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Foliations That Are Not Approximable By Smoother Ones

Christopher Ennis, Morris W. Hirsch¹, and Charles Pugh²§0. Introduction

We give several new examples of foliations that cannot be approximated by smoother ones. Let M be a C^∞ manifold of dimension $m > 0$ and F a foliation of M of codimension k , having smooth leaves, and defined by C^1 foliation boxes. The smoothness of F is measured by the differentiability of its tangent field

$$\tau F : M \longrightarrow G_{m-k}(M)$$

where $G_{m-k}(M)$ is the Grassmann bundle of $(m-k)$ - planes tangent to M , and τF is the section assigning to $x \in M$ the tangent plane $\tau_x F$ of the leaf F_x through x . If τF is a C^s map, we call F a C^s foliation, $0 \leq s < \infty$. This is slightly stronger than requiring F to be defined by C^s foliation boxes. This point is discussed below.

We give the space $C^s(M, G_{m-k}(M))$ the weak C^s topology. The set of C^s foliations is given the induced topology as a subset.

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Let $0 \leq r \leq s \leq t$. We say that a C^s foliation F can be C^r approximated by C^t foliations provided every neighborhood of F in $C^r(M, G_{m-k}(M))$ contains a C^t foliation. A basic approximation question is:

Can every C^s codimension- k foliation of every m -manifold
be C^r approximated by C^{s+1} foliations ($0 \leq r \leq s$) ?

Some answers are:

(i) Yes if $k = m - 1$. This amounts to the Whitney approximation theorem applied to vector fields.

(ii) No if $k \leq m - 1$ and $k, r, s \geq 2$, as shown by Anderson [1] using finite group actions.

(iii) No if $0 = r - s$, $m = 3$ and $k = 1$, proved by Rosenberg and Thurston [13] on the 3-torus T^3 .

(iv) No if $m \geq 3$ and $r = s = k = 1$. In these examples one can take $M = T^m$. See remarks below.

(v) No if $m \geq 3$ and either: $r = 0$ ^{and} $s = 1 = k$, or $r = 1 = s$ and $1 \leq k \leq m - 2$. This is the content of the following theorem, our main result. Let M^m denote a compact m -dimensional manifold without boundary.

0.1 THEOREM. Fix $m \geq 3$.

(a) There exists a C^1 codimension-one foliation of some compact M^m which cannot be C^0 approximated by C^2 foliations.

(b) For each k in the range $1 \leq k \leq m - 1$ there is a C^1 codimension- k foliation of some M^m that cannot be C^1 approximated by C^2 foliations.

(c) The foliations in (a) and (b) can be taken to be orbit foliations of locally free actions by solvable Lie groups.

In §1 we give our basic constructions. In §5 and §6, (b) in §6, and (c) in §2 we describe the main example and prove the weaker result that it cannot be C^1 approximated by C^2 foliations. A different example of this kind is outlined in Remark 2, below. The full result in (a) is proved in §4 and §5. Parts (b) and (c) are proved in §6 and §7, respectively.

Anderson stated without proof that Rosenberg and Thurston's example (see (iii) above) can be improved to $s = 1$. Our example differs in that it has no compact leaf at which to apply Kopell's Lemma (see p. 466 of [13]).

In his paper [11], D. Pixton shows that in the space C^1 actions of $\mathbb{Z} \times \cdots \times \mathbb{Z}$ on S^1 , the set of C^2 actions having at least one compact orbit is nowhere dense. By Hirsch [9] there is a nonempty open set of C^1 actions, each having a compact orbit. Therefore there exists a C^1 action $\alpha \in S$ which cannot be C^1 approximated by C^2 actions. By suspending the k generators of α , we obtain a C^1 locally free action β of $\mathbb{R}^k$ on the k -torus T^{k+1} . Let H denote the orbit foliation of β . It can be shown that any sufficiently close C^1 approximation F to H is the orbit foliation of the suspension of an action $\gamma(F)$ of $\mathbb{Z}^k$ on S^1 which is C^1 close to α ; $\gamma(F) \rightarrow \gamma$ as $F \rightarrow H$ in the C^1 topologies; and $\gamma(F)$ is C^2 when F is C^2 . It follows that H cannot be C^1 approximated by C^2 foliations.

In this way we obtain:

0.2 THEOREM. For every integer $k \geq 2$ there is a locally free C^1 action of $\mathbb{R}^k$ on T^{k+1} whose orbit foliation cannot be C^1 approximated by C^2 foliations.

This result leads to another proof of part (b) of Theorem 0.1 for the case $k = 1$. The two results differ in that all known examples of the actions described in Theorem 0.2 have compact leaves, while our main example (see §2) has no compact leaf.

The preceding theorem also gives examples of part (c) of Theorem 0.1 for the case $k = 1$. But in the proof of part (c) given in §7, we construct actions of nonabelian solvable groups.

The following conjecture is suggested by the foregoing remarks.

0.3 CONJECTURE. Let F be a codimension-one foliation of a compact manifold which comes from a C^1 action of $\mathbb{R}^n$. If F has no compact leaves, then F can be C^1 approximated by C^2 foliations.

In [6] D. Hart proves that if F is a foliation which has C^s foliation-boxes (so that τF may be only C^{s-1}) and if $s \geq 1$ then F is C^s diffeomorphic to a foliation G whose tangent field τG is of class C^s .

In [3] M. Cohen constructs a C^1 codimension-one foliation F of M^3 which cannot be C^1 approximated by C^2 foliations which are C^1 diffeomorphic to F . Our construction in §2 is similar to his. Cohen also shows that any C^1 foliation F is C^1 diffeomorphic to

a foliation G whose tangent field is C^∞ off some closed, nowhere dense set K in M . It remains an interesting question to determine how large K must be. For example, is there some obstruction to making K have codimension two in M ?

§1. Suspending Foliated Manifolds

Before outlining our construction we discuss some technical points.

By our definition, a foliation of class C^s has its field of tangent spaces of class C^s . Thus the set of C^s foliations of M ($s \geq 1$) is a subset of the space of C^s maps $M \rightarrow G_{m-k}(M)$, and we give it the subspace topology. Let us temporarily call these C^s foliations analytical.

A different, inequivalent definition of C^s foliation is in terms of foliation boxes. By a geometrical foliation of M of class C^s and codimension k , we mean a decomposition of M into disjoint subsets, called leaves, with the following property: let F_x denote the leaf through x . There is a covering of M by open sets U_i (called foliation boxes) and there are diffeomorphisms $f_i : U_i \approx \mathbb{R}^{m-k} \times \mathbb{R}^k$ such that if $f_i(x) = (y, z)$, then the component of x in $F_x \cap U_i$ is $f_i^{-1}(\mathbb{R}^{m-k} \times \{z\})$. The set of geometrical foliations has a topology derived from the C^s topology on such maps f_i ; see Hirsch [8] for details.

Every analytical foliation is geometrical, but the converse is not true: a geometrical foliation may have a tangent plane field which is only C^{s-1} , not C^s . This is a technical difficulty which has plagued the subject since its inception.

Thanks to recent work of Hart [6], these difficulties can be evaded rather simply. Let $\underline{A}$, $\underline{G}$ denote respectively the spaces of analytical

and geometrical foliations of M , for some fixed codimension k and differentiability $s > 1$. Give them metrics d_A, d_G . Let $w : \underline{A} \rightarrow \underline{G}$ be the natural map (weakening of structure). Hart proves (in essence) that w has approximate local inverses in the following sense. Fix $G \in \underline{G}$. Then there is a neighborhood $\underline{N} \subset \underline{G}$ of G such that for every $\varepsilon > 0$ there are continuous maps $s_\varepsilon : \underline{N} \rightarrow \underline{A}$ and $h_\varepsilon : \underline{N} \rightarrow \text{Diff}^s(M)$, so that the following hold for all $G \in \underline{N}$:

- (i) $h_\varepsilon(G)$ maps leaves of F to leaves of $s_\varepsilon(G)$;
- (ii) $d(h_\varepsilon, l_M) < \varepsilon$ where l_M is the identity map of M and d is the metric for $\text{Diff}^s(M)$;
- (iii) $d_G(ws_\varepsilon(G), G) < \varepsilon$;
- (iv) if $G = w(A)$ (i.e. if τ_F is C^s), then

$$d_A(s_\varepsilon w(A), A) < \varepsilon.$$

It seems plausible that one can take $\underline{N} = \underline{G}$, but we don't need this.

It follows that for studying approximations one can pass back and forth between $\underline{A}$ and $\underline{G}$ (as Hart does in [6] for studying the Closing Lemma).

We shall actually use only the result of Hart that every geometrical foliation is C^s diffeomorphic to some analytical foliation.

In the rest of the paper we use "foliation" (without adjectives) to mean "analytical foliation."

Our basic method is that of suspension and desuspension. Let f be a C^s diffeomorphism of the m -dimensional manifold M . The suspension

of f is the $(m+1)$ - dimensional manifold

$$M^f = (M \times \mathbb{R})/\mathbb{Z}$$

where the generator of $\mathbb{Z}$ acts on $M \times \mathbb{R}$ by

$$(x, t) \longmapsto (f(x), t - 1) .$$

We give M^f the natural C^s structure making the orbit map

$$\pi : M \times \mathbb{R} \longrightarrow M^f$$

of class C^∞ .

Suppose now that F is a foliation of M of codimension k and differentiability $s \geq 1$, which is preserved by f ; that is,

$$f(F_x) = F_{f(x)}$$

for all $x \in M$. Let $F \times \mathbb{R}$ denote the C^s foliation of $M \times \mathbb{R}$ whose leaves are the sets $F_x \times \mathbb{R}$. Clearly the action of $\mathbb{Z}$ preserves $F \times \mathbb{R}$. Therefore there is a unique foliation F^f of M^f such that π takes each leaf of $F \times \mathbb{R}$ onto a leaf of F^f . We call M^f and F^f the suspensions of M and F by f . Notice that F and F^f have the same codimension.

What is the differentiability of F^f ? The Grassmann bundles of $(m-k+1)$ - planes are related by the commutative diagram

$$\begin{array}{ccc} M \times \mathbb{R} & \xleftarrow{\quad} & G_{m-k+1}(M \times \mathbb{R}) \\ \pi \downarrow & & \downarrow \pi' \\ M^f & \xleftarrow{\quad} & G_{m-k+1}(M^f) \end{array}$$

where π' is the natural map induced by π . The sections corresponding to F and F^f are related by the commutative diagram (suppressing the subscript $m - k + 1$):

$$\begin{array}{ccc} M \times R & \xrightarrow{\tau F} & G(M \times R) \\ \pi \downarrow & & \downarrow \pi' \\ M^f & \xrightarrow{\tau F^f} & G(M^f) \end{array}$$

Let $U \subset M \times R$ be an open set which π maps diffeomorphically onto an open set $V \subset M^f$. Then $\tau F^f|_V$ can be expressed as

$$\pi' \circ (\tau F) \circ (\pi|_U)^{-1} : V \longrightarrow G(M^f).$$

Now π' , expressed in local coordinates, involves the differential of π . Therefore π' is only of class C^{s-1} . In fact the natural differential structure on $G(M^f)$ is only C^{s-1} . Therefore τF^f is only of class C^{s-1} . This means that F^f is merely a C^{s-1} foliation.

On the other hand, it is easy to see that F^f is C^s as a geometrical foliation. Therefore, we can apply Hart's theorem to conclude that there is a C^s diffeomorphism h of M^f and a C^s analytical foliation K such that h takes leaves of F^f to leaves of K .

By construction, M^f has a C^s differential structure. By standard theorem (see e.g. [7]) M^f has a compatible C^∞ differential structure α . Then $h^* \alpha$ is another C^∞ differential structure on M^f , and

$$h : (M^f, f^* \alpha) \longrightarrow (M^f, \alpha)$$

is a C^∞ diffeomorphism. Since K is a C^s foliation of (M^f, α) , it follows that F^f is a C^s foliation of $(M^f, f^* \alpha)$. We have proved:

The natural C^S structure on M^f is compatible with a C^∞ structure in which F^f is a C^S foliation.

In what follows, we shall always assume M^f has been given such a C^∞ structure. Thus, F^f is of class C^S .

Notice that there is a C^∞ embedding of $i : M \longrightarrow M^f$, namely the composition

$$M \xrightarrow{i_0} M \times \mathbb{R} \xrightarrow{\pi} M^f$$

where $i_0(x) = (x, 0)$. We often identify M with $i(M)$. With this identification we can recover F from F^f by

$$F = F^f \cap M ;$$

that is, for each $x \in M \subset M^f$, the leaf F_x is the component of x in $(F^f)_x \cap M$.

We can also recover the C^S diffeomorphism f of M from data on M^f . The action of $\mathbb{Z}$ on $M \times \mathbb{R}$ preserves the vertical vector field $\tilde{V} = (0, \frac{\partial}{\partial t})$ on $M \times \mathbb{R}$ (where t is the coordinate in $\mathbb{R}$). Therefore, $\tilde{V}$ is related by π to a vector field V on M^f . Let $\{\phi_t\}_{t \in \mathbb{R}}$ denote the flow of V on M^f , covered by the flow $\tilde{\phi}_t : (x, s) \longmapsto (x, s + 1)$ on $M \times \mathbb{R}$. It is easy to see that

$$\phi_1(M) = M \quad \text{and} \quad \phi_1|_M = f .$$

Notice that each integral curve of V lies in a leaf of F^f , so that $\{\phi_t\}$ preserves F^f .

Now suppose G is a C^S foliation of M^f which is very C^S close to F^f . We want to C^S perturb V to a C^S vector field W

tangent to G . For this purpose, fix a C^∞ Riemannian metric on M^f (independent of G). Let $TG \subset TM^f$ be the subbundle of the tangent vector bundle of M^f which is tangent to (leaves of) G . Let $(TG)^\perp \subset TM^f$ be its orthogonal complement. Let $P : TM^f \rightarrow TG$ be the vector bundle projection with kernel $(TG)^\perp$. Because the Grassman map $\tau_G : M^f \rightarrow G_{m-k+1}(M^f)$ is C^S , it follows that P is C^S . Therefore the vector field $W = PV$ is C^S . By construction, W is tangent to G , so its integral curves lie in leaves of G . It is easy to see that $W \rightarrow V$ in $C^S(M^f, TM^f)$ as $G \rightarrow F$ in the space of C^S foliations.

Set

$$M_s = \pi(M \times \{s\}) \subset M^f.$$

Notice $M^0 = M$ and $M_s = M_{s+1}$. If G is sufficiently close to F , then the vector field W will be transverse to all the submanifolds M_s . From now on we assume this holds.

We now multiply the vector field W by a positive C^S function $\gamma : M^f \rightarrow \mathbb{R}$ so that the time-one map of the flow of γW preserves M . To define γ , let $q : M^f \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$ be the C^S map covered by the projection $M \times \mathbb{R} \rightarrow \mathbb{R}$. Let Z be the vector field on S^1 covered by $\frac{d}{dt}$ in $\mathbb{R}$. Define $\alpha : M^f \rightarrow \mathbb{R}$ by the equation

$$dq_p W(p) = \alpha(p) Z(q(p)) \quad , \quad p \in M^f.$$

The assumption that W is transverse to the submanifolds M_t ensures that $\alpha > 0$.

Clearly α is continuous. In fact, α is C^S . To prove this, it suffices to prove that the composition

$$M \times \mathbb{R} \xrightarrow{\pi} M^f \xrightarrow{\alpha} \mathbb{R}$$

is C^S . It is easy to see that this map is the same as the composition

$$M \times \mathbb{R} \xrightarrow{\tilde{W}} T(M \times \mathbb{R}) \xrightarrow{\beta} \mathbb{R} ,$$

where $\tilde{W}$ covers W , and β is the natural vector bundle map assigning to each tangent vector to $M \times \mathbb{R}$ its $\mathbb{R}$ -component. (In other words, β is the derivative of the natural projection $M \times \mathbb{R} \rightarrow \mathbb{R}$.) Since $\tilde{W}$ is C^S and β is C^∞ , we find that α is C^S .

Set $\gamma = 1/\alpha$ and set $Y(x) = \gamma(x)W(x)$. Let $\{\theta_t\}$ be the flow in M^f of the vector field Y , and $\{\tilde{\theta}_t\}$ the flow in $M \times \mathbb{R}$ covering $\{\theta_t\}$. It is easy to see that for all $s, t \in \mathbb{R}$,

$$\tilde{\theta}_t(M \times \{s\}) = M \times \{s + t\} .$$

Therefore there is a unique C^S map $H : M \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\tilde{\theta}_t(x, s) = (H(x, s), s + t) .$$

It follows that $\theta_t : M_s \rightarrow M_{s+t}$. In particular, $\theta_1(M) = M$, using the identification $M_0 = M$. It is easy to see that the C^S diffeomorphism $\theta_1 : M \rightarrow M$ is defined by

$$\theta_1(x) = fH(x, 0) .$$

Define $h(x) = fH(x, 0)$. This diffeomorphism $h : M \rightarrow M$ is completely determined by the foliation G of M^f (once we fix the Riemannian metric on M^f). It is easy to see that $h \rightarrow f$ in $\text{Diff}^S(M)$ as $G \rightarrow F$ in the space of C^S foliations of M^f .

Consider now the problem of C^r approximating F^f by a C^{s+1} foliation G , $0 \leq r \leq s$. If this is possible, then by desuspension we obtain a C^{s+1} foliation H which C^r approximates F , together with a C^{s+1} diffeomorphism h which preserves H and C^r approximates f . If we can show that, owing to the nature of F and f , such approximations are impossible, then we will have proved that F^f cannot be C^r approximated by C^{s+1} foliations. This is our basic method.

Now consider the C^s foliation $H = G \cap M$ of M . It is easy to see that $H \longrightarrow F$ as $G \longrightarrow F^f$, in the spaces of C^s foliations.

By construction, $\{\theta_t\}$ preserves G . Therefore, h preserves H . Thus, from the perturbation G of F^f , we obtain the perturbation H of F and the perturbation h of f , such that h preserves H . Moreover, H and h have the same differentiability as G .

It is not hard to show that there is a C^s diffeomorphism $\lambda : M^h \longrightarrow M^f$ taking H^h to G . In fact λ is covered by the map

$$M \times \mathbb{R} \longrightarrow M \times \mathbb{R} ,$$

$$(x, t) \longmapsto \tilde{\theta}_t(x, 0) .$$

Thus (H, h) is a desuspension of G .

The important element in constructing the approximating diffeomorphism $h : M \longrightarrow M$ is the vector field W tangent to G . It is not necessary to use a Riemannian metric for this purpose. An alternative method, to be followed in §4, relies instead on the existence of a foliation K of M which is transverse to F and invariant under f . Then K^f is a foliation of M^f which intersects F^f in a 1-dimensional foliation of M^f whose leaves are the integral curves of the vector

field V . If G is so close to F^f that K^f is transverse to G , then $G \cap F^f$ is a 1-dimensional foliation whose leaves are integral curves of a vector field U . This vector field, suitably scaled, can be used in the same way as W . The time-one map of the flow of U will leave M invariant. Restricted to M , this map provides a C^{s+1} diffeomorphism $\tilde{f}$ of M (if K and G are C^{s+1}) which is C^r close to f (if G is C^r close to F^f). Moreover $\tilde{f}$, by construction, has the important property of preserving the foliation K . In fact, $\tilde{f}$ acts on leaves of K in the same way as f .

If for some reason no such approximation $\tilde{f}$ can exist, then G cannot exist either. This method will be used in §4.

Suppose now that G is a Lie group acting on M by a locally free C^1 action, denoted by $(g, x) \mapsto g \cdot x$, whose orbit foliation is F . Suppose that there is an automorphism ϕ of G such that

$$g \cdot (f(x)) = f(\phi(g) \cdot x)$$

for all $x \in M$, $g \in G$, where $f: M \rightarrow M$ is the diffeomorphism which preserves F . Suppose also that ϕ embeds in a 1-parameter group of automorphism $\{\phi_t\}_{t \in \mathbb{R}}$ of G with $\phi_1 = \phi$. We shall extend G to a group H acting on M^f with orbit foliation F^f .

Let H be the split extension of G by $\mathbb{R}$ where $\mathbb{R}$ acts on G by: $t \cdot g = \phi_t(g)$. By definition the underlying space of H is $G \times \mathbb{R}$ and the group operation is

$$(h, t)(g, r) = (h\phi_t(g), t + r) .$$

Define an operation of H on $M \times \mathbb{R}$ by

$$(g, r) \cdot (x, s) = (\phi_{s-r}(g)x, s - r)$$

for $(g, r) \in G \times \mathbb{R} = H$ and $(x, s) \in M \times \mathbb{R}$. One readily verifies that this defines a homomorphism $H \longrightarrow \text{Diff}^S(M \times \mathbb{R})$, and also that the actions of H and $\mathbb{Z}$ on $M \times \mathbb{R}$ commute. (The action of $\mathbb{Z}$ is such that $(M \times \mathbb{R})/\mathbb{Z} = M^f$, i.e., $n \cdot (x, s) = (h(x), (s - n)0)$.) Therefore there is induced a C^S action of H on M^f . It is evident that the orbit foliation of this action is F^f .

Notice that H is solvable when G is solvable, since there is an exact sequence

$$0 \longrightarrow G \longrightarrow H \longrightarrow \mathbb{R} \longrightarrow 0.$$

§2. A Suspended Solenoid

Let $T^2 = S^1 \times S^1$ be the 2-torus and $F : T^2 \longrightarrow T^2$ the DA-diffeomorphism of Smale [15]. In §3 we give a precise construction of f and prove that f leaves invariant a C^1 foliation F of T^2 , some of whose leaves form the solenoid Σ which is the DA's attractor.

Applying the suspension construction we obtain a foliation F^f of $M^3 = (T^2)^f$.

We claim F^f cannot be C^0 approximated by C^2 foliations of M^3 .

We first prove there is no C^1 approximation by C^2 foliations, since this proof is simpler and can be generalized in several directions.

Suppose G is a C^2 foliation of M^3 which is C^1 near F^f . From the results in §1 we can desuspend G : there is a C^2 foliation

H of T^2 , preserved by a C^2 diffeomorphism h of T^2 , such that H^h is C^1 diffeomorphic to G ; moreover, $H \longrightarrow F$ and $h \longrightarrow f$ as $G \longrightarrow F^f$, in the C^1 topologies.

We now use the well-known fact that the DA-diffeomorphism f is structurally stable. Therefore we can take G so close to F^f that h is topologically conjugate to f .

Let $g : T^2 \longrightarrow T^2$ be a homeomorphism such that $gfg^{-1} = h$. Then g carries Σ into a solenoid Σ' invariant under h . By taking h sufficiently C^1 close to f , we may assume that Σ' is a hyperbolic basic set for h , composed of unstable manifolds of h . Uniqueness of the unstable manifold foliation of a basic set implies that H_x , for $x \in \Sigma'$, is the unstable manifold of x for h . Therefore Σ' is a union of H leaves. Now H is a C^2 foliation of T^2 , so by Denjoy's theorem either every leaf is dense or every leaf is compact. In particular, this holds for leaves of H in Σ' . The same must therefore hold for leaves of F in Σ ; but this is a contradiction. This completes the proof that F^f cannot be C^1 approximated by C^2 foliations of M^3 .

Before proving, in §4, that there is no C^0 approximation, we study the DA in detail in §3.

§3. Foliations Invariant Under the DA-Diffeomorphism

Let f_0 be the linear Anosov diffeomorphism $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on T^2 , let $E_0^u \oplus E_0^s$ be its hyperbolic splitting, and express

$$f_0 = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \quad \text{respecting } E_0^u \oplus E_0^s$$

In fact, $a = \frac{1}{2}(3 + \sqrt{5})$, $E_0^u = \text{span}(2, -1 + \sqrt{5})$, and $E_0^s \perp E_0^u$ everywhere.

Modify f_0 according to Smale's DA construction [15] by pushing outward along E_0^s in a neighborhood of 0. This produces a new map, the DA-diffeomorphism

$$f : T^2 \longrightarrow T^2$$

having two new saddle points on what was originally the stable manifold of 0, while 0 itself becomes a source. The unstable manifold of 0 is open-dense in T^2 . Its complement is a solenoid, call it Σ , which is the closure of the unstable manifolds of the newly produced saddles. It is a hyperbolic set for f which attracts all orbits except 0.

Notice that $a > 1$. Fix a constant α , $1 < \alpha < a$. Let $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ function such that $0 \leq \beta \leq 1$, $\beta(x) \equiv 1$ for x near 0, $\beta(x) = 0$ for $|x| \geq 1$.

Now choose a C^∞ map $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$(1) \quad 0 < k = \sup \gamma'(y) \leq \alpha,$$

$$\gamma(y) \equiv y/a \text{ for } |y| \text{ near } 1,$$

and the graph of γ is transverse to the diagonal and crosses it 3 times: at 0 and $\pm y_0$, $0 < y_0 < 1$. Moreover, set

$$(2) \quad c = \sup_{x,y} |\beta'(x)(\gamma(y) - y/a)|$$

and choose γ to make c so small that

$$a \geq \alpha \left[1 + \left(\frac{c}{1 - \alpha/a} \right)^2 \right]^{\frac{1}{2}}.$$

Now define

$$f(z) = \begin{cases} f_0(z) & \text{if } z \in T^2 \setminus S \\ (ax, \beta(x)\gamma(y) + (1 - \beta(x))y/a) & \text{if } z \in S \end{cases}$$

Then f is the DA-diffeomorphism, and

$$T_z f = \begin{bmatrix} a & 0 \\ c_z & k_z \end{bmatrix} \quad \text{respecting } E_0^u \oplus E_0^s$$

where c_z, k_z are C^∞ functions of $z \in T^2$, and $k_z > 0$. Notice that

$$(3) \quad \begin{aligned} c &= \sup |c_z|, \\ k &= \sup k_z \leq \alpha. \end{aligned}$$

For if $z \in S$, then $c_z = 0$ and $k_z = a^{-1} < 1$; while if $z \in S$ then

$$c_z = \frac{\partial}{\partial x} [\beta(x)\gamma(y) + (1 - \beta(x))y/a] = \beta'(x)(\gamma(y) - y/a)$$

and

$$k_z = \frac{\partial}{\partial y} [\beta(z)\gamma(y) + (1 - \beta(x))y/a] = \beta(x)\gamma'(y) + (1 - \beta(x))/a \leq \alpha$$

by (1).

Let $E^u \oplus E^s$ be the hyperbolic splitting of the tangent bundle of T^2 over the solenoid Σ . The main result of this section is

$$(4) \quad E^u \text{ extends to a } C^1 \text{ Tf-invariant line field } E^u \text{ over } T^2.$$

This immediately implies the result needed for §2:

3.1 PROPOSITION. The integral curves of E^u form an f -invariant C^1 foliation of T^2 .

As remarked in §2, Denjoy showed:

(5) No C^2 1-foliation of T^2 admits a solenoid of leaves.

Thus (5) implies that E^u can only be C^1 , not C^2 .

To prove (4) we use the C^1 Section Theorem from [11, p. 31].

At each $z \in T^2$ consider

$$L_z = L(E_{Oz}^u, E_{Oz}^s) = \text{all linear maps } E_{Oz}^u \longrightarrow E_{Oz}^s.$$

The union of the L_z forms a bundle L over T^2 , and Tf acts naturally on L as follows. Let $P \in L_z$. Then

$$P \longmapsto f_{\#}P = (c_z + k_z P) a^{-1}.$$

See [11, p. 34]. This gives a bundle map

$$\begin{array}{ccc} L & \xrightarrow{f_{\#}} & L \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{f} & T^2 \end{array}$$

Let $L(\ell)$ be the disk bundle of size ℓ in L : $L_z(\ell) = \{P \in L_z : |P| \leq \ell\}$.

Observe from (3) that

$$|f_{\#}P| \leq |c_z| + k_z |P| a^{-1} \leq c + \alpha a^{-1} \ell \leq \ell$$

if $|P| \leq \ell$ and $\ell \geq c/(1 - \alpha/a)$. Also

$$|f_{\#}P_1 - f_{\#}P_2| \leq \alpha a^{-1}|P_1 - P_2| \quad \text{for} \quad P_1, P_2 \in L_z$$

Thus ([11, p. 34]), $f_{\#}$ is a fiber contraction of $L(\ell)$, $L(\ell)$ has a unique $f_{\#}$ -invariant section $\sigma: T^2 \rightarrow L(\ell)$, and σ is continuous. This produces the Tf -invariant line field as

$$E_z^u = \text{graph}(\sigma(z))$$

and proves it is continuous. Since it is an invariant continuous complement to $E_0^s = E^s$ over Σ , it equals E^u on Σ by uniqueness of the hyperbolic splitting there.

To prove E^u is C^1 , it suffices to compare the fiber contraction rate of $f_{\#}$ versus its base contraction rate:

$$\text{fiber contraction at } z = k_z a^{-1},$$

$$\text{base contraction at } z = \|(T_z f)^{-1}\|^{-1}.$$

If

$$(6) \quad \sup_z [(k_z a^{-1}) \|(T_z f)^{-1}\|] < 1$$

then E^u is C^1 [11, p. 31]. This norm $\|\cdot\|$ need not be the standard one -- it can arise from any convenient Finsler on T^2 . Let $\langle \cdot, \cdot \rangle_0$ denote the standard Riemann structure on T^2 as a flat torus and set

$$\langle v, w \rangle = \begin{cases} \langle v, v \rangle_0 & \text{if } v = w \in E_z^u \text{ or } E_{0z}^s \\ 0 & \text{if } v \in E_z^u \text{ and } w \in E_{0z}^s. \end{cases}$$

This gives a C^0 Riemann structure $\langle \cdot, \cdot \rangle$ on T^2 agreeing with $\langle \cdot, \cdot \rangle_0$ on E^u and E_0^s , but making E^u perpendicular to E_0^s . Now

$$T_z f = \begin{bmatrix} a_z & 0 \\ 0 & k_z \end{bmatrix} \quad \text{respecting } E^u \oplus E_0^s$$

so, in the new norm,

$$(7) \quad \|(T_z f)^{-1}\| = \max(k_z^{-1}, a_z^{-1}) .$$

Notice that a_z is the norm of $T_z f$ on E_z^u , i.e., if $\begin{bmatrix} 1 \\ w \end{bmatrix}$ spans E_z^u then

$$a_z = \frac{\left| \begin{bmatrix} a & 0 \\ c_z & k_z \end{bmatrix} \begin{bmatrix} 1 \\ w \end{bmatrix} \right|}{(1+w^2)^{1/2}} = \left[\frac{a^2 + (c_z + k_z w)^2}{1+w^2} \right]^{1/2} \geq \left[\frac{a^2}{1+w^2} \right]^{1/2}$$

But $|w| \leq \ell$ since $E_z^u = \text{graph } \sigma(z)$ and $\sigma(z)$ is a linear map $E_{0z}^u \rightarrow E_{0z}^s$ having slope $\leq \ell$. Besides, we were able to choose $\ell \geq c/(1-\alpha/a)$, so

$$a_z \geq \frac{a}{\left[1 + \left[\frac{c}{1-\alpha/a} \right]^2 \right]^{1/2}} \geq \alpha$$

by (2). Thus by (3) $a_z \geq \alpha \geq k_z$, so (7) becomes

$$(8) \quad \|(T_z f)^{-1}\| = k_z^{-1} .$$

Now (6) follows from (8) since $a^{-1} < 1$. Hence E^u is C^1 and (4)

is proved. Recall that (4) implies (3.1), the fact needed in our construction.

The following general result, which we shall not use, also implies (5), at least near Σ .

EXTENSION THEOREM. Suppose f is a C^2 diffeomorphism of M , $\Lambda \subset M$ is a hyperbolic set for f with hyperbolic splitting $E^u \oplus E^s$, E^s is a line bundle, and Λ is an expanding attractor. Thus E^u extends to a C^1 Tf -invariant bundle $\hat{E}^u$ on some neighborhood of Λ .

REMARKS. (a) When $\Lambda = M$, $\hat{E}^u$ must be E^u since it is known that E^u is C^1 in this case.

(b) To prove this Extension Theorem requires a new, relative version of the C^r Section Theorem [11, p. 31], so we prefer the more direct construction of $\hat{E}^u$ given above.

(c) Whether $\hat{E}^u$ can be chosen to integrate some foliation when $\dim(M) \geq 3$ is a tantalizing question.

In the next section we shall use the fact that the DA-diffeomorphism f preserves the foliation G tangent to E_0^s , as is evident from the formula for $f(z)$ (following (2) above). This foliation is C^2 (even real analytic) and its leaves are dense lines in T^2 .

§4. Foliations Invariant Under Perturbations of the DA

Let $f : T^2 \rightarrow T^2$ be the DA-diffeomorphism, leaving invariant both the C^1 (but not C^2) foliation F and the C^2 foliation K by dense lines transverse to F . Let $M^3 = (T^2)^f$. Let G be a C^2 foliation of M^3 which closely C^0 approximates F^f . We seek a contradiction.

Let $H = G \cap T^2$ be the induced foliation of T^2 . We may assume H and K are transverse to each other.

As explained at the end of §1, we can assume there is a C^2 diffeomorphism $\tilde{f}$ of T^2 which leaves both K and H invariant, and which C^0 approximates f as closely as desired. We assume $\tilde{f}$ is so close to f that it carries some closed disk $D \subset T^2$ onto a neighborhood of itself:

$$\text{Int } \tilde{f}(D) \supset D.$$

Here D is a neighborhood of the fixed point $(0, 0)$ (in the notation at the beginning of §3), in which the DA-diffeomorphism f is constructed by deforming the linear toral automorphism f_0 . Now f is an expansion in a neighborhood of 0 because $Df(0, 0) = \begin{bmatrix} a & 0 \\ 0 & k \end{bmatrix}$ with $a > 1$, $k > 1$. Therefore there is a disk D with this property for any map $\tilde{f}$ of f which is sufficiently C^0 close to f . We call such a disk D a source disk for $\tilde{f}$.

Our desired contradiction is a consequence of:

4.1 THEOREM. Let H, K be 1-dimensional C^1 foliations of T^2 which are mutually transverse. Suppose neither foliation has any exceptional minimal sets. Let g be a homeomorphism of T^2 which preserves H and K and which admits a source disk D . Then the induced endomorphism g_* of $H_1(T^2; \mathbb{R})$ has $+1$ or -1 as an eigenvalue.

To reach a contradiction, assuming 4.1, take $g = \tilde{f}$. Since H is C^2 it has no exceptional minimal sets by Denjoy's theorem; and since K has every orbit dense, it also has no exceptional minimal sets. Then $\tilde{f}_*$ has ± 1 as an eigenvalue, by 4.1. But $\tilde{f}$ is homotopic to the

linear toral automorphism $f_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so the eigenvalues are $\frac{1}{2}(3 \pm \sqrt{5})$.

PROOF OF 4.1. Suppose H or K has a compact leaf C , representing $[C] \in H_1(T^2)$. Then $[C] \neq 0$; this follows easily from the Poincaré-Bendixson theorem applied to the foliations $\hat{H}$, $\hat{K}$ of $\mathbb{R}^2$ induced by the covering space projection $\pi : \mathbb{R}^2 \rightarrow T^2$.

Either $g(C) = C$ or else $g(C) \cap C = \emptyset$, since C and $g(C)$ are leaves of a foliation. In the first case, $g_*[C] = \pm[C]$. In the second case, C and $g(C)$ bound a cylinder in T^2 , and again, $g_*[C] = \pm[C]$.

From now on assume neither H nor K has a compact leaf. It is known that this implies H and K are orientable; we fix orientations for them. Since there are no exceptional minimal sets, H and K are homeomorphic to orbit foliations of irrational flows. It follows that every half-leaf is dense. (The positive half-leaf H_x^+ is obtained by following the positive orientation of H_x , starting at x . Similarly for K .)

Now let D be a source disk for g and put $W = \bigcup_{n \geq 0} g^n(D)$. Then W is a simply connected open set, and $g(W) = W$.

No half-leaf is contained in W . To see this, suppose $H_x^+ \subset W$. Let $\pi(y) = x$. Let $\hat{W} \subset \mathbb{R}^2$ be the component of y in $\pi^{-1}(W)$, and consider $\hat{H}_y^+$. Since every leaf of $\hat{H}$ is closed in $\mathbb{R}^2$, we know that $\hat{H}_y^+$ is closed in $\hat{W}$. Since W is simply connected, $\pi : \hat{W} \rightarrow W$ is a homeomorphism. Therefore $H_x^+ = \pi(\hat{H}_y^+)$ is closed in W , contradicting denseness of half-leaves. Similarly for half-leaves of K .

By Brouwer's theorem D contains a fixed point p for g . Then H_p^+ meets the boundary ∂W of W . Let q be the point of $H_p^+ \cap \partial W$ which is nearest to p (in the natural ordering of H_p^+ as a half-line). Then $g(q) = q$. To see this, observe that $H_p^+ \cap \partial W$ is invariant under g . If $g(q) \neq q$ then either $g(q)$ or $g^{-1}(q)$ would be nearer to p .

Let $A \subset F_p^+$ be the interval from p to q . Since K_p^+ is dense there is a point $r \in A \cap (K_p^+ \setminus \{p\})$ which is nearest to p in the natural ordering of K_p^+ . Then $g(r) = r$. Let $C_1 \subset H_p^+$ be the oriented arc from p to r and let $C_2 \subset K_p^+$ be the oriented arc from r to p . Let C be the oriented loop $C_1 \cup C_2$. Then $g(C) = C$ preserving orientation. The Poincaré-Bendixson theorem implies that $[C] \neq 0$ in $H_1(T^2)$, and evidently $g_*[C] = [C]$. QED

REMARK. In the case where F is C^2 and K has only dense leaves, C. Pugh has shown that the hypothesis of Theorem 4.1 is contradictory.

§5. Higher Dimensional Examples

Let E be a C^s foliation of a compact M^m which cannot be C^r approximated by C^{s+1} foliations. For example, $r = 0$, $s = 1$, $M = (T^2)^f$ and $E = F^f$, constructed in §4. Let V be any C^∞ manifold and let $E \times V$ denote the foliation of $M \times V$ whose leaves are $F_x \times V$, $x \in M$. Then $E \times V$ has the same codimension as E , and the same differentiability. Fix $v \in V$ and identify M with $M \times \{v\}$. If F_n , $n = 1, 2, \dots$, is any sequence of C^{s+1} foliation of $M \times V$ converging C^r to $E \times V$, then for sufficiently large

j each F_j intersects M in a C^2 foliation E_j , and $\lim_{j \rightarrow \infty} E_j = E$ in the C^r topology on C^s foliations, a contradiction. This proves that $E \times V$ cannot be C^r approximated by C^s foliations. The proof of part (a) of Theorem 0.1 is complete.

§6. Higher Codimensional Examples

Let $M^3 = (T^2)^f$ be the 3-manifold from §2 on which we constructed the C^1 foliation F^f . For any compact manifold V we construct a C^1 2-dimensional foliation on $V \times M^3$ that cannot be C^1 approximated by C^2 foliations. Together with the method of §4, this gives (b) of our Main Theorem.

Let $f : T^2 \rightarrow T^2$ be the C^∞ DA-diffeomorphism of §2, §3. Choose a point $p \in V$ and let $\phi : V \rightarrow V$ be a C^∞ diffeomorphism isotopic to the identity having p as an attracting fixed point. The product diffeomorphism

$$\phi \times f : V \times T^2 \longrightarrow V \times T^2$$

leaves invariant the torus $p \times T^2$. We choose ϕ so that $\|(D\phi)_p\| \cdot \|(Df^{-1})_x\|^2 < 1$ for all $x \in T^2$. Then $p \times T^2$ is persistent in the sense that any C^1 approximation to $\phi \times f$ possesses a unique invariant torus $\tilde{T}$ near $p \times T^2$ and $\tilde{T}$ C^1 approximates $p \times T^2$. Besides, if the C^1 approximation to $\phi \times f$ is itself of class C^2 , but does not necessarily C^2 approximate $\phi \times f$, then $\tilde{T}$ is of class C^2 . See [11, pp. 39, 50-51] for the proof of these assertions about $\tilde{T}$.

Now we suspend $V \times T^2$ by $\phi \times f$. Since $\phi \times f$ is isotopic to $1_V \times f$, it follows that

$$(V \times T^2)^{\phi \times f} \approx (V \times T^2)^1 V \times f \approx V \times (T^2)^f = V \times M^3 .$$

The foliation F of T^2 , discussed in §2, extends to a foliation F' of $V \times T^2$:

$$F'_{(x, y)} = \{x\} \times F_y .$$

Thus F' has 1-dimensional leaves. It is clearly preserved by $\phi \times f$.

The foliation K of T^2 extends to the foliation $V \times K$ of $V \times T^2$:

$$(V \times K)_{(x, y)} = V \times K_y .$$

Then $V \times K$ has codimension-one, and is $\phi \times f$ -invariant. Evidently $V \times K$ is transverse to F' .

Now consider the suspended foliation $\hat{F} = (F')^{\phi \times f}$. It is a C^1 foliation of $V \times M^3$ having 2-dimensional leaves. As remarked at the end of §1, if $\hat{F}$ can be C^1 approximated by C^2 foliations G , then $\phi \times f$ can be C^1 approximated by C^2 diffeomorphisms of $V \times T^2$ which preserve both K' and $H' = G \cap (V \times T^2)$. (We are identifying $V \times T^2$ with a submanifold of $(V \times T^2)^{\phi \times f}$ which is transverse to G . The leaves of G intersect this submanifold in leaves of a foliation $G \cap (V \times T^2)$ of $V \times T^2$). To reach a contradiction, it thus suffices to prove that there cannot exist such approximation to $\phi \times f$. For this we will use Theorem 4.1.

As remarked above, any sufficiently C^1 close approximation $g : V \times T^2 \rightarrow V \times T^2$ to $\phi \times f$ has an invariant torus $T \subset V \times T^2$ which is C^1 close to $\{p\} \times T^2$. Thus $g|_{\tilde{T}}$ is a C^1 diffeomorphism of $\tilde{T}$. It preserves the foliation $\tilde{K} = \tilde{T} \cap (V \times K)$.

Next we show that the leaf of H' through any point of $\tilde{T}$ lies wholly in $\tilde{T}$.

If $q \in \tilde{T}$ let H_q be the tangent line to H' at x . It suffices to prove that H_q is tangent to $\tilde{T}$. These lines form a subbundle of $T_{\tilde{T}}(V \times T^2)$ which is invariant under Tg . By normal hyperbolicity of g at $\tilde{T}$, there is a Tg -invariant splitting

$$T_{\tilde{T}}(V \times T^2) = T(\tilde{T}) \oplus \nu.$$

Moreover the backward iterates of Tg drive vectors in $T_{\tilde{T}}(V \times T^2)$ which are not in $T(\tilde{T})$ toward ν (in the sense of angle in some Riemannian metric). We can assume that the angles between H_q and ν_q are bounded above zero. It follows that every H_q must be in $\tilde{T}_q$.

Thus the map $g|_{\tilde{T}}$ preserves $\tilde{K}$ and the foliation $\tilde{H} = H'|_{\tilde{T}}$. Evidently $\tilde{K}$ (like K and $V \times K$) has dense leaves, while $\tilde{H}$ (being C^2) has no exceptional minimal sets.

The diffeomorphism $g|_{\tilde{T}}$ of $\tilde{T}$ will, like the DA f , have a source disk if the approximation of g is sufficiently C^1 close to $\phi \times f$, which happens if G is sufficiently C^1 close to $(F')^{\phi \times f}$. (Here C^0 approximations would suffice. But we already needed C^1 approximations to get the existence and normal hyperbolicity of $\tilde{T}$.)

Applying Theorem 4.1 yields a contradiction, since the induced endomorphism $(g|_{\tilde{T}})_*$ of $H^1(\tilde{T})$ is conjugate to f_* on $H^1(T^2)$. This shows that $\hat{F}$ cannot be C^1 approximated by C^2 foliations. This completes the proof of (b) of Theorem 0.1.

§7. Group Actions

The non-approximable foliations constructed in previous sections can all be obtained, on suitable manifolds, as orbit foliations of locally free actions of solvable groups. Consider first the foliation F of M^3 of §2.

The DA-diffeomorphism f of T^2 preserves two foliations, the C^1 foliation F and the analytic foliation K . The latter is the stable manifold foliation of the linear Anosov diffeomorphism f_0 from which f is derived.

Let X be a C^1 vector field tangent to F , such that in the hyperbolic splitting $E_0^u \oplus E_0^s$ the E^u component of X is identically 1. The DA-diffeomorphism f has the following effect on X :

$$(1) \quad df_p X(p) = aX(f(p)) ,$$

where $a = \frac{1}{2}(3 + \sqrt{5})$.

Let G denote the Lie group $\mathbb{R}$, acting on T^2 by the flow of X ; let $s \in \mathbb{R}$ send $x \in T^2$ to $s \cdot x$. This is a free C^1 action. Define an automorphism Φ of G by $\Phi(s) = as$. Then (1) can be written

$$\frac{d}{ds}(f(s \cdot p)) = a \frac{d}{ds}(s \cdot f(p)) .$$

Integrating both sides gives

$$f(s \cdot p) = (as) \cdot f(p) .$$

Introduce the 1-parameter group of automorphisms $\{\Phi_t\}$ of G :

$$\Phi_t(s) = a^t s ; \quad t \in \mathbb{R} , \quad s \in G .$$

Then $\Phi_1 = \Phi$. It follows from the construction at the end of §1 that the suspended foliation F^f of $(T^2)^f = M$ is the orbit foliation of a C^1 locally free action of the nonabelian two-dimensional solvable Lie group H whose Lie algebra is defined by $[Y, X] = aX$.

The foliation $F^f \times V$ of §5 is an orbit foliation provided V is a Lie group. There is then an obvious action of $H \times V$ on $M \times V$.

In §6 we consider the foliation F' on $V \times T^2$ whose leaf at (x, y) is $\{x\} \times F_y$. This is then suspended by $\phi \times f$ to get a foliation $\hat{F}$ of $V \times (T^2)^f$. Evidently $\hat{F}$ is the orbit foliation of H acting on $V \times (T^2)^f$ via the action of H on $(T^2)^f$ described above.

This completes the proof of part (c) of Theorem 0.1.

§8. Other Examples and a Question

In §2 we gave a proof that the foliation F^f on $(T^2)^f$ could not be C^1 approximated by C^2 foliations, which used the structural stability of the DA-diffeomorphism f of T^2 . This (unlike the later proof that F^f is not C^0 approximable) does not rely on the low dimension of F , only on its low codimension. It can therefore be imitated using DA-diffeomorphisms on other manifolds.

Let g_0 be a codimension-one Anosov diffeomorphism of a compact manifold Q , that is, the unstable manifold foliation of Q has codimension one. Let g be a C^1 DA-diffeomorphism of Q derived from g_0 . It is known that g_0 is structurally stable. Using the methods of §3 one finds a C^1 foliation F of Q , of codimension 1, invariant under g . Imitating the first proof in §1, one shows that the C^1 foliation F^g of Q^g cannot be C^1 approximated by C^2 foliations. (One needs Sacksteder's generalization of Denjoy's theorem, [14, Theorem 5].)

Are there C^1 foliations of the n -sphere that cannot be C^1 (or C^0) approximated by C^2 foliations? None of the methods of this paper are useful for such a problem, since the manifolds they produce always have infinite fundamental group. It is amusing to note that the Reeb foliation of S^3 cannot be approximated by analytic foliations:

Haeffliger proved there are no analytic foliations of codimension one on any manifold with finite fundamental group.

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