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Intrinsic harmonic analysis on manifolds with boundary, and Onsager’s conjecture

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Intrinsic harmonic analysis on manifolds with boundary, and Onsager's conjecture

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Manh Khang Huynh

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ABSTRACT OF THE DISSERTATION

Intrinsic harmonic analysis on manifolds with boundary, and Onsager's conjecture

by

Manh Khang Huynh
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2021
Professor Terence Chi-Shen Tao, Chair

We use Hodge theory and functional analysis to develop a clean approach to heat flows and intrinsic harmonic analysis on Riemannian manifolds with boundary. We also introduce heatable currents as the natural analogue to tempered distributions and justify their importance in Hodge theory. As an application, we prove Onsager's conjecture (energy conservation of ideal fluids), where the weak solution lies in the trace-critical Besov space $B^{\frac{1}{3}}_{3,1}$.

In the second half of the thesis, by applying techniques from geometric microlocal analysis to construct the Hodge-Neumann heat kernel, we obtain off-diagonal decay and local Bernstein estimates, and then use them to extend the result to the Besov space $\widehat{B}^{\frac{1}{3}}_{3,V}$, which generalizes both the space $\widehat{B}^{1/3}_{3,c(\mathbb{N})}$ from [IO14] and the space $\underline{B}^{1/3}_{3,\mathrm{VMO}}$ from [Bar+19b; NNT20] — the best known function space where Onsager's conjecture holds on flat backgrounds.

The dissertation of Manh Khang Huynh is approved.

Mario Bonk

Monica Visan

Rowan Brett Killip

Terence Chi-Shen Tao, Committee Chair

University of California, Los Angeles

2021

To my mother,

my advisor,

and my friends.

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K. M. Huynh, "Construction of the Hodge-Neumann Heat Kernel, Local Bernstein Estimates, and Onsager's Conjecture in Fluid Dynamics," 2020-10-29. http://arxiv.org/abs/2010.14762.

K. M. Huynh, "Hodge-Theoretic Analysis on Manifolds with Boundary, Heatable Currents, and Onsager's Conjecture in Fluid Dynamics," 2020-02-03. http://arxiv.org/abs/1907.05360.

K. M. Huynh, “A Note on the Turing Universality of Homogeneous Potential Wells and Geodesible Flows,” 2020-10-25. http://arxiv.org/abs/2010.06210.

CHAPTER 1

Introduction

1.1 History and motivation

It is a well-known fact that the methods of harmonic analysis can be profitably used to handle dispersive PDEs (e.g. non-linear Schrodinger, nonlinear wave) and the fluid equations (e.g. Navier-Stokes, Euler) (cf. [Tao13; Tao09; Tao06]). For fluid dynamics in particular, the central problem of turbulence underlying the quest for global regularity of 3D Navier-Stokes, can be characterized via harmonic analysis as the transfer of energy towards ever higher frequencies.

In harmonic analysis, we often work on simple geometric settings such as $\mathbb{R}^n$, $\mathbb{T}^n$ or locally compact abelian groups, where we have the Fourier transform. But in applications, the geometric settings are rarely so ideal, and there are challenging problems in fluid dynamics which arise from the boundary or curvature in aerodynamic designs, atmospheric models, etc. Consequently, it is profitable to understand how harmonic analysis techniques can work in different geometric settings.

This turned out to be a very rich and diverse field, with various ideas and approaches.[1] For instance, in [Str83], Strichartz introduced harmonic analysis (and the Riesz transform) in the setting of complete Riemannian manifolds. Then in [KR06], Klainerman and Rodnianski defined the L^2-heat flow by the spectral theorem and used it to obtain the Littlewood-Paley projection on compact 2-surfaces. In [IO14], Isett and Oh tackled Onsager's conjecture on

[1] For scalar functions, much more is known due to the very precise estimates of scalar heat kernels. See, for instance, [KVZ14; Duo90; CD03].

Riemannian manifolds without boundary by using Strichartz's heat flow. All these in turn suggest there should be a workable theory of harmonic analysis for vector fields on a manifold with boundary.

1.2 Hodge theory and functional analysis

There are three main characteristics in our setting: vector fields, curvature and boundary.[2] By necessity, our theory of harmonic analysis will feature much more interplay between analysis and geometry than usual.

An oversimplified description of harmonic analysis on $\mathbb{R}^n$ would be "the spectral theory of the Laplacian" [Str89], where the heat kernel is the Gaussian function. It is only natural then for us to look into Hodge theory, which studies the de Rham cohomology of a manifold via the Laplacian. For an analyst, Hodge theory provides the key information regarding the frequency zero (the kernel of the Laplacian), and how it interacts with the boundary. We also can not forget to mention that the Helmholtz decomposition, originally discovered in a hydrodynamic context, turned out to be a part of Hodge theory.

By assuming standard results such as elliptic regularity, and using tools from functional analysis, the development of harmonic analysis in this thesis can be broken down into certain key steps:

1. Defining the frequency zero as the kernel of the Hodge-Neumann Laplacian. By removing the frequency zero, we obtain the inverse Laplacian and the Poincaré inequality for Sobolev spaces.

2. The heat flow generated by the Hodge-Neumann Laplacian is analytic on L^2 (i.e. the time variable t in $e^{t\Delta}$ can be analytically extended to $z \in \mathbb{C}$ where $\arg(z)$ is small). By

[2]There is a lot of literature out there dealing with heat kernels and harmonic analysis, but when filtered by these three characteristics, there was very little one could cite, and for other technical reasons, it was simpler to re-develop everything and modify them to suit the author's own needs. During the process, the author was able to simplify certain steps substantially and discover new results.

the theory of sectorial operators, all we need for this is that the Laplacian is self-adjoint and negative (trivial to show).

3. The L^2-analyticity of the heat flow is extrapolated to L^p-analyticity for $p \in (1, \infty)$, by a simplified version of Kato-Beurling extrapolation. This step is substantially simpler than traditional developments of the heat flow, since it does not involve establishing the resolvent estimate in *Yosida's half-plane criterion*.[3]

4. The L^p-analyticity implies the $W^{1,p}$-analyticity of the heat flow via the Poincaré inequality, and some abstract tools from functional analysis such as Krein-Smulian and the Vitali holomorphic convergence theorem.[4]

By a simple analogy $P_{\leq \frac{1}{\sqrt{t}}} f \approx e^{t\Delta} f$ where $P_{\leq N}$ is the Littlewood-Paley projection, the analyticity of the heat flow on L^p and $W^{1,p}$ implies the all-important Bernstein estimates in harmonic analysis, as can be found in [Tao06, Appendix A]. As a bonus, the Hodge heat flow also commutes with all the important operators in Hodge theory such as the exterior derivative and the codifferential, so it will preserve the incompressibility of the fluid in the Euler equation.

One attractive feature of the approach is that it does not require heat kernel estimates or resolvent estimates, both of which can be highly non-trivial depending on the geometric setting. Besides elliptic regularity (which can be shown in various ways, and is part of standard Hodge theory), the approach is purely functional-analytic.

1.3 Heatable currents and a global approach to Onsager's conjecture

Recall the incompressible Euler equation in fluid dynamics:

[3] Either via "Agmon's trick" [Agm62] as done in [Miy80] or manual estimates as in [BAE16].

[4] The author is not aware of whether this has ever been done for vector fields on manifolds with boundary.

$$\begin{cases} \partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V} &= -\operatorname{grad} \mathfrak{p} \quad \text{in } M \\ \operatorname{div} \mathcal{V} &= 0 \qquad\qquad \text{in } M \quad \text{(incompressibility)} \\ \langle \mathcal{V}, \nu \rangle &= 0 \qquad\qquad \text{on } \partial M \quad \text{(impermeability)} \end{cases} \tag{1.1}$$

$$\text{where} \begin{cases} (M, g) \text{ is an oriented, compact smooth Riemannian manifold with boundary} \\ \nu \text{ is the outwards unit normal vector field on } \partial M. \\ I \subset \mathbb{R} \text{ is an open interval, } \mathcal{V} : I \to \mathfrak{X}M,\ \mathfrak{p} : I \times M \to \mathbb{R}. \end{cases}$$

Roughly speaking, Onsager's conjecture says that the energy $\|\mathcal{V}(t, \cdot)\|_{L^2(M)}$ is a.e. constant in time when $\mathcal{V}$ is a weak solution whose regularity is at least $\frac{1}{3}$. Making that statement precise is part of the challenge. This problem is interesting because the failure of energy conservation comes from the transfer of energy towards higher frequencies (eventually running off to infinity), and learning how regularity can prevent this sort of energy cascade gives us a better understanding of the problem of turbulence.

In the flat and boundaryless case, the "positive direction" (conservation when regularity is at least $\frac{1}{3}$) has been known for a long time [Eyi94; CET94; Che+08]. The "negative direction" (failure of energy conservation when regularity is less than $\frac{1}{3}$) is substantially harder [DS14; DS13], and was finally settled by Isett in his seminal paper [Ise18a] for 3D Euler on the torus (see the survey in [DS19] for more details and references). Since then, for the positive direction, more attention has been directed towards the case with boundary on flat backgrounds [BT18; DN18; BTW19; NN19; Bar+19b; Bar+19a]. The case of manifolds without boundary was first handled via a heat-flow approach in [IO14].

Consequently, this thesis is an effort to extend the positive side of Onsager's conjecture to manifolds with boundary, hopefully recovering the best results from both the flat case and the boundaryless case.

A conceptual problem now arises: how can we apply the heat flow to the convective term $\nabla_{\mathcal{V}} \mathcal{V}$, which is a distribution? As the heat flow does not preserve compact supports in the interior of M, it is not defined on distributions.

If we recall harmonic analysis on $\mathbb{R}^n$, the same problem appears when we try to apply the Fourier transform to distributions. It is impossible, and we have to restrict to a subclass of distributions called tempered distributions, which then becomes the general setting for harmonic analysis, in which we can define different function spaces such as Sobolev spaces $W^{s,p}$ (via the Bessel potential $\langle\nabla\rangle^s$) and Besov spaces $B^s_{p,q}$, which are essentially the real interpolation spaces between Sobolev spaces (thus allowing us to capture more subtle information regarding regularity and integrability).

This inspired the author to define the notion of (Neumann) **heatable currents** in Part I:

- Let $\mathscr{D}_N\Omega^k := \{\omega \in \Omega^k : \mathbf{n}\Delta^m\omega = 0, \mathbf{n}d\Delta^m\omega = 0\ \forall m \in \mathbb{N}_0\}$ be the space of **heated** k**-forms** with the Frechet C^∞ topology. Here $\mathbf{n}$ denotes the normal part; Δ is the Hodge Laplacian, and d is the exterior derivative (like the gradient). In simpler words, all Neumann conditions are satisfied.
- Let $\mathscr{D}'_N\Omega^k := \left(\mathscr{D}_N\Omega^k\right)^*$ be the space of **heatable** k**-currents** with the weak* topology.

We could then show that this is the correct generalization of Schwartz functions and tempered distributions.[5] In particular, if $w \in \mathscr{D}'_N\Omega^k$ then for any $t > 0 : e^{t\Delta}w \in \mathscr{D}_N\Omega^k$. It can also be showed that the associated Sobolev spaces, defined by the Hodge-Neumann Laplacian within the space of heatable currents, have the same topology as the classical Sobolev spaces (defined by partitions of unity and local coordinates).

With this theory of harmonic analysis based purely on functional analysis, and the definition of heatable currents, the author was then able to prove Onsager's conjecture in the Besov space $B^{1/3}_{3,1}$—the largest Besov space where the trace theorem applies. This is the main goal of Part I. Here is the full technical statement:

Theorem 1 (Onsager's conjecture, 1st version)**.** *Let M be a compact, oriented Riemannian manifold with no or smooth boundary. Let $\mathbb{P}$ be the Leray projection, enforcing incompress-*

[5] The author is not aware of whether vector-valued tempered distributions on manifolds with boundary have ever been defined or used for PDEs.

ibility and impermeability (to be defined later). Let $\mathfrak{X} = \mathfrak{X}M$ be the space of vector fields on M.

Let $\mathcal{V} \in L_t^3 \mathbb{P} B_{3,1}^{\frac{1}{3}} \mathfrak{X}$ be such that $\forall \mathcal{X} \in C_c^\infty(I, \mathbb{P}\mathfrak{X}) : \iint_{I \times M} \langle \mathcal{V}, \partial_t \mathcal{X} \rangle + \langle \mathcal{V} \otimes \mathcal{V}, \nabla \mathcal{X} \rangle = 0$ (Hodge-Leray weak solution).

Then we can show

$$\int_I \eta'(t) \langle\langle \mathcal{V}(t), \mathcal{V}(t) \rangle\rangle \, \mathrm{d}t = 0 \; \forall \eta \in C_c^\infty(I)$$

Consequently, $\langle\langle \mathcal{V}(t), \mathcal{V}(t) \rangle\rangle$ is constant for a.e. $t \in I$.

A very curious fact is that no "strip decay" condition involving the pressure $\mathfrak{p}$ (which is present in different forms for the results on flat spaces) seems to be necessary. This is because our approach is global in nature, without any spatial cut-offs. The trade-off for this improvement is that the Besov space $B_{3,1}^{1/3}$ is a bit smaller than the Besov spaces featured in the best results on flat spaces (typically subspaces of $B_{3,\infty,\mathrm{loc}}^{1/3}$). Still, it is a unique result that does not require assumptions on $\mathfrak{p}$, and more details can be found in [Huy19].

1.4 A local approach to Onsager's conjecture

Following [Huy19], the natural question to ask is whether our theory of harmonic analysis can also facilitate a local approach to Onsager's conjecture, using spatial cut-offs and assuming a "strip decay" condition involving the pressure $\mathfrak{p}$. Ideally, we want to recover the best results on flat spaces, with $\underline{B}_{3,\mathrm{VMO}}^{1/3}-$spatial regularity, as in [Bar+19b; NNT20].[6] We also want to recover the space $\widehat{B}_{3,c(\mathbb{N})}^{1/3}$ from [IO14] (the best result on manifolds without boundary).[7] Is there a possible generalization for both, on manifolds with boundary? The answer is yes [Huy20], and detailed in Part II.

[6] $\underline{B}_{3,\mathrm{VMO}}^{1/3}$ is a VMO-type subspace of $L^3 \cap B_{3,\infty,\mathrm{loc}}^{1/3}$ and can be defined by local convolutions.

[7] $\widehat{B}_{3,c(\mathbb{N})}^{1/3}$ is the closure of C_c^∞ in the $B_{3,\infty}^{1/3}$ topology.

In essence, the absolute Neumann heat flow, created via functional analysis, is a replacement for the usual convolution on flat spaces, with special properties like commutativity with divergence. However, obtaining a pointwise profile of heat kernels for differential forms (let alone their derivatives) is a difficult problem, so it was hard to reconcile the global heat-flow approach on manifolds with local-type convolution arguments as on flat backgrounds. Even the definition of $\underline{B}^{\frac{1}{3}}_{3,\mathrm{VMO}}$ itself is local, and it was not immediately obvious that the heat-flow approach could handle such function spaces.

Construction of the Hodge-Neumann heat kernel

The solution to this is a manual construction of the Hodge-Neumann heat kernel, using techniques from microlocal analysis and index theory (in particular, Melrose's calculus on manifolds with corners [Mel18; Mel92]). The theory mimics the development of pseudodifferential operators, in creating a filtered algebra that quantifies how "nonsingular" an operator is as we approach the edges. In particular, much like the pseudolocality of ΨDOs, the construction yields a precise description near the diagonal, as well as rapid decay away from the diagonal. This enables the use of the heat flow as local convolution, and we obtain **local Bernstein estimates**, which allow us to handle VMO-type function spaces.

The construction is arguably the most technical step of the thesis, and is adapted from the work in [MV13].[8] Initial attempts to stay within the space of smooth kernels would fail due to the boundary. At the root of the problem was Brüning and Seeley's Singular Asymptotics Lemma [GG00; BS85], which warns that logarithmic singularities can develop at the boundary, destroying the smoothness. One needs to step into the space of singular kernels, and this is where Melrose's calculus (dealing with singular functions on manifolds with corners) comes into the picture. We can still construct a singular heat kernel, which can then become smooth by functional-analytic arguments.

[8] The decision was made after discussions involving Daniel Grieser, András Vasy and Rafe Mazzeo. The original plan was to follow the note [Gri04] which is more elementary, but we have decided to clean up the note, modify some steps and publish it at a later date.

1.4.0.1 Technical statement of the main result

For $r > 0$, we define $M_{>r} := \{x \in M : \mathrm{dist}(x, \partial M) > r\}$. For $p \in (1, \infty)$, we say $\mathcal{X} \in L_t^p \widehat{B}_{p,V}^{1/p} \mathfrak{X}(M)$ if $\mathcal{X} \in L_t^p L^p \mathfrak{X}(M)$ and $\forall r > 0 : \left(\frac{1}{\sqrt{s}}\right)^{\frac{1}{p}} \left\| \mathcal{X} - e^{s\Delta}\mathcal{X} \right\|_{L_t^p L^p(M_{>r})} \xrightarrow{s \to 0} 0$.

$L_t^3 \widehat{B}_{3,V}^{1/3}$ contains the space $L_t^3 \widehat{B}_{3,c(\mathbb{N})}^{1/3}$ from [IO14] (with equality when there is no boundary). While on flat backgrounds, $L_t^3 \widehat{B}_{3,V}^{1/3}$ coincides with $L_t^3 \underline{B}_{3,\mathrm{VMO}}^{1/3}$ from [Bar+19b; NNT20; Wie20].

The replacement for the trace theorem is the following "strip decay" hypothesis near the boundary:

$$\left\| \left(\frac{|\mathcal{V}|^2}{2} + \mathfrak{p} \right) \langle \mathcal{V}, \widetilde{\nu} \rangle \right\|_{L_t^1 L^1\left(M_{[\frac{r}{2}, r]}, \mathrm{avg}\right)} \xrightarrow{r \downarrow 0} 0,$$

$$\text{where} \begin{cases} \widetilde{\nu}\text{: the extension of } \nu \text{ near the boundary.} \\ M_{[r/2,r]} := \{x \in M : \mathrm{dist}(x, \partial M) \in [r/2, r]\}. \\ \text{avg: the measure is normalized to become a probability measure.} \end{cases}$$

Theorem 2. *Let M be as in (1.1). Then $\|\mathcal{V}(t, \cdot)\|_{L^2(M)}$ is a.e. constant in time if $(\mathcal{V}, \mathfrak{p})$ is a weak solution with $\mathcal{V} \in L_t^3 \mathbb{P} L^3 \mathfrak{X} \cap L_t^3 \widehat{B}_{3,V}^{\frac{1}{3}} \mathfrak{X}$ and the "strip decay" condition holds true.*

1.5 General outline of the thesis

The global approach to Onsager's conjecture, and the fundamental tools of intrinsic harmonic analysis on manifolds with boundary, are contained in Part I, which is functionally identical to [Huy19] (with necessary modifications for a thesis).

The local approach to Onsager's conjecture (offering the best result in terms of regularity), and the construction of the Neumann heat kernel by geometric microlocal analysis, are contained in Part II, which is functionally identical to [Huy20] (with necessary modifications).

Part I

Hodge-theoretic analysis on manifolds with boundary, heatable currents, and a global approach to Onsager's conjecture in fluid dynamics

CHAPTER 2

Introduction

2.1 Onsager's conjecture

Recall the incompressible Euler equation in fluid dynamics:

$$\begin{cases} \partial_t V + \operatorname{div}(V \otimes V) & = -\operatorname{grad} p \quad \text{in } M \\ \operatorname{div} V & = 0 \quad \text{in } M \\ \langle V, \nu \rangle & = 0 \quad \text{on } \partial M \end{cases} \tag{2.1}$$

$$\text{where} \begin{cases} (M, g) \text{ is an oriented, compact smooth Riemannian manifold with smooth boundary} \\ \nu \text{ is the outwards unit normal vector field on } \partial M. \\ I \subset \mathbb{R} \text{ is an open interval, } V : I \to \mathfrak{X}M,\ p : I \times M \to \mathbb{R}. \end{cases}$$

Observe that the **Neumann condition** $\langle V, \nu \rangle = 0$ means $V \in \mathfrak{X}_N$, where $\mathfrak{X}_N$ is the set of vector fields on M which are tangent to the boundary. Note that when V is not smooth, we need the trace theorem to define the condition (see Section 6.2).

Roughly speaking, Onsager's conjecture says that the energy $\|V(t, \cdot)\|_{L^2}$ is a.e. constant in time when V is a weak solution whose regularity is at least $\frac{1}{3}$. Making that statement precise is part of the challenge.

In the boundaryless case, the "positive direction" (conservation when regularity is at least $\frac{1}{3}$) has been known for a long time [Eyi94; CET94; Che+08]. The "negative direction" (failure of energy conservation when regularity is less than $\frac{1}{3}$) is substantially harder [DS13; DS14], and was finally settled by Isett in his seminal paper [Ise18b] (see the survey in [DS19] for more details and references).

Since then more attention has been directed towards the case with boundary, and its effects in the generation of turbulence. In [BT18], the "positive direction" was proven in the case M is a bounded domain in $\mathbb{R}^n$ and $V \in L_t^3 C^{0,\alpha}\mathfrak{X}_N$ ($\alpha > \frac{1}{3}$). The result was then improved in various ways [DN18; Bar+19a; BTW19]. In [NN19], the conjecture was proven for V in $L_t^3 B_{3,\infty}^{\alpha}\mathfrak{X}$ ($\alpha > \frac{1}{3}$) along with some "strip decay" conditions for V and p near the boundary (more details in Section 4.2). Most recently, the conjecture was proven as part of a more general conservation of entropy law in [Bar+19b], where M is a domain in $\mathbb{R}^n$, $V \in L_t^3 \underline{B}_{3,\mathrm{VMO}}^{1/3}\mathfrak{X}$ (where $\underline{B}_{3,\mathrm{VMO}}^{1/3}\mathfrak{X}$ is a VMO-type subspace of $B_{3,\infty}^{1/3}\mathfrak{X}$), along with a "strip decay" condition involving both V and p near the boundary (see Section 4.2).

Much less is known about the conjecture on general Riemannian manifolds. The key arguments on flat spaces rely on the nice properties of convolution, such as $\operatorname{div}(T * \phi_\varepsilon) = \operatorname{div}(T) * \phi_\varepsilon$ where T is a tensor field and $\phi_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \delta_0$ is a mollifier, or that mollification is essentially local. This "local approach" by convolution does not generalize well to Riemannian manifolds. In [IO14] – the main inspiration for this thesis – Isett and Oh used the heat flow to prove the conjecture on compact Riemannian manifolds without boundary, for $V \in L_t^3 B_{3,c(\mathbb{N})}^{\frac{1}{3}}\mathfrak{X}$ (where $B_{3,c(\mathbb{N})}^{\frac{1}{3}}\mathfrak{X}$ is the $B_{3,\infty}^{\frac{1}{3}}$-closure of compactly supported smooth vector fields). The situation becomes more complicated when the boundary is involved. Most notably, the covariant derivative behaves badly on the boundary (e.g. the second fundamental form), and it is difficult to avoid boundary terms that come from integration by parts. Even applying the heat flow to a distribution might no longer be well-defined. This requires a finer understanding of analysis involving the boundary, as well as the properties of the heat flow.

In this part of the thesis, we will see how we can resolve these issues, and that the conjecture still holds true with the boundary:

Fact. *Assuming M as in Equation (2.1), conservation of energy is true when (V, p) is a weak solution with $V \in L_t^3 B_{3,1}^{\frac{1}{3}}\mathfrak{X}_N$.*

It is not a coincidence that this is also the lowest regularity where the trace theorem holds. We also note a very curious fact that no "strip decay" condition involving p (which is present in different forms for the results on flat spaces) seems to be necessary, and we only need $p \in L^1_{\mathrm{loc}}(I \times M)$ (see Section 4.3 for details). One way to explain this minor improvement is that the "strip decay" condition involving V naturally originates from the trace theorem (see Section 4.3), and is therefore included in the condition $V \in L^3_t B^{\frac{1}{3}}_{3,1} \mathfrak{X}_N$, while the presence of p is more of a technical artifact arising from localization (see [Bar+19b, Section 4]), which typically does not respect the Leray projection. By using the trace theorem and the heat flow, our approach becomes global in nature, and thus avoids the artifact. Another approach is to formulate the conjecture in terms of Leray weak solutions like in [RRS18], without mentioning p at all, and we justify how this is possible in Section 4.3.

A more local approach, where we assume $V \in L^3_t B^{\frac{1}{3}}_{3,c(\mathbb{N})} \mathfrak{X}$ as in [IO14], and the "strip decay" condition as in [Bar+19b, Equation 4.9], is the topic of Part II. Nevertheless, $B^{\frac{1}{3}}_{3,1} \mathfrak{X}_N$ is an interesting space with its own unique results, which keep the exposition simple and allow the boundary condition to be natural.

2.2 Modularity

This part of the thesis is intended to be modular: the chapter dealing with Onsager's conjecture (Chapter 4) is relatively short, while the rest is to detail the tools for harmonic analysis on manifolds we will need (and more). As we will summarize the tools in Chapter 4, they can be read independently.

2.3 Motivation behind the approach

Riemannian manifolds (and their semi-Riemannian counterparts) are among the most important natural settings for modern geometric PDEs and physics, where the objects for analysis are often vector bundles and differential forms. The two fundamental tools for a harmonic analyst – **mollification** and **Littlewood-Paley projection** via the Fourier transform – do not straightforwardly carry over to this setting, especially when the boundary is involved. Even in the case of scalar functions on bounded domains in $\mathbb{R}^n$, mollification arguments often need to stay away from the boundary, which can present a problem when the trace is nonzero. Consider, however, the idea of a special kind of Littlewood-Paley projection which preserves the boundary conditions and commutes with important operators such as divergence and the **Leray projection**, or using the principles of harmonic analysis without translation invariance. It is one among a vast constellation of ideas which have steadily become more popular over the years, with various approaches proposed (and we can not hope to fully recount here).

For our discussion, the starting point of interest is perhaps [Str83], in which Strichartz introduced to analysts what had long been known to geometers, the rich setting of complete Riemannian manifolds, where harmonic analysis (and the **Riesz transform** in particular) can be done via the Laplacian and the **heat semigroup** $e^{t\Delta}$, constructed by **dissipative operators** and Yau's lemma. Then in [KR06], Klainerman and Rodnianski defined the L^2-heat flow by the **spectral theorem** and used it to get the Littlewood-Paley projection on compact 2-surfaces. In [IO14], Isett and Oh successfully tackled Onsager's conjecture on Riemannian manifolds without boundary by using Strichartz's heat flow. These results hint at the central importance of the heat flow for analysis on manifolds. But it is not enough to settle the case with boundary, especially when derivatives are involved. Some pieces of the puzzle are still missing.

To paraphrase James Arthur (in his introduction to the trace formula and the Langlands program), there is an intimate link between geometric objects and "spectral" phenomena, much like how the shape of a drum affects its sounds. For a Riemannian manifold, that link is better known as the Laplacian – the generator of the heat flow – and **Hodge theory** is the study of how the Laplacian governs the cohomology of a Riemannian manifold. An oversimplified description of Fourier analysis on $\mathbb{R}^n$ would be "the spectral theory of the Laplacian" [Str89], where the heat kernel is the Gaussian function, invariant under the Fourier transform and a possible choice of mollifier. Additionally, the **Helmholtz decomposition**, originally discovered in a hydrodynamic context, turned out to be a part of Hodge theory. It should therefore be no surprise that Hodge theory is the natural framework in which we formulate harmonic analysis on manifolds, heat flows and Onsager's conjecture. Wherever there is the Laplacian, there is harmonic analysis. Historically, Milgram managed to establish a subset of Hodge theory by heat flow methods [MR51]. Here, however, we will establish Hodge theory by standard elliptic estimates, from which we develop analysis on manifolds and construct the heat flow. Most notably, Hodge theory greatly simplifies some crucial approximation steps involving the boundary (Corollary 72), and helps predict some key results Onsager's conjecture would require (Theorem 17, Section 8.4, Section 9.3). That such leaps of faith turn out to be true only further underscore how well-made the conjecture is in its anticipation of undiscovered mathematics.

For those familiar with the smoothing properties of Littlewood-Paley projection as well as **Bernstein inequalities** [Tao06, Appendix A], the rough picture is that $e^{t\Delta} \approx P_{\leq \frac{1}{\sqrt{t}}}$. While the introduction of curvature necessitates the change of constants in estimates, and the boundary requires its own considerations, it is remarkable how far we can go with this analogy. Regarding the properties we will need for Onsager's conjecture, there is a satisfying explanation: the theory of **sectorial operators** in functional analysis. This, together with Hodge theory, the theory of **Besov spaces** and **interpolation theory**, allows us to build a basic foundation for global analysis on Riemannian manifolds in general, which will be more than enough to handle Onsager's conjecture.

Hodge theory and sectorial operators, in their various forms, have been used in fluid dynamics for a long time by Fujita, Kato, Giga, Miyakawa *et al.* (cf. [FK64; Miy80; Gig81; GM85; BAE16] and their references). Although we will not use them for this thesis, we also ought to mention the results regarding bisectorial operators, H^∞ functional calculus, and Hodge theory on rough domains developed by Alan McIntosh, Marius Mitrea, Sylvie

Monniaux *et al.* (cf. [McI86; DM96; FMM98; AM04; MM08; MM09a; MM09b; GMM10; She12; MM18] and their references), which generalize many Hodge-theoretic results in this thesis. Alternative formalizations of Littlewood-Paley theory also exist (cf. [HMY08; KP14; FFP16; KW16; BBD18; Tan18] and their references). Here, we are mainly focused on the analogy between the heat flow and the Littlewood-Paley projection on L^p spaces of differential forms (over manifolds with boundary), as well as the interplay with Hodge theory.

Lastly, we also introduce **heatable currents** – the largest space on which the heat flow can be profitably defined – as the analogue to tempered distributions on manifolds (Section 8.4). In doing so, we will realize that the energy-conserving weak solution in Onsager's conjecture solves the Euler equation in the sense of heatable currents. This is an elegant insight that helps show how interconnected these subjects are. For the sake of accessibility, besides providing a gentle introduction to the theory with copious references, this thesis also hopes to convince the reader of the naturality behind the formalism.

2.4 Blackboxes

Since we draw upon many areas, the thesis is intended to be as self-contained as possible, but we will assume familiarity with basic elements of functional analysis, harmonic analysis and complex analysis. Some familiarity with differential and Riemannian geometry is certainly needed (cf. [Lee09; Cha06]), as well as **Penrose notation** (cf. [Wal84, Section 2.4]). In addition, a number of blackbox theorems will be borrowed from the following sources:

1. For interpolation theory: *Interpolation Spaces* [BL76] and "Abstract Stein Interpolation" [Voi92]

2. For harmonic analysis and elements of functional analysis:

   - *Singular Integrals and Differentiability Properties of Functions. (PMS-30)* [Ste71]
   - *Partial Differential Equations I* [Tay11a]

- *Recent Developments in the Navier-Stokes Problem (Chapman & Hall/CRC Research Notes in Mathematics Series)* [Lem02b]

3. For Besov spaces: *Theory of Function Spaces*; *Theory of Function Spaces II* [Tri10; Tri92]

4. For Hodge theory: *Hodge Decomposition—A Method for Solving Boundary Value Problems* [Sch95]

5. For semigroups and sectorial operators: *One-Parameter Semigroups for Linear Evolution Equations* [Eng00] and *Vector-Valued Laplace Transforms and Cauchy Problems: Second Edition (Monographs in Mathematics)* [Are+11]

The first three categories should be familiar with harmonic analysts.

2.5 For the specialists

Some noteworthy characteristics of our approach:

- An alternative development of the (absolute Neumann) heat flow. In particular, the extrapolation of analyticity to L^p spaces does not involve establishing the resolvent estimate in *Yosida's half-plane criterion* (Theorem 41), either via "Agmon's trick" [Agm62] as done in [Miy80] or manual estimates as in [BAE16]. Instead, by abstract Stein interpolation, we only need the local boundedness of the heat flow on L^p, which can follow cleanly from Gronwall and integration by parts (Theorem 73). In short, functional analysis does the heavy lifting. We also managed to attain $W^{1,p}$-analyticity assuming the Neumann condition (Section 8.3), and $B^{\frac{1}{p}}_{p,1}$-analyticity via the Leray projection (Section 9.3).

- We do not focus on the **Stokes operator** in this thesis, but our results (Section 8.3, Section 9.3) do contain the case of the Stokes operator corresponding to the "Navier-type" / "free" boundary condition, as discussed in [Miy80; Gig82; MM09a; MM09b;

BAE16] and others. This should not be confused with the Stokes operator corresponding to the “no-slip” boundary condition, as discussed in [FK64; GM85; MM08] and others. See [HS18] for more references.

- For simplicity, we stay within the smooth and compact setting, which, as Hilbert would say, is that special case containing all the germs of generality. An effort has also been made to keep the material concrete (as opposed to, for instance, using Hilbert complexes).

- Heatable currents are introduced as the analogue to tempered distributions, and we show how they naturally appear in the characterization of the adjoints of d and δ (Section 8.4).

- A refinement of a special case of the fractional Leibniz rule, with the supports of functions taken into account, is given in Theorem 56.

- For the proof of Onsager's conjecture, there are some subtle, but substantial differences with [IO14]:

  – In [IO14], Besov spaces are *defined* by the heat flow, and compatibility with the usual scalar Besov spaces is proven when M is $\mathbb{R}^n$ or $\mathbb{T}^n$. Here we will use the standard scalar Besov spaces as defined by Triebel in [Tri10; Tri92], and prove the appropriate estimates for the heat flow by interpolation.

  – The heat flow used by Isett & Oh (constructed by Strichartz using dissipative operators) is generated by the **Hodge Laplacian**, which is self-adjoint in the no-boundary case. In the case with boundary, there are four different self-adjoint versions for the Hodge Laplacian (see Theorem 63), and we choose the **absolute Neumann** version. There are also heat flows generated by the **connection Laplacian**, but we do not use them in this thesis since the connection Laplacian does not commute with the **exterior derivative** and the Leray projection etc. The theory of dissipative operators is also not sufficient to establish L^p-analyticity and $W^{1,p}$-analyticity for all $p \in (1, \infty)$, so we instead use the theory of sectorial operators, which is made for this purpose.

  – The commutator we will use is a bit different from that in [IO14]. This will help us eliminate some boundary terms. We will also avoid the explicit formula and computations in [IO14, Lemma 4.4], as they also lead to various boundary terms. Generally speaking, the covariant derivative behaves badly on the boundary.

- A calculation of the pressure by negative-order Hodge-Sobolev spaces (Section 9.2).

- More results will be proven for analysis on manifolds than needed for Onsager's conjecture, as they are of independent interest. For the sake of accessibility, we will also review most of the relevant background material, with the assumption that the reader is a harmonic analyst who knows some differential geometry.

It is hard to overstate our indebtedness to all the mathematicians whose work our theory will build upon, from harmonic analysis to Hodge theory and sectorial operators, and yet hopefully each will be able to find within this thesis something new and interesting.

CHAPTER 3

Common notation

It might not be an exaggeration to say the main difficulty in reading a manuscript dealing with Hodge theory is understanding the notation, and an effort has been made to keep our notation as standard and self-explanatory as possible.

Some common notation we use:

- $A \lesssim_{x,\neg y} B$ means $A \leq CB$ where $C > 0$ depends on x and not y. Similarly, $A \sim_{x,\neg y} B$ means $A \lesssim_{x,\neg y} B$ and $B \lesssim_{x,\neg y} A$. When the dependencies are obvious by context, we do not need to make them explicit.
- $\mathbb{N}_0, \mathbb{N}_1$: the set of natural numbers, starting with 0 and 1 respectively.
- DCT: dominated convergence theorem, FTC: fundamental theorem of calculus, PTAS: passing to a subsequence, WLOG: without loss of generality.
- TVS: topological vector space, NVS: normed vector space, SOT: strong operator topology.
- For TVS X, $Y \leq X$ means Y is a subspace of X.
- $\mathcal{L}(X, Y)$: the space of continuous linear maps from TVS X to Y. Also $\mathcal{L}(X) = \mathcal{L}(X, X)$.
- $C^0(S \to Y)$: the space of bounded, continuous functions from metric space S to normed vector space Y. Not to be confused with $C^0_{\text{loc}}(S \to Y)$, which is the space of locally bounded, continuous functions.

- $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$ and $\|x\|^*_{D(A)} = \|Ax\|_X$ where A is an unbounded operator on (real/complex) Banach space X and $x \in D(A)$. Note that $\|\cdot\|^*_{D(A)}$ is not always a norm. Also define $D(A^\infty) = \cap_{k\in\mathbb{N}_1} D(A^k)$.

- For $\delta \in (0, \pi]$, define the open sector $\Sigma^+_\delta = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \delta\}$, $\Sigma^-_\delta = -\Sigma^+_\delta$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also define $\Sigma^+_0 = (0, \infty)$ and $\Sigma^-_0 = -\Sigma^+_0$.

- $B(x, r)$: the open ball of radius r centered at x in a metric space.

- $\mathcal{S}(\mathbb{R}^n)$: the space of Schwartz functions on $\mathbb{R}^n$, $\mathcal{S}(\overline{\Omega})$: restrictions of Schwartz functions to the domain $\Omega \subset \mathbb{R}^n$.

There is also a list of other symbols we will use at the end of Part I.

CHAPTER 4

Onsager's conjecture

4.1 Summary of preliminaries

At the cost of some slight duplication of exposition, we will quickly summarize the key tools we need for the proof, and leave the development of such tools for the rest of the Part I. Alternatively, the reader can read the theory first and come back to this section later.

Definition 3. For the rest of Part I, unless otherwise stated, let M be a compact, smooth, Riemannian n-dimensional manifold, with no or smooth boundary. We also let $I \subset \mathbb{R}$ be an open time interval. We write $M_{<r} = \{x \in M : \operatorname{dist}(x, \partial M) < r\}$ for $r > 0$ small. Similarly define $M_{\geq r}, M_{<r}, M_{[r_1,r_2]}$ etc. Let $\mathring{M}$ denote the interior of M.

By musical isomorphism, we can consider $\mathfrak{X}M$ (the space of **smooth vector fields**) mostly the same as $\Omega^1(M)$ (the space of **smooth 1-forms**), *mutatis mutandis*. We note that $\mathfrak{X}M$, $\mathfrak{X}(\partial M)$ and $\mathfrak{X}M\big|_{\partial M}$ are different. Unless otherwise stated, let the implicit domain be M, so $\mathfrak{X}$ stands for $\mathfrak{X}M$, and similarly Ω^k for $\Omega^k M$. For $X \in \mathfrak{X}$, we write $X^\flat$ as its dual 1-form. For $\omega \in \Omega^1$, we write $\omega^\sharp$ as its dual vector field.

Let $\mathfrak{X}_{00}(M)$ denote the set of smooth vector fields of compact support in $\mathring{M}$. Define $\Omega^k_{00}(M)$ similarly (smooth differential forms with compact support in $\mathring{M}$).

Let ν denote the outwards unit normal vector field on ∂M. ν can be extended via geodesics to a smooth vector field $\widetilde{\nu}$ which is of unit length near the boundary (and cut off at some point away from the boundary).

For $X \in \mathfrak{X}M$, define $\mathbf{n}X = \langle X, \nu\rangle\, \nu \in \mathfrak{X}M|_{\partial M}$ (the **normal part**) and $\mathbf{t}X = X|_{\partial M} - \mathbf{n}X$

(the **tangential part**). We note that $\mathbf{t}X$ and $\mathbf{n}X$ only depend on $X\big|_{\partial M}$, so $\mathbf{t}$ and $\mathbf{n}$ can be defined on $\mathfrak{X}M\big|_{\partial M}$, and $\mathbf{t}\left(\mathfrak{X}M|_{\partial M}\right) \xrightarrow{\sim} \mathfrak{X}(\partial M)$.

For $\omega \in \Omega^k(M)$, define $\mathbf{t}\omega$ and $\mathbf{n}\omega$ by

$$\mathbf{t}\omega(X_1, ..., X_k) := \omega(\mathbf{t}X_1, ..., \mathbf{t}X_k) \ \ \forall X_j \in \mathfrak{X}M, j = 1, ..., k$$

and $\mathbf{n}\omega = \omega|_{\partial M} - \mathbf{t}\omega$. Note that $(\mathbf{n}X)^\flat = \mathbf{n}X^\flat \ \forall X \in \mathfrak{X}$.

Let ∇ denote the **Levi-Civita connection**, d the **exterior derivative**, δ the **codifferential**, and $\Delta = -(d\delta + \delta d)$ the **Hodge-Laplacian**, which is defined on vector fields by the musical isomorphism.

Familiar scalar function spaces such as $L^p, W^{m,p}$ (**Lebesgue-Sobolev spaces**), $B^s_{p,q}$ (**Besov spaces**), $C^{0,\alpha}$ (**Holder spaces**) (see Chapter 6 for precise definitions) can be defined on M by partitions of unity and given a unique topology (Section 6.2, Subsection 7.1.2). Similarly, we define such function spaces for **tensor fields** and **differential forms** on M by partitions of unity and local coordinates (see subsection 7.1). For instance, we can define $L^2\mathfrak{X}$ or $B^{\frac{1}{3}}_{3,1}\mathfrak{X}$.

Fact 4. *$\forall \alpha \in \left(\frac{1}{3}, 1\right), \forall p \in (1, \infty)$: $W^{1,p}\mathfrak{X} \hookrightarrow B^{\frac{1}{p}}_{p,1}\mathfrak{X} \hookrightarrow L^p\mathfrak{X}$ and $C^{0,\alpha}\mathfrak{X} = B^\alpha_{\infty,\infty}\mathfrak{X} \hookrightarrow B^\alpha_{3,\infty}\mathfrak{X} \hookrightarrow B^{\frac{1}{3}}_{3,1}\mathfrak{X}$ (cf. Section 6.2, Section 6.4)*

Definition 5. We write $\langle \cdot, \cdot \rangle$ to denote the **Riemannian fiber metric** for tensor fields on M. We also define the dot product

$$\langle\langle \sigma, \theta \rangle\rangle = \int_M \langle \sigma, \theta \rangle \, \mathrm{vol}$$

where σ and θ are tensor fields of the same type, while vol is the **Riemannian volume form**. When there is no possible confusion, we will omit writing vol.

We define $\mathfrak{X}_N = \{X \in \mathfrak{X} : \ \mathbf{n}X = 0 \ \}$ (**Neumann condition**). Similarly, we can define Ω^k_N. In order to define the Neumann condition for less regular vector fields (and differential forms), we need to use the **trace theorem**.

Fact 6. *(Section 6.2, Subsection 7.1.2) Let* $p \in [1,\infty)$*. Then*

- $B^{\frac{1}{p}}_{p,1}(M) \twoheadrightarrow L^p(\partial M)$ *and* $B^{\frac{1}{p}}_{p,1}\mathfrak{X}M \twoheadrightarrow L^p\mathfrak{X}M\big|_{\partial M}$ *are continuous surjections.*
- $\forall m \in \mathbb{N}_1 : B^{m+\frac{1}{p}}_{p,1}\mathfrak{X}M \twoheadrightarrow B^m_{p,1}\mathfrak{X}M\big|_{\partial M} \hookrightarrow W^{m,p}\mathfrak{X}M\big|_{\partial M}$ *is continuous.*

Also closely related is the **coarea formula**:

Fact 7. *(Theorem 55) Let* $p \in [1,\infty)$*,* $r > 0$ *be small and* f *be in* $B^{\frac{1}{p}}_{p,1}(M)$*:*

1. $\left([0,r) \to \mathbb{R}, \rho \mapsto \|f\|_{L^p(\partial M_{>\rho})}\right)$ *is continuous and bounded by* $C\|f\|_{B^{\frac{1}{p}}_{p,1}}$ *for some* $C > 0$*.*
2. $|M_{<r}| \sim_{M,\neg r} |\partial M|\, r$ *and* $\|f\|_{L^p(M_{\leq r})} \sim_{\neg r} \left\|\|f\|_{L^p(\partial M_{>\rho})}\right\|_{L^p_\rho((0,r))}$*.*
3. $\|f\|_{L^p(M_{\leq r},\mathrm{avg})} \lesssim_{\neg r} \|f\|_{B^{\frac{1}{p}}_{p,1}(M)}$ *and* $\|f\|_{L^p(M_{\leq r},\mathrm{avg})} \xrightarrow{r\downarrow 0} \|f\|_{L^p(\partial M,\mathrm{avg})}$*, where* avg *means normalizing the measure to make it a probability measure.*
4. *Let* $\mathfrak{f} \in L^p(I \to B^{\frac{1}{p}}_{p,1}(M))$*, then* $\|\mathfrak{f}\|_{L^p_t B^{\frac{1}{p}}_{p,1}(M)} \gtrsim_{\neg r} \|\mathfrak{f}\|_{L^p_t L^p(M_{\leq r},\mathrm{avg})} \xrightarrow{r\downarrow 0} \|\mathfrak{f}\|_{L^p_t L^p(\partial M,\mathrm{avg})}$*.*

Analogous results hold if $f \in B^{\frac{1}{p}}_{p,1}\mathfrak{X}$*. (Subsection 7.1.2)*

Therefore, we can define spaces such as $B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N = \{X \in B^{\frac{1}{3}}_{3,1}\mathfrak{X} : \mathbf{n}X = 0\}$ and $W^{1,3}\mathfrak{X}_N$. However, something like $L^2\mathfrak{X}_N$ would not make sense since the trace map does not continuously extend to $L^2\mathfrak{X}$.

Definition 8. We define $\mathbb{P}$ as the **Leray projection** (constructed in Theorem 70), which projects $\mathfrak{X}$ onto $\mathrm{Ker}\left(\mathrm{div}\big|_{\mathfrak{X}_N}\right)$. Note that the Neumann condition is enforced by $\mathbb{P}$.

Fact 9. $\forall m \in \mathbb{N}_0, \forall p \in (1,\infty)$*,* $\mathbb{P}$ *is continuous on* $W^{m,p}\mathfrak{X}$ *and* $\mathbb{P}(W^{m,p}\mathfrak{X}) = W^{m,p}\text{-cl}\left(\mathrm{Ker}\left(\mathrm{div}\big|_{\mathfrak{X}_N}\right)\right)$ *(closure in the* $W^{m,p}$*-topology). (Section 7.4)*

We collect some results regarding our heat flow in one place:

Fact 10 (Absolute Neumann heat flow)**.** *There exists a semigroup of operators* $(S(t))_{t\geq 0}$ *acting on* $\cup_{p\in(1,\infty)}L^p\mathfrak{X}$ *such that*

1. $S(t_1)\,S(t_2) = S(t_1+t_2)\ \forall t_1,t_2\geq 0$ *and* $S(0)=1$.
2. *(Section 8.2)* $\forall p\in(1,\infty), \forall X\in L^p\mathfrak{X}:$
   (a) $S(t)X\in\mathfrak{X}_N$ *and* $\partial_t(S(t)X) = \Delta S(t)X\ \forall t>0$.
   (b) $S(t)X \xrightarrow[t\to t_0]{C^\infty} S(t_0)X\ \forall t_0>0$.
   (c) $\|S(t)X\|_{W^{m,p}} \lesssim_{m,p} \left(\frac{1}{t}\right)^{\frac{m}{2}} \|X\|_{L^p}\ \forall m\in\mathbb{N}_0, \forall t\in(0,1)$.
   (d) $S(t)X \xrightarrow[t\to 0]{L^p} X$.
3. *(Section 8.3)* $\forall p\in(1,\infty), \forall X\in W^{1,p}\mathfrak{X}_N:$
   (a) $\|S(t)X\|_{W^{m+1,p}} \lesssim_{m,p} \left(\frac{1}{t}\right)^{\frac{m}{2}} \|X\|_{W^{1,p}}\ \forall m\in\mathbb{N}_0, \forall t\in(0,1)$.
   (b) $S(t)X \xrightarrow[t\to 0]{W^{1,p}} X$.
4. *(Theorem 78)* $S(t)\,\mathbb{P} = \mathbb{P}S(t)$ *on* $W^{m,p}\mathfrak{X}\ \forall m\in\mathbb{N}_0, \forall p\in(1,\infty), \forall t\geq 0$.
5. *(Section 8.2)* $\langle\langle S(t)X,Y\rangle\rangle = \langle\langle X,S(t)Y\rangle\rangle\ \forall t\geq 0, \forall p\in(1,\infty), \forall X\in L^p\mathfrak{X}, \forall Y\in L^{p'}\mathfrak{X}$.

These estimates precisely fit the analogy $e^{t\Delta}\approx P_{\leq\frac{1}{\sqrt{t}}}$ where P is the **Littlewood-Paley projection**. We also stress that the heat flow preserves the space of tangential, divergence-free vector fields (the range of $\mathbb{P}$), and is intrinsic (with no dependence on choices of local coordinates).

Analogous results hold for scalar functions and differential forms (Chapter 8). We also have commutativity with the exterior derivative and codifferential in the case of differential forms (Theorem 75). Loosely speaking, this allows the heat flow to preserve the overall Hodge structure on the manifold. All these properties would not be possible under standard mollification via partitions of unity.

Note that for $X\in\mathfrak{X}$, $X\otimes X$ is not dual to a differential form. As our heat flow is generated by the Hodge Laplacian, it is less useful in mollifying general tensor fields (for which the connection Laplacian is better suited). Fortunately, we will never actually have to do so in this thesis.

We observe some basic identities (cf. Theorem 60):

- Using **Penrose abstract index notation** (see Section 7.2), for any smooth tensors $T_{a_1 \ldots a_k}$, we define $(\nabla T)_{ia_1 \ldots a_k} = \nabla_i T_{a_1 \ldots a_k}$ and $\operatorname{div} T = \nabla^i T_{ia_2 \ldots a_k}$.

- For all smooth tensors $T_{a_1 \ldots a_k}$ and $Q_{a_1 \ldots a_{k+1}}$:

$$\int_M \nabla_i \left(T_{a_1 \ldots a_k} Q^{ia_1 \ldots a_k} \right) = \int_M \nabla_i T_{a_1 \ldots a_k} Q^{ia_1 \ldots a_k} + \int_M T_{a_1 \ldots a_k} \nabla_i Q^{ia_1 \ldots a_k} = \int_{\partial M} \nu_i T_{a_1 \ldots a_k} Q^{ia_1 \ldots a_k}$$

- For $X \in \mathfrak{X}_N, Y \in \mathfrak{X}, f \in C^\infty(M)$:

    1. $\int_M Xf = \int_M \operatorname{div}(fX) - \int_M f \operatorname{div}(X) = \int_{\partial M} \langle fX, \nu \rangle - \int_M f \operatorname{div} X = -\int_M f \operatorname{div} X$
    2. $\int_M \langle \operatorname{div}(X \otimes X), Y \rangle = -\int_M \langle X \otimes X, \nabla Y \rangle$

- $(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{ij}{}_{kl} = -R_{ab\sigma}{}^i T^{\sigma j}{}_{kl} - R_{ab\sigma}{}^j T^{i\sigma}{}_{kl} + R_{abk}{}^\sigma T^{ij}{}_{\sigma l} + R_{abl}{}^\sigma T^{ij}{}_{k\sigma}$ for any tensor $T^{ij}{}_{kl}$, where R is the **Riemann curvature tensor**. Similar identities hold for other types of tensors. When we do not care about the exact indices and how they contract, we can just write the **schematic identity** $(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{ij}{}_{kl} = R * T$. As R is bounded on compact M, interchanging derivatives is a zeroth-order operation on M. In particular, we have the **Weitzenbock formula**:

$$\Delta X = \nabla_i \nabla^i X + R * X \; \forall X \in \mathfrak{X} M \tag{4.1}$$

- For $X \in \mathbb{P}L^2\mathfrak{X}, Y \in \mathfrak{X}, Z \in \mathfrak{X}, f \in C^\infty(M)$:

    1. $\int_M Xf = 0$
    2. $\int_M \langle \nabla_X Y, Z \rangle = -\int_M \langle Y, \nabla_X Z \rangle$.

There is an elementary lemma which is useful for convergence (the proof is straightforward and omitted):

Lemma 11 (Dense convergence)**.** *Let X, Y be (real/complex) Banach spaces and $X_0 \leq X$ be norm-dense. Let $(T_j)_{j\in\mathbb{N}}$ be bounded in $\mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X,Y)$.*

If $T_j x_0 \to T x_0\ \forall x_0 \in X_0$ then $T_j x \to Tx\ \forall x \in X$.

Definition 12 (Heatable currents)**.** As the heat flow does not preserve compact supports in $\mathring{M}$, it is not defined on distributions. This inspires the formulation of **heatable currents**. Define:

- $\mathscr{D}\Omega^k = \Omega_{00}^k = \operatorname{colim}\{\left(\Omega_{00}^k(K), C^\infty \text{ topo}\right) : K \subset \mathring{M} \text{ compact}\}$ as the space of **test k-forms** with **Schwartz's topology**[1] (colimit in the category of locally convex TVS).

- $\mathscr{D}'\Omega^k = \left(\mathscr{D}\Omega^k\right)^*$ as the space of **k-currents** (or **distributional k-forms**), equipped with the weak* topology.

- $\mathscr{D}_N\Omega^k = \{\omega \in \Omega^k : \mathbf{n}\Delta^m\omega = 0, \mathbf{n}d\Delta^m\omega = 0\ \forall m \in \mathbb{N}_0\}$ as the space of **heated k-forms** with the Frechet C^∞ topology and $\mathscr{D}_N'\Omega^k = \left(\mathscr{D}_N\Omega^k\right)^*$ as the space of **heatable k-currents** (or **heatable distributional k-forms**) with the weak* topology.

- **Spacetime test forms**: $\mathscr{D}\left(I, \Omega^k\right) = C_c^\infty\left(I, \Omega_{00}^k\right) = \operatorname{colim}\{\left(C_c^\infty\left(I_1, \Omega_{00}^k(K)\right), C^\infty \text{ topo}\right) : I_1 \times K \subset I \times \mathring{M} \text{ compact}\}$ and $\mathscr{D}_N\left(I, \Omega^k\right) = \operatorname{colim}\{\left(C_c^\infty\left(I_1, \mathscr{D}_N\Omega^k\right), C^\infty \text{ topo}\right) : I_1 \subset I \text{ compact}\}$.

- **Spacetime distributions** $\mathscr{D}'\left(I, \Omega^k\right) = \mathscr{D}\left(I, \Omega^k\right)^*$, $\mathscr{D}_N'\left(I, \Omega^k\right) = \mathscr{D}_N\left(I, \Omega^k\right)^*$.

In particular, $\mathscr{D}_N\mathfrak{X}$ is defined from $\mathscr{D}_N\Omega^1$ by the musical isomorphism, and it is invariant under our heat flow (much like how the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is invariant under the Littlewood-Paley projection). By that analogy, heatable currents are tempered distributions on manifolds, and we can write

$$\langle\langle S(t)\Lambda, X\rangle\rangle = \langle\langle \Lambda, S(t)X\rangle\rangle\ \ \forall \Lambda \in \mathscr{D}_N'\mathfrak{X}, \forall X \in \mathscr{D}_N\mathfrak{X}, \forall t \geq 0$$

[1] Confusingly enough, "Schwartz's topology" refers to the topology on the space of distributions, not the topology for Schwartz functions.

where the dot product $\langle\langle\cdot,\cdot\rangle\rangle$ is simply abuse of notation.

Fact 13. *Some basic properties of $\mathscr{D}_N\mathfrak{X}$ and $\mathscr{D}'_N\mathfrak{X}$:*

- $\langle\langle\Delta X, Y\rangle\rangle = \langle\langle X, \Delta Y\rangle\rangle\ \forall X, Y \in \mathscr{D}_N\mathfrak{X}$. *(Theorem 60)*

- $S(t)\Lambda \in \mathscr{D}_N\mathfrak{X}\ \forall t > 0, \forall\Lambda \in \mathscr{D}'_N\mathfrak{X}$. *(Section 8.4, a heatable current becomes heated once the heat flow is applied)*

- $\mathfrak{X}_{00} \subset \mathscr{D}_N\mathfrak{X}$ *and is dense in* $L^p\mathfrak{X}\ \forall p \in [1,\infty)$. *Also,* $L^p\mathfrak{X} \hookrightarrow \mathscr{D}'_N\mathfrak{X}$ *is continuous* $\forall p \in [1,\infty]$.

- $\mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X} = \mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$, $\mathbb{P}W^{1,p}\mathfrak{X} = \mathbb{P}W^{1,p}\mathfrak{X}_N$ *and* $\mathbb{P}\mathscr{D}_N\mathfrak{X} \leq \mathscr{D}_N\mathfrak{X}$. *(Section 7.4)*

- $W^{1,p}\text{-cl}\left(\mathscr{D}_N\mathfrak{X}\right) = W^{1,p}\mathfrak{X}_N\ \forall p \in (1,\infty)$ *(Section 8.3),* $B^{\frac{1}{3}}_{3,1}\text{-cl}\left(\mathbb{P}\mathscr{D}_N\mathfrak{X}\right) = \mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$ *(Section 9.3)*

- $\forall X \in \mathscr{D}_N\mathfrak{X} : S(t)X \xrightarrow[t\downarrow 0]{C^\infty} X$ *and* $\partial_t\left(S(t)X\right) = \Delta S(t)X = S(t)\Delta X\ \forall t \geq 0$. *(Theorem 34, Section 8.2)*

- *(Section 8.2, Section 9.3)* $\forall t \in (0,1), \forall m, m' \in \mathbb{N}_0, \forall p \in (1,\infty), \forall X \in \mathscr{D}_N\mathfrak{X}$:

  1. $\|S(t)X\|_{W^{m+m',p}} \lesssim \left(\frac{1}{t}\right)^{\frac{m'}{2}} \|X\|_{W^{m,p}}$.
  2. $\|S(t)X\|_{B^{m+m'+\frac{1}{p}}_{p,1}} \lesssim \left(\frac{1}{t}\right)^{\frac{1}{2p}+\frac{m'}{2}} \|X\|_{W^{m,p}}$.
  3. $t^{\frac{1}{2}\left(m-\frac{1}{p}\right)} \|S(t)X\|_{W^{m,p}} + \|S(t)X\|_{B^{\frac{1}{p}}_{p,1}} \lesssim \|X\|_{B^{\frac{1}{p}}_{p,1}}$ *when* $m \geq 1$ *and* $X \in \mathbb{P}\mathscr{D}_N\mathfrak{X}$.

     By dense convergence (Lemma 11), this means $S(t)X \xrightarrow[t\downarrow 0]{B^{\frac{1}{3}}_{3,1}} X\ \forall X \in \mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$.

Corollary 14 (Vanishing)**.** $\forall X \in \mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N : s^{\frac{1}{3}} \|S(s)X\|_{W^{1,3}} \xrightarrow{s\downarrow 0} 0$.

Remark. So, for $\mathcal{U} \in L^3_t\mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$: $\|\mathcal{U}(t)\|_{L^3_t B^{\frac{1}{3}}_{3,1}} \gtrsim \left\|\left\|\sigma^{\frac{1}{3}} \|S(\sigma)\mathcal{U}(t)\|_{W^{1,3}}\right\|_{L^\infty_\sigma([0,s])}\right\|_{L^3_t} \xrightarrow[DCT]{s\downarrow 0} 0$. This pointwise vanishing property becomes important for the commutator estimate in Onsager's conjecture at the critical regularity level $\frac{1}{3}$, while higher regularity levels have enough room for vanishing in norm (which is better).

Proof. For $Y \in \mathbb{P}\mathscr{D}_N\mathfrak{X}$, as $s > 0$ small: $s^{\frac{1}{3}} \|S(s)Y\|_{W^{1,3}} \lesssim s^{\frac{1}{3}} \|Y\|_{W^{1,3}} \xrightarrow{s\downarrow 0} 0$. Then note $s^{\frac{1}{3}} \|S(s)X\|_{W^{1,3}} \lesssim \|X\|_{B^{\frac{1}{3}}_{3,1}} \; \forall X \in \mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$, so we can apply dense convergence (Lemma 11). $\square$

4.2 Searching for the proper formulation

Onsager's conjecture states that energy is conserved when $\mathcal{V}$ has enough regularity, with appropriate conditions near the boundary. But making this statement precise is half of the challenge.

Definition 15. We say $(\mathcal{V}, \mathfrak{p})$ is a **weak solution** to the Euler equation when

- $\mathcal{V} \in L^2_{\mathrm{loc}}(I, \mathbb{P}L^2\mathfrak{X})$, $\mathfrak{p} \in L^1_{\mathrm{loc}}(I \times M)$
- $\forall \mathcal{X} \in C^\infty_c(I, \mathfrak{X}_{00}) : \iint_{I\times M} \langle \mathcal{V}, \partial_t \mathcal{X}\rangle + \langle \mathcal{V} \otimes \mathcal{V}, \nabla \mathcal{X}\rangle + \mathfrak{p} \operatorname{div} \mathcal{X} = 0$.

The last condition means $\partial_t \mathcal{V} + \operatorname{div}(\mathcal{V} \otimes \mathcal{V}) + \operatorname{grad} \mathfrak{p} = 0$ as spacetime distributions. Note that $\mathcal{V} \otimes \mathcal{V} \in L^1_{\mathrm{loc}}(I, L^1\mathfrak{X})$ so it is a distribution.

The keen reader should notice we use a different font for time-dependent vector fields.

There is not enough time-regularity for FTC, and we cannot say

$$\langle\langle \mathcal{V}(t_1), X\rangle\rangle - \langle\langle \mathcal{V}(t_0), X\rangle\rangle = \int_{t_0}^{t_1} \langle\langle \mathcal{V} \otimes \mathcal{V}, \nabla X\rangle\rangle + \int_{t_0}^{t_1}\int_M \mathfrak{p} \operatorname{div} X \; \forall X \in \mathfrak{X}_{00}$$

But we can still use approximation to the identity (in the time variable) near t_0,t_1, as well as Lebesgue differentiation to get something similar for a.e. t_0, t_1. By using dense convergence (Lemma 11) and modifying I into $I_0 \subset I$ such that $|I \backslash I_0| = 0$, we can say $\mathcal{V} \in C^0_{\mathrm{loc}}(I_0, (L^2\mathfrak{X}, \text{weak})) \leq L^\infty_{\mathrm{loc}}(I, L^2\mathfrak{X})$.

We do not have $\mathcal{V} \in C^0_{\mathrm{loc}}(I, L^2\mathfrak{X})$, so energy conservation only means $\partial_t \left(\|\mathcal{V}(t)\|^2_{L^2\mathfrak{X}}\right) = 0$

as a distribution. In other words, the goal is to show

$$\int_I \eta'(t) \langle\langle \mathcal{V}(t), \mathcal{V}(t)\rangle\rangle \,\mathrm{d}t = 0 \;\forall \eta \in C_c^\infty(I)$$

Next, having the test vector field $\mathcal{X} \in C_c^\infty(I, \mathfrak{X}_{00})$ can be quite restrictive, since the heat flow (much like the Littlewood-Paley projection) does not preserve compact supports in $\mathring{M}$. We need a notion that is more in tune with our theory.

Definition 16. We say $(\mathcal{V}, \mathfrak{p})$ is a **Hodge weak solution** to the Euler equation when $\mathcal{V} \in L^2_{\text{loc}}(I, \mathbb{P}L^2\mathfrak{X})$, $\mathfrak{p} \in L^1_{\text{loc}}(I \times M)$ and

$$\forall \mathcal{X} \in C_c^\infty(I, \mathfrak{X}_N) : \iint_{I\times M} \langle \mathcal{V}, \partial_t \mathcal{X}\rangle + \langle \mathcal{V}\otimes\mathcal{V}, \nabla\mathcal{X}\rangle + \mathfrak{p}\operatorname{div}\mathcal{X} = 0$$

Now this looks better, since $\mathfrak{X}_N$ is invariant under the heat flow. However, this is a leap of faith we will need to justify later (cf. Section 4.3).

As $\mathbb{P}\mathfrak{X} \leq \mathfrak{X}_N$, we can go further and say $\mathcal{V}$ is a **Hodge-Leray weak solution** to the Euler equation when $\mathcal{V} \in L^2_{\text{loc}}(I, \mathbb{P}L^2\mathfrak{X})$ and

$$\forall \mathcal{X} \in C_c^\infty(I, \mathbb{P}\mathfrak{X}) : \iint_{I\times M} \langle \mathcal{V}, \partial_t \mathcal{X}\rangle + \langle \mathcal{V}\otimes\mathcal{V}, \nabla\mathcal{X}\rangle = 0$$

This would help give a formulation of Onsager's conjecture that does not depend on the pressure, similar to [RRS18].

Next, we look at the conditions for $\mathcal{V}$ and $\mathfrak{p}$ near ∂M. In [BT18], they assumed $\mathcal{V} \in L^3_t C^{0,\alpha}\mathfrak{X}_N$ with $\alpha \in \left(\frac{1}{3}, 1\right)$. In [NN19], they assumed $\mathcal{V} \in L^3_t B^{\alpha}_{3,\infty}\mathfrak{X}$ ($\alpha \in \left(\frac{1}{3}, 1\right)$) with a more general "strip decay" condition:

- $\|\mathcal{V}\|^2_{L^3_t L^3(M_{<r},\text{avg})} \|\langle \mathcal{V}, \widetilde{\nu}\rangle\|_{L^3_t L^3(M_{<r},\text{avg})} \xrightarrow{r\downarrow 0} 0$
- $\|\mathfrak{p}\|_{L^{\frac{3}{2}}_t L^{\frac{3}{2}}(M_{<r},\text{avg})} \|\langle \mathcal{V}, \widetilde{\nu}\rangle\|_{L^3_t L^3(M_{<r},\text{avg})} \xrightarrow{r\downarrow 0} 0.$

In [Bar+19b] (the most recent result), they assumed $\mathcal{V} \in L_t^3 \underline{B}^{1/3}_{3,\mathrm{VMO}}\mathfrak{X}$ (see the paper for the full definition), along with a minor relaxation for the "strip decay" condition:

$$\left\| \left(\frac{|\mathcal{V}|^2}{2} + \mathfrak{p} \right) \langle \mathcal{V}, \widetilde{\nu} \rangle \right\|_{L_t^1 L^1\left(M_{[\frac{r}{4},\frac{r}{2}]},\mathrm{avg}\right)} \xrightarrow{r\downarrow 0} 0$$

When $\mathcal{V} \in L_t^3 B^{\frac{1}{3}}_{3,1}\mathfrak{X}$, $\|\langle \mathcal{V}, \widetilde{\nu}\rangle\|_{L_t^3 L^3(M_{<r},\mathrm{avg})} \xrightarrow{r\downarrow 0} \|\langle \mathcal{V}, \nu\rangle\|_{L_t^3 L^3(\partial M,\mathrm{avg})}$ by Fact 7. This motivates our formulation later in Section 4.5, where we put $\mathcal{V} \in L_t^3 \mathbb{P} B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$.

4.3 Justification of formulation

We define the cutoffs

$$\psi_r(x) = \Psi_r\left(\mathrm{dist}\left(x, \partial M\right)\right) \tag{4.2}$$

where $r > 0$ small, $\Psi_r \in C^\infty([0,\infty),[0,\infty))$ such that $\mathbf{1}_{[0,\frac{3}{4}r)} \geq \Psi_r \geq \mathbf{1}_{[0,\frac{r}{2}]}$ and $\|\Psi_r'\|_\infty \lesssim \frac{1}{r}$. Then $\nabla\psi_r(x) = f_r(x)\widetilde{\nu}(x)$ where $|f_r(x)| \lesssim \frac{1}{r}$ and $\operatorname{supp}\psi_r \subset M_{<r}$.

Let $(\mathcal{V}, \mathfrak{p})$ be a weak solution to the Euler equation and $\alpha \in (\frac{1}{3}, 1)$. Define different conditions:

1. $\mathcal{V} \in L_t^3 C^{0,\alpha}\mathfrak{X}_N$.
2. $\mathcal{V} \in L_t^3 B^{\alpha}_{3,\infty}\mathfrak{X}$ and $\|\mathcal{V}\|^2_{L_t^3 L^3(M_{<r},\mathrm{avg})} \|\langle \mathcal{V}, \widetilde{\nu}\rangle\|_{L_t^3 L^3(M_{<r},\mathrm{avg})} \xrightarrow{r\downarrow 0} 0$.
3. $\mathcal{V} \in L_t^3 B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$.
4. $(\mathcal{V}, \mathfrak{p})$ is a Hodge weak solution.
5. $\mathcal{V}$ is a Hodge-Leray weak solution.

Theorem 17. *We have* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5).

Proof. By Fact 4, $C^{0,\alpha}\mathfrak{X}_N = B^{\alpha}_{\infty,\infty}\mathfrak{X}_N \hookrightarrow B^{\alpha}_{3,\infty}\mathfrak{X}_N \hookrightarrow B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$. Then by the coarea formula,

$$\begin{aligned}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|^3_{L^3_tL^3(M_{<r},\mathrm{avg})} &\lesssim \|\mathcal{V}\|^2_{L^3_tL^3(M_{<r},\mathrm{avg})}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|_{L^3_tL^3(M_{<r},\mathrm{avg})}\\ &\lesssim \|\mathcal{V}\|^2_{L^3_tB^{\frac{1}{3}}_{3,1}\mathfrak{X}}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|_{L^3_tL^3(M_{<r},\mathrm{avg})}\end{aligned}$$

So for $\mathcal{V}\in L^3_tB^{\frac{1}{3}}_{3,1}\mathfrak{X}$:

$$\begin{aligned}\|\mathcal{V}\|^2_{L^3_tL^3(M_{<r},\mathrm{avg})}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|_{L^3_tL^3(M_{<r},\mathrm{avg})}\xrightarrow{r\downarrow 0}0 &\iff \|\langle\mathcal{V},\nu\rangle\|_{L^3_tL^3(\partial M)}=0\\ &\iff \mathbf{n}\mathcal{V}=0\end{aligned}$$

As (4) $\implies$ (5) is obvious, the only thing left is to show (3) $\implies$ (4). Recall the cutoffs ψ_r from Equation (4.2).

Let $I_1\subset I$ be bounded and $\mathcal{X}\in C^\infty_c(I_1,\mathfrak{X}_N)$, then $(1-\psi_r)\mathcal{X}\in C^\infty_c(I,\mathfrak{X}_{00})$, and so by the definition of weak solution:

$$\begin{aligned}0 &= \iint_{I\times M}(1-\psi_r)\langle\mathcal{V},\partial_t\mathcal{X}\rangle+\langle\mathcal{V},\nabla_{\mathcal{V}}((1-\psi_r)\mathcal{X})\rangle+\mathfrak{p}\operatorname{div}((1-\psi_r)\mathcal{X})\\ &= \iint_{I\times M}(1-\psi_r)(\langle\mathcal{V},\partial_t\mathcal{X}\rangle+\langle\mathcal{V},\nabla_{\mathcal{V}}\mathcal{X}\rangle+\mathfrak{p}\operatorname{div}\mathcal{X})\\ &\quad -\iint_{I\times M}(\langle\mathcal{V},\nabla\psi_r\rangle\langle\mathcal{V},\mathcal{X}\rangle+\mathfrak{p}\langle\mathcal{X},\nabla\psi_r\rangle)\end{aligned}$$

We are done if the first term goes to zero as $r\downarrow 0$. So we only need to show the second term goes to zero. Since $\nabla\psi_r=f_r\widetilde{\nu}$ and $\operatorname{supp}\psi_r\subset M_{<r}$, we only need to bound

$$\begin{aligned}&\left|\iint_{I_1\times M_{<r}}f_r\langle\mathcal{V},\widetilde{\nu}\rangle\langle\mathcal{V},\mathcal{X}\rangle+\mathfrak{p}f_r\langle\mathcal{X},\widetilde{\nu}\rangle\right|\\ \lesssim&\frac{1}{r}\|\mathcal{V}\|_{L^3_tL^3(M_{<r})}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|_{L^3_tL^3(M_{<r})}\|\mathcal{X}\|_{L^3_tL^3(M_{<r})}\end{aligned}$$

$$+\frac{1}{r}\|\mathfrak{p}\|_{L^1(I_1\times M_{<r})}\|\langle\mathcal{X},\widetilde{\nu}\rangle\|_{L_t^\infty L^\infty(M_{<r})}$$

$$\lesssim\|\mathcal{V}\|_{L_t^3L^3(M_{<r},\mathrm{avg})}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|_{L_t^3L^3(M_{<r},\mathrm{avg})}\|\mathcal{X}\|_{L_t^3L^3(M_{<r},\mathrm{avg})}$$

$$+\|\mathfrak{p}\|_{L^1(I_1\times M_{<r})}\|\langle\mathcal{X},\widetilde{\nu}\rangle\|_{L_t^\infty C^{0,1}(M_{<r})}$$

$$\lesssim\|\mathcal{V}\|_{L_t^3B_{3,1}^{\frac{1}{3}}\mathfrak{X}}\|\langle\mathcal{V},\widetilde{\nu}\rangle\|_{L_t^3L^3(M_{<r},\mathrm{avg})}\|\mathcal{X}\|_{L_t^3B_{3,1}^{\frac{1}{3}}\mathfrak{X}}+\|\mathfrak{p}\|_{L^1(I_1\times M_{<r})}\|\langle\mathcal{X},\widetilde{\nu}\rangle\|_{L_t^\infty C^{0,1}(M_{<r})}$$

$$\xrightarrow{r\downarrow 0}0$$

We used the estimate $\|\langle\mathcal{X},\widetilde{\nu}\rangle\|_{L^\infty(M_{<r})}\lesssim r\|\langle\mathcal{X},\widetilde{\nu}\rangle\|_{C^{0,1}(M_{<r})}$ since $\langle\mathcal{X},\nu\rangle=0$ on ∂M. □

Remark. Interestingly, as Section 4.5 will show, no "strip decay" condition involving $\mathfrak{p}$ seems to be necessary. See the end of Section 2.1 for a discussion of this minor improvement.

We briefly note that when $\partial M=\emptyset$, it is customary to set $\mathrm{dist}(x,\partial M)=\infty$, and $\psi_r=0$, $M_{>r}=M=\mathring{M}$, $M_{<r}=\emptyset$, and $\mathscr{D}_N\mathfrak{X}M=\mathscr{D}\mathfrak{X}M=\mathfrak{X}M$.

4.4 Heating the nonlinear term

Let $U,V\in B_{3,1}^{\frac{1}{3}}\mathfrak{X}$. Then $U\otimes V\in L^1\mathfrak{X}$ and $\mathrm{div}\,(U\otimes V)$ is defined as a distribution. To apply the heat flow to $\mathrm{div}\,(U\otimes V)$, we need to define $(\mathrm{div}\,(U\otimes V))^\flat$ so that it is heatable.
Recall integration by parts:

$$\langle\langle\mathrm{div}\,(Y\otimes Z),X\rangle\rangle=-\langle\langle Y\otimes Z,\nabla X\rangle\rangle+\int_{\partial M}\langle\nu,Y\rangle\langle Z,X\rangle\ \ \forall X,Y,Z\in\mathfrak{X}(M)$$

Observe that for $X\in\mathfrak{X}$, even though $\langle\langle\mathrm{div}\,(U\otimes V),X\rangle\rangle$ is not defined, $\int_{\partial M}\langle\nu,U\rangle\langle V,X\rangle-\langle\langle U\otimes V,\nabla X\rangle\rangle$ is well-defined by the trace theorem. So we will define the heatable 1-current $(\mathrm{div}\,(U\otimes V))^\flat$ by

$$\langle\langle\mathrm{div}\,(U\otimes V),X\rangle\rangle=-\langle\langle U\otimes V,\nabla X\rangle\rangle+\int_{\partial M}\langle\nu,U\rangle\langle V,X\rangle\ \ \forall X\in\mathscr{D}_N\mathfrak{X}\ (X\text{ is heated})$$

It is continuous on $\mathscr{D}_N\mathfrak{X}$ since

$$|\langle\langle \operatorname{div}(U\otimes V), X\rangle\rangle| \lesssim \|U\|_{B^{\frac{1}{3}}_{3,1}} \|V\|_{B^{\frac{1}{3}}_{3,1}} \|X\|_{B^{\frac{1}{3}}_{3,1}} + \|U\|_{L^3} \|V\|_{L^3} \|\nabla X\|_{L^3}.$$

By the same formula and reasoning, we see that $(\operatorname{div}(U\otimes V))^\flat$ is not just heatable, but also a continuous linear functional on $(\mathfrak{X}(M), C^\infty \text{ topo})$.

On the other hand, we can get away with less regularity by assuming $U\in \mathbb{P}L^2\mathfrak{X}$. Then we simply need to define $\langle\langle \operatorname{div}(U\otimes V), X\rangle\rangle = -\langle\langle U\otimes V, \nabla X\rangle\rangle \ \forall X\in\mathfrak{X}$.

In short, $(\operatorname{div}(U\otimes V))^\flat$ is heatable when $U\in\mathbb{P}L^2\mathfrak{X}$ and $V\in L^2\mathfrak{X}$. Consequently, by Theorem 17, when $(\mathcal{V},\mathfrak{p})$ is a weak solution to the Euler equation and $\mathcal{V}\in L^3_t B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$: $(\mathcal{V},\mathfrak{p})$ is a Hodge weak solution and

$$\boxed{\partial_t\mathcal{V} + \operatorname{div}(\mathcal{V}\otimes\mathcal{V}) + \operatorname{grad}\mathfrak{p} = 0 \text{ in } \mathscr{D}'_N(I,\mathfrak{X}).} \tag{4.3}$$

4.5 Proof of Onsager's conjecture

For the rest of the proof, we will write $e^{t\Delta}$ for $S(t)$, as we will not need another heat flow. For $\varepsilon>0$ and vector field X, we will write X^ε for $e^{\varepsilon\Delta}X$.

We opt to formulate the conjecture without mentioning the pressure (see Section 4.3 for the justification).

Theorem 18 (Onsager's conjecture)**.** *Let M be a compact, oriented Riemannian manifold with no or smooth boundary. Let $\mathcal{V}\in L^3_t\mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$ such that $\forall\mathcal{X}\in C^\infty_c(I,\mathbb{P}\mathfrak{X}):\iint_{I\times M}\langle\mathcal{V},\partial_t\mathcal{X}\rangle + \langle\mathcal{V}\otimes\mathcal{V},\nabla\mathcal{X}\rangle = 0$ (Hodge-Leray weak solution).*

Then we can show

$$\int_I \eta'(t)\,\langle\langle\mathcal{V}(t),\mathcal{V}(t)\rangle\rangle\,\mathrm{dt} = 0 \ \forall\eta\in C^\infty_c(I)$$

Consequently, $\langle\langle\mathcal{V}(t),\mathcal{V}(t)\rangle\rangle$ is constant for a.e. $t\in I$.

As usual, there is a **commutator estimate** which we will leave for later:

$$\begin{aligned}&\int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U}\otimes\mathcal{U}\right)^{2\varepsilon}, \mathcal{U}^{2\varepsilon}\right\rangle\right\rangle - \int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U}^{2\varepsilon}\otimes\mathcal{U}^{2\varepsilon}\right), \mathcal{U}^{2\varepsilon}\right\rangle\right\rangle \\ &= \int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U}\otimes\mathcal{U}\right)^{3\varepsilon}, \mathcal{U}^{\varepsilon}\right\rangle\right\rangle - \int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U}^{2\varepsilon}\otimes\mathcal{U}^{2\varepsilon}\right)^{\varepsilon}, \mathcal{U}^{\varepsilon}\right\rangle\right\rangle \xrightarrow{\varepsilon\downarrow 0} 0 \end{aligned} \tag{4.4}$$

for all $\mathcal{U}\in L_t^3\mathbb{P}B_{3,1}^{\frac{1}{3}}\mathfrak{X}_N, \eta\in C_c^\infty(I)$.

Notation: we write $\operatorname{div}\left(\mathcal{U}\otimes\mathcal{U}\right)^{\varepsilon}$ for $\left(\operatorname{div}\left(\mathcal{U}\otimes\mathcal{U}\right)\right)^{\varepsilon}$ and $\nabla\mathcal{U}^{\varepsilon}$ for $\nabla\left(\mathcal{U}^{\varepsilon}\right)$ (recall that the heat flow does not work on tensors $\mathcal{U}\otimes\mathcal{U}$ and $\nabla\mathcal{U}$). Compared with [IO14], our commutator estimate looks a bit different, to ease some integration by parts procedures down the line.

Remark. For any U in $\mathbb{P}L^2\mathfrak{X}$, $\operatorname{div}\left(U\otimes U\right)^{\flat}$ is a heatable 1-current (see Section 4.4). In particular, for $\varepsilon>0$, $\operatorname{div}\left(U\otimes U\right)^{\varepsilon}$ is smooth and

$$\left\langle\left\langle \operatorname{div}\left(U\otimes U\right)^{\varepsilon}, Y\right\rangle\right\rangle = -\left\langle\left\langle U\otimes U, \nabla\left(Y^{\varepsilon}\right)\right\rangle\right\rangle \ \forall Y\in\mathfrak{X} \tag{4.5}$$

Consequently, Equation (4.4) is well-defined.

Theorem 19 (Onsager)**.** *Assume Equation (4.4) is true. Then* $\int_I \eta'(t)\left\langle\left\langle \mathcal{V}(t),\mathcal{V}(t)\right\rangle\right\rangle \mathrm{d}t = 0$.

Proof. Let $\Phi\in C_c^\infty(\mathbb{R})$ and $\Phi_\tau \xrightarrow{\tau\downarrow 0}\delta_0$ be a radially symmetric mollifier. Write $\mathcal{V}^\varepsilon$ for $e^{\varepsilon\Delta}\mathcal{V}$ (spatial mollification) and $\mathcal{V}_\tau$ for $\Phi_\tau * \mathcal{V}$ (temporal mollification). First, we mollify in time and space

$$\frac{1}{2}\int_I \eta'\left\langle\left\langle \mathcal{V},\mathcal{V}\right\rangle\right\rangle \overset{\text{DCT}}{=} \lim_{\varepsilon\downarrow 0}\lim_{\tau\downarrow 0}\frac{1}{2}\int_I \eta'\left\langle\left\langle \mathcal{V}_\tau^\varepsilon,\mathcal{V}_\tau^\varepsilon\right\rangle\right\rangle$$

Then we want to get rid of the time derivative:

$$\frac{1}{2}\int_I \eta'\left\langle\left\langle \mathcal{V}_\tau^\varepsilon,\mathcal{V}_\tau^\varepsilon\right\rangle\right\rangle = -\int_I \eta\left\langle\left\langle \partial_t\mathcal{V}_\tau^\varepsilon,\mathcal{V}_\tau^\varepsilon\right\rangle\right\rangle = -\int_I \left\langle\left\langle \partial_t\left(\eta\mathcal{V}_\tau^\varepsilon\right),\mathcal{V}_\tau^\varepsilon\right\rangle\right\rangle + \int_I \eta'\left\langle\left\langle \mathcal{V}_\tau^\varepsilon,\mathcal{V}_\tau^\varepsilon\right\rangle\right\rangle$$

Then we use the definition of Hodge-Leray weak solution, and exploit the commutativity between spatial and temporal operators:

$$\begin{aligned}\frac{1}{2}\int_I \eta' \langle\langle \mathcal{V}^\varepsilon_\tau, \mathcal{V}^\varepsilon_\tau\rangle\rangle &= \int_I \langle\langle \partial_t\left(\eta \mathcal{V}^\varepsilon_\tau\right), \mathcal{V}^\varepsilon_\tau\rangle\rangle = \int_I \left\langle\left\langle \partial_t \left[\left(\eta \mathcal{V}^{2\varepsilon}_\tau\right)_\tau\right], \mathcal{V}\right\rangle\right\rangle \\ &= -\int_I \left\langle\left\langle \nabla\left[\left(\eta \mathcal{V}^{2\varepsilon}_\tau\right)_\tau\right], \mathcal{V}\otimes\mathcal{V}\right\rangle\right\rangle \\ &= -\int_I \left\langle\left\langle \left[\eta\left(\nabla \mathcal{V}^{2\varepsilon}_\tau\right)\right]_\tau, \mathcal{V}\otimes\mathcal{V}\right\rangle\right\rangle \\ &= -\int_I \eta \left\langle\left\langle \left(\nabla \mathcal{V}^{2\varepsilon}\right)_\tau, \left(\mathcal{V}\otimes\mathcal{V}\right)_\tau\right\rangle\right\rangle\end{aligned}$$

where we used the fact that $\left(\eta \mathcal{V}^{2\varepsilon}_\tau\right)_\tau \in C^\infty_c\left(I, \mathbb{P}\mathfrak{X}\right)$ to pass to the second line.

As there is no longer a time derivative on $\mathcal{V}$, we get rid of τ by letting $\tau \downarrow 0$ (fine as $\mathcal{V}$ is L^3 in time). Recall Equation (4.5):

$$\begin{aligned}\frac{1}{2}\int_I \eta' \langle\langle \mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon\rangle\rangle &= -\int_I \eta \left\langle\left\langle \nabla\left(\mathcal{V}^{2\varepsilon}\right), \mathcal{V}\otimes\mathcal{V}\right\rangle\right\rangle = \int_I \eta \left\langle\left\langle \mathcal{V}^\varepsilon, \operatorname{div}\left(\mathcal{V}\otimes\mathcal{V}\right)^\varepsilon\right\rangle\right\rangle \\ &= \int_I \eta \left\langle\left\langle \mathcal{V}^\varepsilon, \operatorname{div}\left(\mathcal{V}^\varepsilon\otimes\mathcal{V}^\varepsilon\right)\right\rangle\right\rangle + o_\varepsilon(1) \\ &= \int_I \eta \left\langle\left\langle \mathcal{V}^\varepsilon, \nabla_{\mathcal{V}^\varepsilon}\mathcal{V}^\varepsilon\right\rangle\right\rangle + o_\varepsilon(1) = \int_I \eta \int_M \mathcal{V}^\varepsilon\left(\frac{|\mathcal{V}^\varepsilon|^2}{2}\right) + o_\varepsilon(1) = o_\varepsilon(1)\end{aligned}$$

where we used the commutator estimate to pass to the second line, and the fact that $\mathcal{V}^\varepsilon \in \mathbb{P}\mathfrak{X}$ to make the integral vanish.

So $\frac{1}{2}\int_I \eta' \langle\langle \mathcal{V}, \mathcal{V}\rangle\rangle = \lim_{\varepsilon\downarrow 0}\lim_{\tau\downarrow 0}\frac{1}{2}\int_I \eta' \langle\langle \mathcal{V}^\varepsilon_\tau, \mathcal{V}^\varepsilon_\tau\rangle\rangle = \lim_{\varepsilon\downarrow 0}\frac{1}{2}\int_I \eta' \langle\langle \mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon\rangle\rangle = 0.$ □

The proof is short and did not much use the Besov regularity of $\mathcal{V}$. It is the commutator estimate that presents the main difficulty. We proceed similarly as in [IO14].

Let $\mathcal{U} \in L^3_t \mathbb{P} B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$. By setting $\mathcal{U}(t)$ to 0 for t in a null set, WLOG $\mathcal{U}(t) \in \mathbb{P}B^{\frac{1}{3}}_{3,1}\mathfrak{X}_N$ $\forall t \in I$. Define the commutator

$$\mathcal{W}(t,s) = \operatorname{div}\left(\mathcal{U}(t)\otimes\mathcal{U}(t)\right)^{3s} - \operatorname{div}\left(\mathcal{U}\left(t\right)^{2s}\otimes\mathcal{U}\left(t\right)^{2s}\right)^s$$

When t and s are implicitly understood, we will not write them. As $\operatorname{div}(\mathcal{U}(t)\otimes\mathcal{U}(t))^{3s}$ solves $(\partial_s-3\Delta)\mathcal{X}=0$, we define $\mathcal{N}=(\partial_s-3\Delta)\mathcal{W}$. Then $\mathcal{W}$ and $\mathcal{N}$ obey the Duhamel formula:

Lemma 20 (Duhamel formulas)**.**

1. *$\mathcal{W}(t,s)\xrightarrow{s\downarrow 0}0$ in $\mathscr{D}'_N\mathfrak{X}$ and therefore in $\mathscr{D}'\mathfrak{X}$. Furthermore, $\mathcal{W}(\cdot,s)\xrightarrow{s\downarrow 0}0$ in $\mathscr{D}'_N(I,\mathfrak{X})$ and therefore in $\mathscr{D}'(I,\mathfrak{X})$ (spacetime distribution).*
2. *For fixed $t_0\in I$ and $s>0$: $\int_\varepsilon^s\mathcal{N}(t_0,\sigma)^{3(s-\sigma)}\,\mathrm{d}\sigma\xrightarrow{\varepsilon\downarrow 0}\mathcal{W}(t_0,s)$ in $\mathscr{D}'_N\mathfrak{X}$.*

Proof.

1. Let $X\in\mathscr{D}_N\mathfrak{X},\mathcal{X}\in C_c^\infty(I,\mathscr{D}_N\mathfrak{X})$. It is trivial to check (with DCT)

$$\left\langle\left\langle\mathcal{U}(t)\otimes\mathcal{U}(t),\nabla\left(X^{3s}\right)\right\rangle\right\rangle-\left\langle\left\langle\mathcal{U}(t)^{2s}\otimes\mathcal{U}(t)^{2s},\nabla\left(X^{s}\right)\right\rangle\right\rangle\xrightarrow{s\downarrow 0}0$$
$$\int_I\left\langle\left\langle\mathcal{U}\otimes\mathcal{U},\nabla\left(\mathcal{X}^{3s}\right)\right\rangle\right\rangle-\int_I\left\langle\left\langle\mathcal{U}^{2s}\otimes\mathcal{U}^{2s},\nabla\left(\mathcal{X}^{s}\right)\right\rangle\right\rangle\xrightarrow{s\downarrow 0}0$$

2. Let $\varepsilon>0$. By the smoothing effect of $e^{s\Delta}$, $\mathcal{W}(t_0,\cdot)$ and $\mathcal{N}(t_0,\cdot)$ are in $C^0_{\mathrm{loc}}((0,1],\mathscr{D}_N\mathfrak{X})$. As $\left(e^{s\Delta}\right)_{s\geq 0}$ is a C_0 semigroup on $\left(H^m\text{-cl}\left(\mathscr{D}_N\mathfrak{X}\right),\|\cdot\|_{H^m}\right)$ $\forall m\in\mathbb{N}_0$, and a semigroup basically corresponds to an ODE (cf. [Tay11a, Appendix A, Proposition 9.10 & 9.11]), from $\partial_s\mathcal{W}=3\Delta\mathcal{W}+\mathcal{N}$ for $s\geq\varepsilon$ we get the Duhamel formula

$$\forall s>\varepsilon:\mathcal{W}(t_0,s)=\mathcal{W}(t_0,\varepsilon)^{3(s-\varepsilon)}+\int_\varepsilon^s\mathcal{N}(t_0,\sigma)^{3(s-\sigma)}\,\mathrm{d}\sigma$$

So we only need to show $\mathcal{W}(t_0,\varepsilon)^{3(s-\varepsilon)}\xrightarrow[\varepsilon\downarrow 0]{\mathscr{D}'_N\mathfrak{X}}0$. Let $X\in\mathscr{D}_N\mathfrak{X}$.

$$\left\langle\left\langle X,\mathcal{W}(t_0,\varepsilon)^{3(s-\varepsilon)}\right\rangle\right\rangle=\left\langle\left\langle X^{3(s-\varepsilon)},\operatorname{div}\left(\mathcal{U}(t_0)\otimes\mathcal{U}(t_0)\right)^{3\varepsilon}\right\rangle\right\rangle$$
$$-\left\langle\left\langle X^{3(s-\varepsilon)},\operatorname{div}\left(\mathcal{U}(t_0)^{2\varepsilon}\otimes\mathcal{U}(t_0)^{2\varepsilon}\right)^{\varepsilon}\right\rangle\right\rangle$$

$$= -\left\langle\left\langle\nabla\left(X^{3s}\right), \mathcal{U}(t_0)\otimes\mathcal{U}(t_0)\right\rangle\right\rangle$$
$$+\left\langle\left\langle\nabla\left(X^{3s-2\varepsilon}\right), \mathcal{U}(t_0)^{2\varepsilon}\otimes\mathcal{U}(t_0)^{2\varepsilon}\right\rangle\right\rangle \xrightarrow{\varepsilon\downarrow 0} 0.$$

□

From now on, we write $\int_{0+}^{s}$ for $\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{s}$. Then

$$\int_I \mathrm{d}t\ \eta(t)\left\langle\left\langle\mathcal{W}(t,s), \mathcal{U}(t)^s\right\rangle\right\rangle = \int_I \mathrm{d}t\ \eta(t)\int_{0+}^{s}\mathrm{d}\sigma\left\langle\left\langle\mathcal{N}(t,\sigma)^{3(s-\sigma)}, \mathcal{U}(t)^s\right\rangle\right\rangle$$

To clean up the algebra, we will classify the terms that are going to appear but are actually negligible in the end. The following estimates lie at the heart of the problem, showing why the regularity needs to be at least $\frac{1}{3}$, and that our argument barely holds thanks to the pointwise vanishing property (Corollary 14).

Lemma 21 (3 error estimates)**.** *Define the k-jet fiber norm* $|X|_{J^k} = \left(\sum_{j=0}^{k}\left|\nabla^{(j)}X\right|^2\right)^{\frac{1}{2}}$ $\forall X\in\mathfrak{X}$ *(more details in Subsection 7.1.1). Then we have*

1. $\int_I|\eta|\int_{0+}^{s}\mathrm{d}\sigma\int_M\left|\mathcal{U}^{2\sigma}\right|_{J^1}^2\left|\mathcal{U}^{4s-2\sigma}\right|_{J^1}\xrightarrow{s\downarrow 0} 0$
2. $\int_I|\eta|\int_{0+}^{s}\mathrm{d}\sigma\int_{\partial M}\left|\mathcal{U}^{2\sigma}\right|^2\left|\mathcal{U}^{4s-2\sigma}\right|_{J^2}\xrightarrow{s\downarrow 0} 0$
3. $\int_I|\eta|\int_{0+}^{s}\mathrm{d}\sigma\int_{\partial M}\left|\mathcal{U}^{2\sigma}\right|\left|\mathcal{U}^{2\sigma}\right|_{J^1}\left|\mathcal{U}^{4s-2\sigma}\right|_{J^1}\xrightarrow{s\downarrow 0} 0$

Proof. Define $A(t,s) = s^{\frac{1}{3}}\left\|\mathcal{U}(t)^{\frac{s}{2}}\right\|_{W^{1,3}}$. Then for $s>0$ small: $\left\|\mathcal{U}(t)^s\right\|_{B_{3,1}^{1+\frac{1}{3}}}\lesssim\left(\frac{1}{s}\right)^{\frac{1}{6}}\left\|\mathcal{U}(t)^{\frac{s}{2}}\right\|_{W^{1,3}}\lesssim\left(\frac{1}{s}\right)^{\frac{1}{2}}A(t,s)$ and $\left\|\left\|A(t,\sigma)\right\|_{L^\infty_{\sigma\le s}}\right\|_{L^3_t}\xrightarrow{s\downarrow 0} 0$ by Corollary 14. We also note that $\left\|\mathcal{U}(t)^s\right\|_{B_{3,1}^{2+\frac{1}{3}}}\lesssim\left(\frac{1}{s}\right)^{\frac{2}{3}}\left\|\mathcal{U}(t)^{\frac{s}{2}}\right\|_{W^{1,3}}\lesssim\left(\frac{1}{s}\right)A(t,s)$.

Now we can prove the error estimates go to 0:

1.

$$\begin{aligned}
&\int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \int_M \left|\mathcal{U}^{2\sigma}\right|_{J^1}^2 \left|\mathcal{U}^{4s-2\sigma}\right|_{J^1} \lesssim \int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \left\|\mathcal{U}^{2\sigma}\right\|_{W^{1,3}}^2 \left\|\mathcal{U}^{4s-2\sigma}\right\|_{W^{1,3}} \\
&\lesssim \int_I \mathrm{d}t\, |\eta(t)| \int_{0+}^s \mathrm{d}\sigma \left(\frac{1}{\sigma}\right)^{\frac{2}{3}} \left(\frac{1}{2s-\sigma}\right)^{\frac{1}{3}} A\left(t, 2\sigma\right)^2 A\left(t, 4s-2\sigma\right) \\
&\overset{\sigma\mapsto s\sigma}{=} \int_I \mathrm{d}t\, |\eta(t)| \int_{0+}^1 \mathrm{d}\sigma \ \left(\frac{1}{\sigma}\right)^{\frac{2}{3}} \left(\frac{1}{2-\sigma}\right)^{\frac{1}{3}} A\left(t, 2s\sigma\right)^2 A\left(t, 4s-2s\sigma\right) \\
&\lesssim \int_I \mathrm{d}t\ \ |\eta(t)| \left\|A\left(t,\sigma\right)\right\|_{L^\infty_{\sigma\le 4s}}^3 \xrightarrow{s\downarrow 0} 0.
\end{aligned}$$

2.

$$\begin{aligned}
&\int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \int_{\partial M} \left|\mathcal{U}^{2\sigma}\right|^2 \left|\mathcal{U}^{4s-2\sigma}\right|_{J^2} \\
&\lesssim \int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \left\|\mathcal{U}^{2\sigma}\right\|_{L^3\mathfrak{X}M|_{\partial M}}^2 \left\|\mathcal{U}^{4s-2\sigma}\right\|_{W^{2,3}\mathfrak{X}M|_{\partial M}} \\
&\overset{\text{Trace}}{\lesssim} \int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \left\|\mathcal{U}^{2\sigma}\right\|_{B^{\frac{1}{3}}_{3,1}\mathfrak{X}M}^2 \left\|\mathcal{U}^{4s-2\sigma}\right\|_{B^{2+\frac{1}{3}}_{3,1}\mathfrak{X}M} \\
&\lesssim \int_I \mathrm{d}t\ \ |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{B^{\frac{1}{3}}_{3,1}\mathfrak{X}M}^2 \int_{0+}^s \mathrm{d}\sigma \left(\frac{1}{2s-\sigma}\right) A\left(t, 4s-2\sigma\right) \\
&\overset{\sigma\mapsto s\sigma}{=} \int_I \mathrm{d}t\ \ |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{B^{\frac{1}{3}}_{3,1}\mathfrak{X}M}^2 \int_{0+}^1 \mathrm{d}\sigma \left(\frac{1}{2-\sigma}\right) A\left(t, 4s-2s\sigma\right) \\
&\lesssim \int_I \mathrm{d}t\ \ |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{B^{\frac{1}{3}}_{3,1}\mathfrak{X}M}^2 \left\|A\left(t,\sigma\right)\right\|_{L^\infty_{\sigma\le 4s}} \\
&\lesssim \left\|\mathcal{U}\right\|_{L^3_t B^{\frac{1}{3}}_{3,1}(M)}^2 \left\|\left\|A\left(t,\sigma\right)\right\|_{L^\infty_{\sigma\le 4s}}\right\|_{L^3_t} \xrightarrow{s\downarrow 0} 0
\end{aligned}$$

3.

$$\begin{aligned}
&\int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \int_{\partial M} \left|\mathcal{U}^{2\sigma}\right| \left|\mathcal{U}^{2\sigma}\right|_{J^1} \left|\mathcal{U}^{4s-2\sigma}\right|_{J^1} \\
&\overset{\text{Trace}}{\lesssim} \int_I |\eta| \int_{0+}^s \mathrm{d}\sigma \left\|\mathcal{U}^{2\sigma}\right\|_{B^{\frac{1}{3}}_{3,1}\mathfrak{X}M} \left\|\mathcal{U}^{2\sigma}\right\|_{B^{1+\frac{1}{3}}_{3,1}\mathfrak{X}M} \left\|\mathcal{U}^{4s-2\sigma}\right\|_{B^{1+\frac{1}{3}}_{3,1}\mathfrak{X}M} \\
&\lesssim \int_I \mathrm{d}t\ \ |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{B^{\frac{1}{3}}_{3,1}} \int_{0+}^s \mathrm{d}\sigma \left(\frac{1}{\sigma}\right)^{\frac{1}{2}} \left(\frac{1}{2s-\sigma}\right)^{\frac{1}{2}} A\left(t, 2\sigma\right) A\left(t, 4s-2\sigma\right)
\end{aligned}$$

$$\overset{\sigma\mapsto s\sigma}{=} \int_I \mathrm{d}t \ |\eta(t)| \, \|\mathcal{U}(t)\|_{B^{\frac{1}{3}}_{3,1}} \int_{0+}^{1} \mathrm{d}\sigma \left(\frac{1}{\sigma}\right)^{\frac{1}{2}} \left(\frac{1}{2-\sigma}\right)^{\frac{1}{2}} A(t, 2s\sigma)\, A(t, 4s-2s\sigma)$$
$$\lesssim \int_I \mathrm{d}t \ |\eta(t)| \, \|\mathcal{U}(t)\|_{B^{\frac{1}{3}}_{3,1}} \ \|A(t,\sigma)\|^2_{L^\infty_{\sigma\le 4s}}$$
$$\lesssim \|\mathcal{U}\|_{L^3_t B^{\frac{1}{3}}_{3,1}} \left\| \|A(t,\sigma)\|_{L^\infty_{\sigma\le 4s}} \right\|^2_{L^3_t} \xrightarrow{s\downarrow 0} 0$$

□

Note that

$$\mathcal{N}(t,\sigma) = (\partial_\sigma - 3\Delta)\left(-\operatorname{div}\left(\mathcal{U}^{2\sigma}\otimes\mathcal{U}^{2\sigma}\right)^{\sigma}\right)$$
$$= -2\operatorname{div}\left(\Delta\mathcal{U}^{2\sigma}\otimes\mathcal{U}^{2\sigma}\right)^{\sigma} - 2\operatorname{div}\left(\mathcal{U}^{2\sigma}\otimes\Delta\mathcal{U}^{2\sigma}\right)^{\sigma} + 2\Delta\operatorname{div}\left(\mathcal{U}^{2\sigma}\otimes\mathcal{U}^{2\sigma}\right)^{\sigma}$$

Finally, we will show

$$\int_I \eta \left\langle\left\langle \mathcal{W}(s), \mathcal{U}^s \right\rangle\right\rangle = \int_I \mathrm{d}t \ \eta(t) \left\langle\left\langle \mathcal{W}(t,s), \mathcal{U}(t)^s \right\rangle\right\rangle \xrightarrow{s\downarrow 0} 0$$

Proof. Integrate by parts into 3 components:

$$\int_I \eta \left\langle\left\langle \mathcal{W}(s), \mathcal{U}^s \right\rangle\right\rangle = \int_I \mathrm{d}t \ \eta(t) \int_{0+}^{s} \mathrm{d}\sigma \left\langle\left\langle \mathcal{N}(t,\sigma)^{3(s-\sigma)}, \mathcal{U}(t)^s \right\rangle\right\rangle$$
$$= \int_I \mathrm{d}t \ \eta(t) \int_{0+}^{s} \mathrm{d}\sigma \left\langle\left\langle \mathcal{N}(t,\sigma), \mathcal{U}(t)^{4s-3\sigma} \right\rangle\right\rangle$$
$$= 2\int_I \eta \int_{0+}^{s} \mathrm{d}\sigma \left\langle\left\langle \Delta\mathcal{U}^{2\sigma}\otimes\mathcal{U}^{2\sigma}, \nabla\left(\mathcal{U}^{4s-2\sigma}\right) \right\rangle\right\rangle$$
$$+ 2\int_I \eta \int_{0+}^{s} \mathrm{d}\sigma \left\langle\left\langle \mathcal{U}^{2\sigma}\otimes\Delta\mathcal{U}^{2\sigma}, \nabla\left(\mathcal{U}^{4s-2\sigma}\right) \right\rangle\right\rangle$$
$$- 2\int_I \eta \int_{0+}^{s} \mathrm{d}\sigma \left\langle\left\langle \mathcal{U}^{2\sigma}\otimes\mathcal{U}^{2\sigma}, \nabla\left(\Delta\mathcal{U}^{4s-2\sigma}\right) \right\rangle\right\rangle$$

Note that for the third component, we used some properties from Fact 13 to move the Laplacian. It also explains our choice of $\mathcal{W}$.

We now use Penrose notation to estimate the 3 components. To clean up the notation, we only focus on the integral on M, with the other integrals $2\int_I \eta \int_{0+}^s \mathrm{d}\sigma\,(\cdot)$ in variables t and σ implicitly understood. We also use **schematic identities** for linear combinations of similar-looking tensor terms where we do not care how the indices contract (recall Equation (4.1)). By the error estimates above, all the terms with R or ν will be negligible as $s \downarrow 0$, and interchanging derivatives will be a free action. We write $\approx$ to throw the negligible error terms away. Also, when we write $(\nabla_j \mathcal{U}_l)^{4s-2\sigma}$, we mean the heat flow is applied to $\mathcal{U}$, not $\nabla\mathcal{U}$ (which is not possible anyway).

First component:

$$
\begin{aligned}
&\int_M \left\langle \Delta\mathcal{U}^{2\sigma} \otimes \mathcal{U}^{2\sigma}, \nabla\left(\mathcal{U}^{4s-2\sigma}\right)\right\rangle \\
=&\cancel{\int_M R * \mathcal{U}^{2\sigma} * \mathcal{U}^{2\sigma} * \nabla\left(\mathcal{U}^{4s-2\sigma}\right)} + \int_M \left(\nabla_i \nabla^i \mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \mathcal{U}_l\right)^{4s-2\sigma} \\
\approx&\cancel{\int_{\partial M} \left(\nu_i \nabla^i \mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \mathcal{U}_l\right)^{4s-2\sigma}} - \cancel{\int_M \left(\nabla^i \mathcal{U}^j\right)^{2\sigma} \left(\nabla_i \mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \mathcal{U}_l\right)^{4s-2\sigma}} \\
&- \int_M \left(\nabla^i \mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\nabla_i \nabla_j \mathcal{U}_l\right)^{4s-2\sigma}
\end{aligned}
$$

Second component:

$$
\begin{aligned}
&\int_M \left\langle \mathcal{U}^{2\sigma} \otimes \Delta\mathcal{U}^{2\sigma}, \nabla\left(\mathcal{U}^{4s-2\sigma}\right)\right\rangle \\
=&\cancel{\int_M \mathcal{U}^{2\sigma} * R * \mathcal{U}^{2\sigma} * \nabla\left(\mathcal{U}^{4s-2\sigma}\right)} + \int_M \left(\mathcal{U}^j\right)^{2\sigma} \left(\nabla_i \nabla^i \mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \mathcal{U}_l\right)^{4s-2\sigma} \\
\approx&\cancel{\int_{\partial M} \left(\mathcal{U}^j\right)^{2\sigma} \left(\nu_i \nabla^i \mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \mathcal{U}_l\right)^{4s-2\sigma}} - \cancel{\int_M \left(\nabla_i \mathcal{U}^j\right)^{2\sigma} \left(\nabla^i \mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \mathcal{U}_l\right)^{4s-2\sigma}} \\
&- \int_M \left(\mathcal{U}^j\right)^{2\sigma} \left(\nabla^i \mathcal{U}^l\right)^{2\sigma} \left(\nabla_i \nabla_j \mathcal{U}_l\right)^{4s-2\sigma}
\end{aligned}
$$

For the third component, note $\nabla (R * U) = \nabla R * U + R * \nabla U$

$$
\begin{aligned}
& -\int_M \left\langle \mathcal{U}^{2\sigma} \otimes \mathcal{U}^{2\sigma}, \nabla \left(\Delta \mathcal{U}^{4s-2\sigma}\right)\right\rangle \\
= & -\cancel{\int_M \mathcal{U}^{2\sigma} * \mathcal{U}^{2\sigma} * \nabla \left(R * \mathcal{U}^{4s-2\sigma}\right)} - \int_M \left(\mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\nabla_j \nabla^i \nabla_i \mathcal{U}_l\right)^{4s-2\sigma} \\
\approx & -\int_M \left(\mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\cancel{R * \nabla \left(\mathcal{U}^{4s-2\sigma}\right)} + \nabla^i \nabla_j \nabla_i \mathcal{U}_l^{4s-2\sigma}\right) \\
\approx & -\int_M \left(\mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\cancel{\nabla \left(R * \mathcal{U}^{4s-2\sigma}\right)} + \nabla^i \nabla_i \nabla_j \mathcal{U}_l^{4s-2\sigma}\right) \\
\approx & \cancel{-\int_{\partial M} \left(\mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\nu^i \nabla_i \nabla_j \mathcal{U}_l\right)^{4s-2\sigma}} + \int_M \left(\nabla^i \mathcal{U}^j\right)^{2\sigma} \left(\mathcal{U}^l\right)^{2\sigma} \left(\nabla_i \nabla_j \mathcal{U}_l\right)^{4s-2\sigma} \\
& + \int_M \left(\mathcal{U}^j\right)^{2\sigma} \left(\nabla^i \mathcal{U}^l\right)^{2\sigma} \left(\nabla_i \nabla_j \mathcal{U}_l\right)^{4s-2\sigma}
\end{aligned}
$$

Add them up, and we get 0 as $2 \int_I \eta \int_{0+}^s \mathrm{d}\sigma \, (\cdot) \xrightarrow{s \downarrow 0} 0$. □

So we are done and the rest of Part I is to develop the tools we have borrowed for the proof.

CHAPTER 5

Functional analysis

5.1 Common tools

We note a useful inequality:

Theorem 22 (Ehrling's inequality)**.** *Let $X, Y, \widetilde{X}$ be (real/complex) Banach spaces such that X is reflexive and $X \hookrightarrow \widetilde{X}$ is a continuous injection. Let $T : X \to Y$ be a linear compact operator. Then $\forall \varepsilon > 0, \exists C_\varepsilon > 0$:*

$$\|Tx\|_Y \leq \varepsilon \|x\|_X + C_\varepsilon \|x\|_{\widetilde{X}} \ \forall x \in X$$

Remark. Usually, X is some higher-regularity space than $\widetilde{X}$ (e.g. H^1 and L^2). The inequality is useful when the higher-regularity norm is expensive. We will need this for the L^p-analyticity of the heat flow (Theorem 73).

Proof. Proof by contradiction: Assume $\varepsilon > 0$ and there is $(x_j)_{j\in\mathbb{N}}$ such that $\|x_j\|_X = 1$ and $\|Tx_j\|_Y > \varepsilon + j\|x_j\|_{\widetilde{X}}$. Since X is reflexive, by Banach-Alaoglu and PTAS, WLOG assume $x_j \xrightharpoonup{X} x_\infty$. Then $Tx_j \xrightharpoonup{Y} Tx_\infty$ and $x_j \xrightharpoonup{\widetilde{X}} x_\infty$. As T is compact, PTAS, WLOG $Tx_j \to Tx_\infty$. So $\|Tx_\infty\|_Y \geq \limsup_{j\to\infty} \left(\varepsilon + j\|x_j\|_{\widetilde{X}}\right) > 0$ and $x_j \xrightarrow{\widetilde{X}} 0$. Then $x_j \xrightharpoonup{\widetilde{X}} 0$ and $x_\infty = 0$, contradicting $\|Tx_\infty\|_Y > 0$. □

Definition 23 (Banach-valued holomorphic functions)**.** Let $\Omega \subset \mathbb{C}$ be an open set and X be a complex Banach space. Then a function $f : \Omega \to X$ is said to be **holomorphic** (or **analytic**) when $\forall z \in \Omega : f'(z) := \lim_{|h|\to 0} \frac{f(z+h)-f(z)}{h}$ exists. The words "holomorphic" and “analytic” are mostly interchangeable, but “analytic” stresses the existence of power series expansion and can also describe functions on $\mathbb{R}$ for which analytic continuation into the complex plane exists.

Theorem 24 (Identity theorem)**.** *Let X be a complex Banach space and $X_0 \leq X$ closed. Let $\Omega \subset \mathbb{C}$ be connected, open and $f : \Omega \to X$ holomorphic. Assume there is a sequence $(z_j)_{j\in\mathbb{N}}$ such that $z_j \to z \in \Omega$ and $f(z_j) \in X_0$ $\forall j$. Then $f(\Omega) \subset X_0$.*

> *Proof.* Let $\Lambda \in X^*$ such that $\Lambda(X_0) = 0$. Reduce this to the scalar version in complex analysis. □

In fact, many theorems from scalar complex analysis similarly carry over via linear functionals (cf. [Rud91, Theorem 3.31]).

5.2 Interpolation theory

We will quickly review the theory of complex and real interpolation, and state the abstract Stein interpolation theorem. Interpolation theory can be seen as vast generalizations of the Marcinkiewicz and Riesz-Thorin interpolation theorems.

Definition 25. An **interpolation couple** of (real/complex) Banach spaces is a pair (X_0, X_1) of Banach spaces with a Hausdorff TVS $\mathcal{X}$ such that $X_0 \hookrightarrow \mathcal{X}, X_1 \hookrightarrow \mathcal{X}$ are continuous injections. Then $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces under the norms

$$\|x\|_{X_0\cap X_1} = \max\left(\|x\|_{X_0}, \|x\|_{X_1}\right) \text{ and } \|x\|_{X_0+X_1} = \inf_{x=x_0+x_1, x_j\in X_j} \|x_0\|_{X_0} + \|x_1\|_{X_1}$$

Let (Y_0, Y_1) be another interpolation couple. We say $T : (X_0, X_1) \to (Y_0, Y_1)$ is a **morphism** when $T \in \mathcal{L}(X_0 + X_1, Y_0 + Y_1)$ and $T \in \mathcal{L}(X_j, Y_j)$ for $j = 0, 1$ under domain restriction. That implies $T \in \mathcal{L}(X_0 \cap X_1, Y_0 \cap Y_1)$ and we write $T \in \mathcal{L}((X_0, X_1), (Y_0, Y_1))$. We also write $\mathcal{L}((X_0, X_1)) = \mathcal{L}((X_0, X_1), (X_0, X_1))$.

Let $P \in \mathcal{L}((X_0, X_1))$ such that $P^2 = P$. Then we call P a **projection** on the interpolation couple (X_0, X_1).

Definition 26. Let (X_0, X_1) be an interpolation couple of (real/complex) Banach spaces. Then define the J-functional:

$$\begin{aligned} J: \quad (0,\infty) \times X_0 \cap X_1 &\longrightarrow \mathbb{R} \\ (t,x) &\longmapsto \|x\|_{X_0} + t\,\|x\|_{X_1} \end{aligned}$$

For $\theta \in (0,1), q \in [1,\infty]$, define the **real interpolation space**

$$(X_0, X_1)_{\theta,q} = \left\{ \sum_{j\in\mathbb{Z}} u_j : u_j \in X_0 \cap X_1, \left(2^{-j\theta} J(2^j, u_j)\right)_{j\in\mathbb{Z}} \in l^q_j(\mathbb{Z}) \right\}$$

which is Banach under the norm $\|x\|_{(X_0,X_1)_{\theta,q}} = \inf_{x=\sum_{j\in\mathbb{Z}} u_j} \left\|2^{-j\theta} J(2^j, u_j)\right\|_{l^q_j}$. Note that $\sum_{j\in\mathbb{Z}} u_j$ denotes a series that converges in $X_0 + X_1$.

- When $q \in [1,\infty]$ and $x \in X_0 \cap X_1$, note that $\forall j \in \mathbb{Z} : x = \sum_{k\in\mathbb{Z}} \delta_{kj} x$ and

  $$\|x\|_{(X_0,X_1)_{\theta,q}} \le \inf_{j\in\mathbb{Z}} \left|2^{-j\theta} J(2^j, x)\right| = \inf_{j\in\mathbb{Z}} \left|2^{-j\theta} \|x\|_{X_0} + 2^{j(1-\theta)} \|x\|_{X_1}\right| \sim_{\neg\theta,\neg q} \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta}$$

  The last estimate comes from AM-GM and shifting j so that $\|x\|_{X_0} \sim 2^j \|x\|_{X_1}$. Note that the implied constants do not depend on θ and q.

- By considering the finite partial sums $\sum_{|j|<j_0} u_j$, we conclude that $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta,q}$ when $q \in [1,\infty)$.

- Let (Y_0, Y_1) be another interpolation couple and $T \in \mathcal{L}((X_0, X_1), (Y_0, Y_1))$. For $\theta \in (0,1), q \in [1,\infty]$, define $X_{\theta,q} = (X_0, X_1)_{\theta,q}, Y_{\theta,q} = (Y_0, Y_1)_{\theta,q}$. Then $T \in \mathcal{L}(X_{\theta,q}, Y_{\theta,q})$ and

  $$\|T\|_{\mathcal{L}(X_{\theta,q},Y_{\theta,q})} \lesssim_{\neg\theta,\neg q,\neg T} \|T\|_{\mathcal{L}(X_0,Y_0)}^{1-\theta} \|T\|_{\mathcal{L}(X_1,Y_1)}^{\theta}$$

where the implied constant does not depend on θ and q. This can be proved by a simple shifting argument.

- If P is a projection on (X_0, X_1) then $(PX_0, PX_1)_{\theta,q} = P\,(X_0, X_1)_{\theta,q}$.

Remark. There is also an equivalent characterization by the K-functional, which we shall omit. This theory can also be extended to quasi-Banach spaces. We refer to [BL76; Tri10] for more details.

Definition 27. Let (X_0, X_1) be an interpolation couple of complex Banach spaces.

Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. We then define the Banach space of vector-valued holomorphic/analytic functions on the strip:

$$\mathcal{F}_{X_0,X_1} = \{f \in C^0(\overline{\Omega} \to X_0 + X_1) : f \text{ holomorphic in } \Omega,\ \|f(it)\|_{X_0} + \|f(1+it)\|_{X_1} \xrightarrow{|t|\to\infty} 0\}$$

with the norm $\|f\|_{\mathcal{F}_{X_0,X_1}} = \max\left(\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right)$.

For $\theta \in [0,1]$, define the **complex interpolation space** $[X_0, X_1]_\theta = \{f(\theta) : f \in \mathcal{F}_{X_0,X_1}\}$, which is Banach under the norm

$$\|x\|_{[X_0,X_1]_\theta} = \inf_{\substack{f\in\mathcal{F}_{X_0,X_1}\\ f(\theta)=x}} \|f\|_{\mathcal{F}_{X_0,X_1}}$$

- When $x \in X_0 \cap X_1\backslash\{0\}$, $\theta \in [0,1]$, $\varepsilon > 0$, define $f_\varepsilon(z) = e^{\varepsilon(z^2-\theta^2)}\frac{x}{\|x\|_{X_0}^{1-z}\|x\|_{X_1}^{z}}$. By the freedom in choosing ε, we conclude

$$\begin{aligned}\|x\|_{[X_0,X_1]_\theta} &\le \inf_{\varepsilon>0} \|f_\varepsilon\|_{\mathcal{F}_{X_0,X_1}} \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta} \\ &\le \inf_{\varepsilon>0} \max\left(e^{\varepsilon(1-\theta^2)}, e^{-\epsilon\theta^2}\right) \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta} = \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta}\end{aligned}$$

- When $\theta \in [0,1]$, by Poisson summation and Fourier series, we can prove that

$$\mathcal{F}^0_{X_0,X_1} = \{e^{Cz^2}\sum_{j=1}^{N} e^{\lambda_j z}x_j : N \in \mathbb{N}, C > 0, \lambda_j \in \mathbb{R}, x_j \in X_0 \cap X_1\}$$

  is dense in $\mathcal{F}_{X_0,X_1}$ (cf. [BL76, Lemma 4.2.3]). This implies $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$. There is a simple extension of the above density result. Let U be dense in $X_0 \cap X_1$ and define $A(\Omega) = \{\phi \in C^0(\overline{\Omega} \to \mathbb{C}) : \phi \text{ holomorphic in } \Omega\}$. Then

$$\mathcal{F}^U_{X_0,X_1} = \{e^{Cz^2}\sum_{j=1}^{N} \phi_j(z)u_j : N \in \mathbb{N}, C > 0, \phi_j \in A(\Omega), u_j \in U\}$$

  is dense in $\mathcal{F}_{X_0,X_1}$. This will lead to the abstract Stein interpolation theorem.

- Let (Y_0, Y_1) be another interpolation couple and $T \in \mathcal{L}\left((X_0,X_1),(Y_0,Y_1)\right)$. Then for $\theta \in [0,1]$, almost by the definitions, we conclude

$$\|T\|_{\mathcal{L}([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)} \le \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1,Y_1)}$$

- If P is a projection on (X_0, X_1) then $[PX_0, PX_1]_\theta = P[X_0, X_1]_\theta$

Remark. A keen reader would notice that we use square brackets for complex interpolation, and parentheses for real interpolation. One reason is that the real interpolation methods easily extend to quasi-Banach spaces, while the complex interpolation method does not. There is a version of complex interpolation for special quasi-Banach spaces, which is denoted by parentheses (cf. [Tri10, Section 2.4.4]), but we shall omit it for simplicity.

Blackbox 28 (Abstract Stein interpolation)**.** *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples of complex Banach spaces and U dense in $X_0 \cap X_1$. Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and $(T(z))_{z\in\overline{\Omega}}$ be a family of linear mappings $T(z) : U \to Y_0 + Y_1$ such that*

1. *$\forall u \in U : \left(\overline{\Omega} \to Y_0 + Y_1, z \mapsto T(z)u\right)$ is continuous, bounded and analytic in Ω.*

2. *For* $j = 0, 1$ *and* $u \in U$*:* $(\mathbb{R} \to Y_j, t \mapsto T(j+it)u)$ *is continuous and bounded by* $M_j \|u\|_{X_j}$ *for some* $M_j > 0$*.*

Then for $\theta \in [0,1]$*, we can conclude*

$$\|T(\theta)u\|_{[Y_0,Y_1]_\theta} \leq M_0^{1-\theta} M_1^\theta \|u\|_{[X_0,X_1]_\theta} \ \forall u \in U$$

Consequently, by unique extension, we have $T(\theta) \in \mathcal{L}([X_0,X_1]_\theta, [Y_0,Y_1]_\theta)$*.*

Proof. See [Voi92], which is a very short read. □

Remark. We will only use Stein interpolation in Section 5.3.

5.3 Stein extrapolation of analyticity of semigroups

We are inspired by [Fac15, Theorem 3.1.1] (Stein extrapolation) and [Fac15, Theorem 3.1.10] (Kato-Beurling extrapolation), and wish to create variants for our own use. We will focus on Stein extrapolation, since it is simpler to deal with.

There exists a subtle, but very important criterion to establish analyticity/holomorphicity:

Blackbox 29 (Holo on total)**.** *Let* $\Omega \subset \mathbb{C}$ *be open and* X *complex Banach. Let* $f : \Omega \to X$ *be a function. Assume* $N \leq X^*$ *is* ***total*** *(separating points) and* f *is locally bounded.*

Then f *is analytic iff* Λf *is analytic* $\forall \Lambda \in N$*.*

Proof. This is a consequence of Krein-Smulian and the Vitali holomorphic convergence theorem, and we refer to [Are+11, Theorem A.7]. □

Remark. It will quickly become obvious how crucial this criterion is for the rest of the thesis. Let us briefly note that an improvement has just been discovered by Arendt *et al.* [ABK19] (the author thanks Stephan Fackler for bringing this news).

Corollary 30 (Inheritance of analyticity)**.** *Let $\Omega \subset \mathbb{C}$ be open and X, Y be complex Banach spaces where $j : X \hookrightarrow Y$ is a continuous injection. Let $f : \Omega \to X$ be locally bounded. Then f is analytic iff $j \circ f$ is analytic.*

> *Proof.* $\mathrm{Im}(j^*)$ is weak*-dense, therefore total. ☐

Corollary 31 (Evaluation on dense set)**.** *Let X, Y be complex Banach spaces with $X_0 \leq X$. Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathcal{L}(X, Y)$ be a function. Assume $X_0 \leq X$ is weakly dense and f is locally bounded.*

Then f is analytic $\iff \forall x_0 \in X_0,\ f(\cdot)x_0 : \Omega \to Y$ is analytic.

> *Proof.* Consider $N_{X_0} = \mathrm{span}\{y^* \circ \mathrm{ev}_{x_0} : x_0 \in X_0, y^* \in Y^*\} \leq \mathcal{L}(X, Y)^*$. It is total as X_0 is weakly dense. Use Blackbox 29. ☐

5.3.1 Semigroup definitions

As mentioned before, we assume the reader is familiar with basic elements of functional analysis, including semigroup theory as covered in [Tay11a, Appendix A.9].

Unfortunately, definitions vary depending on the authors, so we need to be careful about which ones we are using.

Definition 32. For $\delta \in (0, \pi]$, define $\Sigma_\delta^+ = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \delta\}$, $\Sigma_\delta^- = -\Sigma_\delta^+$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also define $\Sigma_0^+ = (0, \infty)$ and $\Sigma_0^- = -\Sigma_0^+$.

Let X be a complex Banach space.

$(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is called:

- a **semigroup** when $T : [0, \infty) \to \mathcal{L}(X)$ is a monoid homomorphism ($T(0) = 1, T(t_1 + t_2) = T(t_1)T(t_2)$)
- **degenerate** when $T : (0, \infty) \to \mathcal{L}(X)$ is continuous in the SOT (**strong operator topology**).

- **immediately norm-continuous** when $T : (0, \infty) \to \mathcal{L}(X)$ is norm-continuous.

- C_0 (**strongly continuous**) when $T : [0, \infty) \to \mathcal{L}(X)$ is continuous in the SOT.

- **bounded** when $T([0, \infty))$ is bounded in $\mathcal{L}(X)$, and **locally bounded** when $T(K)$ is bounded $\forall K \subset [0, \infty)$ bounded. (so C_0 implies local boundedness by Banach-Steinhaus, and the semigroup property implies we just need to test $K \subset [0, 1)$)

$(T(z))_{z \in \Sigma_\delta^+ \cup \{0\}} \subset \mathcal{L}(X)$ is called

- a **semigroup** when $T : \Sigma_\delta^+ \cup \{0\} \to \mathcal{L}(X)$ is a monoid homomorphism.

- C_0 when $\forall \delta' \in (0, \delta), T : \Sigma_{\delta'}^+ \cup \{0\} \to \mathcal{L}(X)$ is continuous in the SOT.

- **bounded** when $T\left(\Sigma_{\delta'}^+\right)$ is bounded $\forall \delta' \in (0, \delta)$ and **locally bounded** when $T(K)$ is bounded $\forall K \subset \Sigma_{\delta'}^+$ bounded. (so C_0 implies local boundedness, and the semigroup property implies we just need to test $K \subset \mathbb{D} \cap \Sigma_{\delta'}^+$)

- **analytic** when $T : \Sigma_\delta^+ \to \mathcal{L}(X)$ is analytic

We say $(T(t))_{t \geq 0}$ is **analytic** of angle $\delta \in (0, \frac{\pi}{2}]$ if there is an extension $(T(z))_{z \in \Sigma_\delta^+ \cup \{0\}} \subset \mathcal{L}(X)$ which is analytic and locally bounded. If furthermore $(T(z))_{z \in \Sigma_\delta^+ \cup \{0\}}$ is bounded, we say $(T(t))_{t \geq 0}$ is **boundedly analytic** of angle δ.

Remark. A subtle problem is that when $(T(t))_{t \geq 0}$ is bounded and analytic, we cannot conclude $(T(t))_{t \geq 0}$ is boundedly analytic (cf. [Are+11, Definition 3.7.3]).

Blackbox 33. *If $(T(t))_{t \geq 0}$ is a C_0 semigroup which is (boundedly) analytic of angle $\delta \in (0, \frac{\pi}{2}]$, then $(T(z))_{z \in \Sigma_\delta^+ \cup \{0\}}$ is a C_0, (bounded) semigroup.*

Proof. The semigroup property comes from the identity theorem, and C_0 comes from the Vitali holomorphic convergence theorem. We refer to [Are+11, Proposition 3.7.2].

□

Theorem 34 (Sobolev tower). *Let $(e^{tA})_{t\geq 0}$ be a C_0 semigroup on a (real/complex) Banach space X with generator A (implying A is closed and densely defined). Then $\forall m \in \mathbb{N}_1$, $D(A^m)$ is a Banach space under the norm $\|x\|_{D(A^m)} = \|x\|_X + \sum_{k=1}^m \left\|A^k x\right\|_X$, and $D(A^m)$ is dense in X.*

As e^{tA} and A commute on $D(A)$, we conclude that $\left(e^{tA}\right)_{t\geq 0}$, after domain restriction, is also a C_0 semigroup on $D(A^m)$ and $\left\|e^{tA}\right\|_{\mathcal{L}(D(A^m))} \leq \left\|e^{tA}\right\|_{\mathcal{L}(X)}$ $\forall t \geq 0$.

Lastly, if X is a complex Banach space and $\left(e^{tA}\right)_{t\geq 0}$ is (boundedly) analytic on X, $\left(e^{tA}\right)_{t\geq 0}$ is also (boundedly) analytic on $D(A^m)$ after domain restriction.

Proof. Most are just the basics of semigroup theory (cf. [Tay11a, Appendix A.9]). We only prove the last assertion. All we need is commutativity: if $\left(e^{tA}\right)_{t\geq 0}$ is extended to $(e^{zA})_{z\in\Sigma_\delta^+\cup\{0\}}$, we want to show $e^{zA}A = Ae^{zA}$ on $D(A)$.

By Blackbox 33, $(e^{zA})_{z\in\Sigma_\delta^+\cup\{0\}}$ is a C_0 semigroup. Therefore $\forall x \in D(A), \forall z \in \Sigma_\delta^+$:

$$e^{zA}Ax = e^{zA}\left(X\text{-}\lim_{t\downarrow 0}\frac{e^{tA}-1}{t}x\right) = X\text{-}\lim_{t\downarrow 0} e^{zA}\frac{e^{tA}-1}{t}x = X\text{-}\lim_{t\downarrow 0}\frac{e^{tA}-1}{t}e^{zA}x$$

The last term implies $e^{zA}x \in D\left(A\right)$ and $e^{zA}Ax = Ae^{zA}x$. Then use Corollary 30 and Corollary 31 to get analyticity. □

5.3.2 Simple extrapolation (with core)

Lemma 35. *Let U, X be complex Banach spaces and $U \hookrightarrow X$ be a continuous injection with dense image.*

1. *Let $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be locally bounded and $T(t)U \leq U$ $\forall t \geq 0$. Assume $(T(t))_{t\geq 0}$ is a C_0 semigroup on U. Then $(T(t))_{t\geq 0}$ on X is also a C_0 semigroup.*

2. *Let $(T(z))_{z\in\Sigma_\delta^+\cup\{0\}} \subset \mathcal{L}(X)$ (where $\delta \in (0, \frac{\pi}{2}]$) be locally bounded and $T(z)U \leq U$ $\forall z \in \Sigma_\delta^+$. Assume $(T(z))_{z\in\Sigma_\delta^+\cup\{0\}}$ is a C_0, analytic semigroup on U. Then $(T(z))_{z\in\Sigma_\delta^+\cup\{0\}}$ on X is also a C_0, analytic semigroup.*

Remark. The assumption of local boundedness on X is important. We will also use this result in Section 8.3 to establish the $W^{1,p}$-analyticity of the heat flow.

Proof. The semigroup property comes from the density of U in X.

To get C_0 on X, use the local boundedness on X and dense convergence (Lemma 11).

For analyticity in (2), use Corollary 31. □

Lemma 36 (Core)**.** *Let A be an unbounded linear operator on a (real/complex) Banach space X and $E \leq D(A)$. E is called a **core** when E is dense in $\left(D(A), \|\cdot\|_{D(A)}\right)$.*

If A is the generator of a C_0 semigroup on X, E is dense in X and $e^{tA}E \leq E$, then E is a core.

Proof. Let $x \in D(A)$. Then there is $(x_j)_{j\in\mathbb{N}}$ in E such that $x_j \xrightarrow{X} x$. It is trivial to check

$$\frac{1}{t}\int_0^t e^{sA}x_j \, \mathrm{d}s \xrightarrow[j\to\infty]{\|\cdot\|_{D(A)}} \frac{1}{t}\int_0^t e^{sA}x \, \mathrm{d}s \xrightarrow[t\downarrow 0]{\|\cdot\|_{D(A)}} x$$

as $\left(e^{sA}\big|_{D(A)}\right)_{s\geq 0}$ is $\|\cdot\|_{D(A)}$-continuous. Note that $\int_0^t e^{sA}x_j \, \mathrm{d}s$ is in the $\|\cdot\|_{D(A)}$-closure of E by the Riemann integral. □

Theorem 37 (Simple extrapolation with core)**.** *Let (X_0, X_1) be an interpolation couple of complex Banach spaces and $X_\theta = [X_0, X_1]_\theta$ for $\theta \in (0,1]$.*

Let $(T(t))_{t\geq 0} \subset \mathcal{L}\left((X_0, X_1)\right)$. Assume that on X_0, $(T(t))_{t\geq 0}$ is bounded.

Assume that on X_1, $(T(t))_{t\geq 0}$ is a C_0 semigroup, boundedly analytic of angle $\delta \in (0, \frac{\pi}{2}]$ with generator A_1.

Assume $\exists m \in \mathbb{N}_1 :$ $\left(D(A_1^m), \|\cdot\|_{D(A_1^m)}\right) \hookrightarrow \left(X_0 \cap X_1, \|\cdot\|_{X_0\cap X_1}\right) \hookrightarrow X_0$ are continuous injections with dense images.

Then on X_θ, $(T(t))_{t\geq 0}$ is a C_0 semigroup, and boundedly analytic of angle $\theta\delta$.

Remark. The existence of a convenient core like $D(A_1^m)$ is usually a trivial consequence of Sobolev embedding. We can replace bounded analyticity on X_1 and X_θ with analyticity, and boundedness on X_0 with local boundedness via the usual rescaling argument ($\forall \delta' \in (0, \delta) \subset (0, \frac{\pi}{2}), \exists C_{\delta'} > 0 : \left\| e^{-C_{\delta'} z} T(z) \right\|_{\mathcal{L}(X_1)} \lesssim_{\delta'} 1 \; \forall z \in \Sigma_{\delta'}^+$).

The existence of a core allows conditions on X_0 and X_1 to be more general than those in [Fac15, Theorem 3.1.1] (which requires immediate norm-continuity on X_0), and actually be equivalent to those in [Fac15, Theorem 3.1.10] (though Kato-Beurling covers more than just complex interpolation). Once again, the assumption of (local) boundedness on X_0 is important.

We will use this result to establish the L^p-analyticity of the heat flow in Section 8.2.

Proof. Let $U = D(A_1^m)$. Then U is Banach as A_1 is closed. Obviously $U \hookrightarrow X_\theta$ is a continuous injection with dense image, and $(T(z))_{z \in \Sigma_\delta^+ \cup \{0\}}$ is a C_0, bounded, analytic semigroup on U (via Sobolev tower).

By Lemma 35, $(T(t))_{t \geq 0}$ is a C_0, bounded semigroup on X_0. Also by Lemma 35, to get the desired conclusion, we only need to show $(T(z))_{z \in \Sigma_{\theta\delta}^+ \cup \{0\}}$ is locally bounded in $\mathcal{L}(X_\theta)$.

Fix $\delta' \in (0, \delta)$. We use abstract Stein interpolation. Define the strip $\Omega = \{0 < \mathrm{Re} < 1\}$. Let $\alpha \in (-\delta', \delta')$, $\rho > 0, u \in U$ and

$$L(z) = T(\rho e^{i\alpha z}) u \; \forall z \in \overline{\Omega}$$

Note that $U \leq X_0 \cap X_1$ is dense. We check the other conditions for interpolation:

- As $U \hookrightarrow X_0$ and $U \hookrightarrow X_1$ are continuous, $(\overline{\Omega} \to X_0 + X_1, z \mapsto L(z)u)$ is continuous, bounded on $\overline{\Omega}$ and analytic on Ω (as $L(z)u \in X_1 \hookrightarrow X_0 + X_1$).
- For $j = 0, 1$ $(\mathbb{R} \to X_j, s \mapsto L(j + is)u)$ is
  - continuous since $U \hookrightarrow X_j$ is continuous.
  - bounded by $C_{j,T} \|u\|_{X_j}$ for some $C_{j,T} > 0$ since $(T(t))_{t \geq 0}$ is bounded on

X_0 and $(T(te^{i\alpha}))_{t\geq 0}$ is bounded on X_1.

Then by Stein interpolation, we conclude $\{T(\rho e^{i\theta\alpha}) : \rho > 0, \alpha \in (-\delta', \delta')\} = T(\Sigma^+_{\theta\delta'}) \subset \mathcal{L}(X_\theta)$ is bounded. □

5.3.3 Coreless version

There is an alternative version which we will not use, but is of independent interest:

Theorem 38 (Coreless extrapolation)**.** *Let (X_0, X_1) be an interpolation couple of complex Banach spaces and $X_\theta = [X_0, X_1]_\theta$ for $\theta \in (0, 1]$.*

Let $(T(t))_{t\geq 0} \subset \mathcal{L}((X_0, X_1))$ be a semigroup. Assume that on X_0, $(T(t))_{t\geq 0}$ is bounded and degenerate.

Assume that on X_1, $(T(t))_{t\geq 0}$ is a C_0 semigroup, boundedly analytic of angle $\delta \in (0, \frac{\pi}{2}]$ with generator A_1.

Then on X_θ, $(T(t))_{t\geq 0}$ is a C_0 semigroup, boundedly analytic of angle $\theta\delta$.

Remark. The differences with the previous version are underlined. Again, via rescaling we can replace bounded analyticity on X_1 and X_θ with analyticity, and boundedness on X_0 with local boundedness. The conditions on X_0 and X_1 are still a bit more general than those in [Fac15, Theorem 3.1.1], which requires immediate norm-continuity on X_0. In practice local boundedness on X_0 can usually come from global analysis, while degeneracy can come from Sobolev embedding and dense convergence (Lemma 11). Immediate norm-continuity is harder to establish.

Note that Theorem 38 is not as general as [Fac15, Theorem 3.1.10] (which removes the need for degeneracy and covers more than just complex interpolation), though it is markedly easier to prove.

Proof. By interpolation, $(T(t))_{t\geq 0}$ is a bounded semigroup on X_θ.

Let $U = X_0 \cap X_1$. Obviously $(T(t))_{t\geq 0}$ is a bounded semigroup on U.

Then observe that $\forall u \in U, \forall t, t_0 \geq 0$:

$$\begin{aligned}\|(T(t)-T(t_0))\,u\|_{X_\theta} &\leq \|(T(t)-T(t_0))\,u\|_{X_0}^{1-\theta}\,\|(T(t)-T(t_0))\,u\|_{X_1}^{\theta}\\ &\lesssim \|(T(t)-T(t_0))\,u\|_{X_1}^{\theta}\end{aligned}$$

Since $\theta \neq 0$, we have $T(t)u \xrightarrow[t\to t_0]{X_\theta} T(t_0)u$. As $(T(t))_{t\geq 0}$ is bounded on X_θ and U is dense in X_θ, we conclude $(T(t))_{t\geq 0}$ is C_0 on X_θ by dense convergence (Lemma 11).

Fix $\delta' \in (0, \delta)$. We use abstract Stein interpolation. Define the strip $\Omega = \{0 < \mathrm{Re} < 1\}$. Let $\alpha \in (-\delta', \delta')$, $\rho > 0, u \in U$ and

$$L(z) = T(\rho e^{i\alpha z})u \;\forall z \in \overline{\Omega}$$

Note that $U = X_0 \cap X_1$. We check the other conditions for interpolation:

- As $U \hookrightarrow X_0$ and $U \hookrightarrow X_1$ are continuous, $(\overline{\Omega} \to X_0 + X_1, z \mapsto L(z)u)$ is continuous, bounded on $\overline{\Omega}$ and analytic on Ω (as $L(z)u \in X_1 \hookrightarrow X_0 + X_1$).
- For $j = 0, 1$ $(\mathbb{R} \to X_j, s \mapsto L(j + is)u)$ is
  - continuous since $(T(t))_{t\geq 0}$ is degenerate on X_0 and $(T(te^{i\alpha}))_{t\geq 0}$ is C_0 on X_1.
  - bounded by $C_{j,T}\,\|u\|_{X_j}$ for some $C_{j,T} > 0$ since $(T(t))_{t\geq 0}$ is bounded on X_0 and $(T(te^{i\alpha}))_{t\geq 0}$ is bounded on X_1.

By Stein interpolation, $\{T(\rho e^{i\theta\alpha}) : \rho > 0, \alpha \in (-\delta', \delta')\} = T(\Sigma^+_{\theta\delta'}) \subset \mathcal{L}(X_\theta)$ is bounded.

Finally, we just need to show $(T(z))_{z\in\Sigma^+_{\theta\delta}\cup\{0\}}$ is analytic on X_θ. Let $u \in U$. Then

$$\left(\Sigma^+_\delta \to X_1 \hookrightarrow X_0 + X_1, z \mapsto T(z)u\right)$$

is analytic. Therefore $\left(\Sigma_{\theta\delta}^{+} \to X_\theta \hookrightarrow X_0 + X_1, z \mapsto T(z)u\right)$ is analytic. On the other hand, $\left(\Sigma_{\theta\delta}^{+} \to X_\theta, z \mapsto T(z)u\right)$ is locally bounded, so we can use Corollary 30 to conclude $\left(\Sigma_{\theta\delta}^{+} \to X_\theta, z \mapsto T(z)u\right)$ is analytic. As U is dense in X_θ, by corollary 31, we conclude $\left(\Sigma_{\theta\delta}^{+} \to \mathcal{L}\left(X_\theta\right), z \mapsto T(z)\right)$ is analytic. □

5.4 Sectorial operators

Recall that if $(T(t))_{t\geq 0}$ is a C_0 semigroup on a complex Banach space X, then it has a closed, densely defined generator A, and $T(t) = e^{tA}$ is exponentially bounded: $\left\|e^{tA}\right\| \lesssim_{\neg t} e^{Ct}$ for some $C > 0$. Then $\forall \zeta \in \{\mathrm{Re} > C\} : \zeta \in \rho(A)$ and

$$\frac{1}{\zeta - A} x = \int_0^\infty e^{-\zeta t} e^{tA} x \,\mathrm{d}t \;\forall x \in X$$

(cf. [Tay11a, Appendix A, Proposition 9.2])

This means that the resolvent $\frac{1}{\zeta - A}$ is the Laplace transform of the semigroup e^{tA}. This naturally leads to the question when we can perform the inverse Laplace transform, to recover the semigroup from the resolvent. This motivates the definition of sectorial operators, which includes the Laplacian.

Unfortunately, there are wildly different definitions currently in use by authors. The reader should study the definitions closely whenever they consult any literature on sectorial operators (e.g. [Lun95; Haa06; Are+11; Eng00]).

Definition 39. Let A be an unbounded operator on a complex Banach space X. For $\theta \in [0, \pi)$, we say A is

- **sectorial** of angle θ ($A \in \mathrm{Sect}(\theta)$) when $\begin{cases} \sigma(A) \subset \overline{\Sigma_\theta^-} \\ \forall \omega \in [0, \pi - \theta) : M(A, \omega) := \sup\limits_{\lambda \in \Sigma_\omega^+} \left\| \frac{\lambda}{\lambda - A} \right\| < \infty \end{cases}$
- **quasi-sectorial** when $\exists a \in \mathbb{R} : A - a$ is sectorial.
- **acutely sectorial** when $A \in \mathrm{Sect}(\theta)$ for some $\theta \in [0, \frac{\pi}{2})$

- **acutely quasi-sectorial** when $\exists a \in \mathbb{R} : A - a$ is acutely sectorial.

For $r > 0, \eta \in (\frac{\pi}{2}, \pi)$, we define the (counterclockwise-oriented) **Mellin curve**

$$\gamma_{r,\eta} = e^{i\eta}[r, \infty) \cup e^{-i\eta}[r, \infty) \cup re^{i[-\eta,\eta]}$$

Remark. Depending on the author, "sectorial" can mean any of those four, and that is not taking sign conventions into account (some authors want $-\Delta$ to be sectorial), as well as whether A should be densely defined. The term "quasi-sectorial" is taken from [Haa06].

In particular, letting the spectrum be in the left half-plane means we agree with [Eng00; Lun95] and disagree with [Are+11; McI86; Haa06]. This is simply a personal preference, of being able to say "the Laplacian is sectorial", or "generators of C_0 analytic semigroups are acutely sectorial". Also, for bounded holomorphic calculus, $e^{t\Delta}$ morally comes from $(e^{tz})_{z\in\sigma(\Delta)}$ which is bounded in the left half-plane.

In keeping with tradition, here is the usual visualization:

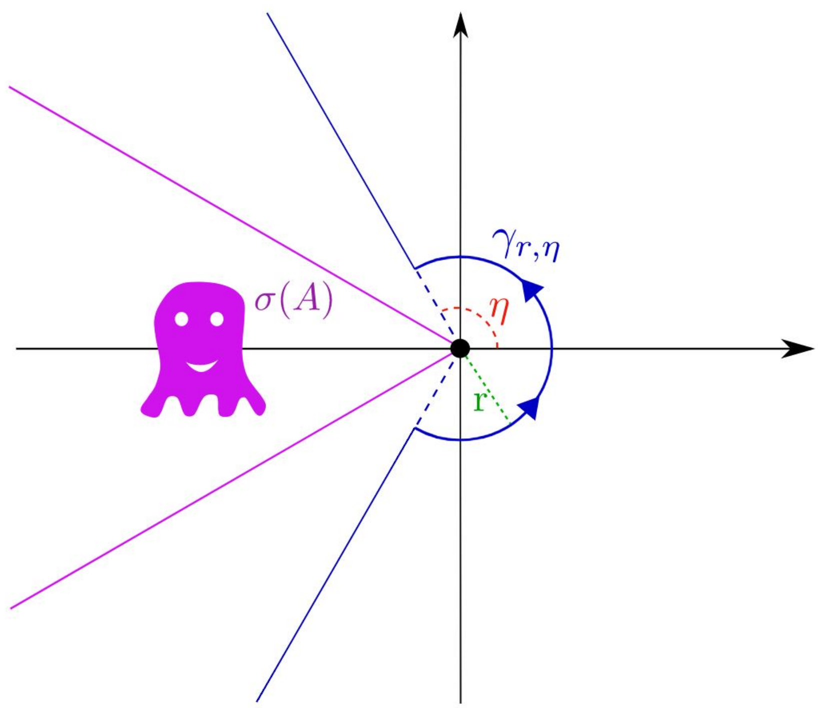

Figure 5.1: Acutely sectorial operators

Blackbox 40. *A generates a C_0, boundedly analytic semigroup on complex Banach space X if and only if A is densely defined and acutely sectorial.*

When that happens, $\exists\delta \in \left(0, \frac{\pi}{2}\right)$ and $\eta \in (\frac{\pi}{2}, \pi)$ such that $\left(e^{tA}\right)_{t\geq 0}$ extends to $\left(e^{\zeta A}\right)_{\zeta\in\Sigma_\delta^+\cup\{0\}}$ and

$$e^{\zeta A} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\zeta z}\frac{1}{z-A}\,dz \quad \forall\zeta \in \Sigma_\delta^+, \forall r > 0$$

Also $\forall t > 0, \forall k \in \mathbb{N}_1 : e^{tA}(X) \leq D(A^\infty)$, $\left\|A^k e^{tA}\right\| \lesssim_{\neg t,\neg k} \frac{k^k}{t^k}$ and $\partial_t^k(e^{tA}x) = A^k e^{tA}x\ \forall x \in X$.

Remark. This is the aforementioned inverse Laplace transform. The Mellin curve and the resolvent estimate in the definition of sectoriality ensure sufficient decay for the integral to make sense. As it is a complex line integral and the resolvent is analytic, the semigroup becomes analytic.

A trivial consequence is that $D(A^\infty)$ is dense in X and therefore a core.

When A is densely defined and acutely quasi-sectorial, a simple rescaling $e^{t(A-a)} = e^{-ta}e^{tA}$ implies $\left(e^{tA}\right)_{t\geq 0}$ is a C_0, analytic semigroup.

Proof. See [Eng00, Section II.4.a]. The curious figure $\frac{k^k}{t^k}$ comes from $A^k e^{tA} = \left(Ae^{\frac{t}{k}A}\right)^k$. □

Theorem 41 (Yosida's half-plane criterion)**.** *A is acutely quasi-sectorial if and only if $\exists C > 0$ such that*

- $\{\mathrm{Re} > C\} \subset \rho(A)$
- $\sup\limits_{\lambda\in\{\mathrm{Re}>C\}} \left\|\frac{\lambda}{\lambda-A}\right\| < \infty$

Remark. This is how the L^p-analyticity of the heat flow is traditionally established. Yet proving the resolvent estimate is nontrivial, as it is quite a refinement of elliptic estimates, so we choose not to do so. Interestingly, we will instead use this for the $B^{\frac{1}{3}}_{3,1}$-analyticity of the heat flow in Section 9.3, though that case is especially easy since we already have analyticity at the two endpoints L^3 and $W^{1,3}$.

Proof. We only need to prove $\Leftarrow$. Recall the proof of how $\rho(A)$ is open: $\forall\lambda \in \rho(A), B\left(\lambda, \left\|\frac{1}{\lambda - A}\right\|^{-1}\right) \subset \rho(A)$. Applying this allows us to open up $\{\mathrm{Re} > C\}$ and get $C + \Sigma^+_\eta \subset \rho(A)$ for some $\eta \in (\frac{\pi}{2}, \pi)$. By choosing η near $\frac{\pi}{2}$, the resolvent estimate is retained. □

Definition 42. Let A be an unbounded operator on a Hilbert space X. Then A is called

- **symmetric** when $\langle Ax, y\rangle = \langle x, Ay\rangle\ \forall x, y \in D(A)$, or equivalently, $A \subset A^*$ (where A and A^* are identified with their graphs).
- **self-adjoint** when $A = A^*$. This implies $\sigma(A) \subset \mathbb{R}$ (cf. [Tay11a, Appendix A, Proposition 8.5]).
- **dissipative** when $\mathrm{Re}\,\langle Ax, x\rangle \le 0\ \forall x \in D(A)$.

When A is dissipative, $\forall\lambda \in \{\mathrm{Re} > 0\}, \forall x \in D(A) : \mathrm{Re}\,\langle(\lambda - A)\,x, x\rangle \ge \mathrm{Re}\,\langle\lambda x, x\rangle$ so $\|(\lambda - A)\,x\| \ge \mathrm{Re}\,\lambda\,\|x\|$.

Recall how $\rho(A)$ is proved to be open: $\forall\lambda \in \rho(A), B\left(\lambda, \left\|\frac{1}{\lambda - A}\right\|^{-1}\right) \subset \rho(A)$. Consequently, if A is dissipative and $\exists\lambda_0 \in \{\mathrm{Re} > 0\} \cap \rho(A)$, we can conclude $\{\mathrm{Re} > 0\} \subset \rho(A)$.

Theorem 43 (Dissipative sectoriality)**.** *Assume X is a complex Hilbert space and A is an unbounded, self-adjoint, dissipative operator on X. Then A is acutely sectorial of angle* 0*.*

Remark. Though standard, this might be the most elegant theorem in the theory, and later on will instantly imply the L^2-analyticity of the heat flow in Section 8.1. The theorem can also be proved by Euclidean geometry. When X is separable, we can also use the spectral theorem for unbounded operators.

Proof. As A is self-adjoint, $\mathbb{C}\backslash\mathbb{R} \subset \rho(A)$. By dissipativity, we conclude $\sigma(A) \subset (-\infty, 0]$. Also by self-adjointness, $\operatorname{Re}\langle Ax, x\rangle = \langle Ax, x\rangle \leq 0 \; \forall x \in D(A)$.

Arbitrarily pick $\theta \in (\frac{\pi}{2}, \pi)$. We want to show $\left\|\frac{z}{z-A}\right\| \lesssim_\theta 1 \; \forall z \in \Sigma_\theta^+$.

Let $x \in X$ and $u = \frac{1}{z-A}x$. As $|\langle u, x\rangle| \leq \|u\|_X \|x\|_X$, we want to show $\|u\|_X^2 \lesssim_\theta \left|\frac{1}{z}\langle u, x\rangle\right|$. Note that

$$\frac{1}{z}\langle u, x\rangle = \frac{1}{z}\langle u, (z-A)u\rangle = \langle u, u\rangle - \frac{1}{z}\langle Au, u\rangle$$

WLOG assume $\|u\|_X = 1$. Then we want $1 \lesssim_\theta \left|1 - \frac{1}{z}\langle Au, u\rangle\right|$. Note that $-\langle Au, u\rangle \geq 0$ and $-\frac{1}{z}\langle Au, u\rangle \in \Sigma_\theta^+$. Then we are done since

$$\left|1 - \frac{1}{z}\langle Au, u\rangle\right| \geq \operatorname{dist}(0, 1 + \Sigma_\theta^+) > 0.$$

By Euclidean geometry, we can even calculate $\operatorname{dist}(0, 1 + \Sigma_\theta^+)$. We will not need it though. □

CHAPTER 6

Scalar function spaces

Throughout this chapter, we work with complex-valued functions.

6.1 On $\mathbb{R}^n$

Definition 44. Here we recall the various (inhomogeneous) function spaces which are particularly suitable for interpolation. They are defined as subspaces of $\mathcal{S}'(\mathbb{R}^n)$ with certain norms being finite:

1. **Lebesgue-Sobolev spaces:** for $m \in \mathbb{N}_0, p \in [1, \infty]$: $\|f\|_{W^{m,p}(\mathbb{R}^n)} \sim \sum_{k=0}^{m} \left\|\nabla^k f\right\|_p$ where $\nabla^k f \in L^p$ are tensors defined by distributions. It is customary to write H^m for $W^{m,2}$.

2. **Bessel potential spaces**: for $s \in \mathbb{R}, p \in [1, \infty]$: $\|f\|_{H^{s,p}(\mathbb{R}^n)} \sim \|\langle\nabla\rangle^s f\|_p$ where $\langle\nabla\rangle^s = (1-\Delta)^{\frac{s}{2}}$ is the Bessel potential.

3. **Besov spaces**: for $s \in \mathbb{R}, p \in [1, \infty], q \in [1, \infty]$:

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|P_{\leq 1} f\|_p + \left\|N^s \|P_N f\|_p\right\|_{l^q_{N>1}}$$

   where P_N and $P_{\leq N}$ (for $N \in 2^{\mathbb{Z}}$) are the standard Littlewood-Paley projections (cf. [Tao06, Appendix A]).

4. **Triebel-Lizorkin spaces**: for $s \in \mathbb{R}, p \in [1, \infty), q \in [1, \infty]$: $\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \sim \|P_{\leq 1} f\|_p + \left\|N^s \|P_N f\|_{l^q_{N>1}}\right\|_p$

Remark. As there are multiple characterizations for the same spaces, we only define up to equivalent norms. Of course, the topologies induced by equivalent norms are the same.

In the literature, "Fractional Sobolev spaces" like $W^{s,p}$ could either refer to $B^s_{p,p}$ (**Sobolev–Slobodeckij spaces**) or $H^{s,p}$. We shall avoid using the term at all. There are also some delicate issues with $F^s_{\infty,q}$ which we do not need to discuss here (cf. [Tri10, Section 2.3.4]).

Blackbox 45. *Recall from harmonic analysis (cf.[Tri10, Section 2.5.6, 2.3.3, 2.11.2] and [Lem02b, Part 1, Chapter 3.1]):*

- $W^{m,p}(\mathbb{R}^n) = H^{m,p}(\mathbb{R}^n)$ *for* $m \in \mathbb{N}_0, p \in (1,\infty)$.
- $F^s_{p,2}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}, p \in (1,\infty)$.
- $B^m_{p,1}(\mathbb{R}^n) \hookrightarrow W^{m,p}(\mathbb{R}^n) \hookrightarrow B^m_{p,\infty}(\mathbb{R}^n)$ *for* $m \in \mathbb{N}_0, p \in [1,\infty]$.
- $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $W^{m,p}(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$ *and* $F^s_{p,q}(\mathbb{R}^n)$ *for* $m \in \mathbb{N}_0,\ s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty)$.
- $B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty]$.
- $\left(B^s_{p,q}(\mathbb{R}^n)\right)^* = B^{-s}_{p',q'}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty)$.
  $\left(F^s_{p,q}(\mathbb{R}^n)\right)^* = F^{-s}_{p',q'}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}, p \in (1,\infty), q \in (1,\infty)$.

6.2 On domains

Definition 46. A C^∞ **domain** Ω in $\mathbb{R}^n$ is defined as an open subset of $\mathbb{R}^n$ with smooth boundary, and scalar function spaces are then defined on Ω. If $\Omega \subset S \subset \overline{\Omega}$, let function spaces on S implicitly refer to function spaces on Ω. This will make it possible to discuss function spaces on, for example, $\overline{\mathbb{R}^n_+} \cap B_{\mathbb{R}^n}(0,1)$, or compact Riemannian manifolds with boundary.

Obviously, Sobolev spaces are still defined on domains by distributions. The big question is finding a good characterization for $B^s_{p,q}$ and $F^s_{p,q}$ on domains, when the Fourier transform is no longer available. This is among the main topics of Triebel's seminal books. Let us review the results:

Definition 47. Let Ω be either $\mathbb{R}^n$, or the half-space $\mathbb{R}^n_+$, or a bounded C^∞ domain in $\mathbb{R}^n$.

Then $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ can simply be defined as the restrictions of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ to Ω and

$$\|f\|_{B^s_{p,q}(\Omega)} = \inf\{\|F\|_{B^s_{p,q}(\mathbb{R}^n)} : F \in B^s_{p,q}(\mathbb{R}^n),\, F|_\Omega = f\} \text{ for } s \in \mathbb{R}; p, q \in [1, \infty]$$

$$\|f\|_{F^s_{p,q}(\Omega)} = \inf\{\|F\|_{F^s_{p,q}(\mathbb{R}^n)} : F \in F^s_{p,q}(\mathbb{R}^n),\, F|_\Omega = f\} \text{ for } s \in \mathbb{R}, p \in [1, \infty), q \in [1, \infty]$$

A more useful characterization is via BMD (**ball mean difference**). Let $\tau_h f(x) = f(x+h)$ be the translation operator and $\Delta_h f = \tau_h f - f$ be the difference operator. Then for $m \in \mathbb{N}_1$, we can define $\Delta_h^m = (\Delta_h)^m$ as the m-th difference operator. As we need to stay on the domain Ω, define

$$V^m(x,t) = \frac{1}{m}\left(B(x, mt) \cap \Omega - x\right) \quad \text{for } x \in \Omega, t > 0, m \in \mathbb{N}_1$$

So $V^m(x,t) \subset B(0,t)$, $x + mV^m(x,t) \subset \Omega$ and $\Delta_h^l f(x)$ is well-defined when $h \in V^m(x,t)$. Also note for $t \in (0,1)$: $|V^m(x,t)| \sim_{\Omega,m} t^n$. Then by [Tri92, Section 3.5.3, 5.2.2]:

1. For $m \in \mathbb{N}_1, p \in [1, \infty], q \in [1, \infty], s \in (0, m), r \in [1, p]$:

$$\|f\|_{B^s_{p,q}(\Omega)} \sim \|f\|_p + \left\| t^{-s} \left\| \|\Delta_h^m f(x)\|_{L^r_h\left(\frac{1}{t^n}\mathrm{d}h, V^m(x,t)\right)} \right\|_{L^p_x(\Omega)} \right\|_{L^q\left(\frac{1}{t}\mathrm{d}t,(0,1)\right)} \tag{6.1}$$

   We carefully note here that $m > s$ (the difference operator must be strictly higher-order than the regularity), and that the variable t is small, which will play a big role in Theorem 56. We also note that this is different from the classical characterization via differences ([Tri10, Section 3.4.2], [Tri92, Section 1.10.3]) which analysts might be more familiar with:

$$\|f\|_{B^s_{p,q}(\Omega)} \sim \|f\|_p + \left\| |h|^{-s} \left\| \Delta^m_{h,\Omega} f(x) \right\|_{L^p_x(\Omega)} \right\|_{L^q\left(\frac{\mathrm{d}h}{|h|^n}, B(0,1)\right)}$$

where $\Delta^m_{h,\Omega} f(x)$ is the same as $\Delta^m_h f(x)$, but zero wherever undefined, and $m \in \mathbb{N}_1, p \in (1,\infty), q \in [1,\infty], s \in (0,m)$.

2. For $m \in \mathbb{N}_1, p \in [1,\infty), q \in [1,\infty], s \in (0,m), r \in [1,p]$:

$$\|f\|_{F^s_{p,q}(\Omega)} \sim \|f\|_p + \left\| t^{-s} \left\| \|\Delta^m_h f(x)\|_{L^r_h\left(\frac{\mathrm{d}h}{t^n}, V^m(x,t)\right)} \right\|_{L^q\left(\frac{\mathrm{d}t}{t},(0,1)\right)} \right\|_{L^p_x(\Omega)}$$

Blackbox 48 (Diffeomorphisms and smooth multipliers)**.** *Every diffeomorphism on $\mathbb{R}^n$ preserves (under pullback) the topology of*

- $W^{k,p}(\mathbb{R}^n)$ *for* $k \in \mathbb{N}_0, p \in [1,\infty]$
- $B^s_{p,q}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}, p \in [1,\infty], q \in [1,\infty]$
- $F^s_{p,q}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty]$

Also on the same spaces, for $\phi \in C^\infty_c(\mathbb{R}^n)$, $f \mapsto \phi f$ is a bounded linear map .

Remark. This allows us to trivially define function spaces on compact Riemannian manifolds with boundary via partitions of unity and give them unique topologies.

Proof. For $W^{k,p}$ it is trivial. For $B^s_{p,q}$ and $F^s_{p,q}$, see [Tri92, Section 4.3, 4.2.2] and [Tri10, Section 2.8.2]. □

Blackbox 49 (Extension and trace)**.** *Let Ω be either the half-space $\mathbb{R}^n_+$ or a bounded C^∞ domain in $\mathbb{R}^n$.*

1. ***Stein extension**: There exists a common (continuous linear) extension operator $\mathfrak{E}$: $W^{k,p}(\Omega) \hookrightarrow W^{k,p}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0, p \in [1,\infty]$*

2. ***Triebel extension***: *For any* $N \in \mathbb{N}_1$, *there exists a common (continuous linear) extension operator* $\mathfrak{E}^N$ *such that*

   (a) $\mathfrak{E}^N : B^s_{p,q}(\Omega) \hookrightarrow B^s_{p,q}(\mathbb{R}^n)$ *for all* $|s| < N, p \in [1,\infty], q \in [1,\infty]$

   (b) $\mathfrak{E}^N : F^s_{p,q}(\Omega) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$ *for all* $|s| < N, p \in [1,\infty), q \in [1,\infty]$

3. ***Trace theorems***: *Let* $n \geq 2$.

   (a) *For* $p \in [1,\infty], q \in [1,\infty], s > \frac{1}{p} : B^s_{p,q}(\Omega) \twoheadrightarrow B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ *is a* ***retraction*** *(continuous surjection with a bounded linear section as a right inverse).*

   (b) *For* $p \in [1,\infty), q \in [1,\infty], s > \frac{1}{p} : F^s_{p,q}(\Omega) \twoheadrightarrow B^{s-\frac{1}{p}}_{p,p}(\partial\Omega)$ *is a retraction.*

   (c) *(Limiting case) For* $p \in [1,\infty)$, $B^{\frac{1}{p}}_{p,1}(\Omega) \twoheadrightarrow L^p(\partial\Omega)$ *and* $W^{1,1}(\Omega) \twoheadrightarrow L^1(\partial\Omega)$ *are continuous surjections.*

Remark. It is important to note that we do not have the trace theorem for, say, $B^{\frac{1}{3}}_{3,2}(\Omega)$ (cf. [Sch11, Section 3])

Proof.

1. See [Ste71, Section VI.3].

2. See [Tri92, Section 4.5, 5.1.3].

3. See [Tri10, Section 2.7.2, 3.3.3] and the remarks.

□

Corollary 50. *Let* Ω *be either the half-space* $\mathbb{R}^n_+$ *or a bounded* C^∞ *domain in* $\mathbb{R}^n$.

- $F^m_{p,2}(\Omega) = W^{m,p}(\Omega)$ *for* $m \in \mathbb{N}_0, p \in (1,\infty)$.
- $B^m_{p,1}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow B^m_{p,\infty}(\Omega)$ *for* $m \in \mathbb{N}_0, p \in [1,\infty]$.
- $\mathcal{S}(\overline{\Omega})$ *is dense in* $W^{m,p}(\Omega)$, $F^s_{p,q}(\Omega)$ *and* $B^s_{p,q}(\Omega)$ *for* $m \in \mathbb{N}_0$, $s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty)$.

- $B^s_{p,\min(p,q)}(\Omega) \hookrightarrow F^s_{p,q}(\Omega) \hookrightarrow B^s_{p,\max(p,q)}(\Omega)$ *for* $s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty]$

Remark. When Ω is a bounded C^∞ domain, $\mathcal{S}(\overline{\Omega}) = C^\infty(\overline{\Omega})$.

> *Proof.* Use Triebel and Stein extensions. □

6.3 Holder & Zygmund spaces

Definition 51. Let Ω be either $\mathbb{R}^n$, the half-space $\mathbb{R}^n_+$ or a bounded C^∞ domain in $\mathbb{R}^n$. Recall some L^∞ type spaces:

- **Holder spaces**: for $k \in \mathbb{N}_0, \alpha \in (0,1]$,

$$\|f\|_{C^{k,\alpha}(\Omega)} = \|f\|_{C^k(\Omega)} + \max_{|\beta|=k} \left[D^\beta f\right]_{C^{0,\alpha}(\Omega)}$$

  where $[g]_{C^{0,\alpha}(\Omega)} = \sup_{x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha}$

- **Zygmund spaces**: for $s > 0$, define $\mathfrak{C}^s(\Omega) = B^s_{\infty,\infty}(\Omega)$. Then for $m \in \mathbb{N}_1, s \in (0,m)$:

$$\begin{aligned}\|f\|_{\mathfrak{C}^s(\Omega)} &\sim \|f\|_{L^\infty(\Omega)} + \left\| t^{-s} \left\| \|\Delta^m_h f(x)\|_{L^\infty_h(V^m(x,t))} \right\|_{L^\infty_x(\Omega)} \right\|_{L^\infty_t((0,1))} \\ &\sim \sup |f| + \sup_{0<|h|\leq 1, x\in\Omega} \frac{\left|\Delta^m_{h,\Omega} f(x)\right|}{|h|^s}\end{aligned}$$

It is well-known (cf. [Tri10, Section 2.2.2, 2.5.7, 2.5.12, 2.8.3]) that

- $\|f\|_{\mathfrak{C}^{k+\alpha}(\Omega)} \sim \|f\|_{C^k} + \max_{|\beta|=k} \left\|D^\beta f\right\|_{\mathfrak{C}^\alpha(\Omega)}$ for $k \in \mathbb{N}_0, \alpha \in (0,1]$
- $\|f\|_{\mathfrak{C}^{k+\alpha}(\Omega)} \sim \|f\|_{C^{k,\alpha}}$ for $k \in \mathbb{N}_0, \alpha \in (0,1)$.
- $\|fg\|_{\mathfrak{C}^s(\Omega)} \lesssim \|f\|_{\mathfrak{C}^s} \|g\|_{\mathfrak{C}^s}$ for $s > 0$.

Note that $C^{0,1}, C^1$ and $\mathfrak{C}^1$ are different.

6.4 Interpolation & embedding

Blackbox 52 (Interpolation)**.** *Let Ω be either $\mathbb{R}^n$, the half-space $\mathbb{R}^n_+$ or a bounded C^∞ domain in $\mathbb{R}^n$. Throughout the theorem, always assume $\theta \in (0,1), s_\theta = (1-\theta)s_0 + \theta s_1$.*

1. $\left(B^{s_0}_{p,q_0}(\Omega), B^{s_1}_{p,q_1}(\Omega)\right)_{\theta,q} = B^{s_\theta}_{p,q}(\Omega)$ *for* $s_0 \neq s_1, s_j \in \mathbb{R}, p \in [1,\infty], q_j, q \in [1,\infty]$.

   $\left(F^{s_0}_{p,q_0}(\Omega), F^{s_1}_{p,q_1}(\Omega)\right)_{\theta,q} = B^{s_\theta}_{p,q}(\Omega)$ *for* $s_0 \neq s_1, s_j \in \mathbb{R}, p \in [1,\infty), q_j, q \in [1,\infty]$.

2. $\left(B^{s_0}_{p_0,q_0}(\Omega), B^{s_1}_{p_1,q_1}(\Omega)\right)_{\theta,p_\theta} = B^{s_\theta}_{p_\theta,p_\theta}(\Omega)$ *for* $s_0 \neq s_1, s_j \in \mathbb{R}, p_j \in [1,\infty], q_j \in [1,\infty], \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$

3. $\left[B^{s_0}_{p_0,q_0}(\Omega), B^{s_1}_{p_1,q_1}(\Omega)\right]_\theta = B^{s_\theta}_{p_\theta,q_\theta}(\Omega)$ *and* $\left[F^{s_0}_{p_0,q_0}(\Omega), F^{s_1}_{p_1,q_1}(\Omega)\right]_\theta = F^{s_\theta}_{p_\theta,q_\theta}(\Omega)$

   for $s_j \in \mathbb{R}, p_j \in (1,\infty), q_j \in (1,\infty), \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

4. $\left[L^{p_0}(\Omega), L^{p_1}(\Omega)\right]_\theta = L^{p_\theta}(\Omega)$ *for* $p_j \in [1,\infty], \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

5. $\left(W^{m_0,p}(\Omega), W^{m_1,p}(\Omega)\right)_{\theta,q} = B^{m_\theta}_{p,q}(\Omega)$ *for* $m_j \in \mathbb{N}_0, m_0 \neq m_1, p \in [1,\infty], q \in [1,\infty]$, $m_\theta = (1-\theta)\, m_0 + \theta m_1$.

Proof.

1. Extension operators and [Tri10, Section 2.4.2].
2. Extension operators and [BL76, Theorem 6.4.5].
3. Extension operators and [Tri10, Section 2.4.7].
4. Extension by zero and [BL76, Section 5.1.1]
5. Recall $B^m_{p,1}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow B^m_{p,\infty}(\Omega)$ for $m \in \mathbb{N}_0, p \in [1,\infty]$. Then apply 1.

□

Blackbox 53 (Embedding)**.** *Let* Ω *be a bounded* C^∞ *domain in* $\mathbb{R}^n$*. Assume* $\infty > s_0 > s_1 > -\infty$*. Then*

1. $B^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1,q_1}(\Omega)$ *is compact for* $p_j \in [1,\infty], q_j \in [1,\infty], \frac{1}{p_1} > \frac{1}{p_0} - \frac{s_0-s_1}{n}$

   $F^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow F^{s_1}_{p_1,q_1}(\Omega)$ *is compact for* $p_j \in [1,\infty), q_j \in [1,\infty], \frac{1}{p_1} > \frac{1}{p_0} - \frac{s_0-s_1}{n}$

2. $B^{s_0}_{p_0,q}(\Omega) \hookrightarrow B^{s_1}_{p_1,q}(\Omega)$ *is continuous for* $p_j \in [1,\infty], q \in [1,\infty], \frac{1}{p_1} = \frac{1}{p_0} - \frac{s_0-s_1}{n}$

   $F^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow F^{s_1}_{p_1,q_1}(\Omega)$ *is continuous for* $p_j \in [1,\infty), q_j \in [1,\infty], \frac{1}{p_1} = \frac{1}{p_0} - \frac{s_0-s_1}{n}$

Proof.

1. See [Tri10, Section 4.3.2, Remark 1] and [Tri10, Section 3.3.1].
2. See [Tri10, Section 3.3.1].

□

Corollary 54. *Let* Ω *be a bounded* C^∞ *domain in* $\mathbb{R}^n$*. Then*

1. *For* $m_j \in \mathbb{N}_0, m_0 > m_1, p_j \in [1,\infty], \frac{1}{p_1} > \frac{1}{p_0} - \frac{m_0-m_1}{n}$*:*
   $W^{m_0,p_0}(\Omega) \hookrightarrow B^{m_0}_{p_0,\infty}(\Omega) \hookrightarrow B^{m_1}_{p_1,1}(\Omega) \hookrightarrow W^{m_1,p_1}(\Omega)$ *is compact.*
2. *For* $m_j \in \mathbb{N}_0, m_0 > m_1, p_0 \in [1,\infty], \alpha \in (0,1), 0 > \frac{1}{p_0} - \frac{m_0-(m_1+\alpha)}{n}$*:*
   $W^{m_0,p_0}(\Omega) \hookrightarrow B^{m_0}_{p_0,\infty}(\Omega) \hookrightarrow B^{m_1+\alpha}_{\infty,\infty}(\Omega) = C^{m_1,\alpha}(\overline{\Omega})$ *is compact.*
3. *For* $m \in \mathbb{N}_1, p \in (1,\infty) : W^{m,p}(\Omega) \hookrightarrow B^m_{p,\infty}(\Omega) \hookrightarrow B^{\frac{1}{p}}_{p,1}(\Omega) \twoheadrightarrow L^p(\partial\Omega)$ *is compact.*

Remark. These include the Rellich-Kondrachov embeddings found in [Ada03, Theorem 6.3], so the Besov embeddings generalize Sobolev embeddings.

6.5 Strip decay

Some notation first: let Ω be a C^∞ domain in $\mathbb{R}^n$ or a compact Riemannian manifold with or without boundary. Define $\Omega_{>r} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$ where $\operatorname{dist}(x, \partial\Omega) = \infty$ if $\partial\Omega = \emptyset$. Similarly define $\Omega_{\geq r}, \Omega_{<r}, \Omega_{[r_1,r_2]}$.

When $|\Omega| < \infty$ and $p \in [1, \infty)$, we write

$$\|f\|_{L^p(\Omega,\mathrm{avg})} = \|f(x)\|_{L^p_x(\frac{\mathrm{d}x}{|\Omega|},\Omega)} = \left(\fint_\Omega |f|^p\right)^{\frac{1}{p}} = \frac{1}{|\Omega|^{1/p}}\left(\int_\Omega |f|^p\right)^{\frac{1}{p}}$$

By convention, we set $\|f\|_{L^\infty(\Omega,\mathrm{avg})} = \|f(x)\|_{L^\infty_x(\frac{\mathrm{d}x}{|\Omega|},\Omega)} = \|f\|_{L^\infty(\Omega)}$. The implicit measure is of course the Riemannian measure. In such mean integrals, the domain becomes a probability space.

Theorem 55 (Coarea formula)**.**

1. *For any $h \in \mathbb{R}^n$, the translation semigroup $(\tau_{th})_{t\geq 0}$ is a C_0 semigroup on $W^{m,p}(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ for $m \in \mathbb{N}_0$, $s \in \mathbb{R}, p \in [1,\infty), q \in [1,\infty)$. Consequently, for $p \in [1,\infty)$ and $f \in B^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$,*

$$\left([0,\infty) \to L^p(\mathbb{R}^{n-1}), t \mapsto \tau_{th} f|_{\mathbb{R}^{n-1}}\right)$$

   is continuous and bounded by $C\|f\|_{B^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)}$ for some $C > 0$.

2. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ (or a compact Riemannian manifold with boundary). Let $p \in [1,\infty)$. Then for $f \in B^{\frac{1}{p}}_{p,1}(\Omega)$ and $r > 0$ small:*

   (a) $\left([0,r) \to \mathbb{R}, \rho \mapsto \|f\|_{L^p(\partial\Omega_{>\rho})}\right)$ is continuous and bounded by $C\|f\|_{B^{\frac{1}{p}}_{p,1}(\Omega)}$ for some $C > 0$.

   (b) $\|f\|_{L^p(\Omega_{\leq r})} \sim_{\neg r} \left\|\|f\|_{L^p(\partial\Omega_{>\rho})}\right\|_{L^p_\rho((0,r))}$

   (c) $\|f\|_{L^p(\Omega_{\leq r},\mathrm{avg})} \lesssim_{\neg r} \|f\|_{B^{\frac{1}{p}}_{p,1}(\Omega)}$ and $\|f\|_{L^p(\Omega_{\leq r},\mathrm{avg})} \xrightarrow{r\downarrow 0} \|f\|_{L^p(\partial\Omega,\mathrm{avg})}$.

(d) *Let* $I \subset \mathbb{R}$ *be an open interval and* $\mathfrak{f} \in L^p(I \to B^{\frac{1}{p}}_{p,1}(\Omega))$*, then* $\|\mathfrak{f}\|_{L^p_t B^{\frac{1}{p}}_{p,1}(\Omega)} \gtrsim_{\neg r}$ $\|\mathfrak{f}\|_{L^p_t L^p(\Omega_{\leq r},\mathrm{avg})} \xrightarrow{r\downarrow 0} \|\mathfrak{f}\|_{L^p_t L^p(\partial\Omega,\mathrm{avg})}$.

3. *Let* $p \in [1,\infty)$, $f \in W^{1,p}(\Omega)$, *show that* $\|f\|_{L^p(\Omega_{<r})} \lesssim_{\neg r} r\,\|f\|_{W^{1,p}(\Omega_{<r})} + r^{\frac{1}{p}}\,\|f\|_{L^p(\partial\Omega)}$ *for* $r > 0$ *small.*

Proof.

1. Use the density of $\mathcal{S}(\mathbb{R}^n)$ and Lemma 11.

2.

   (a) By partition of unity, geodesic normals, diffeomorphisms and the smallness of r, reduce the problem to the half-space case, which is just 1).

   (b) Approximate f in $B^{\frac{1}{p}}_{p,1}$ by $C^\infty(\overline{\Omega})$ functions. This is the well-known coarea formula, which corresponds to Fubini's theorem in the half-space case. Note that $\|f\|_{L^p(\partial\Omega_{>\rho})}$ is defined by the trace theorem. See [Cha06, Section III.5] for more details.

   (c) For r small, $|\Omega_{<r}| \sim |\partial\Omega|\, r$ and $|\partial\Omega_{>r}| \sim |\partial\Omega|$, so

$$\|f\|_{L^p(\Omega_{\leq r},\mathrm{avg})} \sim \left\| \|f\|_{L^p(\partial\Omega_{>\rho},\mathrm{avg})} \right\|_{L^p_\rho((0,r),\mathrm{avg})} \leq \sup_{\rho<r} \|f\|_{L^p(\partial\Omega_{>\rho},\mathrm{avg})}$$

   and $\left\| \|f\|_{L^p(\partial\Omega_{>\rho},\mathrm{avg})} \right\|_{L^p_\rho((0,r),\mathrm{avg})} \xrightarrow{r\downarrow 0} \|f\|_{L^p(\partial\Omega,\mathrm{avg})}$ by continuity in a).

   (d) Dominated convergence.

3. By the trace theorem, WLOG $f \in C^\infty(\overline{\Omega})$. By partition of unity and diffeomorphisms, WLOG $\overline{\Omega} = \overline{\mathbb{R}^n_+} = \{(\mathbf{x}, y) : \mathbf{x} \in \mathbb{R}^{n-1}, y \geq 0\}$. Then

$$\|f\|_{L^p(\Omega_{<r})} \sim \left\| \|f(\mathbf{x},y)\|_{L^p_y([0,r])} \right\|_{L^p_{\mathbf{x}}} \lesssim \left\| \left\| \|\partial_y f(\mathbf{x},\rho)\|_{L^1_\rho([0,y])} + |f(\mathbf{x},0)| \right\|_{L^p_y([0,r])} \right\|_{L^p_{\mathbf{x}}}$$

$$\lesssim \left\| \left\| \|\partial_y f(\mathbf{x},\rho)\|_{L^p_\rho([0,y])} |y|^{\frac{1}{p'}} \right\|_{L^p_y([0,r])} \right\|_{L^p_{\mathbf{x}}} + r^{\frac{1}{p}} \|f(\mathbf{x},0)\|_{L^p_{\mathbf{x}}}$$

The first term $\lesssim \left\| \|\partial_y f(\mathbf{x},\rho)\|_{L^p_\rho([0,r])} \left\| |y|^{\frac{1}{p'}} \right\|_{L^p_y([0,r])} \right\|_{L^p_{\mathbf{x}}} \lesssim r \|\partial_y f\|_{L^p(\Omega_{<r})}$. So we are done.

□

Theorem 56 (Product estimate). *Let M be a bounded C^∞ domain in $\mathbb{R}^n$ (or a compact Riemannian manifold with boundary). Assume $r > 0$ small, $f_r \in C^\infty(\overline{M})$ with support in $M_{<r}$. Then for $p \in (1,\infty)$, $g \in B^{\frac{1}{p}}_{p,1}(M)$:*

$$\|f_r g\|_{B^{1/p}_{p,1}(M)} \lesssim_{M,\neg r} \|f_r\|_{B^{1/p}_{\infty,1}(M)} \|g\|_{L^p(M_{<4r})} + \|f_r\|_{L^\infty(M_{<r})} \|g\|_{B^{1/p}_{p,1}(M)}$$

Remark. The theory of product and commutator estimates (Kato-Ponce, Coifman-Meyer etc.) has a long and rich history which we will not recount here (cf. [KP88; Tao07; GO14; NT19]). However, for our intended application, f_r has very small support and we want to use $\|g\|_{L^p(M_{<4r})}$ instead of $\|g\|_{L^p(M)}$ to control the product. Unfortunately there does not seem to be much, if at all, literature on this issue. This theorem will only be used for Theorem 88, and is not necessary for Onsager's conjecture.

Proof. By diffeomorphisms, partition of unity, and geodesic normals, WLOG assume $M = \overline{\mathbb{R}^n_+}$ with $M_{<r} = \{x \in \mathbb{R}^n : 0 \le x_n < r\}$.
Recall $\|g\|_{L^p(x_n=a)} \lesssim \|g\|_{B^{1/p}_{p,1}(\mathbb{R}^n_+)}$ $\forall 0 \le a < \infty$ where $\|g\|_{L^p(x_n=a)} := \|g\|_{L^p(\{x\in\mathbb{R}^n : x_n = a\})}$ is defined by the trace theorem.

WLOG, assume $\|f_r\|_\infty \le 1$. Recall the characterization of Besov spaces by ball mean difference (BMD) and write $V(x,t)$ for $V^1(x,t)$ (see Equation (6.1)). Then

$$\|f_r g\|_{B^{1/p}_{p,1}(M)} \sim \|f_r g\|_{L^p(M)} + \left\| t^{-\frac{1}{p}-n} \left\| \|\Delta_h(f_r g)(x)\|_{L^1_h(V(x,t))} \right\|_{L^p_x(M)} \right\|_{L^1_t(\frac{\mathrm{d}t}{t},(0,1))}$$

The term $\|f_r g\|_{L^p(M)}$ is easily bounded and thrown away. For the remaining term, we use the identity $\Delta_h(f_r g) = \Delta_h f_r g + \tau_h f_r \Delta_h g$ to bound it by

$$\begin{aligned} &\left\| t^{-\frac{1}{p}-n} \left\| \|\Delta_h f_r(x)\|_{L^1_h(V(x,t))} g(x) \right\|_{L^p_x(M)} \right\|_{L^1(\frac{\mathrm{d}t}{t},(0,1))} \\ + &\left\| t^{-\frac{1}{p}-n} \left\| \|f_r\|_\infty \|\Delta_h g(x)\|_{L^1_h(V(x,t))} \right\|_{L^p_x(M)} \right\|_{L^1(\frac{\mathrm{d}t}{t},(0,1))} \end{aligned}$$

The second term here is just $\|f_r\|_{L^\infty}\|g\|_{B^{1/p}_{p,1}(M)}$, so throw it away. For the remaining term, by using $\|\cdot\|_{L^p(M)} \lesssim \|\cdot\|_{L^p(M_{<4r})} + \|\cdot\|_{L^p(M_{>4r})}$ and

$$\left\| \|\Delta_h f_r(x)\|_{L^1_h(V(x,t))} g(x) \right\|_{L^p_x(M_{<4r})} \lesssim \left\| \|\Delta_h f_r(x)\|_{L^1_h(V(x,t))} \right\|_{L^\infty_x(M_{<4r})} \|g(x)\|_{L^p_x(M_{<4r})}$$

we are left with

$$\|f_r\|_{B^{1/p}_{\infty,1}(M)} \|g\|_{L^p(M_{<4r})} + \left\| t^{-\frac{1}{p}-n} \left\| \|\Delta_h f_r(x)\|_{L^1_h(V(x,t))} g(x) \right\|_{L^p_x(M_{>4r})} \right\|_{L^1(\frac{\mathrm{d}t}{t},(0,1))}$$

Throwing away the first term, we have arrived at the important estimate: what happens on $M_{>4r}$. It will turn out that the values of g on $M_{>4r}$ are well-controlled by $\|g\|_{B^{1/p}_{p,1}(M)}$. To begin, recall f_r is supported on $M_{<r}$ and use the crude geometric estimate

$$\begin{aligned} t^{-n}\|\Delta_h f_r(x)\|_{L^1_h(V(x,t))} =& t^{-n}\|f_r(x+h)\|_{L^1_h(V(x,t))} \\ \lesssim& \frac{|B(x,t)\cap M_{<r}|}{|B(x,t)|} \lesssim \frac{r}{x_n}\mathbf{1}_{t>x_n-r} \quad \forall x \in M_{>4r}, \forall t \in (3r,1) \end{aligned}$$

Note that $t > 3r$ comes from $t > x_n - r > 4r - r$. So we have used the "room" from $4r$ to get an $O(r)$-lower bound for t. By $x_n < r + t$, we now only need to bound

$$\left\| t^{-\frac{1}{p}} \left\| g(x)\frac{r}{x_n} \right\|_{L^p_x(M_{[4r,r+t]})} \right\|_{L^1(\frac{\mathrm{d}t}{t},(3r,1))}$$

Obviously, we will integrate g on x_n-slices (using $p > 1$):

$$\left\| g(x) \frac{r}{x_n} \right\|_{L^p_x(M_{[4r,r+t]})} = r \left\| \frac{1}{\rho} \|g\|_{L^p(x_n=\rho)} \right\|_{L^p_\rho([4r,r+t])} \lesssim \|g\|_{B^{1/p}_{p,1}(M)} r \left\| \frac{1}{\rho} \right\|_{L^p_\rho([4r,\infty))}$$
$$\lesssim r^{\frac{1}{p}} \|g\|_{B^{1/p}_{p,1}(M)}$$

Then we are done (using $p < \infty$):

$$r^{\frac{1}{p}} \left\| t^{-\frac{1}{p}} \right\|_{L^1(\frac{\mathrm{d}t}{t},(3r,1))} = \left\| \left(\frac{r}{t}\right)^{\frac{1}{p}} \right\|_{L^1(\frac{\mathrm{d}t}{t},(3r,1))} = \left\| \left(\frac{1}{t}\right)^{\frac{1}{p}} \right\|_{L^1(\frac{\mathrm{d}t}{t},(3,\frac{1}{r}))} \lesssim_{\neg r} 1$$

$\square$

CHAPTER 7

Hodge theory

We stick closely to the terminology and symbols of [Sch95], with some careful exceptions.

7.1 The setting

Definition 57. Define a **∂-manifold** as a paracompact, Hausdorff, metric-complete, oriented, smooth manifold, with no or smooth boundary.

Note that this means $B_{\mathbb{R}^n}(0,1)$ is not a ∂-manifold (as it is not complete), but $\overline{B_{\mathbb{R}^n}(0,1)}$ is.

For the rest of this thesis, unless mentioned otherwise, we work on M which is a compact Riemannian n-dimensional ∂-manifold (where $n \geq 2$), and use ν to denote the outwards unit normal vector field on ∂M.

As before, define $M_{>r} = \{x \in M : \operatorname{dist}(x, \partial M) > r\}$, and similarly for $M_{\geq r}, M_{<r}, M_{[r_1,r_2]}$ etc.

For $r > 0$ small, the map $(\partial M \times [0,r) \to M_{<r}, (x,t) \mapsto \exp_x(-t\nu))$ is a diffeomorphism, which we call a **Riemannian collar**. Then ν can be extended via geodesics to a smooth vector field $\widetilde{\nu}$ which is of unit length near the boundary (cut off at some point away from the boundary, but we only care about the area near the boundary).

Let vol stand for the Riemannian volume form orienting M and vol_∂ for that of ∂M. Let $\jmath : \partial M \hookrightarrow M$ be the smooth inclusion map and ι stand for interior product (contraction) of differential forms. Note that for a smooth differential form ω, $\jmath^*\omega$ only depends on $\omega\big|_{\partial M}$, so by abuse of notation, we can write

$$\mathrm{vol}_\partial = \jmath^*(\iota_\nu \,\mathrm{vol})$$

where $\iota_\nu \,\mathrm{vol} \in \Omega^{n-1}(M)\big|_{\partial M}$. Additionally, the Stokes theorem reads $\int_M d\omega = \int_{\partial M} \jmath^*\omega$ for $\omega \in \Omega^{n-1}(M)$.

7.1.1 Vector bundles

Let $\mathbb{F}$ be a real vector bundle over M with a Riemannian fiber metric $\langle \cdot, \cdot \rangle_{\mathbb{F}}$.

Define

- $\Gamma(\mathbb{F})$: the space of smooth sections of $\mathbb{F}$
- $\Gamma_c(\mathbb{F})$: smooth sections with compact support (so $\Gamma_c(\mathbb{F}) = \Gamma(\mathbb{F})$ since M is compact)
- $\Gamma_{00}(\mathbb{F})$: smooth sections with compact support in $\mathring{M}$ (the interior of M).

Remark. We are following [Sch95], where Hodge theory is also formulated for non-compact M. In the book, $\Gamma_0\mathbb{F}$ is used instead of $\Gamma_{00}\mathbb{F}$ to denote compact support in $\mathring{M}$. As that can be confused with having zero trace, we opt to write $\Gamma_{00}\mathbb{F}$ instead.

Then on $\Gamma_c(\mathbb{F})$, define the dot product

$$\langle\langle \sigma, \theta \rangle\rangle = \int_M \langle \sigma, \theta \rangle_{\mathbb{F}} \,\mathrm{vol}$$

and $|\sigma|_{\mathbb{F}} = \sqrt{\langle \sigma, \sigma \rangle_{\mathbb{F}}}$. Then for $p \in [1, \infty)$, $L^p\Gamma(\mathbb{F})$ is the completion of $\Gamma_c(\mathbb{F})$ under the norm

$$\|\sigma\|_{L^p\Gamma(\mathbb{F})} = \||\sigma|_{\mathbb{F}}\|_{L^p(M)}$$

Let $\nabla^{\mathbb{F}}$ be a connection on $\mathbb{F}$. Then for $\sigma \in \Gamma(\mathbb{F}), \nabla^{\mathbb{F}}\sigma \in \Gamma(T^*M \otimes \mathbb{F})$ and we can define the fiber metric

$$\langle \alpha \otimes \sigma, \beta \otimes \theta \rangle_{T^*M\otimes\mathbb{F}} = \langle \alpha, \beta \rangle_{T^*M} \langle \sigma, \theta \rangle_{\mathbb{F}}$$

In local coordinates (Einstein notation):

$$\left\langle \nabla^{\mathbb{F}}\sigma, \nabla^{\mathbb{F}}\theta \right\rangle_{T^*M\otimes\mathbb{F}} = \left\langle dx^i \otimes \nabla^{\mathbb{F}}_i \sigma, dx^j \otimes \nabla^{\mathbb{F}}_j \sigma \right\rangle_{\mathbb{F}} = \left\langle dx^i, dx^j \right\rangle_{T^*M} \left\langle \nabla^{\mathbb{F}}_i \sigma, \nabla^{\mathbb{F}}_j \theta \right\rangle_{\mathbb{F}} = g^{ij} \left\langle \nabla^{\mathbb{F}}_i \sigma, \nabla^{\mathbb{F}}_j \theta \right\rangle_{\mathbb{F}}$$

For higher derivatives, define the k-jet fiber metric

$$\langle \sigma, \theta \rangle_{J^k\mathbb{F}} = \sum_{0\le j\le k} \left\langle \left(\nabla^{\mathbb{F}}\right)^{(j)} \sigma, \left(\nabla^{\mathbb{F}}\right)^{(j)} \theta \right\rangle_{\left(\bigotimes^j T^*M\right)\otimes\mathbb{F}}$$

and $|\sigma|_{J^k\mathbb{F}} = \sqrt{\langle \sigma, \sigma \rangle_{J^k\mathbb{F}}}$. Then we have Cauchy-Schwarz: $|\langle \sigma, \theta \rangle_{J^k\mathbb{F}}| \le |\sigma|_{J^k\mathbb{F}} |\theta|_{J^k\mathbb{F}}$.

Then for $m \in \mathbb{N}_0, p \in [1, \infty)$, we define the Sobolev space $W^{m,p}\Gamma(\mathbb{F})$ as the completion of $\Gamma_c(\mathbb{F})$ under the norm

$$\|\sigma\|_{W^{m,p}\Gamma(\mathbb{F})} = \||\sigma|_{J^m\mathbb{F}}\|_{L^p(M)}$$

It is worth noting that $|\sigma|_{J^m\mathbb{F}}$, up to some constants, does not depend on $\nabla^{\mathbb{F}}$. Indeed, assume there is another connection $\widetilde{\nabla}^{\mathbb{F}}$, then $\nabla^{\mathbb{F}} - \widetilde{\nabla}^{\mathbb{F}}$ is tensorial:

$$\left(\nabla^{\mathbb{F}}_X - \widetilde{\nabla}^{\mathbb{F}}_X\right)(f\sigma) = f\left(\nabla^{\mathbb{F}}_X - \widetilde{\nabla}^{\mathbb{F}}_X\right)(\sigma) = \left(\nabla^{\mathbb{F}}_{fX} - \widetilde{\nabla}^{\mathbb{F}}_{fX}\right)(\sigma)$$

for $f \in C^\infty(M), \sigma \in \Gamma(\mathbb{F}), X \in \mathfrak{X}M$.

So there is a $C^\infty(M)$-multilinear map $A : \mathfrak{X}M \otimes_{C^\infty(M)} \Gamma(\mathbb{F}) \to \Gamma(\mathbb{F})$ such that

$$\left(\nabla^{\mathbb{F}}_X - \widetilde{\nabla}^{\mathbb{F}}_X\right)(\sigma) = A(X, \sigma).$$

By the compactness of M and the boundedness of A, we conclude $|\sigma|_{J^m\mathbb{F},\nabla^{\mathbb{F}}} \sim |\sigma|_{J^m\mathbb{F},\widetilde{\nabla}^{\mathbb{F}}}$. Therefore the topology of $W^{m,p}\Gamma(\mathbb{F})$ is uniquely defined.

Definition 58 (Distributions)**.** Set $\mathscr{D}\Gamma(\mathbb{F}) = \Gamma_{00}(\mathbb{F})$ as the space of **test sections** and

$\mathscr{D}'\Gamma(\mathbb{F}) = (\mathscr{D}\Gamma(\mathbb{F}))^*$ the space of **distributional sections**. As usual, in the category of locally convex TVS, $\mathscr{D}\Gamma(\mathbb{F})$ is given **Schwartz's topology** as the colimit of $\{\Gamma(\mathbb{F})_K : K \subset \mathring{M}$ compact$\}$, where $\Gamma(\mathbb{F})_K := \{\sigma \in \Gamma(\mathbb{F}) : \operatorname{supp}\sigma \subset K\}$ has the **Frechet** C^∞ topology.

7.1.2 Compatibility with scalar function spaces

We aim to show that the global definitions of Sobolev spaces in Subsection 7.1.1 are compatible with the definitions of Sobolev spaces by local coordinates.

Let $(\psi_\alpha, U_\alpha)_\alpha$ be a finite partition of unity, where U_α is open in M and ψ_α is supported in U_α. Normally in differential geometry, U_α is diffeomorphic to either $\overline{\mathbb{R}^n_+} \cap B_{\mathbb{R}^n}(0,1)$ or $B_{\mathbb{R}^n}(0,1)$. However, it is problematic that the half-ball does not have C^∞ boundary, so we use some piecewise-linear functions and mollification to create a bounded C^∞ domain.

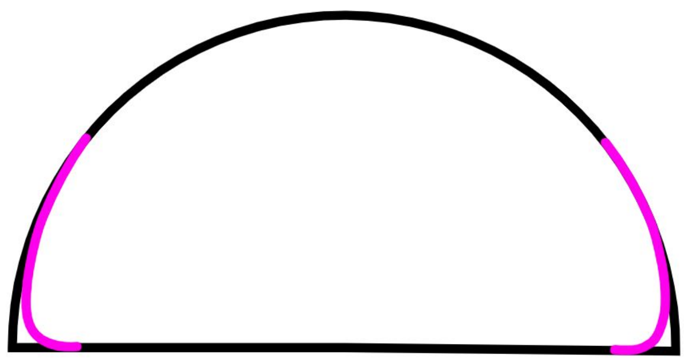

Figure 7.1: Smoothing the corners

So WLOG, $\overline{U_\alpha}$ is diffeomorphic to the closure of a bounded C^∞ domain in $\mathbb{R}^n$, and scalar function spaces are well-defined on U_α (recall Definition 46). Note that $\operatorname{supp}\psi_\alpha$ might intersect with ∂M.

For U_α chosen small enough, the bundle $\mathbb{F}$ on U_α is diffeomorphic to $U_\alpha \times F$ (where F is the typical fiber of $\mathbb{F}$).

Let $\left(e^\alpha_\beta\right)_\beta$ be the coordinate sections on $\operatorname{supp}\psi_\alpha$, and cut off such that $\operatorname{supp}\psi_\alpha \subset \left(\overset{\circ}{\operatorname{supp}} e^\alpha_\beta\right) \subset \operatorname{supp} e^\alpha_\beta \subset U_\alpha$. Let $\sigma \in \Gamma(\mathbb{F})$. Then there exist $c^\alpha_\beta(\sigma) \in C^\infty_c(U_\alpha)$ such that $\operatorname{supp} c^\alpha_\beta(\sigma) \subset \operatorname{supp}\psi_\alpha$, $\psi_\alpha\sigma = \sum_\beta c^\alpha_\beta(\sigma)e^\alpha_\beta$ and

$$\sigma = \sum_{\alpha,\beta} c^\alpha_\beta(\sigma)e^\alpha_\beta$$

Now, observe that $|\sigma|_{\mathbb{F}} \sim \sum_\alpha |\psi_\alpha\sigma|_{\mathbb{F}}$ and

$$|\psi_\alpha\sigma|_{\mathbb{F}} = \left(\sum_{\beta,\beta'} c^\alpha_\beta(\sigma)c^\alpha_{\beta'}(\sigma)\left\langle e^\alpha_\beta, e^\alpha_{\beta'}\right\rangle_{\mathbb{F}}\right)^{\frac{1}{2}} \sim \left(\sum_\beta \left|c^\alpha_\beta(\sigma)\right|^2\right)^{\frac{1}{2}}$$

To see this, let $x \in \operatorname{supp}\psi_\alpha$ and $\left\langle e^\alpha_\beta, e^\alpha_{\beta'}\right\rangle_{\mathbb{F}}(x) = B_{\beta\beta'}(x)$. Then $B_x(u,v) := \sum_{\beta,\beta'} u_\beta v_{\beta'} B_{\beta\beta'}(x)$ is a positive-definite inner product, which induces a norm on a finite-dimensional vector space, where all norms are equivalent. Then simply note $B_x(u,u)$ is continuous in variable $x \in \operatorname{supp}\psi_\alpha$.

Also, in local coordinates, there are $s^\gamma_{i\beta} \in C^\infty_c(U_\alpha)$ such that $\nabla^{\mathbb{F}}_i e^\alpha_\beta = \sum_\gamma s^\gamma_{i\beta}e^\alpha_\gamma$ on $\operatorname{supp}\psi_\alpha$. Then

$$\nabla^{\mathbb{F}}_i(\psi_\alpha\sigma) = \sum_\beta \partial_i c^\alpha_\beta(\sigma)e^\alpha_\beta + \sum_{\beta,\gamma} c^\alpha_\beta(\sigma)s^\gamma_{i\beta}e^\alpha_\gamma = \sum_\beta d^\alpha_{i\beta}(\sigma)e^\alpha_\beta$$

where $d^\alpha_{i\beta}(\sigma) = \partial_i c^\alpha_\beta(\sigma) + \sum_\gamma c^\alpha_\gamma(\sigma)s^\beta_{i\gamma}$.

So $|\sigma|_{J^1\mathbb{F}} \sim \sum_{\alpha,\beta}|c^\alpha_\beta(\sigma)| + \sum_{\alpha,\beta,i}|d^\alpha_{i\beta}(\sigma)| \sim \sum_{\alpha,\beta}|c^\alpha_\beta(\sigma)| + \sum_{\alpha,\beta,i}|\partial_i c^\alpha_\beta(\sigma)|$.

Similarly $|\sigma|_{J^m\mathbb{F}} \sim \sum_{\alpha,\beta}\sum_{k\le m}\left|\nabla^{(k)}c^\alpha_\beta(\sigma)\right|$.

So for $m \in \mathbb{N}_0, p \in [1,\infty)$,

$$\|\sigma\|_{W^{m,p}} \sim \sum_{\alpha,\beta}\left\|c^\alpha_\beta(\sigma)\right\|_{W^{m,p}(U_\alpha,\mathbb{R})}$$

Now define $S\sigma = \left(c^\alpha_\beta(\sigma)\right)_{\alpha,\beta}$ and $R\left(c^\alpha_\beta\right)_{\alpha,\beta} = \sum_{\alpha,\beta} c^\alpha_\beta e^\alpha_\beta$. Then $RS = 1$ on $\Gamma(\mathbb{F})$ and $P := SR$ is a projection on $\prod_{\alpha,\beta} C^\infty(\overline{U_\alpha})$. Note that P depends on the choice of partition of unity. By looking into the definitions of R and S, we can extend this to have $P = SR$ as a continuous projection on $\prod_{\alpha,\beta} L^1(U_\alpha)$ and

$$\left\| P\left(c^\alpha_\beta\right)_{\alpha,\beta} \right\|_{\prod_{\alpha,\beta} W^{m,p}(U_\alpha)} \lesssim \sum_{\alpha,\beta} \left\| c^\alpha_\beta \right\|_{W^{m,p}(U_\alpha)} \text{ for } m \in \mathbb{N}_0, p \in [1,\infty], c^\alpha_\beta \in W^{m,p}(U_\alpha)$$

The keen reader should have noticed we never mentioned the case $p = \infty$ in Subsection 7.1.1 as we defined $W^{m,p}\Gamma(\mathbb{F})$ by the completion of smooth sections, and $C^\infty(M)$ is not dense in $W^{m,\infty}(M)$. Now, however, by using local coordinates, we are justified in defining $W^{m,p}\Gamma(\mathbb{F}) = \{\sum_{\alpha,\beta} c^\alpha_\beta e^\alpha_\beta : c^\alpha_\beta \in W^{m,p}(U_\alpha)\}$ for $m \in \mathbb{N}_0, p \in [1,\infty]$ with the norm defined (up to equivalent norms) as

$$\left\| \sum_{\alpha,\beta} c^\alpha_\beta e^\alpha_\beta \right\|_{W^{m,p}\Gamma\mathbb{F}} := \left\| S \sum_{\alpha,\beta} c^\alpha_\beta e^\alpha_\beta \right\|_{\prod_{\alpha,\beta} W^{m,p}(U_\alpha)}$$

Then $B^s_{p,q}\Gamma(\mathbb{F})$ and $F^s_{p,q}\Gamma(\mathbb{F})$ can be defined similarly. In other words, for $m \in \mathbb{N}_0, p \in [1,\infty], q \in [1,\infty], s \geq 0$:

- $W^{m,p}\Gamma(\mathbb{F}) \simeq P \prod_{\alpha,\beta} W^{m,p}(U_\alpha)$
- $B^s_{p,q}\Gamma(\mathbb{F}) \simeq P \prod_{\alpha,\beta} B^s_{p,q}(U_\alpha)$
- $F^s_{p,q}\Gamma(\mathbb{F}) \simeq P \prod_{\alpha,\beta} F^s_{p,q}(U_\alpha),\ p \neq \infty$

By using Blackbox 48, we can show the Banach topologies of these spaces are uniquely defined (independent of the choices of ψ_α, U_α). For convenience (such as working with Holder's inequality), we still use the Sobolev norms $W^{m,p}$ ($m \in \mathbb{N}_0, p \in [1,\infty)$) defined globally in Subsection 7.1.1.

All theorems from chapter 6 that worked on bounded C^∞ domains carry over to our setting on M, *mutatis mutandis*. For instance, $B^{\frac{1}{3}}_{3,1}\Gamma(\mathbb{F}) \twoheadrightarrow L^3\,\Gamma(\mathbb{F})|_{\partial M}$ is a continuous surjection and

$$B^{\frac{1}{3}}_{3,1}\Gamma(\mathbb{F}) = \left(L^3\Gamma(\mathbb{F}), W^{1,3}\Gamma(\mathbb{F})\right)_{\frac{1}{3},1}$$

Moreover, for $p \in (1, \infty)$, $L^p\Gamma(\mathbb{F})$ is reflexive. By Holder's inequality, $(L^p\Gamma(\mathbb{F}))^* = L^{p'}\Gamma(\mathbb{F})$ for $p \in (1, \infty)$.

7.1.3 Complexification issue

A small step which we omitted is complexification. As $\mathbb{F}$ is a real vector bundle, the previous definitions only give $W^{m,p}\Gamma(\mathbb{F}) \simeq P\prod_{\alpha,\beta} W^{m,p}(U_\alpha, \mathbb{R})$ for $m \in \mathbb{N}_0, p \in [1, \infty]$. In working with real manifolds, differential forms/tensors and their dot products, we always assume real-valued coefficients for sections, but whenever we need to use theorems involving complex Banach spaces or the theory of function spaces, we assume an implicit complexification step. Fortunately, no complications arise from complexification (see Chapter 10 for the full reasoning), so for the rest of the thesis we can ignore this detail. When we want to be explicit, we will specify the scalars we are using, e.g. $\mathbb{R}W^{m,p}\Gamma(\mathbb{F})$ versus $\mathbb{C}W^{m,p}\Gamma(\mathbb{F})$.

7.2 Differential forms & boundary

Unless mentioned otherwise, the metric is the Riemannian metric, and the connection is the **Levi-Civita connection**.

For $X \in \mathfrak{X}M$, define $\mathbf{n}X = \langle X, \nu \rangle \nu \in \mathfrak{X}M|_{\partial M}$ (the **normal part**) and $\mathbf{t}X = X|_{\partial M} - \mathbf{n}X$ (the **tangential part**). We note that $\mathbf{t}X$ and $\mathbf{n}X$ only depend on $X\big|_{\partial M}$, so $\mathbf{t}$ and $\mathbf{n}$ can be defined on $\mathfrak{X}M\big|_{\partial M}$, and by abuse of notation, $\mathbf{t}(\mathfrak{X}M|_{\partial M}) \xrightarrow{\sim} \mathfrak{X}(\partial M)$.

For $\omega \in \Omega^k(M)$, define $\mathbf{t}\omega$ and $\mathbf{n}\omega$ by

$$\mathbf{t}\omega(X_1, ..., X_k) := \omega(\mathbf{t}X_1, ..., \mathbf{t}X_k) \;\; \forall X_j \in \mathfrak{X}M, j = 1, ..., k$$

and $\mathbf{n}\omega = \omega|_{\partial M} - \mathbf{t}\omega$. By abuse of notation, we similarly observe that $\mathbf{t}\left(\Omega^k(M)\big|_{\partial M}\right) \xrightarrow{\sim} \Omega^k(\partial M) = \jmath^*\left(\Omega^k(M)\big|_{\partial M}\right) = \jmath^*\left(\Omega^k(M)\right)$.

Recall the **musical isomorphism**: $X_p^\flat(Y_p) = \langle X_p, Y_p \rangle$ and $\left\langle \omega_p^\sharp, Y_p \right\rangle = \omega_p(Y_p)$ for $p \in$

$M, \omega_p \in T_p^*M, X_p \in T_pM, Y_p \in T_pM$.

Recall the usual **Hodge star operator** $\star : \Omega^k(M) \xrightarrow{\sim} \Omega^{n-k}(M)$, **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$, **codifferential** $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$, and **Hodge Laplacian** $\Delta = -(d\delta + \delta d)$ (cf. [Tay11a, Section 2.10] and [Sch95, Definition 1.2.2]).

We will often use **Penrose abstract index notation** (cf. [Wal84, Section 2.4]), which should not be confused with the similar-looking **Einstein notation** for local coordinates, or the similar-sounding **Penrose graphical notation**. In Penrose notation, we collect the usual identities in differential geometry (cf. [Lee09]):

- For any tensor $T_{a_1\dots a_k}$, define $(\nabla T)_{ia_1\dots a_k} = \nabla_i T_{a_1\dots a_k}$ and $\operatorname{div} T = \nabla^i T_{ia_2\dots a_k}$.
- $(d\omega)_{ba_1\dots a_k} = (k+1)\, \widetilde{\nabla}_{[b}\omega_{a_1\dots a_k]}\ \forall \omega \in \Omega^k(M)$ where $\widetilde{\nabla}$ is any torsion-free connection.
- $(\delta\omega)_{a_1\dots a_{k-1}} = -\nabla^b \omega_{ba_1\dots a_{k-1}} = -(\operatorname{div} w)_{a_1\dots a_{k-1}} \forall \omega \in \Omega^k(M)$
- $(\nabla_a\nabla_b - \nabla_b\nabla_a)\, T^{ij}{}_{kl} = -R_{ab\sigma}{}^{i} T^{\sigma j}{}_{kl} - R_{ab\sigma}{}^{j} T^{i\sigma}{}_{kl} + R_{abk}{}^{\sigma} T^{ij}{}_{\sigma l} + R_{abl}{}^{\sigma} T^{ij}{}_{k\sigma}$ for any tensor $T^{ij}{}_{kl}$, where R is the **Riemann curvature tensor** and ∇ the Levi-Civita connection. Similar identities hold for other types of tensors. When we do not care about the exact indices and how they contract, we can just write the **schematic identity** $(\nabla_a\nabla_b - \nabla_b\nabla_a)\, T^{ij}{}_{kl} = R * T$. As R is bounded on compact M, interchanging derivatives is a zeroth-order operation on M.
- For tensor $T_{a_1\dots a_k}$, define the **Weitzenbock curvature operator**

$$
\begin{aligned}
\operatorname{Ric}(T)_{a_1\dots a_k} &= 2\sum_{j=1}^{k} \nabla_{[i}\nabla_{a_j]} T_{a_1\dots a_{j-1}}{}^{i}{}_{a_{j+1}\dots a_k} \\
&= \sum_j R_{a_j}{}^{\sigma} T_{a_1\dots a_{j-1}\sigma a_{j+1}\dots a_k} - \sum_{j\neq l} R_{a_j}{}^{\mu}{}_{a_l}{}^{\sigma} T_{a_1\dots\sigma\dots\mu\dots a_k}
\end{aligned}
$$

  where $R_{ab} = R_{a\sigma b}{}^{\sigma}$ is the **Ricci tensor**. The invariant form is

$$
\operatorname{Ric}(T)(X_1, \dots X_k) = \sum_a \left(R(\partial_i, X_a)T\right)(X_1, \dots, X_{a-1}, \partial^i, X_{a+1}, \dots, X_k)\ \forall X_j \in \mathfrak{X}M
$$

where $\partial^i = g^{ij}\partial_j$ and $R(\partial_i, \partial_j) = \nabla_i\nabla_j - \nabla_j\nabla_i$ (Penrose notation). Note that $\langle R(\partial_a, \partial_b)\partial_d, \partial_c\rangle = R_{abcd}$. Special cases include $\mathrm{Ric}(f) = 0\ \forall f \in C^\infty(M)$ and $\mathrm{Ric}(X)_a = R_a{}^\sigma X_\sigma\ \forall X \in \mathfrak{X}M$ (justifying the notation Ric).

In local coordinates

$$\mathrm{Ric}\,(\omega) = dx^j \wedge \big(R(\partial_i, \partial_j)\omega \cdot \partial^i\big)\ \ \forall \omega \in \Omega^k(M),$$

where $\cdot$ stands for contraction (interior product). Then we have the **Weitzenbock formula**:

$$\Delta\omega = \nabla_i\nabla^i\omega - \mathrm{Ric}(\omega)\ \forall\omega \in \Omega^k(M)$$

where $\nabla_i\nabla^i\omega = \mathrm{tr}(\nabla^2\omega)$ is also called the **connection Laplacian**, which differs from the Hodge Laplacian by a zeroth-order term. The geometry of M and differential forms are more easily handled by the Hodge Laplacian, while the connection Laplacian is more useful in calculations with tensors and the Penrose notation.

- For tensors $T_{a_1\ldots a_k}$ and $Q_{a_1\ldots a_k}$, the **tensor inner product** is $\langle T, Q\rangle = T_{a_1\ldots a_k}Q^{a_1\ldots a_k}$. But for $\omega, \eta \in \Omega^k(M)$, there is another dot product, called the **Hodge inner product**, where

$$\langle\omega, \eta\rangle_\Lambda = \frac{1}{k!}\langle\omega, \eta\rangle$$

So $|\omega|_\Lambda = \sqrt{\frac{1}{k!}}\,|\omega|$. Then we define $\langle\langle\omega, \eta\rangle\rangle = \int_M \langle\omega, \eta\rangle\,\mathrm{vol}$ and $\langle\langle\omega, \eta\rangle\rangle_\Lambda = \int_M \langle\omega, \eta\rangle_\Lambda\,\mathrm{vol}$. Recall that $\omega \wedge \star\eta = \langle\omega, \eta\rangle_\Lambda\,\mathrm{vol}\ \forall\omega \in \Omega^k(M), \forall\eta \in \Omega^k(M)$. Also

$$\langle\langle d\omega, \eta\rangle\rangle_\Lambda = \langle\langle\omega, \delta\eta\rangle\rangle_\Lambda\ \ \forall\omega \in \Omega^k_{00}(M), \forall\eta \in \Omega^{k+1}_{00}(M)$$

So $\langle\cdot,\cdot\rangle_\Lambda$ is more convenient for integration by parts and the Hodge star. Nevertheless, as they only differ up to a constant factor, we can still define $W^{m,p}\Omega^k(M)$ ($m \in \mathbb{N}_0, p \in [1, \infty)$) by $\langle\cdot,\cdot\rangle$ as in Subsection 7.1. Finally, by the Weitzenbock formula and Penrose notation, we easily get the **Bochner formula**:

$$\frac{1}{2}\Delta\left(|\omega|^2\right) = \frac{1}{2}\nabla_i\nabla^i\left(\langle\omega,\omega\rangle\right) = \langle\Delta\omega,\omega\rangle + |\nabla\omega|^2 + \langle\mathrm{Ric}\left(\omega\right),\omega\rangle$$

Remark. In [Sch95], the conventions are a bit different, with

$$\Delta = \left(d\delta + \delta d\right), \Delta^\Lambda = -\nabla_i\nabla^i, R^W = -\,\mathrm{Ric}$$

and $\mathcal{N}$ the inwards unit normal vector field. Also the difference between $\langle\langle\cdot,\cdot\rangle\rangle$ and $\langle\langle\cdot,\cdot\rangle\rangle_\Lambda$ is not made explicit in the book. We will not use such notation.

Lemma 59. *Some basic identities:*

1. $\forall\omega \in \Omega^k\left(M\right) : \mathbf{t}\omega = 0 \iff \jmath^*\omega = 0$. *Similarly,* $\mathbf{n}\omega = 0 \iff \iota_\nu\omega = 0$.
2. $\left(\mathbf{t}X\right)^\flat = \mathbf{t}(X^\flat)\ \forall X \in \mathfrak{X}M$
3. $\jmath^*\mathbf{t}\omega = \jmath^*\omega$, $\mathbf{t}\omega = \iota_\nu(\nu^\flat\wedge\omega)$, $\mathbf{n}\omega = \nu^\flat\wedge\iota_\nu\omega$, $\mathbf{t}(\omega\wedge\eta) = \mathbf{t}\omega\wedge\mathbf{t}\eta\ \forall\omega \in \Omega^k(M), \forall\eta \in \Omega^l(M)$
4. $\langle\langle\mathbf{t}\omega,\eta\rangle\rangle_\Lambda = \langle\langle\mathbf{t}\omega,\mathbf{t}\eta\rangle\rangle_\Lambda = \langle\langle\omega,\mathbf{t}\eta\rangle\rangle_\Lambda\ \forall\omega,\eta \in \Omega^k(M)$
5. $\mathbf{t}\left(\star\omega\right) = \star\left(\mathbf{n}\omega\right)$, $\mathbf{n}\left(\star\omega\right) = \star\left(\mathbf{t}\omega\right)$, $\star d\omega = \left(-1\right)^{k+1}\delta\star\omega$, $\star\delta\omega = \left(-1\right)^k d\star\omega$, $\star\Delta\omega = \Delta\star\omega$ $\forall\omega \in \Omega^k(M)$
6. $\jmath^*\mathbf{t}d\omega = \jmath^*d\omega = d^{\partial M}\jmath^*\omega = d^{\partial M}\jmath^*\mathbf{t}\omega\ \forall\omega \in \Omega^k(M)$
7. *Let* $\omega \in \Omega^k(M)$. *If* $\mathbf{t}\omega = 0$ *then* $\mathbf{t}d\omega = 0$. *If* $\mathbf{n}\omega = 0$ *then* $\mathbf{n}\delta\omega = 0$.
8. $\iota_\nu\omega = \mathbf{t}\left(\iota_\nu\omega\right) = \iota_\nu\mathbf{n}\omega\ \forall\omega \in \Omega^k(M)$
9. $\jmath^*\left(\omega\wedge\star\eta\right) = \langle\jmath^*\omega,\jmath^*\iota_\nu\eta\rangle_\Lambda \mathrm{vol}_\partial\ \forall\omega \in \Omega^k(M), \forall\eta \in \Omega^{k+1}(M)$

Proof. We will only prove the last assertion. Observe that $\jmath^*\left(\mathrm{vol}\right) = 0$ so $\mathrm{vol}\,|_{\partial M} = \mathbf{n}\,\mathrm{vol} = \nu^\flat\wedge\iota_\nu\,\mathrm{vol}$. Recall $\mathrm{vol}_\partial = \jmath^*(\iota_\nu\,\mathrm{vol})$ and $\mathbf{t}\Omega^k \xrightarrow{\sim} \jmath^*\Omega^k$, so the problem is equivalent to proving

$$\nu^\flat \wedge \mathbf{t}\omega \wedge \mathbf{t} \star \eta = \langle \mathbf{t}\omega, \mathbf{t}\iota_\nu \eta \rangle_\Lambda \operatorname{vol} \text{ on } \partial M$$

Simply observe that $\mathbf{t}\iota_\nu\eta = \iota_\nu\eta$ and

$$\begin{aligned} \nu^\flat \wedge \mathbf{t}\omega \wedge \mathbf{t}\left(\star\eta\right) = \nu^\flat \wedge \mathbf{t}\omega \wedge \star\mathbf{n}\eta = \left\langle \nu^\flat \wedge \mathbf{t}\omega, \mathbf{n}\eta \right\rangle_\Lambda \operatorname{vol} \\ = \left\langle \nu^\flat \wedge \mathbf{t}\omega, \nu^\flat \wedge \iota_\nu\eta \right\rangle_\Lambda \operatorname{vol} = \langle \mathbf{t}\omega, \iota_\nu\eta \rangle_\Lambda \operatorname{vol} \end{aligned}$$

□

Theorem 60 (Integration of tensors and forms by parts)**.**

1. *For tensors* $T_{a_1 \ldots a_k}$ *and* $Q_{a_1 \ldots a_{k+1}}$*,*

$$\begin{aligned} \int_M \nabla_i \left(T_{a_1 \ldots a_k} Q^{i a_1 \ldots a_k} \right) &= \int_M \nabla_i T_{a_1 \ldots a_k} Q^{i a_1 \ldots a_k} + \int_M T_{a_1 \ldots a_k} \nabla_i Q^{i a_1 \ldots a_k} \\ &= \int_{\partial M} \nu_i T_{a_1 \ldots a_k} Q^{i a_1 \ldots a_k} \end{aligned}$$

   In other words, $\int_M \langle \nabla T, Q \rangle \operatorname{vol} + \int_M \langle T, \operatorname{div} Q \rangle \operatorname{vol} = \int_{\partial M} \langle \nu \otimes T, Q \rangle \operatorname{vol}_\partial$*.*

2. *For* $p \in (1,\infty), \omega \in \mathbb{R}W^{1,p}\Omega^k, \eta \in \mathbb{R}W^{1,p'}\Omega^{k+1}$ *:*

$$\langle\langle d\omega, \eta \rangle\rangle_\Lambda = \langle\langle \omega, \delta\eta \rangle\rangle_\Lambda + \langle\langle \jmath^*\omega, \jmath^*\iota_\nu\eta \rangle\rangle_\Lambda \tag{7.1}$$

   where $\langle\langle \jmath^*\omega, \jmath^*\iota_\nu\eta \rangle\rangle_\Lambda = \int_{\partial M} \langle \jmath^*\omega, \jmath^*\iota_\nu\eta \rangle_\Lambda \operatorname{vol}_\partial$*.*

3. *For* $p \in (1,\infty), \omega \in \mathbb{R}W^{2,p}\Omega^k(M), \eta \in \mathbb{R}W^{1,p'}\Omega^k(M)$ *:*

$$\mathcal{D}(\omega, \eta) = \langle\langle -\Delta\omega, \eta \rangle\rangle_\Lambda + \langle\langle \jmath^*\iota_\nu d\omega, \jmath^*\eta \rangle\rangle_\Lambda - \langle\langle \jmath^*\delta\omega, \jmath^*\iota_\nu\eta \rangle\rangle_\Lambda \tag{7.2}$$

   where $\mathcal{D}(\omega, \eta) := \langle\langle d\omega, d\eta \rangle\rangle_\Lambda + \langle\langle \delta\omega, \delta\eta \rangle\rangle_\Lambda$ *is called the* ***Dirichlet integral****.*

Proof.

1. Let $X^i = T_{a_1 \dots a_k} Q^{i a_1 \dots a_k}$. Then it is just the divergence theorem.

2. By approximation, it is enough to prove the smooth case.

$$\int_{\partial M} \langle \jmath^* \omega, \jmath^* \iota_\nu \eta \rangle_\Lambda \operatorname{vol}_\partial = \int_{\partial M} \jmath^* (\omega \wedge \star \eta) = \int_M d(\omega \wedge \star \eta)$$
$$= \int_M d\omega \wedge \star \eta + (-1)^k \int_M \omega \wedge d \star \eta = \langle\langle d\omega, \eta \rangle\rangle_\Lambda - \langle\langle \omega, \delta\eta \rangle\rangle_\Lambda$$

3. Trivial.

□

7.3 Boundary conditions and potential theory

Definition 61. We define:

- $\Omega^k_D(M) = \{\omega \in \Omega^k(M) : \mathbf{t}\omega = 0\}$ (**Dirichlet boundary condition**)
- $\Omega^k_{\mathrm{hom} D}(M) = \{\omega \in \Omega^k(M) : \mathbf{t}\omega = 0, \mathbf{t}\delta\omega = 0\}$ (**relative Dirichlet boundary condition**)
- $\Omega^k_N(M) = \{\omega \in \Omega^k(M) : \mathbf{n}\omega = 0\}$ (**Neumann boundary condition**)
- $\Omega^k_{\mathrm{hom}\, N}(M) = \{\omega \in \Omega^k(M) : \mathbf{n}\omega = 0, \mathbf{n}d\omega = 0\}$ (**absolute Neumann boundary condition**)
- $\Omega^k_0(M) = \Omega^k_D(M) \cap \Omega^k_N(M)$ (**trace-zero boundary condition**)
- $\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : d\omega = 0, \delta\omega = 0\}$ (**harmonic fields**)
- $\mathcal{H}^k_D(M) = \mathcal{H}^k(M) \cap \Omega^k_D(M)$ (**Dirichlet fields**)
- $\mathcal{H}^k_N(M) = \mathcal{H}^k(M) \cap \Omega^k_N(M)$ (**Neumann fields**)

Remark. In writing the function spaces, we omit M when there is no possible confusion. Note that Ω^k_{00} (compact support in $\mathring{M}$) is different from Ω^k_0.

We can readily extend these definitions to less regular spaces by replacing $\omega \in \Omega^k$ with, for example, $\omega \in B^{\frac{1}{3}}_{3,1}\Omega^k$. Boundary conditions are defined via the trace theorem, and therefore require some regularity. For example, $B^{\frac{1}{3}}_{3,1}\Omega^k_N$ makes sense, while $L^2\Omega^k_N$ and $H^1\Omega^k_{\mathrm{hom}N}$ do not make sense.

Observe that $L^2\text{-cl}\left(\Omega^k_N\right)$ (**closure** in the L^2 norm) is just $L^2\Omega^k$ since Ω^k_{00} is dense in $L^2\Omega^k$.

Most of these symbols come from [Sch95]. Note that in [Sch95], the difference between L^2X and $L^2\text{-cl}(X)$ (where X is some space) is not made explicit.

Function spaces of type $p = \infty$ are problematic since the smooth members are not dense (see Corollary 50). For instance, $W^{m,\infty}\Omega^k \neq W^{m,\infty}\text{-cl}(\Omega^k)$ in general.

A special case is when $k = 0$: $\Omega^0_N(M) = \Omega^0(M) = C^\infty(M)$ and $\Omega^0_{\mathrm{hom}D}(M) = \Omega^0_D(M)$. Indeed, the conditions for $\Omega^0_{\mathrm{hom}D}$ and $\Omega^0_{\mathrm{hom}N}$ are what analysts often call "Dirichlet" and "Neumann" boundary conditions respectively.

In fluid dynamics, the condition for Ω^1_N is also called "impermeable", while Ω^1_0 is "no-slip". On the other hand, $\Omega^1_{\mathrm{hom}N}$ is often given various names, such as "Navier-type", "free boundary" or "Hodge" [MM09a; Mon13; BAE16]. The consensus, however, seems to be that $\Omega^1_{\mathrm{hom}N}$ should be called the "absolute boundary condition" [Wu91; Hsu; COQ09; Bau17; Ouy17], which explains our choice of naming.

Lemma 62. *We have* ***Hodge duality****:*

- $\star : \Omega^k_D(M) \xrightarrow{\sim} \Omega^{n-k}_N(M)$, $\star : \Omega^k_{\mathrm{hom}\,D}(M) \xrightarrow{\sim} \Omega^{n-k}_{\mathrm{hom}\,N}(M)$, $\star : \mathcal{H}^k_D(M) \xrightarrow{\sim} \mathcal{H}^{n-k}_N(M)$.
- $\nabla_X(\star\omega) = \star(\nabla_X\omega)$, $|\star\omega|_\Lambda = |\omega|_\Lambda$ *for* $\omega \in \Omega^k, X \in \mathfrak{X}M$.
- *For* $m \in \mathbb{N}_0, p \in [1,\infty)$, *we have* $\star : W^{m,p}\Omega^k_D(M) \xrightarrow{\sim} W^{m,p}\Omega^{n-k}_N(M)$,

$$\star : W^{m,p}\Omega^k_{\mathrm{hom}\,D}(M) \xrightarrow{\sim} W^{m,p}\Omega^{n-k}_{\mathrm{hom}\,N}(M).$$

We stress that harmonic fields are **harmonic forms**, i.e. $\Delta\omega = 0$, but the converse is not true in general.

Theorem 63 (4 versions)**.** *Let $\omega \in \Omega^k(M)$ be a harmonic form. Then ω is a harmonic field if either*

1. $\mathbf{t}\omega = 0, \mathbf{n}\omega = 0$ *(trace-zero)*
2. $\mathbf{t}\omega = 0, \mathbf{t}\delta\omega = 0$ *(relative Dirichlet)*
3. $\mathbf{n}\omega = 0, \mathbf{n}d\omega = 0$ *(absolute Neumann)*
4. $\mathbf{t}\delta\omega = 0, \mathbf{n}d\omega = 0$ ***(Gaffney)***

Proof. Trivial to show $\mathcal{D}(\omega, \omega) = 0$ via integration by parts. □

Remark. The four conditions correspond to four different versions of the Poisson equation $\Delta\omega = \eta$ (cf. [Sch95, Section 3.4]), and four ways we can make Δ self-adjoint. In this thesis, we will just focus on the absolute Neumann Laplacian and the absolute Neumann heat flow.

Gaffney, one of the earliest figures in the field, showed that the Laplacian corresponding to the 4th boundary condition is self-adjoint and called it the "Neumann problem" (cf. [Gaf54; Con54]). We, however, feel the name "Neumann" should only be used when its Hodge dual is Dirichlet-related (for instance, the Dirichlet potential vs the Neumann potential, to be introduced shortly). Therefore, absent a better rationalization or convention, we see no reason not to honor the name of the mathematician.

In the same vein, some authors consider the 1st condition to be the "Dirichlet boundary condition" (following the intuition from the scalar case, where the trace and the tangential part coincide). By the same reasoning as above, we choose not to do so in this thesis.

Blackbox 64 (Dirichlet/Neumann fields)**.** *$\mathcal{H}^k_D(M)$ and $\mathcal{H}^k_N(M)$ are finite-dimensional, and therefore complemented in $\mathbb{R}W^{m,p}\Omega^k(M)$ $\forall m \in \mathbb{N}_0, p \in [1, \infty]$.*

Remark. All norms on $\mathcal{H}^k_N$ are equivalent, so we do not need to specify which norm on $\mathcal{H}^k_N$ we are using at any time.

These are very nice spaces, yet they often prevent uniqueness for boundary value problems. We almost always want to work on their orthogonal complements, where Hodge theory truly shines.

Proof. See [Sch95, Theorem 2.2.6]. □

Corollary 65. *$\forall m \in \mathbb{N}_0, p \in [1,\infty]$, there is a continuous projection $P_{m,p} : \mathbb{R}W^{m,p}\Omega^k \twoheadrightarrow \mathcal{H}^k_N$ such that*

- *it is compatible across different Sobolev spaces, i.e. $P_{m_0,p_0}(\omega) = P_{m_1,p_1}(\omega)$ if $\omega \in W^{m_0,p_0}\Omega^k \cap W^{m_1,p_1}\Omega^k$.*
- *$1 - P_{m,p} : \mathbb{R}W^{m,p}\Omega^k \twoheadrightarrow W^{m,p}\left(\mathcal{H}^k_N\right)^{\perp} := \{\omega \in W^{m,p}\Omega^k : \langle\langle \omega, \phi \rangle\rangle_\Lambda = 0 \ \forall \phi \in \mathcal{H}^k_N\}$ is also a compatible projection.*

Proof. Define the continuous linear map $\mathcal{I}_{m,p} : W^{m,p}\Omega^k \to \left(\mathcal{H}^k_N\right)^*$ where

$$\mathcal{I}_{m,p}\omega(\phi) = \langle\langle \omega, \phi \rangle\rangle_\Lambda \, \forall \phi \in \mathcal{H}^k_N, \forall \omega \in W^{m,p}\Omega^k$$

Then note that $(\phi_1, \phi_2) \mapsto \langle\langle \phi_1, \phi_2 \rangle\rangle_\Lambda$ is a positive-definite inner product on $\mathcal{H}^k_N$, so $\mathcal{I}_{m,p}\big|_{\mathcal{H}^k_N} : \mathcal{H}^k_N \xrightarrow{\sim} \left(\mathcal{H}^k_N\right)^*$. We also observe that $\mathcal{I}_{m,p}\big|_{\mathcal{H}^k_N}$ does not depend on m, p, so we can define the continuous inverse $\mathcal{J} : \left(\mathcal{H}^k_N\right)^* \xrightarrow{\sim} \mathcal{H}^k_N$. Then we can just set $P_{m,p} = \mathcal{J} \circ \mathcal{I}_{m,p}$. As we defined $\mathcal{I}_{m,p}$ by $\langle\langle \cdot, \cdot \rangle\rangle_\Lambda$, $P_{m,p}$ is compatible across different m, p. □

Remark. From now on, for $\omega \in W^{m,p}\Omega^k$, we can decompose $\boxed{\omega = \mathcal{P}^N\omega + \mathcal{P}^{N\perp}\omega}$ where $\mathcal{P}^N\omega = \omega_{\mathcal{H}^k_N} \in \mathcal{H}^k_N$ and $\mathcal{P}^{N\perp}\omega = \omega_{\left(\mathcal{H}^k_N\right)^\perp} \in W^{m,p}\left(\mathcal{H}^k_N\right)^\perp$. The decomposition is **natural**, i.e. continuous and compatible across different Sobolev spaces. By Hodge duality, similarly define $\mathcal{P}^D$ and $\mathcal{P}^{D\perp}$. Note $\mathcal{P}^{N\perp}W^{1,p}\Omega^k_N \leq W^{1,p}\Omega^k_N$ and $\mathcal{P}^{N\perp}W^{2,p}\Omega^k_{\hom N} \leq W^{2,p}\Omega^k_{\hom N}$.

Blackbox 66 (Potential theory)**.** *For $m \in \mathbb{N}_0, p \in (1,\infty)$, we define the* ***injective Neumann Laplacian***

$$\Delta_N : \mathcal{P}^{N\perp} W^{m+2,p}\Omega^k_{\text{hom}\,N} \to \mathcal{P}^{N\perp} W^{m,p}\Omega^k$$

as simply Δ under domain restriction. Then $(-\Delta_N)^{-1}$ is called the ***Neumann potential****, which is bounded (and actually a Banach isomorphism). Δ_N can also be thought of as an unbounded operator on $\mathcal{P}^{N\perp} W^{m,p}\Omega^k$.*

By Hodge duality, we also define the Dirichlet counterparts Δ_D and $(-\Delta_D)^{-1}$.

Proof. See [Sch95, Section 2.2, 2.3] □

Remark. Because duality is involved, we stay away from $p \in \{1,\infty\}$. Amazingly enough, this is the only elliptic estimate we will need for the rest of the thesis. One could say the whole theory is a functional analytic consequence of elliptic regularity (much like how the Nash embedding theorem is a consequence of Schauder estimates, following Günther's approach [Tao16]).

There are many identities which might seem complicated, but are actually trivial to check and helpful for grasping the intuition behind routine operations in Hodge theory, as well as its rich algebraic structure.

Definition. We write d_c as d restricted to $W^{1,p}\Omega^k_D$ and δ_c as δ restricted to $W^{1,p}\Omega^k_N$ for $p \in (1,\infty)$. We will prove in Section 8.4 that they are essentially adjoints of δ and d. Let us note that $\Delta_N = -(d\delta_c + \delta_c d)$ on $\mathcal{P}^{N\perp} W^{2,p}\Omega^k_{\text{hom}\,N}$.

Corollary 67. *Let $p \in (1,\infty)$. Some basic properties:*

1. *$\mathcal{P}^{D\perp}\delta = \delta$ and $\mathcal{P}^{N\perp} d = d$ on $W^{1,p}\Omega^k$.*
   $\mathcal{P}^{N\perp}\delta_c = \delta_c$ on $W^{1,p}\Omega^k_N$ and $\mathcal{P}^{D\perp} d_c = d_c$ on $W^{1,p}\Omega^k_D$.
2. *$(-\Delta_D)^{-1}\delta = \delta(-\Delta_D)^{-1}$ on $\mathcal{P}^{D\perp} W^{1,p}\Omega^k$.*
   $(-\Delta_N)^{-1} d = d(-\Delta_N)^{-1}$ on $\mathcal{P}^{N\perp} W^{1,p}\Omega^k$.

$(-\Delta_N)^{-1}\,\delta_c = \delta_c\,(-\Delta_N)^{-1}$ *on* $\mathcal{P}^{N\perp}W^{1,p}\Omega^k_N$.

$(-\Delta_D)^{-1}\,d_c = d_c\,(-\Delta_D)^{-1}$ *on* $\mathcal{P}^{D\perp}W^{1,p}\Omega^k_D$.

3. $\delta = \delta\mathcal{P}^{D\perp} = \delta\mathcal{P}^{N\perp}$ *and* $d = d\mathcal{P}^{D\perp} = d\mathcal{P}^{N\perp}$ *on* $W^{1,p}\Omega^k$.

4. $d\delta d = d\,(\delta d + d\delta) = d\,(-\Delta)$.

   $\delta d\delta\,(-\Delta_D)^{-1} = \delta$ *on* $\mathcal{P}^{D\perp}W^{1,p}\Omega^k$ *and* $d\delta d(-\Delta_N)^{-1} = d$ *on* $\mathcal{P}^{N\perp}W^{1,p}\Omega^k$.

5. $d\left(W^{2,p}\Omega^k_{\hom N}\right) = d\left(W^{2,p}\Omega^k_N\right) \cap W^{1,p}\Omega^{k+1}_N$, $\delta\left(W^{2,p}\Omega^k_{\hom D}\right) = \delta\left(W^{2,p}\Omega^k_D\right) \cap W^{1,p}\Omega^{k-1}_D$.

   $d\left(W^{3,p}\Omega^k_{\hom N}\right) \leq W^{2,p}\Omega^{k+1}_{\hom N}$, $\delta\left(W^{3,p}\Omega^k_{\hom D}\right) \leq W^{2,p}\Omega^{k-1}_{\hom D}$.

Remark. A good mnemonic device is that Δ_N is formed by d and δ_c, so $(-\Delta_N)^{-1}$ commutes with d and δ_c.

Proof.

1. Integration by parts.

2. Just check that the expressions are well-defined by using 1).

3. This comes from $1 = \mathcal{P}^D + \mathcal{P}^{D\perp}$ and so forth.

4. We simply note that $dd = 0$ and $\delta\delta = 0$.

5. This follows from the definitions of $W^{2,p}\Omega^k_{\hom N}$ and $W^{2,p}\Omega^k_{\hom D}$.

□

7.4 Hodge decomposition

We proceed differently from [Sch95], by using a more algebraic approach in order to derive some results not found in the book. There will be a lot of identities gathered through experience, so their appearances can seem unmotivated at first. Hence, as motivation, let's look at an example of a problem we will need Hodge theory for: is it true that $W^{2,p}\Omega^k_{\hom N}$ is dense in $W^{1,p}\Omega^k_N$ for $p \in (1,\infty)$? The problem is more subtle than it seems, and it is true that the heat flow, once constructed, will imply the answer is yes. But we do not yet have the heat flow, and it turns out this problem is needed for the $W^{1,p}$-analyticity of the heat flow itself. This foundational approximation of boundary conditions can be done easily once we understand Hodge theory and the myriad connections between different boundary conditions.

Let $\omega \in W^{m,p}\Omega^k$ ($m \in \mathbb{N}_0, p \in (1,\infty)$). In one line, the **Hodge-Morrey decomposition algorithm** is

$$\boxed{\omega = d_c\delta\left(-\Delta_D\right)^{-1}\mathcal{P}^{D\perp}\omega + \delta_c d\left(-\Delta_N\right)^{-1}\mathcal{P}^{N\perp}\omega + \omega_{\mathcal{H}^k}}$$

where $\mathcal{P}^{D\perp}\omega = \omega_{\left(\mathcal{H}^k_D\right)^\perp}, \mathcal{P}^{N\perp}\omega = \omega_{\left(\mathcal{H}^k_N\right)^\perp}$ are defined as in Corollary 65, and $\omega_{\mathcal{H}^k}$ is simply defined by subtraction. This is the heart of the matter, and the rest is arguably just bookkeeping.

Note that if $\omega \in W^{1,p}\Omega^k$, $d\omega = d\delta d\left(-\Delta_N\right)^{-1}\mathcal{P}^{N\perp}\omega + d\omega_{\mathcal{H}^k} = d\mathcal{P}^{N\perp}\omega + d\omega_{\mathcal{H}^k} = d\omega + d\omega_{\mathcal{H}^k}$. So $d\omega_{\mathcal{H}^k} = 0$ and similarly $\delta\omega_{\mathcal{H}^k} = 0$, justifying the notation. A mild warning is that we do not yet have $W^{1,p}\mathcal{H}^k = W^{1,p}\text{-cl}\left(\mathcal{H}^k\right)$.

As we will keep referring to this decomposition, let us define

- $\mathcal{P}_1 = d_c\delta\left(-\Delta_D\right)^{-1}\mathcal{P}^{D\perp}$. Then $\mathcal{P}_1 = d_c\left(-\Delta_D\right)^{-1}\delta\mathcal{P}^{D\perp} = d_c\left(-\Delta_D\right)^{-1}\delta$ on $W^{1,p}\Omega^k$.
- $\mathcal{P}_2 = \delta_c d\left(-\Delta_N\right)^{-1}\mathcal{P}^{N\perp}$ Then $\mathcal{P}_2 = \delta_c\left(-\Delta_N\right)^{-1}d$ on $W^{1,p}\Omega^k$.
- $\mathcal{P}_3 = 1 - \mathcal{P}_1 - \mathcal{P}_2$.

We observe that the decomposition $1 = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$ is natural (continuous and compatible across different Sobolev spaces) since all the operations are natural. In particular, $\mathcal{P}_j$ (for $j \in \{1,2,3\}$) is a zeroth-order operator, and if ω is smooth, so is $\mathcal{P}_j\omega$ by Sobolev embedding. Recall that $\mathbf{t}\omega = 0$ implies $\mathbf{t}d\omega = 0$, while $\mathbf{n}\omega = 0$ implies $\mathbf{n}\delta\omega = 0$ (Lemma 59).

Theorem 68 (Smooth decomposition)**.** *Some basic properties of $\mathcal{P}_j$ on Ω^k:*

1. $\mathcal{P}_1\delta = 0$ *on* Ω^{k+1} *and* $\mathcal{P}_2 d = 0$ *on* Ω^{k-1}.

   $\mathcal{P}_1 = \mathcal{P}_2 = 0$ *on* $\mathcal{H}^k$.

2. $\mathcal{P}_3\delta_c = 0$ *on* Ω_N^{k+1} *and* $\mathcal{P}_3 d_c = 0$ *on* Ω_D^{k-1}.

3. $\mathcal{P}_j\mathcal{P}_i = \delta_{ij}\mathcal{P}_i$. *Therefore* $\Omega^k = \bigoplus_{j=1}^3 \mathcal{P}_j\left(\Omega^k\right)$.

4. $\mathcal{P}_1\left(\Omega^k\right) = d_c\left(\Omega_D^{k-1}\right) = d_c\mathcal{P}^{D\perp}\left(\Omega_D^{k-1}\right) = d_c\delta\mathcal{P}^{D\perp}\left(\Omega^k_{\mathrm{hom}\,D}\right) \leq \Omega_D^k$.

   $\mathcal{P}_2\left(\Omega^k\right) = \delta_c\left(\Omega_N^{k+1}\right) = \delta_c\mathcal{P}^{N\perp}\left(\Omega_N^{k+1}\right) = \delta_c d\mathcal{P}^{N\perp}\left(\Omega^k_{\mathrm{hom}\,N}\right) \leq \Omega_N^k$.

   $\mathcal{P}_3\left(\Omega^k\right) = \mathcal{H}^k$.

5. $\Omega^k = \bigoplus_{j=1}^3 \mathcal{P}_j\left(\Omega^k\right)$ *is* $\langle\langle\cdot,\cdot\rangle\rangle_\Lambda$*-orthogonal decomposition.*

Proof.

1. On Ω^{k+1}, $\mathcal{P}_1\delta = d_c\left(-\Delta_D\right)^{-1}\delta\delta = 0$.

   Let $\eta \in \mathcal{H}^k$. Then $\mathcal{P}_1\eta = d_c\left(-\Delta_D\right)^{-1}\delta\eta = 0$.

2. We just need $\mathcal{P}_2\delta_c = \delta_c$ on Ω_N^{k+1}. Indeed, $\mathcal{P}_2\delta_c = \delta_c d\left(-\Delta_N\right)^{-1}\delta_c\mathcal{P}^{N\perp} = \delta_c d\delta_c\left(-\Delta_N\right)^{-1}\mathcal{P}^{N\perp} = \delta_c\mathcal{P}^{N\perp} = \delta_c$.

3. By 1), $\mathcal{P}_2\mathcal{P}_1 = \mathcal{P}_1\mathcal{P}_2 = \mathcal{P}_1\mathcal{P}_3 = \mathcal{P}_2\mathcal{P}_3 = 0$. By 2), $\mathcal{P}_3\mathcal{P}_2 = \mathcal{P}_3\mathcal{P}_1 = 0$. Then observe $\mathcal{P}_2 = (\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)\mathcal{P}_2 = \mathcal{P}_2^2$. Similarly, $\mathcal{P}_1^2 = \mathcal{P}_1$ and $\mathcal{P}_3^2 = \mathcal{P}_3$.

4. Recall $\mathcal{P}_3\left(\Omega^k\right) \leq \mathcal{H}^k$. It becomes an equality since $\mathcal{P}_2\left(\mathcal{H}^k\right) = \mathcal{P}_1\left(\mathcal{H}^k\right) = 0$. Similarly, obviously $\mathcal{P}_1\left(\Omega^k\right) = d_c\delta\mathcal{P}^{D\perp}\left(\Omega^k_{\mathrm{hom}\,D}\right) \leq d_c\left(\Omega_D^{k-1}\right)$. It becomes an equality since $\mathcal{P}_2 d = 0$ and $\mathcal{P}_3 d_c = 0$.

5. Trivial.

□

To extend this to Sobolev spaces, we will need to use distributions and duality.

Corollary 69 (Sobolev version)**.** *Some basic properties of* $\mathcal{P}_j$ *on* $W^{m,p}\Omega^k$ $(m \in \mathbb{N}_0, p \in (1,\infty))$*:*

1. $\langle\langle \mathcal{P}_j \omega, \phi \rangle\rangle_\Lambda = \langle\langle \omega, \mathcal{P}_j \phi \rangle\rangle_\Lambda$ $\forall \omega \in W^{m,p}\Omega^k, \forall \phi \in \Omega^k_{00}, j = 1, 2, 3$

2. $\mathcal{P}_1 \delta = 0$ *on* $W^{m+1,p}\Omega^{k+1}$ *and* $\mathcal{P}_2 d = 0$ *on* $W^{m+1,p}\Omega^{k-1}$.

3. $\mathcal{P}_1 = \mathcal{P}_2 = 0$ *on* $W^{m+1,p}\mathcal{H}^k$ *and* $W^{m,p}\text{-cl}\left(\mathcal{H}^k\right)$.

4. $\mathcal{P}_3 \delta_c = 0$ *on* $W^{m+1,p}\Omega^{k+1}_N$ *and* $\mathcal{P}_3 d_c = 0$ *on* $W^{m+1,p}\Omega^{k-1}_D$.

5. $\mathcal{P}_j \mathcal{P}_i = \delta_{ij} \mathcal{P}_i$. *Therefore* $W^{m,p}\Omega^k = \bigoplus_{j=1}^3 \mathcal{P}_j \left(W^{m,p}\Omega^k\right)$.

6. $\mathcal{P}_3 \left(W^{m,p}\Omega^k\right) = W^{m,p}\mathcal{H}^k$ *for* $m \geq 1$ *and* $W^{m,p}\text{-cl}\left(\mathcal{H}^k\right)$ *for* $m \geq 0$.
   $\mathcal{P}_2 \left(W^{m,p}\Omega^k\right) = \delta_c \left(W^{m+1,p}\Omega^{k+1}_N\right) = \delta_c d \mathcal{P}^{N\perp} \left(W^{m+2,p}\Omega^k_{\mathrm{hom}\, N}\right)$.
   $\mathcal{P}_1 \left(W^{m,p}\Omega^k\right) = d_c \left(W^{m+1,p}\Omega^{k-1}_D\right) = d_c \delta \mathcal{P}^{D\perp} \left(W^{m+2,p}\Omega^k_{\mathrm{hom}\, D}\right)$.

7. $\mathbf{t}\mathcal{P}_1 = 0$ *and* $\mathbf{n}\mathcal{P}_2 = 0$ *on* $W^{m+1,p}\Omega^k$.

8. *For* $p \geq 2$, $W^{m,p}\Omega^k = \bigoplus_{j=1}^3 \mathcal{P}_j \left(W^{m,p}\Omega^k\right)$ *is* $\langle\langle \cdot, \cdot \rangle\rangle_\Lambda$*-orthogonal decomposition.*

9. $W^{m,p}\text{-cl}\left(d_c \left(\Omega^{k-1}_D\right)\right) = d_c \left(W^{m+1,p}\Omega^{k-1}_D\right)$.
   $W^{m,p}\text{-cl}\left(\delta_c \left(\Omega^{k+1}_N\right)\right) = \delta_c \left(W^{m+1,p}\Omega^{k+1}_N\right)$.
   $W^{m+1,p}\text{-cl}\left(\mathcal{H}^k\right) = W^{m+1,p}\mathcal{H}^k$.

10. $d = d\left(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3\right) = d\mathcal{P}_2 = d\mathcal{P}^{N\perp} = \mathcal{P}^{N\perp} d$ *on* $W^{m+1,p}\Omega^k$.
    Consequently, $\mathbf{n} d \mathcal{P}_2 \left(W^{m+2,p}\Omega^k_{\mathrm{hom}\, N}\right) = \mathbf{n} d \left(W^{m+2,p}\Omega^k_{\mathrm{hom}\, N}\right) = 0$, *and*

    $$\mathcal{P}_2 \left(W^{m+2,p}\Omega^k_{\mathrm{hom}\, N}\right) \leq W^{m+2,p}\Omega^k_{\mathrm{hom}\, N}.$$

    We also have

    $$d\left(W^{m+1,p}\Omega^{k-1}\right) = d\mathcal{P}_2 \left(W^{m+1,p}\Omega^{k-1}\right) = d\left(W^{m+1,p}\Omega^{k-1}_N\right) = d\mathcal{P}^{N\perp}\left(W^{m+1,p}\Omega^{k-1}_N\right).$$

11. $\delta_c = \mathcal{P}_2 \delta_c$ *on* $W^{m+1,p}\Omega^k_N$ *and*

    $$\mathcal{P}_2 \left(W^{m+1,p}\Omega^k\right) = \delta_c \left(W^{m+2,p}\Omega^{k+1}_N\right) = \delta_c d \mathcal{P}^{N\perp} \left(W^{m+3,p}\Omega^k_{\mathrm{hom}\, N}\right)$$

$$= \delta_c \mathcal{P}^{N\perp}\left(W^{m+2,p}\Omega^{k+1}_{\mathrm{hom}\,N}\right).$$

Remark. Note that L^p-cl$\left(\mathcal{H}^k\right)$ $(p \in (1,\infty))$ is defined, while $L^p\mathcal{H}^k$ is not.

Proof.

1. Observe $\mathcal{P}_1\omega \in d_c\left(W^{m+1,p}\Omega^{k-1}_D\right), \mathcal{P}_2\omega \in \delta_c\left(W^{m+1,p}\Omega^{k+1}_N\right), \mathcal{P}_3\omega \in W^{m,p}$-cl$\left(\mathcal{H}^k\right)$. Simply show $d_c\left(W^{m+1,p}\Omega^{k-1}_D\right) \perp \delta_c\left(\Omega^{k+1}_N\right)$, $W^{m,p}$-cl$\left(\mathcal{H}^k\right) \perp d_c\left(\Omega^{k-1}_D\right)$, and so forth via integration by parts.

2. $W^{m+1,p}\Omega^{k+1} = W^{m+1,p}$-cl$\left(\Omega^{k+1}\right)$.

3. The case $W^{m,p}$-cl$\left(\mathcal{H}^k\right)$ is trivial. For $\omega \in W^{m+1,p}\mathcal{H}^k$,

$$\langle\langle \mathcal{P}_1\omega, \phi\rangle\rangle_\Lambda = \langle\langle \omega, \mathcal{P}_1\phi\rangle\rangle_\Lambda = 0 \ \forall \phi \in \Omega^k_{00}$$

   since $W^{m+1,p}\mathcal{H}^k \perp d_c(\Omega^{k-1}_D)$ (integration by parts).

4. Let $\omega \in W^{m+1,p}\Omega^{k+1}_N$. Then $\langle\langle \mathcal{P}_3\delta_c\omega, \phi\rangle\rangle_\Lambda = \langle\langle \delta_c\omega, \mathcal{P}_3\phi\rangle\rangle_\Lambda = 0 \ \forall \phi \in \Omega^k_{00}$ since $\delta_c\left(W^{m+1,p}\Omega^{k+1}_N\right) \perp \mathcal{H}^k$.

The rest is trivial. □

To connect Hodge decomposition to fluid dynamics, we will need the **Friedrichs decomposition**:

$$\mathcal{P}_3 = \left(\mathcal{P}^N + \mathcal{P}^{N\perp}\right)\mathcal{P}_3 = \mathcal{P}^N_3 + \mathcal{P}^{\mathrm{ex}}_3$$

where

- $\mathcal{P}^N_3 := \mathcal{P}^N\mathcal{P}_3 = \mathcal{P}^N = \mathcal{P}_3\mathcal{P}^N$ (as $\mathcal{P}^{N\perp}\mathcal{P}_1 = \mathcal{P}_1$ and $\mathcal{P}^{N\perp}\mathcal{P}_2 = \mathcal{P}_2$)
- $\mathcal{P}^{\mathrm{ex}}_3 := \mathcal{P}^{N\perp}\mathcal{P}_3 = \mathcal{P}_3\mathcal{P}^{N\perp}$

We similarly define $\mathcal{P}^D_3, \mathcal{P}^{\mathrm{co}}_3$ via Hodge duality. Note that ex and co stand for "exact" and "coexact" (and we will see why shortly).

Then we define $\mathbb{P} := \mathcal{P}_3^N + \mathcal{P}_2$ as the **Leray projection**. Then $1 = (\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1) + \left(\mathcal{P}_3^N + \mathcal{P}_2\right) = (\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1) + \mathbb{P}$ is called the **Helmholtz decomposition**.

Theorem 70 (Friedrichs decomposition)**.** *Basic properties of $\mathcal{P}_3^N, \mathcal{P}_3^{\text{ex}}$ on $W^{m,p}\Omega^k$ ($m \in \mathbb{N}_0, p \in (1,\infty)$):*

1. $\mathcal{P}_3^{\text{ex}} = d\delta(-\Delta_N)^{-1}\mathcal{P}_3^{\text{ex}}$ *on* $W^{m,p}\Omega^k$.

2. $\mathcal{P}_3^{\text{ex}}\left(W^{m,p}\Omega^k\right) = W^{m,p}\text{-cl}\left(\mathcal{H}^k\right) \cap d\left(W^{m+1,p}\Omega^{k-1}\right)$.

3. $(\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1)\left(W^{m,p}\Omega^k\right) = d\left(W^{m+1,p}\Omega^{k-1}\right) = d\left(W^{m+1,p}\Omega_N^{k-1}\right) = d\mathcal{P}^{N\perp}\left(W^{m+1,p}\Omega_N^{k-1}\right)$.

4. $\mathbb{P}\left(W^{m,p}\Omega^k\right) = \left(\mathcal{P}_3^N + \mathcal{P}_2\right)\left(W^{m,p}\Omega^k\right) = \text{Ker}\left(\delta_c\big|_{W^{m,q}\Omega_N^k}\right)$ *when* $m \geq 1$ *and* $W^{m,p}\text{-cl}\left(\text{Ker}\left(\delta_c\big|_{\Omega_N^k}\right)\right)$ *when* $m \geq 0$.

5. $(\mathcal{P}_3 + \mathcal{P}_2)\left(W^{m,p}\Omega^k\right) = \text{Ker}\left(\delta\big|_{W^{m,q}\Omega^k}\right)$ *when* $m \geq 1$.
   $(\mathcal{P}_3 + \mathcal{P}_2)\left(W^{m,p}\Omega^k\right) = W^{m,p}\text{-cl}\left(\text{Ker}\left(\delta\big|_{\Omega^k}\right)\right)$ *when* $m \geq 0$.

6. $\mathcal{P}^{N\perp}\mathbb{P} = \mathcal{P}_2 = \mathbb{P}\mathcal{P}^{N\perp}$ *on* $W^{m,p}\Omega^k$.
   Therefore $d\mathbb{P} = d\mathcal{P}^{N\perp}\mathbb{P} = d\mathcal{P}_2 = d = d\mathcal{P}^{N\perp} = \mathcal{P}^{N\perp}d$ *on* $W^{m+1,p}\Omega^k$.

7. $\mathbb{P}\left(W^{m+2,p}\Omega^k_{\text{hom}\,N}\right) \leq \mathcal{P}_2\left(W^{m+2,p}\Omega^k_{\text{hom}\,N}\right) \oplus \mathcal{H}_N^k \leq W^{m+2,p}\Omega^k_{\text{hom}\,N}$.

Proof.

1. On Ω^k: $\delta d(-\Delta_N)^{-1}\mathcal{P}_3^{\text{ex}} = \delta(-\Delta_N)^{-1}d\mathcal{P}_3^{\text{ex}} = 0$, so $\mathcal{P}_3^{\text{ex}} = (-\Delta)(-\Delta_N)^{-1}\mathcal{P}_3^{\text{ex}} = d\delta(-\Delta_N)^{-1}\mathcal{P}_3^{\text{ex}}$. Then we are done by density.

2. $\mathcal{P}_3^N d = \mathcal{P}_3\mathcal{P}^N d = 0$ as $\mathcal{P}^{N\perp}d = d$.

3. $\mathcal{P}_2 d = 0$ and $\mathcal{P}_3^N d = 0$.

4. We first prove the smooth version. Let $\omega \in \text{Ker}\left(\delta_c\big|_{\Omega_N^k}\right)$. Then $\langle\langle \mathcal{P}_1\omega, \mathcal{P}_1\omega\rangle\rangle_\Lambda = \langle\langle \mathcal{P}_1\omega, \omega\rangle\rangle_\Lambda = 0$ as $\text{Ker}\left(\delta_c\big|_{\Omega_N^k}\right) \perp d\left(\Omega^{k-1}\right)$, so $\mathcal{P}_1\omega = 0$. Similarly, $\mathcal{P}_3^{\text{ex}}\omega = 0$. Then $\left(\mathcal{P}_3^N + \mathcal{P}_2\right)\Omega^k = \text{Ker}\left(\delta_c\big|_{\Omega_N^k}\right)$.

For $W^{m,p}\Omega^k$, the case $W^{m,p}\text{-cl}\left(\operatorname{Ker}\left(\delta_c\big|_{\Omega^k_N}\right)\right)$ is trivial. Then assume $m \geq 1$ and $\omega \in \operatorname{Ker}\left(\delta_c\big|_{W^{m,q}\Omega^k_N}\right)$. We can show $\mathcal{P}_1\omega = \mathcal{P}_3^{\text{ex}}\omega = 0$ as distributions since $\operatorname{Ker}\left(\delta_c\big|_{W^{m,q}\Omega^k_N}\right) \perp d(\Omega^{k-1})$.

5. Just note that $\operatorname{Ker}\left(\delta\big|_{W^{m,q}\Omega^k}\right) \perp d_c(\Omega^{k-1}_D)$ and argue similarly.

6. Easy to check that $\mathcal{P}^{N\perp}\mathcal{P}_3^N = \mathcal{P}_3^N\mathcal{P}^{N\perp} = 0$ and $\mathcal{P}^{N\perp}\mathcal{P}_2 = \mathcal{P}_2\mathcal{P}^{N\perp} = \mathcal{P}_2$.

7. Trivial.

□

Remark. Similar results for $\mathcal{P}_3^D, \mathcal{P}_3^{\text{co}}$ hold by Hodge duality. When M has no boundary, $\mathcal{H}^k = \mathcal{H}^k_D = \mathcal{H}^k_N$ so $\mathcal{P}_3 = \mathcal{P}_3^N = \mathcal{P}_3^D$.

A simple consequence of the Hodge-Helmholtz decomposition is that

$$\frac{\operatorname{Ker}\left(\delta_c\big|_{\Omega^k_N}\right)}{\delta_c\left(\Omega^{k+1}_N\right)} = \frac{\left(\mathcal{P}_3^N + \mathcal{P}_2\right)\left(\Omega^k\right)}{\mathcal{P}_2\left(\Omega^k\right)} = \mathcal{P}_3^N\left(\Omega^k\right) = \frac{\left(\mathcal{P}_3 + \mathcal{P}_1\right)\left(\Omega^k\right)}{\left(\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1\right)\left(\Omega^k\right)} = \frac{\operatorname{Ker}\left(d\big|_{\Omega^k}\right)}{d\left(\Omega^{k-1}\right)}$$

This can be rewritten as $\boxed{\mathbb{H}^k_a(M) = \mathcal{H}^k_N(M) = \mathbb{H}^k_{\text{dR}}(M,d)}$ (**Hodge isomorphism theorem**) where $\mathbb{H}^k_{\text{dR}}(M,d) := \frac{\operatorname{Ker}\left(d\big|_{\Omega^k}\right)}{d\left(\Omega^{k-1}\right)}$ is called the **k-th de Rham cohomology group**, and $\mathbb{H}^k_a(M) := \frac{\operatorname{Ker}\left(\delta_c\big|_{\Omega^k_N}\right)}{\delta_c\left(\Omega^{k+1}_N\right)}$ is called the **k-th absolute de Rham cohomology group**. In particular, $\beta^k(M) := \dim \mathcal{H}^k_N(M) = \dim \mathbb{H}^k_{\text{dR}}(M,d)$ is called the **k-th Betti number** of M. Note that the Hodge dual of $\mathbb{H}^{n-k}_a(M)$ is $\mathbb{H}^k_r(M) := \frac{\operatorname{Ker}\left(d_c\big|_{\Omega^k_D}\right)}{d_c\left(\Omega^{k-1}_D\right)}$, the **$k$-th relative de Rham cohomology group**.

We can also define right inverses (*potentials*) for d, δ, δ_c, d_c (see Section 9.1).

In many ways, Hodge theory reduces otherwise complicated boundary value problems into purely algebraic calculations. A standard Hodge-theoretic calculation related to the Euler equation is given later in Section 9.2. We can also derive a general form of the Poincare inequality:

Corollary 71 (Poincare-Hodge-Dirac inequality). *Let* $\omega \in \mathcal{P}^{N\perp} W^{m+1,p} \Omega_N^k$ $(m \in \mathbb{N}_0, p \in (1,\infty))$. *Then*

$$\|\omega\|_{W^{m+1,p}} \sim \|d\omega\|_{W^{m,p}} + \|\delta_c \omega\|_{W^{m,p}}$$

and we have a bijection

$$\begin{aligned}\mathcal{P}^{N\perp} W^{m+1,p} \Omega_N^k \xrightarrow{d\oplus\delta_c} & \, d\left(W^{m+1,p}\Omega^k\right) \oplus \delta_c \left(W^{m+1,p}\Omega_N^k\right) \\ &= \left(\mathcal{P}_1 + \mathcal{P}_3^{ex}\right)\left(W^{m,p}\Omega^{k+1}\right) \oplus \mathcal{P}_2(W^{m,p}\Omega^{k-1})\end{aligned}$$

In particular, $\boxed{(d\oplus\delta_c)^{-1}(d\eta, \delta_c \upsilon) = \mathcal{P}_2(\eta - \upsilon) + \upsilon \ \forall \eta, \upsilon \in \mathcal{P}^{N\perp} W^{m+1,p}\Omega_N^k}$.

Proof. Observe that

- $\mathcal{P}^{N\perp} W^{m+1,p} \Omega_N^k \xrightarrow{d\oplus\delta_c} d\mathcal{P}^{N\perp}\left(W^{m+1,p}\Omega_N^k\right) \oplus \delta_c \mathcal{P}^{N\perp}\left(W^{m+1,p}\Omega_N^k\right)$ is a continuous injection.
- $d\mathcal{P}^{N\perp}\left(W^{m+1,p}\Omega_N^k\right) = d\left(W^{m+1,p}\Omega^k\right) = \left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right)\left(W^{m,p}\Omega^{k+1}\right)$ by Corollary 69.
- $\delta_c \mathcal{P}^{N\perp}\left(W^{m+1,p}\Omega_N^k\right) = \delta_c\left(W^{m+1,p}\Omega_N^k\right) = \mathcal{P}_2(W^{m,p}\Omega^{k-1})$ by Corollary 67 and 69.

By open mapping, we only need to prove $d\oplus\delta_c$ (the **injective Hodge-Dirac operator**) is surjective: let $\eta, \upsilon \in \mathcal{P}^{N\perp} W^{m+1,p}\Omega_N^k$. We want to find $\omega \in \mathcal{P}^{N\perp} W^{m+1,p}\Omega_N^k$ such that $d\omega = d\eta, \delta_c\omega = \delta_c \upsilon$. By the restriction $\delta_c \omega = \delta_c \upsilon$, the freedom is in choosing

$$\vartheta := \omega - \upsilon \in \mathcal{P}^{N\perp}\text{Ker}\left(\delta_c\big|_{W^{m+1,p}\Omega_N^k}\right) = \mathcal{P}^{N\perp}\mathbb{P}(W^{m+1,p}\Omega^k) = \mathcal{P}_2(W^{m+1,p}\Omega^k)$$

such that $d\omega = d\upsilon + d\vartheta = d\eta$. In other words, we want ϑ such that $d\vartheta = d(\eta - \upsilon)$ and $\mathcal{P}_2 \vartheta = \vartheta$. Then we are done by setting $\vartheta = \mathcal{P}_2(\eta - \upsilon)$. □

Remark. We note that a less general version of the Poincare inequality was used in [Sch95] to establish the potential estimates in Blackbox 66 as well as Blackbox 64. A more general version [Sch95, Lemma 2.4.10] deals with the case $p \geq 2$. Our version here only requires $p \in (1, \infty)$.

Among other things, the inequality allows the following approximation of boundary conditions, which will play a crucial role for the $W^{1,p}$-analyticity of the heat flow in Section 8.3.

Corollary 72. *Let $p \in (1, \infty)$.*

1. $W^{1,p}\Omega^k_N = d\left(W^{2,p}\Omega^{k-1}_{\text{hom}N}\right) \oplus \operatorname{Ker}\left(\delta_c\big|_{W^{1,p}\Omega^k_N}\right)$ *and* $\Omega^k_N = d\left(\Omega^{k-1}_{\text{hom}N}\right) \oplus \operatorname{Ker}\left(\delta_c\big|_{\Omega^k_N}\right)$.
2. $L^p\text{-cl}\left(d\left(\Omega^k_{\text{hom}N}\right)\right) = d\left(W^{1,p}\Omega^k_N\right) = d\left(W^{1,p}\Omega^k\right)$.
3. $W^{1,p}\text{-cl}\left(W^{2,p}\Omega^k_{\text{hom}N}\right) = W^{1,p}\Omega^k_N$.

Proof.

1. Because $\mathbb{P}W^{1,p}\Omega^k \leq W^{1,p}\Omega^k_N$, we conclude $\mathbb{P}W^{1,p}\Omega^k = \mathbb{P}W^{1,p}\Omega^k_N$. Meanwhile, $\left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right) W^{1,p}\Omega^k_N = (1 - \mathbb{P}) W^{1,p}\Omega^k_N \leq W^{1,p}\Omega^k_N$, so $\left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right) W^{1,p}\Omega^k_N \leq d\left(W^{2,p}\Omega^{k-1}_N\right) \cap W^{1,p}\Omega^k_N = d\left(W^{2,p}\Omega^{k-1}_{\text{hom}N}\right)$.
2. $L^p\text{-cl}\left(\left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right)\Omega^{k+1}_N\right) = \left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right) L^p\text{-cl}\left(\Omega^{k+1}_N\right) = \left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right) L^p\Omega^{k+1}$.
3. We are done if $W^{1,p}\text{-cl}\left(\mathcal{P}^{N\perp}\Omega^k_{\text{hom}N}\right) = \mathcal{P}^{N\perp}W^{1,p}\Omega^k_N$.
   Recall $\mathcal{P}_2\left(\Omega^k_{\text{hom}N}\right) \leq \Omega^k_{\text{hom}N}$ and $\delta_c\left(\Omega^k_N\right) = \delta_c\mathcal{P}^{N\perp}\left(\Omega^k_{\text{hom}\,N}\right)$ by Corollary 69, so by the formula of $(d \oplus \delta_c)^{-1}$ from Corollary 71:

$$
\begin{aligned}
(d \oplus \delta_c)^{-1}\left[d\left(\Omega^k_{\text{hom}N}\right) \oplus \delta_c\left(\Omega^k_N\right)\right] &= (d \oplus \delta_c)^{-1}\left[d\mathcal{P}^{N\perp}\left(\Omega^k_{\text{hom}N}\right) \oplus \delta_c\mathcal{P}^{N\perp}\left(\Omega^k_{\text{hom}\,N}\right)\right] \\
&= \mathcal{P}^{N\perp}\Omega^k_{\text{hom}N}
\end{aligned}
$$

   So

$$
W^{1,p}\text{-cl}\left(\mathcal{P}^{N\perp}\Omega^k_{\text{hom}N}\right) = (d \oplus \delta_c)^{-1}\left[L^p\text{-cl}\left(d\left(\Omega^k_{\text{hom}N}\right)\right) \oplus L^p\text{-cl}\left(\delta_c\left(\Omega^k_N\right)\right)\right]
$$

$$= (d \oplus \delta_c)^{-1} \left[d\left(W^{1,p}\Omega^k\right) \oplus \delta_c\left(W^{1,p}\Omega^k_N\right)\right]$$
$$= \mathcal{P}^{N\perp} W^{1,p}\Omega^k_N$$

□

7.5 An easy mistake

Let $p \in (1,\infty), \omega \in \Omega^k_{\text{hom}N}$. In other words, $\mathbf{n}\omega = 0$ and $\mathbf{n}d\omega = 0$. Using intuition from Euclidean space, it is tempting to conclude $\nabla_\nu \omega = 0$, but this is not true in general.

We will not use Penrose notation but work in local coordinates on ∂M, with $\partial_1, ... \partial_{n-1}$ for directions on ∂M and ∂_n for the direction of $\widetilde{\nu}$. Let $\{a_1, ..., a_k\} \subset \{1, ..., n-1\}$. Observe that $\mathbf{n}d\omega = 0$ implies

$$0 = (d\omega)_{na_1...a_k} = \partial_n \omega_{a_1...a_k} + \sum_i (\pm 1)\partial_{a_i}\omega_{na_1...\widehat{a_i}...a_k} = \partial_n \omega_{a_1...a_k}$$

since $\omega_{na_1...\widehat{a_i}...a_k} = 0$ on ∂M. Then recall $\partial_n \omega_{a_1...a_k} = (\nabla_n \omega)_{a_1...a_k} + \Gamma * \omega$ where $\Gamma * \omega$ is schematic for some terms with the Christoffel symbols. As Γ is bounded on M, we conclude $|\mathbf{t}\nabla_\nu \omega| \lesssim |\omega|$ and $|\mathbf{t}\nabla_\nu \omega|_\Lambda \lesssim |\omega|_\Lambda$ on ∂M. Then

$$\iota_\nu d\left(|\omega|^2\right) = \nabla_\nu \langle \omega, \omega \rangle = 2\langle \nabla_\nu \omega, \omega \rangle = 2\langle \mathbf{t}\nabla_\nu \omega, \omega \rangle$$

so $|\nabla_\nu (|\omega|^2)| \lesssim |\omega|^2$ on ∂M. This will be important in establishing the L^p-analyticity of the heat flow in Section 8.2.

CHAPTER 8

Heat flow

As promised, we now obtain a simple construction of the heat flow. We still work on the same setting as in Subsection 7.1.

8.1 L^2-analyticity

Recall that Δ_N is an unbounded operator on $\mathbb{R}\mathcal{P}^{N\perp}L^2\Omega^k$ and $(-\Delta_N)^{-1}$ is bounded. It is trivial to check that $(-\Delta_N)^{-1}$ is symmetric, therefore self-adjoint. Then Δ_N is also self-adjoint. Then for $\omega \in D(\Delta_N) = \mathcal{P}^{N\perp}H^2\Omega^k_{\hom N}$: $\langle\langle \Delta_N\omega, \omega\rangle\rangle_\Lambda = -\mathcal{D}(\omega,\omega) \leq 0$. So Δ_N is dissipative. Therefore, by a complexification argument , $\Delta_N^{\mathbb{C}}$ is acutely sectorial of angle 0 by Theorem 43 and $\left(e^{t\Delta_N^{\mathbb{C}}}\right)_{t\geq 0}$ is a C_0, analytic semigroup on $\mathbb{C}\mathcal{P}^{N\perp}L^2\Omega^k$. By Blackbox 40, we can derive some basic facts about $e^{t\Delta_N}$:

- For $m \in \mathbb{N}_1$, $D(\Delta_N^m) \leq \mathcal{P}^{N\perp}H^{2m}\Omega^k$ and $\|\Delta_N^m\omega\|_{L^2} \sim \|\omega\|_{H^{2m}} \sim \|\omega\|_{D\left(\Delta_N^m\right)}$ $\forall\omega \in D\left(\Delta_N^m\right)$ by potential estimates. Recall that $\left(e^{t\Delta_N}\right)_{t\geq 0}$ on $(D(\Delta_N^m), \|\cdot\|_{H^{2m}})$ is also a C_0 semigroup by Sobolev tower (Theorem 34).

- For $t > 0$, by either the spectral theorem (with a complexification step) or semigroup theory, $e^{t\Delta_N}$ is a self-adjoint contraction on $\mathbb{R}\mathcal{P}^{N\perp}L^2\Omega^k$, with image in $D(\Delta_N^\infty) \leq \mathcal{P}^{N\perp}\Omega^k$ by the analyticity of $\left(e^{s\Delta_N^{\mathbb{C}}}\right)_{s\geq 0}$.

- $\forall\omega \in \mathcal{P}^{N\perp}L^2\Omega^k$, $\left((0,\infty) \to \mathcal{P}^{N\perp}\Omega^k, t \mapsto e^{t\Delta_N}\omega\right)$ is C^∞-continuous by Sobolev tower. Let $m \in \mathbb{N}_1$, then $\partial_t^m\left(e^{t\Delta_N}\omega\right) = \Delta_N^m e^{t\Delta_N}\omega$ and $\left\|e^{t\Delta_N}\omega\right\|_{H^{2m}} \sim \left\|\Delta_N^m e^{t\Delta_N}\omega\right\|_{L^2} \lesssim_{\neg m,\neg t} \frac{m^m}{t^m}\|\omega\|_{L^2}$

Next we define the **non-injective Neumann Laplacian** $\widetilde{\Delta_N}$ as an unbounded operator on $L^2\Omega^k$ with $D(\widetilde{\Delta_N}^m) = D(\Delta_N^m) \oplus \mathcal{H}_N^k$ and $\widetilde{\Delta_N}^m = \Delta_N^m \oplus 0 \; \forall m \in \mathbb{N}_1$. By using either the spectral theorem or checking the definitions manually, $\widetilde{\Delta_N}$ is also a self-adjoint, dissipative operator. Then we also get an analytic heat flow, and $\widetilde{\Delta_N} = \Delta_N \oplus 0_{\mathcal{H}_N^k}$ with $e^{t\widetilde{\Delta_N}} = e^{t\Delta_N} \oplus \mathrm{Id}_{\mathcal{H}_N^k}$.

Recall that for $m \in \mathbb{N}_0, p \in (1,\infty), \omega \in W^{m,p}\Omega^k : \|\omega\|_{W^{m,p}} \sim \left\|\mathcal{P}^{N\perp}\omega\right\|_{W^{m,p}} + \left\|\mathcal{P}^N\omega\right\|_{\mathcal{H}_N^k}$ where we do not need to specify the norm on $\mathcal{H}_N^k$ as they're all equivalent. Then the previous results for Δ_N can easily be extended to $\widetilde{\Delta_N}$:

- For $m \in \mathbb{N}_1$, $D(\widetilde{\Delta_N}^m) \leq H^{2m}\Omega^k$ and $\forall \omega \in D(\widetilde{\Delta_N}^m)$: $\left\|\widetilde{\Delta_N}^m\omega\right\|_{L^2} \sim \left\|\mathcal{P}^{N\perp}\omega\right\|_{H^{2m}}$ and $\|\omega\|_{D(\widetilde{\Delta_N}^m)} \sim \|\omega\|_{H^{2m}}$. Recall $\left(e^{t\widetilde{\Delta_N}}\right)_{t\geq 0}$ on $D(\widetilde{\Delta_N}^m)$ is also an a C_0 semigroup. (Sobolev tower)

- For $t > 0$, by either the spectral theorem (with a complexification step) or semigroup theory, $e^{t\widetilde{\Delta_N}}$ is a self-adjoint contraction on $\mathbb{R}L^2\Omega^k$, with image in $D\left(\widetilde{\Delta_N}^\infty\right) \leq \Omega^k$.

- $\forall \omega \in L^2\Omega^k$, $\left((0,\infty) \to \Omega^k, t \mapsto e^{t\widetilde{\Delta_N}}\omega\right)$ is C^∞-continuous by Sobolev tower. Let $m \in \mathbb{N}_1$, then $\partial_t^m \left(e^{t\widetilde{\Delta_N}}\omega\right) = \widetilde{\Delta_N}^m e^{t\widetilde{\Delta_N}}\omega$ and

$$\left\|e^{t\widetilde{\Delta_N}}\omega\right\|_{H^{2m}} \sim \left\|e^{t\widetilde{\Delta_N}}\mathcal{P}^{N\perp}\omega\right\|_{H^{2m}} + \left\|\mathcal{P}^N\omega\right\|_{\mathcal{H}_N^k} \lesssim_{\neg m,\neg t} \frac{m^m}{t^m}\left\|\mathcal{P}^{N\perp}\omega\right\|_{L^2} + \left\|\mathcal{P}^N\omega\right\|_{\mathcal{H}_N^k}$$

By these estimates, we conclude that $e^{t\widetilde{\Delta_N}} \xrightarrow{t\to\infty} \mathcal{P}^N$ in $\mathcal{L}\left(L^2\Omega^k\right)$ (**Kodaira projection**). In fact, this is how Hodge decomposition was done historically.

8.2 L^p-analyticity

Though we could use the same symbols Δ_N and $\widetilde{\Delta_N}$ for the Neumann Laplacian on L^p, that can create confusion regarding the domains. Let them still refer to the unbounded operators on $\mathbb{R}\mathcal{P}^{N\perp}L^2\Omega^k$ and $\mathbb{R}L^2\Omega^k$ as before. However, $e^{t\Delta_N}$ and $e^{t\widetilde{\Delta_N}}$ are compatible across all L^p spaces (as we will see).

First we note that $\Omega^k_{00} \le D\left(\widetilde{\Delta_N}^\infty\right)$ so $D\left(\widetilde{\Delta_N}^\infty\right)$ is dense in L^p $\forall p \in (1,\infty)$.

Then for L^p-analyticity, we make a Gronwall-type argument (adapted from [IO14, Appendix A] to handle the boundary).

Theorem 73 (Local boundedness)**.** *For $p \in (1,\infty)\,, s \in (0,1)$ and $u \in D\left(\widetilde{\Delta_N}^\infty\right)$:*

$$\left\|e^{s\widetilde{\Delta_N}}u\right\|_p \lesssim_p \|u\|_p$$

Proof. By duality and the density of $D\left(\widetilde{\Delta_N}^\infty\right)$ in $L^2 \cap L^p$, WLOG assume $p \ge 2$. By complex interpolation (with a complexification step), WLOG assume $p = 4K$ where K is a large natural number.

Let $U(s) = e^{s\Delta_N}u$, so $\partial_s U = \Delta U$ and

$$\begin{aligned}\partial_s\left(|U|^{4K}\right) &= 2K|U|^{4K-2}\langle 2\Delta U, U\rangle \\ &\overset{\text{Bochner}}{=} 2K|U|^{4K-2}\left(\Delta\left(|U|^2\right) - 2\left|\nabla U\right|^2 - 2\left\langle \mathrm{Ric}\left(U\right), U\right\rangle\right)\end{aligned}$$

So

$$\partial_s \int_M |U|^{4K} \le 2K\int_M |U|^{4K-2}\Delta\left(|U|^2\right) + \mathcal{O}_{M,K}\left(\int_M |U|^{4K}\right)$$

Let $f = |U|^2$. As $U \in D\left(\widetilde{\Delta_N}^\infty\right) \le \Omega^k_{\mathrm{hom}\,N}$, $|\nabla_\nu f| \lesssim f$ on ∂M by Section 7.5. By Gronwall, we just need $\int_M f^{2K-1}\Delta f \lesssim \int_M f^{2K}$ (pseudo-dissipativity). Simply integrate by parts:

$$\begin{aligned}\left\langle\left\langle \Delta f, f^{2K-1}\right\rangle\right\rangle &= -\left\langle\left\langle df, d\left(f^{2K-1}\right)\right\rangle\right\rangle + \left\langle\left\langle \nabla_\nu f, f^{2K-1}\right\rangle\right\rangle \\ &= -(2K-1)\int_M |df|^2 f^{2K-2} + \mathcal{O}_M\left(\int_{\partial M} f^{2K}\right) \\ &= -\frac{2K-1}{K^2}\int_M \left|d\left(f^K\right)\right|^2 + \mathcal{O}_M\left(\int_{\partial M} f^{2K}\right)\end{aligned}$$

Let $F = |f|^K$. So for any $\varepsilon > 0$, we want $C_\varepsilon > 0$ such that $\int_{\partial M} F^2 \leq \varepsilon \int_M |dF|^2 + C_\varepsilon \int_M F^2$. This follows from Ehrling's inequality, and the fact that $H^1(M) \to L^2(\partial M)$ is compact. $\square$

So $\left(e^{t\widetilde{\Delta_N}}\right)_{t\geq 0}$ can be uniquely extended by density to $L^2\Omega^k + L^p\Omega^k$ and $e^{t\widetilde{\Delta_N}}\Big|_{L^p\Omega^k} \in \mathcal{L}\left(L^p\Omega^k\right)$. With a complexification step and an appropriate core chosen by Sobolev embedding, local boundedness on L^p implies L^p-analyticity for all $p \in (1,\infty)$ by Theorem 37.

Let A_p be the generator of $\left(e^{t\widetilde{\Delta_N}}\right)_{t\geq 0}$ on $L^p\Omega^k$. By the definition of generator, $A_p = \widetilde{\Delta_N}$ on $D\left(\widetilde{\Delta_N}^\infty\right)$. In our terminology, $A_p^{\mathbb{C}}$ is acutely quasi-sectorial. But we want a more concrete description of $D(A_p)$.

Lemma 74. *Let $p \in (1,\infty)$. Then* $\left(D(A_p), \|\cdot\|_{D(A_p)}\right) \sim \left(W^{2,p}\Omega^k_{\mathrm{hom}\,N}, \|\cdot\|_{W^{2,p}}\right)$ *and*

$$W^{2,p}\text{-cl}\left(D\left(\widetilde{\Delta_N}^\infty\right)\right) = W^{2,p}\Omega^k_{\mathrm{hom}\,N}$$

Proof. Observe that $\forall u \in D\left(\widetilde{\Delta_N}^\infty\right) : \mathcal{P}^{N\perp}u \in D(\Delta_N^\infty)$ and

$$\begin{aligned}\|u\|_{D(A_p)} = \|u\|_p + \left\|\widetilde{\Delta_N}u\right\|_p &\sim \left\|\mathcal{P}^N u\right\|_{\mathcal{H}^k_N} + \left\|\mathcal{P}^{N\perp}u\right\|_p + \left\|\Delta_N \mathcal{P}^{N\perp}u\right\|_p \\ &\sim \left\|\mathcal{P}^N u\right\|_{\mathcal{H}^k_N} + \left\|\mathcal{P}^{N\perp}u\right\|_{W^{2,p}} \sim \|u\|_{W^{2,p}}\end{aligned}$$

Then $\|\cdot\|_{D(A_p)} \sim \|\cdot\|_{W^{2,p}}$ since $D\left(\widetilde{\Delta_N}^\infty\right)$ is a dense core in $\left(D(A_p), \|\cdot\|_{D(A_p)}\right)$ (see Lemma 36). This also implies $D(A_p) = W^{2,p}\text{-cl}\left(D\left(\widetilde{\Delta_N}^\infty\right)\right) = W^{2,p}\text{-cl}\left(D\left(\Delta_N^\infty\right)\right) \oplus \mathcal{H}^k_N$.

Recall that $D\left(\Delta_N^\infty\right) \leq \left(\mathcal{P}^{N\perp}W^{2,p}\Omega^k_{\mathrm{hom}\,N}, \|\cdot\|_{W^{2,p}}\right) \overset{\Delta_N}{\xrightarrow{\sim}} \left(\mathcal{P}^{N\perp}L^p\Omega^k, \|\cdot\|_{L^p}\right)$. Since $L^p\text{-cl}\left(\Delta_N D\left(\Delta_N^\infty\right)\right) = L^p\text{-cl}\left(D\left(\Delta_N^\infty\right)\right) = \mathcal{P}^{N\perp}L^p\Omega^k$, we conclude $W^{2,p}\text{-cl}\left(D\left(\Delta_N^\infty\right)\right) = \left(-\Delta_N\right)^{-1}\left(\mathcal{P}^{N\perp}L^p\Omega^k\right) = \mathcal{P}^{N\perp}W^{2,p}\Omega^k_{\mathrm{hom}\,N}$ and we are done. $\square$

So for $p \in (1,\infty), s \in (0,1)$ and $u \in L^p\Omega^k$: $\left\| e^{s\widetilde{\Delta_N}} u \right\|_{W^{2,p}} \lesssim \frac{1}{s} \|u\|_p$. That implies

$$\left\| e^{s\widetilde{\Delta_N}} u \right\|_{W^{1,p}} \lesssim \frac{1}{\sqrt{s}} \|u\|_p$$

by complex interpolation (with complexification), using $[\mathbb{C}L^p, \mathbb{C}W^{2,p}]_{\frac{1}{2}} = \left[\mathbb{C}F^0_{p,2}, \mathbb{C}F^2_{p,2}\right]_{\frac{1}{2}} = \mathbb{C}F^1_{p,2}$.

Obviously, $D\left(A_p^\infty\right) = \{\omega \in \Omega^k_{\text{hom}N} : \Delta^m \omega \in W^{2,p}\Omega^k_{\text{hom}\,N}\ \forall m \in \mathbb{N}_0\} = D\left(\widetilde{\Delta_N}^\infty\right)$ by Sobolev embedding.

Additionally, by the density of $D\left(A_p^\infty\right)$ in L^p, we can show by approximation that

$$\left\langle\left\langle e^{t\widetilde{\Delta_N}}\omega, \eta\right\rangle\right\rangle_\Lambda = \left\langle\left\langle \omega, e^{t\widetilde{\Delta_N}}\eta\right\rangle\right\rangle_\Lambda \ \forall \omega \in L^p\Omega^k, \eta \in L^{p'}\Omega^k, p \in (1,\infty), t \geq 0$$

This implies that $e^{t\widetilde{\Delta_N}}\mathcal{P}^{N\perp} = \mathcal{P}^{N\perp}e^{t\widetilde{\Delta_N}}$ on $W^{m,p}\Omega^k\ \forall m \in \mathbb{N}_0, \forall p \in (1,\infty)$.

8.3 $W^{1,p}$-analyticity

We first observe that

$$W^{1,p}\text{-cl}\left(D\left(\widetilde{\Delta_N}^\infty\right)\right) = W^{1,p}\text{-cl}\left(W^{2,p}\text{-cl}\left(D\left(\widetilde{\Delta_N}^\infty\right)\right)\right) = W^{1,p}\text{-cl}\left(W^{2,p}\Omega^k_{\text{hom}\,N}\right) = W^{1,p}\Omega^k_N$$

by Corollary 72 and Lemma 74.

Because we will soon be dealing with differential forms of different degrees, define $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$ as the **graded algebra** of differential forms where multiplication is the wedge product. We simply define $W^{m,p}\Omega(M) = \bigoplus_{k=0}^n W^{m,p}\Omega^k(M)$, and similarly for $B^s_{p,q}, F^s_{p,q}$ spaces. Spaces like $\Omega_D(M)$, $\Omega_{00}(M)$ or $W^{m,p}\Omega_{\text{hom}N}$ are also defined by direct sums. The dot products $\langle\cdot,\cdot\rangle_\Lambda$ and $\langle\langle\cdot,\cdot\rangle\rangle_\Lambda$ are also definable as the sum from each degree. Also define $\mathcal{H}(M) = \bigoplus_{k=0}^n \mathcal{H}^k(M)$.

As an example, $\omega \in L^2\Omega(M)$ and $\eta \in L^2\Omega(M)$ would imply $\omega \wedge \eta \in L^1\Omega(M)$. We also recover integration by parts:

$$\langle\langle d\omega,\eta\rangle\rangle_\Lambda=\langle\langle \omega,\delta\eta\rangle\rangle_\Lambda+\langle\langle \jmath^*\omega,\jmath^*\iota_\nu\eta\rangle\rangle_\Lambda\ \forall\omega\in\mathbb{R}W^{1,p}\Omega\left(M\right),\forall\eta\in\mathbb{R}W^{1,p'}\Omega\left(M\right),p\in(1,\infty)$$

Then we can set $D\left(\widetilde{\Delta_N}\right)=H^2\Omega_{\hom N}$ and $D\left(A_p\right)=W^{2,p}\Omega_{\hom N}$ $(p\in(1,\infty))$, and previous results such as sectoriality or the Poincare inequality still hold true in this new degree-independent framework, *mutatis mutandis.*

Theorem 75 (Commuting with derivatives I)**.** *Let* $p\in(1,\infty)$.

1. $\delta_c\left(D\left(A_p^\infty\right)\right)\leq D\left(A_p^\infty\right)$ *and* $d\left(D\left(A_p^\infty\right)\right)\leq D\left(A_p^\infty\right)$
2. *Let* $\omega\in D(A_p)=W^{2,p}\Omega_{\hom N}$ *and* $\mathfrak{D}\in\{d,\delta_c,\delta_c d,d\delta_c\}$*. Then for* $t>0:\mathfrak{D}e^{t\widetilde{\Delta_N}}\omega=e^{t\widetilde{\Delta_N}}\mathfrak{D}\omega$*.*

Proof.

1. Let $\eta\in D\left(A_p^\infty\right)$. Obviously $d\eta\in W^{2,p}\Omega_{\hom N}$, so $d\Delta^m\eta\in W^{2,p}\Omega_{\hom N}$ $\forall m\in\mathbb{N}_0$.
   Observe that $\mathbf{n}\eta=0$ implies $\mathbf{n}\delta\eta=0$, and $\mathbf{n}d\eta=0$ implies $\mathbf{n}\delta d\eta=0$. But $\mathbf{n}\Delta\eta=0$ so $\mathbf{n}d\delta\eta=0$ and we conclude $\delta_c\eta\in W^{2,p}\Omega_{\hom N}$. Similarly, $\delta_c\Delta^m\eta\in W^{2,p}\Omega_{\hom N}$ $\forall m\in\mathbb{N}_0$.

2. Let $t>0$. Note that $\mathfrak{D}e^{t\widetilde{\Delta_N}}\omega\in D\left(A_p^\infty\right)$.
   Then $\frac{e^{h\widetilde{\Delta_N}}-1}{h}e^{t\widetilde{\Delta_N}}\omega\xrightarrow[h\downarrow 0]{C^\infty}\widetilde{\Delta_N}e^{t\widetilde{\Delta_N}}\omega$ so $\partial_t\left(\mathfrak{D}e^{t\widetilde{\Delta_N}}\omega\right)=\mathfrak{D}\widetilde{\Delta_N}e^{t\widetilde{\Delta_N}}\omega=\widetilde{\Delta_N}\mathfrak{D}e^{t\widetilde{\Delta_N}}\omega$.
   Therefore
   $$e^{h\widetilde{\Delta_N}}\mathfrak{D}e^{t\widetilde{\Delta_N}}\omega=\mathfrak{D}e^{(t+h)\widetilde{\Delta_N}}\omega\ \forall t>0,\forall h>0$$
   Note that $\mathfrak{D}e^{(t+h)\widetilde{\Delta_N}}\omega\xrightarrow[t\downarrow 0]{L^p}\mathfrak{D}e^{h\widetilde{\Delta_N}}\omega$ since $e^{(t+h)\widetilde{\Delta_N}}\omega\xrightarrow[t\downarrow 0]{C^\infty}e^{h\widetilde{\Delta_N}}\omega$.
   On the other hand, $e^{h\widetilde{\Delta_N}}\mathfrak{D}e^{t\widetilde{\Delta_N}}\omega\xrightarrow[t\downarrow 0]{L^p}e^{h\widetilde{\Delta_N}}\mathfrak{D}\omega$ as $e^{t\widetilde{\Delta_N}}\omega\xrightarrow[t\downarrow 0]{W^{2,p}}\omega$ (<u>why we need $\omega\in D(A_p)$</u>).
   So $\mathfrak{D}e^{h\widetilde{\Delta_N}}\omega=e^{h\widetilde{\Delta_N}}\mathfrak{D}\omega$ $\forall h>0$.

□

We can extend this via complexification. For $\omega \in \mathbb{C}W^{2,p}\Omega_{\text{hom}N}$, $\mathfrak{D}^{\mathbb{C}} e^{t\widetilde{\Delta_N^{\mathbb{C}}}}\omega = e^{t\widetilde{\Delta_N^{\mathbb{C}}}}\mathfrak{D}^{\mathbb{C}}\omega \ \forall t > 0$.

By L^p-analyticity, $\exists \alpha = \alpha(p) > 0$ such that $\left(e^{z\widetilde{\Delta_N^{\mathbb{C}}}}\right)_{z\in\Sigma_\alpha^+\cup\{0\}}$ is a C_0, locally bounded, analytic semigroup on $\mathbb{C}L^p\Omega$. Then by the identity theorem, $\mathfrak{D}^{\mathbb{C}} e^{z\widetilde{\Delta_N^{\mathbb{C}}}}\omega = e^{z\widetilde{\Delta_N^{\mathbb{C}}}}\mathfrak{D}^{\mathbb{C}}\omega \ \forall z \in \Sigma_\alpha^+$.

Theorem 76 ($W^{1,p}$-analyticity)**.** *$\left(e^{z\widetilde{\Delta_N^{\mathbb{C}}}}\right)_{z\in\Sigma_\alpha^+\cup\{0\}}$ is a C_0, analytic semigroup on $\mathbb{C}W^{1,p}\Omega_N$.*

Proof. Note that $\left(D(A_p^{\mathbb{C}}), \|\cdot\|_{W^{2,p}}\right)$ is dense in $(\mathbb{C}W^{1,p}\Omega_N, \|\cdot\|_{W^{1,p}})$ by Corollary 72.

So by Lemma 35, we just need to show $\left(e^{z\widetilde{\Delta_N^{\mathbb{C}}}}\right)_{z\in\Sigma_\alpha^+\cup\{0\}} \subset \mathcal{L}\left(\mathbb{C}W^{1,p}\Omega_N\right)$ and is locally bounded. So it is enough to show

$$\left\|e^{z\widetilde{\Delta_N^{\mathbb{C}}}}u\right\|_{W^{1,p}} \lesssim \|u\|_{W^{1,p}} \ \forall u \in D\left(A_p^{\mathbb{C}}\right), \forall z \in \mathbb{D} \cap \Sigma_\alpha^+$$

Consider $\mathcal{P}^{N\perp}u$, then we only need $\left\|e^{z\widetilde{\Delta_N^{\mathbb{C}}}}u\right\|_{W^{1,p}} \lesssim \|u\|_{W^{1,p}} \ \forall u \in \mathcal{P}^{N\perp}D\left(A_p^{\mathbb{C}}\right), \forall z \in \mathbb{D} \cap \Sigma_\alpha^+$.

Recall $e^{t\widetilde{\Delta_N}}\mathcal{P}^{N\perp} = \mathcal{P}^{N\perp}e^{t\widetilde{\Delta_N}}$ from Section 8.2. By the Poincare inequality (Corollary 71):

$$\begin{aligned}\left\|e^{z\widetilde{\Delta_N^{\mathbb{C}}}}u\right\|_{W^{1,p}} &\sim \left\|d^{\mathbb{C}}e^{z\widetilde{\Delta_N^{\mathbb{C}}}}u\right\|_p + \left\|\delta_c^{\mathbb{C}}e^{z\widetilde{\Delta_N^{\mathbb{C}}}}u\right\|_p = \left\|e^{z\widetilde{\Delta_N^{\mathbb{C}}}}d^{\mathbb{C}}u\right\|_p + \left\|e^{z\widetilde{\Delta_N^{\mathbb{C}}}}\delta_c^{\mathbb{C}}u\right\|_p \\ &\lesssim \left\|d^{\mathbb{C}}u\right\|_p + \left\|\delta_c^{\mathbb{C}}u\right\|_p \sim \|u\|_{W^{1,p}} \ \forall u \in \mathcal{P}^{N\perp}D\left(A_p^{\mathbb{C}}\right), \forall z \in \mathbb{D} \cap \Sigma_\alpha^+\end{aligned}$$

□

Corollary 77. *Let $\omega \in W^{1,p}\Omega_N$ and $\mathfrak{D} \in \{d, \delta_c\}$. Then for $t > 0 : \mathfrak{D}e^{t\widetilde{\Delta_N}}\omega = e^{t\widetilde{\Delta_N}}\mathfrak{D}\omega$.*

Proof. Same as before, but with $e^{t\widetilde{\Delta_N}}\omega \xrightarrow[t\downarrow 0]{W^{1,p}} \omega$. □

Let $A_{1,p}$ be the generator of $\left(e^{t\widetilde{\Delta_N}}\right)_{t\geq 0}$ on $W^{1,p}\Omega_N$. Then $A_{1,p}$ and A_p agree on $D\left(A_p^2\right)$ by the definition of generators, so $A_{1,p} = \widetilde{\Delta_N}$ on $D\left(\widetilde{\Delta_N}^{\infty}\right)$. By potential estimates, $\|\cdot\|_{D(A_{1,p})} \sim$

$\|\cdot\|_{W^{3,p}}$ on $D\left(\widetilde{\Delta_N}^{\infty}\right)$ and therefore on $\|\cdot\|_{W^{3,p}}\text{-cl}\left(D\left(\widetilde{\Delta_N}^{\infty}\right)\right) = D(A_{1,p})$. By the same argument as in Lemma 74, $D(A_{1,p}) = (-\Delta_N)^{-1}\left(\mathcal{P}^{N\perp}W^{1,p}\Omega_N\right) \oplus \mathcal{H}_N \geq D\left(\widetilde{\Delta_N}^{\infty}\right)$.

Theorem 78 (Compatibility with Hodge-Helmholtz)**.** *Let $m \in \mathbb{N}_0, p \in (1,\infty), t > 0$. By Corollary 77 and Corollary 69:*

- $e^{t\widetilde{\Delta_N}}d\left(W^{m+1,p}\Omega_N\right) = d\left(e^{t\widetilde{\Delta_N}}W^{m+1,p}\Omega_N\right) \leq d(\Omega_N) = d(\Omega)$.
- $e^{t\widetilde{\Delta_N}}\delta_c\left(W^{m+1,p}\Omega_N\right) = \delta_c\left(e^{t\widetilde{\Delta_N}}W^{m+1,p}\Omega_N\right) \leq \delta_c(\Omega_N) = \delta_c(\Omega_{\text{hom}N})$.

As $e^{t\widetilde{\Delta_N}} = 1$ on $\mathcal{H}_N$, we finally conclude $e^{t\widetilde{\Delta_N}}\left(\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1\right) = \left(\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1\right)e^{t\widetilde{\Delta_N}}, e^{t\widetilde{\Delta_N}}\mathcal{P}_2 = \mathcal{P}_2 e^{t\widetilde{\Delta_N}}$ and $e^{t\widetilde{\Delta_N}}\mathcal{P}_3^N = \mathcal{P}_3^N e^{t\widetilde{\Delta_N}} = \mathcal{P}_3^N$ on $W^{m,p}\Omega(M)$. Also, $e^{t\widetilde{\Delta_N}}\mathbb{P} = \mathbb{P}e^{t\widetilde{\Delta_N}}$ on $W^{m,p}\Omega(M)$ where $\mathbb{P}$ is the Leray projection.

By the definition of generators,

$$\widetilde{\Delta_N}\left(\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1\right) = \left(\mathcal{P}_3^{\text{ex}} + \mathcal{P}_1\right)\widetilde{\Delta_N}, \mathcal{P}_3^N\widetilde{\Delta_N} = \widetilde{\Delta_N}\mathcal{P}_3^N = 0, \mathcal{P}_2\widetilde{\Delta_N} = \widetilde{\Delta_N}\mathcal{P}_2 = \widetilde{\Delta_N}\mathbb{P} = \mathbb{P}\widetilde{\Delta_N}$$

on $D(A_p) = W^{2,p}\Omega_{\text{hom}N}$.

We briefly note that in the no-boundary case, we have $\Omega = \Omega_N = \Omega_{\text{hom}N}$, $\widetilde{\Delta_N} = \widetilde{\Delta_D} = \Delta$, $e^{t\Delta}\mathcal{P}_1 = \mathcal{P}_1 e^{t\Delta}$ on $W^{m,p}\Omega$, $\mathcal{P}_1\Delta = \Delta\mathcal{P}_1$ on $W^{2,p}\Omega$.

Remark. The operator $\mathbb{P}\widetilde{\Delta_N}$, with the domain $\mathbb{P}D(A_p)$, is a well-defined unbounded operator on $\mathbb{P}L^p\Omega$. By our arguments, its complexification is acutely sectorial, and $\mathbb{P}\widetilde{\Delta_N} = \widetilde{\Delta_N}, e^{t\mathbb{P}\widetilde{\Delta_N}} = e^{t\widetilde{\Delta_N}}$ on $\mathbb{P}L^p\Omega$. Other authors call it the **Stokes operator** corresponding to the "Navier-type" / "free" boundary condition [Miy80; Gig82; MM09a; MM09b; BAE16].

8.4 Distributions and adjoints

Like the Littlewood-Paley projection, the heat flow does not preserve compact supports in $\mathring{M}$. So applying the heat flow to a distribution is not well-defined. This can be a problem as we will need to heat up the nonlinear term in the Euler equation for Onsager's conjecture. For the Littlewood-Paley projection, we fixed it by introducing tempered distributions. That in turn motivates the following definition.

Definition 79. Let $I \subset \mathbb{R}$ be an open interval. Define

- $\mathscr{D}\Omega^k = \Omega^k_{00} = \operatorname{colim}\{\left(\Omega^k_{00}(K), C^\infty \text{ topo}\right) : K \subset \mathring{M} \text{ compact}\}$ as the space of **test k-forms** with Schwartz's topology (colimit in the category of locally convex TVS).

- $\mathscr{D}'\Omega^k = \left(\mathscr{D}\Omega^k\right)^*$ as the space of **k-currents** (or **distributional k-forms**), equipped with the weak* topology.

- $\mathscr{D}_N\Omega^k = D\left(\widetilde{\Delta_N}^\infty\right)$ as the space of **heated k-forms** with the Frechet C^∞ topology and $\mathscr{D}'_N\Omega^k = \left(\mathscr{D}_N\Omega^k\right)^*$ as the space of **heatable k-currents** (or **heatable distributional k-forms**) with the weak* topology.

- **Spacetime test forms**: $\mathscr{D}\left(I, \Omega^k\right) = C^\infty_c\left(I, \Omega^k_{00}\right) = \operatorname{colim}\{\left(C^\infty_c\left(I_1, \Omega^k_{00}(K)\right), C^\infty \text{ topo}\right) : I_1 \times K \subset I \times \mathring{M} \text{ compact}\}$ and $\mathscr{D}_N\left(I, \Omega^k\right) = \operatorname{colim}\{\left(C^\infty_c\left(I_1, \mathscr{D}_N\Omega^k\right), C^\infty \text{ topo}\right) : I_1 \subset I \text{ compact}\}$.

- **Spacetime distributions** $\mathscr{D}'\left(I, \Omega^k\right) = \mathscr{D}\left(I, \Omega^k\right)^*$, $\mathscr{D}'_N\left(I, \Omega^k\right) = \mathscr{D}_N\left(I, \Omega^k\right)^*$.

Obviously $\mathscr{D}\Omega^k \xhookrightarrow{i} \mathscr{D}_N\Omega^k$, so there is an adjoint $\mathscr{D}'_N\Omega^k \xrightarrow{i^*} \mathscr{D}'\Omega^k$. Unfortunately, $\operatorname{Im}(i)$ is not dense so i^* is not injective. Nevertheless, we will make i^* the implicit canonical map from $\mathscr{D}'_N$ to $\mathscr{D}'$. In particular, $\omega_j \xrightarrow{\mathscr{D}'_N} 0$ implies $\omega_j \xrightarrow{\mathscr{D}'} 0$. Similarly, $\mathscr{D}\left(I, \Omega^k\right) \hookrightarrow \mathscr{D}_N\left(I, \Omega^k\right)$ and $\mathscr{D}'_N\left(I, \Omega^k\right) \to \mathscr{D}'\left(I, \Omega^k\right)$.

By Sobolev tower (Theorem 34), we observe that $e^{t\widetilde{\Delta_N}}\phi \xrightarrow[t\downarrow 0]{C^\infty} \phi\ \forall \phi \in \mathscr{D}_N\Omega^k$.

For $\Lambda \in \mathscr{D}'_N\Omega^k$, $t \geq 0$ and $\phi \in \mathscr{D}_N\Omega^k$, we define $e^{t\widetilde{\Delta_N}}\Lambda(\phi) = \Lambda\left(e^{t\widetilde{\Delta_N}}\phi\right)$. As Λ is continuous, $\exists m_0, m_1 \in \mathbb{N}_0$ such that $|\Lambda(\phi)| \lesssim \|\phi\|_{C^{m_0}} \lesssim \|\phi\|_{H^{m_1}}$. Then for $t > 0$ and $\phi \in \mathscr{D}_N\Omega^k$: $\left|e^{t\widetilde{\Delta_N}}\Lambda(\phi)\right| \lesssim \left\|e^{t\widetilde{\Delta_N}}\phi\right\|_{H^{m_1}} \lesssim_{t,m_1} \|\phi\|_{L^2} \implies e^{t\widetilde{\Delta_N}}\Lambda \in L^2\Omega^k$ and $e^{t\widetilde{\Delta_N}}\Lambda = e^{\frac{t}{2}\widetilde{\Delta_N}}e^{\frac{t}{2}\widetilde{\Delta_N}}\Lambda \in \mathscr{D}_N\Omega^k$.

Also, for $p \in (1, \infty)$ and $\omega \in L^p\Omega^k$, $e^{t\widetilde{\Delta_N}}\omega$ is the same in $L^p\Omega^k$ and $\mathscr{D}'_N\Omega^k$.

Remark. We note an important limitation: though heated forms are closed under d and δ by Theorem 75, because of integration by parts, we cannot naively define δ or Δ on heatable currents.

Analogous concepts such as $\mathscr{D}_D$ and $\mathscr{D}'_D$ can be defined via Hodge duality for the relative Dirichlet heat flow.

Recall the graded algebra $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ from Section 8.3. We can easily define $\mathscr{D}\Omega, \mathscr{D}_N\Omega$ etc. by direct sums.

For $\Lambda \in \mathscr{D}'_N\Omega$ and $\phi \in \mathscr{D}_N\Omega$, we can define $\delta_c^{\mathscr{D}'_N}\Lambda(\phi) = \Lambda(d\phi)$ and $d^{\mathscr{D}'_N}\Lambda(\phi) = \Lambda(\delta_c\phi)$. These will be consistent with the smooth versions, though we take care to note that

$$\left\langle\left\langle \delta_c^{\mathscr{D}'_N}\omega, \phi \right\rangle\right\rangle_\Lambda = \langle\langle \omega, d\phi \rangle\rangle_\Lambda = \langle\langle \delta\omega, \phi \rangle\rangle_\Lambda + \langle\langle \jmath^*\iota_\nu\omega, \jmath^*\phi \rangle\rangle_\Lambda \ \forall\omega \in W^{1,p}\Omega, \phi \in \mathscr{D}_N\Omega, p \in (1,\infty) \tag{8.1}$$

So $\delta_c^{\mathscr{D}'_N}$ agrees with δ_c on $W^{1,p}\Omega_N$ as defined previously. In particular,

$$\widetilde{\Delta_N}^{\mathscr{D}'_N} = -\left(d^{\mathscr{D}'_N}\delta_c^{\mathscr{D}'_N} + \delta_c^{\mathscr{D}'_N}d^{\mathscr{D}'_N}\right)$$

is well-defined on $\mathscr{D}'_N\Omega$.

Note that $\delta^{\mathscr{D}'_N}\Lambda$ cannot be defined since there is $\phi \in \mathscr{D}_N\Omega$ such that $d_c\phi$ is not defined.

For convenience, we also write $\Lambda(\phi) = \langle\langle \Lambda, \phi \rangle\rangle_\Lambda$ (abuse of notation) and $\Lambda^\varepsilon = e^{\varepsilon\widetilde{\Delta_N}}\Lambda$ for $\varepsilon > 0$. Observe that for all $\Lambda \in \mathscr{D}'_N\Omega, \phi \in \mathscr{D}_N\Omega$:

$$\langle\langle d(\Lambda^\varepsilon), \phi \rangle\rangle_\Lambda = \langle\langle \Lambda^\varepsilon, \delta_c\phi \rangle\rangle_\Lambda = \langle\langle \Lambda, (\delta_c\phi)^\varepsilon \rangle\rangle_\Lambda = \langle\langle \Lambda, \delta_c(\phi^\varepsilon) \rangle\rangle_\Lambda = \left\langle\left\langle \left(d^{\mathscr{D}'_N}\Lambda\right)^\varepsilon, \phi \right\rangle\right\rangle_\Lambda$$

Then $d(\Lambda^\varepsilon) = \left(d^{\mathscr{D}'_N}\Lambda\right)^\varepsilon$ and similarly $\delta_c(\Lambda^\varepsilon) = \left(\delta_c^{\mathscr{D}'_N}\Lambda\right)^\varepsilon \forall\Lambda \in \mathscr{D}'_N\Omega$.

Problem (Consistency problem)**.** For $p \in (1,\infty)$, we have $L^p\Omega \hookrightarrow \mathscr{D}'_N\Omega$ and $L^p\Omega \hookrightarrow \mathscr{D}'\Omega$, and we can identify $\mathscr{D}'_N\Omega \cap L^p\Omega = \mathscr{D}'\Omega \cap L^p\Omega = L^p\Omega$. Let $d^{\mathscr{D}'}$ and $d^{\mathscr{D}'_N}$ be d defined on $\mathscr{D}'$ and $\mathscr{D}'_N$ respectively. For $\omega \in L^p\Omega$, if $d^{\mathscr{D}'}\omega \in \mathscr{D}'\Omega \cap L^p\Omega$, the question is whether we can say $d^{\mathscr{D}'_N}\omega \in \mathscr{D}'_N\Omega \cap L^p\Omega$.

More explicitly, if $\alpha, \omega \in L^p\Omega$ and $\langle\langle \alpha, \phi_0 \rangle\rangle_\Lambda = \langle\langle \omega, \delta_c \phi_0 \rangle\rangle_\Lambda \ \forall \phi_0 \in \mathscr{D}\Omega$, can we say $\langle\langle \alpha, \phi \rangle\rangle_\Lambda = \langle\langle \omega, \delta_c \phi \rangle\rangle_\Lambda \ \forall \phi \in \mathscr{D}_N\Omega$? The answer is yes, and the method is analogous to some key steps in Section 4.3 and Section 9.3.

Recall the cutoffs ψ_r from Equation (4.2).

Lemma 80. *Let $p \in (1, \infty)$ and $\phi \in W^{1,p}\Omega^k_N$. Then $(1 - \psi_r)\phi \xrightarrow[r\downarrow 0]{L^p} \phi$ and $\delta_c((1 - \psi_r)\phi) \xrightarrow[r\downarrow 0]{L^p} \delta_c \phi$.*

Proof. In Penrose notation,

$$\delta_c((1-\psi_r)\phi)_{a_1\ldots a_{k-1}} = -\nabla^i((1-\psi_r)\phi)_{ia_1\ldots a_{k-1}} = \nabla^i\psi_r\phi_{ia_1\ldots a_{k-1}} - (1-\psi_r)\nabla^i\phi_{ia_1\ldots a_{k-1}}$$

$$\implies \delta_c((1-\psi_r)\phi) = \iota_{\nabla\psi_r}\phi + (1-\psi_r)\delta_c\phi = f_r\iota_{\widetilde{\nu}}\phi + (1-\psi_r)\delta_c\phi$$

Then we only need $f_r\iota_{\widetilde{\nu}}\phi \xrightarrow[r\downarrow 0]{L^p} 0$. As $\iota_{\widetilde{\nu}}\phi = 0$ on ∂M, by Theorem 55, $\|f_r\iota_{\widetilde{\nu}}\phi\|_{L^p} \lesssim \frac{1}{r}\|\iota_{\widetilde{\nu}}\phi\|_{L^p(M_{<r})} \lesssim \|\iota_{\widetilde{\nu}}\phi\|_{W^{1,p}(M_{<r})} \xrightarrow{r\downarrow 0} 0$. ∎

Then we can conclude $\{\omega \in L^p\Omega(M) : d^{\mathscr{D}'_N}\omega \in L^p\} = \{\omega \in L^p\Omega(M) : d^{\mathscr{D}'}\omega \in L^p\}$.

Recall that for an unbounded operator A, we write $(A, D(A))$ to specify its domain.

Theorem 81 (Adjoints of d, δ)**.** *For $p \in (1, \infty)$, the closure of $(d, \Omega(M))$ as well as $(d, \mathscr{D}_N\Omega(M))$ on $L^p\Omega(M)$ is d_{L^p} where $D(d_{L^p}) = \{\omega \in L^p\Omega(M) : d^{\mathscr{D}'_N}\omega \in L^p\} = \{\omega \in L^p\Omega(M) : d^{\mathscr{D}'}\omega \in L^p\}$.*

By Hodge duality, the closure of $(\delta, \Omega(M))$ as well as $(\delta, \mathscr{D}_D\Omega(M))$ on $L^p\Omega(M)$ is δ_{L^p} where $D(\delta_{L^p}) = \{\omega \in L^p\Omega(M) : \delta^{\mathscr{D}'_D}\omega \in L^p\} = \{\omega \in L^p\Omega(M) : \delta^{\mathscr{D}'}\omega \in L^p\}$.

Define $\boxed{\delta_{c,L^p} = d^*_{L^{p'}}}$ and $\boxed{d_{c,L^p} = \delta^*_{L^{p'}}}$. Then δ_{c,L^p} is the closure of $(\delta, \mathscr{D}_N\Omega(M))$ as well as $(\delta, \mathscr{D}\Omega(M))$. Also, $D(\delta_{c,L^p}) = \{\omega \in L^p\Omega(M) : \delta_c^{\mathscr{D}'_N}\omega \in L^p\}$.

Similarly, d_{c,L^p} is the closure of $(d, \mathscr{D}_D\Omega(M))$ and $(d, \mathscr{D}\Omega(M))$. Also, $D(d_{c,L^p}) = \{\omega \in L^p\Omega(M) : d_c^{\mathscr{D}'_D}\omega \in L^p\}$.

Proof. Firstly, it is trivial to check d_{L^p} is closed and $(d, \Omega(M))$ is closable ($\omega_j \xrightarrow{L^p} 0$ and $d\omega_j \xrightarrow{L^p} \eta$ would imply $\eta = 0$ since $d\omega_j \xrightarrow{\mathscr{D}'} 0$). Then let $\omega \in D(d_{L^p})$. We can conclude $(\omega^\varepsilon, d(\omega^\varepsilon)) = \left(\omega^\varepsilon, \left(d^{\mathscr{D}'_N}\omega\right)^\varepsilon\right) \xrightarrow[\varepsilon\downarrow 0]{L^p \oplus L^p} \left(\omega, d^{\mathscr{D}'_N}\omega\right)$. This also gives the closure of $(d, \mathscr{D}_N\Omega(M))$.

Then let $G(\delta_{c,L^p}) \leq L^p\Omega \oplus L^p\Omega$ be the graph of δ_{c,L^p}. Similarly for $G(d_{L^{p'}}) \leq L^{p'}\Omega \oplus L^{p'}\Omega$. Write $\mathfrak{J}(x,y) = (-y, x)$. By the definition of adjoints, $\mathfrak{J}(G(\delta_{c,L^p})) = G(d_{L^{p'}})^\perp$. Then observe that

$$
\begin{aligned}
&\left((L^p \oplus L^p)\text{-cl}\left\{(-\delta_c\phi, \phi) : \phi \in \mathscr{D}\Omega\right\}\right)^\perp \\
&= \left\{(\omega_1, \omega_2) \in L^{p'} \oplus L^{p'} : \langle\langle \omega_1, \delta_c\phi\rangle\rangle_\Lambda = \langle\langle \omega_2, \phi\rangle\rangle_\Lambda \ \forall \phi \in \mathscr{D}\Omega\right\} \\
&= \left\{(\omega_1, \omega_2) \in L^{p'} \oplus L^{p'} : \omega_2 = d^{\mathscr{D}'}\omega_1\right\} = G(d_{L^{p'}})
\end{aligned}
$$

Then $G(\delta_{c,L^p}) = (L^p \oplus L^p)\text{-cl}\left\{(\phi, \delta_c\phi) : \phi \in \mathscr{D}\Omega\right\}$. Do the same for $\phi \in \mathscr{D}_N\Omega$. Finally, by the definition of adjoints:

$$
\begin{aligned}
D(\delta_{c,L^p}) &= \left\{\omega \in L^p\Omega(M) : \left|\langle\langle \omega, d_{L^{p'}}\phi\rangle\rangle_\Lambda\right| \lesssim \|\phi\|_{L^{p'}} \ \forall \phi \in D(d_{L^{p'}})\right\} \\
&= \Bigg\{\omega \in L^p\Omega(M) : \left|\delta_c^{\mathscr{D}'_N}\omega(\phi^\varepsilon)\right| = |\langle\langle \omega, d\phi^\varepsilon\rangle\rangle_\Lambda| = \left|\langle\langle \omega, (d_{L^{p'}}\phi)^\varepsilon\rangle\rangle_\Lambda\right| \\
&\qquad \lesssim \|\phi^\varepsilon\|_{L^{p'}} \ \forall \phi \in D(d_{L^{p'}}), \forall \varepsilon > 0\Bigg\} \\
&= \left\{\omega \in L^p\Omega(M) : \left|\delta_c^{\mathscr{D}'_N}\omega(\phi)\right| \lesssim \|\phi\|_{L^{p'}} \ \forall \phi \in \mathscr{D}_N\Omega\right\} \\
&= \left\{\omega \in L^p\Omega(M) : \delta_c^{\mathscr{D}'_N}\omega \in L^p\right\}
\end{aligned}
$$

For the third equal sign, we implicitly used the fact that $e^{t\widetilde{\Delta_N}}\phi \xrightarrow[t\downarrow 0]{C^\infty} \phi \ \forall \phi \in \mathscr{D}_N\Omega^k$. $\square$

In particular, $W^{1,p}\Omega_N = W^{1,p}\text{-cl}(\mathscr{D}_N\Omega) \leq D(\delta_{c,L^p})$. Similarly, $W^{1,p}\Omega_D \leq D(d_{c,L^p})$. This makes our choice of notation consistent.

Interestingly, a literature search yields a similar result regarding the adjoints of d and δ in [AM04, Proposition 4.3], where the authors used Lie flows *on the domain* M which is bounded in $\mathbb{R}^n$, as well as zero extensions to $\mathbb{R}^n$ to characterize $D(d_{L^p})$ and $D\left(d^*_{L^p}\right)$. In [MM09a, Equation 2.12], for $\eta \in D(\delta_{L^p})$, the authors defined $\nu \vee \eta \in B^{-\frac{1}{p}}_{p,p}\Omega\left(\partial M\right) = \left(B^{\frac{1}{p}}_{p',p'}\Omega\left(\partial M\right)\right)^*$ ($p \in (1,\infty)$) by

$$\langle\langle \nu \vee \eta, \jmath^*\omega\rangle\rangle_\Lambda = \langle\langle \eta, d\omega\rangle\rangle_\Lambda - \left\langle\left\langle \delta^{\mathscr{D}'}\eta, \omega\right\rangle\right\rangle_\Lambda \ \forall\omega \in \Omega\left(M\right)$$

which is reminiscent of Equation (8.1). Note that $\langle\langle \nu \vee \eta, \jmath^*\omega\rangle\rangle_\Lambda$ is abuse of notation (referring to the natural pairing via duality). Recall from Blackbox 49 that $W^{1,p'}\Omega\left(M\right) = F^1_{p',2}\Omega\left(M\right) \overset{\text{Trace}}{\twoheadrightarrow} B^{\frac{1}{p}}_{p',p'}\Omega\left(M\right)\big|_{\partial M}$ has a bounded linear section Ext, so it is possible to choose ω such that $\|\jmath^*\omega\|_{B^{\frac{1}{p}}_{p',p'}} \sim \|\omega\|_{W^{1,p'}}$ and therefore $\nu \vee \eta$ is well-defined with

$$\|\nu \vee \eta\|_{B^{-\frac{1}{p}}_{p,p}} \sim \sup_{\substack{\omega \in W^{1,p'}\Omega(M) \\ \|\jmath^*\omega\|_{B^{1/p}_{p',p'}}=1}} |\langle\langle \nu \vee \eta, \jmath^*\omega\rangle\rangle_\Lambda| \lesssim \|\eta\|_{L^p} + \left\|\delta^{\mathscr{D}'}\eta\right\|_{L^p}$$

Of course, for $\eta \in W^{1,p}\Omega$, $\nu \vee \eta = \jmath^*\iota_\nu\eta$. We can now show an alternative description of $D\left(\delta_{c,L^p}\right)$:

Theorem 82. *For $p \in (1,\infty)$, $D\left(\delta_{c,L^p}\right) = \{\eta \in L^p\Omega(M) : \delta_c^{\mathscr{D}'_N}\eta \in L^p\} = \{\eta \in L^p\Omega(M) : \delta^{\mathscr{D}'}\eta \in L^p$ and $\nu \vee \eta = 0\}$.*

Proof. Assume $\eta \in L^p\Omega(M)$ and $\delta_c^{\mathscr{D}'_N}\eta \in L^p$. Then $\exists\alpha \in L^p\Omega\left(M\right) : \alpha = \delta_c^{\mathscr{D}'_N}\eta = \delta^{\mathscr{D}'}\eta$. By the definition of $\nu\vee\eta$, $\langle\langle \alpha, \omega\rangle\rangle_\Lambda + \langle\langle \nu \vee \eta, \jmath^*\omega\rangle\rangle_\Lambda = \langle\langle \eta, d\omega\rangle\rangle_\Lambda \ \forall\omega \in \Omega\left(M\right)$. By the definition of $\delta_c^{\mathscr{D}'_N}\eta$, $\langle\langle \alpha, \omega\rangle\rangle_\Lambda = \langle\langle \eta, d\omega\rangle\rangle_\Lambda \ \forall\omega \in \mathscr{D}_N\Omega$. So $\langle\langle \nu \vee \eta, \jmath^*\omega\rangle\rangle_\Lambda = 0 \ \forall\omega \in \mathscr{D}_N\Omega$. Recall that Ext (the right inverse of Trace) is bounded, so $B^{\frac{1}{p}}_{p',p'}\text{-cl}\left(\jmath^*\left(\mathscr{D}_N\Omega\right)\right) \overset{\text{Ext}}{=} \jmath^*\left(W^{1,p}\text{-cl}\left(\mathscr{D}_N\Omega\right)\right) = \jmath^*\left(W^{1,p}\Omega_N\left(M\right)\right) \overset{\text{Ext}}{=} \jmath^*\left(W^{1,p}\Omega\left(M\right)\right) = B^{\frac{1}{p}}_{p',p'}\Omega\left(\partial M\right)$. Therefore $\nu \vee \eta = 0$.

Conversely, now assume $\eta \in L^p\Omega(M)$, $\delta^{\mathscr{D}'}\eta = \alpha \in L^p$ and $\nu \vee \eta = 0$. Then by the definition of $\nu \vee \eta$ for $\eta \in D(\delta_{L^p})$, $\langle\langle \alpha, \omega \rangle\rangle_\Lambda = \langle\langle \eta, d\omega \rangle\rangle_\Lambda$ $\forall \omega \in \Omega(M)$. The formula also holds for $\omega \in \mathscr{D}_N\Omega$, and therefore $\delta_c^{\mathscr{D}'_N}\eta = \alpha \in L^p$. □

This result agrees with [MM09a, Equation 2.17]. Our characterization of the adjoints of d and δ further highlights how heatable currents are truly natural objects in Hodge theory, independent of the theory of heat flows.

In particular, it is trivial to show $\mathbb{P}L^p\Omega = L^p\text{-cl}\,\mathrm{Ker}\left(\delta_c\big|_{\Omega_N}\right) = \{\eta \in D\left(\delta_{c,L^p}\right) : \delta_c^{\mathscr{D}'_N}\eta = 0\}$ for $p \in (1, \infty)$.

Remark. The name "heatable current" simply refers to the largest topological vector space of differential forms (and hence vector fields) for which the heat equation can be solved (i.e. *heatable*), and once we apply the heat flow a heatable current becomes heated. The name "current" for distributional forms was introduced by Georges de Rham [Rha84], likely with its physical equivalents in mind, and has since become standard in various areas of mathematics such as geometric measure theory and complex manifolds.

It is not easy to search for literature dealing with the subject and how it relates to Hodge theory. They are mentioned in a couple of papers [BB97; Tro09] dealing with "tempered currents" or "temperate currents" on $\mathbb{R}^n$ – differential forms with tempered-distributional coefficients. Yet the notion of "tempered" – not growing too fast – does not make sense on a compact manifold with boundary. Arguably, it is the ability to facilitate the heat flow, or the Littlewood-Paley projection, that most characterizes tempered distributions and makes them ideal for harmonic analysis. For scalar functions, much more is known (cf. [KP14; BBD18; Tan18] and their references). In the same vein, various results from harmonic analysis should also hold for heatable currents.

8.5 Square root

We will not need this for the rest of the thesis, but a popular question is the characterization of the square root of the Laplacian.

By the Poincare inequality, $\mathcal{P}^{N\perp}H^1\Omega^k_N$ is a Hilbert space where the H^1-inner product can be replaced by $(\omega,\eta)\mapsto\mathcal{D}(\omega,\eta)$ (the Dirichlet integral). The space is dense in $\mathcal{P}^{N\perp}L^2\Omega^k$. Define $\mathcal{A}$ as an unbounded operator on $\mathcal{P}^{N\perp}L^2\Omega^k$ where

$$D\left(\mathcal{A}\right)=\{\omega\in\mathcal{P}^{N\perp}H^1\Omega^k_N:\left|\mathcal{D}(\omega,\eta)\right|\lesssim_{\omega}\left\|\eta\right\|_2\ \forall\eta\in\mathcal{P}^{N\perp}H^1\Omega^k_N\}$$

and $\langle\langle\mathcal{A}\omega,\eta\rangle\rangle_{\Lambda}=\mathcal{D}(\omega,\eta)\ \forall\omega\in D(\mathcal{A}),\forall\eta\in\mathcal{P}^{N\perp}H^1\Omega^k_N$. Easy to check that $\langle\langle\mathcal{A}\omega,\eta\rangle\rangle_{\Lambda}=\mathcal{D}\left(\left(-\Delta_N\right)^{-1}\mathcal{A}\omega,\eta\right)\ \forall\eta\in\mathcal{P}^{N\perp}H^1\Omega^k_N$. Therefore $\omega=\left(-\Delta_N\right)^{-1}\mathcal{A}\omega\in\mathcal{P}^{N\perp}H^2\Omega^k_{\mathrm{hom}N}$ and $\mathcal{A}\omega=\left(-\Delta_N\right)\omega\ \forall\omega\in D(A)$, so $\mathcal{A}\subset-\Delta_N$. It is trivial to check $D\left(-\Delta_N\right)\leq D\left(\mathcal{A}\right)$, so $\mathcal{A}=-\Delta_N$.

By **Friedrichs extension** (cf. [Tay11a, Appendix A, Proposition 8.7], [Tay11c, Section 8, Proposition 2.2]), we conclude that

$$\begin{aligned}\mathbb{C}\mathcal{P}^{N\perp}H^1\Omega^k_N &= \left[\mathbb{C}\mathcal{P}^{N\perp}L^2\Omega^k,\left(D\left(\Delta^{\mathbb{C}}_N\right),\left\|\cdot\right\|_{D\left(\Delta^{\mathbb{C}}_N\right)}\right)\right]_{\frac{1}{2}}=\left[\mathbb{C}\mathcal{P}^{N\perp}L^2\Omega^k,\mathbb{C}\mathcal{P}^{N\perp}H^2\Omega^k_{\mathrm{hom}N}\right]_{\frac{1}{2}}\\ &= \left(D\left(\sqrt{-\Delta^{\mathbb{C}}_N}\right),\left\|\cdot\right\|_{D\left(\sqrt{-\Delta^{\mathbb{C}}_N}\right)}\right)\end{aligned}$$

By direct summing, we can extend the result to $\widetilde{\Delta_N}$ to get

$$\mathbb{C}H^1\Omega^k_N=\left[\mathbb{C}L^2\Omega^k,\mathbb{C}H^2\Omega^k_{\mathrm{hom}N}\right]_{\frac{1}{2}}=\left(D\left(\sqrt{-\widetilde{\Delta^{\mathbb{C}}_N}}\right),\left\|\cdot\right\|_{D\left(\sqrt{-\widetilde{\Delta^{\mathbb{C}}_N}}\right)}\right)$$

We note that the norms are only defined up to equivalent norms, and $\left\|\cdot\right\|_{D(\mathcal{A})}$ is not the same as $\left\|\cdot\right\|^*_{D(\mathcal{A})}$ (see Chapter 3). This difference is not always made explicit in [Tay11a; Tay11c].

8.6 Some trace-zero results

Although we will not need them for the rest of the thesis, let us briefly delineate some results regarding the trace-zero Laplacian (cf. Theorem 63) which are similar to those obtained above for the absolute Neumann Laplacian. We begin by retracing our steps from

Corollary 65.

Define $\mathcal{H}_0^k(M) = \mathcal{H}_N^k(M) \cap \mathcal{H}_D^k(M)$. Obviously, $\mathcal{H}_0^k(M)$ is finite-dimensional and we can define $\mathcal{P}^0$ and $\mathcal{P}^{0\perp}$ the same way we did for $\mathcal{P}^N$ and $\mathcal{P}^{N\perp}$ in Corollary 65. When M has no boundary, $\mathcal{P}^{0\perp} = \mathcal{P}^{N\perp}$ and $\mathcal{P}^0 = \mathcal{P}^N = \mathcal{P}_3$.

It is a celebrated theorem, following from the **Aronszajn continuation theorem** [AKS62], that $\mathcal{H}_0^k(M) = 0$ when every connected component of M has nonempty boundary (cf. [Sch95, Theorem 3.4.4]). When that happens, $\mathcal{P}^{0\perp} = 1$ and $\mathcal{P}^0 = 0$.

Blackbox 83 (Potential theory)**.** *For $m \in \mathbb{N}_0, p \in (1, \infty)$, we define the* ***injective trace-zero Laplacian***

$$\Delta_0 : \mathcal{P}^{0\perp} W^{m+2,p}\Omega_0^k \to \mathcal{P}^{0\perp} W^{m,p}\Omega^k$$

as simply Δ under domain restriction. Then $(-\Delta_0)^{-1}$ is called the ***trace-zero potential****, which is bounded. Δ_0 can also be thought of as an unbounded operator on $\mathcal{P}^{0\perp} W^{m,p}\Omega_0^k$.*

Proof. We only need to prove the theorem on each connected component of M. So WLOG, M is connected. If $\partial M = \emptyset$, we are back to the absolute Neumann case in Blackbox 66. When $\partial M \neq \emptyset$, $\mathcal{P}^{0\perp} = 1$ and we only need to show the trace-zero Poisson problem $\left(\Delta\omega, \omega\big|_{\partial M}\right) = (\eta, 0)$ is uniquely solvable for each $\eta \in W^{m,p}\Omega^k$. This is [Sch95, Theorem 3.4.10]. □

Consequently, we have a trivial decomposition

$$\omega = \mathcal{P}^{0\perp}\omega + \mathcal{P}^0\omega = d\delta\left(-\Delta_0\right)^{-1}\mathcal{P}^{0\perp}\omega + \delta d\left(-\Delta_0\right)^{-1}\mathcal{P}^{0\perp}\omega + \mathcal{P}^0\omega$$

for $\omega \in W^{m,p}\Omega^k$, $m \in \mathbb{N}_0$, $p \in (1, \infty)$. This decomposition is not as useful as the Hodge-Morrey decomposition (Section 7.4) since the the first two terms are not orthogonal. However, it does mean that, when $\mathcal{P}^0 = 0$, every differential form is a sum of exact and coexact forms.

For $\omega \in \mathcal{P}^{0\perp} W^{m+2,p}\Omega_0^k, m \in \mathbb{N}_0, p \in (1,\infty)$, we also have $\omega = (-\Delta_0)^{-1}(-\Delta_0)\omega = (-\Delta_0)^{-1}(d\delta\omega + \delta d\omega)$, so $\|\omega\|_{W^{m+2,p}} \sim \|\delta\omega\|_{W^{m+1,p}} + \|d\omega\|_{W^{m+1,p}}$. This trick is not enough to get the full Poincare inequality $\|\omega\|_{W^{1,p}} \sim \|\delta\omega\|_p + \|d\omega\|_p$, and therefore [Sch95, Lemma 2.4.10.iv] might be wrong.

As $(-\Delta_0)^{-1}$ is symmetric and bounded on $\mathcal{P}^{0\perp} L^2\Omega^k$, we conclude Δ_0 is a self-adjoint and dissipative operator on $\mathcal{P}^{0\perp} L^2\Omega^k$, with the domain $D(\Delta_0) = \mathcal{P}^{0\perp} H^2\Omega_0^k$. This means $\Delta_0^{\mathbb{C}}$ is acutely sectorial on $\mathbb{C}\mathcal{P}^{0\perp} L^2\Omega^k$.

Next we define the **non-injective trace-zero Laplacian** $\widetilde{\Delta_0}$ as an unbounded operator on $L^2\Omega^k$ with $D\left(\widetilde{\Delta_0}^m\right) = D(\Delta_0^m) \oplus \mathcal{H}_0^k$ and $\widetilde{\Delta_0}^m = \Delta_0^m \oplus 0\ \forall m \in \mathbb{N}_1$. Again, $\widetilde{\Delta_0^{\mathbb{C}}}$ is acutely sectorial on $\mathbb{C}L^2\Omega^k$ and $\|\omega\|_{D\left(\widetilde{\Delta_0}^m\right)} \sim \|\omega\|_{H^{2m}}\ \forall \omega \in D\left(\widetilde{\Delta_0}^m\right), \forall m \in \mathbb{N}_1$. In particular, $D\left(\widetilde{\Delta_0}\right) = \mathcal{P}^{0\perp} H^2\Omega_0^k \oplus \mathcal{H}_0^k = H^2\Omega_0^k$.

For L^p-analyticity, observe that on ∂M: $\left|\nabla_\nu\left(|\omega|^2\right)\right| = 2\left|\langle\nabla_\nu\omega, \omega\rangle\right| = 0 \lesssim |\omega|^2\ \forall \omega \in W^{2,p}\Omega_0^k, \forall p \in (1,\infty)$. So we argue as in Theorem 73, and L^p-analyticity follows.

Remark. The operator $\mathbb{P}\widetilde{\Delta_0}$, with the domain $H^2\Omega_0^k \cap \mathbb{P}L^2\Omega^k$, is a well-defined unbounded operator on $\mathbb{P}L^2\Omega^k$. It is called the **Stokes operator** corresponding to the trace-zero/no-slip boundary condition, as discussed in [FK64; GM85; MM08] and others. It lies outside the scope of this thesis. For more information, see [HS18] and its references.

CHAPTER 9

Results related to the Euler equation

9.1 Hodge-Sobolev spaces

We will have need of negative-order Sobolev spaces when we calculate the pressure in the Euler equation.

Recall the space of heatable currents $\mathscr{D}'_N\Omega$ (defined in Section 8.4). Note that $\mathcal{P}^{N\perp}$ is well-defined on $\mathscr{D}'_N\Omega$ by $\left\langle\left\langle \mathcal{P}^{N\perp}\Lambda, \phi\right\rangle\right\rangle_\Lambda = \left\langle\left\langle \Lambda, \mathcal{P}^{N\perp}\phi\right\rangle\right\rangle\ \forall\Lambda \in \mathscr{D}'_N\Omega, \forall\phi \in \mathscr{D}_N\Omega$. Same for $\mathcal{P}^N$, and we can uniquely identify $\mathcal{P}^N\Lambda \in \mathcal{H}_N\ \forall\Lambda \in \mathscr{D}'_N\Omega$.

Similarly, $\mathbb{P}\left(\mathscr{D}_N\Omega\right) \leq \mathscr{D}_N\Omega$ (use Theorem 78 and Theorem 70), so $\mathbb{P}$, $1-\mathbb{P} = \left(\mathcal{P}_1 + \mathcal{P}_3^{\text{ex}}\right)$ and $\mathcal{P}_2 = \mathbb{P} - \mathcal{P}^N$ are well-defined on $\mathscr{D}'_N\Omega$.

For all $p \in (1,\infty)$, define $D_N = d^{\mathscr{D}'_N} + \delta_c^{\mathscr{D}'_N}$ on $\mathcal{P}^{N\perp}\mathscr{D}'_N\Omega$ and $\widetilde{D_N} = d^{\mathscr{D}'_N} + \delta_c^{\mathscr{D}'_N}$ on $\mathscr{D}'_N\Omega$ as the **injective** and **non-injective (Neumann) Hodge-Dirac operators**.

By the Poincare inequality (Corollary 71), it is easy to check that

$$D_N\big|_{\mathcal{P}^{N\perp}\mathscr{D}_N\Omega} : \mathcal{P}^{N\perp}\mathscr{D}_N\Omega \to \mathcal{P}^{N\perp}\mathscr{D}_N\Omega$$

is bijective. Consequently, so is D_N on $\mathcal{P}^{N\perp}\mathscr{D}'_N\Omega$.

Observe that

$$\forall m \in \mathbb{N}_0, \forall p \in (1,\infty), \forall\alpha \in \mathcal{P}^{N\perp}W^{m,p}\Omega\left(M\right), \exists!\beta = \left(D_N\right)^{-1}\alpha \in \mathcal{P}^{N\perp}W^{m+1,p}\Omega_N$$

and

$$\|\beta\|_{W^{m+1,p}} \sim \|\alpha\|_{W^{m,p}} = \|d\beta + \delta_c\beta\|_{W^{m,p}} \sim \|d\beta\|_{W^{m,p}} + \|\delta_c\beta\|_{W^{m,p}} \tag{9.1}$$

because $\mathcal{P}^{N\perp}W^{m,p}\Omega = d\left(W^{m+1,p}\Omega\right) \oplus \delta_c\left(W^{m+1,p}\Omega_N\right)$ is a direct sum of closed subspaces (corresponding to $\mathcal{P}_1 + \mathcal{P}_3^{\mathrm{ex}}$ and $\mathcal{P}_2$).

Note that we do not have $d^{\mathscr{D}'_N}D_N = D_N d^{\mathscr{D}'_N}$, but $d^{\mathscr{D}'_N}D_N^2 = D_N^2 d^{\mathscr{D}'_N} = -\Delta_N^{\mathscr{D}'_N} d^{\mathscr{D}'_N}$ is true.

Definition 84. For $m \in \mathbb{Z}, p \in (1,\infty)$, let $W^{m,p}(D_N) := (D_N)^{-m}\left(\mathcal{P}^{N\perp}L^p\Omega\right) = \{\alpha \in \mathcal{P}^{N\perp}\mathscr{D}'_N\Omega : (D_N)^m\alpha \in L^p\Omega\}$ and $W^{m,p}\left(\widetilde{D_N}\right) := W^{m,p}(D_N) \oplus \mathcal{H}_N$. They are Banach spaces under the norms $\|\alpha\|_{W^{m,p}(D_N)} := \|(D_N)^m\alpha\|_{L^p\Omega}$ and $\|\beta\|_{W^{m,p}\left(\widetilde{D_N}\right)} := \left\|\mathcal{P}^{N\perp}\beta\right\|_{W^{m,p}(D_N)} + \left\|\mathcal{P}^N\beta\right\|_{\mathcal{H}_N}$.

In a sense, these are comparable to homogeneous and inhomogeneous Bessel potential spaces. We can extend the definitions to fractional powers, but that is outside the scope of this thesis.

It is trivial to check that $\|\alpha\|_{W^{m,p}\left(\widetilde{D_N}\right)} \sim \|\alpha\|_{W^{m,p}\Omega}\ \forall\alpha \in \mathscr{D}_N\Omega, \forall m \in \mathbb{N}_0, \forall p \in (1,\infty)$.

Theorem 85. *Some basic properties of* $W^{m,p}\left(\widetilde{D_N}\right)$*:*

1. $\mathscr{D}_N\Omega$ *is dense in* $W^{m,p}\left(\widetilde{D_N}\right)\ \forall m \in \mathbb{Z}, \forall p \in (1,\infty)$.
2. $W^{m,p}\left(\widetilde{D_N}\right) = W^{m,p}\text{-cl}\left(\mathscr{D}_N\Omega\right)\ \forall m \in \mathbb{N}_0, \forall p \in (1,\infty)$.
3. $\left\|d^{\mathscr{D}'_N}\beta\right\|_{W^{m,p}\left(\widetilde{D_N}\right)} + \left\|\delta_c^{\mathscr{D}'_N}\beta\right\|_{W^{m,p}\left(\widetilde{D_N}\right)} \lesssim \|\beta\|_{W^{m+1,p}\left(\widetilde{D_N}\right)}\ \forall\beta \in W^{m+1,p}\left(\widetilde{D_N}\right), \forall m \in \mathbb{Z}, \forall p \in (1,\infty)$
   Then $\mathcal{P}_2 = \delta_c^{\mathscr{D}'_N} d^{\mathscr{D}'_N}\left(-\Delta_N^{\mathscr{D}'_N}\right)^{-1}\mathcal{P}^{N\perp} = \delta_c^{\mathscr{D}'_N}\left(-\Delta_N^{\mathscr{D}'_N}\right)^{-1} d^{\mathscr{D}'_N}$ *and* $\mathbb{P} = \mathcal{P}_2 + \mathcal{P}^N$ *are of order 0 on* $W^{m,p}\left(\widetilde{D_N}\right)$.
4. $\left(W^{m,p}\left(\widetilde{D_N}\right)\right)^* = W^{-m,p'}\left(\widetilde{D_N}\right)\ \forall m \in \mathbb{Z}, \forall p \in (1,\infty)$ *via the pairing*

$$\langle\alpha,\phi\rangle_{W^{-m,p}\left(\widetilde{D_N}\right),W^{m,p'}\left(\widetilde{D_N}\right)} = \left\langle\left\langle D_N^{-m}\mathcal{P}^{N\perp}\alpha, D_N^m\mathcal{P}^{N\perp}\phi\right\rangle\right\rangle_\Lambda + \left\langle\left\langle\mathcal{P}^N\alpha, \mathcal{P}^N\phi\right\rangle\right\rangle_\Lambda$$

Proof.

1. Because $D_N^m\left(\mathcal{P}^{N\perp}\mathscr{D}_N\Omega\right)=\mathcal{P}^{N\perp}\mathscr{D}_N\Omega$ is dense in $\mathcal{P}^{N\perp}L^p\Omega$.

2. We only need $W^{m,p}\left(D_N\right)=\mathcal{P}^{N\perp}W^{m,p}\left(\widetilde{D_N}\right)\leq W^{m,p}\text{-cl}\left(\mathcal{P}^{N\perp}\mathscr{D}_N\Omega\right)$. Let $\alpha\in\mathcal{P}^{N\perp}W^{m,p}\left(\widetilde{D_N}\right)$ and $\alpha^{\varepsilon}=e^{\varepsilon\widetilde{\Delta_N}}\alpha$ as usual. Then $D_N^m\left(\alpha^{\varepsilon}\right)=\left(D_N^m\alpha\right)^{\varepsilon}\xrightarrow[\varepsilon\downarrow 0]{L^p}D_N^m\alpha$. So $D_N^{-m}D_N^m\left(\alpha^{\varepsilon}\right)=\alpha^{\varepsilon}\xrightarrow[\varepsilon\downarrow 0]{W^{m,p}}\alpha$ by Equation (9.1).

3. Let $D_N^{m+1}\mathcal{P}^{N\perp}\beta\in L^p$. Then $D_N^m\mathcal{P}^{N\perp}\beta\in\mathcal{P}^{N\perp}W^{1,p}\Omega_N$ by Equation (9.1).

   When $m=2k$ ($k\in\mathbb{Z}$):

$$\begin{aligned}\left\|dD_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}+\left\|\delta_c D_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}&\sim\left\|dD_N^{2k}\mathcal{P}^{N\perp}\beta+\delta_c D_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}\\&=\left\|D_N^{2k+1}\mathcal{P}^{N\perp}\beta\right\|_{L^p}\end{aligned}$$

   When $m=2k+1$ ($k\in\mathbb{Z}$): $D_N^{2k}\mathcal{P}^{N\perp}\beta\in\mathcal{P}^{N\perp}W^{2,p}\Omega_{\hom N}$ and

$$\begin{aligned}&\left\|D_N dD_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}+\left\|D_N\delta_c D_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}\\&=\left\|\delta_c dD_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}+\left\|d\delta_c D_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}\\&\sim\left\|\delta_c dD_N^{2k}\mathcal{P}^{N\perp}\beta+d\delta_c D_N^{2k}\mathcal{P}^{N\perp}\beta\right\|_{L^p}=\left\|D_N^{2k+2}\mathcal{P}^{N\perp}\beta\right\|_{L^p}\end{aligned}$$

4. Simply observe that $\left(W^{m,p}\left(D_N\right)\right)^*=W^{-m,p'}\left(D_N\right)$ via the isomorphisms

$$W^{m,p}\left(D_N\right)\overset{D_N^m}{\xrightarrow{\sim}}\mathcal{P}^{N\perp}L^p\Omega$$

   and $W^{-m,p'}\left(D_N\right)\overset{D_N^{-m}}{\xrightarrow{\sim}}\mathcal{P}^{N\perp}L^{p'}\Omega$.

□

Remark. We briefly note that D_N with the domain $\mathcal{P}^{N\perp}H^1\Omega_N$ is self-adjoint on $\mathcal{P}^{N\perp}L^2\Omega$ and its complexification is therefore "bisectorial". For more on this, see [McI86; McI10; MM18].

Corollary 86. *Assume* $U \in \mathbb{P}L^2\mathfrak{X}$. *Define* $\operatorname{div}(U\otimes U) \in \mathscr{D}'_N\mathfrak{X}$ *by* $\langle\langle \operatorname{div}(U\otimes U), X\rangle\rangle_\Lambda := -\langle\langle U\otimes U, \nabla X\rangle\rangle\ \forall X \in \mathscr{D}_N\mathfrak{X}$.

If $p \in (1,\infty)$ *and* $U\otimes U \in L^p\Gamma(TM\otimes TM)$, *then* $\left\|\operatorname{div}(U\otimes U)^\flat\right\|_{W^{-1,p}\left(\widetilde{D_N}\right)} \lesssim \|U\otimes U\|_{L^p}$.

Proof. For $\eta \in \Omega(M)$, write η_k for the part of η in Ω^k. Let $\phi \in \mathscr{D}_N\Omega$, then

$$\left|\left\langle\left\langle D_N^{-1}\mathcal{P}^{N\perp}\operatorname{div}(U\otimes U)^\flat, \phi\right\rangle\right\rangle_\Lambda\right| = \left|\left\langle\left\langle U\otimes U, \nabla\left(D_N^{-1}\mathcal{P}^{N\perp}\phi\right)_1^\sharp\right\rangle\right\rangle\right| \lesssim \|U\otimes U\|_{L^p}\|\phi\|_{L^{p'}}$$

This implies $\operatorname{div}(U\otimes U)^\flat \in W^{-1,p}\left(\widetilde{D_N}\right)$. Then observe $\left|\left\langle\left\langle \operatorname{div}(U\otimes U)^\flat, \phi\right\rangle\right\rangle_\Lambda\right| = \left|\left\langle\left\langle U\otimes U, \nabla(\phi)_1^\sharp\right\rangle\right\rangle\right| \lesssim \|U\otimes U\|_{L^p}\|\phi\|_{W^{1,p'}\left(\widetilde{D_N}\right)}$. □

9.2 Calculating the pressure

In this section, we assume that $\partial_t\mathcal{V} + \operatorname{div}(\mathcal{V}\otimes\mathcal{V}) + \operatorname{grad}\mathfrak{p} \overset{\mathscr{D}'_N(I,\mathfrak{X})}{=\!=} 0$, $\mathcal{V} \in L^2_{\text{loc}}(I, \mathbb{P}L^2\mathfrak{X})$, $\mathfrak{p} \in L^1_{\text{loc}}(I\times M)$. This is true, for instance, in the case of Onsager's conjecture (see Section 4.3 and Section 4.4).

We first note that $\mathcal{H}^0_N = \mathcal{H}^0 = \{\text{locally constant functions}\}$. Then we can show $\mathcal{V}$ uniquely determines $\mathfrak{p}$ by a formula, up to a difference in $\mathcal{H}^0_N$ ($d\mathfrak{p}$ is always unique). It is no loss of generality to <u>set $\mathfrak{p} = \mathcal{P}^{N\perp}\mathfrak{p}$</u> (implying $\int_M \mathfrak{p} = 0$).

1. Assume $\mathcal{V}\otimes\mathcal{V} \in L^q_tW^{m+1,p}\Gamma(TM\otimes TM)$ for some $m \in \mathbb{N}_0, p \in (1,\infty), q \in [1,\infty]$.

   Let $\omega = \operatorname{div}(\mathcal{V}\otimes\mathcal{V})^\flat$. Then $d^{\mathscr{D}'_N}\mathfrak{p} \overset{\mathscr{D}'_N(I,\mathfrak{X})}{=\!=} (\mathbb{P}-1)\,\omega \in L^q_tW^{m,p}\Omega^1$. By the Poincare inequality (Corollary 71), there is a unique $\mathfrak{f} \in L^q_t\mathcal{P}^{N\perp}W^{m+1,p}\Omega^0$ such that $d\mathfrak{f} = (\mathbb{P}-1)\,\omega \overset{\mathscr{D}'_N(I,\mathfrak{X})}{=\!=} d^{\mathscr{D}'_N}\mathfrak{p}$. An explicit formula is $\mathfrak{f} = -R_d\omega$ where

   $$R_d := \mathcal{P}^{N\perp}\delta\left(-\Delta_D\right)^{-1}\mathcal{P}^{D\perp} + \mathcal{P}^{N\perp}\delta\left(-\Delta_N\right)^{-1}\mathcal{P}^{\text{ex}}_3$$

   is the potential for d.

We aim to show $\mathfrak{f}=\mathfrak{p}$. Let $\psi\in C_c^\infty\left(I,\mathscr{D}_N\Omega^0\right)$. Then because $\Omega^0=\mathcal{P}_2\left(\Omega^0\right)\oplus\mathcal{P}_3^N\left(\Omega^0\right)$, we conclude $\mathcal{P}^{N\perp}\psi=\delta_c\phi$ where $\phi:=d\left(-\Delta_N\right)^{-1}\mathcal{P}^{N\perp}\psi\in C_c^\infty\left(I,\mathscr{D}_N\Omega^1\right)$ and

$$\begin{aligned}\int_I\langle\langle\mathfrak{f},\psi\rangle\rangle_\Lambda=\int_I\left\langle\left\langle\mathfrak{f},\mathcal{P}^{N\perp}\psi\right\rangle\right\rangle_\Lambda=\int_I\langle\langle\mathfrak{f},\delta_c\phi\rangle\rangle_\Lambda=\int_I\langle\langle d\mathfrak{f},\phi\rangle\rangle_\Lambda\\=\int_I\left\langle\left\langle d^{\mathscr{D}'_N}\mathfrak{p},\phi\right\rangle\right\rangle_\Lambda=\int_I\langle\langle\mathfrak{p},\psi\rangle\rangle_\Lambda.\end{aligned}$$

Therefore $\mathfrak{p}=\mathfrak{f}$ and $\boxed{\|\mathfrak{p}\|_{L_t^qW^{m+1,p}}\lesssim\|\omega\|_{L_t^qW^{m,p}}\lesssim\|\mathcal{V}\otimes\mathcal{V}\|_{L_t^qW^{m+1,p}}}$.

2. Assume $\mathcal{V}\otimes\mathcal{V}\in L_t^qL^p\Gamma\left(TM\otimes TM\right)$ for some $p\in(1,\infty),q\in[1,\infty]$.
   Let $\omega=\operatorname{div}(\mathcal{V}\otimes\mathcal{V})^\flat$. Then $d^{\mathscr{D}'_N}\mathfrak{p}\overset{\mathscr{D}'_N(I,\mathfrak{X})}{=\!=}\left(\mathbb{P}-1\right)\omega\in L_t^qW^{-1,p}\left(\widetilde{D_N}\right)$ by Corollary 86 and Theorem 85. Then $-\delta_c^{\mathscr{D}'_N}d^{\mathscr{D}'_N}\mathfrak{p}\overset{\mathscr{D}'_N(I,\mathfrak{X})}{=\!=}\delta_c^{\mathscr{D}'_N}\left(1-\mathbb{P}\right)\omega=\delta_c^{\mathscr{D}'_N}\omega\in L_t^qW^{-2,p}\left(\widetilde{D_N}\right)$ and $\mathfrak{p}=-D_N^{-2}\delta_c^{\mathscr{D}'_N}\omega$, so $\|\mathfrak{p}\|_{L_t^qL^p}\lesssim\left\|\delta_c^{\mathscr{D}'_N}\omega\right\|_{L_t^qW^{-2,p}\left(\widetilde{D_N}\right)}\lesssim\|\omega\|_{L_t^qW^{-1,p}\left(\widetilde{D_N}\right)}\lesssim\|\mathcal{V}\otimes\mathcal{V}\|_{L_t^qL^p}$.

Remark. It is also possible to define $R_{\delta_c}:=d^{\mathscr{D}'_N}\left(-\Delta_N^{\mathscr{D}'_N}\right)^{-1}\mathcal{P}^{N\perp}$ on $\mathscr{D}'_N\Omega$ and have $R_d=\left(D_N^{-1}-R_{\delta_c}\right)\mathcal{P}^{N\perp}$ on $\mathscr{D}'_N\Omega$. This would then imply $\|R_d\alpha\|_{W^{m+1,p}\left(\widetilde{D_N}\right)}\lesssim\|\alpha\|_{W^{m,p}\left(\widetilde{D_N}\right)}$ $\forall\alpha\in W^{m,p}\left(\widetilde{D_N}\right),\forall m\in\mathbb{Z},\forall p\in(1,\infty)$.

9.3 On an interpolation identity

Let $p\in(1,\infty)$. We are faced with the difficulty of finding a good interpolation characterization for $B_{p,1}^{\frac{1}{p}}\Omega_N$. We do have $B_{p,1}^{\frac{1}{p}}\Omega=\left(L^p\Omega,W^{1,p}\Omega\right)_{\frac{1}{p},1}$ (complexification, then projection onto the real part), but our heat flow is not analytic on $\mathbb{C}W^{1,p}\Omega$. The hope is that $B_{p,1}^{\frac{1}{p}}\Omega_N=\left(L^p\Omega,W^{1,p}\Omega_N\right)_{\frac{1}{p},1}$, and our first guess is to try to find some kind of projection. Indeed, the Leray projection yields

$$\mathbb{P}B_{p,1}^{\frac{1}{p}}\Omega=\left(\mathbb{P}L^p\Omega,\mathbb{P}W^{1,p}\Omega\right)_{\frac{1}{p},1}\tag{9.2}$$

and the heat flow is well-behaved on $\mathbb{P}W^{1,p}\Omega=\mathbb{P}W^{1,p}\Omega_N$ (Theorem 70, Theorem 78). By interpolation, $\mathbb{P}$ is $B_{p,1}^{\frac{1}{p}}$-continuous, so $\mathbf{n}\mathbb{P}:B_{p,1}^{\frac{1}{p}}\Omega\to L^p\Omega\big|_{\partial M}$ is continuous and $\mathbb{P}B_{p,1}^{\frac{1}{p}}\Omega=$

$\mathbb{P}B^{\frac{1}{p}}_{p,1}\Omega_N$.

This is enough to get all the Besov estimates we will need for Onsager's conjecture.

Additionally, it is true that the heat semigroup is also C_0 and analytic on $\mathbb{CP}B^{\frac{1}{p}}_{p,1}\Omega_N$ by Yosida's half-plane criterion (Theorem 41). Unlike the L^p-analyticity case, here we already have analyticity on the 2 endpoints, so the criterion simply follows by interpolation. Alternatively, observe that there exists $C > 0$ such that $\sup_{t>0} \left\| t\left(\widetilde{\Delta^{\mathbb{C}}_N} - C\right) e^{t\left(\widetilde{\Delta^{\mathbb{C}}_N} - C\right)} \right\|_{\mathcal{L}(V)} < \infty$ for $V \in \{\mathbb{CP}L^p\Omega, \mathbb{CP}W^{1,p}\Omega_N\}$. Therefore it also holds for $V = \mathbb{CP}B^{\frac{1}{p}}_{p,1}\Omega_N$ by interpolation, and that is another criterion for analyticity ([Eng00, Section II, Theorem 4.6.c]).

Unfortunately, this does not tell us about the relationship between $(L^p\Omega, W^{1,p}\Omega_N)_{\frac{1}{p},1}$ and $B^{\frac{1}{p}}_{p,1}\Omega_N$. Obviously $(L^p\Omega, W^{1,p}\Omega_N)_{\frac{1}{p},1} \hookrightarrow B^{\frac{1}{p}}_{p,1}\Omega_N$ by the density of $W^{1,p}\Omega_N$. The other direction is more delicate. Interpolation involving boundary conditions is often nontrivial. The reader can see [Gui91; Lof92; Ama19] to get an idea of the challenges involved, especially at the critical regularity levels $\mathbb{N} + \frac{1}{p}$.

Nevertheless, there are a few interesting things we can say about these spaces.

Definition 87 (Neumann condition on strip)**.** For vector field X and $r > 0$ small, with ψ_r as in Equation (4.2), define

$$\mathbf{n}_r X = \psi_r \langle X, \widetilde{\nu} \rangle \widetilde{\nu} \text{ and } \mathbf{t}_r X = X - \mathbf{n}_r X$$

Then define $\mathfrak{X}_{N,r} = \{X \in \mathfrak{X} : \langle X, \widetilde{\nu} \rangle = 0 \text{ on } M_{<r}\}$. Similarly we can define $W^{m,p}\mathfrak{X}_{N,r}$ and $B^s_{p,q}\mathfrak{X}_{N,r}$ by setting $\|\langle X, \widetilde{\nu} \rangle\|_{L^1(M_{<r})} = 0$. We note that $L^3\mathfrak{X}_{N,r}$ makes sense since the definition does not require the trace theorem, unlike $L^3\mathfrak{X}_N$ which is ill-defined.

Some basic facts:

1. $\mathbf{t}_r\mathfrak{X} \leq \mathfrak{X}_{N,\frac{r}{2}}$
2. $\mathbf{t}_r = 1$ and $\mathbf{n}_r = 0$ on $\mathfrak{X}_{N,r}$

3. $\mathbf{t}_{\frac{r}{2}}\mathbf{t}_r = \mathbf{t}_r$

4. $\|\mathbf{t}_r X\|_{W^{m,p}} \lesssim_{r,m,p} \|X\|_{W^{m,p}}$ for $m \in \mathbb{N}_0, p \in [1,\infty]$

5. $W^{m,p}\mathfrak{X}_{N,r}$ and $B^s_{p,q}\mathfrak{X}_{N,r}$ are Banach for $m \in \mathbb{N}_0, p \in [1,\infty], s \geq 0, q \in [1,\infty]$

6. $B^{m_\theta}_{p,q}\mathfrak{X}_{N,r} \xhookrightarrow{\mathbf{t}_r=1} \left(W^{m_0,p}\mathfrak{X}_{N,\frac{r}{2}}, W^{m_1,p}\mathfrak{X}_{N,\frac{r}{2}}\right)_{\theta,q} \hookrightarrow B^{m_\theta}_{p,q}\mathfrak{X}_{N,\frac{r}{2}}$ for $\theta \in (0,1)\,, m_j \in \mathbb{N}_0, m_0 \neq m_1, p \in [1,\infty], q \in [1,\infty]$, $m_\theta = (1-\theta)\,m_0 + \theta m_1$.

Remark. The last assertion is proven by the definition of the J-method, and it works like partial interpolation. The reader can notice the similarity with the Littlewood-Paley projection $(P_{\leq N}P_{\leq \frac{N}{2}} = P_{\leq \frac{N}{2}})$. The hope is that $\mathbf{t}_r X \xrightarrow{t\downarrow 0} X$ in a good way for $X \in \mathfrak{X}_N$.

A subtle issue is that for $X \in B^{m_\theta}_{p,q}\mathfrak{X}_{N,r}$, $\|X\|_{\left(W^{m_0,p}\mathfrak{X}_{N,\frac{r}{2}},W^{m_1,p}\mathfrak{X}_{N,\frac{r}{2}}\right)_{\theta,q}} \lesssim_r \|X\|_{B^{m_\theta}_{p,q}\mathfrak{X}_{N,r}}$. The implicit constant which depends on r can blow up as $r \downarrow 0$.

Define $B^s_{p,q}\mathfrak{X}_{N,0+} = B^s_{p,q}\text{-cl}\left(\cup_{r>0 \text{ small}} B^s_{p,q}\mathfrak{X}_{N,r}\right)$.

Also define $W^{m,p}\mathfrak{X}_{N,0+} = W^{m,p}\text{-cl}\left(\cup_{r>0 \text{ small}} W^{m,p}\mathfrak{X}_{N,r}\right)$.

Then we recover the usual spaces by results from Section 6.5:

Theorem 88. *Let $p \in (1,\infty)$:*

1. $L^p\mathfrak{X}_{N,0+} = L^p\mathfrak{X}$, $W^{1,p}\mathfrak{X}_{N,0+} = W^{1,p}\mathfrak{X}_N$.

2. $B^{\frac{1}{p}}_{p,1}\mathfrak{X}_{N,0+} = B^{\frac{1}{p}}_{p,1}\mathfrak{X}_N$.

Proof.

1. Let $X \in L^p\mathfrak{X}$. Then $\mathbf{n}_r X \xrightarrow[r\downarrow 0]{L^p} 0$ by shrinking support. If $X \in W^{1,p}\mathfrak{X}_N$, then by Theorem 55

$$\begin{aligned}
\|\mathbf{n}_r X\|_{W^{1,p}} &= \|\psi_r \langle X, \widetilde{\nu}\rangle\|_{W^{1,p}(M_{<r})} \\
&\lesssim \|\psi_r\|_{W^{1,\infty}(M_{<r})} \|\langle X, \widetilde{\nu}\rangle\|_{L^p(M_{<r})} + \|\psi_r\|_{L^\infty} \|\langle X, \widetilde{\nu}\rangle\|_{W^{1,p}(M_{<r})} \\
&\lesssim \frac{1}{r} \|\langle X, \widetilde{\nu}\rangle\|_{L^p(M_{<r})} + \|\langle X, \widetilde{\nu}\rangle\|_{W^{1,p}(M_{<r})} \lesssim \|\langle X, \widetilde{\nu}\rangle\|_{W^{1,p}(M_{<r})} \xrightarrow{r\downarrow 0} 0
\end{aligned}$$

2. Let $Y \in B^{\frac{1}{p}}_{p,1}\mathfrak{X}$. As $B^{\frac{1}{p}}_{\infty,\infty} = (L^\infty, W^{1,\infty})_{\frac{1}{p},\infty}$ and $\psi_r \in W^{1,\infty}$, we conclude $\|\psi_r\|_{B^{\frac{1}{p}}_{\infty,\infty}} \lesssim \|\psi_r\|^{\frac{1}{p'}}_{L^\infty} \|\psi_r\|^{\frac{1}{p}}_{W^{1,\infty}} \lesssim \left(\frac{1}{r}\right)^{\frac{1}{p}}$.

Then by Theorem 55 and Theorem 56 :

$$\begin{aligned}
\|\mathbf{n}_r Y\|_{B^{\frac{1}{p}}_{p,1}} &\lesssim_{\neg r} \|\psi_r\|_{B^{\frac{1}{p}}_{\infty,1}(M)} \|\langle Y, \widetilde{\nu}\rangle\|_{L^p(M_{<4r})} + \|\psi_r\|_{L^\infty} \|\langle Y, \widetilde{\nu}\rangle\|_{B^{\frac{1}{p}}_{p,1}(M)} \\
&\lesssim \left(\frac{1}{r}\right)^{\frac{1}{p}} \|\langle Y, \widetilde{\nu}\rangle\|_{L^p(M_{<4r})} + \|Y\|_{B^{\frac{1}{p}}_{p,1}} \\
&\lesssim \|\langle Y, \widetilde{\nu}\rangle\|_{L^p(M_{<4r},\mathrm{avg})} + \|Y\|_{B^{\frac{1}{p}}_{p,1}(M)} \lesssim_{\neg r} \|Y\|_{B^{\frac{1}{p}}_{p,1}}
\end{aligned}$$

Therefore $\|\mathbf{n}_r Y\|_{B^{1/p}_{p,1}}$ does not blow up as $r \downarrow 0$. Then we make a dense convergence argument: assume $X \in B^{\frac{1}{p}}_{p,1}\mathfrak{X}_N$ and let $X_j \in \mathfrak{X}$ such that $X_j \xrightarrow{B^{1/p}_{p,1}} X$, then $\|\langle X_j, \nu\rangle\|_{L^p(\partial M)} \xrightarrow{j\to\infty} 0$. Note that we do not have $\mathbf{n}X_j = 0$. By Theorem 55:

$$\begin{aligned}
\|\mathbf{n}_r X_j\|_{B^{\frac{1}{p}}_{p,1}} &\lesssim \|\mathbf{n}_r X_j\|^{\frac{1}{p'}}_{L^p} \|\mathbf{n}_r X_j\|^{\frac{1}{p}}_{W^{1,p}} \\
&\lesssim \|\langle X_j, \widetilde{\nu}\rangle\|^{\frac{1}{p'}}_{L^p(M_{<r})} \Big(\|\psi_r\|^{\frac{1}{p}}_{W^{1,\infty}(M_{<r})} \|\langle X_j, \widetilde{\nu}\rangle\|^{\frac{1}{p}}_{L^p(M_{<r})} \\
&\qquad + \|\psi_r\|^{\frac{1}{p}}_{L^\infty} \|\langle X_j, \widetilde{\nu}\rangle\|^{\frac{1}{p}}_{W^{1,p}(M_{<r})} \Big) \\
&\lesssim \|\langle X_j, \widetilde{\nu}\rangle\|_{L^p(M_{<r})} \left(\frac{1}{r}\right)^{\frac{1}{p}} + \|\langle X_j, \widetilde{\nu}\rangle\|^{\frac{1}{p'}}_{L^p(M_{<r})} \|\langle X_j, \widetilde{\nu}\rangle\|^{\frac{1}{p}}_{W^{1,p}(M_{<r})} \\
&\lesssim r^{\frac{1}{p'}} \|\langle X_j, \widetilde{\nu}\rangle\|_{W^{1,p}(M_{<r})} + \|\langle X_j, \nu\rangle\|_{L^p(\partial M)} + \|\langle X_j, \widetilde{\nu}\rangle\|_{W^{1,p}(M_{<r})}.
\end{aligned}$$

So $\limsup_{r\downarrow 0} \|\mathbf{n}_r X_j\|_{B^{\frac{1}{p}}_{p,1}} \lesssim \|\langle X_j, \nu\rangle\|_{L^p(\partial M)}$ and

$$\begin{aligned}
\limsup_{r\downarrow 0} \|\mathbf{n}_r X\|_{B^{\frac{1}{p}}_{p,1}} &\lesssim \limsup_{r\downarrow 0} \|\mathbf{n}_r (X - X_j)\|_{B^{\frac{1}{p}}_{p,1}} + \limsup_{r\downarrow 0} \|\mathbf{n}_r X_j\|_{B^{\frac{1}{p}}_{p,1}} \\
&\lesssim \|X - X_j\|_{B^{\frac{1}{p}}_{p,1}} + \|\langle X_j, \nu\rangle\|_{L^p(\partial M)}
\end{aligned}$$

As j is arbitrary, let $j \to \infty$ and $\limsup_{r\downarrow 0} \|\mathbf{n}_r X\|_{B^{\frac{1}{p}}_{p,1}} = 0.$

$\square$

These results hold not just for vector fields, but also for differential forms once we perform the proper modifications: for differential form ω, define $\mathbf{n}_r\omega = \psi_r\widetilde{\nu}^\flat \wedge (\iota_{\widetilde{\nu}}\omega)$, $\mathbf{t}_r\omega = \omega - \mathbf{n}_r\omega$, $W^{m,p}\Omega^k_r = \{\omega \in W^{m,p}\Omega^k : \iota_{\widetilde{\nu}}\omega = 0 \text{ on } M_{<r}\}$, replace $\langle X, \widetilde{\nu}\rangle$ with $\iota_{\widetilde{\nu}}\omega$ in the proofs etc. In particular, $B^{\frac{1}{p}}_{p,1}\Omega^k_{N,0+} = B^{\frac{1}{p}}_{p,1}\Omega^k_N$ for $p \in (1,\infty)$.

CHAPTER 10

Complexification

Throughout this small chapter, the overline always stands for conjugation, and not topological closure.

Let $\mathbb{R}X$ be a real NVS, then a **complexification** of $\mathbb{R}X$ is a tuple $\left(\mathbb{C}X, \mathbb{R}X \overset{\phi}{\hookrightarrow} \mathbb{C}X\right)$ such that

1. $\mathbb{C}X$ is a complex NVS.

2. ϕ is a linear, continuous injection and $\phi(\mathbb{R}X) \oplus i\phi(\mathbb{R}X) = \mathbb{C}X$.

3. $\|\phi(x)\|_{\mathbb{C}X} = \|x\|_{\mathbb{R}X}$ and $\|\phi(x) + i\phi(y)\|_{\mathbb{C}X} = \|\phi(x) - i\phi(y)\|_{\mathbb{C}X}$ $\forall x, y \in \mathbb{R}X$.

The last property says $\|\cdot\|_{\mathbb{C}X}$ is a **complexification norm**. By treating $\phi(\mathbb{R}X)$ as the real part, $\forall z \in \mathbb{C}X$, we can define $\Re z, \Im z$ as the real and imaginary parts respectively, so $z = \Re z + i\Im z$. Then define $\overline{z} = \Re z - i\Im z$. So $\overline{\lambda z} = \overline{\lambda}\overline{z}$ $\forall z \in \mathbb{C}X, \forall \lambda \in \mathbb{C}$.

Construction A standard construction of such a complexification is $\mathbb{C}X = \mathbb{R}X \otimes_{\mathbb{R}} \mathbb{C}$. As $\mathbb{R}X$ is a flat and free $\mathbb{R}$-module, $0 \to \mathbb{R} \hookrightarrow \mathbb{C} \overset{\Im}{\twoheadrightarrow} \mathbb{R} \to 0$ induces $0 \to \mathbb{R}X \overset{\phi}{\hookrightarrow} \mathbb{C}X \overset{\Im}{\twoheadrightarrow} \mathbb{R}X \to 0$ as a split short exact sequence and $\mathbb{C}X = \phi(\mathbb{R}X) \oplus i\phi(\mathbb{R}X)$. Then we can make ϕ implicit and not write it again. The representation $z = x + iy = (x, y)$ is unique. Easy to see that any two complexifications of $\mathbb{R}X$ must be isomorphic as $\mathbb{C}$-modules.

We define the **minimal complexification norm** (also called **Taylor norm**)

$$\|x + iy\|_T := \sup_{\theta \in [0,2\pi]} \|x\cos\theta - y\sin\theta\|_{\mathbb{R}X} = \sup_{\theta \in [0,2\pi]} \left\|\Re e^{i\theta}(x + iy)\right\|_{\mathbb{R}X} \ \forall x, y \in \mathbb{R}X$$

Any other complexification norm is equivalent to $\|\cdot\|_T$.

Proof. Let $\|\cdot\|_B$ be another complexification norm. Then

$$\left\|\Re e^{i\theta}(x+iy)\right\|_{\mathbb{R}X} = \left\|\Re e^{i\theta}(x+iy)\right\|_B \leq \|x+iy\|_B$$

(minimal) and

$$\|x+iy\|_B \leq \|x\|_{\mathbb{R}X} + \|y\|_{\mathbb{R}X} = \|\Re(x+iy)\|_{\mathbb{R}X} + \|\Re(-i(x+iy))\|_{\mathbb{R}X} \leq 2\|x+iy\|_T.$$

□

So the topology of $\mathbb{C}X$ is unique. It is more convenient, however, to set $\|x+iy\|_{\mathbb{C}X} = \|(x,y)\|_{\mathbb{R}X\oplus\mathbb{R}X} = \left(\|x\|_{\mathbb{R}X}^2 + \|y\|_{\mathbb{R}X}^2\right)^{\frac{1}{2}}$ $\forall x,y \in \mathbb{R}X$. Easy to see that any two complexifications of $\mathbb{R}X$ must be isomorphic as complex NVS, so we write $\boxed{\mathbb{C}X = \mathbb{R}X \otimes_{\mathbb{R}} \mathbb{C}}$ from this point on, and if $\mathbb{R}X$ is normed, so is $\mathbb{C}X$. Obviously, if $\mathbb{R}X$ is Banach, so is $\mathbb{C}X$, and when that happens, we call $(\mathbb{R}X, \mathbb{C}X)$ a **Banach complexification couple**.

Real operators Let $(\mathbb{R}X, \mathbb{C}X)$ and $(\mathbb{R}Y, \mathbb{C}Y)$ be 2 Banach complexification couples.

- An operator $A : D(A) \leq \mathbb{C}X \to \mathbb{C}Y$ is called a **real operator** when $D(A) = \mathbb{C}\Re D(A)$ and $A\Re(D(A)) \leq \mathbb{R}Y$. In particular, $A(x,y) = (Ax, Ay)$ $\forall x,y \in \mathbb{R}X$.
- An unbounded $\mathbb{R}$-linear operator $T : D(T) \leq \mathbb{R}X \to \mathbb{R}Y$ has a natural complexified version $T^{\mathbb{C}} = T \otimes_{\mathbb{R}} 1_{\mathbb{C}} : \mathbb{C}X \to \mathbb{C}Y$ where $D\left(T^{\mathbb{C}}\right) = \mathbb{C}D(T)$. Obviously $T^{\mathbb{C}}$ is a real operator and we write $(\mathbb{R}X, \mathbb{C}X) \xrightarrow{(T,T^{\mathbb{C}})} (\mathbb{R}Y, \mathbb{C}Y)$.
  - $\overline{D\left(T^{\mathbb{C}}\right)} = D\left(T^{\mathbb{C}}\right)$ and $T^{\mathbb{C}}\overline{z} = \overline{T^{\mathbb{C}}z}$ $\forall z \in \mathbb{C}X$.
  - T is closed $\iff$ $T^{\mathbb{C}}$ is closed. Same for bounded, compact, densely defined.
- For any unbounded $\mathbb{C}$-linear operator $A : D(A) \leq \mathbb{C}X \to \mathbb{C}Y$ such that $D(A) =$

$\mathbb{C}\Re\left(D\left(A\right)\right)$, define 2 real operators $\begin{cases} \Re A = \left(\Re \circ \left(A\big|_{\Re D(A)}\right)\right)^{\mathbb{C}} \\ \Im A = \left(\Im \circ \left(A\big|_{\Re D(A)}\right)\right)^{\mathbb{C}} \end{cases}$
Then $A = \Re A + i\Im A$. We can see that A is real $\iff \Re A = A \iff \Im A = 0$. Also, A is bounded $\iff \Re A, \Im A$ are bounded.

Spectrum For $(\mathbb{R}X, \mathbb{C}X) \xrightarrow{(T, T^{\mathbb{C}})} (\mathbb{R}Y, \mathbb{C}Y)$, define

- $\rho(T) := \rho\left(T^{\mathbb{C}}\right), \sigma(T) := \sigma\left(T^{\mathbb{C}}\right)$.
- $\rho_{\mathbb{R}}(T) := \{\lambda \in \mathbb{R} : \lambda - T \text{ is boundedly invertible}\}$ and $\sigma_{\mathbb{R}}(T) := \mathbb{R} \backslash \rho_{\mathbb{R}}(T)$.

If $\zeta \in \mathbb{C}$ and $\zeta - T^{\mathbb{C}}$ is boundedly invertible, so is $\overline{\zeta} - T^{\mathbb{C}}$. So $\overline{\sigma\left(T\right)} = \sigma\left(T\right)$ and $\overline{\rho(T)} = \rho(T)$. For $\lambda \in \mathbb{R}$, $\lambda - T^{\mathbb{C}}$ is boundedly invertible $\iff \lambda - T$ is boundedly invertible. So $\rho_{\mathbb{R}}(T) = \rho(T) \cap \mathbb{R}$ and $\sigma_{\mathbb{R}}\left(T\right) = \sigma\left(T\right) \cap \mathbb{R}$.

Semigroup T generates an $\mathbb{R}$-linear C_0 semigroup $\iff$ $T^{\mathbb{C}}$ generates a $\mathbb{C}$-linear C_0 semigroup. When that happens, $\left(e^{tT}\right)^{\mathbb{C}} = e^{tT^{\mathbb{C}}}$.

Proof. When either happens, T and $T^{\mathbb{C}}$ are densely defined. Also, $T - j$ and $T^{\mathbb{C}} - j$ are boundedly invertible for $j \in \mathbb{N}$ large enough, so T and $T^{\mathbb{C}}$ are closed. Easy to use Hille-Yosida to show both T and $T^{\mathbb{C}}$ must generate C_0 semigroups.

As in the proof of Hille-Yosida, define the Yosida approximations $T_j = T\frac{1}{1-\frac{1}{j}T}$, $T_j^{\mathbb{C}} = T^{\mathbb{C}}\frac{1}{1-\frac{1}{j}T^{\mathbb{C}}} = (T_j)^{\mathbb{C}}$. As T_j and $T_j^{\mathbb{C}}$ are bounded, $\left(e^{tT_j}\right)^{\mathbb{C}} = e^{tT_j^{\mathbb{C}}}$ by power series expansion. Then $\left(e^{tT}\right)^{\mathbb{C}} = e^{tT^{\mathbb{C}}}$ as $e^{tT} = \lim_{j\to\infty} e^{tT_j}$ pointwise. $\square$

Hilbert spaces Let $\mathbb{R}H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then $\mathbb{C}H$ is also Hilbert with the inner product

$$\langle x_1 + iy_1, x_2 + iy_2 \rangle_{\mathbb{C}H} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i\left(\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle\right) \ \forall x_j, y_j \in \mathbb{R}H$$

Then $\|x+iy\|_{\mathbb{C}H} = \left(\|x\|_{\mathbb{R}H}^2 + \|y\|_{\mathbb{R}H}^2\right)^{\frac{1}{2}}$ $\forall x, y \in \mathbb{R}H$, consistent with our previously chosen norm.

Also, $\langle z_1, z_2\rangle_{\mathbb{C}H} = \overline{\langle z_2, z_1\rangle_{\mathbb{C}H}}$ $\forall z_1, z_2 \in \mathbb{C}H$.

Let $\left(A, A^{\mathbb{C}}\right) : (\mathbb{R}H, \mathbb{C}H) \to (\mathbb{R}H, \mathbb{C}H)$ be unbounded.

- A is symmetric $\iff$ $A^{\mathbb{C}}$ is symmetric. When that happens, $\langle Ax + iAy, x+iy\rangle_{\mathbb{C}H} = \langle Ax, x\rangle + \langle Ay, y\rangle$ $\forall x, y \in \mathbb{R}H$.
- $\mathbb{C}\left(\mathbb{R}H \oplus \mathbb{R}H\right) = \mathbb{C}H \oplus \mathbb{C}H$ and $G\left(A^{\mathbb{C}}\right) = \mathbb{C}G\left(A\right)$ (graphs). Also $\mathbb{C}\left(G\left(A\right)^{\perp}\right) = G\left(A^{\mathbb{C}}\right)^{\perp}$.
- A is self-adjoint $\iff$ $A^{\mathbb{C}}$ is self-adjoint. When this happens, $\sigma(A) = \sigma(A^{\mathbb{C}}) \subset \mathbb{R}$.
- A is dissipative $\iff$ $A^{\mathbb{C}}$ is dissipative.

For more information on complexification, see [Glü17, Appendix C].

Nomenclature

ψ_r, f_r cutoffs on M living near the boundary, page 30

$e^{t\Delta}$ the absolute Neumann heat flow, defined for the proof of Onsager's conjecture, page 33

$\mathcal{L}\left(\left(X_0, X_0\right), \left(Y_0, Y_1\right)\right)$ morphisms between interpolation couples, page 43

$\left(X_0, X_1\right)_{\theta,q}$ real interpolation, page 44

$\left[X_0, X_1\right]_{\theta}$ complex interpolation, page 45

$W^{m,p}$ Sobolev spaces, page 60

$B^s_{p,q}$ Besov spaces, page 60

$F^s_{p,q}$ Triebel-Lizorkin spaces, page 60

$\mathfrak{C}^s(\Omega)$ Zygmund spaces, page 65

$\Omega_{<r}$ $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r\}$, page 68

$\|f\|_{L^p(\Omega,\mathrm{avg})}$ integration on probability space , page 68

ν outwards unit normal vector field on ∂M, page 73

$\widetilde{\nu}$ extension of ν near ∂M, page 73

$\jmath$ $\jmath : \partial M \hookrightarrow M$ is the smooth inclusion map, page 73

ι interior product (contraction) of differential forms, page 73

vol_∂ volume form of ∂M, page 74

$\Gamma(\mathbb{F})$, $\Gamma_c(\mathbb{F})$, $\Gamma_{00}(\mathbb{F})$ the space of smooth sections of $\mathbb{F}$ with different support conditions, page 74

$\langle\langle\sigma, \theta\rangle\rangle$ dot product on $\Gamma(\mathbb{F})$, page 74

$\mathbb{R}W^{m,p}$, $\mathbb{C}W^{m,p}$ real and complexified versions of function space, page 79

$\mathfrak{X}M$ set of smooth vector fields on M, page 79

$\mathbf{t}$ tangential part, page 79

$\mathbf{n}$ normal part, page 79

$\Omega^k(M)$ set of smooth differential forms on M, page 79

$X_p^\flat$, $\omega_p^\sharp$ musical isomorphism, page 80

$\star$ Hodge star, page 80

δ codifferential, page 80

Δ Hodge Laplacian, page 80

R_{abcd} Riemann curvature tensor, page 80

Ric Weitzenbock curvature operator, page 80

$\langle T, Q\rangle$ tensor inner product, page 81

$\langle \omega, \eta\rangle_\Lambda, \langle\langle \omega,\ \eta\rangle\rangle_\Lambda$ Hodge inner product, page 81

$\mathcal{D}(\omega, \eta)$ Dirichlet integral, page 83

$\Omega_D^k, \Omega_{\mathrm{hom}D}^k$ different Dirichlet conditions for differential forms, page 85

$\Omega_N^k, \Omega_{\mathrm{hom}\,N}^k$ different Neumann conditions for differential forms, page 85

$\mathcal{H}^k, \mathcal{H}_D^k, \mathcal{H}_N^k$ harmonic fields, then with Dirichlet and Neumann conditions, page 85

$L^2\text{-cl}\,(\cdot)$ closure under L^2 norm, page 85

$\mathcal{P}^N$, $\mathcal{P}^{N\perp}$, $\mathcal{P}^D$, $\mathcal{P}^{D\perp}$ natural orthogonal decomposition, page 87

Δ_N injective Neumann Laplacian, page 88

$(-\Delta_N)^{-1}$, $(-\Delta_D)^{-1}$ Neumann and Dirichlet potentials, page 88

δ_c, d_c adjoints of d and δ, page 88

$\mathcal{P}_1\omega$, $\mathcal{P}_2\omega$, $\mathcal{P}_3\omega$ the component projections in Hodge decomposition, page 90

$\mathcal{P}_3^N$, $\mathcal{P}_3^{\mathrm{ex}}$, $\mathcal{P}_3^D$, $\mathcal{P}_3^{\mathrm{co}}$ Friedrichs decomposition, page 93

$\mathbb{P}$ Leray projection, page 94

$\widetilde{\Delta_N}$ non-injective Neumann Laplacian, page 100

A_p generator of heat flow on L^p, page 102

$A_{1,p}$ generator of heat flow on $W^{1,p}$, page 105

$\mathscr{D}_N\Omega^k, \mathscr{D}'_N\Omega^k$ heated forms and heatable currents, page 107

D_N, $\widetilde{D_N}$ the injective and non-injective (Neumann) Hodge-Dirac operators, page 116

$W^{m,p}(D_N)$, $W^{m,p}\left(\widetilde{D_N}\right)$ Hodge-Sobolev spaces, page 117

$\mathbb{C}Y, T^{\mathbb{C}}$ complexification of spaces and operators , page 126

$e^{t\Delta}$ the absolute Neumann heat flow, defined for the proof of Onsager's conjecture, page 146

ψ_r, f_r cutoffs on M living near the boundary, page 150

Part II

Construction of the Hodge-Neumann heat kernel, local Bernstein estimates, and a local approach to Onsager's conjecture

CHAPTER 11

Introduction

Recall the incompressible Euler equation in fluid dynamics:

$$\begin{cases} \partial_t \mathcal{V} + \operatorname{div}(\mathcal{V} \otimes \mathcal{V}) &= -\operatorname{grad} \mathfrak{p} \quad \text{in } M \\ \operatorname{div} \mathcal{V} &= 0 \quad \text{in } M \\ \langle \mathcal{V}, \nu \rangle &= 0 \quad \text{on } \partial M \end{cases} \tag{11.1}$$

where $\begin{cases} (M, g) \text{ is an oriented, compact smooth Riemannian manifold with boundary} \\ \nu \text{ is the outwards unit normal vector field on } \partial M. \\ I \subset \mathbb{R} \text{ is an open interval, } \mathcal{V} : I \to \mathfrak{X}M,\ \mathfrak{p} : I \times M \to \mathbb{R}. \end{cases}$

We observe that the Neumann condition $\langle \mathcal{V}, \nu \rangle = 0$ means $\mathcal{V} \in \mathfrak{X}_N$, where $\mathfrak{X}_N$ is the set of vector fields which are tangent to the boundary.

The last two conditions can also be rewritten as $\mathcal{V} = \mathbb{P}\mathcal{V}$, where $\mathbb{P}$ is the Leray projection operator.

Roughly speaking, Onsager's conjecture says that the energy $\|\mathcal{V}(t, \cdot)\|_{L^2(M)}$ is a.e. constant in time when $\mathcal{V}$ is a weak solution whose regularity is at least $\frac{1}{3}$. Making that statement precise is part of the challenge.

In the boundaryless case, the "positive direction" (conservation when regularity is at least $\frac{1}{3}$) has been known for a long time [Eyi94; CET94; Che+08]. The "negative direction" (failure of energy conservation when regularity is less than $\frac{1}{3}$) is substantially harder [DS14; DS13], and was finally settled by Isett in his seminal paper [Ise18a] (see the survey in [DS19] for more details and references).

Since then more attention has been directed towards the case with boundary on flat backgrounds [BT18; DN18; BTW19; NN19; Bar+19b; Bar+19a]. The case of manifolds without boundary was first handled via a heat-flow approach in [IO14]. This inspired the consideration of manifolds with boundary in Part I, with the weak solution lying in $L_t^3 B_{3,1}^{\frac{1}{3}}$, the largest space in which the trace theorem applies. However, the best results on flat backgrounds hold in the slightly bigger space $L_t^3 \underline{B}_{3,\mathrm{VMO}}^{\frac{1}{3}}$, so this sequel aims to make that improvement.

In essence, the absolute Neumann heat flow, created via functional analysis, is a replacement for the usual convolution on flat spaces, with special properties like commutativity with divergence. However, obtaining a pointwise profile of heat kernels for differential forms (let alone their derivatives) is a difficult problem, so it was hard to reconcile the heat-flow approach with local-type convolution arguments on flat backgrounds. Even the definition of $\underline{B}_{3,\mathrm{VMO}}^{\frac{1}{3}}$ itself is local, and it was not immediately obvious that the heat-flow approach could handle such function spaces.

The solution to this is a manual construction of the Hodge-Neumann heat kernel (Chapter B), using techniques from microlocal analysis and index theory (in particular, Richard Melrose's calculus on manifolds with corners [Mel18; Mel92]). The theory mimics the development of pseudodifferential operators, in creating a filtered algebra that quantifies how "nonsingular" an operator is as we approach the edges. In particular, much like the pseudolocality of ΨDOs, the construction yields a precise description near the diagonal, as well as rapid decay away from the diagonal. This enables the use of the heat flow as local convolution, and we obtain local Bernstein estimates which allow us to handle VMO-type function spaces.

The addition of local Besov-type estimates also marks another stage of development for the theory of **intrinsic harmonic analysis** for differential forms (including scalar functions and vector fields) on compact Riemannian manifolds with boundary, originally set forth in the prequel with Hodge theory as the foundation. In particular, we have extended the notion of tempered distributions, and the methods of Littlewood-Paley frequency decomposition (e.g. Bernstein-type estimates), which have proved useful on flat backgrounds for problems in fluid dynamics and dispersive PDEs (cf. [Tao09; Tao13; Tao06; Lem02a]), to manifolds with boundary. More history and references can be found in Part I.

CHAPTER 12

Main result

To state the main result, we need some terminology.

The standard Besov spaces $B^s_{p,q}$, and the absolute Neumann heat semigroup $e^{s\widetilde{\Delta_N}}$ were discussed in Part I.

For $r > 0$, we define $M_{>r} := \{x \in M : \operatorname{dist}(x, \partial M) > r\}$. Let $\mathring{M}$ denote the interior of M.

For $p \in (1, \infty)$, we say $X \in \widehat{B}^{1/p}_{p,V}\mathfrak{X}(M)$ if $X \in L^p\mathfrak{X}(M)$ and $\forall r > 0$:

$$\left(\frac{1}{\sqrt{s}}\right)^{\frac{1}{p}} \left\| X - e^{s\widetilde{\Delta_N}} X \right\|_{L^p(M_{>r})} \xrightarrow{s \to 0} 0$$

Or equivalently (by Corollary 118), $(\sqrt{s})^{1-\frac{1}{p}} \left\| e^{s\widetilde{\Delta_N}} X \right\|_{W^{1,p}(M_{>r})} \xrightarrow{s \to 0} 0$

Similarly, for $p \in (1, \infty)$, we say $\mathcal{X} \in L^p_t \widehat{B}^{1/p}_{p,V}\mathfrak{X}(M)$ if $\mathcal{X} \in L^p_t L^p \mathfrak{X}(M)$ and $\forall r > 0$:

$$\left(\frac{1}{\sqrt{s}}\right)^{\frac{1}{p}} \left\| \mathcal{X} - e^{s\widetilde{\Delta_N}} \mathcal{X} \right\|_{L^p_t L^p(M_{>r})} \xrightarrow{s \to 0} 0$$

As shown in Lemma 106, $\widehat{B}^{\frac{1}{3}}_{3,V}$ contains the space $\widehat{B}^{1/3}_{3,c(\mathbb{N})}$ from [IO14] (with equality when $\partial M = \emptyset$). While on flat backgrounds, by Theorem 126, $\widehat{B}^{1/3}_{3,V}$ coincides with $\underline{B}^{\frac{1}{3}}_{3,\mathrm{VMO}}$ from [Bar+19b; NNT20; Wie20].

Let $\mathfrak{X}_{00}$ be the space of smooth vector fields compactly supported in the interior of M. We say $(\mathcal{V}, \mathfrak{p})$ is a **weak solution** to the Euler equation when

- $\mathcal{V} \in L^3\left(I, \mathbb{P}L^3\mathfrak{X}\right)$, $\mathfrak{p} \in L^{\frac{3}{2}}(I, H^{-\beta}(M))$ for any $\beta \in \mathbb{N}_0$
- $\forall \mathcal{X} \in C_c^\infty(I, \mathfrak{X}_{00}) : \iint_{I\times M} \langle \mathcal{V}, \partial_t \mathcal{X}\rangle + \langle \mathcal{V}\otimes\mathcal{V}, \nabla\mathcal{X}\rangle + \mathfrak{p}\operatorname{div}\mathcal{X} = 0.$

The last condition means $\partial_t\mathcal{V} + \operatorname{div}(\mathcal{V}\otimes\mathcal{V}) + \operatorname{grad}\mathfrak{p} = 0$ as spacetime distributions.

Remark 89 (Local elliptic regularity). As $\mathcal{V} \in L_t^3 L^3\mathfrak{X}$, we have $\Delta\mathfrak{p} = -\operatorname{div}(\operatorname{div}(\mathcal{V}\otimes\mathcal{V}))$ in $L_t^{\frac{3}{2}} H^{-2,\frac{3}{2}}(M)$. By embedding, there is $\beta \in \mathbb{N}_1$ such that $\mathfrak{p} \in L^{\frac{3}{2}}\left(I, H^{-\beta,\frac{3}{2}}(M)\right)$. Let $K \subset\subset W \subset\subset \mathring{M}$ where K and W are precompact open sets. Then by interior elliptic regularity (see [Tay11b, Subsection 5.11, Theorem 11.1] and [Tay11d, Subsection 13.6]), we have for a.e. $t \in I$:

$$\|\mathfrak{p}(t)\|_{L^{\frac{3}{2}}(K)} \lesssim_{K,W} \|\Delta\mathfrak{p}(t)\|_{H^{-2,\frac{3}{2}}(W)} + \|\mathfrak{p}(t)\|_{H^{-\beta,\frac{3}{2}}(W)}$$

Then we can conclude $\mathfrak{p} \in L_t^{\frac{3}{2}} L^{\frac{3}{2}}(K)$, for any $K \subset \mathring{M}$ precompact.

As can be seen in [NN19; Bar+19b; NNT20], the correct replacement for the trace theorem is the following "strip decay" hypothesis near the boundary:

$$\left\| \left(\frac{|\mathcal{V}|^2}{2} + \mathfrak{p} \right) \langle \mathcal{V}, \widetilde{\nu} \rangle \right\|_{L_t^1 L^1\left(M_{[\frac{r}{2},r]},\mathrm{avg}\right)} \xrightarrow{r\downarrow 0} 0 \tag{12.1}$$

where $\begin{cases} \widetilde{\nu} \text{ is the extension of } \nu \text{ near the boundary.} \\ M_{[r/2,r]} = \{x \in M : \operatorname{dist}(x, \partial M) \in [r/2, r]\}. \\ \text{avg means the measure is normalized to become a probability measure.} \end{cases}$

Theorem 90. *Let M be as in (11.1). Then $\|\mathcal{V}(t,\cdot)\|_{L^2(M)}$ is a.e. constant in time if $(\mathcal{V},\mathfrak{p})$ is a weak solution with $\mathcal{V} \in L_t^3 \mathbb{P}L^3\mathfrak{X} \cap L_t^3 \widehat{B}^{\frac{1}{3}}_{3,V}\mathfrak{X}$ and (12.1) being true.*

12.1 Outline of Part II

In Chapter 14, we summarize the key tools from Part I, discuss some connections between the heat flow and Besov spaces, and then prove Onsager's conjecture. However, at certain points we will need some local-type estimates involving the heat flow, which are themselves derived from the construction of the heat kernel. To avoid interrupting the flow of the thesis, the local estimates are proved in Chapter A, while the construction of the kernel, arguably the most technical step of the thesis, can be found in Chapter B.

CHAPTER 13

Common notation

Some common notation we use:

- $A \lesssim_{x,\neg y} B$ means $A \le CB$ where $C > 0$ depends on x and not y. Similarly, $A \sim_{x,\neg y} B$ means $A \lesssim_{x,\neg y} B$ and $B \lesssim_{x,\neg y} A$. When the dependencies are obvious by context, we do not need to make them explicit.
- $\mathbb{N}_0, \mathbb{N}_1$: the set of natural numbers, starting with 0 and 1 respectively.
- DCT: dominated convergence theorem, FTC: fundamental theorem of calculus, WLOG: without loss of generality.
- TVS: topological vector space. For TVS X, $Y \le X$ means Y is a subspace of X.
- $\mathcal{L}(X, Y)$: the space of continuous linear maps from TVS X to Y. Also $\mathcal{L}(X) = \mathcal{L}(X, X)$.
- $C^0(S \to Y)$: the space of bounded, continuous functions from metric space S to normed vector space Y. Not to be confused with $C^0_{\mathrm{loc}}(S \to Y)$, which is the space of locally bounded, continuous functions.
- $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$ and $\|x\|^*_{D(A)} = \|Ax\|_X$ where A is an unbounded operator on (real/complex) Banach space X and $x \in D(A)$. Note that $\|\cdot\|^*_{D(A)}$ is not always a norm. We also define $D(A^\infty) = \cap_{k\in\mathbb{N}_1} D(A^k)$.
- $B(x, r) = B_r(x)$: the open ball of radius r centered at x in a metric space.

CHAPTER 14

Onsager's conjecture

14.1 Summary of preliminaries

We will quickly summarize the key tools that we need for the proof (see Section 4.1 for the precise locations where they are proved).

Definition 91. For the rest of the thesis, unless otherwise stated, let M be a compact, smooth, Riemannian n-dimensional manifold, with no or smooth boundary. We also let $I \subset \mathbb{R}$ be an open time interval. We write $M_{<r} = \{x \in M : \operatorname{dist}(x, \partial M) < r\}$ for $r > 0$ small. Similarly define $M_{\geq r}, M_{<r}, M_{[r_1,r_2]}$ etc. Let $\mathring{M}$ denote the interior of M.

By the musical isomorphism, we can consider $\mathfrak{X}M$ (the space of **smooth vector fields**) mostly the same as $\Omega^1(M)$ (the space of **smooth 1-forms**), *mutatis mutandis.* We note that $\mathfrak{X}M$, $\mathfrak{X}(\partial M)$ and $\mathfrak{X}M\big|_{\partial M}$ are different. Unless otherwise stated, let the implicit domain be M, so $\mathfrak{X}$ stands for $\mathfrak{X}M$, and similarly Ω^k for $\Omega^k M$. For $X \in \mathfrak{X}$, we write $X^\flat$ as its dual 1-form.

Let $\mathfrak{X}_{00}(M)$ denote the set of smooth vector fields of compact support in $\mathring{M}$. We define $\Omega^k_{00}(M)$ similarly (smooth differential forms with compact support in $\mathring{M}$).

Let ν denote the outwards unit normal vector field on ∂M. ν can be extended via geodesics to a smooth vector field $\widetilde{\nu}$ which is of unit length near the boundary (and cut off at some point away from the boundary).

For $X \in \mathfrak{X}M$, define $\mathbf{n}X = \langle X, \nu\rangle \nu \in \mathfrak{X}M|_{\partial M}$ (the **normal part**) and $\mathbf{t}X = X|_{\partial M} - \mathbf{n}X$ (the **tangential part**). We note that $\mathbf{t}X$ and $\mathbf{n}X$ only depend on $X\big|_{\partial M}$, so $\mathbf{t}$ and $\mathbf{n}$ can be defined on $\mathfrak{X}M\big|_{\partial M}$, and $\mathbf{t}\left(\mathfrak{X}M|_{\partial M}\right) \xrightarrow{\sim} \mathfrak{X}(\partial M)$.

For $\omega \in \Omega^k\left(M\right)$, define $\mathbf{t}\omega$ and $\mathbf{n}\omega$ by

$$\mathbf{t}\omega(X_1, ..., X_k) := \omega(\mathbf{t}X_1, ..., \mathbf{t}X_k) \;\; \forall X_j \in \mathfrak{X}M, j = 1, ..., k$$

and $\mathbf{n}\omega = \omega|_{\partial M} - \mathbf{t}\omega$. Note that $\left(\mathbf{n}X\right)^\flat = \mathbf{n}X^\flat \; \forall X \in \mathfrak{X}$.

Let ∇ denote the **Levi-Civita connection**, d the **exterior derivative**, δ the **codifferential**, and $\Delta = -\left(d\delta + \delta d\right)$ the **Hodge-Laplacian**, which is defined on vector fields by the musical isomorphism.

Familiar scalar function spaces such as $L^p, W^{m,p}$ (**Lebesgue-Sobolev spaces**), $B^s_{p,q}$ (**Besov spaces**), $C^{0,\alpha}$ (**Holder spaces**) can be defined on M by partitions of unity and given a unique topology. Similarly, we define such function spaces for **tensor fields** and **differential forms** on M by partitions of unity and local coordinates. For instance, we can define $L^2\mathfrak{X}$ or $B^{\frac{1}{3}}_{3,1}\mathfrak{X}$.

Fact 92. *$\forall \alpha \in \left(\frac{1}{3}, 1\right), \forall p \in (1, \infty) : W^{1,p}\mathfrak{X} \hookrightarrow B^{\frac{1}{p}}_{p,1}\mathfrak{X} \hookrightarrow L^p\mathfrak{X}$ and $C^{0,\alpha}\mathfrak{X} = B^{\alpha}_{\infty,\infty}\mathfrak{X} \hookrightarrow B^{\alpha}_{3,\infty}\mathfrak{X} \hookrightarrow B^{\frac{1}{3}}_{3,1}\mathfrak{X} \hookrightarrow B^{\frac{1}{3}}_{3,\infty}\mathfrak{X}$*

Definition 93. We write $\langle\cdot,\cdot\rangle$ to denote the **Riemannian fiber metric** for tensor fields on M. We also define the dot product

$$\langle\langle\sigma, \theta\rangle\rangle = \int_M \langle\sigma, \theta\rangle \operatorname{vol}$$

where σ and θ are tensor fields of the same type, while vol is the **Riemannian volume form**. When there is no possible confusion, we will omit writing vol.

Define $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$ as the **graded algebra** of differential forms where multiplication is the wedge product. We then naturally define $W^{m,p}\Omega(M) = \bigoplus_{k=0}^n W^{m,p}\Omega^k(M)$, and similarly for $B^s_{p,q}, F^s_{p,q}$ spaces. Spaces like $\Omega_N\left(M\right)$, $\Omega_{00}\left(M\right)$ are also defined by direct sums.

We define $\mathfrak{X}_N = \{X \in \mathfrak{X} : \ \mathbf{n}X = 0 \}$ (**Neumann condition**). In order to define the Neumann condition for less regular vector fields, we use the **trace theorem**. We can similarly define Ω^k_N.

Fact 94 (Trace theorem)**.** *Let $p \in [1, \infty)$. Then*

- $B^{\frac{1}{p}}_{p,1}(M) \twoheadrightarrow L^p(\partial M)$ *and* $B^{\frac{1}{p}}_{p,1}\mathfrak{X}M \twoheadrightarrow L^p\mathfrak{X}M\big|_{\partial M}$ *are continuous surjections.*
- $\forall m \in \mathbb{N}_1 : B^{m+\frac{1}{p}}_{p,1}\mathfrak{X}M \twoheadrightarrow B^m_{p,1}\mathfrak{X}M\big|_{\partial M} \hookrightarrow W^{m,p}\mathfrak{X}M\big|_{\partial M}$ *is continuous.*

Definition 95. We define $\mathbb{P}$ as the **Leray projection**, which projects $\mathfrak{X}$ onto $\operatorname{Ker}\left(\operatorname{div}\big|_{\mathfrak{X}_N}\right)$. Note that the Neumann condition is enforced by $\mathbb{P}$.

Fact 96. *$\forall m \in \mathbb{N}_0, \forall p \in (1, \infty)$, $\mathbb{P}$ is continuous on $W^{m,p}\mathfrak{X}$ and $\mathbb{P}(W^{m,p}\mathfrak{X}) = W^{m,p}\text{-cl}(\mathbb{P}\mathfrak{X})$ (closure in the $W^{m,p}$-topology).*

We collect some results regarding our heat flow in one place:

Fact 97 (Absolute Neumann heat flow)**.** *There exists a semigroup of operators $(S(t))_{t\geq 0}$ acting on $\cup_{p\in(1,\infty)}L^p\mathfrak{X}$ such that*

1. $S(t_1)S(t_2) = S(t_1 + t_2)\ \forall t_1, t_2 \geq 0$ *and* $S(0) = 1$.
2. $\forall p \in (1, \infty), \forall X \in L^p\mathfrak{X}$:
   (a) $S(t)X \in \mathfrak{X}_N$ *and* $\partial_t(S(t)X) = \Delta S(t)X\ \forall t > 0$.
   (b) $S(t)X \xrightarrow[t\to t_0]{C^\infty} S(t_0)X\ \forall t_0 > 0$.
   (c) $\|S(t)X\|_{W^{m,p}} \lesssim_{m,p} \left(\frac{1}{t}\right)^{\frac{m}{2}} \|X\|_{L^p}\ \forall m \in \mathbb{N}_0, \forall t \in (0, 1)$.
   (d) $S(t)X \xrightarrow[t\to 0]{L^p} X$.
3. $\forall p \in (1, \infty), \forall X \in W^{1,p}\mathfrak{X}_N$:
   (a) $\|S(t)X\|_{W^{m+1,p}} \lesssim_{m,p} \left(\frac{1}{t}\right)^{\frac{m}{2}} \|X\|_{W^{1,p}}\ \forall m \in \mathbb{N}_0, \forall t \in (0, 1)$.
   (b) $S(t)X \xrightarrow[t\to 0]{W^{1,p}} X$.

4. $S(t)\mathbb{P} = \mathbb{P}S(t)$ *on* $W^{m,p}\mathfrak{X}\ \forall m \in \mathbb{N}_0, \forall p \in (1,\infty), \forall t \geq 0.$

5. $\langle\langle S(t)X, Y\rangle\rangle = \langle\langle X, S(t)Y\rangle\rangle\ \forall t \geq 0, \forall p \in (1,\infty), \forall X \in L^p\mathfrak{X}, \forall Y \in L^{p'}\mathfrak{X}.$

These estimates precisely fit the analogy $e^{t\Delta} \approx P_{\leq \frac{1}{\sqrt{t}}}$ where P is the **Littlewood-Paley projection**.

Analogous results hold for scalar functions and differential forms.

We observe some basic identities from differential geometry:

- Using **Penrose abstract index notation**, for any smooth tensors $T_{a_1\dots a_k}$, we define $(\nabla T)_{ia_1\dots a_k} = \nabla_i T_{a_1\dots a_k}$ and $\operatorname{div} T = \nabla^i T_{ia_2\dots a_k}$.
- For all smooth tensors $T_{a_1\dots a_k}$ and $Q_{a_1\dots a_{k+1}}$:

$$\begin{aligned}\int_M \nabla_i \left(T_{a_1\dots a_k} Q^{ia_1\dots a_k}\right) &= \int_M \nabla_i T_{a_1\dots a_k} Q^{ia_1\dots a_k} + \int_M T_{a_1\dots a_k} \nabla_i Q^{ia_1\dots a_k} \\ &= \int_{\partial M} \nu_i T_{a_1\dots a_k} Q^{ia_1\dots a_k}.\end{aligned}$$

- $(\nabla_a\nabla_b - \nabla_b\nabla_a) T^{ij}{}_{kl} = -R_{ab\sigma}{}^i T^{\sigma j}{}_{kl} - R_{ab\sigma}{}^j T^{i\sigma}{}_{kl} + R_{abk}{}^\sigma T^{ij}{}_{\sigma l} + R_{abl}{}^\sigma T^{ij}{}_{k\sigma}$ for any tensor $T^{ij}{}_{kl}$, where R is the **Riemann curvature tensor**. Similar identities hold for other types of tensors. When we do not care about the exact indices and how they contract, we can just write the **schematic identity** $(\nabla_a\nabla_b - \nabla_b\nabla_a) T^{ij}{}_{kl} = R * T$. As R is bounded on compact M, interchanging derivatives is a zeroth-order operation on M. In particular, we have the **Weitzenbock formula**:

$$\Delta X = \nabla_i \nabla^i X + R * X\ \forall X \in \mathfrak{X}M \tag{14.1}$$

There is an elementary lemma which is useful for convergence (the proof is straightforward and omitted):

Lemma 98 (Dense convergence)**.** *Let* X, Y *be (real/complex) Banach spaces and* $X_0 \leq X$ *be norm-dense. Let* $(T_j)_{j\in\mathbb{N}}$ *be bounded in* $\mathcal{L}(X,Y)$ *and* $T \in \mathcal{L}(X,Y)$.

If $T_j x_0 \to T x_0 \; \forall x_0 \in X_0$ *then* $T_j x \to T x \; \forall x \in X$.

As the heat flow does not preserve compact supports in $\mathring{M}$, it is not defined on distributions. This inspires the formulation of **heatable currents**.

Definition 99 (Heatable currents)**.** Define:

- $\mathscr{D}\Omega^k = \Omega^k_{00} = \operatorname{colim}\{\left(\Omega^k_{00}(K), C^\infty \text{ topo}\right) : K \subset \mathring{M} \text{ compact}\}$ as the space of **test** k**-forms** with **Schwartz's topology**[1] (colimit in the category of locally convex TVS).
- $\mathscr{D}'\Omega^k = \left(\mathscr{D}\Omega^k\right)^*$ as the space of k**-currents** (or **distributional** k**-forms**), equipped with the weak* topology.
- $\mathscr{D}_N\Omega^k = \{\omega \in \Omega^k : \mathbf{n}\Delta^m\omega = 0, \mathbf{n}d\Delta^m\omega = 0 \; \forall m \in \mathbb{N}_0\}$ as the space of **heated** k**-forms** with the Frechet C^∞ topology and $\mathscr{D}'_N\Omega^k = \left(\mathscr{D}_N\Omega^k\right)^*$ as the space of **heatable** k**-currents** (or **heatable distributional** k**-forms**) with the weak* topology.

In particular, $\mathscr{D}_N\mathfrak{X}$ is defined from $\mathscr{D}_N\Omega^1$ by the musical isomorphism, and it is invariant under our heat flow (much like how the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is invariant under the Littlewood-Paley projection). By that analogy, heatable currents are tempered distributions on manifolds, and we can write

$$\langle\langle S(t)\Lambda, X\rangle\rangle = \langle\langle \Lambda, S(t)\, X\rangle\rangle \;\; \forall \Lambda \in \mathscr{D}'_N\mathfrak{X}, \forall X \in \mathscr{D}_N\mathfrak{X}, \forall t \geq 0$$

where the dot product $\langle\langle \cdot, \cdot \rangle\rangle$ is simply abuse of notation.

Fact 100. *Some basic properties of* $\mathscr{D}_N\Omega(M)$ *and* $\mathscr{D}'_N\Omega(M)$*:*

- $\langle\langle \Delta X, Y\rangle\rangle = \langle\langle X, \Delta Y\rangle\rangle \;\; \forall X, Y \in \mathscr{D}_N\mathfrak{X}$.
- $\mathscr{D}\Omega \hookrightarrow \mathscr{D}_N\Omega$ *and* $L^p\Omega \hookrightarrow \mathscr{D}'_N\Omega \;\; \forall p \in (1, \infty)$.

---

[1] Confusingly enough, "Schwartz's topology" refers to the topology on the space of distributions, not the topology for Schwartz functions.

- $S(t)\Lambda \in \mathscr{D}_N\Omega\ \forall t > 0, \forall \Lambda \in \mathscr{D}'_N\Omega$. *(a heatable current becomes heated once the heat flow is applied)*

- $W^{1,p}\text{-cl}\left(\mathscr{D}_N\Omega\right) = W^{1,p}\Omega_N$ *and* $B^{\frac{1}{p}}_{p,1}\text{-cl}\left(\mathbb{P}\mathscr{D}_N\Omega\right) = \mathbb{P}B^{\frac{1}{p}}_{p,1}\Omega_N\ \forall p \in (1,\infty)$

- $\forall X \in \mathscr{D}_N\Omega : S(t)X \xrightarrow[t\downarrow 0]{C^\infty} X$ *and* $\partial_t\left(S(t)X\right) = \Delta S(t)X = S(t)\Delta X\ \forall t \geq 0.$

- $\forall t \in (0,1), \forall m, m' \in \mathbb{N}_0, \forall p \in (1,\infty), \forall X \in \mathscr{D}_N\Omega :$

  1. $\|S(t)X\|_{W^{m+m',p}} \lesssim \left(\frac{1}{t}\right)^{\frac{m'}{2}} \|X\|_{W^{m,p}}$
  2. $\|S(t)X\|_{W^{m,p}} \lesssim \left(\frac{1}{t}\right)^{\frac{1}{2}\left(m-\frac{1}{p}\right)} \|X\|_{B^{\frac{1}{p}}_{p,1}}$ *when* $m \geq 1$
  3. $\|S(t)X\|_{B^{m+m'+\frac{1}{p}}_{p,1}} \lesssim \left(\frac{1}{t}\right)^{\frac{1}{2p}+\frac{m'}{2}} \|X\|_{W^{m,p}}$

14.2 Heating the nonlinear term

Recall integration by parts:

$$\langle\langle \operatorname{div}\left(Y \otimes Z\right), X\rangle\rangle = -\langle\langle Y \otimes Z, \nabla X\rangle\rangle + \int_{\partial M} \langle \nu, Y\rangle \langle Z, X\rangle\ \forall X, Y, Z \in \mathfrak{X}\left(M\right)$$

Let $U, V \in B^{\frac{1}{3}}_{3,1}\mathfrak{X}$. Then $U \otimes V \in L^1\mathfrak{X}$ and $\operatorname{div}\left(U \otimes V\right)$ is defined as a distribution. So we will define the heatable 1-current $\left(\operatorname{div}\left(U \otimes V\right)\right)^\flat$ by

$$\langle\langle \operatorname{div}\left(U \otimes V\right), X\rangle\rangle := -\langle\langle U \otimes V, \nabla X\rangle\rangle + \int_{\partial M} \langle \nu, U\rangle \langle V, X\rangle\ \forall X \in \mathscr{D}_N\mathfrak{X}\ (X \text{ is heated})$$

It is continuous on $\mathscr{D}_N\mathfrak{X}$ since

$$\left|\langle\langle \operatorname{div}\left(U \otimes V\right), X\rangle\rangle\right| \lesssim \|U\|_{B^{\frac{1}{3}}_{3,1}} \|V\|_{B^{\frac{1}{3}}_{3,1}} \|X\|_{B^{\frac{1}{3}}_{3,1}} + \|U\|_{L^3} \|V\|_{L^3} \|\nabla X\|_{L^3}.$$

By the same formula and reasoning, we see that $\left(\operatorname{div}\left(U \otimes V\right)\right)^\flat$ is not just heatable, but also a continuous linear functional on $\left(\mathfrak{X}\left(M\right), C^\infty \text{ topo}\right)$.

On the other hand, we can get away with less regularity by assuming $U \in \mathbb{P}L^2\mathfrak{X}$ and $V \in L^2\mathfrak{X}$. Then $(\operatorname{div}(U \otimes V))^\flat$ is heatable as we simply need to define

$$\langle\langle \operatorname{div}(U \otimes V), X\rangle\rangle = -\langle\langle U \otimes V, \nabla X\rangle\rangle \ \forall X \in \mathfrak{X} \tag{14.2}$$

14.3 Besov spaces

For the rest of the proof, we will write $e^{t\Delta}$ for the absolute Neumann heat flow, as we will not need another heat flow. For $\varepsilon > 0$ and vector field X, we will write X^ε for $e^{\varepsilon\Delta}X$.

Now we define a crude version of the Littlewood-Paley projections: $P_{\leq t} = e^{\frac{1}{t^2}\Delta}$ for $t > 0$ and $P_N = P_{\leq N} - P_{\leq \frac{N}{2}}$ for $N > 1, N \in 2^{\mathbb{Z}}$.

The definition of $P_{\leq t}$ gives a quick Bernstein estimate:

Theorem 101. *For $N \geq 1$ and $X \in \mathscr{D}'_N\Omega^k$,*

$$\|P_N X\|_p \lesssim \frac{1}{N^2}\left\|P_{\leq\sqrt{2}N}X\right\|_{W^{2,p}} \lesssim \frac{1}{N}\|P_{\leq 2N}X\|_{W^{1,p}}.$$

Proof. Recall that $e^{\varepsilon\Delta}X \in \mathscr{D}_N\Omega^k \ \forall\varepsilon > 0$. Then observe that

$$P_N X = \left(\exp\left(\frac{\Delta}{2N^2}\right) - \exp\left(\frac{7\Delta}{2N^2}\right)\right)\exp\left(\frac{\Delta}{2N^2}\right)X = \int_{\frac{7}{2N^2}}^{\frac{1}{2N^2}} \Delta e^{t\Delta}\exp\left(\frac{\Delta}{2N^2}\right)X \, \mathrm{d}t$$

and $P_{\leq\sqrt{2}N} = P_{\leq 2N}P_{\leq 2N}$. □

Definition 102. For $\alpha \in (0,1)$, $p \in (1,\infty)$, $q \in [1,\infty]$, we define the **Besov heat space** $\widehat{B}^\alpha_{p,q}\Omega^k$ as the space of heatable k-currents X where the norm

$$\begin{aligned}\|X\|_{\widehat{B}^\alpha_{p,q}} &= \|X\|_{L^p} + \left\|s^{\frac{1}{2}(1-\alpha)}\left\|e^{s\Delta}X\right\|_{W^{1,p}}\right\|_{L^q\left(\frac{\mathrm{d}s}{s},(0,1)\right)} \\ &\sim \|X\|_{L^p} + \left\|N^{\alpha-1}\|P_{\leq N}X\|_{W^{1,p}}\right\|_{l^q_N(N\in 2^{\mathbb{Z}}, N>1)}\end{aligned}$$

is finite.

Recall the theory of real interpolation. The following fact justifies the name "Besov" in Besov heat space:

Theorem 103. $\left[L^p\Omega^k, W^{1,p}\Omega^k_N\right]_{\theta,q} = \widehat{B}^\theta_{p,q}\Omega^k$ *for* $q \in [1,\infty], p \in (1,\infty), \theta \in (0,1)$.

Proof. By definition, $\widehat{B}^\theta_{p,q}\Omega^k \hookrightarrow L^p\Omega^k$. We first show $\widehat{B}^\theta_{p,q}\Omega^k \hookrightarrow \left[L^p\Omega^k, W^{1,p}\Omega^k_N\right]_{\theta,q}$.

Assume $\|X\|_{\widehat{B}^\theta_{p,q}} \leq 1$. Then we decompose $X \overset{L^p}{=} P_{\leq 1}X + \sum_{N>1, N\in 2^{\mathbb{Z}}} P_N X$. Set $X_0 = P_{\leq 1}X$ and $X_k = P_{2^{-k}}X \ \forall k \in \mathbb{Z}, k \leq -1$. Then by the J-method, and the fact that $X = \sum_{k\leq 0} X_k$, we have

$$\begin{aligned}
\|X\|_{\left[L^p\Omega^k, W^{1,p}\Omega^k_N\right]_{\theta,q}} &\lesssim \left\| 2^{-k\theta}\|X_k\|_{L^p} + 2^{k(1-\theta)}\|X_k\|_{W^{1,p}} \right\|_{l^q_k(k\leq 0)} \\
&\lesssim \|X\|_{L^p} + \left\| \left(\frac{1}{2}\right)^{-m\theta} \|P_{2^m}X\|_{L^p} + \left(\frac{1}{2}\right)^{m(1-\theta)} \|P_{2^m}X\|_{W^{1,p}} \right\|_{l^q_m(m\geq 1)} \\
&\lesssim \|X\|_{L^p} + \left\| \left(\frac{1}{2}\right)^{m(1-\theta)} \|P_{\leq 2^{m+1}}X\|_{W^{1,p}} \right\|_{l^q_m(m\geq 1)} \lesssim 1
\end{aligned}$$

Now we will show $\left[L^p\Omega^k, W^{1,p}\Omega^k_N\right]_{\theta,q} \hookrightarrow \widehat{B}^\theta_{p,q}\Omega^k$. Assume $\|Y\|_{\left[L^p\Omega^k, W^{1,p}\Omega^k_N\right]_{\theta,q}} \leq 1$, then $\|Y\|_{L^p} \lesssim 1$. We will use the K-method: for any $N \geq 1, Y_0 \in L^p\Omega^k, Y_1 \in W^{1,p}\Omega^k_N$ such that $Y = Y_0 + Y_1$, we have

$$\|P_{\leq N}Y\|_{W^{1,p}} \leq \|P_{\leq N}Y_0\|_{W^{1,p}} + \|P_{\leq N}Y_1\|_{W^{1,p}} \lesssim N\|Y_0\|_{L^p} + \|Y_1\|_{W^{1,p}}$$

Note that this is why we need $W^{1,p}\Omega^k_N$ instead of $W^{1,p}\Omega^k$. Then

$$N^{\theta-1}\|P_{\leq N}Y\|_{W^{1,p}} \lesssim \inf_{Y_0+Y_1=Y} N^\theta \|Y_0\|_{L^p} + N^{\theta-1}\|Y_1\|_{W^{1,p}} = N^\theta K\left(\frac{1}{N}, Y\right)$$

so

$$\left\|N^{\theta-1}\|P_{\leq N}Y\|_{W^{1,p}}\right\|_{l^q_N(N\in 2^{\mathbb{Z}}, N>1)} \lesssim \left\|N^{-\theta}K(N,Y)\right\|_{l^q_N(N\in 2^{\mathbb{Z}}, N<1)} \leq 1$$

$\square$

Remark 104. We recover the standard Besov space when the manifold is boundaryless, effectively generalizing the proof in [IO14, Appendix B]. More importantly, in the case with boundary, we have

$$\mathbb{P}B^{\frac{1}{3}}_{3,q}\Omega^k = \left[\mathbb{P}L^3\Omega^k, \mathbb{P}W^{1,p}\Omega^k\right]_{1/3,q} = \left[\mathbb{P}L^3\Omega^k, \mathbb{P}W^{1,p}\Omega^k_N\right]_{1/3,q} = \mathbb{P}\widehat{B}^{\frac{1}{3}}_{3,q}\Omega^k$$

for $q \in [1,\infty]$. The fact that we need to apply the Leray projection is an important technicality.

Definition 105. For $p \in (1,\infty)$, we say $X \in \widehat{B}^{1/p}_{p,V}\mathfrak{X}(M)$ if $X \in L^p\mathfrak{X}(M)$ and $\forall r > 0:$

$$N^{\frac{1}{p}-1}\left\|P_{\leq N}X\right\|_{W^{1,p}(M_{>r})} \xrightarrow{N\to\infty} 0 \tag{14.3}$$

Similarly, we say $\mathcal{X} \in L^p_t\widehat{B}^{1/p}_{p,V}\mathfrak{X}(M)$ if $\mathcal{X} \in L^p_tL^p\mathfrak{X}(M)$ and $\forall r > 0:$

$$N^{\frac{1}{p}-1}\left\|P_{\leq N}\mathcal{X}\right\|_{L^p_tW^{1,p}(M_{>r})} \xrightarrow{N\to\infty} 0 \tag{14.4}$$

Remark. The vanishing property in (14.4) becomes important for the commutator estimate in Onsager's conjecture at the critical regularity $\frac{1}{3}$, while higher regularity has enough room for vanishing in norm (which is better).

It is shown in Corollary 118 that (14.3) is equivalent to

$$N^{\frac{1}{p}}\left\|P_{>N}X\right\|_{L^p(M_{>r})} \xrightarrow{N\to\infty} 0 \ \forall r > 0$$

We briefly note that when $\partial M = \emptyset$, it is customary to set $\operatorname{dist}(x,\partial M) = \infty$, $M_{>r} = M = \mathring{M}$, $M_{<r} = \emptyset$, and $\mathscr{D}_N\mathfrak{X}M = \mathscr{D}\mathfrak{X}M = \mathfrak{X}M$.

Recall the space $\widehat{B}^{1/3}_{3,c(\mathbb{N})}\mathfrak{X} = \widehat{B}^{1/3}_{3,\infty}\text{-cl}\left(\mathscr{D}_N\mathfrak{X}\right)$ from [IO14].

Lemma 106. *$\widehat{B}^{1/3}_{3,c(\mathbb{N})}\mathfrak{X} \hookrightarrow \widehat{B}^{1/3}_{3,V}\mathfrak{X}$. When $\partial M = \emptyset$, $\widehat{B}^{1/3}_{3,V}\mathfrak{X} = \widehat{B}^{1/3}_{3,c(\mathbb{N})}\mathfrak{X}$.*

Proof. Observe that $\mathscr{D}_N \mathfrak{X} \hookrightarrow \widehat{B}^{1/3}_{3,V}\mathfrak{X}$. For any $r > 0, N \geq 1$ and $X \in \widehat{B}^{1/3}_{3,\infty}$,

$$N^{-2/3} \|P_{\leq N} X\|_{W^{1,3}(M_{>r})} \leq N^{-2/3} \|P_{\leq N} X\|_{W^{1,3}(M)} \lesssim \|X\|_{\widehat{B}^{1/3}_{3,\infty}(M)}$$

Because $\left\{ f \in l^\infty (\mathbb{N}) : f(k) \xrightarrow{k \to \infty} 0 \right\}$ is closed in $l^\infty (\mathbb{N})$, we conclude $\widehat{B}^{1/3}_{3,c(\mathbb{N})}\mathfrak{X} \hookrightarrow \widehat{B}^{1/3}_{3,V}\mathfrak{X}$.

On the other hand, when $\partial M = \emptyset$, observe that $M_{>r} = M$. Let $X \in \widehat{B}^{1/3}_{3,V}\mathfrak{X}$. We aim to show $P_{\leq K} X \xrightarrow[K \to \infty]{\widehat{B}^{1/3}_{3,\infty}} X$. For any $N, K \in 2^{\mathbb{N}_0}$:

$$N^{-2/3} \|P_{\leq N} X\|_{W^{1,3}(M)} \gtrsim_{\neg K, \neg N} N^{-2/3} \|P_{\leq N} (1 - P_{\leq K}) X\|_{W^{1,3}(M)} \xrightarrow{K \to \infty} 0$$

Let $N_0 \in 2^{\mathbb{N}_1}$. Then observe that

$$\begin{aligned}
&\limsup_{K \to \infty} \left\| N^{-2/3} \|P_{\leq N} (1 - P_{\leq K}) X\|_{W^{1,3}(M)} \right\|_{l^\infty_{N>1}} \\
&\leq \underbrace{\limsup_{K \to \infty} \left\| N^{-2/3} \|P_{\leq N} (1 - P_{\leq K}) X\|_{W^{1,3}(M)} \right\|_{l^\infty_N \left(N \in 2^{\mathbb{N}_0}, N < N_0\right)}}_{0} \\
&\quad + \left\| N^{-2/3} \|P_{\leq N} X\|_{W^{1,3}(M)} \right\|_{l^\infty_N \left(N \in 2^{\mathbb{N}_0}, N \geq N_0\right)}
\end{aligned}$$

As N_0 is arbitrary, let $N_0 \to \infty$ and we are done. □

Remark 107. On the other hand, Theorem 126 shows that, on flat backgrounds, $\widehat{B}^{1/3}_{3,V}$ coincides with the VMO-type Besov space $\underline{B}^{1/3}_{3,\mathrm{VMO}}$ from [Bar+19b; NNT20].

We will also need to borrow a result from Chapter A, which allows us to employ cutoffs.

Fact 108 (Pointwise multiplier). *If $f \in \mathscr{D}(M)$ and $\mathcal{X} \in L^3_t \widehat{B}^{\frac{1}{3}}_{3,V}\mathfrak{X}$, then $f\mathcal{X} \in L^3_t \widehat{B}^{\frac{1}{3}}_{3,V}\mathfrak{X}$.*

14.4 Proof of Onsager's conjecture

Definition 109. We define the cutoffs

$$\psi_r(x) = \Psi_r\left(\operatorname{dist}(x, \partial M)\right) \tag{14.5}$$

where $r > 0$ small, $\Psi_r \in C^\infty([0,\infty),[0,\infty))$ such that $\mathbf{1}_{[0,\frac{3}{4}r)} \geq \Psi_r \geq \mathbf{1}_{[0,\frac{r}{2}]}$ and $\|\Psi_r'\|_\infty \lesssim \frac{1}{r}$. Then there is f_r smooth such that $\nabla\psi_r(x) = f_r(x)\widetilde{\nu}(x)$ with $|f_r(x)| \lesssim \frac{1}{r}$ and $\operatorname{supp} f_r \subset M_{[\frac{r}{2},\frac{3r}{4}]}$.

Let $\chi_r = 1 - \psi_r$. Then $\nabla\chi_r = -f_r\widetilde{\nu}$. As usual, there is a **commutator estimate** which we will now assume (leaving the proof to later):

$$\begin{aligned}
&\int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U} \otimes \chi_r\mathcal{U}\right)^{2\varepsilon}, \left(\chi_r\mathcal{U}\right)^{2\varepsilon}\right\rangle\right\rangle - \int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U}^{2\varepsilon} \otimes \left(\chi_r\mathcal{U}\right)^{2\varepsilon}\right), \left(\chi_r\mathcal{U}\right)^{2\varepsilon}\right\rangle\right\rangle \\
=&\int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U} \otimes \chi_r\mathcal{U}\right)^{3\varepsilon}, \left(\chi_r\mathcal{U}\right)^{\varepsilon}\right\rangle\right\rangle - \int_I \eta \left\langle\left\langle \operatorname{div}\left(\mathcal{U}^{2\varepsilon} \otimes \left(\chi_r\mathcal{U}\right)^{2\varepsilon}\right)^{\varepsilon}, \left(\chi_r\mathcal{U}\right)^{\varepsilon}\right\rangle\right\rangle \xrightarrow{\varepsilon\downarrow 0} 0
\end{aligned} \tag{14.6}$$

for fixed $r > 0$, $\mathcal{U} \in L_t^3 \widehat{B}^{\frac{1}{3}}_{3,V}\mathfrak{X} \cap L_t^3\mathbb{P}L^3\mathfrak{X}$, $\eta \in C_c^\infty(I)$.

Remark. For any U in $\mathbb{P}L^2\mathfrak{X}$ and $V \in L^2\mathfrak{X}$, $\operatorname{div}(U \otimes V)^\flat$ is a heatable 1-current (see Section 14.2). In particular, for $\varepsilon > 0$, $\operatorname{div}(U \otimes V)^\varepsilon$ is smooth and

$$\left\langle\left\langle \operatorname{div}(U \otimes V)^\varepsilon, Y\right\rangle\right\rangle = -\left\langle\left\langle U \otimes V, \nabla\left(Y^\varepsilon\right)\right\rangle\right\rangle \ \forall Y \in \mathfrak{X} \tag{14.7}$$

Consequently, (14.6) is well-defined.

Notation: we write $\operatorname{div}(\mathcal{U} \otimes \mathcal{V})^\varepsilon$ for $\left(\operatorname{div}(\mathcal{U} \otimes \mathcal{V})\right)^\varepsilon$ and $\nabla\mathcal{U}^\varepsilon$ for $\nabla\left(\mathcal{U}^\varepsilon\right)$ (recall that the heat flow does not work on tensors $\mathcal{U} \otimes \mathcal{V}$ and $\nabla\mathcal{U}$).

Theorem 110 (Onsager's conjecture)**.** *Let M be a compact, oriented Riemannian manifold with no or smooth boundary. Let $(\mathcal{V}, \mathfrak{p})$ be a weak solution and $\mathcal{V} \in L_t^3 \widehat{B}^{\frac{1}{3}}_{3,V}\mathfrak{X} \cap L_t^3\mathbb{P}L^3\mathfrak{X}$.*

Assume (14.6) is true. Also assume strip decay:

$$\left\| \left(\frac{|\mathcal{V}|^2}{2} + \mathfrak{p} \right) \langle \mathcal{V}, \widetilde{\nu} \rangle \right\|_{L^1_t L^1\left(M_{[\frac{r}{2},r]},\text{avg}\right)} \xrightarrow{r\downarrow 0} 0$$

Then we can show

$$\int_I \eta'(t) \langle\langle \mathcal{V}(t), \mathcal{V}(t) \rangle\rangle \, \mathrm{d}t = 0 \; \forall \eta \in C_c^\infty(I)$$

Consequently, $\langle\langle \mathcal{V}(t), \mathcal{V}(t) \rangle\rangle$ *is constant for a.e.* $t \in I$.

Proof. Let $\Phi \in C_c^\infty(\mathbb{R})$ and $\Phi_\tau \xrightarrow{\tau\downarrow 0} \delta_0$ be a radially symmetric mollifier. Write $\mathcal{V}^\varepsilon$ for $e^{\varepsilon\Delta}\mathcal{V}$ (spatial mollification) and $\mathcal{V}_\tau$ for $\Phi_\tau * \mathcal{V}$ (temporal mollification). First, we use the cutoff χ_r and mollify in time and space

$$\frac{1}{2}\int_I \eta' \langle\langle \mathcal{V}, \mathcal{V} \rangle\rangle \overset{\text{DCT}}{=} \lim_{r\downarrow 0} \lim_{\varepsilon\downarrow 0} \lim_{\tau\downarrow 0} \frac{1}{2}\int_I \eta' \langle\langle (\chi_r \mathcal{V})^\varepsilon_\tau, (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle$$

Then for ε, τ small, we want to get rid of the time derivative:

$$\begin{aligned} \frac{1}{2}\int_I \eta' \langle\langle (\chi_r \mathcal{V})^\varepsilon_\tau, (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle &= -\int_I \eta \langle\langle \partial_t (\chi_r \mathcal{V})^\varepsilon_\tau, (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle \\ &= -\int_I \langle\langle \partial_t \left(\eta (\chi_r \mathcal{V})^\varepsilon_\tau\right), (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle + \int_I \eta' \langle\langle (\chi_r \mathcal{V})^\varepsilon_\tau, (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle \end{aligned}$$

We now use the definition of weak solution (WS), and exploit the commutativity between spatial and temporal operators. For the sake of exposition, we will freely cancel the error terms that go to zero upon taking the limits. At the end of the proof, we will show why they can be cancelled.

$$\begin{aligned} &\frac{1}{2}\int_I \eta' \langle\langle (\chi_r \mathcal{V})^\varepsilon_\tau, (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle = \int_I \langle\langle \partial_t \left(\eta (\chi_r \mathcal{V})^\varepsilon_\tau\right), (\chi_r \mathcal{V})^\varepsilon_\tau \rangle\rangle \\ &= \int_I \left\langle\left\langle \partial_t \left[\left(\eta (\chi_r \mathcal{V})^{2\varepsilon}_\tau\right)_\tau \chi_r \right], \mathcal{V} \right\rangle\right\rangle \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{WS}}{=\!=} -\int_I \langle\langle \nabla\left[\left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau \chi_r\right], \mathcal{V}\otimes\mathcal{V}\rangle\rangle - \langle\langle \operatorname{div}\left[\left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau \chi_r\right], \mathfrak{p}\rangle\rangle \\
&= -\int_I \langle\langle \nabla\left[\left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau\right] \chi_r, \mathcal{V}\otimes\mathcal{V}\rangle\rangle - \cancel{\langle\langle \left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau \otimes\nabla\chi_r, \mathcal{V}\otimes\mathcal{V}\rangle\rangle} \\
&\quad - \langle\langle \operatorname{div}\left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau \chi_r, \mathfrak{p}\rangle\rangle - \cancel{\langle\langle \left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau \cdot\nabla\chi_r, \mathfrak{p}\rangle\rangle} \\
&= -\int_I \langle\langle \left(\eta\nabla\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau \chi_r, \mathcal{V}\otimes\mathcal{V}\rangle\rangle - \cancel{\left\langle\left\langle \left(\eta\operatorname{div}\left(\left(\chi_r\mathcal{V}\right)^{2\varepsilon}\right)_\tau\right)_\tau \chi_r, \mathfrak{p}\right\rangle\right\rangle} \\
&= -\int_I \eta\langle\langle \nabla\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau, \chi_r\left(\mathcal{V}\otimes\mathcal{V}\right)_\tau\rangle\rangle
\end{aligned}
$$

As there is no longer a time derivative on $\mathcal{V}$, we will get rid of τ by letting $\tau\downarrow 0$ (fine as $\mathcal{V}$ is L^3 in time). Also recall Equation (14.7):

$$
\begin{aligned}
&\lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2}\int_I \eta'\langle\langle(\chi_r\mathcal{V})^\varepsilon, (\chi_r\mathcal{V})^\varepsilon\rangle\rangle = -\lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_I \eta\langle\langle \nabla\left(\chi_r\mathcal{V}\right)^{2\varepsilon}, \mathcal{V}\otimes\chi_r\mathcal{V}\rangle\rangle \\
&= \lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_I \eta\langle\langle(\chi_r\mathcal{V})^\varepsilon, \operatorname{div}\left(\mathcal{V}\otimes\chi_r\mathcal{V}\right)^\varepsilon\rangle\rangle \\
&= \lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_I \eta\langle\langle(\chi_r\mathcal{V})^\varepsilon, \operatorname{div}\left(\mathcal{V}^\varepsilon\otimes(\chi_r\mathcal{V})^\varepsilon\right)\rangle\rangle \\
&= \lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_I \eta\langle\langle(\chi_r\mathcal{V})^\varepsilon, \nabla_{\mathcal{V}^\varepsilon}\left(\chi_r\mathcal{V}\right)^\varepsilon\rangle\rangle = \lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_I \eta\int_M \mathcal{V}^\varepsilon\left(\frac{\left|(\chi_r\mathcal{V})^\varepsilon\right|^2}{2}\right) = 0
\end{aligned}
$$

where we used the commutator estimate to pass to the second line, and the fact that $\mathcal{V}^\varepsilon\in\mathbb{P}\mathfrak{X}$ to make the integral vanish.

We are done. As promised, we now show why we could cancel the error terms previously. Let us calculate

$$
\begin{aligned}
-\lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\lim_{\tau\downarrow 0}\int_I &\langle\langle \left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau\otimes\nabla\chi_r, \mathcal{V}\otimes\mathcal{V}\rangle\rangle \\
&+ \left\langle\left\langle \left(\eta\left(\chi_r\mathcal{V}\right)^{2\varepsilon}_\tau\right)_\tau\cdot\nabla\chi_r + \left(\eta\operatorname{div}\left(\left(\chi_r\mathcal{V}\right)^{2\varepsilon}\right)_\tau\right)_\tau \chi_r, \mathfrak{p}\right\rangle\right\rangle
\end{aligned} \tag{14.8}
$$

Recall from Part I that $\delta_c = \delta\restriction_{\Omega_N}$ and $\delta_c^{\mathscr{D}'_N}$ is the extension of δ_c to heatable currents, defined by

$$\delta_c^{\mathscr{D}_N'} \Lambda(\phi) = \Lambda(d\phi) \ \forall \Lambda \in \mathscr{D}_N'\Omega, \forall \phi \in \mathscr{D}_N\Omega$$

Then the fact that $\mathbb{P}\mathcal{V}^\flat = \mathcal{V}^\flat$ is equivalent to $\delta_c^{\mathscr{D}_N'}\mathcal{V}^\flat = 0$. This implies:

$$\begin{aligned} -\operatorname{div}\left((\chi_r\mathcal{V})^{2\varepsilon}\right) &= \delta_c\left(\left(\chi_r\mathcal{V}^\flat\right)^{2\varepsilon}\right) = \left(\delta_c^{\mathscr{D}_N'}\left(\chi_r\mathcal{V}^\flat\right)\right)^{2\varepsilon} \\ &= \left(-\nabla\chi_r\cdot\mathcal{V} + \chi_r\delta_c^{\mathscr{D}_N'}\mathcal{V}^\flat\right)^{2\varepsilon} = \left(f_r\widetilde{\nu}\cdot\mathcal{V}\right)^{2\varepsilon} \end{aligned} \tag{14.9}$$

With that simplification, and the lack of any time derivatives, (14.8) becomes

$$\begin{aligned} &\lim_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_I \eta\left\langle\left\langle(\chi_r\mathcal{V})^{2\varepsilon}\otimes f_r\widetilde{\nu}, \mathcal{V}\otimes\mathcal{V}\right\rangle\right\rangle + \eta\left\langle\left\langle(\chi_r\mathcal{V})^{2\varepsilon}\cdot f_r\widetilde{\nu}, \mathfrak{p}\right\rangle\right\rangle + \eta\left\langle\left\langle\left(f_r\widetilde{\nu}\cdot\mathcal{V}\right)^{2\varepsilon}, \chi_r\mathfrak{p}\right\rangle\right\rangle \\ =&\lim_{r\downarrow 0}\int_I \eta\left\langle\left\langle\mathcal{V}\cdot\mathcal{V}, \chi_r f_r\widetilde{\nu}\cdot\mathcal{V}\right\rangle\right\rangle + 2\eta\left\langle\left\langle\mathcal{V}\cdot\chi_r f_r\widetilde{\nu}, \mathfrak{p}\right\rangle\right\rangle = \lim_{r\downarrow 0}\int_I 2\eta\left\langle\left\langle\frac{|\mathcal{V}|^2}{2} + \mathfrak{p}, \chi_r f_r\widetilde{\nu}\cdot\mathcal{V}\right\rangle\right\rangle \\ =&\lim_{r\downarrow 0} O\left(\int_I |\eta|\int_{M_{[\frac{r}{2},r]}}\left|\frac{|\mathcal{V}|^2}{2} + \mathfrak{p}\right|\frac{1}{r}\left|\langle\widetilde{\nu}, \mathcal{V}\rangle\right|\right) = 0 \end{aligned}$$

where we used the strip decay hypothesis. □

Remark 111. The proof did not much use the Besov regularity of $\mathcal{V}$, which is mainly used for the commutator estimate.

It is the commutator estimate that presents the main difficulty. We proceed similarly as in [IO14].

<u>Note that from this point on $r > 0$ is fixed.</u>

Let $\mathcal{U} \in L_t^3\widehat{B}_{3,V}^{\frac{1}{3}}\mathfrak{X} \cap L_t^3\mathbb{P}L^3\mathfrak{X}$ and χ_r be as before.

By setting $\mathcal{U}(t)$ to 0 for t in a null set, WLOG we assume $\mathcal{U}(t) \in \mathbb{P}L^3\mathfrak{X} \cap \widehat{B}_{3,V}^{1/3}\mathfrak{X}\ \forall t \in I$.

Define the commutator

$$\mathcal{W}(t,s) = \operatorname{div}\left(\mathcal{U}(t)\otimes\chi_r\mathcal{U}(t)\right)^{3s} - \operatorname{div}\left(\mathcal{U}^{2s}\otimes\left(\chi_r\mathcal{U}(t)\right)^{2s}\right)^s$$

When t and s are implicitly understood, we will not write them. As $\operatorname{div}(\mathcal{U}(t)\otimes\mathcal{U}(t))^{3s}$ solves $(\partial_s - 3\Delta)\,\mathcal{X} = 0$, we define $\mathcal{N} = (\partial_s - 3\Delta)\,\mathcal{W}$. Then $\mathcal{W}$ and $\mathcal{N}$ obey the Duhamel formula.

Lemma 112 (Duhamel)**.** *For fixed $t_0 \in I$ and $s > 0$: $\int_\varepsilon^s \mathcal{N}(t_0,\sigma)^{3(s-\sigma)}\,\mathrm{d}\sigma \xrightarrow{\varepsilon\downarrow 0} \mathcal{W}(t_0,s)$ in $\mathscr{D}'_N\mathfrak{X}$.*

Proof. Let $\varepsilon > 0$. By the smoothing effect of $e^{s\Delta}$, $\mathcal{W}(t_0,\cdot)$ and $\mathcal{N}(t_0,\cdot)$ are in $C^0_{\mathrm{loc}}((0,1],\mathscr{D}_N\mathfrak{X})$. As $\left(e^{s\Delta}\right)_{s\geq 0}$ is a C_0 semigroup on $(H^m\text{-cl}(\mathscr{D}_N\mathfrak{X}),\|\cdot\|_{H^m})$ $\forall m\in\mathbb{N}_0$, and a semigroup basically corresponds to an ODE (cf. [Tay11b, Appendix A, Proposition 9.10 & 9.11]), from $\partial_s\mathcal{W} = 3\Delta\mathcal{W} + \mathcal{N}$ for $s \geq \varepsilon$ we get the Duhamel formula

$$\forall s > \varepsilon : \mathcal{W}(t_0,s) = \mathcal{W}(t_0,\varepsilon)^{3(s-\varepsilon)} + \int_\varepsilon^s \mathcal{N}(t_0,\sigma)^{3(s-\sigma)}\,\mathrm{d}\sigma$$

So we only need to show $\mathcal{W}(t_0,\varepsilon)^{3(s-\varepsilon)} \xrightarrow[\varepsilon\downarrow 0]{\mathscr{D}'_N\mathfrak{X}} 0$. Let $X \in \mathscr{D}_N\mathfrak{X}$.

$$\begin{aligned}
&\left\langle\!\left\langle X, \mathcal{W}(t_0,\varepsilon)^{3(s-\varepsilon)}\right\rangle\!\right\rangle \\
&= \left\langle\!\left\langle X^{3(s-\varepsilon)}, \operatorname{div}(\mathcal{U}(t_0)\otimes\chi_r\mathcal{U}(t_0))^{3\varepsilon} - \operatorname{div}\left(\mathcal{U}(t_0)^{2\varepsilon}\otimes(\chi_r\mathcal{U}(t_0))^{2\varepsilon}\right)^{\varepsilon}\right\rangle\!\right\rangle \\
&= -\left\langle\!\left\langle \nabla\left(X^{3s}\right), \mathcal{U}(t_0)\otimes\chi_r\mathcal{U}(t_0)\right\rangle\!\right\rangle + \left\langle\!\left\langle \nabla\left(X^{3s-2\varepsilon}\right), \mathcal{U}(t_0)^{2\varepsilon}\otimes(\chi_r\mathcal{U}(t_0))^{2\varepsilon}\right\rangle\!\right\rangle \xrightarrow{\varepsilon\downarrow 0} 0.
\end{aligned}$$

□

From now on, we write $\int_{0+}^s$ for $\lim_{\varepsilon\downarrow 0}\int_\varepsilon^s$. Then

$$\int_I \mathrm{d}t\,\eta(t)\,\langle\langle \mathcal{W}(t,s), \mathcal{U}(t)^s\rangle\rangle = \int_I \mathrm{d}t\,\eta(t)\int_{0+}^s \mathrm{d}\sigma\,\left\langle\!\left\langle \mathcal{N}(t,\sigma)^{3(s-\sigma)}, \mathcal{U}(t)^s\right\rangle\!\right\rangle$$

Definition 113. Define the k-jet fiber norm $|X|_{J^k} = \left(\sum_{j=0}^k \left|\nabla^{(j)}X\right|^2\right)^{\frac{1}{2}}$ $\forall X \in \mathfrak{X}$.

Let $K(\sigma,x,y)$ be the kernel of the heat flow at time $\sigma > 0$. Then by Chapter B, we obtain off-diagonal decay for all derivatives:

Fact 114 (Off-diagonal decay)**.** *For any multi-index* γ *and* $x \neq y$, $D^{\gamma}_{\sigma,x,y} K(\sigma, x, y) = O(\sigma^{\infty})$ *as* $\sigma \downarrow 0$, *locally uniform in* $\{x \neq y\}$.

For convenience, we will write $\boxed{\mathcal{Y} = \chi_r \mathcal{U}}$. Then for $r > 0, \sigma \in (0,1)$ and $x \in M_{<r/4}$:

$$|\mathcal{Y}^{\sigma}(t,x)|_{J^2} \lesssim_{M,r} O_r(\sigma^{\infty}) \|\mathcal{U}(t)\|_{L^3\left(M_{>r/2}\right)} \tag{14.10}$$

which implies $\|\mathcal{Y}^{\sigma}(t)\|_{W^{2,3}\left(M_{<r/4}\right)} + \|\mathcal{Y}^{\sigma}(t)\|_{W^{2,3}\mathfrak{X}M|_{\partial M}} \lesssim_{M,r} O_r(\sigma^{\infty}) \|\mathcal{U}(t)\|_{L^3\left(M_{>r/2}\right)}$.

We now handle the most important error estimates that will appear in our analysis.

Lemma 115 (2 error estimates)**.** *For fixed* $r > 0$ *small, we have*

$$\lim_{s\downarrow 0} \int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \int_M \left|\mathcal{U}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{4s-2\sigma}\right|_{J^1} = 0 \tag{14.11}$$

and

$$\lim_{s\downarrow 0} \int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \int_{\partial M} \left|\mathcal{U}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{4s-2\sigma}\right|_{J^2} = 0 \tag{14.12}$$

.

Proof. We split (14.11) into 2 regions: $M_{<r/4}$ and $M_{\geq r/4}$. Observe that

$$\begin{aligned}
&\int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \int_{M_{<r/4}} \left|\mathcal{U}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{4s-2\sigma}\right|_{J^1} \\
\lesssim &\int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \ \left\|\mathcal{U}^{2\sigma}\right\|_{W^{1,3}\left(M_{<r/4}\right)} \left\|\mathcal{Y}^{2\sigma}\right\|_{W^{1,3}\left(M_{<r/4}\right)} \left\|\mathcal{Y}^{4s-2\sigma}\right\|_{W^{1,3}\left(M_{<r/4}\right)} \\
\lesssim &O_r(s^{\infty}) \int_I \mathrm{d}t \ |\eta(t)| \, \|\mathcal{U}(t)\|^3_{L^3(M)} \int_{0+}^{s} \mathrm{d}\sigma \left(\frac{1}{\sigma}\right)^{1/2} \xrightarrow{s\downarrow 0} 0.
\end{aligned}$$

Define $B(t,s) = s^{\frac{1}{3}} \|\mathcal{U}(t)^s\|_{W^{1,3}\left(M_{\geq r/4}\right)}$ and $C(t,s) = s^{\frac{1}{3}} \|\mathcal{Y}(t)^s\|_{W^{1,3}\left(M_{\geq r/4}\right)}$.

By Fact 108, $\mathcal{Y} \in L^3_t \widehat{B}^{1/3}_{3,V}\mathfrak{X}$.

Therefore, $\|B(t,s)\|_{L^3_t}$ and $\|C(t,s)\|_{L^3_t}$ are continuous in s and converge to 0 as

$s \to 0$ by (14.4). Observe that

$$
\begin{aligned}
&\int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \int_{M_{\geq r/4}} \left|\mathcal{U}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{4s-2\sigma}\right|_{J^1} \\
&\lesssim \int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \left\|\mathcal{U}^{2\sigma}\right\|_{W^{1,3}\left(M_{\geq r/4}\right)} \left\|\mathcal{Y}^{2\sigma}\right\|_{W^{1,3}\left(M_{\geq r/4}\right)} \left\|\mathcal{Y}^{4s-2\sigma}\right\|_{W^{1,3}\left(M_{\geq r/4}\right)} \\
&= \int_I \mathrm{d}t \, |\eta(t)| \int_{0+}^{s} \mathrm{d}\sigma \left(\frac{1}{\sigma}\right)^{\frac{2}{3}} \left(\frac{1}{2s-\sigma}\right)^{\frac{1}{3}} B\left(t, 2\sigma\right) C\left(t, 2\sigma\right) C\left(t, 4s-2\sigma\right) \\
&\lesssim_\eta \int_{0+}^{s} \mathrm{d}\sigma \, \left(\frac{1}{\sigma}\right)^{\frac{2}{3}} \left(\frac{1}{2s-\sigma}\right)^{\frac{1}{3}} \left\|B\left(t, 2\sigma\right)\right\|_{L_t^3} \left\|C\left(t, 2\sigma\right)\right\|_{L_t^3} \left\|C\left(t, 4s-2\sigma\right)\right\|_{L_t^3} \\
&\overset{\sigma = s\tau}{=} \int_{0+}^{1} \mathrm{d}\tau \, \left(\frac{1}{\tau}\right)^{\frac{2}{3}} \left(\frac{1}{2-\tau}\right)^{\frac{1}{3}} \left\|B\left(t, 2s\tau\right)\right\|_{L_t^3} \left\|C\left(t, 2s\tau\right)\right\|_{L_t^3} \left\|C\left(t, 4s-2s\tau\right)\right\|_{L_t^3} \\
&\xrightarrow[\text{DCT}]{s\downarrow 0} 0
\end{aligned}
$$

So (14.11) is proven. For (14.12), observe that

$$
\begin{aligned}
&\int_I |\eta| \int_{0+}^{s} \mathrm{d}\sigma \int_{\partial M} \left|\mathcal{U}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{2\sigma}\right|_{J^1} \left|\mathcal{Y}^{4s-2\sigma}\right|_{J^2} \\
&\lesssim O_r\left(s^\infty\right) \int_I \mathrm{d}t \;\; |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{L^3(M)}^2 \int_{0+}^{s} \mathrm{d}\sigma \;\; \left\|\mathcal{U}\left(t\right)^{2\sigma}\right\|_{W^{1,3}\mathfrak{X}M|_{\partial M}} \\
&\lesssim O_r\left(s^\infty\right) \int_I \mathrm{d}t \;\; |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{L^3(M)}^2 \int_{0+}^{s} \mathrm{d}\sigma \;\; \left\|\mathcal{U}\left(t\right)^{2\sigma}\right\|_{B_{3,1}^{1+1/3}(M)} \\
&\lesssim O_r\left(s^\infty\right) \int_I \mathrm{d}t \;\; |\eta\left(t\right)| \left\|\mathcal{U}\left(t\right)\right\|_{L^3(M)}^3 \int_{0+}^{s} \mathrm{d}\sigma \;\; \left(\frac{1}{\sigma}\right)^{2/3} \\
&\xrightarrow{s\downarrow 0} 0
\end{aligned}
$$

where we used (14.10) to pass to the second line, and the trace theorem to pass to the third line. □

Note that

$$
\begin{aligned}
\mathcal{N}\left(t, \sigma\right) = \left(\partial_\sigma - 3\Delta\right)\left(-\operatorname{div}\left(\mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}\right)^\sigma\right) &= -2\operatorname{div}\left(\Delta\mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}\right)^\sigma - 2\operatorname{div}\left(\mathcal{U}^{2\sigma} \otimes \Delta\mathcal{Y}^{2\sigma}\right)^\sigma \\
&\quad + 2\Delta\operatorname{div}\left(\mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}\right)^\sigma
\end{aligned}
$$

Now, we finally show

$$\int_I \eta \langle\langle \mathcal{W}(s), \mathcal{Y}^s \rangle\rangle = \int_I \mathrm{d}t \ \eta(t) \langle\langle \mathcal{W}(t,s), \mathcal{Y}(t)^s \rangle\rangle \xrightarrow{s\downarrow 0} 0$$

Proof of the commutator estimate. First we integrate by parts into three components:

$$\int_I \eta \langle\langle \mathcal{W}(s), \mathcal{Y}^s \rangle\rangle = \int_I \mathrm{d}t \ \eta(t) \int_{0+}^s \mathrm{d}\sigma \left\langle\left\langle \mathcal{N}(t,\sigma)^{3(s-\sigma)}, \mathcal{Y}(t)^s \right\rangle\right\rangle$$
$$= \int_I \mathrm{d}t \ \eta(t) \int_{0+}^s \mathrm{d}\sigma \left\langle\left\langle \mathcal{N}(t,\sigma), \mathcal{Y}(t)^{4s-3\sigma} \right\rangle\right\rangle$$
$$= 2\int_I \eta \int_{0+}^s \mathrm{d}\sigma \left\langle\left\langle \Delta\mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}, \nabla\left(\mathcal{Y}^{4s-2\sigma}\right) \right\rangle\right\rangle + 2\int_I \eta \int_{0+}^s \mathrm{d}\sigma \left\langle\left\langle \mathcal{U}^{2\sigma} \otimes \Delta\mathcal{Y}^{2\sigma}, \nabla\left(\mathcal{Y}^{4s-2\sigma}\right) \right\rangle\right\rangle$$
$$- 2\int_I \eta \int_{0+}^s \mathrm{d}\sigma \left\langle\left\langle \mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}, \nabla\left(\Delta\mathcal{Y}^{4s-2\sigma}\right) \right\rangle\right\rangle$$

Note that for the third component, we used Fact 100 to move the Laplacian.

We now use the Penrose abstract index notation to estimate the three components. To clean up the notation, we only focus on the integral on M, with the other integrals $2\int_I \eta \int_{0+}^s \mathrm{d}\sigma\,(\cdot)$ in variables t and σ implicitly understood. We also use **schematic identities** for linear combinations of similar-looking tensor terms where we do not care how the indices contract (recall Equation (14.1)).

By Lemma 115, it is easy to check that all the terms with R or ν will be negligible (going to 0 in the limit), and interchanging derivatives will be a negligible action. We write $\approx$ to throw the negligible error terms away.

First component:

$$\int_M \left\langle \Delta\mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}, \nabla\left(\mathcal{Y}^{4s-2\sigma}\right) \right\rangle$$
$$= \int_M R * \mathcal{U}^{2\sigma} * \mathcal{Y}^{2\sigma} * \nabla\left(\mathcal{Y}^{4s-2\sigma}\right) + \int_M \nabla_i \nabla^i \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l$$

$$
\begin{aligned}
&\approx \cancel{\int_{\partial M} \nu_i \nabla^i \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l} - \cancel{\int_M \nabla^i \left(\mathcal{U}^{2\sigma}\right)^j \nabla_i \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l} \\
&\quad - \int_M \nabla^i \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_i \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l
\end{aligned}
$$

Second component:

$$
\begin{aligned}
&\int_M \left\langle \mathcal{U}^{2\sigma} \otimes \Delta \mathcal{Y}^{2\sigma}, \nabla \left(\mathcal{Y}^{4s-2\sigma}\right) \right\rangle \\
&= \cancel{\int_M \mathcal{U}^{2\sigma} * R * \mathcal{Y}^{2\sigma} * \nabla \left(\mathcal{Y}^{4s-2\sigma}\right)} + \int_M \left(\mathcal{U}^{2\sigma}\right)^j \nabla_i \nabla^i \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l \\
&\approx \cancel{\int_{\partial M} \left(\mathcal{U}^{2\sigma}\right)^j \nu_i \nabla^i \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l} - \cancel{\int_M \nabla_i \left(\mathcal{U}^{2\sigma}\right)^j \nabla^i \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l} \\
&\quad - \int_M \left(\mathcal{U}^{2\sigma}\right)^j \nabla^i \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_i \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l
\end{aligned}
$$

For the third component, we use the identity $\nabla (R * U) = \nabla R * U + R * \nabla U$ to compute:

$$
\begin{aligned}
&- \int_M \left\langle \mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma}, \nabla \left(\Delta \mathcal{Y}^{4s-2\sigma}\right) \right\rangle \\
&= -\cancel{\int_M \mathcal{U}^{2\sigma} * \mathcal{Y}^{2\sigma} * \nabla \left(R * \mathcal{Y}^{4s-2\sigma}\right)} - \int_M \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_j \nabla^i \nabla_i \left(\mathcal{Y}^{4s-2\sigma}\right)_l \\
&\approx \cancel{\int_M \mathcal{U}^{2\sigma} * \mathcal{Y}^{2\sigma} * R * \nabla \left(\mathcal{Y}^{4s-2\sigma}\right)} - \int_M \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla^i \nabla_j \nabla_i \left(\mathcal{Y}^{4s-2\sigma}\right)_l \\
&\approx \cancel{\int_M \mathcal{U}^{2\sigma} * \mathcal{Y}^{2\sigma} * \nabla \left(R * \mathcal{Y}^{4s-2\sigma}\right)} - \int_M \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla^i \nabla_i \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l \\
&\approx -\cancel{\int_{\partial M} \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nu^i \nabla_i \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l} + \int_M \nabla^i \left(\mathcal{U}^{2\sigma}\right)^j \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_i \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l \\
&\quad + \int_M \left(\mathcal{U}^{2\sigma}\right)^j \nabla^i \left(\mathcal{Y}^{2\sigma}\right)^l \nabla_i \nabla_j \left(\mathcal{Y}^{4s-2\sigma}\right)_l
\end{aligned}
$$

By adding them up, we are done. □

APPENDIX A

Local analysis

Let M be as in Equation (11.1). Throughout this chapter, we write $e^{t\Delta}$ for the absolute Neumann heat flow, as we will not need another heat flow.

Assume the absolute Neumann heat kernel is already constructed, with off-diagonal decay (Fact 114).

As before, define $P_{\leq N} = e^{\frac{1}{N^2}\Delta}$ for $N > 0$ and $P_N = P_{\leq N} - P_{\leq \frac{N}{2}}$ for $N > 1, N \in 2^{\mathbb{Z}}$.

Let $\chi_r = 1 - \psi_r$ (see Equation (14.5)).

Then we have the localized Bernstein estimates:

Theorem 116. *For any $r > 0$; $m_1, m_2 \in \mathbb{N}_0$; $p \in (1, \infty)$; $N \geq 1$ and $X \in W^{m_1,p}\Omega(M)$:*

$$\|P_{\leq N} X\|_{W^{m_1+m_2,p}(M_{\geq 2r})} \lesssim_{r,m_1,m_2,p} N^{m_2} \|X\|_{W^{m_1,p}(M_{\geq r})} + O_r\left(\tfrac{1}{N^\infty}\right) \|X\|_{L^p(M_{\leq 3r})}$$

Proof. Observe that $1 - \chi_{2r} = \psi_{2r} = \psi_{2r}\psi_{4r}$. Then:

$$\begin{aligned}
&\|P_{\leq N} X\|_{W^{m_1+m_2,p}(M_{\geq 2r})} \\
&\lesssim \|P_{\leq N}(\chi_{2r} X)\|_{W^{m_1+m_2,p}(M_{\geq 2r})} + \|P_{\leq N}(\psi_{2r}\psi_{4r} X)\|_{W^{m_1+m_2,p}(M_{\geq 2r})} \\
&\lesssim_{m_1,m_2} N^{m_2} \|\chi_{2r} X\|_{W^{m_1,p}(M)} + O_r\left(\frac{1}{N^\infty}\right) \|\psi_{4r} X\|_{L^p(M)} \\
&\lesssim_r N^{m_2} \|X\|_{W^{m_1,p}(M_{\geq r})} + O_r\left(\frac{1}{N^\infty}\right) \|X\|_{L^p(M_{\leq 3r})}
\end{aligned}$$

where we have used the standard Bernstein estimate (Theorem 101) and the off-diagonal decay of the heat kernel to pass from the first line to the second line (supp $\psi_{2r} \subseteq M_{\leq \frac{3}{2}r}$ which does not intersect $M_{\geq 2r}$). ☐

Corollary 117. *For any $r, C_1, C_2 > 0$; $N \geq 1; p \in (1, \infty)$ and $X \in \mathscr{D}'_N \Omega(M)$:*

$$\begin{aligned}
&\left\|\left(P_{\leq C_1 N} - P_{\leq C_2 N}\right) X\right\|_{L^p\left(M_{\geq 2r}\right)} \\
&\lesssim_{C_1,C_2,r,p} \frac{1}{N^2}\left\|P_{\leq 2\max(C_1,C_2)N} X\right\|_{W^{2,p}\left(M_{\geq r}\right)} + O_{C_1,C_2,r}\left(\frac{1}{N^\infty}\right)\|X\|_{L^p(M)} \\
&\lesssim_{C_1,C_2,r,p} \frac{1}{N}\left\|P_{\leq 3\max(C_1,C_2)N} X\right\|_{W^{1,p}\left(M_{\geq r/2}\right)} + O_{C_1,C_2,r}\left(\frac{1}{N^\infty}\right)\|X\|_{L^p(M)}
\end{aligned}$$

Proof. WLOG $C_1 > C_2 > 0$. Let $C = 2\max(C_1, C_2)$. Then by FTC:

$$\begin{aligned}
&\left\|\left(P_{\leq C_1 N} - P_{\leq C_2 N}\right) X\right\|_{L^p\left(M_{\geq 2r}\right)} \\
&\leq \int_{\frac{1}{C_1^2 N^2} - \frac{1}{C^2 N^2}}^{\frac{1}{C_2^2 N^2} - \frac{1}{C^2 N^2}} \mathrm{d}t \left\|e^{\left(t + \frac{1}{C^2 N^2}\right)\Delta} X\right\|_{W^{2,p}\left(M_{\geq 2r}\right)} \\
&\lesssim_{C_1,C_2,r,p} \int_{\frac{1}{C_1^2 N^2} - \frac{1}{C^2 N^2}}^{\frac{1}{C_2^2 N^2} - \frac{1}{C^2 N^2}} \mathrm{d}t \left(\left\|e^{\frac{\Delta}{C^2 N^2}} X\right\|_{W^{2,p}\left(M_{\geq r}\right)} + O_r\left(t^\infty\right)\left\|e^{\frac{\Delta}{2N^2}} X\right\|_{L^p(M)}\right) \\
&\lesssim_{C_1,C_2} \frac{1}{N^2}\left\|P_{\leq CN} X\right\|_{W^{2,p}\left(M_{\geq r}\right)} + O_{C_1,C_2,r}\left(\frac{1}{N^\infty}\right)\|X\|_{L^p(M)}
\end{aligned}$$

We have used Theorem 116 to pass to the second line.

The rest is trivial. ☐

Corollary 118. *Let $p \in (1, \infty)$ and $X \in L^p\Omega(M)$. Then the following conditions are equivalent:*

1. $N^{\frac{1}{p}-1}\left\|P_{\leq N} X\right\|_{W^{1,p}(M_{>r})} \xrightarrow{N \to \infty} 0 \ \forall r > 0$
2. $N^{\frac{1}{p}}\left\|\left(P_{\leq C_1 N} - P_{\leq C_2 N}\right) X\right\|_{L^p(M_{>r})} \xrightarrow{N \to \infty} 0 \ \forall r, C_1, C_2 > 0$
3. $N^{\frac{1}{p}}\left\|P_{> N} X\right\|_{L^p(M_{>r})} \xrightarrow{N \to \infty} 0 \ \forall r > 0$

Proof. It is trivial to show (3) $\implies$ (2) as $P_{\le C_1 N} - P_{\le C_2 N} = P_{>C_2 N} - P_{>C_1 N}$.

Next, we show (2) $\implies$ (3). Let

$$w(N) = N^{\frac{1}{p}} \|P_N X\|_{L^p(M_{>r})} = N^{\frac{1}{p}} \left\| \left(P_{\le N} - P_{\le N/2}\right) X \right\|_{L^p(M_{>r})} \xrightarrow{N\to\infty} 0$$

Then:

$$\begin{aligned} N^{\frac{1}{p}} \|P_{>N} X\|_{L^p(M_{>r})} &\le N^{\frac{1}{p}} \sum_{\substack{K\in 2^{\mathbb{Z}} \\ K>N}} \|P_K X\|_{L^p(M_{>r})} = N^{\frac{1}{p}} \sum_{\substack{K\in 2^{\mathbb{Z}} \\ K>N}} K^{-\frac{1}{p}} w(K) \\ &\lesssim \|w(\kappa)\|_{l^\infty(\kappa>N,\kappa\in 2^{\mathbb{Z}})} \xrightarrow{N\to\infty} 0 \end{aligned}$$

We proceed to show (1) $\implies$ (2). By Corollary 117:

$$\begin{aligned} & N^{\frac{1}{p}} \|(P_{\le C_1 N} - P_{\le C_2 N}) X\|_{L^p(M_{>r})} \\ & \lesssim_{C_1,C_2} N^{\frac{1}{p}-1} \left\| P_{\le 3\max(C_1,C_2) N} X \right\|_{W^{1,p}\left(M_{\ge r/4}\right)} + O_{C_1,C_2,r}\left(\frac{1}{N^\infty}\right) \|X\|_{L^p(M)} \\ & \xrightarrow{N\to\infty} 0 \end{aligned}$$

Finally, we show (2) $\implies$ (1). Let $N_0 \ge 1$ and $N_0 \in 2^{\mathbb{Z}}$. There are constants $C_1, C_2 > 0$ such that $P_N = P_{\le 2N}\left(P_{\le C_1 N} - P_{\le C_2 N}\right)$.

$$\begin{aligned} & \limsup_{N\to\infty} N^{\frac{1}{p}-1} \|P_{\le N} X\|_{W^{1,p}(M_{>r})} = \limsup_{N\to\infty} N^{\frac{1}{p}-1} \|(P_{\le N} - P_{\le N_0}) X\|_{W^{1,p}(M_{>r})} \\ & \lesssim \limsup_{N\to\infty} N^{\frac{1}{p}-1} \sum_{\substack{K\in 2^{\mathbb{Z}} \\ N_0<K\le N}} \|P_K X\|_{W^{1,p}(M_{>r})} \\ & \lesssim \limsup_{N\to\infty} N^{\frac{1}{p}-1} \sum_{\substack{K\in 2^{\mathbb{Z}} \\ N_0<K\le N}} \left(K \|(P_{\le C_1 K} - P_{\le C_2 K}) X\|_{L^p\left(M_{>r/2}\right)} + O_r\left(\frac{1}{K^\infty}\right) \|X\|_{L^p(M)} \right) \end{aligned}$$

$$\lesssim \limsup_{N\to\infty} N^{\frac{1}{p}-1} \sum_{\substack{K\in 2^{\mathbb{Z}} \\ N_0<K\leq N}} K^{1-1/p} w(K) + \underbrace{\limsup_{N\to\infty} N^{\frac{1}{p}-1} O_r\left(\frac{1}{N_0^{\infty}}\right) \|X\|_{L^p(M)}}_{0} \tag{A.1}$$

where $w(K) := K^{1/p} \left\|\left(P_{\leq C_1 K} - P_{\leq C_2 K}\right) X\right\|_{L^p\left(M_{>r/2}\right)} \xrightarrow{K\to\infty} 0$. Then we can bound (A.1) by

$$\limsup_{N\to\infty} \sum_{\substack{K\in 2^{\mathbb{Z}} \\ N_0<K\leq N}} \left(\frac{K}{N}\right)^{1-1/p} \|w(\kappa)\|_{l^\infty(\kappa\geq N_0,\kappa\in 2^{\mathbb{Z}})} \lesssim \|w(\kappa)\|_{l^\infty(\kappa\geq N_0,\kappa\in 2^{\mathbb{Z}})}$$

But N_0 is arbitrary. Let $N_0 \to \infty$ and we are done. □

Remark 119. By repeating the proof, for $\mathcal{X} \in L_t^p L^p \Omega(M)$:

$$\forall r > 0 : N^{\frac{1}{p}-1} \|P_{\leq N}\mathcal{X}\|_{L_t^p W^{1,p}(M_{>r})} \xrightarrow{N\to\infty} 0$$
$$\iff \forall r > 0 : N^{\frac{1}{p}} \|P_{>N}\mathcal{X}\|_{L_t^p L^p(M_{>r})} \xrightarrow{N\to\infty} 0$$

We now prove a simple lemma from functional analysis.

Lemma 120 (Loss of norm)**.** *Let X, Y be Banach spaces and $T : X \hookrightarrow Y$ is continuous injection. Let $(f_j)_{j\in\mathbb{N}_1}$ be a sequence in X and $f \in X$. If $Tf_j \rightharpoonup Tf$ then*

$$\|f\|_X \leq \liminf_{j\to\infty} \|f_j\|_X$$

Proof. Note that $T^* : Y^* \to X^*$ has dense image. Then

$$\begin{aligned}\|f\|_X &= \sup_{\substack{\|x^*\|_{X^*}=1 \\ x^*\in X^*}} |\langle f, x^*\rangle| = \sup_{\substack{y^*\in Y^* \\ \|T^*y^*\|_{X^*}=1}} |\langle f, T^*y^*\rangle| = \sup_{\substack{y^*\in Y^* \\ \|T^*y^*\|_{X^*}=1}} \lim_{j\to\infty} |\langle Tf_j, y^*\rangle| \\ &= \sup_{\substack{y^*\in Y^* \\ \|T^*y^*\|_{X^*}=1}} \lim_{j\to\infty} |\langle f_j, T^*y^*\rangle| \leq \sup_{\substack{y^*\in Y^* \\ \|T^*y^*\|_{X^*}=1}} \liminf_{j\to\infty} \|f_j\|_X = \liminf_{j\to\infty} \|f_j\|_X\end{aligned}$$

□

Theorem 121. *Let $p \in (1,\infty)$, $f \in \mathscr{D}_N(M)$, and $X \in \widehat{B}^{1/p}_{p,V}\mathfrak{X}(M)$ (as in Definition 105). Then $fX \in \widehat{B}^{1/p}_{p,V}\mathfrak{X}$.*

Proof. To show $fX \in \widehat{B}^{1/p}_{p,V}\mathfrak{X}$, we just need to show a commutator estimate (much like in the proof of Onsager's conjecture):

$$\begin{cases} \mathcal{W}(s) & := f^s X^s - (fX)^s \\ (\sqrt{s})^{1-\frac{1}{p}}\mathcal{W}(s) & \xrightarrow[s\downarrow 0]{W^{1,p}(M)} 0 \end{cases}$$

where X^s is short for $e^{s\Delta}X$. Indeed, assuming this commutator estimate holds true, $\forall r>0$:

$$\begin{aligned} &\limsup_{t\downarrow 0}\left(\sqrt{t}\right)^{1-\frac{1}{p}}\left\|e^{t\Delta}(fX)\right\|_{W^{1,p}(M_{>r})} \\ \leq &\limsup_{t\downarrow 0}\left(\sqrt{t}\right)^{1-\frac{1}{p}}\left\|f^tX^t\right\|_{W^{1,p}(M_{>r})} + \underbrace{\limsup_{t\downarrow 0}\left(\sqrt{t}\right)^{1-\frac{1}{p}}\left\|\mathcal{W}(t)\right\|_{W^{1,p}(M_{>r})}}_{0} \\ \lesssim &\limsup_{t\downarrow 0}\left(\sqrt{t}\right)^{1-\frac{1}{p}}\left\|f^t\right\|_{C^1(M)}\left\|X^t\right\|_{W^{1,p}(M_{>r})} = 0 \end{aligned}$$

where we have used the fact that $e^{t\Delta}f \xrightarrow[t\to 0]{C^\infty} f$, as $f \in \mathscr{D}_N(M)$.

Now we prove the commutator estimate. Define $\mathcal{N}(s) = (\partial_s - \Delta)\mathcal{W}(s) = (\Delta f^s)X^s + f^s(\Delta X^s) - \Delta(f^sX^s)$. By the Weitzenbock formula, we get

$$\mathcal{N}(s) = \left(D^1 f^s\right) * \left(D^1 X^s\right)$$

where D^1 is schematic for some differential operator of order at most 1, with smooth coefficients (independent of s), and $(D^1 f^s) * (D^1 X^s)$ is schematic for a linear combination of similar-looking tensor terms.

On the other hand, by the Duhamel formula for semigroups (cf. [Tay11b, Appendix A, Proposition 9.10 & 9.11]), for any $s > \varepsilon > 0$ we get

$$\mathcal{W}(s) = \mathcal{W}(\varepsilon)^{s-\varepsilon} + \int_\varepsilon^s \mathcal{N}(\sigma)^{s-\sigma} \, \mathrm{d}\sigma$$

It is trivial to show that $\mathcal{W}(\varepsilon)^{s-\varepsilon} \xrightarrow[\varepsilon\downarrow 0]{L^p} 0$. Indeed, let $Y \in L^{p'}\mathfrak{X}(M)$. Then

$$\left\langle\left\langle \mathcal{W}(\varepsilon)^{s-\varepsilon}, Y \right\rangle\right\rangle = \left\langle\left\langle f^\varepsilon X^\varepsilon - (fX)^\varepsilon, Y^{s-\varepsilon} \right\rangle\right\rangle \xrightarrow{\varepsilon\downarrow 0} \langle\langle fX - fX, Y^s \rangle\rangle = 0$$

Then $\int_\varepsilon^s \mathcal{N}(\sigma)^{s-\sigma} \, \mathrm{d}\sigma \xrightarrow[\varepsilon\downarrow 0]{L^p} \mathcal{W}(s)$, and by Lemma 120, we conclude

$$\begin{aligned}
\|\mathcal{W}(s)\|_{W^{1,p}(M)} &\leq \liminf_{\varepsilon\downarrow 0} \left\| \int_\varepsilon^s \mathcal{N}(\sigma)^{s-\sigma} \, \mathrm{d}\sigma \right\|_{W^{1,p}(M)} \\
&\leq \int_0^s \left\| e^{(s-\sigma)\Delta} \left(D^1 f^\sigma * D^1 X^\sigma \right) \right\|_{W^{1,p}(M)} \mathrm{d}\sigma \\
&\lesssim \int_0^s \left(\frac{1}{s-\sigma} \right)^{\frac{1}{2}} \left\| D^1 f^\sigma * D^1 X^\sigma \right\|_{L^p(M)} \mathrm{d}\sigma \\
&\lesssim_f \int_0^s \left(\frac{1}{s-\sigma} \right)^{\frac{1}{2}} \|X^\sigma\|_{W^{1,p}(M)} \, \mathrm{d}\sigma \\
&\lesssim \|X\|_{L^p(M)} \int_0^s \left(\frac{1}{s-\sigma} \right)^{\frac{1}{2}} \left(\frac{1}{\sigma} \right)^{\frac{1}{2}} \mathrm{d}\sigma \\
&\overset{\sigma=s\tau}{=} \|X\|_{L^p(M)} \int_0^1 \left(\frac{1}{1-\tau} \right)^{\frac{1}{2}} \left(\frac{1}{\tau} \right)^{\frac{1}{2}} \mathrm{d}\tau \lesssim_p \|X\|_{L^p(M)}
\end{aligned}$$

This obviously implies $\left(\sqrt{s}\right)^{1-\frac{1}{p}} \mathcal{W}(s) \xrightarrow[s\downarrow 0]{W^{1,p}(M)} 0$. □

Remark. By repeating the proof, with necessary modifications, for any $f \in \mathscr{D}_N(M)$, and $\mathcal{X} \in L_t^p \widehat{B}_{p,V}^{1/p} \mathfrak{X}(M)$ (as in Definition 105), we have:

$$f\mathcal{X} \in L_t^p \widehat{B}_{p,V}^{1/p} \mathfrak{X}$$

A.1 On flat backgrounds

Remark 122. When M is a bounded domain in $\mathbb{R}^n$, the third condition in Corollary 118 takes on a more familiar form. Indeed, let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int \phi = 1$ and $\phi_\varepsilon = \frac{1}{\varepsilon^n}\phi\left(\frac{\cdot}{\varepsilon}\right)$. Then we have the analogy

$$P_{\leq \frac{1}{\sqrt{t}}} f = e^{t\Delta} f \approx \phi_{\sqrt{t}} * f$$

This means

$$N^{\frac{1}{p}} \left\| P_{>N} X \right\|_{L^p(M_{>r})} \xrightarrow{N\to\infty} 0 \tag{A.2}$$

is analogous to

$$\frac{1}{\varepsilon^{1/p}} \left\| X - \phi_\varepsilon * X \right\|_{L^p(M_{>r})} \xrightarrow{\varepsilon\to 0} 0 \tag{A.3}$$

Definition 123. As in [Bar+19b; NNT20], for $p \in (1,\infty)$, we say $X \in \underline{B}^{1/p}_{p,\mathrm{VMO}} \mathfrak{X}(M)$ if $X \in L^p\mathfrak{X}(M)$ and $\forall r > 0:$

$$A_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \left\| X(x-\varepsilon h) - X(x) \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})} \xrightarrow{\varepsilon\downarrow 0} 0 \tag{A.4}$$

Similarly, we say $\mathcal{X} \in L^p_t \underline{B}^{1/p}_{p,\mathrm{VMO}} \mathfrak{X}(M)$ if $X \in L^p_t L^p \mathfrak{X}(M)$ and $\forall r > 0:$

$$A_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \left\| \mathcal{X}(t, x-\varepsilon h) - \mathcal{X}(t,x) \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_t L^p_x(M_{>r})} \xrightarrow{\varepsilon\downarrow 0} 0 \tag{A.5}$$

Remark 124. In (A.4), note that $A_r(\varepsilon)$ is continuous for $\varepsilon \in [0,r)$. Define

$$\widetilde{A_r}(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \left\| \mathbf{1}_{M_{>r}}(x-\varepsilon h)\left(X(x-\varepsilon h) - X(x)\right) \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})}$$

for $\varepsilon \in (0,1]$ (well-defined). Then $\widetilde{A_r}(\varepsilon)$ is also continuous in ε, with $\widetilde{A_r}(\varepsilon) \leq A_r(\varepsilon)$ $\forall \varepsilon \in (0,r)$ and $\widetilde{A_r}(\varepsilon) \lesssim_{r,p} \|X\|_{L^p(M)}$ $\forall \varepsilon \in [\frac{r}{2}, 1]$. By Section 6.2, we conclude

$$\|X\|_{B^{1,p}_{p,\infty}(M_{>r})} \sim \|X\|_{L^p(M_{>r})} + \left\| \widetilde{A_r}(\varepsilon) \right\|_{L^\infty_\varepsilon((0,1))} \lesssim_{r,p} \|X\|_{L^p(M)} + \|A_r(\varepsilon)\|_{L^\infty_\varepsilon([0,r/2])}$$

From this we conclude $\underline{B}^{1/p}_{p,\mathrm{VMO}} \hookrightarrow B^{1/p}_{p,\infty,\mathrm{loc}}$ and $L^p_t \underline{B}^{1/p}_{p,\mathrm{VMO}} \hookrightarrow L^p_t B^{1/p}_{p,\infty,\mathrm{loc}}$ where

$$B^{1/p}_{p,\infty,\mathrm{loc}}(M) := L^p(M) \cap \left(\bigcap_{r>0} B^{1/p}_{p,\infty}(M_{>r}) \right)$$

and $L^p_t B^{1/p}_{p,\infty,\mathrm{loc}}(M) := L^p_t L^p(M) \cap \left(\bigcap_{r>0} L^p_t B^{1/p}_{p,\infty}(M_{>r}) \right)$.

We observe that (A.4) trivially implies (A.3). To relate (A.4) to (A.2), we now borrow some results from the construction of the heat kernel (to be proven in Chapter B).

Fact 125. *Fix $r > 0$. Let $K(t,x,y)$ be the Hodge-Neumann heat kernel as constructed in Chapter B.*

For $r' > 0$, let $E_{r'} = \{(x,y) \in M \times M : d(x,y) \geq r'\}$. Then $E_{r'}$ is compact, and by the locally uniform off-diagonal decay of the heat kernel, we conclude

$$\forall x, y \in E_{r'}, \forall t \leq 1 : |K(t,x,y)| = O_{r',\neg x,\neg y}(t^{\infty}) \tag{A.6}$$

Now let $F_{r,r'} = \{(x,y) \in M_{\leq r} \times M : d(x,y) \leq r'\}$. Then $F_{r,r'}$ is compact. By interior blow-up, there is $r' = r'(r) \in \left(0, \frac{r}{4}\right)$ such that

$$\forall x, y \in F_{r,r'}, \forall t \leq 1 : |K(t,x,y)| = O_{r,\neg x,\neg y}\left(\frac{1}{t^{n/2}} \left\langle \frac{x-y}{\sqrt{t}} \right\rangle^{-\infty} \right) \tag{A.7}$$

Theorem 126. *Let M be a bounded C^{∞}-domain in $\mathbb{R}^n$, $p \in (1,\infty)$ and $X \in L^p\mathfrak{X}(M)$. Then*

$$\forall r > 0 : A_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \|X(x - \varepsilon h) - X(x)\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})} \xrightarrow{\varepsilon \downarrow 0} 0 \tag{A.8}$$

is equivalent to

$$\forall r > 0 : N^{\frac{1}{p}} \|P_{>N} X\|_{L^p(M_{>r})} \xrightarrow{N \to \infty} 0 \tag{A.9}$$

Remark. The proof actually shows for $N \geq 1$:

$$N^{\frac{1}{p}} \left\| P_{>N} X \right\|_{L^p(M_{>r})} = O_r \left(\left\| A_r \right\|_{L^\infty\left(\left[0,\frac{r}{2}\right]\right)} + \frac{\left\| X \right\|_{L^p(M)}}{N^\infty} \right)$$

Proof. We first show (A.8) implies (A.9). Fix $r > 0$. Let $r' = r'(r) \in \left(0, \frac{r}{4}\right)$ as in (A.7). By (A.6), we can disregard the region $\{d(x,y) > r'\}$, and just need to show

$$\left(\frac{1}{\sqrt{t}}\right)^{\frac{1}{p}} \left\| \int_{d(y,x) \leq r'} \mathrm{d}y \; K(t,x,y) \left(X(y) - X(x)\right) \right\|_{L^p_x(M_{>r})} \xrightarrow{t \to 0} 0$$

But by (A.7), the left-hand side is bounded by

$$\begin{aligned} &\left(\frac{1}{\sqrt{t}}\right)^{\frac{1}{p}} \left\| \left\| O_r \left(\frac{1}{t^{n/2}} \left\langle \frac{x-y}{\sqrt{t}} \right\rangle^{-\infty} \right) \left| X(y) - X(x) \right| \right\|_{L^1_y(B_{r'}(x))} \right\|_{L^p_x(M_{>r})} \\ \lesssim_r &\left(\frac{1}{\sqrt{t}}\right)^{\frac{1}{p}} \left\| \left\| \langle \zeta \rangle^{-\infty} \left| X(x - \sqrt{t}\zeta) - X(x) \right| \right\|_{L^1_{|\zeta| \leq \frac{r'}{\sqrt{t}}}} \right\|_{L^p_x(M_{>r})} \end{aligned} \tag{A.10}$$

where we made the change of variable $\zeta = \frac{x-y}{\sqrt{t}}$. By (A.8) and Holder's inequality, we can disregard the region $\{|\zeta| \leq 1\}$. Then we split $1 < |\zeta| \leq \frac{r'}{\sqrt{t}}$ into dyadic rings:

$$\begin{aligned} (A.10) &\lesssim \left(\frac{1}{\sqrt{t}}\right)^{\frac{1}{p}} \sum_{N \in 2^{\mathbb{N}_0}, N \lesssim \frac{r'}{\sqrt{t}}} \frac{1}{N^\infty} \left\| \left\| X(x - \sqrt{t}\zeta) - X(x) \right\|_{L^1_{|\zeta| \sim N}} \right\|_{L^p_x(M_{>r})} \\ &\lesssim \left(\frac{1}{\sqrt{t}}\right)^{\frac{1}{p}} \sum_{N \in 2^{\mathbb{N}_0}, N \lesssim \frac{r'}{\sqrt{t}}} \frac{1}{N^\infty} \left\| \left\| X(x - \sqrt{t}\zeta) - X(x) \right\|_{L^p_{|\zeta| \sim N}} \right\|_{L^p_x(M_{>r})} \\ &\lesssim \left(\frac{1}{\sqrt{t}}\right)^{\frac{n+1}{p}} \sum_{N \in 2^{\mathbb{N}_0}, N \lesssim \frac{r'}{\sqrt{t}}} \frac{1}{N^\infty} \left\| \left\| X(x - \tau) - X(x) \right\|_{L^p_{|\tau| \sim \sqrt{t}N}} \right\|_{L^p_x(M_{>r})} \end{aligned}$$

where we made the change of variable $\tau = \sqrt{t}\zeta$. Now observe that (A.8) implies that for $\varepsilon \leq r/2$:

$$\left\| \left\| X\left(x-\tau\right) - X(x) \right\|_{L^p_{|\tau|\leq\varepsilon}} \right\|_{L^p_x(M_{>r})} = \varepsilon^{\frac{n+1}{p}} A_r\left(\varepsilon\right)$$

where $0 \leq A_r(\varepsilon) \leq \|A_r\|_{L^\infty([0,r/2])}$ and $A_r\left(\varepsilon\right) \xrightarrow{\varepsilon\downarrow 0} 0$. Then

$$\begin{aligned}
(A.10) &\lesssim \left(\frac{1}{\sqrt{t}}\right)^{\frac{n+1}{p}} \sum_{N\in 2^{\mathbb{N}_0}, N\lesssim \frac{r'}{\sqrt{t}}} \frac{1}{N^\infty} \left(\sqrt{t}N\right)^{\frac{n+1}{p}} A_r\left(\sqrt{t}N\right) \\
&\lesssim \sum_{N\in 2^{\mathbb{N}_0}, N\lesssim \frac{r'}{\sqrt{t}}} \frac{1}{N^\infty} A_r\left(\sqrt{t}N\right) \\
&\xrightarrow[\text{DCT}]{t\downarrow 0} 0
\end{aligned}$$

Now we show (A.9) implies (A.8). Observe that by Corollary 118, (A.9) is equivalent to

$$N^{\frac{1}{p}-1} \left\| P_{\leq N} X \right\|_{W^{1,p}(M_{>r})} \xrightarrow{N\to\infty} 0 \ \forall r > 0$$

Now fix $r > 0$. Then for $\varepsilon \in \left(0, \min\left(1, \frac{r}{2}\right)\right)$, define $N = \frac{1}{\varepsilon}$, and we have:

$$\begin{aligned}
A_r\left(\varepsilon\right) \leq N^{\frac{1}{p}} &\left\| \left\| P_{\leq N} X\left(x - \frac{1}{N}h\right) - P_{\leq N}X\left(x\right) \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})} \\
&+ N^{\frac{1}{p}} \left\| \left\| P_{>N} X\left(x - \frac{1}{N}h\right) \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})} \\
&+ N^{\frac{1}{p}} \left\| \left\| P_{>N} X\left(x\right) \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})} \\
&\lesssim N^{\frac{1}{p}-1} \left\| \left\| \left\| \nabla P_{\leq N} X\left(x - \tau\frac{1}{N}h\right) \right\|_{L^1_\tau([0,1])} \right\|_{L^p_{|h|\leq 1}} \right\|_{L^p_x(M_{>r})} + N^{\frac{1}{p}} \left\| P_{>N} X \right\|_{L^p\left(M_{>\frac{r}{2}}\right)} \\
&\lesssim N^{\frac{1}{p}-1} \left\| P_{\leq N} X \right\|_{W^{1,p}\left(M_{>\frac{r}{2}}\right)} + N^{\frac{1}{p}} \left\| P_{>N} X \right\|_{L^p\left(M_{>\frac{r}{2}}\right)} \xrightarrow{\varepsilon\to 0} 0
\end{aligned}$$

Note that we used Minkowski's inequality, in passing to the last line. ∎

APPENDIX B

Construction of the heat kernel

Recall the Japanese bracket notation $\langle a \rangle = \sqrt{1+|a|^2} \sim 1+|a|$. We also write $a = O(b^\infty)$ or $|a| \lesssim b^\infty$ to mean $|a| \lesssim_l b^l \; \forall l \in \mathbb{N}$.

Let (M,g) be a compact Riemannian n-manifold with boundary. A differential k-form is a member of $C^\infty(M; \Lambda^k M)$.

In this chapter, unless otherwise noted, we write Δ for the Hodge Laplacian on forms. We also let (t,x,y) be the standard local coordinates for $[0,\infty) \times M \times M$. When x or y is near the boundary, we can stipulate that x_n and y_n stand for the Riemannian distance to the boundary (**geodesic normal coordinates**).

We aim to construct a unique Hodge-Neumann heat kernel with the absolute Neumann boundary condition. In particular, define $\mathrm{END}(\Lambda^k M) = \mathrm{Hom}(\pi_2^* \Lambda^k M, \pi_1^* \Lambda^k M)$, where π_i is the projection from $(0,\infty) \times M \times M$ onto the i-th M. We want

$$K \in C^\infty_{\mathrm{loc}}\left((0,\infty) \times M \times M; \mathrm{END}\left(\Lambda^k M\right)\right)$$

such that

$$\begin{aligned}
(\partial_t - \Delta_x) K(t,x,y) &= 0 && \\
\mathbf{n}_x K(t,x,y) &= 0 && \text{for } x \in \partial M \\
\mathbf{n}_x d_x K(t,x,y) &= 0 && \text{for } x \in \partial M \\
\lim_{t \downarrow 0} K(t,x,y) &= \delta_y(x) &&
\end{aligned}$$

where the last condition means $\forall u \in \mathscr{D}\left(M;\Lambda^k M\right), \int K\left(t,x,y\right)u(y)\,\mathrm{d}y \xrightarrow{t\downarrow 0} u(x)$.

During the construction, we will be able to prove certain properties of the kernel, such as off-diagonal decay for all derivatives.

The construction of the heat kernel comes from [MV13], and we simply discuss the modifications required for our case, to handle the Hodge-Neumann Laplacian on a smooth manifold with smooth boundary.[1]

B.1 Kernel in Einstein sum notation

Let $A \in C^\infty_{\text{loc}}\left((0,\infty)\times M^2; \text{END}\left(\Lambda^k M\right)\right)$. Let $U \subset M$ be a coordinate patch. Then, by using Einstein notation, locally for $x, y \in U$ we have:

$$A\left(t,x,y\right) = {A_I}^J\left(t,x,y\right) dx^I \otimes \partial_{y^J}$$

where $I, J \in \mathcal{I}_k = \{(i_1, ..., i_k) : i_1 < i_2 < ... < i_k\}$ and ∂_{y^J} is dual to the form dy^J. (also in Einstein notation, we write x^n instead of x_n)

- Note that we are abusing notation, as dx^I here is a local section of $\pi_1^*\Lambda^k M \twoheadrightarrow (0,\infty)\times M^2$, defined by pulling back the actual form dx^I on M. We can explicitly write ${A_I}^J\left(t,x,y\right)\, dx^I\big|_x \otimes \partial_{y^J}\big|_y$ to emphasize the pullback.
- Observe that $d_x A\left(t,x,y\right) = d_x\left({A_I}^J\left(t,x,y\right)dx^I\right)\otimes\partial_{y^J} = \partial_{x^i}{A_I}^J\left(t,x,y\right)\left(dx^i \wedge dx^I\right)\otimes \partial_{y^J}$.
- If $u(y) = u_J(y)dy^J$ is a differential form on M, we write

$$A\left(t,x,y\right)u(y) = {A_I}^J\left(t,x,y\right)u_J(y)dx^I$$

[1]The author thanks Daniel Grieser, András Vasy and Rafe Mazzeo for discussing these ideas. The original plan was to follow the note [Gri04] which is simpler and does not rely on Melrose's calculus, but we have decided to clean up the note, modify some steps and publish it at a later date.

which is a section of $\pi_1^* \Lambda^k M$.

As agreed above, when U touches the boundary, ∂_{x^n} is the inwards normal direction, so for $x \in \partial M$: $\mathbf{n}_x A(t,x,y) = 1_{n \in I} A_I{}^J(t,x,y)\, dx^I \otimes \partial_{y^J}$.

- If $\mathbf{n}_x A = 0$ for all $x \in \partial M$, then

$$\mathbf{n}_x d_x A = 1_{n \notin I} \partial_{x^n} A_I{}^J(t,x,y) \left(dx^n \wedge dx^I\right) \otimes \partial_{y^J}$$

  So $\mathbf{n}_x d_x A = 0 \iff \partial_{x^n} A_I{}^J(t,x,y) = 0$ whenever $x \in \partial M, n \notin I$. In other words, $\mathbf{n}_x A = 0$ and $\mathbf{n}_x d_x A = 0$ mean the normal part obeys the Dirichlet boundary condition, while the tangential part obeys the Neumann boundary condition. This will inspire the choice of leading terms later on.

B.2 Heat calculus

Let $x = (x', x_n)$ and $y = (y', y_n)$ be points in $\mathbb{R}^n$. Recall:

1. The scalar heat kernel on $\mathbb{R}^n$: $K(t,x,y) = \left(\frac{1}{4\pi}\right)^{n/2} \tau^{-n} e^{-\frac{|\zeta|^2}{4}}$ where $\tau = \sqrt{t}, \zeta = \frac{x-y}{\tau}$.

2. The Dirichlet scalar heat kernel on $\mathbb{R}^{n-1} \times [0,\infty)$:

$$K(t,x,y) = \left(\frac{1}{4\pi}\right)^{n/2} \tau^{-n} e^{-\frac{|\zeta'|^2}{4}} \left(e^{-\frac{1}{4}|\xi_n - \eta_n|^2} - e^{-\frac{1}{4}|\xi_n + \eta_n|^2}\right)$$

   where $\xi_n = \frac{x_n}{\tau}, \eta_n = \frac{y_n}{\tau}, \zeta' = \frac{x'-y'}{\tau}$.

3. The Neumann scalar heat kernel on $\mathbb{R}^{n-1} \times [0,\infty)$:

$$K(t,x,y) = \left(\frac{1}{4\pi}\right)^{n/2} \tau^{-n} e^{-\frac{|\zeta'|^2}{4}} \left(e^{-\frac{1}{4}|\xi_n - \eta_n|^2} + e^{-\frac{1}{4}|\xi_n + \eta_n|^2}\right)$$

They will inspire the formulation of our boundary heat calculus, which describes heat-type kernels on manifolds.

We assume the reader is familiar with the spaces of conormal and polyhomogeneous distributions on a manifold with corners [Mel18].

B.2.1 Blown-up heat space

We first construct the blown-up heat space M_h^2, with the faces lf, ff, td, tf as defined in [MV13] (though our case is simpler).

We start with $[0,\infty) \times M \times M$, with faces tf (temporal face), rf (right face), lf (left face) being defined as $\{0\} \times M \times M$, $[0,\infty) \times \partial M \times M$, $[0,\infty) \times M \times \partial M$ respectively. Then we perform a parabolic blow-up [Mel18, Section 7.4] on the submanifold $\{0\} \times \partial M \times \partial M$ in the time direction dt, to create the face ff (front face)[2]. This creates an intermediate manifold that we will call M_1.

After that, we perform another parabolic blow-up on the lift of the submanifold $\{0\} \times \Delta(M)$ to M_1 (to be more precisely defined in (B.5)), which creates another face td (time diagonal). This is the space M_h^2 we need.

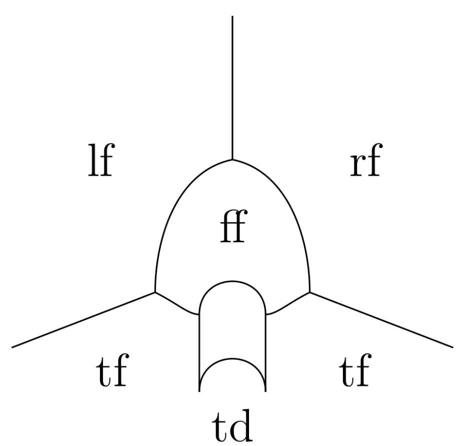

Figure B.1: The blown-up heat space M_h^2

[2]We are following [MV13] by letting rf be defined by $x_n = 0$. Other authors might prefer $y_n = 0$.

B.2.2 Local coordinates

By letting $\tau = \sqrt{t}$, we call (τ, x, y) the **ts-coordinate system** (time-rescaled) for $[0, \infty) \times M \times M$.

- On M_1, near rf and away from lf (i.e. $y_n > 0$), we use the **rf-coordinate system**

$$T = \frac{t}{y_n^2}, \theta' = \frac{x' - y'}{y_n}, \theta_n = \frac{x_n}{y_n}, y', y_n \tag{B.1}$$

where θ_n, y_n, T are respectively the boundary defining functions for rf, ff, tf. For blow-ups, it is also useful to define the (time-rescaled) **tsrf-coordinate system**

$$\varsigma = \sqrt{T}, \theta', \theta_n, y', y_n \tag{B.2}$$

We observe that as $(\varsigma, \theta', \theta_n, y', y_n) \to (\varsigma, \theta', \theta_n, y', 0)$ in the tsrf-coordinate, in the ts-coordinate we have

$$(0, (y', 0), (y', 0)) + y_n (\varsigma, (\theta', \theta_n), (0, 1)) \to (0, (y', 0), (y', 0))$$

The (time-rescaled) tangent vector $(\varsigma, (\theta', \theta_n), (0, 1))$[3] at $(0, (y', 0), (y', 0))$ (modulo vectors tangent to $\{0\} \times \partial M \times \partial M$, and modulo positive scalar multiplication) corresponds to a point on ff, which is $[(\varsigma, (\theta', \theta_n), (0, 1))] = [(\varsigma, (0, \theta_n), (-\theta', 1))]$. This is what allows us to extend the (ts)rf-coordinate systems from $[0, \infty) \times M \times M$ to M_1, with $\{y_n = 0\}$ being the face ff.

- On M_1, near ff and away from tf, we use the **ff-coordinate system**

$$\tau = \sqrt{t}, x', \xi_n = \frac{x_n}{\sqrt{t}}, \zeta' = \frac{x' - y'}{\sqrt{t}}, \eta_n = \frac{y_n}{\sqrt{t}} \tag{B.3}$$

[3]Explicitly, the tangent vector is $\varsigma \partial_\tau + (\theta', \theta_n) \cdot \partial_x + (0, 1) \cdot \partial_y$.

where τ, ξ_n, η_n are respectively the boundary defining functions for ff, rf, lf. As $(\tau, x', \xi_n, \zeta', \eta_n) \to (0, x', \xi_n, \zeta', \eta_n)$ in the ff-coordinate, in the ts-coordinate we have

$$(0, (x', 0), (x', 0)) + \tau (1, (0, \xi_n), (-\zeta', \eta_n)) \to (0, (x', 0), (x', 0))$$

The (time-rescaled) tangent vector $(1, (0, \xi_n), (-\zeta', \eta_n))$ at $(0, (x', 0), (x', 0))$ corresponds to a point on ff, which is $[(1, (0, \xi_n), (-\zeta', \eta_n))]$.

- On M_h^2, near td, near ff, away from lf, away from tf, we can use the rf-coordinate system from (B.1) to define the **fftd-coordinate system**

$$\vartheta = \sqrt{T}, \sigma' = \frac{\theta'}{\sqrt{T}}, \sigma_n = \frac{\theta_n - 1}{\sqrt{T}}, y', y_n \tag{B.4}$$

where ϑ is the defining function for td. Note that as $(\vartheta, \sigma', \sigma_n, y', y_n) \to (0, \sigma', \sigma_n, y', y_n)$ in the fftd-coordinate, in the tsrf-coordinate we have

$$(0, 0, 1, y', y_n) + \vartheta (1, \sigma', \sigma_n, 0, 0) \to (0, 0, 1, y', y_n) \tag{B.5}$$

We observe that the points $(0, 0, 1, y', y_n)$ in the tsrf-coordinate, are precisely the lift of the submanifold $D_0 := \{0\} \times \Delta(M)$ to M_1, which we will write as D_1. By blowing up D_1, we create the face td and M_h^2. Note that $\theta_n = 1 > 0$, so td does not intersect rf (or lf). Also, the (time-rescaled) tangent vector $(1, \sigma', \sigma_n, 0, 0)$ at $(0, 0, 1, y', y_n)$ corresponds to a point on the face td.

On the other hand, the point $(\vartheta, \sigma', \sigma_n, y', 0)$ in the fftd-coordinate on M_h^2 maps down to the point $(\vartheta, \vartheta\sigma', \vartheta\sigma_n + 1, y', 0)$ in the tsrf-coordinate on M_1 (the map being injective on $\{\vartheta > 0\}$), which in turn corresponds to the point $[(\vartheta, (0, \vartheta\sigma_n + 1), (-\vartheta\sigma', 1))]$ on ff.

- The points $(0, 0, 1, y', 0)$ in the (ts)rf-coordinate are precisely the intersection ff $\cap D_1$ in M_1.

The points $(0, \sigma', \sigma_n, y', 0)$ in the fftd-coordinate are precisely the intersection ff $\cap$ td in M_h^2.

- On M_h^2, near td, away from ff and away from tf, we use the **td-coordinate system**

$$\tau = \sqrt{t}, x, \zeta = \frac{x-y}{\sqrt{t}} \tag{B.6}$$

where τ is the defining function for td. As $(\tau, x, \zeta) \to (0, x, \zeta)$ in td-coordinate, in ts-coordinate we have

$$(0, x, x) + \tau (1, 0, -\zeta) \to (0, x, x)$$

So we identify the point $(0, x, \zeta)$ in td-coordinate with the (time-rescaled) tangent vector $(1, 0, -\zeta)$ at $(0, x, x) \in D_0$, which gives a point of td (or to be precise, away from the edges, D_1 and D_0 are locally diffeomorphic, and td being defined as a bundle over D_1 is also locally defined over D_0).

- Wherever we have both the td-coordinate system and the fftd-coordinate system, the point $(\tau, x, \zeta) = (\tau, (x', x_n), (\zeta', \zeta_n))$ in the td-coordinate (with $x_n > 0, x_n - \tau\zeta_n > 0$) corresponds to the point $\left(\frac{\tau}{x_n - \tau\zeta_n}, \zeta', \zeta_n, x' - \tau\zeta', x_n - \tau\zeta_n\right)$ in the fftd-coordinate. Conversely, $(\vartheta, \sigma', \sigma_n, y', y_n)$ in the fftd-coordinate corresponds to

$$(\vartheta y_n, (y' + \vartheta y_n \sigma', y_n + \vartheta y_n \sigma_n), (\sigma', \sigma_n))$$

in the td-coordinate. Consequently,

$$(0, (x', x_n), (\zeta', \zeta_n)) \text{ in td-coordinate corresponds to } (0, \zeta', \zeta_n, x', x_n) \text{ in fftd-coordinate} \tag{B.7}$$

and we identify the tangent vector $(1, 0, -\zeta)$ at $(0, x, x) \in D_0$ (in the ts-coordinate) with the tangent vector $(1, \zeta', \zeta_n, 0, 0)$ at $(0, 0, 1, x', x_n) \in D_1$ (in the tsrf-coordinate), as the same point in td.

Remark 127 (Compatibility condition at $\mathrm{ff} \cap \mathrm{td}$). For any smooth functions u on ff and v on td, the following are equivalent:

1. In the fftd-coordinate:

$$u\left(\vartheta,\sigma',\sigma_n,y',0\right) \xrightarrow{\vartheta\to 0} v\left(0,\sigma',\sigma_n,y',0\right) \tag{B.8}$$

2. There is a smooth function f on M_h^2 such that $N_{\mathrm{ff}}^0(f) = u$, $N_{\mathrm{td}}^0(f) = v$.

B.2.3 Edge calculus

Definition 128. For $\alpha, \alpha' \in -\mathbb{N}_0$, we define $\Psi_{\mathrm{e-h}}^{\alpha,\alpha',E_{\mathrm{lf}},E_{\mathrm{rf}}}\left(M;\Lambda^k M\right)$[4] as the space of Schwartz kernels K that are pushforwards of polyhomogeneous kernels $\widetilde{K}$ on M_h^2 (though we will abuse notation and also write K for $\widetilde{K}$) such that:

- the index sets at lf and rf are $E_{\mathrm{lf}} = \left(E_{\mathrm{lf}}^{\mathbf{t}}, E_{\mathrm{lf}}^{\mathbf{n}}\right)$ and $E_{\mathrm{rf}} = \left(E_{\mathrm{rf}}^{\mathbf{t}}, E_{\mathrm{rf}}^{\mathbf{n}}\right)$. Here $E_{\mathrm{lf}}^{\mathbf{t}}, E_{\mathrm{rf}}^{\mathbf{t}}$ describe the local coefficients of $\mathbf{t}_x K$ (the tangent component), while $E_{\mathrm{lf}}^{\mathbf{n}}, E_{\mathrm{rf}}^{\mathbf{n}}$ describe the local coefficients of $\mathbf{n}_x K$.
- the index set at ff is $\{(j-(n+2+\alpha),0) : j \in \mathbb{N}_0\}$ (expansion in τ from (B.3))
- the index set at td is $\{(j-(n+2+\alpha'),0) : j \in \mathbb{N}_0\}$ (expansion in τ from (B.6)). By convention, it is $\emptyset$ when $\alpha' = -\infty$.
- the index set at tf is $\emptyset$ (off-diagonal decay).

Theorem 129. *The absolute Neumann heat kernel H lies in $\Psi_{\mathrm{e-h}}^{-2,-2,E_{\mathrm{lf}},E_{\mathrm{rf}}}\left(M;\Lambda^k M\right)$ where*

- $E_{\mathrm{lf}}^{\mathbf{t}}, E_{\mathrm{lf}}^{\mathbf{n}}, E_{\mathrm{rf}}^{\mathbf{t}}, E_{\mathrm{rf}}^{\mathbf{n}} \subseteq \mathbb{N}_0 \times \{0\}$[5]

[4]To translate to the definition of $\Psi_{\mathrm{e-h}}^{l,p,E_{\mathrm{lf}},E_{\mathrm{rf}}}$ from [MV13, Section 3.2], we can use the formulas $\alpha = -l, \alpha' = -p-2, n = m, n-1 = b$.

[5]In fact, due to symmetry, we must have $E_{\mathrm{lf}} = E_{\mathrm{rf}}$.

- $\mathbf{n}_x H = 0$ *and* $\mathbf{n}_x d_x H = 0$.

We can also write $\Psi_{\text{e-h}}^{-2,-2,\mathbb{N}_0,\mathbb{N}_0}$ *to describe smoothness at* lf *and* rf*.*

B.3 Proof of Theorem 129

We proceed exactly as in [MV13, Section 3.2].

For any $A \in \Psi_{\text{e-h}}^{\alpha,\alpha',E_{\text{lf}},E_{\text{rf}}}\left(M;\Lambda^k M\right)$, we can expand w.r.t. ff (with coordinates as in (B.3))

$$A = A^{\text{ff}}_{-n-2-\alpha}\left(x,\xi_n,\zeta',\eta_n\right)\tau^{-n-2-\alpha} + A^{\text{ff}}_{-n-2-\alpha+1}\left(x,\xi_n,\zeta',\eta_n\right)\tau^{-n-2-\alpha+1} + \ldots.$$

We write $N_{\text{ff}}^{-n-2-\alpha}(A)$ for the leading coefficient $A^{\text{ff}}_{-n-2-\alpha}$. We can expand similarly w.r.t. td and define $N_{\text{td}}^{-n-2-\alpha'}(A)$.

Then we note that $t\left(\partial_t - \Delta_x\right)$ is a b-operator which could be restricted to ff and td. In particular,

$$\begin{cases} N_{\text{ff}}^{-n-2-\alpha}\left(t\left(\partial_t-\Delta_x\right)A\right) &= N_{\text{ff}}^{-n-2-\alpha}\left(t\left(\partial_t-\Delta_x\right)\right)N_{\text{ff}}^{-n-2-\alpha}(A) \\ N_{\text{td}}^{-n-2-\alpha'}\left(t\left(\partial_t-\Delta_x\right)A\right) &= N_{\text{td}}^{-n-2-\alpha'}\left(t\left(\partial_t-\Delta_x\right)\right)N_{\text{td}}^{-n-2-\alpha'}(A) \end{cases}$$

where, in the td-coordinate system from (B.6) and the ff-coordinate system from (B.3):

$$\begin{cases} N_{\text{td}}^{-n-2-\alpha}\left(t\left(\partial_t-\Delta_x\right)\right) &= -\Delta_\zeta(x) - \frac{1}{2}\zeta\cdot\partial_\zeta - \frac{n+2+\alpha}{2} \\ N_{\text{ff}}^{-n-2-\alpha}\left(t\left(\partial_t-\Delta_x\right)\right) &= -\Delta_{(\zeta',\xi_n)}(x',0) - \frac{1}{2}\left(\zeta',\xi_n,\eta_n\right)\cdot\partial_{(\zeta',\xi_n,\eta_n)} - \frac{n+2+\alpha'}{2} \end{cases} \tag{B.9}$$

Here we have written $\zeta\cdot\partial_\zeta = \sum_i \zeta_i\partial_{\zeta_i}$ and $\Delta_\zeta(x) = \sum_{i,j} g^{ij}(x)\partial_{\zeta_i}\partial_{\zeta_j}$.

Then we have $t\left(\partial_t-\Delta_x\right)\Psi_{\text{e-h}}^{\alpha,\alpha',E_{\text{lf}},E_{\text{rf}}} \subseteq \Psi_{\text{e-h}}^{\alpha,\alpha',\mathbb{N}_0,\mathbb{N}_0}$.

From this point on, we fix $E_{\text{lf}}, E_{\text{rf}}$ to be as in Theorem 129.

Claim 130. There is an element $H^{(1)} \in \Psi_{\text{e-h}}^{-2,-2,E_{\text{lf}},E_{\text{rf}}}\left(M;\Lambda^k M\right)$ such that

$$\begin{cases} P^{(1)} & := t\left(\partial_t - \Delta_x\right) H^{(1)} \in \Psi_{\text{e-h}}^{-3,-\infty,\mathbb{N}_0,\mathbb{N}_0} \\ \lim_{t\downarrow 0} H^{(1)}\left(t,x,y\right) & = \delta_y\left(x\right) \end{cases}$$

Proof. To prove this claim, we construct $A \in \Psi_{\text{e-h}}^{-2,-2,E_{\text{lf}},E_{\text{rf}}}$ such that

$$N_{\text{td}}^{-n}\left(A\right)\left(x,\zeta\right) \tag{B.10}$$

$$= \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|\zeta|^2_{g(x)}}{4}} \text{Id} = \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|\zeta|^2_{g(x)}}{4}}\, dx^I\big|_x \otimes \partial_{y^I}\big|_x \tag{B.11}$$

$$N_{\text{ff}}^{-n}\left(A\right)\left(x',\xi_n,\zeta',\eta_n\right)$$

$$= \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|\zeta'|^2_{g(x',0)}}{4}} \left(e^{-\frac{1}{4}|\xi_n-\eta_n|^2}\left(\mathbf{t}+\mathbf{n}\right) + e^{-\frac{1}{4}|\xi_n+\eta_n|^2}\left(\mathbf{t}-\mathbf{n}\right)\right) \tag{B.12}$$

$$= 1_{n\notin I}\left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|\zeta'|^2_{g(x',0)}}{4}} \left(e^{-\frac{1}{4}|\xi_n-\eta_n|^2} + e^{-\frac{1}{4}|\xi_n+\eta_n|^2}\right) dx^I\big|_{(x',0)} \otimes \partial_{y^I}\big|_{(x',0)}$$

$$+ 1_{n\in I}\left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|\zeta'|^2_{g(x',0)}}{4}} \left(e^{-\frac{1}{4}|\xi_n-\eta_n|^2} - e^{-\frac{1}{4}|\xi_n+\eta_n|^2}\right) dx^I\big|_{(x',0)} \otimes \partial_{y^I}\big|_{(x',0)}$$

This choice satisfies the compatibility condition from (B.8) (with $N_{\text{td}}^{-n}\left(A\right) = N_{\text{td}}^{0}\left(t^{\frac{n}{2}}A\right)$ and $N_{\text{ff}}^{-n}\left(A\right) = N_{\text{ff}}^{0}\left(t^{\frac{n}{2}}A\right)$), since

$$e^{-\frac{1}{4}|\sigma'|^2_{g(y',0)}}\left(e^{-\frac{1}{4}|\sigma_n|^2}\left(\mathbf{t}+\mathbf{n}\right) + e^{-\frac{1}{4}|\sigma_n+\frac{2}{\vartheta}|^2}\left(\mathbf{t}-\mathbf{n}\right)\right) \xrightarrow{\vartheta\to 0} e^{-\frac{1}{4}|\sigma'|^2_{g(y',0)}}\left(e^{-\frac{1}{4}|\sigma_n|^2}\right)$$
$$= e^{-\frac{1}{4}|(\sigma',\sigma_n)|^2_{g(y',0)}}.$$

We note that A is smooth on $(0,\infty)\times M\times M$, and we can make A have the same index set for rf as $N_{\text{ff}}^{-n}\left(A\right)$. More is true: as in Section B.1, by Taylor expansion in ξ_n, we note that $N_{\text{ff}}^{-n}\left(A\right)$ satisfies the absolute Neumann condition, and so does A. Off-diagonal decay is also explicit from these formulas (when $x \neq y$ stay fixed and $t \to 0$, we have $\zeta = \frac{x-y}{\sqrt{t}} \to \infty$).

By direct calculations, $N_{\mathrm{ff}}^{-n}\left(t\left(\partial_t-\Delta_x\right)A\right)=0$ and $N_{\mathrm{td}}^{-n}\left(t\left(\partial_t-\Delta_x\right)A\right)=0$. Therefore $t\left(\partial_t-\Delta_x\right)A\in\Psi_{\mathrm{e-h}}^{-3,-3,\mathbb{N}_0,\mathbb{N}_0}$. We then observe two facts:

- In the expansion of A at td, A_j^{td} for $j>-n$ can be freely changed.
- For any smooth $f\left(x,\zeta\right)$ that is Schwartz in ζ (rapidly decaying) and $j\geq 1$, there is a unique $F\left(x,\zeta\right)$ rapidly decaying in ζ such that

$$N_{\mathrm{td}}^{-n+j}\left(t\left(\partial_t-\Delta_x\right)\right)F\left(x,\zeta\right)=f\left(x,\zeta\right)$$

In particular, by using the Fourier transform $\zeta\mapsto z$ (with the convention $\widehat{F}(z)=\int_{\mathbb{R}^n}F(\zeta)e^{-i2\pi\zeta\cdot z}\,\mathrm{d}\zeta$):

$$\widehat{F}\left(x,z\right)=\int_0^1\mathrm{d}s\;2s^{j-1}\hat{f}\left(x,sz\right)e^{-\left(4\pi^2\right)\left(1-s^2\right)|z|_{g(x)}^2}$$

See also [Alb17, Section 6.2] for an explanation of this. It boils down to the fact that Δ_ζ is smoothing (elliptic) for ζ.

Therefore it is possible to change $\left(A_j^{\mathrm{td}}\right)_{j>-n}$to make $t\left(\partial_t-\Delta_x\right)A$ vanish to infinite order at td. It boils down to solving

$$N_{\mathrm{td}}^{j}\left(t\left(\partial_t-\Delta_x\right)\right)A_j^{\mathrm{td}}=B_j,\ j>-n$$

where B_j is an inhomogeneous term depending on $A_{-n}^{\mathrm{td}},...,A_{j-1}^{\mathrm{td}}$. Changing $\left(A_j^{\mathrm{td}}\right)_{j>-n}$ will not affect the index set of A at rf, since td does not intersect rf and lf, by the above reasoning with (B.4). A is smooth at rf and lf, and we therefore obtain $t\left(\partial_t-\Delta_x\right)A\in\Psi_{\mathrm{e-h}}^{-3,-\infty,\mathbb{N}_0,\mathbb{N}_0}$.

We finally note that $\lim_{t\downarrow 0}A\left(t,x,y\right)=\delta_y\left(x\right)$ due to (B.10), which is the "universal" formula for the expansion of heat kernels in the interior of manifolds. The claim is then proven. We refer to [MV13, Proposition 3.2] for more details. □

So we have solved away the leading coefficient of $t(\partial_t - \Delta_x) H$ at ff, as well as all the coefficients at td.

Next, we solve away all the coefficients at rf.

Claim 131. There is an element $H^{(2)} \in \Psi_{\mathrm{e-h}}^{-2,-2,E_{\mathrm{lf}},E_{\mathrm{rf}}}\left(M;\Lambda^k M\right)$ such that

$$\begin{cases} P^{(2)} & := t(\partial_t - \Delta_x) H^{(2)} \in \Psi_{\mathrm{e-h}}^{-3,-\infty,\mathbb{N}_0,\emptyset} \\ \lim_{t\downarrow 0} H^{(2)}(t,x,y) & = \delta_y(x) \end{cases}$$

Proof. Let $r(x)$ be a boundary-defining function for rf such that $r(x) = \operatorname{dist}(x,\partial M) = x_n$ near rf. We observe that $r^2(\partial_t - \Delta_x)$ is a b-operator which can be restricted to rf (defined by $\theta_n = 0$ in the rf-coordinate system from (B.1)). On the other hand, in the ff-coordinate system from (B.3), $r = \tau\xi_n$, so $r^2(\partial_t - \Delta_x)$ is also a b-operator w.r.t. ff.

We observe that $(\partial_t - \Delta_x) H^{(1)} \in \Psi_{\mathrm{e-h}}^{-1,-\infty,\mathbb{N}_0,\mathbb{N}_0}$ and we want $(\partial_t - \Delta_x) H^{(2)} \in \Psi_{\mathrm{e-h}}^{-1,-\infty,\mathbb{N}_0,\emptyset}$. Therefore it is enough to find $J \in \Psi_{\mathrm{e-h}}^{-3,-3,E_{\mathrm{lf}},E_{\mathrm{rf}}}\left(M;\Lambda^k M\right)$ such that $r^2(\partial_t - \Delta_x)\left(H^{(1)} - J\right)$ vanishes to infinite order at rf.

Let $B = r^2(\partial_t - \Delta_x) H^{(1)} \in \Psi_{\mathrm{e-h}}^{-3,-\infty,\mathbb{N}_0,\mathbb{N}_0+2}$. We note that $B_0^{\mathrm{rf}} = B_1^{\mathrm{rf}} = 0$, so it is fine to set $J_0^{\mathrm{rf}} = J_1^{\mathrm{rf}} = 0$.

Recall that $\Delta_x = \sum_{ij} g^{ij}(x)\,\partial_{x_i}\partial_{x_j} + \sum_i b_i \partial_{x_i} + c$ where b_i, c are smooth. Then by translating $r^2(\partial_t - \Delta_x)$ into rf-coordinates, we have to solve the formal expansion at rf:

$$\theta_n^2\left(\partial_T - \sum_{i,j\neq n} g^{ij}\partial_{\theta_i}\partial_{\theta_j} - \sum_{i\neq n} y_n b_i \partial_{\theta_i} - \partial_{\theta_n}^2 - y_n b_n \partial_{\theta_n} - c y_n^2\right)\left(\sum_{j\geq 2} J_j^{\mathrm{rf}}\theta_n^j\right) = \sum_{j\geq 2} B_j^{\mathrm{rf}}\theta_n^j \tag{B.13}$$

Note that near rf, because we have chosen the geodesic normal coordinates, $g^{in} = \delta^{in}$

for any $i \in \{1, ..., n\}$. Then, (B.13) boils down to solving

$$N_{\mathrm{rf}}^{j}\left(r^{2}\left(\partial_{t}-\Delta_{x}\right)\right) J_{j}^{\mathrm{rf}}\left(T, \theta', y', y_{n}\right)=C_{j}\left(T, \theta', y', y_{n}\right),\ j \geq 2$$

where

- $N_{\mathrm{rf}}^{j}\left(r^{2}\left(\partial_{t}-\Delta_{x}\right)\right)=-j(j-1),\ j \geq 1$.
- C_j is an inhomogeneous term depending on B_j^{rf} and $J_2^{\mathrm{rf}}, ..., J_{j-1}^{\mathrm{rf}}$. In particular, $C_2 = B_2^{\mathrm{rf}}$.

Solving this is trivial (with unique solutions), since for $j \geq 2$, $N_{\mathrm{rf}}^{j}\left(r^{2}\left(\partial_{t}-\Delta_{x}\right)\right)$ is a nonzero constant. We note that $\left(J_j^{\mathrm{rf}}\right)_{j\geq 2}$ inherits many properties from $\left(B_j^{\mathrm{rf}}\right)_{j\geq 2}$ by induction:

- In the rf-coordinate system, B_j^{rf} is defined from $\frac{1}{j!}\partial_{\theta_n}^{j}\big|_{\theta_n=0} B$ (abuse of notation). But y_n is the defining function for ff, so the index set of B_j^{rf} at ff is the same as that of B, and therefore this is also true for J_2^{rf}. This extends to J_j^{rf} $\forall j \geq 2$, because we can explicitly derive C_j from (B.13), and see that the powers of y_n never get lowered (no ∂_{y_n} or $\frac{1}{y_n}$).
- The index sets of B at td and tf are empty (i.e. $B = O\left(T^{\infty}\right)$ as $T \to 0$), which implies $J_j^{\mathrm{rf}} = O\left(T^{\infty}\right)$.

Note that we also have to solve for J_j^{rf} where y is away from the boundary (which means there is no rf-coordinate system). In that case, we use the ts-coordinate system and solve the formal expansion at rf. This proceeds in the same fashion (but it is even simpler, since we are far away from ff).

Consequently, constructing J from $\left(J_j^{\mathrm{rf}}\right)_{j\geq 0}$ gives us $J \in \Psi_{\mathrm{e-h}}^{-3,-\infty,\mathbb{N}_0,\mathbb{N}_0+2}\left(M;\Lambda^k M\right)$ such that $B - r^2\left(\partial_t - \Delta_x\right) J$ vanishes to infinite order at rf.

With the index set at rf being $\mathbb{N}_0 + 2$, J trivially satisfies the absolute Neumann boundary condition. Also, because the index sets of J at ff and td are higher than those of $H^{(1)}$, we conclude

$$\begin{cases} N_{\mathrm{ff}}^{-n}\left(H^{(1)} - J\right) & = N_{\mathrm{ff}}^{-n}\left(H^{(1)}\right) \\ N_{\mathrm{td}}^{-n}\left(H^{(1)} - J\right) & = N_{\mathrm{td}}^{-n}\left(H^{(1)}\right) \end{cases}$$

By setting $H^{(2)} = H^{(1)} - J$, the claim is proven. □

For the last step, we consider the formal Volterra series:

$$H = H^{(2)} + H^{(2)} * R^{(2)} + H^{(2)} * R^{(2)} * R^{(2)} + \ldots.$$

where $R^{(2)} := -\left(\partial_t - \Delta_x\right) H^{(2)} \in \Psi_{\mathrm{e-h}}^{-1,-\infty,\mathbb{N}_0,\emptyset}$, and the composition $A * B$ is defined by

$$A * B\,(t,x,y) = \int_0^t \mathrm{d}s \int_M \mathrm{dvol}_g\,(z)\;\; A(t-s,x,z)B(s,z,y)$$

By [MV13, Theorem 5.3], if $Q_{\mathrm{lf}} + Q'_{\mathrm{rf}} > -1$; $\alpha, \gamma, \beta \in -\mathbb{N}_1$, we have the formula

$$\Psi_{\mathrm{e-h}}^{\alpha,\gamma,Q_{\mathrm{lf}},Q_{\mathrm{rf}}} * \Psi_{\mathrm{e-h}}^{\beta,-\infty,Q'_{\mathrm{lf}},Q'_{\mathrm{rf}}} \subset \Psi_{\mathrm{e-h}}^{\alpha+\beta,-\infty,P_{\mathrm{lf}},P_{\mathrm{rf}}}$$

where $P_{\mathrm{lf}} = Q'_{\mathrm{lf}}\overline{\cup}\left(Q_{\mathrm{lf}} - \beta\right)$; $P_{\mathrm{rf}} = Q_{\mathrm{rf}}\overline{\cup}\left(Q'_{\mathrm{rf}} - \alpha\right)$. This means that for $N \in \mathbb{N}_1$:

$$H^{(2)} * \left(R^{(2)}\right)^{*N} \in \Psi_{\mathrm{e-h}}^{-2-N,-\infty,E_{\mathrm{lf},N},E_{\mathrm{rf}}}$$

where $E_{\mathrm{lf,N}}$ is defined inductively by $E_{\mathrm{lf},1} = \mathbb{N}_0\overline{\cup}\left(\mathbb{N}_0 + 1\right)$ and $E_{\mathrm{lf},N+1} = \mathbb{N}_0\overline{\cup}\left(E_{\mathrm{lf},N} + 1\right)$ for $N \geq 1$.

Letting $\mathbb{N}_j = \{x \in \mathbb{N} : x \geq j\}$ and $\mathscr{N} = \overline{\bigcup}_{j\in\mathbb{N}_0}\mathbb{N}_j$, we conclude that

$$\forall N : E_{\mathrm{lf},N} \subset \mathscr{N} = \left\{(x,y) \in \mathbb{N}_0^2 : y \leq x\right\}$$

which is a well-defined index set.

A common property of Volterra series is that they converge. We can observe this from the fact that $\forall m \in \mathbb{N}_2$, $L^{*m}(t,x,y)$ is equal to

$$\int_{M^{m-1}} \mathrm{dvol}_g\left(z_1,...,z_{m-1}\right) \int_{\Delta^t_{m-1}} \mathrm{d}\left(s_1,...,s_{m-1}\right)\ L\left(t-s_1-...-s_{m-1},x,z_{m-1}\right)...L\left(s_1,z_1,y\right)$$

where Δ^t_{m-1} is the simplex defined by $\{0 \leq s_1 \leq s_1+s_2 \leq ... \leq s_1+...+s_{m-1} \leq t\}$. As the volume of Δ^t_{m-1} is $\frac{t^{m-1}}{(m-1)!}$, the factorial factor $\frac{1}{(m-1)!}$ ultimately forces strong convergence as $m \to \infty$. See [BGV04, Section 2.4], [MV13, Section 3.2], and [Mel18] for more details and estimates.

Consequently, we obtain $H \in \Psi^{-2,-2,\mathscr{N},E_{\mathrm{rf}}}_{\mathrm{e-h}}$. Because of the identity

$$\left(\partial_t - \Delta_x\right)\left(H^{(2)} * \left(R^{(2)}\right)^{*N}\right) = \left(R^{(2)}\right)^{*N} - \left(R^{(2)}\right)^{*(N+1)},$$

we conclude

$$\left(\partial_t - \Delta_x\right) H = 0$$

Let us check that H is the true Hodge-Neumann heat kernel.

- The absolute Neumann boundary condition comes from the strong convergence of the Volterra series.
- For any $u \in L^2\left(M;\Lambda^k M\right)$:

  $$H(t)u(x) := \int_M H\left(t,x,y\right)u(y)\,\mathrm{dvol_g}y \in C^\infty\left(\left(0,\infty\right),\Omega^k_{\mathrm{hom}N}\right)$$

  and satisfies $\left(\partial_t - \Delta_x\right)\left(H(t)u(x)\right) = 0$ on $\{t>0\}$. In particular, $H\left(t\right) \in \mathrm{End}\left(L^2\right)$ for all $t>0$ and

  $$\partial_t\left(\|H(t)u\|^2_{L^2}\right) \leq 0 \tag{B.14}$$

  because the Neumann Laplacian $\widetilde{\Delta_N}$ is self-adjoint and dissipative.

- We have $N_{\mathrm{td}}^{-n}(H) = N_{\mathrm{td}}^{-n}\left(H^{(1)}\right)$, therefore $\lim_{t\downarrow 0} H(t,x,y) = \delta_y(x)$. For any $u \in \Omega_{00}^k(M) : H(t)u \xrightarrow[t\downarrow 0]{L^2} u$, which, along with (B.14), implies $\|H(t)u\|_{L^2} \leq \|u\|_{L^2}$. By density, we conclude the same for $u \in L^2\left(M;\Lambda^k M\right)$. Recall that $e^{t\widetilde{\Delta_N}}$ is the heat semigroup defined by functional analysis. For any $u \in L^2\left(M;\Lambda^k M\right)$, $U(t) := H(t)u - e^{t\widetilde{\Delta_N}}u$ is a $C_t^0 L_x^2$ solution of

$$\begin{cases} (\partial_t - \Delta_x)\, U(t,x) = & 0\ \forall t > 0 \\ U(t) \xrightarrow[t\downarrow 0]{L^2} & 0 \end{cases}$$

By an energy argument just like (B.14), we must have $U(t) = 0$ for all t. Then, $H(t) = e^{t\widetilde{\Delta}_N}$.

So H is the true heat kernel, which must be smooth on $(0,\infty)\times M\times M$ by standard parabolic theory. Another way to see this is that the heat kernel must be symmetric, therefore smoothness in x implies smoothness in y. Either way, because we have smoothness, there are no log terms on lf, and we conclude $H \in \Psi_{\mathrm{e-h}}^{-2,-2,\mathbb{N}_0,\mathbb{N}_0}$.

B.4 Relevant properties

We extract some key properties from Theorem 129 that we need for this thesis, and write them in a language more familiar with analysts.

1. (**off-diagonal decay**) For any multi-index γ and $x \neq y$,

$$D_{t,x,y}^{\gamma} H(t,x,y) = O\left(t^{\infty}\right) \tag{B.15}$$

   as $t \downarrow 0$, locally uniform in $(x,y) \notin \Delta(M)$.

2. (**interior blow-up**) For $x \in \operatorname{int}(M)$, locally in projective coordinates $(\tau,x,\zeta) = \left(\sqrt{t}, x, \frac{x-y}{\sqrt{t}}\right)$, with $\widetilde{H}$ being the pullback of $t^{\frac{n}{2}}H$ in these coordinates, we have

(a) $\widetilde{H}$ smooth in τ, x, ζ, up to $\{\tau = 0\}$.

(b) (**rapid decay**) For any multi-index γ and bounded τ:

$$D^{\gamma}_{\tau,x,\zeta}\widetilde{H}\left(\tau, x, \zeta\right) = O\left(\langle\zeta\rangle^{-\infty}\right) \tag{B.16}$$

Remark 132. Both (B.15) and (B.16) come from the empty index set at tf. We also refer to [Kot16, Section 2.3.3] for an explanation of (B.16).

There are more specific properties from Theorem 129, which we do not currently need.

[ABK19] Wolfgang Arendt, Manuel Bernhard, and Marcel Kreuter. “Elliptic Problems and Holomorphic Functions in Banach Spaces”. 2019.

[Ada03] R. A. Adams. *Sobolev Spaces*. 2003.

[Agm62] Shmuel Agmon. “On the Eigenfunctions and on the Eigenvalues of General Elliptic Boundary Value Problems”. *Communications on Pure and Applied Mathematics*. URL: http://doi.wiley.com/10.1002/cpa.3160150203. 1962.

[AKS62] N. Aronszajn, A. Krzywicki, and J. Szarski. “A Unique Continuation Theorem for Exterior Differential Forms on Riemannian Manifolds”. *Arkiv för Matematik*. Aug. 1962.

[Alb17] Pierre Albin. *Linear Analysis on Manifolds*. URL: https://www.ams.org/open-math-notes/omn-view-listing?listingId=110667. Jan. 18, 2017.

[AM04] Andreas Axelsson and Alan McIntosh. “Hodge Decompositions on Weakly Lipschitz Domains”. 2004.

[Ama19] Herbert Amann. *Linear and Quasilinear Parabolic Problems*. 2019.

[Are+11] Wolfgang Arendt et al. *Vector-Valued Laplace Transforms and Cauchy Problems: Second Edition (Monographs in Mathematics)*. 2011.

[BAE16] Hind Baba, Chérif Amrouche, and Miguel Escobedo. “Analyticity of the Semi-Group Generated by the Stokes Operator with Navier-Type Boundary Conditions on Lp Spaces”. *Contemporary Mathematics*. URL: http://www.ams.org/conm/666. 2016.

[Bar+19a] Claude Bardos et al. “On the Extension of Onsager’s Conjecture for General Conservation Laws”. *Journal of Nonlinear Science*. URL: https://doi.org/10.1007/s00332-018-9496-4. Apr. 1, 2019.

[Bar+19b] Claude Bardos et al. *Onsager's Conjecture in Bounded Domains for the Conservation of Entropy and Other Companion Laws*. URL: http://arxiv.org/abs/1902.07120. Feb. 19, 2019.

[Bau17] Fabrice Baudoin. *Stochastic Analysis and Related Topics : A Festschrift in Honor of Rodrigo Bañuelos*. 2017.

[BB97] Alexander Braverman and Maxim Braverman. "Tempered Currents and the Cohomology of the Remote Fiber of a Real Polynomial Map". 1997.

[BBD18] Huy-Qui Bui, The Anh Bui, and Xuan Thinh Duong. "Weighted Besov and Triebel–Lizorkin Spaces Associated to Operators". 2018.

[BGV04] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat Kernels and Dirac Operators*. 2004.

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation Spaces*. 1976.

[BS85] Jochen Brüning and Robert Seeley. "Regular Singular Asymptotics". *Advances in Mathematics*. URL: https://linkinghub.elsevier.com/retrieve/pii/0001870885901148. Nov. 1985.

[BT18] Claude Bardos and Edriss S. Titi. "Onsager's Conjecture for the Incompressible Euler Equations in Bounded Domains". *Archive for Rational Mechanics and Analysis*. URL: https://doi.org/10.1007/s00205-017-1189-x. Apr. 1, 2018.

[BTW19] Claude Bardos, Edriss S. Titi, and Emil Wiedemann. "Onsager's Conjecture with Physical Boundaries and an Application to the Vanishing Viscosity Limit". *Communications in Mathematical Physics*. URL: https://doi.org/10.1007/s00220-019-03493-6. Aug. 1, 2019.

[CD03] Thierry Coulhon and Xuan Thinh Duong. "Riesz Transform and Related Inequalities on Non-Compact Riemannian Manifolds". *Communications on Pure and Applied Mathematics*. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.3040. 2003.

[CET94] Peter Constantin, Weinan E, and Edriss S. Titi. "Onsager's Conjecture on the Energy Conservation for Solutions of Euler's Equation". *Communications in Mathematical Physics*. URL: https://projecteuclid.org/euclid.cmp/1104271041. 1994.

[Cha06] Isaac Chavel. *Riemannian Geometry : A Modern Introduction*. 2006.

[Che+08] A Cheskidov et al. "Energy Conservation and Onsager's Conjecture for the Euler Equations". *Nonlinearity*. URL: https://iopscience.iop.org/article/10.1088/0951-7715/21/6/005. June 1, 2008.

[Con54] P. E. Conner. "The Greens and Neumanns Problems for Differential Forms on Riemannian Manifolds". *Proceedings of the National Academy of Sciences*. Dec. 1954.

[COQ09] Gui-Qiang Chen, Dan Osborne, and Zhongmin Qian. "The Navier-Stokes Equations with the Kinematic and Vorticity Boundary Conditions on Non-Flat Boundaries". *Acta Mathematica Scientia*. July 2009.

[DM96] Xuan Thinh Duong and Alan McIntosh. "Functional Calculi of Second-Order Elliptic Partial Differential Operators with Bounded Measurable Coefficients". *The Journal of Geometric Analysis*. June 1996.

[DN18] Theodore D. Drivas and Huy Q. Nguyen. "Onsager's Conjecture and Anomalous Dissipation on Domains with Boundary". *SIAM Journal on Mathematical Analysis*. URL: https://epubs.siam.org/doi/10.1137/18M1178864. Jan. 1, 2018.

[DS13] Camillo De Lellis and László Székelyhidi. "Dissipative Continuous Euler Flows". *Inventiones mathematicae*. URL: https://doi.org/10.1007/s00222-012-0429-9. Aug. 1, 2013.

[DS14] Camillo De Lellis and László Székelyhidi Jr. "Dissipative Euler Flows and Onsager's Conjecture". *Journal of the European Mathematical Society*. URL: https://www.ems-ph.org/journals/show_abstract.php?issn=1435-9855&vol=16&iss=7&rank=5. Aug. 23, 2014.

[DS19] Camillo De Lellis and László Székelyhidi. "On Turbulence and Geometry: From Nash to Onsager". *Notices of the American Mathematical Society*. URL: http://www.ams.org/notices/201905/rnoti-p677.pdf. May 1, 2019.

[Duo90] Xuan Thinh Duong. "Hînf Functional Calculus of Second Order Elliptic Partial Differntial Operators on Lp Spaces". URL: https://projecteuclid.org/euclid.pcma/1416335063. 1990.

[Eng00] Engel. *One-Parameter Semigroups for Linear Evolution Equations*. 2000.

[Eyi94] Gregory L. Eyink. "Energy Dissipation without Viscosity in Ideal Hydrodynamics I. Fourier Analysis and Local Energy Transfer". *Physica D: Nonlinear Phenomena*. URL: https://linkinghub.elsevier.com/retrieve/pii/0167278994901171. Nov. 1994.

[Fac15] Stephan Fackler. "Regularity Properties of Sectorial Operators: Extrapolation, Counterexamples and Generic Classes". URL: https://oparu.uni-ulm.de/xmlui/handle/123456789/3295. 2015.

[FFP16] Hans G. Feichtinger, Hartmut Führ, and Isaac Z. Pesenson. "Geometric Space–Frequency Analysis on Manifolds". *Journal of Fourier Analysis and Applications*. Jan. 2016.

[FK64] Hiroshi Fujita and Tosio Kato. "On the Navier-Stokes Initial Value Problem. I". *Archive for Rational Mechanics and Analysis*. URL: http://link.springer.com/10.1007/BF00276188. 1964.

[FMM98] Eugene Fabes, Osvaldo Mendez, and Marius Mitrea. "Boundary Layers on Sobolev–Besov Spaces and Poissons Equation for the Laplacian in Lipschitz Domains". *Journal of Functional Analysis*. Nov. 1998.

[Gaf54] Matthew P. Gaffney. "The Heat Equation Method of Milgram and Rosenbloom for Open Riemannian Manifolds". *The Annals of Mathematics*. Nov. 1954.

[GG00] Daniel Grieser and Michael J. Gruber. *Singular Asymptotics Lemma and Push-Forward Theorem*. URL: http://arxiv.org/abs/math/0009161. Sept. 15, 2000.

[Gig81] Yoshikazu Giga. "Analyticity of the Semigroup Generated by the Stokes Operator inLrspaces". *Mathematische Zeitschrift*. URL: https://doi.org/10.1007/BF01214869. Sept. 1, 1981.

[Gig82] Yoshikazu Giga. "The Nonstationary Navier-Stokes System with Some First Order Boundary Condition". *Proceedings of the Japan Academy, Series A, Mathematical Sciences*. 1982.

[Glü17] Jochen Glück. "Invariant Sets and Long Time Behaviour of Operator Semigroups". URL: https://oparu.uni-ulm.de/xmlui/handle/123456789/4277. 2017.

[GM85] Yoshikazu Giga and Tetsuro Miyakawa. "Solutions in Lr of the Navier-Stokes Initial Value Problem". *Archive for Rational Mechanics and Analysis*. URL: https://doi.org/10.1007/BF00276875. Sept. 1, 1985.

[GMM10] V. Gol'dshtein, I. Mitrea, and M. Mitrea. "Hodge Decompositions with Mixed Boundary Conditions and Applications to Partial Differential Equations on Lipschitz Manifolds". *Journal of Mathematical Sciences*. Dec. 2010.

[GO14] Loukas Grafakos and Seungly Oh. "The Kato-Ponce Inequality". *Communications in Partial Differential Equations*. May 2014.

[Gri04] D. Grieser. *NOTES ON HEAT KERNEL ASYMPTOTICS*. URL: https://www.semanticscholar.org/paper/NOTES-ON-HEAT-KERNEL-ASYMPTOTICS-Grieser/81e2265316899f63118a185135f1e20e5bd0d32d. 2004.

[Gui91] Davide Guidetti. "On Interpolation with Boundary Conditions". *Mathematische Zeitschrift*. May 1991.

[Haa06] Markus Haase. *The Functional Calculus for Sectorial Operators*. 2006.

[HMY08] Yongsheng Han, Detlef Müller, and Dachun Yang. "A Theory of Besov and Triebel-Lizorkin Spaces on Metric Measure Spaces Modeled on Carnot-Carathéodory Spaces". *Abstract and Applied Analysis*. 2008.

[HS18] Matthias Hieber and Jürgen Saal. "The Stokes Equation in the Lp-Setting: Well-Posedness and Regularity Properties". 2018.

[Hsu] Elton P Hsu. "Multiplicative Functional for the Heat Equation on Manifolds with Boundary".

[Huy19] Khang Manh Huynh. *Hodge-Theoretic Analysis on Manifolds with Boundary, Heatable Currents, and Onsager's Conjecture in Fluid Dynamics*. URL: http://arxiv.org/abs/1907.05360. July 14, 2019.

[Huy20] Khang Manh Huynh. *Construction of the Hodge-Neumann Heat Kernel, Local Bernstein Estimates, and Onsager's Conjecture in Fluid Dynamics*. URL: http://arxiv.org/abs/2010.14762. Oct. 26, 2020.

[IO14] Philip Isett and Sung-Jin Oh. *A Heat Flow Approach to Onsager's Conjecture for the Euler Equations on Manifolds*. URL: http://arxiv.org/abs/1310.7947. Jan. 17, 2014.

[Ise18a] Philip Isett. "A Proof of Onsager's Conjecture". *Annals of Mathematics*. 2018.

[Ise18b] Philip Isett. "A Proof of Onsagers Conjecture". *Annals of Mathematics*. 2018.

[Kot16] Chris Kottke. *Linear Analysis on Manifolds: Notes for Math 7376, Spring 2016*. URL: https://ckottke.ncf.edu/neu/7376_sp16/lacm.pdf. June 14, 2016.

[KP14] Gerard Kerkyacharian and Pencho Petrushev. "Heat Kernel Based Decomposition of Spaces of Distributions in the Framework of Dirichlet Spaces". *Transactions of the American Mathematical Society*. June 2014.

[KP88] Tosio Kato and Gustavo Ponce. "Commutator Estimates and the Euler and Navier-Stokes Equations". *Communications on Pure and Applied Mathematics*. Oct. 1988.

[KR06] S. Klainerman and I. Rodnianski. "A Geometric Approach to the Littlewood–Paley Theory". *Geometric & Functional Analysis GAFA*. URL: https://doi.org/10.1007/s00039-006-0551-1. Feb. 1, 2006.

[KVZ14] Rowan Killip, Monica Visan, and Xiaoyi Zhang. *Riesz Transforms Outside a Convex Obstacle*. URL: http://arxiv.org/abs/1205.5784. Dec. 11, 2014.

[KW16] Christoph Kriegler and Lutz Weis. "Paley-Littlewood Decomposition for Sectorial Operators and Interpolation Spaces". *Mathematische Nachrichten*. Feb. 2016.

[Lee09] Jeffrey M. Lee. *Manifolds and Differential Geometry*. 2009.

[Lem02a] Pierre Gilles Lemarie-Rieusset. *Recent Developments in the Navier-Stokes Problem*. Apr. 26, 2002.

[Lem02b] Pierre Gilles Lemarie-Rieusset. *Recent Developments in the Navier-Stokes Problem (Chapman & Hall/CRC Research Notes in Mathematics Series)*. URL: https://www.xarg.org/ref/a/1584882204/. Apr. 2002.

[Lof92] Jorgen Lofstrom. "Interpolation of Boundary Value Problems of Neumann Type on Smooth Domains". *Journal of the London Mathematical Society*. Dec. 1992.

[Lun95] Alessandra Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. 1995.

[McI10] Alan McIntosh. "Operator Theory-Spectra and Functional Calculi". 2010.

[McI86] Alan McIntosh. "Operators Which Have an Hinf Functional Calculus". 1986.

[Mel18] Richard B Melrose. *The Atiyah-Patodi-Singer Index Theorem*. URL: http://public.ebookcentral.proquest.com/choice/publicfullrecord.aspx?p=5437055. 2018.

[Mel92] Richard B. Melrose. "Calculus of Conormal Distributions on Manifolds with Corners". *International Mathematics Research Notices*. URL: https://academic.oup.com/imrn/article/1992/3/51/824177. Jan. 1, 1992.

[Miy80] Tetsuro Miyakawa. "The \$L\spp\$ Approach to the Navier-Stokes Equations with the Neumann Boundary Condition". *Hiroshima Mathematical Journal*. URL: https://projecteuclid.org/euclid.hmj/1206134338. 1980.

[MM08] Marius Mitrea and Sylvie Monniaux. "The Regularity of the Stokes Operator and the Fujita–Kato Approach to the Navier–Stokes Initial Value Problem in Lipschitz Domains". *Journal of Functional Analysis*. URL: http://www.sciencedirect.com/science/article/pii/S0022123607004429. Mar. 15, 2008.

[MM09a] Marius Mitrea and Sylvie Monniaux. "On the Analyticity of the Semigroup Generated by the Stokes Operator with Neumann-Type Boundary Conditions on Lipschitz Subdomains of Riemannian Manifolds". *Transactions of the American Mathematical Society*. URL: http://www.ams.org/jourcgi/jour-getitem?pii=S0002-9947-08-04827-7. June 1, 2009.

[MM09b] Marius Mitrea and Sylvie Monniaux. "The Nonlinear Hodge-Navier-Stokes Equations in Lipschitz Domains". *Differential and Integral Equations*. URL: https://projecteuclid.org/euclid.die/1356019778. Mar. 2009.

[MM18] Alan Mcintosh and Sylvie Monniaux. "Hodge-Dirac, Hodge-Laplacian and Hodge-Stokes Operators in Lp Spaces on Lipschitz Domains". *Revista Matemática Iberoamericana*. URL: https://hal.archives-ouvertes.fr/hal-01351604. 2018.

[Mon13] Sylvie Monniaux. “Various Boundary Conditions for Navier-Stokes Equations in Bounded Lipschitz Domains”. *Discrete & Continuous Dynamical Systems - S*. URL: https://www.aimsciences.org/article/doi/10.3934/dcdss.2013.6.1355. 2013.

[MR51] A. N. Milgram and P. C. Rosenbloom. “Harmonic Forms and Heat Conduction: I: Closed Riemannian Manifolds”. *Proceedings of the National Academy of Sciences*. Mar. 1951.

[MV13] Rafe Mazzeo and Boris Vertman. *Analytic Torsion on Manifolds with Edges*. URL: http://arxiv.org/abs/1103.0448. July 13, 2013.

[NN19] Quoc-Hung Nguyen and Phuoc-Tai Nguyen. “Onsager’s Conjecture on the Energy Conservation for Solutions of Euler Equations in Bounded Domains”. *Journal of Nonlinear Science*. URL: https://doi.org/10.1007/s00332-018-9483-9. Feb. 1, 2019.

[NNT20] Quoc-Hung Nguyen, Phuoc-Tai Nguyen, and Bao Quoc Tang. “Energy Conservation for Inhomogeneous Incompressible and Compressible Euler Equations”. *Journal of Differential Equations*. URL: http://www.sciencedirect.com/science/article/pii/S0022039620302904. Oct. 15, 2020.

[NT19] Virginia Naibo and Alexander Thomson. “Coifman-Meyer Multipliers: Leibniz-Type Rules and Applications to Scattering of Solutions to PDEs”. *Transactions of the American Mathematical Society*. Apr. 2019.

[Ouy17] Cheng Ouyang. “Multiplicative Functional for the Heat Equation on Manifolds with Boundary”. *Stochastic Analysis and Related Topics*. 2017.

[Rha84] Rham. *Differentiable Manifolds : Forms, Currents, Harmonic Forms*. 1984.

[RRS18] James C. Robinson, José L. Rodrigo, and Jack W.D. Skipper. “Energy Conservation for the Euler Equations on T2xR+ for Weak Solutions Defined without Reference to the Pressure”. *Asymptotic Analysis*. Nov. 2018.

[Rud91] Walter Rudin. *Functional Analysis*. 1991.

[Sch11] Cornelia Schneider. "Traces of Besov and Triebel-Lizorkin Spaces on Domains". *Mathematische Nachrichten*. Mar. 2011.

[Sch95] Günter Schwarz. *Hodge Decomposition—A Method for Solving Boundary Value Problems*. URL: http://link.springer.com/10.1007/BFb0095978. 1995.

[She12] Zhongwei Shen. "Resolvent Estimates in l p for the Stokes Operator in Lipschitz Domains". *Archive for Rational Mechanics and Analysis*. Mar. 2012.

[Ste71] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions. (PMS-30)*. URL: https://www.xarg.org/ref/a/0691080798/. Feb. 1971.

[Str83] Robert S Strichartz. "Analysis of the Laplacian on the Complete Riemannian Manifold". *Journal of Functional Analysis*. URL: https://linkinghub.elsevier.com/retrieve/pii/0022123683900903. June 1983.

[Str89] Robert S. Strichartz. "Harmonic Analysis as Spectral Theory of Laplacians". *Journal of Functional Analysis*. URL: https://linkinghub.elsevier.com/retrieve/pii/0022123689900049. Nov. 1989.

[Tan18] Koichi Taniguchi. "Besov Spaces Generated by the Neumann Laplacian". *European Journal of Mathematics*. Mar. 2018.

[Tao06] Terence Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*. 2006.

[Tao07] Terence Tao. "LECTURE NOTES 6 FOR 247B". 2007.

[Tao09] Terence Tao. *A Quantitative Formulation of the Global Regularity Problem for the Periodic Navier-Stokes Equation*. URL: http://arxiv.org/abs/0710.1604. May 21, 2009.

[Tao13] Terence Tao. "Localisation and Compactness Properties of the Navier–Stokes Global Regularity Problem". *Analysis & PDE*. URL: https://projecteuclid.org/euclid.apde/1513731290. 2013.

[Tao16] Terrence Tao. “Notes on the Nash Embedding Theorem”. URL: https://terrytao.wordpress.com/2016/05/11/notes-on-the-nash-embedding-theorem/. May 2016.

[Tay11a] Michael E. Taylor. *Partial Differential Equations I*. 2011.

[Tay11b] Michael E. Taylor. *Partial Differential Equations I: Basic Theory*. URL: http://link.springer.com/10.1007/978-1-4419-7055-8. 2011.

[Tay11c] Michael E. Taylor. *Partial Differential Equations II*. 2011.

[Tay11d] Michael E. Taylor. *Partial Differential Equations III: Nonlinear Equations*. URL: http://link.springer.com/10.1007/978-1-4419-7049-7. 2011.

[Tri10] H. Triebel. *Theory of Function Spaces*. URL: https://books.google.com/books?id=fMUMl6iLMGwC. 2010.

[Tri92] H. Triebel. *Theory of Function Spaces II*. URL: https://books.google.com/books?id=-0vvAAAAMAAJ. 1992.

[Tro09] M. Troyanov. “On the Hodge Decomposition in Rn”. *Moscow Mathematical Journal*. 2009.

[Voi92] Jürgen Voigt. “Abstract Stein Interpolation”. *Mathematische Nachrichten*. 1992.

[Wal84] Robert Wald. *General Relativity*. 1984.

[Wie20] Emil Wiedemann. *Conserved Quantities and Regularity in Fluid Dynamics*. URL: http://arxiv.org/abs/2003.07807. Mar. 17, 2020.

[Wu91] Siye Wu. “Topological Quantum Field Theories on Manifolds with a Boundary”. *Communications in Mathematical Physics*. URL: https://projecteuclid.org:443/euclid.cmp/1104202263. 1991.