4.2 so we are done with this case. Now suppose that $[\mathbf{F}, \mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] = (k\lambda_i + \lambda_j)\mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}$ then

$$\begin{aligned}
[\mathbf{F}, \mathcal{L}^{k+1}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] &= [\mathbf{F}, [\mathbf{\Phi}_i, \mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}]] \\
&= -[\mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}, [\mathbf{F}, \mathbf{\Phi}_i]] - [\mathbf{\Phi}_i, [\mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}, \mathbf{F}]] \\
&= [[\mathbf{F}, \mathbf{\Phi}_i], \mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] + [\mathbf{\Phi}_i, [\mathbf{F}, \mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}]] \qquad (4.49) \\
&= \lambda_i[\mathbf{\Phi}_i, \mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] + (k\lambda_i + \lambda_j)[\mathbf{\Phi}_i, \mathcal{L}^{k}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}] \\
&= ((k+1)\lambda_i + \lambda_j)\mathcal{L}^{k+1}_{\mathbf{\Phi}_i}\mathcal{L}^{1}_{\mathbf{\Phi}_j}
\end{aligned}$$

This completes the induction and shows that $[\mathbf{F}, \mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j}] = (n_1\lambda_i + m_1\lambda_j)[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$

Next we show that

$$[\mathbf{F}, \mathcal{L}^{n_2}_{\mathbf{\Phi}_i}\mathcal{L}^{m_2}_{\mathbf{\Phi}_j}\mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j}] = ((n_1+n_2)\lambda_i + (m_1+m_2)\lambda_j)\mathcal{L}^{n_2}_{\mathbf{\Phi}_i}\mathcal{L}^{m_2}_{\mathbf{\Phi}_j}\mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j}$$

This is shown by induction on the two integers n_2 and m_2 which, in its entirety, consists of four proofs corresponding to 2 base cases and 2 induction cases. Note that if $m_2 = 0$ then

$$\mathcal{L}^{n_2}_{\mathbf{\Phi}_i}\mathcal{L}^{m_2}_{\mathbf{\Phi}_j}\mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j} = \mathcal{L}^{n_2}_{\mathbf{\Phi}_i}\mathcal{L}^{n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j} = \mathcal{L}^{n_2+n_1}_{\mathbf{\Phi}_i}\mathcal{L}^{m_1}_{\mathbf{\Phi}_j}$$

which is the case previously considered. Thus we take take $n_2 = 0$, $m_2 = 1$ as the base cases and to reduce notational clutter we hereby denote $\mathcal{L}^{n_i}_{\mathbf{\Phi}_i}\mathcal{L}^{n_j}_{\mathbf{\Phi}_j}$ as $\mathcal{L}^{n_i,n_j}_{\mathbf{\Phi}_{ij}}$.

(Base Case: $n_2 = 0$, $m_2 = 1$)

$$\begin{aligned}
[\mathbf{F}, \mathcal{L}^1_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] &= [\mathbf{F}, [\mathbf{\Phi}_j, \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= -[\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, [\mathbf{F}, \mathbf{\Phi}_j]] - [\mathbf{\Phi}_j, [\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, \mathbf{F}]] \\
&= [[\mathbf{F}, \mathbf{\Phi}_j], \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + [\mathbf{\Phi}_j, [\mathbf{F}, \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= \lambda_j[\mathbf{\Phi}_j, \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + (n_1\lambda_i + m_1\lambda_j)[\mathbf{\Phi}_j, \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] \\
&= (n_1\lambda_i + (m_1+1)\lambda_j)\mathcal{L}^1_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}} \\
&= ((n_1+n_2)\lambda_i + (m_1+m_2)\lambda_j)\mathcal{L}^1_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}
\end{aligned}$$

(Inductive Case: $n_2 = 0$, $m_2 = k+1$)

$$\begin{aligned}
[\mathbf{F}, \mathcal{L}^{k+1}_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] &= [\mathbf{F}, [\mathbf{\Phi}_j, \mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= -[\mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, [\mathbf{F}, \mathbf{\Phi}_j]] - [\mathbf{\Phi}_j, [\mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, \mathbf{F}]] \\
&= [[\mathbf{F}, \mathbf{\Phi}_j], \mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + [\mathbf{\Phi}_j, [\mathbf{F}, \mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= \lambda_j[\mathbf{\Phi}_j, \mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + (n_1\lambda_i + (m_1+k)\lambda_j)[\mathbf{\Phi}_j, \mathcal{L}^k_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] \\
&= (n_1\lambda_i + (m_1+k+1)\lambda_j)\mathcal{L}^{k+1}_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}} \\
&= ((n_1+n_2)\lambda_i + (m_1+m_2)\lambda_j)\mathcal{L}^{k+1}_{\mathbf{\Phi}_j}\mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}
\end{aligned}$$

(Base Case: $n_2 = k+1$, $m_2 = 1$)

$$\begin{aligned}
[\mathbf{F}, \mathcal{L}^{k+1,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] &= [\mathbf{F}, [\mathbf{\Phi}_i, \mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= -[\mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, [\mathbf{F}, \mathbf{\Phi}_i]] - [\mathbf{\Phi}_i, [\mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, \mathbf{F}]] \\
&= [[\mathbf{F}, \mathbf{\Phi}_i], \mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + [\mathbf{\Phi}_i, [\mathbf{F}, \mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= \lambda_i [\mathbf{\Phi}_i, \mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + ((n_1+k)\lambda_i + (m_1+1)\lambda_i)[\mathbf{\Phi}_i, \mathcal{L}^{k,1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] \\
&= ((n_1+k+1)\lambda_i + (m_1+1)\lambda_j) \mathcal{L}^{k+1}_{\mathbf{\Phi}_i} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}} \\
&= ((n_1+n_2)\lambda_i + (m_1+m_2)\lambda_j) \mathcal{L}^{k+1}_{\mathbf{\Phi}_i} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}
\end{aligned}$$

(Inductive Case: $n_2 = k+1$, $m_2 = k+1$)

$$\begin{aligned}
[\mathbf{F}, \mathcal{L}^{k+1,k+1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] &= [\mathbf{F}, [\mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}, \mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= -[\mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, [\mathbf{F}, \mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}]] - [\mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}, [\mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}, \mathbf{F}]] \\
&= [[\mathbf{F}, \mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}], \mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] + [\mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}, [\mathbf{F}, \mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}]] \\
&= (\lambda_i + \lambda_j)[\mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}, \mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] \\
&+ ((n_1+k)\lambda_i + (m_1+k)\lambda_j)[\mathcal{L}^{1,1}_{\mathbf{\Phi}_{ij}}, \mathcal{L}^{k,k}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}] \\
&= ((n_1+k+1)\lambda_i + (m_1+k+1)\lambda_j) \mathcal{L}^{k+1,k+1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}} \\
&= ((n_1+n_2)\lambda_i + (m_1+m_2)\lambda_j) \mathcal{L}^{k+1,k+1}_{\mathbf{\Phi}_{ij}} \mathcal{L}^{n_1,m_1}_{\mathbf{\Phi}_{ij}}
\end{aligned}$$

This completes the four inductive arguments needed to show

$$[\mathbf{F}, \mathcal{L}^{n_2}_{\mathbf{\Phi}_i} \mathcal{L}^{m_2}_{\mathbf{\Phi}_j} \mathcal{L}^{n_1}_{\mathbf{\Phi}_i} \mathcal{L}^{m_1}_{\mathbf{\Phi}_j}] = ((n_1+n_2)\lambda_i + (m_1+m_2)\lambda_j) \mathcal{L}^{n_2}_{\mathbf{\Phi}_i} \mathcal{L}^{m_2}_{\mathbf{\Phi}_j} \mathcal{L}^{n_1}_{\mathbf{\Phi}_i} \mathcal{L}^{m_1}_{\mathbf{\Phi}_j}$$

.

At this point it should be clear to see that that the arguments can be repeated to show the claim holds. The proofs for the group and the discrete time case can also be carried by induction and again using the fact that

$$g^*[\mathbf{F}(x), \mathbf{G}(x)] = [g^*\mathbf{F}(x), g^*\mathbf{G}(x)] \tag{4.50}$$

holds for $\mathbf{F}, \mathbf{G} \in \mathfrak{X}(t\mathcal{M})$ and a diffeomorphism $g : \mathcal{M} \to \mathcal{M}$.

Similar to theorem 2.4 we required the eigensections form an algebra for the above results to hold. For this reason, smoothness was required because r-times differentiable vector fields do not form a Lie algebra under the Lie bracket.

Algebraic Properties of Eigenforms

In order to discuss the algebraic properties for eigen k-forms one must consider an eigen k-form as a member of the entire exterior algebra.

Definition 4.4.2. *An algebra $\mathcal{A}$ is said to be a* ***unital algebra*** *if it contains an element I such that $Ia = a = aI$ for all $a \in \mathcal{A}$. The element I is called the* ***unit*** *or* ***identity****.*

Definition 4.4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras and $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that $f(a_1a_2) = f(a_1)f(a_2)$ for all $a_1, a_2 \in \mathcal{A}$. The mapping f is said to be a* ***homomorphism of algebras****. If a homomorphism of algebras is a bijection then it is said to be an* ***isomorphism of algebras****.*

Definition 4.4.4. *A linear operation $D : \mathcal{A} \to \mathcal{A}$ on an algebra $\mathcal{A}$ is said to be a* ***derivation*** *if*

$$D(a_1a_2) = D(a_1)a_2 + a_1D(a_2) \tag{4.51}$$

for all $a_1, a_2 \in \mathcal{A}$.

Definition 4.4.5. *An algebra $\mathcal{A}$ is said to be a* ***differential algebra*** *if it is equipped with a derivation D.*

Definition 4.4.6. *Let $\Omega_k(T^*\mathcal{M})$ denote the space of smooth differential k-forms for every k. The **exterior algebra of differential forms**, denoted $\Omega(T^*\mathcal{M})$, is the direct sum*

$$\Omega_0(T^*\mathcal{M}) \oplus \Omega_1(T^*\mathcal{M}) \oplus \Omega_2(T^*\mathcal{M}) \oplus \cdots \oplus \Omega_n(T^*\mathcal{M}) \tag{4.52}$$

together with the vector space structure of each $\Omega_k(T^*\mathcal{M})$, the exterior derivative $\boldsymbol{d} : \Omega_k(T^*\mathcal{M}) \to \Omega_{k+1}(T^*\mathcal{M})$ and the exterior product $\wedge : \Omega(T^*\mathcal{M}) \times \Omega(T^*\mathcal{M}) \to \Omega(T^*\mathcal{M})$. It is a unital, graded and differential algebra. The derivation is provided by the exterior derivative.

Theorem 4.4.1. *For a continuous-time or discrete-time dynamical system. Denote by $\mathcal{A}_U$ the subset of eiegnforms that belong to the exterior algebra $\Omega(T*\mathcal{M})$. Then $\mathcal{A}_U$ forms an anticommutative monoid under pointwise exterior products and is also a differential algebra.*

Proof: Associativity and anticommutativity follow from properties of the wedge product. The existence of an identity follows from the fact that the exterior algebra $\Omega(T^*\mathcal{M})$ is a unital algebra. The unit, being a member of the zero grade $\Omega_0(T^*\mathcal{M})$, is the constant function equal to 1 everywhere; which is trivially a Koopman eigenfunction. The only thing left to show is that the wedge product of two eigenforms is also an eigenform and that eigenforms are closed under exterior differentiation.

In the continuous-time case, we make use of the following facts

$$\mathcal{L}_{\mathbf{F}}(\boldsymbol{\gamma}_1(x) \wedge \boldsymbol{\gamma}_2(x)) = \mathcal{L}_{\mathbf{F}}\boldsymbol{\gamma}_1(x) \wedge \boldsymbol{\gamma}_2(x) + \boldsymbol{\gamma}_1(x) \wedge \mathcal{L}_{\mathbf{F}}\boldsymbol{\gamma}_2(x) \tag{4.53}$$

and

$$\mathcal{L}_{\mathbf{F}}\boldsymbol{d}\boldsymbol{\gamma}_1(x) = \boldsymbol{d}\mathcal{L}_{\mathbf{F}}\boldsymbol{\gamma}_1(x) \tag{4.54}$$

for a k-form $\boldsymbol{\gamma}_1(x)$ and l-form $\boldsymbol{\gamma}_2(x)$, see [AMR83] page 361 and 372. Thus, if $\boldsymbol{\alpha}_i(x)$ and

$\boldsymbol{\beta}_j(x)$ are eigenforms of $\mathcal{L}_{\mathbf{F}}$, we have

$$\begin{aligned}\mathcal{L}_{\mathbf{F}}(\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x)) &= \mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x) + \boldsymbol{\alpha}_i(x) \wedge \mathcal{L}_{\mathbf{F}}\boldsymbol{\beta}_j(x) \\ &= \lambda^{\boldsymbol{\alpha}_i}\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x) + \boldsymbol{\alpha}_i(x) \wedge \lambda^{\boldsymbol{\beta}_j}\boldsymbol{\beta}_j(x) \\ &= (\lambda^{\boldsymbol{\alpha}_i} + \lambda^{\boldsymbol{\beta}_j})\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x)\end{aligned} \tag{4.55}$$

and

$$\mathcal{L}_{\mathbf{F}}\boldsymbol{d}\boldsymbol{\alpha}_i(x) = \boldsymbol{d}\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_i(x) = \boldsymbol{d}\lambda^{\boldsymbol{\alpha}_i}\boldsymbol{\alpha}_i(x) = \lambda^{\boldsymbol{\alpha}_i}\boldsymbol{d}\boldsymbol{\alpha}_i(x) \tag{4.56}$$

For the group we make use of the fact that the pull-back operation is a homomorphism of differential algebras, see [AMR83] page 360. Thus, have

$$\begin{aligned}W^t_{\mathbf{F}}(\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x)) &= \boldsymbol{S}^{t*}_{\mathbf{F}}((\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x)) \\ &= \boldsymbol{S}^{t*}_{\mathbf{F}}\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{S}^{t*}_{\mathbf{F}}\boldsymbol{\beta}_j(x) \\ &= e^{\lambda^{\boldsymbol{\alpha}}_i t}\boldsymbol{\alpha}_i(x) \wedge e^{\lambda^{\boldsymbol{\beta}}_j t}\boldsymbol{\beta}_j(x) \\ &= e^{(\lambda^{\boldsymbol{\alpha}}_i + \lambda^{\boldsymbol{\beta}}_j)t}(\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x))\end{aligned} \tag{4.57}$$

and

$$W^t_{\mathbf{F}}\boldsymbol{d}\boldsymbol{\alpha}_j(x) = \boldsymbol{S}^{t*}_{\mathbf{F}}\boldsymbol{d}\boldsymbol{\alpha}_j(x) = \boldsymbol{d}\boldsymbol{S}^{t*}_{\mathbf{F}}\boldsymbol{\alpha}_j(x) = \boldsymbol{d}e^{\lambda^{\boldsymbol{\alpha}}_j t}\boldsymbol{\alpha}_j(x) = e^{\lambda^{\boldsymbol{\alpha}}_j t}\boldsymbol{d}\boldsymbol{\alpha}_j(x) \tag{4.58}$$

In the discrete time case we have that

$$\begin{aligned}W_S(\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x)) &= \boldsymbol{S}^*((\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x)) \\ &= \boldsymbol{S}^*\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{S}^*\boldsymbol{\beta}_j(x) \\ &= \lambda^{\boldsymbol{\alpha}}_i\boldsymbol{\alpha}_i(x) \wedge \lambda^{\boldsymbol{\beta}}_j\boldsymbol{\beta}_j(x) \\ &= \lambda^{\boldsymbol{\alpha}}_i\lambda^{\boldsymbol{\beta}}_j(\boldsymbol{\alpha}_i(x) \wedge \boldsymbol{\beta}_j(x))\end{aligned} \tag{4.59}$$

and

$$W_S \boldsymbol{d}\boldsymbol{\alpha}_j(x) = \boldsymbol{S}^* \boldsymbol{d}\boldsymbol{\alpha}_j(x) = \boldsymbol{d}\boldsymbol{S}^* \boldsymbol{\alpha}_j(x) = \boldsymbol{d}\lambda_j^{\boldsymbol{\alpha}} \boldsymbol{\alpha}_j(x) = \lambda_j^{\boldsymbol{\alpha}} \boldsymbol{d}\boldsymbol{\alpha}_j(x) \tag{4.60}$$

Corollary 4.4.1.1. *For a continuous-time or discrete-time dynamical system let* $\boldsymbol{\omega}_i(x)$ *be a set of linearly independent eigencosections.*

1. *In the continuous-time case, the differential* k*-form* $\boldsymbol{\alpha}_{i_1,\dots,i_k}(x) = \boldsymbol{\omega}_{i_1}(x) \wedge \overset{k}{\cdots} \wedge \boldsymbol{\omega}_{i_k}(x)$*, for* $i_1, \dots, i_k$ *distinct, is an eigen* k*-form of* $\mathcal{L}_{\mathbf{F}}$ *at eigenvalue* $\lambda_{i_1,\dots,i_k} = \sum_{j=1}^{k} \lambda_{i_j}$ *equivalently, at eigenvalue* $e^{\lambda_{i_1,\dots,i_k} t}$ *for the group.*

2. *In the discrete-time case, the differential* k*-form* $\boldsymbol{\alpha}_{i_1,\dots,i_k}(x) = \boldsymbol{\omega}_{i_1}(x) \wedge \overset{k}{\cdots} \wedge \boldsymbol{\omega}_{i_k}(x)$*, for* $i_1, \dots, i_k$ *distinct, is an eigen* k*-form of* W_S *at eigenvalue* $\lambda_{i_1,\dots,i_k} = \prod_{j=1}^{k} \lambda_{i_j}$ *for the group.*

Proof: In both the discrete and continuous-time cases the claim that $\boldsymbol{\alpha}_{i_1,\dots,i_k}(x)$ is an eigen k-form follows from theorem 4.4.1. The fact that $\boldsymbol{\alpha}_{i_1,\dots,i_k}(x)$ is a well-defined eigen k-form follows from the fact that $\boldsymbol{\omega}_{i_1}, \dots, \boldsymbol{\omega}_{i_k}$, for $i_1, \dots, i_k$ distinct, are linearly independent and this implies that $\boldsymbol{\omega}_{i_1} \wedge \overset{k}{\cdots} \wedge \boldsymbol{\omega}_{i_k} \neq 0$, see [GMM13] page 108.

We note, that the original case of pointwise products of Koopman eigenfunctions is included in this description since the exterior product of two functions is simply their pointwise product.

Notice that since $\boldsymbol{d}^2 = 0$ and by convention one would not consider the zero k-form to be an eigen k-form, it could thus be said that by convention, the subalgebra of eigen k-forms is not a differential algebra. These, of course, are simply consequences of conventions. Interestingly, the fact that non-closed eigenforms can be lifted by one degree via the exterior derivative can be seen as the opposite direction of Proposition 4.3.3 which states that eigenforms can be brought down by one degree via the interior product. As a particular case, we have the following corollary.

Corollary 4.4.1.2. *For a continuous or discrete-time dynamical system the the exterior derivative* $\boldsymbol{d}\psi_i(x)$ *of a differentiable eigenfunction* $\psi_i(x)$ *of* $U^t_{\mathbf{F}}$ *or* U^n_S*, at eigenvalue* $e^{\lambda_i^\psi}$ *or* λ_i^ψ*, is an eigencosection at eigenvalue* $e^{\lambda_i^\psi}$ *or* λ_i^ψ*.*

The Module Structure of Eigensections and Eigenforms

As a final consideration, we note that the algebraic properties of Koopman eigenfunctions make it possible to generate nonlinear eigenfunctions out of linear eigenfunctions. For example the 1-dimensional dynamical system generated by the vector field $\mathbf{F}(x) = \lambda x$ has $\psi(x) = x$ as a linear eigenfunction at eigenvalue λ. By utilizing the algebraic structure we can then construct the nonlinear eigenfunctions $\psi^n(x) = x^n$ at eigenvalues λ^n. It is also clear that the exterior products of linear eigenforms will generate nonlinear eigenforms of a higher grade. However, this is not the situation for the eigensections. Namely, the Lie bracket of two linear eigensections, or any sections for that matter, will never generate a nonlinear eigensection. At first glance, this may seem to simply be a difference between the algebraic properties of eigensections and eigenforms. However, upon further investigation, we realized that this observation highlights how one should carefully note the underlying algebra. Thus, while it is true that the spaces $\mathfrak{X}(T\mathcal{M})$ and $\Omega(T^*\mathcal{M})$ are a Lie and exterior algebra, respectively, they are also modules over $C(\mathcal{M}, \mathbb{F})$. It is the module structure that motivates the following result.

Proposition 4.4.4. *For a continuous-time or discrete-time dynamical system, let* $\psi_i(x)$ *be an eigenfunction,* $\boldsymbol{\Phi}_j(x)$ *be an eigensection and* $\boldsymbol{\alpha}_k(x)$ *be an eigen* k*-form. Then, in the continuous-time case,*

1. $\psi_i(x)\boldsymbol{\Phi}_j(x)$ *is also an eigensection of* $\mathcal{L}_{\mathbf{F}}$ *at eigenvalue* $\lambda_i^\psi + \lambda_j^{\boldsymbol{\Phi}}$ *equivalently, at eigenvalue* $e^{(\lambda_i^\psi + \lambda_j^{\boldsymbol{\Phi}})t}$ *for* $V^t_{\mathbf{F}}$*.*

2. $\psi_i(x)\boldsymbol{\alpha}_k(x)$ *is also an eigen* k*-form of* $\mathcal{L}_{\mathbf{F}}$ *at eigenvalue* $\lambda_i^\psi + \lambda_k^{\boldsymbol{\alpha}}$ *equivalently, at*

eigenvalue $e^{(\lambda_i^\psi+\lambda_k^{\boldsymbol{\omega}})t}$ *for* $W_{\mathbf{F}}^t$.

in the discrete-time case,

1. $\psi_i(x)\mathbf{\Phi}_j(x)$ *is also an eigensection of* V_S *at eigenvalue* $\lambda_i^\psi\lambda_j^{\mathbf{\Phi}}$.

2. $\psi_i(x)\boldsymbol{\alpha}_k(x)$ *is also an eigen k-form of* W_S *at eigenvalue* $\lambda_i^\psi\lambda_k^{\boldsymbol{\alpha}}$.

Proof: In the continuous-time case we have

$$\begin{aligned}\mathcal{L}_{\mathbf{F}}(\psi_i(x)\mathbf{\Phi}_j(x)) &= \mathcal{L}_{\mathbf{F}}(\psi_i(x))\mathbf{\Phi}_j(x) + \psi_i(x)\mathcal{L}_{\mathbf{F}}\mathbf{\Phi}_j(x) \\ &= \lambda_i^\psi\psi_i(x)\mathbf{\Phi}_j(x) + \lambda_j^{\mathbf{\Phi}}\psi_j(x)\mathbf{\Phi}_i(x) \\ &= (\lambda_i^\psi + \lambda_j^{\mathbf{\Phi}})\psi_i(x)\mathbf{\Phi}_j(x)\end{aligned} \tag{4.61}$$

and for the group we have

$$\begin{aligned}V_{\mathbf{F}}^t\psi_i(x)\mathbf{\Phi}_j(x) &= \nabla S_{\mathbf{F}}^{-t}|_{S_{\mathbf{F}}^t(x)}(\psi_i(x)\mathbf{\Phi}_j(x)) \circ S_{\mathbf{F}}^t(x) \\ &= \psi_i(S_{\mathbf{F}}^t(x))\nabla S_{\mathbf{F}}^{-t}|_{S_{\mathbf{F}}^t(x)}\mathbf{\Phi}_j(S_{\mathbf{F}}^t(x)) \\ &= \boldsymbol{S}_{\mathbf{F}}^{t*}\psi(x)\boldsymbol{S}_{\mathbf{F}}^{t*}\mathbf{\Phi}_j(x) \\ &= e^{\lambda_i^\psi t}\psi_i(x)e^{\lambda_j^{\mathbf{\Phi}}t}\mathbf{\Phi}_j(x) \\ &= e^{(\lambda_i^\psi+\lambda_j^{\mathbf{\Phi}})t}\psi_i(x)\mathbf{\Phi}_j(x)\end{aligned} \tag{4.62}$$

The second claim is a special case of theorem 4.4.1 since $\psi_i(x)\boldsymbol{\alpha}_k(x) = \psi_i(x) \wedge \boldsymbol{\alpha}_k(x)$.

For the discrete-time case

$$\begin{aligned} V_S \psi_i(x) \mathbf{\Phi}_j(x) &= \nabla S^{-n}(\psi_i(x)\mathbf{\Phi}_j(x)) \circ S_{\mathbf{F}}(x) \\ &= \psi_i(S_{\mathbf{F}}(x)) \nabla S \mathbf{\Phi}_j(S(x)) \\ &= \boldsymbol{S}^* \psi(x) \boldsymbol{S}^* \mathbf{\Phi}_j(x) \\ &= \lambda_i^{\psi} \psi_i(x) \lambda_j^{\mathbf{\Phi}} \mathbf{\Phi}_j(x) \\ &= \lambda_i^{\psi} \lambda_j^{\mathbf{\Phi}} \psi_i(x) \mathbf{\Phi}_j(x) \end{aligned} \tag{4.63}$$

4.4.2 Differential Topological Properties of Eigensections and Eigenforms

we summarize the behavior of eigensections and eigenforms in the presence of a diffeomorphic conjugacy. Consider, again, two dynamical systems, one defined on the space $\mathcal{M}$ generated by a vector field $\mathbf{F}(x) \in \mathfrak{X}(T\mathcal{M})$ or a mapping $S(x)$ and another on the space $\mathcal{N}$ generated by the vector field $\mathbf{G}(y) \in \mathfrak{X}(T\mathcal{N})$ or a mapping $T(y)$. Recall definition 2.4.1 which states that two dynamical systems are said to be topologically conjugate if there exists a homeomorphism $h : \mathcal{M} \to \mathcal{N}$ such that

$$\begin{aligned} h \circ S_{\mathbf{F}}^t(x) &= S_{\mathbf{G}}^t \circ h(x) \\ h \circ S(x) &= T \circ h(x) \end{aligned} \tag{4.64}$$

If h is a diffeomorphism then it is said to be a diffeomorphic conjugacy. We saw in chapter 2 that if and $\psi(y)$ is an eigenfunction of $U_{\mathbf{G}}^t$ then $\psi \circ h(x)$ is an an eigenfunction of $U_{\mathbf{F}}^t$. In other words, eigenfunctions pull back to eigenfunctions under a conjugacy. Thus, it would seem natural to conjecture if the same is true for the spectrum of the other induced operators.

Before stating the results, we remark that perhaps it may be more traditional to ask if eigensections on $\mathcal{M}$ push forward to eigensections on N given that one typically pushes vectors and pulls forms. Of course, when h is a diffeomorphism, if one can push, then one can also pull the spectral objects. Thus, while it may seem to be a matter of choice, one could argue that it is preferable to have spectral objects pull-back to spectral objects. The reason for this is that the notion of a conjugacy is typically used to transform a complex dynamical system into a simpler, but equivalent, dynamical system [PAD16; Har60; Gro59; Ste57]. If, for example, one can find a conjugacy of a nonlinear system to a linear system then the spectral objects of the nonlinear systems can be determined by pulling back the spectrum of the linear system. See sections 4.5 and 2.5 for computing the spectrum of the induced operators of a linear dynamical system.

Theorem 4.4.2. *Let $h : \mathcal{M} \to \mathcal{N}$ be diffeomorphic conjugacy between two, continuous-time or discrete-time, dynamical systems. Also, let $\mathbf{\Phi}_i(y) \in \mathfrak{X}(T\mathcal{N})$ and $\boldsymbol{\alpha}_j(y) \in \Omega^k(T^*\mathcal{N})$ be an eigensection and eigen k-form at eigenvalues $e^{\lambda_i^{\mathbf{\Phi}} t}$ or $\lambda_i^{\mathbf{\Phi}}$ and $e^{\lambda_j^{\boldsymbol{\alpha}} t}$ or $\lambda_j^{\boldsymbol{\alpha}}$. Then*

1. *in the continuous-time case, $\mathbf{\Phi}_i(y)$ pulls back to an eigensection $h^*\mathbf{\Phi}_i(y) \in \mathfrak{X}(T\mathcal{M})$ of $V_{\mathbf{F}}^t$ at eigenvalue $e^{\lambda_i^{\mathbf{\Phi}} t}$ and $\boldsymbol{\alpha}_j(y) \in \Omega_k(T^*\mathcal{M})$ pulls back to an eigen k-form $h^*\boldsymbol{\alpha}_j(y)$ of $W_{\mathbf{F}}^t$ at eigenvalue $e^{\lambda_j^{\boldsymbol{\alpha}} t}$.*

2. *in the discrete-time case $\mathbf{\Phi}_i(y)$ pulls back to an eigensection $h^*\mathbf{\Phi}_i(y) \in \mathfrak{X}(T\mathcal{M})$ of V_S^n at eigenvalue $\lambda_i^{\mathbf{\Phi}}$ and $\boldsymbol{\alpha}_j(y)$ pulls back to an eigen k-form $h^*\boldsymbol{\alpha}_j(y) \in \Omega_k(T^*\mathcal{M})$ of W_S^n at eigenvalue $\lambda_j^{\boldsymbol{\alpha}_j}$.*

Proof: We will prove the result for the continuous-time case and the proof can be repeated for the discrete-time case.

To prove the claim for the eigensection, we note that the negative time conjugacy relation $h(S_{\mathbf{F}}^{-t}(x)) = S_{\mathbf{G}}^{-t}(h(x))$ implies $S_{\mathbf{F}}^{-t}(x) = h^{-1}(S_{\mathbf{G}}^{-t}(h(x)))$ and by taking a derivative we arrive at the following formula

$$\nabla S_{\mathbf{F}}^{-t}|_x = \nabla h^{-1}|_{S_{\mathbf{G}}^{-t}(h(x))} \nabla S_{\mathbf{G}}^{-t}|_{h(x)} \nabla h|_x \tag{4.65}$$

We can then evaluate (4.65) at the point $S_{\mathbf{F}}^t(x)$ and obtain

$$\begin{aligned} \nabla S_{\mathbf{F}}^{-t}|_{S_{\mathbf{F}}^t(x)} &= \nabla h^{-1}|_{S_{\mathbf{G}}^{-t}(h(S_{\mathbf{F}}^t(x)))} \nabla S_{\mathbf{G}}^{-t}|_{h(S_{\mathbf{F}}^t(x))} \nabla h|_{S_{\mathbf{F}}^t(x)} \\ &= \nabla h^{-1}|_{S_{\mathbf{G}}^{-t}(S_{\mathbf{G}}^t(h(x)))} \nabla S_{\mathbf{G}}^{-t}|_{S_{\mathbf{G}}^t(h(x))} \nabla h|_{S_{\mathbf{F}}^t(x)} \\ &= \nabla h^{-1}|_{h(x)} \nabla S_{\mathbf{G}}^{-t}|_{S_{\mathbf{G}}^t(h(x))} \nabla h|_{S_{\mathbf{F}}^t(x)} \end{aligned} \tag{4.66}$$

Equation (4.66) will be useful in the following computation.

$$\begin{aligned} V_{\mathbf{F}}^t h^* \mathbf{\Phi}_i(y) &= V_{\mathbf{F}}^t \nabla h^{-1}|_{h(x)} \mathbf{\Phi}_i \circ h(x) \\ &= \nabla S_{\mathbf{F}}^{-t}|_{S_{\mathbf{F}}^t(x)} \nabla h^{-1}|_{h(S_{\mathbf{F}}^t(x))} \mathbf{\Phi}_i \circ h(S_{\mathbf{F}}^t(x)) \\ &= \nabla h^{-1}|_{h(x)} \nabla S_{\mathbf{G}}^{-t}|_{S_{\mathbf{G}}^t(h(x))} \nabla h|_{S_{\mathbf{F}}^t(x)} \nabla h^{-1}|_{h(S_{\mathbf{F}}^t(x))} \mathbf{\Phi}_i \circ h(S_{\mathbf{F}}^t(x)) \\ &= \nabla h^{-1}|_{h(x)} \nabla S_{\mathbf{G}}^{-t}|_{S_{\mathbf{G}}^t(h(x))} \mathbf{\Phi}_i \circ S_G^t(h(x)) \\ &= \nabla h^{-1}|_{h(x)} V_{\mathbf{G}}^t \mathbf{\Phi}_i(h(x)) \\ &= e^{\lambda_i^{\mathbf{\Phi}} t} \nabla h^{-1}|_{h(x)} \mathbf{\Phi}_i \circ h(x) = e^{\lambda_i^{\mathbf{\Phi}} t} h^* \mathbf{\Phi}_i(y) \end{aligned} \tag{4.67}$$

To prove the claim for the eigen k-form we take a derivative of the conjugacy relation to arrive at

$$\nabla h|_{S_{\mathbf{F}}^t(x)} \nabla S_{\mathbf{F}}^t|_x = \nabla S_{\mathbf{G}}^t|_{h(x)} \nabla h|_x \tag{4.68}$$

Now, let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ denote k tangent vectors at x. Equation (4.68) will be useful in the following computation.

$$
\begin{aligned}
W_{\mathbf{F}}^t h^* \boldsymbol{\alpha}_j(y)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) &= W_{\mathbf{F}}^t \boldsymbol{\alpha}_j \circ h(x)(\nabla h|_x \boldsymbol{v}_1, \ldots, \nabla h|_x \boldsymbol{v}_k) \\
&= \boldsymbol{\alpha}_j \circ h(S_{\mathbf{F}}^t(x))(\nabla h|_{S_{\mathbf{F}}^t(x)} \nabla S_{\mathbf{F}}^t|_x \boldsymbol{v}_1, \ldots, \nabla h|_{S_{\mathbf{F}}^t(x)} \nabla S_{\mathbf{F}}^t|_x \boldsymbol{v}_k) \\
&= \boldsymbol{\alpha}_j \circ S_{\mathbf{G}}^t(h(x))(\nabla S_{\mathbf{G}}^t|_{h(x)} \nabla h|_x \boldsymbol{v}_1, \ldots, \nabla S_{\mathbf{G}}^t|_{h(x)} \nabla h|_x \boldsymbol{v}_k) \\
&= W_{\mathbf{G}}^t \boldsymbol{\alpha}_j \circ h(x)(\nabla h|_x \boldsymbol{v}_1, \ldots, \nabla h|_x \boldsymbol{v}_k) \\
&= e^{\lambda_j^{\boldsymbol{\alpha}} t} \boldsymbol{\alpha}_j \circ h(x)(\nabla h|_x \boldsymbol{v}_1, \ldots, \nabla h|_x \boldsymbol{v}_k) = e^{\lambda_j^{\boldsymbol{\alpha}} t} h^* \boldsymbol{\alpha}_j
\end{aligned}
\tag{4.69}
$$

4.5 Linear Systems With Simple Spectrum

For this section we will consider a linear continuous-time dynamical system generated by the vector field $\mathbf{F}(x) = \mathbf{A}x$ or a linear discrete-time dynamical system generated by the transformation $S(x) = \mathbf{A}x$ with $\mathbf{A} \in \mathbb{R}^{n \times n}$. We will assume that the matrix $\mathbf{A}$ has distinct eigenvalues λ_i and a full set of linearly independent right and left eigenvectors.

4.5.1 The Eigensections of A Linear Dynamical System

We begin by finding an explicit expression for the Lie derivative of a linear section $\mathbf{G}(x) = \mathbf{B}x$ in the direction of $\mathbf{F}(x) = \mathbf{A}x$. Since the flow of a linear dynamical system is well known we utilize theorem 4.1.3.

$$
\begin{aligned}
\mathcal{L}_{\mathbf{F}} \mathbf{G}(x) &= \frac{d}{dt}\bigg|_{t=0} \boldsymbol{S}_{\mathbf{F}}^{t*} \mathbf{G}(x) \\
&= \frac{d}{dt}\bigg|_{t=0} \nabla S^{-t}|_{S_{\mathbf{F}}^t(x)} \cdot \mathbf{G} \circ S_{\mathbf{F}}^t(x) \\
&= \frac{d}{dt}\bigg|_{t=0} e^{-\mathbf{A}t} \mathbf{B} e^{At} x \\
&= (-\mathbf{A} e^{-\mathbf{A}t} \mathbf{B} e^{At} x + e^{-\mathbf{A}t} \mathbf{B} \mathbf{A} e^{At} x)\bigg|_{t=0} \\
&= (\mathbf{BA} - \mathbf{AB}) x
\end{aligned}
\tag{4.70}
$$

Equation (4.70) shows that the Lie derivative of two linear sections $\mathbf{F}(x) = \mathbf{A}x$ and $\mathbf{G}(x) = \mathbf{B}x$ is again a linear section $\mathbf{H}(x) = (\mathbf{BA} - \mathbf{AB})x$.

For the induced Koopman operator on functions, the space of all constant functions are trivially eigenfunctions of the operator. For the induced operator on sections, the situation is different in that not every constant section is an eigensection but only those which are invariant under the tangent flow $\nabla S^t_{\mathbf{F}}$. From this, it is clear that the only the constant eigensections are those which everywhere point in the direction of the eigenvectors of $\mathbf{A}$.

Proposition 4.5.1. *Let λ_i, v_i denote the eigenvalues and right eigenvectors of the matrix $\mathbf{A}$, corresponding to the linear continuous-time or discrete time dynamical system. Then,*

1. *for the continuous-time case, the constant vector fields $\mathbf{\Phi}_i(x) = v_i$ are eigensections of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $-\lambda_i$ equivalently, at eigenvalue $e^{-\lambda_i t}$ for $V^t_{\mathbf{F}}$.*

2. *for the discrete-time case, the constant vector fields $\mathbf{\Phi}_i(x) = v_i$ are eigensections of V_S at eigenvalue $\frac{1}{\lambda_i}$ for $\lambda_i \neq 0$.*

Proof: In the continuous-time case, we have

$$\begin{aligned} V^t_{\mathbf{F}}\mathbf{\Phi}_i(x) &= V^t_{\mathbf{A}} v_i \\ &= \nabla(e^{-\mathbf{A}t}x) \cdot v_i \circ e^{At}x \\ &= e^{-\mathbf{A}t} v_i \\ &= e^{-\lambda_i t} v_i = e^{-\lambda_i t}\mathbf{\Phi}_i(x) \end{aligned} \tag{4.71}$$

for the group. For the generator we apply theorem 4.1.2.

$$\begin{aligned} \mathcal{L}_{\mathbf{F}}\mathbf{\Phi}_i(x) &= \nabla\mathbf{G}(x) \cdot \mathbf{F}(x) - \nabla\mathbf{F}(x) \cdot \mathbf{G}(x) \\ &= \nabla(v_i) \cdot \mathbf{A}x - \nabla(\mathbf{A}x) \cdot v_i \\ &= -\mathbf{A}v_i = -\lambda_i v_i = -\lambda_i \mathbf{\Phi}_i(x) \end{aligned} \tag{4.72}$$

In the discrete-time case we have

$$\begin{aligned} V_S\mathbf{\Phi}_i(x) &= V_{\mathbf{A}}v_i \\ &= \nabla(\mathbf{A}^{-1}x)\cdot v_i \circ \mathbf{A}x \\ &= \mathbf{A}^{-1}v_i \\ &= \frac{1}{\lambda_i}v_i = \frac{1}{\lambda_i}\mathbf{\Phi}_i(x) \end{aligned} \tag{4.73}$$

Since a linear, non-constant, eigensection would have a corresponding matrix representation we attempt to construct a matrix from the spectrum of $\mathbf{A}$. The most natural way to do so is to consider the outer product of eigenvectors of $\mathbf{A}$.

Proposition 4.5.2. *Let $\lambda_i, v_i, w_i^\intercal$ denote the eigenvalues, right eigenvectors and left eigenvectors of the matrix $\mathbf{A}$, corresponding to the linear continuous-time dynamical system or the discrete time system. Then,*

1. *for the continuous-time case, the linear vector fields $\mathbf{\Phi}_{ij}(x) = v_i \otimes w_j x$ are linear eigensections of $\mathcal{L}_{\mathbf{F}}$ at eigenvalues $\lambda_{ij} = \lambda_j - \lambda_i$ equivalently, at eigenvalues $e^{\lambda_{ij}t}$ for $V_{\mathbf{F}}^t$.*

2. *for the discrete-time case, the linear vector fields $\mathbf{\Phi}_{ij}(x) = v_i \otimes w_j x$ are linear eigensections of V_S at eigenvalue $\lambda_{ij} = \frac{\lambda_j}{\lambda_i}$ for $\lambda_i \neq 0$.*

Proof: In the continuous-time case, we have

$$\begin{aligned} V_{\mathbf{F}}^t\mathbf{\Phi}_{ij}(x) &= V_{\mathbf{A}}^t v_i \otimes w_j x \\ &= \nabla(e^{-\mathbf{A}t}x)\cdot v_i w_j^\intercal \circ e^{\mathbf{A}t}x \\ &= e^{-\mathbf{A}t}v_i w_j^\intercal e^{\mathbf{A}t}x \\ &= e^{-\lambda_i t}v_i w_j^\intercal e^{\lambda_j t}x \\ &= e^{(\lambda_j-\lambda_i)t}v_i w_j^\intercal x = e^{\lambda_{ij}t}\mathbf{\Phi}_{ij}(x) \end{aligned} \tag{4.74}$$

and for the generator we have

$$\begin{aligned}
\mathcal{L}_{\mathbf{F}}\mathbf{\Phi}_{ij}(x) &= (\mathbf{\Phi}_{ij}\mathbf{A} - \mathbf{A}\mathbf{\Phi}_{ij})(x) \\
&= (v_i w_j^\intercal \mathbf{A} - \mathbf{A} v_i w_j^\intercal)x \\
&= (\lambda_j v_i w_j^\intercal - \lambda_i v_i w_j^\intercal)x \\
&= (\lambda_j - \lambda_i) v_i w_j^\intercal x = \lambda_{ij}\mathbf{\Phi}_{ij}(x)
\end{aligned} \tag{4.75}$$

In the discrete-time case we have,

$$\begin{aligned}
V_S\mathbf{\Phi}_{ij}(x) &= V_{\mathbf{A}} v_i \otimes w_j x \\
&= \nabla(\mathbf{A}^{-1}x) \cdot v_i w_j^\intercal \mathbf{A}x \\
&= \mathbf{A}^{-1} v_i w_j^\intercal \mathbf{A}x \\
&= \frac{\lambda_j}{\lambda_i} v_i w_j^\intercal x = \lambda_{ij}\mathbf{\Phi}_{ij}(x)
\end{aligned} \tag{4.76}$$

we saw in section 2.5 that the linear Koopman eigenfunctions $\psi_i(x)$ associated with a linear dynamical system can be used to obtain a spectral expansion of an arbitrary, vector-valued, linear function $\boldsymbol{f}(x) = \mathbf{C}x$, for any $\mathbf{C} \in \mathbb{R}^{n\times n}$. Moreover, the spectral expansion can be utilized to provide a complete description of the time evolution of the function $\boldsymbol{f}(x)$. In a similar manner we utilize the linear eigensections to derive the corresponding spectral expansion of an arbitrary linear vector field $\mathbf{G}(x) = \mathbf{B}x$. To do so we will require the notion of an inner product on the space $\mathbb{F}^{n\times n}$ of n by n matrices for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Specifically, we utilize the Frobenius inner product of two n by n matrices $\mathbf{C}, \mathbf{D}$ defined as $\langle \mathbf{C}, \mathbf{D}\rangle_{\mathbb{F}} = \mathrm{Tr}(\mathbf{C}^\intercal \mathbf{D})$, where Tr is the trace of a matrix and if $\mathbb{F} = \mathbb{C}$ then $(\cdot)^\intercal$ is the Hermitian transpose.

Proposition 4.5.3. *The time evolution of an arbitrary linear vector field* $\mathbf{G}(x) = \mathbf{B}x$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ *under the dynamics of a continuous-time or discrete-time dynamical system, is* *given by the following spectral expansion.*

1. *In the continuous-time case*

$$V_{\mathbf{F}}^t \mathbf{G}(x) = \sum_{ij=1}^{n} e^{\lambda_{ij} t} \mathbf{\Phi}_{ij}(x) b_{ij} \tag{4.77}$$

for the group and for the generator

$$\mathcal{L}_{\mathbf{F}}^n \mathbf{G}(x) = \sum_{ij=1}^{n} \lambda_{ij}^n \mathbf{\Phi}_{ij}(x) b_{ij} \tag{4.78}$$

2. *In the discrete-time case*

$$V_S^n \mathbf{G}(x) = \sum_{ij=1}^{n} \lambda_{ij}^n \mathbf{\Phi}_{ij}(x) b_{ij} \tag{4.79}$$

Where b_{ij} is the projection of $\mathbf{B}$, under the Frobenius inner product, onto the eigenspace spanned by $\mathbf{\Phi}_{ij}(x)$.

Proof: By our assumption, $\mathbf{A}$ has a full set of right eigenvectors v_i, left eigenvectors w_i and distinct eigenvalues λ_i. Now, since the right and left eigenvectors form a basis for $\mathbb{F}^n$ then their tensor product also forms a basis for $\mathbb{F}^n \otimes \mathbb{F}^n \simeq \mathbb{F}^{n\times n}$, see [AMR83] page 272. Hence we can expand $\mathbf{B}$ in the eigensection basis as $\mathbf{B} = \sum_{ij=1}^{n} b_{ij} \mathbf{\Phi}_{ij}$. From this we can compute the time evolution of $\mathbf{G}(x) = \mathbf{B}x$. In the continuous-time case we have

$$V_{\mathbf{F}}^t \mathbf{G}(x) = V_{\mathbf{F}}^t \sum_{ij=1}^{n} b_{ij} \mathbf{\Phi}_{ij} = \sum_{ij=1}^{n} b_{ij} V_{\mathbf{F}}^t \mathbf{\Phi}_{ij} x = \sum_{ij=1}^{n} b_{ij} e^{\lambda_{ij} t} \mathbf{\Phi}_{ij} x \tag{4.80}$$

for the group and for the generator we have

$$\mathcal{L}_{\mathbf{F}}^n \mathbf{G}(x) = \mathcal{L}_{\mathbf{F}}^n \sum_{ij=1}^{n} b_{ij} \mathbf{\Phi}_{ij} = \sum_{ij=1}^{n} b_{ij} \mathcal{L}_{\mathbf{F}}^n \mathbf{\Phi}_{ij} x = \sum_{ij=1}^{n} b_{ij} \lambda_{ij}^n \mathbf{\Phi}_{ij} x \tag{4.81}$$

For the discrete-time case

$$V_S^n \mathbf{G}(x) = V_S^n \sum_{ij=1}^{n} b_{ij} \mathbf{\Phi}_{ij} = \sum_{ij=1}^{n} b_{ij} V_S^n \mathbf{\Phi}_{ij} x = \sum_{ij=1}^{n} b_{ij} \lambda_{ij}^n \mathbf{\Phi}_{ij} x \tag{4.82}$$

We note that if instead of the matrix $\mathbf{B}$ we had the matrix $\mathbf{C} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ then only the b_{ij} coefficients in the expansion would change and the same is true if instead of $\mathbf{B}$ we had $\mathbf{C} = \mathbf{D}\mathbf{B}$ or $\mathbf{C} = \mathbf{B}\mathbf{D}$. Which indicates that the eigenvalues and eigensections $(\lambda_{ij}, \mathbf{\Phi}_{ij})$ are intrinsic to the dynamics $\mathbf{F}(x) = \mathbf{A}x$ while the coefficients b_{ij} are not and depend on the choice of $\mathbf{B}$.

4.5.2 The Eigenforms of A Linear Dynamical System

A linear cosection

$$\boldsymbol{\gamma}(x) = \begin{bmatrix} (b_{11}x_1, + \quad \cdots +, b_{1n}x_n)\boldsymbol{d}x_1 & \cdots & (b_{n1}x_1, + \quad \cdots +, b_{nn}x_n)\boldsymbol{d}x_n \end{bmatrix} \tag{4.83}$$

viewed as a linear functional $\boldsymbol{\gamma} : T\mathcal{M} \to \mathbb{F}$, has the following matrix representation

$$\boldsymbol{\gamma}(x) = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}x_1 & & \\ & \ddots & \\ & & \boldsymbol{d}x_n \end{bmatrix} = x^{\mathsf{T}} \mathbf{B} \boldsymbol{d}x$$

and the Lie derivative of the linear differential one-form $\boldsymbol{\gamma}(x)$ in the direction of $\mathbf{F}(x) = \mathbf{A}x$ can be computed, again, by applying theorem 4.1.4.

$$\begin{aligned}
\mathcal{L}_{\mathbf{F}}\boldsymbol{\gamma}(x) &= \frac{d}{dt}\Big|_{t=0} \boldsymbol{S}_{\mathbf{F}}^{t*}\boldsymbol{\gamma}(x) \\
&= \frac{d}{dt}\Big|_{t=0} \boldsymbol{\gamma} \circ S_{\mathbf{F}}^{t}(x) \cdot \nabla S_{\mathbf{F}}^{t} \boldsymbol{d}x \\
&= \frac{d}{dt}\Big|_{t=0} (e^{\mathbf{A}t}x)^{\intercal}\mathbf{B} \cdot \nabla(e^{\mathbf{A}t}x)\boldsymbol{d}x \\
&= \frac{d}{dt}\Big|_{t=0} x^{\intercal} e^{\mathbf{A}^{\intercal}t}\mathbf{B}e^{\mathbf{A}t}\boldsymbol{d}x \\
&= (x^{\intercal}\mathbf{A}^{\intercal}e^{\mathbf{A}^{\intercal}t}\mathbf{B}e^{\mathbf{A}t}\boldsymbol{d}x + x^{\intercal}e^{\mathbf{A}^{\intercal}t}\mathbf{B}\mathbf{A}e^{\mathbf{A}t}\boldsymbol{d}x)|_{t=0} \\
&= x^{\intercal}(\mathbf{A}^{\intercal}\mathbf{B} + \mathbf{B}\mathbf{A})\boldsymbol{d}x
\end{aligned} \tag{4.84}$$

We saw in section 2.5 that the linear eigenfunctions associated with a linear dynamical system are given by $\psi_i(x) = \langle x, w_i \rangle$ where $w_i^{\intercal}$ is a left eigenvector of the matrix $\mathbf{A}$. Furthermore, according to propositions 4.5.1-4.5.2 the eigensections for a linear system were also built from the spectrum of $\mathbf{A}$. One would expect a similar story to play out for the eigenforms. As before, we first sort out the constant eigenforms.

Proposition 4.5.4. *Let λ_i, w_i denote the eigenvalues and left eigenvectors of the matrix $\mathbf{A}$, corresponding to a linear, continuous-time or discrete-time, dynamical system. Then,*

1. *for the continuous-time case, the constant differential 1-forms $\boldsymbol{\omega}_i(x) = w_i\boldsymbol{d}x$ are eigencosections of $\mathcal{L}_{\mathbf{F}}$ with eigenvalues λ_i equivalently, at eigenvalue $e^{\lambda_i t}$ for $W_{\mathbf{F}}^t$.*

2. *for the discrete-time case, the constant differential 1-forms $\boldsymbol{\omega}_i(x) = w_i\boldsymbol{d}x$ are eigencosections of W_S at eigenvalue λ_i.*

Proof: In the continuous-time case, we have

$$\begin{aligned} W_{\mathbf{F}}^t \boldsymbol{\omega}_i(x) &= W_{\mathbf{A}}^t w_i \boldsymbol{d}x \\ &= w_i \circ e^{\mathbf{A}t} x \nabla (e^{\mathbf{A}t} x) \boldsymbol{d}x \\ &= w_i e^{\mathbf{A}t} \boldsymbol{d}x \\ &= e^{\lambda_i t} w_i \boldsymbol{d}x = e^{\lambda_i t} \boldsymbol{\omega}_i(x) \end{aligned} \tag{4.85}$$

and for the generator we apply definition 4.1.38

$$\begin{aligned} \mathcal{L}_{\mathbf{F}} \boldsymbol{\omega}_i(x) &= \mathbf{F}^{\intercal}(x) \cdot (\nabla \boldsymbol{\omega}_i^{\intercal}(x))^{\intercal} + \boldsymbol{\omega}_i(x) \cdot \nabla \mathbf{F}(x) \\ &= x^{\intercal} \mathbf{A}^{\intercal} \cdot (\nabla (\boldsymbol{d}x w_i^{\intercal}))^{\intercal} + w_i \nabla (\mathbf{A}x) \boldsymbol{d}x \\ &= w_i \mathbf{A} \boldsymbol{d}x = \lambda_i w_i \boldsymbol{d}x = \lambda_i \boldsymbol{\omega}_i(x) \end{aligned} \tag{4.86}$$

In the discrete-time case we have

$$\begin{aligned} W_S \boldsymbol{\omega}_i(x) &= W_{\mathbf{A}} w_i \boldsymbol{d}x \\ &= w_i \circ \mathbf{A}x \cdot \nabla (\mathbf{A}x) \boldsymbol{d}x \\ &= w_i \mathbf{A} \boldsymbol{d}x \\ &= \lambda_i w_i \boldsymbol{d}x = \lambda_i \boldsymbol{\omega}(x) \end{aligned} \tag{4.87}$$

Similar to the linear eigensections, the linear eigencosections can be built out of tensor products of the constant eigencosections.

Proposition 4.5.5. *Let $\lambda_i, w_i^{\intercal}$ denote the eigenvalues and left eigenvectors of the matrix $\mathbf{A}$, corresponding to a linear, continuous-time or discrete-time, dynamical system. Then,*

1. *for the continuous-time case, the linear differential 1-forms $\boldsymbol{\omega}_{ij}(x) = x^{\intercal} w_i \otimes w_j \boldsymbol{d}x$ are linear eigencosections of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $\lambda_{ij} = \lambda_i + \lambda_j$ equivalently, at eigenvalue $e^{\lambda_{ij} t}$ for $W_{\mathbf{F}}^t$.*

2. *for the discrete-time case, the linear differential* 1*-forms* $\boldsymbol{\omega}_{ij}(x) = x^\intercal w_i \otimes w_j \boldsymbol{d}x$ *are linear eigencosections of* W_S *at eigenvalue* $\lambda_i \lambda_j$*.*

Proof: In the continuous-time case, we have

$$\begin{aligned} W^t_{\mathbf{F}} \boldsymbol{\omega}_{ij}(x) &= W^t_{\mathbf{A}} x^\intercal w_i \otimes w_j \boldsymbol{d}x \\ &= x^\intercal w_i w_j^\intercal \circ e^{\mathbf{A}t} \cdot \nabla (e^{\mathbf{A}t} x) \boldsymbol{d}x \\ &= (e^{\mathbf{A}t} x)^\intercal w_i w_j^\intercal e^{\mathbf{A}t} \boldsymbol{d}x \\ &= x^\intercal e^{\mathbf{A}^\intercal t} w_i w_j^\intercal e^{\mathbf{A}t} \boldsymbol{d}x \\ &= x^\intercal e^{\lambda_i t} w_i w_j^\intercal e^{\lambda_j t} \boldsymbol{d}x \\ &= e^{(\lambda_i + \lambda_j)t} x^\intercal w_i w_j^\intercal \boldsymbol{d}x = e^{\lambda_{ij} t} \boldsymbol{\omega}_{ij}(x) \end{aligned} \tag{4.88}$$

for the group and for the generator we have

$$\begin{aligned} \mathcal{L}_{\mathbf{F}} \boldsymbol{\omega}_{ij}(x) &= x^\intercal (\mathbf{A}^\intercal \boldsymbol{\omega}_{ij} + \boldsymbol{\omega}_{ij} \mathbf{A}) \boldsymbol{d}x \\ &= x^\intercal (\mathbf{A}^\intercal w_i w_j^\intercal + w_i w_j^\intercal \mathbf{A}) \boldsymbol{d}x \\ &= x^\intercal (\lambda_i w_i w_j^\intercal + \lambda_j w_i w_j^\intercal) \boldsymbol{d}x \\ &= (\lambda_i + \lambda_j) x^\intercal w_i w_j^\intercal \boldsymbol{d}x = \lambda_{ij} \boldsymbol{\omega}_i(x) \end{aligned} \tag{4.89}$$

In the discrete-time case we have

$$\begin{aligned} W_S \boldsymbol{\omega}_{ij}(x) &= W_{\mathbf{A}} x^\intercal w_i \otimes w_j \boldsymbol{d}x \\ &= (\mathbf{A}x)^\intercal w_i w_j^\intercal \nabla (\mathbf{A}x) \boldsymbol{d}x \\ &= x^\intercal \mathbf{A}^\intercal w_i w_j^\intercal \mathbf{A} \boldsymbol{d}x \\ &= \lambda_i \lambda_j x^\intercal w_i w_j^\intercal \boldsymbol{d}x = \lambda_i \lambda_j \boldsymbol{\omega}_{ij}(x) \end{aligned} \tag{4.90}$$

We can now build the higher-order linear eigenforms out of the eigencosections we just computed via the algebraic properties. However, as mentioned in section 4.4, the exterior product of two linear eigencosections will in general, produce nonlinear eigenforms. We thus consider the constant eigenforms to create the higher degree linear eigenforms.

Proposition 4.5.6. *Let λ_i, w_i denote the eigenvalues, and left eigenvectors of the matrix $\mathbf{A}$, corresponding to the linear continuous-time or discrete-time dynamical system. Also, let $\boldsymbol{\omega}_i(x) = w_i \boldsymbol{d}x$ be the constant eigencosection at eigenvalue $e^{\lambda_i t}$ or λ_i. Then*

1. *for the continuous-time case, the differential k-form $\boldsymbol{\alpha}_{i_1,\ldots,i_k}(x) = x^\intercal \boldsymbol{\omega}_{i_1} \wedge \overset{k}{\cdots} \wedge \boldsymbol{\omega}_{i_k}$, for $i_1, \ldots, i_k$ distinct, is an eigen k-form of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $\lambda_{i_1,\ldots,i_k} = \sum_{j=1}^{k} \lambda_{i_j}$ equivalently, at eigenvalue $e^{\lambda_{i_1,\ldots,i_k} t}$ for the group.*

2. *for the discrete-time case, the differential k-form $\boldsymbol{\alpha}_{i_1,\ldots,i_k}(x) = x^\intercal \boldsymbol{\omega}_{i_1} \wedge \overset{k}{\cdots} \wedge \boldsymbol{\omega}_{i_k}$, for $i_1, \ldots, i_k$ distinct, is an eigen k-form of W_S at eigenvalue $\lambda_{i_1,\ldots,i_k} = \prod_{j=1}^{k} \lambda_{i_j}$ for the group.*

Proof: In both the discrete and continuous-time cases the claim that $\boldsymbol{\alpha}_{i_1,\ldots,i_k}(x)$ is an eigen k-form follows from theorem 4.4.1 and corollary 4.4.1.1.

4.6 Properties of Closed and Exact Eigenforms

In this brief section, we attempt to characterize the closed and exact properties of an eigenform. Now, every exact differential form is a closed differential form, the other way around depends on the cohomology groups of $\mathcal{M}$. Assuming the kth cohomology group is non-trivial, the pertinent question is whether it is possible to obtain a closed eigen k-form, which is not exact. The following proposition demonstrates that this can not happen

Proposition 4.6.1. *Consider a continuous-time dynamical system generated by $\mathbf{F}(x) \in \mathfrak{X}(T\mathcal{M})$. Then,*

1. *Every closed eigen k-form $\boldsymbol{\alpha}_i(x)$ with $\lambda_i \neq 0$ is exact.*

2. *Every exact eigen k-form is the exterior derivative of a non-closed eigen $(k-1)$-form.*

Proof: To prove the first claim let $\boldsymbol{\alpha}_i(x)$, be a closed eigen k-form of $\mathcal{L}_{\mathbf{F}}$ at eigenvalue $\lambda_i^{\boldsymbol{\alpha}}$. Cartan's formula shows that

$$\mathcal{L}_{\mathbf{F}}\boldsymbol{\alpha}_i(x) = \boldsymbol{d}\iota_{\mathbf{F}}\boldsymbol{\alpha}_i(x) + \iota_{\mathbf{F}}\boldsymbol{d}\boldsymbol{\alpha}_i(x) = \boldsymbol{d}\iota_{\mathbf{F}}\boldsymbol{\alpha}_i(x) = \lambda_i\boldsymbol{\alpha}_i(x) \tag{4.91}$$

which shows that $\boldsymbol{\alpha}_i(x)$ is the exterior derivative of $\frac{1}{\lambda_i}\iota_{\mathbf{F}}\boldsymbol{\alpha}_i(x)$ and hence exact.

The second claim follows from theorem 4.4.1 and the convention that the zero k-form is not a proper eigen k-form.

4.7 Integrability of Eigendistributions and Their Foliations

4.7.1 The Vector fields Point of View

Loosely speaking, the fundamental theorem on flows [Lee12] tells us that we can "integrate" a complete vector field $\mathbf{F}(x)$ and the resulting integral curves will be disjoint, immersed 1-dimensional submanifolds of $\mathcal{M}$. Furthermore, the integral curves are everywhere tangent to $\mathbf{F}(x)$ and foliate $\mathcal{M}$. One can then ask if it is possible to "integrate" a collection of d linearly independent vector fields to obtain a d-dimensional immersed submanifold whose tangent bundle is everywhere spanned by the d vector fields. To make these statements more precise, we collect some necessary definitions and results.

Relevant Definitions

Definition 4.7.1. *A d-dimensional* ***distribution****, denoted Δ, is a d-dimensional subbundle of $T\mathcal{M}$. If it is a smooth subbundle then it is called a* ***smooth distribution****.*

Definition 4.7.2. *Let Δ be a smooth distribution on $\mathcal{M}$. A **local section** of Δ, defined on an open subset of $\mathcal{M}$, is a smooth vector field $\mathbf{H}(x)$ such that $\mathbf{H}(x) \in \Delta_x$ at every x.*

Definition 4.7.3. *Let Δ be a d-dimensional distribution. If every $x \in \mathcal{M}$ has a local neighborhood U on which there are smooth vector fields $\mathbf{G}_1(x), \ldots, \mathbf{G}_d(x)$ such that at every point $y \in U$, $\mathbf{G}_1(y), \ldots, \mathbf{G}_d(y)$ form a basis for Δ_y then Δ is said to be **spanned by the vector fields** $\mathbf{G}_1(x), \ldots, \mathbf{G}_d(x)$.*

Definition 4.7.4. *A distribution Δ is said to be **involutive** if it is closed under the Lie bracket.*

Definition 4.7.5. *Let Δ be a smooth distribution on $\mathcal{M}$. An immersed submanifold $\mathcal{N} \subset \mathcal{M}$ is called an **integral manifold of** Δ if $T_x\mathcal{N} = \Delta_x$ for every $x \in \mathcal{N}$.*

Definition 4.7.6. *A distribution Δ is said to be **integrable** if every $x \in \mathcal{M}$ is contained in an integral manifold of Δ.*

Definition 4.7.7. *Let $\mathfrak{F} = \{\mathfrak{L}_a\}_{a\in A}$ be a partition of $\mathcal{M}$ into disjoint connected sets called **leaves**. $\mathfrak{F}$ is called a p-dimensional, class C^r **foliation** if every point of $\mathcal{M}$ has a neighborhood U with a local C^r coordinate chart $x = (x_1, \ldots, n_n) : U \to \mathbb{R}^n$ such that for each leaf $\mathfrak{L}_a$ the connected components $(U \cap \mathfrak{L}_a)^b$ of $(U \cap \mathfrak{L}_a)$ are given by $x((U \cap \mathfrak{L}_a)^b) = (x_1, \ldots, x_p, x_{p+1} = c_1^{a,b}, \ldots, x_n = c_{n-p}^{a,b})$, $c_i^{a,b}$ are constants for every $a \in A$ and b.*

Definition 4.7.8. *Let $\{\mathbf{\Phi}_1(x), \ldots, \mathbf{\Phi}_d(x)\}$ be a collection of linearly independent eigensections of a dynamical system. The distribution $\Delta_{\mathbf{\Phi}} = span\{\mathbf{\Phi}_1(x), \ldots, \mathbf{\Phi}_d(x)\}$ is said to be an **eigendistribution**.*

The Eigendistributions of A Hyperbolic Dynamical System

In the context of this work, we are interested in determining under what conditions can an eigendistribution be integrable. If so, what are the resulting integral manifolds, and do they also foliate $\mathcal{M}$? The answer to these questions will undoubtedly involve the contents of the Frobenius theorem which has two (dual) formulations, one in terms of vector fields and the other in terms of differential forms.

Proposition 4.7.1. *Let $\mathfrak{F} = \{\mathfrak{L}_a\}_{a \in A}$ be a foliation of $\mathcal{M}$. The set $\{\cup_{a\in A} \cup_{x\in a} T_x\mathfrak{L}_a\}$ is a subbundle of $T\mathcal{M}$ called the* ***tangent bundle to the foliation*** *and is denoted $T(\mathcal{M}, \mathfrak{F})$*

Proof: See [AMR83] page 266.

Theorem 4.7.1 (**Frobenius** (Vector Field Formulation))**.** *Let Δ be a d-dimensional distribution on $\mathcal{M}$. The following are equivalent*

1. *Δ is involutive.*
2. *Δ is integrable.*
3. *There exists a d-dimensional foliation $\mathfrak{F}$ of $\mathcal{M}$ such that $\Delta = T(\mathcal{M}, \mathfrak{F})$.*

The vector fields formulation of the Frobenius theorem shows that our previous questions are equivalent to determining under what conditions is an eigendistribution involutive. The following theorem shows that if the dynamical system is uniformly hyperbolic and some dimensionality conditions are placed on the eigendistribution, one can show the eigendistribution is guaranteed to be involutive. Recall that uniform hyperbolicity for a discrete-time system guarantees a splitting of the tangent bundle into the direct sum $T\mathcal{M} = TE^s \oplus TE^u$. For a continuous-time dynamical system the definition is similar.

Definition 4.7.9. *A continuous-time dynamical system $(\mathcal{M}, S^t_{\mathbf{F}})$ is said to be* ***hyperbolic***

1. *If $\mathbf{F}(x) \neq 0$ for all $x \in \mathcal{M}$.*

2. *If there exists two subbundles E^S and E^u such that tangent bundle of $\mathcal{M}$ splits as a continuous Whitney sum $T\mathcal{M} = E^s \oplus E^u \oplus [\mathbf{F}]$. Where $[\mathbf{F}]$ denotes the one-dimensional subbundle spanned by $\mathbf{F}(x)$.*

3. *For every positive $t \in \mathbb{R}$ and for some Riemannian metric g:*

$$\begin{aligned} &\|\nabla S^t_{\mathbf{F}} \boldsymbol{v}\|_g \geq a \cdot e^{\lambda_i t}\|\boldsymbol{v}\|_g, \quad \|\nabla S^{-t}_{\mathbf{F}} \boldsymbol{v}\|_g \leq b \cdot e^{-\lambda_i t}\|\boldsymbol{v}\|_g, \ \text{if} \quad \boldsymbol{v} \in E^u \\ &\|\nabla S^t_{\mathbf{F}} \boldsymbol{v}\|_g \leq b \cdot e^{-\lambda_i t}\|\boldsymbol{v}\|_g, \quad \|\nabla S^{-t}_{\mathbf{F}} \boldsymbol{v}\|_g \geq a \cdot e^{\lambda_i t}\|\boldsymbol{v}\|_g, \ \text{if} \quad \boldsymbol{v} \in E^s \end{aligned} \tag{4.92}$$

The constants a, b, λ are positive and independent of t and $\boldsymbol{v}$ but a and b depend on the metric g. The subbundle E^u is said to be ***the unstable subbundle*** *and the subbundle E^s is said to be* ***the stable subbundle****.*

Theorem 4.7.2. *Consider a uniformly hyperbolic continuous-time or discrete-time dynamical system. Let n_s, n_u denote the dimensions of the stable E^s and unstable E^u subbundles and let $\{\boldsymbol{\Phi}_1(x), \ldots, \boldsymbol{\Phi}_{n_s+n_u}(x)\}$ be a collection of smooth, linearly independent, eigensections of $V^t_{\mathbf{F}}$ or V^n_S at eigenvalues $\{e^{\lambda_1 t}, \ldots, e^{\lambda_{n_s+n_u} t}\}$ or $\{\lambda_1, \ldots, \lambda_{n_s+n_u}\}$.*
In the continuous-time case define the following distributions

1. $\Delta^s_{\boldsymbol{\Phi}} = span\{\boldsymbol{\Phi}_k | V^t_{\mathbf{F}} \boldsymbol{\Phi}_k = e^{\lambda^{\boldsymbol{\Phi}}_k t},\ \Re(\lambda^{\boldsymbol{\Phi}}_k) > 0\},\ 1 < k < n_s$.

2. $\Delta^u_{\boldsymbol{\Phi}} = span\{\boldsymbol{\Phi}_l | V^t_{\mathbf{F}} \boldsymbol{\Phi}_l = e^{\lambda^{\boldsymbol{\Phi}}_l t},\ \Re(\lambda^{\boldsymbol{\Phi}}_l) < 0\},\ 1 < l < n_u$.

and in the discrete-time case

1. $\Delta^s_{\boldsymbol{\Phi}} = span\{\boldsymbol{\Phi}_k | V_S \boldsymbol{\Phi}_k = \lambda^{\boldsymbol{\Phi}}_k,\ |\lambda^{\boldsymbol{\Phi}}_k| > 0\},\ 1 < k < n_s$.

2. $\Delta^u_{\boldsymbol{\Phi}} = span\{\boldsymbol{\Phi}_l | V_S \boldsymbol{\Phi}_l = \lambda^{\boldsymbol{\Phi}}_l,\ |\lambda^{\boldsymbol{\Phi}}_l| < 0\},\ 1 < l < n_u$.

called the ***stable*** *and* ***unstable eigendistributions****, respectively. Then there exist two foliations $\mathfrak{F}^s$, $\mathfrak{F}^u$ of M such that $\Delta^s_{\boldsymbol{\Phi}}$, $\Delta^u_{\boldsymbol{\Phi}}$ are the tangent bundles to the foliations $\mathfrak{F}^s$, $\mathfrak{F}^u$.*

Moreover, these are the the stable and unstable foliations of the dynamical system namely, the foliations tangent to E^s and E^u.

Proof: To prove the claim, we show that it holds for the stable eigendistribution, under continuous-time dynamics and the proof can repeated and modifeid for teh other cases.

First, recall that hyperbolicity of the dynamical system implies hyperbolicity of $V_{\mathbf{F}}^t$, which implies that the stable and unstable eigendistributions defined above exist [Mat68; CS81b; CS80]. Now we show that an eigensection in the stable eigendistribution is necessarily in E^s for every $x \in \mathcal{M}$. Recall the following formula that eigensections satisfy and rearrange the terms.

$$V_{\mathbf{F}}^t \mathbf{\Phi}_i(x) = \nabla S_{\mathbf{F}}^{-t}|_{S_{\mathbf{F}}^t(x)} \cdot \mathbf{\Phi}_i(x) \circ S_{\mathbf{F}}^t(x) = e^{\lambda_i t} \mathbf{\Phi}_i(x) \implies e^{-\lambda_i t} \mathbf{\Phi}_i(x) \circ S_{\mathbf{F}}^t(x) = \nabla S^t|_x \cdot \mathbf{\Phi}_i(x)$$

Now fix a metric g and take the induced norm of the above expression.

$$\begin{aligned} \|\nabla S^t|_x \cdot \mathbf{\Phi}_i(x)\|_g &= \|e^{-\lambda_i t} \mathbf{\Phi}_i(x) \circ S_{\mathbf{F}}^t(x)\|_g \\ &= |e^{-\lambda_i t}| \|\mathbf{\Phi}_i(x) \circ S_{\mathbf{F}}^t(x)\|_g \\ &= e^{-\Re(\lambda_i) t} \|\mathbf{\Phi}_i(x) \circ S_{\mathbf{F}}^t(x)\|_g \\ &\leq e^{-\Re(\lambda_i) t} M_g \|\mathbf{\Phi}_i(x)\|_g \end{aligned} \tag{4.93}$$

where $M_g = \|\mathbf{\Phi}_i\|$ is the norm of $\mathbf{\Phi}_i$ as an element of the Banach space $\mathfrak{X}(\mathcal{M})$. Specifically,

$$\|\mathbf{\Phi}_i\| = \sup_{x \in \mathcal{M}} |\mathbf{\Phi}_i(x)| = \sup_{x \in \mathcal{M}} g(\mathbf{\Phi}_i(x), \mathbf{\Phi}_i(x))$$

Thus, according to equation (4.93) and definition 4.7.9, since $\Re(\lambda_k) > 0$ for $\mathbf{\Phi}_k \in \Delta_{\mathbf{\Phi}}^u$ then

$$\mathbf{\Phi}_k \in \Delta_{\mathbf{\Phi}}^s \implies \mathbf{\Phi}_k \in E^s$$

for every $k \in \{1, \cdots, n_s\}$

We now proceed by showing that the stable eigendistribution $\Delta^s_{\mathbf{\Phi}}$ is involutive and the argument can be repeated for the unstable eigendistribution. Let $\mathbf{G}(x), \mathbf{H}(x)$ be sections of $\Delta^s_{\mathbf{\Phi}}$ which means they can be expressed as

$$\mathbf{G}(x) = \sum_{i=1}^{n_s} g_i(x)\mathbf{\Phi}_i(x) \text{ and } \mathbf{H}(x) = \sum_{j=1}^{n_s} h_j(x)\mathbf{\Phi}_j(x) \tag{4.94}$$

for some differentiable functions $g_i(x)$, $h_j(x) \in C(M, \mathbb{F})$. A straightforward computation yields the following:

$$\mathcal{L}_{\mathbf{G}}\mathbf{H}(x) = \sum_{j=1}^{n_s} \hat{h}_j(x)\mathbf{\Phi}_j(x) - \sum_{i=1}^{n_s} \hat{g}_i(x)\mathbf{\Phi}_i(x) + \sum_{ij=1}^{n_s} g_i(x)h_j(x)[\mathbf{\Phi}_i(x), \mathbf{\Phi}_j(x)]. \tag{4.95}$$

Where $\hat{h}_j(x) = \sum_{i=1}^{n_S} g_i(x)\mathbf{\Phi}_i(x)(h_j(x))$ and $\hat{g}_i(x) = \sum_{j=1}^{n_s} h_j(x)\mathbf{\Phi}_j(x)(g_i(x))$. The details of the computation can be referenced in [Lee12] page 188.

Now, clearly the first two terms of the last line in (4.95) are in $\Delta^s_{\mathbf{\Phi}}$. By proposition 4.4.2 we are guaranteed that $\mathcal{L}_{\mathbf{F}}[\mathbf{\Phi}_i, \mathbf{\Phi}_j] = (\lambda^{\mathbf{\Phi}}_i + \lambda^{\mathbf{\Phi}}_j)[\mathbf{\Phi}_i, \mathbf{\Phi}_j] = \lambda^{\mathbf{\Phi}}_{ij}[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$. Thus, $[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$ is also an eigensection with $\Re(\lambda^{\mathbf{\Phi}}_{ij}) > 0$. However, since $\Delta^s_{\mathbf{\Phi}}$ is spanned by n_s linearly independent eigensections and $\dim(E^s) = n_s$, then $[\mathbf{\Phi}_i, \mathbf{\Phi}_j]$ must be linearly dependent on the $\mathbf{\Phi}_k \in \Delta^s_{\mathbf{\Phi}}$. Therefore, $[\mathbf{G}(x), \mathbf{H}(x)] \in \Delta^s_{\mathbf{\Phi}}$ which makes $\Delta^s_{\mathbf{\Phi}}$ an involutive distribution and by the vector fields formulation of the Frobenius theorem $\Delta^s_{\mathbf{\Phi}}$ is integrable. Furthermore, the Frobenius theorem guarantees the existence of a foliation $\mathfrak{F}^s$ such that $\Delta^s_{\mathbf{\Phi}}$ is the tangent bundle to $\mathfrak{F}^s$.

Theorem 4.7.2 demonstrates that the eigendistributions of a uniformly hyperbolic system are tangent to the stable and unstable foliations of a dynamical system. The existence of such foliations for a uniformly hyperbolic system is not a new result. Their existence has already been guaranteed by the Arnold-Sinai theorem [AA68; AS62]. The contribution of theorem 4.7.2 is that the tangent distributions of the foliations are contained in the spectrum of the induced operator on sections of the tangent bundle.

4.7.2 The Differential Forms Point of View

The Arnold-Sinai theorem guarantees the existence of the stable and unstable foliations of a hyperbolic dynamical system. One could then ask how to go about determining those foliations. One method lies in the well-known fact that the proof of the Frobenius theorem is constructive [Lee12], in that it provides a technique for explicitly finding the integral manifolds. Briefly, this involves using a coordinate transformation to find commuting vector fields that span the same distribution and then composing the commuting vector fields' flows to construct a chart for the integral manifold. While this construction is indeed useful and provides an explicit parameterization of the integral manifolds, it requires calculating the flows.

Alternatively, one could seek an implicit description of the integral manifolds via the level sets of a function. It is then reasonable to ask if such a function can be found spectrally from certain Koopman operators. Before proceeding, we again collect some necessary definitions and results.

Relevant Definitions

Definition 4.7.10. *A differential k-form $\boldsymbol{\alpha}(x)$ is said to* ***annihilate*** *a distribution Δ if $\Delta \subset \ker\{\boldsymbol{\alpha}(x)\}$. In other words, $\boldsymbol{\alpha}(\mathbf{G}_1, \ldots, \mathbf{G}_k) = 0$ for any local sections $\mathbf{G}_1(x), \ldots, \mathbf{G}_k(x)$ of Δ. The k-**annihilator** of Δ, denoted $\Delta^{\perp}(k)$, is the collection of all differential k-forms which annihilate Δ. We will denote the 1-annihilator by $\Delta^{\perp}$.*

Definition 4.7.11. *A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be a* ***left ideal*** *if $i_1 + i_2 \in \mathcal{I}$ and $i_1 a \in \mathcal{I}$ for any $i_1, i_2 \in \mathcal{I}$ and $a \in \mathcal{A}$. It is said to be a* ***right ideal*** *if $i_1 + i_2 \in \mathcal{I}$ and*

*$ai_1 \in \mathcal{I}$ for any $i_1, i_2 \in \mathcal{I}$ and $a \in \mathcal{A}$. If an ideal is both a left and right ideal it is said to be a **two sided ideal** or simply an **ideal**. If $\mathcal{A}$ is also a differential algebra and $D\mathcal{I} \subset \mathcal{I}$ then the ideal $\mathcal{I}$ is said to be a **differential ideal**.*

Definition 4.7.12. *Let $\mathcal{M}$ be an n-dimensional manifold and $\mathcal{I} \subset \Omega(T^*\mathcal{M})$ be an ideal. $\mathcal{I}$ is said to be **locally generated by** $n-k$ **independent one-forms**, if every $x \in \mathcal{M}$ has a neighborhood U and $n-k$ pointwise linearly independent one-forms $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_{n-k}$ such that*

1. *If $\boldsymbol{\theta} \in \mathcal{I}$, then $\boldsymbol{\theta}|_U = \sum_{i=1}^{n-k} \boldsymbol{\theta}_i \wedge \boldsymbol{\gamma}_i$ for some $\boldsymbol{\theta}_i \in \Omega(T^*\mathcal{M})$*

2. *If $\boldsymbol{\theta} \in \Omega(T^*\mathcal{M})$ and $\mathcal{M}$ is covered by open sets $\{U_k\}$ such that for each U_k in the cover,*

$$\boldsymbol{\theta}|_{U_k} = \sum_{i=1}^{n-k} \boldsymbol{\theta}_i \wedge \boldsymbol{\gamma}_i \tag{4.96}$$

for some $\boldsymbol{\theta}_i \in \Omega(T^*\mathcal{M})$. Then $\boldsymbol{\theta} \in \mathcal{I}$.

Note that the following direct sum of annihilators of a distribution Δ

$$\mathcal{I}(\Delta) = \Delta^{\perp}(0) \oplus \Delta^{\perp}(1) \oplus \cdots \oplus \Delta^{\perp}(n) \tag{4.97}$$

is an ideal in $\Omega(T^*\mathcal{M})$, see [AMR83] page 374.

Definition 4.7.13. *The ideal $\mathcal{I}(\Delta)$ corresponding to the direct sum of annihilators of a distribution Δ will be called the **ideal generated by** Δ.*

The Foliations Tangent to an Eigendistribution

According to the definitions above, the direct sum of a distribution's annihilators will always be an ideal. Whether the ideal is also a differential ideal is the contents of the differential forms formulation of the Frobenius Theorem.

Theorem 4.7.3 (**Frobenius** (Differential Forms Formulation)). *Let Δ be d-dimensional a distribution on $\mathcal{M}$ and $\mathcal{I}(\Delta)$ be the ideal generated by Δ. The following are equivalent*

1. *Δ is involutive.*

2. *Δ is integrable.*

3. *$\mathcal{I}(\Delta)$ is a differential ideal locally generated by $n-d$ linearly independent one-forms $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_{n-d}$.*

4. *For every $x \in \mathcal{M}$ there exists an open set U and one-forms $\boldsymbol{\gamma}_i, \dots, \boldsymbol{\gamma}_{n-d}$ generating $\mathcal{I}(\Delta)$ such that*

$$\boldsymbol{d}\boldsymbol{\gamma}_i = \sum_{j=1}^{n-d} \boldsymbol{\gamma}_{ij} \wedge \boldsymbol{\gamma}_j, \text{ for some } \boldsymbol{\gamma}_{ij} \in \Omega^1(T^*U) \tag{4.98}$$

5. *$\boldsymbol{d}\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_1 \wedge \cdots \wedge \boldsymbol{\gamma}_{n-d} = 0$*

6. *There exists a $\boldsymbol{\theta} \in \Omega(T^*U)$ such that $\boldsymbol{d}(\boldsymbol{\gamma}_1 \wedge \cdots \wedge \boldsymbol{\gamma}_{n-d}) = \boldsymbol{\theta} \wedge \boldsymbol{\gamma}_1 \wedge \cdots \wedge \boldsymbol{\gamma}_{n-d}$*

Essentially the condition that the ideal generated by a distribution Δ be a differential ideal is equivalent to the integrability of Δ. A useful corollary is the following.

Corollary 4.7.3.1. *A distribution Δ is integrable if and only if for all open subsets U of $\mathcal{M}$*

$$\boldsymbol{\theta} \in \Delta^{\perp}(1) \implies \boldsymbol{d}\boldsymbol{\theta} \in \Delta^{\perp}(2) \tag{4.99}$$

To motivate one of the results of this section, consider a collection of linearly independent differential 1-forms $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_{n-d}$ which span the annihilator $\Delta_{\boldsymbol{\Phi}}^{\perp}$ of an eigendistribution $\Delta_{\boldsymbol{\Phi}}$. By definition the 1-forms are such that

$$\boldsymbol{\gamma}_j(x)(\boldsymbol{\Phi}_i(x)) = 0 \tag{4.100}$$

for all $1 \leq i \leq d$ and, $1 \leq j \leq n-d$. Now if we further imposed that each of the $\boldsymbol{\gamma}_j(x)$ are exact differential 1-forms, then each $\boldsymbol{\gamma}_j = \boldsymbol{d}\psi_j$ for some differentiable function $\psi_j \in C(\mathcal{M}, \mathbb{F})$ (referred to as the potential function of $\boldsymbol{\gamma}_j$). Then rewriting (4.100) for these exact 1-forms results in the following

$$\boldsymbol{\gamma}_j(x)(\boldsymbol{\Phi}_i(x)) = \boldsymbol{\Phi}_i^\intercal(x)\nabla\psi_j^\intercal(x) = 0 \tag{4.101}$$

for all $1 \leq i \leq d$ and, $1 \leq j \leq n-d$. It is by no accident that we chose to label the potential functions as $\psi_j(x)$. The choice was obviously motivated by the fact that equation (4.101) shows that the potential functions $\psi_j(x)$ are joint Koopman eigenfunctions at eigenvalue $\lambda_j^\psi = 0$ of the eigensections $\boldsymbol{\Phi}_1(x), \ldots, \boldsymbol{\Phi}_d(x)$. Lastly, viewing equation (4.101) as a tangency condition between the eigenfunction level sets and the sections of the distribution $\Delta_{\boldsymbol{\Phi}}$ we would reasonably suspect that the d-dimensional integral manifold could be prescribed implicitly as the level sets of the $n-d$ linearly independent eigenfunctions.

To motivate another result of this section, let us return to the situation in which the integrability of a d-dimensional eigendistribution was yet to be determined. While in the case of uniformly hyperbolic dynamics, this was possible to show, with mild effort, for general dynamics, it may not be. It would seem reasonable to investigate whether there are alternative conditions that can establish the integrability of an eigendistribution corresponding to a, not necessarily hyperbolic, dynamical system. Perhaps the existence (or lack of existence) of the $n-d$ joint Koopman eigenfunctions of the eigendistributions, previously discussed, can yield such conditions. We will make use of the following lemma.

Lemma 4.7.4. *Let $\Delta \subset T\mathcal{M}$ be a d-dimensional distribution over an n-dimensional manifold $\mathcal{M}$, spanned by the smooth vector fields $\mathbf{F}_1(x), \ldots, \mathbf{F}_d(x)$. The annihilator $\Delta^\perp$ is spanned by $n-d$ linearly independent cosections $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_{n-d}$.*

Proof: Let $\Delta^n = span\{\mathbf{F}_1, \ldots, \mathbf{F}_d, \mathbf{F}_{d+1}, \ldots, \mathbf{F}_n\}$ be the completion of Δ to a complete basis for $T\mathcal{M}$ and let $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_n\}$ be the dual basis to Δ^n. We then have the following facts:

1. Any $\boldsymbol{\theta} \in \Omega_1(T^*\mathcal{M})$ can be written in the basis as $\boldsymbol{\theta} = \sum_{i=1}^n f_i \boldsymbol{\gamma}_i$, for smooth functions $f_i \in C(\mathcal{M}, \mathbb{F})$ and since $\boldsymbol{\theta}(\mathbf{F}_i) = f_i \boldsymbol{\gamma}_i(\mathbf{F}_i) = f_i$ we can write $\boldsymbol{\theta} = \sum_{i=1}^n \boldsymbol{\theta}(\mathbf{F}_i)\boldsymbol{\gamma}_i$.

2. If $\mathbf{G} \in \Delta$ then we can write $\mathbf{G} = \sum_{i=1}^d g_i \mathbf{F}_i(x)$, for smooth functions $g_i \in C(\mathcal{M}, \mathbb{F})$.

3. If $i \in \{d+1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$ then we have that $\boldsymbol{\gamma}_i(\mathbf{F}_j) = 0$.

First suppose that $\mathbf{G} \in \Delta$ then $\boldsymbol{\gamma}_i(\mathbf{G}) = 0$ for all $i \in \{d+1, \ldots, n\}$ which implies that $span\{\boldsymbol{\gamma}_{d+1}, \ldots, \boldsymbol{\gamma}_n\} \subset \Delta^\perp$.

Now suppose that $\boldsymbol{\theta} \in \Delta^\perp$ by definition $\boldsymbol{\theta}(\mathbf{F}_i) = 0$ for $i \in \{1, \ldots, d\}$ then we can write $\boldsymbol{\theta}$ as $\boldsymbol{\theta} = \sum_{i=d+1}^n \boldsymbol{\theta}(\mathbf{F}_i)\boldsymbol{\gamma}_i$ which means $\boldsymbol{\theta} \in span\{\boldsymbol{\gamma}_{d+1}, \ldots, \boldsymbol{\gamma}_n\}$ and this implies that $\Delta^\perp \subset span\{\boldsymbol{\gamma}_{d+1}, \ldots, \boldsymbol{\gamma}_n\}$.

Thus $\Delta^\perp = span\{\boldsymbol{\gamma}_{d+1}, \ldots, \boldsymbol{\gamma}_n\}$ and being basis vectors the $\boldsymbol{\gamma}_i$'s are linearly independent.

Theorem 4.7.5. *Let $\Delta_{\boldsymbol{\Phi}}$ be an eigendistribution spanned by d linearly independent eigensections $\boldsymbol{\Phi}_1(x), \ldots, \boldsymbol{\Phi}_d(x)$ of a continuous-time or discrete-time dynamical system. Suppose, there exists $n-d$ joint, linearly independent, non-constant, eigenfunctions $\psi_1(x), \ldots, \psi_{n-d}(x)$ at eigenvalue 1 of the operators $U^t_{\boldsymbol{\Phi}_i}$. Then*

1. *the distribution $\Delta_{\boldsymbol{\Phi}}$ is integrable.*

2. *the joint level sets of $\psi_j(x)$, corresponding to joint regular values, implicitly define the integral manifolds of $\Delta_{\boldsymbol{\Phi}}$.*

Proof: Again, we can prove the result for continuous-time dynamics, and the proof can be repeated for the discrete-time case.

To prove the first claim we first note that by lemma 4.7.4 the annihilator $\Delta_{\boldsymbol{\Phi}}^{\perp}$ is spanned by $n-d$ linearly independent one-forms. Consider the eigencosections defined by $\boldsymbol{\omega}_j(x) = \boldsymbol{d}\psi_j(x)$ which are linearly independent by the assumption that the eigenfunctions $\psi_j(x)$ are linearly independent. We have that

$$\boldsymbol{\omega}_j(x)(\boldsymbol{\Phi}_i(x)) = \boldsymbol{d}\psi_j(x)(\boldsymbol{\Phi}_i(x)) = \mathcal{L}_{\boldsymbol{\Phi}_i}\psi_j(x) = 0 \tag{4.102}$$

for every $1 < i < d$, $1 < j < n-d$ which implies that $\Delta_{\boldsymbol{\Phi}}^{\perp} = span\{\boldsymbol{\omega}_1(x), \ldots, \boldsymbol{\omega}_{n-d}(x)\}$. Moreover,

$$\boldsymbol{d}\boldsymbol{\omega}_j(x)(\boldsymbol{\Phi}_i(x)) = 0 \tag{4.103}$$

for every $1 < i < d$, $1 < j < n-d$ and by the differential forms formulation of the Frobenius theorem and corollary 4.7.3.1 this is equivalent to the integrability of $\Delta_{\boldsymbol{\Phi}}$ and the existence of a foliation which is tangent to Δ.

To prove the second claim we observe that under the conditions assumed on the $\psi_j(x)$ if c is a joint regular value of all the $\psi_j(x)$ then $\boldsymbol{c} = [c, \ldots, c]^{\intercal} \in \mathbb{F}^{n-d}$ is also a regular value of the vector valued function $\boldsymbol{\psi}(x) = [\psi_1(x), \ldots, \psi_{n-d}(x)]^{\intercal} \in C(\mathcal{M}, \mathbb{F}^{n-d})$. Therefore, the level sets $L_{\boldsymbol{c}} = \{x \mid \boldsymbol{\psi}(x) = \boldsymbol{c}\}$ are $n - (n-d) = d$ dimensional submanifolds of $\mathcal{M}$ [Spi99]. As such, the tangent space of $L_{\boldsymbol{c}}$ at every point is also d-dimensional and $\nabla\boldsymbol{\psi}^{\intercal}(x)$ is everywhere normal to $L_{\boldsymbol{c}}$. Now, since $\mathcal{L}_{\boldsymbol{\Phi}_i}\boldsymbol{\psi}(x) = \boldsymbol{\Phi}_i^{\intercal}(x) \cdot \nabla\boldsymbol{\psi}^{\intercal}(x) = 0$ for every $1 < j < d$ this implies that the eigensections are everywhere tangent to $L_{\boldsymbol{c}}$ and being linearly independent implies that they span $TL_{\boldsymbol{c}}$. These, by definition 4.7.5, are the conditions for $L_{\boldsymbol{c}}$ to be an integral manifold of $\Delta_{\boldsymbol{\Phi}}$.

4.8 Examples and Applications

In this section, we provide some applications and examples of the results obtained within this chapter.

4.8.1 Application to Differential Geometry

In this application, we set aside dynamical systems theory for the moment and simply consider an n-dimensional manifold $\mathcal{M}$ along with the space of smooth, complete vector fields $\mathfrak{X}(T\mathcal{M})$. We begin by recalling some definitions and two well-known theorems of differential geometry.

Relevant Definitions & Results

Definition 4.8.1. *Two vector fields* $\mathbf{F}$ *and* $\mathbf{G}$ *are said to* ***commute*** *if*

$$[\mathbf{F}(x), \mathbf{G}(x)] := \mathcal{L}_{\mathbf{F}}\mathbf{G}(x) = 0 \tag{4.104}$$

Definition 4.8.2. *If* $S^t_{\mathbf{F}}$ *is a smooth flow, a vector field* $\mathbf{G}(x)$ *is said to be* ***invariant under*** $S^t_{\mathbf{F}}$ *if*

$$\nabla S^t_{\mathbf{F}}|_x \cdot \mathbf{G}(x) = \mathbf{G} \circ S^t_{\mathbf{F}}(x) \tag{4.105}$$

for all t, x *defined in the domain of* $S^t_{\mathbf{F}}$.

Theorem 4.8.1. *For two smooth vector fields* $\mathbf{F}(x)$ *and* $\mathbf{G}(x)$*, the following are equivalent,*

1. $\mathbf{F}(x)$ *and* $\mathbf{G}(x)$ *commute.*

2. $\mathbf{F}(x)$ *is invariant under* $S^t_{\mathbf{G}}(x)$.

3. $\mathbf{G}(x)$ *is invariant under* $S^t_{\mathbf{F}}(x)$.

Proof: *See [Lee12] pages 231-232.*

Definition 4.8.3. *The two flows $S^t_{\mathbf{F}}$ and $S^t_{\mathbf{G}}$ are said to* ***commute*** *if for every $x \in \mathcal{M}$ and all $s, t \in \mathbb{R}$ we have that*

$$S^t_{\mathbf{F}} \circ S^s_{\mathbf{G}}(x) = S^t_{\mathbf{G}} \circ S^s_{\mathbf{F}}(x) \tag{4.106}$$

Theorem 4.8.2. *Smooth vector fields commute if and only if their flows commute.*

Proof: See [Lee12] page 233.

Lemma 4.8.3 (Rescaling Lemma)**.** *For any scalar $a \in \mathbb{R}$ the flow $S^{at}_{\mathbf{F}}$ is the flow of the vector field $a \cdot \mathbf{F}(x)$.*

Proof: See [Lee12] page 208.

Commutability of Eigensections

In the context of this work definitions 4.8.1-4.8.3 and theorems 4.8.1-4.8.2 are very specific statements regarding the eigenspace at $\lambda = 0$ of the infinitesimal generator $\mathcal{L}$. When $\lambda \neq 0$ then, $[\mathbf{F}(x), \mathbf{G}(x)] = \lambda \mathbf{G}(x)$ means that the direction of $\mathbf{G}(x)$ is still unchanged along the flow of $\mathbf{F}(x)$ but its magnitude, and hence the "speed" of the flow $S^t_{\mathbf{G}}$, is scaled in some proportion prescribed by λ. Geometrically, this means that the two sides of the parallelogram, generated by the composition of the flows of $\mathbf{F}(x)$ and $\mathbf{G}(x)$, travel in the correct directions so that they should close but fail to do so due to the scaling caused by λ.

We can leverage the facts mentioned above to show that the flows of two vector fields $\mathbf{F}(x), \mathbf{G}(x)$ commute, subject to an appropriate rescaling of the flow time, if and only if one vector field is an eigensection of the other vector field; $[\mathbf{F}(x), \mathbf{G}(x)] = \lambda \mathbf{G}(x)$ or vice versa.

Theorem 4.8.4. *Let $\mathbf{F}, \mathbf{\Phi} \in \mathfrak{X}(\mathcal{M})$ and let $S^t_{\mathbf{F}}(x), S^t_{\mathbf{\Phi}}(x)$ denote their respective flows. The following are equivalent.*

1. $[\mathbf{F}(x), \mathbf{\Phi}(x)] = \lambda \mathbf{\Phi}(x)$
2. $V^t_{\mathbf{F}} \mathbf{\Phi}(x) = S^{t*}_{\mathbf{F}} \mathbf{\Phi}(x) = e^{\lambda t} \mathbf{\Phi}(x)$
3. $V^t_{\mathbf{\Phi}} \mathbf{F}(x) = S^{t*}_{\mathbf{\Phi}} \mathbf{F}(x) = e^{-\lambda t} \mathbf{F}(x)$
4. $S^{\bar{t}}_{\mathbf{\Phi}} \circ S^{\bar{s}}_{\mathbf{F}}(x) = S^{\bar{s}}_{\mathbf{F}} \circ S^{e^{(\lambda \bar{s})} \bar{t}}_{\mathbf{\Phi}}(x)$
5. $S^{\bar{t}}_{\mathbf{F}} \circ S^{\bar{s}}_{\mathbf{\Phi}}(x) = S^{\bar{s}}_{\mathbf{\Phi}} \circ S^{e^{(-\lambda \bar{s})} \bar{t}}_{\mathbf{F}}(x)$

where $\bar{t}$ and $\bar{s}$ denote a specified flow time.

Proof:

(4) $\iff$ (2)

Assume (2) holds. Then we have that

$$\nabla S^{-t}_{\mathbf{F}}|_{S^t_{\mathbf{F}}(x)} (\mathbf{\Phi} \circ S^t_{\mathbf{F}}(x)) = e^{\lambda t} \mathbf{\Phi}(x) \iff e^{-\lambda t} (\mathbf{\Phi} \circ S^t_{\mathbf{F}}(x)) = \nabla S^t_{\mathbf{F}}|_x \mathbf{\Phi}(x) \tag{4.107}$$

Now consider the curve $C(t) = S^{\bar{s}}_{\mathbf{F}} \circ S^{e^{(\lambda \bar{s})} t}_{\mathbf{\Phi}}(x)$ and compute the velocity of the curve.

$$\begin{aligned}
\frac{d}{dt}C(t) &= \frac{d}{dt}(S_{\mathbf{F}}^{\bar{s}} \circ S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) \\
&= \nabla S_{\mathbf{F}}^{\bar{s}}|_{S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)} \cdot (\frac{d}{dt}S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) \\
&= \nabla S_{\mathbf{F}}^{\bar{s}}|_{S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)} \cdot (\frac{d}{dt}(e^{(\lambda\bar{s})}t) \cdot \frac{d}{dt}S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) \\
&= e^{\lambda\bar{s}}\nabla S_{\mathbf{F}}^{\bar{s}}|_{S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)} \cdot (\mathbf{\Phi} \circ S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) \\
&= e^{\lambda\bar{s}}e^{-\lambda\bar{s}}(\mathbf{\Phi} \circ S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) \circ S_{\mathbf{F}}^{\bar{s}}(x) \\
&= (\mathbf{\Phi} \circ S_{\mathbf{F}}^{\bar{s}} \circ S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) \\
&= \mathbf{\Phi}(S_{\mathbf{F}}^{\bar{s}} \circ S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}t}(x)) = \mathbf{\Phi}(C(t))
\end{aligned}$$

Which implies that the curve $C(t)$ is an integral curve of $\mathbf{\Phi}(x)$ with initial condition $C(0) = S_{\mathbf{F}}^{s}(x)$. However, the flow of the initial condition $S_{\mathbf{F}}^{\bar{s}}(x)$ under the vector field $\mathbf{\Phi}(x)$ is given by $S_{\mathbf{\Phi}}^{t}(S_{\mathbf{F}}^{\bar{s}}(x))$ and thus by uniqueness of the flow we have the following

$$C(\bar{t}) = S_{\mathbf{F}}^{\bar{s}} \circ S_{\mathbf{\Phi}}^{e^{(\lambda\bar{s})}\bar{t}}(x) = S_{\mathbf{\Phi}}^{\bar{t}}(S_{\mathbf{F}}^{\bar{s}}(x)) = S_{\mathbf{\Phi}}^{\bar{t}} \circ S_{\mathbf{F}}^{\bar{s}}(x)$$

Assume (4) holds and differentiate the left hand side w.r.t t to obtain

$$\left.\frac{d}{dt}\right|_{t=0}(S_{\mathbf{\Phi}}^{t} \circ S_{\mathbf{F}}^{\bar{s}}(x)) = \mathbf{\Phi} \circ S_{\mathbf{F}}^{\bar{s}}(x) \tag{4.108}$$

which is necessarily equal to the time derivative of the right hand side of of (4). Thus

$$\begin{aligned}
\mathbf{\Phi} \circ S_{\mathbf{F}}^{\bar{s}}(x) &= \left.\frac{d}{dt}\right|_{t=0}(S_{\mathbf{F}}^{\bar{s}} \circ S_{\mathbf{\Phi}}^{e^{(\lambda s)}t}(x)) \\
&= \left.(\nabla S_{\mathbf{F}}^{\bar{s}}|_{S_{\mathbf{\Phi}}^{e^{(\lambda s)}t}(x)} \cdot \frac{d}{dt}S_{\mathbf{\Phi}}^{e^{(\lambda s)}t}(x))\right|_{t=0} \\
&= \left.(\nabla S_{\mathbf{F}}^{\bar{s}}|_{S_{\mathbf{\Phi}}^{e^{(\lambda s)}t}(x)} \cdot e^{\lambda s}\mathbf{\Phi} \circ S_{\mathbf{\Phi}}^{e^{(\lambda s)}t}(x))\right|_{t=0} \\
&= e^{\lambda s}\nabla S_{\mathbf{F}}^{\bar{s}}|_{x} \cdot \mathbf{\Phi}(x)
\end{aligned}$$

Now if we multiply both sides by $\nabla S_{\mathbf{F}}^{-\bar{s}}|_{S_{\mathbf{F}}^{\bar{s}}(x)}$ we obtain the desired result

$$S_{\mathbf{F}}^{\bar{s}*}\mathbf{\Phi}(x) = \nabla S_{\mathbf{F}}^{-\bar{s}}|_{S_{\mathbf{F}}^{\bar{s}}(x)}\mathbf{\Phi} \circ S_{\mathbf{F}}^{\bar{s}}(x) = e^{\lambda\bar{s}}\mathbf{\Phi}(x) \tag{4.109}$$

(2) $\iff$ (1)

Assume (2) holds then

$$[\mathbf{F}(x), \mathbf{\Phi}(x)] = \frac{d}{dt}\Big|_{t=0} S_{\mathbf{F}}^{t*}\mathbf{\Phi}(x) = \frac{d}{dt}\Big|_{t=0} e^{\lambda t}\mathbf{\Phi}(x) = (\lambda e^{\lambda t}\mathbf{\Phi}(x))\Big|_{t=0} = \lambda\mathbf{\Phi}(x)$$

Assume (1) holds and compute

$$\begin{aligned}
\frac{d}{dt}(S_{\mathbf{F}}^{t*}\mathbf{\Phi}(x)) &= \frac{d}{ds}\Big|_{s=0}(S_{\mathbf{F}}^{(t+s)*}\mathbf{\Phi}(x)) \\
&= \frac{d}{ds}\Big|_{s=0}((S_{\mathbf{F}}^{t} \circ S_{\mathbf{F}}^{s})^{*}\mathbf{\Phi}(x)) \\
&= S_{\mathbf{F}}^{t*}\frac{d}{ds}\Big|_{s=0}(S_{\mathbf{F}}^{s*}\mathbf{\Phi}(x)) \\
&= S_{\mathbf{F}}^{t*}([\mathbf{F}(x), \mathbf{\Phi}(x)]) = \lambda S_{\mathbf{F}}^{t*}(\mathbf{\Phi}(x))
\end{aligned}$$

From the above computation we see that $\frac{d}{dt}(S_{\mathbf{F}}^{t*}\mathbf{\Phi}(x)) = \lambda S_{\mathbf{F}}^{t*}(\mathbf{\Phi}(x))$ which is an ordinary differential equation with initial condition $\mathbf{\Phi}(x)$, since $S_{\mathbf{F}}^{t*}|_{t=0}\mathbf{\Phi}(x) = \mathbf{\Phi}(x)$. The differential equation just described has the well-known solution $S_{\mathbf{F}}^{t*}\mathbf{\Phi}(x) = e^{\lambda t}S_{\mathbf{F}}^{t*}|_{t=0}\mathbf{\Phi}(x) = e^{\lambda t}\mathbf{\Phi}(x)$

We have shown (4) $\iff$ (2) $\iff$ (1), the exact same arguments can be used to show (5) $\iff$ (3) $\iff$ $[\mathbf{\Phi}(x), \mathbf{F}(x)] = -\lambda\mathbf{\Phi}(x)$ and $[\mathbf{\Phi}(x), \mathbf{F}(x)] = -\lambda\mathbf{\Phi}(x) \iff (1)$ is trivially true.

4.8.2 Arnold's Cat Map

The next application will deal with a spectral study, from the perspective of the induced group of operators on sections, of the hyperbolic toral automorphism known as

Arnold's cat map. Throughout this example the space $\mathcal{M} = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ will be the torus. The automorphism $S(x) : \mathcal{M} \to \mathcal{M}$ given by

$$S(x) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mod 1 \tag{4.110}$$

is called the **cat map**. We consider the discrete-time dynamical system generated by the cat map and note the following properties.

1. Since the determinant $\det \nabla S(x) = 1$ the dynamical system generated by the cat map is measure-preserving.

2. The dynamical system generated by the cat map has a countable Lebesgue spectrum [AA68], which implies it is a mixing and ergodic dynamical system.

3. Property (2) also implies that the only eigenfunctions of U_S are the constant functions.

4. The dynamical system generated by the cat map is a uniformly hyperbolic dynamical system [AA68].

5. The matrix $\nabla S(x)$ has two eigenvalues λ_s, λ_u with $|\lambda_u| > 1$ and $|\lambda_s| < 1$

6. The Kolmogorov-Sinai entropy of the cat map is $\ln(|\lambda_u|)$. See [Sin59; AA68].

7. The Lyapunov exponents of the cat map are $\ln(|\lambda_s|)$ and $\ln(|\lambda_u|)$.

Due to the properties mentioned above, a spectral analysis of the cat map via U_S forces one to consider the continuous spectrum of U_S. Unfortunately, studying the continuous spectrum is often a complex task. A spectral analysis via U_S for the cat map has been done in a few cases, but only numerically [Gov+19; KPM18].

Denote by v_s and v_u the eigenvectors of $\nabla S(x)$ corresponding to λ_s, λ_u and consider the two vector fields $\mathbf{\Phi}_s(x) = v_s$ and $\mathbf{\Phi}_u(x) = v_u$. The vector fields are constant on $\mathcal{M}$ and everywhere point in the direction of the stable and unstable eigenvectors of $\nabla S(x)$. A straightforward calculation shows that these vector fields are eigensections of V_s.

$$V_S \mathbf{\Phi}_s(x) = (\nabla S(x))^{-1} \cdot v_s \circ s(x) = (\nabla S(x))^{-1} \cdot v_s = \frac{1}{\lambda_s} v_s \tag{4.111}$$

$$V_S \mathbf{\Phi}_u(x) = (\nabla S(x))^{-1} \cdot v_u \circ s(x) = (\nabla S(x))^{-1} \cdot v_u = \frac{1}{\lambda_u} v_u \tag{4.112}$$

We note that since $\log \lambda_u = |\log(\frac{1}{\lambda_u})|$, the above calculation shows that the Lyapunov exponents and the Kolmogorov-Sinai entropy, up to a minus sign, are in the spectrum of V_S. This is significant since the entropy was an invariant not captured by the induced operator on functions.

Now since the cat map is a hyperbolic dynamical system there is a splitting of the tangent bundle $T\mathcal{M} = E^s \oplus E^u$ and the contents of theorem 4.7.2 apply. Thus, we consider the corresponding stable $\Delta^s_{\mathbf{\Phi}} = span\{\mathbf{\Phi}_s\}$ and unstable $\Delta^s_{\mathbf{\Phi}} = span\{\mathbf{\Phi}_u\}$ eigendistributions. Furthermore, according to theorem 4.7.5 we can obtain the stable $\mathfrak{F}_s$ and unstable $\mathfrak{F}_u$ foliations as the level sets of the eigenfunctions $\psi_s(x)$ and $\psi_u(x)$ at eigenvalue 1 of $U^t_{\mathbf{\Phi}_s}$ and $U^t_{\mathbf{\Phi}_u}$.

Now, let w_s and w_u be the left eigenvectors of $\nabla S(x)$ which are dual to v_s and v_u, respectively. Set $\psi_s(x) = \langle x, w_u \rangle$ and $\psi_u(x) = \langle x, w_s \rangle$ and compute $U^t_{\mathbf{\Phi}_s} \psi_s(x)$ and

$U^t_{\mathbf{\Phi}_u}\psi_u(x)$.

$$\begin{aligned} U^t_{\mathbf{\Phi}_s}\psi_s(x) &= \langle x, w_u\rangle \circ S^t_{\mathbf{\Phi}_s} \\ &= \langle x + v_s t, w_u\rangle \\ &= \langle x, w_u\rangle + t\langle v_s, w_u\rangle \\ &= \langle x, w_u\rangle = \psi_s(x) \end{aligned} \tag{4.113}$$

$$\begin{aligned} U^t_{\mathbf{\Phi}_u}\psi_u(x) &= \langle x, w_s\rangle \circ S^t_{\mathbf{\Phi}_u} \\ &= \langle x + v_u t, w_s\rangle \\ &= \langle x, w_s\rangle + t\langle v_u, w_s\rangle \\ &= \langle x, w_s\rangle = \psi_u(x) \end{aligned} \tag{4.114}$$

Thus, by theorem 4.7.5, $\mathfrak{F}_s = \{L^s_c\}$ and $\mathfrak{F}_u = \{L^u_c\}$ where $L^u_c\{x|\psi_u(x) = c\}$ and $L^s_c\{x|\psi_s(x) = c\}$. A plot of the level sets of said functions is shown below in figure 4.1. It can be seen that they are everywhere tangent (parallel) to the stable and unstable eigendistributions, as a reference, we have also included a plot of the stable and unstable eigenvectors of $\nabla S(x)$.

For completeness we also compute the spectrum of W_S. Set $\boldsymbol{\omega}_s(x) = w_s\boldsymbol{d}x$ and $\boldsymbol{\omega}_u(x) = w_u\boldsymbol{d}x$

$$W_S\boldsymbol{\omega}_s(x) = w_s \circ s(x)\cdot\nabla S(x)\boldsymbol{d}x = w_s\cdot\nabla S(x)\boldsymbol{d}x = \lambda_s w_s\boldsymbol{d}x = \lambda_s\boldsymbol{\omega}_s(x) \tag{4.115}$$

$$W_S\boldsymbol{\omega}_u(x) = w_u \circ s(x)\cdot\nabla S(x)\boldsymbol{d}x = w_u\cdot\nabla S(x)\boldsymbol{d}x = \lambda_u w_u\boldsymbol{d}x = \lambda_u\boldsymbol{\omega}_u(x) \tag{4.116}$$

Thus the operator W_S contains the Lyapunov exponents and entropy exactly. Of course the minus sign appearing in the spectrum of V_S is simply due to the fact that we are working with the pull-back operator rather than the push-forward.

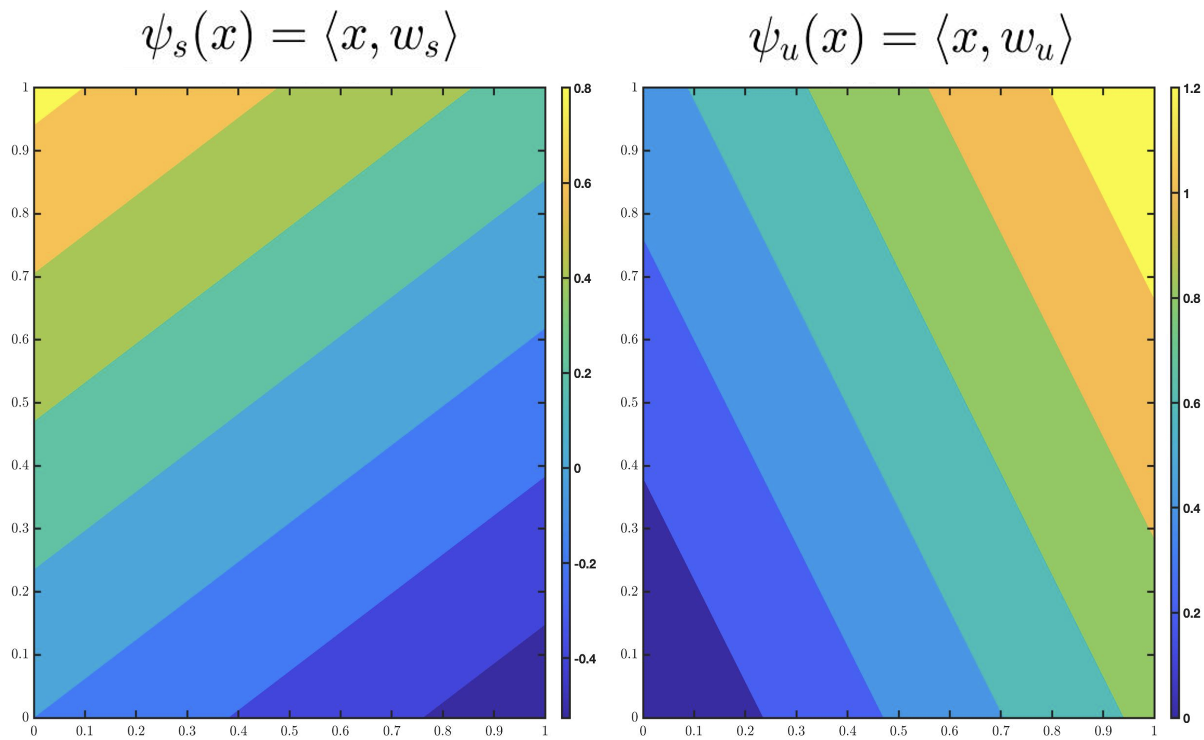

Figure 4.1: Obtained From the Spectrum of $U^t_{\mathbf{\Phi}_s}$ and $U^t_{\mathbf{\Phi}_u}$.

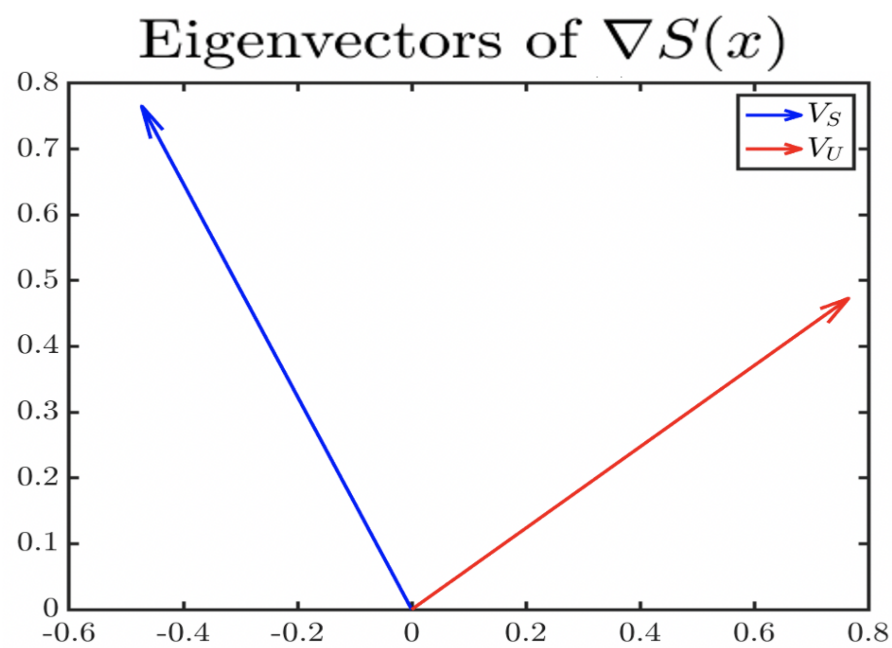
Eigenvectors of $\nabla S(x)$
V_S
V_U
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Figure 4.2:

4.8.3 Spectral Expansion of a Linear System

Our final two examples are mostly for illustrative purposes. First we generate a random 2×2 matrix $\mathbf{A}$, with integer entries, and consider the linear dynamical system $\mathbf{F}(x) = \mathbf{A}x$. The matrix generated is

$$\mathbf{A} = \begin{bmatrix} 1 & 16 \\ -20 & 17 \end{bmatrix}$$

and the eigensections are plotted below.

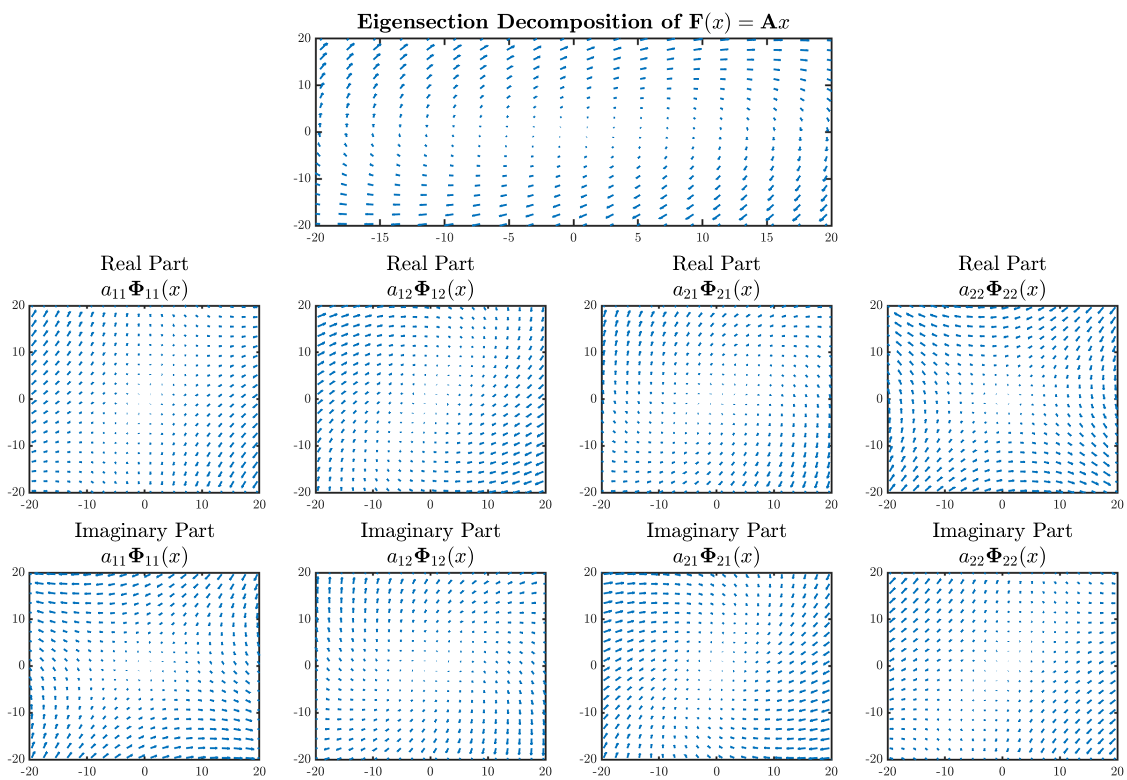

Figure 4.3:

The next example will illustrate the spectral expansion derived in proposition 4.5.3 by randomly generating two 8×8 matrices $\mathbf{A}$ and $\mathbf{B}$ and considering the time evolution of the dynamical system $\mathbf{G}(x) = \mathbf{B}x$ under the dynamics of $\mathbf{F}(x) = \mathbf{A}x$. The matrices generated are

$$\mathbf{A} = \begin{bmatrix} 20 & 20 & 20 & -17 & -19 & -9 & -4 & 19 \\ 20 & 10 & 13 & 7 & -14 & -18 & -3 & 18 \\ 20 & -4 & -11 & -11 & 10 & -17 & 9 & -4 \\ -10 & 11 & 5 & -20 & 0 & 14 & -2 & 7 \\ -12 & -15 & -8 & 5 & -11 & -5 & 18 & -3 \\ 4 & -4 & -19 & 19 & 2 & -12 & 11 & 15 \\ 5 & 17 & 13 & 16 & 16 & 0 & 3 & -2 \\ 12 & 12 & -3 & 14 & -3 & -2 & -20 & 15 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 5 & 3 & 3 & 13 & 16 & 12 & 10 & 7 \\ 13 & 7 & 11 & 18 & 20 & 1 & 7 & 5 \\ 14 & 11 & 20 & 6 & 16 & 3 & 15 & 7 \\ 6 & 5 & 11 & 20 & 7 & 17 & 4 & 3 \\ 4 & 20 & 8 & 13 & 12 & 12 & 12 & 20 \\ 5 & 4 & 15 & 18 & 12 & 18 & 9 & 13 \\ 17 & 9 & 12 & 13 & 9 & 17 & 2 & 15 \\ 20 & 14 & 13 & 14 & 4 & 9 & 2 & 9 \end{bmatrix}$$

and the comparison between the expansion and $V^t_{\mathbf{F}}\mathbf{G}(x)$ is shown below. In every subfigure of figure 4.4 the matrix corresponding to $V^t_{\mathbf{F}}\mathbf{G}$ is plotted on the left and the matrix corresponding to $\sum_{ij} e^{\lambda_{ij}t}\mathbf{\Phi}_{ij}b_{ij}$ is plotted on the right.

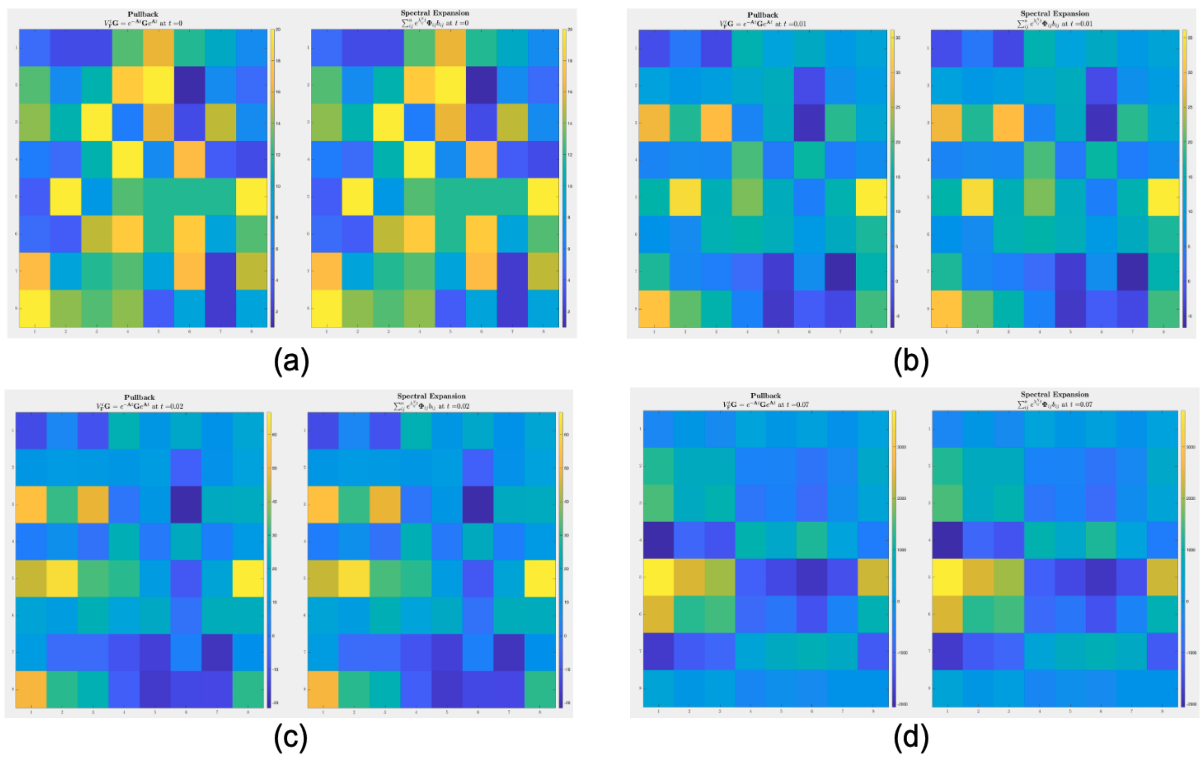

Figure 4.4: (a) Comparison at $t = 0$. (b) Comparison $t = .01$ (c) Comparison at $t = .02$ (d) Comparison at $t = .07$

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Chapter 5

Conclusion

5.1 Discussion

5.1.1 Summary of Chapter 3

In chapter 3, we proposed a data-driven method, based on the Koopman operator's spectral properties, which provides a platform for identifying and analyzing spatiotemporal traffic patterns. We were able to distinguish between the various types of patterns previously proposed by Ahn, Laval, and others [Ahn05; AC07; SH07] (MLC, PLC, SGW, "pumping effect"). We identify new patterns with standing wave node-like features, spatial harmonic features, growing/decaying patterns, multi-lane MLC patterns, multi-lane PLC patterns, multi-lane patterns with combined lateral and longitudinal (zig-zag) travel, multi-lane patterns associated with the merging effects of on-ramps, and novel patterns that exist within a network of highways. Specifically, a network mode analysis can reveal the order of congestion, synchrony of congestion, and indicate which highways are occupied the most. Every pattern we uncovered is global (across all detectors) and oscillates with a single corresponding frequency, in contrast to Fourier transform methods that would detect different frequencies for different sensors.

Furthermore, via Koopman eigenvalue analysis, we provide objective means for extracting temporal characteristics of traffic patterns and further analyze the eigenvalues to provide evidence for the works of Kim, Gartner, Dendrinos [TM04; NMR01; Den10; Che+12; TBO12] and others. We have demonstrated intra-week (supplementary figures 14-15) and intra-day patterns within highway traffic in addition to the well-known daily and weekly cycles. Lastly, we have shown how superimposing the modes can decompose the data into the corresponding decaying, growing, and persisting sub-patterns.

We have also developed an accurate and efficient platform for the real-time forecasting of highway traffic conditions. Our method demonstrates that the wave-like trends account for most of the dynamics and yield accurate forecasts, despite the unpredictability involved in a traffic system. As opposed to previous approaches, we have not filtered, smoothed, or aggregated our data. We have not distinguished between weekday/weekend, or adverse weather conditioned traffic, nor have we limited our analysis to single or few detectors. Our method's performance does not rely on extensive historical training nor parameter tuning or selection. We showed the method's capability to generalize to the challenging scenario of multi-lane highways and multi-lane networks of highways, without any loss to its performance or efficiency. The robustness, efficiency, and versatility of the algorithm make it possible to implement with real-time monitoring systems to provide cost-efficient forecasts. This is in strong contrast to many state-of-the-art benchmarks that depend heavily on the proper pre-processing of data, tuning of parameters, and training over large historical data only to produce case-specific (weekday/weekend/holiday), location-specific and limited (single-lane, single highway) forecasts. The information uncovered by the KMD can be relayed to autonomous vehicle control units and dynamic on-ramp metering algorithms to mitigate traffic. Lastly, we emphasize that our methodology makes no assumptions on the physical nature of the underlying system. We only assume to have time-ordered data arising from observations of a linear or nonlinear dynamical system. The forecasting methodology we have developed is, in fact, quite general and can be applied across different fields of study beyond highway traffic.

5.1.2 Summary of Chapter 4

In chapter 4 we saw how a dynamical system induces, in addition to the Koopman group of operators U^t, a group of operators on sections of the tangent bundle. Prior works on such operators exist and have given rise to spectral invariants that were not reflected in the spectrum of the induced Koopman operators. In this work, we demonstrated that the induced operators on sections of the tangent bundle are a proper generalization of the induced Koopman operators. We then provided the appropriate construction for the induced operators on sections of arbitrary tensor bundles of the state space and proposed their study.

In this dissertation, we have focused on studying the spectrum of the induced operator V^t on sections of the tangent and the operator W^t on sections of the kth cotangent bundles. We sort out the spectral relations by demonstrating how knowledge of eigenfunctions, eigensections, and eigenforms, or combinations of them, can be used to generate additional spectral objects of a dynamical system. We can show that the eigenforms are preserved under Lie differentiation in the direction of an eigensection. Specifically, for eigenfunctions, the Lie derivative of an eigenfunction in the direction of an eigensection is again an eigenfunction. This result provides a route for showing that in certain cases the Koopman operators induced by the system and the Koopman operators induced by the eigensections share a common eigenfunction. The result is analogous to the statement that commuting matrices share at least one eigenvector. We then show that knowledge of an eigen k-form and an eigensection can be combined to produce an eigen $(k-1)$-form via their interior product. A corollary to this is that the interior product between an eigencosection and an eigensection produces an eigenfunction.

We then work out the algebraic properties of eigensections and eigenforms by showing that the spectrum of the induced operators on the space of smooth vector fields and differential forms is a subalgebra of the Lie and exterior algebra, respectively. Also, the the exterior derivative of a non-closed eigen k-form is an eigen $k+1$-form. Now, since the only closed function is the zero function, the exterior derivative of n-linearly independent, differentiable, eigenfunctions will generate n linearly independent eigencosections. From there, the exterior product of k-distinct eigencosections will allow one to generate $\binom{n}{k}$ linearly independent eigen k-forms. The algebraic results we derive for eigenforms coincides with the well-known algebraic properties of Koopman eigenfunctions. This result is expected since functions are forms of zero grade.

We also show that eigensections and eigenforms pull back to eigensections and eigenforms in the presence of a diffeomorphic conjugacy between two dynamical systems. This is in the same spirit of the well-known result that eigenfunctions pull back to eigenfunctions under a conjugacy. The main difference is that eigenfunctions will pullback under an arbitrary mapping, vector fields require a diffeomorphic conjugacy to pull back, and forms require a differentiable homeomorphism.

We then investigated the eigensections and eigenforms of a linear dynamical system with a simple spectrum. At first, this may seem unnecessary since the dynamics of a linear dynamical system are well understood. However, the fact remains that there exists conditions for which a nonlinear dynamical system can be shown to be conjugate to a linear system, and these are the contents of the celebrated Hart-Grobman, Sternberg, and Poincare linearization theorems. These facts, coupled with the conjugacy results, beckon the characterization of the spectrum of the induced operators of a linear system. This will provide a method for computing the spectrum of a nonlinear system conjugate to a linear system by pulling back the linear system's induced spectrum. Additionally, we have shown that the linear eigensections can be used to obtain a spectral expansion to represent the time evolution of an arbitrary linear vector field.

Additionally, we have introduced the notion of an eigendistribution and studied their integrability properties. For the class of uniformly hyperbolic dynamical systems, we consider the stable and unstable eigendistributions corresponding to the span of eigensections, which experience exponential growth or decay in time. The stable and unstable eigendistributions are shown to be integrable so long as they altogether span the stable and unstable subbundles. This result can be seen as a spectral counterpart to the Arnold-Sinai theorem, which guarantees the existence of the stable and unstable foliations.

The result is complemented by showing that the integral manifolds, and hence the foliations, can be obtained via the joint level sets of $n - \dim(\Delta_{\mathbf{\Phi}})$ Koopman eigenfunctions of the eigensections which span the eigendistribution. The result provides a constructive method of determining the foliations from the induced operators' spectrum on sections and functions. Furthermore, for a not necessarily hyperbolic system, the existence of $n - \dim(\Delta_{\mathbf{\Phi}})$ joint Koopman eigenfunctions of the spanning eigensections is used to show that the corresponding eigendistribution is integrable.

Lastly, we demonstrated two applications of the results developed in this dissertation. One application was to the field of differential geometry, where we prove a generalization of the well-known theorem on the commutativity of the flows of commuting vector fields. Specifically, we show that the flows of two vector fields $\mathbf{F}(x), \mathbf{G}(x)$ commute, subject to an appropriate rescaling of the flow time, if and only if one vector field is an eigensection of the other vector field, $[\mathbf{F}(x), \mathbf{G}(x)] = \lambda \mathbf{G}(x)$ or vice versa. The required time scaling is, as one would expect, prescribed by the eigenvalue λ. Notably, when $\lambda = 0$, we recover the original statement that the flows of commuting vector fields commute as a particular case of our result.

Our second application was the spectral analysis of the dynamical system known as the cat map. We utilize the spectrum of the induced operators on the space of sections and cosections to show that the linearization's eigenvalues are contained in the spectrum of the induced operators. This indicates that the Lyapunov exponents and entropy, which are invariants of the system not captured by the spectrum of the induce Koopman operators, are in these operators' spectrum. Furthermore, we utilize our result and determine the eigenfunctions of the stable and unstable eigensections to recover the stable and unstable foliations. The main takeaway is that while the cat map itself had no eigenfunctions, the eigensections of the cat map do have eigenfunctions. Moreover, the level sets of the eigenfunctions of the eigensections provide the foliations of the cat map.

5.2 Future Works

There are several intended directions for our future works. Of course, there is a desire to implement numerical algorithms that can compute the spectrum of the induced operators on vector fields and forms. Such algorithms are currently in the works but are yet to be finalized.

There is also an intention to address the induced operators' spectrum for linear systems with degenerate spectrum. The ideas should carry over with sufficient care. Additionally, deriving a spectral expansion for the time evolution of a nonlinear vector field under linear dynamics would be the next step. For the standard Koopman operator, this involved the consideration of real-analytic observables. It seems reasonable to start with a similar assumption for vector fields and proceed from there. Next would be a spectral expansion for a nonlinear vector field under nonlinear dynamics. This will entail careful consideration of which specific space of vector fields we are attempting to work in. For a measure-preserving system, the space of square-integrable vector fields and forms seems to be a reasonable choice.

In terms of a specific class of dynamical systems, the non-uniformly hyperbolic systems are our next interest. Specifically, we are reasonably confident that these operators' spectrum contains the exponents of a non-uniformly hyperbolic system's growth or decay. However, in the non-uniform case, the exponents are, of course, state-dependent. This motivates our desire to study the following type of relation.

$$\mathcal{L}_{\mathbf{F}}\mathbf{\Phi}(x) = \lambda(x)\mathbf{\Phi}(x) \tag{5.1}$$

where $\mathcal{L}_{\mathbf{F}}\lambda(x) = 0$. The relation is similar to the generalized eigenfunctions that Von Neumann and Halmos set forth. The difference is that the space of sections of a (tangent, cotangent, tensor) bundle over $\mathcal{M}$ carries the additional structure of being a module over functions. Loosely speaking, this is equivalent to saying that functions are to fields what scalars are to functions. Our point is the following. Functions are an algebra over scalars as such the eigenvalues for an operator on functions are typically defined to be complex numbers. Analogously, the module structure of fields seems to justify, at least intuitively, taking eigenvalues of the induced operators on sections from the ring of functions. We would call the function $\lambda(x)$ and eigenvalue function of $\mathcal{L}_{\mathbf{F}}$.

Results on such types of expressions exist but not in the language of the spectrum of an operator. In the context of differential geometry, such a function $\lambda(x)$ would correspond to the negative of what is called a normalizer and results showing the existence of integrals of motion when $[\mathbf{F}, \mathbf{\Phi}] = \lambda(x)\mathbf{F} + \mu(x)\mathbf{\Phi}$ have been obtained by others.

We believe the eigenvalue functions capture the state dependence of the exponents of a non uniformly hyperbolic system but this warrants further investigation. Furthermore, if the conjecture is true, a result equivalent to Mather's statement regarding the uniformly hyperbolic case is expected and should be within reach.

There is, of course, an intent to carry out the spectral study to higher-order tensors. It is expected that understanding the spectrum on vector fields and forms will allow us to build results for tensor fields via the tensor algebra. One example that comes to mind is the notion of a Killing vector field that arises in physics. A Killing vector field satisfies

$$\mathcal{L}_{\mathbf{F}} g = 0 \tag{5.2}$$

where g is a Riemannian metric. In physics, one usually fixes a metric g and searches for a field $\mathbf{F}$. From our perspective, this is the inverse of the spectral problem and a precise statement regarding the $\lambda = 0$ case. We propose the spectral problem of finding tensor fields which satisfy

$$\mathcal{L}_{\mathbf{F}} g = \lambda(x) g \tag{5.3}$$

In physics, such vector fields $\mathbf{F}$ are called conformal Killing fields and have been studied. Our proposition is related to those works but again we are interested in fixing the vector field and finding the associated eigentensor.