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Toric Degenerations of Calabi-Yau Manifolds in Grassmannians

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Karl Strom Fredrickson

Committee in charge:

Professor Mark L. Gross, Chair
Professor Benjamin Grinstein
Professor J. William Helton
Professor Kenneth A. Intriligator
Professor Dragos Oprea

2011

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Chair

University of California, San Diego

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VITA

2005	B. A. in Mathematics and Chemistry, University of Washington
2005-2011	Teaching Assistant, Instructor, and Graduate Student Researcher, University of California, San Diego
2011	Ph. D. in Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

Toric Degenerations of Calabi-Yau Manifolds in Grassmannians

by

Karl Strom Fredrickson

Doctor of Philosophy in Mathematics

University of California San Diego, 2011

Professor Mark L. Gross, Chair

We describe methods for constructing toric degenerations of Calabi-Yau manifolds in Grassmannians. Toric degenerations were introduced by Gross and Siebert in their work on mirror symmetry, and consist of a one-parameter family of algebraic varieties with a certain type of singular fiber. Gross showed that toric degenerations give, in a certain sense, a complete description of Calabi-Yau manifolds that arise from the Batyrev-Borisov construction. This thesis focuses on Calabi-Yau complete intersections in Grassmannians, which in general cannot be obtained from the Batyrev-Borisov construction. We completely work out the details of the simplest example, that of a quartic hypersurface in $G(2, 4)$, and discuss how a similar strategy might be used in higher-dimensional cases.

Chapter 1

Introduction

Most of this thesis is concerned with studying a particular geometric object, an example of a Calabi-Yau manifold, using the method of toric degenerations. In this chapter I will try to briefly introduce the ideas of Calabi-Yau manifolds, mirror symmetry, and toric degenerations, as well as provide some motivation for the thesis problem and methods used to approach it.

1.1 Calabi-Yau Manifolds and Mirror Symmetry

Calabi-Yau manifolds are a type of complex manifold, a geometric object locally modeled on complex analytic space $\mathbb{C}^n$. They satisfy the additional property of having a trivial canonical bundle, which is difficult to understand intuitively but gives them some very important mathematical properties. Eugenio Calabi made a conjecture in differential geometry about such manifolds which was later proven by Shing-Tung Yau in 1978.

Mathematically, a Calabi-Yau manifold is usually defined as a compact Kähler manifold with trivial canonical bundle. Since a manifold is, by definition, non-singular, this definition would not allow the possibility of a Calabi-Yau manifold with singularities. However, such a concept does make sense in algebraic geometry, provided that the singularities are mild enough. In this thesis, all examples of Calabi-Yau manifolds will be projective varieties over $\mathbb{C}$, the field of complex numbers. We will sometimes talk about Calabi-Yau varieties that have mild sin-

gularities, such as isolated nodes in the three dimensional case. However, the main examples, complete intersections in Grassmannians, will be nonsingular.

In one complex dimension, the only examples of nonsingular Calabi-Yau manifolds are elliptic curves. Considered as smooth manifolds, elliptic curves have two real dimensions and are diffeomorphic to $S^1 \times S^1$, i.e., a donut. In two dimensions, assuming that the manifold is simply connected, all nonsingular Calabi-Yau manifolds are examples of so-called K3 surfaces. As in the one-dimensional case, these are all diffeomorphic to each other. In three dimensions, things get more complicated, and many different topological types of Calabi-Yau threefolds are possible.

Calabi-Yau threefolds are well known today for the role they play in string theory. String theory suggests that in addition to time and the three spatial dimensions, there are extra dimensions which are “curled up” (in mathematical language, compactified) and thus hidden from everyday experience. In some versions of the theory, these extra dimensions take the form of a Calabi-Yau threefold, providing six extra real dimensions.

Mirror symmetry, which is now a large mathematical subject, was originally discovered by physicists studying string theory. It is a dual relationship between pairs of Calabi-Yau manifolds of the same dimension, known as “mirror pairs”. It was believed that two different formulations of string theory would result in the same physics if they used an appropriate pair of Calabi-Yau manifolds to supply the extra dimensions. These pairs of Calabi-Yaus that yield the same physics are said to be mirror pairs.

One of the first mathematical implications of mirror symmetry was that a certain sequence of numbers related to one Calabi-Yau manifold, its Gromov-Witten invariants, could be found by a different type of calculation on the mirror which was relatively well understood. (Gromov-Witten invariants are geometric invariants that attempt to count the number of holomorphic curves on a threefold satisfying certain conditions. Their definition is complicated and they will not be formally involved in this thesis, although they play an important role in mirror symmetry.) This insight allowed physicists to calculate, for the first time, all the

Gromov-Witten invariants of one of the most well-studied Calabi-Yau threefolds, the quintic threefold. For an introduction to this calculation, see Mark Gross's chapter in [HGJ93]. Although the physicists' calculations did not give a complete proof, this was a very impressive accomplishment to mathematicians, and inspired many attempts to find a rigorous mathematical explanation of mirror symmetry.

1.2 Mathematical Approaches to Mirror Symmetry

Important mathematical approaches to mirror symmetry include the Batyrev-Borisov construction, the SYZ conjecture, and homological mirror symmetry. Homological mirror symmetry was first conjectured by Kontsevich and makes heavy use of homological algebra methods such as derived categories. It is a very active area of research, but less related to this thesis than the other two and will not be discussed.

The Batyrev-Borisov construction, which is described for hypersurfaces in the paper [Bat94] and more generally in [BB94], is a fundamental method for constructing Calabi-Yau manifolds and their mirrors as complete intersections in certain toric varieties. In the hypersurface case, the Calabi-Yau pairs are associated to dual pairs of reflexive polytopes. We let M be a free abelian group, isomorphic to $\mathbb{Z}^n$ for some positive integer n , and let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual group and likewise $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. An n -dimensional compact convex polytope $\Delta \subseteq M_{\mathbb{R}}$ with the origin in its interior is said to be *reflexive* if it and its dual, $\Delta^* \subseteq N_{\mathbb{R}}$, are both lattice polytopes. A polytope in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$ is a lattice polytope if it can be written as the convex hull of points in the lattices M and N , respectively. Here Δ^* is defined as the set

$$\{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq -1 \text{ for all } m \in \Delta\},$$

where $\langle n, m \rangle$ is the real number given by the natural pairing between $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$.

Given a reflexive polytope, or indeed any n -dimensional lattice polytope $\Delta \subseteq M_{\mathbb{R}} \cong \mathbb{R}^n$, one can construct a related algebraic variety called a *toric variety*.

Although the construction can be done over any base field, for us the base field will always be $\mathbb{C}$, in which case the variety will have n complex dimensions. The toric variety is often written as $\mathbb{P}(\Delta)$. Familiar examples of toric varieties include complex projective space $\mathbb{P}^n$, complex affine space $\mathbb{C}^n$, and $\mathbb{P}^n$ or $\mathbb{C}^n$ blown up at a point. Any product of toric varieties is also a toric variety. For a complete introduction to toric varieties, see [Ful93].

The toric variety associated to a polytope is given by gluing together affine spaces associated to each vertex of the polytope. A quick way to give the complete definition is to let $C_\Delta \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$ be the cone

$$C_\Delta = \{(rv, r) \mid r \in \mathbb{R}, r \geq 0, v \in \Delta\}.$$

The ring $R = \mathbb{C}[C_\Delta \cap (M \oplus \mathbb{Z})]$ is then naturally graded by the $\mathbb{Z}$ -coordinate, so we can define $\mathbb{P}(\Delta)$ as $\text{Proj}(R)$. Global sections of the line bundle $\mathcal{O}_{\mathbb{P}(\Delta)}(1)$, which naturally comes from the Proj construction, consist of homogeneous elements of R of degree 1. But because of the definition of R and C_Δ , we see that a basis of these elements consists of lattice points in the polytope Δ .

We will frequently use this correspondence between global sections of a line bundle and lattice points in a polytope. More generally, whenever we have a cone D in a real vector space such as $M_{\mathbb{R}}$, and are considering the ring $\mathbb{C}[D \cap M]$, we will write z^m for the ring element associated to a lattice element $m \in D \cap M$. The ring elements z^m can be thought of as generalizations of monomials, so that we have a correspondence between monomials and elements of the lattice.

Returning to mirror symmetry, a basic example of a reflexive polytope in $\mathbb{R}^n$ is given by

$$\Delta = \text{Conv}(v, v + (n+1)e_1, v + (n+1)e_2, \dots, v + (n+1)e_n).$$

Here $e_1, \dots, e_n$ are the unit vectors in $\mathbb{R}^n$, and $v = -e_1 - e_2 - \dots - e_n$. The reflexive dual of Δ is $\Delta^* = \text{Conv}(v, e_1, e_2, \dots, e_n)$ (where we identify $\mathbb{R}^n$ with its dual by using the standard inner product $\langle e_i, e_j \rangle = \delta_{ij}$). The toric variety associated to Δ is complex projective space $\mathbb{P}^n$, and Δ can be identified with the Newton polytope of degree $n+1$ monomials on $\mathbb{P}^n$. These monomials make up a basis for the vector space of global sections of the line bundle $\mathcal{O}_{\mathbb{P}^n}(n+1)$. On the other hand, the toric

variety $\mathbb{P}(\Delta^*)$ associated to Δ^* is distinct from $\mathbb{P}^n$ and will have singularities for $n \geq 2$.

If we apply the Batyrev-Borisov construction to this pair of reflexive polytopes, one of the Calabi-Yau manifolds will be a hypersurface in $\mathbb{P}^n$ given by the zero set of a generic section of $\mathcal{O}_{\mathbb{P}^n}(n+1)$, that is, a linear combination of degree $n+1$ monomials with sufficiently general coefficients. To obtain the mirror Calabi-Yau manifold, we also take the hypersurface in $\mathbb{P}(\Delta^*)$ given by a generic section of the line bundle associated to Δ^* . However, unlike a generic hypersurface in $\mathbb{P}^n$, this hypersurface will be singular, because $\mathbb{P}(\Delta^*)$ is singular. This is where the idea of MPCP functions and an MPCP resolution of $\mathbb{P}(\Delta^*)$ comes into play.

MPCP stands for “maximal partially crepant projective”. If $\Delta \subseteq M_{\mathbb{R}}$ is an n -dimensional reflexive polytope, then we define an “integral piecewise linear function on a subdivision of Δ ” as a function $h : M_{\mathbb{R}} \rightarrow \mathbb{R}$ such that each point of $M_{\mathbb{R}}$ is contained in an n -dimensional polyhedral cone on which h is integral linear, and each such polyhedral cone is contained in the cone over a face of Δ . (Integral linear means linear and taking integer values on elements of the lattice M . By cone over a set S in a real vector space, we always mean $C(S) = \{rs \mid s \in S, r \in \mathbb{R}, r \geq 0\}$.) Suppose h is integral piecewise linear on a subdivision of Δ . Then h is an MPCP function if its maximal domains of linearity are cones over elementary simplices contained in boundary faces of Δ . An elementary simplex is a lattice simplex that contains no other lattice points besides its vertices. Proposition 4 of [GZK89] provides an algorithm for constructing an MPCP function for any reflexive polytope.

The importance of this definition is not very clear, but it allows us to resolve all singularities of the original Calabi-Yau hypersurface in $\mathbb{P}(\Delta^*)$, at least in the threefold case. The cones over the faces of Δ fit together to form a combinatorial object called a “fan”, Σ , in $M_{\mathbb{R}}$. The fan Σ carries enough information from the polytope Δ^* to be able to construct the toric variety $\mathbb{P}(\Delta^*)$. (In fact, toric varieties are often defined by giving a fan rather than a polytope.) If Σ and Σ' are fans contained in the same vector space, there is an obvious notion of Σ' being a subdivision of Σ , and if $X_{\Sigma'}$ and X_{Σ} are the associated toric varieties, then we get

a map $X_{\Sigma'} \rightarrow X_{\Sigma}$. Let $\Sigma(\Delta)$ be the fan consisting of cones over the faces of Δ , and let $\Sigma'(\Delta)$ be the subdivision of $\Sigma(\Delta)$ induced by an MPCP function h . Then $X_{\Sigma'(\Delta)} \rightarrow X_{\Sigma(\Delta)} = \mathbb{P}(\Delta^*)$ is a partial resolution of singularities of $\mathbb{P}(\Delta^*)$, and it completely resolves the singularities of the generic hypersurface in $\mathbb{P}(\Delta^*)$, provided the hypersurface is three dimensional. All this is proven in [Bat94].

If we use the polytopes Δ and Δ^* defined above and set $n = 4$, then one Calabi-Yau manifold will be the quintic threefold in $\mathbb{P}(\Delta) = \mathbb{P}^4$ and the other, after an MPCP resolution of $\mathbb{P}(\Delta^*)$, will be its mirror. This is a very well studied, and historically important, case of mirror symmetry.

Later we will make use of MPCP functions to define degenerations of the Calabi-Yau manifolds we are interested in. For us, their importance is less with resolving singularities than their relationship with “simplicity” as described in Theorem 3.16 of [Gro05].

Now we move on to a brief description of the SYZ conjecture. Named after its conjecturers Strominger, Yau, and Zaslow, the SYZ conjecture asserts that mirror Calabi-Yau pairs can be realized as fibrations over the same base space B , which has real dimension equal to one-half the real dimension of the Calabi-Yaus. Moreover, the generic fiber is a torus and the fibrations are dual in a certain sense. There are many different versions of the SYZ conjecture with varying levels of strength, and the original version made a strong assertion about the differential geometry of the fibration which is probably not true in general. (See [Gro08] for more on this, and other aspects of the SYZ conjecture relevant to toric degenerations.)

We will focus on the case where the base space is an affine manifold with singularities, which is the most relevant for our situation. To define an affine manifold with singularities, first we need the concept of an affine manifold.

Definition 1.1. An affine manifold is a topological manifold B together with a covering of charts to $\mathbb{R}^n$, (U_i, φ_i) , such that for any two charts (U_i, φ_i) , (U_j, φ_j) with $U_i \cap U_j$ nonempty, the transition map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

is the restriction of an affine map from $\mathbb{R}^n$ to itself, that is, the composition of a translation and a linear isomorphism.

If all the transition maps can be written as the composition of a integral linear map and translation by an integer vector (in other words, an affine automorphism of $\mathbb{Z}^n$), then the affine structure is said to be integral, and B is an integral affine manifold.

In this paper, all affine structures will be integral. Now we define an affine manifold with singularities, which is a topological manifold with an affine structure on a well-behaved open subset.

Definition 1.2. An (integral) affine manifold with singularities is a topological manifold B with an (integral) affine structure on an open subset $B_0 \subseteq B$, such that $B \setminus B_0$ is a locally finite union of locally closed submanifolds of codimension ≥ 2 .

Let B be an integral affine manifold with singularities. We will describe how to construct canonical dual torus fibrations over the nonsingular subset B_0 . Let (U, φ) be one of the affine charts on B_0 . Then the tangent bundle of U is isomorphic to $\varphi(U) \times \mathbb{R}^n$, which has a subbundle $\varphi(U) \times \mathbb{Z}^n$. Given any two intersecting affine charts (U, φ) and (U', φ') , the two corresponding subbundles will agree on $U \cap U'$ because of the condition that the transition maps are integral affine. Thus we get a subbundle of the whole tangent bundle of B_0 such that the local sections are as above. Modding out by the subbundle will produce a torus bundle, since each fiber will be isomorphic to $\mathbb{R}^n / \mathbb{Z}^n$. The same procedure on the cotangent bundle produces a dual torus fibration. We denote the torus fibration obtained from the tangent bundle by $X(B_0)$, and the one obtained from the cotangent bundle by $\check{X}(B_0)$.

Given an MPCP subdivision of the polytope Δ defined above (the Newton polytope of quintics on $\mathbb{P}^4$), there is a natural way to give the boundary of Δ , $B = \partial\Delta$, the structure of an affine manifold with singularities. We start by taking the first barycentric subdivision of all polytopes in $\partial\Delta$ arising from the MPCP subdivision; denote the resulting collection of polytopes by P_1 . Then, define the

singular locus $B \setminus B_0$ as the union of all one-dimensional simplices in P_1 which do not contain any elements of the lattice M . The affine structure is defined locally at every lattice point v contained in the boundary. Specifically, if we take P_v to be the union of all polytopes in P_1 containing v , and let $\text{Int}(P_v)$ be the interior of P_v in the subspace topology of $\partial\Delta$, then the affine structure exists locally on $\text{Int}(P_v)$. It is induced by the quotient map $\pi : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/\text{Span}_{\mathbb{R}}(v)$, which will define a homeomorphism between $\text{Int}(P_v)$ and a neighborhood of the origin in the quotient space $M_{\mathbb{R}}/\text{Span}_{\mathbb{R}}(v)$. We then use the natural affine structure on $M_{\mathbb{R}}/\text{Span}_{\mathbb{R}}(v)$.

The affine structure on B_0 actually extends across the interiors of maximal boundary faces of Δ , so that the singular locus can be reduced to a smaller graph contained in two-dimensional boundary faces of Δ , and B_0 as defined in the previous paragraph can be enlarged to a set B'_0 which includes the interiors of maximal boundary faces.

Since B'_0 is an integral affine manifold, a fact which is also relatively easy to prove, we can construct the dual torus fibrations $X(B'_0) \rightarrow B'_0$ and $\check{X}(B'_0) \rightarrow B'_0$ defined above. Somewhat surprisingly, it is possible to show that these fibrations can be compactified to fibrations $X(B) \rightarrow B$ and $\check{X}(B) \rightarrow B$, such that $X(B)$ is homeomorphic to the quintic and $\check{X}(B)$ is homeomorphic to its mirror. Thus we have a sort of topological version of the SYZ conjecture. The paper [Gro01] proves this and gives an explicit description of the compactifications. See also section 4 of [Gro08] for an introductory discussion. This connection between the Batyrev-Borisov construction and the SYZ conjecture, via the polytope Δ , was one of the motivations for the Gross-Siebert approach to mirror symmetry.

1.3 Toric Degenerations and Affine Manifolds

The main ingredient in the Gross-Siebert approach to mirror symmetry is a *toric degeneration* of Calabi-Yau varieties. The most general definition of a toric degeneration is given in [GS06] and somewhat long, but the following simplified definition will suffice for our purposes.

Definition 1.3. A toric degeneration of Calabi-Yau varieties is a proper flat map

$p : \mathcal{X} \rightarrow \mathbb{C}$, where $\mathcal{X}$ is a normal variety, and:

1. The generic fiber of p is an irreducible normal Calabi-Yau variety, and the singular fiber $p^{-1}(0)$ is reduced and a union of toric varieties intersecting along toric strata.
2. There exists a closed subset $Z \subseteq \mathcal{X}$ of relative codimension ≥ 2 satisfying the following: Z contains no toric stratum of the singular fiber $p^{-1}(0)$, and for every geometric point $x \in \mathcal{X} \setminus Z$, the map p is locally equivalent, up to an étale change of coordinates, to the evaluation of a monomial on some open subset of an affine toric variety Y_x . The monomial must vanish precisely once on every toric divisor of Y_x . In other words, there exists an étale neighborhood of x , $U_x \rightarrow \mathcal{X} \setminus Z$, and an étale map $f : U_x \rightarrow Y_x$, such that $p : U_x \rightarrow \mathbb{C}$ is equal to $m \circ f : U_x \rightarrow \mathbb{C}$, $m : Y_x \rightarrow \mathbb{C}$ being evaluation of the monomial.

This definition is different from the one in [GS06] or [Gro05] in two ways. First, the more general definition allows self-intersection of components of the singular fiber, which never occurs in this paper (or in [Gro05]). Second, in the original definition, $\mathcal{X}$ is a family over the spectrum of a discrete valuation ring, rather than affine space $\mathbb{C}$. However, $\mathbb{C}$ can be replaced with a discrete valuation ring by basechange, as in the comment before Theorem 3.10 in [Gro05].

The importance of this definition is that it allows us to construct an affine manifold with singularities from any toric degeneration by gluing together lattice polytopes associated to the toric varieties Y_x . Specifically, the affine toric variety Y_x can be defined by a fan consisting of a single cone, and the associated lattice polytope σ_x is given by taking the convex hull of the primitive integral generators of boundary rays of the cone. The affine manifold with singularities obtained from the gluing, B , also has a “polyhedral decomposition” P given by union of the polytopes σ_x . Together (B, P) is referred to as the *dual intersection complex* of the toric degeneration. Note that the dual intersection complex involves both an affine manifold, as in the SYZ conjecture, and lattice polytopes, as in the Batyrev-Borisov construction, suggesting that toric degenerations are a sort of combination of both.

To formally define a polyhedral decomposition of an affine manifold with singu-

larities, first we need to define polyhedral decompositions of regions of $\mathbb{R}^n$. (This definition is the same as Definition 1.21 of [GS06].)

Definition 1.4. A polyhedral decomposition of a closed set $R \subseteq \mathbb{R}^n$ is a locally finite covering P of R by compact convex polytopes (called “cells”) with the property that:

- (1) if $\sigma \in P$ and $\tau \subseteq \sigma$ is a face of σ then $\tau \in P$;
- (2) if $\sigma, \sigma' \in P$, then $\sigma \cap \sigma'$ is a common face of σ and σ' .

The decomposition is said to be integral if all vertices (points of $\mathbb{R}^n$ contained in P) are contained in $\mathbb{Z}^n$.

Now we define polyhedral decompositions of integral affine manifolds with singularities. This definition is again simpler than the more general one in [GS06], because we do not need to allow self-intersecting cells.

Definition 1.5. A polyhedral decomposition of an integral affine manifold with singularities B is a collection P of closed subsets of B , called cells, which covers B and satisfies the following properties. For any $v \in P$ a zero-dimensional cell (i.e., point), we must have:

- (1) v is not contained in the singular locus $B \setminus B_0$.
- (2) For each v there exists a polyhedral decomposition P_v of a closed neighborhood R of the origin in $\mathbb{R}^n$, and a map $\varphi_v : R \rightarrow B$ which is a homeomorphism onto its image, such that if $\sigma \in P_v$, then $\varphi_v(\sigma) \in P$.
- (3) φ_v is an integral affine map in some neighborhood of the origin.
- (4) If $\sigma \in P_v$ is a top dimensional cell, then $\varphi_v(Int(\sigma))$ is contained in B_0 , i.e., does not intersect the singular locus, and φ_v restricted to $Int(\sigma)$ is integral affine.

Given a toric degeneration, the dual intersection complex is constructed using the procedure in Construction 1.26 of [GS06]. Namely, we give the interiors of the lattice polytopes σ_x , associated to zero dimensional toric strata x in the singular fiber, the affine structure inherited from affine space. Then we glue them together using the combinatorics of the singular fiber. To get an affine manifold with singularities, the affine structure near vertices of the σ_x still needs to be specified. Each vertex of a polytope corresponds to a component of the singular fiber, so to

obtain an affine structure we use the fan that defines the associated component of the singular fiber as a toric variety.

As shown in [Gro05], there is a canonical way to construct toric degenerations for Calabi-Yau manifolds arising from the Batyrev-Borisov construction. The dual intersection complexes of these degenerations are closely related to the reflexive polytopes used in the Batyrev-Borisov construction. As a very simple example, consider the family of quintic threefolds given by

$$tf + z_0 z_1 z_2 z_3 z_4 = 0$$

in $\mathbb{P}^4 \times \mathbb{C}$, where t is the coordinate on $\mathbb{C}$, $z_0, \dots, z_4$ are homogeneous coordinates on $\mathbb{P}^4$, and f is a generic quintic. This is a toric degeneration whose dual intersection complex can be constructed from the polytope Δ^* defined earlier ($\Delta^* = \text{Conv}((-1, -1, -1, -1), e_1, e_2, e_3, e_4)$ where the e_i are the unit vectors in $\mathbb{R}^4$). B can be identified with the boundary $\partial\Delta^*$. The affine structure on interiors of the maximal faces is inherited from $\mathbb{R}^4$, while the affine structure near a vertex v (note the vertices are the only lattice points in the boundary) is induced by the quotient map $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^4 / \text{Span}_{\mathbb{R}}(v)$. The polyhedral decomposition is simply given by the union of the five boundary faces. We can identify the vertices of Δ^* with components of the singular fiber, which is given by $z_0 z_1 z_2 z_3 z_4 = 0$ and thus consists of five $\mathbb{P}^3$ s. The maximal faces can be identified with zero-dimensional toric strata in the singular fiber.

If the toric degeneration is also polarized, then it is possible to dualize the polyhedral decomposition on B to obtain a new pair $(\check{B}, \check{P})$. This operation is called the *discrete Legendre transform* and in the Gross-Siebert program, it is where mirror symmetry actually happens. That is, mirror Calabi-Yau manifolds should have toric degenerations with dual intersection complexes that are discrete Legendre transforms of each other. Thus, mirror symmetry can be done directly on the singular fiber in a way that is easier to understand than on the Calabi-Yaus themselves. The Gross-Siebert approach has also made significant progress towards calculation of Gromov-Witten invariants by analyzing piecewise linear “curves” directly on the dual intersection complex (so-called “tropical geometry”).

The reverse process of constructing a degeneration (and thus a smooth Calabi-

Yau fiber) from its dual intersection complex is covered in [GS07]. Also see [GS08] for a less technical introduction. This is a complicated process of great theoretical interest, and it is not directly involved in my thesis work. However, the smoothing process does require a condition on the monodromy of the dual intersection complex called *simplicity*. Defined in [GS06], it says that certain lattice polytopes arising from the monodromy of (B, P) must be elementary simplices. To get as much information as possible, it is important to be able to construct a toric degeneration that gives a simple dual intersection complex.

The example toric degeneration of the quintic given above, although it has a “simple” definition, does not result in a simple dual intersection complex. To obtain a simple degeneration, one must use the more complicated methods in [Gro05]. This involves choosing an MPCP function on the polytope Δ and defining a degeneration in a toric variety associated to the MPCP function. While the generic fiber is the same, it degenerates in a much more complicated way, giving a singular fiber with more components and a more complicated dual intersection complex.

One of the main purposes of the paper [Gro05] was to show that the Batyrev-Borisov construction of mirror pairs can be completely described by toric degenerations. That is, given a pair of Calabi-Yau manifolds arising from the Batyrev-Borisov construction, it is possible to construct toric degenerations of them whose dual intersection complexes are simple and are the discrete Legendre transforms of each other. The Batyrev-Borisov construction describes mirrors of complete intersections in toric varieties. The question arises, can toric degenerations be used to understand other classes of Calabi-Yau manifolds? This is the focus of my thesis work.

1.4 Mirror Symmetry for Calabi-Yau Complete Intersections in Grassmannians

One such class is the class of Calabi-Yau complete intersections in Grassmannians. The Grassmannian $G(k, n)$ is a variety that parametrizes k -dimensional

subspaces of the complex vector space $\mathbb{C}^n$. There is a standard embedding of $G(k, n)$ into $\mathbb{P}^{\binom{n}{k}-1}$ called the Plücker embedding. The homogenous ideal that defines $G(k, n)$ as a projective variety via this embedding is generated by quadratic polynomials. For instance, $G(2, 4)$ as a subvariety of $\mathbb{P}^5$ is defined by the equation $z_0z_1 - z_2z_3 + z_4z_5 = 0$. Although $G(2, 4)$ is a complete intersection in $\mathbb{P}^5$, the general Grassmannian is not, meaning that its homogeneous ideal may have more generators than its codimension.

To obtain a Calabi-Yau complete intersection in $G(k, n)$, one can embed it via the Plücker embedding, then intersect it with generic hypersurfaces in $\mathbb{P}^{\binom{n}{k}-1}$ whose degree adds up to n . (The simplest such example is the intersection of a quartic hypersurface with $G(2, 4)$ in $\mathbb{P}^5$.) Since a Grassmannian with its Plücker embedding is (in general) not a complete intersection, this is a class of Calabi-Yau manifolds for which the Batyrev-Borisov construction cannot be directly applied.

In [BCKv97], Batyrev et al. described a possible mirror construction for Calabi-Yau threefolds of this type by combining the Batyrev-Borisov construction with so-called conifold transitions. Briefly, a conifold transition involves degenerating a Calabi-Yau threefold X to a singular Calabi-Yau manifold X_0 with isolated nodes. (Here, a node refers to a singularity that locally looks like the singularity at the origin in the hypersurface $xy = zw$ in $\mathbb{C}^4$.) Then the nodes are resolved via a “small resolution” of singularities, which replaces each node with a $\mathbb{P}^1$. This produces another Calabi-Yau manifold Y , and X and Y are said to be connected by a conifold transition.

A long-standing idea in mirror symmetry, due to Morrison in [Mor99], was that if two Calabi-Yaus were related by a conifold transition, their mirrors should also be related by a conifold transition. Suppose we can perform a conifold transition on X to get a Calabi-Yau Y which is a complete intersection in a toric variety. Then the mirror of Y , Y^* , is known from the Batyrev-Borisov construction, and if we can perform a conifold transition on Y^* then theoretically the resulting Calabi-Yau should be a mirror for X . Since we never applied the Batyrev-Borisov construction to X , X need not be a complete intersection in a toric variety.

[BCKv97] conjectures that this method can be applied to get the mirror of

X when X is any complete intersection Calabi-Yau threefold in a Grassmannian. Because of technical difficulties with performing a conifold transition on the mirror manifold Y^* , [BCKv97] only fully carried out the construction in some low-dimensional cases. However, they show how to obtain Y and Y^* in all cases.

The conifold transition in [BCKv97] was given by degenerating the ambient Grassmannian $G(k, n)$ to a toric variety called $P(k, n)$. This degeneration was first described by Sturmfels in [Stu95], using Gröbner basis methods. To try to construct a toric degeneration for such a Calabi-Yau, it makes sense to also use the degeneration of $G(k, n)$ to $P(k, n)$. However, the equations for the generic hypersurfaces must also degenerate, because otherwise the singular fiber will not be a union of toric varieties. In the simplest example, that of a generic quartic intersected with $G(2, 4) \subseteq \mathbb{P}^5$, the degeneration used by Batyrev et al. is

$$tz_0z_1 - z_2z_3 + z_4z_5 = 0$$

$$f = 0$$

where $z_0, \dots, z_5$ are the coordinates on $\mathbb{P}^5$, t is the degeneration parameter, and f is a generic quartic. The first equation gives the degeneration of $G(2, 4)$ to $P(2, 4)$, since $P(2, 4) \subseteq \mathbb{P}^5$ is defined by $z_2z_3 = z_4z_5$. To change this to a toric degeneration, the equation $f = 0$ is changed to

$$tf + z_0z_1z_2z_3 = 0.$$

Here, the singular fiber $t = 0$ is evidently a union of toric varieties, and I was able to prove that the “locally toric” condition (2) of Definition 1.3 also holds, so this is indeed a toric degeneration. However, the dual intersection complex is not simple. The most serious problem is that this degeneration is indistinguishable from the one for a singular Calabi-Yau hypersurface in $P(2, 4)$, and singular Calabi-Yau threefolds cannot have simple degenerations. To come up with a simple degeneration, the methods of [Gro05] must be used. Of course, since [Gro05] deals exclusively with the Batyrev-Borisov construction, the approach needs to be modified in several ways.

The main purpose of this thesis is to describe a construction that gives a simple degeneration for the case of a quartic hypersurface in $G(2, 4)$. A simple degenera-

tion of this Calabi-Yau is already known by the methods of [Gro05], since it can be deformed to a generic $(2,4)$ complete intersection in $\mathbb{P}^5$, but the new degeneration is quite different, being based on the degeneration of $G(2,4)$ to $P(2,4)$. In Chapter 5, we discuss how a similar approach might be used to give simple degenerations of Calabi-Yau manifolds in higher dimensional Grassmannians that are not complete intersections.

The main difficulty is dealing with the smoothing of the conifold singularities that occur in a quartic hypersurface in $P(2,4)$. The methods of [Gro05] produce a degeneration that is very close to being simple, except for a small part of the dual intersection complex corresponding to these singularities. The polytopes used to define simplicity are squares in this region, whereas only simplices are allowed by the definition of simplicity. To deal with this, we define the degeneration in a toric variety associated to two, rather than one, MPCP functions h and h' . h and h' differ only on the lattice points associated to the monomials $z_0^3 z_1$, $z_0^2 z_1^2$ and $z_0 z_1^3$, which represents the smoothing of $P(2,4) \subseteq \mathbb{P}^5$ given by the equation

$$t^{n_1} z_0^2 + t^{n_2} z_0 z_1 + t^{n_3} z_1^2 - z_2 z_3 + z_4 z_5 = 0.$$

Here the n_i are some positive integers coming from h and h' . This can be viewed as a more complicated version of the degeneration of $G(2,4)$ into $P(2,4)$ given above. Given certain conditions on h and h' , this will force the monodromy squares to be broken up into horizontal and vertical line segments, which is allowed in a simple dual intersection complex.

There are some technical difficulties we have to deal with in order to prove that our degeneration is toric. Most of them are caused by the fact that the total space of the degeneration $\mathcal{X}$ does not seem to be a complete intersection in a toric variety (at least in any natural way), unlike the degenerations in [Gro05]. This makes proving the properties of the singular fiber and proving normality of $\mathcal{X}$ messier and more difficult than one would like. The same problem could cause difficulties in higher dimensional cases, although perhaps easier proofs can be found.

There are many possible directions for future research. One would be to simply carry out the construction in higher dimensional cases, as mentioned above and discussed in Chapter 5. Another would be to try to use the dual intersection

complexes to better understand the conifold transitions described by [BCKv97]. My advisor Mark Gross has described what the dual intersection complexes should locally look like for Calabi-Yau threefolds related by a conifold transition, and their mirrors. This could be used to better understand the mirrors of Calabi-Yau manifolds in Grassmannians, or perhaps help prove that the conifold transitions predicted by [BCKv97] exist.

Chapter 2

Preliminaries

Later on we will need the following lemma concerning toric varieties:

Proposition 2.1. *Let $C \subseteq M_{\mathbb{R}}$ be a cone, and let $X = \text{Spec } \mathbb{C}[C \cap M]$ be the corresponding affine toric variety, with its open torus $T = \text{Spec } \mathbb{C}[M]$. Let $m_1, \dots, m_n$ and $m'_1, \dots, m'_n$ be elements of M such that $m_1 - m'_1, \dots, m_n - m'_n$ form a $\mathbb{Z}$ -basis for $M \cap \text{Span}_{\mathbb{R}}(m_1 - m'_1, \dots, m_n - m'_n)$. We consider the subset $Y \subseteq T$ consisting of points satisfying all n equations $z^{m_1} = z^{m'_1}, \dots, z^{m_n} = z^{m'_n}$ of regular functions on T , and let $\bar{Y}$ be its closure in X in the Zariski topology. Let $\pi : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/\text{Span}_{\mathbb{R}}(m_1 - m'_1, \dots, m_n - m'_n)$ be the quotient map, and let $i : \text{Spec } \mathbb{C}[\pi(C) \cap \pi(M)] \rightarrow X$ be the map induced by the ring homomorphism $\pi_* : \mathbb{C}[C \cap M] \rightarrow \mathbb{C}[\pi(C) \cap \pi(M)]$. Then*

- a) the closure of the image of i is equal to $\bar{Y}$;*
- b) The following conditions are equivalent:*
 - 1. i is a closed embedding;*
 - 2. $\bar{Y}$ is a normal variety;*
 - 3. $\pi(C \cap M) = \pi(C) \cap \pi(M)$.*

Proof. For part (a), the closure of the image of i consists of points in the subvariety of X defined by the kernel of the ring homomorphism π_* (the kernel is prime since its image lies in a ring with no zero divisors). So we must show that the kernel of the ring homomorphism π_* , the ideal defining the image of i , is equal to the ideal $I(\bar{Y})$ of functions vanishing on $\bar{Y}$ in $\mathbb{C}[C \cap M]$. That $I(\bar{Y})$ lies in the kernel is clear from

the definition of π_* . For the other direction, suppose $\pi_*(a_1 z^{j_1} + \cdots + a_k z^{j_k}) = 0$. This will happen if and only if the sub-sums of $a_1 z^{j_1} + \cdots + a_k z^{j_k}$, consisting of all monomials z^{j_i} for which $\pi(j_i)$ is a given monomial, map to zero. Thus it suffices to show that these sub-sums are in $I(\overline{Y})$.

First of all we claim that such a linear combination can be written as a sum

$$\sum_{b=1}^c a_b (z^{j_b} - z^{j'_b}).$$

for some $j_1, \dots, j_c$ and $j'_1, \dots, j'_c \in M$. This can be shown using a straightforward induction argument. Then we just have to show that if $j, j' \in C \cap M$ are such that $\pi(j) = \pi(j')$, then $z^j - z^{j'} \in I(\overline{Y})$. We write $j - j'$ as a $\mathbb{Z}$ -linear combination $c_1(m_1 - m'_1) + c_2(m_2 - m'_2) + \cdots + c_n(m_n - m'_n)$. Then we can obtain the equation $z^{j-j'} = 1$ in $\mathbb{C}[M]$ by multiplying and taking powers of the equations $z^{m_1 - m'_1} = 1, \dots, z^{m_n - m'_n} = 1$, so points on Y must satisfy $z^{j-j'} = 1$. Multiplying both sides by $z^{j'}$ gives $z^j = z^{j'}$ as needed.

For part (b), i is a closed embedding if and only if the ring map π_* is surjective, which clearly is equivalent to the condition that $\pi(C \cap M) = \pi(C) \cap \pi(M)$. If i is a closed embedding, then condition 2 that $\overline{Y}$ is normal is satisfied, since then $\overline{Y} \cong \text{Spec } \mathbb{C}[\pi(C) \cap \pi(M)]$ and toric varieties are normal. Conversely, if $\overline{Y}$ is normal then the ring of functions on $\overline{Y}$ must be normally closed in its quotient field. But the ring of functions on $\overline{Y}$ is isomorphic to the image of π_* , and the normal closure of the image of π_* in its quotient field (whether $\overline{Y}$ is normal or not) must be all of $\mathbb{C}[\pi(C) \cap \pi(M)]$. Thus if $\overline{Y}$ is normal, we must have that $\pi(C \cap M) = \pi(C) \cap \pi(M)$. $\square$

Example 2.2. As an example of how part (b) of Proposition 2.1 might fail, let $C \subseteq \mathbb{R}^2$ be the cone given by the first quadrant, $C = \{(x, y) \mid x \geq 0, y \geq 0\}$, and let the toric variety X in the Proposition be given by $\text{Spec } \mathbb{C}[C \cap \mathbb{Z}^2]$, so that $X \cong \mathbb{C}^2$. Now let $m_1 = (0, 2)$ and $m'_1 = (3, 0)$. Because $m_1 - m'_1 = (-3, 2)$ is a primitive integral vector, it is a generator for the intersection of $\mathbb{Z}^2$ with $\text{Span}_{\mathbb{R}}(-3, 2)$, so the hypothesis in the first part of the Proposition is satisfied.

One can check that the map $i : \mathbb{C} \rightarrow \mathbb{C}^2$ is given by $i(w) = (w^2, w^3)$, whose image is equal to the cusp curve $y^2 = x^3$ (where y corresponds to the monomial

$z^{(0,1)}$ and x corresponds to $z^{(1,0)}$. We see that i fails to be a closed embedding and its image $\overline{Y}$ is non-normal. To see that the third condition about surjectivity on lattice points fails, note that if π is the map $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 / \text{Span}_{\mathbb{R}}(-3, 2)$, then $\pi(-1, 1)$ is a lattice point in the quotient lattice, and contained in $\pi(C)$, since $(-1, 1) - (-3, 2)/2 = (1/2, 0) \in C$, but no lattice point in C maps to $\pi(-1, 1)$ under π .

For an example where all the conditions of part (b) hold, just choose $m_1 = (0, 1)$ and $m'_1 = (1, 0)$. Then i will be the diagonal map from $\mathbb{C}$ into $\mathbb{C}^2$.

Let $M = \mathbb{Z}^5$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Also let $e_1, e_2, \dots, e_5$ be the unit vectors in $\mathbb{Z}^5$. Let L be a lattice polytope in $M_{\mathbb{R}}$ representing the Newton polytope of quartics on complex projective space $\mathbb{P}^5$, for instance, we can take L to be the convex hull of the origin together with the five vectors $4e_1, \dots, 4e_5$. If $\pi : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}} / \text{Span}(e_2 + e_3 - e_4 - e_5)$ is the projection, then $\pi(L)$ is the reflexive polytope for the toric variety $P(2, 4)$ (although not properly centered at the origin), since $P(2, 4)$ is the zero locus of $z_2 z_3 - z_4 z_5$ in $\mathbb{P}^5$. Throughout the following, we will let Δ be a translation of L by some vector such that $\pi(\Delta)$ is the reflexive polytope for $P(2, 4)$ centered at the origin; for instance, we could take $\Delta = L - (1, 1, 1, 0, 0)$. We will write $\pi(\Delta)$ as $\Delta_{P(2,4)}$.

A few more conventions are as follows. We will abbreviate $M_{\mathbb{R}} / \text{Span}(e_2 + e_3 - e_4 - e_5)$ as $M'_{\mathbb{R}}$. Given a real-valued function h on $M'_{\mathbb{R}}$, we can pull it back via precomposition with π to a real-valued function on $M_{\mathbb{R}}$. This function will be written as π^*h . When P is a polytope in a real vector space V (for us, typically $M_{\mathbb{R}}$ or $M'_{\mathbb{R}}$) and h is a real valued function on P , we write $\tilde{P}(h)$ for the set

$$\{(v, t) \mid v \in P, t \geq h(v)\} \subseteq V \oplus \mathbb{R},$$

the vertical rays over the graph of h restricted to P . If h is piecewise linear, then $\tilde{P}(h)$ is a (noncompact) polytope. In this case, we refer to a face f of $\tilde{P}(h)$ as “vertical” if $f + (0, r) \subseteq f$ for any $r \in \mathbb{R}$ nonnegative. All other faces will be referred to as “lower” faces.

Lemma 2.3. *Let h be any convex real-valued piecewise linear function on $M'_{\mathbb{R}}$. Then the maximal proper faces of $\tilde{\Delta}(\pi^*h)$ consist of:*

1. Vertical faces over the maximal proper faces of Δ , of the form

$$\{(v, t) \mid v \in f, t \geq h(v)\}$$

where f is a maximal proper face of Δ ;

2. Inverse images of maximal lower faces of $\tilde{\Delta}_{P(2,4)}(h)$; that is, sets of the form

$$\tilde{\Delta}(\pi^*h) \cap (\pi \oplus Id)^{-1}(f)$$

where f is a maximal lower face of $\tilde{\Delta}_{P(2,4)}(h)$.

Proof. Clearly, the maximal lower faces of $\tilde{\Delta}(\pi^*h)$ will be the graphs of π^*h over maximal linear domains of π^*h . Since π is linear, these maximal linear domains will just be sets of the form $\{v \mid v \in \Delta, \pi(v) \in F\}$ where F is a maximal linear domain of h . As the graph of h over any such region F is a maximal face of $\tilde{\Delta}_{P(2,4)}(h)$, like f in (2) of the lemma, this shows that maximal lower faces of $\tilde{\Delta}(\pi^*h)$ are exactly the sets in (2). $\square$

We can use the preceding results to describe the rays in the fan $\Sigma(\tilde{\Delta}(\pi^*h))$ in $N_{\mathbb{R}} \oplus \mathbb{R}$, dual to $\tilde{\Delta}(\pi^*h)$ (where $N_{\mathbb{R}}$ is as usual the dual space to $M_{\mathbb{R}}$). As in [Gro05], for h piecewise linear convex on $M'_{\mathbb{R}}$ (or any other real vector space), we define

$$\nabla^h = \{n \in N'_{\mathbb{R}} \mid \langle m, n \rangle \geq -h(m) \forall m \in M'_{\mathbb{R}}\}$$

the Newton polytope of h , where $N'_{\mathbb{R}}$ is the dual space to $M'_{\mathbb{R}}$.

Because M' is the quotient of M under the map π , we have an inclusion $i : N' \rightarrow N$ dual to π , and after tensoring with $\mathbb{R}$, $i : N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$. We also use C for the cone over a set with vertex the origin.

In [Gro05], the fan in $N'_{\mathbb{R}} \oplus \mathbb{R}$ dual to the polytope $\tilde{\Delta}_{P(2,4)}(h)$ is completely described. The “horizontal” rays in the fan are of the form $C((r, 0))$, where $r \in N'_{\mathbb{R}}$ generates a ray in the fan for $P(2, 4)$. These are dual to maximal proper vertical faces of $\tilde{\Delta}_{P(2,4)}(h)$. The other rays are of the form $C((v, 1))$ where v is a vertex of ∇^h , which are dual to the maximal lower faces. An exactly analogous statement holds for $\tilde{\Delta}(\pi^*h)$, and the Newton polytope ∇^{π^*h} is simply the inclusion of ∇^h into $N_{\mathbb{R}}$ via the map $i : N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$. Thus we have:

Proposition 2.4. *Suppose that h is an integral convex piecewise linear function. The rays in the fan $\Sigma(\tilde{\Delta}(\pi^*h))$, written as cones over their primitive integral generators, consist of:*

1. $C((r, 0))$ where $r \in N_{\mathbb{R}}$ is a primitive generator of a ray in the fan dual to Δ , i.e., the fan for $\mathbb{P}^5$. These are the rays dual to the maximal proper vertical faces of $\tilde{\Delta}(\pi^*h)$.

2. $C((i(v), 1))$ where v is a vertex of ∇^h . These are the rays dual to the maximal lower faces of $\tilde{\Delta}(\pi^*h)$. (v is integral by integrality of h , and since i is an integral linear map, $i(v)$ is an integral vector. Thus $(i(v), 1)$ is a primitive integral vector.)

In general, when we have a fan in $N_{\mathbb{R}} \oplus \mathbb{R}$ or $N'_{\mathbb{R}} \oplus \mathbb{R}$, we will describe rays lying in $N_{\mathbb{R}}$ or $N'_{\mathbb{R}}$ as “horizontal” rays, and upward-pointing rays (like in case 2 of the Proposition) as “upper” rays.

We will also often use p_1 to denote the canonical projection onto the first component of a direct sum (such as $M \oplus \mathbb{Z}$, $M_{\mathbb{R}} \oplus \mathbb{R}$, etc.).

Chapter 3

Main Construction

Now we describe the main object of our consideration, a degeneration of a certain Calabi-Yau threefold in $\mathbb{P}^5$. The ambient space of the degeneration will be the toric variety given by the polytope $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ (plus being Minkowski sum). From here on this toric variety will be denoted as $X(h, h')$. h and h' are two integral convex piecewise linear functions which will differ on a certain subset of lattice points in $\Delta_{P(2,4)}$. We let t denote the monomial $(0, 1) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$. The same argument as in Proposition 3.4 of [Gro05] shows that t defines a regular function on $X(h, h')$, giving a map $p : X(h, h') \rightarrow \mathbb{C}$, and all fibers of this map except the singular fiber are isomorphic to $\mathbb{P}^5$.

Recall that under our conventions $P(2, 4)$ is given by the equation $z_2 z_3 = z_4 z_5$ in $\mathbb{P}^5$, where $z_0, \dots, z_5$ are homogeneous coordinates on $\mathbb{P}^5$.

Construction 3.1. Let h and h' be integral convex piecewise linear functions on $M'_{\mathbb{R}}$ with the following properties:

1. $h' \leq h$, and if $h'(m) \neq h(m)$ for $m \in \Delta_{P(2,4)} \cap M'$, then m corresponds to one of the monomials $z_0^3 z_1$, $z_0^2 z_1^2$, $z_0 z_1^3$. (Here we use the correspondence between lattice points in $\Delta_{P(2,4)}$ and global sections of the line bundle of quartics on $P(2, 4)$.)

2. $\tilde{\Delta}(\pi^*h)$ and $\tilde{\Delta}(\pi^*h')$ are lattice polytopes.

Now we let n be any positive integer, let $c_m \in \mathbb{C}$ for $m \in \Delta \cap M$ be a set of generic coefficients, and let Y be the set of points in the open torus of $X(h, h')$ satisfying

$$z_0 z_1 z_2 z_3 + t^n \left(\sum_{m \in \Delta \cap M} c_m t^{\pi^* h(m)} z^m \right) = 0$$

and

$$z_0 z_1 z_4 z_5 + t^n \left(\sum_{m \in \Delta \cap M} c_m t^{\pi^* h'(m)} z^m \right) = 0.$$

We let $\mathcal{X} = \overline{Y}$, the closure of Y in the Zariski topology of $X(h, h')$, and by restricting p we obtain a degeneration $p : \mathcal{X} \rightarrow \mathbb{C}$.

Lemma 3.2. *If $h = h'$, then the generic fiber of p in Construction 3.1 is a quartic hypersurface in $P(2, 4)$. If $h \neq h'$ then the generic fiber is a smooth quartic hypersurface in a deformation of $P(2, 4)$.*

Proof. Subtracting the second equation defining Y from the first equation, and recalling that $h = h'$ except possibly on the monomials in the set $S = \{z_0^3 z_1, z_0^2 z_1^2, z_0 z_1^3\}$, we obtain:

$$z_0 z_1 z_2 z_3 - z_0 z_1 z_4 z_5 + t^n \left(\sum_{z^m \in S} c_m (t^{\pi^* h(m)} - t^{\pi^* h'(m)}) z^m \right) = 0.$$

As Y lies in the big torus of $X(h, h')$, $z_0 z_1$ is invertible on Y and we may factor it out to obtain

$$z_2 z_3 - z_4 z_5 + t^n (c_0 (t^{k_0} - t^{k'_0}) z_0^2 + c_1 (t^{k_1} - t^{k'_1}) z_0 z_1 + c_2 (t^{k_2} - t^{k'_2}) z_1^2) = 0$$

where k_i and k'_i for $i = 0, 1, 2$ are the integers arising from h and h' . We see that if $h = h'$ then this equation collapses to $z_2 z_3 = z_4 z_5$ and thus all generic fibers lie in $P(2, 4)$, with the generic quartic given by either of the two equations defining Y in Construction 3.1. On the other hand, if $h \neq h'$ then for a generic value of t the above equation will have at most one singular point, which will be avoided by a generic quartic, and thus its intersection with a generic quartic will be smooth. $\square$

We will often abbreviate the long quadratic expression in parentheses in the above equation as d , so the equation can be written as

$$z_2 z_3 - z_4 z_5 + t^n d = 0.$$

We will also abbreviate the sums in the two defining equations of Construction 3.1 as f and g , so they can be written as $t^n f + z_0 z_1 z_2 z_3 = 0$ and $t^n g + z_0 z_1 z_4 z_5 = 0$.

We note that Construction 3.1 is very similar to, and was motivated by, the toric degenerations in [Gro05]. In the Batyrev-Borisov construction, a codimension two Calabi-Yau manifold in the toric variety $\mathbb{P}(\Delta)$ comes from a nef partition $\Delta_r + \Delta_s = \Delta$ of compact lattice polytopes. Additionally, for the construction in [Gro05] we have h_r and h_s convex piecewise linear functions on Δ_r and Δ_s , respectively, such that $\tilde{\Delta}_r(h_r)$ and $\tilde{\Delta}_s(h_s)$ are lattice polytopes. The toric degeneration is then given by the equations

$$tr + r_0 = 0$$

$$ts + s_0 = 0$$

in the toric line bundles induced by $\tilde{\Delta}_r(h_r)$ and $\tilde{\Delta}_s(h_s)$ on $\mathbb{P}(\tilde{\Delta}_r(h_r) + \tilde{\Delta}_s(h_s))$. (t^n could also be substituted for t with similar results.) Here r and s are generic sections of $\tilde{\Delta}_r(h_r)$ and $\tilde{\Delta}_s(h_s)$, and r_0 and s_0 are distinguished sections corresponding to certain monomials. The distinguished sections define piecewise linear functions φ_r and φ_s on the fan for $\mathbb{P}(\tilde{\Delta}_r(h_r) + \tilde{\Delta}_s(h_s))$, and t defines a linear function which is just projection to the second component of $N_{\mathbb{R}} \oplus \mathbb{R}$. The dual intersection complex of the degeneration is then the set of solutions to $t = \varphi_r = \varphi_s = 1$ in $N_{\mathbb{R}} \oplus \mathbb{R}$ (or $t = 1$, $\varphi_r = \varphi_s = n$ if t^n instead of t is used in the defining equations).

In Construction 3.1, very similar equations are used, with Δ_r and Δ_s both equal to the Newton polytope of quartics on $\mathbb{P}^5$ (this is of course not a nef partition), and h and h' playing the role of h_r and h_s . The distinguished sections r_0 and s_0 correspond to $z_0 z_1 z_2 z_3$ and $z_0 z_1 z_4 z_5$. Although the monomials the two defining equations for Y lie in the polytopes $\tilde{\Delta}(\pi^* h)$ and $\tilde{\Delta}(\pi^* h')$, so that the equations could be interpreted as sections of line bundles and not just functions on the open torus, this would result in undesired extra components in the degeneration because of the presence of $z_0 z_1$ in both distinguished sections $z_0 z_1 z_2 z_3$ and $z_0 z_1 z_4 z_5$. Nonetheless, it is still useful to keep this similarity in mind, because the distinguished sections define piecewise linear functions φ and φ' on the fan for $X(h, h')$, and we will show that the dual intersection complex is still the solution set of $\varphi = \varphi' = n$, $t = 1$ in the fan, just as in the Batyrev-Borisov case.

The main goal of what follows will be to prove the following theorem:

Theorem 3.3. *There exists a positive integer n , and height functions h and h' satisfying the conditions of Construction 3.1, such that the degeneration $p : \mathcal{X} \rightarrow \mathbb{C}$ is toric, and its dual intersection complex is simple. The dual intersection complex is given by the solutions to $\varphi = \varphi' = n$, $t = 1$ in $N_{\mathbb{R}} \oplus \mathbb{R}$, where φ and φ' are piecewise linear functions on the fan for $X(h, h')$, induced by the sections $z_0 z_1 z_2 z_3$ and $z_0 z_1 z_4 z_5$ of the line bundles given by $\tilde{\Delta}(\pi^* h)$ and $\tilde{\Delta}(\pi^* h')$, respectively.*

Remark 3.4. Recall from [GS06] that the dual intersection complex is defined as an *affine manifold with singularities* together with a *polyhedral decomposition* of the manifold into affine polyhedra (for the precise definitions, see chapter 1 of [GS06]). Thus, to fully specify the dual intersection complex in Theorem 3.3 we must describe the affine structure and polyhedral decomposition. Because our dual intersection complex is defined by setting piecewise linear functions equal to a constant, it is automatically a union of polyhedra, which defines the polyhedral decomposition. The interiors of the polyhedra also inherit an affine structure from $N_{\mathbb{R}} \oplus \mathbb{R}$. The remaining piece of data needed is an affine structure in a neighborhood of each vertex. This will be given later on, but roughly, it comes from modding out by the span of certain rays in the fan of $X(h, h')$ and using the affine structure on the quotient space.

Also note that like in [Gro05], there will be no self-intersections of polyhedra in the dual intersection complex, so the techniques used to deal with this can be avoided.

Next, we will establish a number of combinatorial results needed for analyzing the singular fiber, and eventually, proving that $p : \mathcal{X} \rightarrow \mathbb{C}$ is a toric degeneration. Recall that we have a linear quotient map $\pi \oplus \text{Id} : M_{\mathbb{R}} \oplus \mathbb{R} \rightarrow M'_{\mathbb{R}} \oplus \mathbb{R}$. Because Minkowski sum commutes with linear maps,

$$(\pi \oplus \text{Id})(\tilde{\Delta}(\pi^* h) + \tilde{\Delta}(\pi^* h')) = \tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h'). \quad (3.1)$$

It is clear that $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ will consist of the vertical rays over the graph of a piecewise *affine* linear (not necessarily piecewise linear) function defined on

$2\Delta_{P(2,4)}$. We will call this function $f_{h,h'}$. The following result is in some ways similar to Proposition 3.5 of [Gro05], showing that the Minkowski sum of the two polytopes in which the defining equations lie has desirable properties.

Lemma 3.5. $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h') = \tilde{2\Delta}(\pi^*f_{h,h'})$.

Proof. If v is any point in 2Δ , then the height of the lower boundary of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ at v is given by

$$\min \{ \pi^*h(v_1) + \pi^*h'(v_2) \mid v_1, v_2 \in \Delta, v_1 + v_2 = v \}.$$

Since $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ is a lattice polytope, it suffices to show that for $v \in 2\Delta \cap M$, the above minimum is equal to $\pi^*f_{h,h'}(v)$. To complete the proof, we will need some more information about the function $f_{h,h'}$:

Claim. If x is any point in $\Delta_{P(2,4)}$, then $f_{h,h'}(x) = h'(x)$.

Proof. As above, we have that

$$f_{h,h'}(x) = \min \{ h(x_1) + h'(x_2) \mid x_1, x_2 \in \Delta_{P(2,4)}, x_1 + x_2 = x \}.$$

If we take $x_1 = 0$ and $x_2 = x$ then we get $h(0) + h'(x) = h'(x)$, so the minimum is no larger than $h'(x)$. On the other hand, suppose we have any $x_1, x_2 \in \Delta_{P(2,4)}$ such that $x_1 + x_2 = x$. Then $h(x_1) + h'(x_2) \geq h'(x_1) + h'(x_2) \geq h'(x_1 + x_2) = h'(x)$ (the first inequality is because $h \geq h'$ and the second is because h' is upper convex). Thus, the minimum is equal to $h'(x)$ as desired. $\square$

Continuing with the main proof, if v is a lattice point in 2Δ , then $\pi(v)$ is a lattice point in $2\Delta_{P(2,4)}$. Since $\Delta_{P(2,4)}$ is reflexive this means that either $\pi(v) \in \Delta_{P(2,4)}$ or $\pi(v) \in 2\partial\Delta_{P(2,4)}$, where $\partial\Delta_{P(2,4)}$ is the boundary.

First suppose that we are in the latter case, and also that $\pi(v)$ is contained in one of the maximal boundary faces of $2\Delta_{P(2,4)}$ corresponding to the divisors $z_0 = 0$ or $z_1 = 0$. Because $h = h'$ on the faces $z_0 = 0$ and $z_1 = 0$ of $\Delta_{P(2,4)}$ (they do not contain any of the monomials on which h and h' differ), it follows that the height of the lower boundary of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ at $\pi(v)$ (i.e. $\pi^*f_{h,h'}(v)$) is just $2h(\pi(v)/2) = 2h'(\pi(v)/2)$. Then if we observe that $(v/2, \pi^*h(v/2)) \in \tilde{\Delta}(\pi^*h)$, $(v/2, \pi^*h'(v/2)) \in \tilde{\Delta}(\pi^*h')$ and their sum $(v, 2h(\pi(v)/2)) \in \tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$,

we get that the height of the lower boundary of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ at v is also $2h(\pi(v)/2)$ as needed.

Now suppose that $\pi(v)$ is still contained in $2\partial\Delta_{P(2,4)}$ but contained in one of the boundary faces $z_i = z_j = 0$, $i \in \{2, 3\}$ and $j \in \{4, 5\}$. Because the restriction of π to 2Δ is bijective on all these faces, $\pi(v)$ can have only one preimage in 2Δ , namely v . The height of the lower boundary of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ at v is therefore forced to be the same as the lower boundary of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ at $\pi(v)$.

Finally, we must deal with the case where $\pi(v)$ is not on the boundary of $2\Delta_{P(2,4)}$ and thus a lattice point in $\Delta_{P(2,4)}$. By the Claim, we know that $\pi^*f_{h,h'}(v) = \pi^*h'(v)$. It suffices to show that $v = v_1 + v_2$, $v_1, v_2 \in \Delta$, where $\pi(v_1) = 0$, since then we can take $(v_2, \pi^*h'(v_2)) \in \tilde{\Delta}(\pi^*h')$ and $(v_1, 0) \in \tilde{\Delta}(\pi^*h)$ which will sum to the desired $(v, \pi^*h'(v))$. (This shows that the lower boundary of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ is no higher than $\pi^*h'(v)$ at v , while we already know it is no lower than $\pi^*h'(v)$ by the combination of the Claim and equation (3.1).) To do this, start by writing $v = u_1 + u_2$ with $u_1, u_2 \in \Delta \cap M$. $\pi(u_1), \pi(u_2) \in \Delta_{P(2,4)} \cap M$, and if either one of $\pi(u_1)$ or $\pi(u_2)$ equals 0 then we are done. If not, then $\pi(u_1), \pi(u_2) \in \partial\Delta_{P(2,4)}$ and must not be contained in any common boundary face of $\Delta_{P(2,4)}$, since then $\pi(u_1) + \pi(u_2)$ would be on the boundary of $2\Delta_{P(2,4)}$.

At this point it is helpful to think of z^{u_1} and z^{u_2} as quartic monomials in $z_0, \dots, z_5$ (which can be done unambiguously since they lie in Δ , the Newton polytope of quartics on $\mathbb{P}^5$). Then $z^v = z^{u_1+u_2}$ is their product, and it suffices to show that either $z_0z_1z_2z_3$ or $z_0z_1z_4z_5$ factors into $z^{u_1+u_2}$, as these both map to the origin under π and then v can be written in the desired form. Suppose by contradiction that the variable z_i with $i = 0$ or 1 does not appear in $z^{u_1+u_2}$. Then neither z^{u_1} nor z^{u_2} contain the variable z_i and $\pi(u_1)$ and $\pi(u_2)$ are contained in the common face $z_i = 0$, a contradiction. Thus z_0z_1 divides $z^{u_1+u_2}$. Now suppose a variable z_i with $i = 2, 3$ does not appear in $z^{u_1+u_2}$ and also a variable z_j with $j = 4, 5$ does not appear in $z^{u_1+u_2}$. Then $\pi(u_1)$ and $\pi(u_2)$ are both contained in the face $z_i = z_j = 0$, a contradiction. It follows that either z_2z_3 or z_4z_5 factors into $z^{u_1+u_2}$, and we have the result. $\square$

Remark 3.6. Even if h and h' are integral, it is not clear whether the affine linear

functions making up $f_{h,h'}$ will have integral linear parts. However, a straightforward argument shows that if we scale h and h' by an integer m , we will get that $\tilde{\Delta}_{P(2,4)}(mh) + \tilde{\Delta}_{P(2,4)}(mh') = 2\tilde{\Delta}_{P(2,4)}(mf_{h,h'})$. Assuming h and h' have been scaled up by the same large integer m will thus guarantee integrality of all the affine linear functions making up $f_{h,h'}$, as well as $\pi^*f_{h,h'}$. We will need this property later on, so from now on we refer to h, h' satisfying this property, as well as the conditions of Construction 3.1, as a “good” choice of h and h' .

We will also need the closely related result:

Corollary 3.7. *Suppose that v_1 is a vertex of $\tilde{\Delta}(\pi^*h')$, v_2 is a vertex of $\tilde{\Delta}(\pi^*h)$, and $v_1 + v_2$ is a vertex of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$. Then at least one of the following holds:*

1. $v_1 = v_2$.
2. When projected to M , one of the v_i is one of the monomials $z_0^3z_1$, $z_0^2z_1^2$, or $z_0z_1^3$, and the other is contained in one of the codimension 2 boundary faces $z_i = z_j = 0$ of Δ , $i \in \{2, 3\}$, $j \in \{4, 5\}$.
3. One of the v_i is such that $(\pi \oplus Id)(v_i) = 0$.

Proof. Write u_i for $p_1(v_i)$, where p_1 is projection onto M . Suppose that $\pi(u_1)$ and $\pi(u_2)$ lie in the same boundary face of $\Delta_{P(2,4)}$. If it is one of the faces $z_i = 0$, $i = 0$ or 1, then u_1 and u_2 are both contained in the corresponding face $z_i = 0$ of Δ . Because $h = h'$ on these faces, we get that $v_1 = v_2$, giving the first possibility.

Now suppose that $\pi(u_1)$ and $\pi(u_2)$ still lie in one of the same boundary faces of $\Delta_{P(2,4)}$, but it is one of the faces $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$. If either one of $\pi(u_1)$ or $\pi(u_2)$ is $z_0^3z_1$, $z_0^2z_1^2$, or $z_0z_1^3$, then Case 2 holds and we are done. If this is not the case but $\pi(u_1) = \pi(u_2)$, then we are in Case 1, since then h and h' are equal, and $u_1 = u_2$ (because π is a bijection on these faces). Thus suppose that $\pi(u_1) \neq \pi(u_2)$ but neither is equal to $z_0^3z_1$, $z_0^2z_1^2$, or $z_0z_1^3$. Consider the points $a_1 = ((u_1 + u_2)/2, \pi^*h'(u_1 + u_2)/2) \in \tilde{\Delta}(\pi^*h')$ and $a_2 = ((u_1 + u_2)/2, \pi^*h(u_1 + u_2)/2) \in \tilde{\Delta}(\pi^*h)$. Then

$$a_1 + a_2 = (u_1 + u_2, (\pi^*h'(u_1 + u_2) + \pi^*h(u_1 + u_2))/2).$$

We claim that

$$(\pi^*h'(u_1 + u_2) + \pi^*h(u_1 + u_2))/2 \leq \pi^*h'(u_1) + \pi^*h(u_2).$$

If true, this is a contradiction because then $v_1 + v_2$ is not a vertex, as it either lies above the lower boundary or can be written as $a_1 + a_2$ for the different pair of points a_1, a_2 . Now, $\pi^*h'(u_1 + u_2) \leq \pi^*h'(u_1) + \pi^*h'(u_2)$ by convexity and similarly for $\pi^*h(u_1 + u_2)$. So we have

$$(\pi^*h'(u_1 + u_2) + \pi^*h(u_1 + u_2))/2 \leq (\pi^*h'(u_1) + \pi^*h'(u_2) + \pi^*h(u_1) + \pi^*h(u_2))/2.$$

By assumption $\pi^*h'(u_1) = \pi^*h(u_1)$ and $\pi^*h'(u_2) = \pi^*h(u_2)$, so after using this fact and simplifying, the RHS becomes $(2\pi^*h'(u_1) + 2\pi^*h(u_2))/2$ and we have the result.

Now suppose that $\pi(u_1)$ and $\pi(u_2)$ are contained in different boundary faces of $\Delta_{P(2,4)}$. Then $\pi(u_1 + u_2) \in \Delta_{P(2,4)}$. This is exactly the situation that we dealt with in the last part of the proof of Lemma 3.5, and essentially the fact that Case 3 holds was already established there. By the Claim in the proof of Lemma 3.5, the height of the lower boundary of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ at $\pi(u_1 + u_2)$ is $\pi^*h'(u_1 + u_2)$, as is the height of the lower boundary of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ at $u_1 + u_2$. The last part of the proof of Lemma 3.5 shows that there exists $t_1, t_2 \in \Delta$ with $t_1 + t_2 = u_1 + u_2$ and $\pi(t_2) = 0$. Then we have that $(t_1, h'(t_1)) \in \tilde{\Delta}(\pi^*h')$, $(t_2, 0) \in \tilde{\Delta}(\pi^*h)$, and $(t_1, \pi^*h'(t_1)) + (t_2, 0) = (t_1 + t_2, \pi^*h'(t_1 + t_2)) = v_1 + v_2$. Since the decomposition of vertices of a Minkowski sum into a sum of vertices of the summands is unique, we have $v_1 = t_1$ and $v_2 = t_2$, which proves the result, since then $(\pi \oplus \text{Id})(v_2) = 0$. $\square$

The following strengthening of Lemma 3.5 will be needed for analyzing the singular fiber of the degeneration.

Lemma 3.8. *If $r \subseteq N_{\mathbb{R}} \oplus \mathbb{R}$ is a ray in the fan dual to $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ that does not appear in the fans dual to $\tilde{\Delta}(\pi^*h)$ or $\tilde{\Delta}(\pi^*h')$, then $r \subseteq C(\nabla^{\pi^*h} \times \{1\})$.*

Proof. Because the lower boundary of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ is defined by $\pi^*f_{h,h'}$, the composition of a function with π , it automatically follows that any rays dual to maximal lower faces will be of the form $(i \oplus \text{Id})(r')$, where i is the map dual to

π and r' is a ray dual to a maximal lower face of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$. So it suffices to show that r' is contained in $C(\nabla^h \times \{1\})$.

Note that if Δ_1 and Δ_2 are any polytopes, and f_i is a face of Δ_i , then $f_1 + f_2$ is a face of $\Delta_1 + \Delta_2$ if and only if the dual cones C_1 and C_2 are the minimal pair of cones with intersection $C_1 \cap C_2$. (If this holds, then $C_1 \cap C_2$ is the dual cone to $f_1 + f_2$.) We will be done if we can show that it is impossible for faces of $\tilde{\Delta}_{P(2,4)}(h)$ and $\tilde{\Delta}_{P(2,4)}(h')$ that are both contained in proper vertical faces to sum to a maximal lower face. This is because a face is contained in a proper vertical face iff its dual cone contains a ray of the form $r \times \{0\}$ where r is a ray in the fan for $\Delta_{P(2,4)}$. Then if two faces sum to a maximal lower face, at least one of them will not be contained in a proper vertical face and therefore its dual cone will be contained in $C(\nabla^h \times \{1\})$ or $C(\nabla^{h'} \times \{1\})$. $C(\nabla^{h'} \times \{1\}) \subseteq C(\nabla^h \times \{1\})$ because of the condition that $h' \leq h$. $C_1 \cap C_2$ will then be contained in $C(\nabla^h \times \{1\})$, as needed.

Suppose by contradiction that $f_1 + f_2$ is a maximal lower face, but both f_1 and f_2 are contained in proper vertical faces. They cannot be contained in the same proper vertical face, because then $f_1 + f_2$ would be contained in the boundary of $2\Delta_{P(2,4)}$ when projected to $M'_\mathbb{R}$ and could not be maximal dimension. Thus, let v_i be a vertex of f_i such that v_1 and v_2 are not contained in a common proper vertical face. Then $v_1 + v_2$ must be contained in $\Delta_{P(2,4)}$ when projected to $M'_\mathbb{R}$, call the projection v_3 . We must have $v_1 + v_2 = (v_3, h'(v_3))$ because $h'(v_3)$ is the height of the lower boundary of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ at v_3 by the Claim in Lemma 3.5. But we also have that $(v_3, h'(v_3)) = (v_3, h'(v_3)) + 0$. This is a contradiction, because generally if $f_1 + f_2$ is a face of $\Delta_1 + \Delta_2$, f_i a face of Δ_i , and $v_1 + v_2 \in f_1 + f_2$ with $v_i \in \Delta_i$, then $v_i \in f_i$. 0 cannot be an element of f_1 or f_2 if both are contained in proper vertical faces. $\square$

Let $Y(h, h')$ be the toric variety associated to the polytope

$$\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h').$$

Then $\pi \oplus \text{Id}$ induces a map on toric varieties

$$i : Y(h, h') \rightarrow X(h, h').$$

Proposition 3.9. *i is a closed embedding.*

Proof. For v a vertex of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$, we let C_v be the cone over $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ translated by $-v$, so that v moves to the origin. Then the affine toric varieties given by C_v , $\text{Spec } \mathbb{C}[C_v \cap (M \oplus \mathbb{Z})]$, as v ranges over all vertices constitute an affine open cover of $X(h, h')$. As in Proposition 2.1, to show i is an embedding, it suffices to show for each C_v that $(\pi \oplus \text{Id})(C_v \cap (M \oplus \mathbb{Z})) = (\pi \oplus \text{Id})(C_v) \cap (\pi \oplus \text{Id})(M \oplus \mathbb{Z})$, in other words, all lattice points in $(\pi \oplus \text{Id})(C_v)$ are images of lattice points in C_v .

After removing the singular fiber $p^{-1}(0)$, i is just the inclusion of $P(2, 4) \times \mathbb{C}^*$ into $\mathbb{P}^5 \times \mathbb{C}^*$, which we know is a closed embedding. This implies that for any lattice point $(l', m') \in (\pi \oplus \text{Id})(C_v)$, there exists a lattice point $(l, m) \in C_v$ with $\pi(l) = l'$. But by Lemma 3.5 the lower boundary of C_v will be the pullback of the lower boundary of $(\pi \oplus \text{Id})(C_v)$ by π . This implies that if m' is greater than or equal to the height of the lower boundary of $(\pi \oplus \text{Id})(C_v)$ at $\pi(l) = l'$, then $(l, m') \in C_v$ also. Since this is true for m' by definition, we have that $(l, m') \in C_v$ and $(\pi \oplus \text{Id})(l, m') = (l', m')$ as needed. $\square$

The next goal is to analyze the singular fiber and show it satisfies the properties of a toric degeneration. Eventually we will show that the singular fiber is reduced and equal to the union

$$W = \bigcup_{f \in S} i(V(f)),$$

where S is the set of lower faces $f \subseteq \tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ such that $p_1(f) \subseteq 2\partial\Delta_{P(2,4)}$, and $V(f)$ is the toric stratum associated to the face f . As a scheme, the singular fiber is by definition the subscheme of $\mathcal{X}$ given by $t = 0$. In any affine open set U_v given by the cone C_v at a vertex v of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$, if we let $I_{\mathcal{X}}$ be the defining ideal of $\mathcal{X} \cap U_v$, then the singular fiber is given by the ideal $I_{\mathcal{X}} + (t)$. Thus it suffices to show that $I_W = I_{\mathcal{X}} + (t)$ where I_W is the ideal of all functions vanishing on $W \cap U_v$.

In order to prove this we will need the following lemma, which shows that if $m \in M$ and a and b are lattice points in M representing z_2z_3 and z_4z_5 in the Newton polytope for $\mathcal{O}_{\mathbb{P}^5}(2)$, and $z^m z^a$ and $z^m z^b$ are regular on an open affine

toric subset of $\mathbb{P}^5$, then $z^m s$ is also regular, where s is any section of $\mathcal{O}_{\mathbb{P}^5}(2)$. The importance of this is that if we have a representative of $z_2 z_3 - z_4 z_5 + t^n d$ on the open torus such that the $z_2 z_3 - z_4 z_5$ part is regular on an open affine toric subset of $X(h, h')$, then the $t^n d$ part will also be regular, given that n is sufficiently high.

Lemma 3.10. *Suppose that $\Sigma \subseteq N_{\mathbb{R}}$ is the fan for $\mathbb{P}^5$, and $Q \subseteq M_{\mathbb{R}}$ is the Newton polytope of $\mathcal{O}_{\mathbb{P}^5}(2)$, well defined up to translation. Let $C \subseteq M_{\mathbb{R}}$ be the dual cone to some cone in Σ . Suppose there exists a translate of Q , $Q + x$ for $x \in M_{\mathbb{R}}$, such that C contains both $a + x$ and $b + x$, where $a \in Q$ represents $z_2 z_3$ and $b \in Q$ represents $z_4 z_5$. Then C contains $Q + x$.*

Proof. We can assume that $Q = \text{Conv}(0, 2e_1, \dots, 2e_5)$, where $e_1, \dots, e_5$ is a basis for $M \cong \mathbb{Z}^5$. Then z_0^2 corresponds to $0 \in Q$ and z_i^2 corresponds to $2e_i$ for $1 \leq i \leq 5$. The primitive generators of the rays in Σ are then $\check{e}_i$ for $1 \leq i \leq 5$ (the dual basis) and $-\check{e}_1 - \check{e}_2 - \dots - \check{e}_5$. A cone in Σ consists of the convex hull of any five or fewer of these rays, and an element of $M_{\mathbb{R}}$ is in the dual cone C iff it evaluates nonnegatively on all the rays making up the cone.

Thus suppose that $a + x$ and $b + x$ are in C . We have $a = (0, 1, 1, 0, 0)$ and $b = (0, 0, 0, 1, 1)$, and let $x = (x_1, \dots, x_5)$. If $\check{e}_i$ is a ray in the cone, then if $i \in \{1, 2, 3\}$ we have $\langle \check{e}_i, b + x \rangle = x_i$, while if $i \in \{1, 4, 5\}$ we have $\langle \check{e}_i, a + x \rangle = x_i$. Since i must be contained in one of these sets, we have $x_i \geq 0$, and $\langle \check{e}_i, q + x \rangle \geq 0$ for any $q \in Q$, since all coordinates of q are ≥ 0 in the basis $e_1, \dots, e_5$. Now suppose that $-\check{e}_1 - \check{e}_2 - \dots - \check{e}_5$ is a ray in the cone. The fact that $a + x \in C$ implies that $-x_1 - \dots - x_5 - 2 \geq 0$, so we have $\langle -\check{e}_1 - \dots - \check{e}_5, x \rangle \geq 2$. For any $q \in Q$, $\langle -\check{e}_1 - \dots - \check{e}_5, q \rangle \geq -2$, so combining these facts yields $\langle -\check{e}_1 - \dots - \check{e}_5, q + x \rangle \geq 0$, which implies the result. $\square$

First we show the containment:

Lemma 3.11. *For sufficiently high values of n , we have $I_W \subseteq I_X + (t)$.*

Proof. First we give a set of generators for I_W . The local image of the map i is given by the vanishing of all equations of the form $z^{m_1} = z^{m_2}$, where $m_1, m_2 \in C_v \cap (M \oplus \mathbb{Z})$ are such that $m_1 - m_2 = e_2 + e_3 - e_4 - e_5$, the vector corresponding to

the binomial $z_2z_3 - z_4z_5$. Thus, $z^{m_1} - z^{m_2} \in I_W$ for all such pairs m_1, m_2 . Now, the image of the map i is isomorphic to an affine toric variety by Proposition 3.9, the toric variety given by the cone $(\pi \oplus \text{Id})(C_v)$, which we abbreviate as C'_v . $W \cap U_v$ is a union of toric strata in this toric variety, so the ideal of functions vanishing on it can be generated by monomials in C'_v . Suppose we have a monomial z^m with $m \in C'_v$ that vanishes on $W \cap U_v$. Looking at the projection $p_1(m) \subseteq M'_\mathbb{R}$, first suppose it lies on the boundary of $p_1(C'_v)$. Then z^m cannot vanish on the toric strata corresponding to the boundary faces that $p_1(m)$ touches, unless it lies above the lower boundary of C'_v , in which case it is contained in the ideal (t) , by the integrality condition of Remark 3.6. Conversely, everything in (t) clearly vanishes on W , so $(t) \subseteq I_W$.

Now suppose that $p_1(m)$ lies in the interior of $p_1(C'_v)$. Then m vanishes at least once on each toric divisor represented by a ray of the dual cone $(C'_v)^*$ that lies in $N'_\mathbb{R} \subseteq N'_\mathbb{R} \oplus \mathbb{R}$ (“horizontal” rays). If we can show that there is a monomial z^s such that s evaluates to one on the primitive generators of all “horizontal” boundary rays in $(C'_v)^*$ and evaluates to zero on all its other rays, then $m - s$ is still nonnegative on all rays in $(C'_v)^*$ and therefore contained in C'_v , and $z^m = z^s z^{m-s}$, so $z^m \in (z^s)$.

The needed z^s is given by the local representative of the section associated to the lattice point at the origin, in the line bundle corresponding to the polytope $\widetilde{\Delta}_{P(2,4)}(h)$. This line bundle section vanishes precisely once on each toric divisor of $P(2,4)$. Furthermore, the piecewise linear function ν associated to it will evaluate to zero on the entire set $C(\nabla^h \times \{1\})$, since the lattice point at the origin touches each hyperplane in the graph of h . By Lemma 3.8, all non-horizontal rays in $(C'_v)^*$ are contained in $C(\nabla^h \times \{1\})$, so ν evaluates to zero on them as well. Since the linear function s is by definition equal to ν on $(C'_v)^*$, z^s is the needed monomial. If r is any lattice point in C_v such that $(\pi \oplus \text{Id})(r) = s$, then it follows that I_W is generated by the pairs $z^{m_1} - z^{m_2}$ above, together with t and z^r .

We now check that each of these generators of I_W is contained in $I_\mathcal{X} + (t)$. For t this is obvious. For z^r , just note that the defining equation $t^n f + z_0 z_1 z_2 z_3 = 0$ gives a section of the line bundle associated to the polytope $\widetilde{\Delta}(\pi^* h)$. By the definition of $\mathcal{X}$ any local representative of this section will be contained in $I_\mathcal{X}$, and since all

terms that are multiples of (t) already lie in $I_{\mathcal{X}} + (t)$, we get that the monomial associated to the remaining term $z_0 z_1 z_2 z_3 \in I_{\mathcal{X}} + (t)$. This is the desired monomial since $z_0 z_1 z_2 z_3$ maps to the zero section of $\widetilde{\Delta}_{P(2,4)}(h)$ under $\pi \oplus \text{Id}$.

The sections of the form $z^{m_1} - z^{m_2}$ require more work. Because (t) includes all monomials lying above the lower boundary of C_v (by the integrality assumption of Remark 3.6), we need only check that if $z^{m_1} - z^{m_2}$ is a pair with m_1 or m_2 lying on the lower boundary, then $z^{m_1} - z^{m_2} \in I_{\mathcal{X}} + (t)$. (In fact, by Lemma 3.5 if either one of m_1 or m_2 lie on the lower boundary then both do.)

We can choose $u_0 \in \mathbb{C}[M \oplus \mathbb{Z}]$ representing the line bundle section

$$z_2 z_3 - z_4 z_5 + t^n d$$

on $\mathbb{P}^5 \times \mathbb{C}^*$. (Recall d is defined after Lemma 3.2 and consists of monomials quadratic in z_0 and z_1 .) Then we will have $u_0 = z^{k_1} - z^{k_2} + t^n d_0$ where $k_1 - k_2 = e_2 + e_3 - e_4 - e_5$, and d_0 is the representative of d . Now if m_1, m_2 is a pair as above then there exists $m \in M \oplus \mathbb{Z}$ such that $z^m(z^{k_1} - z^{k_2} + t^n d_0) = z^{m_1} - z^{m_2} + z^m t^n d_0$. Since z^{m_1} and z^{m_2} are regular on U_v , we can apply Lemma 3.10. This means that as long as the powers of t in $z^m t^n d_0$ are sufficiently high that its monomials do not lie below the lower boundary of C_v , $z^m u_0$ will define a regular function on U_v and by the definition of $\mathcal{X}$ as the closure of a set in the open torus, we will have that $z^m u_0 \in I_{\mathcal{X}}$. Furthermore, if the monomials of $z^m t^n d_0$ lie above the lower boundary of C_v , they will be contained in the ideal (t) and then we will have that the remaining part of $z^m u_0$, $z^{m_1} - z^{m_2}$, is contained in $I_{\mathcal{X}} + (t)$. It suffices to show that there is a value of n that will accomplish this for any such z^m .

The lower boundary of C_v is defined by the graphs of real-valued linear functions on a finite number of cones in $M_{\mathbb{R}}$. We denote these linear functions by $l_1, \dots, l_j$. Now if z^q and $z^{q'}$ are any two nonzero monomials in u_0 , then the difference in the heights of the lower boundary at $m+q$ and $m+q'$ is no more than $\max_{1 \leq i \leq j} l_i((m+q') - (m+q)) = \max_{1 \leq i \leq j} l_i(q' - q)$, which is some finite number independent of m . If we take n to be greater than these maxima over all such pairs of nonzero monomials, then the monomials in $z^m t^n d_0$ will lie above the lower boundary of C_v as long as z^{m_1} and z^{m_2} do. $\square$

Now we deal with the “locally toric” part of the definition (part 2 of Definition 1.3). Recall that outside a certain set Z , p must be locally étale equivalent to evaluation of a monomial on an affine toric variety, and the monomial must vanish precisely once on each toric divisor of the affine toric variety. The main difficulty is to establish this condition on the singular fiber. Note that unlike in [Gro05], the components of the singular fiber W are in general not toric strata of the ambient toric variety $X(h, h')$, but rather of the embedded toric variety $Y(h, h')$ (given by the image of the map i). Because of this, we will check that the locally toric condition holds for any point outside a closed subset C of the singular fiber, with the property that C does not contain any toric strata of $Y(h, h')$.

Proposition 3.12. *For any good choice of h, h' and any sufficiently large value of n , there is a closed set $C \subseteq p^{-1}(0)$ such that at any geometric point $x \in p^{-1}(0)$ outside C , p is locally étale equivalent to evaluation of a monomial on an open subset of a toric variety Y . C contains no toric strata of $Y(h, h') \subseteq X(h, h')$.*

Proof. The proof will be divided into Steps 1-3.

Step 1. At well-behaved points in the singular fiber, we show the existence of an étale map that transforms the defining equations of $\mathcal{X}$ to binomial equations of the type in Proposition 2.1.

We work on an affine toric variety U_v containing x , corresponding to the cone C_v , where $v = v_1 + v_2$ is a vertex of $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$. Suppose that $y_0, \dots, y_4$ are elements of $M \oplus \mathbb{Z}$ such that $t, y_0, \dots, y_4$ is a $\mathbb{Z}$ -basis of $M \oplus \mathbb{Z}$ and S, T are complex-valued functions equal to 1 at x . Then the map Φ given by $z^{y_0} \mapsto Sz^{y_0}$, $z^{y_1} \mapsto Tz^{y_1}$, $t \mapsto t$, and $z^{y_i} \mapsto z^{y_i}$ for $i \neq 0$ or 1 will locally define an isomorphism from a neighborhood of x to some other neighborhood of x that takes points in the open torus to points in the open torus. Also, the local isomorphism will preserve t and hence the projection map p of the degeneration. Let y_0 and y_1 be the elements of $M \oplus \mathbb{Z}$ such that z^{y_0} and z^{y_1} are the local representatives of the sections $z_0 z_1 z_2 z_3$ (of the line bundle corresponding to $\tilde{\Delta}(\pi^*h)$) and $z_0 z_1 z_4 z_5$ (of the line bundle corresponding to $\tilde{\Delta}(\pi^*h')$). Then on U_v , the defining equations in Construction 3.1 are given by

$$t^n f + z^{y_0} = 0$$

$$t^n g + z^{y_1} = 0.$$

To show t, y_0, y_1 can be extended to a $\mathbb{Z}$ -basis for $M \oplus \mathbb{Z}$, we use Corollary 3.7 and give an argument for each of the three possibilities.

In Case 1, we have $v = v_1 + v_2$ with $v_1 = v_2$. t, y_0, y_1 can be extended to a $\mathbb{Z}$ -basis because since $v_1 = v_2$, y_0 and y_1 are obtained by translating the lattice points corresponding to the sections $z_0 z_1 z_2 z_3$ and $z_0 z_1 z_4 z_5$ by the same vector, so their difference is preserved and we have $y_0 - y_1 = e_2 + e_3 - e_4 - e_5$. Letting u_i denote the projection of y_i to M , $\pi(u_0) = \pi(u_1)$ is a lattice point in $\Delta_{P(2,4)}$. Since this is a reflexive polytope, we get that $\pi(u_0)$ is either zero or a primitive vector. If it is a primitive vector, then we have established that t, y_0, y_1 can be extended to a $\mathbb{Z}$ -basis. If it is zero, then $z^{y_0} = z^{y_1} = 1$ on the image of i in C_v , and the defining equations $tf - z^{m_0} = 0$ and $tg - z^{m_1} = 0$ will collapse to $1 = 0$ when $t = 0$. Lemma 3.11 shows that all geometric points of the singular fiber lie in the image of i , so in this case U_v contains no points of the singular fiber and there is nothing to prove.

In Case 2, $v = v_1 + v_2$ where v_1 and v_2 are contained in one of the faces $z_i = z_j = 0$ of Δ , $i \in \{2, 3\}$, $j \in \{4, 5\}$, upon projection to M . (This holds because, according to Case 2, one of the v_i is contained in such a face, while the other corresponds to one of the monomials $z_0^3 z_1$, $z_0^2 z_1^2$, or $z_0 z_1^3$, which are contained in all such faces.) We then have that the dual cone to C_v will contain the rays corresponding to the toric divisors $z_i = 0$ and $z_j = 0$ on $\mathbb{P}^5$. Then y_0 (corresponding to $z_0 z_1 z_2 z_3$) evaluates to 1 on the primitive generator of the ray $z_i = 0$, and 0 on the primitive generator of the ray $z_j = 0$, while y_1 does the reverse. Since the primitive generators of $z_i = 0$ and $z_j = 0$ can be extended to a $\mathbb{Z}$ -basis, this is enough to show that y_0 and y_1 can be as well. A simple extension of the argument shows t, y_0, y_1 can be extended to a $\mathbb{Z}$ -basis.

Finally, in Case 3, we have $v = v_1 + v_2$ with the property that $(\pi \oplus \text{Id})(v_i) = 0$ for either $i = 1$ or 2. The same argument as in Case 1 shows that when $t = 0$ one of the defining equations collapses to $1 = 0$ on the image of i , and since the singular fiber is contained in the image of i , there is nothing to prove.

If we can arrange in a neighborhood of x that

$$f(t, Sz^{y_0}, Tz^{y_1}, z^{y_2}, \dots, z^{y_4}) = \alpha S$$

$$g(t, Sz^{y_0}, Tz^{y_1}, z^{y_2}, \dots, z^{y_4}) = \beta T$$

for nonzero constants α, β , then after applying Φ the defining equations will transform to

$$\alpha St^n + Sz^{y_0} = 0$$

$$\beta Tt^n + Tz^{y_1} = 0$$

and after factoring out we obtain

$$\alpha t^n + z^{y_0} = 0 \tag{3.2}$$

$$\beta t^n + z^{y_1} = 0. \tag{3.3}$$

Since Φ preserves the open torus, it will map points of $\mathcal{X}$ in the open torus to points satisfying the equations (3.2)-(3.3) in a neighborhood of x . Since Φ is a homeomorphism, it will thus map a neighborhood of $\mathcal{X}$ isomorphically to a neighborhood of the closure (in U_v) of the points in the open torus satisfying (3.2)-(3.3). We will analyze this closure later using Proposition 2.1. First, we prove that the map Φ , given by suitable functions S and T , exists. S and T will be found using the following Implicit Function Theorem (see Lemma 4.6 in [Mil08] for a reference):

Theorem 3.13. *Let A be a Henselian local ring with maximal ideal m , $w = (w_1, \dots, w_n)$ be a system of polynomials in the variables $V = (V_1, \dots, V_n)$ over A , and let $\Delta_w \in A[V]$ be the determinant of the Jacobian matrix $(\partial w_i / \partial V_j)$. Suppose that $(v_1, \dots, v_n) \in A^n$ is an “approximate solution” to $w = 0$ in the sense that $w_i(v_1, \dots, v_n) \in m$ for all i , and also that $\Delta_w(v_1, \dots, v_n) \notin m$. Then there exists a solution $(\tilde{v}_1, \dots, \tilde{v}_n) \in A^n$ of $w = 0$ with $v_i = \tilde{v}_i \pmod{m}$ for all i .*

Let $\mathcal{O}_{X, \bar{x}}$ be the Henselian local ring of $X(h, h')$ at x , which will play the role of A in the theorem, and m be its maximal ideal. Define α and β to be the values of f and g , respectively, at x . We let

$$w_1 = f(t, Sz^{y_0}, Tz^{y_1}, z^{y_2}, \dots, z^{y_4}) - \alpha S$$

$$w_2 = g(t, Sz^{y_0}, Tz^{y_1}, z^{y_2}, \dots, z^{y_4}) - \beta T$$

so that $w_i \in \mathcal{O}_{X,\bar{x}}[S, S^{-1}, T, T^{-1}]$. $S, T = 1$ is clearly a solution to $w_i = 0$ modulo m . (Note that although the Implicit Function Theorem as stated only applies to polynomials in S and T , one can easily extend the result to Laurent polynomials as well, as long as the approximate solutions are not equal to zero mod m .) The Jacobian of (w_1, w_2) with respect to S, T is

$$\begin{pmatrix} z^{y_0} f_{y_0}(t, z) - \alpha & z^{y_1} f_{y_1}(t, z) \\ z^{y_0} g_{y_0}(t, z) & z^{y_1} g_{y_1}(t, z) - \beta \end{pmatrix}$$

where z in the function arguments is an abbreviation for $Sz^{y_0}, Tz^{y_1}, z^{y_2}, \dots, z^{y_4}$. After substituting $S, T = 1$ and $f = \alpha, g = \beta$, and taking the determinant, we obtain

$$\Delta_w = (z^{y_0} f_{y_0} - f)(z^{y_1} g_{y_1} - g) - z^{y_0+y_1} f_{y_1} g_{y_0}$$

which is a regular function on U_v . The desired functions S, T will exist in an étale neighborhood of any geometric point $x \in U_v$ for which $\Delta_w(x) \neq 0$.

For any vertex v , we let O_v be the open set of U_v corresponding to $\Delta_w \neq 0$. If O is the union of the O_v over all vertices v , then we let $B = X(h, h') \setminus O$. To show that B contains no toric stratum of $Y(h, h')$, it suffices to show that each zero-dimensional toric stratum of $Y(h, h')$ is contained in one of the sets O_v for some v . Hence given a zero-dimensional toric stratum $x \in Y(h, h')$, choose any U_v containing x . To evaluate Δ_w at x , we simply substitute 1 for all monomials z^q in Δ_w such that $(\pi \oplus \text{Id})(q) = 0$ and substitute 0 for all others. Notice that if c_i and c_j are the generic coefficients giving the constant terms of f and g respectively on U_v , then c_i and c_j appear only in f and g in the expression for Δ_w because they are canceled everywhere else by the derivatives. Considering $\Delta_w(x)$ as a polynomial in the generic coefficients $c_1, \dots, c_k$, we get that $\Delta_w(x) = c_i c_j + P(c_1, \dots, c_k)$ where P is some polynomial with no terms containing a factor of $c_i c_j$. So $\Delta_w(x)$ is a nonzero polynomial in the generic coefficients. There are only a finite number of zero dimensional toric strata, and by the generic condition, we may assume that $c_1, \dots, c_k$ do not lie in the zero sets of any finite number of nonzero polynomials, so we have shown B contains no toric stratum of $Y(h, h')$.

To establish the locally toric condition, we will also need that α and β , the values of the local representatives of f and g at the point x , are nonzero. Thus,

we also throw out the zero sets of f and g on the singular fiber (which will not contain any toric strata of $Y(h, h')$ by a similar type of argument as in the previous paragraph). The set C of the Proposition statement is thus defined as $C = B \cup \{f = t = 0\} \cup \{g = t = 0\}$.

Step 2. We apply Proposition 2.1 to the equations (3.2)-(3.3). First we check that the $\mathbb{Z}$ -basis condition holds.

As long as f and $g \neq 0$ at x , the equations (3.2)-(3.3) can be transformed to

$$t^n - z^{y_0} = 0 \tag{3.4}$$

$$t^n - z^{y_1} = 0 \tag{3.5}$$

by a simple rescaling of coordinates. To understand the closure of points in the open torus satisfying equations (3.4)-(3.5), we apply Proposition 2.1. In order to do so we must prove that $(y_0, -n), (y_1, -n)$ is a basis for $\text{Span}_{\mathbb{R}}((y_0, -n), (y_1, -n)) \cap (M \oplus \mathbb{Z})$. Since this property is equivalent to showing that $(y_0, -n), (y_1, -n)$ can be extended to a $\mathbb{Z}$ -basis for $M \oplus \mathbb{Z}$, and we have already checked that t, y_0, y_1 can be extended to a basis, there is nothing to prove. By part (b) of Proposition 2.1, to show that the closure is isomorphic to a toric variety, it suffices to prove that it is normal. This will be done using Serre's criterion. Again we argue separately for Cases 1 and 2 of Corollary 3.7; since we have shown that in Case 3 U_v contains no points of the singular fiber, it can be ignored.

Step 3. We check that part (b) of Proposition 2.1 holds, by applying Serre's criterion to show that the closure of the solutions to equations (3.4)-(3.5) in the open torus is normal. This shows that the closure is isomorphic to a toric variety Y_v . We then check in each case that the monomial t vanishes precisely once on each toric divisor of Y_v , thus proving all the conditions in part 2 of Definition 1.3 hold.

In Case 1, we start by replacing the equations (3.4)-(3.5) with the equivalent $t^n - z^{y_0} = 0, z^{y_0} - z^{y_1} = 0$. Because $y_0 - y_1 = \pm(e_2 + e_3 - e_4 - e_5)$, the solution of $z^{y_0} - z^{y_1} = 0$ in the open torus of $X(h, h')$ is the open torus of $Y(h, h')$. We may therefore apply Proposition 2.1 to the affine toric variety $Y(h, h') \cap U_v$, given by the cone $C'_v = (\pi \oplus \text{Id})(C_v)$, and the single equation given by the restriction

of $t^n - z^{y_0} = 0$. If we let $y' = (\pi \oplus \text{Id})(y_0)$, then its restriction to $Y(h, h') \cap U_v$ is $t^n - z^{y'} = 0$. Let Y_v be the closure of the zero set of this equation in the open torus, as in Proposition 2.1.

The vector $nt - y'$ is primitive, and y' evaluates to 1 on all horizontal rays of the cone dual to C'_v , since it represents the section of quartics on $P(2, 4)$ which vanishes precisely once on every toric divisor. Thus, $t^n - z^{y'}$ defines an irreducible nonsingular subset of the open torus of $Y(h, h')$ away from the singular fiber $t = 0$. When $t = 0$, we get that $z^{y'} = 0$, so the zero locus on the singular fiber will be the union of toric strata defined by the monomial ideal $(t, z^{y'})$ in $Y(h, h') \cap U_v$. By Lemma 3.8, y' evaluates to zero on all upper rays in the cone dual to C'_v . t evaluates to zero on all horizontal rays in the cone dual to C'_v . A toric stratum on which the monomials t and $z^{y'}$ both vanish will be represented by a face of the dual cone containing both a ray on which t (as a linear function) evaluates positively and a ray on which y' evaluates positively, and since these must be two separate rays, the face will have dimension ≥ 2 . The toric stratum will therefore have dimension ≤ 3 , as the dimension of the cone is 5. Thus if the subscheme of $Y(h, h') \cap U_v$ defined by $t^n - z^{y'}$ has any components, including embedded components, outside of the open torus, they must be of dimension ≤ 3 . But since $Y(h, h') \cap U_v$ is Cohen-Macaulay, this subscheme can have no components of dimension less than four by the Unmixedness Theorem. Therefore, the ideal generated by $t^n - z^{y'}$ is prime, and equal to the ideal of functions vanishing on Y_v .

This shows Y_v is a complete intersection in the affine toric variety $Y(h, h') \cap U_v$, and since toric varieties are Cohen-Macaulay, implies the S_2 condition for Y_v . To show R_1 , we note that Y_v is nonsingular when $t \neq 0$, so it suffices to show that the ideal (t) is radical, or equivalently that the ideal $I = (t, z^{y'})$ in the coordinate ring of $Y(h, h') \cap U_v$ is radical (the same ideal as considered in the previous paragraph). Since its zero set is a union of toric strata of $Y(h, h') \cap U_v$, its radical can be generated by monomials. It therefore suffices to show that if $z^{am} \in I$ for a a positive integer and $m \in M' \oplus \mathbb{Z}$, then $z^m \in I$. t evaluates to 1 (by the integrality assumption of Remark 3.6) on all upper rays in the dual cone to C'_v , and evaluates to zero on all other horizontal rays on the dual cone. y' does the exact opposite.

Therefore, if z^m is a monomial on $Y(h, h') \cap U_v$, $z^m \in I$ if and only if the set of rays that m evaluates positively on contains all of the upper rays or all of the lower rays (or both). Clearly if am has this property then m does as well. This completes the proof of *R1*.

We have thus shown that Y_v is isomorphic to a toric variety. It remains to show that t vanishes precisely once on each of its toric divisors. Because (t) is a radical ideal, t cannot vanish more than once on any toric divisor. On the other hand, the cone in $N'_\mathbb{R} \oplus \mathbb{R}$ for Y_v is given by $(nt - y')^\perp \cap (C'_v)^*$, and since y' evaluates positively on any nonzero element of $(C'_v)^* \cap N'_\mathbb{R}$ whereas the linear function t evaluates to zero, this intersection cannot include any nonzero elements of $N'_\mathbb{R}$. So t must vanish at least once, and therefore exactly once, on all toric divisors of Y_v .

In Case 2, we use a similar argument, except that it will take place directly on the toric variety U_v instead of $Y(h, h') \cap U_v$. We let Y_v be the closure in U_v of the zero set of equations (3.4)-(3.5) on the open torus. Because we already know the relevant vectors obtained from (3.4)-(3.5) form a basis, the zero set on the open torus is irreducible. If z^{y_0} is the section corresponding to $z_0 z_1 z_2 z_3$, then let $y_2 = y_0 - (e_2 + e_3 - e_4 - e_5)$. y_0 evaluates to zero on all upper rays of the dual cone to C_v , and since the upper rays are contained in the subspace orthogonal to $\text{Span}_\mathbb{R}(e_2 + e_3 - e_4 - e_5)$, y_2 will as well. Because of the condition in Case 2 that one of the vectors v_1 or v_2 projects to $z_0^3 z_1$, $z_0^2 z_1^2$ or $z_0 z_1^3$, we know that the horizontal rays are a subset of the rays $z_i = 0$ for $i = 2, 3, 4, 5$. Clearly y_2 evaluates to zero on the rays given by $i = 2, 3$, and to one for $i = 4, 5$. We see that y_2 is nonnegative on all rays in the dual cone to C_v , so $y_2 \in C_v$ and z^{y_2} is regular on U_v .

Now $z^{y_2 - y_1}$ (y_1 as in equation (3.5), corresponding to the section $z_0 z_1 z_4 z_5$) may not be regular on U_v , but $z^{y_2 - y_1} z^{y_1} = z^{y_2}$ is, and $t^n z^{y_2 - y_1}$ will be for sufficiently high n . This is because $y_2 - y_1$ is clearly zero on all horizontal rays so only possibly negative on upper rays. Then we claim Y_v is defined by the equations

$$t^n - z^{y_0} = 0 \tag{3.6}$$

$$t^n z^{y_2 - y_1} - z^{y_2} = 0. \tag{3.7}$$

If $t \neq 0$, then t^n and $t^n z^{y_2 - y_1}$ are both nonzero (since they do not vanish on any toric divisors corresponding to horizontal rays) so we get that z^{y_0} and z^{y_2} are both

nonzero, which implies that solutions of (3.6)-(3.7) when $t \neq 0$ lie in the open torus. Thus for $t \neq 0$ the solutions of (3.6)-(3.7) define an irreducible, four dimensional smooth subset of the open torus. When $t = 0$ we get the ideal (t, z^{y_2}, z^{y_0}) . y_2 and y_0 evaluate to zero on all upper rays of the dual cone and evaluate positively on disjoint sets of horizontal rays (respectively $z_4, z_5 = 0$ and $z_2, z_3 = 0$). Thus any face of the dual cone representing a stratum on which all three monomials t , z^{y_2} and z^{y_0} vanish must be at least three dimensional, and the corresponding toric stratum must be of dimension ≤ 3 . The same argument as in Case 1 now shows that (3.6)-(3.7) generate the ideal of functions vanishing on Y_v , so we have $S2$.

We now must establish the $R1$ condition. Essentially the same proof as in Case 1 goes through, by showing that the ideal (t, z^{y_2}, z^{y_0}) is radical. A regular monomial z^m on U_v will be contained in the ideal if and only if the set of rays that m evaluates positively on contains at least one of the following sets: the horizontal rays $z_2, z_3 = 0$; the horizontal rays $z_4, z_5 = 0$; and the upper rays. This condition is true for m if it is true for am , a a positive integer. This proves $R1$ and shows Y_v is toric, and to show that t vanishes exactly once on each toric divisor of Y_v , the same proof as in Case 1 is valid. $\square$

The next few results will be needed for showing normality, and for showing that the singular fiber is reduced and equal to W . We let R be the subvariety of $X(h, h')$ given by the closure of the zero set of $z_2 z_3 - z_4 z_5 + t^n d$ in the open torus. Let v be any lattice point in $\tilde{\Delta}(\pi^* h) + \tilde{\Delta}(\pi^* h')$, which can be represented as a monomial $t^q z$, where z is a degree eight monomial in $z_0, \dots, z_5$. As usual we can form an affine toric subset $U_v \subseteq X(h, h')$ by translating $\tilde{\Delta}(\pi^* h) + \tilde{\Delta}(\pi^* h')$ so that v lies at the origin, and then forming the cone C_v . In the next two lemmas, when we write $z_2 z_3 - z_4 z_5 + t^n d$, we will assume we have chosen a fixed representative of this line bundle section on the open torus, so that it refers to a fixed element of $\mathbb{C}[M \oplus \mathbb{Z}]$.

Lemma 3.14. *Let I_R be the ideal of functions vanishing on $R \cap U_v$. If n is sufficiently high that Lemma 3.11 holds, then $I_R + (t)$ is generated by t along with all binomials $z^{m_1} - z^{m_2}$ such that $m_1 - m_2 = e_2 + e_3 - e_4 - e_5$ and $m_1, m_2 \in C_v$.*

Proof. I_R can be generated as a vector space by all elements $q(z_2z_3 - z_4z_5 + t^nd)$ where q is regular on the open torus and the product is regular on U_v . $q(z_2z_3 - z_4z_5 + t^nd)$ is regular on U_v if and only if its Newton polytope is contained in C_v . As the Newton polytope is the Minkowski sum of the Newton polytopes of q and $z_2z_3 - z_4z_5 + t^nd$, if it is regular then we know that $z^m(z_2z_3 - z_4z_5 + t^nd)$ is regular when z^m is any nonzero monomial in q . As in the last paragraph of the proof of Lemma 3.11, we can choose n sufficiently high so that whenever z^m is such that $z^m(z_2z_3 - z_4z_5 + t^nd)$ is regular on U_v , then z^mt^nd is in (t) . Since the other term $z^m(z_2z_3 - z_4z_5)$ is one of the binomials in the Lemma statement, this shows we can choose generators for $I_R + (t)$ of the desired form.

In the other direction, we need to check that t , and all binomials $z^{m_1} - z^{m_2}$ such that $m_1 - m_2 = e_2 + e_3 - e_4 - e_5$ and $m_1, m_2 \in C_v$, are contained in $I_R + (t)$. For t this is obvious. Suppose we have a binomial $z^{m_1} - z^{m_2}$ as above. Then we can choose z^m such that $z^{m_1} - z^{m_2}$ is equal to the $z^m(z_2z_3 - z_4z_5)$ part of $z^m(z_2z_3 - z_4z_5 + t^nd)$. The argument in the last two paragraphs of Lemma 3.11 again shows that we can choose n sufficiently high so that whenever z^m is such that $z^m(z_2z_3 - z_4z_5)$ is regular on U_v , z^mt^nd is also regular on U_v , and in fact contained in (t) . So we have that $z^m(z_2z_3 - z_4z_5 + t^nd)$ is regular on U_v and therefore contained in I_R . Since the last term z^mt^nd is contained in (t) , this forces the remaining part, $z^{m_1} - z^{m_2}$, to be contained in $I_R + (t)$. $\square$

Lemma 3.15. *Suppose v and z , as above, are such that z contains a factor of z_2z_3 or z_4z_5 , and n is sufficiently high. Then the ideal of the subvariety $U_v \cap R$ can be generated by a single function on U_v .*

Proof. As the proofs for z_2z_3 and z_4z_5 are identical, we assume z contains a factor of z_4z_5 . We can multiply $z_2z_3 - z_4z_5 + t^nd$ by a function z^m , $m \in M \oplus \mathbb{Z}$, such that $z^m z_4z_5 = 1 = z^0$. Then $z^m z_2z_3$ will be regular on U_v , because the vector $e_2 + e_3 - e_4 - e_5$ will be contained in the cone C_v by the hypothesis. (Indeed, the possible rays in the dual cone to C_v are the horizontal rays making up the fan for $\mathbb{P}^5$, and upper rays. As a linear function $e_2 + e_3 - e_4 - e_5$ evaluates to zero on all upper rays by Lemma 3.5. Since z contains a factor of z_4z_5 , z_4 and z_5 are locally invertible and the dual cone will not contain the horizontal rays generated by $\check{e}_4$

and $\check{e}_5$ corresponding to divisors $z_4 = 0$ and $z_5 = 0$. Then we see that $e_2 + e_3 - e_4 - e_5$ is nonnegative on all remaining horizontal rays, which are generated by $\check{e}_1, \check{e}_2, \check{e}_3$, and $-\check{e}_1 - \cdots - \check{e}_5$.) By Lemma 3.10, for sufficiently high n , $z^m t^n d$ is then also regular on U_v and contained in the ideal (t) .

The ideal $(t, z^m(z_2 z_3 - z_4 z_5 + t^n d))$ on U_v will then contain all binomials $z^{m_1} - z^{m_2}$ such that m_1 and m_2 lie on the lower boundary of C_v and $m_1 - m_2 = e_2 + e_3 - e_4 - e_5$. (Just multiply $z^m(z_2 z_3 - z_4 z_5 + t^n d)$ by z^{m_2} , and use the fact that $z^{m_2} z^m t^n d \in (t)$.) The components of the subscheme given by $(t, z^m(z_2 z_3 - z_4 z_5 + t^n d))$ are therefore equal to the toric varieties given by cones in the lower boundary of $(\pi \oplus \text{Id})(C_v)$, which are all codimension 2 in U_v . Thus any components of the subscheme defined by $z^m(z_2 z_3 - z_4 z_5 + t^n d) = 0$ lying in the singular fiber, including embedded components, would have to be codimension ≥ 2 in U_v . Since $z^m(z_2 z_3 - z_4 z_5 + t^n d) = 0$ defines $U_v \cap R$ away from the singular fiber, the usual argument with the Unmixedness Theorem shows that $z^m(z_2 z_3 - z_4 z_5 + t^n d)$ generates the ideal of the subvariety $U_v \cap R$. $\square$

As the proof of the next lemma is not very transparent, we give a brief explanation of the idea behind it. Suppose we take the subvariety of $\mathbb{P}^5 \times \mathbb{C}$ defined by $z_2 z_3 - z_4 z_5 + t^n d = 0$, and look at its intersection with the open affine set $z_2 \neq 0$ in $\mathbb{P}^5 \times \mathbb{C}$. Then setting $z_2 = 1$ we get $z_3 - z_4 z_5 + t^n d = 0$, or $z_3 = z_4 z_5 - t^n d$. This is just a graph over affine space $\mathbb{C}^5$, and thus isomorphic to $\mathbb{C}^5$. The idea is to try to extend this result to the more complicated toric variety $X(h, h')$.

Lemma 3.16. *Suppose v and z , as above, are such that z contains one of the variables $\{z_2, z_3\}$ but not the other, or one of the variables $\{z_4, z_5\}$ but not the other, and n is sufficiently high. Then $R \cap U_v$ is isomorphic to $Y(h, h') \cap U_v$.*

Proof. We assume without loss that z contains z_3 but not z_2 . We will show the existence of an automorphism $\eta : U_v \rightarrow U_v$ that maps $Y(h, h') \cap U_v$ to $R \cap U_v$. The cone $p_1(C_v) \subseteq M_{\mathbb{R}}$ can be described as $C(\text{Conv}(b_2, s_1, \dots, s_j))$, where $b_1, \dots, b_5$ is a $\mathbb{Z}$ -basis of M such that z^{b_2} is a local representative of the section $z_2 z_3$ of quadratics on $\mathbb{P}^5$, and $s_i \in \{\pm b_1, \pm b_3, \pm b_4, \pm b_5\}$. Thus, if $c_1 b_1 + c_2 b_2 + \cdots + c_5 b_5 \in p_1(C_v)$, then we must have that $c_2 \geq 0$ and also $c_1 b_1 + c_3 b_3 + c_4 b_4 + c_5 b_5 \in p_1(C_v)$. (All

of this can be seen by noting that b_2 evaluates to 1 on the ray $z_2 = 0$ in the dual cone to $p_1(C_v)$ and zero on all others.) We can choose a local representative of $z_2z_3 - z_4z_5 + t^nd$ on $U_v \setminus \{t = 0\}$ such that the z_2z_3 term is represented by z^{b_2} . We denote by z^a the representative of the z_4z_5 term, and by t^nd_0 the representative of the t^nd term.

Let z^m be regular on U_v with $m = c_0t + c_1b_1 + \cdots + c_5b_5$. Then we have $z^m = z^{c_2b_2}z^{m'}$ where $m' = m - c_2b_2$. We define $\eta(z^m) = (z^{b_2} + t^nd_0)^{c_2}z^{m'}$. As long as we check that the RHS is regular on U_v it will define an automorphism of U_v , in particular because c_2 is always nonnegative. (The inverse is $\eta^{-1}(z^m) = (z^{b_2} - t^nd)^{c_2}z^{m'}$.) We already know $\eta(z^m)$ is regular on $U_v \setminus \{t = 0\}$, so it suffices to check that all powers of t in the monomials of $\eta(z^m)$ are high enough so that they lie in C_v .

For a particular z^m , this can clearly be achieved by picking a sufficiently high value of n . However, since there are an infinite number of possible monomials z^m , we must show it is possible to choose an n that works for every one. This is essentially the same situation we faced at the end of the proof of Lemma 3.11, although slightly more complicated: since the power c_2 could be arbitrarily high, there could be a large number of terms in the expansion of $\eta(z^m)$. So we need to combine the argument of Lemma 3.11 with an induction argument.

The terms in the expansion of $\eta(z^m)$ will be of the form $z^{(c_2-i)b_2}(t^nd_0)^iz^{m'}$ for $0 \leq i \leq c_2$. Note that if $j = i + 1$ then we can obtain the next term $z^{(c_2-j)b_2}(t^nd_0)^jz^{m'}$ by multiplying the previous one by t^nd_0/z^{b_2} . Therefore, all monomials in the $i + 1$ term can be obtained by multiplying monomials in the previous term by a monomial of the form z^{q-b_2} , where z^q is a nonzero monomial in t^nd_0 . This is a fixed, finite set of monomials that is completely independent of m , so the same argument given at the end of Lemma 3.11 shows that for sufficiently high n , the $i + 1$ term is regular on U_v if the i term is (in fact, the same choice of n given there is adequate). Since the $i = 0$ term is regular, an induction argument shows that all terms for $0 \leq i \leq c_2$ are regular. Thus η is well defined on U_v .

Now we check that η maps the ideal of $Y(h, h') \cap U_v$ to the ideal of $R \cap U_v$. The former ideal can be generated as a vector space by elements of the form

$z^m(z^{b_2} - z^a)$, $m \in M \oplus \mathbb{Z}$, that are regular on U_v (note neither factor necessarily is). Now, m cannot evaluate negatively on the ray corresponding to the divisor $z_2 = 0$, because a evaluates to zero on this ray and then z^{m+a} would not be regular on U_v . Thus if c_2 is the coefficient on b_2 in the basis expansion of m , and $m = c_2 b_2 + m'$, then $c_2 \geq 0$ and we have $\eta(z^m) = z^{m'}(z^{b_2} + t^n d_0)^{c_2}$. Therefore $\eta(z^m(z^{b_2} - z^a)) = z^{m'}(z^{b_2} + t^n d_0)^{c_2}((z^{b_2} + t^n d_0) - z^a)$. Since $c_2 \geq 0$, this is a product of a regular function on the open torus with $z^{b_2} - z^a + t^n d_0$. Since the latter function defines R on the open torus, it therefore vanishes on the intersection of R with the open torus, and since it is regular on U_v , on $R \cap U_v$. An identical argument shows that η^{-1} takes the ideal of $Y(h, h') \cap U_v$ to the ideal of $R \cap U_v$. $\square$

For the following proposition we again assume that fixed representatives of $t^n f + z_0 z_1 z_2 z_3$, $t^n g + z_0 z_1 z_4 z_5$, and $z_2 z_3 - z_4 z_5 + t^n d$ have been chosen on the open torus, so that they, and each of their terms, refer to definite elements of $\mathbb{C}[M \oplus \mathbb{Z}]$.

Proposition 3.17. *Again let v and z be as above, and suppose that z is not one of the monomials z_0^8 or z_1^8 . Then $\mathcal{X} \cap U_v$ is isomorphic to a complete intersection in a toric variety. Furthermore, the singular fiber of $\mathcal{X} \cap U_v$ is reduced and equal to $W \cap U_v$.*

Proof. If z contains any of the variables z_2, z_3, z_4 , or z_5 , then we are in the case of either Lemma 3.15 or 3.16. Otherwise, it contains only z_0 and z_1 . First suppose we are in this latter case.

By assumption z contains both z_0 and z_1 and no other variables. The defining equation $t^n f + z_0 z_1 z_2 z_3 = 0$ lies in a line bundle defined on $X(h, h')$, so there is a monomial z^m such that $z^m(t^n f + z_0 z_1 z_2 z_3)$ represents the line bundle section on U_v . Picking a local representative of $z_2 z_3 - z_4 z_5 + t^n d$ on U_v , we choose $z^{m'}$ such that $z^{m'} z_2 z_3 = z^m z_0 z_1 z_2 z_3$. Then $z^{m'} z_4 z_5$ is also regular on U_v . We see this by noting that the lattice elements corresponding to $z^{m'} z_4 z_5$ and $z^{m'} z_2 z_3$ differ by the vector $-e_2 - e_3 + e_4 + e_5$. Since $z^{m'} z_2 z_3$ locally represents the line bundle section $z_0 z_1 z_2 z_3$ on $\mathbb{P}^5$, the associated lattice element evaluates to one on $\check{e}_2$ and $\check{e}_3$ in the dual cone, and zero on $\check{e}_4$ and $\check{e}_5$. (These vectors generate the only horizontal rays in the dual cone since we have assumed that z_0 and z_1 are both invertible.) After

adding $-e_2 - e_3 + e_4 + e_5$ to this function, it will clearly remain nonnegative on horizontal rays in the dual cone, while the value on upper rays will be unchanged, since they lie in the subspace orthogonal to $-e_2 - e_3 + e_4 + e_5$ by Lemma 3.5.

By Lemma 3.10, since $z^{m'} z_2 z_3$ and $z^{m'} z_4 z_5$ are both regular on U_v , so is $z^{m'} t^n d$ for sufficiently high n . We claim $\mathcal{X} \cap U_v$ is defined by

$$z^m(t^n f + z_0 z_1 z_2 z_3) = 0$$

$$z^{m'}(z_2 z_3 - z_4 z_5 + t^n d) = 0.$$

Using the fact that when $t \neq 0$, z_0 and $z_1 \neq 0$ on U_v , one can see that these equations define $\mathcal{X} \cap U_v$ for $t \neq 0$.

When $t = 0$, we get that $z^m z_0 z_1 z_2 z_3 = z^{m'} z_2 z_3 = 0$, and from the second equation, $z^{m'} z_4 z_5 = 0$. $z^m z_0 z_1 z_2 z_3 = z^{m'} z_2 z_3$ vanishes on no divisors corresponding to upper rays in the dual cone to C_v by Lemma 3.8, so neither does $z^{m'} z_4 z_5 = 0$, since they differ by the vector $e_2 + e_3 - e_4 - e_5$. We can now see that any toric strata contained in the zero set of the ideal $(t, z^{m'} z_2 z_3, z^{m'} z_4 z_5)$ must be of dimension ≤ 3 , since $z^{m'} z_2 z_3$ and $z^{m'} z_4 z_5$ respectively vanish on the disjoint sets of divisors $z_2, z_3 = 0$ and $z_4, z_5 = 0$, and no divisors corresponding to upper rays. t vanishes only on divisors corresponding to upper rays. By the Unmixedness Theorem, the ideal generated by the above two equations is prime and generates the ideal of $\mathcal{X} \cap U_v$, $I_{\mathcal{X}}$.

To check the statement about the singular fiber in this case, or that $I_W = I_{\mathcal{X}} + (t)$, note that we already have $I_W \subseteq I_{\mathcal{X}} + (t)$ by Lemma 3.11. To show the reverse inclusion we just need to check that the ideal generated by the above two generators and t is contained in I_W . The first equation gives $z^m z_0 z_1 z_2 z_3$ when $t = 0$, which is one of the generators of I_W given in Lemma 3.11. The second equation gives $z^{m'}(z_2 z_3 - z_4 z_5)$, which is also contained in I_W .

Now suppose we are in the case of Lemma 3.15. We let I be the ideal generated by the single equation in Lemma 3.15 generating I_R , together with $z^m(t^n f + z_0 z_1 z_2 z_3)$. We claim that $I = I_{\mathcal{X}}$. One easily checks that I defines $\mathcal{X}$ when $t \neq 0$. Now consider the zero set of I intersected with the singular fiber $t = 0$. By Lemma 3.14, the components of R when $t = 0$ are the components in the singular

fiber of $Y(h, h') \cap U_v$, so we can assume we are operating on the quotient cone $C'_v = (\pi \oplus \text{Id})(C_v)$ that defines $Y(h, h')$. When $t = 0$, the other generator becomes $z^m(z_0 z_1 z_2 z_3)$. This monomial vanishes on all divisors corresponding to horizontal rays in the dual cone to C'_v , and on none of those corresponding to upper rays (by Lemma 3.8). t vanishes only on upper rays, so once again we have that all components are dimension ≤ 3 (since C'_v is five dimensional), and by the Unmixedness Theorem, $I = I_{\mathcal{X}}$, the prime ideal that defines $\mathcal{X}$.

For the singular fiber, by the same reasoning as in the previous case, we just need to check that the two generators are contained in I_W when $t = 0$. This is immediate.

Finally suppose we are in the case of Lemma 3.16. Consider the subscheme of $R \cap U_v$ given by the equation $z^m(t^n f + z_0 z_1 z_2 z_3) = 0$. The same reasoning as in the previous case shows that when $t = 0$ the components of this subscheme are dimension ≤ 3 . Since the isomorphism from Lemma 3.16 shows that $R \cap U_v$ is isomorphic to $Y(h, h') \cap U_v$ in this case, $R \cap U_v$ is Cohen-Macaulay and the Unmixedness Theorem applies to show that the single equation $z^m(t^n f + z_0 z_1 z_2 z_3)$ defines $\mathcal{X} \cap U_v$ as a subvariety of $R \cap U_v$. Thus $\mathcal{X} \cap U_v$ is isomorphic to a complete intersection in the toric variety $Y(h, h') \cap U_v$.

The proof that $I_W = I_{\mathcal{X}} + (t)$ is the same as in the previous cases. $\square$

We are now prepared to show that $p : \mathcal{X} \rightarrow \mathbb{C}$ is a toric degeneration. By Proposition 3.17, we know that all points of W are contained in the singular fiber of $\mathcal{X}$, except possibly for the two zero-dimensional toric strata corresponding to the vertices z_0^8 and z_1^8 . This is because the toric varieties making up W arise from a lattice subdivision of the boundary of $2\Delta_{P(2,4)}$ into polytopes. Any face of the polytopes, other than the vertices z_0^8 or z_1^8 , will contain a lattice point whose preimage under π contains one of the lattice points v where Proposition 3.17 applies. But the singular fiber of $\mathcal{X}$ obviously has to be closed as a set, which will force it to contain these two points as well. On any affine open set $U_v \subseteq X(h, h')$, the ideal I_W by definition consists of all functions vanishing on $W \cap U_v$. Since, as we just established, all elements $I_{\mathcal{X}} + (t)$ also must vanish on points of W , we get that $I_{\mathcal{X}} + (t) \subseteq I_W$. But by Lemma 3.11, we also have

$I_W \subseteq I_{\mathcal{X}} + (t)$, so $I_W = I_{\mathcal{X}} + (t)$.

This shows that the singular fiber is reduced and equal to W , which is a union of toric varieties meeting along toric strata. Since the singular fiber is reduced and only has components of dimension 3, it follows that p is flat.

To show normality, first we establish the $R1$ condition. Away from the singular fiber, we know from Lemma 3.2 that the fibers are either smooth Calabi-Yaus or, if $h = h'$, Calabi-Yaus with four nodes. Hence by flatness of p , the singular locus of $\mathcal{X}$ is of codimension 3 when $t \neq 0$. At the singular fiber, we know from flatness of p that singularities of $\mathcal{X}$ are contained in the singular locus of W , which is codimension one in W and therefore codimension 2 in $\mathcal{X}$.

The $S2$ condition only needs to be checked on the singular fiber. Again by Proposition 3.17, we know that a neighborhood of every point in the singular fiber is isomorphic to a complete intersection in a toric variety, except for the two zero-dimensional toric strata corresponding to the vertices z_0^8 and z_1^8 . We get that all local rings at points on the singular fiber, except possibly at these two zero dimensional toric strata, are normal by Serre's criterion. However, by the "locally toric" result of Proposition 3.12 we know that there is an étale map from the local ring at these points to the local ring on a toric variety, which is normal. This implies that the two remaining local rings on $\mathcal{X}$ are normal, as desired.

Finally we must establish the locally toric condition outside a closed set $Z \subseteq \mathcal{X}$ of relative codimension 2. We set $Z = \text{Sing}(\mathcal{X}) \cap C$ where C is the closed subset of the singular fiber given in Proposition 3.12. Since the locally toric condition is automatic at nonsingular points of $\mathcal{X}$, it will hold at all points outside Z . Z is clearly of relative codimension 2 away from the singular fiber. On the singular fiber, note that we know C contains no toric strata of the singular fiber from Proposition 3.12. All singular points of $\mathcal{X}$ in a given component of the singular fiber are contained in the complement of its open torus, which is a union of codimension one toric strata. Since C cannot contain any of these strata, its intersection with any given stratum will be codimension 2. This establishes that Z is codimension ≤ 2 in the singular fiber. Thus we can finally state:

Theorem 3.18. *For n sufficiently high and a good choice of h and h' , the degen-*

eration $p : \mathcal{X} \rightarrow \mathbb{C}$ defined in Construction 3.1 is a toric degeneration.

Chapter 4

Affine Manifold Structure

As described in [GS06], given any toric degeneration, we may construct the associated dual intersection complex (B, P) . This is an affine manifold with singularities, B , together with a polyhedral decomposition of B into affine polyhedra, P . Theorem 3.18 states that the degeneration of Construction 3.1, $p : \mathcal{X} \rightarrow \mathbb{C}$, is toric, as long as the choice of h and h' is good and n is sufficiently high. For the rest of this section, we will always assume these hypotheses are true.

Throughout this section, φ and φ' will be piecewise linear functions on the fan for $X(h, h')$, induced by the sections $z_0 z_1 z_2 z_3$ and $z_0 z_1 z_4 z_5$ of the line bundles given by $\tilde{\Delta}(\pi^* h)$ and $\tilde{\Delta}(\pi^* h')$, respectively. In addition to assuming that the value of n is high enough for all results of the previous section to hold, we will assume n is high enough that $n > \varphi'(u)$ for all primitive integral generators of upper rays in the fan for $X(h, h')$. (By Lemma 3.8, we already know that $\varphi(u)$ is zero for any such u .)

The next proposition gives a partial description of the dual intersection complex (B, P) of $p : \mathcal{X} \rightarrow \mathbb{C}$. Note that to be complete, we also need to provide an affine structure at the vertices of (B, P) (the so-called “fan structure”). This will be done later on.

Proposition 4.1. *As a topological manifold, B is given by the set of solutions to $\varphi = \varphi' = n$, $t = 1$ in $N_{\mathbb{R}} \oplus \mathbb{R}$. Since φ, φ' and t are each piecewise linear, B is naturally a union of polyhedra in $N_{\mathbb{R}} \oplus \mathbb{R}$. This defines the polyhedral decomposition*

P. The interiors of the maximal (3-dimensional) polyhedra inherit a natural affine structure from $N_{\mathbb{R}} \oplus \mathbb{R}$, which defines the affine structure on B at these points.

Proof. The dual intersection complex is defined by gluing together polytopes associated to each zero-dimensional toric stratum in the singular fiber. Specifically, if v is such a zero-dimensional toric stratum, we know that near v the map p is étale equivalent to evaluation of a monomial on an affine toric variety Y_v . Y_v is defined by a fan consisting of a single cone, and the polytope associated to v is defined by taking the convex hull of the integral generators of all boundary rays in the cone.

Suppose that D is a cone in the fan for $X(h, h')$. We must show that, if the affine toric variety associated to D contains a zero-dimensional stratum of the singular fiber, v , then the solutions to $\varphi = \varphi' = n, t = 1$ in D are isomorphic to the polytope associated to Y_v . First we show that the cone defining Y_v is isomorphic to the solutions of $\varphi - nt = \varphi' - nt = 0$ in D . Y_v is defined, at least on the open torus of $X(h, h')$, by the solutions to the equations $t^n - z^{y_0} = t^n - z^{y_1} = 0$ (equations (3.4)-(3.5) in the proof of Proposition 3.12). y_0 and y_1 are elements of $M \oplus \mathbb{Z}$ representing φ and φ' , respectively, on D , so this is true.

Now we must show that the solutions to $\varphi = \varphi' = n, t = 1$ on D are equal to the convex hull of the primitive integral generators of the boundary rays in the cone for Y_v . Since we have shown that p is a toric degeneration, we know that t vanishes precisely once on each toric divisor of Y_v , so t evaluates to one on the primitive integral generators of each ray. Substituting $t = 1$ into $\varphi - nt = \varphi' - nt = 0$ gives $\varphi = \varphi' = n$.

We now explain why the polytopes for the toric varieties Y_v will be glued together as in [GS06] and why there are no “extra” unwanted points in the solution set. One can show that the smallest toric strata of $X(h, h')$ containing entire components of the singular fiber are the toric strata associated to cones of the form $C(\text{Conv}(r_i, r_j, u))$ (which we will refer to as “type 1”) or $C(\text{Conv}(r_i, u))$ (type 2). Here u is an upper ray, and the type 1 case, r_i and r_j generate the rays corresponding to the divisors $z_i = 0$ and $z_j = 0$ on $\mathbb{P}^5$, where $i \in \{2, 3\}$ and $j \in \{4, 5\}$. In the type 2 case, r_i generates a ray corresponding to the divisor $z_i = 0$ for $i = 0$ or 1 . In both cases the toric strata contain precisely one component of

the singular fiber.

This fact can be shown by observing that the singular fiber consists of toric strata of $Y(h, h')$, specifically, the toric strata that correspond to bounded faces of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ which project to the boundary of $2\Delta_{P(2,4)}$ under the map $p_1 : M'_{\mathbb{R}} \oplus \mathbb{R} \rightarrow M'_{\mathbb{R}}$. Maximal faces of this form are in bijective correspondence with cones in the fan for $Y(h, h')$ that consist of exactly one horizontal ray and exactly one upper ray, since they can be obtained by intersecting a maximal vertical face of $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$ with a maximal lower face. If the horizontal ray corresponds to one of the divisors $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$, then the cone will include to a type 1 cone in the fan for $X(h, h')$, while if it corresponds to one of the divisors $z_i = 0$, $i = 0$ or 1 , it will include to a type 2 cone.

One can also show that, provided n is sufficiently large, the cones containing exactly one point of the solution set are the same as the ones just described. (Specifically, we need $n > \varphi'(u)$ for all upper rays u .) We will address this in the following claim:

Claim. The cones in the fan for $X(h, h')$ containing one point of the solution set are exactly the cones of type 1 and type 2 given above.

Proof. First we show that type 1 and type 2 cones do contain exactly one point of the solution set.

On a cone of type C_1 , we have that $\varphi(r_i) = \varphi'(r_j) = 1$ and $\varphi(r_j) = \varphi'(r_i) = 0$. We also have $\varphi(u) = 0$, $t(r_i) = t(r_j) = 0$, and $t(u) = 1$ (the first equality coming from Lemma 3.8). $\varphi'(u)$ is not necessarily zero, but by assumption $n \geq \varphi'(u)$. The unique solution is then equal to $nr_i + (n - \varphi'(u))r_j + u$.

On a cone of type C_2 , we claim that $\varphi(u) = \varphi'(u) = 0$. We know $\varphi(u) = 0$ from Lemma 3.8. On the other hand, h and h' are equal on the faces of $\Delta_{P(2,4)}$ corresponding to $z_0 = 0$ and $z_1 = 0$, so we are in Case 1 of Corollary 3.7. This means φ' and φ will differ by $e_2 + e_3 - e_4 - e_5 \in M$ on the cone, and since u is contained in the subspace orthogonal to this vector, we have $\varphi'(u) = \varphi(u)$. We also have that $t(u) = 1$, and $\varphi(r_i) = \varphi'(r_i) = 1$. Altogether, this implies that the unique solution is $nr_i + u$.

Now we show that a cone containing exactly one point of the solution set must

be of type 1 or type 2. First of all, we claim that for a cone to contain any solutions, the cone must contain at least one horizontal ray on which φ is nonzero and one horizontal ray on which φ' is nonzero. Suppose otherwise. If the cone contains only horizontal rays on which φ is zero, then φ is identically zero on the entire cone, because φ is also zero on all upper rays by Lemma 3.8. Now suppose the cone contains only horizontal rays on which φ' is zero. The subset of the cone on which $t = 1$ is equal to $\text{Conv}(u_1, \dots, u_r) + H$ where $u_1, \dots, u_r$ are the primitive integral generators of its upper rays, and H is the convex hull of all its horizontal rays. Because φ' is identically zero on H , and by assumption strictly less than n on $\text{Conv}(u_1, \dots, u_r)$, we see that the cone cannot contain any points where $t = 1$ and $\varphi' = n$.

This establishes that a cone containing any solutions must contain one horizontal ray on which φ is nonzero and one horizontal ray on which φ' is nonzero. Thus it must contain either the ray corresponding to $z_0 = 0$, the ray corresponding to $z_1 = 0$, or one each of the rays corresponding to $z_2, z_3 = 0$ and the rays corresponding to $z_4, z_5 = 0$. Since $t = 1$ on solutions, the cone must also contain at least one upper ray. Thus the cone must contain a cone of type 1 or type 2 as a subcone if it contains any solutions. On the other hand, it is easy to check that any cone larger than a type 1 or type 2 cone would be forced to contain more than a single point as a solution. $\square$

This completes the proof that the vertices of the solution set are in bijective correspondence with components of the singular fiber. It is straightforward to show that the solution set on any given cone C is bounded, since one of the functions φ, φ' or t is positive on any given ray of the fan for $X(h, h')$. It follows that the solution set is the convex hull of the vertices contained in C , i.e., the vertices corresponding to components of the singular fiber that the affine toric variety associated to C intersects. This is exactly how the dual intersection complex is constructed by gluing.

By definition, the affine structure of B at points in the interior of maximal polytopes is inherited from the polytopes, in other words, from $N_{\mathbb{R}} \oplus \mathbb{R}$. $\square$

Now we turn to the fan structure. As stated above, the smallest cones in the

fan for $X(h, h')$ that contain the vertices of the dual intersection complex are of the first type, $C_1 = C(\text{Conv}(r_i, r_j, u))$, or the second type, $C_2 = C(\text{Conv}(r_i, u))$. We claim that the affine structure near the vertex is obtained by modding out by $\text{Span}_{\mathbb{R}}(C_i)$ in both cases, $i = 1$ or 2 . To prove the claim we must show that, under the quotient map, a neighborhood of B near the vertex v maps bijectively to a neighborhood of the origin in the fan for the component of the singular fiber associated to v . Note that $\text{Span}_{\mathbb{R}}(C_i)$ is two dimensional when $i = 2$ and three dimensional when $i = 1$. Thus when $i = 2$ the fan will be codimension one in the quotient space. When $i = 1$ the fan will fill up the whole quotient space.

Proposition 4.2. *For a vertex v of the dual intersection complex contained in a cone C_i , of type 1 or 2, the fan structure at v is induced by the quotient map $q : N_{\mathbb{R}} \oplus \mathbb{R} \rightarrow (N_{\mathbb{R}} \oplus \mathbb{R})/\text{Span}_{\mathbb{R}}(C_i)$.*

Proof. We start with the type 1 case. Let $p \in N_{\mathbb{R}} \oplus \mathbb{R}$ be a point in a cone D of the fan for $X(h, h')$ which contains v . Since we need only check bijectivity near a small neighborhood of the origin in the quotient fan, we may assume p is close to the origin of $N_{\mathbb{R}} \oplus \mathbb{R}$ and therefore that $\varphi(p)$, $\varphi'(p)$ and $t(p)$ are arbitrarily small. It suffices to show there is a unique point p' in $B \cap D$ such that $q(p) = q(p')$, or that there are unique nonnegative reals c_1, c_2, c_3 satisfying $\varphi(p + c_1 r_i + c_2 r_j + c_3 u) = \varphi'(p + c_1 r_i + c_2 r_j + c_3 u) = n$, $t(p + c_1 r_i + c_2 r_j + c_3 u) = 1$. We have that $\varphi(r_i) = \varphi'(r_j) = t(u) = 1$ and $\varphi(r_j) = \varphi'(r_i) = 0$. We also have $\varphi(u) = 0$ and $t(r_i) = t(r_j) = 0$ (the first equality coming from Lemma 3.8). This implies that we must have $c_1 = n - \varphi(p)$, $c_2 = n - (1 - t(p))\varphi'(u) - \varphi'(p)$, and $c_3 = 1 - t(p)$. These numbers are all nonnegative by the assumption that p is close to the origin and $n > \varphi'(u)$ for all upper rays u .

For the type 2 case, we note that since the singular fiber is the toric stratum of $Y(h, h')$ defined by C_i , the fan for the singular fiber will consist of images of points in the fan for $Y(h, h')$ under q . This means the fan will lie in the image of the subspace $N'_{\mathbb{R}} \oplus \mathbb{R}$ under q (recall $N'_{\mathbb{R}}$ is defined as the subspace of $N_{\mathbb{R}}$ orthogonal to $e_2 + e_3 - e_4 - e_5$). Since r_i and u both lie in $N'_{\mathbb{R}}$ in this case, the fan will lie in a codimension one subspace of the quotient space.

Thus, again let $p \in N'_{\mathbb{R}} \oplus \mathbb{R}$ be a point in a cone D of the fan for $X(h, h')$ which

contains v . We assume p is close to the origin. We have $\varphi(r_i) = \varphi'(r_i) = t(u) = 1$ and $\varphi(u) = \varphi(u') = t(r_i) = 0$. We know $\varphi'(u) = 0$, unlike in the previous case, because h and h' are equal on the faces of $\Delta_{P(2,4)}$ corresponding to $z_0 = 0$ and $z_1 = 0$, so we are in Case 1 of Corollary 3.7. This means φ' and φ will differ by $e_2 + e_3 - e_4 - e_5 \in M$ on the cone, and since u is contained in the subspace orthogonal to this vector, we have $\varphi'(u) = \varphi(u)$. This also implies $\varphi(p) = \varphi'(p)$. Solving $\varphi(p + c_1 r_i + c_2 u) = \varphi(p + c_1 r_i + c_2 u) = n$, $t(p + c_1 r_i + c_2 u) = 1$ yields $c_1 = n - \varphi(p) = n - \varphi'(p)$, and $c_2 = 1 - t(p)$, which will be nonnegative as long as p is sufficiently close to the origin.

We should also check that the image of B near v lies in the aforementioned codimension one subspace of the quotient space. This is immediate since B will automatically lie in $N'_\mathbb{R} \oplus \mathbb{R}$ for all maximal faces containing v , since $\varphi' = \varphi$ on B and φ and φ' differ by $e_2 + e_3 - e_4 - e_5 \in M$ on all cones containing v , as pointed out above. $\square$

We can now calculate monodromy. We let v_1 and v_2 be two vertices of the dual intersection complex, and P_1 and P_2 be two maximal polytopes containing them. We calculate the monodromy around a loop starting at v_1 , passing through the interior of P_1 to v_2 , and back through the interior of P_2 to v_1 . First we deal with the case where the vertices are contained in cones of the same type (cones of type 1 or 2 as described above).

Proposition 4.3. *Let v_i and P_i be as above for $i = 1, 2$. Suppose that v_i is contained in the cone $D_i = C(\text{Conv}(r_i, u_i))$, i.e., both cones are of type 1, and suppose that φ is represented by $-m_i$ on the cone containing P_i . Then identifying the tangent space at v_1 with $N'_\mathbb{R}/\text{Span}_\mathbb{R}(r_1)$, the monodromy transformation is*

$$n \mapsto n + \langle m_2 - m_1, n \rangle (r_2 - r_1) \text{ mod } \text{Span}_\mathbb{R}(r_1)$$

for $n \in N'_\mathbb{R}$.

Proof. The proof is the same as the proof of Proposition 2.13 in [Gro05]. Note that since the linear representatives of φ and φ' on both cones are equal on $N'_\mathbb{R}$, φ' could have been used equally well in the proposition statement. $\square$

Proposition 4.4. *Let v_i and P_i be as above for $i = 1, 2$. Suppose that v_i is contained in the cone $D_i = C(\text{Conv}(r_i, r'_i, u_i))$, i.e., both cones are of type 2, and suppose that φ and φ' are represented by $-m_i$ and $-m'_i$, respectively, on the cones containing P_i . Then identifying the tangent space at v_1 with $N_{\mathbb{R}}/\text{Span}_{\mathbb{R}}(r_1, r'_1)$, the monodromy transformation is*

$$n \mapsto n + \langle m_2 - m_1, n \rangle (r_2 - r_1) + \langle m'_2 - m'_1, n \rangle (r'_2 - r'_1) \text{ mod } \text{Span}_{\mathbb{R}}(r_1, r'_1)$$

for $n \in N_{\mathbb{R}}$.

Proof. Same as the proof of Proposition 2.13 in [Gro05]. $\square$

Next we suppose that v_1 is in a cone of type 1 and v_2 is in a cone of type 2.

Proposition 4.5. *Let v_i and P_i be as above for $i = 1, 2$. Suppose that v_1 is contained in $D_1 = C(\text{Conv}(r_1, u_1))$ and v_2 is contained in $D_2 = C(\text{Conv}(r_2, r'_2, u_2))$, and suppose that φ and φ' are represented by $-m_i$ and $-m'_i$, respectively, on the cones containing P_i . Identifying the tangent space at v_1 with $N'_{\mathbb{R}}/\text{Span}_{\mathbb{R}}(r_1)$, the monodromy transformation is*

$$n \mapsto n + \langle m_2 - m_1, n \rangle (r_2 + r'_2) \text{ mod } \text{Span}_{\mathbb{R}}(r_1)$$

for $n \in N'_{\mathbb{R}}$.

Proof. Applying the proof of Proposition 2.13 in [Gro05], we would get that the monodromy transformation is

$$n \mapsto n + \langle m_2 - m_1, n \rangle r_2 + \langle m'_2 - m'_1, n \rangle r'_2 \text{ mod } \text{Span}_{\mathbb{R}}(r_1).$$

However, m_i and m'_i will differ by $e_2 + e_3 - e_4 - e_5$, because the cones containing P_i both contain the vertex v which is in the cone D_1 of type 1, as in the proof of Proposition 4.2 and 4.3. Since $n \in N'_{\mathbb{R}}$, we get that $\langle m_1, n \rangle = \langle m'_1, n \rangle$ and similarly with m_2 and m'_2 . Thus $\langle m_2 - m_1, n \rangle = \langle m'_2 - m'_1, n \rangle$. This is consistent with the fact that the image of n under the monodromy transformation should still lie in the subspace $N'_{\mathbb{R}}$ (modulo the span of r_1). $\square$

With these calculations in hand, we can now prove that the dual intersection complex is simple, given two conditions on h and h' . The first condition is that h and h' are MPCP functions on a subdivision of $\Delta_{P(2,4)}$. In other words, their maximal domains of linearity consist of cones over elementary simplices (which are lattice simplices that contain no lattice points other than their vertices).

The second condition is that the slopes of h and h' on the line from z_0^4 to z_1^4 in $\Delta_{P(2,4)}$ consist of eight distinct values. (See Figure 4.1 below.) This will

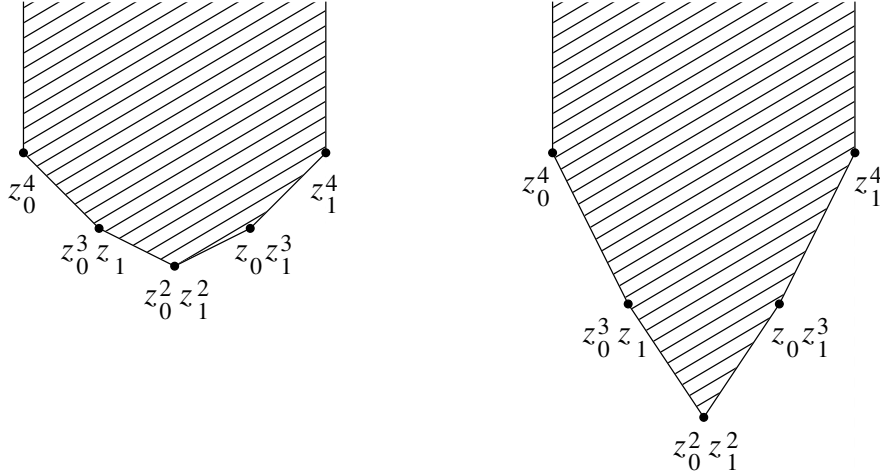


Figure 4.1: Cross sections of $\tilde{\Delta}_{P(2,4)}(h)$ (left) and $\tilde{\Delta}_{P(2,4)}(h')$ (right) for a pair h, h' satisfying the “eight distinct slopes” condition.

ensure that in the Minkowski sum $\tilde{\Delta}_{P(2,4)}(h) + \tilde{\Delta}_{P(2,4)}(h')$, the graph of the line from z_0^8 to z_1^8 in $2\Delta_{P(2,4)}$ consists of eight distinct line segments. The line from z_0^4 to z_1^4 (or rather its dual in the dual intersection complex) is where forbidden quadrivalent monodromy would occur if we applied the construction of [Gro05] to a quartic hypersurface in $P(2, 4)$ without resolving its nodal singularities. Even if the line was broken up into four distinct segments by an MPCP function, their duals would have quadrivalent monodromy, meaning that certain polytopes used in the definition of simplicity would be squares, while simplicity requires that all such polytopes are elementary simplices. Breaking the line up into eight segments breaks each square into a horizontal and vertical line segment, which the definition

of simplicity does allow.

The general definition of simplicity, given in Definition 1.60 of [GS06], is complicated, but in the 3-dimensional case it simplifies considerably. For a complete discussion see Example 1.62 of [GS06]. We must show that for all one-dimensional faces τ of the dual intersection complex, the polytope $\check{\Delta}(\tau)$ is an elementary simplex of some dimension. Likewise for τ a two-dimensional face of the dual intersection, we must have $\Delta(\tau)$ an elementary simplex of some dimension. The polytopes $\Delta(\tau)$ and $\check{\Delta}(\tau)$ are both defined by taking monodromy around loops passing through vertices of τ (see Definition 1.58 of [GS06]).

We will give a definition of $\Delta(\tau)$ and $\check{\Delta}(\tau)$ that is valid for our situation, namely, a three-dimensional dual intersection complex with no self-intersecting cells. To give the definitions, we first need to recall a few facts about monodromy of the dual intersection complex (B, P) . Suppose we are calculating monodromy by taking a loop ℓ that only touches the vertices of τ and the interiors of maximal faces containing τ , where $\tau \in P$ is a one-dimensional cell of the dual intersection complex. At any point p in the interior of τ not contained in the singular locus $B \setminus B_0$, there is a distinguished one-dimensional subspace of the tangent space Λ_p at p , $\Lambda_\tau \subseteq \Lambda_p$, which consists of vectors parallel to τ . If the loop ℓ is based at the vertex v of τ , then we can define $\Lambda_\tau \subseteq \Lambda_v$ by parallel transporting the subspace Λ_τ from a point in the interior of τ close to v . See section 1.5 of [GS06].

Since Λ_τ is an integral subspace of Λ_v , we can also choose a generator d_τ of the one-dimensional sublattice contained in Λ_τ .

Now, Propositions 1.29 and 1.32 of [GS06] show that the monodromy map $T_\ell : \Lambda_v \rightarrow \Lambda_v$ must have the form

$$T_\ell(m) = m + \langle n_\ell, m \rangle d_\tau \quad (4.1)$$

where $n_\ell \in \Lambda_v^*$ is contained in $\Lambda_\tau^\perp$.

Now suppose that $\tau \in P$ is a two-dimensional cell, and ℓ is a loop based at a vertex v of τ , which only touches other vertices of τ and the interiors of maximal cells containing τ . Similar to the case when τ was one-dimensional, we can define a two-dimensional integral subspace $\Lambda_\tau \subseteq \Lambda_v$, and choose a generator $\check{d}_\tau$ of the one-dimensional integral subspace $\Lambda_\tau^\perp \subseteq \Lambda_v^*$. The results of [GS06] then show that

the map $T_\ell : \Lambda_v \rightarrow \Lambda_v$ is given by

$$T_\ell(m) = m + \langle \check{d}_\tau, m \rangle m_\ell \quad (4.2)$$

where $m_\ell \in \Lambda_v$ is contained in Λ_τ .

With these results we can now define $\Delta(\tau)$ and $\check{\Delta}(\tau)$.

Definition 4.6. Let $\tau \in P$ be a one-dimensional face of the dual intersection complex (B, P) . We choose a vertex v of τ and generator of the sublattice contained in Λ_τ , d_τ , and fix them for use in all monodromy calculations. Given two maximal faces σ_1 and σ_2 containing τ , we let $\ell(\sigma_1, \sigma_2)$ be a loop which passes from v into the interior of σ_1 , from the interior of σ_1 to the other vertex of τ , v' , from v' to the interior of σ_2 , and back to v .

Now fix a maximal face containing τ , σ . We define $\check{\Delta}(\tau)$ as the convex hull of the vectors $n_{\ell(\sigma, \sigma_i)} \subseteq \Lambda_\tau^\perp$ in equation (4.1) as σ_i ranges over all maximal faces containing τ .

Definition 4.7. Let $\tau \in P$ be a two-dimensional face of the dual intersection complex (B, P) . τ is contained by two unique maximal faces of the dual intersection complex, σ_1 and σ_2 . We choose a vertex of τ , v , and generator of the sublattice contained in $\Lambda_\tau^\perp$, $\check{d}_\tau$, and fix them for use in all monodromy calculations. Given another vertex of τ , v' , we let $\ell(v, v')$ be the loop that passes from v to the interior of σ_1 , from the interior of σ_1 to v' , and from v' to the interior of σ_2 and back to v .

We define $\Delta(\tau)$ as the convex hull of the vectors $m_{\ell(v, v')} \subseteq \Lambda_\tau$ in equation (4.2) as v' ranges over all vertices of τ' .

The polytopes $\Delta(\tau)$ and $\check{\Delta}(\tau)$ are independent of the choices in the definitions, up to translation.

The faces of the dual intersection complex are dual to the faces in a lattice subdivision of the boundary of $2\Delta_{P(2,4)}$, specifically, the subdivision induced by $\check{\Delta}_{P(2,4)}(h) + \check{\Delta}_{P(2,4)}(h')$ which describes the singular fiber as a union of toric varieties. This is what will be meant by the “dual” of a face of the dual intersection complex.

For convenience, Figures 4.2 and 4.3 below show the boundary faces of $\Delta_{P(2,4)}$, since their configuration will be important in the proof. Note that the diagrams are

not strictly correct since the vertices of $\Delta_{P(2,4)}$ do not all lie in a three-dimensional hyperplane of $\mathbb{R}^4$, but they do show the combinatorics correctly. Figure 4.2 shows the four tetrahedral faces corresponding to the divisors $z_i = z_j = 0$, $z_i \in \{2, 3\}$, $z_j \in \{4, 5\}$, and Figure 4.3 shows the two faces corresponding to the divisors $z_0 = 0$ and $z_1 = 0$, which are pyramids over a square base.

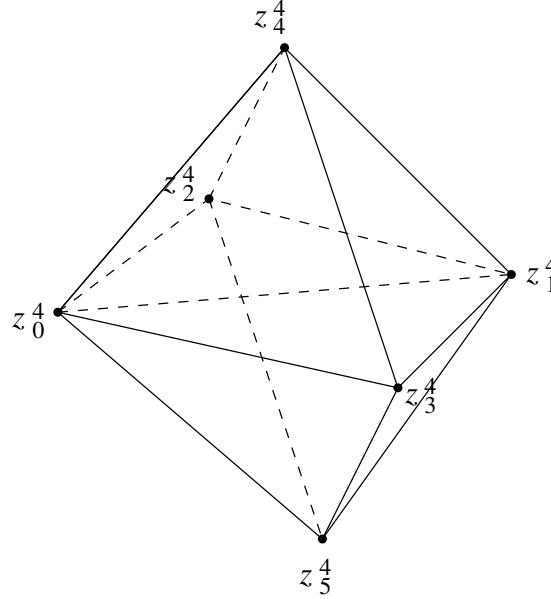


Figure 4.2: The four maximal boundary faces of $\Delta_{P(2,4)}$ corresponding to the divisors $z_i = z_j = 0$, $z_i \in \{2, 3\}$, $z_j \in \{4, 5\}$.

Proposition 4.8. *If τ is a one-dimensional face of the dual intersection complex, then $\check{\Delta}(\tau)$ is an elementary simplex of some dimension.*

Proof. Recall that $\check{\Delta}(\tau)$ is defined by calculating monodromy around loops that pass through the vertices of τ and the interiors of pairs of maximal faces containing τ . Since the vertices of τ are fixed, all monodromy calculations for a given τ will fall under one of Propositions 4.3, 4.4, or 4.5, depending on the types of the cones containing the vertices of τ .

In the case of Proposition 4.3, the $m_i \in M'_{\mathbb{R}} \oplus \mathbb{R}$ representing φ on maximal faces P_i containing τ will arise from a two-dimensional face dual to τ in the subdivision of $\Delta_{P(2,4)}$ on which h is piecewise linear. (Technically the dual of τ lies in a

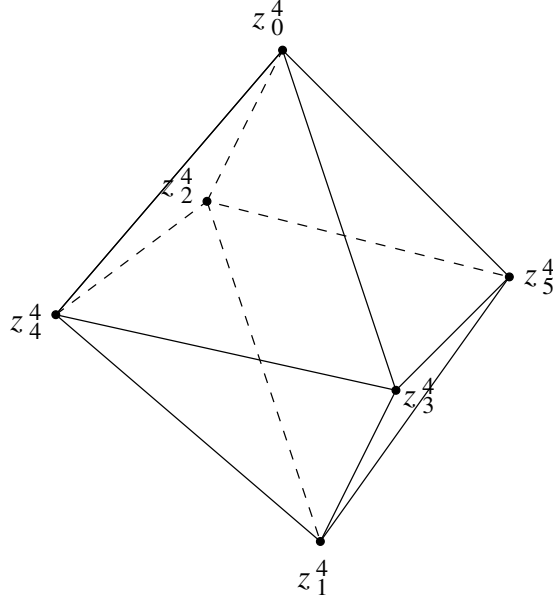


Figure 4.3: The two maximal boundary faces of $\Delta_{P(2,4)}$ corresponding to $z_0 = 0$ and $z_1 = 0$.

subdivision of $2\Delta_{P(2,4)}$, but since $h = h'$ on these faces, the subdivision is just a subdivision of $\Delta_{P(2,4)}$ scaled by a factor of 2.) This will be an elementary simplex by the MPCP assumption. The vector $r_2 - r_1$ is either primitive or zero, so the result follows.

In the case of Proposition 4.4, first note that one of the vectors $r_2 - r_1$ or $r'_2 - r'_1$ must be zero. This is because, by examining Figure 4.2, we see that a subface of the boundary face corresponding to $z_i = z_j = 0$ in $2\Delta_{P(2,4)}$ can only intersect a subface of $z_{i'} = z_{j'} = 0$ in a two-dimensional face f (dual to τ) if either $i = i'$ or $j = j'$. Therefore one of the terms in the monodromy mapping will be zero, let us assume without loss the $r_2 - r_1$ term. The m'_i in the other term then arise from a face in the subdivision of $\Delta_{P(2,4)}$ on which h' is piecewise linear. (Specifically, it is the face f_1 for which $f_1 + f_2 = f$ for some other face f_2 in the subdivision of $\Delta_{P(2,4)}$ corresponding to h .) By the MPCP assumption this is an elementary simplex, and again $r'_2 - r'_1$ is either primitive or zero.

In the third case (Proposition 4.5), the proof is the same as in the first case. In this case $\check{\Delta}(\tau)$ will be an elementary 2-simplex. $\square$

Proposition 4.9. *If τ is a two-dimensional face of the dual intersection complex, then $\Delta(\tau)$ is an elementary simplex of some dimension.*

Proof. Since τ is two-dimensional, its dual face τ' will be a one-dimensional face in a subdivision of the boundary of $2\Delta_{P(2,4)}$. The possible types of monodromy around loops passing through the vertices of τ will depend on which maximal-dimensional boundary faces of $2\Delta_{P(2,4)}$ contain τ' . An analysis of Figures 4.2 and 4.3 shows that the possibilities are:

1. τ' is contained by at most one face of the form $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$, and zero, one, or both of the faces $z_k = 0$ for $k = 0$ or 1 . In this case the monodromy will be of the type in Propositions 4.3 and 4.5. This situation occurs whenever τ' is not contained in two or more faces of the form $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$.

2. τ' is contained by exactly two faces of the form $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$, and at most one face of the form $z_k = 0$, $k = 0$ or 1 . The monodromy will be of the type in Propositions 4.4 and 4.5.

3. τ' is contained by all four of the faces of the form $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$. The monodromy is given by Proposition 4.4. This occurs when τ' is contained in the line segment from z_0^8 to z_1^8 .

(Note case 1 includes the possibility that τ' is contained by only one maximal boundary face, in which case there will be no monodromy.)

In the first case, the vector $m_2 - m_1$ in both Propositions 4.3 and 4.5 will be the same primitive vector by the MPCP assumption, since m_2 and m_1 are just translates of the vertices of a scaling of τ' , $\tau'/2$. (The scaling is because the subdivision of $2\Delta_{P(2,4)}$ in this region is an MPCP subdivision of $\Delta_{P(2,4)}$ scaled by a factor of two, since h and h' are equal on the faces corresponding to $z_0 = 0$ and $z_1 = 0$. The face τ' is itself a line segment containing three lattice points.) If there is any monodromy in this case we must have that τ' is contained in one of the faces $z_0 = 0$ or $z_1 = 0$; without loss assume $z_0 = 0$. Let r_0 and r_1 be the generators of the rays corresponding to these faces. If τ' is contained in one of the faces $z_i = z_j = 0$, then let r_2 and r'_2 be the generators of the rays for the divisors $z_i = 0$ and $z_j = 0$ (in the fan for $\mathbb{P}^5$). If we base the monodromy calculations at

the vertex of τ corresponding to $z_0 = 0$, then $\Delta(\tau)$ will be the image of one of $\text{Conv}(0, r_1)$, $\text{Conv}(0, r_1, r_2 + r'_2)$, or $\text{Conv}(0, r_2 + r'_2)$ in $N'_\mathbb{R}/\text{Span}_\mathbb{R}(r_0)$. These are all elementary simplices as needed.

In the second case, first suppose that τ' is contained in a face $z_k = 0$ with $k = 0$ or 1. Then we base the monodromy calculations at the vertex of τ corresponding to $z_k = 0$; again assume without loss that $k = 0$. Let r_0 be the generator of the ray corresponding to this divisor. Now if τ is contained in the faces $z_{i_1} = z_{j_1} = 0$ and $z_{i_2} = z_{j_2} = 0$, let r_1, r'_1, r_2, r'_2 respectively be the generators of the rays corresponding to $z_{i_1} = 0, z_{j_1} = 0, z_{i_2} = 0, z_{j_2} = 0$. Then $\Delta(\tau)$ will be equal to the image of $\text{Conv}(0, r_1 + r'_1, r_2 + r'_2)$ in $N'_\mathbb{R}/\text{Span}_\mathbb{R}(r_0)$, which is an elementary simplex as needed.

Now suppose that τ' is not contained in a face $z_k = 0$ with $k = 0$ or 1. Again let τ' be contained in the faces $z_{i_1} = z_{j_1} = 0$ and $z_{i_2} = z_{j_2} = 0$, and let r_1, r'_1, r_2, r'_2 respectively be the generators of the rays corresponding to $z_{i_1} = 0, z_{j_1} = 0, z_{i_2} = 0, z_{j_2} = 0$. The monodromy will be of the type in Proposition 4.4. If we base the monodromy calculations at the vertex of τ corresponding to $z_{i_1} = z_{j_1} = 0$, then $\Delta(\tau)$ will be equal to the image of $\text{Conv}(0, r_2 + r'_2)$ in $N_\mathbb{R}/\text{Span}_\mathbb{R}(r_1, r'_1)$, which is an elementary line segment.

In the third case, we will have that one of the expressions $\langle m_2 - m_1, n \rangle (r_2 - r_1)$ or $\langle m'_2 - m'_1, n \rangle (r'_2 - r'_1)$ in the monodromy map of Proposition 4.4 is zero for all monodromy calculations. To see this, note that τ' lies on the line from z_0^8 to z_1^8 in $2\Delta_{P(2,4)}$, where the hypothesis about h and h' having different slopes on the line from z_0^4 to z_1^4 applies. This means that τ' must be the Minkowski sum of a point and a primitive line segment in the boundary of $\Delta_{P(2,4)}$, since the eight distinct line segments in the subdivision of $2\Delta_{P(2,4)}$ must all be elementary, i.e., contain no vertices other than their endpoints. Thus one of the pairs m_2, m_1 or m'_2, m'_1 will arise from a single point, and one of $m_2 - m_1$ or $m'_2 - m'_1$ will be zero. The other expression will be a primitive vector. This implies that $\Delta(\tau)$ is a primitive line segment in this case: either the image of $\text{Conv}(0, r_2)$ or $\text{Conv}(0, r'_2)$ in $N_\mathbb{R}/\text{Span}_\mathbb{R}(r_1, r'_1)$. $\square$

Provided we can find a choice of h and h' such that both are MPCP and

they satisfy the condition on slopes and the conditions of Construction 3.1, this completes the proof of our main theorem, Theorem 3.3. However, condition 2 in Construction 3.1, that $\tilde{\Delta}(\pi^*h)$ and $\tilde{\Delta}(\pi^*h')$ are lattice polytopes, implies a slightly subtle restriction on h and h' that needs to be addressed. The images of faces of Δ under π are not necessarily faces of $\Delta_{P(2,4)}$. Thus, even if $\tilde{\Delta}_{P(2,4)}(h)$ is a lattice polytope, $\tilde{\Delta}(\pi^*h)$ might not be, because the values of $\pi^*(h)$ on a face of Δ might be affected by lattice points outside the face. Put a different way, the subdivision of $\Delta_{P(2,4)}$ on which h is linear might not be a refinement of the subdivision of $\Delta_{P(2,4)}$ induced by the images of faces of Δ . (This would be defined as the coarsest subdivision such that the image of each face of Δ is a union of cells in the subdivision.)

It is not difficult to see that the induced subdivision is given by the polytopes

$$\pi(\text{Conv}(z_i^4, z_j^4, z_k^4, z_2 z_3 z_4 z_5)),$$

with $i \in \{2, 3\}$, $j \in \{4, 5\}$, and $k \in \{0, 1\}$, together with the remaining faces corresponding to the divisors $z_i = z_j = 0$, $i \in \{2, 3\}$, $j \in \{4, 5\}$ (these faces remain intact since π is bijective here). The first faces are subfaces of the faces corresponding to the divisors $z_0 = 0$ and $z_1 = 0$. The original faces are three dimensional “pyramids” over the same square base, with the subdivision dividing them into four tetrahedra each. Thus, the two original faces corresponding to $z_0 = 0$ and $z_1 = 0$ are divided into a total of eight faces in the subdivision. The square is divided into four triangles by its diagonals, with a new vertex added at the center where the diagonals cross. See Figure 4.4.

We need the following lemma:

Lemma 4.10. *There exists an MPCP function h on $\Delta_{P(2,4)}$ which is piecewise linear on a refinement of the subdivision induced by the faces of Δ .*

Proof. We use the method for constructing MPCP functions given in Proposition 4 of [GZK89]. Namely, we start with a piecewise linear convex function h , and alter it by decreasing its values at the lattice points in the boundary of $\Delta_{P(2,4)}$ by small rational numbers (then obtain a new piecewise linear function by taking the convex hull of the graph of the altered function). At each stage the subdivision on

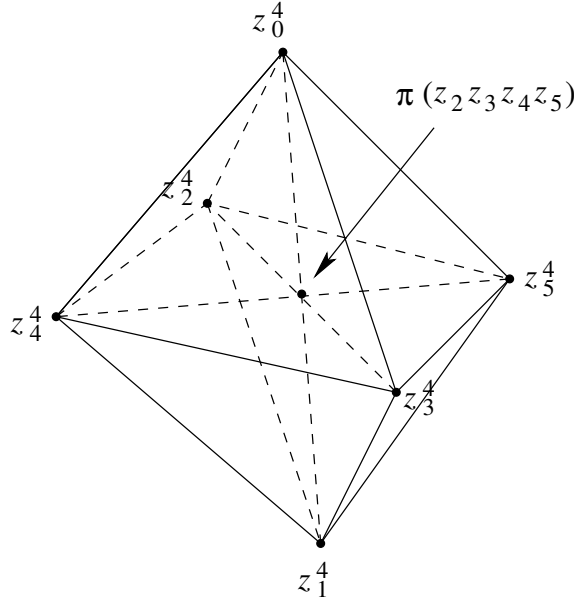


Figure 4.4: The subdivision of faces of $\Delta_{P(2,4)}$ induced by images of faces of Δ under the map π .

which h is piecewise linear is a refinement of the subdivisions at the previous steps. After sufficiently many iterations, the faces in the subdivision become elementary simplices, as required for MPCP functions.

Thus, let h be the piecewise linear function that is uniformly equal to one on each boundary face of $\Delta_{P(2,4)}$. We alter h by subtracting a small rational number from its value at $\pi(z_2 z_3 z_4 z_5)$. After this alteration, one can check that h is piecewise linear and strictly convex on the subdivision induced by faces of Δ . All subsequent subdivisions will be refinements of this subdivision, as required. $\square$

After we have obtained h , we obtain h' by subtracting small rational values from the three vertices corresponding to $z_0^3 z_1^3$, $z_0^2 z_1^2$, and $z_0 z_1^3$. This does not alter the value of h at other lattice points, since h is already MPCP, so there are four lattice points in each maximal face of the subdivision, constituting a basis for $M'_{\mathbb{R}}$. We can then scale up h and h' by the same integer, sufficiently large to make them integral.

This completes the proof of Theorem 3.3.

Remark 4.11. We make a brief comment about applications to mirror symmetry. To obtain the dual intersection complex of the mirror Calabi-Yau manifold, it is necessary to take the discrete Legendre transform of the dual intersection complex of $p : \mathcal{X} \rightarrow \mathbb{C}$. This requires a polarization of the family $\mathcal{X}$. The entire family is contained in the toric variety $X(h, h')$, which is naturally polarized by the polytope $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$, and this induces a polarization of $\mathcal{X}$. The dual intersection complex obtained from the discrete Legendre transform will also be simple. Theoretically at least, the mirror degeneration could then be constructed by the smoothing process of [GS07]. One could also attempt to construct a mirror degeneration by hand, using the dual intersection complex as a guide, as done in [Gro05].

Of course, the mirror manifold of a generic quartic hypersurface in $G(2, 4)$, or at least a smooth deformation of it, is already known by other methods, for instance, the Batyrev-Borisov construction. Since the manifold can be considered as a product of the Batyrev-Borisov construction, the main result of [Gro05] also shows the existence of dual toric degenerations of it and its mirror (both of which have simple dual intersection complexes). Nonetheless, finding a degeneration of the mirror which is dual to the degeneration p is a problem that could help with understanding conifold transitions and Calabi-Yau manifolds in higher-dimensional Grassmannians.

Chapter 5

Higher Dimensional Cases

This section is meant to informally illustrate how the approach in this paper could be applied to cases in higher-dimensional Grassmannians. Mainly, we want to show that the techniques used for the $G(2, 4)$ case could be applied to non-complete-intersection Grassmannians, even though $G(2, 4) \subseteq \mathbb{P}^5$ is a complete intersection. We will limit the discussion to the case of a quintic hypersurface in the six-dimensional Grassmannian $G(2, 5) \subseteq \mathbb{P}^9$.

Letting z_{ij} for $1 \leq i < j \leq 5$ be homogeneous coordinates on $\mathbb{P}^9$, the Plücker embedding of $G(2, 5)$ is defined by the five equations

$$z_{23}z_{45} - z_{24}z_{35} + z_{25}z_{34} = 0,$$

$$z_{13}z_{45} - z_{14}z_{35} + z_{15}z_{34} = 0,$$

$$z_{12}z_{45} - z_{14}z_{25} + z_{15}z_{24} = 0,$$

$$z_{12}z_{35} - z_{13}z_{25} + z_{15}z_{23} = 0,$$

$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0.$$

Substituting the variables z_{12} and z_{45} with tz_{12} and tz_{45} defines a family $\mathcal{X}_0$ in $\mathbb{P}^9 \times \mathbb{C}$ where the fibers for $t \neq 0$ are isomorphic to $G(2, 5)$ and the singular fiber is isomorphic to the toric variety $P(2, 5)$, defined by the five equations

$$z_{24}z_{35} = z_{25}z_{34}, z_{14}z_{35} = z_{15}z_{34}, z_{14}z_{25} = z_{15}z_{24},$$

$$z_{13}z_{25} = z_{15}z_{23}, z_{13}z_{24} = z_{14}z_{23}.$$

(See section 2.2 of [BCKv97].)

If we then intersect $\mathcal{X}_0$ with the additional equation

$$tf + z_{12}z_{45}z_{13}z_{34}z_{25} = 0,$$

where f is a generic quintic, we obtain a family $\mathcal{X}$ where the singular fiber is a smooth five-dimensional Calabi-Yau manifold. Because the monomial

$$z_{12}z_{45}z_{13}z_{34}z_{25}$$

vanishes precisely once on each toric divisor of $P(2, 5)$, it should be possible to prove that $\mathcal{X}$ is a toric degeneration. Here, the dual intersection complex would be the reflexive dual of the Newton polytope of quintics on $P(2, 5)$, with the affine structure given in Example 1.18 of [GS06].

This dual intersection complex is far from being simple, and would be the same if the degeneration consisted of singular Calabi-Yau hypersurfaces in $P(2, 5)$. We need a strategy for modifying the degeneration $\mathcal{X}$ to “detect” the smoothing of the ambient toric variety $P(2, 5)$ into $G(2, 5)$. The degeneration of $G(2, 5)$ into $P(2, 5)$ obtained with the substitutions $z_{12} \mapsto tz_{12}$ and $z_{45} \mapsto tz_{45}$ is unlikely to be sufficiently general for these purposes, much as the degeneration $tz_0z_1 + z_2z_3 - z_4z_5 = 0$ of $G(2, 4)$ into $P(2, 4)$ had to be modified by adding z_0^2 and z_1^2 terms.

First suppose we have a height function h on $\Delta_{P(2,5)}$, the Newton polytope of quintics on $P(2, 5)$. Using the method of [Gro05], we can define a degeneration in $\tilde{\Delta}_{P(2,5)}(h)$ with the equation

$$z_{12}z_{45}z_{13}z_{34}z_{25} + t^n \left(\sum_{m \in \Delta_{P(2,5)} \cap M'} t^{h(m)} c_m z^m \right) = 0,$$

where n is a positive integer and c_m for $m \in \Delta_{P(2,5)} \cap M'$ is a set of generic coefficients. (Here, M' is a free abelian group isomorphic to $\mathbb{Z}^6$ and $\Delta \subseteq M'_{\mathbb{R}}$.) Now let $\Delta \subseteq M_{\mathbb{R}}$ be the Newton polytope of quintics on $\mathbb{P}^9$. Similar to the $P(2, 4)$ case, we can regard $\Delta_{P(2,5)}$ as the image of Δ under a linear quotient map π , which quotients out by the binomial equations defining $P(2, 5)$. We can “pull back” the above equation to $\mathbb{P}^9$ by defining a degeneration in $\tilde{\Delta}(\pi^*h)$,

$$z_{12}z_{45}z_{13}z_{34}z_{25} + t^n \left(\sum_{m \in \Delta \cap M} t^{\pi^*h(m)} d_m z^m \right) = 0, \quad (5.1)$$

where now d_m for $m \in \Delta \cap M$ is another set of generic coefficients. Adding the five binomial equations for $P(2, 5)$ would then embed the previous degeneration in the larger toric variety associated to $\tilde{\Delta}(\pi^*h)$.

So far, this would give the same degeneration as in [Gro05]. If we wanted to attempt to “smooth” the degeneration, we would add equations defining a smoothing of $P(2, 5)$ into $G(2, 5)$, rather than the binomial equations defining $P(2, 5)$. As mentioned above, although substituting $z_{12} \mapsto tz_{12}$ and $z_{45} \mapsto tz_{45}$ does smooth $P(2, 5)$ into $G(2, 5)$, it is unlikely to be sufficient for our purposes. What would probably be necessary is to map z_{12} to something of the form $\sum_{m \in \Delta' \cap M} t^{h_1(m)} z^m$ (perhaps also adding generic coefficients), where Δ' is the Newton polytope of $\mathcal{O}_{\mathbb{P}^9}(1)$ and h_1 is some height function defined on Δ' . Likewise, we would map z_{45} to something of the form $\sum_{m \in \Delta' \cap M} t^{h_2(m)} z^m$, with h_2 being another height function. As an example, the first defining equation for the Grassmannian would alter to:

$$z_{23} \left(\sum_{m \in \Delta' \cap M} t^{h_2(m)} z^m \right) - z_{24}z_{35} + z_{25}z_{34} = 0. \quad (5.2)$$

If we were to multiply equation (5.2) by $-z_{12}z_{45}z_{13}$ and add it to equation (5.1), we would obtain a new equation

$$z_{12}z_{45}z_{13}z_{24}z_{35} - z_{12}z_{45}z_{13}z_{23} \left(\sum_{m \in \Delta' \cap M} t^{h_2(m)} z^m \right) + t^n \left(\sum_{m \in \Delta \cap M} t^{\pi^*h(m)} d_m z^m \right) = 0.$$

Here we have a new initial monomial term in which $z_{25}z_{34}$ has been “swapped out” for $z_{24}z_{35}$ (much in the same way that z_2z_3 is swapped for z_4z_5 in the $G(2, 4)$ case). Under the appropriate conditions, the second term in which the function h_2 appears can be viewed as altering the height function h in the third term at certain lattice points. So this equation really lies in the toric variety defined by $\tilde{\Delta}(\pi^*g_1)$, where g_1 is the function h with its values replaced by those of $h_2 - n$ in certain regions. The other defining equations for the Grassmannians would produce similar alterations of h , g_2 , g_3 , etc., and the entire construction would lie in the toric variety associated to a Minkowski sum of the form

$$\tilde{\Delta}(\pi^*g_1) + \tilde{\Delta}(\pi^*g_2) + \cdots$$

which is analogous to the sum $\tilde{\Delta}(\pi^*h) + \tilde{\Delta}(\pi^*h')$ in the $G(2,4)$ case. Obviously there are many combinatorial details to be worked out here, but the core idea remains the same as for the $G(2,4)$ case, and this should produce at least a partial smoothing of the dual intersection complex.

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