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UNIVERSITY OF CALIFORNIA SAN DIEGO

Geometric and Representation Theoretic Aspects of p -adic Motives

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Xin Tong

Committee in charge:

Professor Kiran Kedlaya, Chair
Professor Kenneth Inrtiligator
Professor James McKernan
Professor Cristian Popescu
Professor Claus Sorensen

2021

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University of California San Diego

2021

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ABSTRACT OF THE DISSERTATION

Geometric and Representation Theoretic Aspects of p -adic Motives

by

Xin Tong

Doctor of Philosophy in Mathematics

University of California San Diego, 2021

Professor Kiran Kedlaya, Chair

In this dissertation, we discuss mainly the corresponding geometric and representation theoretic aspects of relative p -adic Hodge theory and p -adic motives. To be more precise, we study the corresponding analytic geometry of the corresponding spaces over and attached to period rings in the relative p -adic Hodge theory, including derived topological de Rham complexes and derived topological logarithmic de Rham complexes after Bhatt, Gabber, Guo and Illusie which is in some sense equivalent to the derived prismatic cohomology of Bhatt-Scholze as shown in the work of Li-Liu, $\mathcal{O}_{\mathbb{B}_{\text{dR}}}$ -sheaves after Scholze, φ - $\tilde{\mathcal{C}}_X$ -sheaves and relative- B -pairs after Kedlaya-Liu, multidimensional rings after Carter-Kedlaya-Zábrádi and Pal-Zábrádi and many other possible general universal motivic rings or sheaves. Many contexts are expected to be sheafified, such as over Scholze's pro-étale sites of the considered analytic spaces by using perfectoids or the quasisyntomic sites by using quasiregular semiperfectoids as in the work of Bhatt-Morrow-Scholze and Bhatt-Scholze. The main motivation comes from the corresponding noncommutative Tamagawa Number conjectures after Burns-Flach-Fukaya-Kato, relative version of the generalized version of the period rings as in the work of Carter-Kedlaya-Zábrádi and Pal-Zábrádi, arithmetic families of the representations of fundamental groups in analytic geometry

such as for analytification of the moduli stacks of algebraic curves after Reinecke, arithmetic families of general motivic structures in analytic geometry such as in the work of Andreatta-Brinon, Andreatta-Iovita, Berger, Bhatt-Morrow-Scholze, Bhatt-Scholze, Fargues-Fontaine, Fargues-Scholze, Fontaine and Kedlaya-Liu, noncommutative analytic geometry and noncommutative deformation, derived noncommutative analytic geometry and derived noncommutative deformation, Langlands programs, analytic approach to algebraic topology and so on. Due to the natural though not functorial correspondence between the linear topology and the one induced by a Banach norm, we do not restrict ourselves to the functional analytic point of view when we take completion after Bambozzi-Ben-Bassat-Kremnizer, Clausen-Scholze, Gabber-Ramero, Huber, Kedlaya-Liu and Scholze.

Chapter 1

Preliminary and Introduction

[CS1], [CS2] and the corresponding derived I -complete objects are really more relevant in the arithmetic geometry in our mind such as the corresponding objects in [BMS2], [BS1], [BS2], [Lu3], [NS], [Po] and [Ye]¹. In fact on the ∞ -categorical level, [BMS2] and [NS] considered the I -completion in the ∞ -category theory as in [Lu3, Chapter 7.3]. The corresponding derived Nygaard-complete, derived Hodge-complete or other derived filtered complete objects are relevant while the corresponding derived Nygaard-incomplete, derived Hodge-incomplete or other derived filtered incomplete objects may be also relevant in certain situations.

¹These already include many different types of I -adic completion in the derived sense, and the corresponding localization and completion in the algebraic topology and ∞ -categorical theory. We would believe that the corresponding noncommutative consideration as these will be robust enough to deal with the main problems in Iwasawa theory and noncommutative derived analytic geometry.

1.2 Introduction

1.2.1 The Motivation

Noncommutative Tamagawa number conjectures and noncommutative Iwasawa main conjectures are main topics in the field of arithmetic geometry. They give us the explicit approaches to study the motives after we take the realizations². The associated Galois representations for instance could be used to define the corresponding L -functions, as well as local factors. Roughly speaking, the conjectures mention the following style isomorphism up to certain factor³:

$$\text{Determinant}_A(R\Gamma(\pi_1(*), p\text{LieDef}_A(V))) \longrightarrow \text{Determinant}_A(0) \quad (1.2.1)$$

when taking some possibly trivial Iwasawa deformation along some p -adic Lie group tower and one could achieve the zero homotopy from the determinant of the cohomology in certain categories⁴, which should be closely related to special values of L -function and generalized functional equation.

Our study was significantly inspired by the pictures of [BF1], [BF2], [FK], [Na1], [Wit1], [Wit2], [Wit3]⁵. On the other hand, work of Kedlaya-Liu [KL1], [KL2] and work of Kedlaya-Pottharst [KP] also significantly inspired us to consider at least the geometrization or generalization of the work of [BF1], [BF2], [FK], [Na1], [Wit1], [Wit2], [Wit3]. In fact we have other important p -adic cohomology theories, especially in the integral setting we have the prismatic cohomology from Bhatt-Scholze [BS1]. Therefore one might want to ask if we could have a well-posed prismatic Iwasawa deformation theory. To be more precise, whenever we have a well-defined p -adic cohomology theory, we should be able to consider some interesting Iwasawa deformation theory, possibly carrying some Banach coefficients as in [Na1]. It is actually very straightforward to first consider the integral picture, and it is very confusing to actually first understand the picture given certain Fréchet-Stein algebra in [ST] since the classical Iwasawa main conjecture happens over

²One might want to consider motives over \mathbb{Z} after some reasonable p -adic cohomology theories are established such as in [Ked3] and [Sch5]. Doing so might be relevant in some p -adic cohomological approach to Riemann Hypothesis over \mathbb{Z} after [Ked2] which is parallel to [De1], [De2], [Wei].

³ $\pi_1(*)$ is the profinite fundamental group of certain arithmetic scheme or analytic space.

⁴As those categories in [De3], [FK], [MT], [Wit3].

⁵The commutative picture was already in [Ka1], [Ka2], [PR1], [PR2].

integral Iwasawa algebra Λ as in [Iw].

1.2.2 The Results

In fact our study is just as described in the consideration above, namely we actually study some non-étale objects which will happen over certain period sheaves over analytic spaces carrying Frobenius actions. And then we can take the corresponding Banach deformation as well as the Iwasawa deformation. Note that the previous deformation will be very crucial such as in [KPX] and [Na1]. Meanwhile we can regard them as some certain sheaves over Fargues-Fontaine stacks in some deformed way after [FF], [KL1] and [KL2]. And we need to work with noncommutative rings as in [BF1], [BF2], [FK], [Wit1], [Wit2], [Wit3], [Z]. These are reflected in the following:

1. Hodge-Iwasawa Deformation: proposition 2.3.8, proposition 2.3.10, proposition 3.4.25, proposition 3.4.26 and proposition 3.5.51.
2. Multidimensional Frobenius Modules: theorem 4.1.5 and theorem 4.1.6.
3. Hodge-Iwasawa Cohomology: proposition 6.4.3, proposition 6.4.26, proposition 6.4.27, proposition 6.4.28, proposition 6.4.44, proposition 6.4.45, proposition 6.4.50, proposition 6.4.51.
4. Noncommutative Hodge-Iwasawa Deformation: theorem 7.3.15, theorem 7.3.16, corollary 7.3.17, theorem 7.3.18, theorem 7.3.21, theorem 7.4.12, proposition 7.5.13, theorem 8.1.1, theorem 8.1.2, theorem 8.1.3, theorem 8.1.4, theorem 8.1.5, theorem 8.1.6, theorem 8.1.7, theorem 9.1.1, theorem 9.1.2.
5. Derived Noncommutative Hodge-Iwasawa Deformation: theorem 10.1.1, theorem 10.1.2, theorem 10.1.3, theorem 10.1.4, theorem 10.5.10.

Remark 1.2.1. Our goal is to find certain derived Iwasawa ∞ -categories as in [De3], [FK], [MT], [Wit3] in analytic geometry over certain deformed sheaves or rings. Note that this is very complicated and particularly far beyond Iwasawa deformation of regular motives. What we have achieved is literally some generalization of Kedlaya-Liu abelian categories of pseudocoherent sheaves in [KL2].

We now discuss some examples of the spaces and local charts in our study. As mention above, they at least will be some spaces over or attached to some period rings or sheaves, which sometimes are the corresponding local charts of some stacks such as the Fargues-Fontaine spaces or the spaces before taking the quotients by equivariance coming from the motivic structures.

Example 1.2.2. (∞ -Categorical Rings and ∞ -Categorical Spaces)

1. The rigid analytic affinoids and spaces in [Ta] are some key examples, which is actually some initial goal in our main consideration mentioned above inspired by [BF1], [BF2], [FK], [Wit1], [Wit2], [Wit3], [Z];
2. The pseudorigid analytic affinoids and spaces over \mathbb{Z}_p are also interesting to study, which is a geometric version of the arithmetic family in [Bel1] and [Bel2];
3. We will consider the noncommutative I -adically complete rings as in [FK] in the Iwasawa consideration, as well as simplicial commutative rings which are derived I -adically complete rings over any interesting period rings, such as the prisms in [BS1], for instance the $\mathbb{A}_{\text{inf}}(\mathcal{O}_{\mathbb{C}_p^\times})$, $W(\mathbb{F}_p)[[u]]$, $\mathbb{Z}_p[[q-1]]$ and Robba rings in [KL1] and [KL2]. One can consider for instance the topological rings over these period rings carrying the topology induced from the period rings. Tate adic Banach rings in [KL1] and [KL2] produce certain topological adic rings satisfying the open mapping property as in [CBCKSW].

In chapter 2 and 3, we study the Frobenius modules over Robba rings carrying rigid analytic coefficients and Fréchet-Stein coefficients, in both equal-characteristic situation and mixed-characteristic situation. We call the theory Hodge-Iwasawa since the study of the Frobenius modules over the Robba rings and sheaves are significant in Galois deformation theory, deformation of representations of fundamental groups of analytic spaces and our generalizations of the picture in [BF1], [BF2], [FK], [KP], [Na1], [Wit1], [Wit2], [Wit3]. We show the equivalence between categories of finite projective or pseudocoherent (φ, Γ) -modules over Robba rings with rigid analytic coefficients and Fréchet-Stein coefficients, which also are compared to sheaves over schematic and adic Fargues-Fontaine

curves in some deformed sense. Especially when we are working over analytic fields, the picture is already interesting and significant enough in the Galois representation theory and Galois deformation theory.

In chapter 4 and 5, we study multidimensional Robba rings and multidimensional (φ, Γ) -modules. This point of view of taking multidimensional analogs of the Robba rings and multidimensional (φ, Γ) -modules is actually motivated from some programs in making progress of the local Tamagawa number conjecture of Nakamura in [Na1] literally proposed in [PZ]. Following [CKZ] and [PZ], we define the corresponding multidimensional Robba rings and multidimensional (φ, Γ) -modules by taking analytic function rings over p -adic rigid affinoids in rigid geometry. And we define the multidimensional (φ, Γ) -cohomologies, multidimensional (ψ, Γ) -cohomologies and multidimensional ψ -cohomologies. We carefully study the complexes of the multidimensional (φ, Γ) -cohomologies, multidimensional (ψ, Γ) -cohomologies and multidimensional ψ -cohomologies, and show that they are actually living in the derived category of the bounded perfect complexes. Chapter 4 mainly focuses on imperfect Robba rings, while in chapter 5 we define perfection of Robba rings in several variables and study the comparison of multidimensional (φ, Γ) -modules in certain situations carefully, which is literally following [KL1], [KL2], [KPX] and [Ked1].

In chapter 6, we apply the main results in chapter 3 to study the cohomologies and categories of relative (φ, Γ) -modules over Robba sheaves over certain analytic spaces. We mainly discuss three applications which are crucial in our project of Iwasawa deformation of motivic structures over some higher dimensional spaces. The first is the study of abelian property of the categories of relative (φ, Γ) -modules over Robba sheaves over rigid analytic spaces after [KL2], which is important whenever one would like to construct some K -theoretic objects to formulate Iwasawa main conjectures as in [Wit1], [Wit2], [Wit3]. The second is the study of families of Riemann-Hilbert correspondences after [LZ] which is crucial in further application to the arithmetic geometry along our consideration. The last one is the consideration of the equivariant version of the Iwasawa main conjecture of Nakamura [Na2].

In chapter 7, 8 and 9, we consider generalization of our work of the generalization

of the work of Kedlaya-Liu presented in chapter 2, 3 and 6, which is also motivated by our consideration in chapter 4 and 5 after [CKZ], [Na1], [PZ] and [Z] in order to make further progress. We consider the deformation in possibly noncommutative general Banach rings such as perfectoid rings, preperfectoid rings, general quotient of the noncommutative free Tate rings and so on. Certainly one thing we have to deal with is the sheafiness of the deformed rings, which will produce some difficulty to apply Kedlaya-Liu's descent in [KL1], [KL2] for vector bundles and stably-pseudocoherent sheaves. Even in the noncommutative coefficient situation we have not worked out a theory on the noncommutative analytic toposes, which implies there is no geometric method for us to study and apply. So finding new ideas is very important. In fact, we on the representation theoretic level have the result due to Kedlaya to have the descent for vector bundles. And one can in the commutative situation use Clausen-Scholze space [CS2] to achieve the similar result by embedding Huber spaces to Clausen-Scholze spaces in [CS2] and apply Clausen-Scholze descent. Also we could consider the derived analytic spaces from Bambozzi-Kremnizer in [BK]. In the noncommutative situation we generalize results in [KL1], [KL2] and [CBCKSW] to deform the structure sheaves directly in analytic topology, étale topology and pro-étale but not v -topology, which allows us to compare certain stably-pseudocoherent sheaves and modules carrying Banach deformed coefficients even if they are noncommutative, which certainly provides possibility to make further progress in the study of the noncommutative situation in chapter 4 and 5.

In chapter 10, we initiate the project on some topics on the geometric and representation theoretic aspects of period rings. In this first paper, we consider more general base spaces. To be more precise we discuss more general perfectoid rings. Distinguished deformation of rings is a generalization notion of the Fontaine-Wintenberger idempotent correspondence. For instance in [BS1], for any quasiregular semiperfectoid ring A one can canonically associate a prism (P_A, I_A) . This is a very general correspondence generalized from the notions in [BMS1] and [GR]. Therefore in our consideration we will consider the adic Banach rings in [KL1], [KL2] which are not necessarily analytic in the sense of Kedlaya's AWS lecture notes in [CBCKSW]. Assuming certain perfectoidness after [KL1] and [KL2] we study the deformed Robba rings associated. Also after [KL1] and [KL2] we studied derived deformation of the Robba rings and the descent of finite projective mod-

ule spectra over them, which we will believe has some application to conjectural derived eigenvarieties and derived Galois deformation for instance in [GV].

In chapter 11 and 12, we consider more widely the process of taking topological and functional analytic completions, in the derived sense coming from [B1], [BBBK], [BK], [BMS2], [BS1], [CS1]. Maybe this derived consideration will let us see the hidden ∞ -categorical structures of the representations of the Iwasawa algebras in our Iwasawa-Prismatic theory, since we are taking the deformations in some derived sense. Besides the application in mind to the prismatic or derived de Rham Iwasawa theory, chapter 11 and chapter 12 initiate also the study of some relative p -adic motive and Hodge theory over general derived I -adic spaces after [An1], [An2], [B1], [B2], [Bei], [BMS2], [BS1], [CBCKSW], [DLLZ1], [DLLZ2], [Dr1], [GL], [G1], [Hu2], [III1], [III2], [KL1], [KL2], [NS], [O], [Qui], [Ro], [Sch2] such as the pseudorigid analytic spaces and more general spaces carrying some derived I -adic topology, as well as the prelog simplicial commutative rings in [B1] carrying some derived I -adic topology as well.

1.2.3 The Picture and The Future Consideration

Let us try to discuss a little bit how one should think about our main motivation. We start from the following two settings of possible geometrizations of Iwasawa theory.

1. **∞ -Categorical Iwasawa-Prismatic Theory**⁶: After Bhatt-Lurie, Bhatt-Scholze and Drinfeld [BS1], [Dr], [Sch4];

2. **∞ -Categorical Hodge-Iwasawa Theory**⁷: After Kedlaya-Liu and Kedlaya-Pottharst [KL1], [KL2], [KP].

Setting 1.2.3. (∞ -Categorical Iwasawa-Prismatic Theory) Just like in [KP], [Wit1], [Wit2], [Wit3], and by using prismatic cohomology theory in [BS1], [Sch4] we can take a quasisyntomic formal ring R^8 over \mathbb{Z}_p and have two sites of $\mathrm{Spf}R$. The first site is the

⁶This should be slightly more relevant in the geometrization of integral Iwasawa theory, but one can invert p as well.

⁷This should be slightly more relevant in the geometrization of rational Iwasawa theory, but one can not invert p as well.

⁸As in [Sch4], one can take a p -adic formal scheme which is quasisyntomic. Certainly this is already a

corresponding prismatic site $(X_{\text{prim}}, \mathcal{O}_{X_{\text{prim}}})$. The second site is the quasisyntomic site $(X_{\text{qsyn}}, \mathcal{O}_{X_{\text{qsyn}}})$. Recall for the second one, for any quasiregular semiperfectoid affinoid A in X_{qsyn} we have that one can canonically associate a prism (P_A, I_A) to A where we have $\mathcal{O}_{X_{\text{qsyn}}}(A) := P_A$. Now by Bhatt-Scholze [Sch4] we have more sheaves from the structure sheaves here namely we have:

$$\mathcal{O}_{X_{\text{prim}}}[1/I_{\mathcal{O}_{X_{\text{prim}}}}]_p^\wedge, \mathcal{O}_{X_{\text{prim}}}[1/I_{\mathcal{O}_{X_{\text{prim}}}}]_p^\wedge[1/p], \mathcal{O}_{X_{\text{qsyn}}}[1/I_{\mathcal{O}_{X_{\text{qsyn}}}}]_p^\wedge, \mathcal{O}_{X_{\text{qsyn}}}[1/I_{\mathcal{O}_{X_{\text{qsyn}}}}]_p^\wedge[1/p]. \quad (1.2.2)$$

Carrying some integral Iwasawa algebra $\mathbb{Z}_p[[G]]$ for some compact p -adic Lie group, and after taking the derived completion⁹ we have:

$$\mathcal{O}_{X_{\text{prim}}}[1/I_{\mathcal{O}_{X_{\text{prim}}}}]_p^\wedge \otimes^{\text{Lcom}} \mathbb{Z}_p[[G]], \quad (1.2.3)$$

$$\mathcal{O}_{X_{\text{prim}}}[1/I_{\mathcal{O}_{X_{\text{prim}}}}]_p^\wedge \otimes^{\text{Lcom}} \mathbb{Z}_p[[G]][1/p], \quad (1.2.4)$$

$$\mathcal{O}_{X_{\text{qsyn}}}[1/I_{\mathcal{O}_{X_{\text{qsyn}}}}]_p^\wedge \otimes^{\text{Lcom}} \mathbb{Z}_p[[G]], \quad (1.2.5)$$

$$\mathcal{O}_{X_{\text{qsyn}}}[1/I_{\mathcal{O}_{X_{\text{qsyn}}}}]_p^\wedge \otimes^{\text{Lcom}} \mathbb{Z}_p[[G]][1/p]. \quad (1.2.6)$$

Then one might want to ask if one can use such style deformation to establish the parallel story in [BF1], [BF2], [FK], [KP], [Wit1], [Wit2], [Wit3]. For instance taking the $J \subset \mathbb{Z}_p[[G]]$ -adic quotient we have the Koszul complexes parametrized by such J :

$$\text{Kos}_J \mathcal{O}_{X_{\text{prim}}}[1/I_{\mathcal{O}_{X_{\text{prim}}}}]_p^\wedge \otimes \mathbb{Z}_p[[G]], \quad (1.2.7)$$

$$\text{Kos}_J \mathcal{O}_{X_{\text{qsyn}}}[1/I_{\mathcal{O}_{X_{\text{qsyn}}}}]_p^\wedge \otimes \mathbb{Z}_p[[G]]. \quad (1.2.8)$$

$$(1.2.9)$$

Then one may define the ∞ -category of pro-systems of the Iwasawa complexes over these \mathbb{E}_1 -rings, and consider the associated Waldhausen categories as in [Wit1], [Wit2], [Wit3].

Setting 1.2.4. (∞ -Categorical Iwasawa-Prismatic Theory) Within the same framework we take R to be \mathcal{O}_K for some p -adic local field K . Bhatt-Scholze [BS1], [Sch4] showed type of spaces which is interesting enough including a point situation and many situations where motivic comparisons could happen as in [BMS2], [BS1].

⁹At this moment we are assuming that the derived completion is possible in our current setting. We want to mention that in the noncommutative setting there are many ways to do the completion in the derived sense, which is already subtle in the commutative setting.

that we have the category of Galois representations of \mathbb{Z}_p -coefficients of Gal_K is equivalent to the category of prismatic F -crystals over

$$(X_{\text{prim}}, \mathcal{O}_{X_{\text{prim}}} [1/I_{\mathcal{O}_{X_{\text{prim}}}]_p^\wedge]),$$

while the category of Galois representations of \mathbb{Q}_p -coefficients of Gal_K is equivalent to the category of prismatic F -crystals over $(X_{\text{prim}}, \mathcal{O}_{X_{\text{prim}}} [1/I_{\mathcal{O}_{X_{\text{prim}}}]_p^\wedge [1/p])$. Then one could ask if we could consider some Iwasawa deformation through the some p -adic Lie quotient of Gal_K to establish the parallel story in [KP], [Wit1], [Wit2], [Wit3]. Namely for any such F -crystal M with associated representation V over \mathbb{Z}_p or \mathbb{Q}_p we take the Iwasawa deformation $p\text{DfLie}(M)$ by some Iwasawa F -crystal¹⁰ through the quotient from G_K to some compact p -adic Lie group G , then we can ask if the following:

$$R\Gamma(X_{\text{prim}}, p\text{DfLie}(M)), R\Gamma(X_{\text{qsyn}}, p\text{DfLie}(M)) \quad (1.2.10)$$

recover the classical Iwasawa theory by using the Galois cohomology

$$R\Gamma(\text{Gal}_K, p\text{DfLie}(V))$$

of $p\text{DfLie}(V)$, as well as the étale cohomology $R\Gamma(\text{Spec}K, p\text{DfLie}(\tilde{V}))$ of the local system $p\text{DfLie}(\tilde{V})$ attached to $p\text{DfLie}(V)$.

Beyond the somehow étale situations in the above picture, one could consider the corresponding category of prismatic crystals, which will be beyond the Galois representation theoretic consideration. Also one could regard these objects as certain sheaves over the prismatic stacks in [Dr].

In our study, we have the following picture. We will consider picture beyond étale situation, and we will study the Frobenius sheaves and Frobenius modules in the very general situation. And we will have the chance to regard the sheaves and modules with Frobenius actions as certain sheaves over Fargues-Fontaine stacks after [FF], [KL1] and [KL2]. Actually we conjecture that the quasisyntomic descent and étale comparison results of Bhatt-Scholze [BS1] will imply equivalence in some accurate sense beyond the vague similarity.

¹⁰How one should define this crystal will be determined by how one forms the completed tensor product in the Iwasawa deformation.

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Chapter 2

Hodge-Iwasawa Theory I

2.1 Introduction

2.1.1 Hodge-Iwasawa Theory and Higher Dimensional Modeling of the Weil Conjectures

This paper deals with some simultaneous generalization of the noncommutative Iwasawa theory after Kedlaya-Pottharst and the relative p -adic Hodge theory in the style of Kedlaya-Liu, with further sophisticated philosophy in mind dated back to Fukaya-Kato and Kato. To make the motivations and the applications more serious and clear to the readers, let us start from recalling what happens along the intersection of the two pictures above.

Example 2.1.1. (After Kedlaya-Pottharst) Let K be a finite extension of \mathbb{Q}_p . In [KP1], the authors proposed that one could consider a family $\{L\}_L$ parametrized by perfectoid Galois fields of intrinsic descriptions of the category $\widetilde{\Phi}\Gamma_{K,A}$ (with the notations in [KP1]) in terms of φ -modules over $\widetilde{\Pi}_{L,A}$ in the family with the further action of the groups $G(L/K)$ for each L in the family. These actually for a fixed L/K will give rise to some perfectoid p -adic Hodge theory immediately. Then they considered some geometrization which is just some application of the theory in [KL15, Chapter 9], namely one first forgets the action of the group $G_{L/K}$ but replaces this by pro-étale topology over $\mathrm{Spa}(K, \mathcal{O}_K)$. Then one defines an object \mathcal{M} in the category $\widetilde{\Phi}\Gamma_{K,A}$ which has no a priori relation to any tower L/K . One recovers the Iwasawa theory from the sheaf attached to the Iwasawa algebra attached to the group $G(L/K)$. This makes more sense if we know clearly the structure of $G(L/K)$. The memorization of the Iwasawa structure back from the Iwasawa deformation (instead of taking the limit throughout some rigidized towers) will be some key observation in our development.

On the other hand, we have the following example:

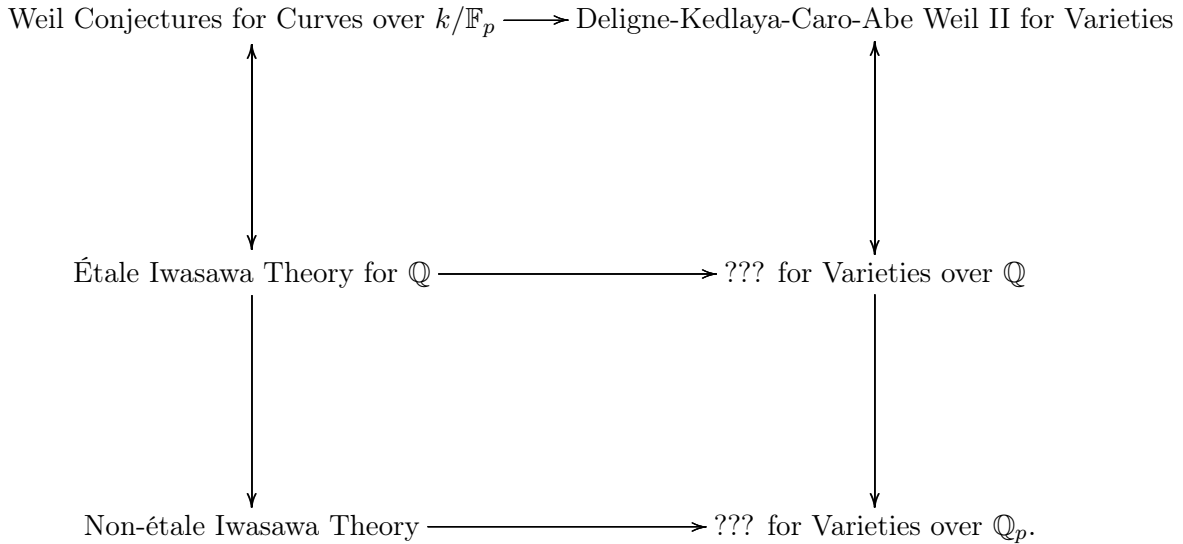
Example 2.1.2. Let A be now an Artin algebra over \mathbb{Q}_p , in this situation one could actually consider the local systems \underline{A} over some specific smooth proper rigid analytic space X/\mathbb{Q}_p . Therefore one could consider the situation where this deforms from some \mathbb{Q}_p -local systems. Then one should be able to relate constructible \underline{A} -local systems to the corresponding deformation of the corresponding \mathbb{Q}_p representation of the corresponding étale

fundamental groups. One could then as in [KL15] and [KL16] consider the corresponding Galois groups of affinoids in both rigid geometry, adic geometry or perfectoid geometry. Then in these situations, one could consider the corresponding deformation problem. Furthermore one could then consider more general rigid affinoids A/\mathbb{Q}_p and consider the corresponding families of representations of the Galois groups mentioned above over A . In order to study these one could then construct the corresponding A -relative relative period rings perfect generalizing [KL15] and [KL16].

Therefore now we consider the combination of the two pictures and the corresponding simultaneous generalization of the pictures in the relative p -adic Hodge theory and the Iwasawa theory in the sense mentioned above. Recall that the toric chart in [KL16] could admit some cyclotomic deformation with the corresponding Γ -action where Γ will be just \mathbb{Z}_p^d for some rank d . Then one could consider the corresponding picture in the Iwasawa theory to consider the corresponding (φ, Γ) -cohomology of the corresponding multivariate cyclotomic deformations. We will call this generalized Iwasawa cohomology and the generalized Iwasawa theory, since over the perfectoid covering of the base space over $\mathbb{Q}_p(\widehat{p^{1/p^\infty}})$, one has the corresponding relative de Rham period rings. Therefore one could consider the generalized Bloch-Kato dual exponential maps and so on.

One should somehow regard that Iwasawa theory is mimicking the celebrated Weil conjectures, namely the Riemann Hypothesis for the arithmetic of the algebraic curves. The conjectures themselves actually do not have too much sort of motivic motivation a priori, for instance one could just ask the questions on the counting problems within the arithmetic of algebraic curves. However the conjectures to some extent are definitely admitting deep and fundamental motivic point of views. Since the issue is that one could define the corresponding L -functions in terms of some Hodge structure for algebraic varieties over finite fields, namely the Frobenius actions on the ℓ -adic étale cohomologies. Also one could actually directly reinterpret the corresponding number theoretic properties in term of the corresponding cohomological properties, for instance the zeros of zeta functions and the functional equations, which is further adopted by Perrin-Riou in her celebrated work for instance. Iwasawa established the corresponding point of view successfully in the context of number fields, which is actually to some extent very fascinating and deep. Although the original pictures from Iwasawa admit more class group formulation

and the corresponding motivation (namely more Galois theoretic or more étale), which is similar to the original motivation of Weil conjectures, but following later philosophy like from Kato everything could be formulated quite cohomologically, namely due to some deep relationship and similarization between the étale cohomology and Galois cohomology.



As shown in the diagram above, one could see that actually Weil conjectures are quite general, in the sense that first they are actually above varieties over finite fields, which was originally proved by Deligne in terms of étale cohomology. By p -adic method, Kedlaya reproved the conjectures with actually more general point of view, since there one considers general smooth coefficients which is to say the isocrystals. The key issue is to establish some sort of finiteness of the rigid cohomology with such smooth non-étale coefficients. The method adopted by Kedlaya is through deep p -adic local monodromy theorem established in [Ked1] which proves the affine curve cases directly, then through clever geometric method to reduce everything to affine spaces, and then to affine curves with induction. Actually the picture can be made more general in the sense that one could consider more general p -adic constructible coefficients, which was considered by Abe and etc, for instance the arithmetic D -modules.

In the mixed-characteristic situation, Nakamura established some general Iwasawa theory for non-étale coefficients (by using Kedlaya-Liu’s language these are pro-étale non-étale Hodge-Frobenius sheaves) over $X = \mathrm{Spa}(\mathbb{Q}_p, \mathcal{O}_{\mathbb{Q}_p})$. Through Kedlaya-Pottharst’s point of view, namely the analytic cohomology of the cyclotomic deformation of pro-étale Hodge structures. We will give our understanding on the possible ??? in the entries of the graph above, which is already represented at the beginning of this subsection. Replacing the cyclotomic towers or toric towers by pro-étale sites allows us to define the generalized Iwasawa cohomology. Therefore deformation of the Hodge sheaves over pro-étale sites will remember the Iwasawa theoretic consideration, which is observed as mentioned above by Kedlaya-Pottharst.

The first main topic exhibited in our main body of work is on the deformation over a reduced affinoid algebra A the corresponding categories of vector bundles and the corresponding Frobenius modules (bundles) over the ring $\tilde{\Pi}_{R,A}^\infty$ and $\tilde{\Pi}_{R,A}$ (with the notations in proposition 2.3.8 as below):

Theorem 2.1.3. *The categories of A -relative quasicoherent finite locally free sheaves over the schematic relative Fargues-Fontaine curve $\mathrm{Proj}P_{R,A}$, the Frobenius modules over $\tilde{\Pi}_{R,A}^\infty$, the Frobenius modules over $\tilde{\Pi}_{R,A}$ and the Frobenius bundles over $\tilde{\Pi}_{R,A}$ are equivalent. Moreover for those Fréchet-Stein distribution algebra $A_\infty(G)$ attached to a p -adic Lie group G in the style of the Hodge-Iwasawa consideration assumed in setting 2.3.11, the corresponding statement holds as well (see section 2.3.2).*

This is actually considered in [KP1] in the situation where R is a perfectoid field. The main goal in our mind at this moment is essentially some belief (encoded in some possible application) that the deformations (both from some representation reason or Iwasawa-Tamagawa reason) share some rigid equivalence among the vector bundles over relative Fargues-Fontaine curves, B -pairs in families and finally the Frobenius-Hodge modules in relative sense. Of course if one focuses on the corresponding Galois representation context, then this belief will be somehow complicated while we believe the categories mentioned here are big enough to eliminate the difference from the deformations and the original absolute situation, without touching the étaleness. There are many contexts

where such deformation on the level of φ -modules has been already proved to be important and convenient, such as [Nak4] and [KPX]. [KPX] considered essentially the deformation theory of φ -modules by using the tools from Fréchet algebras. Note that B -pairs are generalizations of Galois representations from the point of view of the coefficients, Frobenius modules over ind-Fréchet sheaves are generalizations of Galois representations from the point of view of the slope theory and the corresponding purity. The relative deformation of the Hodge structures in such generality looks challenging, for instance the parallel story to Deligne-Laumon, [AM], [G1] and [G2] looks quite mysterious.

We actually discuss in this paper the corresponding pseudocoherent objects as well in the sense of [SGA6] and the corresponding pseudocoherent Frobenius modules studied in [KL16]:

Theorem 2.1.4. *The category of the pseudocoherent sheaves over the schematic Fargues-Fontaine curve $\text{Proj}P_{R,A}$ and the category of the pseudocoherent modules over $\tilde{\Pi}_{R,A}$ endowed with isomorphisms coming from the Frobenius pullbacks are equivalent.*

One of our current *main conjectures* (in the étale setting) in this article is quite general over a rigid analytic space X (over \mathbb{Q}_p) separated and of finite type (in the category of adic spaces) with respect to an adic ring T as in [Wit1], [Wit2] and [Wit3, 1.4], and we consider the corresponding categories $\mathbb{D}_{\text{perf}}(X_{\sharp}, T)$ ($\sharp = \text{ét}, \text{proét}$) of compatible families $(\mathcal{F}_I^{\bullet})_{I \subset T}$ complexes perfect and dg-flat of T/I constructible flat étale or pro-étale sheaves and we conjecture that we have the corresponding Waldhausen enrichment. To be more precise:

Conjecture 2.1.5. *Assume now that p is a unit in the ring T . Assume that we have $\mathbb{D}_{\text{perf}}(X_{\sharp}, T)$ ($\sharp = \text{ét}, \text{proét}$) are Waldhausen categories and we have the corresponding continuous maps from the attached K -theory topological spaces to the one attached to $\mathbb{D}_{\text{perf}}(T)$ induced by the direct image functor (which will make the space X more restrictive). Then we have that the maps are null-homotopic in some canonical way. (See conjecture 2.4.14)*

Using relatively homotopical language, this could be directly related to the formulation adopted by Fukaya-Kato in Deligne's category when we focus on the corresponding *spaces of 1 type* where the corresponding homotopy groups will vanish above the degree

one. This has actually been considered extensively already in [Wit1] in the context in terms of SGAIV and V. The corresponding categories we mentioned above looks very complicated to study in the general situation. Since our consideration is highly globalized due to the fact that we are considering the whole categories of all the corresponding interesting objects, instead of each single object. We have discussed some further development beyond the étale setting (in the p -adic setting) at the end of the paper.

2.1.2 Why Relative p -adic Hodge theory?

Since our motivation comes from the deformation of representations of étale fundamental groups of higher dimensional spaces and the geometrization of Iwasawa theory, it is important to look inside the intrinsic Hodge structures through some coherent tools, which is why we need to consider the corresponding relative p -adic Hodge theory in some deformed setting. The main goal of the study around the fundamental groups as illustrated above actually is usually of the following aspects:

Setting 2.1.6.

- I. Try to understand the intrinsic geometric structures and its reconstruction effect on the varieties.
- II. Try to understand the representations.
- III. Try to understand the deformation of representations.
- IV. Try to understand the Iwasawa sheaves with coefficients in some equivariant setting (where the equivariance sometimes factors through the étale fundamental groups).

Essentially and especially in our setting, the generalization is actually more complicated than the usual situation, namely the representations are relative in the sense that it is varying over a variety which is somewhat as general as possible with some generalized coefficients. In the usual setting, one could regard everything as local systems in most of desired cases. Then using Scholze's pro-étale sites and pushing down one could define the corresponding period functors. In our setting one can definitely rely on the advantage of the original ideas of Kedlaya-Liu [KL15] and [KL16], namely transforming the problem of representations of fundamental groups not to the local systems over pro-étale sites but

to the modules over relative Robba rings over quasi pro-étale sites. Under general philosophy from Berger, everything on the p -adic Hodge module structure will be encoded in the module structure over relative Robba rings.

2.1.3 Further Study

In our current establishment, we have touched some foundational results along our consideration towards general Hodge-Iwasawa theory. Deformations of the corresponding Hodge structures in our consideration (namely the analytic module structures over Robba sheaves and rings) have been shown by us to have some well-behaved properties. The aspect of deformation of such general Hodge structures will be definitely one of our long-term goal which should be parallel to those current existing comparison theorems and the existing various types of equivariant cohomological projects (for instance in the setting of étale cohomology and rigid cohomology). Over a rigid analytic space, the investigation of the corresponding families of the representations of the étale fundamental group seems to be a very hard problem for instance, especially when one would like to consider the equivariant setting, namely the corresponding equivariance coming from the quotient of the étale fundamental groups. We actually expect our insight should work in more general setting where one considers some general enough quotients here.

On the Iwasawa side, we expect that we can follow and generalize the original ideas in [KP1] to geometrize the corresponding interesting towers in Tamagawa-Iwasawa theory. This is actually different from some standard construction in the original Iwasawa theory and Λ -adic Hodge theory. We expect this will give us a more uniform approach to recover many resulting Iwasawa theory of the general p -adic Hodge structures, for instance the corresponding exponential maps and the dual exponential maps, and the corresponding Iwasawa interpolation throughout some interesting character varieties, carrying some relativization. One another thing we would like to mention is that actually we have already motivated the corresponding study of the Iwasawa theory in the pseudocoherent setting beyond the vector bundle non-étale context.

Maybe a tour through the topological method in p -adic analysis should be pos-

sible in a long term consideration. Carrying some reasonable type of relativization and equivariance, one can apply higher categorical approach to study the corresponding relative p -adic Hodge theory in our context. We only here touched upon a piece of the story since our main goal here is Iwasawa deformation of general p -adic Hodge structure. But there is no reason to only restrict the consideration to this context. In other word the higher categorical investigation we limited here could motivate some further deeper understanding. Especially when we introduce the corresponding non-commutativity and higher-dimensional equivariance, where the context of the study of the rational p -adic Hodge theory looks quite challenging. On the other hand, actually the corresponding discussion above may also shed some light on the corresponding equivariant program in other contexts, such as the arithmetic D -modules. We would like to mention that actually the higher categorical aspects of p -adic Hodge theory are already encoded in some contexts in the literature, for instance the work [BMS].

2.1.4 Notations

Notation	Description
p	A prime number.
(R, R^+)	A perfect adic Banach algebra uniform over \mathbb{F}_{p^h} .
A	A reduced affinoid algebra over \mathbb{Q}_p in the rigid geometry after Tate. In general we consider the spectral norms on A unless specified otherwise.
T	An adic ring. Note that this is considered in the work for instance [Wit1], [Wit2] and [Wit3, Section 1.4].
$A_\infty(G)$	Fréchet-Stein algebra attached to some nice p -adic group G of type Lie.
$\Omega_R^{\text{int}}, \Omega_R$	Kedlaya-Liu's perfect period rings, over R . See [KL15] and [KL16].

$\tilde{\Pi}_R^{\text{int},r}, \tilde{\Pi}_R^{\text{bd},r}, \tilde{\Pi}_R^I, \tilde{\Pi}_R^+, \tilde{\Pi}_R^\infty, \tilde{\Pi}_R$	Kedlaya-Liu's perfect period rings, over R . See [KL15] and [KL16].
$\Omega_{R,A}^{\text{int}}$	A -relative version of Kedlaya-Liu's perfect period rings, $\Omega_{R,A}^{\text{int}}$.
$\Omega_{R,A}$	A -relative version of Kedlaya-Liu's perfect period rings, $\Omega_{R,A}$.
$\tilde{\Pi}_{R,A}^I$	A -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R,A}^r$	A -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R,A}^\infty$	A -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R,A}^+$	A -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R,A}^{\text{int},r}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{int},+}$	A -relative version of Kedlaya-Liu's relative perfect integral Robba rings.
$\tilde{\Pi}_{R,A}^{\text{bd},r}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}^{\text{bd},+}$	A -relative version of Kedlaya-Liu's relative perfect bounded Robba rings.
$\underline{\underline{\Omega}}_A^{\text{int}}, \underline{\underline{\Omega}}_A$	A -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\underline{\underline{\tilde{\Pi}}}_A^{\text{int},r}, \underline{\underline{\tilde{\Pi}}}_A^{\text{int}}$	A -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\underline{\underline{\tilde{\Pi}}}_A^{\text{bd},r}, \underline{\underline{\tilde{\Pi}}}_A^{\text{bd}}$	A -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\underline{\underline{\tilde{\Pi}}}_A^I, \underline{\underline{\tilde{\Pi}}}_A^\infty, \underline{\underline{\tilde{\Pi}}}_A^r, \underline{\underline{\tilde{\Pi}}}_A^+$	A -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\Omega_{R,A_\infty(G)}^{\text{int}}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's perfect period rings, $\Omega_{R,A_\infty(G)}^{\text{int}}$.
$\Omega_{R,A_\infty(G)}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's perfect period rings, $\Omega_{R,A_\infty(G)}$.
$\tilde{\Pi}_{R,A_\infty(G)}^I$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect Robba rings.

$\tilde{\Pi}_{R, A_\infty(G)}^r$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R, A_\infty(G)}^\infty$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R, A_\infty(G)}^+$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect Robba rings.
$\tilde{\Pi}_{R, A_\infty(G)}^{\text{int}, r}, \tilde{\Pi}_{R, A_\infty(G)}^{\text{int}}, \tilde{\Pi}_{R, A_\infty(G)}^{\text{int}, +}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect integral Robba rings.
$\tilde{\Pi}_{R, A_\infty(G)}^{\text{bd}, r}, \tilde{\Pi}_{R, A_\infty(G)}^{\text{bd}}, \tilde{\Pi}_{R, A_\infty(G)}^{\text{bd}, +}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect bounded Robba rings.
$\underline{\Omega}_{A_\infty(G)}^{\text{int}}, \underline{\Omega}_{A_\infty(G)}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\tilde{\underline{\Pi}}_{A_\infty(G)}^{\text{int}, r}, \tilde{\underline{\Pi}}_{A_\infty(G)}^{\text{int}}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\tilde{\underline{\Pi}}_{A_\infty(G)}^{\text{bd}, r}, \tilde{\underline{\Pi}}_{A_\infty(G)}^{\text{bd}}$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
$\tilde{\underline{\Pi}}_{A_\infty(G)}^I, \tilde{\underline{\Pi}}_{A_\infty(G)}^\infty, \tilde{\underline{\Pi}}_{A_\infty(G)}, \tilde{\underline{\Pi}}_{A_\infty(G)}^r$	$A_\infty(G)$ -relative version of Kedlaya-Liu's relative perfect sheaves over $\text{Spa}(R, R^+)$.
X	Locally v -ringed spaces or preadic spaces.
$X, X_{\text{ét}}, X_{\text{proét}}$	The corresponding analytic sites, étale sites, pro-étale sites of X .

2.2 Frobenius Modules over Ind-Fréchet Algebras

Now we define the Frobenius modules in our setting, which are in some sense the generalized Hodge structures. Following [KL15] and [KL16] we know that actually the main tool in the study will be to find the links between the local systems or representations of fundamental groups with the Frobenius modules. Note that this is actually a common feature of p -adic cohomology theories or if you wish the p -adic weights after Deligne et al, which is to say in more detail: initially one has not the chance to see the intrinsic Hodge structures from the coefficient systems or the representations which is to say for instance the Frobenius structure, instead one can only have the chance to observe this after the application of Fontaine's functors. This is actually not the same as the situation in the initial Hodge theory and the ℓ -adic weights theory in the sense again Deligne et al.

So let us consider the following notations of Frobenius modules and Frobenius bundles, where the latter will be some collection of kind of sections with respect to each interval.

Setting 2.2.1. Following [KL15, Chapter 5], in the notation of both [KL16] and [KL15] we will in our context consider the situation where $h = 1$ in our initial consideration on the finite field \mathbb{F}_{p^h} . And note that we could have another parameter a in our initial setting of the definitions, so we are going to consider a -th power Frobenius φ^a in the following. This means that in the characteristic 0 situation we will consider \mathbb{Q}_{p^a} and in the characteristic $p > 0$ we consider working over the finite field \mathbb{F}_{p^a} for some chosen positive integer $a \geq 1$.

Definition 2.2.2. For general A which is a reduced affinoid algebra over \mathbb{Q}_p with integral subring \mathfrak{o}_A , we define the following various A -relative version of the corresponding Period rings from [KL15, Chapter 5]:

$$\tilde{\Pi}_{R,A} := \bigcup_r \tilde{\Pi}_{R,A}^r, \quad \tilde{\Pi}_{R,A}^r := \tilde{\Pi}_R^r \widehat{\otimes}_{\mathbb{Q}_p} A, \quad (2.2.1)$$

$$\tilde{\Pi}_{R,A}^I := \tilde{\Pi}_R^I \widehat{\otimes}_{\mathbb{Q}_p} A, \quad \tilde{\Pi}_{R,A}^\infty := \tilde{\Pi}_R^\infty \widehat{\otimes}_{\mathbb{Q}_p} A. \quad (2.2.2)$$

$$\tilde{\Pi}_{R,A}^{\text{bd}} := \bigcup_r \tilde{\Pi}_R^{\text{bd},r} \widehat{\otimes}_{\mathbb{Q}_p} A, \quad \tilde{\Pi}_{R,A}^{\text{bd},r} := \tilde{\Pi}_R^{\text{bd},r} \widehat{\otimes}_{\mathbb{Q}_p} A, \quad (2.2.3)$$

$$\tilde{\Pi}_{R,A}^{\text{int}} := \bigcup_r \tilde{\Pi}_R^{\text{int},r} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{o}_A, \quad \tilde{\Pi}_{R,A}^{\text{int},r} := \tilde{\Pi}_R^{\text{int},r} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{o}_A. \quad (2.2.4)$$

Remark 2.2.3. One can also consider more deformed version of the period rings as in [KL15] in some natural way which we do not explicitly write.

Example 2.2.4. To be more concrete, in the situation where A is $\mathbb{Q}_p \langle T_1, \dots, T_d \rangle$ for some d , we can consider the following more explicit definitions. First form the ring of Witt vectors $W(R)$, then we consider the algebra $W(R)_A$ which is defined by taking the corresponding completed tensor product whose elements have the form of:

$$\sum_{n \geq 0, i_1 \geq 0, \dots, i_n \geq 0} p^n [\bar{x}_{n, i_1, \dots, i_d}] T_1^{i_1} T_2^{i_2} \dots T_d^{i_d}.$$

Over $W(R)_A$ we have the Gauss norm $\|\cdot\|_{\alpha^r, A}$ for each $r > 0$ which is defined by

$$\|\cdot\|_{\alpha^r, A} \left(\sum_{n \geq 0, i_1 \geq 0, \dots, i_n \geq 0} p^n [\bar{x}_{n, i_1, \dots, i_d}] T_1^{i_1} T_2^{i_2} \dots T_d^{i_d} \right) := \sup_{n \geq 0, i_1 \geq 0, \dots, i_n \geq 0} \{p^{-n} \alpha(\bar{x}_{n, i_1, \dots, i_d})^r\}.$$

Then we define $\tilde{\Pi}_{R,A}^{\text{int},r}$ to be the subring of $W(R)_A$ consisting of all the elements satisfying:

$$\lim_{n, i_1, \dots, i_d} \|\cdot\|_{\alpha^r, A} (p^{-n} \alpha(\bar{x}_{n, i_1, \dots, i_d})^r) = 0.$$

And we put $\tilde{\Pi}_{R,A}^{\text{int}}$ to be the union of $\tilde{\Pi}_{R,A}^{\text{int},r}$ for all $r > 0$. We then define $\tilde{\Pi}_{R,A}^{\text{bd},r}$ to be $\tilde{\Pi}_{R,A}^{\text{int},r}[1/p]$, and furthermore define the ring $\tilde{\Pi}_{R,A}^{\text{bd}}$ to be the corresponding union throughout all $r > 0$. Then we define the Robba ring $\tilde{\Pi}_{R,A}^I$ for some interval I as the Fréchet completion of the ring $W(R)_A[1/p]$ by using the family of norms $\|\cdot\|_{\alpha^t, A}$ for all $t \in I$. Then one could take specific intervals $(0, r]$ or $(0, \infty)$ to define the corresponding Robba rings $\tilde{\Pi}_{R,A}^r$ and $\tilde{\Pi}_{R,A}^\infty$. Finally we can define the whole Robba ring $\tilde{\Pi}_{R,A}$ as the union of all the $\tilde{\Pi}_{R,A}^r$ throughout all $r > 0$.

Setting 2.2.5. We need to specify the corresponding Frobenius in our setting, where we will consider the Frobenius action which is A -linear but acts trivially over the A -part in the products appeared as above in the definitions.

Proposition 2.2.6. *For A a reduced affinoid algebra over \mathbb{Q}_p as defined above, we have the corresponding equality:*

$$\tilde{\Pi}_{R,A}^{[s_1, r_1]} \cap \tilde{\Pi}_{R,A}^{[s_2, r_2]} = \tilde{\Pi}_{R,A}^{[s_1, r_2]}.$$

Here we assume that $0 < s_1 \leq s_2 \leq r_1 \leq r_2$.

Proof. This could be proved by using the strategy which mimicks [KL15, Section 5.2], or one could consider any representation taking the form of $\sum_i r_i \otimes a_i$ with r_i in the non-relative Robba rings involved and $a_i \in A$, then the result follows from the situation without considering the algebra A . \square

Definition 2.2.7. Consider the A -relative ring of periods

$$\Delta : \Omega_{R,A}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^+, \tilde{\Pi}_{R,A}^\infty.$$

Then we are going to define the A -relative φ^a -module over Δ to be a finite locally free Δ -module carrying semilinear Frobenius action of φ^a . Moreover now for $\Delta : \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}$ we have for each radius $r > 0$ a module M_r over Δ^r descending the module M and the isomorphism

$$\varphi^* M_r \xrightarrow{\sim} M_r \otimes_{\Delta^r} \Delta^{r/q}.$$

We will call the module M_r over Δ^r a model of M . One can also define any A -relative Frobenius module to be one as above coming from some base change of some model M_r .

Then one has the corresponding bundles in our setting as in the following generalizing the situation in [KL15, Section 6.1].

Definition 2.2.8. Consider the period ring $\Delta := \tilde{\Pi}_{R,A}$. We define a φ^a -module M_I over Δ_I for some closed interval $I \subset (0, \infty)$ having the form of $[s, r]$ with $0 < s \leq r/q$ to be a finite locally free module over Δ^I carrying the semilinear Frobenius and the isomorphism

$$\varphi^* M_I \otimes_{\Delta^{[s/q, r/q]}} \Delta^{[s, r/q]} \xrightarrow{\sim} M_I \otimes_{\Delta^{[s, r]}} \Delta^{[s, r/q]}.$$

Then under this restriction on the interval we define a φ^a -bundle to be a collection $\{M_I\}_I$ of finite locally free modules M_I over each Δ^I such that for each pair of intervals such that $I \subset I'$ we have an isomorphism $\psi_{I', I} : M_{I'} \otimes \Delta^I \xrightarrow{\sim} M_I$ and for a triple of intervals such that $I \subset I' \subset I''$ we have the obvious cocycle condition $\psi_{I'', I} \circ \psi_{I', I'} = \psi_{I'', I'}$. Note that we have only finished the definitions under the restriction on the intervals. Now as in [KL15, Section 6.1] we extend this to any more general interval I by taking any pair $I \subset I'$ of intervals and extracting some module $M_{I'}$ where I' satisfies the restriction in our previous definition. Then also we extend the definition of φ^a -bundles to all the general intervals in our setting. Finally we define global section of a φ^a -bundle to be a collection

of elements having the general form $\{v_I\}_I$ such that for each pair $I \subset I'$ we have that the image of $v_{I'}$ under the image of $\psi_{I',I}$ is v_I .

Now as in the usual situation it is natural to now consider the finiteness issue as in the following:

Proposition 2.2.9. (After Kedlaya-Liu [KL15, Lemma 6.1.4]) *Consider now an arbitrary φ^a -bundle $\{M_I\}$ over $\tilde{\Pi}_{R,A}$. Suppose now that for any interval $I \subset (0, \infty)$ we have that there exists a finite number generating elements $\mathbf{e}_1, \dots, \mathbf{e}_n$ for some uniform $n \in \mathbb{Z}$ for the finitely locally free module M_I . Then this set of elements could be promoted to be a generating set of the global section as a module but now over $\tilde{\Pi}_{R,A}^\infty$.*

Proof. This is generalization of the corresponding results in [KL15, Lemma 6.1.4] in the usual Hodge structures. Basically the main issue in the proof is to exhibit the desired finitely generatedness by some approximating process as in the usual situation. As in the usual situation we consider first the morphism $M_{[p^{-\ell}, p^\ell]} \rightarrow (\tilde{\Pi}_{R,A}^{[p^{-\ell}, p^\ell]})^n$ by using the corresponding basis as in the assumption whose composite with the inverse $(\tilde{\Pi}_{R,A}^{[p^{-\ell}, p^\ell]})^n \rightarrow M_{[p^{-\ell}, p^\ell]}$ is the identity map. And one could then bound the gap between the subspace norm and the quotient norm of $M_{[p^{-\ell}, p^\ell]}^n$ by some constant e_ℓ as in [KL15, Lemma 6.1.4]. Now let $\mathbf{v} \in M$. Then the point is to extract for each $i = 1, \dots, n$ and $\ell = 0, 1, \dots$ the following element $B_{i,\ell} \in \tilde{\Pi}_{R,A}^{[p^{-\ell}, p^\ell]}$ and $A_{i,\ell} \in \tilde{\Pi}_{R,A}^\infty$ by the following fashion following [KL15, Lemma 6.1.4], first for any $j < \ell$ we choose the corresponding $B_{i,j}, j < \ell$ such that $\mathbf{v} - \sum_{i=1}^n \sum_{j \leq \ell} B_{i,j} \mathbf{e}_i = \sum_{i=1}^n A_{i,\ell} \mathbf{e}_i$ after which we choose $B_{i,\ell}$ to ensure that $\|\cdot\|_{\alpha^t, A}(B_{i,\ell} - A_{i,\ell}) \leq p^{-1} e_\ell^{-1} \|\cdot\|_{\alpha^t, A}(A_{i,\ell})$ for each $i = 1, \dots, n$ with $t \in [p^{-\ell}, p^\ell]$. Then one could then finish as in [KL15, Lemma 6.1.4] since by the suitable choices we will have desired convergence which gives rise to a desired expression of \mathbf{v} in terms of the basis given. □

Lemma 2.2.10. *Consider the corresponding projection map from a φ^a -modules M over $\tilde{\Pi}_{R,A}$ to the corresponding module M_I over $\tilde{\Pi}_{R,A}^I$ for some interval I . Then we have that this operation is a tensor equivalence. (Here we assume that the ring $\tilde{\Pi}_{R,A}^I$ is sheafy.)*

Proof. See [KL15, Lemma 6.1.5]. □

Then we study the Frobenius invariance in our setting which generalizes the absolute situation where we do not have any deformation at all. Here we are going to use the notation $M(n)$ to denote the corresponding twist of M where the twist is defined to be the one where φ acts by p^{-n} .

Proposition 2.2.11. (After Kedlaya-Liu [KL15, Proposition 6.2.2]) *Now we consider a φ^a -bundle M over $\tilde{\Pi}_{R,A}$. Then we have that there exists some integer $N \geq 1$ such that for $n \geq N$ we have that $\varphi^a - 1 : M_{[s,rq]}(n) \rightarrow M_{[s,r]}(n)$ is surjective. One could also have the chance to take the integer to be 1 if the module could be derived from some module defined over $\tilde{\Pi}_{R,A}^{\text{int}}$.*

Proof. Following [KL15, Proposition 6.2.2], we consider the quotient $r/s \leq q^{1/2}$. Then by considering the Frobenius actions one could further assume that $r \in [1, q]$. Then we will use the notation $A_{i,j}$ and $B_{i,j}$ to denote the corresponding matrices for the actions of φ^{-a} and φ^a respectively. Here $A_{i,j}$ has entries in $\tilde{\Pi}_{R,A}^{[s,rq]}$ while the latter $B_{i,j}$ has entries in $\tilde{\Pi}_{R,A}^{[s/q,r]}$. Then we put:

$$c_1 = \sup\{\|\cdot\|_{\alpha^t, A}(A_{i,j}) | t \in [s, rq]\}, \quad (2.2.5)$$

$$c_2 = \sup\{\|\cdot\|_{\alpha^t, A}(B_{i,j}) | t \in [s/q, r]\}. \quad (2.2.6)$$

The goal is to extract from any element \mathbf{w} in the target module a preimage \mathbf{v} which amounts to solving the following equation:

$$\varphi^a \mathbf{v} - \mathbf{v} = \mathbf{w}.$$

Strictly speaking here we need some twisted version of this equation. By using our new c_1 and c_2 one chooses a suitable integer N which shares the same property as the one in [KL15, Proposition 6.2.2]. Then we adapt the argument therein to our A -relative setting. We start by setting the coordinate of \mathbf{w} to be (x_1, \dots, x_r) where r is the rank. First we take suitable constant $c > 0$ such that we have for all $n \geq N$:

$$\varepsilon := \sup\{p^{-n} c_1 c^{-(q-1)r/q}, p^n c_2 c^{(q-1)s}\} < 1.$$

Then as in [KL15, Proposition 6.2.2] one seeks suitable decomposition of x_i taking the form of $x_i := y_i + z_i$ where $y_i \in \tilde{\Pi}_{R,A}^{[s/q,r]}$ and $z_i \in \tilde{\Pi}_{R,A}^{[s,rq]}$ such that for the well-located

t as [KL15, Lemma 5.2.9] with the estimates therein through the relative version of the equalities in [KL15, Lemma 5.2.9]:

$$\|\cdot\|_{\alpha^t, A}(\varphi^{-a}(y_i)) \leq c^{-(q-1)t/q} \|\cdot\|_{\alpha^t, A}(x_i), \quad (2.2.7)$$

$$\|\cdot\|_{\alpha^t, A}(\varphi^a(z_i)) \leq c^{(q-1)t} \|\cdot\|_{\alpha^t, A}(x_i), \quad (2.2.8)$$

$$(2.2.9)$$

with the corresponding:

$$\|\cdot\|_{\alpha^t, A}(y_i), \|\cdot\|_{\alpha^t, A}(z_i) \leq \|\cdot\|_{\alpha^t, A}(x_i).$$

(Note that this is actually A -relative version of the [KL15, Lemma 5.2.9], where one considers the product norm $\|\cdot\|_{\alpha^t, A}$.)

Then we set as in [KL15, Proposition 6.2.2] and compute:

$$\sum_{i=1}^r x'_i \mathbf{f}_i := p^n \varphi^{-a} \left(\sum_{i=1}^r y_i \mathbf{f}_i \right) + p^{-n} \varphi^a \left(\sum_{i=1}^r z_i \mathbf{f}_i \right), \quad (2.2.10)$$

which could also be deduced from the kind of definition by using the matrices $A_{i,j}$ and $B_{i,j}$. As in [KL15, Proposition 6.2.2] we further consider the corresponding coordinate vectors y, z, x' as functions of x . And set the following iteration $x_{(\ell+1)} := x'(x_{(\ell)})$ for each $\ell \geq 0$. To extract the desired \mathbf{v} one just sets now:

$$\mathbf{v} := \sum_{\ell \geq 0} -p^n \varphi^{-a} \left(\sum_{i=1}^r y_i(x_{(\ell)}) \mathbf{f}_i \right) + \sum_{i=1}^r z_i(x_{(\ell)}) \mathbf{f}_i.$$

As in the usual situation and by our construction above we have that this element will converge to an desired element in the domain module since we have

$$\mathbf{v} - p^{-n} \varphi^a \mathbf{v} = \sum_{i=1}^r x_{(0),i} \mathbf{f}_i.$$

Finally one could then follow the strategy in [KL15, Proposition 6.2.2] to tackle the situation in more general setting by taking some suitable

$$t = \max\{r/q^{1/2}, s\}$$

for those radii which do not satisfy the condition as above and extract suitable preimage \mathbf{v} and then show that one can put this to be the desired element. To be more precise one considers the corresponding radius $t = \max\{r/q^{1/2}, s\}$ where the corresponding radii s, r

may not satisfy the corresponding conditions as above. Then we have $t \geq r/q^{1/2}$ which implies that $r/t \leq q^{1/2}$ in this situation, which further implies (by the previous situation) one can extract a suitable element \mathbf{v}_0 . To finish we still need to show that this element \mathbf{v}_0 is a desired preimage in our situation. Actually what we know in our situation is that the corresponding element \mathbf{v}_0 lives in the corresponding section $M_{[t,rq]}$, where this gives rise to the solution of the equation:

$$\mathbf{v} - p^{-n}\varphi^a\mathbf{v} = \sum_{i=1}^r x_{(0),i}\mathbf{f}_i,$$

which further gives rise to the corresponding element in the corresponding section:

$$M_{[t/q,r]}.$$

Suppose now we a priori have the corresponding situation that:

$$t/q > s,$$

then we take the corresponding intersection in this situation with respect to the pair:

$$[t/q, r], [t, rq]$$

which gives rise to the fact that:

$$v_0 \in M_{[t/q,rq]}$$

where if we have $t/q = s$ then we are done, otherwise we repeat the construction above again to produce the corresponding element gives rise to the situation:

$$v_0 \in M_{[t/q^2,rq]}, \tag{2.2.11}$$

$$v_0 \in M_{[t/q^3,rq]}, \tag{2.2.12}$$

$$\dots, \tag{2.2.13}$$

$$v_0 \in M_{[t/q^n,rq]}, \dots, \tag{2.2.14}$$

until there is some integer $N \geq 1$ such that we have $t/q^N \leq s$ then we are done. □

Lemma 2.2.12. *For $0 < s \leq r$, we consider the Robba ring $\Pi_{R,A}^{[s,r]}$. Any multiplicative seminorm of this ring is bounded by $\max\{\|\cdot\|_{\alpha^s,A}, \|\cdot\|_{\alpha^s,A}\}$ if and only if this seminorm is bounded by the norm $\|\cdot\|_{\alpha^t,A}$ for some unique $t \in [s, r]$ coming from [KL15, Lemma 5.3.4], by restricting the seminorm to the ring $\Pi_R^{[s,r]}$.*

Proof. As in [KL15, Lemma 5.3.4], one direction is trivial by the property of the norm. So suppose now that we have that the seminorm is bounded by $\max\{\|\cdot\|_{\alpha^s, A}, \|\cdot\|_{\alpha^s, A}\}$. Suppose we denote the seminorm by $|\cdot|$, then we have that $|\cdot|([\bar{r}] \otimes 1)$ for any $\bar{r} \in R$ is bounded by $\|\cdot\|_{\alpha^t, A}([\bar{r}])$ which could be proved in the same way as in [KL15, Lemma 5.3.4] (where the number t is determined uniquely by [KL15, Lemma 5.3.4]), while for any $a \in A$ we have $|\cdot|(1 \otimes a)$ is bounded by $\max\{\|\cdot\|_{\alpha^s, A}, \|\cdot\|_{\alpha^s, A}\}(1 \otimes a)$ which is just $\|\cdot\|_A(a)$. Therefore we have $|\cdot|([\bar{r}] \otimes 1)|\cdot|(1 \otimes a) = |\cdot|([\bar{r}] \otimes a)$ is then bounded by the product of $\|\cdot\|_{\alpha^t, A}([\bar{r}])$ and $\|\cdot\|_A(a)$, which gives the result. \square

Proposition 2.2.13. (After Kedlaya-Liu [KL15, Proposition 6.2.4]) *Now again working over the ring $\tilde{\Pi}_{R,A}$ and consider a φ^a -bundle which is denoted by M . Then we have that there exists some positive integer N such that for all $n \geq N$ one could find finitely many φ^a -invariant global sections of $M(n)$ that generate M .*

Proof. We follow [KL15, Proposition 6.2.4]. In our generalized setting, first we take s to be $rq^{-1/2}$. Then as in the situation of [KL15, Proposition 6.2.4] one chooses an element $\bar{\pi}$ whose norm is inside $(0, 1)$. Then we consider the corresponding product which we will denote it by $[\bar{\pi}]$, and choose suitable rational number $z \in \mathbb{Z}[1/p]$ such that $[\bar{\pi}^z]$ gives rise to the value c in the proof of the previous proposition. Then also as in the previous proposition one constructs (using φ^a instead) the corresponding element \mathbf{v} but in our situation we will denote this by \mathbf{v}'_i for each i suitable. To be more precise from the data above we put \mathbf{v}'_i as the definition of \mathbf{v} by using $x_{(0)} = 0$ and $(y_{(0),j}, z_{(0),j}) = (-[\bar{\pi}^z], [\bar{\pi}^z])$ for $j = i$ and zero identically otherwise.

Then we have the corresponding set of elements $\{\mathbf{v}'_i\}$ in $M_{[rq^{-1/2}, rq]}$ consisting those elements which are invariant under the action of φ^a . We repeat the argument in the proof of the previous proposition as in the following. First one uses the notation $X_{i,j}$ to denote the corresponding matrix of the operator φ^{-a} (certainly under some chosen set of generators with the specific rank r in our situation) while one uses the notation $Y_{i,j}$ to denote the corresponding matrix of the operator φ^a , where from the corresponding matrix elements one could compute the following two numbers:

$$b_1 := \sup_t \{\|\cdot\|_{\alpha^t, A}(X_{i,j}), i, j \in \{1, 2, \dots, r\}, t \in [s, rq]\} \quad (2.2.15)$$

$$b_2 := \sup_t \{\|\cdot\|_{\alpha^t, A}(Y_{i,j}), i, j \in \{1, 2, \dots, r\}, t \in [s/q, r]\}. \quad (2.2.16)$$

Now by using our new scalars b_1, b_2 one could have the chance to extract some suitable $N \geq 1$ as in our previous proposition and in the corresponding result of [KL15, Proposition 6.2.2]. Then we consider the following construction through the corresponding iterated induction as in the following. The corresponding initial coordinates have been constructed in our situation as mentioned in the previous paragraph. Then we consider some general coordinate (x_1, \dots, x_r) for r the rank. Then as in [KL15, Lemma 5.2.9] for each $i = 1, \dots, r$ one can make the following decomposition, namely putting $x_i = y_i + z_i$ with the following estimate with some chosen constant $c > 0$:

$$\|\cdot\|_{\alpha^t, A}(\varphi^{-a}(y_i)) \leq c^{-(q-1)t/q} \|\cdot\|_{\alpha^t, A}(x_i), \quad (2.2.17)$$

$$\|\cdot\|_{\alpha^t, A}(\varphi^a(z_i)) \leq c^{(q-1)t} \|\cdot\|_{\alpha^t, A}(x_i), \quad (2.2.18)$$

$$(2.2.19)$$

and:

$$\|\cdot\|_{\alpha^t, A}(y_i), \|\cdot\|_{\alpha^t, A}(z_i) \leq \|\cdot\|_{\alpha^t, A}(x_i).$$

This will be the key step in the iterating process, while the scalar c will be chosen to guarantee that we have the following situation:

$$\delta := \max\{p^{-n}b_1c^{-(q-1)r/q}, p^n b_2 c^{(q-1)s}\} < 1.$$

Then the corresponding iterating process will be further conducted in the following way, namely we consider the corresponding (well-defined due to our construction above):

$$\sum_{i=1}^r x'_i \mathbf{f}_i := p^n \varphi^{-a} \sum_{i=1}^r y_i \mathbf{f}_i + p^{-n} \varphi^a \sum_{i=1}^r z_i \mathbf{f}_i.$$

Note here that from each $x_i, i = 1, \dots, r$ one can produce the corresponding elements $x'_i, i = 1, \dots, r$ which further produces the corresponding elements $y_i, z_i, i = 1, \dots, r$, which implies that one can regard the corresponding elements $y_i, z_i, i = 1, \dots, r$ as functions $y_i(x_1, \dots, x_r), z_i(x_1, \dots, x_r), i = 1, \dots, r$ of the original elements $x_i, i = 1, \dots, r$. The whole iterating process is conducted through putting $x_{(\ell+1)} := x'(x_{(\ell)})$ for each $\ell = 0, 1, \dots$ (note that in this situation $x_{(1)}$ is constructed directly from $x_{(0)}$), which produces the desired new elements by considering the following series:

$$\mathbf{v}'_i := \sum_{\ell \geq 0} -p^n \varphi^{-a} \left(\sum_{i=1}^r y_i(x_{(\ell)}) \mathbf{f}_i \right) + \sum_{i=1}^r z_i(x_{(\ell)}) \mathbf{f}_i.$$

As in our previous situation this process produces the desired elements which are invariant under the Frobenius operators in the sense that:

$$\mathbf{v}'_i - p^{-n} \varphi^a \mathbf{v}'_i = \sum_{i=0}^r x_{(0),i} \mathbf{f}_i.$$

Then we write \mathbf{v}'_i as $[\bar{\pi}^s] \mathbf{f}_i + M \mathbf{f}$ with some matrix $\{M_{i,j}\}$ satisfying the condition that $\|\cdot\|_{\alpha^t, A}(M_{i,j}) \leq \varepsilon \alpha(\bar{\pi})^{st}$ for each $t \in [rq^{-1/2}, r]$. Therefore up to here we conclude that the matrix $[\bar{\pi}^s] + M$ will then consequently be invertible over the period ring $\tilde{\Pi}_{R,A}^{[rq^{-1/2}, r]}$ which implies the corresponding finite generating property for the interval $[rq^{-1}, r]$ (by further repeating the corresponding construction above for the interval $[rq^{-1}, rq^{-1/2}]$). Then to finish one could then iterate as in [KL15, Proposition 6.2.4] and apply the Frobenius action from φ^a . Note that this will rely on the analog in our situation of [KL15, Lemma 5.3.4] in the A -relative setting (in our setting this could be resolved by considering the corresponding seminorms on the period rings in the usual sense and A , as in the previous lemma). \square

Then we consider the Frobenius invariants acting on an exact sequence, generalizing the corresponding result from [KL15, Corollary 6.2.3]:

Corollary 2.2.14. *Suppose now we have an exact sequence of φ^a -modules over the corresponding period ring $\tilde{\Pi}_{R,A}$ taking the form of $0 \rightarrow M_\alpha \rightarrow M \rightarrow M_\beta \rightarrow 0$. Then there exists a positive integer N such that for all $n \geq N$ we have the following exact sequence:*

$$0 \longrightarrow M_\alpha(n)^{\varphi^a=1} \longrightarrow M(n)^{\varphi^a=1} \longrightarrow M_\beta(n)^{\varphi^a=1} \longrightarrow 0.$$

Proof. This is a direct consequence of the previous proposition 2.2.11. \square

The finite projective objects are considered in the results above, actually one can naturally extend the discussion above to the setting of pseudocoherent objects. Following [KL16, 4.6.9] we consider the following results around the finite generated objects:

Proposition 2.2.15. *Let M be a Frobenius module defined over $\tilde{\Pi}_{R,A}$ (namely endowed with a semilinear action of the Frobenius operator) which is assumed to be finitely generated. Then we have that one can find then an integer $N \geq 0$ such that for all $n \geq N$, $H_{\varphi^a}^0(M(n))$ generates the module M itself and we have that the $H_{\varphi^a}^1(M(n))$ vanishes. Also let M be a Frobenius module defined over $\tilde{\Pi}_{R,A}$ (namely endowed with a semilinear action*

of the Frobenius operator) which is assumed to be finite projective. Then we have that one can find then an integer $N \geq 0$ such that for all $n \geq N$, $H_{\varphi^a}^0(M(n))$ generates the module M itself and we have that the $H_{\varphi^a}^1(M(n))$ vanishes.

Proof. In this situation one can follow the proof of [KL16, 4.6.9] and the strategy in the proof of the previous two propositions. \square

2.3 Contact with Schematic Relative Fargues-Fontaine Curves

2.3.1 Rigid Analytic Deformation of Schematic Relative Fargues-Fontaine curves

In this section, we study some relationship between the vector bundles over Fargues-Fontaine curves and the generalized Frobenius modules defined in the previous section. This is relative version of the established results essentially in [KL15]. The original picture could be dated back to the work of Fargues-Fontaine, where they established the classification of the Galois equivariant vector bundles over the so-called Fargues-Fontaine curves. For the convenience of the reader we first recall the definition of the scheme $\text{Proj}P$ in the current picture.

Setting 2.3.1. Recall some algebraic geometry from [KL15, Definition 6.3.1] that first the ring P is a graded commutative ring taking the form of $\bigoplus_{n \geq 0} P_n$ where each P_n is defined to be a subring of $\tilde{\Pi}_R^+$ which consists of all the elements which is invariant under the Frobenius φ^a (up to scalars p^n for each n). Then for each element f of degree $d > 0$ in P_d we consider the local affine scheme $\text{Spec}(P[1/f]_0)$, then we glue these to get the projective spectrum of P as in [KL15, Definition 6.3.1] which is denoted by $\text{Proj}P$. For more algebraic geometric discussion see [KL15, Definition 6.3.1].

Then we could generalize the situation in [KL15, Definition 6.3.1] to our A -relative situation.

Setting 2.3.2. In our situation, we consider the ring $\tilde{\Pi}_{R,A}^+$ or the ring $\tilde{\Pi}_{R,A}$ which is used to defined the corresponding graded ring as in the previous setting (namely, taking the corresponding $\varphi^a = p^n$ invariants for each $n \geq 0$), which will be denoted by $P_{R,A}$ or even just P_A . To be slightly more precise $P_{R,A}$ is defined to be the corresponding direct sum $\bigoplus_{n \geq 0} P_{n,A}$.

Setting 2.3.3. First we are going to use the notation f to denote both the corresponding elements with specific degrees d in P_d and the corresponding elements in the tensor product of the local affine rings. And then we consider now the following invariance for any M ,

which is any φ^a -bundle over $\tilde{\Pi}_{R,A}$. The invariance M_f is now defined in our setting as $M[1/f]^{\varphi^a}$. To be more explicit we have $M_f = \bigcup_{n \geq 0} f^{-n} M(dn)^{\varphi^a}$. As in the usual situation one could regard this as a module over the tensor product $P_A[1/f]_0$, which then for any I an interval induces the following map:

$$M_f \otimes_{P_A[1/f]_0} \tilde{\Pi}_{R,A}^I[1/f] \rightarrow M_f \otimes_{\tilde{\Pi}_{R,A}^I} \tilde{\Pi}_{R,A}^I[1/f].$$

To study more, we first consider the following corollary after Kedlaya-Liu:

Corollary 2.3.4. *Consider an arbitrary exact sequence of φ^a -modules taking the form of*

$$0 \longrightarrow M_A \longrightarrow M \longrightarrow M_B \longrightarrow 0.$$

Then we have the derived corresponding exact sequence:

$$0 \longrightarrow M_{A,f} \longrightarrow M_f \longrightarrow M_{B,f} \longrightarrow 0.$$

Proof. Just consider the structure of M_f , then one could derive the results from our previous discussion. \square

Then we have the following relative version of [KL15, Corollary 6.3.4]:

Lemma 2.3.5. *Let M be a Frobenius bundle as above, and take M_1 and M_2 two Frobenius subbundles whose summation is also a subobject. Then we actually have an equality $(M_1 + M_2)_f \xrightarrow{\sim} M_{1,f} + M_{2,f}$.*

Proof. This is basically an A -relative version of the corresponding result in [KL15, Corollary 6.3.4]. \square

Then we consider the following key proposition for our further study:

Proposition 2.3.6. (After Kedlaya-Liu [KL15, Theorem 6.3.9]) *Now let M again be a φ^a -bundle over $\tilde{\Pi}_{R,A}$. Then we have that*

$$M_f \otimes_{P_A[1/f]_0} \tilde{\Pi}_{R,A}^I[1/f] \rightarrow M_f \otimes_{\tilde{\Pi}_{R,A}^I} \tilde{\Pi}_{R,A}^I[1/f]$$

is a bijection, and M_f is finite projective over $P_A[1/f]_0$.

Proof. We follow [KL15, Theorem 6.3.9] which gives basically from proposition 2.2.13 that there exists $\mathbf{e}_1, \dots, \mathbf{e}_k$ which are the global sections of $M(dn)$ for some n which generate M itself. Then we have that considering $f^{-n}\mathbf{e}_1, \dots, f^{-n}\mathbf{e}_k$ could give rise to the corresponding surjectivity as in [KL15, Theorem 6.3.9]. Then we consider the corresponding presentation by using these generating elements from $\tilde{\Pi}_{R,A}(-dn)^k(dn)$ mapping to $M(dn)$ which gives rise to a mapping which is surjective mapping from $M_1 \rightarrow M$ (certainly here we have to consider the corresponding bundle in our context corresponding to the module $\tilde{\Pi}_{R,A}(-dn)^k$ and then twist to establish the corresponding map as above), which furthermore gives rise to (by the corresponding observation parallel to [KL15, Theorem 6.3.9]) an exact sequence taking the form of

$$0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M \longrightarrow 0 .$$

Then we consider the corresponding induced exact sequence taking the form of:

$$0 \longrightarrow M_{2,f} \longrightarrow M_{1,f} \longrightarrow M_f \longrightarrow 0 ,$$

from the discussion above. Then to finish we look at the corresponding commutative diagram as in [KL15, Theorem 6.3.9] in the following:

$$\begin{array}{ccccccc}
M_{2,f} \otimes_{P_A[1/f]_0} \tilde{\Pi}_{R,A}^I[1/f] & \longrightarrow & M_{1,f} \otimes_{P_A[1/f]_0} \tilde{\Pi}_{R,A}^I[1/f] & \longrightarrow & M_f \otimes_{P_A[1/f]_0} \tilde{\Pi}_{R,A}^I[1/f] & \longrightarrow & 0 , \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow M_{2,f} \otimes_{\tilde{\Pi}_{R,A}^I} \tilde{\Pi}_{R,A}^I[1/f] & \longrightarrow & M_{1,f} \otimes_{\tilde{\Pi}_{R,A}^I} \tilde{\Pi}_{R,A}^I[1/f] & \longrightarrow & M_f \otimes_{\tilde{\Pi}_{R,A}^I} \tilde{\Pi}_{R,A}^I[1/f] & \longrightarrow & 0
\end{array}$$

which finishes the proof on the bijectivity as in the absolute situation in [KL15, Theorem 6.3.9]. The finite projectiveness could be prove exactly by the same fashion in [KL15, Theorem 6.3.9] (also one can similarly derive the corresponding finitely-presentedness). To be more precise one can look at the exact sequence constructed above:

$$0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M \longrightarrow 0 .$$

Then by taking suitable higher degree twist in our situation (by considering the corresponding vanishing of the cohomology $H^1(M^\vee \otimes M_2(dN))$) one can get the following

push-out diagram just as in [KL15, Theorem 6.3.9]:

$$\begin{array}{ccccc}
 M_2 & \longrightarrow & M_1 & \longrightarrow & M, \\
 \downarrow & & \downarrow & & \downarrow \\
 f^{-N}M_2 & \longrightarrow & M'_1 & \longrightarrow & M
 \end{array}$$

which gives rise to that $M_{1,f} \xrightarrow{\sim} M'_{1,f}$. Then by our construction, since $M_{1,f}$ is free we are done. □

We then have the following results after suitable localization.

Corollary 2.3.7. *Pick the element f as in the previous proposition, we have that the following four categories are equivalent:*

- I. *The category of finite locally free $P_A[1/f]_0$ -modules over the relative coordinate ring of the localization of the Fargues-Fontaine curve $\text{Proj}P_A$;*
- II. *The category of A -relative φ^α -modules over $\tilde{\Pi}_{R,A}^\infty[1/f]$;*
- III. *The category of A -relative φ^α -modules over $\tilde{\Pi}_{R,A}[1/f]$;*
- IV. *The category of A -relative φ^α -bundles over $\tilde{\Pi}_{R,A}[1/f]$.*

Proof. This is essentially by iteratedly applying the base change functor and the previous proposition. □

Proposition 2.3.8. (After Kedlaya-Liu [KL15, Theorem 6.3.12]) *We have that the following four categories are equivalent:*

- I. *The category of quasicoherent finite locally free sheaves of $\mathcal{O}_{\text{Proj}P_A}$ -modules (namely the A -relative vector bundles) over the Fargues-Fontaine curve $\text{Proj}P_A$;*
- II. *The category of A -relative φ^α -modules over $\tilde{\Pi}_{R,A}^\infty$;*
- III. *The category of A -relative φ^α -modules over $\tilde{\Pi}_{R,A}$;*
- IV. *The category of A -relative φ^α -bundles over $\tilde{\Pi}_{R,A}$.*

Proof. Let us mention briefly the corresponding functors involved since essentially the process is as in the same fashion of the construction of [KL15, Theorem 6.3.12]. First from II to III, this is just the base change, and the III to IV is the functor which maps

any φ^a -module to the associated φ^a -bundle. Then for the functor from I to II one considers the process which associates any vector bundle V in the first category some finite locally free sheaf \mathcal{V} over the corresponding space associated to $\tilde{\Pi}_{R,A}^\infty$ by considering the localization and glueing through the consideration by f mentioned above, for the glueing we apply the direct analog of [KL15, Lemma 6.3.7] by considering proposition 2.2.9 and proposition 2.2.13. Then one takes the global section to get desired module in the second category. Then in our situation the previous proposition gives us the final equivalence after the well-established glueing process under the consideration of the further relativization coming from the algebra A . \square

One can then further discuss the corresponding results on the comparison of pseudocoherent objects (just as in [KL16, Definition 4.4.4]), as in [KL16, Section 4.6]. The results for the pseudocoherent objects are bit more complicated than the results established above in the context of just vector bundles. We first have the following analog of [KL16, Corollary 4.6.10]:

Proposition 2.3.9. *Let M_α, M, M_β be three finitely generated Frobenius modules over the ring $\tilde{\Pi}_{R,A}$ (namely endowed with a semilinear action from the Frobenius operator). And now we put the modules then in an exact sequence:*

$$0 \longrightarrow M_\alpha \longrightarrow M \longrightarrow M_\beta \longrightarrow 0.$$

Then we have first the following exact sequence for sufficiently large integer $n \geq 0$:

$$0 \longrightarrow M_\alpha(n)^{\varphi^a} \longrightarrow M(n)^{\varphi^a} \longrightarrow M_\beta(n)^{\varphi^a} \longrightarrow 0.$$

Then we have for any element f which is a $\varphi^a = p^\ell$ invariance, after forming the following module for each $M_* := M_\alpha, M, M_\beta$:

$$M_{*,f} := \bigcup_{n \in \mathbb{Z}} f^{-n} M_*(\ell n)^{\varphi^a},$$

the following exact sequence of Frobenius modules over $(\tilde{\Pi}_{R,A}[1/f])^{\varphi^a}$:

$$0 \longrightarrow M_{\alpha,f} \longrightarrow M_f \longrightarrow M_{\beta,f} \longrightarrow 0.$$

Eventually we have the fact that if M' is then a pseudocoherent Frobenius module over $\tilde{\Pi}_{R,A}$ then the module M_f is also a pseudocoherent module over $(\tilde{\Pi}_{R,A}[1/f])^{\varphi^a}$.

Proof. Apply proposition 2.2.15 as in corollary 2.2.14 to derive the first two consequences. As in [KL16, Corollary 4.6.10] one can further prove the last statement by taking the corresponding projective resolutions and by using the previous statement repeatedly. \square

Proposition 2.3.10. (After Kedlaya-Liu [KL16, Theorem 4.6.12]) *Taking pullbacks along the map from the scheme associated to $\tilde{\Pi}_{R,A}^\infty$ (by the analog of [KL15, Lemma 6.3.7] by using proposition 2.2.13 and proposition 2.2.9) to the (schematic) Fargues-Fontaine curve (as in [KL16, Definition 4.6.11]) gives rise to an exact functor which establishes the equivalence between the category of pseudocoherent sheaves over the Fargues-Fontaine curve and the category of pseudocoherent Frobenius modules over $\tilde{\Pi}_{R,A}$ (with isomorphism by the Frobenius pullbacks). This equivalence respects the corresponding comparison on the sheaf cohomologies and the Frobenius cohomologies.*

Proof. We follow the strategy of the proof of [KL16, Theorem 4.6.12] to prove this. First in our context we have that the above functor is actually just initially known to be right exact under the base change. Now suppose V is a pseudocoherent sheaf over the schematic Fargues-Fontaine curve in our context. As in the beginning of the proof of [KL16, Theorem 4.6.12] one has the following exact sequence:

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V \longrightarrow 0,$$

where V_1 is pseudocoherent and V_2 is assumed and set to be a vector bundle. Then we apply the pullback construction (as mentioned in the statement of the proposition) we have the corresponding (after taking the further base change) exact sequence of the pseudocoherent Frobenius modules over the ring $\tilde{\Pi}_{R,A}$:

$$W_1 \longrightarrow W_2 \longrightarrow W \longrightarrow 0,$$

where W_2 is finite projective and W_1 is finitely generated. We then have the following exact sequence:

$$0 \longrightarrow * \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W \longrightarrow 0,$$

where $*$ is the kernel of the map $W_1 \rightarrow \text{Kernel}(W_2 \rightarrow W)$. Then choose some f with some specific degree as what we did before, by the previous proposition we have the following exact sequence:

$$0 \longrightarrow *_f \longrightarrow W_{1,f} \longrightarrow W_{2,f} \longrightarrow W_f \longrightarrow 0,$$

then by taking the corresponding section we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & V_1|_{\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}} & \longrightarrow & V_2|_{\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}} & \longrightarrow & V|_{\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}} \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & *_f & \longrightarrow & W_{1,f} & \longrightarrow & W_{2,f} \longrightarrow W_f \longrightarrow 0
\end{array}$$

where by our above results on the comparison on the finite projective objects we have that the second vertical morphism is isomorphism which implies that the third vertical map is surjective. Then we repeat the corresponding argument and the construction by using V_1 as our V we can also derive the fact that the first vertical map is also at this situation surjective. Then as in [KL16, Theorem 4.6.12] the five lemma implies that the third vertical map is also injective. Then we apply the same construction and argument to the situation where we use V_1 to be our V we can derive the fact that the first vertical map is also injective. Then we have that all the vertical maps are then in this situation isomorphism, which further implies that $*_f$ is trivial, as in [KL16, Theorem 4.6.12] by using the previous proposition we have that $*$ is trivial. The functor send finite projective objects to the corresponding finite projective objects so then we have the pseudocoherent objects in the corresponding essential image. Then we have a well-defined exact functor in our situation where the quasi-inverse is just taking the corresponding Frobenius invariance. The functor from the modules over the Robba ring to the sheaves over the Fargues-Fontaine curve will have the corresponding composition with the quasi-inverse being equivalence. On the other hand, to show that the functor from the sheaves over the Fargues-Fontaine curve will have the corresponding composition with the quasi-inverse being equivalence will eventually reduce to the corresponding statement in the finite projective setting as what we did before. \square

2.3.2 Fréchet-Stein Deformation of Schematic Relative Fargues-Fontaine curves

For our purpose and motivation, we would like to consider the corresponding situation where A is replaced by the distribution algebras attached to some specific p -adic

Lie groups or some mixed-type algebra by taking the product of A with these interesting distribution algebras. Let us recall the following construction:

Setting 2.3.11. Let G be a p -adic Lie group in the fashion picked in the next setting, then we will use the notation $\mathcal{O}_K[[G]]$ to denote the corresponding integral completed group algebra over some finite extension K/\mathbb{Q}_p . Then we use the notation $K[[G]]$ to denote the base change of the integral ring to K . From here we consider the following integral version of the distribution we are considering:

$$A_n := \mathcal{O}_K[[G]][\mathfrak{m}^n/p]_{(p)}^\wedge[1/p]$$

then take the inverse limit we get the ring A_∞ , where \mathfrak{m} is the Jacobson radical. However this is not Noetherian, which makes life complicated. It is expected that one should consider this ring in order to do the equivariant or even more general noncommutative Iwasawa theory and the corresponding Tamagawa conjectures, in stead of just the usual Iwasawa algebra. For instance in the situation where G is just \mathbb{Z}_p^\times and $K = \mathbb{Q}_p$ this is just the ring $\Pi^\infty(\Gamma_K)$ which is just the Robba ring corresponding to the radius ∞ . Also see [Zah1] where this is discussed in more detail.

Setting 2.3.12. In our current context we are going to focus on those groups in the following form:

$$G = \mathbb{Z}_p^d, \mathbb{Z}_p^\times, \mathbb{Z}_p^\times \rtimes \mathbb{Z}_p^n.$$

In this situation we just define the corresponding period rings deformed over the distribution algebra $A_\infty(G)$:

$$\tilde{\Omega}_{R,A_\infty(G)}^{\text{int}} := \varprojlim_{n \rightarrow \infty} \tilde{\Omega}_{R,A_n^{\text{int}}(G)}^{\text{int}}, \tilde{\Omega}_{R,A_\infty(G)} := \varprojlim_{n \rightarrow \infty} \tilde{\Omega}_{R,A_n(G)}, \quad (2.3.1)$$

$$\tilde{\Pi}_{R,A_\infty(G)}^{\text{int},r} := \varprojlim_{n \rightarrow \infty} \tilde{\Pi}_{R,A_n^{\text{int}}(G)}^{\text{int},r}, \tilde{\Pi}_{R,A_\infty(G)}^{\text{int}} := \varprojlim_{n \rightarrow \infty} \tilde{\Pi}_{R,A_n^{\text{int}}(G)}^{\text{int}}, \quad (2.3.2)$$

$$\tilde{\Pi}_{R,A_\infty(G)}^{\text{bd},r} := \varprojlim_{n \rightarrow \infty} \tilde{\Pi}_{R,A_n(G)}^{\text{bd},r}, \tilde{\Pi}_{R,A_\infty(G)}^{\text{bd}} := \varprojlim_{n \rightarrow \infty} \tilde{\Pi}_{R,A_n(G)}^{\text{bd}}, \quad (2.3.3)$$

$$\tilde{\Pi}_{R,A_\infty(G)}^I := \varprojlim_{n \rightarrow \infty} \tilde{\Pi}_{R,A_n(G)}^I, \quad (2.3.4)$$

from which one can define furthermore the period rings

$$\tilde{\Pi}_{R,A_\infty(G)}^r, \tilde{\Pi}_{R,A_\infty(G)}^\infty, \tilde{\Pi}_{R,A_\infty(G)}. \quad (2.3.5)$$

Then we work over the corresponding finite locally free modules or bundles over the corresponding period rings defined above with the same definitions we used in definition 2.2.7 to define those modules over the period rings associated to $A_\infty(G)$. We also endow the rings defined above with partial Frobenius where the action is induced from the Witt vector part, not the coefficient algebra part. We then define the corresponding families of $A_\infty(G)$ -relative φ^a -module M over the ring $\tilde{\Pi}_{R,A_\infty(G)}^\infty$ or $\tilde{\Pi}_{R,A_\infty(G)}$ to be a projective system $(M_n)_n$ where each M_n for each n is a corresponding $A_n(G)$ -relative φ^a -modules over $\tilde{\Pi}_{R,A_n(G)}^\infty$ or $\tilde{\Pi}_{R,A_n(G)}$ such that we have the base change requirement for each n :

$$A_n(G) \otimes_{A_{n+1}(G)} M_{n+1} \xrightarrow{\sim} M_n.$$

Similarly we define the Frobenius bundles in this compatible way. And finally we define the corresponding global section of any families of the Frobenius modules over the corresponding period rings to be the corresponding projective limit taking the form of $\varprojlim_{n \rightarrow \infty} M_n$.

Remark 2.3.13. One can safely extend the assumption on the group G to more general setting such that the corresponding algebra $A_\infty(G)$ are inverse limit of reduced (commutative) affinoid algebras.

Definition 2.3.14. We will in this situation use the notation $P_{A_\infty(G)}$ to mean the ring $\varprojlim_n P_{A_n(G)}$, therefore in our situation we have the corresponding ind-scheme $\text{Proj}P_{A_\infty(G)}$ at the infinite level. Therefore we define the corresponding vector bundles over $\text{Proj}P_{A_\infty(G)}$ to be the corresponding quasi-coherent finite locally free sheaves over the infinite level scheme $\text{Proj}P_{A_\infty(G)}$. We then define the families of vector bundles to be those families taking the form of $(M_n)_n$ where each M_n is a quasicohherent finite locally free sheaves M_n over each $\text{Proj}P_{A_n(G)}$ in the compatible way as in the previous setting. In this situation we define the corresponding global section of a family of vector bundles to be $\varprojlim_{n \rightarrow \infty} M_n$.

Remark 2.3.15. Here we actually have different notions of the corresponding modules over the corresponding Stein style. Since as in the usual story of the Iwasawa theory we have to study the corresponding derived categories, K -group spectra, the corresponding determinant category and so on, so we need to be careful when we would like to use the corresponding notions defined above. Here the main subtle point in the consideration is that actually it is nontrivial if the global sections of the corresponding sheaves are finitely generated.

For the corresponding families of the modules and the bundles we have the following comparison theorem:

Proposition 2.3.16. *We have that the following four categories are equivalent:*

- I. *The category of families of quasicoherent finite locally free sheaves of $\mathcal{O}_{\text{Proj}P_{A_\infty(G)}}$ -modules (namely the $A_\infty(G)$ -relative vector bundles) over the Fargues-Fontaine curve $\text{Proj}P_{A_\infty(G)}$;*
- II. *The category of families of $A_\infty(G)$ -relative φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)}^\infty$;*
- III. *The category of families of $A_\infty(G)$ -relative φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)}$;*
- IV. *The category of families of $A_\infty(G)$ -relative φ^a -bundles over $\tilde{\Pi}_{R,A_\infty(G)}$.*

Proof. This will be further application of our established comparison above for $A_n(G)$ for each n . □

And furthermore one could then further consider the following generality:

Corollary 2.3.17. *We have that the following four categories are equivalent:*

- I. *The category of families of quasicoherent finite locally free sheaves of $\mathcal{O}_{\text{Proj}P_{A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A}}$ -modules (namely the $A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A$ -relative vector bundles) over the Fargues-Fontaine curve $\text{Proj}P_{A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A}$;*
- II. *The category of families of $A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A$ -relative φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A}^\infty$;*
- III. *The category of families of $A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A$ -relative φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A}$;*
- IV. *The category of families of $A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A$ -relative φ^a -bundles over $\tilde{\Pi}_{R,A_\infty(G)\widehat{\otimes}_{\mathbb{Q}_p}A}$.*

Then we will study in some generality established below the cohomology which will generalize the situation in [KL15, Proposition 6.3.19]. In our setting actually one could consider more general A . The algebras we are considering in the relative setting will be again the general Iwasawa theoretic level, namely the Fréchet algebra $A\widehat{\otimes}A_\infty(G)$. One can study the corresponding complexes of the Frobenius modules, or one can study the corresponding complexes of the corresponding families of the Frobenius modules and bundles.

Proposition 2.3.18. *Consider now a φ^a -module M over $\tilde{\Pi}_{R,A}^\infty$ (which is always assumed to be finite projective). Then we have the following statements in our situation which generalize the corresponding results in [KL15, Proposition 6.3.19]:*

I. Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \xrightarrow{\varphi^a - 1} & M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_r & \xrightarrow{\varphi^a - 1} & M_{r/q} & \longrightarrow & 0.
\end{array}$$

Then we have that the vertical maps give rise to the quasi-isomorphism of the two complexes involved;

II. In our situation, the natural base change $M \rightarrow M \otimes_{\tilde{\Pi}_{R,A}^\infty} \tilde{\Pi}_{R,A}$ gives rise to a quasi-isomorphism;

III. Consider the following commutative diagram for some interval $[s, r]$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \xrightarrow{\varphi^a - 1} & M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{[s,r]} & \xrightarrow{\varphi^a - 1} & M_{[s,r/q]} & \longrightarrow & 0.
\end{array}$$

Then again we have that the vertical map is a quasi-isomorphism of the two complexes involved. Here the corresponding radii satisfy the condition that $0 < s \leq r/q$.

Proof. Largely this follows the same idea as in [KL15, Proposition 6.3.19] as below. The resulting complexes in our situation are those in the derived category of modules over ring A . We only need to prove I and III, then taking the limit will give us the second statement. Now for the first statement, as in the situation of [KL15, Proposition 6.3.19] the injectivity of both the maps

$$H_{\varphi^a}^0(M) \rightarrow H_{\varphi^a}^0(M_r),$$

$$H_{\varphi^a}^0(M) \rightarrow H_{\varphi^a}^0(M_I)$$

follows due to the property of the finite projectiveness. Then we consider the corresponding surjectivity. In this case, we just need to consider as in [KL15, Proposition 6.3.19] the equation:

$$v = \varphi^{-na}v$$

for each $n \geq 1$, for instance considering those $v \in H_{\varphi^a}^0(M_r)$ and considering this will give rise to some element in $M_{q^n r}$, then taking the bootstrap like this to derived the result,

which is the same for the situation in III. Then for the first order cohomology, one could just follow the idea in [KL15, Proposition 6.3.19] with the help of the previous comparison to lift the corresponding extension. We do not repeat the argument again. \square

Remark 2.3.19. Here we only consider the corresponding comparison carrying the deformation to A instead of more general situation where the deformation is over simultaneously A and $A_\infty(G)$. But one could easily have the idea on what should be established in more general setting.

2.4 Hodge-Iwasawa Sheaves

2.4.1 Constructible p -adic Iwasawa Sheaves

It is actually quite natural to consider the corresponding constructible p -adic sheaves in Iwasawa theory as those considered in [Wit1] on the interpolation of L -functions after Grothendieck from [SGA5]. On the other hand, in the situation where one considers the Weil conjectures, it is actually more natural to consider both constructible and non-étale objects as in all types of Weil-II considered by Deligne [DWeI] and [DWeII], Kedlaya [KWeII], Caro, Abe [CAWeII] and etc. Therefore we actually would like to consider the commutative equivariant version of the Hodge sheaves with Hodge structures considered in [KL15]. We will not consider the generality just as in the previous section here, but we will consider the generality instead more general than [KP1].

Setting 2.4.1. All the flat constructible p -adic sheaves (both the \mathbb{Z}_p ones or the \mathbb{Z}_p -isogeny ones), and all the Frobenius modules over the period Hodge sheaves will be considered in terms of some Iwasawa deformations, with the deformation in the corresponding reduced affinoids A over \mathbb{Q}_p , or some adic ring T (we will consider the situation considered by Witte in his noncommutative projects [Wit1] and [Wit3]), or a general Fréchet-Stein algebra $A_\infty(G)$ attached to some nice group G . In this section we assume that each adic ring is defined over \mathbb{Z}_p and each quotient T/I by some open two-sided ideal I is of order a power p^n of p .

Remark 2.4.2. The Fréchet-Stein deformation over a point is essentially the one proposed by Kedlaya-Pottharst [KP1], which means that actually it is very important to consider the corresponding DfmLie namely the p -adic Lie deformation of the corresponding relative A Hodge-Frobenius sheaves over the corresponding pro-étale site of a pre-adic space X , which will be a natural generalization of the corresponding constructions proposed in [KP1] and established in [KL15].

We now start from the spaces in our setting, namely the pre-adic spaces X over \mathbb{Q}_p in the sense of those considered in [KL15]. Then we will consider the corresponding homotopical categories of p -adic sheaves over the sites $X_{\text{ét}}$ and $X_{\text{proét}}$. As the setting considered by Witte in [Wit1] and [Wit3], we will also consider the families of complexes with finite coefficients parametrized by the corresponding set of radicals \mathfrak{J} of an adic ring

T .

In the style of the setting in Grothendieck's SGA IV and V, Witte studies the corresponding Iwasawa theory in the Fukaya-Kato setting, which is actually noncommutative. This is really general since the corresponding coefficients of the sheaves are actually in the adic ring mentioned above. The corresponding interpolation happens in the corresponding algebraic K -groups associated to some Waldhausen category. Waldhausen categories are natural places where one could do algebraic K -theory through Waldhausen S -constructions. We work out some generalization to the setting in our situations. We will consider from the following generality:

Definition 2.4.3. (After Witte, [Wit1],[Wit3, Chapter 5]) Let \mathfrak{J} be the set of all the two-sided ideals open of the adic ring T . Let $\mathbb{D}_{\text{perf}}(X, T)$ denote the following category. Each object $(\mathcal{M}_I^\bullet)_{I \in \mathfrak{J}}$ of the category is now a family (to be more precise an inverse system) of the corresponding perfect (which are quasi-isomorphic to those strictly-perfect ones) complexes of constructible flat étale sheaves over $X_{\text{ét}}$ parametrized by the corresponding open two-sided ideals of the adic ring T , such that: I. For each member of such family $(\mathcal{M}_I^\bullet)_{I \in \mathfrak{J}}$, we have that the complex \mathcal{M}_I^\bullet is now supposed to be dg-flat over the ring T/I (degreewise flat and the tensor product with any acyclic complex will be again acyclic, note here that the corresponding tensor product is taken over the ring T/I when we are talking about each individual complex in a single family); II. We have the corresponding transition map $\psi_{I,J} : \mathcal{M}_I^\bullet \rightarrow \mathcal{M}_J^\bullet$ for any two open two-sided ideals $I \subset J$ with the basic base change requirement:

$$\mathcal{M}_I^\bullet \otimes_{T/I} T/J \xrightarrow{\sim} \mathcal{M}_J^\bullet.$$

All the modules over any quotient T/I is assumed to be left T/I -modules, while note that then the tensor product will happen when we have another right T/I -module.

Remark 2.4.4. The information of the homotopy type comes from the weak-equivalence which is defined to be the quasi-isomorphism and the cofibration which is defined to be the injection with kernel in the original category.

Note that Fukaya-Kato originally considered Deligne's virtual categories (strictly speaking through more transparent approach) where they formulated their celebrated ζ -isomorphism conjecture and the local ε -conjectures. Witte used Waldhausen construction

to give more K -theoretic pictures, which could be regarded as some reinterpretation of the Fukaya-Kato's picture in the corresponding homotopical categories. We are somehow deeply inspired by this point of view. Concentrating everything in the zero-th degree we have the following definition:

Definition 2.4.5. We consider the subcategory $\mathbb{D}_{\text{perf}}^{\text{const}}(X, T)$ of $\mathbb{D}_{\text{perf}}(X, T)$ which consists of all the families of complexes degenerated to zeroth degree of flat constructible sheaves over $X_{\text{ét}}$, again parametrized by elements in \mathfrak{J} , which to be more precise means that the corresponding coefficients will live in the coefficient ring taking the form of T/I . For each element we will then use the corresponding notation $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{J}}$ to denote each family of flat constructible étale sheaves.

This subcategory plays the role exactly the same as those flat constructible ℓ -adic sheaves considered by Witte's picture following Deligne's construction and equivariant Abe-Caro's arithmetic D -module point of views. If one is more Hodge theoretic, then one should degenerate again to define:

Definition 2.4.6. We will use the notation $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X, T)$ to denote the subcategory of $\mathbb{D}_{\text{perf}}^{\text{const}}(X, T)$ consists of all the families of complexes degenerated to zeroth degree of étale local systems $(\mathbb{L}_I)_{I \in \mathfrak{J}}$ parametrized by the elements in \mathfrak{J} .

These will be generalization of our consideration in the previous section, but restricting to the integral p -adic Hodge theory.

Definition 2.4.7. We call each object $(\mathbb{L}_I)_{I \in \mathfrak{J}}$ in the corresponding category

$$\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X, T)$$

a noncommutative étale local system.

Then we could use this point of view to generalize the corresponding Fontaine's style equivalence in the previous section over perfectoid spaces immediately after [KL15, Section 8.5]:

Proposition 2.4.8. *We have the following categories are equivalent for $X = \text{Spa}(R, R^+)$ where R is a perfect adic Banach uniform algebra over \mathbb{F}_p with associated perfectoid space $X' = \text{Spa}(A, A^+)$ under the perfectoid correspondence:*

- I. The category $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X, T)$ of the noncommutative étale local systems over X ;
 II. The category $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X', T)$ of the noncommutative étale local systems over X' .

Moreover, we have that there is a fully faithful embedding from the category $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X, T)$ (or $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X', T)$ respectively) of all families of noncommutative étale local systems over X or X' into the corresponding category of the families $(M_I)_I$ of φ -modules over $\tilde{\Pi}_R^{\text{int}}$ with deformation to T where each M_I is a φ -module over $\tilde{\Pi}_R^{\text{int}} \otimes_{\mathbb{Z}_p} (T/I)$ in the usual sense such that we have the transition map $\psi_{I,J}$ as above with the additional base change properties.

Proof. In this situation this is derived from the corresponding result for the categories without Iwasawa deformation as proved in [KL15, Section 8.5]. \square

Definition 2.4.9. We define the following key categories over the site $X_{\text{proét}}$ just as in the way as above by replacing X with $X_{\text{proét}}$:

$$\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X_{\text{proét}}, T) \subset \mathbb{D}_{\text{perf}}^{\text{const}}(X_{\text{proét}}, T) \subset \mathbb{D}_{\text{perf}}(X_{\text{proét}}, T).$$

Then we glue everything in the previous two propositions over the corresponding pro-étale sites. Let $Y \in X_{\text{proét}}$ be a perfectoid subdomain in the basis of the neighbourhood of this site, as in [KL15, Section 9.2] taking the form of $U = \varprojlim_{n \rightarrow \infty} U_n$. Then we use the corresponding notations \mathcal{O}_X , \mathcal{O}_X^+ , \mathcal{O}_X° to denote the corresponding sheaves evaluating over U as in [KL15, Section 9.2] as below:

$$\mathcal{O}_X(U) := \varinjlim_n \mathcal{O}_{U_n}(U_n) \tag{2.4.1}$$

$$\mathcal{O}_X^+(U) := \varinjlim_n \mathcal{O}_{U_n}^+(U_n) \tag{2.4.2}$$

$$\mathcal{O}_X^\circ(U) := \varinjlim_n \mathcal{O}_{U_n}^\circ(U_n). \tag{2.4.3}$$

Recall that from [KL15, Section 9.2] we have the corresponding completed sheaves which we use the notations $\widehat{\mathcal{O}}_X$, $\widehat{\mathcal{O}}_X^+$ to denote them, then under the perfectoid correspondence we have the corresponding version of the sheaves over the pro-étale sites. Recall that these are denoted in [KL15, Section 9.2] by $\overline{\mathcal{O}}_X$ and $\overline{\mathcal{O}}_X^+$ following Andreatta-Iovita, by taking suitable tilting under the evaluation on any perfectoid subdomain U as above. Then we are now at the position to glue the previous proposition over $X_{\text{proét}}$.

Definition 2.4.10. Following [KL15, Section 9.3], we are going to use the notation $\widetilde{\mathbf{A}}_X \widehat{\otimes}_{\mathbb{Z}_p} T$ and the notation $\widetilde{\mathbf{A}}_X \otimes_{\mathbb{Z}_p} (T/I)$ to be the sheafifications of the corresponding presheaves over $X_{\text{proét}}$ sending each $U \in X_{\text{proét}}$ to

$$W(\overline{\mathcal{O}}_X(U)) \widehat{\otimes}_{\mathbb{Z}_p} T, W(\overline{\mathcal{O}}_X(U)) \otimes_{\mathbb{Z}_p} (T/I), \quad (2.4.4)$$

for each element $I \in \mathfrak{J}$. And we use the notations:

$$\widetilde{\mathbf{A}}_X^\dagger \widehat{\otimes}_{\mathbb{Z}_p} T, \widetilde{\mathbf{A}}_X^\dagger \otimes_{\mathbb{Z}_p} (T/I)$$

to denote the corresponding sheafification of the presheaves sending each element U in $X_{\text{proét}}$ to:

$$\widetilde{\Pi}_{\overline{\mathcal{O}}_X(U)}^{\text{int}} \widehat{\otimes}_{\mathbb{Z}_p} T, \widetilde{\Pi}_{\overline{\mathcal{O}}_X(U)}^{\text{int}} \otimes_{\mathbb{Z}_p} (T/I). \quad (2.4.5)$$

Then we have the following generalization of our previous proposition to more general setting over the pro-étale sites, again immediately after [KL15, Theorem 9.3.7] and its proof.

Proposition 2.4.11. *We have the following categories are equivalent for X defined above (as in the previous proposition):*

- I. *The category $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X_{\text{proét}}, T)$ of the noncommutative étale local systems over $X_{\text{proét}}$;*
- II. *The category $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X'_{\text{proét}}, T)$ of the noncommutative étale local systems over $X'_{\text{proét}}$.*

Moreover, we have that there is a fully faithful embedding from the category $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X_{\text{proét}}, T)$ (or $\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X'_{\text{proét}}, T)$ respectively) of all families of noncommutative pro-étale local systems over X or X' into the corresponding category of the families $(M_I)_I$ of φ -modules over $\widetilde{\mathbf{A}}_X^\dagger$ with deformation to T where each M_I is a φ -module over $\widetilde{\mathbf{A}}_X^\dagger \otimes_{\mathbb{Z}_p} (T/I)$ in the usual sense such that we have the transition map $\psi_{I,J}$ as above with the additional base change properties.

2.4.2 Noncommutative-Equivariant K -Theory in Waldhausen Categories

Witte used K -theory to have had formulated some conjectures related to the corresponding Fukaya-Kato's key conjectures in Deligne's virtual category. Note that we

defined key categories in the previous subsection for our further study on the Hodge-Iwasawa theory in the integral setting:

$$\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X, T) \subset \mathbb{D}_{\text{perf}}^{\text{const}}(X, T) \subset \mathbb{D}_{\text{perf}}(X, T).$$

Also we have the following pro-étale version of the corresponding categories:

$$\mathbb{D}_{\text{perf}}^{\text{const,smooth}}(X_{\text{proét}}, T) \subset \mathbb{D}_{\text{perf}}^{\text{const}}(X_{\text{proét}}, T) \subset \mathbb{D}_{\text{perf}}(X_{\text{proét}}, T).$$

In the corresponding K -theory it is actually natural to ask for instance whether we have the corresponding special isomorphisms or homotopies as considered by Fukaya-Kato and Witte in the category of rigid analytic spaces or more general Huber's adic spaces. For instance one can consider the corresponding category $\mathbb{D}_{\text{perf}}(\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p}), T)$ over a point. We actually put our observation into the following conjecture which actually predicts that over rigid analytic spaces we have the corresponding well-defined Waldhausen theory where in the algebraic geometry this is conjectured and studied by Witte in [Wit1], [Wit2] and [Wit3, Section 1.4]. First let us consider the following definition:

Definition 2.4.12. (After Witte, [Wit1],[Wit3, Chapter 5]) Let \mathfrak{J} be the set of all the two-sided ideals open of the adic ring T where we assume that p is a unit in T . Let $\mathbb{D}_{\text{perf}}(X, T)$ denote the following category. Each object $(\mathcal{M}_I^\bullet)_{I \in \mathfrak{J}}$ of the category is now a family (to be more precise an inverse system) of the corresponding perfect (which are quasi-isomorphic to those strictly-perfect ones) complexes of constructible flat étale sheaves over $X_{\text{ét}}$ parametrized by the corresponding open two-sided ideals of the adic ring T , such that: I. For each member of such family $(\mathcal{M}_I^\bullet)_{I \in \mathfrak{J}}$, we have that the complex \mathcal{M}_I^\bullet is now supposed to be dg-flat over the ring T/I (degreewise flat and the tensor product with any acyclic complex will be again acyclic, note here that the corresponding tensor product is taken over the ring T/I when we are talking about each individual complex in a single family); II. We have the corresponding transition map $\psi_{I,J} : \mathcal{M}_I^\bullet \rightarrow \mathcal{M}_J^\bullet$ for any two open two-sided ideals $I \subset J$ with the basic base change requirement:

$$\mathcal{M}_I^\bullet \otimes_{T/I} T/J \xrightarrow{\sim} \mathcal{M}_J^\bullet.$$

All the modules over any quotient T/I is assumed to be left T/I -modules, while note that then the tensor product will happen when we have another right T/I -module. And we also have the corresponding categories in the pro-étale setting.

Remark 2.4.13. Note that here we consider the corresponding situation where p is a unit in the adic ring T which is different from the previous section.

Conjecture 2.4.14. (After Witte) *With the assumption in the previous definition, for X a rigid analytic space over the p -adic field \mathbb{Q}_p which is separated and of finite type (when considered as some adic space locally of finite type, see [Huber1]) such that for $\sharp = \acute{e}t, \text{pro}\acute{e}t$ the categories $\mathbb{D}_{\text{perf}}(X_{\sharp}, T)$ could be endowed with the structure of Waldhausen categories, and we have there is a Waldhausen exact functor $R\Gamma(X_{\sharp}, \cdot)$ (induced by the corresponding direct image functor as in [Wit3, Chapter 4-5]) as below for $\sharp = \acute{e}t, \text{pro}\acute{e}t$:*

$$\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{R\Gamma(X_{\sharp}, \cdot)} \mathbb{D}_{\text{perf}}(T)$$

which induces the corresponding the morphism $\mathbb{K}R\Gamma(X_{\sharp}, \cdot)$ as:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{\mathbb{K}R\Gamma(X_{\sharp}, \cdot)} \mathbb{K}\mathbb{D}_{\text{perf}}(T),$$

we conjecture that there is a homotopy between this morphism and the zero map:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{0} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

And we conjecture this will be compatible with the higher direct image functor $Rf_!$ between different spaces (see Huber's book [Huber1, Chapter 5]) and the changing coefficient morphism for a pair T, T' of adic rings.

Remark 2.4.15. We are actually not quite for sure how much higher categorical information one could see through the potential enrichment. But one might want to ask this even in the scheme situation.

We would like to start from here (in the future consideration) to study the corresponding K -theoretic construction after the work of [Wit1] and [Wit3]. We would also like to start from in the future consideration the corresponding schematic version of the construction in Witte's thesis for instance in [Wit3], where [Wit3] has already constructed the corresponding categories of étale sheaves and the corresponding functors and properties.

Example 2.4.16. Suppose now everything is just over the point associated to the point \mathbb{Q}_p , namely the space $\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})$. Then the story collapses to the corresponding Galois

cohomology and the corresponding Waldhausen category of the derived Galois representations, which is the Waldhausen categorical generalization of the usual Galois theoretic Tamagawa-Iwasawa theory. To be more precise we have first of all that the categories

$$\mathbb{D}_{\text{perf}}(\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})_{\sharp}, T)$$

could be endowed with structure of Waldhausen categories as shown in [Wit3, Chapter 5, 6]. Moreover we have the following Waldhausen exact functor (see [Wit3, Chapter 5]):

$$R\Gamma(\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})_{\sharp}, \cdot) : \mathbb{D}_{\text{perf}}(\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})_{\sharp}, T) \longrightarrow \mathbb{D}_{\text{perf}}(T)$$

which gives rise to the following map on the spectra:

$$KR\Gamma(\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})_{\sharp}, \cdot) : K\mathbb{D}_{\text{perf}}(\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})_{\sharp}, T) \longrightarrow K\mathbb{D}_{\text{perf}}(T).$$

Remark 2.4.17. Recall that actually in the categories of the virtual objects, Nakamura obtains partial results on the existence of the corresponding ε -isomorphism for étale (φ, Γ) -modules, for more detail see [Nak1] and [Nak2]. We only expect that one can construct the corresponding homotopy for the Waldhausen categorical consideration. Therefore in the situation where p is not a unit in T we still conjecture that there should be such Waldhausen category and the corresponding induced maps on the K -theoretic spaces, and even the null-homotopy. Note that this is some generalization of the corresponding picture proposed in [KP1] in the étale situation. Here we are considering derived étale φ -sheaves over the pro-étale sites in the corresponding Waldhausen categories. The situation of a point as above corresponding to \mathbb{Q}_p is already interesting to some extent. We will continue our discussion in this situation below.

2.4.3 Hodge-Iwasawa Period Sheaves and Hodge-Iwasawa Vector Bundles

We consider here vector bundles over the corresponding adic version of the Fargues-Fontaine curves and consider the relationship between these and the ones we encountered in the last section. We are going to first recall the basics about the Fargues-Fontaine curves in the adic setting. We start from the following picture in the local situation, namely over some perfectoid ring R which is some member in the basis of topology of an adic spaces in some well-established category.

Setting 2.4.18. We recall some adic geometry from [KL15, Definition 8.7.4] around the corresponding geometry of the adic version of the Fargues-Fontaine curve with respect to the algebra R as in our discussion above, which is to be denoted as in [KL15, Definition 8.7.4] FF_R . For the convenience of the readers we recall the basic construction of the curves on the level of the adic spaces. In this local setting the curve FF_R is defined to be the quotient of the corresponding adic space:

$$O_R := \bigcup_{0 < s < r} \mathrm{Spa}(\tilde{\Pi}_R^{[s,r]}, \tilde{\Pi}_R^{[s,r],+}).$$

See [KL15, Definition 8.7.4]. Then by the basic construction we could have the chance to find a link between the adic Fargues-Fontaine curve and our originally considered schematic Fargues-Fontaine curve. To be more precise, we consider the following map:

$$P_R \rightarrow \tilde{\Pi}_R^I$$

which naturally induces a morphism in the category of locally ringed spaces:

$$\mathrm{FF}_R \rightarrow \mathrm{Proj} P_R$$

coming from the one attached to the space O_R . Then we could consider the Iwasawa deformation of the vector bundles over these different locally-ringed spaces in the large categories of locally-ringed spaces.

The corresponding deformed version of the adic Fargues-Fontaine curves could be defined in the same way. But note that here we need to require the corresponding deformed period rings are sheafy in the category of adic rings. This is in general not a trivial statement. For instance if we are in the situation where the affinoid algebra A is a Tate algebra then we know that this is indeed the case. However in the full generality this is not that easy to see to be true. We now choose to work around this by just putting some assumption on the algebra A which is already applicable enough in some interesting enough situations. Recall from [KH] that when a ring is sousperfectoid then we should have the following nice result. Recall that (from [KH, Introduction]) a Huber ring S is called sousperfectoid if it admits a split continuous morphism to a perfectoid Huber ring P . Note that this morphism is considered in the category of S -modules which are topological.

Proposition 2.4.19. (Kedlaya-Hansen) *The sousperfectoid rings are sheafy.*

Proof. See [KH, Corollary 7.4]. □

Example 2.4.20. Sousperfectoid rings are not rare, for instance one can just take the corresponding perfectoid field $\mathbb{Q}_p(p^{p^{-\infty}})^\wedge$ then \mathbb{Q}_p will be then sousperfectoid in this larger perfectoid ring which admits a splitting. Also the ring $\tilde{\Pi}_R^I$ is then sousperfectoid with respect to $\tilde{\Pi}_R^I \hat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(p^{p^{-\infty}})^\wedge$. More general, smooth affinoid algebras are sousperfectoid.

Assumption 2.4.21. From now on, we assume that A is sousperfectoid.

Corollary 2.4.22. *For any closed interval I we have that the ring $\tilde{\Pi}_{R,A}^{[s,r]}$ is sheafy.*

Remark 2.4.23. Base on the observation above, we actually make the following conjecture:

Conjecture 2.4.24. *The ring $\tilde{\Pi}_{R,B}^{[s,r]}$ is stably uniform for any reduced affinoid algebra B , where the corresponding observation is as in the following.*

Note that a Tate affinoid algebra over a uniform Banach ring is uniform, so we have that the ring $\tilde{\Pi}_{R,T_d}^{[s,r]}$ is uniform, but it is not the case that the rational localization of $\tilde{\Pi}_{R,T_d}^{[s,r]}$ is the product of rational localizations of $\tilde{\Pi}_R^{[s,r]}$ and T_d , which is to say that one does not have the chance to prove this directly along this. However since the ring $\tilde{\Pi}_{R,T_d}^{[s,r]}$ is sousperfectoid, so we have that the corresponding ring $\tilde{\Pi}_{R,T_d}^{[s,r]}$ in the situation of Tate algebra is stably uniform. Now for general A we suspect this is also true.

Setting 2.4.25. In the same fashion, in our situation we consider the corresponding space

$$O_{R,A} := \bigcup_{0 < s < r} \mathrm{Spa}(\tilde{\Pi}_{R,A}^{[s,r]}, \tilde{\Pi}_{R,A}^{[s,r],+})$$

which naturally gives rise to the corresponding quotient $\mathrm{FF}_{R,A}$ as mentioned above. This will allow us to consider the corresponding comparison in our generalized context as in the following.

Setting 2.4.26. In our current situation, we have our adic version of the corresponding Fargues-Fontaine curves in some rigid analytic deformation context, which includes

the situation where the rigid analytic spaces is some Stein space. In our situation the corresponding space will be:

$$O_{R,A_\infty(G)} := \varinjlim_{n \rightarrow \infty} \bigcup_{0 < s < r} \text{Spa}(\tilde{\Pi}_{R,A_n(G)}^{[s,r]}, \tilde{\Pi}_{R,A_n(G)}^{[s,r],+}).$$

Then we have the obvious notion of vector bundles over this space, which is Stein in some generalized sense, this is because the corresponding building rings are not noetherian. Then through taking the corresponding quotient we have the corresponding quotient space $\text{FF}_{R,A_\infty(G)}$. We can also define the corresponding vector bundles over the ∞ -level space while moreover we have the corresponding notions of families of vector bundles $(M_n)_n$ where each M_n is a quasicohherent finite locally free sheaf over $\text{FF}_{R,A_n(G)}$.

Then we have the following relative version of [KL15, Theorem 8.7.7]:

Proposition 2.4.27. *The categories of A -relative vector bundles over the two spaces $\text{FF}_{R,A}$ and $\text{Proj}P_{R,A}$ defined above are equivalent. The categories of families of A -relative vector bundles over the two spaces $\text{FF}_{R,A_\infty(G)}$ and $\text{Proj}P_{R,A_\infty(G)}$ defined above are equivalent.*

Proof. One could derive this by the same way as in [KL15, Theorem 8.7.7]. □

Remark 2.4.28. One interesting question here is that we can study the corresponding global sections taking the form of $\varprojlim_{n \rightarrow \infty} M_n$ of the families of vector bundles and the corresponding families of Frobenius modules through the equivalences we established so far.

First we have the following corollary of the previous proposition:

Corollary 2.4.29. *Categories of the families of Frobenius modules over $\tilde{\Pi}_{R,A_\infty(G)}$, the families of Frobenius bundles over $\tilde{\Pi}_{R,A_\infty(G)}$ and the families of vector bundles over the adic Fargues-Fontaine curve $\text{FF}_{R,A_\infty(G)}$ are equivalent.*

Remark 2.4.30. The main issue here is that the corresponding families of Frobenius modules and vector bundles might not have finite generated global sections since in the whole process of taking inverse limit the numbers of the generators might be blowing up. At this moment we do not know whether the global sections of a family of vector bundles over $\text{FF}_{R,A_\infty(G)}$ are finitely generated. In the situation where one can get the

chance to control the largest number of generators during the whole process of taking the inverse limit, one might be able to have the chance to control the finiteness of the global sections. However, this is not trivially correct. The Fargues-Fontaine curves involved are defined over some Fréchet-Stein algebras which behave well due to the well-established theory of Fréchet-Stein algebras in the noetherian setting. However the Robba rings in the relative setting are highly not noetherian. The corresponding cohomologies of these big modules or families of big modules will eventually then be the corresponding modules or families of modules over Fréchet-Stein algebras, which will be definitely easier to control and study since the theory of Fréchet-Stein algebras should be able to help us to construct the corresponding Iwasawa theoretic or K -theoretic objects.

Furthermore we consider the following Hodge-Iwasawa construction under the idea from Kedlaya-Pottharst through the perfectoid subdomain covering mentioned above over any preadic space X defined over \mathbb{Q}_p .

Definition 2.4.31. We define the following A -relative sheaves of period rings after [KL15, Definition 9.3.3]:

$$\underline{\Omega}_{X,A}, \underline{\Omega}_{X,A}^{\text{int}}, \underline{\Pi}_{X,A}^{\text{int},r}, \underline{\Pi}_{X,A}^{\text{int}}, \underline{\Pi}_{X,A}^{\text{bd},r}, \underline{\Pi}_{X,A}^{\text{bd}}, \underline{\Pi}_{X,A}^r, \underline{\Pi}_{X,A}^I, \underline{\Pi}_{X,A}^\infty, \underline{\Pi}_{X,A}, \quad (2.4.6)$$

by locally taking the suitable completed product and then glueing. Similarly we have the following Iwasawa sheaves as well:

$$\underline{\Omega}_{X,A\widehat{\otimes}A_\infty(G)}, \underline{\Omega}_{X,A\widehat{\otimes}A_\infty(G)}^{\text{int}}, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^{\text{int},r}, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^{\text{int}}, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^{\text{bd},r}, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^{\text{bd}}, \quad (2.4.7)$$

$$\underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^r, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^I, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}^\infty, \underline{\Pi}_{X,A\widehat{\otimes}A_\infty(G)}, \quad (2.4.8)$$

Then we have the following A -relative version of [KL15, Theorem 9.3.12]:

Proposition 2.4.32.

The categories of A -relative vector bundles over $\text{FF}_{X,A}$ and Frobenius-equivariant A -relative Hodge-Iwasawa finite locally free sheaves over the period sheaves $\underline{\Pi}_{X,A}^\infty, \underline{\Pi}_{X,A}$ are all equivalent.

Proof. This is by considering the perfectoid covering and the local equivalence we established before. \square

2.4.4 Higher Homotopical Geometrized Tamagawa-Iwasawa Theory of Hodge-Iwasawa Modules

The structure of the corresponding Hodge-Iwasawa modules in our mind has the potential to give us the chance to use them to study the corresponding rational Iwasawa theory. The corresponding relationship we established above could show us the corresponding Iwasawa theory of the relative Frobenius modules over the pro-étale site is equivalent to that of the corresponding B -pairs (we will also discuss this more in our further study). For instance the latter gives us directly some kind of family version of exponential maps and dual-exponential maps in the higher dimensional situation. We now give some further discussion around our main goals in mind. We first start from the following context:

Setting 2.4.33. Over a separated rigid analytic space over \mathbb{Q}_p (we assume it to be of finite type as an adic space) X we have the category of the pseudocoherent φ -sheaves over the sheaf of ring $\tilde{\underline{\Pi}}_X$ over the pro-étale site $X_{\text{pro-étale}}$. We then use the notation $ChQCoh\Pi\Phi_{X_{\text{pro-étale}}}$ to denote the category of all the chain complexes of objects in the category $QCoh\Pi\Phi_{X_{\text{pro-étale}}}$ which we define it to be the category of all the quasicohherent φ -sheaves (as in the pseudo-coherent situations) over the ring $\tilde{\underline{\Pi}}_X$.

Remark 2.4.34. One should actually regard the category $QCoh\Pi\Phi_{X_{\text{pro-étale}}}$ as the category of all the direct limits of the objects in the category of all the pseudocoherent φ -sheaves over the sheaf of ring $\tilde{\underline{\Pi}}_X$ over the pro-étale site $X_{\text{pro-étale}}$. During taking the direct limit we assume that we are actually always carrying the corresponding Frobenius action.

Definition 2.4.35. A complex of objects of $ChQCoh\Pi\Phi_{X_{\text{pro-étale}}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in $\Pi\Phi_{X_{\text{pro-étale}}}$ (the category of all the pseudo-coherent φ -sheaves over $\tilde{\underline{\Pi}}_X$)

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

A complex of objects of $ChQCoh\Pi\Phi_{X_{\text{pro-étale}}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in $\Pi\Phi_{X_{\text{pro-étale}}}$ (the category of all the pseudo-coherent φ -sheaves over $\underline{\underline{\tilde{X}}}$ as those in [KL16, Chapter 8])

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{pseudo}}\Pi\Phi_{X_{\text{pro-étale}}}$ to denote the category (not the derived one) of all the pseudocoherent complexes of objects in $QCoh\Pi\Phi_{X_{\text{pro-étale}}}$ in the above sense. And we use the notation $D_{\text{perf}}\Pi\Phi_{X_{\text{pro-étale}}}$ for the perfect complexes. We use the notation $D_{\text{pseudo}}^{\text{cb}}\Pi\Phi_{X_{\text{pro-étale}}}$ to denote the category of cohomology bounded pseudocoherent complexes. And we use the notations $D_{\text{perf}}^{\text{dg-flat}}\Pi\Phi_{X_{\text{pro-étale}}}$ to denote the the subcategories consisting of dg-flat complexes. And we use the notations $D_{\text{perf}}^{\text{str}}\Pi\Phi_{X_{\text{pro-étale}}}$ to denote the subcategories consisting of strictly perfect complexes.

Proposition 2.4.36. *There is a Waldhausen structure over each of*

$$D_{\text{pseudo}}\Pi\Phi_{X_{\text{pro-étale}}}, D_{\text{pseudo}}^{\text{cb}}\Pi\Phi_{X_{\text{pro-étale}}}, \quad (2.4.9)$$

$$D_{\text{perf}}\Pi\Phi_{X_{\text{pro-étale}}}, D_{\text{perf}}^{\text{dg-flat}}\Pi\Phi_{X_{\text{pro-étale}}}, D_{\text{perf}}^{\text{str}}\Pi\Phi_{X_{\text{pro-étale}}}. \quad (2.4.10)$$

Proof. We only show this for the first category. We are talking about the Waldhausen category in the sense considered by [Wit3, Proposition 3.1.1] and [TT1] (namely the corresponding complicial biWaldhausen categories). Recalling from the work [Wit3, Proposition 3.1.1], the corresponding criterion is to check that the corresponding smaller full additive subcategory of complexes extracted from an abelian category has the stability under the corresponding shifts and the extension through the exact sequences. But we are talking about pseudocoherent complexes, two out of three property follows from [SP, Tag, 064R] where one can show this directly by applying [SP, Derived Category, Lemma 16.11]. Of course note that we are not talking about derived category. In this situation we define the corresponding cofibration to be the corresponding degreewise monomorphism with kernel in the original category in which we are considering, while we define the corresponding weak-equivalences to be quasi-isomorphisms. Then one can check the corresponding conditions in the criterion of Waldhausen category in the sense of [Wit3, Proposition 3.1.1] and [TT1] satisfy. \square

We now consider the corresponding objects over the schematic Fargues-Fontaine curves. By our previous comparison theorems (and the corresponding theorems in [KL16]) on the deformations of the objects we can only consider the scheme theory to extract the corresponding information on the Tamagawa-Iwasawa deformations for the relative p -adic Hodge theory over the Robba rings which to some extent very complicated. We can now make the following discussion:

Setting 2.4.37. Consider any uniform perfect adic Banach algebra R as in our previous consideration. We have the schematic version of the Fargues-Fontaine curve which is denoted by $\text{Proj}P_R$. We now use the notation $\text{Mod}\mathcal{O}_{\text{Proj}P_R}$ to denote the corresponding category of all the sheaves of $\mathcal{O}_{\text{Proj}P_R}$ -modules over the scheme $\text{Proj}P_R$. Then we will use the notation $\text{ChMod}\mathcal{O}_{\text{Proj}P_R}$ to denote the category of all the complexes of sheaves of $\mathcal{O}_{\text{Proj}P_R}$ -modules over the scheme $\text{Proj}P_R$, with the corresponding derived category $D\text{Mod}\mathcal{O}_{\text{Proj}P_R}$.

Definition 2.4.38. A complex of objects of $\text{Mod}\mathcal{O}_{\text{Proj}P_R}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of all the sheaves of $\mathcal{O}_{\text{Proj}P_R}$ -modules over the schematic Fargues-Fontaine curve

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{pseudo}}\text{Proj}P_R$ to denote the category (not the derived one) of all the pseudocoherent complexes of objects in $\text{Mod}\mathcal{O}_{\text{Proj}P_R}$ in the above sense. A complex of objects of $\text{Mod}\mathcal{O}_{\text{Proj}P_R}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in the category of all the sheaves of $\mathcal{O}_{\text{Proj}P_R}$ -modules over the schematic Fargues-Fontaine curve

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{perf}}\text{Proj}P_R$ to denote the category (not the derived one) of all the perfect complexes of objects in $\text{Mod}\mathcal{O}_{\text{Proj}P_R}$ in the above sense.

Proposition 2.4.39. *The category $D_{\text{pseudo}}\text{Proj}P_R$ is a category admitting Waldhausen structure.*

Proof. This is standard such as in [TT1, Section 1-3]. We present the proof as above for the convenience of the readers. We are now talking about again the context of [Wit3, Proposition 3.1.1] and [TT1]. The corresponding criterion in [Wit3] requires the corresponding stability of the corresponding shifts and extension in the category of the corresponding complexes in our context. Again we can consider the corresponding results of [SP, Tag, 064R] to prove this. Then the corresponding cofibration will be the corresponding monomorphism with kernel in the original category of the complexes while the corresponding weak-equivalences are taken to be the corresponding quasi-isomorphisms. \square

Then we can define the corresponding category of pseudocoherent sheaves over the adic Fargues-Fontaine curves:

Setting 2.4.40. After [KL16], we consider the category $Mod\mathcal{O}_{\text{FF}\tilde{R}_\psi}$ of all sheaves of $\mathcal{O}_{\text{FF}\tilde{R}_\psi}$ -modules. Here the ring \tilde{R}_ψ is the perfect adic uniform Banach ring attached to the toric tower considered in [KL16]. We then use the notation $ChMod\mathcal{O}_{\text{FF}\tilde{R}_\psi}$ to denote the category of all the chain complexes of objects in the previous category $Mod\mathcal{O}_{\text{FF}\tilde{R}_\psi}$.

Definition 2.4.41. A complex of objects of $ChMod\mathcal{O}_{\text{FF}\tilde{R}_\psi}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of all the pseudocoherent sheaves over the Fargues-Fontaine curve

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{pseudo,alg}}\text{FF}\tilde{R}_\psi$ to denote the category (not the derived one) of all the pseudocoherent complexes of objects in $Mod\mathcal{O}_{\text{FF}\tilde{R}_\psi}$ in the above sense. We also have the category $D_{\text{pseudo,alg}}^{\text{cb}}\text{FF}\tilde{R}_\psi$ consisting of all the cohomology bounded pseudocoherent complexes.

A complex of objects of $ChMod\mathcal{O}_{\text{FF}\tilde{R}_\psi}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in the category of all the pseudocoherent sheaves over the Fargues-Fontaine curve

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{perf,alg}}\text{FF}_{\tilde{R}_\psi}$ to denote the category (not the derived one) of all the perfect complexes of objects in $\text{Mod}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi}}$ in the above sense. We also have the subcategories $D_{\text{perf,alg}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi}$ and $D_{\text{perf,alg}}^{\text{str}}\text{FF}_{\tilde{R}_\psi}$ of dg-flat objects and strictly perfect objects.

Proposition 2.4.42. *The categories*

$$D_{\text{pseudo,alg}}\text{FF}_{\tilde{R}_\psi}, D_{\text{pseudo,alg}}^{\text{cb}}\text{FF}_{\tilde{R}_\psi} \quad (2.4.11)$$

$$D_{\text{perf,alg}}\text{FF}_{\tilde{R}_\psi}, D_{\text{perf,alg}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi}, D_{\text{perf,alg}}^{\text{str}}\text{FF}_{\tilde{R}_\psi} \quad (2.4.12)$$

admit structure of Waldhausen categories.

Proof. The corresponding category involved in this situation is not that far from being equivalent to the situation where we considered in the Frobenius sheaves situation. On the other hand, one can then following the proof to give the chance to prove the corresponding statement here. \square

Now we consider the more general analytic objects:

Setting 2.4.43. Following [KL16], we have the category $\text{QCoh}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi}}$ of the quasi-coherent sheaves (direct limits of the pseudocoherent objects) over the Fargues-Fontaine curve $\text{FF}_{\tilde{R}_\psi}$. We then use the notation $\text{ChQCoh}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi}}$ to denote the category of all the chain complexes of objects in the previous category $\text{QCoh}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi}}$.

Definition 2.4.44. A complex of objects of $\text{ChQCoh}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of all the pseudocoherent sheaves over the Fargues-Fontaine curve:

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{pseudoFF}}\tilde{\mathcal{R}}_\psi$ to denote the category (not the derived one) of all the pseudocoherent complexes of objects in $QCoh\mathcal{O}_{\text{FF}}\tilde{\mathcal{R}}_\psi$ in the above sense. And we use the notations $D_{\text{pseudoFF}}^{\text{cb}}\tilde{\mathcal{R}}_\psi$ to denote the subcategories of cohomology bounded complexes. A complex of objects of $ChQCoh\mathcal{O}_{\text{FF}}\tilde{\mathcal{R}}_\psi$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in the category of all the pseudocoherent sheaves over the Fargues-Fontaine curve:

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{perfFF}}\tilde{\mathcal{R}}_\psi$ to denote the category (not the derived one) of all the perfect complexes of objects in $QCoh\mathcal{O}_{\text{FF}}\tilde{\mathcal{R}}_\psi$ in the above sense. And we use the notations $D_{\text{perf}}^{\text{dg-flat}}\tilde{\mathcal{R}}_\psi$ and $D_{\text{perf}}^{\text{str}}\tilde{\mathcal{R}}_\psi$ to denote the subcategories of dg-flat complexes and strictly perfect complexes.

Proposition 2.4.45. *The categories*

$$D_{\text{pseudoFF}}\tilde{\mathcal{R}}_\psi, D_{\text{pseudoFF}}^{\text{cb}}\tilde{\mathcal{R}}_\psi, \tag{2.4.13}$$

$$D_{\text{perfFF}}\tilde{\mathcal{R}}_\psi, D_{\text{perf}}^{\text{dg-flat}}\tilde{\mathcal{R}}_\psi, D_{\text{perf}}^{\text{str}}\tilde{\mathcal{R}}_\psi \tag{2.4.14}$$

admit the structure of Waldhausen categories.

Proof. See the proof for the previous proposition. □

Now one can further discuss the corresponding deformed version of the picture above, we mainly then focus on the schematic version of the Fargues-Fontaine curve:

Setting 2.4.46. Let A be a reduced affinoid algebra as before. Consider any uniform perfect adic Banach algebra R as in our previous consideration. We have the schematic version of the Fargues-Fontaine curve which is denoted by $\text{Proj}P_{R,A}$. We now use the notation $\text{Mod}\mathcal{O}_{\text{Proj}P_{R,A}}$ to denote the corresponding category of all the sheaves of $\mathcal{O}_{\text{Proj}P_{R,A}}$ -modules over the scheme $\text{Proj}P_R$. Then we will use the notation $Ch\text{Mod}\mathcal{O}_{\text{Proj}P_{R,A}}$ to denote the category of all the complexes of sheaves of $\mathcal{O}_{\text{Proj}P_{R,A}}$ -modules over the scheme $\text{Proj}P_{R,A}$, with the corresponding derived category $D\text{Mod}\mathcal{O}_{\text{Proj}P_{R,A}}$.

Definition 2.4.47. A complex of objects of $Mod\mathcal{O}_{\text{Proj}P_{R,A}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of all the quasicohherent sheaves over the schematic Fargues-Fontaine curve

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{pseudo}}\text{Proj}P_{R,A}$ to denote the category (not the derived one) of all the pseudocoherent complexes of objects in $Mod\mathcal{O}_{\text{Proj}P_{R,A}}$ in the above sense. We can also define the corresponding category $D_{\text{pseudo}}^{\text{cb}}\text{Proj}P_{R,A}$ as above.

A complex of objects of $Mod\mathcal{O}_{\text{Proj}P_{R,A}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in the category of all the quasicohherent sheaves over the schematic Fargues-Fontaine curve

$$\longrightarrow \dots \longrightarrow F^{m-2} \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow 0.$$

We then use the notation $D_{\text{perf}}\text{Proj}P_{R,A}$ to denote the category (not the derived one) of all the perfect complexes of objects in $Mod\mathcal{O}_{\text{Proj}P_{R,A}}$ in the above sense. We can also define the corresponding category $D_{\text{perf}}^{\text{dg-flat}}\text{Proj}P_{R,A}$ of dg-flat perfect complexes and the corresponding category $D_{\text{pseudo}}^{\text{str}}\text{Proj}P_{R,A}$ of strictly perfect complexes.

Proposition 2.4.48. *The categories*

$$D_{\text{pseudo}}\text{Proj}P_{R,A}, D_{\text{pseudo}}^{\text{cb}}\text{Proj}P_{R,A}, \tag{2.4.15}$$

$$D_{\text{perf}}\text{Proj}P_{R,A}, D_{\text{perf}}^{\text{dg-flat}}\text{Proj}P_{R,A}, D_{\text{perf}}^{\text{str}}\text{Proj}P_{R,A}, \tag{2.4.16}$$

are categories admitting Waldhausen structure.

Proof. We only prove the statement for the first category. See the proof above for $D_{\text{pseudo}}\text{Proj}P_R$ without the deformation. \square

Conjecture 2.4.49. *Assume that ψ is the cyclotomic tower. Then the corresponding total derived section functor (by using the Godement resolution) induces a map on the K -theory space $\mathbb{K}D_{\text{perf}}^{\text{dg-flat}}\text{Proj}P_{\tilde{R}_\psi, A}$ to that of the category $\mathbb{K}D_{\text{perf}}^{\text{dg-flat}}(A)$ of all the dg-flat perfect complexes of A -modules. And we conjecture that this is homotopic to zero.*

Assume that ψ is the cyclotomic tower. Then the corresponding total derived section functor (by using the Godement resolution) induces a map on the K -theory space $\mathbb{K}D_{\text{perf}}^{\text{str}}\text{Proj}P_{\tilde{R}_\psi, A}$ to that of the category $\mathbb{K}D_{\text{perf}}^{\text{str}}(A)$ of all the strictly perfect complexes of A -modules. And we conjecture that in this situation this is homotopic to zero.

Although not equivalent to the previous picture in the deformed setting we should still consider the following picture over the adic Fargues-Fontaine curve:

Setting 2.4.50. Let A be as above such that we have the well-defined adic space $\text{FF}_{\tilde{R}_\psi, A}$ where the corresponding tower ψ is the same as the previous one. Then we have the corresponding category $\text{Mod}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi, A}}$ as above by taking the corresponding sheaves of $\mathcal{O}_{\text{FF}_{\tilde{R}_\psi, A}}$ -modules over the deformed version of the Fargues-Fontaine curve. Then we have the corresponding category of all the chain complexes namely the category $\text{ChMod}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi, A}}$ and the corresponding derived one $\text{DMod}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi, A}}$.

Definition 2.4.51. A complex in the category $\text{ChMod}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi, A}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is now called pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of all the pseudocoherent objects defined over the deformed version of adic version of the Fargues-Fontaine curve in the current situation.

Correspondingly we use the notation $D_{\text{pseudo,alg}}\text{FF}_{\tilde{R}_\psi, A}$ to denote the corresponding category of all the pseudocoherent complexes defined above. We also have the corresponding category as above $D_{\text{pseudo,alg}}^{\text{cb}}\text{FF}_{\tilde{R}_\psi, A}$. A complex in the category $\text{ChMod}\mathcal{O}_{\text{FF}_{\tilde{R}_\psi, A}}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is now called perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in the category of all the pseudocoherent objects defined over the deformed version of adic version of the Fargues-Fontaine curve in the current situation. Correspondingly

we use the notation $D_{\text{perf,alg}}\text{FF}_{\tilde{R}_\psi,A}$ to denote the corresponding category of all the pseudocoherent complexes defined above. We also have the corresponding categories of dg-flat complexes and the strictly perfect complexes which will be denoted by $D_{\text{perf,alg}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi,A}$ and $D_{\text{perf,alg}}^{\text{str}}\text{FF}_{\tilde{R}_\psi,A}$.

Proposition 2.4.52. *The categories*

$$D_{\text{pseudo,alg}}\text{FF}_{\tilde{R}_\psi,A}, D_{\text{pseudo,alg}}^{\text{cb}}\text{FF}_{\tilde{R}_\psi,A}, \quad (2.4.17)$$

$$D_{\text{perf,alg}}\text{FF}_{\tilde{R}_\psi,A}, D_{\text{perf,alg}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi,A}, D_{\text{perf,alg}}^{\text{str}}\text{FF}_{\tilde{R}_\psi,A} \quad (2.4.18)$$

admit Waldhausen structure.

Proof. See the proof before where we do not have the corresponding deformation with respect to the algebra A . \square

Setting 2.4.53. We use the notation $QCoh\text{FF}_{\tilde{R}_\psi,A}$ to denote the corresponding quasi-coherent sheaves of $\mathcal{O}_{\text{FF}_{\tilde{R}_\psi,A}}$ -modules (direct limits of the pseudocoherent ones which are assumed to form an abelian category) over the adic space $\text{FF}_{\tilde{R}_\psi,A}$. Then we use the notation $ChQCoh\text{FF}_{\tilde{R}_\psi,A}$ to denote the category of chain complexes consisting of all the objects in the category $QCoh\text{FF}_{\tilde{R}_\psi,A}$. And we use the notation $DQCoh\text{FF}_{\tilde{R}_\psi,A}$ to denote the corresponding derived category.

Definition 2.4.54. A complex in the category $ChQCoh\text{FF}_{\tilde{R}_\psi,A}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called then pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of all the pseudocoherent objects over the deformed version of the adic version of the Fargues-Fontaine curve. The corresponding whole category of all such objects is denoted now by the notation $D_{\text{pseudo}}\text{FF}_{\tilde{R}_\psi,A}$. We also have the category as above $D_{\text{pseudo}}^{\text{cb}}\text{FF}_{\tilde{R}_\psi,A}$.

Definition 2.4.55. A complex in the category $ChQCoh\text{FF}_{\tilde{R}_\psi,A}$ taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called then perfect if it is quasi-isomorphic to a bounded complex of finite projective objects in the category of all the pseudocoherent objects over the deformed version of the

adic version of the Fargues-Fontaine curve. The corresponding whole category of all such objects is denoted now by the notation $D_{\text{perf}}\text{FF}_{\tilde{R}_\psi, A}$. We also have the categories of dg-flat complexes and strictly perfect complexes, namely $D_{\text{perf}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi, A}$ and $D_{\text{perf}}^{\text{str}}\text{FF}_{\tilde{R}_\psi, A}$.

Remark 2.4.56. The consideration here is in some sense different from the corresponding consideration in the schematic Fargues-Fontaine curve situation where we use the machinery of scheme theory and it is sometimes not stable under the analytic operations. But the latter should have its own interests from algebraic geometric point of view, in the general situation. For instance this gives us the chance to study the corresponding mysterious algebraic modules over the Robba rings with Frobenius pullbacks as isomorphisms. We remind the readers of the fact that over the cyclotomic tower many difficulties disappears. For instance more general algebraic approaches could be available.

Proposition 2.4.57. *The categories*

$$D_{\text{pseudo}}\text{FF}_{\tilde{R}_\psi, A}, D_{\text{pseudo}}^{\text{cb}}\text{FF}_{\tilde{R}_\psi, A}, \quad (2.4.19)$$

$$D_{\text{perf}}\text{FF}_{\tilde{R}_\psi, A}, D_{\text{perf}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi, A}, D_{\text{perf}}^{\text{str}}\text{FF}_{\tilde{R}_\psi, A} \quad (2.4.20)$$

admit Waldhausen structure.

Proof. See the proof of the previous propositions around the same issue. \square

Conjecture 2.4.58. *Assume that ψ is the cyclotomic tower. Then the corresponding derived section functor induces a map on the K -theory space $\mathbb{K}D_{\text{perf}}^{\text{dg-flat}}\text{FF}_{\tilde{R}_\psi, A}$ to that of the category $D_{\text{perf}}^{\text{dg-flat}}(A)$ of all the dg-flat perfect complexes of A -modules. We conjecture that this is homotopic to zero.*

Assume that ψ is the cyclotomic tower. Then the corresponding derived section functor induces a map on the K -theory space $\mathbb{K}D_{\text{perf}}^{\text{str}}\text{FF}_{\tilde{R}_\psi, A}$ to that of the category $D_{\text{perf}}^{\text{str}}(A)$ of all the strictly perfect complexes of A -modules. We conjecture that this is homotopic to zero.

Remark 2.4.59. These are conjectured in the situation where ψ is just the cyclotomic tower, in general one could easily guess what the conjectures look like.

We make some discussion here on the picture in the étale setting. The conjecture is much easier in the ℓ -adic situation. This represents the difference between the ℓ -adic

situation and the p -adic setting in some situation. For instance let us try to consider the following proposition after Witte (see [Wit3, Proposition 6.1.5]):

Proposition 2.4.60. *Let T be \mathbb{Z}_ℓ where $\ell \neq p$. For X a rigid analytic space over the p -adic field \mathbb{Q}_p which is separated and of finite type (when considered as some adic space locally of finite type, see [Huber1]), suppose we have that for $\sharp = \text{ét}$ the category $\mathbb{D}_{\text{perf}}(X_\sharp, T)$ could be endowed with the structure of Waldhausen categories and there is a Waldhausen exact functor $R\Gamma_?(X_\sharp, \cdot)$ (induced by the direct image functor in this context as in [Wit3, Chapter 4-5]) as below for $\sharp = \text{ét}$ and $? = \emptyset$:*

$$\mathbb{D}_{\text{perf}}(X_\sharp, T) \xrightarrow{R\Gamma_?(X_\sharp, \cdot)} \mathbb{D}_{\text{perf}}(T)$$

which induces the corresponding the morphism $\mathbb{K}R\Gamma_?(X_\sharp, \cdot)$ as:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_\sharp, T) \xrightarrow{\mathbb{K}R\Gamma_?(X_\sharp, \cdot)} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

Then there is a homotopy between this morphism and the zero map:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_\sharp, T) \xrightarrow{0} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

Proof. We follow the strategy of [Wit3, Proposition 6.1.5] to prove this. The idea is to pullback all the things back along the following morphism:

$$f : \text{Spa}(\mathbb{Q}_p^{\text{ur}}, \mathfrak{o}_{\mathbb{Q}_p^{\text{ur}}}) \rightarrow \text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p}).$$

Now for each complex M^\bullet involved we put:

$$R\Gamma(X_{\mathbb{Q}_p^{\text{ur}}, \sharp}, M^\bullet) := \Gamma(\text{Spa}(\mathbb{Q}_p^{\text{ur}}, \mathfrak{o}_{\mathbb{Q}_p^{\text{ur}}}), f^* R\pi_* M^\bullet), \quad (2.4.21)$$

where the morphism $\pi : X \rightarrow \text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})$ is the corresponding structure morphism of X . We have the corresponding finiteness due to our assumption here. One then looks at the corresponding Frobenius Fr over the corresponding field \mathbb{Q}_p^{ur} . Then as in [Wit3, Proposition 6.1.5] we have that then the following cofibration sequence for each complex M^\bullet :

$$R\Gamma_?(X_{\mathbb{Q}_p, \sharp}, M^\bullet) \longrightarrow R\Gamma_?(X_{\mathbb{Q}_p^{\text{ur}}, \sharp}, M^\bullet) \xrightarrow{1-\text{Fr}} R\Gamma_?(X_{\mathbb{Q}_p^{\text{ur}}, \sharp}, M^\bullet),$$

as in [Wit3, Proposition 6.1.5] (note that here we need the corresponding flabbiness of the corresponding derived direct images), by the corresponding Waldhausen's additive theorem we have the corresponding homotopy:

$$\mathbb{K}R\Gamma_?(X_{\mathbb{Q}_p^{\text{ur}}, \#}, M^\bullet) \oplus \mathbb{K}R\Gamma_?(X_{\mathbb{Q}_p, \#}, M^\bullet) \rightsquigarrow \mathbb{K}R\Gamma_?(X_{\mathbb{Q}_p^{\text{ur}}, \#}, M^\bullet) \quad (2.4.22)$$

which implies that we have then as in [Wit3, Proposition 6.1.5]

$$\pi_i \mathbb{K}R\Gamma_?(X_\#, \cdot) = 0. \quad (2.4.23)$$

□

We now show how this recovers the results of Witte at a point:

Corollary 2.4.61. *Suppose now the space X is just the point $\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})$, and keep the assumption that T is the ring \mathbb{Z}_ℓ . Then we have for $\# = \text{ét}$ the Waldhausen category $\mathbb{D}_{\text{perf}}(X_\#, T)$ which induces the Waldhausen exact functor (induced from the total derived direct image functor) $R\Gamma_?(X_\#, \cdot)$ where $? = \emptyset$ and $\#$ as above:*

$$\mathbb{D}_{\text{perf}}(X_\#, T) \xrightarrow{R\Gamma_?(X_\#, \cdot)} \mathbb{D}_{\text{perf}}(T)$$

which induces the corresponding the morphism $\mathbb{K}R\Gamma_?(X_\#, \cdot)$ as:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_\#, T) \xrightarrow{\mathbb{K}R\Gamma_?(X_\#, \cdot)} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

And moreover we have that this map is homotopic to zero:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_\#, T) \xrightarrow{0} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

Proof. One can apply results of [Wit3] to recover the corresponding structures here or construct the corresponding structures here directly. Then we can apply the previous proposition. □

Let us analyze the corresponding p -adic setting in the following sense. Since in the proof used above the corresponding results on the finiteness the cohomological dimension over the space attached to the maximal unramified extension of \mathbb{Q}_p are crucial. We first consider the integral version of the picture we considered above.

Setting 2.4.62. We are going to use the notation $QCoh\Omega^{\text{int}}\Phi_{X_{\text{pro-étale}}}^a$ (where X is assumed to be separated and of finite type as an adic space) to denote the corresponding quasi-coherent (direct limits of pseudocoherent ones) φ^a -sheaves over the sheaf of ring $\underline{\Omega}_{X_{\text{pro-étale}}}^{\text{int}}$, and we use the corresponding notation $ChQCoh\Omega^{\text{int}}\Phi_{X_{\text{pro-étale}}}^a$ to denote the corresponding category of the chain complexes and use the corresponding notation $DQCoh\Omega^{\text{int}}\Phi_{X_{\text{pro-étale}}}^a$ to denote the corresponding derived category.

Definition 2.4.63. A complex considered in the previous setting taking the form of

$$\longrightarrow \dots \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \dots$$

is called then pseudocoherent if it is quasi-isomorphic to a bounded above complex of finite projective objects in the category of pseudocoherent φ^a -modules over $\underline{\Omega}_X^{\text{int}}$.

The corresponding setting of local systems gives us the following consideration after Witte:

Conjecture 2.4.64. *Assume that T is an adic ring over \mathbb{Z}_p such that we can find a two-sided ideal I such that each T/I^n for $n \geq 0$ is finite of order a power of p . Suppose that X is a rigid analytic space which is assumed to be separated and of finite type over the p -adic number field \mathbb{Q}_p . Assume that the corresponding category $\mathbb{D}_{\text{perf}}(X_{\sharp}, T)$ for $\sharp = \text{ét}$ admits Waldhausen structure and we have the corresponding Waldhausen exact functor (induced from the corresponding direct image functor as in [Wit3, Chapter 4-5]) $R\Gamma_{?}(X_{\sharp}, \cdot)$ where $? = \emptyset$:*

$$\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{R\Gamma_{?}(X_{\sharp}, \cdot)} \mathbb{D}_{\text{perf}}(T)$$

which induces the corresponding the morphism $\mathbb{K}R\Gamma_{?}(X_{\sharp}, \cdot)$ as:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{\mathbb{K}R\Gamma_{?}(X_{\sharp}, \cdot)} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

Then we have that this map is homotopic to the zero map:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{0} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

Remark 2.4.65. The proof should be actually the same as in the previous result in the corresponding ℓ -adic situation, the difference here is that one uses the corresponding finiteness and cohomological dimension results in the p -adic setting by applying [KL,

8.1,10.1] for instance, which is to say in our setting the corresponding finiteness and cohomological dimension results over the point attached to \mathbb{Q}_p^{ur} would produce us the corresponding cofibration sequence involved. Also we need to consider a general ring T here. Following the idea in [FK, Proposition 2.1.3], we quotient out the corresponding Jacobson radical of each finite quotient T/I which is a finite ring. We can then reduce to semi-simple case, and then reduce to finite field case.

Conjecture 2.4.66. *When T is as in the previous proposition the statement in the previous corollary holds as well in our current situation, which is to say over $\text{Spa}(\mathbb{Q}_p, \mathfrak{o}_{\mathbb{Q}_p})$ we have the Waldhausen category $\mathbb{D}_{\text{perf}}(X_{\sharp}, T)$ for $\sharp = \text{ét}$ with the corresponding Waldhausen exact functor $R\Gamma_?(X_{\sharp}, \cdot)$ for $? = \emptyset$:*

$$\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{R\Gamma_?(X_{\sharp}, \cdot)} \mathbb{D}_{\text{perf}}(T)$$

which induces the corresponding the morphism $\mathbb{K}R\Gamma_?(X_{\sharp}, \cdot)$ as:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{\mathbb{K}R\Gamma_?(X_{\sharp}, \cdot)} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

And we have that this map is homotopic to the zero map:

$$\mathbb{K}\mathbb{D}_{\text{perf}}(X_{\sharp}, T) \xrightarrow{0} \mathbb{K}\mathbb{D}_{\text{perf}}(T).$$

Remark 2.4.67. We used the corresponding context of Waldhausen categories after Witte, but certain cases of the corresponding ϵ -isomorphism conjectures in the étale setting are due to many people who studied the corresponding ϵ -isomorphism conjectures, mainly in [FK], [F],[BB], [LVZ] and [Nak3].

This chapter is based on the following paper, where the author of this dissertation is the main author:

- Tong, Xin. "Hodge-Iwasawa Theory I." arXiv preprint arXiv:2006.03692 (2020).

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Chapter 3

Hodge-Iwasawa Theory II

3.1 Introduction

3.1.1 Introduction

In our previous paper [T1] on the Hodge-Iwasawa theory we studied some deformation of p -adic Hodge structures over some general perfectoid spaces and adic spaces. We mainly focused on the corresponding perfect setting which to some extent gives us some suitable and desired functoriality with respect to some general adic spaces. This reflects the corresponding motivation in some geometric and categorical way since our initial goal on the corresponding Hodge-Iwasawa consideration is to establish some possibility to deform the functorial geometric structures.

In our current situation we still emphasize higher dimensional geometric structures. We will generalize the corresponding context of [KL16] to our context which on one hand produces more general tools to study the deformation of Hodge structures and on the other hand we will be more flexible over some specific towers both coming from geometry and Iwasawa theory. The context of [KL16] is already over some deep generalization. First off the towers considered in [KL16] are axiomatic before some more concrete study. Also the corresponding context is within the consideration of generalized Witt vectors. Also the context considers the corresponding pseudocoherent modules and sheaves instead of the objects initially considered in the context of [KL15]. We consider the corresponding deformations of these generalized structures over some affinoid algebras.

In our current consideration we will do some kind of globalization to some extent which will generalize the corresponding story in some more rigid framework. To be more precise we will consider the situation where the deformation happens over some affinoid rigid analytic space. The sheafiness will be non-trivial here but will be true in this specific context.

We will work in more general setting than the settings of our previous work and [KL16]. Although the setting of [KL16] is already more general than [KL15], we choose to work in the setting where the coefficient field E is ultrametric with perfect residue field of characteristic $p > 0$ (even in the nondiscrete valued situation). When E is not discrete

valued this is actually quite hard to deal with, where we only consider the situation where we do have some stable unique expression in the variable π (for elements in the ring of Witt vectors), which we hope go back to this later to work within more general setting as long as one can define the corresponding power-multiplicative norms to do various completion under the hypothesis that R is at least uniform Banach. Therefore our work will provide some additional reference to the study on this level of generality. Please be careful enough on our notations.

3.1.2 Main Results

We have managed to do some generalization of our previous work [T1], and the corresponding context in [KL15], [KL16], [KP]. As mentioned in the previous subsection the corresponding imperfect setting of the Hodge-Iwasawa theory is one main goal of our development. Also we would like to discuss to some extent the corresponding globalization of the deformed information which the rings in our consideration are carrying. To summarize we have:

I. We have now considered the corresponding equal characteristic analog of the corresponding comparison between the vector bundles, pseudocoherent sheaves over the deformed schematic Fargues-Fontaine curves and the corresponding Frobenius modules over the deformed Robba rings, generalizing [T1], [KL16] and [KP]. The corresponding motivation comes from certainly not only [KP], but also the corresponding t -motivic Hodge theory in the relative setting such as in Hartl-Kim [HK].

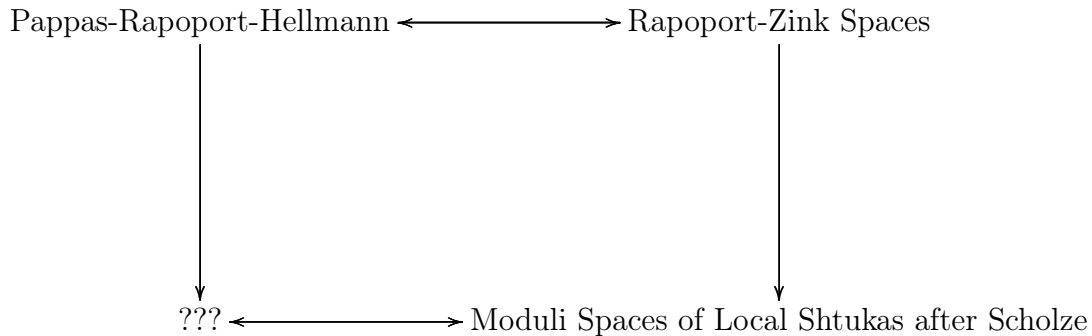
II. We have now considered the corresponding equal characteristic analog of the corresponding comparison between the vector bundles, pseudocoherent sheaves over the deformed adic Fargues-Fontaine curves and the corresponding Frobenius modules over the deformed Robba rings [T1], [KL15], [KL16]. Again the corresponding motivation comes from certainly not only [KP], but also the corresponding t -motivic Hodge theory in the relative setting such as in Hartl-Kim [HK]. This will lead to some sort of globalization in the category of adic spaces.

III. We have discussed the corresponding relationship between Frobenius modules over the perfect period rings and the Frobenius modules over the corresponding imperfection of the period rings in our deformed setting. This is to some extent important since the corresponding imperfection might lead to the corresponding noetherian objects which might be easy to control. This will have further application to the study of our deformation of the étale local systems.

IV. We have discussed the corresponding relationship between Frobenius modules over the perfect period rings and the Frobenius modules over the corresponding imperfection of the period rings in our deformed setting, but over the corresponding noncommutative Banach rings. This is to some extent important since the corresponding imperfection might lead to the corresponding noncommutative noetherian objects which might be easy to control. This will have further application to the study of our noncommutative deformation of the étale local systems. It is very natural to consider noncommutative setting in our development since we are considering the corresponding deformation into the noncommutative Fréchet-Stein algebras.

3.1.3 Further Consideration

We should mention that actually the corresponding ideas we mentioned within the globalization are not quite new since this bears some similarity to Pappas-Rapoport [PR1] and Hellmann's work [Hel1] on the arithmetic stacks of Frobenius modules. The corresponding comparison in our context could be morally conveyed in the following diagram:



For instance in our consideration we will consider some arithmetic deformation of vector bundles over the arithmetic deformation of Fargues-Fontaine curves. It is very

natural to consider the corresponding moduli problem around the corresponding vector bundles over these deformed spaces. Certainly these are not directly diamonds, but at least they should be stacks over the corresponding categories of certain adic spaces.

In this paper, we have built some well-posed equivariant versions of many results in the relative p -adic Hodge theory in [KL15] and [KL16] which have already reached beyond our previous consideration. We expect everything could be further applied to the corresponding study on the cohomologies of all the corresponding objects we defined and studied here. For instance if one applies these to the t -motivic setting, then immediately one has the chance to construct the corresponding t -adic local Tamagawa number conjecture after Nakamura [Nakamura1] (also inspired by the recent work [FGHP]), which certainly is a relative version in rigid family (again after Nakamura relying on the results from Kedlaya-Pottharst-Xiao on the finiteness of the corresponding cohomology of (φ, Γ) -modules over relative Robba rings).

We have reached some noncommutative Iwasawa deformation of the relative p -adic Hodge structures. At least the corresponding hope is directly targeted at the corresponding noncommutative Tamagawa number conjectures after Burns-Flach-Fukaya-Kato [BF1], [BF2] and [FK1], Nakamura and Zähringer [Zah1]. Our belief is that although we have not shown that coherent sheaves and modules over noncommutative deformation of full Robba rings are equivalent, but at least for coherent sheaves one could have a coherent Iwasawa theory in the noncommutative setting.

3.1.4 Lists of Notations

The arrangement of the corresponding notations getting involved in this paper is bit complicated, so we have made some indication to indicate where they mainly emerge into the corresponding discussion.

Notation	Description (mainly in section 2, section 3, section 4, section 6.1, section 6.2, section 6.3)
E	A complete discrete valued field which is nonarchimedean with residue field k perfect in characteristic $p > 0$.
A	A reduced affinoid algebra in rigid analytic geometry in the sense of Tate, or a noncommutative Banach affinoid algebra.
W_π	The ring of generalized Witt vectors with respect to some finite extension E of \mathbb{Q}_p with pseudo-uniformizer π .
(R, R^+)	Adic perfect uniform algebra over \mathbb{F}_{p^h} for some $h \geq 1$.
$\tilde{\Omega}_{R,A}^{\text{int}}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Omega}_{R,A}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}^{\text{int},r}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}^{\text{int}}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}^{\text{bd},r}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}^{\text{bd}}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}^r$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Pi}_{R,A}^\infty$	Deformed version of the period rings in the style of [KL15] and [KL16].

$\tilde{\Pi}_{R,A}^I$	Deformed version of the period rings in the style of [KL15] and [KL16].
$\tilde{\Omega}_{*,A}^{\text{int}}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Omega}_{*,A}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^{\text{int},r}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^{\text{int}}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^{\text{bd},r}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^{\text{bd}}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^r$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^\infty$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$\tilde{\Pi}_{*,A}^I$	Deformed version of the period sheaves in the style of [KL15] and [KL16].
$W_{\pi,\infty}$	The ring of generalized Witt vectors with respect to some finite extension E of \mathbb{Q}_p with pseudo-uniformizer π , but with base change to E_∞ . In this case each element admits unique expression $\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{x}_n]$.
$\tilde{\Omega}_{R,\infty}^{\text{int}}, \tilde{\Omega}_{*,\infty}^{\text{int}}$	Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .
$\tilde{\Omega}_{R,\infty}, \tilde{\Omega}_{*,\infty}$	Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

$\tilde{\Pi}_{R,\infty}^{\text{int},r}, \tilde{\Pi}_{*,\infty}^{\text{int},r}$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}^{\text{int}}, \tilde{\Pi}_{*,\infty}^{\text{int}}$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}^{\text{bd},r}, \tilde{\Pi}_{*,\infty}^{\text{bd},r}$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}^{\text{bd}}, \tilde{\Pi}_{*,\infty}^{\text{bd}}$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}^r, \tilde{\Pi}_{*,\infty}^r$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}, \tilde{\Pi}_{*,\infty}$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}^\infty, \tilde{\Pi}_{*,\infty}^\infty$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty}^I, \tilde{\Pi}_{*,\infty}^I$

Period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Omega}_{R,\infty,A}^{\text{int}}, \tilde{\Omega}_{*,\infty,A}^{\text{int}}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Omega}_{R,\infty,A}, \tilde{\Omega}_{*,\infty,A}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}^{\text{int},r}, \tilde{\Pi}_{*,\infty,A}^{\text{int},r}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}^{\text{int}}, \tilde{\Pi}_{*,\infty,A}^{\text{int}}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}^{\text{bd},r}, \tilde{\Pi}_{*,\infty,A}^{\text{bd},r}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

$\tilde{\Pi}_{R,\infty,A}^{\text{bd}}, \tilde{\Pi}_{*,\infty,A}^{\text{bd}}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}^r, \tilde{\Pi}_{*,\infty,A}^r$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}, \tilde{\Pi}_{*,\infty,A}$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}^\infty, \tilde{\Pi}_{*,\infty,A}^\infty$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

 $\tilde{\Pi}_{R,\infty,A}^I, \tilde{\Pi}_{*,\infty,A}^I$

Deformed period rings or sheaves in the style of [KL15] and [KL16], with base change to E_∞ .

Notation

Description (mainly in section 5, section 6.4, section 6.5, section 6.6, section 6.7)

 (H_\bullet, H_\bullet^+)

A tower as in [KL16, Chapter 5].

 $\Pi_{H,A}^{\text{int},r}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\Pi_{H,A}^{\text{int},\dagger}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\Omega_{H,A}^{\text{int}}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\Omega_{H,A}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

$\Pi_{H,A}^{\text{bd},r}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\Pi_{H,A}^{\text{bd},\dagger}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\Pi_{H,A}^{[s,r]}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\Pi_{H,A}^r$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\Pi_{H,A}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\check{\Pi}_{H,A}^{\text{int},r}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\check{\Pi}_{H,A}^{\text{int},\dagger}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\check{\Omega}_{H,A}^{\text{int}}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\check{\Omega}_{H,A}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.
$\check{\Pi}_{H,A}^{\text{bd},r}$	Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

$\check{\Pi}_{H,A}^{\text{bd},\dagger}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\check{\Pi}_{H,A}^{[s,r]}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\check{\Pi}_{H,A}^r$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\check{\Pi}_{H,A}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\widehat{\Pi}_{H,A}^{\text{int},r}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\widehat{\Pi}_{H,A}^{\text{int},\dagger}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\widetilde{\Omega}_{H,A}^{\text{int}}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\widetilde{\Omega}_{H,A}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\widehat{\Pi}_{H,A}^{\text{bd},r}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

 $\widehat{\Pi}_{H,A}^{\text{bd},\dagger}$

Imperfect relative period ring corresponding to (H_\bullet, H_\bullet^+) which is deformed analog of Kedlaya-Liu's imperfect ring.

3.2 Arithmetic Families of Period Rings

3.2.1 Basic Settings and Basic Definitions of Period Rings

In this section we first define the corresponding period rings and period sheaves involved in our study which is generalized version of the study in our previous paper, where we consider the corresponding ring of usual p -typical Witt vectors. For the convenience of the readers we present the corresponding construction in detail to some extent. First following [KL16, Hypothesis 3.1.1] we consider the following setting:

Setting 3.2.1. In our context we consider the corresponding generalized Witt vectors in the style of [KL16, 3.1] where we first fix some complete nonarchimedean (normalized as in [KL16, Hypothesis 3.1.1]) discrete valued field E where we will use the notation π to denote a chosen pseudo-uniformizer of E , and we use the notation \mathfrak{o}_E to denote the corresponding ring of integers of E . Slightly generalizing the context [KL16, Hypothesis 3.1.1] of [KL16] we consider the corresponding situation where k is a perfect field of characteristic $p > 0$, and we assume the field k to contain \mathbb{F}_{p^h} . We now assume that the ultrametric field E has residue field k . Again as in our previous paper we will use the notation (R, R^+) to denote a perfect adic Banach uniform algebra over k . Following [KL16] we will use the notation α to denote the spectral seminorms on perfect adic Banach uniform algebras. When E is of equal-characteristic we assume that E contains the field $\mathbb{F}_p((\eta))$.

Remark 3.2.2. The field E is assumed to be discrete valued. One can actually consider more general E . The most general situation (namely just assume E is ultrametric field) is actually a little bit hard to manage due to the fact that the element in the generalized Witt vector will have no unique expression as a power series, but in some nice situations this is not that hard for instance one can consider the base change to the perfectoid field E_∞ .

Then we could consider the following definitions of the period rings in our situation which is just the corresponding deformation over A of the corresponding rings defined in [KL16, Definition 4.1.1] in some generalized fashion:

Setting 3.2.3. Recall that in our situation we have the corresponding rings of generalized Witt vectors associated to some perfect ring R or R^+ . Generalizing the situation in [KL16,

Definition 4.1.1] and [KL16, Setting 3.1.1] we denote them by $W_\pi(R)$ or $W_\pi(R^+)$ where π is the chosen pseudo-uniformizer as defined. To be more precise here $W_\pi(R)$ (same to $W_\pi(R^+)$) is defined to be

$$W_\pi(R) := W(R) \otimes_{W(k)} \mathcal{O}_E.$$

Then we have the following ring of period rings from this notation:

$$\tilde{\Omega}_R^{\text{int}} = W_\pi(R), \tilde{\Omega}_R = W_\pi(R)[1/\pi].$$

Each element of $W_\pi(R)$ could be expressed as some power series:

$$\sum_{k \geq 0} \pi^k [\bar{x}_k]$$

where one has the corresponding Gauss norm coming from the norm α in the style of for each $r > 0$

$$\|\cdot\|_{\alpha^r} \left(\sum_{k \geq 0} \pi^k [\bar{x}_k] \right) := \sup_{k \geq 0} \{p^{-k} \alpha(\bar{x}_k)^r\}.$$

Then one can define the corresponding integral Robba ring $\tilde{\Pi}_R^{\text{int}, r}$ by taking the completion of the ring $W_\pi(R^+)[[R]]$ by using the norm defined above. From this one has the corresponding bounded Robba ring $\tilde{\Pi}_R^{\text{bd}, r}$ which is defined just to be $\tilde{\Pi}_R^{\text{int}, r}[1/\pi]$. Then we consider the corresponding Robba $\tilde{\Pi}_R^I$ with respect to some interval $I \subset (0, \infty)$ which is defined to be the corresponding ind-Fréchet completion of $W_\pi(R^+)[[R]][1/\pi]$ (namely $W_\pi(R^+)[[r], r \in R][1/\pi]$) with respect to the family of norms $\|\cdot\|_{\alpha^r}, \forall r \in I$. By taking specific intervals like $(0, r)$ or $(0, \infty)$ we have the corresponding Robba ring $\tilde{\Pi}_R^r$ and $\tilde{\Pi}_R^\infty$. Then by taking the corresponding union throughout all the radius $r > 0$ we have the corresponding Robba rings $\tilde{\Pi}_R^{\text{int}}, \tilde{\Pi}_R^{\text{bd}}, \tilde{\Pi}_R$. As in [KL16, 4.1] we have also the corresponding integral versions of some of the rings defined above.

Our consideration is to extend the scope of the consideration mentioned above by extending the corresponding dimension of the ring involved to some extent. To be more precise one considers the corresponding power or Laurent series over the corresponding functional analytic algebras, and then take suitable quotient.

Setting 3.2.4. In characteristic zero situation, we are going to use the notation $\mathbb{Q}_p\{T_1, \dots, T_d\}$ to denote a Tate algebra in rigid analytic geometry after Tate, and use in general A to represent the corresponding quotients of the Tate algebras over \mathbb{Q}_p as

above. Throughout we assume A to be reduced, carrying the spectral seminorm. While in positive characteristic situation we are going to use the notation A to denote a general affinoid algebra which is assumed to be reduced.

We first deform the corresponding constructions in [KL16, Definition 4.1.1]:

Definition 3.2.5. We first consider the corresponding deformation of the above rings over $\mathbb{Q}_p\{T_1, \dots, T_d\}$. We are going to use the notation $W_\pi(R)_{\mathbb{Q}_p\{T_1, \dots, T_d\}}$ to denote the complete tensor product of $W_\pi(R)$ with the Tate algebra $\mathbb{Q}_p\{T_1, \dots, T_d\}$ consisting of all the element taking the form as:

$$\sum_{k \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} T_2^{i_2} \dots T_d^{i_d}$$

over which we have the Gauss norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for any $r > 0$ which is defined by:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}} \left(\sum_{k \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} T_2^{i_2} \dots T_d^{i_d} \right) := \sup_{k \geq 0, i_1 \geq 0, \dots, i_d \geq 0} p^{-k} \alpha(\bar{x}_{k, i_1, \dots, i_d})^r.$$

Then we define the corresponding convergent rings:

$$\tilde{\Omega}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}} = W_\pi(R)_{\mathbb{Q}_p\{T_1, \dots, T_d\}}, \tilde{\Omega}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}} = W_\pi(R)_{\mathbb{Q}_p\{T_1, \dots, T_d\}}[1/\pi].$$

Then as in the previous setting we define the corresponding $\mathbb{Q}_p\{T_1, \dots, T_d\}$ -relative integral Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r}$ as the completion of the ring $W_\pi(R^+)_{\mathbb{Q}_p\{T_1, \dots, T_d\}}[[R]]$ by using the Gauss norm defined above. Then one can take the corresponding union of all $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r}$ throughout all $r > 0$ to define the integral Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}}$. For the bounded Robba rings we just set $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}, r} = \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r}[1/\pi]$ and take the union throughout all $r > 0$ to define the corresponding ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}}$. Then we define the corresponding Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^I$ with respect to some interval $I \subset (0, \infty)$ by taking the Fréchet completion of $W_\pi(R^+)_{\mathbb{Q}_p\{T_1, \dots, T_d\}}[[R]][1/\pi]$ with respect to all the norms $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for all $r \in I$ which means that the corresponding equivalence classes in the completion procedure will be simultaneously Cauchy with respect to all the norms $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for all $r \in I$. Then we take suitable intervals such as $(0, r]$ and $(0, \infty)$ to define the corresponding Robba rings $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^r$ and $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^\infty$, respectively. Then taking the union throughout all $r > 0$ one can define the corresponding Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}$. Again as in [KL16] one can define the corresponding integral rings of the similar types.

Then one can define the corresponding period rings in our context deformed over some affinoid A which is isomorphic to some quotient of $\mathbb{Q}_p\{T_1, \dots, T_d\}$, again deforming from the context of [KL16, Definition 4.1.1].

Definition 3.2.6. In the characteristic 0, we define the following period rings:

$$\tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Omega}_{R,A}, \tilde{\Pi}_{R,A}^{\text{int},r}, \tilde{\Pi}_{R,A}^{\text{bd},r}, \tilde{\Pi}_{R,A}^I, \tilde{\Pi}_{R,A}^r, \tilde{\Pi}_{R,A}^\infty$$

by taking the suitable quotient of the following period rings defined above:

$$\tilde{\Omega}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{\text{int}}, \tilde{\Omega}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}, \tilde{\Pi}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{\text{int},r}, \tilde{\Pi}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{\text{bd},r}, \tilde{\Pi}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}^I, \quad (3.2.1)$$

$$\tilde{\Pi}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}^r, \tilde{\Pi}_{R,\mathbb{Q}_p\{T_1,\dots,T_d\}}^\infty \quad (3.2.2)$$

with respect to the structure of the affinoid algebra A in the sense of Tate. The corresponding constructions do not depend on the corresponding choices of the presentations. Therefore they carry the corresponding quotient seminorms of the above Gauss norms defined in the previous definition, note that these are not something induced from the corresponding spectral seminorms from A chosen at the very beginning of our study. We use the notation $\overline{\|\cdot\|}_{\alpha^r, \mathbb{Q}_p\{T_1,\dots,T_d\}}$ to denote the corresponding quotient Gauss norm which induces then the corresponding spectral seminorm $\|\cdot\|_{\alpha^r, A}$ for each $r > 0$. Then we define the corresponding period rings:

$$\tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A} \quad (3.2.3)$$

by taking suitable union throughout all $r > 0$.

Definition 3.2.7. In positive characteristic situation, when we are working over general affinoid algebra A , we use the same notations as in the previous definition, but by using $W(R)_{\mathbb{F}_p[[\eta]]\{T_1,\dots,T_d\}}$ and $W(R)_A$ as the starting rings, namely here A is isomorphic to a quotient of $\mathbb{F}_p((\eta))\{T_1, \dots, T_d\}$.

We can then define the corresponding rings over the ring E_∞ which is defined to be the completion of the ring $E(\pi^{p^{-\infty}})$. We first consider the corresponding undeformed versions (after [KL16, Definition 4.1.1]):

Definition 3.2.8. Consider now the base change $W_{\pi,\infty}(R)$ which is defined now to be the completed tensor product $W_\pi(R) \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$. Then the point is that each element in

this ring admits a unique expression taking the form of $\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{x}_n]$ which allows us to perform the construction mentioned above. First we can define for some radius $r > 0$ the corresponding period ring $\tilde{\Pi}_{R,\infty}^{\text{int},r}$ by taking the completion of the ring

$$W_{\pi,\infty}(R^+) [[R]]$$

with respect to the following Gauss type norm:

$$\|\cdot\|_{\alpha^r} \left(\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{x}_n] \right) := \sup_{n \in \mathbb{Z}[1/p]_{\geq 0}} \{p^{-n} \alpha(\bar{x}_n)^r\}.$$

Then we can define the union $\tilde{\Pi}_{R,\infty}^{\text{int}}$ throughout all the radius $r > 0$. Then we just define the bounded Robba ring $\tilde{\Pi}_{R,\infty}^{\text{bd},r}$ by $\tilde{\Pi}_{R,\infty}^{\text{int},r}[1/\pi]$ and also we could define the union $\tilde{\Pi}_{R,\infty}^{\text{bd}}$ throughout all the radius $r > 0$. Then for any interval in $(0, \infty)$ which is denoted by I we can define the corresponding Robba rings $\tilde{\Pi}_{R,\infty}^I$ by taking the Fréchet completion of

$$W_{\pi,\infty}(R^+) [[R]] [1/\pi]$$

with respect to all the norms $\|\cdot\|_{\alpha^t}$ for all $t \in I$. Then by taking suitable specified intervals one can define the rings $\tilde{\Pi}_{R,\infty}^r$ and $\tilde{\Pi}_{R,\infty}^\infty$ as before, and finally one can define the corresponding union $\tilde{\Pi}_{R,\infty}$ throughout all the radius $r > 0$. Again we have the corresponding integral version of the rings defined over E_∞ as [KL16].

Then we can define the following affinoid deformations (after [KL16, Definition 4.1.1] in the flavor as above):

Definition 3.2.9. Consider now the base change $W_{\pi,\infty,\mathbb{Q}_p\{T_1,\dots,T_d\}}(R)$ which is defined now to be the completed tensor product $(W_\pi(R) \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p\{T_1, \dots, T_d\}$. Then the point is that each element in this ring admits a unique expression taking the form of

$$\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1 \geq 0, \dots, i_d \geq 0} \pi^n [\bar{x}_{n,i_1,\dots,i_d}] T_1^{i_1} \dots T_d^{i_d}$$

which allows us to perform the construction mentioned above. First we can define for some radius $r > 0$ the corresponding period ring $\tilde{\Pi}_{R,\infty,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{\text{int},r}$ by taking the completion of the ring

$$W_{\pi,\infty,\mathbb{Q}_p\{T_1,\dots,T_d\}}(R^+) [[R]]$$

with respect to the following Gauss type norm:

$$\|\cdot\|_{\alpha^r} \left(\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1 \geq 0, \dots, i_d \geq 0} \pi^n [\bar{x}_{n, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d} \right) := \sup_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1 \geq 0, \dots, i_d \geq 0} \{p^{-n} \alpha(\bar{x}_{n, i_1, \dots, i_d})^r\}.$$

Then we can define the union $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}}$ throughout all the radius $r > 0$. Then we just define the bounded Robba ring $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}, r}$ by $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r}[1/\pi]$ and also we could define the union $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}}$ throughout all the radius $r > 0$. Then for any interval in $(0, \infty)$ which is denoted by I we can define the corresponding Robba rings $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^I$ by taking the Fréchet completion of

$$W_{\pi, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(R^+)[[R]][1/\pi]$$

with respect to all the norms $\|\cdot\|_{\alpha^t}$ for all $t \in I$. Then by taking suitable specified intervals one can define the rings $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^r$ and $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\infty}$ as before, and finally one can define the corresponding union $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ throughout all the radius $r > 0$. Again we have the corresponding integral version of the rings defined over E_{∞} as [KL16]. Finally over A we can define in the same way as above to deform all the rings over E_{∞} , and we do not repeat the construction again.

3.2.2 Basic Properties of Period Rings

Then we do some reality checks over the investigation of the properties of the above period rings in the style taken in [KL15, Section 5.2] and [KL16].

Proposition 3.2.10. (After Kedlaya-Liu [KL15, Lemma 5.2.1]) *The function $t \mapsto \|x\|_{\alpha^t, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^r$ is continuous log convex for the corresponding variable $t \in (0, r]$ for any $r > 0$.*

Proof. Adapt the corresponding argument in the proof of 5.2.1 of [KL15, Lemma 5.2.1] to our situation we then first look at the situation where the element is just of the form of $\pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}$ where the corresponding norm function in terms of $t > 0$ is just affine. Then one focuses on the finite sums of these kind of elements which gives rise to the log convex directly. Finally by taking the approximation we get the desired result. \square

Proposition 3.2.11. (After Kedlaya-Liu [KL15, Lemma 5.2.2]) *For any element $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ we have that $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}}$ if and only if we have the situation*

where x actually lives in $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^r$ (for some specific $r > 0$) such that x itself is bounded under the norm $\|\cdot\|_{\alpha^t, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for each $t \in (0, r]$.

Proof. One direction of the proof is easy, so we only choose to present the proof of the implication in the other direction as in the original proof of 5.2.2 of [KL15, Lemma 5.2.2] as in the following. First choose some radius $r > 0$ such that the element could be assumed to be living in the ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^r$. The idea is to transfer the original question to the question about showing the integrality of x when we add some hypothesis on the norm by taking suitable powers of p (since the norm is bounded for each $t \in (0, r]$ so we are reduced to the situation where the norm is bounded by 1). Then we argue as in [KL15, Lemma 5.2.2] to choose some approximating sequence $\{x_i\}$ living in $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}}$ of x . Therefore we have for any $j \geq 1$ one can find then some integer $N_j \geq 1$ such that for any $i \geq N_j$ we have the estimate:

$$\|\cdot\|_{\alpha^t, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x_i - x) \leq p^{-j}, \forall t \in [p^{-j}r, r].$$

Then the idea is to consider the integral decomposition of the element x_i which has the form of $\sum_{k=n(x_i), i_1 \geq 0, i_2 \geq 0, \dots, i_d \geq 0} \pi^k [\bar{x}_{i, k, i_1, \dots, i_d}]$ into the following two parts:

$$x_i := y_i + z_i \tag{3.2.4}$$

$$:= \sum_{k=0, i_1 \geq 0, i_2 \geq 0, \dots, i_d \geq 0} \pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d} + z_i \tag{3.2.5}$$

from which we actually have the corresponding estimate over the residual part of the decomposition above:

$$\|\cdot\|_{\alpha^{p^{-j}r}, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}) \leq 1, \forall k < 0$$

which implies by direct computation:

$$\alpha^{p^{-j}r}(\bar{x}_{i, k, i_1, \dots, i_d}) \leq p^k \tag{3.2.6}$$

$$\alpha(\bar{x}_{i, k, i_1, \dots, i_d}) \leq p^{kp^j/r} \tag{3.2.7}$$

which implies that we have the following estimate:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x_i - y_i) \leq p^{-k} p^{kp^j/r} \tag{3.2.8}$$

$$\leq p^{1-p^j} \tag{3.2.9}$$

which implies that $y_i \rightarrow x$ under the norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ which furthermore under the norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ due the property of the norm, which finishes the proof of the desired result. \square

Proposition 3.2.12. (After Kedlaya-Liu [KL15, Corollary 5.2.3]) *We have the following identity:*

$$(\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}})^\times = (\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{bd}})^\times.$$

Proof. See 5.2.3 of [KL15, Corollary 5.2.3]. \square

Proposition 3.2.13. (After Kedlaya-Liu [KL15, Lemma 5.2.6]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r_2]} = \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}.$$

Proof. We adapt the argument in [KL15, Lemma 5.2.6] 5.2 to prove this in the situation where $r_1 < r_2$ (otherwise this is trivial), again one direction is easy where we only present the implication in the other direction. We take any element

$$x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r_2]}$$

and take suitable approximating elements $\{x_i\}$ living in the bounded Robba ring such that for any $j \geq 1$ one can find some integer $N_j \geq 1$ we have whenever $i \geq N_j$ we have the following estimate:

$$\|\cdot\|_{\alpha^t, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x_i - x) \leq p^{-j}, \forall t \in [r_1, r_2].$$

Then we consider the corresponding decomposition of x_i for each $i = 1, 2, \dots$ into a form having integral part and the rational part $x_i = y_i + z_i$ by setting

$$y_i = \sum_{k=0, i_1, \dots, i_d} \pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}$$

out of

$$x_i = \sum_{k=n(x_i), i_1, \dots, i_d} \pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}.$$

Note that by our initial hypothesis we have that the element x lives in the ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1}$ which further implies that

$$\|\cdot\|_{\alpha^{r_1}, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}) \leq p^{-j}.$$

Therefore we have $\alpha(\bar{x}_{i,k,i_1,\dots,i_d}) \leq p^{(k-j)/r_1}$ directly from this through computation, which implies that then:

$$\|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{T_1, \dots, T_d\}} (\pi^k [\bar{x}_{i,k,i_1,\dots,i_d}] T_1^{i_1} \dots T_d^{i_d}) \leq p^{-k} p^{(k-j)r_2/r_1} \quad (3.2.10)$$

$$\leq p^{1+(1-j)r_1/r_1}. \quad (3.2.11)$$

Then one can read off the result directly from this estimate since under this estimate we can have the chance to modify the original approximating sequence $\{x_i\}$ by $\{y_i\}$ which are initially chosen to be in the integral Robba ring, which implies that actually the element x lives in the right-hand side of the identity in the statement of the proposition. \square

Proposition 3.2.14. (After Kedlaya-Liu [KL15, Lemma 5.2.6]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\tilde{\Pi}_{R,A}^{\text{int}, r_1} \cap \tilde{\Pi}_{R,A}^{[r_1, r_2]} = \tilde{\Pi}_{R,A}^{\text{int}, r_2}.$$

Here A is some reduced affinoid algebra over \mathbb{Q}_p .

Proof. See the proof of proposition 3.5.7. \square

Then we have the following analog of the corresponding result of [KL15, Lemma 5.2.8]:

Proposition 3.2.15. (After Kedlaya-Liu [KL15, Lemma 5.2.8]) *Consider now in our situation the radii $0 < r_1 \leq r_2$, and consider any element $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r_2]}$. Then we have that for each $n \geq 1$ one can decompose x into the form of $x = y + z$ such that $y \in \pi^n \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$ with $z \in \bigcap_{r \geq r_2} \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r]}$ with the following estimate for each $r \geq r_2$:*

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}} (z) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{T_1, \dots, T_d\}} (z)^{r/r_2}.$$

Proof. As in [KL15, Lemma 5.2.8] and in the proof of our previous proposition we first consider those elements x living in the bounded Robba rings which could be expressed in general as

$$\sum_{k=n(x), i_1, \dots, i_d} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}.$$

In this situation the corresponding decomposition is very easy to come up with, namely we consider the corresponding y_i as the corresponding series:

$$\sum_{k \geq n, i_1, \dots, i_d} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}$$

which give us the desired result since we have in this situation when focusing on each single term:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}} (\pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d}) = p^{-k} \alpha(\bar{x}_{k, i_1, \dots, i_d})^r \quad (3.2.12)$$

$$= p^{-k(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{T_1, \dots, T_d\}} (\pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d})^{r/r_2} \quad (3.2.13)$$

$$\leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{T_1, \dots, T_d\}} (\pi^k [\bar{x}_{k, i_1, \dots, i_d}] T_1^{i_1} \dots T_d^{i_d})^{r/r_2} \quad (3.2.14)$$

for all those suitable k . Then to tackle the more general situation we consider the approximating sequence consisting of all the elements in the bounded Robba ring as in the usual situation considered in [KL15, Lemma 5.2.8], namely we inductively construct the following approximating sequence just as:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}} (x - x_0 - \dots - x_i) \leq p^{-i-1} \|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}} (x), \quad i = 0, 1, \dots, r \in [r_1, r_2]. \quad (3.2.15)$$

Here all the elements x_i for each $i = 0, 1, \dots$ are living in the bounded Robba ring, which immediately gives rise to the suitable decomposition as proved in the previous case namely we have for each i the decomposition $x_i = y_i + z_i$ with the desired conditions as mentioned in the statement of the proposition. We first take the series summing all the elements y_i up for all $i = 0, 1, \dots$, which first of all converges under the norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for all the radius $r \in [r_1, r_2]$, and also note that all the elements y_i within the infinite sum live inside the corresponding integral Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$, which further implies the corresponding convergence ends up in $\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$. For the elements z_i where $i = 0, 1, \dots$ also sum up to a converging series in the desired ring since combining all the estimates above we have:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{T_1, \dots, T_d\}} (z_i) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{T_1, \dots, T_d\}} (x)^{r/r_2}.$$

□

Proposition 3.2.16. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_1, r_1]} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_2, r_2]} = \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_1, r_2]},$$

here the radii satisfy $s_1 \leq s_2 \leq r_1 \leq r_2$.

Proof. In our situation one direction is obvious while on the other hand we consider any element x in the intersection on the left, then by the previous proposition we have the decomposition $x = y + z$ where $y \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1}$ and $z \in \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_1, r_2]}$. Then as in [KL15, Lemma 5.2.10] section 5.2 we look at $y = x - z$ which lives in the intersection:

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_2, r_2]} = \tilde{\Pi}_{R, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$$

which finishes the proof. □

Proposition 3.2.17. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\tilde{\Pi}_{R, A}^{[s_1, r_1]} \cap \tilde{\Pi}_{R, A}^{[s_2, r_2]} = \tilde{\Pi}_{R, A}^{[s_1, r_2]},$$

here the radii satisfy $s_1 \leq s_2 \leq r_1 \leq r_2$.

Proof. See the proof of proposition 3.5.7. □

Remark 3.2.18. This subsection is finished so far only for the situation where E is of mixed characteristic. But everything uniformly carries over for our original assumption on the field E and A . We will not repeat the proof again.

3.3 Period Modules and Period Sheaves

We now consider the corresponding Frobenius modules over the corresponding period rings defined in the previous section. Also one can consider the corresponding period sheaves in the style of [KL16]. We would like to point out that actually the corresponding sheaves in our context could mean the following two different objects. First the corresponding period rings defined in the previous section themselves are sheafy, which means that one can study the corresponding analytic geometry over the relative affinoid spaces over for instance the ring $\tilde{\Pi}_R^{\text{int},r}$ or the ring $\tilde{\Pi}_R^{\text{int},r}$ which has its own interests. On the other hand we have the situation where one can glue along the algebra R over corresponding étale, corresponding pro-étale and corresponding v-sites but leaving the algebra A unglued. We point out that both could have the chance to be compared in a coherent way, to the corresponding representation theoretic consideration in the pseudocoherent setting.

Setting 3.3.1. We will work in the categories of the pseudocoherent, fpd and finite projective modules over the period rings defined above. First we specify the Frobenius in our setting. The rings involved are:

$$\tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Omega}_{R,A}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{int},r}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}^{\text{bd},r}, \tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^r, \tilde{\Pi}_{R,A}^+, \tilde{\Pi}_{R,A}^\infty, \tilde{\Pi}_{R,A}^I. \quad (3.3.1)$$

We are going to endow these rings with the Frobenius induced by continuation from the Witt vector part only, which is to say the corresponding Frobenius induced by the p^h -power absolute Frobenius over R . Note all the rings above are defined by taking the product of Δ where each Δ representing one of the following rings (over E):

$$\tilde{\Omega}_R^{\text{int}}, \tilde{\Omega}_R, \tilde{\Pi}_R^{\text{int}}, \tilde{\Pi}_R^{\text{int},r}, \tilde{\Pi}_R^{\text{bd}}, \tilde{\Pi}_R^{\text{bd},r}, \tilde{\Pi}_R, \tilde{\Pi}_R^r, \tilde{\Pi}_R^+, \tilde{\Pi}_R^\infty, \tilde{\Pi}_R^I \quad (3.3.2)$$

with the affinoid ring A . The Frobenius acts on A trivially and we assume that the action is A -linear.

First we consider the following sheafification as in [KL16, Definition 4.4.2]:

Setting 3.3.2. Consider the space $X = \text{Spa}(R, R^+)$, over this perfectoid space there were sheaves:

$$\tilde{\Omega}^{\text{int}}, \tilde{\Omega}, \tilde{\Pi}^{\text{int}}, \tilde{\Pi}^{\text{int},r}, \tilde{\Pi}^{\text{bd}}, \tilde{\Pi}^{\text{bd},r}, \tilde{\Pi}, \tilde{\Pi}^r, \tilde{\Pi}^+, \tilde{\Pi}^\infty, \tilde{\Pi}^I. \quad (3.3.3)$$

defined over this space through the corresponding adic, étale, pro-étale or v -topology, we consider the corresponding sheaves defined over the same Grothendieck sites but with further deformed consideration:

$$\widetilde{\Omega}_{*,A}^{\text{int}}, \widetilde{\Omega}_{*,A}, \widetilde{\Pi}_{*,A}^{\text{int}}, \widetilde{\Pi}_{*,A}^{\text{int},r}, \widetilde{\Pi}_{*,A}^{\text{bd}}, \widetilde{\Pi}_{*,A}^{\text{bd},r}, \widetilde{\Pi}_{*,A}, \widetilde{\Pi}_{*,A}^r, \widetilde{\Pi}_{*,A}^+, \widetilde{\Pi}_{*,A}^\infty, \widetilde{\Pi}_{*,A}^I. \quad (3.3.4)$$

Remark 3.3.3. The consideration in the previous setting in some sense reflects some kind of rigidity. Since the corresponding construction is just that the ring A acts through the completed tensor product directly on the sheaves after [KL16], in most cases when one would like to compare those sheaves of modules over the sheaves of rings above and the corresponding modules over the period rings defined above, one needs to work through the corresponding vanishing results and so on in this deformed context. We work around this by considering the corresponding globalized version of this.

Setting 3.3.4. In our context we can make the following parallel discussion to that in [KL16]. First we can use the corresponding notation E_∞ to denote the corresponding completion of $E(\pi^{1/p^\infty})$ and we denote the corresponding integral ring by \mathcal{O}_{E_∞} . Then we have could have that the corresponding splitting of the ring \mathcal{O}_E in this bigger perfectoid ring. Then we can consider the corresponding extended period rings $\widetilde{\Omega}_R^{\text{int}} \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ and $\widetilde{\Pi}_R^{\text{int},r} \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ in the same fashion as in [KL16, Definition 4.1.11].

Proposition 3.3.5.

(After Kedlaya-Liu [KL16, Lemma 4.1.12, Corollary 4.1.14])

The ring $\widetilde{\Pi}_R^{\text{int},r} \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ is perfectoid, so stably uniform and sheafy. By using the same notation in [KL16, Lemma 4.1.12, Corollary 4.1.14] we have that the corresponding tilting perfectoid ring is $R\{(\overline{\pi}/p^{-1/r})^{1/p^\infty}\}$ which is the completion of $R[(\overline{\pi}/p^{-1/r})^{1/p^\infty}]$.

Proof. This is the same as the proof for [KL16, Lemma 4.1.12, Corollary 4.1.14] after we choose a suitable topologically nilpotent element. \square

Proposition 3.3.6.

(After Kedlaya-Liu [KL16, Proposition 4.1.13, Corollary 4.1.14])

The ring $\widetilde{\Pi}_R^I \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ for some closed interval $I = [s, r]$ is perfectoid so stably uniform and sheafy. Then in our situation this perfectoid admits tilting ring

$$R\{(\overline{\pi}/p^{-1/r})^{1/p^\infty}, (p^{-1/s}/\overline{\pi})^{1/p^\infty}\}$$

which is the completion of $R[(\bar{\pi}/p^{-1/r})^{1/p^\infty}, (p^{-1/s}/\bar{\pi})^{1/p^\infty}]$.

Proof. This is the same as the proof of [KL16, Proposition 4.1.13, Corollary 4.1.14] by considering the previous proposition. \square

Setting 3.3.7. We are going to use the notation \mathfrak{X} to denote a rigid analytic space in rigid geometry. Then we can consider the corresponding relative period rings or sheaves over $\mathcal{O}_{\mathfrak{X}}$. The period rings and the corresponding period sheaves over the sheaf $\mathcal{O}_{\mathfrak{X}}$ are defined to be the following sheaves (one takes the complete tensor product of the undeformed period rings with the corresponding exact sequence for the sheaf $?$, which again gives the corresponding exact sequence due to the fact that the undeformed period rings admit Schauder bases as in [KP, Definition 6.1]):

$$\tilde{\Pi}_{R,?}, \tilde{\Pi}_{R,?}^r, \tilde{\Pi}_{R,?}^\infty, \tilde{\Pi}_{R,?}^I, \quad (3.3.5)$$

with

$$\tilde{\Pi}_{*,?}, \tilde{\Pi}_{*,?}^r, \tilde{\Pi}_{*,?}^\infty, \tilde{\Pi}_{*,?}^I, \quad (3.3.6)$$

where $*$ is some étale site, proét site or v -site and etc, and $?$ represents $\mathcal{O}_{\mathfrak{X}}$ which implies that we treat these sheaves of rings as sheaves over $*$ in some relative sense. One should understand this as the corresponding sheaves essentially with respect $*$.

Remark 3.3.8. It is hard to consider the corresponding comparison between the representation theories over these sheaves (deformed) with the rings (without any sheafified consideration), since the corresponding global section functor is a little bit hard to study. But on the other hand comparing this among the sheaves themselves and with the Fargues-Fontaine curves are very natural and interesting.

Assumption 3.3.9. From now on, we are going to assume that the affinoid ring A is splitting in some perfectoid covering A^{perf} , but only when the corresponding sheafiness of the deformed period rings is essentially relevant. For instance if A is just some analytic field as those in [KL16] then this is satisfied. This will guarantee the corresponding period rings over A (we now assume them to be sousperfectoid) will then be sheafy by Kedlaya-Hansen's sheafiness criterion [KH].

We first consider the following result which is slight generalization of the corresponding result from [KL16, 4.3.1]. Note that our E is general than the basic setting of [KL16, Setting 3.1.1].

Remark 3.3.10. In what follows, there will be some situation we will consider the perfectoid deformed version of the period rings and sheaves. All the definitions are parallel to the base level definitions (including the corresponding Frobenius modules and sheaves).

Lemma 3.3.11. (After Kedlaya-Liu [KL16, 4.3.1]) *Let M be any étale-stably pseudocoherent or fpd module defined over $\tilde{\Pi}_R^{\text{int},r}$ and $\tilde{\Pi}_{R,\infty}^{\text{int},r}$ for some $r > 0$ or $\tilde{\Pi}_R^I$ and $\tilde{\Pi}_{R,\infty}^I$ for some I which is assumed to be closed, and let \tilde{M} be the corresponding sheaf attached to the module M over the sheaf $\tilde{\Pi}_*^{\text{int},r}$ for some $r > 0$ or $\tilde{\Pi}_*^I$ for some I which is assumed to be closed, over the étale and the pro-étale sites of $\text{Spa}(R, R^+)$. Then we have the following statement:*

$$H^i(X, \tilde{M}) = H^i(X_{\text{ét}}, \tilde{M}) = H^i(X_{\text{proét}}, \tilde{M}) \quad (3.3.7)$$

vanish beyond the 0-th degree and give the module M at the degree 0. Also furthermore we have that in the projective situation we have that the same holds under the v -topology.

Proof. The corresponding results could be transformed to perfectoid spaces as in [KL16, 4.3.1] by considering the splitting after passing to perfectoid field E_∞ and perfectoid ring $\mathcal{O}_{\mathfrak{X}^{\text{perf}}}$, then by applying the results in [KL16, 2.5.1, 2.5.11, 3.4.6, 3.5.6] we can get the desired results. \square

Definition 3.3.12. Now working over some perfect uniform and adic space over k , as in [KL16, Definition 4.3.2] we define the corresponding pseudocoherent sheaves or fpd sheaves over $\tilde{\Pi}_*^{\text{int},r}$ or $\tilde{\Pi}_*^I$ (where I is closed) where $*$ represents one of X , $X_{\text{ét}}$ and $X_{\text{proét}}$ to be the sheaves associated to étale-stably pseudocoherent or fpd modules in some local manner.

Proposition 3.3.13. (After Kedlaya-Liu [KL16, Theorem 4.3.3]) *The global section functor establishes an equivalence between the categories of the pseudocoherent sheaves over $\tilde{\Pi}_*^{\text{int},r}$ or $\tilde{\Pi}_*^I$ (and $\tilde{\Pi}_{*,\infty}^{\text{int},r}$ or $\tilde{\Pi}_{*,\infty}^I$) over the pro-étale site of $X = \text{Spa}(R, R^+)$ and the étale-stably pseudocoherent modules over $\tilde{\Pi}_R^{\text{int},r}$ or $\tilde{\Pi}_R^I$ (and $\tilde{\Pi}_{R,\infty}^{\text{int},r}$ or $\tilde{\Pi}_{R,\infty}^I$). The global section functor establishes an equivalence between the categories of the fpd sheaves over*

$\tilde{\Pi}_*^{\text{int},r}$ or $\tilde{\Pi}_*^I$ over the pro-étale site of $X = \text{Spa}(R, R^+)$ and the étale-stably fpd modules over $\tilde{\Pi}_R^{\text{int},r}$ or $\tilde{\Pi}_R^I$. The following morphisms are effective descent morphisms for pseudocoherent Banach modules: I. $\tilde{\Pi}_{R,A}^r \rightarrow \tilde{\Pi}_{R,A}^s \oplus \tilde{\Pi}_{R,A}^{[s,r]}$; II. $\tilde{\Pi}_{R,A}^I \rightarrow \bigoplus_{i=1}^k \tilde{\Pi}_{R,A}^{I_i}$. here the corresponding set of interval $\{I_i\}_{i=1,\dots,k}$ consists of finitely many closed intervals covering the interval I .

Proof. As in [KL16, Theorem 4.3.3], by taking the splitting base change to \mathcal{O}_{E_∞} and considering the corresponding properties of being perfectoid we can transform the proofs around the global section functor to the corresponding statement for juts perfectoid spaces, which could be finished by [KL16, 2.5.5,2.5.14,3.4.9]. The rest statements are further consequences of the fact that the period rings $\tilde{\Pi}_{R,A}^r$ and $\tilde{\Pi}_{R,A}^I$ are sheafy, see [KL16, 2.5.5,2.5.14]. \square

Then we generalize the corresponding definitions of Frobenius action and Frobenius modules in [KL16, Definition 4.4.2-4.4.4] to our situation.

Definition 3.3.14. In our situation we consider the corresponding Frobenius action on the following period rings and sheaves:

$$\tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^\infty, \tilde{\Omega}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^+, \tilde{\Omega}_{*,A}, \tilde{\Pi}_{*,A}^{\text{bd}}, \tilde{\Pi}_{*,A}^\infty, \tilde{\Pi}_{*,A}^+, \tilde{\Pi}_{*,A}, \quad (3.3.8)$$

which is defined by considering the corresponding lift of the absolute Frobenius of p^h -power in characteristic $p > 0$ induced from R , which will be denoted by φ . We then introduce more general consideration by taking E_a to be some unramified extension of E of degree a divisible by h , the corresponding Frobenius will be denoted by φ^a .

Then we generalize the corresponding Frobenius modules in [KL16, Definition 4.4.4] to our situation as in the following.

Definition 3.3.15. Over the period rings and sheaves (each is denoted by Δ in this definition) defined in the previous definition we define as in [KL16, Definition 4.4.4] the corresponding φ^a -modules over Δ which are respectively projective, pseudocoherent or fpd to be the corresponding finite projective, pseudocoherent or fpd modules over Δ with further assigned semilinear action of the operator φ^a . Here we define in our situation the corresponding φ^a -cohomology to be the (hyper)-cohomology of the following complex:

$$0 \longrightarrow M \xrightarrow{\varphi^{-1}} M \longrightarrow 0.$$

We also require that the modules are complete for the natural topology involved in our situation and for any module over $\widetilde{\Pi}_{*,A}$ to be some base change of some module over $\widetilde{\Pi}_{*,A}^r$ (which will be defined in the following) over each perfectoid subdomain Y (in this situation we are considering the pro-étale site).

Now we define the corresponding modules over the rings which are the domains in the following morphisms induced from the Frobenius map φ^a :

$$\widetilde{\Pi}_{R,A}^{\text{int},r} \rightarrow \widetilde{\Pi}_{R,A}^{\text{int},rp^{-ha}}, \widetilde{\Pi}_{R,A}^{\text{bd},r} \rightarrow \widetilde{\Pi}_{R,A}^{\text{bd},rp^{-ha}}, \widetilde{\Pi}_{R,A}^r \rightarrow \widetilde{\Pi}_{R,A}^{rp^{-ha}} \quad (3.3.9)$$

$$\widetilde{\Pi}_{*,A}^{\text{int},r} \rightarrow \widetilde{\Pi}_{*,A}^{\text{int},rp^{-ha}}, \widetilde{\Pi}_{*,A}^{\text{bd},r} \rightarrow \widetilde{\Pi}_{*,A}^{\text{bd},rp^{-ha}}, \widetilde{\Pi}_{*,A}^r \rightarrow \widetilde{\Pi}_{*,A}^{rp^{-ha}}. \quad (3.3.10)$$

Definition 3.3.16. Over each rings Δ which are the domains in the morphisms as mentioned just before this definition, we define the corresponding projective, pseudocoherent or fpd φ^a -module over any Δ listed above to be the corresponding finite projective, pseudocoherent or fpd module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*}M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one of the targets listed above. Also as in [KL16, Definition 4.4.4] we assume that the module over $\widetilde{\Pi}_{*,A}^I$ is then complete for the natural topology and the corresponding base change to $\widetilde{\Pi}_{*,A}^I$ for any interval which is assumed to be closed $I \subset [0, r)$ gives rise to a module over $\widetilde{\Pi}_{*,A}^I$ with specified conditions which will be specified below. Also the cohomology of any module under this definition will be defined to be the (hyper)cohomology of the complex in the following form:

$$0 \longrightarrow M \xrightarrow{\varphi^{-1}} M \otimes_{\Delta} \square \longrightarrow 0.$$

Then we consider the following morphisms of specific period rings induced by the Frobenius.

$$\widetilde{\Pi}_{R,A}^{[s,r]} \rightarrow \widetilde{\Pi}_{R,A}^{[sp^{-ah}, rp^{-ah}]} \quad (3.3.11)$$

$$\widetilde{\Pi}_{*,A}^{[s,r]} \rightarrow \widetilde{\Pi}_{*,A}^{[sp^{-ah}, rp^{-ah}]} \quad (3.3.12)$$

with the corresponding morphisms in the following:

$$\widetilde{\Pi}_{R,A}^{[s,r]} \rightarrow \widetilde{\Pi}_{R,A}^{[s, rp^{-ah}]} \quad (3.3.13)$$

$$\widetilde{\Pi}_{*,A}^{[s,r]} \rightarrow \widetilde{\Pi}_{*,A}^{[s, rp^{-ah}]} \quad (3.3.14)$$

Definition 3.3.17. Again as in [KL16, Definition 4.4.4], we define the corresponding projective, pseudocoherent and fpd φ^a -modules over the domain rings or sheaves of rings in the morphisms just before this definition to be the finite projective, pseudocoherent and fpd modules (which will be denoted by M) over the domain rings in the morphism just before this definition additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\tilde{\Pi}_{R,A}^{[sp^{-ah}, rp^{-ah}]}} \tilde{\Pi}_{R,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\tilde{\Pi}_{R,A}^{[s,r]}} \tilde{\Pi}_{R,A}^{[s, rp^{-ah}]}, \quad (3.3.15)$$

$$\varphi^{a*} M \otimes_{\tilde{\Pi}_{*,A}^{[sp^{-ah}, rp^{-ah}]}} \tilde{\Pi}_{*,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\tilde{\Pi}_{*,A}^{[s,r]}} \tilde{\Pi}_{*,A}^{[s, rp^{-ah}]}. \quad (3.3.16)$$

We now assume that the modules are complete with respect to the natural topology. And we assume that for any perfectoid subdomain Y in the corresponding topology (defining the sheaves in this situation) the corresponding global sections over Y give rise to étale-stably pseudocoherent modules.

Also one can further define the corresponding bundles carrying semilinear Frobenius in our context as in the situation of [KL16, Definition 4.4.4]:

Definition 3.3.18. Over the ring $\tilde{\Pi}_{R,A}$ we define a corresponding projective, pseudocoherent and fpd Frobenius bundle to be a family $(M_I)_I$ of finite projective, étale stably pseudocoherent and étale stably fpd modules over each $\tilde{\Pi}_{R,A}^I$ carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\tilde{\Pi}_{R,A}^J} \tilde{\Pi}_{R,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, r]$ involved we have $s \leq rp^{ah}$.

We then have the following analog of [KL16, Lemma 4.4.8]:

Proposition 3.3.19. (After Kedlaya [KL15, Lemma 4.4.8]) *Consider the corresponding finite generated Frobenius modules over the following period rings or sheaves defined above:*

$$\tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^{\infty}, \tilde{\Omega}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^+, \tilde{\Omega}_{*,A}, \tilde{\Pi}_{*,A}^{\text{bd}}, \tilde{\Pi}_{*,A}^{\infty}, \tilde{\Pi}_{*,A}^+, \tilde{\Pi}_{*,A},$$

and

$$\tilde{\Pi}_{R,A}^{\text{int},r} \rightarrow \tilde{\Pi}_{R,A}^{\text{int},rp^{-ha}}, \tilde{\Pi}_{R,A}^{\text{bd},r} \rightarrow \tilde{\Pi}_{R,A}^{\text{bd},rp^{-ha}}, \tilde{\Pi}_{R,A}^r \rightarrow \tilde{\Pi}_{R,A}^{rp^{-ha}} \quad (3.3.17)$$

$$\tilde{\Pi}_{*,A}^{\text{int},r} \rightarrow \tilde{\Pi}_{*,A}^{\text{int},rp^{-ha}}, \tilde{\Pi}_{*,A}^{\text{bd},r} \rightarrow \tilde{\Pi}_{*,A}^{\text{bd},rp^{-ha}}, \tilde{\Pi}_{*,A}^r \rightarrow \tilde{\Pi}_{*,A}^{rp^{-ha}}. \quad (3.3.18)$$

Then we have these are the quotients of finite projective ones again endowed with the corresponding Frobenius actions.

Proof. See 1.5.2 of [KL15, Lemma 1.5.2]. □

3.4 Comparison Theorems

3.4.1 The Comparison between the Local Systems and the Period Modules

We now in our context study the corresponding local systems in the generalized setting. The objects to compare will be definitely the Frobenius modules defined in the previous section. However this is not that far from the situation studied in [KL16] since the undeformed period rings are actually finite projective over the rings defined in [KL16] in the most simplified situation. We will also use the notation \underline{L} to denote the local system associated to some topological ring L . We will consider the pro-étale site $X_{\text{proét}}$. As in [KL16, Definition 4.5.1] we have the notion of projective, pseudocoherent and fpd L -local systems in our context over the site $X_{\text{proét}}$.

Proposition 3.4.1. (After Kedlaya-Liu [KL16, Lemma 4.5.4]) *The Frobenius invariances give rise to the following exact sequences:*

$$\begin{aligned}
 0 &\longrightarrow \underline{\mathcal{O}}_{E_a^{\varphi^a}} \longrightarrow \tilde{\Omega}_*^{\text{int}} \longrightarrow \tilde{\Omega}_*^{\text{int}} \longrightarrow 0, \\
 0 &\longrightarrow \underline{\mathcal{O}}_{E_a^{\varphi^a}} \longrightarrow \tilde{\Pi}_*^{\text{int}} \longrightarrow \tilde{\Pi}_*^{\text{int}} \longrightarrow 0, \\
 0 &\longrightarrow \underline{E}_a^{\varphi^a} \longrightarrow \tilde{\Pi}_*^{\text{bd}} \longrightarrow \tilde{\Pi}_*^{\text{bd}} \longrightarrow 0, \\
 0 &\longrightarrow \underline{E}_a^{\varphi^a} \longrightarrow \tilde{\Pi}_* \longrightarrow \tilde{\Pi}_* \longrightarrow 0,
 \end{aligned}$$

where in each exact sequence the third arrow represents the morphism $\varphi^a - 1$.

Proof. It is actually [KL16, Lemma 4.5.4], although we consider some bigger base k , we reproduce the argument for the convenience of the reader. So it is obviously we have the exactness at the first positions which are not zero. The exactness in the middle could actually be derived from [KL15, Lemma 5.2.4]. Then for the exactness in between the third and the last arrows one may follow the proof in [KL16, Lemma 4.5.3] to prove this. For the first sequence see [KL16, Lemma 4.5.3]. For the second sequence, relying on the first sequence one only needs to prove that suppose image of some $y \in \tilde{\Omega}_R^{\text{int}}$ under the map $\varphi^a - 1$ is x living in $\tilde{\Pi}_R^{\text{int}} \subset \tilde{\Omega}_R^{\text{int}}$ then we have that the preimage y is not only in $\tilde{\Omega}_R^{\text{int}}$ but also actually in $\tilde{\Pi}_R^{\text{int}}$. This is because we have for each n and the corresponding expressions

$x = \sum_{n \geq 0} \pi^n [\bar{x}_n]$ and $y = \sum_{n \geq 0} \pi^n [\bar{y}_n]$, the coefficients for y could be controlled by those of x . The corresponding statement for the last sequence is proved by reducing to the second one, namely put:

$$x = y + (\varphi^a - 1) \left(- \sum_{k=0} \sum_{\ell=0} \pi^{i_k} [\bar{x}_{i_k}^{p^\ell}] \right),$$

where one may beforehand put x into a summation of some element in the bounded Robba ring and some element of the form of $\sum_{n_k} \pi^{n_k} [\bar{x}_{i_k}]$ with \bar{x}_{i_k} living in $R^{\circ\circ}$. \square

We then make the following discussion as well, as in [KL16, Lemma 4.5.4] in the following:

Proposition 3.4.2. (After Kedlaya-Liu [KL16, Lemma 4.5.4])

We consider the finitely presented Frobenius modules defined over the ring $\tilde{\Omega}_R^{\text{int}}$ or $\tilde{\Pi}_R^{\text{int}}$. Then in our context, we have: (1) First the Fitting ideal of M is generated by the elements which are in general form of $\pi^n [\bar{x}_n]$ where \bar{x}_n is some idempotent. (2) And we have that the π -torsion submodule T of M is finitely generated, therefore could be annihilated by some power π^n with some integer $n \geq 0$. (3) And the quotient M/T is then projective which implies that the module T is finitely presented. (4) And the submodule T is then finite union of Frobenius modules taking the form of $?/\pi^k?$ for integer $k \geq 0$, where $?$ represents the ring $\tilde{\Omega}_R^{\text{int}}$ or $\tilde{\Pi}_R^{\text{int}}$. (5) Finally we have that the corresponding module M is strictly pseudocoherent.

Proof. Again this is basically [KL16, Lemma 4.5.4], although we consider a larger base k . We reproduce the argument for the convenience of the reader. Then corresponding property of the Fitting ideal follows from the applying [KL15, Proposition 3.2.13]. Then we apply proposition 3.3.19 to choose a projective covering of M taking the form of $P \rightarrow M$ which further induces a map from $P \rightarrow M/T$, then we use the notation N to denote the kernel. Taking quotient by π we can form the exact sequence in our context due to the π -torsion freeness:

$$0 \longrightarrow N/\pi N \longrightarrow P/\pi P \longrightarrow (M/T)/\pi(M/T) \longrightarrow 0.$$

Then as in [KL16, Lemma 4.5.4] one can then in the situation where the base is $\tilde{\Omega}_R^{\text{int}}$ under the π -adic topology lift the generators of $N/\pi N$ by using the converging series of finite

sums of the generators from the quotient. Then in the situation where the base is $\tilde{\Pi}_R^{\text{int}}$ we use the Fréchet topology to run the same argument. T is a union of each T_n for each n , which are the corresponding π^n -torsion components of T . Then by using [KL16, Lemma 1.1.5] one derives that the module M/T is then finitely presented, which further implies that the corresponding module T is finite, which is just equal to some T_n . Then we apply [KL16, Lemma 1.1.5] to finish the proof. \square

Corollary 3.4.3. *Let M be a pseudocoherent Frobenius module over the period ring $\tilde{\Omega}_R^{\text{int}}$ or the period ring $\tilde{\Pi}_R^{\text{int}}$. Then we have that M admits a projective resolution of length at most 1.*

Then we generalize the comparison of local systems and Frobenius modules in [KL16, Theorem 4.5.7] to our context as in the following:

Theorem 3.4.4. (After Kedlaya-Liu [KL16, Theorem 4.5.7]) *The following categories are equivalent:*

- I. *The category of projective, pseudocoherent or fpd $\mathcal{O}_{E_a^{\varphi_a}}$ -local systems over $X_{\text{proét}}$;*
- II. *The category of projective, pseudocoherent or fpd Frobenius modules over the period ring $\tilde{\Omega}_R^{\text{int}}$;*
- III. *The category of projective, pseudocoherent or fpd Frobenius modules over the period ring $\tilde{\Pi}_R^{\text{int}}$;*
- IV. *The category of projective, pseudocoherent or fpd Frobenius modules over the period sheaf $\tilde{\Omega}_{*}^{\text{int}}$ over $X, X_{\text{ét}}, X_{\text{proét}}$;*
- V. *The category of projective, pseudocoherent or fpd Frobenius modules over the period sheaf $\tilde{\Pi}_{*}^{\text{int}}$ over $X, X_{\text{ét}}, X_{\text{proét}}$.*

Under further deformation we do not have the corresponding equivalence as above but we have the following statement:

1. *There is a fully faithful embedding functor from the category of the projective, pseudocoherent or fpd $\mathcal{O}_{E_a^{\varphi_a}} \hat{\otimes} \mathcal{O}_A$ -local systems over $X_{\text{proét}}$ to the following two categories:*
 - 1(a). *The category of projective, pseudocoherent or fpd Frobenius modules over the period sheaf $\tilde{\Omega}_{*,A}^{\text{int}}$ over $X, X_{\text{ét}}, X_{\text{proét}}$;*
 - 1(b). *The category of projective, pseudocoherent or fpd Frobenius modules over the period sheaf $\tilde{\Pi}_{*,A}^{\text{int}}$ over $X, X_{\text{ét}}, X_{\text{proét}}$.*
2. *We have that the following two categories are equivalent:*

2(a). The category of projective, pseudocoherent or fpd Frobenius modules over the period ring $\tilde{\Omega}_{R,\infty}^{\text{int}}$;

2(b). The category of projective, pseudocoherent or fpd Frobenius modules over the period sheaf $\tilde{\Omega}_{*,\infty}^{\text{int}}$ over X , $X_{\acute{e}t}$, $X_{\text{pro}\acute{e}t}$.

3. Furthermore we have the following two categories are equivalent:

3(a). The category of projective, pseudocoherent or fpd Frobenius modules over the period ring $\tilde{\Pi}_{R,\infty}^{\text{int}}$;

3(b). The category of projective, pseudocoherent or fpd Frobenius modules over the period sheaf $\tilde{\Pi}_{*,\infty}^{\text{int}}$ over X , $X_{\acute{e}t}$, $X_{\text{pro}\acute{e}t}$.

Proof. The first main statement, see [KL16, Theorem 4.5.7]. Then we proceed to consider the rest three main statements along the line of the proof of [KL16, Theorem 4.5.7]. For the first statement the corresponding functor is just the base change functor:

$$\mathbb{L} \mapsto \mathbb{L} \otimes_{\mathcal{O}_{\underline{E}_a^{\varphi^a} \hat{\otimes} \mathcal{O}_A}} \tilde{\Omega}_{*,A}^{\text{int}}$$

and

$$\mathbb{L} \mapsto \mathbb{L} \otimes_{\mathcal{O}_{\underline{E}_a^{\varphi^a} \hat{\otimes} \mathcal{O}_A}} \tilde{\Pi}_{*,A}^{\text{int}},$$

these are basically fully faithful by applying the corresponding deformed versions (which is just directly taking the complete tensor product with A) of the corresponding exact sequences in proposition 3.4.1. For the second and the last statements we follow the strategy of the proof of [KL16, Theorem 4.5.7] to proceed as in the following. First for the second one, this is direct consequence of lemma 3.3.11 and proposition 3.3.13. Then the third statement could be reduced to the statement by applying the second statement to the corresponding completion of $R[T^{\pm p^{-\infty}}]$ or $R[T^{p^{-\infty}}]$. \square

Then as in [KL16, Corollary 4.5.8] we consider the setting of more general adic spaces:

Proposition 3.4.5. (After Kedlaya-Liu [KL16, Corollary 4.5.8]) *1. Let X be a preadic space over E_a . Then we have that there is fully faithful embedding functor from the category of all the $\mathcal{O}_{\underline{E}_a^{\varphi^a} \hat{\otimes} \mathcal{O}_A}$ -local systems (projective, pseudocoherent or fpd) over X , $X_{\acute{e}t}$ and $X_{\text{pro}\acute{e}t}$ to the category of all the projective, pseudocoherent or fpd Frobenis modules over the sheaves over $\tilde{\Pi}_{R,A}^{\text{int}}$ over the corresponding sites;*

2. Let X be a preadic space over $E_{\infty,a}$. Then we have that there is fully faithful embedding functor from the category of all the $\mathcal{O}_{E_{\infty,a}^{\varphi^a}} \widehat{\otimes} \mathcal{O}_A$ -local systems (projective, pseudocoherent or fpd) over X , $X_{\acute{e}t}$ and $X_{\text{pro}\acute{e}t}$ to the category of all the projective, pseudocoherent or fpd Frobenius modules over the sheaves over $\widetilde{\Pi}_{R,\infty,A}^{\text{int}}$ over the corresponding sites.

Proof. For one, apply the previous theorem. For two, repeat the argument in the proof of the previous theorem. \square

The following definition is kind of generalization of the corresponding one in [KL16, Definition 4.5.9]:

Definition 3.4.6. Now over the ring $\widetilde{\Pi}_{R,A}$ or $\widetilde{\Pi}_{R,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the ring $\widetilde{\Pi}_{R,A}^{\text{int}}$. Now over the ring $\widetilde{\Pi}_{R,\infty,A}$ or $\widetilde{\Pi}_{R,\infty,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the ring $\widetilde{\Pi}_{R,\infty,A}^{\text{int}}$.

Now over the sheaf $\widetilde{\Pi}_{*,A}$ or $\widetilde{\Pi}_{*,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the sheaf $\widetilde{\Pi}_{*,A}^{\text{int}}$. Now over the ring $\widetilde{\Pi}_{*,\infty,A}$ or $\widetilde{\Pi}_{*,\infty,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the sheaf $\widetilde{\Pi}_{*,\infty,A}^{\text{int}}$.

Proposition 3.4.7. (After Kedlaya-Liu [KL16, Theorem 4.5.11]) 1. Let X be a preadic space over $E_{,a}$. Then we have that there is a fully faithful embedding of the category of the corresponding $E_{,a}^{\varphi^a} \widehat{\otimes} A$ -local systems (in the projective setting) into the category of the corresponding projective Frobenius φ^a -modules over $\widetilde{\Pi}_{*,A}$;

2. Let X be a preadic space over $E_{\infty,a}$. Then we have that there is a fully faithful embedding of the category of the corresponding $E_{\infty,a}^{\varphi^a} \widehat{\otimes} A$ -local systems (in the projective setting) into the category of the corresponding projective Frobenius φ^a -modules over $\widetilde{\Pi}_{*,A}$;

Proof. As in [KL16, Theorem 4.5.11], we consider the corresponding base change of the corresponding exact sequence in proposition 3.4.1 which reflects an exact sequence on the sheaves. \square

Remark 3.4.8. We actually did not get the corresponding v -topology involved here, which is due to the fact that the pseudocoherent modules might possibly behavior annoyingly in such topology. But on the other hand, at least one will have the corresponding

parallel statements as above over v -topology for finite projective objects.

3.4.2 The Comparison beyond the Étale Setting

In this subsection we are going to study the corresponding relationship between the corresponding sheaves over the certain Fargues-Fontaine curves and the corresponding Frobenius modules, which is beyond the corresponding consideration of just étale objects as what we considered in the previous subsection. This is to some extent in our situation important since we would like to use different point of views to study essentially equivalent objects. The following result is actually directly reduced to [KL16, Theorem 4.6.1]:

Theorem 3.4.9. ([KL16, Theorem 4.6.1])

In the setting where the space X is $\mathrm{Spa}(R, R^+)$ we have the following categories are equivalent:

1. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_*^\infty$;*
2. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_*$;*
3. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_*^I$, for some closed interval I ;*
4. *The category of all the strictly-pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_R^\infty$ such that for any closed interval I the corresponding base change of any such module M to the module over $\tilde{\Pi}_R^I$ is étale-stably pseudocoherent;*
5. *The category of all the pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_R$ such that each M of such modules comes from a base change from some strictly-pseudocoherent Frobenius φ^a -module M' over $\tilde{\Pi}_R^r$ for some radius $r > 0$ where the corresponding base change to (for any closed interval $I \subset (0, r]$) the module over $\tilde{\Pi}_R^I$ is assumed to be étale-stably pseudocoherent;*
6. *The category of all the pseudocoherent φ^a -bundles over $\tilde{\Pi}_R$;*
7. *The category of all the Frobenius φ^a -modules over $\tilde{\Pi}_R^{[s, r]}$ where we have $0 < s \leq r/p^{ha}$;*
8. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\mathrm{FF}_{X, \text{ét}}$ in the étale setting;*
9. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\mathrm{FF}_{X, \text{proét}}$ in the pro-étale setting.*

Proof. This is [KL16, Theorem 4.6.1]. We remind the readers of the corresponding functors involved. First the functors from 4 to 5 and to 7, from 4 to 1 and to 3 are base changes, and the functor from 9 to 8 is the pullback along the corresponding morphism of the sites. The functors from 9 to 7 and 6 are restrictions of the corresponding objects involved. \square

We do the first generalization to the general ultrametric field:

Theorem 3.4.10. (After Kedlaya-Liu [KL16, Theorem 4.6.1]) *In the setting where the space X is $\mathrm{Spa}(R, R^+)$ we have the following categories are equivalent:*

1. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,\infty}^\infty$;*
2. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,\infty}$;*
3. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,\infty}^I$, for some closed interval I ;*
4. *The category of all the strictly-pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,\infty}^\infty$ such that for any closed interval I the corresponding base change of any such module M to the module over $\tilde{\Pi}_{R,\infty}^I$ is étale-stably pseudocoherent;*
5. *The category of all the pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,\infty}$ such that each M of such modules comes from a base change from some strictly-pseudocoherent Frobenius φ^a -module M' over $\tilde{\Pi}_{R,\infty}^r$ for some radius $r > 0$ where the corresponding base change to (for any closed interval $I \subset (0, r]$) the module over $\tilde{\Pi}_{R,\infty}^I$ is assumed to be étale-stably pseudocoherent;*
6. *The category of all the pseudocoherent φ^a -bundles over $\tilde{\Pi}_{R,\infty}$;*
7. *The category of all the Frobenius φ^a -modules over $\tilde{\Pi}_{R,\infty}^{[s,r]}$ where we have $0 < s \leq r/p^{ha}$;*
8. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\mathrm{FF}_{X,\infty,\acute{e}t}$ in the étale setting;*
9. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\mathrm{FF}_{X,\infty,\mathrm{pro}\acute{e}t}$ in the pro-étale setting.*

Proof. The functors are the same as in the previous theorem. The proof is actually encoded into the corresponding proof of the following theorem. \square

We are then going to focus on the deformed version of the corresponding correspondences established above:

Theorem 3.4.11. (After Kedlaya-Liu [KL16, Theorem 4.6.1]) *In the setting where the space X is $\mathrm{Spa}(R, R^+)$ we have the following first group of categories are equivalent:*

1. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A}^\infty$;*
2. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A}$;*
3. *The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A}^I$, for some closed interval I .*

Then we have the second group of categories which are equivalent:

4. *The category of all the strictly-pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,A}^\infty$ such that for any closed interval I the corresponding base change of any such module M to the module over $\tilde{\Pi}_{R,A}^I$ is étale-stably pseudocoherent;*
5. *The category of all the pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,A}$ such that each M of such modules comes from a base change from some strictly-pseudocoherent Frobenius φ^a -module M' over $\tilde{\Pi}_{R,A}^r$ for some radius $r > 0$ where the corresponding base change to (for any closed interval $I \subset (0, r)$) the module over $\tilde{\Pi}_{R,A}^I$ is assumed to be étale-stably pseudocoherent;*
6. *The category of all the pseudocoherent φ^a -bundles over $\tilde{\Pi}_{R,A}$;*
7. *The category of all the Frobenius φ^a -modules over $\tilde{\Pi}_{R,A}^{[s,r]}$ where we have $0 < s \leq r/p^{ha}$;*
8. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\mathrm{FF}_{X,A,\acute{e}t}$ in the étale setting;*
9. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\mathrm{FF}_{X,A,\mathrm{pro}\acute{e}t}$ in the pro-étale setting.*

Proof. First the functors are the same as in the proof of the previous theorem. For the proof, we can follow the idea of the proof of [KL16, Theorem 4.6.1] to do so. First the equivalences between 9,8 and 6 could be proved by considering proposition 3.3.13. By considering the corresponding Frobenius action, one can establish the corresponding equivalences among them and further 7. Now as in the proof of [KL16, Theorem 4.6.1], we can now prove the equivalence between 4 and 6 as in the following. First as in the proof of [KL16, Theorem 4.6.1] one considers the exact base change from the key period ring in 4 to the corresponding key period ring in 6, which implies that the corresponding base change functor is then fully faithful. To show it is also essentially surjective, we pick now an arbitrary Frobenius bundle in the category 6, and consider the correspond-

ing reified quasi-Stein covering of the total space by spaces taking the general form as $\mathrm{Spa}(\Pi_{R,A}^{[sp^{nah},rp^{nah}]}, \Pi_{R,A}^{[sp^{nah},rp^{nah}],\mathrm{Gr}})$ with well-located radii r and s . This will imply that the corresponding global section of the Frobenius bundle is finitely generated after applying the [KL16, Proposition 2.7.16] since the application of Frobenius action will imply the uniform bound of the numbers of generators over each such covering subspace as in [KL16, Theorem 4.6.1]. Then having shown this we can as in [KL16, Theorem 4.6.1] extract the corresponding desired property of being pseudocoherent for the corresponding desired object in the category 4 by considering the kernel of some finite free covering of the Frobenius bundle. Then as in [KL16, Theorem 4.6.1] one can show that we have now the categories 4,5 and 6 are equivalent, and furthermore in the same way one may prove that the categories 1,2 and 3 are equivalent. \square

Remark 3.4.12. We did not manage to compare among then the two groups of categories within this theorem, but we believe this could be achieved by further careful analysis. Please note that the corresponding sheaves in the categories 1,2,3 are actually required to be étale-stably pseudocoherent over the ring A as well.

Now over smooth Fréchet-Stein algebra $A_\infty(G)$ (as in our previous paper) in the commutative setting where each $A_n(G)$ in the family for each $n \geq 0$ is smooth reduced affinoid algebra in rigid analytic space, we have the following comparison:

Theorem 3.4.13. (After Kedlaya-Liu [KL16, Theorem 4.6.1]) *In the setting where the space X is $\mathrm{Spa}(R, R^+)$ and the field E is of mixed-characteristic we have the following categories are equivalent:*

1. *The category of all the families of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A_\infty(G)}^\infty$;*
2. *The category of all the families of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A_\infty(G)}$;*
3. *The category of all the families of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A_\infty(G)}^I$, for some closed interval I .*

Then we have the second group of categories which are equivalent:

4. *The category of all the families of all the strictly-pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)}^\infty$ such that for any closed interval I the corresponding base change of any such module M to the module over $\tilde{\Pi}_{R,A_\infty(G)}^I$ is étale-stably pseudocoherent;*

5. The category of all the families of all the pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)}$ such that each M of such modules comes from a base change from some strictly-pseudocoherent Frobenius φ^a -module M' over $\tilde{\Pi}_{R,A_\infty(G)}^r$ for some radius $r > 0$ where the corresponding base change to (for any closed interval $I \subset (0, r]$) the module over $\tilde{\Pi}_{R,A_\infty(G)}^I$ is assumed to be étale-stably pseudocoherent;
6. The category of all the families of all the pseudocoherent φ^a -bundles over $\tilde{\Pi}_{R,A_\infty(G)}$;
7. The category of all the families of all the Frobenius φ^a -modules over $\tilde{\Pi}_{R,A_\infty(G)}^{[s,r]}$ where we have $0 < s \leq r/p^{ha}$;
8. The category of all the families of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\text{FF}_{X,A_\infty(G),\text{ét}}$ in the étale setting;
9. The category of all the families of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\text{FF}_{X,A_\infty(G),\text{proét}}$ in the pro-étale setting.

Proof. This is the direct consequence of the previous theorem if one considers the corresponding systems of all the objects defined over each $A_n(G)$. \square

We now work over a perfectoid field R and drop the assumption on the affinoid algebras we imposed before:

Theorem 3.4.14. (After Kedlaya-Liu [KL16, Theorem 4.6.1]) *In the setting where the space X is $\text{Spa}(R, R^+)$ where R is further assumed to be an analytic field, and we let A be a general reduced affinoid algebra, then we have the following categories are equivalent:*

1. The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A}^\infty$;
2. The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A}$;
3. The category of all the pseudocoherent Frobenius φ^a -modules over the sheaf $\tilde{\Pi}_{*,A}^I$, for some closed interval I .

Then we have the second group of categories which are equivalent:

4. The category of all the strictly-pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,A}^\infty$ such that for any closed interval I the corresponding base change of any such module M to the module over $\tilde{\Pi}_{R,A}^I$ is étale-stably pseudocoherent;
5. The category of all the pseudocoherent Frobenius φ^a -modules over $\tilde{\Pi}_{R,A}$ such that each M of such modules comes from a base change from some strictly-pseudocoherent Frobenius φ^a -module M' over $\tilde{\Pi}_{R,A}^r$ for some radius $r > 0$ where the corresponding base change to (for any closed interval $I \subset (0, r]$) the module over $\tilde{\Pi}_{R,A}^I$ is assumed to be étale-stably

pseudocoherent;

6. *The category of all the pseudocoherent φ^a -bundles over $\tilde{\Pi}_{R,A}$;*
7. *The category of all the Frobenius φ^a -modules over $\tilde{\Pi}_{R,A}^{[s,r]}$ where we have $0 < s \leq r/p^{ha}$;*
8. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\text{FF}_{X,A,\acute{e}t}$ in the étale setting;*
9. *The category of all the pseudocoherent sheaves over the adic version Fargues-Fontaine curve $\text{FF}_{X,A,\text{pro}\acute{e}t}$ in the pro-étale setting.*

Proof. See the proof of the previous theorem. □

3.4.3 The Comparison with the Schematic Fargues-Fontaine Curve

In this section we are going to study the corresponding relationship between the Frobenius modules over period rings and the certain sheaves over the deformed version of schematic Fargues-Fontaine curves. We actually discussed in our previous paper the corresponding results in the setting where E is just \mathbb{Q}_p . Now we consider the more general setting and the corresponding story in the equal characteristic. First we consider the following key lemmas which are analogs in our context of [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9] and [T1, Proposition 2.11]:

Lemma 3.4.15. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) (And also see [T1, Proposition 2.11]) *For any Frobenius φ^a -bundle M over $\tilde{\Pi}_{R,A}$, then we have that for any interval $I = [s, r]$ where $0 < s \leq r$ the map $\varphi^a - 1 : M_{[s,rq]} \rightarrow M_{[s,r]}$ is surjective after taking some Frobenius twist as in [KL15, 6.2.2] and our previous work namely the new morphism $\varphi^a - 1 : M(n)_{[s,rq]} \rightarrow M(n)_{[s,r]}$ for sufficiently large $n \geq 1$. As in the previous established version one may have the chance to take the number to be 1 if the bundle initially comes from the corresponding base change from the integral Robba ring.*

Proof. See the proof of [T1, Proposition 2.11]. □

Lemma 3.4.16. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) (And also see [T1, Proposition 2.11]) *For any finitely generated bundle M carrying the Frobenius action over $\tilde{\Pi}_{R,A}$, then we have that for any interval $I = [s, r]$ where $0 < s \leq r$ the map $\varphi^a - 1 : M_{[s,rq]} \rightarrow M_{[s,r]}$ is surjective after taking some Frobenius twist as in [KL15, 6.2.2] and our previous work namely the new morphism $\varphi^a - 1 : M(n)_{[s,rq]} \rightarrow M(n)_{[s,r]}$ for sufficiently large $n \geq 1$. As in the previous established version one may have the chance to take the number to be 1 if the bundle initially comes from the corresponding base change from the integral Robba ring.*

Proof. See the proof of [T1, Proposition 2.11]. □

Lemma 3.4.17. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) (And also see [T1, Proposition 2.12]) *For any Frobenius φ^a -bundle M over $\tilde{\Pi}_{R,A}$, then we have that for sufficiently large number $n \geq 0$ the module M could be generated by finitely many Frobenius φ^a -invariant elements of the global sections of $M(n)$.*

Proof. See the proof of [T1, Proposition 2.12]. □

Lemma 3.4.18. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) (And also see [T1, Proposition 2.12, Proposition 2.11]) *For any finitely generated bundle M carrying Frobenius action over $\tilde{\Pi}_{R,A}$, then we have that for sufficiently large number $n \geq 0$ the module M could be generated by finitely many Frobenius φ^a -invariant elements of the global sections of $M(n)$.*

Proof. See the proof of [T1, Proposition 2.12, Proposition 2.11]. □

Lemma 3.4.19. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) (And also see [T1, Proposition 2.12, Proposition 2.11, Proposition 2.14]) *For any Frobenius φ^a -module M over $\tilde{\Pi}_{R,A}$, then we have that for sufficiently large number $n \geq 1$ the space $H_{\varphi^a}^0(M(n))$ generates M and the space $H_{\varphi^a}^1(M(n))$ vanishes.*

Proof. See the proof of [T1, Proposition 2.12, Proposition 2.11, Proposition 2.14]. □

Lemma 3.4.20. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) (And also see [T1, Proposition 2.12, Proposition 2.11, Proposition 2.14]) *For any finitely generated module M carrying Frobenius action over $\tilde{\Pi}_{R,A}$, then we have that for sufficiently large number $n \geq 1$ the space $H_{\varphi^a}^0(M(n))$ generates M and the space $H_{\varphi^a}^1(M(n))$ vanishes.*

Proof. See the proof of [T1, Proposition 2.12, Proposition 2.11, Proposition 2.14]. \square

Then we have the following key corollary which is analog of [KL15, Corollary 6.2.3, Lemma 6.3.3], [KL16, Corollary 4.6.10] and [T1, Corollary 2.13, Corollary 3.4, Proposition 3.9]:

Corollary 3.4.21. *I. When M_α, M, M_β are three Frobenius φ^a -bundles over the ring $\tilde{\Pi}_{R,A}$ then we have that for sufficiently large $n \geq 0$ we have the following exact sequence:*

$$0 \longrightarrow M_{\alpha,I}(n)^{\varphi^a} \longrightarrow M_I(n)^{\varphi^a} \longrightarrow M_{\beta,I}(n)^{\varphi^a} \longrightarrow 0,$$

for each interval I (same holds true for finite objects).

II. When M_α, M, M_β are three Frobenius φ^a -modules over the ring $\tilde{\Pi}_{R,A}$ then we have that for sufficiently large $n \geq 0$ we have the following exact sequence:

$$0 \longrightarrow M_{\alpha,I}(n)^{\varphi^a} \longrightarrow M_I(n)^{\varphi^a} \longrightarrow M_{\beta,I}(n)^{\varphi^a} \longrightarrow 0,$$

for each interval I (same holds true for finite objects).

III. For a Frobenius φ^a -bundle M over $\tilde{\Pi}_{R,A}$, and for each module M_I over some $\tilde{\Pi}_{R,A}^I$, we put for any element f such that $\varphi^a f = p^d f$:

$$M_{I,f} := \bigcup_{n \in \mathbb{Z}} f^{-n} M_I(dn)^{\varphi^a}.$$

Then with this convention suppose we have three Frobenius φ^a -bundles taking the form of M_α, M, M_β over $\tilde{\Pi}_{R,A}$, then for each closed interval I we have the following is an exact sequence:

$$0 \longrightarrow M_{\alpha,I,f} \longrightarrow M_{I,f} \longrightarrow M_{\beta,I,f} \longrightarrow 0,$$

where each module in the exact sequence is now a module over $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$ (same holds true for finite objects).

IV. For a Frobenius φ^a -module M over $\tilde{\Pi}_{R,A}$, we put for any element f such that $\varphi^a f = p^d f$:

$$M_f := \bigcup_{n \in \mathbb{Z}} f^{-n} M(dn)^{\varphi^a}.$$

Then with this convention suppose we have three Frobenius φ^a -modules taking the form of M_α, M, M_β over $\tilde{\Pi}_{R,A}$, then we have the following is an exact sequence:

$$0 \longrightarrow M_{\alpha,f} \longrightarrow M_f \longrightarrow M_{\beta,f} \longrightarrow 0,$$

where each module in the exact sequence is now a module over $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$ (same holds true for finite objects).

V. Suppose M is now a pseudocoherent Frobenius φ^a -module over $\tilde{\Pi}_{R,A}$, then we have that the corresponding module M_f then is also pseudocoherent over the ring $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$. For a Frobenius φ^a -bundle M over $\tilde{\Pi}_{R,A}$, and for each module M_I over some $\tilde{\Pi}_{R,A}^I$, we put for any element f such that $\varphi^a f = p^d f$:

$$M_{I,f} := \bigcup_{n \in \mathbb{Z}} f^{-n} M_I(dn)^{\varphi^a}.$$

Then with this convention suppose we have three Frobenius φ^a -bundles taking the form of M_α, M, M_β over $\tilde{\Pi}_{R,A}$, then for each closed interval I we have the following is an exact sequence:

$$0 \longrightarrow M_{\alpha,I,f} \longrightarrow M_{I,f} \longrightarrow M_{\beta,I,f} \longrightarrow 0,$$

where each module in the exact sequence is now in our situation a pseudocoherent module over $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$.

Proof. See the proof of [KL15, Corollary 6.2.3, Lemma 6.3.3], [KL16, Corollary 4.6.10] and [T1, Corollary 2.13, Corollary 3.4, Proposition 3.9]. \square

Then we recall the following construction both from our previous consideration and the corresponding consideration in [KL15, Definition 6.3.1] and [KL16, Definition 4.6.11]:

Setting 3.4.22. Now we consider the graded commutative ring $P_{R,A}$ which is constructed from the ring $\tilde{\Pi}_{R,A}^+$, or $\tilde{\Pi}_{R,A}$, or $\tilde{\Pi}_{R,A}^\infty$ by taking each π^d -eigenfunction of the operator φ^a , where d corresponds to the degree of each element in $P_{d,R,A}$. Then for each element f in this graded commutative ring, one considers the affine charts taking the form of $\text{Spec} P_{R,A}[1/f]_0$ which further glues to a projective scheme which is denoted by $\text{Proj} P_{R,A}$ and called the schematic deformed version of the schematic Fargues-Fontaine curve.

And we have the following analog of [KL15] and the corresponding construction in [T1, Setting 3.3]:

Setting 3.4.23. Starting from any φ^a -bundle M over the ring $\tilde{\Pi}_{R,A}$, one has by the construction in the previous corollary the corresponding module $M_{I,f}$ over the ring $P_{R,A}[1/f]_0$, which then by considering the natural base change construction gives rise to the map:

$$M_{I,f} \otimes_{P_{R,A}[1/f]_0} \tilde{\Pi}_{R,A}^I[1/f] \rightarrow M_{I,f} \otimes_{\tilde{\Pi}_{R,A}^I} \tilde{\Pi}_{R,A}^I[1/f]$$

for each closed interval I .

Proposition 3.4.24. (After Kedlaya-Liu [KL15, Theorem 6.3.9]) *The map defined in the previous setting is a bijection and the corresponding module $M_{I,f}$ for each specific interval I is then in our situation projective of finite type.*

Proof. See [T1, Proposition 3.6]. □

Proposition 3.4.25. (After Kedlaya-Liu [KL15, Theorem 6.3.12]) *We have the following categories are equivalent:*

- I. *The category of all the φ^a -bundles over $\tilde{\Pi}_{R,A}$;*
- II. *The category of all the φ^a -modules over $\tilde{\Pi}_{R,A}$;*
- III. *The category of all the φ^a -modules over $\tilde{\Pi}_{R,A}^\infty$;*
- IV. *The category of all the quasicoherent finite locally free sheaves over the scheme $\text{Proj}P_{R,A}$.*

Proof. Based on the previous proposition, we only need to recall the corresponding functor involved. Starting from an object in the category IV one locally considers the corresponding pullbacks along the natural map from the localized scheme attached to the key ring in the category III, which globally glues to a well-defined object in the corresponding module with Frobenius structure in the category III, by applying the corresponding analog of [KL15, Lemma 6.3.7]. Then the natural base change functor sends the resulting object to some well-defined object in the category II. Finally we associate the corresponding φ^a -bundle to derive the corresponding object in the first category I. □

Proposition 3.4.26. (After Kedlaya-Liu [KL16, Theorem 4.6.12]) *The natural pullback functor from the Fargues-Fontaine curve (which is schematic) to the scheme associated to the Robba ring $\tilde{\Pi}_{R,A}^\infty$ establishes an equivalence between the category of all the pseudocoherent sheaves over the deformed version of the schematic Fargues-Fontaine curve (which is schematic) and the category of all the pseudocoherent modules over $\tilde{\Pi}_{R,A}$ with isomorphisms established by the Frobenius pullbacks.*

Proof. As in [KL16, Theorem 4.6.12] and our previous paper [T1, Proposition 3.10], we start from an object V in the category of the pseudocoherent sheaves over the schematic Fargues-Fontaine curve, which gives rise to the following exact sequence:

$$0 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow V \longrightarrow 0,$$

where the sheaf V_1 is projective of finite type. Then the corresponding functor mentioned in the statement of the proposition establishes then the following exact sequence:

$$W_2 \longrightarrow W_1 \longrightarrow W \longrightarrow 0,$$

in the category of all the pseudocoherent modules over the Robba ring with isomorphisms established by the Frobenius pullbacks, where then W_1 is finite projective and W_2 is finite generated. Then as in [KL16, Theorem 4.6.12] and [T1, Proposition 3.10] we consider the kernel K of the map $W_2 \rightarrow \text{Ker}(W_1 \rightarrow W)$, which gives rise to the following exact sequence:

$$0 \longrightarrow K \longrightarrow W_2 \longrightarrow W_1 \longrightarrow W \longrightarrow 0,$$

which then gives rise to the following exact sequence by corollary 3.4.21:

$$0 \longrightarrow K_f \longrightarrow W_{2,f} \longrightarrow W_{1,f} \longrightarrow W_f \longrightarrow 0,$$

over the ring $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$. Then by taking the corresponding section over the ring $P_{R,A}[1/f]_0$ we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_2 & \Big|_{P_{R,A}[1/f]_0} & \longrightarrow & V_1 & \Big|_{P_{R,A}[1/f]_0} & \longrightarrow & V & \Big|_{P_{R,A}[1/f]_0} & \longrightarrow & 0 \\ & & \downarrow & & & \downarrow & & & \downarrow & & & \\ 0 & \longrightarrow & K_f & \longrightarrow & W_{2,f} & \longrightarrow & W_{1,f} & \longrightarrow & W_f & \longrightarrow & 0. \end{array}$$

Now the middle vertical arrow is isomorphism which shows that the right vertical arrow is surjective. Then apply the corresponding construction to the situation where we replace V by V_2 we will have the parallel result which imply that the first vertical arrow is also surjective. Now by five lemma we have that the rightmost vertical arrow is injective. Then by applying the corresponding consideration to the situation where we replace V by V_2 we could derive the fact that actually the first vertical arrow is also injective. Up

to here, we have that the corresponding module K_f is zero which implies by considering corollary 3.4.21 K itself is zero. Then by the previous results on the corresponding finite projective objects we have that functor in our situation sends the pseudocoherent objects to pseudocoherent objects. And from the argument above we have that the functor is exact. Then to finish the composition of the functors from category of Frobenius modules to the sheaves and back from sheaves to the category of Frobenius modules is an equivalence. And the other direction could be directly deduced from the results on the vector bundles. \square

3.5 Deformation of Imperfect Period Rings

3.5.1 Key Rings

We now consider the corresponding imperfectization of the corresponding constructions we considered above, after [KL16]. The construction in [KL16, Chapter 5] is actually quite general, which is sort of direct imperfection of the corresponding perfect period rings and modules off the tower. We will consider the corresponding towers in [KL16, Chapter 5], so we recall the setting as in the following:

Assumption 3.5.1. In this section, we are going to assume that k is just \mathbb{F}_{p^h} .

Setting 3.5.2. Recall from [KL16, Chapter 5], we have the following setting up. The base will be a Banach adic ring (H, H^+) over \mathfrak{o}_E which is assumed to be uniform and carrying the corresponding spectral norm α . Then we consider the tower:

$$\dots \longrightarrow H_{n-1} \longrightarrow H_n \longrightarrow H_{n+1} \longrightarrow \dots,$$

where each map linking two adjacent rings is the corresponding morphism of the corresponding Banach uniform adic rings whose corresponding induced map on the adic spaces is surjective as in the corresponding construction in [KL16, Definition 5.1.1] with the corresponding spectral norm α_n . The infinite level of the tower will be denoted by H_∞ . This is not actually necessarily complete under the multiplicative extension of all the finite level spectral norms α_n namely α_∞ , so we need to consider the completed ring \overline{H}_∞ . In the situation where the tower is Fontaine perfectoid, we have that following [KL16, Definition 5.1.1] that this is called the corresponding perfectoid tower, which gives rise the equal characteristic counterpart \overline{H}'_∞ with the spectral norm $\overline{\alpha}_\infty$ correspondingly under the perfectoid correspondence in the sense of [KL16, Theorem 3.3.8]. Recall that from [KL16, Definition 5.1.1] the tower is called finite étale if each transition map is finite étale.

Now we describe the corresponding imperfect period rings which we will deform. These rings are those introduced in [KL16, Definition 5.2.1] by using series of imperfection processes. Recall in more detail we have:

Setting 3.5.3. Fix a perfectoid tower (H_\bullet, H_\bullet^+) Recall from [KL16, Definition 5.2.1] we have the following different imperfect constructions:

- A. First we have the ring $\overline{H'}_\infty$, which could give us the corresponding ring $\widetilde{\Omega}_{\overline{H'}_\infty}^{\text{int}}$;
- B. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^{\text{int},r}$ coming from $\overline{H'}_\infty$;
- C. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^{\text{int}}$ coming from $\overline{H'}_\infty$;
- D. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^{\text{bd},r}$ coming from $\overline{H'}_\infty$;
- E. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^{\text{bd}}$ coming from $\overline{H'}_\infty$;
- F. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^r$ coming from $\overline{H'}_\infty$;
- G. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^{[s,r]}$ coming from $\overline{H'}_\infty$;
- H. We then have the ring $\widetilde{\Pi}_{\overline{H'}_\infty}$ coming from $\overline{H'}_\infty$;
- I. $\Pi_H^{\text{int},r}$ comes from the ring $\widetilde{\Pi}_{\overline{H'}_\infty}^{\text{int},r}$ consisting of those elements of $\widetilde{\Pi}_{\overline{H'}_\infty}^{\text{int},r}$ with the requirement that whenever we have n an integer such that $nh > -\log_p r$ we have then $\theta(\varphi^{-n}(x)) \in H_n$;
- J. $\Pi_H^{\text{int},\dagger}$ is defined to be the corresponding union of the rings in I;
- K. Ω_H^{int} is defined to be the corresponding period ring coming from the corresponding π -adic completion of the ring $\Pi_H^{\text{int},\dagger}$ in J;
- L. $\check{\Omega}_H^{\text{int}}$ is the ring which is defined to be the union of all the $\varphi^{-n}\Omega_H^{\text{int}}$;
- M. $\check{\Pi}_H^{\text{int},r}$ is then the ring which is defined to be the union of all the $\varphi^{-n}\Pi_H^{\text{int},p^{hn}r}$;
- N. $\check{\Pi}_H^{\text{int},\dagger}$ is define to be union of all the $\varphi^{-n}\check{\Pi}_H^{\text{int},\dagger}$;
- O. $\widehat{\Omega}_H^{\text{int}}$ is defined to be the π -completion of $\check{\Omega}_H^{\text{int}}$;
- P. $\widehat{\Pi}_H^{\text{int},r}$ is defined to be the $\max\{\|\cdot\|_{\overline{\alpha}_\infty^r}, \|\cdot\|_{\pi\text{-adic}}\}$ -completion of $\check{\Pi}_H^{\text{int},r}$;
- Q. We then have $\widehat{\Pi}_H^{\text{int},\dagger}$ by taking the union over $r > 0$;
- R. Correspondingly we have $\check{\Omega}_H, \check{\Pi}_H^{\text{bd},r}, \check{\Pi}_H^{\text{bd},\dagger}$ by inverting the element π ;
- S. Correspondingly we also have $\widehat{\Omega}_H, \widehat{\Pi}_H^{\text{bd},r}, \widehat{\Pi}_H^{\text{bd},\dagger}$ by inverting the corresponding element π ;
- T. We also have $\Omega_H, \Pi_H^{\text{bd},r}, \Pi_H^{\text{bd},\dagger}$ again by inverting the element π ;
- U. Taking the $\max\{\|\cdot\|_{\overline{\alpha}_\infty^s}, \|\cdot\|_{\overline{\alpha}_\infty^r}\}$ (for $0 < s \leq r$) completion of the ring $\Pi_H^{\text{bd},r}$ we have the ring $\Pi_H^{[s,r]}$, while taking the Fréchet completion with respect to the norm $\|\cdot\|_{\overline{\alpha}_\infty^t}$ for each $0 < t \leq r$ we have the ring Π_H^r ;
- V. Taking the union we have the ring Π_H ;
- W. We use the notation $\check{\Pi}_H$ to denote the corresponding union of all the $\varphi^{-n}\Pi_H$;
- X. We use the notation $\check{\Pi}_H^{[s,r]}$ to be the corresponding union of all the $\varphi^{-n}\Pi_H^{[p^{-hn}s, p^{-hn}r]}$;
- Y. We use the notation $\check{\Pi}_H^r$ to be the union of all the $\varphi^{-n}\check{\Pi}_H^{p^{-hn}r}$.

Then we have the following direct analog of the relative version of the ring defined

above (here as before the ring A denotes a reduced affinoid algebra):

Setting 3.5.4. Now we consider the deformation of the rings above:

I. We have the first group of the period rings in the deformed setting:

$$\Pi_{H,A}^{\text{int},r}, \Pi_{H,A}^{\text{int},\dagger}, \Omega_{H,A}^{\text{int}}, \Omega_{H,A}, \Pi_{H,A}^{\text{bd},r}, \Pi_{H,A}^{\text{bd},\dagger}, \Pi_{H,A}^{[s,r]}, \Pi_{H,A}^r, \Pi_{H,A}. \quad (3.5.1)$$

II. We also have the second group of desired rings in the desired setting:

$$\check{\Pi}_{H,A}^{\text{int},r}, \check{\Pi}_{H,A}^{\text{int},\dagger}, \check{\Omega}_{H,A}^{\text{int}}, \check{\Omega}_{H,A}, \check{\Pi}_{H,A}^{\text{bd},r}, \check{\Pi}_{H,A}^{\text{bd},\dagger}, \check{\Pi}_{H,A}^{[s,r]}, \check{\Pi}_{H,A}^r, \check{\Pi}_{H,A}. \quad (3.5.2)$$

III. We also have the third group:

$$\widehat{\Pi}_{H,A}^{\text{int},r}, \widehat{\Pi}_{H,A}^{\text{int},\dagger}, \widehat{\Omega}_{H,A}^{\text{int}}, \widehat{\Omega}_{H,A}, \widehat{\Pi}_{H,A}^{\text{bd},r}, \widehat{\Pi}_{H,A}^{\text{bd},\dagger}. \quad (3.5.3)$$

Then we have the following globalization as what we did before in the perfect setting:

Definition 3.5.5. Now consider the sheaf $\mathcal{O}_{\mathfrak{X}}$ of a rigid analytic space \mathfrak{X} in rigid analytic geometry. We have the following three groups of sheaves of rings over \mathfrak{X} :

$$\Pi_{H,\mathfrak{X}}^{[s,r]}, \Pi_{H,\mathfrak{X}}^r, \Pi_{H,\mathfrak{X}}, \quad (3.5.4)$$

$$\check{\Pi}_{H,\mathfrak{X}}^{[s,r]}, \check{\Pi}_{H,\mathfrak{X}}^r, \check{\Pi}_{H,\mathfrak{X}} \quad (3.5.5)$$

to be:

$$\Pi_{H,\mathcal{O}_{\mathfrak{X}}}^{[s,r]}, \Pi_{H,\mathcal{O}_{\mathfrak{X}}}^r, \Pi_{H,\mathcal{O}_{\mathfrak{X}}}, \quad (3.5.6)$$

$$\check{\Pi}_{H,\mathcal{O}_{\mathfrak{X}}}^{[s,r]}, \check{\Pi}_{H,\mathcal{O}_{\mathfrak{X}}}^r, \check{\Pi}_{H,\mathcal{O}_{\mathfrak{X}}}. \quad (3.5.7)$$

Now we discuss some properties of the corresponding deformed version of the imperfect rings in our context, which is parallel to the corresponding discussion we made in the perfect setting and generalizing the corresponding discussion in [KL16]:

Proposition 3.5.6. (After Kedlaya-Liu [KL16, Lemma 5.2.10]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\Pi_{H,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{\text{int},r_1} \bigcap \Pi_{H,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{[r_1,r_2]} = \Pi_{H,\mathbb{Q}_p\{T_1,\dots,T_d\}}^{\text{int},r_2}.$$

Proof. We adapt the argument in [KL16, Lemma 5.2.10] to prove this in the situation where $r_1 < r_2$ (otherwise this is trivial), again one direction is easy where we only present the implication in the other direction. We take any element

$$x \in \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1} \cap \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r_2]}$$

and take suitable approximating elements $\{x_i\}$ living in the bounded Robba ring such that for any $j \geq 1$ one can find some integer $N_j \geq 1$ we have whenever $i \geq N_j$ we have the following estimate:

$$\|\cdot\|_{\bar{\alpha}_\infty^t, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x_i - x) \leq p^{-j}, \forall t \in [r_1, r_2].$$

Then we consider the corresponding decomposition of x_i for each $i = 1, 2, \dots$ into a form having integral part and the rational part $x_i = y_i + z_i$ by setting

$$y_i = \sum_{k=0, i_1, \dots, i_d} \pi^k x_{i, k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}$$

out of

$$x_i = \sum_{k=n(x_i), i_1, \dots, i_d} \pi^k x_{i, k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}.$$

Note that by our initial hypothesis we have that the element x lives in the ring

$$\Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1}$$

which further implies that

$$\|\cdot\|_{\bar{\alpha}_\infty^{r_1}, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{i, k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}) \leq p^{-j}.$$

Therefore we have $\bar{\alpha}_\infty(\bar{x}_{i, k, i_1, \dots, i_d}) \leq p^{(k-j)/r_1}, \forall k < 0$ directly from this through computation, which implies that then:

$$\|\cdot\|_{\bar{\alpha}_\infty^{r_2}, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{i, k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}) \leq p^{-k} p^{(k-j)r_2/r_1} \quad (3.5.8)$$

$$\leq p^{1+(1-j)r_1/r_1}. \quad (3.5.9)$$

Then one can read off the result directly from this estimate since under this estimate we can have the chance to modify the original approximating sequence $\{x_i\}$ by $\{y_i\}$ which are initially chosen to be in the integral Robba ring, which implies that actually the element x lives in the right-hand side of the identity in the statement of the proposition. \square

Proposition 3.5.7. (After Kedlaya-Liu [KL16, Lemma 5.2.10]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\Pi_{H,A}^{\text{int},r_1} \cap \Pi_{H,A}^{[r_1,r_2]} = \Pi_{H,A}^{\text{int},r_2}.$$

Here A is some reduced affinoid algebra over \mathbb{Q}_p .

Proof. This is actually not a direct corollary from the corresponding result as in the previous proposition. But since the map from the corresponding Tate algebra to A is strict, which will remain to be so when tensor (completely) with the corresponding Robba rings in the previous proposition (see [BGR84, 2.1.8, Proposition 6]). Then for any element \bar{x} in the intersection on the left hand side, one can consider the corresponding construction to any lift x of \bar{x} with the corresponding approximating estimate by some sequence:

$$\|\cdot\|_{\bar{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x_i - x) \leq p^{-j}, \forall t \in [r_1, r_2],$$

for any $i \geq N_j$ with some N_j when j is arbitrarily chosen. Then we to each x_i we associate y_i and z_i as in the proof of the previous proposition. Also for x we have that it is living in the integral Robba ring which implies that we have:

$$\|\cdot\|_{\bar{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{i,k,i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}) \leq p^{-j}.$$

Therefore we have $\bar{\alpha}_\infty(\bar{x}_{i,k,i_1, \dots, i_d}) \leq p^{(k-j)/r_1}, \forall k < 0$ directly from this through computation, which implies that then:

$$\|\cdot\|_{\bar{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{i,k,i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}) \leq p^{-k} p^{(k-j)r_2/r_1} \tag{3.5.10}$$

$$\leq p^{1+(1-j)r_1/r_1}. \tag{3.5.11}$$

This shows that actually one can rearrange the corresponding lifting to be some lifting with respect to the ring on the right hand side, which will finish the proof then by the strictness again after the corresponding projection. \square

Proposition 3.5.8. (After Kedlaya-Liu [KL15, Lemma 5.2.8] and [KL16]) *Consider now in our situation the radii $0 < r_1 \leq r_2$, and consider any element $x \in \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r_2]}$. Then we have that for each $n \geq 1$ one can decompose x into the form of $x = y + z$ such that $y \in \pi^n \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$ with $z \in \bigcap_{r \geq r_2} \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[r_1, r]}$ with the following estimate for each $r \geq r_2$:*

$$\|\cdot\|_{\bar{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(z) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\bar{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(z)^{r/r_2}.$$

Proof. As in [KL15, Lemma 5.2.8] and [KL16] and in the proof of our previous proposition we first consider those elements x lives in the bounded Robba rings which could be expressed in general as

$$\sum_{k=n(x), i_1, \dots, i_d} \pi^k x_{k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}.$$

In this situation the corresponding decomposition is very easy to come up with, namely we consider the corresponding y_i as the corresponding series:

$$\sum_{k \geq n, i_1, \dots, i_d} \pi^k x_{k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}$$

which give us the desired result since we have in this situation when focusing on each single term:

$$\|\cdot\|_{\overline{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d}) = p^{-k} \overline{\alpha}_\infty(\overline{x}_{k, i_1, \dots, i_d})^r \quad (3.5.12)$$

$$= p^{-k(1-r/r_2)} \|\cdot\|_{\overline{\alpha}_\infty^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d})^{r/r_2} \quad (3.5.13)$$

$$\leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\overline{\alpha}_\infty^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(\pi^k x_{k, i_1, \dots, i_d} T_1^{i_1} \dots T_d^{i_d})^{r/r_2} \quad (3.5.14)$$

for all those suitable k . Then to tackle the more general situation we consider the approximating sequence consisting of all the elements in the bounded Robba ring as in the usual situation considered in [KL15, Lemma 5.2.8] and [KL16], namely we inductively construct the following approximating sequence just as:

$$\|\cdot\|_{\overline{\alpha}_\infty^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x - x_0 - \dots - x_i) \leq p^{-i-1} \|\cdot\|_{\overline{\alpha}_\infty^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x), i = 0, 1, \dots, r \in [r_1, r_2]. \quad (3.5.15)$$

Here all the elements x_i for each $i = 0, 1, \dots$ are living in the bounded Robba ring, which immediately gives rise to the suitable decomposition as proved in the previous case namely we have for each i the decomposition $x_i = y_i + z_i$ with the desired conditions as mentioned in the statement of the proposition. We first take the series summing all the elements y_i up for all $i = 0, 1, \dots$, which first of all converges under the norm $\|\cdot\|_{\overline{\alpha}_\infty, \mathbb{Q}_p\{T_1, \dots, T_d\}}$ for all the radius $r \in [r_1, r_2]$, and also note that all the elements y_i within the infinite sum live inside the corresponding integral Robba ring $\Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$, which further implies the corresponding convergence ends up in $\Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$. For the elements z_i where $i = 0, 1, \dots$ also sum up to a converging series in the desired ring since combining all the estimates above we have:

$$\|\cdot\|_{\overline{\alpha}_\infty^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(z_i) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\overline{\alpha}_\infty^r, \mathbb{Q}_p\{T_1, \dots, T_d\}}(x)^{r/r_2}.$$

□

Proposition 3.5.9. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_1, r_1]} \cap \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_2, r_2]} = \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_1, r_2]},$$

here the radii satisfy $s_1 < s_2 \leq r_1 \leq r_2$.

Proof. In our situation one direction is obvious while on the other hand we consider any element x in the intersection on the left, then by the previous proposition we have the decomposition $x = y + z$ where $y \in \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1}$ and $z \in \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_1, r_2]}$. Then as in [KL15, Lemma 5.2.10] section 5.2 we look at $y = x - z$ which lives in the intersection:

$$\Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_1} \cap \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{[s_2, r_2]} = \Pi_{H, \mathbb{Q}_p\{T_1, \dots, T_d\}}^{\text{int}, r_2}$$

which finishes the proof. □

Proposition 3.5.10. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\Pi_{H, A}^{[s_1, r_1]} \cap \Pi_{H, A}^{[s_2, r_2]} = \Pi_{H, A}^{[s_1, r_2]},$$

here the radii satisfy $s_1 < s_2 \leq r_1 \leq r_2$.

Proof. See the proof of proposition 3.5.7. □

Remark 3.5.11. Again one can follow the same strategy to deal with the corresponding equal-characteristic situation.

3.5.2 Modules and Bundles

Now we consider the modules and bundles over the rings introduced in the previous subsection. First we make the following assumption:

Setting 3.5.12. Recall that from [KL16, Definition 5.2.3] any tower (H_\bullet, H_\bullet^+) is called weakly decompleting if we have that first the density of the perfection of H_∞ in \overline{H}_∞ . Here the ring H_∞ is the ring coming from the mod- π construction of the ring Ω_H^{int} , also at the same time one can find some $r > 0$ such that the corresponding modulo π operation from the ring Ω_H^{int} to the ring H_∞ is actually surjective strictly.

Assumption 3.5.13. We now assume that we are basically in the situation where (H_\bullet, H_\bullet^+) is actually weakly decompleting. Also as in [KL16, Lemma 5.2.7] we assume we fix some radius $r_0 > 0$, for instance this will correspond to the corresponding index in the situation we have the corresponding noetherian tower. Recall that a tower is called noetherian if we have some specific radius as above such that we have the strongly noetherian property on the ring $\Pi_H^{[s,r]}$ with $[s, r] \subset (0, r_0]$. Under this condition we have that actually we have the corresponding strongly noetherian property on the ring $\Pi_{H,A}^{[s,r]}$ with $[s, r] \subset (0, r_0]$, therefore consequently we have that the ring $(\Pi_{H,A}^{[s,r]}, \Pi_{H,A}^{[s,r],+})$ is sheafy. Here the ring $\Pi_{H,A}^{[s,r],+}$ is defined by taking the product construction between A and the ring $\Pi_H^{[s,r],+}$ which is defined in [KL16, Definition 5.3.2]. We now assume that the tower is then noetherian. For more examples, see [KL16, 5.3.3].

Then we can start to discuss the corresponding modules over the rings in our deformed setting, first as in [KL16, Lemma 5.3.3] the following result should be derived from our construction:

Lemma 3.5.14. (After Kedlaya-Liu [KL16, Lemma 5.3.3]) *We have the following strict isomorphism in our setting, where the corresponding notations are as in [KL16, Lemma 5.3.3]:*

$$\Pi_{H,A}^{[s,r]} \{X/p^{-t}\} / (\pi X - 1) \rightarrow \Pi_{H,A}^{[t,r]} \quad (3.5.16)$$

$$\Pi_{H,A}^{[s,r]} \{X/p^{-t}\} / (X - \pi) \rightarrow \Pi_{H,A}^{[s,t]} \quad (3.5.17)$$

$$\Pi_{H,A}^r \{X/p^{-s}\} / (X - \pi) \rightarrow \Pi_{H,A}^s \quad (3.5.18)$$

$$\Pi_{H,A}^r \{X/p^{-s}\} / (\pi X - 1) \rightarrow \Pi_{H,A}^{[s,r]}, \quad (3.5.19)$$

where the corresponding radii satisfy $0 < s \leq t \leq r \leq r_0$.

Proof. See [KL16, Lemma 5.3.3]. □

We then as in [KL16, Lemma 5.3.4] have the following:

Lemma 3.5.15. (After Kedlaya-Liu [KL16, Lemma 5.3.4])

We have the 2-pseudoflatness of the following maps:

$$\Pi_{H,A}^{[r_1,r_2]} \rightarrow \Pi_{H,A}^{[r_1,t]}, \Pi_{H,A}^{[r_1,r_2]} \rightarrow \Pi_{H,A}^{[t,r_2]} \quad (3.5.20)$$

where we have $0 < r_1 \leq t \leq r_2$.

Proof. See [KL16, Lemma 5.3.4]. □

Remark 3.5.16. In effect, one can make more strong statement here, due to the fact that we are working in the noetherian setting. To be more precise see the corresponding result (and the proof) of [KL16, Corollary 2.6.9], one can actually have even the flatness of the corresponding maps in the previous lemma as long as we are working over noetherian rings.

Then we have the following implication:

Proposition 3.5.17. (After Kedlaya-Liu [KL16, Proposition 5.3.5]) *Suppose that we have a module M over the ring $\Pi_{H,A}^I$ (which is assumed to be stably pseudocoherent) where the closed interval I admits a covering by finite many closed intervals $I = \cup_i I_i$. Then we have in our situation the following exact augmented Čech complex:*

$$0 \rightarrow M \rightarrow \bigoplus_i M_i \rightarrow \dots,$$

where the module M_i is defined to be the base change of the module M to the ring $\Pi_{H,A}^{I_i}$ with respect to each i . Also we have that the morphism $\Pi_{H,A}^I \rightarrow \bigoplus_i \Pi_{H,A}^{I_i}$ is in our situation effective descent with respect to the corresponding categories of étale-stably pseudocoherent modules in the Banach setting and with respect to the corresponding categories of finite projective modules in the Banach setting.

Proof. As in [KL16, Proposition 5.3.5] we only have to look at the situation of two intervals. Then as in [KL16, Proposition 5.3.5] by using the previous lemma one can argue as in [KL16, Proposition 5.3.5]. □

Then we deform the basic notation of bundles in [KL16, Definition 5.3.6]:

Definition 3.5.18. (After Kedlaya-Liu [KL16, Definition 5.3.6]) We define the bundle over the ring $\Pi_{H,A}^{r_0}$ to be a collection $(M_I)_I$ of finite projective modules over each $\Pi_{H,A}^I$ with $I \subset (0, r_0]$ closed subintervals of $(0, r_0]$ such that we have the following requirement in the glueing fashion. First for any $I_1 \subset I_2$ two closed intervals we have $M_{I_2} \otimes_{\Pi_{H,A}^{I_2}} \Pi_{H,A}^{I_1} \xrightarrow{\sim} M_{I_1}$ with the obvious cocycle condition with respect to three closed subintervals of $(0, r_0]$ namely taking the form of $I_1 \subset I_2 \subset I_3$.

We define the pseudocoherent sheaf over the ring $\Pi_{H,A}^{r_0}$ to be a collection $(M_I)_I$ of

étale-stably pseudocoherent modules over each $\Pi_{H,A}^I$ with $I \subset (0, r_0]$ closed subintervals of $(0, r_0]$ such that we have the following requirement in the glueing fashion. First for any $I_1 \subset I_2$ two closed intervals we have $M_{I_2} \otimes_{\Pi_{H,A}^{I_2}} \Pi_{H,A}^{I_1} \xrightarrow{\sim} M_{I_1}$ with the obvious cocycle condition with respect to three closed subintervals of $(0, r_0]$ namely taking the form of $I_1 \subset I_2 \subset I_3$.

We make the following discussion around the corresponding module and sheaf structures defined above.

Lemma 3.5.19. (After Kedlaya-Liu [KL16, Lemma 5.3.8]) *We have the isomorphism between the ring $\Pi_{H,A}^r$ and the inverse limit of the ring $\Pi_{H,A}^{[s,r]}$ with respect to the radius s by the map $\Pi_{H,A}^r \rightarrow \Pi_{H,A}^{[s,r]}$.*

Proof. As in [KL16, Lemma 5.3.8] it is injective by the isometry, and then use the corresponding elements $x_n, n = 0, 1, \dots$ in the dense ring $\Pi_{H,A}^r$ to approximate any element x in the ring $\Pi_{H,A}^{[s,r]}$ in the same way as in [KL16, Lemma 5.3.8]:

$$\|\cdot\|_{\bar{\alpha}_{\infty,A}^t}(x - x_n) \leq p^n$$

for any radius t now living in the corresponding interval $[r2^{-n}, r]$. This will establish Cauchy sequence which finishes the proof as in [KL16, Lemma 5.3.8]. \square

Proposition 3.5.20. (After Kedlaya-Liu [KL16, Lemma 5.3.9]) *For some radius $r \in (0, r_0]$. Suppose we have that M is a vector bundle in the general setting or M is a pseudocoherent sheaf in the setting where the tower is noetherian. Then we have that the corresponding global section is actually dense in each section with respect to some closed interval. And then we have the corresponding vanishing result of the first derived inverse limit functor.*

Proof. See [KL16, Lemma 5.3.9]. \square

The interesting issue here as in [KL16] is the corresponding finitely generatedness of the global section of a pseudocoherent sheaf which is actually not guaranteed in general. Therefore as in [KL16] we have to distinguish the corresponding well-behaved sheaves out from the corresponding category of all the corresponding pseudocoherent sheaves.

Proposition 3.5.21. (After Kedlaya-Liu [KL16, Lemma 5.3.10]) *As in the previous proposition we choose some $r \in (0, r_0]$. Now assume that the corresponding tower (H_\bullet, H_\bullet^+) is noetherian. Now for any pseudocoherent sheaf M defined above we have the following three statements are equivalent. A. The first statement is that one can find a sequence of positive integers x_1, x_2, \dots such that for any closed interval living inside $(0, r]$ the section of the sheaf with respect this closed interval admits a projective resolution of modules with corresponding ranks bounded by the sequence of integer x_1, x_2, \dots . B. The second statement is that for any locally finite covering of the corresponding interval $(0, r]$ which takes the corresponding form of $\{I_i\}$ one can find a sequence of positive integers x_1, x_2, \dots such that for any closed interval living inside $\{I_i\}$ the section of the sheaf with respect this closed interval admits a projective resolution of modules with corresponding ranks bounded by the sequence of integer x_1, x_2, \dots . C. Lastly the third statement is that the corresponding global section is a pseudocoherent module over the ring $\Pi_{H,A}^r$.*

Proof. See [KL16, Lemma 5.3.10]. □

As in [KL16, Definition 5.3.11] we call the sheaf satisfying the corresponding equivalent conditions in the proposition above uniform pseudocoherent sheaf. Then we have the following analog of [KL16, Lemma 5.3.12]:

Proposition 3.5.22. (After Kedlaya-Liu [KL16, Lemma 5.3.12]) *The global section functor defines the corresponding equivalence between the categories of the following two sorts of objects. The first ones are the corresponding uniform pseudocoherent sheaves over $\Pi_{H,A}^r$. The second ones are those pseudocoherent modules over the ring $\Pi_{H,A}^r$.*

Proof. See [KL16, Lemma 5.3.12]. □

3.5.3 Frobenius Structure and Γ -Structure on Hodge-Iwasawa Modules

Now we consider the corresponding Frobenius actions over the corresponding imperfect rings we defined before, note that the corresponding Frobenius actions are induced from the corresponding imperfect rings in the undeformed situation from [KL16] which is to say that the Frobenius action on the ring A is actually trivial.

First we consider the corresponding Frobenius modules:

Definition 3.5.23. (After Kedlaya-Liu [KL16, Definition 5.4.2]) Over the period rings $\Pi_{H,A}$ or $\check{\Pi}_{H,A}$ (which is denoted by Δ in this definition) we define the corresponding φ^a -modules over Δ which are respectively projective, pseudocoherent or fpd to be the corresponding finite projective, pseudocoherent or fpd modules over Δ with further assigned semilinear action of the operator φ^a . We also require that the modules are complete for the natural topology involved in our situation and for any module over $\Pi_{H,A}$ to be some base change of some module over $\Pi_{H,A}^r$ (which will be defined in the following). We also require that the modules are complete for the natural topology involved in our situation and any module over $\check{\Pi}_{H,A}$ to be some base change of some module over $\check{\Pi}_{H,A}^r$ (which will be defined in the following).

Definition 3.5.24. (After Kedlaya-Liu [KL16, Definition 5.4.2]) Over each rings $\Delta = \Pi_{H,A}^r, \check{\Pi}_{H,A}^r$ we define the corresponding projective, pseudocoherent or fpd φ^a -module over any Δ to be the corresponding finite projective, pseudocoherent or fpd module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*}M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \Pi_{H,A}^{r/p}, \check{\Pi}_{H,A}^{r/p}$. Also as in [KL16, Definition 5.4.2] we assume that the module over $\Pi_{H,A}^r$ is then complete for the natural topology and the corresponding base change to $\Pi_{H,A}^I$ for any interval which is assumed to be closed $I \subset [0, r)$ gives rise to a module over $\Pi_{H,A}^I$ with specified conditions which will be specified below. Also as in [KL16, Definition 5.4.2] we assume that the module over $\check{\Pi}_{H,A}^r$ is then complete for the natural topology and the corresponding base change to $\check{\Pi}_{H,A}^I$ for any interval which is assumed to be closed $I \subset [0, r)$ gives rise to a module over $\check{\Pi}_{H,A}^I$ with specified conditions which will be specified below.

Definition 3.5.25. (After Kedlaya-Liu [KL16, Definition 5.4.2]) Again as in [KL16, Definition 5.4.2], we define the corresponding projective, pseudocoherent and fpd φ^a -modules over ring $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ to be the finite projective, pseudocoherent and fpd modules (which will be denoted by M) over $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ respectively additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*}M \otimes_{\Pi_{H,A}^{[sp^{-ah}, rp^{-ah}]}} \Pi_{H,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\Pi_{H,A}^{[s,r]}} \Pi_{H,A}^{[s, rp^{-ah}]} \quad (3.5.21)$$

and

$$\varphi^{a*} M \otimes_{\check{\Pi}_{R,A}^{[sp^{-ah}, rp^{-ah}]}} \check{\Pi}_{R,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\check{\Pi}_{R,A}^{[s,r]}} \check{\Pi}_{R,A}^{[s, rp^{-ah}]}.$$

$$(3.5.23)$$

We now assume that the modules are complete with respect to the natural topology and étale stably pseudocoherent.

Also one can further define the corresponding bundles carrying semilinear Frobenius in our context as in the situation of [KL16, Definition 5.4.10]:

Definition 3.5.26. (After Kedlaya-Liu [KL16, Definition 5.4.10]) Over the ring $\Pi_{H,A}^r$ or $\check{\Pi}_{H,A}^r$ we define a corresponding projective, pseudocoherent and fpd Frobenius bundle to be a family $(M_I)_I$ of finite projective, étale stably pseudocoherent and étale stably fpd modules over each $\Pi_{H,A}^I$ or $\check{\Pi}_{H,A}^I$ respectively carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\Pi_{H,A}^J} \Pi_{H,A}^I \xrightarrow{\sim} M_I$$

and

$$M_J \otimes_{\check{\Pi}_{H,A}^J} \check{\Pi}_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, u]$ involved we have $s \leq u/p^{ah}$. Then as in [KL16, Definition 5.4.10], we can consider the corresponding direct 2-limit to achieve the corresponding objects in the categories over full Robba rings.

We can then compare the corresponding objects defined above:

Proposition 3.5.27. (After Kedlaya-Liu [KL16, Lemma 5.4.11])

I. Consider the following objects for some radius r_0 in our situation. The first group of objects are those finite projective φ^a -modules over the Robba ring $\Pi_{H,A}^{r_0}$. The second group of objects are those finite projective φ^a -bundles over the Robba ring $\Pi_{H,A}^{r_0}$. The third group of objects are those finite projective φ^a -modules over the Robba ring $\Pi_{H,A}^{[s,r]}$ for some $[s,r] \in (0, r_0)$. Then we have that the corresponding categories of the three groups of objects are equivalent.

II. Consider the following objects for some radius r_0 in our situation. The first group of objects are those pseudocoherent φ^a -modules over the Robba ring $\Pi_{H,A}^{r_0}$. The second group of objects are those pseudocoherent φ^a -bundles over the Robba ring $\Pi_{H,A}^{r_0}$. The third group of objects are those pseudocoherent φ^a -modules over the Robba ring $\Pi_{H,A}^{[s,r]}$ for some $[s,r] \in (0,r_0)$. Then we have that the corresponding categories of the three groups of objects are equivalent.

III. Consider the following objects for some radius r_0 in our situation. The first group of objects are those finite projective dimensional φ^a -modules over the Robba ring $\Pi_{H,A}^{r_0}$. The second group of objects are those finite projective dimensional φ^a -bundles over the Robba ring $\Pi_{H,A}^{r_0}$. The third group of objects are those finite projective dimensional φ^a -modules over the Robba ring $\Pi_{H,A}^{[s,r]}$ for some $[s,r] \in (0,r_0)$. Then we have that the corresponding categories of the three groups of objects are equivalent.

Proof. See the proof of [KL16, Lemma 5.4.11]. In our situation we need to use the corresponding proposition 3.5.17 to prove the corresponding equivalences between the corresponding categories of bundles in the second groups and modules in the third groups. The corresponding equivalences between the corresponding categories of bundles in the second groups and modules in the first groups are following [KL16, Lemma 5.4.11] after applying the Frobenius pullbacks to compare those sections over different intervals, and note that we have the corresponding uniform pseudocoherent objects in our development above which actually restricts further to the finite projective objects as in [KL16, Proposition 2.7.16]. \square

Now we define the corresponding Γ -modules over the period rings attached to the tower (H_\bullet, H_\bullet^+) . The corresponding structures are actually abstractly defined in the same way as in [KL16]. First we consider the deformation of the corresponding complex $*_{H_\bullet}$ for any ring $*$ in setting 3.5.4.

Assumption 3.5.28. Recall that the corresponding tower is called decompleting if it is weakly decompleting, finite étale on each finite level and having the exact sequence $\bar{\varphi}^{-1}H'_{H_\bullet}/H'_{H_\bullet}$ is exact. We now assume that the tower (H_\bullet, H_\bullet^+) is then decompleting.

Setting 3.5.29. Assume now Γ is a topological group as in [KL16, Definition 5.5.5] acting on the corresponding period rings in the setting 3.5.3 in the original context of

setting 3.5.3. Then we consider the corresponding induced continuous action over the corresponding deformed version in our context namely in setting 3.5.4. Assume now that the tower is Galois with the corresponding Galois group Γ .

Definition 3.5.30. We now consider the corresponding inhomogeneous continuous cocycles of the group Γ , as in [KL16, Definition 5.5.5] we use the following notation to denote the corresponding complex extracted from a single tower for a given period ring $*_{H,A}$ in setting 3.5.4 for each $k > 0$:

$$*_{H^k,A} := C_{\text{con}}(\Gamma^k, *_{H}) \widehat{\otimes}_{\#} ?$$

where $\# = \mathbb{Q}_p, \mathbb{Z}_p$ and $? = A, \mathfrak{o}_A$ respectively, which forms the corresponding complex $(*_{H^\bullet,A}, d^\bullet)$ with the corresponding differential as in [KL16, Definition 5.5.5] in the sense of continuous group cohomology.

Definition 3.5.31. Having established the corresponding meaning of the Γ -structure we now consider the corresponding definition of Γ -modules. Such modules called the corresponding Γ -modules are defined over the corresponding rings in setting 3.5.4. Again we allow the corresponding modules to be finite projective, or pseudocoherent or fpd over the rings in setting 3.5.4. And the modules are defined to be carrying the corresponding continuous semilinear action from the group Γ .

Proposition 3.5.32. (After Kedlaya-Liu [KL16, Corollary 5.6.5])

The complex $\varphi^{-1} \Pi_{H_{\geq n}^\bullet, A}^{[sp^{-h}, rp^{-h}]} / \Pi_{H_{\geq n}^\bullet, A}^{[s,r]}$ and the complex $\widetilde{\Pi}_{H_{\geq n}^\bullet, A}^{[s,r]} / \Pi_{H_{\geq n}^\bullet, A}^{[s,r]}$ are strict exact for any truncation index n . The corresponding radii satisfy the corresponding relation $0 < s \leq r \leq r_0$.

Proof. See [KL16, Corollary 5.6.5], and consider the corresponding Schauder basis of A . □

Proposition 3.5.33. (After Kedlaya-Liu [KL16, Lemma 5.6.6]) *The complex*

$$M \otimes_{\Pi_{H,A}^{[s,r]}} \varphi^{-(\ell+1)} \Pi_{H^\bullet, A}^{[sp^{-h(\ell+1)}, rp^{-h(\ell+1)}]} / \varphi^{-\ell} \Pi_{H^\bullet, A}^{[sp^{-h\ell}, rp^{-h\ell}]}$$

and the complex

$$M \otimes_{\Pi_{H,A}^{[s,r]}} \widetilde{\Pi}_{H^\bullet, A}^{[s,r]} / \varphi^{-\ell} \Pi_{H^\bullet, A}^{[sp^{-h(\ell)}, rp^{-h(\ell)}]}$$

are strict exact for any truncation index n . The corresponding radii satisfy the corresponding relation $0 < s \leq r \leq r_0$, and ℓ is bigger than some existing truncated integer $\ell_0 \geq 0$. Here M is any Γ -module in our context.

Proof. See [KL16, Lemma 5.6.6]. □

Proposition 3.5.34. (After Kedlaya-Liu [KL16, Lemma 5.6.9]) *With the corresponding notations as above we have that the corresponding base change from the ring taking the form of $\check{\Pi}_{H,A}^{[s,r]}$ to the ring taking the form of $\tilde{\Pi}_{H,A}^{[s,r]}$ establishes the corresponding equivalence on the corresponding categories of Γ -modules.*

Proof. This is a relative version of the corresponding result in [KL16, Lemma 5.6.9] we adapt the corresponding argument here. Indeed the corresponding fully faithfulness comes from the previous proposition the idea to prove the corresponding essential surjectivity comes from writing the module M (the corresponding differential) over the ring taking form of $\tilde{\Pi}_{H,A}^{[s,r]}$ as the base change from $\varphi^{-k}(\Pi_{H,A}^{[s,r]})$ of a module M_0 (the corresponding differential) after the corresponding [KL16, Lemma 5.6.8]. Then as in [KL16] we consider the corresponding norms on the corresponding M and the base change of M_0 which could be controlled up to some constant which could be further modified to be zero by reducing each time positive amount of constant from the constant represented by the difference of the norms. □

Definition 3.5.35. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the period rings $\Pi_{H,A}$ or $\check{\Pi}_{H,A}$ (which is denoted by Δ in this definition) we define the corresponding (φ^a, Γ) -modules over Δ which are respectively projective, pseudocoherent or fpd to be the corresponding finite projective, pseudocoherent or fpd Γ -modules over Δ with further assigned semilinear action of the operator φ^a with the isomorphism defined by using the Frobenius. We also require that the modules are complete for the natural topology involved in our situation and for any module over $\Pi_{H,A}$ to be some base change of some module over $\Pi_{H,A}^r$ (which will be defined in the following).

Definition 3.5.36. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over each rings $\Delta = \Pi_{H,A}^r, \check{\Pi}_{H,A}^r$ we define the corresponding projective, pseudocoherent or fpd (φ^a, Γ) -module over any Δ to be the corresponding finite projective, pseudocoherent or fpd Γ -module M over Δ with additionally endowed semilinear Frobenius action from φ^a such

that we have the isomorphism $\varphi^{a*} M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \Pi_{H,A}^{r/p}, \check{\Pi}_{H,A}^{r/p}$. Also as in [KL16, Definition 5.7.2] we assume that the module over $\Pi_{H,A}^r$ is then complete for the natural topology and the corresponding base change to $\Pi_{H,A}^I$ for any interval which is assumed to be closed $I \subset [0, r)$ gives rise to a module over $\Pi_{H,A}^I$ with specified conditions which will be specified below.

Definition 3.5.37. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Again as in [KL16, Definition 5.7.2], we define the corresponding projective, pseudocoherent and fpd (φ^a, Γ) -modules over ring $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ to be the finite projective, pseudocoherent and fpd Γ -modules (which will be denoted by M) over $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ respectively additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\Pi_{H,A}^{[sp^{-ah}, rp^{-ah}]}} \Pi_{H,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\Pi_{H,A}^{[s,r]}} \Pi_{H,A}^{[s, rp^{-ah}]} \quad (3.5.24)$$

and

$$\varphi^{a*} M \otimes_{\check{\Pi}_{R,A}^{[sp^{-ah}, rp^{-ah}]}} \check{\Pi}_{R,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\check{\Pi}_{R,A}^{[s,r]}} \check{\Pi}_{R,A}^{[s, rp^{-ah}]} \quad (3.5.25)$$

$$(3.5.26)$$

We also require the corresponding topological conditions as we considered in the Frobenius module situation.

Definition 3.5.38. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the ring $\Pi_{H,A}^r$ or $\check{\Pi}_{H,A}^r$ we define a corresponding projective, pseudocoherent and fpd (φ^a, Γ) bundle to be a family $(M_I)_I$ of finite projective, pseudocoherent and fpd Γ -modules over each $\Pi_{H,A}^I$ or $\check{\Pi}_{H,A}^I$ respectively carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\Pi_{H,A}^J} \Pi_{H,A}^I \xrightarrow{\sim} M_I$$

and

$$M_J \otimes_{\check{\Pi}_{H,A}^J} \check{\Pi}_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [v, u]$ involved we have $v \leq u/p^{ah}$. We put the corresponding topological conditions as before when we consider the corresponding Frobenius bundles. And one can take the corresponding 2-limit in the direct sense to define the corresponding objects over the full Robba rings.

Definition 3.5.39. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the period rings $\tilde{\Pi}_{H,A}$ (which is denoted by Δ in this definition) we define the corresponding (φ^a, Γ) -modules over Δ which are respectively projective, pseudocoherent or fpd to be the corresponding finite projective, pseudocoherent or fpd Γ -modules over Δ with further assigned semilinear action of the operator φ^a with the isomorphism defined by using the Frobenius. We also require that the modules are complete for the natural topology involved in our situation and for any module over $\tilde{\Pi}_{H,A}$ to be some base change of some module over $\tilde{\Pi}_{H,A}^r$ (which will be defined in the following).

Definition 3.5.40. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over each ring $\Delta = \tilde{\Pi}_{H,A}^r$ we define the corresponding projective, pseudocoherent or fpd (φ^a, Γ) -module over any Δ to be the corresponding finite projective, pseudocoherent or fpd Γ -module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*} M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \tilde{\Pi}_{H,A}^{r/p}$. Also as in [KL16, Definition 5.7.2] we assume that the module over $\tilde{\Pi}_{H,A}^r$ is then complete for the natural topology and the corresponding base change to $\tilde{\Pi}_{H,A}^I$ for any interval which is assumed to be closed $I \subset [0, r)$ gives rise to a module over $\tilde{\Pi}_{H,A}^I$ with specified conditions which will be specified below.

Definition 3.5.41. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Again as in [KL16, Definition 5.7.2], we define the corresponding projective, pseudocoherent and fpd (φ^a, Γ) -modules over ring $\tilde{\Pi}_{H,A}^{[s,r]}$ to be the finite projective, pseudocoherent and fpd Γ -modules (which will be denoted by M) over $\tilde{\Pi}_{H,A}^{[s,r]}$ additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\tilde{\Pi}_{H,A}^{[sp-ah, rp-ah]}} \tilde{\Pi}_{H,A}^{[s, rp-ah]} \xrightarrow{\sim} M \otimes_{\tilde{\Pi}_{H,A}^{[s,r]}} \tilde{\Pi}_{H,A}^{[s, rp-ah]}. \quad (3.5.27)$$

We also require the corresponding topological conditions as we considered in the Frobenius module situation.

Definition 3.5.42. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the ring $\tilde{\Pi}_{H,A}^r$ we define a corresponding projective, pseudocoherent and fpd (φ^a, Γ) bundle to be a family $(M_I)_I$ of finite projective, pseudocoherent and fpd Γ -modules over each $\tilde{\Pi}_{H,A}^I$ carrying the natural Frobenius action coming from the operator φ^a such that for any two

involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\tilde{\Pi}_{H,A}^J} \tilde{\Pi}_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [v, u]$ involved we have $v \leq u/p^{ah}$. We put the corresponding topological conditions as before when we consider the corresponding Frobenius bundles. And one can take the corresponding 2-limit in the direct sense to define the corresponding objects over the full Robba rings.

Remark 3.5.43. In the following proposition, we assume that the corresponding ring $\tilde{\Pi}_{\overline{H}'_\infty, A}^{[s,r]}$ is sheafy.

Proposition 3.5.44. (After Kedlaya-Liu [KL16, Theorem 5.7.5]) *We have now the following categories are equivalence for the corresponding radii $0 < s \leq r \leq r_0$ (with the further requirement as in [KL16, Theorem 5.7.5] that $s \in (0, r/q]$):*

1. *The category of all the finite projective sheaves over the ring $\tilde{\Pi}_{\text{Spa}(H_0, H_0^+), A}$, carrying the φ^a -action;*
2. *The category of all the finite projective sheaves over the ring $\tilde{\Pi}_{\text{Spa}(H_0, H_0^+), A}^r$, carrying the φ^a -action;*
3. *The category of all the finite projective sheaves over the ring $\tilde{\Pi}_{\text{Spa}(H_0, H_0^+), A}^{[s,r]}$, carrying the φ^a -action.*

Then we have the second group of categories which are equivalent:

4. *The category of all the finite projective quasi-coherent sheaves over corresponding adic Fargues-Fontaine curve in the deformed setting $\text{FF}_{\overline{H}'_\infty, A}$, carrying the corresponding action from the group Γ which is assumed to be semilinear and continuous over each section over any affinoid subspace of the whole space which is assumed to be Γ -invariant;*
5. *The category of all the finite projective modules over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
6. *The category of all the finite projective bundles over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
7. *The category of all the finite projective modules over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*
8. *The category of all the finite projective bundles over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*

9. The category of all the finite projective modules over the ring $\check{\Pi}_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action;
10. The category of all the finite projective modules over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;
11. The category of all the finite projective bundles over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;
12. The category of all the finite projective modules over the ring $\tilde{\Pi}_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action.

Proof. The corresponding comparisons on the sheaves and bundles in 1-4 and 10-12 are derived in the corresponding context in the perfect setting as what we did in the previous sections. The rest ones are proved exactly the same as [KL16, Theorem 5.7.5] by using our development. \square

Remark 3.5.45. This proposition generalizes the corresponding results in [KP] including the situation therein considered by Chojecki-Gaisin, while note that we have also included the situation in the equal characteristic situation.

Furthermore if one considers the corresponding context where the ring H'_H is further assumed to be F -(finite projective) then we can discuss the level of pseudocoherent objects.

Lemma 3.5.46. (After Kedlaya-Liu [KL16, Lemma 5.8.7]) *For any radii in our situation namely $0 < s \leq r \leq r_0$ we have the following isomorphism:*

$$\tilde{\Pi}_{H,A}^{[s,r]} \xrightarrow{\sim} \Pi_{H,A}^{[s,r]} \oplus \left(\bigoplus_{\ell=0}^{\infty} \varphi^{-(\ell+1)} \Pi_{H,A}^{[sp^{-h(\ell+1)}, rp^{-h(\ell+1)}]} / \varphi^{-\ell} \Pi_{H,A}^{[sp^{-h\ell}, rp^{-h\ell}]} \right)^\wedge.$$

Proof. This is by considering the Schauder basis of A . See [KL16, Lemma 5.8.7]. \square

Corollary 3.5.47.

(After Kedlaya-Liu [KL16, Corollary 5.8.8, Corollary 5.8.11]) *The map for the radii as in the previous lemma is 2-pseudoflat:*

$$\Pi_{H,A}^{[s,r]} \rightarrow \tilde{\Pi}_{H,A}^{[s,r]}.$$

Also we have that the corresponding base change along this map will preserve the corresponding étale stably pseudocoherence.

Lemma 3.5.48. (After Kedlaya-Liu [KL16, Lemma 5.8.14]) *The complex*

$$M \otimes_{\Pi_{H,A}^{[s,r]}} \varphi^{-(\ell+1)} \Pi_{H^\bullet, A}^{[sp^{-h(\ell+1)}, rp^{-h(\ell+1)}]} / \varphi^{-\ell} \Pi_{H^\bullet, A}^{[sp^{-h\ell}, rp^{-h\ell}]}$$

and the complex

$$M \otimes_{\Pi_{H,A}^{[s,r]}} \widetilde{\Pi}_{H^\bullet, A}^{[s,r]} / \varphi^{-\ell} \Pi_{H^\bullet, A}^{[sp^{-h(\ell)}, rp^{-h(\ell)}]}$$

are strict exact for any truncation index n . The corresponding radii satisfy the corresponding relation $0 < s \leq r \leq r_0$, and ℓ is bigger than some existing truncated integer $\ell_0 \geq 0$. Here M is any pseudocoherent Γ -module in our context.

Proposition 3.5.49. (After Kedlaya-Liu [KL16, Lemma 5.8.17]) *The corresponding base change along the following map is fully faithful for the corresponding pseudocoherent modules carrying the corresponding structure of Γ -action:*

$$\check{\Pi}_{H,A}^{[s,r]} \rightarrow \widetilde{\Pi}_{H,A}^{[s,r]}.$$

The image in the essential sense consists of those modules descending to the corresponding ring in the domain of this map when forgetting the corresponding Γ -action.

Proof. See [KL16, Lemma 5.8.17]. □

Proposition 3.5.50. (After Kedlaya-Liu [KL16, Lemma 5.9.2]) *Keep the notations as above. For any finitely generated in general module over the ring $\widetilde{\Pi}_{H,A}^{[s,r]}$ carrying the action of Γ , one can find another module over $\check{\Pi}_{H,A}^{[s,r]}$ which is now pseudocoherent carrying the action of Γ which covers the previous module through a surjective map after taking the corresponding base change to the ring $\widetilde{\Pi}_{H,A}^{[s,r]}$.*

Proof. As in the proof of [KL16, Lemma 5.9.2], one can prove this in the similar fashion by considering the corresponding isomorphism respecting the cocycle requirement coming from the free module $\widetilde{\Lambda}$ in the presentation of a given module $\widetilde{\Delta}$ over the perfect Robba ring:

$$\widetilde{\Lambda} \otimes_{i_{0,0}} \widetilde{\Pi}_{H^0, A}^{[s,r]} \rightarrow \widetilde{\Lambda} \otimes_{i_{0,1}} \widetilde{\Pi}_{H^1, A}^{[s,r]}$$

coming from our prescribed Γ -action on the module $\widetilde{\Delta}$ over the perfect Robba ring in the way that one considers a converging sequence of different desired lifts (as in [KL16, Lemma 5.6.9]) of

$$\Lambda \otimes_{i_{0,0}} \check{\Pi}_{H^0, A}^{[s,r]} \rightarrow \Lambda \otimes_{i_{0,1}} \check{\Pi}_{H^1, A}^{[s,r]}.$$

To get the corresponding desired covering of $\tilde{\Delta}$ one considers the corresponding finitely generated submodule of the corresponding kernel of the map from Λ to $\tilde{\Delta}$ then take the quotient of Λ through this corresponding submodule. Note that in our situation the ring $\check{\Pi}_{H,A}^{[s,r]}$ is also coherent which finishes the proof as in [KL16, Lemma 5.9.2]. \square

Proposition 3.5.51. (After Kedlaya-Liu [KL16, Theorem 5.9.4]) *We have now the following categories are equivalence for the corresponding radii $0 < s \leq r \leq r_0$ (with the further requirement as in [KL16, Theorem 5.7.5] that $s \in (0, r/q]$):*

1. *The category of all the pseudocoherent sheaves over the ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+), A}$, carrying the φ^a -action;*
2. *The category of all the pseudocoherent sheaves over the ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+), A}^r$, carrying the φ^a -action;*
3. *The category of all the pseudocoherent sheaves over the ring $\tilde{\Pi}_{\mathrm{Spa}(\overline{H}_\infty, \overline{H}_\infty^+), A}^{[s,r]}$, carrying the φ^a -action.*

Then we have the second group of categories which are equivalent:

4. *The category of all the pseudocoherent quasi-coherent sheaves over corresponding adic Fargues-Fontaine curve in the deformed setting $\mathrm{FF}_{\overline{H}_\infty^+, A}$, carrying the corresponding action from the group Γ which is assumed to be semilinear and continuous over each section over any affinoid subspace of the whole space which is assumed to be Γ -invariant;*
5. *The category of all the pseudocoherent modules over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
6. *The category of all the pseudocoherent bundles over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
7. *The category of all the pseudocoherent modules over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*
8. *The category of all the pseudocoherent bundles over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*
9. *The category of all the pseudocoherent modules over the ring $\check{\Pi}_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action;*
10. *The category of all the pseudocoherent modules over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*
11. *The category of all the pseudocoherent bundles over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*

12. The category of all the pseudocoherent modules over the ring $\tilde{\Pi}_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action.

Proof. This is by using the A -relative version of [KL16, Lemma 5.9.3] which states that actually the base change functor along

$$\check{\Pi}_{H,A}^{[s,r]} \rightarrow \tilde{\Pi}_{H,A}^{[s,r]}$$

is then not only fully faithful on the category of pseudocoherent objects as above but also essential surjective. Indeed by the previous proposition in our situation we have that a surjective covering of any module $\tilde{\Lambda}$ over $\tilde{\Pi}_{H,A}^{[s,r]}$:

$$\tilde{\Lambda}' \rightarrow \tilde{\Lambda} \rightarrow 0$$

where $\tilde{\Lambda}'$ descend to the ring $\check{\Pi}_{H,A}^{[s,r]}$ carrying Γ -action. Then by applying the same process above to the corresponding kernel of the covering above we have another exact sequence in the following form:

$$\tilde{\Lambda}'' \rightarrow \tilde{\Lambda}' \rightarrow \tilde{\Lambda} \rightarrow 0$$

Here $\tilde{\Lambda}'$ and $\tilde{\Lambda}$ are finitely presented as in [KL16, Lemma 5.9.3]. Then we can now consider the corresponding cokernel of the map $\tilde{\Lambda}'' \rightarrow \tilde{\Lambda}'$ which will present a corresponding desired object carrying Γ -action over the ring $\check{\Pi}_{H,A}^{[s,r]}$ which is again pseudocoherent in our context. \square

3.6 Organizaion of Noncommutative Setting

3.6.1 Noncommutative Period Rings in Perfect Setting

Now we assume A to be some noncommutative Banach affinoid algebra over the local fields we consider above. Examples of such rings could be coming from the corresponding context of [Soi1], or the corresponding noncommutative Fréchet-Stein algebras as in [ST1]. We will use the notation $E\{Z_1, \dots, Z_n\}$ to denote the corresponding noncommutative non-noetherian Tate algebra as in the commutative setting.

Remark 3.6.1. The noncommutative consideration is not quite new, for instance the aspects rooted in [Zah1], [Wit1] and [Wit2] (and even [KL16]), but obviously the corresponding noncommutative setting is more complicated than the corresponding commutative setting which we discussed extensively in the previous context, therefore we do not have the chance to reach all the corresponding noncommutative version of the results above.

Remark 3.6.2. We choose to closely in some parallel way present the corresponding construction in the noncommutative setting, which is parallel in some aspects to the commutative setting we presented above. We definitely won't have the chance to see the complete picture as in the commutative setting and even more consideration and effort has to be made during the corresponding development.

We first deform the corresponding constructions in [KL16, Definition 4.1.1]:

Definition 3.6.3. We first consider the corresponding deformation of the above rings over $\mathbb{Q}_p\{Z_1, \dots, Z_d\}$. We are going to use the notation $W_\pi(R)_{\mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ to denote the complete tensor product of $W_\pi(R)$ with the Tate algebra $\mathbb{Q}_p\{Z_1, \dots, Z_d\}$ consisting of all the element taking the form as:

$$\sum_{k \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} Z_2^{i_2} \dots Z_d^{i_d}$$

over which we have the Gauss norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for any $r > 0$ which is defined by:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}} \left(\sum_{k \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} Z_2^{i_2} \dots Z_d^{i_d} \right) := \sup_{k \geq 0, i_1 \geq 0, \dots, i_d \geq 0} p^{-k} \alpha(\bar{x}_{k, i_1, \dots, i_d})^r.$$

Then we define the corresponding convergent rings:

$$\tilde{\Omega}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}} = W_\pi(R)_{\mathbb{Q}_p\{Z_1, \dots, Z_d\}}, \tilde{\Omega}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}} = W_\pi(R)_{\mathbb{Q}_p\{Z_1, \dots, Z_d\}}[1/\pi].$$

Then as in the previous setting we define the corresponding $\mathbb{Q}_p\{Z_1, \dots, Z_d\}$ -relative integral Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r}$ as the completion of the ring $W_\pi(R^+)_{\mathbb{Q}_p\{Z_1, \dots, Z_d\}}[[R]]$ by using the Gauss norm defined above. Then one can take the corresponding union of all $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r}$ throughout all $r > 0$ to define the integral Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}}$. For the bounded Robba rings we just set $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}, r} = \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r}[1/\pi]$ and taking the union throughout all $r > 0$ to define the corresponding ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}}$. Then we define the corresponding Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^I$ with respect to some interval $I \subset (0, \infty)$ by taking the Fréchet completion of $W_\pi(R^+)_{\mathbb{Q}_p\{Z_1, \dots, Z_d\}}[[R]][1/\pi]$ with respect to all the norms $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for all $r \in I$ which means that the corresponding equivalence classes in the completion procedure will be simultaneously Cauchy with respect to all the norms $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for all $r \in I$. Then we take suitable intervals such as $(0, r]$ and $(0, \infty)$ to define the corresponding Robba rings $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^r$ and $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^\infty$, respectively. Then taking the union throughout all $r > 0$ one can define the corresponding Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$. Again as in [KL16] one can define the corresponding integral rings of the similar types.

Then one can define the corresponding period rings in our context deformed over some affinoid A which is isomorphic to some quotient of $\mathbb{Q}_p\{Z_1, \dots, Z_d\}$, again deforming from the context of [KL16, Definition 4.1.1].

Definition 3.6.4. In the characteristic 0, we define the following period rings:

$$\tilde{\Omega}_{R, A}^{\text{int}}, \tilde{\Omega}_{R, A}, \tilde{\Pi}_{R, A}^{\text{int}, r}, \tilde{\Pi}_{R, A}^{\text{bd}, r}, \tilde{\Pi}_{R, A}^I, \tilde{\Pi}_{R, A}^r, \tilde{\Pi}_{R, A}^\infty$$

by taking the suitable quotient of the following period rings defined above:

$$\tilde{\Omega}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}}, \tilde{\Omega}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}, \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r}, \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}, r}, \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^I, \quad (3.6.1)$$

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^r, \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^\infty \quad (3.6.2)$$

with respect to the structure of the affinoid algebra A in the sense of Tate. Therefore they carry the corresponding quotient seminorms of the above Gauss norms defined in the previous definition, note that these are not something induced from the corresponding

spectral seminorms from A chosen at the very beginning of our study. We use the notation $\overline{\|\cdot\|}_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ to denote the corresponding quotient Gauss norm which induces then the corresponding spectral seminorm $\|\cdot\|_{\alpha^r, A}$ for each $r > 0$. Then we define the corresponding period rings:

$$\tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A} \quad (3.6.3)$$

by taking suitable union throughout all $r > 0$.

Definition 3.6.5. In positive characteristic situation, when we are working over general affinoid algebra A , we use the same notations as in the previous definition, but by using $W(R)_{\mathbb{F}_p[[\eta]]\{Z_1, \dots, Z_d\}}$ and $W(R)_A$ as the starting rings, namely here A is isomorphic to a quotient of $\mathbb{F}_p((\eta))\{Z_1, \dots, Z_d\}$. Again the corresponding Tate algebra is defined in term of free variables.

Then we can define the following affinoid deformations (after [KL16, Definition 4.1.1] in the flavor as above):

Definition 3.6.6. Consider now the base change $W_{\pi, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(R)$ which is defined now to be the completed tensor product $(W_{\pi}(R) \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_{\infty}}) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p\{Z_1, \dots, Z_d\}$. Then the point is that each element in this ring admits a unique expression taking the form of

$$\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1 \geq 0, \dots, i_d \geq 0} \pi^n [\bar{x}_{n, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}$$

which allows us to perform the construction mentioned above. First we can define for some radius $r > 0$ the corresponding period ring $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r}$ by taking the completion of the ring

$$W_{\pi, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(R^+) [[R]]$$

with respect to the following Gauss type norm:

$$\|\cdot\|_{\alpha^r} \left(\sum_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1 \geq 0, \dots, i_d \geq 0} \pi^n [\bar{x}_{n, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d} \right) := \sup_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1 \geq 0, \dots, i_d \geq 0} \{p^{-n} \alpha(\bar{x}_{n, i_1, \dots, i_d})^r\}.$$

Then we can define the union $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}}$ throughout all the radius $r > 0$. Then we just define the bounded Robba ring $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}, r}$ by $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r}[1/\pi]$ and also we could define the union $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}}$ throughout all the radius $r > 0$. Then for any

interval in $(0, \infty)$ which is denoted by I we can define the corresponding Robba rings $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^I$ by taking the Fréchet completion of

$$W_{\pi, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(R^+)[[R]][1/\pi]$$

with respect to all the norms $\|\cdot\|_{\alpha^t}$ for all $t \in I$. Then by taking suitable specified intervals one can define the rings $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^r$ and $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^\infty$ as before, and finally one can define the corresponding union $\tilde{\Pi}_{R, \infty, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ throughout all the radius $r > 0$. Again we have the corresponding integral version of the rings defined over E_∞ as [KL16]. Finally over A we can define in the same way as above to deform all the rings over E_∞ , and we do not repeat the construction again.

3.6.2 Basic Properties of Period Rings

Then we do some reality checks over the investigation of the properties of the above period rings in the style taken in [KL15, Section 5.2] and [KL16].

Proposition 3.6.7. (After Kedlaya-Liu [KL15, Lemma 5.2.1]) *The function $t \mapsto \|x\|_{\alpha^t, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^r$ is continuous log convex for the corresponding variable $t \in (0, r]$ for any $r > 0$.*

Proof. Adapt the corresponding argument in the proof of 5.2.1 of [KL15, Lemma 5.2.1] to our situation we then first look at the situation where the element is just of the form of $\pi^k[\bar{x}_{k, i_1, \dots, i_d}]Z_1^{i_1} \dots Z_d^{i_d}$ where the corresponding norm function in terms of $t > 0$ is just affine. Then one focuses on the finite sums of these kind of elements which gives rise to the log convex directly. Finally by taking the approximation we get the desired result. \square

Proposition 3.6.8. (After Kedlaya-Liu [KL15, Lemma 5.2.2]) *For any element $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ we have that $x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}}$ if and only if we have the situation where x actually lives in $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^r$ (for some specific $r > 0$) such that x itself is bounded under the norm $\|\cdot\|_{\alpha^t, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for each $t \in (0, r]$.*

Proof. One direction of the proof is easy, so we only choose to present the proof of the implication in the other direction as in the original proof of 5.2.2 of [KL15, Lemma 5.2.2] as in the following. First choose some radius $r > 0$ such that the element could be assumed to be living in the ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^r$. The idea is to transfer the original question to the

question about showing the integrality of x when we add some hypothesis on the norm by taking suitable powers of p (since the norm is bounded for each $t \in (0, r]$ so we are reduced to the situation where the norm is bounded by 1). Then we argue as in [KL15, Lemma 5.2.2] to choose some approximating sequence $\{x_i\}$ living in $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}}$ of x . Therefore we have for any $j \geq 1$ one can find then some integer $N_j \geq 1$ such that for any $i \geq N_j$ we have the estimate:

$$\|\cdot\|_{\alpha^t, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x_i - x) \leq p^{-j}, \forall t \in [p^{-j}r, r].$$

Then the idea is to consider the integral decomposition of the element x_i which has the form of $\sum_{k=n(x_i), i_1 \geq 0, i_2 \geq 0, \dots, i_d \geq 0} \pi^k[\bar{x}_{i,k,i_1, \dots, i_d}]$ into the following two parts:

$$x_i := y_i + z_i \tag{3.6.4}$$

$$:= \sum_{k=0, i_1 \geq 0, i_2 \geq 0, \dots, i_d \geq 0} \pi^k[\bar{x}_{i,k,i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d} + z_i \tag{3.6.5}$$

from which we actually have the corresponding estimate over the residual part of the decomposition above:

$$\|\cdot\|_{\alpha^{p^{-j}r}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k[\bar{x}_{i,k,i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}) \leq 1, \forall k < 0$$

which implies by direct computation:

$$\alpha^{p^{-j}r}(\bar{x}_{i,k,i_1, \dots, i_d}) \leq p^k \tag{3.6.6}$$

$$\alpha(\bar{x}_{i,k,i_1, \dots, i_d}) \leq p^{kp^j/r} \tag{3.6.7}$$

which implies that we have the following estimate:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x_i - y_i) \leq p^{-k} p^{kp^j r/r} \tag{3.6.8}$$

$$\leq p^{1-p^j} \tag{3.6.9}$$

which implies that $y_i \rightarrow x$ under the norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ which furthermore under the norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ due the property of the norm, which finishes the proof of the desired result. \square

Proposition 3.6.9. (After Kedlaya-Liu [KL15, Corollary 5.2.3]) *We have the following identity:*

$$(\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}})^\times = (\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{bd}})^\times.$$

Proof. See 5.2.3 of [KL15, Corollary 5.2.3]. □

Proposition 3.6.10. (After Kedlaya-Liu [KL15, Lemma 5.2.6]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[r_1, r_2]} = \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}.$$

Proof. We adapt the argument in [KL15, Lemma 5.2.6] 5.2 to prove this in the situation where $r_1 < r_2$ (otherwise this is trivial), again one direction is easy where we only present the implication in the other direction. We take any element

$$x \in \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[r_1, r_2]}$$

and take suitable approximating elements $\{x_i\}$ living in the bounded Robba ring such that for any $j \geq 1$ one can find some integer $N_j \geq 1$ we have whenever $i \geq N_j$ we have the following estimate:

$$\|\cdot\|_{\alpha^t, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x_i - x) \leq p^{-j}, \forall t \in [r_1, r_2].$$

Then we consider the corresponding decomposition of x_i for each $i = 1, 2, \dots$ into a form having integral part and the rational part $x_i = y_i + z_i$ by setting

$$y_i = \sum_{k=0, i_1, \dots, i_d} \pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}$$

out of

$$x_i = \sum_{k=n(x_i), i_1, \dots, i_d} \pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}.$$

Note that by our initial hypothesis we have that the element x lives in the ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1}$ which further implies that

$$\|\cdot\|_{\alpha^{r_1}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}) \leq p^{-j}.$$

Therefore we have $\alpha(\bar{x}_{i, k, i_1, \dots, i_d}) \leq p^{(k-j)/r_1}$ directly from this through computation, which implies that then:

$$\|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k [\bar{x}_{i, k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}) \leq p^{-k} p^{(k-j)r_2/r_1} \tag{3.6.10}$$

$$\leq p^{1+(1-j)r_1/r_1}. \tag{3.6.11}$$

Then one can read off the result directly from this estimate since under this estimate we can have the chance to modify the original approximating sequence $\{x_i\}$ by $\{y_i\}$ which are

initially chosen to be in the integral Robba ring, which implies that actually the element x lives in the right-hand side of the identity in the statement of the proposition. \square

Proposition 3.6.11. (After Kedlaya-Liu [KL15, Lemma 5.2.6]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\tilde{\Pi}_{R,A}^{\text{int},r_1} \cap \tilde{\Pi}_{R,A}^{[r_1,r_2]} = \tilde{\Pi}_{R,A}^{\text{int},r_2}.$$

Here A is some noncommutative Banach affinoid algebra over \mathbb{Q}_p .

Proof. See the proof of proposition 3.5.7. \square

Then we have the following analog of the corresponding result of [KL15, Lemma 5.2.8]:

Proposition 3.6.12. (After Kedlaya-Liu [KL15, Lemma 5.2.8]) *Consider now in our situation the radii $0 < r_1 \leq r_2$, and consider any element $x \in \tilde{\Pi}_{R,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{[r_1,r_2]}$. Then we have that for each $n \geq 1$ one can decompose x into the form of $x = y + z$ such that $y \in \pi^n \tilde{\Pi}_{R,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{\text{int},r_2}$ with $z \in \bigcap_{r \geq r_2} \tilde{\Pi}_{R,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{[r_1,r]}$ with the following estimate for each $r \geq r_2$:*

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(z) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(z)^{r/r_2}.$$

Proof. As in [KL15, Lemma 5.2.8] and in the proof of our previous proposition we first consider those elements x lives in the bounded Robba rings which could be expressed in general as

$$\sum_{k=n(x), i_1, \dots, i_d} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}.$$

In this situation the corresponding decomposition is very easy to come up with, namely we consider the corresponding y_i as the corresponding series:

$$\sum_{k \geq n, i_1, \dots, i_d} \pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}$$

which give us the desired result since we have in this situation when focusing on each single term:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(\pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d}) = p^{-k} \alpha(\bar{x}_{k, i_1, \dots, i_d})^r \quad (3.6.12)$$

$$= p^{-k(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(\pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d})^{r/r_2} \quad (3.6.13)$$

$$\leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(\pi^k [\bar{x}_{k, i_1, \dots, i_d}] Z_1^{i_1} \dots Z_d^{i_d})^{r/r_2} \quad (3.6.14)$$

for all those suitable k . Then to tackle the more general situation we consider the approximating sequence consisting of all the elements in the bounded Robba ring as in the usual situation considered in [KL15, Lemma 5.2.8], namely we inductively construct the following approximating sequence just as:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x - x_0 - \dots - x_i) \leq p^{-i-1} \|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x), i = 0, 1, \dots, r \in [r_1, r_2]. \quad (3.6.15)$$

Here all the elements x_i for each $i = 0, 1, \dots$ are living in the bounded Robba ring, which immediately gives rise to the suitable decomposition as proved in the previous case namely we have for each i the decomposition $x_i = y_i + z_i$ with the desired conditions as mentioned in the statement of the proposition. We first take the series summing all the elements y_i up for all $i = 0, 1, \dots$, which first of all converges under the norm $\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for all the radius $r \in [r_1, r_2]$, and also note that all the elements y_i within the infinite sum live inside the corresponding integral Robba ring $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$, which further implies the corresponding convergence ends up in $\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$. For the elements z_i where $i = 0, 1, \dots$ also sum up to a converging series in the desired ring since combining all the estimates above we have:

$$\|\cdot\|_{\alpha^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(z_i) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\alpha^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x)^{r/r_2}.$$

□

Proposition 3.6.13. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_1, r_1]} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_2, r_2]} = \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_1, r_2]},$$

here the radii satisfy $s_1 \leq s_2 \leq r_1 \leq r_2$.

Proof. In our situation one direction is obvious while on the other hand we consider any element x in the intersection on the left, then by the previous proposition we have the decomposition $x = y + z$ where $y \in \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1}$ and $z \in \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_1, r_2]}$. Then as in [KL15, Lemma 5.2.10] section 5.2 we look at $y = x - z$ which lives in the intersection:

$$\tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1} \cap \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_2, r_2]} = \tilde{\Pi}_{R, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$$

which finishes the proof. □

Proposition 3.6.14. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\tilde{\Pi}_{R,A}^{[s_1,r_1]} \cap \tilde{\Pi}_{R,A}^{[s_2,r_2]} = \tilde{\Pi}_{R,A}^{[s_1,r_2]},$$

here the radii satisfy $s_1 \leq s_2 \leq r_1 \leq r_2$.

Proof. See the proof of proposition 3.5.7. □

Remark 3.6.15. Again this subsection is finished so far only for the situation where E is of mixed characteristic. But everything uniformly carries over for our original assumption on the field E and A . We will not repeat the proof again.

3.6.3 Noncommutative Period Rings and Period Sheaves

Setting 3.6.16. We will work in the categories of the pseudocoherent, fpd and finite projective modules over the period rings defined above. First we specify the Frobenius in our setting. The rings involved are:

$$\tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Omega}_{R,A}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{int},r}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}^{\text{bd},r}, \tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^r, \tilde{\Pi}_{R,A}^+, \tilde{\Pi}_{R,A}^\infty, \tilde{\Pi}_{R,A}^I. \quad (3.6.16)$$

We are going to endow these rings with the Frobenius induced by continuation from the Witt vector part only, which is to say the corresponding Frobenius induced by the p^h -power absolute Frobenius over R . Note all the rings above are defined by taking the product of Δ where each Δ representing one of the following rings (over E):

$$\tilde{\Omega}_R^{\text{int}}, \tilde{\Omega}_R, \tilde{\Pi}_R^{\text{int}}, \tilde{\Pi}_R^{\text{int},r}, \tilde{\Pi}_R^{\text{bd}}, \tilde{\Pi}_R^{\text{bd},r}, \tilde{\Pi}_R, \tilde{\Pi}_R^r, \tilde{\Pi}_R^+, \tilde{\Pi}_R^\infty, \tilde{\Pi}_R^I \quad (3.6.17)$$

with the affinoid ring A . The Frobenius acts on A trivially and we assume that the action is A -linear.

First we consider the following sheafification as in [KL16, Definition 4.4.2]:

Setting 3.6.17. Consider the space $X = \text{Spa}(R, R^+)$, over this perfectoid space there were sheaves:

$$\tilde{\Omega}^{\text{int}}, \tilde{\Omega}, \tilde{\Pi}^{\text{int}}, \tilde{\Pi}^{\text{int},r}, \tilde{\Pi}^{\text{bd}}, \tilde{\Pi}^{\text{bd},r}, \tilde{\Pi}, \tilde{\Pi}^r, \tilde{\Pi}^+, \tilde{\Pi}^\infty, \tilde{\Pi}^I. \quad (3.6.18)$$

defined over this space through the corresponding adic, étale, pro-étale or v -topology, we consider the corresponding sheaves defined over the same Grothendieck sites but with further deformed consideration:

$$\tilde{\Omega}_{*,A}^{\text{int}}, \tilde{\Omega}_{*,A}, \tilde{\Pi}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^{\text{int},r}, \tilde{\Pi}_{*,A}^{\text{bd}}, \tilde{\Pi}_{*,A}^{\text{bd},r}, \tilde{\Pi}_{*,A}, \tilde{\Pi}_{*,A}^r, \tilde{\Pi}_{*,A}^+, \tilde{\Pi}_{*,A}^\infty, \tilde{\Pi}_{*,A}^I. \quad (3.6.19)$$

Remark 3.6.18. This is well defined, since we have orthogonal basis for the ring A .

Definition 3.6.19. In our situation we consider the corresponding Frobenius action on the following period rings and sheaves:

$$\tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Omega}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{int}}, \tilde{\Pi}_{R,A}^{\text{bd}}, \tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^\infty, \tilde{\Omega}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^{\text{int}}, \tilde{\Pi}_{*,A}^+, \tilde{\Omega}_{*,A}, \tilde{\Pi}_{*,A}^{\text{bd}}, \tilde{\Pi}_{*,A}^\infty, \tilde{\Pi}_{*,A}^+, \tilde{\Pi}_{*,A}. \quad (3.6.20)$$

which is defined by considering the corresponding lift of the absolute Frobenius of p^h -power in characteristic $p > 0$ induced from R , which will be denoted by φ . We then introduce more general consideration by taking E_a to be some unramified extension of E of degree a divisible by h , the corresponding Frobenius will be denoted by φ^a .

Then we generalize the corresponding Frobenius modules in [KL16, Definition 4.4.4] to our situation as in the following.

Definition 3.6.20. Over the period rings and sheaves (each is denoted by Δ in this definition) defined in the previous definition we define as in [KL16, Definition 4.4.4] the corresponding φ^a -modules over Δ which are respectively projective to be the corresponding finite projective modules over Δ with further assigned semilinear action of the operator φ^a . Here we define in our situation the corresponding φ^a -cohomology to be the (hyper)-cohomology of the following complex:

$$0 \longrightarrow M \xrightarrow{\varphi^{-1}} M \longrightarrow 0.$$

All modules are right over the noncommutative rings.

Now we define the corresponding modules over the rings which are the domains in the following morphisms induced from the Frobenius map φ^a :

$$\tilde{\Pi}_{R,A}^{\text{int},r} \rightarrow \tilde{\Pi}_{R,A}^{\text{int},rp^{-ha}}, \tilde{\Pi}_{R,A}^{\text{bd},r} \rightarrow \tilde{\Pi}_{R,A}^{\text{bd},rp^{-ha}}, \tilde{\Pi}_{R,A}^r \rightarrow \tilde{\Pi}_{R,A}^{rp^{-ha}} \quad (3.6.21)$$

$$\tilde{\Pi}_{*,A}^{\text{int},r} \rightarrow \tilde{\Pi}_{*,A}^{\text{int},rp^{-ha}}, \tilde{\Pi}_{*,A}^{\text{bd},r} \rightarrow \tilde{\Pi}_{*,A}^{\text{bd},rp^{-ha}}, \tilde{\Pi}_{*,A}^r \rightarrow \tilde{\Pi}_{*,A}^{rp^{-ha}}. \quad (3.6.22)$$

Definition 3.6.21. Over each rings Δ which are the domains in the morphisms as mentioned just before this definition, we define the corresponding projective φ^a -module over any Δ listed above to be the corresponding finite projective module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*}M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one of the targets listed above. Also the cohomology of any module under this definition will be defined to be the (hyper)cohomology of the complex in the following form:

$$0 \longrightarrow M \xrightarrow{\varphi^{-1}} M \otimes_{\Delta} \square \longrightarrow 0.$$

All modules are right over the noncommutative rings.

Then we consider the following morphisms of specific period rings induced by the Frobenius.

$$\tilde{\Pi}_{R,A}^{[s,r]} \rightarrow \tilde{\Pi}_{R,A}^{[sp^{-ah},rp^{-ah}]} \quad (3.6.23)$$

$$\tilde{\Pi}_{*,A}^{[s,r]} \rightarrow \tilde{\Pi}_{*,A}^{[sp^{-ah},rp^{-ah}]} \quad (3.6.24)$$

with the corresponding morphisms in the following:

$$\tilde{\Pi}_{R,A}^{[s,r]} \rightarrow \tilde{\Pi}_{R,A}^{[s,rp^{-ah}]} \quad (3.6.25)$$

$$\tilde{\Pi}_{*,A}^{[s,r]} \rightarrow \tilde{\Pi}_{*,A}^{[s,rp^{-ah}]} \quad (3.6.26)$$

Definition 3.6.22. Again as in [KL16, Definition 4.4.4], we define the corresponding projective φ^a -modules over the domain rings or sheaves of rings in the morphisms just before this definition to be the finite projective modules (which will be denoted by M) over the domain rings in the morphism just before this definition additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*}M \otimes_{\tilde{\Pi}_{R,A}^{[sp^{-ah},rp^{-ah}]}} \tilde{\Pi}_{R,A}^{[s,rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\tilde{\Pi}_{R,A}^{[s,r]}} \tilde{\Pi}_{R,A}^{[s,rp^{-ah}]}, \quad (3.6.27)$$

$$\varphi^{a*}M \otimes_{\tilde{\Pi}_{*,A}^{[sp^{-ah},rp^{-ah}]}} \tilde{\Pi}_{*,A}^{[s,rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\tilde{\Pi}_{*,A}^{[s,r]}} \tilde{\Pi}_{*,A}^{[s,rp^{-ah}]}. \quad (3.6.28)$$

All modules are right over the noncommutative rings.

Also one can further define the corresponding bundles carrying semilinear Frobenius in our context as in the situation of [KL16, Definition 4.4.4]:

Definition 3.6.23. Over the ring $\tilde{\Pi}_{R,A}$ we define a corresponding projective Frobenius bundle to be a family $(M_I)_I$ of finite projective modules over each $\tilde{\Pi}_{R,A}^I$ carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\tilde{\Pi}_{R,A}^J} \tilde{\Pi}_{R,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, r]$ involved we have $s \leq rp^{ah}$. All modules are right over the noncommutative rings.

Then we generalize the comparison of local systems and Frobenius modules in [KL16, Theorem 4.5.7] to our noncommutative context as in the following:

Theorem 3.6.24. (After Kedlaya-Liu [KL16, Theorem 4.5.7]) *1. There is a fully faithful embedding functor from the category of the projective right $\mathcal{O}_{E_a^{\varphi^a}} \hat{\otimes} \mathcal{O}_A$ -local systems over $X_{\text{proét}}$ to the following two categories:*

1(a). The category of projective right Frobenius modules over the period sheaf $\tilde{\Omega}_{,A}^{\text{int}}$ over $X, X_{\text{ét}}, X_{\text{proét}}$;*

1(b). The category of projective right Frobenius modules over the period sheaf $\tilde{\Pi}_{,A}^{\text{int}}$ over $X, X_{\text{ét}}, X_{\text{proét}}$.*

Proof. See theorem 3.4.4. □

Then as in [KL16, Corollary 4.5.8] we consider the setting of more general adic spaces:

Proposition 3.6.25. (After Kedlaya-Liu [KL16, Corollary 4.5.8]) *1. Let X be a preadic space over E_a . Then we have that there is fully faithful embedding functor from the category of all the right $\mathcal{O}_{E_a^{\varphi^a}} \hat{\otimes} \mathcal{O}_A$ -local systems (projective) over $X, X_{\text{ét}}$ and $X_{\text{proét}}$ to the category of all the projective right Frobenius modules over the sheaves over $\tilde{\Pi}_{R,A}^{\text{int}}$ over the corresponding sites;*

2. Let X be a preadic space over $E_{\infty,a}$. Then we have that there is fully faithful embedding functor from the category of all the right $\mathcal{O}_{E_{\infty,a}^{\varphi^a}} \hat{\otimes} \mathcal{O}_A$ -local systems (projective) over $X, X_{\text{ét}}$ and $X_{\text{proét}}$ to the category of all the projective right Frobenius modules over the sheaves over $\tilde{\Pi}_{R,\infty,A}^{\text{int}}$ over the corresponding sites.

Proof. For one, apply the previous theorem. For two, repeat the argument in the proof of the previous theorem. \square

The following definition is kind of generalization of the corresponding one in [KL16, Definition 4.5.9]:

Definition 3.6.26. Now over the ring $\tilde{\Pi}_{R,A}$ or $\tilde{\Pi}_{R,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the ring $\tilde{\Pi}_{R,A}^{\text{int}}$. Now over the ring $\tilde{\Pi}_{R,\infty,A}$ or $\tilde{\Pi}_{R,\infty,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the ring $\tilde{\Pi}_{R,\infty,A}^{\text{int}}$. Now over the sheaf $\tilde{\Pi}_{*,A}$ or $\tilde{\Pi}_{*,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the sheaf $\tilde{\Pi}_{*,A}^{\text{int}}$. Now over the ring $\tilde{\Pi}_{*,\infty,A}$ or $\tilde{\Pi}_{*,\infty,A}^{\text{bd}}$ we call the corresponding Frobenius modules globally étale if they arise from the Frobenius modules over the sheaf $\tilde{\Pi}_{*,\infty,A}^{\text{int}}$.

Proposition 3.6.27. (After Kedlaya-Liu [KL16, Theorem 4.5.11]) 1. *Let X be a preadic space over E_a . Then we have that there is a fully faithful embedding of the category of the corresponding right $E_a^{\varphi^a} \hat{\otimes} A$ -local systems (in the projective setting) into the category of the corresponding projective right Frobenius φ^a -modules over $\tilde{\Pi}_{*,A}$;*
 2. *Let X be a preadic space over $E_{\infty,a}$. Then we have that there is a fully faithful embedding of the category of the corresponding right $E_{\infty,a}^{\varphi^a} \hat{\otimes} A$ -local systems (in the projective setting) into the category of the corresponding projective right Frobenius φ^a -modules over $\tilde{\Pi}_{*,A}$;*

Proof. As in [KL16, Theorem 4.5.11], we consider the corresponding base change of the corresponding exact sequence in proposition 3.4.1 which reflects an exact sequence on the sheaves. \square

Lemma 3.6.28. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9]) **(And also see [T1, Proposition 2.11])** *For any Frobenius φ^a -bundle M over $\tilde{\Pi}_{R,A}$, then we have that for any interval $I = [s, r]$ where $0 < s \leq r$ the map $\varphi^a - 1 : M_{[s,rq]} \rightarrow M_{[s,r]}$ is surjective after taking some Frobenius twist as in [KL15, 6.2.2] and our previous work namely the new morphism $\varphi^a - 1 : M(n)_{[s,rq]} \rightarrow M(n)_{[s,r]}$ for sufficiently large $n \geq 1$. As in the previous established version one may have the chance to take the number to be 1*

is the bundle initially comes from the corresponding base change from the integral Robba ring.

Proof. See the proof of [T1, Proposition 2.11]. □

Lemma 3.6.29. (After Kedlaya-Liu [KL15, 6.2.2-6.2.4], [KL16, Lemma 4.6.9])
(And also see [T1, Proposition 2.11, Proposition 2.14])

For any Frobenius φ^a -module M over $\tilde{\Pi}_{R,A}$, we have that for sufficiently large number $n \geq 1$ the space $H_{\varphi^a}^1(M(n))$ vanishes.

Proof. See the proof of [T1, Proposition 2.11, Proposition 2.14]. □

Then we have the following key corollary which is analog of [KL15, Corollary 6.2.3, Lemma 6.3.3], [KL16, Corollary 4.6.10] and [T1, Corollary 2.13, Corollary 3.4, Proposition 3.9]:

Corollary 3.6.30. *I.* When M_α, M, M_β are three Frobenius φ^a -bundles over the ring $\tilde{\Pi}_{R,A}$ then we have that for sufficiently large $n \geq 0$ we have the following exact sequence:

$$0 \longrightarrow M_{\alpha,I}(n)^{\varphi^a} \longrightarrow M_I(n)^{\varphi^a} \longrightarrow M_{\beta,I}(n)^{\varphi^a} \longrightarrow 0,$$

for each interval I .

II. When M_α, M, M_β are three Frobenius φ^a -modules over the ring $\tilde{\Pi}_{R,A}$ then we have that for sufficiently large $n \geq 0$ we have the following exact sequence:

$$0 \longrightarrow M_{\alpha,I}(n)^{\varphi^a} \longrightarrow M_I(n)^{\varphi^a} \longrightarrow M_{\beta,I}(n)^{\varphi^a} \longrightarrow 0,$$

for each interval I .

III. For a Frobenius φ^a -bundle M over $\tilde{\Pi}_{R,A}$, and for each module M_I over some $\tilde{\Pi}_{R,A}^I$, we put for any element f such that $\varphi^a f = p^d f$:

$$M_{I,f} := \bigcup_{n \in \mathbb{Z}} f^{-n} M_I(dn)^{\varphi^a}.$$

Then with this convention suppose we have three Frobenius φ^a -bundles taking the form of M_α, M, M_β over $\tilde{\Pi}_{R,A}$, then for each closed interval I we have the following is an exact sequence:

$$0 \longrightarrow M_{\alpha,I,f} \longrightarrow M_{I,f} \longrightarrow M_{\beta,I,f} \longrightarrow 0,$$

where each module in the exact sequence is now a module over $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$;

IV. For a Frobenius φ^a -module M over $\tilde{\Pi}_{R,A}$, we put for any element f such that $\varphi^a f = p^d f$:

$$M_f := \bigcup_{n \in \mathbb{Z}} f^{-n} M (dn)^{\varphi^a}.$$

Then with this convention suppose we have three Frobenius φ^a -modules taking the form of M_α, M, M_β over $\tilde{\Pi}_{R,A}$, then we have the following is an exact sequence:

$$0 \longrightarrow M_{\alpha,f} \longrightarrow M_f \longrightarrow M_{\beta,f} \longrightarrow 0,$$

where each module in the exact sequence is now a module over $\tilde{\Pi}_{R,A}[1/f]^{\varphi^a}$.

Proof. See the proof of [KL15, Corollary 6.2.3, Lemma 6.3.3], [KL16, Corollary 4.6.10] and [T1, Corollary 2.13, Corollary 3.4, Proposition 3.9]. \square

3.6.4 Noncommutative Imperfect Setting

We now consider the corresponding imperfectization of the corresponding constructions we considered above, after [KL16]. We will consider the corresponding towers in [KL16, Chapter 5], so we keep the assumption on the towers as in the commutative setting, namely:

Assumption 3.6.31. In this section, we are going to assume that k is just \mathbb{F}_{p^h} .

Now we describe the corresponding imperfect period rings which we will deform. These rings are those introduced in [KL16, Definition 5.2.1] by using series of imperfection processes. Recall in more detail in our noncommutative setting we have:

Setting 3.6.32. Fix a perfectoid tower (H_\bullet, H_\bullet^+) Recall from [KL16, Definition 5.2.1] we have the following different imperfect constructions:

- A. First we have the ring \overline{H}'_∞ , which could give us the corresponding ring $\tilde{\Omega}_{\overline{H}'_\infty}^{\text{int}}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- B. We then have the ring $\tilde{\Pi}_{\overline{H}'_\infty}^{\text{int},r}$ coming from \overline{H}'_∞ . Taking the corresponding product with A we have the corresponding deformed period ring;
- C. We then have the ring $\tilde{\Pi}_{\overline{H}'_\infty}^{\text{int}}$ coming from \overline{H}'_∞ . Taking the corresponding product

- with A we have the corresponding deformed period ring;
- D. We then have the ring $\widetilde{\Pi}_{\overline{H}'_\infty}^{\text{bd},r}$ coming from \overline{H}'_∞ . Taking the corresponding product with A we have the corresponding deformed period ring;
- E. We then have the ring $\widetilde{\Pi}_{\overline{H}'_\infty}^{\text{bd}}$ coming from \overline{H}'_∞ . Taking the corresponding product with A we have the corresponding deformed period ring;
- F. We then have the ring $\widetilde{\Pi}_{\overline{H}'_\infty}^r$ coming from \overline{H}'_∞ . Taking the corresponding product with A we have the corresponding deformed period ring;
- G. We then have the ring $\widetilde{\Pi}_{\overline{H}'_\infty}^{[s,r]}$ coming from \overline{H}'_∞ . Taking the corresponding product with A we have the corresponding deformed period ring;
- H. We then have the ring $\widetilde{\Pi}_{\overline{H}'_\infty}$ coming from \overline{H}'_∞ . Taking the corresponding product with A we have the corresponding deformed period ring;
- I. $\Pi_H^{\text{int},r}$ comes from the ring $\widetilde{\Pi}_{\overline{H}'_\infty}^{\text{int},r}$ consisting of those elements of $\widetilde{\Pi}_{\overline{H}'_\infty}^{\text{int},r}$ with the requirement that whenever we have n an integer such that $nh > -\log_p r$ we have then $\theta(\varphi^{-n}(x)) \in H_n$. Taking the corresponding product with A we have the corresponding deformed period ring;
- J. $\Pi_H^{\text{int},\dagger}$ is defined to be the corresponding union of the rings in I . Taking the corresponding product with A we have the corresponding deformed period ring;
- K. Ω_H^{int} is defined to be the corresponding period ring coming from the corresponding π -adic completion of the ring $\Pi_H^{\text{int},\dagger}$ in J . Taking the corresponding product with A we have the corresponding deformed period ring;
- L. $\check{\Omega}_H^{\text{int}}$ is the ring which is defined to be the union of all the $\varphi^{-n}\Omega_H^{\text{int}}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- M. $\check{\Pi}_H^{\text{int},r}$ is then the ring which is defined to be the union of all the $\varphi^{-n}\Pi_H^{\text{int},p^{hn}r}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- N. $\check{\Pi}_H^{\text{int},\dagger}$ is define to be union of all the $\varphi^{-n}\check{\Pi}_H^{\text{int},\dagger}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- O. $\widehat{\Omega}_H^{\text{int}}$ is defined to be the π -completion of $\check{\Omega}_H^{\text{int}}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- P. $\widehat{\Pi}_H^{\text{int},r}$ is defined to be the $\max\{\|\cdot\|_{\overline{\alpha}_\infty}, \|\cdot\|_{\pi\text{-adic}}\}$ -completion of $\check{\Pi}_H^{\text{int},r}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- Q. We then have $\widehat{\Pi}_H^{\text{int},\dagger}$ by taking the union over $r > 0$. Taking the corresponding product with A we have the corresponding deformed period ring;

- R. Correspondingly we have $\check{\Omega}_H, \check{\Pi}_H^{\text{bd},r}, \check{\Pi}_H^{\text{bd},\dagger}$ by inverting the element π . Taking the corresponding product with A we have the corresponding deformed period ring;
- S. Correspondingly we also have $\widehat{\Omega}_H, \widehat{\Pi}_H^{\text{bd},r}, \widehat{\Pi}_H^{\text{bd},\dagger}$ by inverting the corresponding element π . Taking the corresponding product with A we have the corresponding deformed period ring;
- T. We also have $\Omega_H, \Pi_H^{\text{bd},r}, \Pi_H^{\text{bd},\dagger}$ again by inverting the element π . Taking the corresponding product with A we have the corresponding deformed period ring;
- U. Taking the $\max\{\|\cdot\|_{\check{\alpha}_\infty^s}, \|\cdot\|_{\check{\alpha}_\infty^r}\}$ (for $0 < s \leq r$) completion of the ring $\Pi_H^{\text{bd},r}$ we have the ring $\Pi_H^{[s,r]}$, while taking the Fréchet completion with respect to the norm $\|\cdot\|_{\check{\alpha}_\infty^t}$ for each $0 < t \leq r$ we have the ring Π_H^r . Taking the corresponding product with A we have the corresponding deformed period ring;
- V. Taking the union we have the ring Π_H . Taking the corresponding product with A we have the corresponding deformed period ring;
- W. We use the notation $\check{\Pi}_H$ to denote the corresponding union of all the $\varphi^{-n}\Pi_H$. Taking the corresponding product with A we have the corresponding deformed period ring;
- X. We use the notation $\check{\Pi}_H^{[s,r]}$ to be the corresponding union of all the $\varphi^{-n}\Pi_H^{[p^{-hn}s, p^{-hn}r]}$. Taking the corresponding product with A we have the corresponding deformed period ring;
- Y. We use the notation $\check{\Pi}_H^r$ to be the union of all the $\varphi^{-n}\check{\Pi}_H^{p^{-hn}r}$. Taking the corresponding product with A we have the corresponding deformed period ring.

Then we have the following direct analog of the relative version of the ring defined above (here as before the ring A denotes a Banach affinoid algebra in the noncommutative setting):

Setting 3.6.33. Now we consider the deformation of the rings above:

I. We have the first group of the period rings in the deformed setting:

$$\Pi_{H,A}^{\text{int},r}, \Pi_{H,A}^{\text{int},\dagger}, \check{\Omega}_{H,A}^{\text{int}}, \check{\Omega}_{H,A}, \Pi_{H,A}^{\text{bd},r}, \Pi_{H,A}^{\text{bd},\dagger}, \Pi_{H,A}^{[s,r]}, \Pi_{H,A}^r, \Pi_{H,A}. \quad (3.6.29)$$

II. We also have the second group of desired rings in the desired setting:

$$\check{\Pi}_{H,A}^{\text{int},r}, \check{\Pi}_{H,A}^{\text{int},\dagger}, \check{\check{\Omega}}_{H,A}^{\text{int}}, \check{\check{\Omega}}_{H,A}, \check{\Pi}_{H,A}^{\text{bd},r}, \check{\Pi}_{H,A}^{\text{bd},\dagger}, \check{\Pi}_{H,A}^{[s,r]}, \check{\Pi}_{H,A}^r, \check{\Pi}_{H,A}. \quad (3.6.30)$$

III. We also have the third group:

$$\widehat{\Pi}_{H,A}^{\text{int},r}, \widehat{\Pi}_{H,A}^{\text{int},\dagger}, \widehat{\Omega}_{H,A}^{\text{int}}, \widehat{\Omega}_{H,A}, \widehat{\Pi}_{H,A}^{\text{bd},r}, \widehat{\Pi}_{H,A}^{\text{bd},\dagger}. \quad (3.6.31)$$

Now we discuss some properties of the corresponding deformed version of the imperfect rings in our context, which is parallel to the corresponding discussion we made in the perfect setting and generalizing the corresponding discussion in [KL16], again in our noncommutative setting:

Proposition 3.6.34. (After Kedlaya-Liu [KL16, Lemma 5.2.10]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\Pi_{H,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{\text{int},r_1} \cap \Pi_{H,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{[r_1,r_2]} = \Pi_{H,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{\text{int},r_2}.$$

Proof. We adapt the argument in [KL16, Lemma 5.2.10] to prove this in the situation where $r_1 < r_2$ (otherwise this is trivial), again one direction is easy where we only present the implication in the other direction. We take any element

$$x \in \Pi_{H,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{\text{int},r_1} \cap \Pi_{H,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{[r_1,r_2]}$$

and take suitable approximating elements $\{x_i\}$ living in the bounded Robba ring such that for any $j \geq 1$ one can find some integer $N_j \geq 1$ we have whenever $i \geq N_j$ we have the following estimate:

$$\|\cdot\|_{\overline{\alpha}_\infty^t, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(x_i - x) \leq p^{-j}, \forall t \in [r_1, r_2].$$

Then we consider the corresponding decomposition of x_i for each $i = 1, 2, \dots$ into a form having integral part and the rational part $x_i = y_i + z_i$ by setting

$$y_i = \sum_{k=0, i_1, \dots, i_d} \pi^k x_{i,k,i_1,\dots,i_d} Z_1^{i_1} \dots Z_d^{i_d}$$

out of

$$x_i = \sum_{k=n(x_i), i_1, \dots, i_d} \pi^k x_{i,k,i_1,\dots,i_d} Z_1^{i_1} \dots Z_d^{i_d}.$$

Note that by our initial hypothesis we have that the element x lives in the ring $\Pi_{H,\mathbb{Q}_p\{Z_1,\dots,Z_d\}}^{\text{int},r_1}$ which further implies that

$$\|\cdot\|_{\overline{\alpha}_\infty^{r_1}, \mathbb{Q}_p\{Z_1,\dots,Z_d\}}(\pi^k x_{i,k,i_1,\dots,i_d} Z_1^{i_1} \dots Z_d^{i_d}) \leq p^{-j}.$$

Therefore we have $\bar{\alpha}_\infty(\bar{x}_{i,k,i_1,\dots,i_d}) \leq p^{(k-j)/r_1}, \forall k < 0$ directly from this through computation, which implies that then:

$$\|\cdot\|_{\bar{\alpha}_\infty^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k x_{i,k,i_1,\dots,i_d} Z_1^{i_1} \dots Z_d^{i_d}) \leq p^{-k} p^{(k-j)r_2/r_1} \quad (3.6.32)$$

$$\leq p^{1+(1-j)r_1/r_1}. \quad (3.6.33)$$

Then one can read off the result directly from this estimate since under this estimate we can have the chance to modify the original approximating sequence $\{x_i\}$ by $\{y_i\}$ which are initially chosen to be in the integral Robba ring, which implies that actually the element x lives in the right-hand side of the identity in the statement of the proposition. \square

Remark 3.6.35. The following result cannot be derived from the previous proposition but the proof could be made essentially in the fashion, which is the same as the corresponding situation we encountered in the commutative situation, for instance see proposition 3.5.7.

Proposition 3.6.36. (After Kedlaya-Liu [KL16, Lemma 5.2.10]) *For any $0 < r_1 \leq r_2$ we have the following equality on the corresponding period rings:*

$$\Pi_{H,A}^{\text{int}, r_1} \bigcap \Pi_{H,A}^{[r_1, r_2]} = \Pi_{H,A}^{\text{int}, r_2}.$$

Here A is some Banach affinoid algebra over \mathbb{Q}_p in the noncommutative setting.

Proof. See the proof of proposition 3.5.7. \square

Proposition 3.6.37. (After Kedlaya-Liu [KL15, Lemma 5.2.8] and [KL16]) *Consider now in our situation the radii $0 < r_1 \leq r_2$, and consider any element*

$$x \in \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[r_1, r_2]}.$$

Then we have that for each $n \geq 1$ one can decompose x into the form of $x = y + z$ such that $y \in \pi^n \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$ with $z \in \bigcap_{r \geq r_2} \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[r_1, r]}$ with the following estimate for each $r \geq r_2$:

$$\|\cdot\|_{\bar{\alpha}_\infty^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(z) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\bar{\alpha}_\infty^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(z)^{r/r_2}.$$

Proof. As in [KL15, Lemma 5.2.8] and [KL16] and in the proof of our previous proposition we first consider those elements x lives in the bounded Robba rings which could be expressed in general as

$$\sum_{k=n(x), i_1, \dots, i_d} \pi^k x_{k, i_1, \dots, i_d} Z_1^{i_1} \dots Z_d^{i_d}.$$

In this situation the corresponding decomposition is very easy to come up with, namely we consider the corresponding y_i as the corresponding series:

$$\sum_{k \geq n, i_1, \dots, i_d} \pi^k x_{k, i_1, \dots, i_d} Z_1^{i_1} \dots Z_d^{i_d}$$

which give us the desired result since we have in this situation when focusing on each single term:

$$\|\cdot\|_{\bar{\alpha}_\infty^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k x_{k, i_1, \dots, i_d} Z_1^{i_1} \dots Z_d^{i_d}) = p^{-k} \bar{\alpha}_\infty(\bar{x}_{k, i_1, \dots, i_d})^r \quad (3.6.34)$$

$$= p^{-k(1-r/r_2)} \|\cdot\|_{\bar{\alpha}_\infty^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k x_{k, i_1, \dots, i_d} Z_1^{i_1} \dots Z_d^{i_d})^{r/r_2} \quad (3.6.35)$$

$$\leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\bar{\alpha}_\infty^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(\pi^k x_{k, i_1, \dots, i_d} Z_1^{i_1} \dots Z_d^{i_d})^{r/r_2} \quad (3.6.36)$$

for all those suitable k . Then to tackle the more general situation we consider the approximating sequence consisting of all the elements in the bounded Robba ring as in the usual situation considered in [KL15, Lemma 5.2.8] and [KL16], namely we inductively construct the following approximating sequence just as:

$$\|\cdot\|_{\bar{\alpha}_\infty^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x - x_0 - \dots - x_i) \leq p^{-i-1} \|\cdot\|_{\bar{\alpha}_\infty^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x), i = 0, 1, \dots, r \in [r_1, r_2]. \quad (3.6.37)$$

Here all the elements x_i for each $i = 0, 1, \dots$ are living in the bounded Robba ring, which immediately gives rise to the suitable decomposition as proved in the previous case namely we have for each i the decomposition $x_i = y_i + z_i$ with the desired conditions as mentioned in the statement of the proposition. We first take the series summing all the elements y_i up for all $i = 0, 1, \dots$, which first of all converges under the norm $\|\cdot\|_{\bar{\alpha}_\infty^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}$ for all the radius $r \in [r_1, r_2]$, and also note that all the elements y_i within the infinite sum live inside the corresponding integral Robba ring $\Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$, which further implies the corresponding convergence ends up in $\Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$. For the elements z_i where $i = 0, 1, \dots$ also sum up to a converging series in the desired ring since combining all the estimates above we have:

$$\|\cdot\|_{\bar{\alpha}_\infty^r, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(z_i) \leq p^{(1-n)(1-r/r_2)} \|\cdot\|_{\bar{\alpha}_\infty^{r_2}, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}(x)^{r/r_2}.$$

□

Proposition 3.6.38. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_1, r_1]} \cap \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_2, r_2]} = \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_1, r_2]},$$

here the radii satisfy $s_1 < s_2 \leq r_1 \leq r_2$.

Proof. In our situation one direction is obvious while on the other hand we consider any element x in the intersection on the left, then by the previous proposition we have the decomposition $x = y + z$ where $y \in \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1}$ and $z \in \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_1, r_2]}$. Then as in [KL15, Lemma 5.2.10] section 5.2 we look at $y = x - z$ which lives in the intersection:

$$\Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_1} \cap \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{[s_2, r_2]} = \Pi_{H, \mathbb{Q}_p\{Z_1, \dots, Z_d\}}^{\text{int}, r_2}$$

which finishes the proof. □

Proposition 3.6.39. (After Kedlaya-Liu [KL15, Lemma 5.2.10]) *We have the following identity:*

$$\Pi_{H, A}^{[s_1, r_1]} \cap \Pi_{H, A}^{[s_2, r_2]} = \Pi_{H, A}^{[s_1, r_2]},$$

here the radii satisfy $s_1 < s_2 \leq r_1 \leq r_2$.

Proof. See the proof of proposition 3.5.7. □

Remark 3.6.40. Again one can follow the same strategy to deal with the corresponding equal-characteristic situation.

3.6.5 Modules and Bundles over Noncommutative Rings

Now we consider the modules and bundles over the rings introduced in the previous subsection. First we make the following assumption:

Setting 3.6.41. Recall that from [KL16, Definition 5.2.3] any tower (H_\bullet, H_\bullet^+) is called weakly decompleting if we have that first the density of the perfection of H_∞ in \overline{H}_∞ . Here the ring H_∞ is the ring coming from the mod- π construction of the ring Ω_H^{int} , also at the same time one can find some $r > 0$ such that the corresponding modulo π operation from the ring Ω_H^{int} to the ring H_∞ is actually surjective strictly.

Assumption 3.6.42. We now assume that we are basically in the situation where (H_\bullet, H_\bullet^+) is actually weakly decompleting. Also as in [KL16, Lemma 5.2.7] we assume we fix some radius $r_0 > 0$, for instance this will correspond to the corresponding index in the situation we have the corresponding noetherian tower. Recall that a tower is called noetherian if we have some specific radius as above such that we have the strongly noetherian property on the ring $\Pi_H^{[s,r]}$ with $[s,r] \subset (0, r_0]$. We now assume that the tower is then noetherian and any ring $\Pi_{H,A}^{[s,r]}$ with $[s,r] \subset (0, r_0]$ is noetherian in the noncommutative setting.

Example 3.6.43. This assumption could be achieved as in the following. First naively we can consider a noncommutative affinoid algebra which is finite over some commutative affinoid algebra over \mathbb{Q}_p , then tensor with the corresponding the Robba ring $\Pi_H^{[s,r]}$ which is assumed to be strongly noetherian. Then for some deeper case, we can take the Robba ring $D_{[a,b]}(\mathbb{Z}_p, \mathbb{Q}_p)$ attached to the group \mathbb{Z}_p in the sense of [Z1, Proposition 4.1] (namely the usual Robba ring over \mathbb{Q}_p , here as in [Z1, Proposition 4.1] we assume that $a, b \in p^\mathbb{Q}$), and we take the corresponding complete tensor product with the local chart ring of the distribution algebra attached to a uniform p -adic Lie group G as in [Z1, Proposition 4.1]:

$$D_\rho(G, \mathbb{Q}_p)$$

with some radius $\rho > 0$ living in $p^\mathbb{Q}$. The whole product:

$$D_{[a,b]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} D_\rho(G, \mathbb{Q}_p)$$

is noetherian. Indeed, as in [Z1, Proposition 4.1] we apply the criterion in [LVO, Proposition I.7.1.2] to check that actually the graded ring induced by the product norm:

$$gr_{\|\cdot\|_\otimes}^\bullet(D_{[a,b]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} D_\rho(G, \mathbb{Q}_p)) \xrightarrow{\sim} gr_{\|\cdot\|_\otimes}^\bullet(D_{[a,b]}(\mathbb{Z}_p, \mathbb{Q}_p) \otimes D_\rho(G, \mathbb{Q}_p))$$

admits a surjection map from:

$$gr_{\|\cdot\|_{[a,b]}}^\bullet(D_{[a,b]}(\mathbb{Z}_p, \mathbb{Q}_p)) \otimes_{gr_{\bullet \cdot \mathbb{Q}_p}} gr_{\|\cdot\|_\rho}^\bullet(D_\rho(G, \mathbb{Q}_p)) \tag{3.6.38}$$

As in the proof of [Z1, Proposition 4.1] one can show that this is a tensor product of a Laurent polynomial ring and a polynomial ring, which is noetherian.

Then we can start to discuss the corresponding modules over the rings in our deformed setting, first as in [KL16, Lemma 5.3.3] the following result should be derived from our construction:

Then we deform the basic notation of bundles in [KL16, Definition 5.3.6]:

Definition 3.6.44. (After Kedlaya-Liu [KL16, Definition 5.3.6]) We define the bundle over the ring $\Pi_{H,A}^{r_0}$ to be a collection $(M_I)_I$ of finite projective right modules over each $\Pi_{H,A}^I$ with $I \subset (0, r_0]$ closed subintervals of $(0, r_0]$ such that we have the following requirement in the glueing fashion. First for any $I_1 \subset I_2$ two closed intervals we have $M_{I_2} \otimes_{\Pi_{H,A}^{I_2}} \Pi_{H,A}^{I_1} \xrightarrow{\sim} M_{I_1}$ with the obvious cocycle condition with respect to three closed subintervals of $(0, r_0]$ namely taking the form of $I_1 \subset I_2 \subset I_3$.

We define the pseudocoherent sheaf over the ring $\Pi_{H,A}^{r_0}$ to be a collection $(M_I)_I$ of pseudocoherent right modules over each $\Pi_{H,A}^I$ with $I \subset (0, r_0]$ closed subintervals of $(0, r_0]$ such that we have the following requirement in the glueing fashion. First for any $I_1 \subset I_2$ two closed intervals we have $M_{I_2} \otimes_{\Pi_{H,A}^{I_2}} \Pi_{H,A}^{I_1} \xrightarrow{\sim} M_{I_1}$ with the obvious cocycle condition with respect to three closed subintervals of $(0, r_0]$ namely taking the form of $I_1 \subset I_2 \subset I_3$.

We make the following discussion around the corresponding module and sheaf structures defined above.

Lemma 3.6.45. (After Kedlaya-Liu [KL16, Lemma 5.3.8]) *We have the isomorphism between the ring $\Pi_{H,A}^r$ and the inverse limit of the ring $\Pi_{H,A}^{[s,r]}$ with respect to the radius s by the map $\Pi_{H,A}^r \rightarrow \varprojlim \Pi_{H,A}^{[s,r]}$.*

Proof. As in [KL16, Lemma 5.3.8] it is injective by the isometry, and then use the corresponding elements $x_n, n = 0, 1, \dots$ in the dense ring $\Pi_{H,A}^r$ to approximate any element x in the ring $\varprojlim \Pi_{H,A}^{[s,r]}$ in the same way as in [KL16, Lemma 5.3.8]:

$$\|\cdot\|_{\bar{\alpha}_{\infty}^t, A}(x - x_n) \leq p^n$$

for any radius t now living in the corresponding interval $[r2^{-n}, r]$. This will establish Cauchy sequence which finishes the proof as in [KL16, Lemma 5.3.8]. \square

Proposition 3.6.46. (After Kedlaya-Liu [KL16, Lemma 5.3.9]) *For some radius $r \in (0, r_0]$. Suppose we have that M is a vector bundle in the general setting or M is a pseudocoherent sheaf in the setting where the tower is noetherian. Then we have that the*

corresponding global section is actually dense in each section with respect to some closed interval. And then we have the corresponding vanishing result of the first derived inverse limit functor.

Proof. See [KL16, Lemma 5.3.9]. In our setting, note that we are actually in the formalism of the corresponding (noncommutative) Fréchet-Stein algebras as in [ST1]. \square

The interesting issue here as in [KL16] is the corresponding finitely generated of the global section of a pseudocoherent sheaf which is actually not guaranteed in general. Therefore as in [KL16] we have to distinguish the corresponding well-behaved sheaves out from the corresponding category of all the corresponding pseudocoherent sheaves. But we are not quite for sure how more complicated the noncommutative situation is than the commutative setting. For the completeness of the presentation we present some conjectures in our mind.

Conjecture 3.6.47. (After Kedlaya-Liu [KL16, Lemma 5.3.10]) *As in the previous proposition we choose some $r \in (0, r_0]$. Now assume that the corresponding tower (H_\bullet, H_\bullet^+) is noetherian. Now for any pseudocoherent sheaf M defined above we have the following three statements are equivalent. A. The first statement is that one can find a sequence of positive integers x_1, x_2, \dots such that for any closed interval living inside $(0, r]$ the section of the sheaf with respect this closed interval admits a projective resolution of modules with corresponding ranks bounded by the sequence of integer x_1, x_2, \dots . B. The second statement is that for any locally finite covering of the corresponding interval $(0, r]$ which takes the corresponding form of $\{I_i\}$ one can find a sequence of positive integers x_1, x_2, \dots such that for any closed interval living inside $\{I_i\}$ the section of the sheaf with respect this closed interval admits a projective resolution of modules with corresponding ranks bounded by the sequence of integer x_1, x_2, \dots . C. Lastly the third statement is that the corresponding global section is a pseudocoherent module over the ring $\Pi_{H,A}^r$.*

As in [KL16, Definition 5.3.11] we call the sheaf satisfies the corresponding equivalent conditions in the proposition above uniform pseudocoherent sheaf. Then we have the following analog of [KL16, Lemma 5.3.12]:

Conjecture 3.6.48. (After Kedlaya-Liu [KL16, Lemma 5.3.12]) *The global section functor defines the corresponding equivalence between the categories of the following two*

sorts of objects. The first ones are the corresponding uniform pseudocoherent sheaves over $\Pi_{H,A}^r$. The second ones are those pseudocoherent modules over the ring $\Pi_{H,A}^r$.

Conjecture 3.6.49. (After Kedlaya-Liu [KL16, Lemma 5.3.13]) *The global section functor defines the corresponding equivalence between the categories of the following two sorts of objects. The first ones are the corresponding finite projective uniform pseudocoherent sheaves over $\Pi_{H,A}^r$. The second ones are those finite projective modules over the ring $\Pi_{H,A}^r$.*

3.6.6 φ -Module Structures and Γ -Module Structures over Noncommutative Rings

Now we consider the corresponding Frobenius actions over the corresponding imperfect rings we defined before, note that the corresponding Frobenius actions are induced from the corresponding imperfect rings in the undeformed situation from [KL16] which is to say that the Frobenius action on the ring A is actually trivial.

Assumption 3.6.50. We now drop the corresponding requirement on A which makes the noncommutative deformation of the Robba ring over some interval both left and right noetherian, in assumption 3.6.42.

First we consider the corresponding Frobenius modules:

Definition 3.6.51. (After Kedlaya-Liu [KL16, Definition 5.4.2]) Over the period rings $\Pi_{H,A}$ or $\check{\Pi}_{H,A}^r$ (which is denoted by Δ in this definition) we define the corresponding φ^a -modules over Δ which are respectively projective to be the corresponding finite projective right modules over Δ with further assigned semilinear action of the operator φ^a .

Definition 3.6.52. (After Kedlaya-Liu [KL16, Definition 5.4.2]) Over each rings $\Delta = \Pi_{H,A}^r, \check{\Pi}_{H,A}^r$ we define the corresponding projective φ^a -module over any Δ to be the corresponding finite projective right module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*} M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \Pi_{H,A}^{r/p}, \check{\Pi}_{H,A}^{r/p}$.

Definition 3.6.53. (After Kedlaya-Liu [KL16, Definition 5.4.2]) Again as in [KL16, Definition 5.4.2], we define the corresponding projective φ^a -modules over ring $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ to be the finite projective right modules (which will be denoted by M) over $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ respectively additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\Pi_{H,A}^{[sp^{-ah}, rp^{-ah}]}} \Pi_{H,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\Pi_{H,A}^{[s,r]}} \Pi_{H,A}^{[s, rp^{-ah}]} \quad (3.6.39)$$

and

$$\varphi^{a*} M \otimes_{\check{\Pi}_{R,A}^{[sp^{-ah}, rp^{-ah}]}} \check{\Pi}_{R,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\check{\Pi}_{R,A}^{[s,r]}} \check{\Pi}_{R,A}^{[s, rp^{-ah}]}.$$

$$(3.6.41)$$

Also one can further define the corresponding bundles carrying semilinear Frobenius in our context as in the situation of [KL16, Definition 5.4.10]:

Definition 3.6.54. (After Kedlaya-Liu [KL16, Definition 5.4.10]) Over the ring $\Pi_{H,A}^I$ or $\check{\Pi}_{H,A}^I$ we define a corresponding projective Frobenius bundle to be a family $(M_I)_I$ of finite projective right modules over each $\Pi_{H,A}^I$ or $\check{\Pi}_{H,A}^I$ respectively carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\Pi_{H,A}^J} \Pi_{H,A}^I \xrightarrow{\sim} M_I$$

and

$$M_J \otimes_{\check{\Pi}_{H,A}^J} \check{\Pi}_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, r]$ involved we have $s \leq r/p^{ah}$. As in [KL16, Definition 5.4.10] one can take the corresponding 2-limit in the direct sense to define the corresponding objects over the full Robba rings.

We can then compare the corresponding objects defined above:

Conjecture 3.6.55. (After Kedlaya-Liu [KL16, Lemma 5.4.11])

I. Consider the following objects for some radius r_0 in our situation. The first group of objects are those finite projective φ^a -modules over the Robba ring $\Pi_{H,A}^{r_0}$. The second group

of objects are those finite projective φ^a -bundles over the Robba ring $\Pi_{H,A}^{r_0}$. The third group of objects are those finite projective φ^a -modules over the Robba ring $\Pi_{H,A}^{[s,r]}$ for some radii $0 < s \leq r \leq r_0$. Then we have that the corresponding categories of the two groups of objects are equivalent.

Now we define the corresponding Γ -modules over the period rings attached to the tower (H_\bullet, H_\bullet^+) . The corresponding structures are actually abstractly defined in the same way as in [KL16] and parallel to our commutative setting. First we consider the deformation of the corresponding complex $*_{H^\bullet}$ for any ring $*$ in setting 3.6.33.

Assumption 3.6.56. Recall that the corresponding tower is called decompleting if it is weakly decompleting, finite étale on each finite level and having the exact sequence $\bar{\varphi}^{-1}H'_{H^\bullet}/H'_{H^\bullet}$ is exact. We now assume that the tower (H_\bullet, H_\bullet^+) is then decompleting.

Setting 3.6.57. Assume now Γ is a topological group as in [KL16, Definition 5.5.5] acting on the corresponding period rings in the setting 3.6.32 in the original context of setting 3.6.32. Then we consider the corresponding induced continuous action over the corresponding deformed version in our context namely in setting 3.6.33. Assume now that the tower is Galois with the corresponding Galois group Γ .

Definition 3.6.58. We now consider the corresponding inhomogeneous continuous cocycles of the group Γ , as in [KL16, Definition 5.5.5] we use the following notation to denote the corresponding complex extracted from a single tower for a given period ring $*_{H,A}$ in setting 3.6.33 for each $k > 0$:

$$*_{H^k,A} := C_{\text{con}}(\Gamma^k, *_{H}) \widehat{\otimes}_{\#} ?$$

where $\# = \mathbb{Q}_p, \mathbb{Z}_p$ and $? = A, \mathfrak{o}_A$ respectively, which forms the corresponding complex $(*_{H^\bullet,A}, d^\bullet)$ with the corresponding differential as in [KL16, Definition 5.5.5] in the sense of continuous group cohomology.

Definition 3.6.59. Having established the corresponding meaning of the Γ -structure we now consider the corresponding definition of Γ -modules. Such modules called the corresponding right Γ -modules are defined over the corresponding rings in setting 3.6.33. Again we allow that the corresponding right modules to be finite projective, or pseudocoherent or fpd over the rings in setting 3.6.33. And the modules are defined to be carrying the corresponding continuous semilinear action from the group Γ .

Proposition 3.6.60. (After Kedlaya-Liu [KL16, Corollary 5.6.5])

The complex $\varphi^{-1}\Pi_{H_{\geq n}, A}^{[sp^{-h}, rp^{-h}]} / \Pi_{H_{\geq n}, A}^{[s, r]}$ and the complex $\tilde{\Pi}_{H_{\geq n}, A}^{[s, r]} / \Pi_{H_{\geq n}, A}^{[s, r]}$ are strictly exact for any truncation index n . The corresponding radii satisfy the corresponding relation $0 < s \leq r \leq r_0$.

Proof. See [KL16, Corollary 5.6.5], and consider the corresponding Schauder basis of A . □

Proposition 3.6.61. (After Kedlaya-Liu [KL16, Lemma 5.6.6]) *The complex*

$$M \otimes_{\Pi_{H, A}^{[s, r]}} \varphi^{-(\ell+1)} \Pi_{H^{\bullet}, A}^{[sp^{-h(\ell+1)}, rp^{-h(\ell+1)}]} / \varphi^{-\ell} \Pi_{H^{\bullet}, A}^{[sp^{-h\ell}, rp^{-h\ell}]}$$

and the complex

$$M \otimes_{\Pi_{H, A}^{[s, r]}} \tilde{\Pi}_{H^{\bullet}, A}^{[s, r]} / \varphi^{-\ell} \Pi_{H^{\bullet}, A}^{[sp^{-h(\ell)}, rp^{-h(\ell)}]}$$

are strictly exact for any truncation index n . The corresponding radii satisfy the corresponding relation $0 < s \leq r \leq r_0$, and ℓ is bigger than some existing truncated integer $\ell_0 \geq 0$.

Proof. See [KL16, Lemma 5.6.6]. □

Lemma 3.6.62. (After Kedlaya-Liu [KL16, Lemma 5.6.8]) *Suppose we have a direct system of Banach rings (B_i, ι_i) where the corresponding map ι_i as in [KL16, Lemma 5.6.8] is submetric for each i . As in [KL16, Lemma 5.6.8] we assume the direct limit is endowed with the corresponding infimum seminorm. Now assume that $B \rightarrow C$ is an isometry and of the corresponding image which is assumed to be dense into a Banach ring C . Then we have that one can have the chance to have that any finitely generated right projective module over the ring C could come from the corresponding one finitely generated right projective module over some ring B_i after the corresponding base change.*

Proof. We need to work in the corresponding noncommutative setting. We adapt the argument in the proof of [KL16, Lemma 5.6.8] with some possible modification due to the fact that the rings are noncommutative. Let O be the matrix attached to the corresponding projector associated to the module over the ring C . Now choose P such that $\|O - P\| < \|O\|^{-3}$ (and we need to guarantee that this P could live over some B_i). Now

we consider the corresponding iteration where $Q_0 := P$ and set $Q_{k+1} = 3Q_k^2 - 2Q_k^3$, so we have:

$$Q_{k+1} - Q_k = (1 - 2Q_k)(Q_k^2 - Q_k), \quad (3.6.42)$$

$$Q_{k+1}^2 - Q_{k+1} = (4Q_k^2 - 4Q_k - 3)(Q_k^2 - Q_k)^2, \quad (3.6.43)$$

which implies that by induction (see the estimate on $P^2 - P$ below):

$$\|Q_{k+1} - O\| \leq \|O\| \|P^2 - P\|, \quad (3.6.44)$$

$$\|Q_k^2 - Q_k\| \leq (\|P^2 - P\| \|O\|^2)^{2^k} \|O\|^{-2}. \quad (3.6.45)$$

We estimate $P^2 - P$ in the following way:

$$P^2 - P = P^2 - P - O^2 + O \quad (3.6.46)$$

$$= (P - O)(P + O - 1) + OP - PO \quad (3.6.47)$$

$$= (P - O)(P + O - 1) + OP - O^2 + O^2 - PO \quad (3.6.48)$$

$$= (P - O)(P + O - 1) + O(P - O) + (O - P)O \quad (3.6.49)$$

which implies that actually:

$$\|P^2 - P\| \leq \sup\{\|(P - O)(P + O - 1)\|, \|O(P - O)\|, \|(O - P)O\|\} \quad (3.6.50)$$

$$\leq \sup\{\|(P - O)(P + O - 1)\|, \|O\| \|(P - O)\|, \|(O - P)\| \|O\|\} \quad (3.6.51)$$

$$< \|O\|^{-3} \|O\| \quad (3.6.52)$$

$$< \|O\|^{-2} \quad (3.6.53)$$

which by taking the limit gives rise to the desired matrix Q which proves the result. Indeed one can consider the corresponding projection induced by Q from some free module over B_i , which gives rise to the desired projective module over B_i whose base change to C gives the initially chosen finite projective module over C since one can compute:

$$OQ + (1 - O)(1 - Q) - 1 = OQ - O^2 - Q^2 + OQ \quad (3.6.54)$$

$$= O(Q - O) - (Q - O)Q \quad (3.6.55)$$

which implies that $\|OQ + (1 - O)(1 - Q) - 1\| \leq 1$.

□

Remark 3.6.63. The bound for $Q_k - O$ is not the same as in [KL16, Lemma 5.6.8], we would like to thank Professor Kedlaya for letting us know this correction.

Proposition 3.6.64. (After Kedlaya-Liu [KL16, Lemma 5.6.9]) *With the corresponding notations as above we have that the corresponding base change from the ring taking the form of $\check{\Pi}_{H,A}^{[s,r]}$ to the ring taking the form of $\tilde{\Pi}_{H,A}^{[s,r]}$ establishes the corresponding equivalence on the corresponding categories of Γ -modules.*

Proof. This is a relative version of the corresponding result in [KL16, Lemma 5.6.9] we adapt the corresponding argument here. Indeed the corresponding fully faithfulness comes from the previous proposition the idea to prove the corresponding essential surjectivity comes from writing the module M over the ring taking form of $\tilde{\Pi}_{H,A}^{[s,r]}$ as the base change from $\varphi^{-k}(\Pi_{H,A}^{[s,r]})$ of a module M_0 after the corresponding analog of [KL16, Lemma 5.6.8] as above. Then as in [KL16, Lemma 5.6.9] we consider the corresponding norms on the corresponding M (the corresponding differentials) and the base change of M_0 (the corresponding differentials) which could be controlled up to some constant which could be further modified to be zero by reducing each time positive amount of constant from the constant represented by the difference of the norms. \square

Definition 3.6.65. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the period rings $\Pi_{H,A}$ or $\check{\Pi}_{H,A}$ (which is denoted by Δ in this definition) we define the corresponding (φ^a, Γ) -modules over Δ which are respectively projective to be the corresponding finite projective right Γ -modules over Δ with further assigned semilinear action of the operator φ^a with the isomorphism defined by using the Frobenius.

Definition 3.6.66. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over each rings $\Delta = \Pi_{H,A}^r, \check{\Pi}_{H,A}^r$ we define the corresponding projective (φ^a, Γ) -module over any Δ to be the corresponding finite projective right Γ -module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*} M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \Pi_{H,A}^{r/p}, \check{\Pi}_{H,A}^{r/p}$.

Definition 3.6.67. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Again as in [KL16, Definition 5.7.2], we define the corresponding projective (φ^a, Γ) -modules over ring $\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ to be the finite projective right Γ -modules (which will be denoted by M) over

$\Pi_{H,A}^{[s,r]}$ or $\check{\Pi}_{H,A}^{[s,r]}$ respectively additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\Pi_{H,A}^{[sp^{-ah},rp^{-ah}]}} \Pi_{H,A}^{[s,rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\Pi_{H,A}^{[s,r]}} \Pi_{H,A}^{[s,rp^{-ah}]} \quad (3.6.56)$$

and

$$\varphi^{a*} M \otimes_{\check{\Pi}_{R,A}^{[sp^{-ah},rp^{-ah}]}} \check{\Pi}_{R,A}^{[s,rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\check{\Pi}_{R,A}^{[s,r]}} \check{\Pi}_{R,A}^{[s,rp^{-ah}]} \quad (3.6.57)$$

Definition 3.6.68. (After Kedlaya-Liu [KL16, Definition 5.7.2])

Over the ring $\Pi_{H,A}^t$ or $\check{\Pi}_{H,A}^t$ we define a corresponding projective (φ^a, Γ) bundle to be a family $(M_I)_I$ of finite projective right Γ -modules over each $\check{\Pi}_{H,A}^I$ carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\Pi_{H,A}^J} \Pi_{H,A}^I \xrightarrow{\sim} M_I$$

and

$$M_J \otimes_{\check{\Pi}_{H,A}^J} \check{\Pi}_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, r]$ involved we have $s \leq r/p^{ah}$. We require the corresponding topological conditions as we did for the corresponding Frobenius bundles. one can take the corresponding 2-limit in the direct sense to define the corresponding objects over the full Robba rings.

Definition 3.6.69. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the period rings $\check{\Pi}_{H,A}$ (which is denoted by Δ in this definition) we define the corresponding (φ^a, Γ) -modules over Δ which are respectively projective to be the corresponding finite projective right Γ -modules over Δ with further assigned semilinear action of the operator φ^a with the isomorphism defined by using the Frobenius.

Definition 3.6.70. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over each ring $\Delta = \check{\Pi}_{H,A}^r$ we define the corresponding projective (φ^a, Γ) -module over any Δ to be the corresponding finite projective right Γ -module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*} M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \check{\Pi}_{H,A}^{r/p}$.

Definition 3.6.71. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Again as in [KL16, Definition 5.7.2], we define the corresponding projective (φ^a, Γ) -modules over ring $\widetilde{\Pi}_{H,A}^{[s,r]}$ to be the finite projective right Γ -modules (which will be denoted by M) over $\widetilde{\Pi}_{H,A}^{[s,r]}$ additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\widetilde{\Pi}_{H,A}^{[sp^{-ah}, rp^{-ah}]}} \widetilde{\Pi}_{H,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\widetilde{\Pi}_{H,A}^{[s,r]}} \widetilde{\Pi}_{H,A}^{[s, rp^{-ah}]}.$$
 (3.6.58)

Definition 3.6.72. (After Kedlaya-Liu [KL16, Definition 5.7.2])

Over the ring $\widetilde{\Pi}_{H,A}^t$ we define a corresponding projective (φ^a, Γ) bundle to be a family $(M_I)_I$ of finite projective right Γ -modules over each $\widetilde{\Pi}_{H,A}^I$ carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\widetilde{\Pi}_{H,A}^J} \widetilde{\Pi}_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, r]$ involved we have $s \leq r/p^{ah}$. We require the corresponding topological conditions as we did for the corresponding Frobenius bundles. one can take the corresponding 2-limit in the direct sense to define the corresponding objects over the full Robba rings.

Conjecture 3.6.73. (After Kedlaya-Liu [KL16, Theorem 5.7.5]) *We have now the following categories are equivalence for the corresponding radii $0 < s \leq r \leq r_0$ (with the further requirement as in [KL16, Theorem 5.7.5] that $s \in (0, r/q]$):*

1. *The category of all the finite projective sheaves over the ring $\widetilde{\Pi}_{\text{Spa}(H_0, H_0^+), A}$, carrying the φ^a -action;*
2. *The category of all the finite projective sheaves over the ring $\widetilde{\Pi}_{\text{Spa}(H_0, H_0^+), A}^r$, carrying the φ^a -action;*
3. *The category of all the finite projective sheaves over the ring $\widetilde{\Pi}_{\text{Spa}(H_0, H_0^+), A}^{[s,r]}$, carrying the φ^a -action;*
4. *The category of all the finite projective modules over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
5. *The category of all the finite projective bundles over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
6. *The category of all the finite projective modules over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;*

7. The category of all the finite projective bundles over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;
8. The category of all the finite projective modules over the ring $\check{\Pi}_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action;
9. The category of all the finite projective modules over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;
10. The category of all the finite projective bundles over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;
11. The category of all the finite projective modules over the ring $\tilde{\Pi}_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action.

Proposition 3.6.74. (After Kedlaya-Liu [KL16, Theorem 5.7.5]) *We have now the corresponding equivalence among categories described as below:*

1. The category of all the finite projective modules over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;
2. The category of all the finite projective modules over the ring $\check{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action;
3. The category of all the finite projective modules over the ring $\tilde{\Pi}_{H,A}$, carrying the (φ^a, Γ) -action.

Proof. These are proved exactly the same as [KL16, Theorem 5.7.5] by using our development. □

3.6.7 Noncommutative Descent

Motivated by the corresponding comparison between the bundles and modules carrying the corresponding action coming from the Frobenius and the group Γ we study some noncommutative descent, which takes the form which is more representation theoretic and topos theoretic. Note that the corresponding localization and descent are very complicated issues within the corresponding noncommutative geometry.

Here we study the corresponding noncommutative descent in the style coming from [KL15, Section 1.3]. The corresponding story in our setting will be a noncommutative analog of [KL15, Section 1.3]. Also we mention that [DLLZ1] has already considered some

interesting generalization of [KL15, Section 1.3] along the other direction within the study of the corresponding logarithmic spaces.

Setting 3.6.75. (Noncommutative Glueing) Consider the following noncommutative analog of [KL15, Section 1.3]. First we have a square diagram taking the following form:

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi_2 \\ \downarrow & & \downarrow \\ \Pi_1 & \longrightarrow & \Pi_{12}, \end{array}$$

which expands to the corresponding short exact sequence:

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \xrightarrow{*-} \Pi_{12} \longrightarrow 0$$

in the corresponding category of Π -modules. The corresponding datum consists also of three right modules M_1, M_2, M_{12} over the corresponding rings Π_1, Π_2, Π_{12} respectively, with the corresponding base change isomorphisms from the module M_1, M_2 to the module M_{12} . Take the corresponding kernel we have the corresponding exact sequence:

$$0 \longrightarrow M \longrightarrow M_1 \oplus M_2 \longrightarrow M_{12}.$$

We are going to call the corresponding datum in our situation to be *coherent, pseudocoherent, finite, finitely presented, finite projective* if the corresponding modules involved are so over the corresponding rings Π_1, Π_2, Π_{12} , which is to say *coherent, pseudocoherent, finite, finitely presented, finite projective*.

Lemma 3.6.76. (After Kedlaya-Liu [KL15, Lemma 1.3.8]) *For finite datum defined in the corresponding sense, suppose we further assume that the corresponding map $M \otimes \Pi_1 \rightarrow M_1$ is surjective. Then we have that the corresponding maps $M \otimes \Pi_2 \rightarrow M_2$ and $f_1 - f_2 : M_1 \oplus M_2 \rightarrow M_{12}$ are also being surjective, and one can find a finite submodule of M such that the corresponding base change to the R_1 or R_2 maps through surjective maps to the corresponding module M_1 or M_2 respectively.*

Proof. The proof in our noncommutative setting is actually parallel to [KL15, Lemma 1.3.8]. To be more precise the map $f_1 - f_2 : M_1 \oplus M_2 \rightarrow M_{12}$ is then surjective is just because we can derive this from the corresponding surjectivity of $M \otimes \Pi_1 \rightarrow M_1$. The

surjectivity of $M \otimes \Pi_1 \rightarrow M_1$ implies that the corresponding surjectivity of $M \otimes (\Pi_1 \oplus \Pi_2) \rightarrow M_{12}$ which factor through the corresponding desired map in our situation. Then after proving this one can show the corresponding surjectivity of the map $M \otimes \Pi_2 \rightarrow M_2$. Indeed suppose we start from some element $m \in M_2$ then by taking the corresponding base change to Π_{12} we can have corresponding elements m_1 and m_2 living in the image of the maps:

$$g_1 : M \otimes \Pi_1 \rightarrow M_1, \tag{3.6.59}$$

$$g_2 : M \otimes \Pi_2 \rightarrow M_2, \tag{3.6.60}$$

which gives the corresponding element $f_1(m_1) - f_2(m_2)$ which is just the corresponding image of the element m in Π_{12} . Then look at the element $(m_2, m_2 + m)$ in the direct sum, then one can see that the image of the map in our mind will contain actually the element m_2 and $m + m_2$ at the same times, which implies the result. Finally the last statement then follows from this. \square

Lemma 3.6.77. (After Kedlaya-Liu [KL15, Lemma 1.3.9]) *For finitely projective datum, suppose that we have the surjectivity of the map $M \otimes \Pi_1 \rightarrow M_1$. Then we have the kernel M is finitely presented and we have the isomorphisms $M \otimes \Pi_1 \rightarrow M_1$ and $M \otimes \Pi_2 \rightarrow M_2$.*

Proof. By applying the previous lemma one can have the chance to find a submodule M' of M which is finite which admits a covering from some finite free module G . Then set G_1, G_2, G_{12} to be the corresponding base change of G to the corresponding rings

Π_1, Π_2, Π_{12} . Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F_{12} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & G_1 \oplus G_2 & \longrightarrow & G_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Here the module F, F_1, F_2, F_{12} are the corresponding kernel of the corresponding maps in the diagram, namely $G \rightarrow M, G_1 \rightarrow M_1, G_2 \rightarrow M_2, G_{12} \rightarrow M_{12}$ respectively. Now for each $j = \{1, 2\}$ we have the corresponding exact sequence:

$$0 \longrightarrow F_j \otimes_{\Pi_j} \Pi_{12} \longrightarrow G_{12} \longrightarrow M_{12} \longrightarrow 0$$

since the corresponding module M_j ($j \in \{1, 2\}$) is finite projective. We then have the situation that $F_j \otimes_{\Pi_j} \Pi_{12} \xrightarrow{\sim} F_{12}$ for each $j \in \{1, 2\}$ and the corresponding modules F_1, F_2 are finitely generated and projective. Now apply the previous lemma we have that

$F_1 \oplus F_2 \rightarrow F_{12}$ is then surjective which gives the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & F_1 \oplus F_2 & \xrightarrow{\gamma} & F_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & G_1 \oplus G_2 & \xrightarrow{\beta} & G_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \xrightarrow{\alpha} & M_{12} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Then we claim that then the map $G \rightarrow M$ is surjective. Indeed this is by direct diagram chasing. First take any element in $M_1 \oplus M_2$, which is denoted by (m_1, m_2) for which we assume that $(m_1, m_2) \in \text{Ker}\alpha$, then take any element in $(f_1, f_2) \in G_1 \oplus G_2$ lifting this element. Let $\overline{f_1 - f_2}$ be the image of (f_1, f_2) under β . By the commutativity of the diagram we have $\overline{f_1 - f_2}$ dies in M_{12} , so we can find $g_{12} \in F_{12}$ whose image in G_{12} is $\overline{f_1 - f_2}$. Then take any element element (g_1, g_2) in $F_1 \oplus F_2$ which lifts f_{12} . Then since the diagram is commutative we then have that (g_1, g_2) takes image in $G_1 \oplus G_2$ which must be living in same equivalence class with (f_1, f_2) with respect to G , so we have $(f_1, f_2) = (g_1, g_2) \oplus f$ where $f \in G$. But then we have f which is an element in the kernel of β maps to (m_1, m_2) which finishes the proof of the claim. Then repeat this we can have the chance to derive the finiteness of the module F . Then to finish we look at the corresponding commutative diagram:

$$\begin{array}{ccccccc}
 \text{Ker}(G \rightarrow M) \otimes \Pi_i & \longrightarrow & G_i & \longrightarrow & M \otimes \Pi_i & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Ker}(G_i \rightarrow M_i) & \longrightarrow & G_i & \longrightarrow & M_i \longrightarrow 0,
 \end{array}$$

which implies that the corresponding map from $M \otimes \Pi_i$ to M_i is injective as well by five lemma (note that the corresponding map $\text{Ker}(G \rightarrow M) \otimes \Pi_i$ to $\text{Ker}(G_i \rightarrow M_i)$ is surjective). So we have the corresponding desired isomorphisms. \square

The following is a similar noncommutative version of [DLLZ1, Lemma A.3]:

Lemma 3.6.78. (After Kedlaya-Liu [KL15, Lemma 1.3.9]) *For finitely presented datum, suppose that we have the surjectivity of the map $M \otimes \Pi_1 \rightarrow M_1$. And suppose we have the corresponding additional requirement that $\Pi_1 \rightarrow \Pi_{12}$ and $\Pi_2 \rightarrow \Pi_{12}$ are now assumed to be flat. Then we have the kernel M is finitely presented and we have the isomorphisms $M \otimes \Pi_1 \rightarrow M_1$ and $M \otimes \Pi_2 \rightarrow M_2$.*

Proof. By applying the previous lemma one can have the chance to find a submodule M' of M which is finite which admits a covering from some finite free module G . Then set G_1, G_2, G_{12} to be the corresponding base change of G to the corresponding rings Π_1, Π_2, Π_{12} . Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F_{12} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & G_1 \oplus G_2 & \longrightarrow & G_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Here the module F, F_1, F_2, F_{12} are the corresponding kernel of the corresponding maps in the diagram, namely $G \rightarrow M, G_1 \rightarrow M_1, G_2 \rightarrow M_2, G_{12} \rightarrow M_{12}$ respectively. Now for each $j = \{1, 2\}$ we have the corresponding exact sequence:

$$0 \longrightarrow F_j \otimes_{\Pi_j} \Pi_{12} \longrightarrow G_{12} \longrightarrow M_{12} \longrightarrow 0$$

since we have the corresponding additional requirement that $\Pi_1 \rightarrow \Pi_{12}$ and $\Pi_2 \rightarrow \Pi_{12}$ are now assumed to be flat. We then have the situation that $F_j \otimes_{\Pi_j} \Pi_{12} \xrightarrow{\sim} F_{12}$ for each $j \in \{1, 2\}$. The corresponding modules F_1, F_2 are finitely generated by assumption. Now applying the previous lemma we have that $F_1 \oplus F_2 \rightarrow F_{12}$ is then surjective which gives the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & F_1 \oplus F_2 & \xrightarrow{\gamma} & F_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & G_1 \oplus G_2 & \xrightarrow{\beta} & G_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \xrightarrow{\alpha} & M_{12} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Then we claim that then the map $G \rightarrow M$ is surjective. Indeed this is by direct diagram chasing. First take any element in $M_1 \oplus M_2$, which is denoted by (m_1, m_2) for which we assume that $(m_1, m_2) \in \text{Ker}\alpha$, then take any element in $(f_1, f_2) \in G_1 \oplus G_2$ lifting this element. Let $\overline{f_1 - f_2}$ be the image of (f_1, f_2) under β . By the commutativity of the diagram we have $\overline{f_1 - f_2}$ dies in M_{12} , so we can find $g_{12} \in F_{12}$ whose image in G_{12} is $\overline{f_1 - f_2}$. Then take any element element (g_1, g_2) in $F_1 \oplus F_2$ which lifts f_{12} . Then since the diagram is commutative we then have that (g_1, g_2) takes image in $G_1 \oplus G_2$ which must be living in same equivalence class with (f_1, f_2) with respect to G , so we have $(f_1, f_2) = (g_1, g_2) \oplus f$ where $f \in G$. But then we have f which is an element in the kernel of β maps to (m_1, m_2) which finishes the proof of the claim. Then repeat this we can have the chance to derive the finiteness of the module F . Then to finish we look at the

corresponding commutative diagram:

$$\begin{array}{ccccccc}
& \text{Ker}(G \rightarrow M) \otimes \Pi_i & \longrightarrow & G_i & \longrightarrow & M \otimes \Pi_i & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Ker}(G_i \rightarrow M_i) & \longrightarrow & G_i & \longrightarrow & M_i \longrightarrow 0,
\end{array}$$

which implies that the corresponding map from $M \otimes \Pi_i$ to M_i is injective as well by five lemma (note that the corresponding map $\text{Ker}(G \rightarrow M) \otimes \Pi_i$ to $\text{Ker}(G_i \rightarrow M_i)$ is surjective). So we have the corresponding desired isomorphisms. \square

Now we consider the corresponding Banach structures as well. In this case we consider the following glueing process. First we have the following analog of [KL15, Lemma 2.7.1]:

Lemma 3.6.79. (After Kedlaya-Liu [KL15, Lemma 2.7.1]) *Suppose we have the following two homomorphism of Banach rings (which are not necessarily commutative in our situation):*

$$f_1 : \Pi_1 \rightarrow \Gamma \tag{3.6.61}$$

$$f_2 : \Pi_2 \rightarrow \Gamma \tag{3.6.62}$$

such that they are bounded and we have the following sum map induced which is surjective in the strict sense:

$$\Pi_1 \oplus \Pi_2 \rightarrow \Gamma.$$

Then one can find some constant c with the fact that for any positive integer $n \geq 1$, we have the corresponding decomposition of any invertible matrix $I \in \Gamma^{n \times n}$ such that $\|I - 1\| \leq c$ in the following form:

$$f_1(I_1)f_2(I_2) \tag{3.6.63}$$

with invertible matrices I_1, I_2 living in $\Pi_1^{n \times n}$ and $\Pi_2^{n \times n}$ respectively.

Proof. Following [KL15, Lemma 2.7.1], we consider the corresponding lifting process. First by the conditions of the lemma we have that one can lift $I - 1$ to a pair matrices

A, B living in $\text{GL}(\Pi_1)$ and $\text{GL}(\Pi_2)$ with the corresponding estimate on the entries:

$$\|A_{ij}\| \leq c_0 \|(I - 1)_{ij}\| \|B_{ij}\| \leq c_0 \|(I - 1)_{ij}\|$$

for some constant c_0 . Then consider the corresponding iteration process, namely from what we have set I' to be $f_1(1 - A)If_2(1 - B)$ such that we have the corresponding estimate in the form of $\|I' - 1\| \leq d\|I - 1\|^2$. Then to finish run the construction process above step by step, we will be done. \square

Now in the following setting, we set all rings involved to be Banach:

Setting 3.6.80. (Noncommutative Glueing) Consider the following noncommutative analog of [KL15, Section 1.3]. First we have a square diagram taking the following form which is commutative:

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi_2 \\ \downarrow & & \downarrow \\ \Pi_1 & \longrightarrow & \Pi_{12}, \end{array}$$

which expands to the corresponding short exact sequence:

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \xrightarrow{*-} \Pi_{12} \longrightarrow 0$$

in the corresponding category of Π -modules. The corresponding datum consists also of three right modules M_1, M_2, M_{12} over the corresponding rings Π_1, Π_2, Π_{12} respectively, with the corresponding base change isomorphisms from the module M_1, M_2 to the module M_{12} . Take the corresponding kernel we have the corresponding exact sequence:

$$0 \longrightarrow M \longrightarrow M_1 \oplus M_2 \longrightarrow M_{12}.$$

We are going to call the corresponding datum in our situation to be *coherent, pseudocoherent, finite, finitely presented, finite projective* if the corresponding modules involved are so over the corresponding rings Π_1, Π_2, Π_{12} , which is to say *coherent, pseudocoherent, finite, finitely presented, finite projective*, and we require that the corresponding exact sequence:

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \xrightarrow{*-} \Pi_{12} \longrightarrow 0$$

is surjective in the strict sense, and we require that the map:

$$\Pi_2 \rightarrow \Pi_{12}$$

has the image which is assumed in our situation to be dense.

Lemma 3.6.81. (After Kedlaya-Liu [KL15, Lemma 2.7.3]) *Consider a finite glueing datum as above, we then have the fact that $M \otimes \Pi_1 \rightarrow M_1$ and $M \otimes \Pi_2 \rightarrow M_2$ are surjective and the map $M_1 \oplus M_2 \rightarrow M_{12}$ is also surjective.*

Proof. This is a noncommutative version of [KL15, Lemma 2.7.3]. We just need to check that in our situation the condition of lemma 3.6.76 holds. Now consider the bases of M_1 and M_2 respectively:

$$e_1, e_2, \dots, e_m, \tag{3.6.64}$$

$$f_1, f_2, \dots, f_m. \tag{3.6.65}$$

And we consider the corresponding expansion with respect to these bases after taking the image under the maps $s_1 : M_1 \otimes \Pi_{12} \rightarrow M_{12}$ and $s_2 : M_2 \otimes \Pi_{12} \rightarrow M_{12}$:

$$s_2(f_j) = \sum_i I_{ij} s_1(e_i), s_1(f_j) = \sum_i J_{ij} s_2(f_i).$$

Then we have from the assumption on the dense image some matrix J' such that $I(J' - J)$ could be satisfying the condition $\|I(J' - J)\| \leq c$ where c is the constant in the previous lemma. Then by using the previous lemma we have the following decomposition:

$$1 + I(J' - J) = K_1 K_2^{-1}$$

with $K_1 \in \text{GL}(\Pi_1), K_2 \in \text{GL}(\Pi_2)$. Then we consider the element:

$$(a_j, b_j) := \left(\sum_i K_{1,ij} e_i, \sum_i (J' K_2)_{ij} f_i \right).$$

We compute as in the following:

$$s_1(a_j) - s_2(b_j) = \sum_i K_{1,ij} s_1(e_i) - \sum_i J' K_{2,ij} s_2(f_i) \tag{3.6.66}$$

$$= \sum_i K_{1,ij} s_1(e_i) - (I J' K_2)_{ij} s_1(e_i) \tag{3.6.67}$$

$$= \sum_i ((1 - I J) K_2)_{ij} s_1(e_i) \tag{3.6.68}$$

$$= 0. \tag{3.6.69}$$

Therefore the element we defined is now living in the kernel M . Consider just the first component we can see that the map $M \otimes \Pi_1 \rightarrow M_1$ is now surjective. \square

Lemma 3.6.82. (After Kedlaya-Liu [KL15, Lemma 2.7.4]) *For finitely projective datum, we then in our situation have the surjectivity of the map $M \otimes \Pi_1 \rightarrow M_1$. Then we have the kernel M is finitely presented and we have the isomorphisms $M \otimes \Pi_1 \rightarrow M_1$ and $M \otimes \Pi_2 \rightarrow M_2$.*

Proof. This is by the previous lemma and lemma 3.6.77. \square

The following is a similar noncommutative version of [DLLZ1, Lemma A.7]:

Lemma 3.6.83. (After Kedlaya-Liu [KL15, Lemma 2.7.4]) *For finitely presented datum, we then in our situation have the surjectivity of the map $M \otimes \Pi_1 \rightarrow M_1$. Furthermore suppose we are in the situation where $\Pi_1 \rightarrow \Pi_{12}$ and $\Pi_2 \rightarrow \Pi_{12}$ are simultaneously assumed to be flat. Then we have the kernel M is finitely presented and we have the isomorphisms $M \otimes \Pi_1 \rightarrow M_1$ and $M \otimes \Pi_2 \rightarrow M_2$.*

Proof. See lemma above and lemma 3.6.78. \square

Definition 3.6.84. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over the period rings $\Pi_{H,A}$ (which is denoted by Δ in this definition) we define the corresponding (φ^a, Γ) -modules over Δ which are pseudocoherent or fpd to be the corresponding pseudocoherent or fpd right Γ -modules over Δ with further assigned semilinear action of the operator φ^a with the isomorphism defined by using the Frobenius. We also require that the modules are complete for the natural topology involved in our situation and for any module over $\Pi_{H,A}$ to be some base change of some module over $\Pi_{H,A}^r$ (which will be defined in the following).

Definition 3.6.85. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Over each ring $\Delta = \Pi_{H,A}^r$ we define the corresponding pseudocoherent or fpd (φ^a, Γ) -module over any Δ to be the corresponding pseudocoherent or fpd right Γ -module M over Δ with additionally endowed semilinear Frobenius action from φ^a such that we have the isomorphism $\varphi^{a*} M \xrightarrow{\sim} M \otimes \square$ where the ring \square is one $\Delta = \Pi_{H,A}^{r/p}$. Also as in [KL16, Definition 5.7.2] we assume that the module over $\Pi_{H,A}^r$ is then complete for the natural topology and the corresponding base change to $\Pi_{H,A}^I$ for any interval which is assumed to be closed $I \subset [0, r)$ gives rise to a module over $\Pi_{H,A}^I$ with specified conditions which will be specified below.

Definition 3.6.86. (After Kedlaya-Liu [KL16, Definition 5.7.2]) Again as in [KL16, Definition 5.7.2], we define the corresponding pseudocoherent and fpd (φ^a, Γ) -modules over ring $\Pi_{H,A}^{[s,r]}$ to be the pseudocoherent and fpd right Γ -modules (which will be denoted by M) over $\Pi_{H,A}^{[s,r]}$ additionally endowed with semilinear Frobenius action from φ^a with the following isomorphisms:

$$\varphi^{a*} M \otimes_{\Pi_{H,A}^{[sp^{-ah}, rp^{-ah}]}} \Pi_{H,A}^{[s, rp^{-ah}]} \xrightarrow{\sim} M \otimes_{\Pi_{H,A}^{[s,r]}} \Pi_{H,A}^{[s, rp^{-ah}]}.$$
 (3.6.70)

We now assume that the modules are complete with respect to the natural topology.

Definition 3.6.87. (After Kedlaya-Liu [KL16, Definition 5.7.2])

Over the ring $\Pi_{H,A}^t$ we define a corresponding pseudocoherent and fpd (φ^a, Γ) bundle to be a family $(M_I)_I$ of pseudocoherent and fpd right Γ -modules over each $\Pi_{H,A}^I$ carrying the natural Frobenius action coming from the operator φ^a such that for any two involved intervals having the relation $I \subset J$ we have:

$$M_J \otimes_{\Pi_{H,A}^J} \Pi_{H,A}^I \xrightarrow{\sim} M_I$$

with the obvious cocycle condition. Here we have to propose condition on the intervals that for each $I = [s, r]$ involved we have $s \leq r/p^{ah}$. We require the corresponding topological conditions as above. one can take the corresponding 2-limit in the direct sense to define the corresponding objects over the full Robba rings.

Remark 3.6.88. We have the obvious notion of coherent objects in the noetherian setting. Also note that in the noetherian setting we do not have to worry about the corresponding completion issue, which to say the finiteness is all enough. In our current situation and furthermore also in the noetherian setting, by using the corresponding open mapping, one can also mimick the proof in [BGR84, 3.7.3/2,3] to prove this by considering the corresponding presentation and take the corresponding kernel.

Proposition 3.6.89. *Keeping assumption 3.6.42 and under the assumption of conjecture 3.6.73, consider the following categories:*

1. *The category of all the pseudocoherent bundles over the ring $\Pi_{H,A}$, carrying the (φ^a, Γ) -action;*
2. *The category of all the pseudocoherent modules over the ring $\Pi_{H,A}^{[s,r]}$, carrying the (φ^a, Γ) -action.*

Then in our situation they are equivalent, if we have for any intervals permitted $I \subset I'$ we have $\Pi_{H,A}^{I'} \rightarrow \Pi_{H,A}^I$ is flat.

Proof. First note that we are working over noetherian rings. The functor from the category of the Frobenius bundles to that of the Frobenius modules is just projection of the family of Frobenius modules to the corresponding Frobenius module with respect to the specific interval. In our our situation we have

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \xrightarrow{*-}*} \Pi_{12} \longrightarrow 0$$

is exact in the strict sense due to the existence of Schauder basis, and we have the image of $\Pi_2 \rightarrow \Pi_{12}$ is dense as well in the same way. Therefore we can see that the corresponding categories are equivalent since we can show the corresponding essential surjectivity by using the Frobenius action and the glueing process established above to reach any closed interval involved. \square

This chapter is based on the following paper, where the author of this dissertation is the main author:

- Tong, Xin. "Hodge-Iwasawa Theory II." arXiv preprint arXiv:2010.06093 (2020).

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Chapter 4

Analytic Geometry and Hodge-Frobenius Structure

4.1 Introduction

4.1.1 Higher Dimensional Analytic Geometry

Higher dimensional analytic geometry in the nonarchimedean setting and corresponding p -adic analysis received very rapid development in the past decades, dated back to the last century to some point. Its complex analog, which is to say the complex analysis in higher dimension of several variables or the study of complex analytic varieties could be dated back even earlier. For instance searching for the automorphism groups of higher dimensional domains is still attracting extensive attention. On the other hand in the p -adic setting, considering higher dimensional analytic domains will be a very natural generalization (especially in the rigid analytic geometry, or more general analytic geometry). However, the approaches do not generalize very smoothly to the higher dimension, even, one will immediately need to find new approaches. This is actually already clear in the complex setting, for instance one may believe that complex analysis of several variables over complex polydiscs is natural generalization of one variable situation, for instance the Cauchy integration formula and maximal value issues. However in the p -adic setting, this might be more complicated as discussed in the following.

The famous p -adic local monodromy theorem, proved by Tsuzuki in the unit-root setting, and then proved by André-Kedlaya-Mebkhout [Ked04], [And1] and [Me] in the full general setting, is already a very difficult theorem over one-dimensional p -adic annulus, which roughly speaks that over such annulus, each finite locally free sheaf with connection and Frobenius (not necessarily absolute) structure could be made simpler with respect to the connection structure. For instance in the unit-root setting, the picture is even simpler, since then we only need to consider Galois representations with finite monodromy which is to say potentially unramified somewhat: we have an equivalence between the category of unit-root overconvergent isocrystals and the one of all the potentially unramified étale \mathbb{Q}_p -local systems, over some space. Actually Kedlaya's approach established in [Ked04] used very deep theorem on the slope filtration over one-dimensional p -adic annulus to reduce to Tsuzuki's approach.

To generalize to the higher dimensional situation, we have many ways to do so

which are encoded in the following projects. First one could consider the picture in the relative p -adic Hodge theory, for instance established in [KL15] and [KL16], or the multidimensional program of [CKZ18] and [PZ19]. Also we would like to mention some relativization programs in [BC1], [Ked10], [Liu1] where some attempt to generalize the slope filtration theorem and the local monodromy theorem to the relative Frobenius situation had been successful. The applications from [Ked04] to the rigid cohomology and isocrystals have already considered some sort of multidimensional construction as in [Ked06],[Ked07],[Ked08],[Ked11],[Ked09],[Ked07]. However, one could still only get information deduced from the monodromy theorem or the slope filtration theorem essentially from the one mentioned above. The theory of partial differential equations have been regarded as non-trivial and significant generalization of the story relevant above. To be more precise, these are reflected in the following work: [Xiao11], [Xiao12] and [KX10], [Ked10], [Ked11], [Ked13].

That being all said, we will only focus on mixed-characteristic situation in this paper. Here let us do some summarization in a uniform way on the known pictures which one might want to follow being local or global, being étale or non-étale:

Setting 4.1.1. Mixed-characteristic local étale: Berger’s theorem for potentially semi-stable Galois representations [Ber1].

Setting 4.1.2. Mixed-characteristic local non-étale: Again Berger’s theorem for (φ, Γ) -modules.

Setting 4.1.3. Mixed-characteristic global étale: Relative p -adic Hodge theory after Faltings, Andreatta, Iovita, Brinon, Scholze, Kedlaya, Liu and so on, in [Fal], [A1], [AI1], [AI2], [AB1], [AB2], [AB3], [Sch1], [KL15] and [KL16].

Setting 4.1.4. Mixed-characteristic global non-étale: Relative p -adic Hodge theory in the style of Kedlaya-Liu [KL15] and [KL16].

In this paper, we first consider the program of [CKZ18] and [PZ19], and we will work in the generality of [KPX14]. The generalization itself is already interesting enough and actually we need to address some fundamental issues around. The multidimensionalization we studied here represents some similarity to the globalization and multidimensionalization mentioned above in the relative p -adic Hodge theory and some globalization

of the study of isocrystals in the style of Kedlaya's theorem on Shiho's conjecture. Actually we consider slightly different actions within the module structures, which is to say that we will have multi-Frobenius actions, multi-Lie group actions, multi-differentials.

4.1.2 Results and Applications

Here is our first main result, while the corresponding proof will be given in section 4.4.1:

Theorem 4.1.5. *With the same assumptions as in assumption 4.4.1 as below, we set M to be the global section of a locally free coherent sheaf of module over the sheaf of the Robba ring $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ over an affinoid domain X attached to an affinoid algebra A in the rigid analytic geometry, carrying mutually commuting actions of $(\varphi_I, \Gamma_{K_I})$. Then we have the corresponding Herr complex $C_{\varphi_I, \Gamma_{K_I}}^\bullet(M)$ lives then in the derived category $\mathbb{D}_{\text{perf}}^b(A)$, and we have that the corresponding complex $C_{\psi_I}^\bullet(M)$ has all the cohomology groups which are coadmissible over $\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})$. Moreover when $A = \mathbb{Q}_p$, $C_{\psi_I}^\bullet(M)$ lives in $\mathbb{D}_{\text{perf}}^b(\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I}))$.*

Another motivation in our situation comes from the equivariant Iwasawa theory proposed by Kedlaya-Pottharst in [KP1]. Here the idea will be working over more general distribution algebra after Berthelot, Schneider and Teitelbaum in the following sense. The idea is from the following observation. Now suppose G is a nice p -adic Lie group for instance as above we considered the product of \mathbb{Z}_p or \mathbb{Z}_p^\times . Then we consider the corresponding distribution algebra which we will use the notation $\Pi_{\text{an},\infty,A}(G)$ to denote this ring (note that here we have already considered the even more general relative setting). Now consider L/K a p -adic Lie extension. Then we have that one could consider the projection $G_K \rightarrow G(L/K)$ to define the corresponding Galois representation with coefficients in $\Pi_{\text{an},\infty,A}(G)$, which then gives rise to a (φ, Γ_K) -module over $\Pi_{\text{an},\infty,A}(G)$. Then we take the external tensor product of any (φ, Γ) -module M with this p -adic Lie deformation **DfmLie** we will get a module $M \boxtimes \mathbf{DfmLie}$ which is actually (φ, Γ) -module over $\Pi_{\text{an},\infty,A}(G)$. Then we are going to define the Herr complex of this module, which will be called the G -equivariant Iwasawa cohomology. It is natural to investigate the fundamental properties of this cohomology which is over $\text{Max}A \times \text{Max}\Pi_{\text{an},\infty}(G)$. For instance over $\mathbb{Z}_p^{\times,d}$ as above, we will have to study the cohomology of the big (φ, Γ) -module over the

product of the ring $\Pi_{\text{an,con},A}(\pi_K)$ and the ring $\Pi_{\text{an},\infty}(G)$. Therefore through the previous theorem we have the following coherence over more general Fréchet-Stein algebras, while the corresponding proof will be given in section 4.4.1:

Theorem 4.1.6. *Let $X_\infty(G)$ be the quasi-Stein rigid analytic space attached to the p -adic Lie group $G := \Gamma_{K_J}$ for some set J possibly different from I as in setting 4.3.1. Let A be an affinoid algebra over \mathbb{Q}_p . Suppose \mathcal{F} is a locally free coherent sheaf over the sheaf of Robba ring $\Pi_{\text{an,con},I,X_\infty(G)} \widehat{\otimes}_A (\pi_{K_I})$, where we use the same notation to denote the global section and we assume that the global section is finite projective, carrying mutually commuting actions of $(\varphi_I, \Gamma_{K_I})$. Then we have that the cohomology groups of the Herr complex $C_{\varphi_I, \Gamma_I}^\bullet(\mathcal{F})$ are coadmissible over $X_\infty(G) \widehat{\otimes}_A$, and the corresponding cohomology groups of Iwasawa complex $C_{\psi_I}^\bullet(\mathcal{F})$ are coadmissible over $\Pi_{\text{an},\infty,I,X_\infty(G)} \widehat{\otimes}_A (\Gamma_{K_I})$. Moreover when $A = \mathbb{Q}_p$, $C_{\psi_I}^\bullet(\mathcal{F})$ lives in $\mathbb{D}_{\text{perf}}^b(\Pi_{\text{an},\infty,I,X_\infty(G)} \widehat{\otimes}_A (\Gamma_{K_I}))$ and $C_{\varphi_I, \Gamma_I}^\bullet(\mathcal{F})$ lives in $\mathbb{D}_{\text{perf}}^b(X_\infty(G) \widehat{\otimes}_A)$.*

Remark 4.1.7. The two theorems are around commutative period rings. Actually we conjectured that the corresponding analogs in the noncommutative setting should hold. In other words not only we would expect that one can follow the ideas in this work to establish some finiteness theorems when we have the coefficients that are noetherian noncommutative Banach rings as those discussed in section 4.5.1, but also we should as well have some analogs of the results on the (noncommutative) equivariant Iwasawa cohomology. For more detail, see conjecture 4.5.22, conjecture 4.5.23, conjecture 4.5.24.

It is actually not expected (due to some difficulty in high dimensional p -adic analysis) that one could derive the finiteness or perfectness in some ambient derived categories by using analogue of Kedlaya's key ingredient to prove the theorem mentioned in setting 1.2 above, which is to say there is no dévissage possibility transparent for us to use in order to reduce all the proof in the non-étale setting to the main results of [CKZ18] and [PZ19], in the étale setting. The coherence of the cohomology is significant in our theory, for instance if one would like to study the global triangulation of (φ, Γ) -modules over any rigid analytic spaces, for more application see the discussion below.

Now we discuss the applications of our results. The multidimensionalization represented in the program of [PZ19] and [CKZ18] is aimed at some motivation from p -adic

Local Langlands program and relative Fukaya-Kato-Nakamura's local Tamagawa number conjecture or the ε -isomorphism conjecture. One could somehow regard the two conjectures as closely related to each other to some extent. Roughly speaking we have the following conjecture:

Conjecture 4.1.8. (Nakamura [Nak2]) *In the determinant category attached to any (φ, Γ) -modules over relative Robba rings, for each such M there exists an isomorphism between the object $\mathbf{1}$ with the fundamental line attached to M . Moreover this isomorphism is compatible with algebraic functional equation, duality and the de Rham isomorphism.*

The program initiated in [PZ19] and [CKZ18] gives rise to some possible approach to this conjecture, by first generalizing this to higher dimensional situation, and then consider some sort of diagonal embedding in the group theoretic consideration. Again there should be a corresponding relative version of the conjectures of Pal-Zábrádi [PZ19].

Another application will be deformation theory and the variation of the triangulation property represented already in the one-dimensional situation, see [KPX14]. As mentioned above, local triangulation or global triangulation are significant properties in the study of Galois theoretic eigenvarieties. We should mention that actually our approach which is parallel to [KPX14] is different from the one in [Liu1]. The latter used essentially some local spreading properties of the slope filtrations, which is tested by considering the sensibility of the neighbourhood around a good point on the slope filtrations. All of these are not directly expected in the context of [KPX14] and the one we are considering in this paper. For the extended and detailed exposition, see the corresponding section 4.4.2.

4.1.3 What's Next

The framework we studied here is just some partial aspects of the whole story in the higher dimensional analytic geometry. One can definitely study more general analytic geometry even in the archimedean setting where one can search for more very typical multi-Hodge structures such as those coming from the hyper-Kähler geometry. We have only worked over polydisks and polyannuli in the strictly affinoid setting in regular rigid analytic geometry which we believe of course should be the most similar setting to the

classical complex analysis of several variables. But that being said, these restrictions are not necessary at all. One can obviously ask how general the spaces which we could handle are. We certainly believe in some very general sense that the spaces we could potentially handle are those allowing us to do the suitable p -adic functional analysis and p -adic complex analysis of several variables.

The corresponding construction we established here targets directly at the noncommutative Tamagawa number conjectures. We should then follow [Nak1], [Nak2], [Zah1], [PZ19] to amplify our programs in our mind from the philosophy in [BF1], [BF2] and [FK]. And we should believe that our approach could be also applied in many other contexts where [Kie1] and [KL] will provide insights and applications, which certainly might not have to be Iwasawa theoretic. Certainly in the context of [PZ19] one still has to investigate in more detail the corresponding p -adic Hodge theoretic properties of objects studied in this paper, such as the corresponding possible analogs of the main results in [Ked04], [And1], [Me] and [Ber1].

Notation	Description (These are mainly starting participating in the discussion in Chapter 2 in the commutative setting, while in Chapter 5 in the noncommutative setting)
p	A prime number which is > 2 .
I	A finite set.
K_α	Finite extension of \mathbb{Q}_p for each $\alpha \in I$, with chosen $\pi_{K,\alpha}$.
G_{K_α}	The corresponding Galois group of $\text{Gal}(\overline{\mathbb{Q}_p}/K_\alpha)$ for each $\alpha \in I$.
Γ_{K_α}	The corresponding Γ group of K_α for each $\alpha \in I$.
s_I, r_I	Multidimensional radii $0 < s_I \leq r_I$ with coordinates which are rational numbers.
A	An affinoid algebra (which could be noncommutative).
$\Pi_{r_I, I}$	Multidimensional convergent ring with respect to the radius r_I .
$\Pi_{[s_I, r_I], I}$	Multidimensional analytic ring with respect to the radii in $[s_I, r_I]$.
$\Pi_{\text{an}, r_I, I}$	Multidimensional analytic ring with respect to the radius r_I .
$\Pi_{\text{an}, \text{con}, I}$	Multidimensional Robba rings.

$\Pi_{r_I, I, A}$	Multidimensional convergent ring with respect to the radius r_I with coefficients in an affinoid algebra A .
$\Pi_{[s_I, r_I], I, A}$	Multidimensional analytic ring with respect to the radii in $[s_I, r_I]$ with coefficients in an affinoid algebra A .
$\Pi_{\text{an}, r_I, I, A}$	Multidimensional analytic ring with respect to the radius r_I with coefficients in an affinoid algebra A .
$\Pi_{\text{an}, \text{con}, I, A}$	Multidimensional Robba rings with coefficients in an affinoid algebra A .
$(C_\alpha, l_\alpha)_{\alpha \in I}$	Collections of pair (C_α, l_α) each of which consists of a multiplicative topological group which is abelian with finite torsion part and free part isomorphic to \mathbb{Z}_p , and a continuous homomorphism from C_α to \mathbb{Q}_p for each $\alpha \in I$.
$\Pi_{\text{an}, r_I, I}(C_I)$	Multidimensional analytic ring with respect to the radius r_I , with new variables from C_I .
$\Pi_{\text{an}, \text{con}, I}(C_I)$	Multidimensional Robba rings, with new variables from C_I .
$\Pi_{\text{an}, r_I, I, A}(C_I)$	Multidimensional analytic ring with respect to the radius r_I with coefficients in an affinoid algebra A , with new variables from C_I .
$\Pi_{\text{an}, \text{con}, I, A}(C_I)$	Multidimensional Robba rings with coefficients in an affinoid algebra A , with new variables from C_I .
$\Pi_{\text{an}, \infty, I}(C_I)$	Multidimensional analytic ring with respect to the radius r_I , with new variables from C_I .
$\Pi_{\text{an}, \infty, I, A}(C_I)$	Multidimensional analytic ring with respect to the radius r_I with coefficients in an affinoid algebra A , with new variables from C_I .

Notation	Description (These are mainly starting participating in the discussion in Chapter 3 and 4 in the commutative setting, while in Chapter 5 in the noncommutative setting)
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$\Pi_{\text{an}, r_I, I}(\pi_{K, I})$	Multidimensional analytic ring with respect to the radius r_I , with new variables in $\pi_{K, I}$.
$\Pi_{[s_I, r_I], I}(\pi_{K, I})$	Multidimensional analytic ring with respect to the radii in $[s_I, r_I]$, with new variables in $\pi_{K, I}$.

$\Pi_{\text{an,con},I}(\pi_{K,I})$	Multidimensional Robba rings, with new variables in $\pi_{K,I}$.
$\Pi_{\text{an},r_I,I,A}(\pi_{K,I})$	Multidimensional analytic ring with respect to the radius r_I with coefficients in an affinoid algebra A , with new variables in $\pi_{K,I}$.
$\Pi_{[s_I,r_I],I,A}(\pi_{K,I})$	Multidimensional analytic ring with respect to the radii in $[s_I, r_I]$ with coefficients in an affinoid algebra A , with new variables in $\pi_{K,I}$.
$\Pi_{\text{an,con},I,A}(\pi_{K,I})$	Multidimensional Robba rings with coefficients in an affinoid algebra A , with new variables in $\pi_{K,I}$.
$\Pi_{\text{an},\infty,I}(\Gamma_{K,I})$	Multidimensional analytic ring with respect to the radius r_I , with new variables from Γ_K .
$\Pi_{\text{an},\infty,I,A}(\Gamma_{K,I})$	Multidimensional analytic ring with respect to the radius r_I with coefficients in an affinoid algebra A , with new variables from Γ_K .

4.2 Fréchet Rings and Sheaves in the Commutative Case

4.2.1 Fréchet Objects

In this subsection we are going to define the main objects we are going to study ring theoretically. Everything will be modeling the one-dimensional situation. As indicated in the title of the section, we will consider Fréchet-Stein objects in the sense of [ST03]. We first proceed by defining the following rings:

Definition 4.2.1. (After KPX [KPX14, Notation 2.1.1]) We consider the multi radii as mentioned in the notation list, where $0 < s_I \leq r_I$ are all rational numbers or ∞ . Under this assumption first let I is a finite set. Then we consider the multidimensional polynomial rings generalizing the one-dimensional situation with variables $T_1^{\pm 1}, \dots, T_I^{\pm 1}$ which is to say the ring:

$$\mathbb{Q}_p[T_1^{\pm 1}, \dots, T_I^{\pm 1}].$$

Now we endow this rings with some specific Gauss valuations and norms as in the following way for any multi radii $t_I > 0$:

$$v_{t_I}(f) = v_{t_I}\left(\sum_{n_I} a_{n_I} T_I^{i_I}\right) := \inf_{n_I} \left\{ v_p(a_{n_I}) + \sum_{\alpha \in I} t_{\alpha} i_{\alpha} \right\},$$

$$\|f\|_{t_I} = \left\| \sum_{n_I} a_{n_I} T_I^{i_I} \right\|_{t_I} := \sup_{n_I} \left\{ |a_{n_I}| p^{-\sum_{\alpha \in I} t_{\alpha} i_{\alpha} / (p-1)} \right\}.$$

We are going to use the Gauss valuations and norms with respect to the multi-interval $[s_I, r_I]$ in the following sense:

$$v_{[s_I, r_I]}(f) = v_{[s_I, r_I]}\left(\sum_{n_I} a_{n_I} T_I^{i_I}\right) := \min_{t_{\alpha} \in \{s_I, r_I\}} \left\{ v_p(a_{n_I}) + \sum_{\alpha \in I} t_{\alpha} i_{\alpha} \right\},$$

$$\|f\|_{[s_I, r_I]} = \left\| \sum_{n_I} a_{n_I} T_I^{i_I} \right\|_{[s_I, r_I]} := \max_{t_{\alpha} \in \{s_I, r_I\}} \left\| \sum_{n_I} a_{n_I} T_I^{i_I} \right\|_{t_{\alpha}}.$$

The latter will give rise to the following definition:

$$\Pi_{[s_I, r_I], I}$$

which is defined to the completion of the ring $\mathbb{Q}_p[T_1^{\pm 1}, \dots, T_I^{\pm 1}]$ with respect to the norms defined above with respect to the multi-interval $[s_I, r_I]$. For the polynomial rings:

$$\mathbb{Q}_p[T_1, \dots, T_I]$$

we then instead define for some $t_I > 0$:

$$v_{t_I}(f) = v_{t_I}\left(\sum_{n_I} a_{n_I} T_I^{i_I}\right) := \inf_{n_I} \left\{ v_p(a_{n_I}) + \sum_{\alpha \in I} t_\alpha i_\alpha \right\},$$

$$\|f\|_{t_I} = \left\| \sum_{n_I} a_{n_I} T_I^{i_I} \right\|_{t_I} := \sup_{n_I} \left\{ |a_{n_I}| p^{-\sum_{\alpha \in I} t_\alpha i_\alpha / (p-1)} \right\}.$$

This will then give rise to the rings corresponding to the closed polydiscs and open polydiscs:

$$\Pi_{[t_I, \infty_I], I}, \Pi_{\text{an}, \infty_I, I} := \bigcap_{t_I > 0} \Pi_{[t_I, \infty_I], I}.$$

Then correspondingly we define the following analytic rings corresponding to the full polyannuli:

$$\Pi_{\text{an}, r_I, I} := \bigcap_{0 < s_I \leq r_I} \Pi_{[s_I, r_I], I}, \Pi_{\text{an}, \text{con}, I} := \bigcup_{r_I > 0} \bigcap_{0 < s_I \leq r_I} \Pi_{[s_I, r_I], I}.$$

Remark 4.2.2. One could actually define the rings by considering all the (rigid) analytic functions over the corresponding open polydiscs, closed polydiscs, polyannuli and full polyannuli, which will be more transparent.

Remark 4.2.3. Actually one could see that in the definition, one could find that especially in the multi-interval situation, the rightmost radii could be mixed-type with respect to the infinity which is to say that one could have a genuine proper subset of indexes where the rightmost radii are exactly infinite. We will mainly focus on the situation where such proper subset does not emerge in our discussion later, if not specified otherwise.

Now we consider the relative version of the definition as in the one dimensional situation in [KPX14, Notation 2.1.1].

Definition 4.2.4. (After KPX [KPX14, Notation 2.1.1]) Let A be an affinoid algebra over \mathbb{Q}_p in the sense of rigid analytic spaces. Then we are going to define the relative version of the rings defined as above in the absolute situation:

$$\Pi_{[s_I, r_I], I, A},$$

as the analytic functions relative to A over the products:

$$\text{Max}A \times A^I[s_I, r_I], 0 < s_I \leq r_I.$$

Then we define:

$$\Pi_{\text{an}, r_I, I, A} := \bigcap_{0 < s_I \leq r_I} \Pi_{[s_I, r_I], I, A}, \Pi_{\text{an}, \text{con}, I, A} := \bigcup_{r_I > 0} \bigcap_{0 < s_I \leq r_I} \Pi_{[s_I, r_I], I, A}.$$

Correspondingly almost in the parallel and similar way one could define the following ring of analytic functions relative to the space A :

$$\Pi_{[s_I, \infty_I], I, A}, \Pi_{\text{an}, \infty_I, I, A} := \bigcap_{s_I > 0} \Pi_{[s_I, \infty_I], I, A}.$$

Eventually we are going to work within the framework of Fréchet-Stein algebras over rigid analytic spaces. Now we are going to consider the construction over the affinoid algebras. First let $r_{I,0}$ be a fixed radii (where we do not consider the situation that there is a finite proper set of I where the radii approach the infinity). Then we consider a decreasing sequence of multi radii $\{r_{I,n}\}_{n \geq 0}$ which is to say that for each $\alpha \in I$, $r_{\alpha, n+1} \leq r_{\alpha, n}$, and when $n = 0$ the radii is the one fixed above. And we require that for each $\alpha \in I$, $r_{\alpha, 1} < r_{\alpha, 0}$.

Remark 4.2.5. Actually this looks a little bit strange, but one could eventually choose in the following way that for each $n \geq 1$ one just set $r_{\alpha, n} = r_n$ for each α .

Then we consider the following result which in some non-trivial way generalizes the one dimensional situation:

Proposition 4.2.6. *For all $n \geq 1$, the rings $\Pi_{[r_{I,n}, r_{I,0}], I, A}$ and $\Pi_{[r_{I,n}, \infty_I], I, A}$ are noetherian as Banach A -algebras.*

Proof. We are going to follow ideas in [PZ19] and [Ked19] to prove this. This will be some application of a theorem due to Fulton in [Ful1]. We are going to show the statement in the situation where A is just the field \mathbb{Q}_p , and leave the general situation to the reader. We are going to only choose to prove the statement for the ring $\Pi_{[r_{I,n}, r_{I,0}], I, A}$, and we leave the check for the other ring to the readers. We are going to show that the ring $\Pi_{[r_{I,n}, r_{I,0}], I, A}$ is weakly complete algebras over the corresponding adic completion of the

ring $\mathbb{Z}_l[T_\alpha, \alpha \in I]$ with the ideal (p) generated by the element p , and a theorem due to Fulton will imply that the ring $\Pi_{[r_{I,n}, r_{I,0}], I, A}$ is actually noetherian.

Following Monsky-Washnitzer, we recall as in the following the basic definition of a ring R being relatively weakly complete algebras over A . This means that for any element $f \in R$ and any degree $d \geq 0$, one could find k elements t_1, \dots, t_k and a polynomial p_d with k -variables with the coefficients in $(p)^d$, such that there exists a constant $C > 0$ with $\deg p_d \leq C(d+1)$ and with that:

$$f = \sum_{d \geq 0} p_d(t_1, \dots, t_k).$$

Therefore we are going to find some essential estimate over the degree of the polynomial with respect to some given element f . To get start for any such given f and any given d in the condition recalled as above, we consider a degree ∂_d positive which is maximal in the sense that f is inside $T_I^{-\partial_d} \mathbb{Z}_p / (p)^d$ and the index is the maximal throughout all such indexes. Then we are going to use the notation ∂_I to denote the corresponding multi-index (which is possibly not the same as $-\partial_d^I$). Now we take a constant $c_1 > 1$ such that $\|f\|_{[r_{I,n}, r_{I,0}]} \leq c_1$. Then we compute:

$$c_1 \geq \|f\|_{[r_{I,n}, r_{I,0}]} \geq \|f\|_{r_{I,0}} \geq |\text{coeff}_{\partial_I}(f)| p^{\partial_d r_{\alpha,0}/(p-1)} \prod_{\beta \in I \setminus \{\alpha\}} p^{-r_{\beta,0} i_\beta / (p-1)}$$

which gives rise to that for some $c_2 > 0$:

$$|\text{coeff}_{\partial_I}(f)| p^{\partial_d r_{\alpha,0}/(p-1)} \leq c_2.$$

Then we compute as in the following:

$$|\text{coeff}_{\partial_I}(f)| p^{\partial_d r_{\alpha,0}/(p-1)} \leq c_2 \tag{4.2.1}$$

$$\log_p |\text{coeff}_{\partial_I}(f)| + \partial_d r_{\alpha,0}/(p-1) \leq \log_p c_2 \tag{4.2.2}$$

$$(1-d) + \partial_d r_{\alpha,0}/(p-1) \leq \log_p c_2 \tag{4.2.3}$$

$$\partial_d r_{\alpha,0} \leq (p-1) \log_p c_2 + (d-1)(p-1) \tag{4.2.4}$$

$$\partial_d \leq (p-1) \log_p c_2 / r_{\alpha,0} + (d-1)(p-1) / r_{\alpha,0} \tag{4.2.5}$$

which gives rise to the desired estimate, which then further finishes the proof by considering Fulton's theorem in [Full] which says that the desired relative weakly complete algebra is now noetherian. \square

Then we consider the following statement which sends us to the framework of [ST03], which generalizes the one dimensional situation in [KPX14, Section 2.1]:

Proposition 4.2.7. *For all $n \geq 1$, the rings $\Pi_{\text{an},r_{I,0},I,A}$ and $\Pi_{\text{an},\infty_I,I,A}$ are Fréchet-Stein in the sense of [ST03].*

Proof. We keep the notations simple so we just consider the proof for the first ring and leave the task for checking the second case to the reader. Now indeed, we first note that $\Pi_{[r_{I,n},r_{I,0}],I,A}$ is Fréchet algebra in the sense that it is the ring of the analytic functions over a union of the relative annuli as in the following:

$$\bigcup_{0 < r_{I,n} \leq r_{I,0}} \text{Max}A \times A^I[r_{I,n}, r_{I,0}].$$

Furthermore we note that this is also the increasing union which is of the form of:

$$\bigcup_{0 < r_{I,n} \leq r_{I,0}} \text{Max}A \times A^I[r_{I,n}, r_{I,0}],$$

where $\text{Max}A \times A^I[r_{I,n}, r_{I,0}] \subset \text{Max}A \times A^I[r_{I,n+1}, r_{I,0}]$. The corresponding transition map over the algebras then with respect to this increasing union is actually flat since it is étale by the production construction and the base change properties. Finally the transition map is of topologically dense image again by the product construction. \square

Before we proceed to study more in more detail on the representation and structure of the algebras defined above, we first define the vector bundles over polyannuli generalizing the one dimensional result in [KPX14, Definition 2.1.3]. We mimick the definition, which gives rise to the following:

Definition 4.2.8. (After KPX [KPX14, Definition 2.1.3]) We define coherent sheaves of modules and the global sections over the ring $\Pi_{\text{an},r_{I_0},I,A}$ in the following way. First a coherent sheaf \mathcal{M} over the relative multidimensional Robba ring $\Pi_{\text{an},r_{I_0},I,A}$ is defined to be a family $(M_{[s_I,r_I]})_{[s_I,r_I] \subset (0,r_{I,0}]}$ of modules of finite type over each relative $\Pi_{[s_I,r_I],I,A}$ satisfying the following two conditions as in the more classical situation:

$$M_{[s'_I,r'_I]} \otimes_{\Pi_{[s'_I,r'_I],I,A}} \Pi_{[s''_I,r''_I],I,A} \xrightarrow{\sim} M_{[s''_I,r''_I]}$$

for any multi radii satisfying $0 < s'_I \leq s''_I \leq r''_I \leq r'_I \leq r_{I,0}$, with furthermore

$$(M_{[s'_I,r'_I]} \otimes_{\Pi_{[s'_I,r'_I],I,A}} \Pi_{[s''_I,r''_I],I,A}) \otimes_{\Pi_{[s''_I,r''_I],I,A}} \Pi_{[s'''_I,r'''_I],I,A} \xrightarrow{\sim} \Pi_{[s'''_I,r'''_I],I,A},$$

for any multi radii satisfying the following condition:

$$0 < s'_I \leq s''_I \leq s'''_I \leq r'''_I \leq r''_I \leq r'_I \leq r_{I,0}.$$

Then we define the corresponding global section of the coherent sheaf \mathcal{M} , which is usually denoted by M which is defined to be the following inverse limit:

$$M := \varprojlim_{s_I \rightarrow 0^+} M_{[s_I, r_{I,0}]}$$

We are going to follow [KPX14, Definition 2.1.3] to call any module defined over $\Pi_{\text{an}, r_{I_0}, I, A}$ coadmissible if it comes from a global section of a coherent sheaf \mathcal{M} in the sense defined above.

Then we could collect the following properties from more general theory from [ST03] as in [KPX14, Lemma 2.1.4] in the one dimensional situation.

Proposition 4.2.9. (After KPX [KPX14, Lemma 2.1.4]) *We have the following properties for the coherent sheaves defined above over the relative multi-dimensional Robba rings $\Pi_{\text{an}, r_{I_0}, I, A}$:*

I. *For each multi radii $0 < s_I \leq r_I \leq r_{I,0}$ as above, we have that the global section M is dense in the corresponding section $M_{[s_I, r_I]}$ where $r_{I,0}$ is finite, where the statement is also true for $M[1/T_I]$ for $r_{I,0}$ infinite but with r_I not;*

II. *For each multi radii $0 < s_I \leq r_I \leq r_{I,0}$ as above, we have the following comparison:*

$$M \otimes_{\Pi_{\text{an}, r_{I_0}, I, A}} \Pi_{[s_I, r_I], I, A} \xrightarrow{\sim} M_{[s_I, r_I]}$$

III. *For each multi radii $0 < s_I \leq r_{I,0}$ we have the following higher vanishing result:*

$$R^1 \varprojlim_{s_I \rightarrow 0^+} M_{[s_I, r_{I,0}]} = 0;$$

IV. *The kernel or cokernel of a morphism between coadmissible modules over $\Pi_{\text{an}, r_{I_0}, I, A}$ is again coadmissible;*

V. *Any finite presented module over $\Pi_{\text{an}, r_{I_0}, I, A}$ is coadmissible;*

VI. *Any submodule of finite type of any coadmissible $\Pi_{\text{an}, r_{I_0}, I, A}$ -module is again coadmissible;*

VII. *For each multi radii $0 < s_I \leq r_I \leq r_{I,0}$, and for the global section defined above, we have that $M \otimes_{\Pi_{\text{an}, r_{I_0}, I, A}} \Pi_{[s_I, r_I], I, A}$ is flat over $\Pi_{[s_I, r_I], I, A}$.*

Proof. See [KPX14, Lemma 2.1.4] and [ST03]. Again let us mention that actually in this setting initially we only have the corresponding results for r_I fixed which could for instance be $r_{I,0}$, however this could be overcome by the same method as in the one dimensional situation in [KPX14, Lemma 2.1.4]. \square

We then follow the ideas and approaches from [KPX14] to study further the structure theory of the coherent sheaves defined above. The following is then a generalization of the corresponding result in the one-dimensional situation in [KPX14, Corollary 2.1.5].

Lemma 4.2.10. (After KPX [KPX14, Corollary 2.1.5]) *The rings $\Pi_{\text{an},r_{I_0},I,A}$, and $\Pi_{\text{an,con},I,A}$ are flat over A as in [KPX14, Corollary 2.1.5].*

Proof. We adapt the corresponding method of proof in [KPX14, Corollary 2.1.5] to prove this generalization. Indeed one could then prove this by the same method of [KPX14, Corollary 2.1.5] since then by taking a suitable basis we will have $\Pi_{\text{an},r_{I_0},I,\mathbb{Q}_p} \xrightarrow{\cong} \widehat{\bigoplus}_{\eta} \mathbb{Q}_p e_{\eta}$. Then one could just finish as in [KPX14, Corollary 2.1.5]. \square

Definition 4.2.11. (After KPX [KPX14, Definition 2.1.3])

Consider a coherent sheaf $M_{r_{I,0}}$ over $\Pi_{\text{an},r_{I_0},I,A}$ defined as above, then we are going to call this sheaf a vector bundle if for each $[s_I, r_I]$ suitably located multi interval we have $M_{[s_I, r_I]}$ is flat over $\Pi_{\text{an},[s_I, r_I],I,A}$.

This will give us a chance to consider the following result:

Proposition 4.2.12. (After KPX [KPX14, Lemma 2.1.6]) *For any coherent sheaf $M_{r_{I,0}}$ over $\Pi_{\text{an},r_{I_0},I,A}$ which admits a vector bundle structure in our situation, we then have that the global section M of it is finitely projective if and only if it is finitely presented. (Actually one can even derive the corresponding sufficient and necessary condition to be just being finitely generated by applying [KL16, Corollary 2.6.8]).*

Proof. Basic representation theory tells us that the module involved is finitely projective if and only if it is finitely presented and flat. Therefore back to our proposition one direction of the implications is trivial, which is to say that we could now assume that the corresponding module is finitely presented, and it suffices to show that it is flat. Therefore as in [KPX14, Lemma 2.1.6] we consider the map $I \otimes_{\Pi_{\text{an},r_{I_0},I,A}} M \rightarrow M$ for some finitely generated ideal I . Then we are going to show that this is injective. First choose some

presentation of M which is of type (m, n) over the base ring $\Pi_{\text{an}, r_{I_0}, I, A}$ then translate this to the corresponding statement for the ideal I which is to say $I^m \rightarrow I^n$. This actually gives us a chance to derive the statement that $I \otimes_{\Pi_{\text{an}, r_{I_0}, I, A}} M$ is also coadmissible by the proposition 4.2.9. Now again by proposition 4.2.9 we have that it suffices to show that:

$$I \otimes_{\Pi_{\text{an}, r_{I_0}, I, A}} M \rightarrow M$$

is injective over some base change to some $\Pi_{\text{an}, [s_I, r_I], I, A}$. Then as in [KPX14, Lemma 2.1.6] one could finish the proof since this is nothing but the following map:

$$(I \otimes_{\Pi_{\text{an}, r_{I_0}, I, A}} \Pi_{\text{an}, [s_I, r_I], I, A}) \otimes M_{[s_I, r_I]} \rightarrow M_{[s_I, r_I]}$$

which is injective by the flatness from [KPX14, Lemma 2.1.6]. \square

We then have the following corollary which is a direct analog of the corresponding results in [KPX14, Corollary 2.1.7]:

Corollary 4.2.13. *Let A be an affinoid algebra which is reduced. Now consider a module M over the Robba ring with respect to a specific radius r_I , defined over $\text{Max}(A) \times \mathbb{A}^n(0, r_I]$ and we assume that for any maximal ideal \mathfrak{m}_η of A the fiber $M_{\mathfrak{m}_\eta}$ has the same rank. Then we have that M is finite projective if we further assume that M is finitely presented (as mentioned in the previous proposition one can even assume that it is finitely generated).*

Proof. This is a corollary of the previous proposition and the corresponding noetherian property as in [KPX14, Corollary 2.1.7]. \square

Before we study the relationship between the finiteness of the global sections and the coherent sheaves, we generalize the uniform finiteness in the one dimensional situation which is established in [KPX14, Definition 2.1.9].

Definition 4.2.14. (After KPX [KPX14, Definition 2.1.9]) We first generalize the notion of admissible covering in our setting, which is higher dimensional generalization of the situation established in [KPX14, Definition 2.1.9]. For any covering $\{[s_I, r_I]\}$ of $(0, r_{I,0}]$, we are going to specify those admissible coverings if the given covering admits refinement finite locally (and each corresponding member in the covering has the corresponding interior part which is assumed to be not empty with respect to each $\alpha \in I$).

Then it is very natural in our setting that we have the corresponding notations of (m, n) -finitely presentedness and n -finitely generatedness for any m, n positive integers. Then based on these definitions we have the following generalization of the corresponding uniform finiteness. First we are going to call a coherent sheaf $\{M_{[s_I, r_I]}\}_{[s_I, r_I]}$ over $\Pi_{\text{an}, r_I, 0, I, A}$ uniformly (m, n) -finitely presented if there exists an admissible covering $\{[s_I, r_I]\}$ and a pair of positive integers (m, n) such that each module $M_{[s_I, r_I]}$ defined over $\Pi_{\text{an}, [s_I, r_I], I, A}$ is (m, n) -finitely presented. Also we have the notion of uniformly n -finitely generated for any positive integer n by defining that to mean under the existence of an admissible covering we have that each member $M_{[s_I, r_I]}$ in the family defined over $\Pi_{\text{an}, [s_I, r_I], I, A}$ is n -finitely generated. Sometimes we are also going to use the notions of being uniformly finitely generated and being uniformly finitely presented to mean the corresponding objects when the corresponding uniform numbers of the generators are well-understood.

After defining these notions one could consider the following generalization of the corresponding results in [KPX14, Lemma 2.1.10] specifically in the one-dimensional situation. Before establishing this kind of generalization we first consider the following:

Lemma 4.2.15. (After KPX [KPX14, Lemma 2.1.10]) *Consider now the following short exact sequence of coherent sheaves over $\Pi_{\text{an}, r_I, 0, I, A}$:*

$$0 \longrightarrow (M_{[s_I, r_I]}^1)_{\{[s_I, r_I]\}} \longrightarrow (M_{[s_I, r_I]})_{\{[s_I, r_I]\}} \longrightarrow (M_{[s_I, r_I]}^2)_{\{[s_I, r_I]\}} \longrightarrow 0.$$

Then we have that if $(M_{[s_I, r_I]}^1)_{\{[s_I, r_I]\}}$ and one of the rest two sheaves are uniformly finitely presented so is the third, moreover we have that if $(M_{[s_I, r_I]})_{\{[s_I, r_I]\}}, (M_{[s_I, r_I]}^2)_{\{[s_I, r_I]\}}$ are uniformly finitely presented then the third one will be uniformly finitely generated.

Proof. See [KPX14, Lemma 2.1.10]. □

Lemma 4.2.16. (After KPX [KPX14, Lemma 2.1.11]) *Let $\{M_{[s_I, r_I]}\}$ be a coherent sheaf over $\Pi_{\text{an}, r_I, 0, I, A}$ with global section which we will denote it by M (with respect to an admissible covering in our context). Suppose that we have a set of generators $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ which generates $M_{[s_I, r_I]}$ for each $[s_I, r_I]$ involved, then we have that this set of generators will generate M as well.*

Proof. As in the one dimensional situation (see [KPX14, Lemma 2.1.11]) one could prove this by first exhibiting a map by using the given basis $(\Pi_{\text{an}, r_I, 0, I, A})^n \rightarrow M$, then arguing

that this will give rise to the desired generating set by considering the corresponding projection which is again a well-defined surjection and comparison under the assumption mentioned above. \square

Proposition 4.2.17. (After KPX [KPX14, Proposition 2.1.13]) *Consider any arbitrary coherent sheaf $\{M_{[s_I, r_I]}\}$ over $\Pi_{\text{an}, r_I, 0, I, A}$ with the global section M . Then we have the following corresponding statements:*

- I. The coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely generated iff the global section M is finitely generated;*
- II. The coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely generated iff the global section M is finitely presented;*
- III. The coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely presented vector bundle iff the global section M is finite projective.*

Proof. One could derive the the second and the third statements by using the lemma 4.2.15 and proposition 4.2.12. Therefore it suffices for us to prove the first statement. Indeed one could see that one direction in the statement is straightforward, therefore it suffices now to consider the other direction. So now we assume that the coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely generated for some n for instance. Then we are going to show that one could find corresponding generators for the global section M from the given uniform finitely generatedness.

Now by definition we consider an admissible covering taking the form of $\{[s_I, r_I]\}$. In [KPX14, Proposition 2.1.13] the corresponding argument goes by using special decomposition of the covering in some refined way, mainly by reorganizing the initial admissible covering into two single ones, where each of them will be made into the one consisting of those well-established intervals with no overlap. In our setting, we need to use instead the corresponding upgraded nice decomposition of the given admissible covering in our setting, which is extensively discussed in [KL16, 2.6.14-2.6.17]. Namely as in [KL16, 2.6.14-2.6.17] there is a chance for us to extract $2^{|I|}$ families $\{\{[s_{I, \delta_i}, r_{I, \delta_i}]\}_{i=0,1,\dots}\}_{\{1_i\}, \{2_i\}, \dots, \{N_i\}}$, ($N = 2^{|I|}$) of intervals, where there is a chance to have the situation where for each such family the corresponding intervals in it could be made to be disjoint in pairs. Then we are in the situation of [KL16, Proposition 2.6.17], namely whenever we have a $2^{|I|}$ -uniform covering, then the uniform finiteness throughout all the coverings involved will guarantee the

corresponding finiteness of the corresponding global section which proves the proposition. To be more precise we make the argument in [KL16, Proposition 2.6.17] more precise in our setting. First we write the corresponding space $\Lambda^I(0, r_{I,0}]_A$ into the following form:

$$\Lambda^I(0, r_{I,0}]_A := \bigcup_{\alpha \geq 0} \Lambda^I[s_{I,\alpha}, r_{I,0}]_A.$$

For $\Lambda^I[s_{I,-1}, r_{I,0}]_A$ we choose that to be empty. Then for a chosen family $\{[s_{I,\delta_i}, r_{I,\delta_i}]\}_{i=0,1,\dots}$, we use the notation $\{[s_{I,\gamma}, r_{I,\gamma}]\}_\Gamma$ to denote this specific family where Γ is endowed with the corresponding well ordering as in [KL16, proposition 2.6.17]. Now we consider for each γ the corresponding index $\alpha(\gamma)$ such that the intersection between $\Lambda^I[s_{I,\alpha(\gamma)}, r_{I,0}]_A$ and $\Lambda^I[s_{I,\gamma}, r_{I,\gamma}]_A$ is empty and meanwhile this index is the largest among all the candidates such that the intersection like this is empty. Then we consider the corresponding subspace in the following way:

$$B_\gamma := \Lambda^I[s_{I,\alpha(\gamma)}, r_{I,0}]_A \cup \bigcup_{\gamma' < \gamma, \gamma' \in \Gamma} \Lambda^I[s_{I,\gamma'}, r_{I,\gamma'}]_A.$$

Then we can see that one can find an element in the section $\mathcal{O}(B_\gamma)$ which by considering the corresponding restriction gives rise to some element which represents an invertible of an element which is topologically nilpotent over the section over $\Lambda^I[s_{I,\gamma}, r_{I,\gamma}]_A$ but meanwhile vanishes over the section over B_γ . Then by the density of the global section one can see that one can upgrade this element to be some element in $\mathcal{O}(\Lambda^I(0, r_{I,0}]_A)$ which which represents an invertible of an element which is topologically nilpotent over the section over $\Lambda^I[s_{I,\gamma}, r_{I,\gamma}]_A$ but meanwhile represents an element which is topologically nilpotent over the section over B_γ . We denote this element by x_γ . Now we can perform the corresponding iteration process in [KL16, proposition 2.6.17] to generate the following families of elements in M :

$$\dots, \tag{4.2.6}$$

$$e_{\gamma,1}, \dots, e_{\gamma,k}, \tag{4.2.7}$$

$$\dots \tag{4.2.8}$$

whose limit will be the corresponding finite generating set of the global section. We first start from a generating set of $M_{[s_{I,\gamma}, r_{I,\gamma}]}$ such e_1, \dots, e_k , then we are considering will be the sequence defined in the following way:

$$e_{\gamma,i} := e_{\gamma,i} + x_\gamma^\eta e_i,$$

where η is sufficiently large integer. This process guarantee that we can have converging limit set of generators. The generators actually generate the corresponding section $M_{[s_{I,\gamma}, r_{I,\gamma}]}$ since in our situation we can control the corresponding difference between the restriction of our produced set of elements and the actually generator over such section as in [KL16, proposition 2.6.17] and [KPX14, Lemma 2.1.12].

□

4.2.2 Further Discussion and Residue Pairings

We collect some further continuation of the discussion in the previous section before we consider the residue pairings in our setting, these are higher dimensional generalization of the corresponding results in [KPX14].

Lemma 4.2.18. (After KPX [KPX14, Lemma 2.1.16]) *Consider a morphism $f : M \rightarrow N$ of modules over $\Pi_{\text{an}, r_{I,0}, I, A}$ with kernel and cokernel of finite types. Suppose $f : M \rightarrow N$ is injective over $\Pi_{\text{an}, \text{con}, I, A}$. Then f is injective over $\Pi_{\text{an}, r_I, I, A}$ for some radii $r_I \leq r_{I,0}$. Furthermore if $f : M \rightarrow N$ is surjective over $\Pi_{\text{an}, \text{con}, I, A}$, then f is surjective over $\Pi_{\text{an}, r_I, I, A}$ for some radii $r_I \leq r_{I,0}$.*

Proof. See the proof of [KPX14, Lemma 2.1.16] which is a reinterpretation of the properties of being of finite type of the corresponding kernel and cokernel involved. □

Proposition 4.2.19. (After KPX [KPX14, Lemma 2.1.18]) *Suppose now that M is a module defined over $\Pi_{\text{an}, r_{I,0}, I, A}$ and is assumed to be finite over A . Then we have that there exists some multi radii $r_I \leq r_{I,0}$ such that the base change of M to $\Pi_{\text{an}, r_I, I, A}$ vanishes.*

Proof. This is a higher dimensional generalization of the corresponding results in [KPX14, Lemma 2.1.18]. The argument is parallel, we adapt the argument to our setting as in the following. First we consider the situation where A is Artin, then it is easy to see that the proposition is true. Then more generally we consider the copy $A \langle T_I \rangle \subset \Pi_{\text{an}, r_{I,0}, I, A}$, which allows us to go back to the previous situation by taking the reduction. Note that we still satisfy the condition since then the module is also finite over $A \langle T_I \rangle$ since it is so over A . Then one could finish as in the one dimensional situation, see [KPX14, Lemma 2.1.18]. □

Then we define the following higher dimensional analog of the differential Ω over the one dimensional Robba ring relative to the ring A . This amounts to considering a higher dimensional residue formula for a function with I variables.

Definition 4.2.20. (After KPX) We first generalize the definition of the differentials of the Robba rings to higher dimensional situation. First consider our full Robba rings $\Pi_{\text{an,con},I,A}$, then we are going to define the corresponding n -th differential of this ring relative to A as the following:

$$\Omega_{\Pi_{\text{an,con},I,A}/A} := \wedge^{|I|} \Omega_{\Pi_{\text{an,con},I,A}/A}^1$$

which is nothing but the $|I|$ -th differential of the ring $\Pi_{\text{an,con},I,A}$ with respect to the affinoid A . Here the differentials are assumed to be the continuous ones.

Then for any element in the $|I|$ -th differential we could define the corresponding residue of this chosen element as in the following:

Definition 4.2.21. (After KPX) For any $f \in \Omega_{\Pi_{\text{an,con},I,A}/A}$ of the general form of $f = \sum_{n_I} a_{n_1, \dots, n_I} T_1^{n_1} \dots T_I^{n_I}$, we define the residue of this differential of $|I|$ -th order by the following formula:

$$\text{res}(f) := \text{res}\left(\sum_{n_I} a_{n_1, \dots, n_I} T_1^{n_1} \dots T_I^{n_I}\right) := a_{-\mathbf{1}}.$$

Here the symbol $\mathbf{1}$ is defined to be the index of $(1, \dots, 1)$ in a uniform way.

Proposition 4.2.22. (After KPX [KPX14, Lemma 2.1.19]) *The residue map defined above in our setting is well-defined, and the differential module $\Omega_{\Pi_{\text{an,con},I,A}/A}$ is free of rank one. We define the following differential pairing over the relative differential module $\Omega_{\Pi_{\text{an,con},I,A}/A}$:*

$$h : \Pi_{\text{an,con},I,A} \times \Pi_{\text{an,con},I,A} \rightarrow A \tag{4.2.9}$$

$$(f, g) \mapsto \text{res}(f(T_I)g(T_I)dT_1 \wedge \dots \wedge dT_I). \tag{4.2.10}$$

Then the pairing now induces the following canonical isomorphisms (where the topology depends on the choices of the variables):

$$\text{Hom}_{\text{con}}(\Pi_{\text{an,con},I,A}, A) \xrightarrow{\sim} \Pi_{\text{an,con},I,A} \tag{4.2.11}$$

$$\text{Hom}_{\text{con}}(\Pi_{\text{an,con},I,A} / \sum_k \Pi_{\text{an},k,I,A}, A) \xrightarrow{\sim} \Pi_{\text{an},\infty,I,A}. \tag{4.2.12}$$

$$\tag{4.2.13}$$

Here the ring $\Pi_{\text{an},k,I,A}$ is defined as

$$\Pi_{\text{an},k,I,A} := \bigcup_{r_i > 0, i \neq k, r_k = \infty, k=1, \dots, |I|} \Pi_{\text{an},r_I,I,A}. \quad (4.2.14)$$

Proof. As in the one dimensional situation, from the very definitions, one could derive that the residue map is well-defined and the fact that the differential module is generated by dT_I . Consider our product construction and the descent process in the one dimensional situation we have that one could then check the issue of well-definedness over the corresponding multivariate convergent rings \mathcal{E}_Δ in [CKZ18] and [PZ19]. Then for the first statement in the isomorphisms, first one could see that one direction (from right to left) which is induced directly from the residue pairing. On the other hand, we consider the following map which is a generalization of the one dimensional situation:

$$\text{Hom}_{\text{con}}(\Pi_{\text{an},\text{con},I,A}, A) \rightarrow \Pi_{\text{an},\text{con},I,A} \quad (4.2.15)$$

$$\mu_I \mapsto \sum_{n_I} \mu_I(T_I^{-1-n_I})T_1^{n_1} \dots T_I^{n_I}, \quad (4.2.16)$$

which is easy to see that this gives an inverse mapping by the definitions of the residue pairings. For the second isomorphism, under our construction, see [Cre1, Proposition 5.5] and the corresponding [Ked06, Lemma 8.5.1].

□

Then as below we are going to contact with the multi Lie groups structures which is a higher dimensional generalization of the usual pictures in [KPX14, Definition 2.1.20]. To achieve so we need to generalize the constructions in [KPX14, Definition 2.1.20] in the following sense:

Definition 4.2.23. (After KPX [KPX14, Definition 2.1.20]) Uniformizing the notations as in our list of notations at the beginning, we use the notation $(C_\alpha, \ell_\alpha)_{\alpha \in I}$ to denote I pairs consisting of I commutative topological groups with finite torsion parts and with the free part uniformly isomorphic to \mathbb{Z}_p , and I morphism from $\ell_\alpha : C_\alpha \rightarrow \mathbb{Q}_p$ for each $\alpha \in I$. Then based on these we consider the following generalization of the constructions in [KPX14, Definition 2.1.20]. First we are going to choose now for each $\alpha \in I$ a generator taking the form of c_α of the free part of the α -th topological group for all $\alpha \in I$. Then we generalize the corresponding definitions in the usual situation

as in the following. First assume that we only have torsion free topological groups in our setting for each $\alpha \in I$. Then we will use the notations $\Pi_{\text{an},r_I,I,A}(C_I), \Pi_{[s_I,r_I],I,A}(C_I)$ respectively to denote the corresponding higher dimensional period rings defined above namely $\Pi_{\text{an},r_I,I,A}, \Pi_{[s_I,r_I],I,A}$ formally modified by considering the replacement of the variables with $c_\alpha - 1$ for each dimension $\alpha \in I$. As in the usual situations in this situation we do not have the issue of the dependence of the choices of the generators. Also we know since by the corresponding one dimensional result we will have that

$$\|f(\dots, (1 + T_\alpha)^p - 1, \dots)\|_{r_I} = \|f(T_1, \dots, T_I)\|_{\dots, pr_\alpha, \dots}$$

for each $\alpha \in I$ which implies that certainly for all the $\alpha \in I$ together. Therefore from this fact we know that for each $\alpha \in I$:

$$\Pi_{\text{an},(\dots, pr_\alpha, \dots), I, A}(\dots, pC_\alpha, \dots) \otimes_{\mathbb{Z}_p[(\dots, pC_\alpha, \dots)]} \mathbb{Z}_p[C_I] \xrightarrow{\sim} \Pi_{\text{an},r_I,I,A}(C_I),$$

with

$$\Pi_{\text{an},(\dots, p[s_\alpha, r_\alpha], \dots), I, A}(\dots, pC_\alpha, \dots) \otimes_{\mathbb{Z}_p[(\dots, pC_\alpha, \dots)]} \mathbb{Z}_p[C_I] \xrightarrow{\sim} \Pi_{[s_I, r_I], I, A}(C_I).$$

As in [KPX14, Definition 2.1.20] one could then consider more general situation where C_I may not be torsion free. In this case one considers simultaneously all small $r_I < \mathbf{1}$ or all small s_I when r_I are all infinity, and any D_α a torsion free open subgroup of C_α for each $\alpha \in I$, one has:

$$\Pi_{\text{an},([C_1:D_1]_{r_1}, \dots, [C_I:D_I]_{r_I}), I, A}(D_I) \otimes_{\mathbb{Z}_p[D_I]} \mathbb{Z}_p[C_I] \xrightarrow{\sim} \Pi_{\text{an},r_I,I,A}(C_I),$$

with

$$\Pi_{\text{an},([C_\alpha:D_\alpha]_{[s_\alpha, r_\alpha]}, \dots), I, A}(\dots, D_\alpha, \dots) \otimes_{\mathbb{Z}_p[(\dots, D_\alpha, \dots)]} \mathbb{Z}_p[C_I] \xrightarrow{\sim} \Pi_{[s_I, r_I], I, A}(C_I).$$

Then we consider the situation for some general open subgroups D'_I of the groups D_I which are torsion free then for each $\alpha \in I$, and for each $r_\alpha \leq 1/\sharp D_{\text{tor}}$, or for each $s_\alpha \leq 1/\sharp D_{\text{tor}}$ for r_I infinite, we define:

$$\Pi_{\text{an},([D_1:D'_1]_{r_1}, \dots, [D_I:D'_I]_{r_I}), I, A}(D'_I) \otimes_{\mathbb{Z}_p[D'_I]} \mathbb{Z}_p[D_I] \xrightarrow{\sim} \Pi_{\text{an},r_I,I,A}(D_I),$$

with

$$\Pi_{\text{an},([D_\alpha:D'_\alpha]_{[s_\alpha, r_\alpha]}, \dots), I, A}(\dots, D'_\alpha, \dots) \otimes_{\mathbb{Z}_p[(\dots, D'_\alpha, \dots)]} \mathbb{Z}_p[D_I] \xrightarrow{\sim} \Pi_{[s_I, r_I], I, A}(D_I).$$

As in the one dimensional situation everything is well-defined with respect to the choices involved. Then based on this observation we start to consider the generalization of the differentials defined above as in the following. In our situation we consider the differential ω_{ℓ_I} which is defined to be the wedge product $\omega_{\ell_1} \wedge \dots \wedge \omega_{\ell_I}$, where each ω_{ℓ_α} is defined as in the situation in [KPX14, Definition 2.1.20]. Then we could then consider any element $f \in \Omega_{\Pi_{\text{an,con},I,A}(C_I)/A}$, and define:

$$\text{res}(f) := a_{-1}.$$

Then we define pairing as above in more general setting as:

$$h : \Pi_{\text{an,con},I,A}(C_I) \times \Pi_{\text{an,con},I,A}(C_I) \rightarrow A \tag{4.2.17}$$

$$(f, g) \mapsto \text{res}(fg\omega_{\ell_I}). \tag{4.2.18}$$

And we also have the following analog:

$$\text{Hom}_{\text{con}}(\Pi_{\text{an,con},I,A}(C_I), A) \xrightarrow{\sim} \Pi_{\text{an,con},I,A}(C_I). \tag{4.2.19}$$

4.3 Relative Frobenius Sheaves over Affinoids and Fréchet Rings

Now we introduce main objects in our study and our consideration. We first generalize the setting in [KPX14] to the situation we have multi Hodge-Frobenius structure which is quite natural since we consider already some higher dimensional spaces. Then we generalize everything to the setting in [KP1], which is also very natural in the consideration of noncommutative Iwasawa theory in the style of [KP1]. The latter will be around the Iwasawa cohomology of some p -adic Lie deformation of the usual (φ, Γ) -modules. Therefore in the second situation we will consider the usual Hodge structure but we consider the p -adic Lie extension, for instance one could consider the product of \mathbb{Z}_p^\times with some power of \mathbb{Z}_p^\times or \mathbb{Z}_p . This will be then highly relative since we will work over some relative Robba rings or sheaves taking the form of $\Pi_{\text{an,con},I,\Pi_{\text{an},\infty,A}(G)}$, where $\Pi_{\text{an},\infty}(G)$ for instance could be Berthelot's A_∞ -algebra, namely the distribution over G .

Setting 4.3.1. We will work over the ring $\Pi_{\text{an},\infty,A}(G)$ which is just the ring defined in the previous section which is abelian and which is the corresponding completed tensor product of the corresponding rings. To be more precise we will consider the Frobenius modules over $\Pi_{\text{an,con},I,\Pi_{\text{an},\infty,J,A}(\Gamma_{K_J})}$. Note that here J could be different from I , but we assume that we have the relation $I \subset J$, and for those K_α ($\alpha \in J \setminus I$) we have that $K_\alpha = \mathbb{Q}_p$. Now set G be Γ_{K_J} (see the discussion below for the definitions). By considering the philosophy of Burns-Flach-Fukaya-Kato, this is actually already some noncommutative type Hodge structures.

Remark 4.3.2. The noncommutative setting is considered in the 2017 thesis [Zah1].

Let K_I be I finite extension of the p -adic number field \mathbb{Q}_p . Now for each $\alpha \in I$, we consider the cyclotomic field $K_\alpha(\mu_{p^\infty})$, which gives rise to the extension $K_\alpha(\mu_{p^\infty})/K_\alpha$ with the Galois group $G(K_\alpha(\mu_{p^\infty})/K_\alpha)$ which is just the corresponding Γ_{K_α} in our situation. Note that for each K_α we have the associated cyclotomic character χ_α whose kernel will be denoted as usual by H_{K_α} , for each $\alpha \in I$. And for each α we will use the notation $\tilde{e}_{K_\alpha,\infty}$ to denote the ramification indexes along the towers, namely for each α this is the quotient $[K_\alpha(\mu_{p^\infty}) : \mathbb{Q}_p(\mu_{p^\infty})]/[k_{K_\alpha(\mu_{p^\infty})} : k_{K_\alpha}]$. Then first step now is to add some

Hodge-Frobenius structures over the period rings we established in the previous section. To do so we consider the following situation:

Definition 4.3.3. (After KPX [KPX14, Definition 2.2.2]) For each $\alpha \in I$, we choose suitable uniformizer π_{K_α} in our consideration. Then we are going to use the notation $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ to denote the corresponding period ring constructed from $\Pi_{\text{an,con},I,A}$ just by directly replacing the variables by the corresponding uniformizers as above. And similarly for other period rings we use the corresponding notations taking the same form namely $\Pi_{\text{an,con},I,A}(\pi_{K_I})$, $\Pi_{\text{an},r_I,0,I,A}(\pi_{K_I})$, $\Pi_{\text{an},[s_I,r_I],I,A}(\pi_{K_I})$. As in the one dimensional situation we consider the situation where the radii are all sufficiently small. Then one could define the multiple Frobenius actions from the multi Frobenius φ_I over each the ring mentioned above. For suitable radii r_I we define a φ_I -module to be a finite projective $\Pi_{\text{an},r_I,I,A}(\pi_{K_I})$ -module with the requirement that for each $\alpha \in I$ we have $\varphi_\alpha^* M_{r_I} \xrightarrow{\sim} M_{\dots,r_\alpha/p,\dots}$ (after suitable base changes). Then we define $M := M_{r_I} \otimes_{\Pi_{\text{an},r_I,I,A}(\pi_{K_I})} \Pi_{\text{an,con},I,A}(\pi_{K_I})$ to define a φ_I -module over the full relative Robba ring in our situation. Furthermore we have the notion of φ_I -bundles in our situation which consists of a family of φ_I -modules $\{M_{[s_I,r_I]}\}$ where each module $M_{[s_I,r_I]}$ is defined to be finite projective over $\Pi_{\text{an},[s_I,r_I],I,A}(\pi_{K_I})$ satisfying the action formula taking the form of $\varphi_\alpha^* M_{[s_I,r_I]} \xrightarrow{\sim} M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}$ (after suitable base changes). Furthermore for each α we have the corresponding operator $\varphi_\alpha : M_{[s_I,r_I]} \rightarrow M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}$ and we have the corresponding operator ψ_α which is defined to be $p^{-1}\varphi_\alpha^{-1} \circ \text{Trace}_{M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}} / \varphi_\alpha(M_{[s_I,r_I]})$. Certainly we have the corresponding operator ψ_α over the global section M_{r_I} . Note that here we require that the Hodge-Frobenius structures are commutative in the sense that all the Frobenius are commuting with each other, and they are semilinear.

So now based on the definitions above we can consider the corresponding objects over the ring $\Pi_{\text{an},\infty,A}(G)$, which are actually complicated. The basic reason is that over the infinite level we will lose some control of the corresponding finiteness, also very significantly we lose the control of the behavior of the radius throughout the whole variation over the Stein space attached to $\Pi_{\text{an},\infty,A}(G)$. Let $X(\Pi_{\text{an},\infty,A}(G))$ be the corresponding Fréchet-

Stein space attached to the group G , which gives rise to the following representation:

$$X(\Pi_{\text{an},\infty,A}(G)) = \varinjlim_k X(\Pi_{\text{an},\infty,A}(G))_k, \quad (4.3.1)$$

$$\mathcal{O}_{X(\Pi_{\text{an},\infty,A}(G))} = \varprojlim_k \mathcal{O}_{X(\Pi_{\text{an},\infty,A}(G))_k}, \quad (4.3.2)$$

$$\Pi_{\text{an},\infty,A}(G) =: \varprojlim_k \Pi_{\text{an},\infty,A}(G)_k. \quad (4.3.3)$$

Definition 4.3.4. Over $\square := \Pi_{\text{an},\text{con},I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$, or

$\Pi_{\text{an},r_I,0,I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$, or $\Pi_{\text{an},[s_I,r_I],I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$ as in the above (with the notations for the radii as considered above) we have the notion of φ_I -modules over \square . Be careful that we will define a φ_I -module M over \square to be the following projective limit:

$$M := \varprojlim_k M_k \quad (4.3.4)$$

where M_k is a corresponding object over \square_k as defined above as a corresponding φ_I -module M (finite projective) over \square_k . Here $\square_k := \Pi_{\text{an},\text{con},I,\Pi_{\text{an},\infty,A}(G)_k}(\pi_{K_I})$, or $\Pi_{\text{an},r_I,0,I,\Pi_{\text{an},\infty,A}(G)_k}(\pi_{K_I})$, or $\Pi_{\text{an},[s_I,r_I],I,\Pi_{\text{an},\infty,A}(G)_k}(\pi_{K_I})$ respectively. One defines the φ_I -bundles over

$$\Pi_{\text{an},\text{con},I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$$

in the same fashion.

We also have the corresponding notions of families of the corresponding objects considered above in the Fréchet-Stein situation.

Definition 4.3.5. Over $\square := \Pi_{\text{an},\text{con},I,X(\Pi_{\text{an},\infty,A}(G))}(\pi_{K_I})$, or $\Pi_{\text{an},r_I,0,I,X(\Pi_{\text{an},\infty,A}(G))}(\pi_{K_I})$, or $\Pi_{\text{an},[s_I,r_I],I,X(\Pi_{\text{an},\infty,A}(G))}(\pi_{K_I})$ as in the above (with the notations for the radii as considered above) we have the notion of φ_I -modules over \square . Be careful that we will define a φ_I -module M over \square to be the following the sheaf taking each quasi-compact $Y \subset X(\Pi_{\text{an},\infty,A}(G))$ to $M(Y)$ such that this is a corresponding object over \square_Y as defined above as a corresponding φ_I -module M (finite projective) over \square_Y . Here $\square_Y := \Pi_{\text{an},\text{con},I,\mathcal{O}_{X(\Pi_{\text{an},\infty,A}(G))}(Y)}(\pi_{K_I})$, or $\Pi_{\text{an},r_I,0,I,\mathcal{O}_{X(\Pi_{\text{an},\infty,A}(G))}(Y)}(\pi_{K_I})$, or $\Pi_{\text{an},[s_I,r_I],I,\mathcal{O}_{X(\Pi_{\text{an},\infty,A}(G))}(Y)}(\pi_{K_I})$ respectively. One defines the families of φ_I -bundles over $\Pi_{\text{an},\text{con},I,X(\Pi_{\text{an},\infty,A}(G))}(\pi_{K_I})$ in the same fashion. Here one can take the corresponding Y as some member in the filtration:

$$X(\Pi_{\text{an},\infty,A}(G)) = \varinjlim_k X(\Pi_{\text{an},\infty,A}(G))_k.$$

Proposition 4.3.6. (After KPX [KPX14, Proposition 2.2.7]) *We have that natural functor from families of φ_I -modules (in the sense of the previous definition without the finite condition on the global section) over $\Pi_{\text{an},r_{I,0},I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$ to the corresponding families of φ_I -bundles (in the sense of the previous definition without the finite condition on the global section) is an equivalence. Here by a family of φ_I -module we mean a φ_I -module over $\Pi_{\text{an},r_{I,0},I,X(\Pi_{\text{an},\infty,A}(G))}(\pi_{K_I})$, which is the same to a family of φ_I -bundles over $\Pi_{\text{an},r_{I,0},I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$.*

Proof. Here we need to consider the corresponding Fréchet sheaves as in the previous definition since in our situation the relative Robba rings are defined over some non-noetherian Stein space. The statement is obviously true for each M_n as in the usual situation. First the faithfulness is straightforward, then for the surjectivity part, we only need to show that the global section is finite projective, which is a directly consequence of proposition 4.5.19 as long as in our situation one considers the simultaneous application of our multiple Frobenius to shift around to vary through all the intervals. \square

Definition 4.3.7. (After KPX [KPX14, Definition 2.2.12]) Then we add the corresponding more Lie group action in our setting, namely the multi semilinear Γ_{K_I} -action. Again in this situation we require that all the actions of the Γ_{K_I} are commuting with each other and with all the semilinear Frobenius actions defined above. We define the corresponding (φ_I, Γ_I) -modules over $\Pi_{\text{an},r_{I,0},I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$ to be finite projective module with mutually commuting semilinear actions of φ_I and Γ_{K_I} . For the latter action we require that to be continuous.

The continuity condition here could be made more explicit, as in [KPX14, Proposition 2.2.14]. In our setting, we just need to generalize the result in [KPX14, Proposition 2.2.14] to multiple actions from Γ_{K_I} .

Lemma 4.3.8. (After KPX [KPX14, Proposition 2.2.14])

For suitable $r_{I,0}$, consider $M_{r_{I,0}}$ a module (carrying a semilinear action of Γ_{K_I}) of finite type over $\Pi_{\text{an},r_{I,0},I,\Pi_{\text{an},\infty,A}(G)}(\pi_{K_I})$ with the generating set $\mathbf{e}_1, \dots, \mathbf{e}_m$ (namely we look at a sheaf over $\Pi_{\text{an},r_{I,0},I,X(\Pi_{\text{an},\infty,A}(G))}(\pi_{K_I})$ with global section finitely generated). We will follow [KPX14, Proposition 2.2.14] to differentiate the corresponding Banach module norm $|\cdot|_{[s_I, r_{I,0}]}$ and the operator norm $\|\cdot\|_{[s_I, r_{I,0}]}$ when we consider the module over the polydisc

with respect to the corresponding multi-interval (and the corresponding member in the Fréchet family). Then we have: I. The action of the group Γ_{K_I} is continuous if and only if that for each $\gamma \in \Gamma_{K_I}$ and each element in the generating set we have $\lim_{\gamma \rightarrow 1} \gamma \mathbf{e}_j = \mathbf{e}_j$; The continuity of the action implies that: II. $\lim_{\gamma \rightarrow 1} \|\gamma - 1\|_{[s_I, r_I, 0]} = 0$ for any s_I ; III. Whenever we have the subgroups (with the notations in [KPX14, Proposition 2.2.14]) $\Gamma_{n_I} \subset \Gamma_{K_I}$ ($\forall n_I \geq 0$), one can find separately g_{α, s_I} with the fact that $\|(\gamma'_{n_\alpha} - 1)^{g_{\alpha, s_I}}\|_{[s_I, r_I, 0]} < p^{-1/(p-1)}$ for each $\alpha \in I$.

Proof. First we need to work over some affinoid domain with respect to the Fréchet-Stein algebra in the coefficient, but this will be reduced to the corresponding affinoid situation. We then follow the strategy of [KPX14, Proposition 2.2.14], where we just need to consider each group in the product Γ_{K_I} . The first statement could be derived in the same way as in [KPX14, Proposition 2.2.14]. For the second statement therefore we assume the limit identity in I, then prove the corresponding results on the identity in II. As in the proof in [KPX14, Proposition 2.2.14], for I and II, we reduce ourselves to the situation where the base field is \mathbb{Q}_p for each K_α from the induction properties, and correspondingly we reduce us to the situation where all the uniformizers π_I are just same as a product taking the form of π^I as considered in the proof in [KPX14, Proposition 2.2.14]. First is the following observation on the Newton-Leibniz style formula for each decomposition of elements with respect to given basis $\sum r_i e_i$ and for any $\gamma \in \Gamma_{K_I}$:

$$(\gamma - 1)(r_i e_i) = \gamma(r_i)(\gamma - 1)(e_i) + (\gamma - 1)r_i e_i.$$

It suffices to look at the corresponding last term. We then further reduce ourselves to the situation where the module is just free of rank one over the base ring, which directly implies it suffices to consider the element in the ring taking the general form of $\pi_1^{n_1} \dots \pi_I^{n_I}$. Since we consider further the group element γ_α separately for each $\alpha \in I$, the result follows from then directly observation that $\gamma_\alpha(\pi_\beta) - \pi_\beta = f(\pi_\beta)((1 + \pi_\beta)^{ap^k} - 1)$ as well as $\gamma_\alpha(\pi_\beta^{-1}) - \pi_\beta^{-1} = f'(\pi_\beta)((1 + \pi_\beta)^{ap^k} - 1)$ ($\alpha = \beta$ otherwise we have trivial action) where f' actually depends on k but could be controlled by some g' which does not depend. Then from this for each β, α one gets the result. For the third statement we again argue by separately considering the corresponding group Γ_α where $\alpha \in I$. By first stage of reduction, we reduce ourselves to the situation where the base fields are all \mathbb{Q}_p by induction and the uniformizer are all π for each $\alpha \in I$. Since for each α , as in [KPX14,

Proposition 2.2.14] the Banach module norm $|\cdot|_{[s_I, r_I, 0]}$ is invariant under the action of Γ_{n_α} for some n_α . Note that this is based on the corresponding observation that:

$$\gamma_\alpha(\pi_1^{i_1} \dots \pi_\alpha \dots \pi_I^{i_I}) - \pi_1^{i_1} \dots \pi_\alpha \dots \pi_I^{i_I} = \pi_1^{i_1} \dots f(\pi_\alpha)((1 + \pi_\alpha)^{ap^k} - 1) \dots \pi_I^{i_I} \quad (4.3.5)$$

$$\gamma_\alpha(\pi_1^{i_1} \dots \pi_\alpha^{-1} \dots \pi_I^{i_I}) - \pi_1^{i_1} \dots \pi_\alpha^{-1} \dots \pi_I^{i_I} = \pi_1^{i_1} \dots f'(\pi_\alpha)((1 + \pi_\alpha)^{ap^k} - 1) \dots \pi_I^{i_I}, \quad (4.3.6)$$

and note that $|\cdot|_{[s_I, r_I, 0]}(\pi_1^{i_1} \dots ((1 + \pi_\alpha)^{ap^k} - 1) \dots \pi_I^{i_I}) \leq |\cdot|_{[s_I, r_I, 0]}(\pi_1^{i_1} \dots \pi_\alpha \dots \pi_I^{i_I})$. Then we consider the corresponding average which is invariant as in [KPX14, Proposition 2.2.14] for some $n'_\alpha \geq n_\alpha$:

$$|\cdot|_{[s_I, r_I, 0]}^- := \frac{1}{[\Gamma_{n_\alpha} : \Gamma_{n'_\alpha}]} \sum_{\tilde{\gamma}_\alpha \in \Gamma_{n_\alpha} / \Gamma_{n'_\alpha}} |\cdot|_{[s_I, r_I, 0]} \circ \tilde{\gamma}_\alpha$$

which is equivalent to the original one:

$$C^{-1} |\cdot|_{[s_I, r_I, 0]}^- \leq |\cdot|_{[s_I, r_I, 0]} \leq C |\cdot|_{[s_I, r_I, 0]}^-$$

for some positive number $C > 0$. First for the average norm we have that by taking suitable n'_α one could get $\left\| \gamma'_{n'_\alpha} - 1 \right\|_{[s_I, r_I, 0]}^-$ is small enough, then take the corresponding $p^{n'_\alpha - n_\alpha}$ -power of the element $\gamma'_{n'_\alpha} - 1$ one could further make this term smaller than $p^{-1/(p-1)}$ under the corresponding operator norm, namely the term $(\gamma'_{n'_\alpha} - 1)^{p^{n'_\alpha - n_\alpha}}$ as in [KPX14, Proposition 2.2.14]. Then by the inequality above between the original norms and the average ones, we have that by taking suitable power g_{s_α} one could make $\left\| (\gamma'_{n_\alpha} - 1)^{g_{s_\alpha}} \right\|_{[s_I, r_I, 0]}$ smaller than $p^{-1/(p-1)}$. To be more precise by raising the power g_{n_α} one will have $\left\| (\gamma'_{n_\alpha} - 1)^{g_{s_\alpha}} \right\|_{[s_I, r_I, 0]} \leq C \left\| (\gamma'_{n_\alpha} - 1)^{g_{s_\alpha}} \right\|_{[s_I, r_I, 0]}^- \leq CC_{g_{s_\alpha}}$ where $C_{g_{s_\alpha}}$ could be made sufficiently small along this process. \square

4.4 Der Endlichkeitssatz

4.4.1 Finiteness through p -adic Functional Analysis

In this section we are going to study the cohomology of our relative Frobenius modules defined in the previous sections. We start from a (φ_I, Γ_I) -module over $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ as well a (ψ_I, Γ_I) -module over $\Pi_{\text{an,con},I,A}(\pi_{K_I})$. We are going to use M to denote such a module. First we are going to use the notation Δ_I to denote the p -torsion part of Γ_I in the product form, which will be used in the definitions of the corresponding cohomologies as below.

Assumption 4.4.1. We choose to consider the story in which I consists of two elements. We note that actually everything could be carried over to more general setting. We will then use the notation $\{1, 2\}$ to denote our set which indicates the corresponding Frobenius actions and Lie group actions with respect to each index.

Definition 4.4.2. (After KPX [KPX14, 2.3]) For any (φ_I, Γ_I) -module M over $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ we define the complex $C_{\Gamma_I}^\bullet(M)$ of M to be the total complex of the following complex through the induction:

$$[C_{\Gamma_{I \setminus \{I\}}}^\bullet(M) \xrightarrow{\Gamma_I} C_{\Gamma_I \setminus \{I\}}^\bullet(M)].$$

Then we define the corresponding double complex $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$ again by taking the corresponding totalization of the following complex through induction:

$$[C_{\varphi_{I \setminus \{I\}}}^\bullet C_{\Gamma_I}^\bullet(M) \xrightarrow{\varphi_I} C_{\varphi_I \setminus \{I\}}^\bullet C_{\Gamma_I}^\bullet(M)].$$

Then we define the complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the totalization of the double complex defined above, which is called the complex of a (φ_I, Γ_I) -module M . Similarly we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the totalization of the double complex defined in the same way as above by replacing φ_I by then the operators ψ_I . For later comparison we also need the corresponding ψ_I -cohomology, which is to say the complex $C_{\psi_I}^\bullet(M)$ which could be defined by the totalization of the following complex through induction:

$$[C_{\psi_{I \setminus \{I\}}}^\bullet(M) \xrightarrow{\psi_I} C_{\psi_I \setminus \{I\}}^\bullet(M)].$$

One can actually compare the two total complexes defined above in the following way, as in the situation where I is a singleton. First for the (φ_I, Γ_I) -complex we have the following initial double complex:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M^{\Delta_I} & \xrightarrow{(\varphi_1-1, \varphi_2-1)} & M^{\Delta_I} \oplus M^{\Delta_I} & \xrightarrow{(\varphi_2-1)+(1-\varphi_1)} & M^{\Delta_I} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & (\gamma_1-1, \gamma_2-1) & & & & \\
 0 & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} \oplus M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & (\gamma_2-1)+(1-\gamma_1) & & & & \\
 0 & \longrightarrow & M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Here we followed the corresponding convention as in [KPX14]. And then correspondingly we have the following double complex for the corresponding (ψ_I, Γ_I) -module:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M^{\Delta_I} & \xrightarrow{(\psi_1-1, \psi_2-1)} & M^{\Delta_I} \oplus M^{\Delta_I} & \xrightarrow{(\psi_2-1)+(1-\psi_1)} & M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\gamma_1-1, \gamma_2-1) & & & & \\
0 & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} \oplus M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\gamma_2-1)+(1-\gamma_1) & & & & \\
0 & \longrightarrow & M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0.
\end{array}$$

In our situation in order to relate the two double complexes we consider the corresponding map induced from the following maps for the two first lines of two big complexes where one complex is given as above, while the other is the following one:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M^{\Delta_I} & \xrightarrow{(\varphi_1-1, \psi_2-1)} & M^{\Delta_I} \oplus M^{\Delta_I} & \xrightarrow{(\psi_2-1)+(1-\varphi_1)} & M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\gamma_1-1, \gamma_2-1) & & & & \\
0 & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} \oplus M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\gamma_2-1)+(1-\gamma_1) & & & & \\
0 & \longrightarrow & M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \oplus M^{\Delta_I} & \longrightarrow & M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

(which is a direct generalization of the corresponding morphisms in one dimensional situation in [KPX14, Definition 2.3.3]):

$$\begin{array}{ccccccc}
0 & \longrightarrow & M^{\Delta_I} & \xrightarrow{(\varphi_1-1, \gamma_1-1)} & M^{\Delta_I} \oplus M^{\Delta_I} & \xrightarrow{(\gamma_1-1)+(1-\varphi_1)} & M^{\Delta_I} \longrightarrow 0 \\
& & \downarrow \text{Id} & & \downarrow (-\psi_1, \text{Id}) & & \downarrow -\psi_1 \\
0 & \longrightarrow & M^{\Delta_I} & \xrightarrow{(\psi_1-1, \gamma_1-1)} & M^{\Delta_I} \oplus M^{\Delta_I} & \xrightarrow{(\gamma_1-1)+(1-\psi_1)} & M^{\Delta_I} \longrightarrow 0.
\end{array}$$

We will denote the corresponding morphism in between the two total complex by:

$$\Psi_1 : C_{\varphi_1, \psi_2, \Gamma_1, \Gamma_2}^\bullet(M) \longrightarrow C_{\psi_1, \psi_2, \Gamma_1, \Gamma_2}^\bullet(M).$$

Proposition 4.4.3. (After KPX [KPX14, Proposition 2.3.6, Lemma 3.1.2]) *The morphism*

$$\Psi'_1 : C_{\varphi_1, \Gamma_1}^\bullet(M) \longrightarrow C_{\psi_1, \Gamma_1}^\bullet(M)$$

is a quasi-isomorphism. This will imply that Ψ_1 is a quasi-isomorphism.

Proof. First the problem is now reduced to the checking of the injectivity of the morphism Ψ'_1 defined above. Note now that the kernel of the above morphism is now just the corresponding complex which consists of the invariance taking the form of $M^{\Delta_I, \psi_1=0}$, so it suffices to prove that the operator $\gamma_1 - 1$ in our situation has the corresponding property of being invertible. Therefore we will follow the argument in [KPX14, Proposition 2.3.6], which will be just a direct generalization of the corresponding one dimensional result in [KPX14, Proposition 2.3.6]. First we consider the reduction of the proof to the situation where all the finite extensions K_I over \mathbb{Q}_p are all just the fields \mathbb{Q}_p , which could be checked through the induction. Then we assume that we have a corresponding $\Pi_{\text{an}, r_I, 0, I, A}(\pi^I)$ -module $M_{r_I, 0}^{\Delta_{\mathbb{Q}_p}}$ and the corresponding invariance $M_{r_I, 0}^{\Delta_{\mathbb{Q}_p}, \psi_1=0}$. Then we consider the suitable multi integers $n_I \geq 1$ which gives rise to (for suitable multi radii s_I and r_I smaller than the corresponding original multi radii $r_{I,0}$) the corresponding decomposition of the section

$$M_{r_I, 0}^{\Delta_{\mathbb{Q}_p}, \psi_1=0} \otimes_{\Pi_{\text{an}, r_I, 0, I, A}(\pi^I)} \Pi_{\text{an}, [s_I/p^{n_I}, r_I/p^{n_I}], I, A}(\pi^I)$$

into the corresponding summation of the corresponding components taking the form of:

$$(1 + \pi_1)^{i_1} \varphi_1^{n_1} \varphi_2^{n_2} M_{[s_I, r_I]}^{\Delta_{\mathbb{Q}_p}},$$

for suitable integers i_1 in our generalized situation. Then it now suffices to show that for each n_I , the action of the corresponding operators $\gamma_{n_I} - 1$ are invertible over each member in the decomposition as above, which is to say the action of each $\gamma_\alpha - 1$ is invertible over each $(1 + \pi_1)\varphi_1^{n_1} M_{[s_I, r_I]}^{\Delta_{\mathbb{Q}_p}}$, which implies that the $A[\gamma_{n_I}]$ -module structure could be promoted by continuation to the corresponding $\Pi_{\text{an}, r_I, 0, I, A}(\Gamma_{n_I})$ -module structure. Now we look at each member in the decomposition as above, and do the standard computation as in [KPX14, Lemma 3.1.2] as below (which is generalization since we are working with higher dimensional situation) for each m in each member participating in the decomposition as above:

$$(\gamma_{n_1} - 1)(1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}(m) \quad (4.4.1)$$

$$= (\gamma_{n_1} - 1)(1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}(m) \quad (4.4.2)$$

$$= (1 + \pi_1)^{p^{n_1+1}}\varphi_1^{n_1}\varphi_2^{n_2}\gamma_{n_1}(m) - (1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}(m) \quad (4.4.3)$$

$$= (1 + \pi_1)((1 + \pi_1)^{p^{n_1}}\varphi_1^{n_1}\varphi_2^{n_2}\gamma_{n_1} - \varphi_1^{n_1}\varphi_2^{n_2})(m) \quad (4.4.4)$$

$$= (1 + \pi_1)\varphi_1^{n_1}((1 + \pi_1)\varphi_2^{n_2}\gamma_{n_1} - \varphi_2^{n_2})(m) \quad (4.4.5)$$

$$= (1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}\pi_1(1 + (1 + \pi_1)/\pi_1(\gamma_{n_1} - 1))(m) \quad (4.4.6)$$

which implies that the inverse could be taken to be:

$$[(1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}\pi_1(1 + (1 + \pi_1)/\pi_1(\gamma_{n_1} - 1))]^{-1}(m) \quad (4.4.7)$$

$$= \frac{1}{(1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}\pi_1} \cdot \frac{1}{1 + (1 + \pi_1)/\pi_1(\gamma_{n_1} - 1)} m \quad (4.4.8)$$

$$= \frac{1}{(1 + \pi_1)\varphi_1^{n_1}\varphi_2^{n_2}\pi_1} \cdot \sum_{k \geq 0} (-1)^k ((1 + \pi_1)/\pi_1(\gamma_{n_1} - 1))^k m. \quad (4.4.9)$$

□

Now in the same fashion one can define the corresponding morphism:

$$\Psi_2 : C_{\varphi_1, \varphi_2, \Gamma_I}^\bullet(M) \longrightarrow C_{\varphi_1, \psi_2, \Gamma_I}^\bullet(M)$$

Lemma 4.4.4. *This morphism Ψ_2 is also a quasi-isomorphism. Therefore by composition we have the following quasi-isomorphism:*

$$\Psi_I : C_{\varphi_1, \varphi_2, \Gamma_I}^\bullet(M) \longrightarrow C_{\psi_1, \psi_2, \Gamma_I}^\bullet(M).$$

Then now we are at the position to consider the finiteness of the corresponding involved cohomologies. In our situation we consider the following derived categories:

Definition 4.4.5. (After KPX [KPX14, Notation 4.1.2]) Now we consider the following derived categories. The first one is the sub-derived category consisting of all the objects in $\mathbb{D}(A)$ which are quasi-isomorphic to those bounded above complexes of finite projective modules over the ring A . We denote this category (which is defined in the same way as in [KPX14, Notation 4.1.2]) by $\mathbb{D}_{\text{perf}}^-(A)$. Over the larger ring $\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})$ we also have the sub-derived category of $\mathbb{D}(\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I}))$ consisting of all those objects in $\mathbb{D}(\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I}))$ which are quasi-isomorphic to those bounded above complexes of finite projective modules now over the ring $\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})$. We denote this by $\mathbb{D}_{\text{perf}}^-(\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I}))$.

Then we will investigate the complexes attached to any finite projective (φ_I, Γ_I) -module M over $\Pi_{\text{an},\text{con},I,A}(\pi_{K_I})$:

$$C_{\varphi_I, \Gamma_I}^\bullet(M), C_{\psi_I, \Gamma_I}^\bullet(M), C_{\psi_I}^\bullet(M),$$

which are living inside the corresponding derived categories:

$$\mathbb{D}(A), \mathbb{D}(A), \mathbb{D}(\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})).$$

Proposition 4.4.6. (After KPX [KPX14, Proposition 4.2.3]) *Now for any multiplicative character η_I of the group Γ_{K_I} , we have the following isomorphism in the corresponding derived categories defined above:*

$$C_{\psi_I}^\bullet(M) \otimes^{\mathbb{L}} \Pi_{\text{an},\infty,I,A}(\Gamma_{K_I}) / \mathfrak{m}_\eta \Pi_{\text{an},\infty,I,A}(\Gamma_{K_I}) \xrightarrow{\sim} C_{\psi_I, \Gamma_I}^\bullet(M \otimes M_{\eta^{-1}}) \quad (4.4.10)$$

$$\xrightarrow{\sim} C_{\varphi_I, \Gamma_I}^\bullet(M \otimes M_{\eta^{-1}}). \quad (4.4.11)$$

Then we are going to study the finiteness of the cohomologies defined above in more details. The methods in [KPX14] requires some deep results on the slope filtrations for modules over the one-dimensional Robba rings, where allows [KPX14] to reduce the proof in some sense to the étale situation. It looks like we have no chance to apply the ideas in [KPX14]. Here we adopt the ideas in some development from Kedlaya-Liu in [KL]. To be more precise we are going to apply some ideas in [KL] around the application of some p -adic functional analysis.

Proposition 4.4.7. *With the notation and setting up as above we have $h^\bullet(C_{\psi_I}(M))$ are coadmissible over $\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})$.*

Proof. The original situation for $A = \mathbb{Q}_p$ and M is étale is proved by [CKZ18] and [PZ19], which is a direct consequence of Galois cohomologies. Here we adopt the Cartan-Serre method involving the completely continuous maps within the p -adic functional analysis. By simultaneous induction to the base cases for the field \mathbb{Q}_p from all K_I , we can now assume that we are considering the situation where all K_I are \mathbb{Q}_p . Then for the complex $C_{\psi_I}^\bullet(M)$, by definition one could then regard this complex as the double complex taking the following form (after considering the corresponding totalization):

$$\begin{array}{ccc}
 M & \xrightarrow{\psi_1-1} & M \\
 \downarrow \psi_2-1 & & \downarrow \\
 M & \longrightarrow & M.
 \end{array}$$

The boundedness of the complex could be observed from the definition and the corresponding ideas in [KL, Theorem 3.3, Theorem 3.5]. For the finiteness we construct suitable completely continuous maps comparing the corresponding quasi-isomorphic complexes for M as below. First we consider some model M_{r_I} over some $\Pi_{\text{an},r_I,0,I,A}(\pi_{K_I})$. Then as in [KL, Theorem 3.3, Theorem 3.5] we could regard the corresponding complex for M_{r_I} as equivalently the corresponding complex over $\Pi_{\text{an},r_I,0,I,\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})}(\pi_{K_I})$. We are then reduced to considering the corresponding complex over $\Pi_{\text{an},r_{I_0},I,\Pi_{\text{an},[t_I,\infty],I,A}(\Gamma_{K_I})}(\pi_{K_I})$ for some t_I .

Then we consider the complex for some section over some multi interval taking the form of $[s_I, r_I]$ which gives rise to the corresponding quasi-isomorphic complex now over

$$\Pi_{\text{an},[s_I,r_{I_0}],I,\Pi_{\text{an},[t_I,\infty],I,A}(\Gamma_{K_I})}(\pi_{K_I})$$

which gives rise to the complex:

$$\begin{array}{ccc}
M_{[s_1, r_1] \times [s_2, r_2]} & \xrightarrow{\psi_1 - 1} & M_{[ps_1, r_1] \times [s_2, r_2]} \\
\downarrow \psi_2 - 1 & & \downarrow \\
M_{[s_1, r_1] \times [ps_2, r_2]} & \longrightarrow & M_{[ps_1, r_1] \times [ps_2, r_2]}.
\end{array}$$

Here the corresponding quasi-isomorphism among different boxes (e.g. multi intervals) is not actually trivial. For the serious argument after the knowledge on the equivalence between the categories of φ_I -modules over the full Robba ring, φ_I -Frobenius bundles and φ_I -modules over the Robba ring with respect to some single multi interval, which could be realized in our context by using proposition 4.3.6 as in [KP1, Lemma 2.10], we refer to [KP1, Lemma 5.2] and [KL16, Proposition 5.12, Theorem 5.7.11] for the corresponding comparison on the cohomology groups.

To show that the cohomology groups do not change when we switch from a multi-interval $[s'_1, r'_1] \times [s'_2, r'_2]$ to a multi-interval $[s''_1, r''_1] \times [s''_2, r''_2]$, we choose to look at an intermediate multi-interval $[s''_1, r''_1] \times [s'_2, r'_2]$ namely we consider the following in order rings over multi-intervals:

$$\Pi_{\text{an}, [s'_1, r'_1] \times [s'_2, r'_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}) \quad (4.4.12)$$

$$\Pi_{\text{an}, [s''_1, r''_1] \times [s'_2, r'_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}) \quad (4.4.13)$$

$$\Pi_{\text{an}, [s''_1, r''_1] \times [s''_2, r''_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}) \quad (4.4.14)$$

and compare the first two and the last two. To compare with respect to the first two multi-intervals we first consider the ψ_1 -complex, the corresponding cohomology groups could be regarded as the corresponding Yoneda extension groups over the corresponding twisted polynomial ring in variable of ψ_1 with coefficients in the Robba rings, which shows that the ψ_I -cohomology groups do not change when we change the boxes. Then for the remaining comparison we argue in the parallel way.

To show that the cohomology groups do not change when we switch from Robba ring over some fixed multi-radii $r_1 \times r_2$ to some interval $[s_1, r_1] \times [s_2, r_2]$, we consider the

corresponding intermediate Robba ring:

$$\bigcap_{s_2 > 0} \Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}) \quad (4.4.15)$$

and then in order the following rings:

$$\bigcap_{s_1 > 0, s_2 > 0} \Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}), \quad (4.4.16)$$

$$\bigcap_{s_2 > 0} \Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}), \quad (4.4.17)$$

$$\Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, \Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})}(\pi_{K_I}). \quad (4.4.18)$$

Note that finite projective modules over these are equivalent when carrying the corresponding φ_I -structure. To compare with respect to the first two situations, we first consider the ψ_1 -cohomology complex, then by using the corresponding Yoneda extension we have the corresponding ψ_1 -cohomology groups are the same, which implies further that the corresponding ψ_I -cohomology groups are the same. Then for the remaining comparison we argue in the parallel way.

Then to finish, we choose two well-located multi intervals such that one embeds into another (which exists trivially in our situation which is a little bit different from the situation in [KL]):

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{\psi_{I \setminus \{I\}}}^\bullet(M_{\dots, [s'_I, r'_I]}) & \xrightarrow{\psi_I - 1} & C_{\psi_{I \setminus \{I\}}}^\bullet(M_{\dots, [ps'_I, r'_I]}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{\psi_{I \setminus \{I\}}}^\bullet(M_{\dots, [s''_I, r''_I]}) & \xrightarrow{\psi_I - 1} & C_{\psi_{I \setminus \{I\}}}^\bullet(M_{\dots, [ps''_I, r''_I]}) & \longrightarrow & 0, \end{array}$$

or more explicit:

$$\begin{array}{ccccc}
M_{[s'_1, r'_1] \times [s'_2, r'_2]} & \longrightarrow & M_{[ps'_1, r'_1] \times [s'_2, r'_2]} \oplus M_{[s'_1, r'_1] \times [ps'_2, r'_2]} & \longrightarrow & M_{[ps'_1, r'_1] \times [ps'_2, r'_2]} \\
\downarrow & & \downarrow & & \downarrow \\
M_{[s''_1, r''_1] \times [s''_2, r''_2]} & \longrightarrow & M_{[ps''_1, r''_1] \times [s''_2, r''_2]} \oplus M_{[s''_1, r''_1] \times [ps''_2, r''_2]} & \longrightarrow & M_{[ps''_1, r''_1] \times [ps''_2, r''_2]}
\end{array}$$

Here the corresponding maps between objects over different multi intervals are induced exactly from the corresponding inclusion of multi intervals. Then by the fundamental lemma for in [Kie1, Satz 5.2] and [KL, Lemma 1.10] on the continuous completely maps we get the corresponding finiteness when we restrict to $[t_I, \infty]$, since the map according to the embeddings of the multi intervals are quasi-isomorphisms. Then the corresponding cohomology groups could be arranged to be a coherent sheaf over each section with respect to each such $[t_I, \infty]$, then the coadmissibility follows. \square

Proposition 4.4.8. *Based on the previous proposition we have:*

$$C_{\psi_I}(M) \in \mathbb{D}_{\text{perf}}^-(\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})),$$

for $A = \mathbb{Q}_p$.

Proof. We have the corresponding coadmissibility from the previous proposition, then we work in the category of all the coadmissible modules over the ring $\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})$. Then by applying the corresponding result [Zab1, Theorem 3.10] we can show that the complex is quasi-isomorphic to a corresponding complex being finite projective at each degree in the following. By base change to some $\Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})$ for some t_I we have that any such complex over $\Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})$ is quasi-isomorphic to a complex being finitely generated at each degree. Therefore the corresponding original complex is quasi-isomorphic to a corresponding complex being coadmissible at each degree. Then in fact at each degree this latter complex is also projective (in the category of all the coadmissible modules over $\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})$), since this will be further equivalent to that each section over $\Pi_{\text{an}, [t_I, \infty], I, A}(\Gamma_{K_I})$ is projective for some t_I , which is the case in our situation. Then we are

done by [Zab1, Theorem 3.10] which implies that coadmissible and projective modules are finitely generated. □

We then give a direct proof on the finiteness of the cohomology of (φ_I, Γ_I) -modules:

Proposition 4.4.9. *With the notations above, we have that*

$$C_{\varphi_I, \Gamma_I}^\bullet(M) \in \mathbb{D}_{\text{perf}}^-(A), C_{\psi_I, \Gamma_I}^\bullet(M) \in \mathbb{D}_{\text{perf}}^-(A).$$

Proof. This is proved in the étale setting in [PZ19] and [CKZ18] in the absolute situation. We use the corresponding Cartan-Serre method to prove this in the following fashion, modeled on the corresponding context of [KL]. By simultaneous induction to the base cases for the field \mathbb{Q}_p from all K_I , we can now assume that we are considering the situation where all K_I are \mathbb{Q}_p . We choose to prove the corresponding result for (ψ_I, Γ_I) -modules. In our situation we look at the corresponding continuous complexes coming from the corresponding Γ_I -action, which gives us the corresponding complex $C_{\text{con}}^\bullet(\Gamma_{K_I}, M)$. Then the idea is to look at the following double complex of the complex $C_{\text{con}}^\bullet(\Gamma_{K_I}, M)$ (to be more precise we need to look at the corresponding totalization):

$$\begin{array}{ccc} C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{[s_1, r_1] \times [s_2, r_2]}) & \xrightarrow{\psi_1 - 1} & C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{[ps_1, r_1] \times [s_2, r_2]}) \\ \downarrow \psi_2 - 1 & & \downarrow \\ C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{[s_1, r_1] \times [ps_2, r_2]}) & \longrightarrow & C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{[ps_1, r_1] \times [ps_2, r_2]}) \end{array}$$

Here the corresponding quasi-isomorphism among different boxes (e.g. multi intervals) is not actually trivial. For the serious argument after the knowledge on the equivalence between the categories of φ_I -modules over the full Robba ring, φ_I -Frobenius bundles and φ_I -modules over the Robba ring with respect to some single multi interval, which could be realized in our context by using proposition 4.3.6 as in [KP1, Lemma 2.10], we refer to [KP1, Lemma 5.2] and [KL16, Proposition 5.12, Theorem 5.7.11] for the corresponding comparison on the cohomology groups.

To show that the cohomology groups do not change when we switch from a multi-interval $[s'_1, r'_1] \times [s'_2, r'_2]$ to a multi-interval $[s''_1, r''_1] \times [s''_2, r''_2]$, we choose to look at an intermediate multi-interval $[s''_1, r''_1] \times [s'_2, r'_2]$ namely we consider the following in order rings over multi-intervals:

$$\Pi_{\text{an}, [s'_1, r'_1] \times [s'_2, r'_2], I, A}(\pi_{K_I}) \quad (4.4.19)$$

$$\Pi_{\text{an}, [s''_1, r''_1] \times [s'_2, r'_2], I, A}(\pi_{K_I}) \quad (4.4.20)$$

$$\Pi_{\text{an}, [s''_1, r''_1] \times [s''_2, r''_2], I, A}(\pi_{K_I}) \quad (4.4.21)$$

and compare the first two and the last two. To compare with respect to the first two multi-intervals we first consider the ψ_1 -complex, the corresponding cohomology groups could be regarded as the corresponding Yoneda extension groups over the corresponding twisted polynomial ring in variable of ψ_1 with coefficients in the Robba rings, which shows that the ψ_I -cohomology groups do not change when we change the boxes. Then for the remaining comparison we argue in the parallel way.

To show that the cohomology groups do not change when we switch from Robba ring over some fixed multi-radii $r_1 \times r_2$ to some interval $[s_1, r_1] \times [s_2, r_2]$, we consider the corresponding intermediate Robba ring:

$$\bigcap_{s_2 > 0} \Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, A}(\pi_{K_I}) \quad (4.4.22)$$

and then in order the following rings:

$$\bigcap_{s_1 > 0, s_2 > 0} \Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, A}(\pi_{K_I}), \quad (4.4.23)$$

$$\bigcap_{s_2 > 0} \Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, A}(\pi_{K_I}), \quad (4.4.24)$$

$$\Pi_{\text{an}, [s_1, r_1] \times [s_2, r_2], I, A}(\pi_{K_I}). \quad (4.4.25)$$

Note that finite projective modules over these are equivalent when carrying the corresponding φ_I -structure. To compare with respect to the first two situations, we first consider the ψ_1 -cohomology complex, then by using the corresponding Yoneda extension we have the corresponding ψ_1 -cohomology groups are the same, which implies further that the corresponding ψ_I -cohomology groups are the same. Then for the remaining comparison we argue in the parallel way.

Then in order to apply the corresponding fundamental lemma namely [KL, Lemma 1.10], we need to choose suitable internals in our multi setting where one embeds into the other which gives rise to the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_{\psi_{I \setminus \{I\}}}^\bullet(C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{\dots, [s'_I, r'_I]})) & \xrightarrow{\psi_I - 1} & C_{\psi_{I \setminus \{I\}}}^\bullet(C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{\dots, [ps'_I, r'_I]})) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{\psi_{I \setminus \{I\}}}^\bullet(C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{\dots, [s''_I, r''_I]})) & \xrightarrow{\psi_I - 1} & C_{\psi_{I \setminus \{I\}}}^\bullet(C_{\text{con}}^\bullet(\Gamma_{K_I}, M_{\dots, [ps''_I, r''_I]})) & \longrightarrow & 0.
\end{array}$$

To be more explicit this gives rise to the following diagram:

$$\begin{array}{ccccccc}
C_{\Gamma_I}^\bullet(M_{[s'_1, r'_1] \times [s'_2, r'_2]}) & \longrightarrow & C_{\Gamma_I}^\bullet(M_{[ps'_1, r'_1] \times [s'_2, r'_2]}) \oplus C_{\Gamma_I}^\bullet(M_{[s'_1, r'_1] \times [ps'_2, r'_2]}) & \longrightarrow & C_{\Gamma_I}^\bullet(M_{[ps'_1, r'_1] \times [ps'_2, r'_2]}) \\
\downarrow & & \downarrow & & \downarrow \\
C_{\Gamma_I}^\bullet(M_{[s''_1, r''_1] \times [s''_2, r''_2]}) & \longrightarrow & C_{\Gamma_I}^\bullet(M_{[ps''_1, r''_1] \times [s''_2, r''_2]}) \oplus C_{\Gamma_I}^\bullet(M_{[s''_1, r''_1] \times [ps''_2, r''_2]}) & \longrightarrow & C_{\Gamma_I}^\bullet(M_{[ps''_1, r''_1] \times [ps''_2, r''_2]}).
\end{array}$$

Here the corresponding maps between objects over different multi intervals are induced exactly from the corresponding inclusion of multi intervals. Then we consider the corresponding fundamental lemma [Kie1, Satz 5.2] and [KL, Lemma 1.10] which implies the corresponding finiteness. And by our basic construction the result follows. \square

Now we consider a finite projective (φ_I, Γ_I) -module M over the ring $\Pi_{\text{an, con}, I, \Pi_{\text{an}, \infty, A}(G)}$, namely we have:

$$M = \varprojlim_k M_k,$$

where each object M_k satisfies the corresponding framework in the above discussion.

Definition 4.4.10. (After KPX [KPX14, 2.3]) For any (φ_I, Γ_I) -module M over $\Pi_{\text{an, con}, I, \Pi_{\text{an}, \infty, A}(G)}$ we define the complex $C_{\Gamma_I}^\bullet(M)$ of M to be the total complex of the

following complex through the induction:

$$[C_{\Gamma_I \setminus \{I\}}^\bullet(M) \xrightarrow{\Gamma_I} C_{\Gamma_I \setminus \{I\}}^\bullet(M)].$$

Then we define the corresponding double complex $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$ again by taking the corresponding totalization of the following complex through induction:

$$[C_{\varphi_I \setminus \{I\}}^\bullet C_{\Gamma_I}^\bullet(M) \xrightarrow{\varphi_I} C_{\varphi_I \setminus \{I\}}^\bullet C_{\Gamma_I}^\bullet(M)].$$

Then we define the complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the totalization of the double complex defined above, which is called the complex of a (φ_I, Γ_I) -module M . Similarly we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the totalization of the double complex defined in the same way as above by replacing φ_I by then the operators ψ_I . For later comparison we also need the corresponding ψ_I -cohomology, which is to say the complex $C_{\psi_I}^\bullet(M)$ which could be defined by the totalization of the following complex through induction:

$$[C_{\psi_I \setminus \{I\}}^\bullet(M) \xrightarrow{\psi_I} C_{\psi_I \setminus \{I\}}^\bullet(M)].$$

Proposition 4.4.11. *With the notation and setting up as above we have $h^\bullet(C_{\psi_I}(M))$ are coadmissible over $\Pi_{\text{an}, \infty, A} \widehat{\otimes} \Pi_{\text{an}, \infty}(\Gamma_{K_I})(G)$.*

Proof. By simultaneous induction to the base cases for the field \mathbb{Q}_p from all K_I , we can now assume that we are considering the situation where all K_I are \mathbb{Q}_p . Then for the complex $C_{\psi_I}^\bullet(M)$, by definition one could then regard this complex as the double complex taking the following form (after considering the corresponding totalization):

$$\begin{array}{ccc} M & \xrightarrow{\psi_1-1} & M \\ \downarrow \psi_2-1 & & \downarrow \\ M & \longrightarrow & M. \end{array}$$

The boundedness of the complex could be observed from the definition and the corresponding ideas in [KL, Theorem 3.3, Theorem 3.5]. For the finiteness we construct suitable completely continuous maps comparing the corresponding quasi-isomorphic complexes for M as below. Write $\Pi_{\text{an}, \infty, A}(G)$ as the following form:

$$\Pi_{\text{an}, \infty, A}(G) := \varprojlim_k \Pi_{\text{an}, \infty, A}(G)_k,$$

as a Fréchet-Stein algebra. Then take some $\Pi_{\text{an},\infty,A}(G)_k$ in the family with the corresponding M_k . By our result above for M_k proposition 4.4.7 we have that $h^\bullet(C_{\psi_I}(M_k))$ are coadmissible over $\Pi_{\text{an},\infty,\Pi_{\text{an},\infty,A}(G)_k}(\Gamma_{K_I})$. Therefore we have the the corresponding cohomology groups $h^\bullet(C_{\psi_I}(M))$ are actually coming from (the global sections of) some coherent sheaves over $\Pi_{\text{an},\infty,\Pi_{\text{an},\infty,A}(G)}(\Gamma_{K_I})$. □

Proposition 4.4.12. *Based on the previous proposition we have:*

$$C_{\psi_I}(M) \in \mathbb{D}_{\text{perf}}^-(\Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G)),$$

for $A = \mathbb{Q}_p$.

Proof. We have the corresponding coadmissibility from the previous proposition, then we work in the category of all the coadmissible modules over the ring $\Pi_{\text{an},\infty,I,A}(\Gamma_{K_I})$. Then by applying the corresponding result [Zab1, Theorem 3.10] we can show that the complex is quasi-isomorphic to a corresponding complex being finite projective at each degree in the following. Set:

$$\Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G) := \varprojlim_k \Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G)_k. \quad (4.4.26)$$

By base change to some $\Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G)_k$ for some k we have that any such complex over $\Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G)_k$ is quasi-isomorphic to a complex being finitely generated at each degree. Therefore the corresponding original complex is quasi-isomorphic to a corresponding complex being coadmissible at each degree. Then in fact at each degree this latter complex is also projective (in the category of all the coadmissible modules over $\varprojlim_k \Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G)_k$), since this will be further equivalent to that each section over $\varprojlim_k \Pi_{\text{an},\infty,A\hat{\otimes}\Pi_{\text{an},\infty}(\Gamma_{K_I})}(G)_k$ is projective for some k , which is the case in our situation. Then we are done by [Zab1, Theorem 3.10] which implies that coadmissible and projective modules are finitely generated. □

We then give a direct proof on the finiteness of the cohomology of (φ_I, Γ_I) -modules:

Proposition 4.4.13. *With the notations above, we have that*

$$C_{\varphi_I,\Gamma_I}^\bullet(M) \in \mathbb{D}_{\text{perf}}^-(\Pi_{\text{an},\infty,A}(G)), C_{\psi_I,\Gamma_I}^\bullet(M) \in \mathbb{D}_{\text{perf}}^-(\Pi_{\text{an},\infty,A}(G)).$$

Here again we assume that the corresponding ring A is \mathbb{Q}_p .

Proof. We choose to prove the corresponding result for (ψ_I, Γ_I) -modules. As above we only have to consider the corresponding situation where K_I are all \mathbb{Q}_p by induction in our current context. Write $\Pi_{\text{an}, \infty, A}(G)$ as the limit:

$$\Pi_{\text{an}, \infty, A}(G) = \varprojlim_k \Pi_{\text{an}, \infty, A}(G)_k.$$

Working over the ring $\Pi_{\text{an}, \infty, A}(G)_k$ we have the corresponding finiteness by the result from proposition 4.4.9, which will show that $C_{\psi_I, \Gamma_I}^\bullet(M)$ are all coadmissible modules over $\varprojlim_k \Pi_{\text{an}, \infty, A}(G)_k$. By base change to some $\Pi_{\text{an}, \infty, A}(G)_k$ for some k we have that any such complex over $\Pi_{\text{an}, \infty, A}(G)_k$ is quasi-isomorphic to a complex being finitely generated at each degree. Therefore the corresponding original complex is quasi-isomorphic to a corresponding complex being coadmissible at each degree. Then in fact at each degree this latter complex is also projective (in the category of all the coadmissible modules over $\varprojlim_k \Pi_{\text{an}, \infty, A}(G)_k$), since this will be further equivalent to that each section over $\varprojlim_k \Pi_{\text{an}, \infty, A}(G)_k$ is projective for some k , which is the case in our situation. Then we are done by [Zab1, Theorem 3.10] which implies that coadmissible and projective modules are finitely generated. □

4.4.2 Application to Triangulation over Rigid Analytic Spaces

We are now going to establish some application of the finiteness of the corresponding complexes with respect to the action (φ_I, Γ_I) . We will generalize the interesting globalization of the triangulation established in [KPX14] to our generalized setting. This has its own interests since it will be very natural to ask about the behavior of some triangulation in our context over the corresponding interesting rigid families of the generalized Hodge structures.

Since we only proved the corresponding finiteness in the situation where the set I consists of two elements, therefore in this section we are going to make the same assumption. Then we consider the corresponding finite free rank one (φ_I, Γ_I) -modules coming from the corresponding algebraic characters from the product of Γ_{K_I} , and moreover from the product of the groups K_I^\times . Actually in the one dimensional situation this consideration finishes the corresponding construction of all the finite free rank one (φ_I, Γ_I) -modules

in a reasonably well-defined way. We follow largely the corresponding notations in the one dimensional situation. We will assume that over some strong (since we do not have a well-established classifications as mentioned above) pointwisely we have the corresponding strong triangulation (meaning that we have the triangulations by using the parameters strictly coming from the corresponding continuous characters of the group $\Gamma_{K_I}^\times$), then we would like to study the corresponding behavior of the corresponding global triangulations. This means that in order to have some coherent triangulations and spreading effect, one only needs to check things pointwisely.

Setting 4.4.14. In this section we are going to work over a general rigid analytic space X defined over suitable finite extension of L , where we assume that L is large enough with respect to the fields K_I generalizing the corresponding situation in [KPX14, Chapter 6].

We start from the following geometric setting:

Definition 4.4.15. (After KPX [KPX14, Definition 6.2.1]) First we are going to consider the corresponding sheaves $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ with respect to the analytic space X over L , which is defined to be glueing the rings of analytic functions over each affinoid $\text{Max}A$ for the space X , namely glueing the rings having the form of $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ with respect to this affinoid. Also as in [KPX14, Definition 6.2.1] one can define the corresponding sheaf with respect to some specific radii which is to say the sheaves taking the form of $\Pi_{\text{an},r_I,I,X}(\pi_{K_I})$. Then we define the corresponding (φ_I, Γ_I) -modules over these two kinds of sheaves by considering the corresponding families of finite locally free sheaves of modules over $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ or $\Pi_{\text{an},r_I,I,X}(\pi_{K_I})$ in the well-defined and compatible sense as in the one-dimensional situation in [KPX14, Definition 6.2.1].

Setting 4.4.16. We choose to now consider the following situation where all the fields K_I are assumed to be \mathbb{Q}_p . Therefore now the p -adic Lie groups we are considering take the following form namely the product of \mathbb{Q}_p^\times .

Under this assumption namely where all the fields K_I are \mathbb{Q}_p one could directly consider the corresponding construction as generalized below from the one dimensional situation considered in [KPX14, Notation 6.2.2]:

Definition 4.4.17. (After KPX [KPX14, Notation 6.2.2]) Consider the product of the p -adic Lie group \mathbb{Q}_p^\times which is to say $\prod_{\alpha \in I} \mathbb{Q}_p^\times$. We are going to define the corresponding free rank one (φ_I, Γ_I) -modules of the character types attached to any character taking the form of $\delta_I : \prod_{\alpha \in I} \mathbb{Q}_p^\times \rightarrow \Gamma(X, \mathcal{O}_X)^\times$, to be the free of rank one module $\Pi_{\text{an,con},I,X}(\pi_{K_I})\mathbf{e}$ such that we have for each $\alpha \in I$, the Frobenius φ_α acts via $\delta_I(p_\alpha)$ while the group Γ_α acts via $\delta_I(\chi_\alpha(\gamma_\alpha))$. Note that when we have that the set I is a singleton the corresponding definition recovers the one defined in [KPX14, Notation 6.2.2]. We will use the notation $\Pi_{\text{an,con},I,X}(\pi_{K_I})(\delta_I)$ to denote such free rank one object in the general setting.

On the other hand actually we have no direct classification literally as in the one dimensional situation on the free rank one objects at least up to this definition. Therefore we have to at this moment to fix our objects considered, which is to say that we will focus on all the free rank one objects of character types as defined above, then to study the corresponding geometrization and the variation of the corresponding triangulation. We start from the following basic definition which is a direct generalization of the corresponding one in the one dimensional situation.

Definition 4.4.18. (After KPX [KPX14, Definition 6.3.1]) We define the *triangulated* (φ_I, Γ_I) -modules over the higher dimensional Robba rings over a rigid analytic space X taking the form of $\Pi_{\text{an,con},I,X}(\pi_{K_I})$. Such a module M is defined to be a usual (φ_I, Γ_I) -module over $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ carrying a triangulation with n -parameters $\delta_1, \dots, \delta_n$ which are n characters (we will assume that these are continuous ones) of the group $\prod_I \mathbb{Q}_p^\times$ with the values in the corresponding group $\Gamma(X, \mathcal{O}_X)^\times$, which means that we have the following filtration on M by the corresponding sub (φ_I, Γ_I) -modules over the sheaf $\Pi_{\text{an,con},I,X}(\pi_{K_I})$:

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M$$

where n is assumed in our situation to be the rank of M , moreover we have that the corresponding each graded piece M_i/M_{i-1} for $i = 1, \dots, n$ is a rank one (φ_I, Γ_I) -module with the corresponding parameter δ_i defined above with some twisted line bundle $\mathbb{L}_i, i = 1, \dots, n$:

$$M_i/M_{i-1} \xrightarrow{\sim} \Pi_{\text{an,con},I,X}(\pi_{K_I})(\delta_i) \otimes \mathbb{L}_i, i = 1, \dots, n.$$

Definition 4.4.19. (After KPX [KPX14, Definition 6.3.1]) As in [KPX14, Definition 6.3.1] we generalize the corresponding *strictly triangulated modules* to the higher

dimensional situation by defining them to be over $X = \text{Max}(L)$ the triangulated modules as above in a manner of building from each i -th filtration to the $i + 1$ -th uniquely through the corresponding parameters.

We are going to consider the corresponding integrated behavior of a dense subset of pointwise triangulations. First recall that a subspace U is *Zariski dense* if for each member in an admissible covering of the space X the space U is dense inside (see [KPX14, Chapter 6.3]). Then we can now consider the corresponding densely pointwise strictly triangulated families:

Definition 4.4.20. (After KPX [KPX14, Definition 6.3.2]) We are going to call a (φ_I, Γ_I) -module over the sheaf $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ a *densely pointwise strictly triangulated family* if there exists a Zariski-dense subspace X_{dpt} inside X such that the corresponding the restriction to X_{dpt} of the sheaf M is pointwise strictly triangulated in the sense defined above.

Lemma 4.4.21. (After KPX [KPX14, 6.3.3]) *In our situation, we have the cohomology groups $H_{\varphi_I, \Gamma_I}^\bullet(M)$ are coherent sheaves over the rigid analytic space X for M any (φ_I, Γ_I) -modules over the Frobenius sheaves $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ with the corresponding non-vanishing throughout the degrees in $[0, 2I]$.*

Proof. The proof of this result is parallel to the corresponding result in the one dimensional situation, see [KPX14, 6.3.3]. Essentially one has this result as a direct consequence of the results in the local setting in section 4.4.1. \square

We then consider our main results in this section, which is a direct generalization of the results of [KPX14] (chapter 6) to the situation where we have higher dimensional Hodge-Frobenius structure.

Proposition 4.4.22. (After KPX [KPX14, Theorem 6.3.9]) *Suppose that we have a (φ_I, Γ_I) -module M over the ring $\Pi_{\text{an,con},I,X}(\pi_{K_I})$, where we assume that this module is of rank k . Suppose that we have a continuous character from the group $\prod_I \mathbb{Q}_p^\times$ into the group $\Gamma(X, \mathcal{O}_X)^\times$. Suppose that there exists a subset $X_{pdt} \subset X$ over which we have that the cohomology group $H_{\varphi_I, \Gamma_I}^0(M_x^\vee(\delta_x))$ is of dimension one (which is to say for each $x \in X_{pdt}$), with the further assumption that the image of the corresponding ring $\Pi_{\text{an,con},I,X}(\pi_{K_I})$ is*

saturated under this assumption (namely under the corresponding consideration of the basis). Assume this subset is Zariski-dense. Assume now the space X is reduced, then one could be able to find a morphism from another blow-up $b : Y \rightarrow X$ which is birational and proper and a corresponding map $f : b^*M \rightarrow \Pi_{\text{an,con},I,X}(\pi_{K_I})(\delta) \otimes_{\mathcal{O}_X} \mathbb{L}$ (with some well-defined line bundle \mathbb{L} as in the situation of [KPX14, Theorem 6.3.9]).

Proof. Indeed, we can adapt the corresponding strategy of [KPX14, Theorem 6.3.9] to prove this. First by taking the well-established (see [KPX14, Theorem 6.3.9] the geometry actually) normalization we are reduced to assume that the space X is normal and connected, and consequently the ranks of the corresponding coherent sheaves preserve. Then we consider the corresponding local coherence of the corresponding cohomology complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ (for all $\alpha \in I$), and the corresponding local base change results (namely the direct analog of the theorem 4.4.3 of [KPX14]), together with the 6.3.6 of [KPX14] we have the corresponding groups $H_{\varphi_I, \Gamma_I}^\bullet(M)$ are flat for $\bullet = 0$ and having Tor dimension less than or equal to 1 above, after we replace M by some M_0 which is the pullback of $M^\vee(\delta)$ along some $f_0 : Y_0 \rightarrow X$ which is birational and proper. We then glue along the corresponding local construction above as in [KPX14, Theorem 6.3.9]. The first task for us then will be the corresponding construction of the morphism over this Y_0 which will be later be promoted to be the desired Y by taking the corresponding blow-up. The construction goes as in the following way which is modeled on [KPX14, Theorem 6.3.9]. One first considers the corresponding subset $Y_1'^c \subset Y_0$ that consists of all the points at which the corresponding forth term in the following exact sequence (by considering the corresponding base change spectral sequence)

$$0 \longrightarrow H_{\varphi_I, \Gamma_I}^0(M_0) \otimes \kappa_y \longrightarrow H_{\varphi_I, \Gamma_I}^0(M_{0,y}) \longrightarrow \text{Tor}_1(H_{\varphi_I, \Gamma_I}^1(M_0), \kappa_y) \longrightarrow 0.$$

vanishes in our situation. Note that this is dense and open in the Zariski topology. Over the subset Y_1' the resulting exact sequence degenerates just to be:

$$0 \longrightarrow H_{\varphi_I, \Gamma_I}^0(M_0) \otimes \kappa_y \longrightarrow H_{\varphi_I, \Gamma_I}^0(M_{0,y}) \longrightarrow \text{Tor}_1(H_{\varphi_I, \Gamma_I}^1(M_0), \kappa_y) \longrightarrow 0,$$

where the forth term does not vanish. Then we note that over the subset $Y_1'^c$ we have that the corresponding resulting cohomology group $H_{\varphi_I, \Gamma_I}^0(M_0)$ is then of dimensional one, which implies that by the topological property of the subspace it is coherent of rank one throughout the whole space Y_0 . We then have the chance to extract the desired

morphism in our situation as below: the corresponding line bundle \mathbb{L} is chosen to be the corresponding dual line bundle. Then we have the result morphism $\Pi_{\text{an,con},I,X}(\pi_{K_I}) \otimes \mathbb{L}^\vee \rightarrow M_0$ and consequently we have the corresponding desired morphism which is to say $f_0^*M \rightarrow \Pi_{\text{an,con},I,X}(\pi_{K_I})(\delta) \otimes \mathbb{L}$.

Then up to this point, one can then consider 6.3.6 of [KPX14] to extract the desired blow-up $Y \rightarrow Y_0$ and define the desired data through the pullback along blow-up, which is to say the corresponding \mathbb{L} , f (see [KPX14, Theorem 6.3.9]).

□

Remark 4.4.23. The corresponding morphism established by the previous proposition is actually expected to satisfy the corresponding properties parallel to the corresponding situation in [KPX14, Theorem 6.3.9, Corollary 6.3.10]. Namely in the context mentioned in the proof of the previous proposition we are now at the position to (thanks to the previous proposition) conjecture:

A. We have the corresponding subset $Y_1^c \subset Y$ of all the points y where the corresponding map $f_y : M_y \rightarrow \Pi_{\text{an,con},I,\kappa_y}(\pi_{K_I})(\delta_y) \otimes_{\mathcal{O}_X} \mathbb{L}_y$ is surjective (as the original context of [KPX14] we need to assume that the corresponding element generates the zero-th cohomology of $M_y^\vee(\delta_y)$ as well), is open and dense in the Zariski topology; B. The corresponding kernel $\mathbf{Ker} f$ is of rank just $k - 1$.

4.5 Noncommutative Coefficients

4.5.1 Frobenius Modules over Noncommutative Rings

We now consider the coefficient where A is a noetherian noncommutative Banach affinoid algebra over \mathbb{Q}_p namely quotient of noncommutative Tate algebra with free variables. The finiteness even in the univariate situation with noncommutative coefficients is actually not known in [Zah1] which is key ingredient in the noncommutative local p -adic Tamagawa number conjecture in [Zah1]. We make the corresponding discussion in our situation.

Example 4.5.1. A nontrivial example of such noetherian noncommutative Banach affinoid algebra could be the corresponding noncommutative analog of the commutative Tate algebra \mathbb{T}_2 considered in [So1, Section 3]. To be more precise one considers first the corresponding twisted polynomial ring $\mathbb{Q}_p[X_1, X_2]'$ such that $X_1X_2 = aX_2X_1$ for some nonzero element $a \in \mathbb{Q}_p$, then takes the corresponding completion with respect to the Gauss norm $\|\cdot\|$ which is defined by:

$$\left\| \sum_{i_1, i_2} a_{i_1, i_2} X_1^{i_1} X_2^{i_2} \right\| := \sup_{i_1, i_2} \{|a_{i_1, i_2}|_p\}.$$

Note that as in [So1, Proposition 3.1] the corresponding categories of finite left A -modules and the corresponding Banach ones are actually equivalent, under the forgetful functor.

We then consider the families of the corresponding Frobenius modules with coefficients in noncommutative affinoid algebra A considered as above.

Setting 4.5.2. For the Robba rings with respect to some closed interval $[s_I, r_I] \subset (0, \infty]^{|I|}$ we defined above namely $\Pi_{\text{an}, [s_I, r_I], I, \mathbb{Q}_p}$, we take the completed tensor product with the corresponding ring A in this section to define the corresponding noncommutative version of the Robba rings. We keep all the notations compatible with the ones in the commutative setting in the previous discussion in the previous sections.

Definition 4.5.3. (After KPX [KPX14, Definition 2.2.2]) We keep all the notations (except the coefficient A which is now noncommutative) compatible with the ones in the commutative setting in the previous discussion in the previous sections. For each $\alpha \in I$, we choose suitable uniformizer π_{K_α} in our consideration. Then we are going

to use the notation $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ to denote the corresponding period ring constructed from $\Pi_{\text{an,con},I,A}$ just by directly replacing the variables by the corresponding uniformizers as above. And similarly for other period rings we use the corresponding notations taking the same form namely $\Pi_{\text{an,con},I,A}(\pi_{K_I})$, $\Pi_{\text{an},r_I,0,I,A}(\pi_{K_I})$, $\Pi_{\text{an},[s_I,r_I],I,A}(\pi_{K_I})$. As in the one dimensional situation we consider the situation where the radii are all sufficiently small. Then one could define the multiple Frobenius actions from the multi Frobenius φ_I over each the ring mentioned above. For suitable radii r_I we define a φ_I -module to be a finite projective left $\Pi_{\text{an},r_I,I,A}(\pi_{K_I})$ -module with the requirement that for each $\alpha \in I$ we have $\varphi_\alpha^* M_{r_I} \xrightarrow{\sim} M_{\dots,r_\alpha/p,\dots}$ (after suitable base changes). Then we define $M := \Pi_{\text{an,con},I,A}(\pi_{K_I}) \otimes_{\Pi_{\text{an},r_I,I,A}(\pi_{K_I})} M_{r_I}$ to define a φ_I -module over the full relative Robba ring in our situation. Furthermore we have the notion of φ_I -bundles in our situation which consists of a family of φ_I -modules $\{M_{[s_I,r_I]}\}$ where each module $M_{[s_I,r_I]}$ is defined to be finite projective over $\Pi_{\text{an},[s_I,r_I],I,A}(\pi_{K_I})$ satisfying the action formula taking the form of $\varphi_\alpha^* M_{[s_I,r_I]} \xrightarrow{\sim} M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}$ (after suitable base changes). Furthermore for each α we have the corresponding operator $\varphi_\alpha : M_{[s_I,r_I]} \rightarrow M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}$ and we have the corresponding operator ψ_α which is defined to be $p^{-1}\varphi_\alpha^{-1} \circ \text{Trace}_{M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}/\varphi_\alpha(M_{[s_I,r_I]})}$. Certainly we have the corresponding operator ψ_α over the global section M_{r_I} . Note that here we require that the Hodge-Frobenius structures are commutative in the sense that all the Frobenius are commuting with each other, and they are semilinear.

Definition 4.5.4. (After KPX [KPX14, Definition 2.2.2]) We keep all the notations (except the coefficient A which is now noncommutative) compatible with the ones in the commutative setting in the previous discussion in the previous sections. For each $\alpha \in I$, we choose suitable uniformizer π_{K_α} in our consideration. Then we are going to use the notation $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ to denote the corresponding period ring constructed from $\Pi_{\text{an,con},I,A}$ just by directly replacing the variables by the corresponding uniformizers as above. And similarly for other period rings we use the corresponding notations taking the same form namely $\Pi_{\text{an,con},I,A}(\pi_{K_I})$, $\Pi_{\text{an},r_I,0,I,A}(\pi_{K_I})$, $\Pi_{\text{an},[s_I,r_I],I,A}(\pi_{K_I})$. As in the one dimensional situation we consider the situation where the radii are all sufficiently small. Then one could define the multiple Frobenius actions from the multi Frobenius φ_I over each the ring mentioned above. For suitable radii r_I we define a coherent φ_I -module to be a finitely presented left $\Pi_{\text{an},r_I,I,A}(\pi_{K_I})$ -module with the requirement that for each $\alpha \in I$ we have $\varphi_\alpha^* M_{r_I} \xrightarrow{\sim} M_{\dots,r_\alpha/p,\dots}$ (after suitable base changes). Then we

define $M := \Pi_{\text{an,con},I,A}(\pi_{K_I}) \otimes_{\Pi_{\text{an},r_I,I,A}(\pi_{K_I})} M_{r_I}$ to define a coherent φ_I -module over the full relative Robba ring in our situation. Furthermore we have the notion of coherent φ_I -bundles in our situation which consists of a family of coherent φ_I -modules $\{M_{[s_I,r_I]}\}$ where each module $M_{[s_I,r_I]}$ is defined to be finitely presented over $\Pi_{\text{an},[s_I,r_I],I,A}(\pi_{K_I})$ satisfying the action formula taking the form of $\varphi_\alpha^* M_{[s_I,r_I]} \xrightarrow{\sim} M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}$ (after suitable base changes). Furthermore for each α we have the corresponding operator $\varphi_\alpha : M_{[s_I,r_I]} \rightarrow M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}$ and we have the corresponding operator ψ_α which is defined to be $p^{-1}\varphi_\alpha^{-1} \circ \text{Trace}_{M_{\dots,[s_\alpha/p,r_\alpha/p],\dots}/\varphi_\alpha(M_{[s_I,r_I]})}$. Certainly we have the corresponding operator ψ_α over the global section M_{r_I} . Note that here we require that the Hodge-Frobenius structures are commutative in the sense that all the Frobenius are commuting with each other, and they are semilinear.

Definition 4.5.5. (After KPX [KPX14, Definition 2.2.12]) Then we add the corresponding more Lie group action in our setting, namely the multi semilinear Γ_{K_I} -action. Again in this situation we require that all the actions of the Γ_{K_I} are commuting with each other and with all the semilinear Frobenius actions defined above. We define the corresponding (φ_I, Γ_I) -modules over $\Pi_{\text{an},r_I,0,I,A}(\pi_{K_I})$ to be finite projective module with mutually commuting semilinear actions of φ_I and Γ_{K_I} . For the latter action we require that to be continuous. We can also define the corresponding actions for coherent Frobenius modules.

Definition 4.5.6. (After KPX [KPX14, 2.3]) For any (φ_I, Γ_I) -module M over $\Pi_{\text{an,con},I,A}(\pi_{K_I})$ we define the complex $C_{\Gamma_I}^\bullet(M)$ of M to be the total complex of the following complex through the induction:

$$[C_{\Gamma_I \setminus \{I\}}^\bullet(M) \xrightarrow{\Gamma_I} C_{\Gamma_I \setminus \{I\}}^\bullet(M)].$$

Then we define the corresponding double complex $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$ again by taking the corresponding totalization of the following complex through induction:

$$[C_{\varphi_I \setminus \{I\}}^\bullet C_{\Gamma_I}^\bullet(M) \xrightarrow{\varphi_I} C_{\varphi_I \setminus \{I\}}^\bullet C_{\Gamma_I}^\bullet(M)].$$

Then we define the complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the totalization of the double complex defined above, which is called the complex of a (φ_I, Γ_I) -module M . Similarly we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the totalization of the double complex define

d in the same way as above by replacing φ_I by then the operators ψ_I . For later comparison we also need the corresponding ψ_I -cohomology, which is to say the complex $C_{\psi_I}^\bullet(M)$ which could be defined by the totalization of the following complex through induction:

$$[C_{\psi_I \setminus \{I\}}^\bullet(M) \xrightarrow{\psi_I} C_{\psi_I \setminus \{I\}}^\bullet(M)].$$

In order to proceed we now make the following assumption:

Assumption 4.5.7. We assume that the ring A in our situation satisfies the following assumption. For any closed interval $[s_I, r_I]$ we assume that the Robba ring $\Pi_{\text{an}, [s_I, r_I], I, A}$ is left and right noetherian, namely noetherian in the noncommutative situation. In the following we will discuss some samples.

Example 4.5.8. First note we point out that the corresponding Tate algebra with several free variables is actually not noetherian, even the corresponding polynomial rings with several free variables are not noetherian at all. For instance in the situation where we have two variables, as in [K1, Introduction] and [GW1, Exercise 1E] we know that the ring $k[x, y]^{\text{nc}}$ noncommutative polynomial ring contains specific corresponding ideal generated by xy^k for each nonnegative integer power $k \geq 0$. Even in the commutative setting, suppose we have a commutative noetherian Banach algebra B , it is also definitely not trivial that the affinoid algebra:

$$B\{T_1/a_1, \dots, T_m/a_m, b_1/T_1, \dots, b_m/T_m\}$$

is noetherian or not. Actually there does exist some counterexample which makes this not true in general as one might not expect, as in [FGK, 8.3].

Example 4.5.9. One can actually produce another exemplification of the assumption 4.5.7, namely we consider the following situation. We look at the corresponding (generalized version of) twisted polynomial rings in the sense of example 4.5.1 (see [So1, Section 3]). For instance we take a such ring with three variables:

$$\mathbb{Q}_p[X_1, X_2, X_3]' \tag{4.5.1}$$

with:

$$X_i X_j = a_{i,j} X_j X_i, i, j \in \{1, 2, 3\} \tag{4.5.2}$$

such that $a_{3,1} = a_{1,3} = 1$, $a_{3,2} = a_{2,3} = 1$ and $a_{2,1} = a_{1,2} \neq 1$. Then after taking the corresponding completion under the Gauss norm we will have the corresponding noncommutative affinoid which is noetherian and satisfies the corresponding assumption 4.5.7 by suitable rational localization.

Now consider the corresponding rings in the style considered in [Zab1, Proposition 4.1]:

$$D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p), D_{[\rho_1, \rho_2]}(G, K)$$

with G a pro- p group as in [Zab1, Chapter 4] (which is to say we will also assume G to be uniform). The corresponding ring $D_{[\rho_1, \rho_2]}(G, K)$ could be defined through suitable microlocalization with respect to some noncommutative polyannulus with respect to the radii ρ_1, ρ_2 which is direct noncommutative generalization of the usual Robba ring we considered in this paper including obviously the corresponding multivariate ones. With the notation in [Zab1, Proposition 4.1] namely for sufficiently large $s, r, \rho_1, \rho_2 \in p^{\mathbb{Q}}$ and $s, r, \rho_1, \rho_2 < 1$ we have:

Theorem 4.5.10. (Zábrádi [Zab1, Proposition 4.1])

The noncommutative Robba rings $D_{[\rho_1, \rho_2]}(G, K)$ is noetherian.

We now take the corresponding completed tensor product we have then:

$$D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} D_{[\rho_1, \rho_2]}(G, K)$$

which corresponds directly to our interested situation, namely the noncommutative deformation of the usual Robba ring in one variable.

Proposition 4.5.11. *For suitable radii $s, r, \rho_1, \rho_2 \in p^{\mathbb{Q}}$ such that $s, r, \rho_1, \rho_2 < 1$ and:*

$$D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p), D_{[\rho_1, \rho_2]}(G, K)$$

are noetherian as in [Zab1, Proposition 4.1]. Then we have that the corresponding completed tensor product

$$D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} D_{[\rho_1, \rho_2]}(G, K)$$

is also noetherian.

Remark 4.5.12. Before the proof, let us mention that this actually provides a very typical example for situation required in assumption 4.5.7.

Proof. One can follow the proof of [Zab1, Proposition 4.1] to do so by applying the criterion in [LVO, Proposition I.7.1.2], namely we look at the the graded ring of

$$gr_{\|\cdot\|_{\otimes}}^{\bullet} \left(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} D_{[\rho_1, \rho_2]}(G, K) \right),$$

defined by using the bounds of the norm $\|\cdot\|_{\otimes}$, and if this is noetherian then we will have by [LVO, Proposition I.7.1.2] that the ring $D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} D_{[\rho_1, \rho_2]}(G, K)$ is also noetherian. It suffices to look at the corresponding graded ring of the dense subring of tensor product (see [ST03, Lemma 4.3])

$$gr_{\|\cdot\|_{\otimes}}^{\bullet} \left(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{[\rho_1, \rho_2]}(G, K) \right).$$

We do not have the corresponding isomorphism in mind to split the corresponding components to get a product of two single graded rings but we have the corresponding surjective map (see [LVO, I.6.13]):

$$gr^{\bullet} \left(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \right) \otimes_{gr^{\bullet} \mathbb{Q}_p} gr^{\bullet} \left(D_{[\rho_1, \rho_2]}(G, K) \right) \rightarrow gr_{\|\cdot\|_{\otimes}}^{\bullet} \left(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{[\rho_1, \rho_2]}(G, K) \right).$$

This is defined by:

$$\sigma_k(x) \otimes \sigma_l(y) \mapsto \sigma_{k+l}(x \otimes y),$$

where the principal symbol $\sigma_k(x)$ is defined to be $x + F^{k+} D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p)$ if we we have $x \in F^k D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \setminus F^{k+} D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p)$, in the corresponding algebraic microlocalization. The construction for $\sigma_l(y)$ and $\sigma_{k+l}(x \otimes y)$ is the same. Therefore we look at then the corresponding product:

$$gr^{\bullet} \left(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \right) \otimes gr^{\bullet} \left(D_{[\rho_1, \rho_2]}(G, K) \right)$$

The corresponding elements of $gr^{\bullet} \left(D_{[\rho_1, \rho_2]}(G, K) \right)$ are those Laurent polynomials over $gr^{\bullet} K = k[\pi_2, \pi_2^{-1}]$ with variables:

$$t_1, \dots, t_d$$

where $t_i = \sigma(b_i) / \pi_2^l$. The corresponding elements of $gr^{\bullet} \left(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \right)$ are those Laurent polynomials over $gr^{\bullet} \mathbb{Q}_p = \mathbb{F}_p[\pi_1, \pi_1^{-1}]$ with variable:

$$t = \sigma(b) / \pi_1^k.$$

We followed the notations in [Zab1, Proposition 4.1], where b is the corresponding local coordinate for $D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p)$ while σ represents the corresponding principal symbol in the

corresponding nonarchimedean microlocalization, while we have that b_1, \dots, b_d are local coordinates for $D_{[\rho_1, \rho_2]}(G, K)$. The idea is that this ring is actually noetherian then we have the ring

$$gr^\bullet(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \otimes D_{[\rho_1, \rho_2]}(G, K))$$

is also noetherian. Then one can endow

$$D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} D_{[\rho_1, \rho_2]}(G, K)$$

with the corresponding filtration which will imply that the ring

$$D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} D_{[\rho_1, \rho_2]}(G, K)$$

is noetherian as well by [LVO, Proposition I.7.1.2]. Indeed as in [Zab1, Proposition 4.1] what we could do is to consider the corresponding projection to the corresponding Laurent polynomial ring over k in the variables $\pi_2, t, t_1, \dots, t_d$ as in the proof of [Zab1, Proposition 4.1]:

$$\mathbb{F}_p[\pi_1, \pi_1^{-1}, t, t^{-1}] \otimes_{\mathbb{F}_p[\pi_1, \pi_1^{-1}]} k[\pi_2, \pi_2^{-1}, t_1, t_1 t_2^{-1}, \dots, t_1 t_d^{-1}, t_2 t_1^{-1}, \dots, t_d t_1^{-1}]$$

which is noetherian, then the corresponding ring

$$gr^\bullet(D_{[s,r]}(\mathbb{Z}_p, \mathbb{Q}_p)) \otimes_{gr^\bullet \mathbb{Q}_p} gr^\bullet(D_{[\rho_1, \rho_2]}(G, K))$$

is noetherian since one can then lift the generators for the corresponding projection of any ideal through this construction as in the same way of [Zab1, Proposition 4.1]. \square

Setting 4.5.13. Now under the corresponding noetherian assumption in assumption 4.5.7, we actually have that the corresponding multivariate Robba ring $\Pi_{\text{an}, r_{I_0}, I, A}$ is Fréchet-Stein in the sense of [ST03, Chapter 3].

Definition 4.5.14. (After KPX [KPX14, Definition 2.1.3])

We define coherent sheaves of left modules and the global sections over the ring $\Pi_{\text{an}, r_{I_0}, I, A}$ in the following way. First a coherent sheaf \mathcal{M} over the relative multidimensional Robba ring $\Pi_{\text{an}, r_{I_0}, I, A}$ is defined to be a family $(M_{[s_I, r_I]})_{[s_I, r_I] \subset (0, r_{I_0}]}$ of left modules of finite type over each relative $\Pi_{[s_I, r_I], I, A}$ satisfying the following two conditions as in the more classical situation:

$$\Pi_{[s_I'', r_I''], I, A} \otimes_{\Pi_{[s_I', r_I'], I, A}} M_{[s_I', r_I']} \xrightarrow{\sim} M_{[s_I'', r_I'']}$$

for any multi radii satisfying $0 < s'_I \leq s''_I \leq r''_I \leq r'_I \leq r_{I,0}$, with furthermore

$$\Pi_{[s'''_I, r'''_I], I, A} \otimes_{\Pi_{[s''_I, r''_I], I, A}} (\Pi_{[s''_I, r''_I], I, A} \otimes_{\Pi_{[s'_I, r'_I], I, A}} M_{[s'_I, r'_I]}) \xrightarrow{\sim} \Pi_{[s'''_I, r'''_I], I, A},$$

for any multi radii satisfying the following condition:

$$0 < s'_I \leq s''_I \leq s'''_I \leq r'''_I \leq r''_I \leq r'_I \leq r_{I,0}.$$

Then we define the corresponding global section of the coherent sheaf \mathcal{M} , which is usually denoted by M which is defined to be the following inverse limit:

$$M := \varprojlim_{s_I \rightarrow 0_I^+} M_{[s_I, r_{I,0}]}.$$

We are going to follow [KPX14, Definition 2.1.3] to call any module defined over $\Pi_{\text{an}, r_{I,0}, I, A}$ coadmissible if it comes from a global section of a coherent sheaf \mathcal{M} in the sense defined above.

Remark 4.5.15. Since we are in the framework of [ST03, Chapter 3], so now by [ST03, Chapter 3, Theorem] for $r_{I,0}$ finite (namely for all $\alpha \in I$ the radius $r_{\alpha,0}$ is finite) we have that the corresponding global section in the previous definition is dense in each member participating in the coherent sheaf over some multi interval. We also have the same result for the situation where $r_{I,0} = \infty^{|I|}$ after inverting the corresponding variables in the multivariate Robba rings as in the commutative situation we considered in proposition 4.2.9. Also we have the higher vanishing of the corresponding derived inverse limit on coherent sheaves in the sense defined above.

Definition 4.5.16. (After KPX [KPX14, Definition 2.1.9]) We first generalize the notion of admissible covering in our setting, which is higher dimensional generalization of the situation established in [KPX14, Definition 2.1.9]. For any covering $\{[s_I, r_I]\}$ of $(0, r_{I,0}]$, we are going to specify those admissible coverings if the given covering admits refinement finite locally (and each corresponding member in the covering has the corresponding interior part which is assumed to be not empty with respect to each $\alpha \in I$). Then it is very natural in our setting that we have the corresponding notations of (m, n) -finitely presentedness and n -finitely generatedness for any m, n positive integers. Then based on these definitions we have the following generalization of the corresponding uniform finiteness. First we are going to call a coherent sheaf $\{M_{[s_I, r_I]}\}_{[s_I, r_I]}$ over $\Pi_{\text{an}, r_{I,0}, I, A}$

uniformly (m, n) -finitely presented if there exists an admissible covering $\{[s_I, r_I]\}$ and a pair of positive integers (m, n) such that each module $M_{[s_I, r_I]}$ defined over $\Pi_{\text{an}, [s_I, r_I], I, A}$ is (m, n) -finitely presented. Also we have the notion of uniformly n -finitely generated for any positive integer n by defining that to mean under the existence of an admissible covering we have that each member $M_{[s_I, r_I]}$ in the family defined over $\Pi_{\text{an}, [s_I, r_I], I, A}$ is n -finitely generated.

Lemma 4.5.17. (After KPX [KPX14, Lemma 2.1.11]) *Suppose that for a coherent sheaf $\{M_{s_I, r_I}\}_{[s_I, r_I]}$ defined above and for an admissible covering $\{[s_I, r_I]\}$ defined above in our noncommutative setting we have that any module M_{s_I, r_I} is finitely generated uniformly by finite many elements of global section. Then the global section will be also finitely generated by these set of elements as well.*

Proof. As in [KPX14, Lemma 2.1.11], we use the set of elements to present a map from the free module generated by them to the global section, then after base change to each $\Pi_{\text{an}, [s_I, r_I], I, A}$ we do have that the map is corresponding presentation over $\Pi_{\text{an}, [s_I, r_I], I, A}$, then this will give rise to the corresponding presentation for the global section since we have the vanishing of the derived inverse limit functor. \square

Lemma 4.5.18. (After KPX [KPX14, Lemma 2.1.12]) *Let H be a noncommutative Banach ring, and consider some finite module M over H generated by $\{f_1, \dots, f_n\}$ and any another set $\{f'_1, \dots, f'_n\}$ of elements such that $f_i - f'_i$ could be made sufficiently small uniformly for each $i = 1, \dots, n$. Then we have that the set $\{f'_1, \dots, f'_n\}$ could also finitely generate the module M .*

Proof. As in [KPX14, Lemma 2.1.12], one looks at the corresponding matrix $G_{i,j}$ defined by $f_i - f'_i = \sum_k G_{i,k} f_k$. Then by using the open mapping theorem, one can control the norm of each entry $G_{i,j}$ to make them small enough below a uniform small bound, which makes the corresponding matrix $1 + G$ invertible. \square

Proposition 4.5.19. (After KPX [KPX14, Proposition 2.1.13]) *Assume now I consists of exactly one index, consider any arbitrary coherent sheaf $\{M_{[s_I, r_I]}\}$ over $\Pi_{\text{an}, r_I, 0, I, A}$ with the global section M . Then we have the following corresponding statements:*

I. The coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely generated iff the global section M is finitely generated;

II. The coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely presented iff the global section M is finitely presented.

Proof. One could derive the the second statement by using the analog of lemma 4.2.15 (see [L, Lemma 4.54]) and the first statement. Therefore it suffices for us to prove the first statement. Indeed one could see that one direction in the statement is straightforward, therefore it suffices now to consider the other direction. So now we assume that the coherent sheaf $\{M_{[s_I, r_I]}\}$ is uniformly finitely generated for some n for instance. Then we are going to show that one could find corresponding generators for the global section M from the given uniform finitely generatedness.

Now by definition we consider an admissible covering taking the form of $\{[s_I, r_I]\}$. In [KPX14, Proposition 2.1.13] the corresponding argument goes by using special decomposition of the covering in some refined way, mainly by reorganizing the initial admissible covering into two single ones, where each of them will be made into the one consisting of those well-established intervals with no overlap. We can also use instead the corresponding upgraded nice decomposition of the given admissible covering in our setting, which is extensively discussed in [KL16, 2.6.14-2.6.17]. Namely as in [KL16, 2.6.14-2.6.17] there is a chance for us to extract $2^{|I|}$ families $\{\{[s_{I, \delta_i}, r_{I, \delta_i}]\}_{i=0,1,\dots}\}_{\{1_i\}, \{2_i\}, \dots, \{N_i\}}$, ($N = 2^{|I|}$) of intervals, where there is a chance to have the situation where for each such family the corresponding intervals in it could be made to be disjoint in pairs.

Then for a chosen family $\{[s_{I, \delta_i}, r_{I, \delta_i}]\}_{i=0,1,\dots}$, we now for each i involved we consider the corresponding generating set $\mathbf{e}_{\delta_i, 1}, \dots, \mathbf{e}_{\delta_i, n}$ where $i = 0, 1, \dots$ and $n \in \{1, 2, \dots, n\}$. Note that these should be generated over the integral rings, which is not the case when we work with the infinite radii. In the latter situation one should first invert T_I then consider some large powers which could eliminate the different denominators. Then by the previous results we know that one could actually arrange these generators to be global sections by the density of the global sections in the sections over any chosen interval. Now we consider the following step where we for each $j = 1, \dots, n$ form the following sum with coefficients to be chosen:

$$\mathbf{e}_j := \sum_{i=0}^{\infty} a_{\delta_i, j} T_I^{\lambda_{I, \delta_i, j}} \mathbf{e}_{\delta_i, j}.$$

Note that in our situation actually I is a singleton, so we can actually regard δ_i as $2i$.

To check the suitable convergence in this situation we consider the following esti-

mate with bounds given as below to be chosen from the conditions:

$$\forall i'' < \delta_i - 1, \left\| a_{\delta_i, j} T^{\lambda_{I, \delta_i, j}} \mathbf{e}_{\delta_i, j} \right\|_{[s_{I, i''}, r_{I, i''}]} \leq p^{-i}, \quad (4.5.3)$$

$$\forall i' < i, \left\| a_{\delta_i, j} T^{\lambda_{I, \delta_i, j}} \mathbf{e}_{\delta_i, j} / (a_{\delta_{i'}, j} T^{\lambda_{I, \delta_{i'}, j}}) \right\|_{[s_{I, \delta_{i'}}, r_{I, \delta_{i'}}]} \leq \varepsilon_{\delta_{i'}, j} \quad (4.5.4)$$

$$\forall i' < i, \left\| a_{\delta_{i'}, j} T^{\lambda_{I, \delta_{i'}, j}} \mathbf{e}_{\delta_{i'}, j} / (a_{\delta_i, j} T^{\lambda_{I, \delta_i, j}}) \right\|_{[s_{I, \delta_i}, r_{I, \delta_i}]} \leq \varepsilon_{\delta_i, j}. \quad (4.5.5)$$

As in the commutative situation in [KPX14, Proposition 2.1.13] in order to get the desired quantity expressed in the estimate above we seek the following situation from what we presented above:

$$\forall i'' < \delta_i - 1, \left\| a_{\delta_i, j} T^{\lambda_{I, \delta_i, j}} \mathbf{e}_{\delta_i, j} \right\|_{[s_{I, i''}, r_{I, i''}]} \leq p^{-i}, \quad (4.5.6)$$

$$\forall i' < i, \left\| a_{\delta_i, j} T^{\lambda_{I, \delta_i, j}} \mathbf{e}_{\delta_i, j} \right\|_{[s_{I, \delta_{i'}}, r_{I, \delta_{i'}}]} \leq \varepsilon_{\delta_{i'}, j} \left\| a_{\delta_{i'}, j} T^{\lambda_{I, \delta_{i'}, j}} \right\|_{[s_{I, \delta_{i'}}, r_{I, \delta_{i'}}]} \quad (4.5.7)$$

$$\forall i' < i, \left\| a_{\delta_{i'}, j} T^{\lambda_{I, \delta_{i'}, j}} \mathbf{e}_{\delta_{i'}, j} \right\|_{[s_{I, \delta_i}, r_{I, \delta_i}]} \leq \varepsilon_{\delta_i, j} \left\| a_{\delta_i, j} T^{\lambda_{I, \delta_i, j}} \right\|_{[s_{I, \delta_i}, r_{I, \delta_i}]} . \quad (4.5.8)$$

Then what we can achieve up to now is to consider sufficiently large powers on the both sides in order to make the inequalities hold, which is possible due to our initial choice on the intervals, where the first statement could also be proved in the similar way (also see [KPX14, Proposition 2.1.13]). To be more precise, one chooses the corresponding coefficients and the powers of the monomials by induction, where the initial coefficient for $i = 0$ is set to be 1 with order 0. Then one enlarges the corresponding difference between the power of the monomials on the both sides of the inequalities above while evaluating the first and second inequalities at the left most radii and evaluating the last one at right most radii. Then actually this will prove the statement since up to some powers the difference between $\mathbf{e}_{\delta_i, j}$ and \mathbf{e}_j will have sufficiently small norm, which could give rise to suitable finitely generatedness through the lemma 4.5.18. Finally one could then finish the whole proof in the odd situation in the same fashion. □

Proposition 4.5.20. (After KPX [KPX14, Proposition 2.2.7])

In the situation where I consists of exactly one element. We have that natural functor from coherent φ_I -modules (in the sense of the previous definition 4.5.4) over $\Pi_{\text{an}, r_{I, 0}, I, A}(\pi_{K_I})$ to the corresponding coherent φ_I -bundles (in the sense of the previous definition 4.5.4) is an equivalence.

Proof. See proposition 4.3.6, by using the previous proposition. \square

Definition 4.5.21. (After KPX [KPX14, Notation 4.1.2]) Now we consider the following derived categories. All modules will be assumed to be left. The first one is the sub-derived category consisting of all the objects in $\mathbb{D}_{\text{left}}(A)$ which are quasi-isomorphic to those bounded above complexes of finite projective modules over the ring A . We denote this category (which is defined in the same way as in [KPX14, Notation 4.1.2]) by $\mathbb{D}_{\text{perf, left}}^-(A)$. Over the larger ring $\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})$ we also have the sub-derived category of $\mathbb{D}_{\text{left}}(\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I}))$ consisting of all those objects in $\mathbb{D}_{\text{left}}(\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I}))$ which are quasi-isomorphic to those bounded above complexes of finite projective modules now over the ring $\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})$. We denote this by $\mathbb{D}_{\text{perf, left}}^-(\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I}))$.

Then we will investigate the complexes attached to any finite projective or coherent (φ_I, Γ_I) -module M over $\Pi_{\text{an}, \text{con}, I, A}(\pi_{K_I})$:

$$C_{\varphi_I, \Gamma_I}^\bullet(M), C_{\psi_I, \Gamma_I}^\bullet(M), C_{\psi_I}^\bullet(M),$$

which are living inside the corresponding derived categories:

$$\mathbb{D}_{\text{left}}(A), \mathbb{D}_{\text{left}}(A), \mathbb{D}_{\text{left}}(\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})).$$

Conjecture 4.5.22. *With the notation and setting up as above we have $h^\bullet(C_{\psi_I}^\bullet(M))$ are coadmissible over $\Pi_{\text{an}, \infty, I, A}(\Gamma_{K_I})$.*

Conjecture 4.5.23. *With the notations above, we have that*

$$C_{\varphi_I, \Gamma_I}^\bullet(M) \in \mathbb{D}_{\text{perf, left}}^-(A), C_{\psi_I, \Gamma_I}^\bullet(M) \in \mathbb{D}_{\text{perf, left}}^-(A).$$

Conjecture 4.5.24. *Let $A_\infty(H)$ be the Fréchet-Stein algebra attached to any compact p -adic Lie group H which could be written as inverse limit of noncommutative noetherian Banach affinoid algebras over \mathbb{Q}_p . Suppose \mathcal{F} is a family of finite projective left modules over the Robba ring $\Pi_{\text{an}, \text{con}, I, A_\infty(H)}(\pi_{K_I})$, where we use the same notation to denote the global section and we assume that the global section is finite projective, carrying mutually commuting actions of $(\varphi_I, \Gamma_{K_I})$. Then we have that $C_{\psi_I}^\bullet(\mathcal{F})$ lives in $\mathbb{D}_{\text{perf, left}}^b(\Pi_{\text{an}, \infty, I, A_\infty(H)}(\Gamma_{K_I}))$ and $C_{\varphi_I, \Gamma_I}^\bullet(\mathcal{F})$ lives in $\mathbb{D}_{\text{perf, left}}^b(A_\infty(H))$.*

Remark 4.5.25. We conjecture that the corresponding Cartan-Serre approach holds in the noncommutative setting as in [KL, Lemma1.10], which may possibly provide proof of the conjectures above.

This chapter is based on the following paper which is submitted, where the author of this dissertation is the main author:

- Tong, Xin. "Analytic Geometry and Hodge-Frobenius Structure." arXiv preprint arXiv:2011.08358 (2020).

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Chapter 5

Analytic Geometry and Hodge-Frobenius Structure Continued

5.1 Introduction

5.1.1 Introduction and Summary

Generally speaking analytic geometry studies very general manifolds or varieties which are locally related to analytic functions of several variables. In complex analytic geometry, the special domains (like those classical domains) form a very important subject in the corresponding understanding of higher dimensional analytic geometry and their application to other subjects such as number theory. One definitely would like to understand the corresponding story in nonarchimedean geometry, namely what are the corresponding significant special domains. Another issue is that if we could have the chance to find the links with other geometries, such as formal ones and pure algebraic geometry. In the study of analytic geometry, certainly not only the analytic methods will be definitely applied, but also we have algebraic ones. Algebraic ones certainly work well in some very significant cases, for instance those in the Stein cases in classical complex geometry, quasi-Stein cases within the generality of Kedlaya-Liu [KL2] and Fréchet-Stein cases after Schneider-Teitelbaum [ST].

While the corresponding nonarchimedean analytic geometry has its own very significant geometric rights, many useful applications also exist extensively as well. For instance in [T1] we followed [CKZ] and [PZ] to have investigated some relative cohomologies corresponding to very specific multi Frobenius structures. The picture in [T1] was a very practical first step towards more general pictures and more interesting pictures.

The tools in [T1] extend to our current paper since in more general framework of geometry as in [KL1] and [KL2] the methods and ideas could be applied directly to our situation with possible suitable more detailization. For instance in our situation we could consider nonnoetherian base rings. Note that this does not mean they are necessarily preperfectoid or perfect, since in our situation we can consider some very interesting mixed-type Robba rings in very general sense. For instance if we have two variables, we could have one living in some preperfectoid part while the other one living in the some rigid analytic affinoid component. Note that we do not have to maintain in the corresponding strictly analytic situation, especially if one would like to work with Kedlaya's

reified adic spaces [Ked1].

We will also consider the corresponding generalized B -pairs after Berger and Nakamura. However for some deep study we could introduce some mixed-type Hodge structures. Namely we could introduce (φ, Γ) - B -objects. For instance in the situation where we have two factors we can defined the corresponding period rings:

$$B_{\mathrm{dR}, \mathbb{Q}_p} \widehat{\otimes}_{\mathbb{Q}_p} \Pi_{\mathrm{an}, \mathrm{con}, A}, B_{\mathrm{dR}, \mathbb{Q}_p}^+ \widehat{\otimes}_{\mathbb{Q}_p} \Pi_{\mathrm{an}, \mathrm{con}, A}$$

with finite projective objects defined over them, carrying one partial action from the Galois groups and carrying one partial action from the corresponding (φ, Γ) operators.

To make summary, we have:

A. Over really general Banach rings, we have defined many useful mixed type Robba rings by perfection along some partial variables. In fact we also have defined some ∞ -period rings following the recent work from Bambozzi-Kremnizer [BK]. The corresponding issue we encountered is certainly some common issue around sheafiness in adic geometry in some very general classical sense. These directly will be expected to give the chance for one to compare Hodge structures over multivariate imperfect Robba rings with the corresponding Hodge structures over multivariate perfect Robba rings as those with accents \checkmark and $\tilde{\sim}$ in [KL2], see sections from 2-3.

B. Over really general Banach rings, we defined many useful (φ_I, Γ_I) -modules over the mixed type Robba rings by perfection along some partial variables. This directly gives the chance for one to compare multivariate (φ_I, Γ_I) -modules over partially imperfect Robba rings with the corresponding multivariate (φ_I, Γ_I) -modules over partially perfect Robba rings as those with accents \checkmark and $\tilde{\sim}$ in [KL2], see section 4.

C. Over really general Banach rings over \mathbb{Q}_p , we defined many useful mixed type big period rings by taking product with Fontaine's de Rham period ring along some partial variables. First of all, we then immediately have the definition for some B'_I - (φ_I, Γ_I) -modules. Therefore relying on these rings (although they are very interesting in their rights) proved the

equivalence between multivariate B'_I -pairs and multivariate (φ_I, Γ_I) -modules generalizing Berger's work and [KP], see section 5.

D. In sections 6-8, we defined the corresponding B'_I - (φ_I, Γ_I) -cohomologies for B'_I - (φ_I, Γ_I) -modules, and we promote the equivalence to some quasi-isomorphisms within the derived category $D^b(A)$, where A is the base Banach relative ring.

5.1.2 Comments on the Notation

Our notations on the corresponding multivariate mixed type big Hodge structures are inspired by essentially some Langlands programs in ℓ -adic situation such as in [VL], namely the Drinfeld's Lemma. However the work [PZ] and [CKZ] use Δ which is inspired by essentially some Langlands programs in p -adic situation such as that rooted in the work of Zabradi and is related to reductive datum. We remind the readers that these are actually not the corresponding intervals for the Robba rings in order to eliminate the corresponding possible confusion.

5.1.3 Future Study

The current geometric discussion covered in this paper is basically around the commutative analytic geometry. We have not made it to add the corresponding discussion on the noncommutative setting, but this will be pushed to our further study. Certainly the corresponding noncommutative deformation will require some further well-established understanding on the foundational issues, such as the corresponding noncommutative descent as in the commutative situation from [KL1] and [KL2]. Definitely any good understanding on the noncommutative settings of these sorts will be essential to the corresponding good understanding on noncommutative analytic geometry and noncommutative Tamagawa number conjectures after [FK1], [BF1] and [BF2].

Since our dreams will be really those where we can handle very general analytic spaces. Certainly adic spaces need extensively restrictive requirement on the sheafiness

of the Banach rings. However this might be resolved completely by considering [BK] (or possibly equivalently the work of Clausen-Scholze [CS]). We have already defined many interesting ∞ -analytic stacks and the ∞ -Robba rings. We will study Kedlaya-Liu glueing on this level after [KL1] and [KL2] in future work.

We have discussed the corresponding higher dimensional B -pairs by introducing the corresponding higher dimensional de Rham period rings. We certainly hope to amplify the corresponding discussion in more p -adic Hodge theoretic sense. However one could definitely maintain in the world of (φ_I, Γ_I) -modules in some very flexible way, literally after Berger [Ber2]. We believe that we have more rigidity in the current context. In fact, we should mention that this higher dimensionalization of [Ber2] is literally motivated by the work [KL1] and [KL2], as well as the work [CKZ] and [PZ].

5.2 Big Robba Rings over Rigid Analytic Affinoids and Fréchet Objects in Mixed-characteristic Case

5.2.1 Big Robba Rings over Rigid Analytic Affinoids

We follow [T1] to give the thorough definition and discussion of those very big period rings we will need in the further discussions in the following body of the paper, where we consider as in [T1] a finite set I with some subset J . Our Robba rings in this paper will be depending on the I and J simultaneously.

Definition 5.2.1. Let A be any affinoid algebra over \mathbb{Q}_p in rigid analytic geometry. We consider the corresponding multi intervals $[\omega^{r_I}, \omega^{s_I}]$. Recall we have the corresponding Robba rings defined in [T1, Definition 2.4]:

$$\Pi_{[s_I, r_I], I, A}$$

which is defined to be the corresponding affinoid:

$$A \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p \{ \omega^{r_1}/T_1, \dots, \omega^{r_I}/T_I, T_1/\omega^{s_1}, \dots, T_I/\omega^{s_I} \}.$$

Then we have the corresponding rings:

$$\Pi_{\text{an}, r_I, I, A} := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, A}.$$

with

$$\Pi_{\text{an}, \text{con}, I, A} := \bigcup_{r_I} \varprojlim_{s_I} \Pi_{[s_I, r_I], I, A}.$$

However we will in this paper to consider some more complicated version of the rings. We will use some partial Frobenius to perfectize partially the rings defined above. Therefore we will in some more uniform way to denote the rings in the following different way:

$$\Pi_{[s_I, r_I], I, I, \emptyset, A} := \Pi_{[s_I, r_I], I, A} \tag{5.2.1}$$

$$\Pi_{\text{an}, r_I, I, I, \emptyset, A} := \Pi_{\text{an}, r_I, I, A} \tag{5.2.2}$$

$$\Pi_{\text{an}, \text{con}, I, I, \emptyset, A} := \Pi_{\text{an}, \text{con}, I, \emptyset, A}. \tag{5.2.3}$$

Now we follow the idea in [Ked2, Definition 5.2.1] to define some extended version of the rings. We will have the following rings to be:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} \tag{5.2.4}$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} \tag{5.2.5}$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A} \tag{5.2.6}$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A} \tag{5.2.7}$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A} \tag{5.2.8}$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A} \tag{5.2.9}$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A} \tag{5.2.10}$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A} \tag{5.2.11}$$

and

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A} \tag{5.2.12}$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A} \tag{5.2.13}$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A} \tag{5.2.14}$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A} \tag{5.2.15}$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A} \tag{5.2.16}$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A} \tag{5.2.17}$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A} \tag{5.2.18}$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A} \tag{5.2.19}$$

and

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A} \quad (5.2.20)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A} \quad (5.2.21)$$

$$\Pi_{\text{an,con},I,J,I\check{\setminus}J,A} \quad (5.2.22)$$

$$\Pi_{\text{an,con},I,\check{J},I\check{\setminus}J,A} \quad (5.2.23)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\check{\setminus}J,A} \quad (5.2.24)$$

$$\Pi_{\text{an,con},I,J,\widetilde{I\setminus J},A} \quad (5.2.25)$$

$$\Pi_{\text{an,con},I,\check{J},\widetilde{I\setminus J},A} \quad (5.2.26)$$

$$\Pi_{\text{an,con},I,\tilde{J},\widetilde{I\setminus J},A} \quad (5.2.27)$$

Definition 5.2.2. (After Kedlaya-Liu, [KL2, Definition 5.2.1]) We first define the corresponding first group of the rings. The corresponding rings in groups as mentioned above are defined by using the corresponding partial Frobenius $\varphi_1, \dots, \varphi_I$ and the corresponding Fréchet completion. For the ring $\Pi_{[s_I, r_I], I, \check{J}, I\setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{J}, I\setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in J} \prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I\setminus J, A}. \quad (5.2.28)$$

Note that for the corresponding rings getting involved in the corresponding definition above we consider the corresponding various Fréchet norms for each $t_I > 0$:

$$\|\cdot\|_{\prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I\setminus J, A}, t_I}$$

Then we define the corresponding ring $\Pi_{[s_I, r_I], I, \tilde{J}, I\setminus J, A}$, this is defined by the following Fréchet completion process:

$$\Pi_{[s_I, r_I], I, \tilde{J}, I\setminus J, A} := \left(\varinjlim_{n_\alpha \geq 0, \alpha \in J} \prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I\setminus J, A} \right)_{\|\cdot\|_{\prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I\setminus J, A}, t_I}, t_I \in [s_I, r_I]}^\wedge. \quad (5.2.29)$$

Then in the corresponding symmetric way we have the following definition:

Definition 5.2.3. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

For the ring $\Pi_{[s_I, r_I], I, J, I\check{\setminus}J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, J, I\check{\setminus}J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in I \setminus J} \prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I\setminus J, A}. \quad (5.2.30)$$

Note that for the corresponding rings getting involved in the corresponding definition above we consider the corresponding various Fréchet norms for each $t_I > 0$:

$$\|\cdot\|_{\prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}$$

Then we define the corresponding ring $\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}$, this is defined by the following Fréchet completion process:

$$\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A} := \left(\varinjlim_{n_\alpha \geq 0, \alpha \in I \setminus J} \prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} \right)^\wedge_{\|\cdot\|_{\prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}, t_I \in [s_I, r_I]} . \quad (5.2.31)$$

Then we do the following one:

Definition 5.2.4. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

For the ring $\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in I} \prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}. \quad (5.2.32)$$

Note that for the corresponding rings getting involved in the corresponding definition above we consider the corresponding various Fréchet norms for each $t_I > 0$:

$$\|\cdot\|_{\prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}$$

Then we define the corresponding ring $\Pi_{[s_I, r_I], I, \check{\check{J}}, \widetilde{I \setminus J}, A}$, this is defined by the following Fréchet completion process:

$$\Pi_{[s_I, r_I], I, \check{\check{J}}, \widetilde{I \setminus J}, A} := \left(\varinjlim_{n_\alpha \geq 0, \alpha \in I} \prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A} \right)^\wedge_{\|\cdot\|_{\prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}, t_I \in [s_I, r_I]} . \quad (5.2.33)$$

Then we consider the following definition building on the definitions above:

Definition 5.2.5. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

For the ring $\Pi_{[s_I, r_I], I, \check{\check{J}}, I \setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{\check{J}}, I \setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in I \setminus J} \prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, \check{\check{J}}, I \setminus J, A}. \quad (5.2.34)$$

For the ring $\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in J} \prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}. \quad (5.2.35)$$

Then we have the following definitions:

Definition 5.2.6. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

$$\Pi_{\text{an},r_I,I,\check{J},I\setminus J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}, \quad (5.2.36)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\setminus J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}, \quad (5.2.37)$$

$$\Pi_{\text{an},r_I,I,J,I\check{\setminus}J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}, \quad (5.2.38)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\check{\setminus}J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}, \quad (5.2.39)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\check{\setminus}J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}, \quad (5.2.40)$$

$$\Pi_{\text{an},r_I,I,J,I\widetilde{\setminus}J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}, \quad (5.2.41)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\widetilde{\setminus}J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}, \quad (5.2.42)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\widetilde{\setminus}J,A} := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}. \quad (5.2.43)$$

and

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}, \quad (5.2.44)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}, \quad (5.2.45)$$

$$\Pi_{\text{an,con},I,J,I\check{\setminus}J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}, \quad (5.2.46)$$

$$\Pi_{\text{an,con},I,\check{J},I\check{\setminus}J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}, \quad (5.2.47)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\check{\setminus}J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}, \quad (5.2.48)$$

$$\Pi_{\text{an,con},I,J,I\widetilde{\setminus}J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}, \quad (5.2.49)$$

$$\Pi_{\text{an,con},I,\check{J},I\widetilde{\setminus}J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}, \quad (5.2.50)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\widetilde{\setminus}J,A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}. \quad (5.2.51)$$

$$(5.2.52)$$

Setting 5.2.7. The corresponding construction we established above is also motivated from the corresponding construction of multivariate Robba rings in [Ked2].

5.2.2 Fréchet Objects

Now we perform some of the corresponding construction parallel to the corresponding Fréchet-Stein construction used originally in [KPX], which is essentially developed in [KL2, Section 2.6]. In effect, the corresponding construction was already used in [T1], although the paper behaves as if we are not fully after [KL2]. The reason of such impression left for the readers is essentially due to the fact that we are in the noetherian situation. Now we are not definitely in the noetherian situation, but [KL2, Section 2.6] has already tackled this issue.

Setting 5.2.8. (After Kedlaya-Liu, [KL2, Definition 2.6.2]) Following [KL2, Definition 2.6.2] we call a Banach uniform adic ring (R, R^+) quasi-Stein if it could be written as the following inverse limit with the corresponding transition map of dense image for some inverse system $\{\alpha\}$ of indexes:

$$(R, R^+) := \varprojlim_{\alpha} (R_{\alpha}, R_{\alpha}^+).$$

And we call the ring ind-Fréchet-Stein if it is further could be written the following injective-projective limit:

$$(R, R^+) := \varinjlim_{\alpha'} \varprojlim_{\alpha} (R_{\alpha, \alpha'}, R_{\alpha, \alpha'}^+).$$

After this axiomization we could now study the corresponding sheaves over these rings. In what follow, we consider the corresponding radii living in the set of all the rational numbers.

Definition 5.2.9. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.10. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \tilde{J}, I \setminus J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.11. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, J, I \check{\setminus} J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.12. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \check{J}, I \check{\setminus} J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.13. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \tilde{J}, I \check{\setminus} J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.14. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, J, \widetilde{I \setminus J}, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.15. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \check{J}, \widetilde{I \setminus J}, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.2.16. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Remark 5.2.17. There is some overlap and repeating on some objects within the eight categories. Therefore in the following we are going to then work with only 1st, 2nd, 4th, 5th, 8th categories.

Proposition 5.2.18. (After Kedlaya-Liu, [KL2, Corollary 2.6.8]) *The corresponding pseudocoherent finitely projective bundles over*

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.53)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.54)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.55)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.56)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A} \quad (5.2.57)$$

defined as above have global sections which are finite projective if and only if the global sections are finitely generated.

Proof. Just apply [KL2, Corollary 2.6.8]. □

Proposition 5.2.19. (After Kedlaya-Liu, [KL2, Proposition 2.6.17]) *The corresponding pseudocoherent sheaves over*

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.58)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.59)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.60)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}, \quad (5.2.61)$$

$$\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A} \quad (5.2.62)$$

defined as above have global sections which are finitely generated as long as we have the uniform bound on the rank of the bundles over quasi-compacts with respect to closed multi-intervals taking the general form of $[s_I, r_I]$.

Proof. This is actually a direct consequence of [KL2, Proposition 2.6.17], where the space $(0, r_{I,0}]$ admits $2^{|I|}$ -uniform covering. \square

Proposition 5.2.20.

(After Kedlaya-Liu, [KL2, Corollary 2.6.8, Proposition 2.6.17]) *The corresponding pseudocoherent sheaves over*

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \tag{5.2.63}$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, I \setminus J, A}, \tag{5.2.64}$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \tag{5.2.65}$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, I \setminus J, A}, \tag{5.2.66}$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, \widetilde{I \setminus J}, A} \tag{5.2.67}$$

defined as above have global sections which are finite projective as long as we have the uniform bound on the rank of the bundle over each quasi-compact with respect to each closed multi-interval $[s_I, r_I]$, and we have that the sheaves admits section actually finite projective over each quasi-compact with respect to each closed multi-interval $[s_I, r_I]$.

Proof. This is actually a direct corollary of the previous two propositions. \square

5.3 Big Robba Rings over General Banach Affinoids and Fréchet Objects in Mixed-characteristic Case

5.3.1 Big Robba Rings over General Banach Affinoids

Since we have already considered the corresponding foundation from [KL2] on the quasi-Stein nonnoetherian adic Banach uniform algebra over \mathbb{Q}_p , we hope then now study more general p -adic analysis of several variables. Certainly as mentioned in [KPX] one could carry some strongly noetherian coefficients, where everything is sheafy, but one might be very curious about the situation where we do not have so strong condition on the noetherianness. Actually we could then apply the derived analytic geometry in [BK].

Definition 5.3.1. Let A be any commutative Banach algebra over \mathbb{Q}_p . We consider the corresponding multi intervals $[\omega^{r_I}, \omega^{s_I}]$. We have the corresponding Robba rings defined as in [T1, Definition 2.4]:

$$\Pi_{[s_I, r_I], I, A}$$

which is defined to be the corresponding affinoid:

$$A \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p \{ \omega^{r_1}/T_1, \dots, \omega^{r_I}/T_I, T_1/\omega^{s_1}, \dots, T_I/\omega^{s_I} \}.$$

Then we have the corresponding rings:

$$\Pi_{\text{an}, r_I, I, A} := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, A}.$$

with

$$\Pi_{\text{an}, \text{con}, I, A} := \bigcup_{r_I} \varprojlim_{s_I} \Pi_{[s_I, r_I], I, A}.$$

However we will in this paper to consider some more complicated version of the rings. We will use some partial Frobenius to perfectize partially the rings defined above. Therefore we will in some more uniform way to denote the rings in the following different way:

$$\Pi_{[s_I, r_I], I, I, \emptyset, A} := \Pi_{[s_I, r_I], I, A} \tag{5.3.1}$$

$$\Pi_{\text{an}, r_I, I, I, \emptyset, A} := \Pi_{\text{an}, r_I, I, A} \tag{5.3.2}$$

$$\Pi_{\text{an}, \text{con}, I, I, \emptyset, A} := \Pi_{\text{an}, \text{con}, I, \emptyset, A}. \tag{5.3.3}$$

Now we follow the idea in [Ked2, Definition 5.2.1] to define some extended version of the rings. We will have the following rings to be:

$$\Pi_{[s_I, r_I], I, ?, ?', A, ?} = J, \tilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}, \quad (5.3.4)$$

$$(5.3.5)$$

and

$$\Pi_{\text{an}, r_I, I, ?, ?', A, ?} = J, \tilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}, \quad (5.3.6)$$

$$(5.3.7)$$

and

$$\Pi_{\text{an}, \text{con}, I, ?, ?', A, ?} = J, \tilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}. \quad (5.3.8)$$

$$(5.3.9)$$

Definition 5.3.2. (After Kedlaya-Liu, [KL2, Definition 5.2.1]) We first define the corresponding first group of the rings. The corresponding rings in groups as mentioned above are defined by using the corresponding partial Frobenius $\varphi_1, \dots, \varphi_I$ and the corresponding Fréchet completion. For the ring $\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in J} \prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}. \quad (5.3.10)$$

Note that for the corresponding rings getting involved in the corresponding definition above we consider the corresponding various Fréchet norms for each $t_I > 0$:

$$\|\cdot\|_{\prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}$$

Then we define the corresponding ring $\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}$, this is defined by the following Fréchet completion process:

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} := \left(\varinjlim_{n_\alpha \geq 0, \alpha \in J} \prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A} \right)^\wedge_{\|\cdot\|_{\prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}, t_I \in [s_I, r_I]}. \quad (5.3.11)$$

Then in the corresponding symmetric way we have the following definition:

Definition 5.3.3. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

For the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, J, I \setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in I \setminus J} \prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}. \quad (5.3.12)$$

Note that for the corresponding rings getting involved in the corresponding definition above we consider the corresponding various Fréchet norms for each $t_I > 0$:

$$\|\cdot\|_{\prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}$$

Then we define the corresponding ring $\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}$, this is defined by the following Fréchet completion process:

$$\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A} := \left(\varinjlim_{n_\alpha \geq 0, \alpha \in I \setminus J} \prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A} \right)^\wedge_{\|\cdot\|_{\prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}, t_I \in [s_I, r_I]}. \quad (5.3.13)$$

Then we do the following one:

Definition 5.3.4. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

For the ring $\Pi_{[s_I, r_I], I, J, \check{I} \setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, J, \check{I} \setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in I} \prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}. \quad (5.3.14)$$

Note that for the corresponding rings getting involved in the corresponding definition above we consider the corresponding various Fréchet norms for each $t_I > 0$:

$$\|\cdot\|_{\prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}$$

Then we define the corresponding ring $\Pi_{[s_I, r_I], I, J, \widetilde{\check{I} \setminus J}, A}$, this is defined by the following Fréchet completion process:

$$\Pi_{[s_I, r_I], I, J, \widetilde{\check{I} \setminus J}, A} := \left(\varinjlim_{n_\alpha \geq 0, \alpha \in I} \prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A} \right)^\wedge_{\|\cdot\|_{\prod_{\alpha \in I} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, t_I}, t_I \in [s_I, r_I]}. \quad (5.3.15)$$

Then we consider the following definition building on the definitions above:

Definition 5.3.5. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

For the ring $\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in I \setminus J} \prod_{\alpha \in I \setminus J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}. \quad (5.3.16)$$

For the ring $\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}$, this is defined by:

$$\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A} := \varinjlim_{n_\alpha \geq 0, \alpha \in J} \prod_{\alpha \in J} \varphi_\alpha^{n_\alpha} \Pi_{[s_I, r_I], I, J, I \setminus J, A}. \quad (5.3.17)$$

Then we have the following definitions:

Definition 5.3.6. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

$$\Pi_{\text{an}, r_I, I, ?, ?', A} := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, ?, ?', A}, \quad ? = J, \check{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus J, \quad (5.3.18)$$

$$(5.3.19)$$

and

$$\Pi_{\text{an}, \text{con}, I, ?, ?', A} := \varinjlim_{r_I} \varprojlim_{s_I} \Pi_{[s_I, r_I], I, ?, ?', A}, \quad ? = J, \check{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus J. \quad (5.3.20)$$

$$(5.3.21)$$

5.3.2 ∞ -Robba Rings over General Banach Affinoids

We now apply the construction of [BK] to the rings defined in the previous section. Recall from [BK], for any Banach adic algebra R over \mathbb{Q}_p we have the derived spectrum $\text{Spa}^h(R) := \text{Spa}_{\text{Rat}}^h(R)$.

Definition 5.3.7. Consider the following rings we defined in the previous section:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} \tag{5.3.22}$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} \tag{5.3.23}$$

$$\Pi_{[s_I, r_I], I, J, I \check{J}, A} \tag{5.3.24}$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{J}, A} \tag{5.3.25}$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{J}, A} \tag{5.3.26}$$

$$\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A} \tag{5.3.27}$$

$$\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A} \tag{5.3.28}$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, \widetilde{I \setminus J}, A} \tag{5.3.29}$$

We then take the corresponding derived spectrum from Bambozzi-Kremnizer to defined the following ∞ -analytic stacks:

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} \tag{5.3.30}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} \tag{5.3.31}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, J, I \check{J}, A} \tag{5.3.32}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, \check{J}, I \check{J}, A} \tag{5.3.33}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, \tilde{J}, I \check{J}, A} \tag{5.3.34}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A} \tag{5.3.35}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A} \tag{5.3.36}$$

$$\mathrm{Spa}^h \Pi_{[s_I, r_I], I, \tilde{J}, \widetilde{I \setminus J}, A} \tag{5.3.37}$$

Taking the global section we have the following ring spectra:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}^h \quad (5.3.38)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}^h \quad (5.3.39)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}^h \quad (5.3.40)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}^h \quad (5.3.41)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}^h \quad (5.3.42)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}^h \quad (5.3.43)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}^h \quad (5.3.44)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}^h \quad (5.3.45)$$

Then we have the following definitions:

Definition 5.3.8. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}^h \quad (5.3.46)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}^h \quad (5.3.47)$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}^h \quad (5.3.48)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}^h \quad (5.3.49)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}^h \quad (5.3.50)$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}^h \quad (5.3.51)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}^h \quad (5.3.52)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A}^h := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}^h \quad (5.3.53)$$

and

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}^h, \quad (5.3.54)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}^h, \quad (5.3.55)$$

$$\Pi_{\text{an,con},I,J,I\setminus J,A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,J,I\setminus J,A}^h, \quad (5.3.56)$$

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}^h, \quad (5.3.57)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}^h, \quad (5.3.58)$$

$$\Pi_{\text{an,con},I,J,\widetilde{I\setminus J},A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,J,\widetilde{I\setminus J},A}^h, \quad (5.3.59)$$

$$\Pi_{\text{an,con},I,\check{J},\widetilde{I\setminus J},A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},\widetilde{I\setminus J},A}^h, \quad (5.3.60)$$

$$\Pi_{\text{an,con},I,\tilde{J},\widetilde{I\setminus J},A}^h := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},\widetilde{I\setminus J},A}^h. \quad (5.3.61)$$

$$(5.3.62)$$

5.3.3 Fréchet Objects in the Sheafy Situation

Assumption 5.3.9. Assume the followings are sheafy adic Banach uniform algebra over \mathbb{Q}_p :

$$\Pi_{[s_I,r_I],I,?,\check{J},A}, ? = J, \tilde{J}, \check{J}, ?' = I\setminus J, \widetilde{I\setminus J}, I\setminus J. \quad (5.3.63)$$

In what follow, we consider the corresponding radii living in the set of all the rational numbers.

Definition 5.3.10. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an},r_I,0,I,\check{J},I\setminus J,A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I,r_I]})$ over each $\Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.11. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an},r_I,0,I,\tilde{J},I\setminus J,A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably

pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.12. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, J, I \setminus J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, J, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.13. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \check{J}, I \setminus J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.14. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \tilde{J}, I \setminus J, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.15. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, J, \widetilde{I \setminus J}, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.16. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \check{J}, \widetilde{I \setminus J}, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Definition 5.3.17. (After KPX, [KPX, Definition 2.1.3]) Over $\Pi_{\text{an}, r_I, 0, I, \tilde{J}, \widetilde{I \setminus J}, A}$ we define the corresponding stably pseudocoherent sheaves to mean a collection of stably

pseudocoherent modules $(M_{[s_I, r_I]})$ over each $\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}$ satisfying the corresponding compatibility condition and the obvious cocycle condition with respect to the family of the corresponding multi-intervals $\{[s_I, r_I]\}$.

Remark 5.3.18. There is some overlap and repeating on some objects within the eight categories. Therefore in the following we are going to then work with only 1st, 2nd, 4th, 5th, 8th categories.

Proposition 5.3.19. (After Kedlaya-Liu, [KL2, Corollary 2.6.8]) *The corresponding pseudocoherent finitely projective bundles over*

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.64)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.65)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.66)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.67)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, \widetilde{I \setminus J}, A} \quad (5.3.68)$$

defined as above have global sections which are finite projective if and only if the global sections are finitely generated.

Proof. Just apply [KL2, Corollary 2.6.8]. □

Proposition 5.3.20. (After Kedlaya-Liu, [KL2, Proposition 2.6.17]) *The corresponding pseudocoherent sheaves over*

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.69)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.70)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.71)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \quad (5.3.72)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, \widetilde{I \setminus J}, A} \quad (5.3.73)$$

defined as above have global sections which are finitely generated as long as we have the uniform bound on the rank of the bundles over quasi-compacts with respect to closed multi-intervals taking the general form of $[s_I, r_I]$.

Proof. This is actually a direct consequence of [KL2, Proposition 2.6.17], where the space $(0, r_{I,0}]$ admits $2^{|I|}$ -uniform covering. \square

Proposition 5.3.21.

(After Kedlaya-Liu, [KL2, Corollary 2.6.8, Proposition 2.6.17]) *The corresponding pseudocoherent sheaves over*

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}, \tag{5.3.74}$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, I \setminus J, A}, \tag{5.3.75}$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, \check{I} \setminus J, A}, \tag{5.3.76}$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, \tilde{I} \setminus J, A}, \tag{5.3.77}$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, \widetilde{I \setminus J}, A} \tag{5.3.78}$$

defined as above have global sections which are finite projective as long as we have the uniform bound on the rank of the bundle over each quasi-compact with respect to each closed multi-interval $[s_I, r_I]$, and we have that the sheaves admits section actually finite projective over each quasi-compact with respect to each closed multi-interval $[s_I, r_I]$.

Proof. This is actually a direct corollary of the previous two propositions. \square

5.4 Cyclotomic Multivariate (φ_I, Γ_I) -Modules over Rigid Analytic Affinoids in Mixed-characteristic Case

5.4.1 Fundamental Definitions

In the situation where A is a rigid analytic affinoid over \mathbb{Q}_p . Recall from [T1], suppose we have $|I|$ finite extensions of \mathbb{Q}_p , where we denote them as K_1, \dots, K_I . Then we have the corresponding uniformizers $\pi_{K_1}, \dots, \pi_{K_I}$, the corresponding Frobenius operators $\varphi_1, \dots, \varphi_I$ and the corresponding groups $\Gamma_{K_1}, \dots, \Gamma_{K_I}$. Recall from [T1], by adding the corresponding variables from $\pi_{K_1}, \dots, \pi_{K_I}$ and $\Gamma_{K_1}, \dots, \Gamma_{K_I}$ we have the following rings:

$$\Pi_{[s_I, r_I], I, I, \emptyset, A}(\pi_{K_I}) := \Pi_{[s_I, r_I], I, A}(\pi_{K_I}) \quad (5.4.1)$$

$$\Pi_{\text{an}, r_I, I, I, \emptyset, A}(\pi_{K_I}) := \Pi_{\text{an}, r_I, I, A}(\pi_{K_I}) \quad (5.4.2)$$

$$\Pi_{\text{an}, \text{con}, I, I, \emptyset, A}(\pi_{K_I}) := \Pi_{\text{an}, \text{con}, I, \emptyset, A}(\pi_{K_I}) \quad (5.4.3)$$

and

$$\Pi_{[s_I, r_I], I, I, \emptyset, A}(\Gamma_{K_I}) := \Pi_{[s_I, r_I], I, A}(\Gamma_{K_I}) \quad (5.4.4)$$

$$\Pi_{\text{an}, r_I, I, I, \emptyset, A}(\Gamma_{K_I}) := \Pi_{\text{an}, r_I, I, A}(\Gamma_{K_I}) \quad (5.4.5)$$

$$\Pi_{\text{an}, \text{con}, I, I, \emptyset, A}(\Gamma_{K_I}) := \Pi_{\text{an}, \text{con}, I, \emptyset, A}(\Gamma_{K_I}). \quad (5.4.6)$$

Definition 5.4.1. By taking the direct base change we have the following rings in mixed-characteristic situation:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.7)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.8)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.4.9)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.4.10)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.4.11)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.4.12)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.4.13)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}). \quad (5.4.14)$$

By taking the direct base change we have the following rings in mixed-characteristic situation:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\Gamma_{K_I}), \quad (5.4.15)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\Gamma_{K_I}), \quad (5.4.16)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\Gamma_{K_I}), \quad (5.4.17)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\Gamma_{K_I}), \quad (5.4.18)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\Gamma_{K_I}), \quad (5.4.19)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\Gamma_{K_I}), \quad (5.4.20)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\Gamma_{K_I}), \quad (5.4.21)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\Gamma_{K_I}). \quad (5.4.22)$$

Definition 5.4.2. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.23)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.24)$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.4.25)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.4.26)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.4.27)$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.4.28)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.4.29)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}). \quad (5.4.30)$$

and

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.31)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.32)$$

$$\Pi_{\text{an,con},I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.33)$$

$$\Pi_{\text{an,con},I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.34)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.35)$$

$$\Pi_{\text{an,con},I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.36)$$

$$\Pi_{\text{an,con},I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.37)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}). \quad (5.4.38)$$

$$(5.4.39)$$

Definition 5.4.3. (After Kedlaya-Liu, [KL2, Definition 5.2.1])

$$\Pi_{\text{an},r_I,I,\check{J},I\setminus J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\Gamma_{K_I}), \quad (5.4.40)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\setminus J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\Gamma_{K_I}), \quad (5.4.41)$$

$$\Pi_{\text{an},r_I,I,J,I\check{\setminus}J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.42)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\check{\setminus}J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.43)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\check{\setminus}J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.44)$$

$$\Pi_{\text{an},r_I,I,J,I\widetilde{\setminus}J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.45)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.46)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}). \quad (5.4.47)$$

and

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\Gamma_{K_I}), \quad (5.4.48)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\Gamma_{K_I}), \quad (5.4.49)$$

$$\Pi_{\text{an,con},I,J,I\check{\setminus}J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.50)$$

$$\Pi_{\text{an,con},I,\check{J},I\check{\setminus}J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.51)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\check{\setminus}J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.52)$$

$$\Pi_{\text{an,con},I,J,I\widetilde{\setminus}J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.53)$$

$$\Pi_{\text{an,con},I,\check{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}), \quad (5.4.54)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}) := \lim_{\substack{\longrightarrow \\ r_I}} \lim_{\substack{\longleftarrow \\ s_I}} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\Gamma_{K_I}). \quad (5.4.55)$$

Now we consider the corresponding definition of the (φ_I, Γ_I) -modules over the corresponding period rings defined above.

Setting 5.4.4. For the corresponding φ_I -modules over the rings:

$$\Pi_{\text{an},r_I,I,\check{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.56)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.57)$$

$$\Pi_{\text{an},r_I,I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.58)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.59)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.60)$$

$$\Pi_{\text{an},r_I,I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.61)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.62)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) \quad (5.4.63)$$

we assume we have the sufficiently small radius as in [KPX, Definition 2.2.6]. We will keep this assumption in all similar situation involving the corresponding φ_I -modules.

Definition 5.4.5. (After KPX [KPX, Definition 2.2.6]) We define in the following way the corresponding φ_I -modules over the following period rings:

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.64)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.65)$$

$$\Pi_{\text{an,con},I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.66)$$

$$\Pi_{\text{an,con},I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.67)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.68)$$

$$\Pi_{\text{an,con},I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.69)$$

$$\Pi_{\text{an,con},I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.70)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}). \quad (5.4.71)$$

These are the corresponding base change of the corresponding φ_I -modules coming from

the the ones over the following rings:

$$\Pi_{\text{an},r_I,I,\check{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.72)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.73)$$

$$\Pi_{\text{an},r_I,I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.74)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.75)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.76)$$

$$\Pi_{\text{an},r_I,I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.77)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.78)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}). \quad (5.4.79)$$

For this latter group of period rings, we define a corresponding pseudocoherent or finite projective φ_I -module to be a corresponding pseudocoherent or finite projective module M over this latter group of period rings carrying the corresponding semilinear partial Frobenius action coming from each Frobenius operator φ_α , $\alpha \in I$ such that for each α we have

$$\varphi_\alpha^* M \otimes_{\Pi_{\text{an},\{\dots,r_\alpha/p,\dots\},I,*,*,A}(\pi_{K_I})} \Pi_{\text{an},\{\dots,r_\alpha/p,\dots\},I,*,*,A}(\pi_{K_I}) \quad (5.4.80)$$

$$\xrightarrow{\sim} M \otimes_{\Pi_{\text{an},\{\dots,r_\alpha,\dots\},I,*,*,A}(\pi_{K_I})} \Pi_{\text{an},\{\dots,r_\alpha/p,\dots\},I,*,*,A}(\pi_{K_I}). \quad (5.4.81)$$

And we assume that altogether the partial Frobenius operators are commuting with each other. We assume all the modules involved are complete for the natural topology (mainly

in the pseudocoherent situation) whose base changes to the following rings:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.82)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.83)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.4.84)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.4.85)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.4.86)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \widetilde{J}, A}(\pi_{K_I}), \quad (5.4.87)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}), \quad (5.4.88)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}) \quad (5.4.89)$$

give rise to the corresponding φ_I -modules defined over these rings which defined in the following way. We define a corresponding pseudocoherent or finite projective φ_I -module over:

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.90)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.4.91)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.4.92)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.4.93)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.4.94)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \widetilde{J}, A}(\pi_{K_I}), \quad (5.4.95)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}), \quad (5.4.96)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}) \quad (5.4.97)$$

to be a corresponding stably-pseudocoherent or finite projective module M over this latter group of period rings carrying the corresponding semilinear partial Frobenius action coming from each Frobenius operator φ_α , $\alpha \in I$ such that for each α we have

$$\varphi_\alpha^* M \otimes_{\Pi_{\text{an}, \dots, [s_\alpha/p, r_\alpha/p], \dots, I, *, *, A}(\pi_{K_I})} \Pi_{\text{an}, \dots, [s_\alpha, r_\alpha/p], \dots, I, *, *, A}(\pi_{K_I}) \quad (5.4.98)$$

$$\xrightarrow{\sim} M \otimes_{\Pi_{\text{an}, \dots, [s_\alpha, r_\alpha], \dots, I, *, *, A}(\pi_{K_I})} \Pi_{\text{an}, \dots, [s_\alpha, r_\alpha/p], \dots, I, *, *, A}(\pi_{K_I}). \quad (5.4.99)$$

And we assume that altogether the partial Frobenius operators are commuting with each other. We assume all the modules involved are complete for the natural topology (mainly in the pseudocoherent situation).

Finally we define the pseudocoherent or finite projective φ_I -modules over the corresponding period rings:

$$\Pi_{\text{an,con},I,\check{J},I\setminus J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.100)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.101)$$

$$\Pi_{\text{an,con},I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.102)$$

$$\Pi_{\text{an,con},I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.103)$$

$$\Pi_{\text{an,con},I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.104)$$

$$\Pi_{\text{an,con},I,J,\widetilde{I\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,J,\widetilde{I\setminus}J,A}(\pi_{K_I}), \quad (5.4.105)$$

$$\Pi_{\text{an,con},I,\check{J},\widetilde{I\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\check{J},\widetilde{I\setminus}J,A}(\pi_{K_I}), \quad (5.4.106)$$

$$\Pi_{\text{an,con},I,\tilde{J},\widetilde{I\setminus}J,A}(\pi_{K_I}) := \lim_{r_I} \lim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},\widetilde{I\setminus}J,A}(\pi_{K_I}) \quad (5.4.107)$$

to be the corresponding base changes of some φ_I -modules over the rings:

$$\Pi_{\text{an},r_I,I,\check{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.108)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.109)$$

$$\Pi_{\text{an},r_I,I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.110)$$

$$\Pi_{\text{an},r_I,I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.111)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.112)$$

$$\Pi_{\text{an},r_I,I,J,\widehat{I\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,\widehat{I\setminus}J,A}(\pi_{K_I}), \quad (5.4.113)$$

$$\Pi_{\text{an},r_I,I,\check{J},\widehat{I\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},\widehat{I\setminus}J,A}(\pi_{K_I}), \quad (5.4.114)$$

$$\Pi_{\text{an},r_I,I,\tilde{J},\widehat{I\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},\widehat{I\setminus}J,A}(\pi_{K_I}), \quad (5.4.115)$$

with the corresponding requirement that they are basically complete with respect to the natural topology and the partial Frobenius operators are commuting with each other.

Definition 5.4.6. (After KPX [KPX, Definition 2.2.6]) Then we define a corresponding pseudocoherent or finite projective φ_I -sheaf F over one of the following period

rings:

$$\Pi_{\text{an},r_I,0,I,\check{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.116)$$

$$\Pi_{\text{an},r_I,0,I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.117)$$

$$\Pi_{\text{an},r_I,0,I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.118)$$

$$\Pi_{\text{an},r_I,0,I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.119)$$

$$\Pi_{\text{an},r_I,0,I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.120)$$

$$\Pi_{\text{an},r_I,0,I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.121)$$

$$\Pi_{\text{an},r_I,0,I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.122)$$

$$\Pi_{\text{an},r_I,0,I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) \quad (5.4.123)$$

to be the corresponding compatible family of the φ_I -modules over any one $\Pi_{[s_I,r_I],I,*,*,A}(\pi_{K_I})$ of the following rings:

$$\Pi_{[s_I,r_I],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.124)$$

$$\Pi_{[s_I,r_I],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.4.125)$$

$$\Pi_{[s_I,r_I],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.126)$$

$$\Pi_{[s_I,r_I],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.127)$$

$$\Pi_{[s_I,r_I],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.4.128)$$

$$\Pi_{[s_I,r_I],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.129)$$

$$\Pi_{[s_I,r_I],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.4.130)$$

$$\Pi_{[s_I,r_I],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) \quad (5.4.131)$$

satisfying the corresponding restriction requirement and the corresponding cocycle condition as in [KPX, Definition 2.2.6], such that $[s_I, r_I] \subset (0, r_{I,0}]$.

Definition 5.4.7. (After KPX [KPX, Definition 2.2.12]) As in [KPX, Definition 2.2.12] we impose the corresponding Γ_I -structure by adding the corresponding semilinear continuous action of Γ_I on the modules induced from that on the period rings, which are assumed to be commuting with the action from φ_I .

5.4.2 The Comparison Theorems

Now we establish some results on the comparison on the corresponding φ_I -modules we defined above.

Theorem 5.4.8. (After KPX [KPX, Proposition 2.2.7]) *Consider the following categories:*

1. *The category of all the finite projective φ_I -modules over the ring $\Pi_{\text{an}, r_{I,0}, I, ?, ?, A}(\pi_{K_I})$;*
2. *The category of all the finite projective φ_I -sheaves over the ring $\Pi_{\text{an}, r_{I,0}, I, ?, ?, A}(\pi_{K_I})$.*

Then we have that the two categories are equivalent.

Proof. The base change gives rise to the corresponding fully faithful functor from the first category to the second one, while to show the corresponding essential surjectivity, consider the corresponding multi-interval $[r_{1,0}/p, r_{1,0}] \times \dots \times [r_{I,0}/p, r_{I,0}]$ and use the corresponding Frobenius to reach all the corresponding intervals taking the general form of:

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.4.132)$$

This forms a $2^{|I|}$ -uniform covering of the whole space. And the corresponding uniform finiteness of the modules over each

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.4.133)$$

could be achieved by using the corresponding partial Frobenius actions. Then we are done by applying proposition 5.2.20. \square

Theorem 5.4.9. (After KPX [KPX, Proposition 2.2.7]) *Consider the following categories:*

1. *The category of all the pseudocoherent φ_I -modules over the ring $\Pi_{\text{an}, r_{I,0}, I, ?, ?, A}(\pi_{K_I})$;*
2. *The category of all the pseudocoherent φ_I -sheaves over the ring $\Pi_{\text{an}, r_{I,0}, I, ?, ?, A}(\pi_{K_I})$.*

Then we have that the two categories are equivalent.

Proof. The base change gives rise to the corresponding fully faithful functor from the first category to the second one, while to show the corresponding essential surjectivity, consider the corresponding multi-interval $[r_{1,0}/p, r_{1,0}] \times \dots \times [r_{I,0}/p, r_{I,0}]$ and use the corresponding Frobenius to reach all the corresponding intervals taking the general form of:

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.4.134)$$

This forms a $2^{|I|}$ -uniform covering of the whole space. And the corresponding uniform finiteness of the modules over each

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.4.135)$$

could be achieved by using the corresponding partial Frobenius actions. Then we are done by applying proposition 5.2.19. Then one choose finite free covering to promote the finiteness to pseudocoherence as in [KL2, Theorem 4.6.1, Lemma 5.4.11]. \square

We now consider the corresponding vertical comparison for $I = \{1, 2\}$:

Theorem 5.4.10. (After Kedlaya-Liu [KL2, Theorem 5.7.5]) *Consider the following categories:*

1. *The category of all the finite projective (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$;*
 2. *The category of all the finite projective (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \checkmark J, A}(\pi_{K_I})$;*
 3. *The category of all the finite projective (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I})$.*
- Then we have that these categories are equivalent. Here $0 < s_\alpha < r_\alpha < \infty$ for any $\alpha \in I$.*

This is the consequence of the following theorem:

Theorem 5.4.11. (After Kedlaya-Liu [KL2, Theorem 5.7.5]) *Consider the following categories:*

1. *The category of all the finite projective (φ_2, Γ_2) -modules over the ring $\Pi_{[s_I, r_I], I, 1, I \setminus J, A}(\pi_{K_I})$;*
 2. *The category of all the finite projective (φ_2, Γ_2) -modules over the ring $\Pi_{[s_I, r_I], I, 1, I \checkmark J, A}(\pi_{K_I})$;*
 3. *The category of all the finite projective (φ_2, Γ_2) -modules over the ring $\Pi_{[s_I, r_I], I, 1, I \widetilde{\setminus} J, A}(\pi_{K_I})$.*
- Then we have that these categories are equivalent. Here $0 < s_\alpha < r_\alpha < \infty$ for any $\alpha \in I$.*

Proof. In this situation this is just the corresponding relative comparison for finite projective (φ, Γ) modules, which is [KP, Theorem 4.4]. \square

We now consider the corresponding vertical comparison in the following context for $I = \{1, 2\}$:

Theorem 5.4.12. (After Kedlaya-Liu [KL2, Theorem 5.7.5]) *Consider the following categories:*

1. *The category of all the pseudocoherent (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$;*
2. *The category of all the pseudocoherent (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \checkmark J, A}(\pi_{K_I})$;*

3. The category of all the pseudocoherent (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$.
Then we have that these categories are equivalent. Here $0 < s_\alpha < r_\alpha < \infty$ for any $\alpha \in I$.

This is the consequence of the following theorem:

Theorem 5.4.13. (After Kedlaya-Liu [KL2, Theorem 5.7.5]) *Consider the following categories:*

1. The category of all the pseudocoherent (φ_2, Γ_2) -modules over the ring $\Pi_{[s_I, r_I], I, 1, I \setminus J, A}(\pi_{K_I})$;
 2. The category of all the pseudocoherent (φ_2, Γ_2) -modules over the ring $\Pi_{[s_I, r_I], I, 1, I \setminus J, A}(\pi_{K_I})$;
 3. The category of all the pseudocoherent (φ_2, Γ_2) -modules over the ring $\Pi_{[s_I, r_I], I, 1, I \setminus J, A}(\pi_{K_I})$.
- Then we have that these categories are equivalent. Here $0 < s_\alpha < r_\alpha < \infty$ for any $\alpha \in I$.

Proof. In this situation this is just the corresponding relative comparison for finite projective (φ, Γ) modules, which is [T2, Proposition 5.44, Proposition 5.51]. \square

Corollary 5.4.14. *Assume $I = \{1, 2\}$. Consider the following categories:*

1. The category of all the pseudocoherent (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$;
 2. The category of all the pseudocoherent (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$;
 3. The category of all the pseudocoherent (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$.
- Then we have that these categories are equivalent. Here $0 < s_\alpha \leq r_\alpha/p < \infty$ for any $\alpha \in I$.

Consider the following categories:

1. The category of all the finite projective (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$;
 2. The category of all the finite projective (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$;
 3. The category of all the finite projective (φ_I, Γ_I) -modules over the ring $\Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I})$.
- Then we have that these categories are equivalent. Here $0 < s_\alpha \leq r_\alpha/p < \infty$ for any $\alpha \in I$.

5.5 B_I -Pairs and Intermediate Objects in Rigid Family

5.5.1 Mixed-type Hodge Structures

Now we work with the corresponding B -pairs and some mixed-type objects as in [Ber1] and [Nak1].

Definition 5.5.1. Define $B_{\mathrm{dR},I}^+ := \mathbb{C}_p[[t_1, \dots, t_I]]$, and define

$B_{\mathrm{dR},I} := \mathbb{C}_p[[t_1, \dots, t_I]][t_1^{-1}, \dots, t_I^{-1}]$, and similarly we have the obvious higher dimensional analog $B_{e,I}$ of the corresponding period ring B_e .

Then after Bloch-Kato [BK1] we have the following fundamental sequence:

Proposition 5.5.2. *We have the higher dimensional generalization of the corresponding Bloch-Kato fundamental sequence:*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{e,I} \oplus B_{\mathrm{dR},I}^+ \longrightarrow B_{\mathrm{dR},I} \longrightarrow 0$$

induced by the corresponding short exact sequence:

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{e,I} \oplus \mathbb{C}_p[[t_1, \dots, t_I]] \longrightarrow \mathbb{C}_p[[t_1, \dots, t_I]][t_1^{-1}, \dots, t_I^{-1}] \longrightarrow 0.$$

Setting 5.5.3. In what follows, we assume that the corresponding A to be a rigid affinoid in rigid analytic geometry over \mathbb{Q}_p .

Definition 5.5.4. We now consider the following rings:

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.1)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.2)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.3)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.4)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.5)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.6)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}, \quad (5.5.7)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}, \quad (5.5.8)$$

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}, \quad (5.5.9)$$

with

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.10)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.11)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.12)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.13)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.14)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.15)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.16)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.17)$$

$$B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.18)$$

with

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.19)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.20)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}, \quad (5.5.21)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}, \quad (5.5.22)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}, \quad (5.5.23)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}, \quad (5.5.24)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}, \quad (5.5.25)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}, \quad (5.5.26)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}, \quad (5.5.27)$$

with

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.28)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.29)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}, \quad (5.5.30)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}, \quad (5.5.31)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}, \quad (5.5.32)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}, \quad (5.5.33)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \widetilde{J}, A}, \quad (5.5.34)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \widetilde{J}, A}, \quad (5.5.35)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \widetilde{J}, A}, \quad (5.5.36)$$

with

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.37)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.38)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.39)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.40)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.41)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.42)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.43)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.44)$$

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.45)$$

with

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.46)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.47)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}, \quad (5.5.48)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}, \quad (5.5.49)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}, \quad (5.5.50)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}, \quad (5.5.51)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}, \quad (5.5.52)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}, \quad (5.5.53)$$

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}, \quad (5.5.54)$$

with

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.55)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.56)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}, \quad (5.5.57)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}, \quad (5.5.58)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}, \quad (5.5.59)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}, \quad (5.5.60)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}, \quad (5.5.61)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}, \quad (5.5.62)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}, \quad (5.5.63)$$

with

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.64)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.65)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.66)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.67)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.68)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.69)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.70)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.71)$$

$$\lim_{r_I} \lim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \widetilde{I \setminus J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.5.72)$$

with

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.5.73)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.5.74)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}, \quad (5.5.75)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}, \quad (5.5.76)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}, \quad (5.5.77)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}, \quad (5.5.78)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus \widetilde{J}, A}, \quad (5.5.79)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \widetilde{J}, A}, \quad (5.5.80)$$

$$\varinjlim_{r_I} \varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \widetilde{J}, A}. \quad (5.5.81)$$

Now we combine the construction in the following coherent way following [Ber1, Section 2], [Nak1, Definition 2.2] and [KPX, Definition 2.2.6].

Definition 5.5.5. We define a $B_{I'}(\varphi_I, \Gamma_I)$ -module over

$$(B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}[t_1^{-1}, \dots, t_I'^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A})$$

to be a triplet of finite projective modules:

$$(M_e, M_{\text{dR}}, M_{\text{dR}}^+) \quad (5.5.82)$$

over

$$(B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}, B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}[t_1^{-1}, \dots, t_I'^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A})$$

such that we have glueing datum along :

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} \rightarrow B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}[t_1^{-1}, \dots, t_I'^{-1}] \leftarrow B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}. \quad (5.5.83)$$

And this carry the corresponding relative Galois action of:

$$\mathrm{Gal}_{\mathbb{Q}_p,1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p,I'} \quad (5.5.84)$$

on the multi de Rham period rings which is semilinear. And we have that the three modules involved are relative (φ_I, Γ_I) -modules relative to

$$B_{\mathrm{dR},I'}^+, B_{\mathrm{dR},I'}, B_{e,I'}. \quad (5.5.85)$$

Definition 5.5.6. We define a pseudocoherent $B_{I'}(\varphi_I, \Gamma_I)$ -module over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} [t_1^{-1}, \dots, t_I'^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A})$$

to be a triplet of stably-pseudocoherent modules:

$$(M_e, M_{\mathrm{dR}}, M_{\mathrm{dR}}^+) \quad (5.5.86)$$

over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} [t_1^{-1}, \dots, t_I'^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A})$$

such that this is glueing datum along :

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} \rightarrow B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} [t_1^{-1}, \dots, t_I'^{-1}] \leftarrow B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A}. \quad (5.5.87)$$

And this carry the corresponding relative Galois action of:

$$\mathrm{Gal}_{\mathbb{Q}_p,1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p,I'} \quad (5.5.88)$$

on the multi de Rham period rings which is semilinear. And we have that the three modules involved are relative pseudocoherent (φ_I, Γ_I) -modules relative to

$$B_{\mathrm{dR},I'}^+, B_{\mathrm{dR},I'}, B_{e,I'}. \quad (5.5.89)$$

Definition 5.5.7. We define a $B_{I'}(\varphi_I, \Gamma_I)$ -module over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an}, r_I, I, J, I \setminus J, A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} [t_1^{-1}, \dots, t_I'^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an}, r_I, I, *, *, A})$$

to be a triplet of finite projective modules:

$$(M_e, M_{\mathrm{dR}}, M_{\mathrm{dR}}^+) \quad (5.5.90)$$

over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

such that this is glueing datum along :

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A} \rightarrow B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}] \leftarrow B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}. \quad (5.5.91)$$

And this carry the corresponding relative Galois action of:

$$\mathrm{Gal}_{\mathbb{Q}_p,1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p,I'} \quad (5.5.92)$$

on the multi de Rham period rings which is semilinear. And we have that the three modules involved are relative (φ_I, Γ_I) -modules relative to

$$B_{\mathrm{dR},I'}^+, B_{\mathrm{dR},I'}, B_{e,I'}. \quad (5.5.93)$$

Definition 5.5.8. We define a pseudocoherent $B_{I'}(\varphi_I, \Gamma_I)$ -module over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,J,I \setminus J,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I,r_I],I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

to be a triplet of stably pseudocoherent modules:

$$(M_e, M_{\mathrm{dR}}, M_{\mathrm{dR}}^+) \quad (5.5.94)$$

over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

such that this is glueing datum along :

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A} \rightarrow B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}] \leftarrow B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}. \quad (5.5.95)$$

And this carry the corresponding relative Galois action of:

$$\mathrm{Gal}_{\mathbb{Q}_p,1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p,I'} \quad (5.5.96)$$

on the multi de Rham period rings which is semilinear. And we have that the three modules involved are relative pseudocoherent (φ_I, Γ_I) -modules relative to

$$B_{\mathrm{dR},I'}^+, B_{\mathrm{dR},I'}, B_{e,I'}. \quad (5.5.97)$$

5.5.2 Fundamental Comparison on the Mixed-Type Objects

Proposition 5.5.9. (After Berger [Ber1, Théorème A]) *Let I' be a set consisting of two elements and I is empty, then we have that the category of all the $B_{I'}-(\varphi_I, \Gamma_I)$ modules over*

$$(B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}[t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A})$$

is equivalent to the category of all the $(\varphi_{I'}, \Gamma_{I'})$ -modules in the finite projective setting.

Proof. One has the result after the following two propositions. □

We first consider the following comparison:

Proposition 5.5.10. *Let $I' = \{1, 2\}$ be a set consisting of two elements and I is empty, then we have that the category of all the $B_{I'}-(\varphi_I, \Gamma_I)$ modules over*

$$(B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}[t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A})$$

is equivalent to the category of all the $B_{\{1\}}-(\varphi_{\{2\}}, \Gamma_{\{2\}})$ modules.

Proof. This will be the corresponding consequence of the following. Let $I' = \{1, 2\}$ be a set consisting of two elements and I is empty, then we have that the category of all the $B_{I'}-(\varphi_I, \Gamma_I)$ modules over

$$(B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A}, B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A}[t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A})$$

is equivalent to the category of all the $B_{\{1\}}-(\varphi_{\{2\}}, \Gamma_{\{2\}})$ modules over the corresponding perfected rings with $\widetilde{}$ accent. However this could be proved as in [KP, Theorem 2.18] as long as one works with mod $t^k, k \in \mathbb{Z}$ coefficients (also see the corresponding proof of [T3, Proposition 3.8]). To be more precise first we consider the corresponding base change of any $B_{I'}-(\varphi_I, \Gamma_I)$ module over

$$(B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A}, B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A}[t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A})$$

to $B_{\text{dR}, \{1\}}$, which then by the strategy above could be associated a $B_{\{1\}}-(\varphi_{\{2\}}, \Gamma_{\{2\}})$ module over the corresponding perfected rings with $\widetilde{}$ accent (see [KP, Theorem 2.18], [T3, Proposition 3.8]). As in [KP, Theorem 2.18] we will have the situation where the category of all the $B_{I'}-(\varphi_I, \Gamma_I)$ modules over

$$(B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A}, B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A} [t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A})$$

is equivalent to the category of all the $B_{\{1\}}\text{-}(\varphi_{\{2\}}, \Gamma_{\{2\}})$ modules over the corresponding perfected rings with $\tilde{\cdot}$ accent. However by the proof of [KP, Theorem 4.4] we further have the situation where the category of all the $B_{I'}\text{-}(\varphi_I, \Gamma_I)$ modules over

$$(B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A}, B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A} [t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A})$$

is equivalent to the category of all the $B_{\{1\}}\text{-}(\varphi_{\{2\}}, \Gamma_{\{2\}})$ modules over the corresponding perfected rings with no accent. \square

Proposition 5.5.11. *Let $I = \{1, 2\}$ be a set consisting of two elements and I' is empty, then we have that the category of all the $B_{I'}\text{-}(\varphi_I, \Gamma_I)$ modules over*

$$(B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A} [t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A})$$

is equivalent to the category of all the $B_{\{1\}}\text{-}(\varphi_{\{2\}}, \Gamma_{\{2\}})$ modules.

Proof. This will be the corresponding consequence of the following. Let $I = \{1, 2\}$ be a set consisting of two elements and I' is empty, then we have that the category of all the $B_{I'}\text{-}(\varphi_I, \Gamma_I)$ modules over

$$(B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A}, B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A} [t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \tilde{J}, \tilde{I} \setminus J, A})$$

is equivalent to the category of all the $B_{\{1\}}\text{-}(\varphi_{\{2\}}, \Gamma_{\{2\}})$ modules over the corresponding perfected rings with $\tilde{\cdot}$ accent. However this is proved in [KP, Theorem 2.18] as in the proof of the previous proposition. \square

Now we combine the construction in the following coherent way following [Ber1, Section 2], [Nak1, Definition 2.2] and [KPX, Definition 2.2.6].

Definition 5.5.12. We define a $B_{I'}\text{-}(\varphi_I, \Gamma_I)$ -bundle over

$$(B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{\text{an}, r_I, I, J, I \setminus J, A}, B_{\text{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, *, *, A} [t_1^{-1}, \dots, t_I^{-1}], B_{e, I'} \widehat{\otimes} \Pi_{\text{an}, r_I, I, *, *, A})$$

to be a compatible family (with respect to the Robba rings) of triplets of finite projective modules:

$$(M_e, M_{\text{dR}}, M_{\text{dR}}^+) \tag{5.5.98}$$

over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

such that this is glueing datum along :

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A} \rightarrow B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}] \leftarrow B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}. \quad (5.5.99)$$

And this carry the corresponding relative Galois action of:

$$\mathrm{Gal}_{\mathbb{Q}_p,1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p,I'} \quad (5.5.100)$$

on the multi de Rham period rings which is semilinear. And we have that the three modules involved are relative (φ_I, Γ_I) -bundles relative to

$$B_{\mathrm{dR},I'}^+, B_{\mathrm{dR},I'}, B_{e,I'}. \quad (5.5.101)$$

Definition 5.5.13. We define a pseudocoherent $B_{I'}(\varphi_I, \Gamma_I)$ -bundle over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,J,I \setminus J,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{[s_I,r_I],I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

to be a compatible family of triplets of stably pseudocoherent modules:

$$(M_e, M_{\mathrm{dR}}, M_{\mathrm{dR}}^+) \quad (5.5.102)$$

over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

such that this is glueing datum along :

$$B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A} \rightarrow B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}] \leftarrow B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}. \quad (5.5.103)$$

And this carry the corresponding relative Galois action of:

$$\mathrm{Gal}_{\mathbb{Q}_p,1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p,I'} \quad (5.5.104)$$

on the multi de Rham period rings which is semilinear. And we have that the three modules involved are relative pseudocoherent (φ_I, Γ_I) -bundles relative to

$$B_{\mathrm{dR},I'}^+, B_{\mathrm{dR},I'}, B_{e,I'}. \quad (5.5.105)$$

Proposition 5.5.14. (After KPX [KPX, Proposition 2.2.7]) *The category of all the finite projective $B_{I'}(\varphi_I, \Gamma_I)$ -bundle over*

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

is equivalent to the category of all the finite projective $B_{I'}(\varphi_I, \Gamma_I)$ -modules over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I'^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}).$$

Proof. Without considering the corresponding Galois actions for the $B_{I'}$ -pair components we could prove this as in the relative situation carrying just A -coefficient. To be more precise, the base change gives rise to the corresponding fully faithful functor from the first category to the second one, while to show the corresponding essential surjectivity, consider the corresponding multi-interval $[r_{1,0}/p, r_{1,0}] \times \dots \times [r_{I,0}/p, r_{I,0}]$ and use the corresponding Frobenius to reach all the corresponding intervals taking the general form of:

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.5.106)$$

This forms a $2^{|I|}$ -uniform covering of the whole space. And the corresponding uniform finiteness of the modules over each

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.5.107)$$

could be achieved by using the corresponding partial Frobenius actions. Then we are done by applying proposition 5.2.20. \square

Proposition 5.5.15. (After KPX [KPX, Proposition 2.2.7]) *The category of all the pseudocoherent $B_{I'}(\varphi_I, \Gamma_I)$ -bundle over*

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A})$$

is equivalent to the category of all the pseudocoherent $B_{I'}(\varphi_I, \Gamma_I)$ -modules over

$$(B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}, B_{\mathrm{dR},I'}^+ \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}[t_1^{-1}, \dots, t_I'^{-1}], B_{e,I'} \widehat{\otimes} \Pi_{\mathrm{an},r_I,I,*,*,A}).$$

Proof. Without considering the corresponding Galois actions for the $B_{I'}$ -pair components we could prove this as in the relative situation carrying just A -coefficient. To be more precise, the base change gives rise to the corresponding fully faithful functor from the first category to the second one, while to show the corresponding essential surjectivity, consider the corresponding multi-interval $[r_{1,0}/p, r_{1,0}] \times \dots \times [r_{I,0}/p, r_{I,0}]$ and use the corresponding Frobenius to reach all the corresponding intervals taking the general form of:

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.5.108)$$

This forms a $2^{|I|}$ -uniform covering of the whole space. And the corresponding uniform finiteness of the modules over each

$$[r_{1,0}/p^{k_1}, r_{1,0}/p^{k_1-1}] \times \dots \times [r_{I,0}/p^{k_I}, r_{I,0}/p^{k_I-1}], k_\alpha = 1, 2, \dots, \forall \alpha \in I. \quad (5.5.109)$$

could be achieved by using the corresponding partial Frobenius actions. Then we are done by applying proposition 5.2.19. \square

5.6 Cohomologies of Cyclotomic Multivariate (φ_I, Γ_I) -Modules over Rigid Analytic Affinoids in Mixed-characteristic Case

Now we define the corresponding cohomologies of the multivariate (φ_I, Γ_I) -modules over the following groups of rings:

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.1)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.2)$$

$$\Pi_{\text{an}, r_I, I, J, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I}), \quad (5.6.3)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.4)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.5)$$

$$\Pi_{\text{an}, r_I, I, J, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \setminus J, A}(\pi_{K_I}), \quad (5.6.6)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.7)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}). \quad (5.6.8)$$

and

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\Gamma_{K_I}), \quad (5.6.9)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\Gamma_{K_I}), \quad (5.6.10)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}(\Gamma_{K_I}), \quad (5.6.11)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}(\Gamma_{K_I}), \quad (5.6.12)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}(\Gamma_{K_I}), \quad (5.6.13)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \widetilde{J}, A}(\Gamma_{K_I}), \quad (5.6.14)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \widetilde{J}, A}(\Gamma_{K_I}), \quad (5.6.15)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \widetilde{J}, A}(\Gamma_{K_I}). \quad (5.6.16)$$

Definition 5.6.1. (After KPX, [KPX, Definition 2.3.3]) We define by induction the corresponding φ_I -complex $C_{\varphi_I}^\bullet$ of a corresponding (φ_I, Γ_I) -module M over

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.17)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.18)$$

$$\Pi_{\text{an}, r_I, I, J, I \setminus \check{J}, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.6.19)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus \check{J}, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.6.20)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus \check{J}, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.6.21)$$

$$\Pi_{\text{an}, r_I, I, J, I \setminus \widetilde{J}, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \setminus \widetilde{J}, A}(\pi_{K_I}), \quad (5.6.22)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}), \quad (5.6.23)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \widetilde{J}, A}(\pi_{K_I}) \quad (5.6.24)$$

to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, r_{|I|}}) \xrightarrow{\varphi_{|I|}^{-1}} C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, r_{|I|}/p}) \longrightarrow 0$$

as long as $C_{\varphi_{I \setminus |I|}}^\bullet(M)$ is constructed. We define by induction the corresponding φ_I -complex

$C_{\varphi_I}^\bullet(M)$ of a corresponding (φ_I, Γ_I) -module M over

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.25)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.26)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.27)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.28)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.29)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.30)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.31)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.32)$$

to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, [s_{|I|}, r_{|I|}]}) \xrightarrow{\varphi_{|I|}^{-1}} C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, [s_{|I|}, r_{|I|}/p]}) \longrightarrow 0$$

as long as $C_{\varphi_{I \setminus |I|}}^\bullet(M)$ is constructed. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.6.2. (After KPX, [KPX, Definition 2.3.3]) We define by induction the corresponding ψ_I -complex $C_{\psi_I}^\bullet$ of a corresponding (φ_I, Γ_I) -module M over

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.33)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.34)$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.35)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.36)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.37)$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.38)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.39)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.40)$$

to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, r_{|I|}}) \xrightarrow{\psi_{|I|}^{-1}} C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, pr_{|I|}}) \longrightarrow 0$$

as long as $C_{\psi_{I \setminus |I|}}^\bullet(M)$ is constructed. We define by induction the corresponding ψ_I -complex $C_{\psi_I}^\bullet(M)$ of a corresponding (φ_I, Γ_I) -module M over

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.41)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.42)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.6.43)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.6.44)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.6.45)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \tilde{J}, A}(\pi_{K_I}), \quad (5.6.46)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \tilde{J}, A}(\pi_{K_I}), \quad (5.6.47)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \tilde{J}, A}(\pi_{K_I}) \quad (5.6.48)$$

to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, [s_{|I|}, r_{|I|}]}) \xrightarrow{\psi_{|I|}^{-1}} C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, [ps_{|I|}, r_{|I|}]}) \longrightarrow 0$$

as long as $C_{\psi_{I \setminus |I|}}^\bullet(M)$ is constructed. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.6.3. (After KPX, [KPX, Definition 2.3.3]) We define by induction the

corresponding Γ_I -complex $C_{\Gamma_I}^\bullet$ of a corresponding (φ_I, Γ_I) -module M over

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.49)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.50)$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.51)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.52)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.53)$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.54)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.55)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.56)$$

to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\Gamma_{I \setminus |I|}}^\bullet(M) \xrightarrow{\gamma_{|I|-1}} C_{\Gamma_{I \setminus |I|}}^\bullet(M) \longrightarrow 0$$

as long as $C_{\Gamma_{I \setminus |I|}}^\bullet(M)$ is constructed. We define by induction the corresponding Γ_I -complex $C_{\Gamma_I}^\bullet(M)$ of a corresponding (φ_I, Γ_I) -module M over

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.57)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.58)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.59)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.60)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.61)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.62)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.63)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.64)$$

to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\Gamma_{I \setminus |I|}}^\bullet(M) \xrightarrow{\gamma_{|I|-1}} C_{\Gamma_{I \setminus |I|}}^\bullet(M) \longrightarrow 0$$

as long as $C_{\Gamma_{I \setminus |I}}^\bullet(M)$ is constructed. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.6.4. (After KPX, [KPX, Definition 2.3.3]) For any (φ_I, Γ_I) -module M over

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.65)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.66)$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.67)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.68)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.69)$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.70)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.71)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.72)$$

we define the corresponding complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. For any (φ_I, Γ_I) -module M over

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.73)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.74)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.75)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.76)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.77)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.78)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.79)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.80)$$

we define the corresponding complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.6.5. (After KPX, [KPX, Definition 2.3.3]) For any (ψ_I, Γ_I) -module M over

$$\Pi_{\text{an}, r_I, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.81)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.82)$$

$$\Pi_{\text{an}, r_I, I, J, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.83)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.84)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.85)$$

$$\Pi_{\text{an}, r_I, I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.86)$$

$$\Pi_{\text{an}, r_I, I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.87)$$

$$\Pi_{\text{an}, r_I, I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.88)$$

we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\psi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. For any (φ_I, Γ_I) -module M over

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.89)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.6.90)$$

$$\Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.91)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.92)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.6.93)$$

$$\Pi_{[s_I, r_I], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.94)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.6.95)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) \quad (5.6.96)$$

we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\psi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

5.7 Cohomologies of B_I -pairs and Mixed-Type Objects over Rigid Analytic Affinoids in Mixed-characteristic Case

5.7.1 Partial (φ_I, Γ_I) -Cohomology and Partial (ψ_I, Γ_I) -Cohomology

Now we define the corresponding cohomologies of the multivariate (φ_I, Γ_I) -modules over the following two groups of rings:

$$\varprojlim_{s_I} B_{\mathrm{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}} \quad (5.7.1)$$

$$(5.7.2)$$

with

$$\varprojlim_{s_I} B_{\mathrm{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A[t_1^{-1}, \dots, t_{I'}^{-1}], ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}}, \quad (5.7.3)$$

$$(5.7.4)$$

with

$$\varprojlim_{s_I} B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}}, \quad (5.7.5)$$

$$(5.7.6)$$

and

$$B_{\mathrm{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}} \quad (5.7.7)$$

with

$$B_{\mathrm{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A[t_1^{-1}, \dots, t_{I'}^{-1}], ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}} \quad (5.7.8)$$

with

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, \check{I \setminus J}}. \quad (5.7.9)$$

Definition 5.7.1. (After KPX, [KPX, Definition 2.3.3]) We define by induction the corresponding φ_I -complex $C_{\varphi_I}^\bullet$ of a corresponding $B_{I'} - (\varphi_I, \Gamma_I)$ -module M over the first three groups of rings to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, r_{|I|}}) \xrightarrow{\varphi_{|I|}^{-1}} C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, r_{|I|}/p}) \longrightarrow 0$$

as long as $C_{\varphi_{I \setminus |I|}}^\bullet(M)$ is constructed. We define by induction the corresponding φ_I -complex $C_{\varphi_I}^\bullet(M)$ of a corresponding $B_{I'} - (\varphi_I, \Gamma_I)$ -module M over the second three groups of rings to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, [s_{|I|}, r_{|I|}]}) \xrightarrow{\varphi_{|I|}^{-1}} C_{\varphi_{I \setminus |I|}}^\bullet(M_{\dots, [s_{|I|}, r_{|I|}/p]}) \longrightarrow 0$$

as long as $C_{\varphi_{I \setminus |I|}}^\bullet(M)$ is constructed. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.7.2. (After KPX, [KPX, Definition 2.3.3]) We define by induction the corresponding ψ_I -complex $C_{\psi_I}^\bullet$ of a corresponding $B_{I'} - (\varphi_I, \Gamma_I)$ -module M over the first three groups of rings to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, r_{|I|}}) \xrightarrow{\psi_{|I|}^{-1}} C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, pr_{|I|}}) \longrightarrow 0$$

as long as $C_{\psi_{I \setminus |I|}}^\bullet(M)$ is constructed. We define by induction the corresponding ψ_I -complex $C_{\psi_I}^\bullet(M)$ of a corresponding $B_{I'} - (\varphi_I, \Gamma_I)$ -module M over the second three groups of rings to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, [s_{|I|}, r_{|I|}]}) \xrightarrow{\psi_{|I|}^{-1}} C_{\psi_{I \setminus |I|}}^\bullet(M_{\dots, [ps_{|I|}, r_{|I|}]}) \longrightarrow 0$$

as long as $C_{\psi_{I \setminus |I|}}^\bullet(M)$ is constructed. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.7.3. (After KPX, [KPX, Definition 2.3.3]) We define by induction the corresponding Γ_I -complex $C_{\Gamma_I}^\bullet$ of a corresponding $B_{I'} - (\varphi_I, \Gamma_I)$ -module M over the first three groups of rings to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\Gamma_{I \setminus |I|}}^\bullet(M) \xrightarrow{\gamma_{|I|}^{-1}} C_{\Gamma_{I \setminus |I|}}^\bullet(M) \longrightarrow 0$$

as long as $C_{\Gamma_{I \setminus |I|}}^\bullet(M)$ is constructed. We define by induction the corresponding Γ_I -complex $C_{\Gamma_I}^\bullet(M)$ of a corresponding $B_{I'}(\varphi_I, \Gamma_I)$ -module M over the second three groups of rings to be the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\Gamma_{I \setminus |I|}}^\bullet(M) \xrightarrow{\gamma_{|I|-1}} C_{\Gamma_{I \setminus |I|}}^\bullet(M) \longrightarrow 0$$

as long as $C_{\Gamma_{I \setminus |I|}}^\bullet(M)$ is constructed. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.7.4. (After KPX, [KPX, Definition 2.3.3])

For any $B_{I'}(\varphi_I, \Gamma_I)$ -module M over the first three groups of rings we define the corresponding complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. For any (φ_I, Γ_I) -module M over the second three groups of rings we define the corresponding complex $C_{\varphi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\varphi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

Definition 5.7.5. (After KPX, [KPX, Definition 2.3.3])

For any $B_{I'}(\varphi_I, \Gamma_I)$ -module M over the first three groups of rings we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\psi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. For any $B_{I'}(\varphi_I, \Gamma_I)$ -module M over the second three groups of rings we define the corresponding complex $C_{\psi_I, \Gamma_I}^\bullet(M)$ to be the corresponding totalization of $C_{\psi_I}^\bullet C_{\Gamma_I}^\bullet(M)$. Here we assume $0 < s_\alpha \leq r_\alpha/p$ for each $\alpha \in I$.

5.7.2 Partial $B_{I'}$ -Cohomology

We now define the corresponding partial $B_{I'}$ -cohomology in the situation

Definition 5.7.6. (After Nakamura, [Nak1, Appendix 5]) For any

$B_{I'}(\varphi_I, \Gamma_I)$ -module M over

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}}, \quad (5.7.10)$$

$$(5.7.11)$$

with

$$\varprojlim_{s_I} B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A [t_1^{-1}, \dots, t_{I'}^{-1}], ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}}, \quad (5.7.12)$$

$$(5.7.13)$$

with

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?, A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus J}, \quad (5.7.14)$$

$$(5.7.15)$$

and

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.7.16)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.7.17)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, I \setminus J, A}, \quad (5.7.18)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.7.19)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.7.20)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, I \setminus J, A}, \quad (5.7.21)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}, \quad (5.7.22)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}, \quad (5.7.23)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A} \quad (5.7.24)$$

with

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.25)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.26)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.27)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.28)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.29)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, I \setminus J, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.30)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.31)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.32)$$

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus J}, A} [t_1^{-1}, \dots, t_{I'}^{-1}], \quad (5.7.33)$$

with

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \setminus J, A}, \quad (5.7.34)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}, \quad (5.7.35)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, I \setminus J, A}, \quad (5.7.36)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, I \check{\setminus} J, A}, \quad (5.7.37)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, I \check{\setminus} J, A}, \quad (5.7.38)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, I \check{\setminus} J, A}, \quad (5.7.39)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, J, \widetilde{I \setminus} J, A}, \quad (5.7.40)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus} J, A}, \quad (5.7.41)$$

$$B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, \widetilde{J}, \widetilde{I \setminus} J, A} \quad (5.7.42)$$

we define the corresponding $B_{I'}$ -complex $C_{B_{I'}}^\bullet(M)$ to be

$$0 \longrightarrow *_1 \times *_2 \xrightarrow{?-?'} *_3 \longrightarrow 0,$$

where

$$*_1 := C^\bullet(\mathrm{Gal}_{\mathbb{Q}_p, 1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p, I'}, M_{\mathrm{dR}}^+), \quad (5.7.43)$$

$$*_2 := C^\bullet(\mathrm{Gal}_{\mathbb{Q}_p, 1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p, I'}, M_e), \quad (5.7.44)$$

$$*_3 := C^\bullet(\mathrm{Gal}_{\mathbb{Q}_p, 1} \times \dots \times \mathrm{Gal}_{\mathbb{Q}_p, I'}, M_{\mathrm{dR}}). \quad (5.7.45)$$

Definition 5.7.7. Now we consider any $B_{I'}(\varphi_I, \Gamma_I)$ -module M over

$$\varprojlim_{s_I} B_{\mathrm{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus} J, I \check{\setminus} J} \quad (5.7.46)$$

$$(5.7.47)$$

with

$$\varprojlim_{s_I} B_{\mathrm{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, [t_1^{-1}, \dots, t_{I'}^{-1}], ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus} J, I \check{\setminus} J}, \quad (5.7.48)$$

$$(5.7.49)$$

with

$$\varprojlim_{s_I} B_{e,I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}}, \quad (5.7.50)$$

$$(5.7.51)$$

and

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}} \quad (5.7.52)$$

with

$$B_{\text{dR}, I'}^+ \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, [t_1^{-1}, \dots, t_{I'}^{-1}], ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}} \quad (5.7.53)$$

with

$$B_{e, I'} \widehat{\otimes} \Pi_{[s_I, r_I], I, ?, ?', A, ? = J, \widetilde{J}, \check{J}, ?' = I \setminus J, \widetilde{I \setminus J}, I \setminus \check{J}} \quad (5.7.54)$$

Then we can define the corresponding $C_{B_{I'}, \varphi_I, \Gamma_I}^\bullet$ -cohomology complex by taking the corresponding totalization of the corresponding double complex $C_{B_{I'}}^\bullet C_{\varphi_I, \Gamma_I}^\bullet(M)$.

5.8 The Results on the Cohomologies

5.8.1 Comparisons for $C_{\varphi_I, \Gamma_I}^\bullet, C_{\psi_I, \Gamma_I}^\bullet, C_{\psi_I}^\bullet$

We now consider the following categories for A a rigid affinoid over \mathbb{Q}_p :

A. The corresponding category of all the (φ_I, Γ_I) -modules over the corresponding rings carrying the corresponding cohomologies $C_{(\varphi_I, \Gamma_I)}^\bullet, C_{(\psi_I, \Gamma_I)}^\bullet, C_{\psi_I}^\bullet$ (for sufficiently small $r_{I,0}$):

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.8.1)$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, I \setminus J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.8.2)$$

$$\Pi_{\text{an}, r_{I,0}, I, J, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, J, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.8.3)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, \check{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.8.4)$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, \tilde{J}, I \check{\setminus} J, A}(\pi_{K_I}), \quad (5.8.5)$$

$$\Pi_{\text{an}, r_{I,0}, I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, J, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.8.6)$$

$$\Pi_{\text{an}, r_{I,0}, I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, \check{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}), \quad (5.8.7)$$

$$\Pi_{\text{an}, r_{I,0}, I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}) := \varprojlim_{s_I} \Pi_{[s_I, r_{I,0}], I, \tilde{J}, I \widetilde{\setminus} J, A}(\pi_{K_I}). \quad (5.8.8)$$

B. The corresponding category of all the (φ_I, Γ_I) -modules over the corresponding rings carrying the corresponding cohomologies $C_{(\varphi_I, \Gamma_I)}^\bullet, C_{(\psi_I, \Gamma_I)}^\bullet, C_{\psi_I}^\bullet$ (where $0 < s_\alpha \leq r_\alpha/p \leq r_{\alpha,0}$ for each $\alpha \in I$):

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.8.9)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus J, A}(\pi_{K_I}), \quad (5.8.10)$$

$$\Pi_{[s_I, r_I], I, J, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.8.11)$$

$$\Pi_{[s_I, r_I], I, \check{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.8.12)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, I \setminus \check{J}, A}(\pi_{K_I}), \quad (5.8.13)$$

$$\Pi_{[s_I, r_I], I, J, \widetilde{I \setminus J}, A}(\pi_{K_I}), \quad (5.8.14)$$

$$\Pi_{[s_I, r_I], I, \check{J}, \widetilde{I \setminus J}, A}(\pi_{K_I}), \quad (5.8.15)$$

$$\Pi_{[s_I, r_I], I, \tilde{J}, \widetilde{I \setminus J}, A}(\pi_{K_I}). \quad (5.8.16)$$

Theorem 5.8.1. *Let I be a set of two elements. Let M be some object over $\Pi_{\text{an}, r_I, 0, I, *, *, A}(\pi_{K_I})$ in the corresponding category A and let $M_{[s_I, r_I]}$ be the corresponding (under the horizontal equivalence of the categories for the rings of the same type) object over $\Pi_{[s_I, r_I], I, *, *, A}(\pi_{K_I})$ in the category B . Then we have the following quasi-isomorphisms:*

$$C_{\varphi_I, \Gamma_I}^\bullet(M) \xrightarrow{\sim} C_{\varphi_I, \Gamma_I}^\bullet(M_{[s_I, r_I]}), \quad (5.8.17)$$

$$C_{\psi_I, \Gamma_I}^\bullet(M) \xrightarrow{\sim} C_{\psi_I, \Gamma_I}^\bullet(M_{[s_I, r_I]}), \quad (5.8.18)$$

$$C_{\psi_I}^\bullet(M) \xrightarrow{\sim} C_{\psi_I}^\bullet(M_{[s_I, r_I]}), \quad (5.8.19)$$

$$(5.8.20)$$

in the bounded derived category of A -modules $D^b(A)$.

Proof. The proof relies on the following intermediate (φ_I, Γ_I) -modules. Let $I = \{1, 2\}$,

we now defined the following rings:

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,\check{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,\check{J},I\setminus J,A}(\pi_{K_I}), \quad (5.8.21)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,\tilde{J},I\setminus J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,\tilde{J},I\setminus J,A}(\pi_{K_I}), \quad (5.8.22)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,J,I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,J,I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.8.23)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,\check{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.8.24)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,\tilde{J},I\check{\setminus}J,A}(\pi_{K_I}), \quad (5.8.25)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,J,I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.8.26)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,\check{J},I\widetilde{\setminus}J,A}(\pi_{K_I}), \quad (5.8.27)$$

$$\Pi_{\text{an},r_{1,0},[s_2,r_2],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}) := \varprojlim_{s_1} \Pi_{[s_1,r_{1,0}] \times [s_2,r_2],I,\tilde{J},I\widetilde{\setminus}J,A}(\pi_{K_I}). \quad (5.8.28)$$

This category serves as a corresponding intermediate category which factors through the corresponding original equivalence on the categories involved. From A to this category we have the equivalence on the cohomology groups by considering the (φ_2, Γ_2) -module structure, then by regarding the corresponding (φ_2, Γ_2) -cohomology groups as Yoneda extension groups we have the isomorphism on (φ_2, Γ_2) -cohomology groups which further shows the equivalence on the full cohomology groups. Similarly one compares this category with the category B to finish. □

This chapter is based on the following paper, where the author of this dissertation is the main author:

- Tong, Xin. "Analytic Geometry and Hodge-Frobenius Structure Continued." arXiv preprint arXiv:2012.07336 (2020).

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Chapter 6

Category and Cohomology of Hodge-Iwasawa Modules

6.1 Introduction

6.1.1 The Main Scope of the Discussion

The corresponding Hodge-Iwasawa theory was studied in our papers [XT1] and [XT2]. We in the corresponding deformed setting discussed the corresponding relative p -adic Hodge theory after [KP], [KL1] and [KL2]. We call the theory partially Iwasawa in the corresponding sense that the corresponding deformation will reflect the corresponding Iwasawa theoretic information which was observed as in [KP]. On the Hodge-theoretic side, the corresponding cohomology in the deformed setting will also reflect the corresponding towers related to quotients (even nonabelian) of fundamental groups.

Therefore in our current consideration in this paper, we apply all we have developed to the corresponding categories and the corresponding cohomologies. For instance, very interesting application would be the corresponding pro-étale cohomology of general local systems over some smooth proper rigid analytic spaces as in [KL3]. Definitely we expect more applications in the corresponding Iwasawa theoretic consideration.

The other interesting things we would like to pursue in the application of our previously developed theory is the corresponding categorical and K -theoretic study on the corresponding (φ, Γ) -modules over very relative Robba rings. The corresponding categories should form some well-established abelian ones which could further have more derived enrichment (carrying some six functors).

The second goal of this paper is also to consider the Hodge-Structure on more general spaces. Although we will not systematically consider some general framework, since it will impossible to handle very general adic spaces. To be more general than the corresponding rigid analytic spaces, we are going to consider the corresponding spaces admitting atlas of the so-called k_Δ affinoid spaces as in [TC] and [DFN], where Δ is some valuation group larger than $|k^\times|$.

We briefly mention now the corresponding structures of the paper. In the second section we are going to study the corresponding Hodge-Iwasawa modules over logarithmic

towers which generalizes for instance in the mixed-characteristic case the corresponding consideration in [KL2, Chapter 7]. Then in the third section we study after that the corresponding categories and cohomologies of the corresponding Hodge-Iwasawa modules involved. Then we are going to study the corresponding cohomology of more general analytic spaces. In fourth section we consider the Hodge-Iwasawa module over rigid analytic spaces, k_Δ -analytic spaces and we contact the corresponding arithmetic Riemann-Hilbert correspondence after [LZ]. The fifth section is for the corresponding equivariant Iwasawa theory of de Rham (φ, Γ) -modules which is a equivariant version of the corresponding Iwasawa theory in [Nakamura1]. In such context, we still emphasize the corresponding geometrization aspect along [KP] on the reconstruction effect of the corresponding Iwasawa deformation in different contexts. Note that the story is equivariant and relative inspired by Nakamura [Nakamura3], Kedlaya-Pottharst [KP], as well as some further motivation from [BF1], [BF2] and [FK].

6.1.2 Results Involved

We searched many interesting directions in this paper to apply the Hodge-Iwasawa consideration in [XT1] and [XT2]. We present in some form in this current introduction part of this paper some results along the main body of this paper below. First motivation comes from directly following [KL2] where the relative p -adic Hodge theory is essentially applied in the corresponding context of rigid analytic spaces in the strictly situation. [KL2] proved many very deep results around the corresponding categories of pseudocoherent relative (φ, Γ) -modules, namely [KL2] proved that these are abelian categories although working in Banach categories does cause essential difficulties. In our situation we seek the same goal when we deform the corresponding Frobenius modules in arithmetic family. In the context and notation in proposition 6.4.3 we have:

Proposition 6.1.1. (After Kedlaya-Liu, [KL2, Theorem 8.10.6]) *Working over period ring $\tilde{\Pi}_{X,A}$, we have the corresponding category of (φ^a, Γ) -modules is abelian, which is basically compatible with the corresponding category of all the sheaves of modules over $\tilde{\Pi}_{X,A}$ when one would like to form the corresponding kernels and cokernels. A is assumed to be sousperfectoid as those considered in [KH].*

Then following Temkin's notes [DFN, Part I Chapter I] we consider the corresponding analytic spaces which could admit an atlas consisting of all the k_Δ -affinoids namely those quotients of k_Δ strictly rational localizations of Tate algebras. Over such space X :

Proposition 6.1.2. (After Kedlaya-Liu, [KL2, Theorem 8.10.6]) *Working over period ring $\tilde{\Pi}_{X,A}$, we have the corresponding category of (φ^a, Γ) -modules is abelian, which is basically compatible with the corresponding category of all the sheaves of modules over $\tilde{\Pi}_{X,A}$ when one would like to form the corresponding kernel and cokernel. Here A is \mathbb{Q}_p .*

Proposition 6.1.3. (After Kedlaya-Liu, [KL2, Theorem 8.10.6]) *Working over period ring $\tilde{\Pi}_{X,A}$, we have the corresponding category of (φ^a, Γ) -modules is abelian, which is basically compatible with the corresponding category of all the sheaves of modules over $\tilde{\Pi}_{X,A}$ when one would like to form the corresponding kernel and cokernel. Here X is assumed to be smooth. A is assumed to be sousperfectoid as those considered in [KH].*

Then we consider arithmetic family version of Higgs bundles after [LZ] where a version of Riemann-Hilbert correspondence in the arithmetic setting is achieved. The idea in our situation will be definitely the corresponding version of such correspondence for general A -relative (φ, Γ) -modules M in the 'geometric' setting. We also consider the corresponding equivalent version of B -pairs in the A -relative setting after [KL2]. With the notation in later discussion (see proposition 6.4.50 and proposition 6.4.44) we have for a de Rham A -relative (φ, Γ) -module M or equivalently a de Rham A -relative B -pair M :

Proposition 6.1.4. (After Kedlaya-Liu, [KL3, below Definition 10.10]) *Over a general rigid analytic space X , consider a de Rham A -relative (φ, Γ) -modules M or equivalently a de Rham A -relative B -pair M . We then have that the corresponding each higher de Rham derived cohomology group $D_{\text{dR}}^i(M)$ for each $i \geq 0$ is coherent sheaf over X . The projectivity could be achieved if we assume that X is smooth. A is assumed to be sousperfectoid as those considered in [KH].*

Proposition 6.1.5. (After Kedlaya-Liu, [KL3, below Definition 10.10]) *Over a general rigid analytic space X , consider a de Rham A -relative (φ, Γ) -modules M or equivalently a de Rham A -relative B -pair M . We then have that the corresponding each higher*

de Rham derived cohomology group $E_{\text{dR}}^i(M)$ for each $i \geq 0$ is coherent sheaf over X . The projectivity could be achieved if we assume that X is smooth. A is assumed to be sousperfectoid as those considered in [KH].

The other corresponding application we want to search for while again along some idea proposed in [KP] applications in equivariant Iwasawa theory. We choose to consider equivariant version of the Iwasawa theory of de Rham (φ, Γ) -module M .

Remark 6.1.6. We believe (while motivated by [KP]) our idea should eventually produce more than cyclotomic consideration by deforming Berger's $(\varphi, \nabla, \Gamma)$ -modules over pro-étale site which could absorb the Γ -action (which could be reconstructed through the Iwasawa deformation).

Definition 6.1.7.

(After Perrin-Riou and Nakamura, [Nakamura1, Definition 3.7]) With the notation in section 6.5.1 especially the corresponding abelian Fréchet-Stein space we are working on, we have the following derived big Perrin-Riou-Nakamura Exponential map:

$$\text{Exp}_{\mathcal{F}(M), m}^{\Pi^\infty(\Gamma'), \bullet} : R^\bullet \Gamma_{\text{sheaf}}(\varprojlim \mathcal{F}(\Theta_{\text{Ber}, \text{dif}}(M))_p) \rightarrow R^\bullet \Gamma_{\text{sheaf}}(\varprojlim \mathcal{F}(M)_p),$$

which is corresponding to the one for (φ, Γ) -modules:

$$\text{Exp}_{M, m}^{\Pi^\infty(\Gamma'), \bullet} : R^\bullet \Gamma_{\varphi, \Gamma}(\varprojlim \Theta_{\text{Ber}, \text{dif}}(M)_p) \rightarrow R^\bullet \Gamma_{\varphi, \Gamma}(\varprojlim M_p).$$

We actually conjectured that the corresponding this equivariant Perrin-Riou-Nakamura map in algebraic way produces the equivariant p -adic L -functions (in some not necessarily explicit way) which relate directly to the corresponding characteristic ideals. See section 6.5.2.

6.1.3 Future Study

There are many interesting possible extensions of the current paper. We would like to study more in the future, although here we mainly emphasize the corresponding Iwasawa theoretic aspects. The first scope of topics we would like to study further is the

relative p -adic Hodge theory in rigid family and the arithmetic Riemann-Hilbert correspondence in rigid family for B -pairs after [LZ], [TT] and [Shi], along the corresponding foundation we established here.

The second scope of the topics we would like to study in the future is to extend the corresponding discussion to more general analytic spaces. In fact here we do not go beyond [KL2, Chapter 8], although we touched the corresponding k_Δ -analytic spaces as those considered in Temkin's lecture notes [DFN, Part I Chapter I]. Namely we still restrict ourselves to the corresponding varieties covered by strictly or non-strictly affinoids.

The Iwasawa consideration we considered here is actually also worthwhile to be amplified and to be noncommutativized. Certainly we want to study noncommutative deformation of the sheaves in the future not only in all kinds of Iwasawa theories, but also we would like to study the noncommutative deformation of the corresponding the relative p -adic Hodge theory in rigid family and the arithmetic Riemann-Hilbert correspondence in rigid family for B -pairs after [LZ], [TT] and [Shi] in the sense of many existing noncommutative geometric contexts.

6.2 Logarithmic Hodge-Iwasawa Structures in Mixed-Characteristic Case

6.2.1 Γ -Modules over Ramified Towers

Setting 6.2.1. Let E be \mathbb{Q}_p . We use the corresponding π to denote the corresponding uniformizer in a uniform notation. Let A be an affinoid algebra over the corresponding field E in our current situation defined through some strict quotient from the corresponding Tate algebras.

We first consider the corresponding standard ones for (P, P^+) a perfectoid adic Banach uniform pair over \mathcal{O}_E , containing the perfectoid field $\mathbb{Q}_p(\zeta_{p^\infty})^\wedge$.

Setting 6.2.2. In the mixed characteristic situation, we consider the setting in the following fashion:

$$H_0 = P\{x_1/t_1, \dots, x_k/t_k, t_1/y_1, \dots, t_\ell/y_\ell\}, \quad (6.2.1)$$

$$H_0^+ = P^+\{x_1/t_1, \dots, x_k/t_k, t_1/y_1, \dots, t_\ell/y_\ell\}, \quad (6.2.2)$$

$$H_n = P\{(x_1/t_1)^{1/p^n}, \dots, (x_k/t_k)^{1/p^n}, (t_1/y_1)^{1/p^n}, \dots, (t_\ell/y_\ell)^{1/p^n}\}, \quad (6.2.3)$$

$$n = 0, 1, \dots, 0 < k \leq \ell. \quad (6.2.4)$$

And here the corresponding quantities $x_i, y_j, i = 1, \dots, k, j = 1, \dots, \ell$ could be allowed to be real numbers.

Lemma 6.2.3. (Kedlaya-Liu [KL2, Lemma 7.3.3]) *The corresponding standard ramified toric towers defined as above are weakly decompleting.*

Proof. See [KL2, Lemma 7.3.3]. □

The corresponding tower is not finite étale, so the corresponding machinery in [KL2, Chapter 5] needs to be modified, which is already done in the corresponding context in [KL2, Chapter 7]. As in [KL2, Chapter 7, Section 3, Section 4] one can also consider direct semilinear Γ -action and define the corresponding Γ -modules to be certain ones carrying such semilinear action.

When we in some situations have the corresponding topological group Γ (just as in the corresponding logarithmic setting we are considering here) we can simply consider the corresponding definition as in [KL2, Definition 7.3.5]:

Definition 6.2.4. (After Kedlaya-Liu [KL2, Definition 7.3.5]) In our current situation, we consider the corresponding action of the corresponding group Γ in our situation, namely the ℓ -fold product of the corresponding additive group \mathbb{Z}_p on the corresponding period rings taking the form of $*_{H,A}$. We now define the corresponding Γ -modules in the finite projective, pseudocoherent or fpd setting to the corresponding modules over the corresponding period rings as above carrying the corresponding action coming from the group Γ which is assumed to as in [KL2, Definition 7.3.5] particularly semilinear.

Remark 6.2.5. Again the deformation only happens over the rings in mixed-characteristic situation.

Proposition 6.2.6. (Kedlaya-Liu [KL2, Lemma 7.3.6]) *Let $\Gamma_n := p^n \Gamma$ for $n \geq 0$, then we have in our situation the corresponding result that the corresponding complex $C^\bullet(\Gamma_n, \bar{\varphi}^{-1} R_H / R_H)$ is strict exact.*

Proof. See [KL2, Lemma 7.3.6]. □

Proposition 6.2.7. (Kedlaya-Liu [KL2, Lemma 5.6.4]) *Let $\Gamma_n := p^n \Gamma$ for $n \geq 0$, then we have in our situation the corresponding result that the corresponding complex $C^\bullet(\Gamma_n, \bar{R}_H / R_H)$ is strict exact.*

Proof. See [KL2, Lemma 5.6.4]. □

Lemma 6.2.8. (After Kedlaya-Liu [KL2, Corollary 5.6.5]) *We need to fix some suitable $r_0 > 0$ as in [KL2, Corollary 5.6.5] and for the corresponding radii s, r in $(0, r_0]$ we will have the corresponding complexes $C^\bullet(\Gamma_n, \varphi^{-1} \Pi_H^{\text{int},r/p} / \Pi_H^{\text{int},r})_A$ and $C^\bullet(\Gamma_n, \tilde{\Pi}_H^{\text{int},r} / \Pi_H^{\text{int},r})_A$ are strict exact. Moreover we have that the corresponding complexes $C^\bullet(\Gamma_n, \varphi^{-1} \Pi_H^{[s/p,r/p]} / \Pi_H^{[s,r]})_A$ and $C^\bullet(\Gamma_n, \tilde{\Pi}_H^{[s/p,r/p]} / \Pi_H^{[s,r]})_A$ are then in our situation strict exact as well.*

Proof. See [KL2, Corollary 5.6.5]. □

Lemma 6.2.9. (After Kedlaya-Liu [KL2, Lemma 5.6.6]) *Let M be a Γ -module defined over the period rings involved below. Then one can find a sufficiently large integer $\ell^* \geq 0$ such that we have that for any $\ell \geq \ell^*$, the following complexes are correspondingly*

strict exact:

$$M \otimes_{R_H} C^\bullet(\Gamma_n, \overline{\varphi}^{-\ell-1} R_H / \overline{\varphi}^{-\ell} R_H), M \otimes_{R_H} C^\bullet(\Gamma_n, \overline{R}_H / \overline{\varphi}^{-\ell} R_H), \quad (6.2.5)$$

$$M \otimes_{\Pi_{H,A}^{\text{int},r}} C^\bullet(\Gamma_n, \varphi^{-(\ell+1)} \Pi_H^{\text{int},r/p^{\ell+1}} / \varphi^{-\ell} \Pi_H^{\text{int},r/p^\ell})_A, M \otimes_{\Pi_H^{\text{int},r}} C^\bullet(\Gamma_n, \widetilde{\Pi}_H^{\text{int},r} / \varphi^{-\ell} \Pi_H^{\text{int},r/p^\ell})_A \quad (6.2.6)$$

$$M \otimes_{\Pi_{H,A}} C^\bullet(\Gamma_n, \varphi^{-(\ell+1)} \Pi_H^{[s/p^{\ell+1}, r/p^{\ell+1}]} / \varphi^{-\ell} \Pi_H^{[s/p^\ell, r/p^\ell]})_A, \quad (6.2.7)$$

$$M \otimes_{\Pi_H^{\text{int},r}} C^\bullet(\Gamma_n, \widetilde{\Pi}_H / \varphi^{-\ell} \Pi_H^{[s/p^\ell, r/p^\ell]})_A. \quad (6.2.8)$$

Proof. This is the corresponding corollary of proposition 6.2.6, proposition 6.2.7 and lemma 6.2.8, along the corresponding argument in [KL2, Lemma 5.6.6]. \square

Theorem 6.2.10. (After Kedlaya-Liu [KL2, Lemma 5.6.9]) *The corresponding base change functor from the ring \check{R}_H to \widetilde{R}_H is in our situation an equivalence between the corresponding Γ -modules. The corresponding base change functor from the ring $\check{\Pi}_{H,A}^{\text{int}}$ to $\widetilde{\Pi}_{H,A}^{\text{int}}$ is in our situation an equivalence between the corresponding Γ -modules. The corresponding base change functor from the ring $\check{\Pi}_{H,A}$ to $\widetilde{\Pi}_{H,A}$ is in our situation an equivalence between the corresponding Γ -modules.*

Proof. We briefly mention the corresponding argument since this is just analog of the corresponding argument for the proof of the corresponding result [KL2, Lemma 5.6.9]. The corresponding fully faithfulness comes from the section 6.2.1. Then we apply the corresponding [KL2, Lemma 5.6.8] to descend the corresponding modules over the bigger rings with accent $\check{*}$ to the base rings after some Frobenius action. Then the corresponding differentials on the original modules over the bigger rings could be made have norms just a small quantity twist away from the corresponding differentials on the corresponding base changes from the corresponding base rings after Frobenius twists. Then to finish we apply the corresponding to modify this quantity to finish the proof as in [KL2, Lemma 5.6.9]. \square

For the relative logarithmic toric towers, we consider the corresponding following

mixed characteristic situation. We consider the following tower (H, H^+) :

$$H_0 = \mathbb{Q}_p\{x_1/t_1, \dots, x_k/t_k, t_1/y_1, \dots, t_\ell/y_\ell\}, \quad (6.2.9)$$

$$H_0^+ = H_0^\circ, \quad (6.2.10)$$

$$H_n = \mathbb{Q}_p(\zeta_{p^n})\{(x_1/t_1)^{1/p^n}, \dots, (x_k/t_k)^{1/p^n}, (t_1/y_1)^{1/p^n}, \dots, (t_k/y_k)^{1/p^n}\}, n = 0, 1, \dots \quad (6.2.11)$$

The corresponding parameters x_i, y_i are in $p^\mathbb{Q}$, and $\ell \geq k \geq 0$. The corresponding tower is again not Galois. Again when we in some situations have the corresponding topological group Γ (just as in the corresponding logarithmic setting we are considering here) we can simply consider the corresponding definition as in [KL2, Definition 7.3.5]:

Definition 6.2.11. (After Kedlaya-Liu [KL2, Definition 7.3.5]) In our current situation, we consider the corresponding action of the corresponding group Γ in our situation, namely the semidirect product taking the form of $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^\ell$ on the corresponding period rings taking the form of $*_{H,A}$. We now define the corresponding Γ -modules in the finite projective, pseudocoherent or fpd setting to the corresponding modules over the corresponding period rings as above carrying the corresponding action coming from the group Γ which is assumed to as in [KL2, Definition 7.3.5] particularly semilinear.

Proposition 6.2.12. (After Kedlaya-Liu [KL2, Lemma 7.3.6]) *Let $\Gamma_n := p^n\Gamma$ for $n \geq 0$, then we have in our situation the corresponding result that the corresponding complex $C^\bullet(\Gamma_n, \overline{\varphi}^{-1}R_H/R_H)$ is strict exact.*

Proof. See proposition 6.2.6. □

Proposition 6.2.13. (After Kedlaya-Liu [KL2, Lemma 5.6.4]) *Let $\Gamma_n := p^n\Gamma$ for $n \geq 0$, then we have in our situation the corresponding result that the corresponding complex $C^\bullet(\Gamma_n, \overline{R}_H/R_H)$ is strict exact.*

Proof. See proposition 6.2.7. □

Lemma 6.2.14. (After Kedlaya-Liu [KL2, Corollary 5.6.5]) *We need to fix some suitable $r_0 > 0$ as in [KL2, Corollary 5.6.5] and for the corresponding radii s, r in $(0, r_0]$ we will have the corresponding complexes $C^\bullet(\Gamma_n, \varphi^{-1}\Pi_H^{\text{int},r/p}/\Pi_H^{\text{int},r})_A$ and $C^\bullet(\Gamma_n, \widetilde{\Pi}_H^{\text{int},r}/\Pi_H^{\text{int},r})_A$ are strict exact. Moreover we have that the corresponding complexes $C^\bullet(\Gamma_n, \varphi^{-1}\Pi_H^{[s/p,r/p]}/\Pi_H^{[s,r]})_A$ and $C^\bullet(\Gamma_n, \widetilde{\Pi}_H^{[s/p,r/p]}/\Pi_H^{[s,r]})_A$ are then in our situation strict exact as well.*

Proof. See lemma 6.2.8. □

Lemma 6.2.15. (After Kedlaya-Liu [KL2, Lemma 5.6.6]) *Let M be a Γ -module defined over the period rings involved below. Then one can find a sufficiently large integer $\ell^* \geq 0$ such that we have that for any $\ell \geq \ell^*$, the following complexes are correspondingly strict exact:*

$$M \otimes_{R_H} C^\bullet(\Gamma_n, \overline{\varphi}^{-\ell-1} R_H / \overline{\varphi}^{-\ell} R_H), M \otimes_{R_H} C^\bullet(\Gamma_n, \overline{R}_H / \overline{\varphi}^{-\ell} R_H), \quad (6.2.12)$$

$$M \otimes_{\Pi_{H,A}^{\text{int},r}} C^\bullet(\Gamma_n, \varphi^{-(\ell+1)} \Pi_H^{\text{int},r/p^{\ell+1}} / \varphi^{-\ell} \Pi_H^{\text{int},r/p^\ell})_A, M \otimes_{\Pi_H^{\text{int},r}} C^\bullet(\Gamma_n, \tilde{\Pi}_H^{\text{int},r} / \varphi^{-\ell} \Pi_H^{\text{int},r/p^\ell})_A \quad (6.2.13)$$

$$M \otimes_{\Pi_{H,A}} C^\bullet(\Gamma_n, \varphi^{-(\ell+1)} \Pi_H^{[s/p^{\ell+1}, r/p^{\ell+1}]} / \varphi^{-\ell} \Pi_H^{[s/p^\ell, r/p^\ell]})_A, \quad (6.2.14)$$

$$M \otimes_{\Pi_H^{\text{int},r}} C^\bullet(\Gamma_n, \tilde{\Pi}_H / \varphi^{-\ell} \Pi_H^{[s/p^\ell, r/p^\ell]})_A. \quad (6.2.15)$$

Proof. See lemma 6.2.9. □

Theorem 6.2.16. (After Kedlaya-Liu [KL2, Lemma 5.6.9]) *The corresponding base change functor from the ring \check{R}_H to \tilde{R}_H is in our situation an equivalence between the corresponding Γ -modules. The corresponding base change functor from the ring $\check{\Pi}_{H,A}^{\text{int}}$ to $\tilde{\Pi}_{H,A}^{\text{int}}$ is in our situation an equivalence between the corresponding Γ -modules. The corresponding base change functor from the ring $\check{\Pi}_{H,A}$ to $\tilde{\Pi}_{H,A}$ is in our situation an equivalence between the corresponding Γ -modules.*

Proof. See theorem 6.2.10. □

Remark 6.2.17. The corresponding statements for the mixed characteristic version of the relative log towers are established in [KL2, Chapter 7.3, Chapter 7.4] without touching the corresponding large coefficient A here, therefore for more detailed discussion see [KL2, Chapter 7.3, Chapter 7.4]. We should mention that the corresponding [KL2, Chapter 7.3, Chapter 7.4] target at the corresponding finite projective objects, essentially in the generality above we should come across essentially similar difficulties to possible ones in [KL2] when one would like to consider the pseudocoherent sheaves. Things in our current section are basically the complementary discussion to the discussion we made in [XT2] for the unramified towers.

6.2.2 (φ, Γ) -Modules

Now we study the corresponding Γ -modules carrying further Frobenius action, but we will again mainly focus on the corresponding finite projective objects (see remark 6.2.17), which is parallel to the corresponding consideration in [KL2]. First recall the following definition:

Setting 6.2.18. (After Kedlaya-Liu [KL2, Definition 7.5.3]) We will consider the corresponding Kummer étale topology and the corresponding pro-Kummer étale topology considered in [KL2, Definition 7.5.3]. Let (H, H^+) be either the standard ramified toric tower or the relative ramified toric tower we considered as above. Here we recall that a covering of the adic space $\mathrm{Spa}(H_0, H_0^+)$ is called étale if it is so after considering the corresponding pullback along for some integer $v \geq 0$ the map $T_1 \mapsto T_1^{p^{-v}}, \dots, T_\ell \mapsto T_\ell^{p^{-v}}$. And we recall that a covering of the adic space $\mathrm{Spa}(H_0, H_0^+)$ is called finite étale if it is so after considering the corresponding pullback along for some integer $v \geq 0$ the map $T_1 \mapsto T_1^{p^{-v}}, \dots, T_\ell \mapsto T_\ell^{p^{-v}}$. And we recall that a covering of the adic space $\mathrm{Spa}(H_0, H_0^+)$ is called faithfully finite étale if it is so after considering the corresponding pullback along for some integer $v \geq 0$ the map $T_1 \mapsto T_1^{p^{-v}}, \dots, T_\ell \mapsto T_\ell^{p^{-v}}$.

Definition 6.2.19. (After Kedlaya-Liu [KL2, Definition 5.7.2]) Consider the corresponding framework in the previous subsection. Consider the corresponding period rings in our situation (with the corresponding notations in [XT1] and [XT2]):

$$\Omega_{H,A}^{\mathrm{int}}, \check{\Omega}_{H,A}^{\mathrm{int}}, \widehat{\Omega}_{H,A}^{\mathrm{int}}, \widetilde{\Omega}_{H,A}^{\mathrm{int}}, \quad (6.2.16)$$

$$\Pi_{H,A}^{\mathrm{int}}, \check{\Pi}_{H,A}^{\mathrm{int}}, \widehat{\Pi}_{H,A}^{\mathrm{int}}, \widetilde{\Pi}_{H,A}^{\mathrm{int}}, \quad (6.2.17)$$

$$\Pi_{H,A}^{\mathrm{bd}}, \check{\Pi}_{H,A}^{\mathrm{bd}}, \widehat{\Pi}_{H,A}^{\mathrm{bd}}, \widetilde{\Pi}_{H,A}^{\mathrm{bd}}, \quad (6.2.18)$$

$$\Pi_{H,A}, \check{\Pi}_{H,A}, \widetilde{\Pi}_{H,A}, \quad (6.2.19)$$

we have the corresponding notions of finite projective or pseudocoherent or finite projective dimension Γ -modules, where we can then add the corresponding semilinear actions as in [KL2, Definition 5.7.2] to define the corresponding (φ, Γ) -modules satisfying the corresponding condition on the corresponding Frobenius pullbacks. Then furthermore we can define the same objects for those rings having some radius $r > 0$ of interval $[s, r] \in (0, \infty)$ as in [KL2, Definition 5.7.2], again satisfying the corresponding Frobenius

pullback condition taking the general form:

$$\varphi^* \Delta \otimes_{*_{H,A}^{r/p}} *_{H,A}^{r/p} \xrightarrow{\sim} \Delta \otimes *_{H,A}^{r/p}, \quad (6.2.20)$$

$$\varphi^* \Delta \otimes_{*_{H,A}^{[s/p, r/p]}} *_{H,A}^{[s, r/p]} \xrightarrow{\sim} \Delta \otimes_{*_{H,A}^{[s, r]}} *_{H,A}^{[s, r/p]}. \quad (6.2.21)$$

Proposition 6.2.20. (After Kedlaya-Liu [KL2, Theorem 5.7.4]) *Consider the following period rings in the rational setting:*

$$\Omega_H, \check{\Omega}_H, \widehat{\Omega}_H, \widetilde{\Omega}_H, \quad (6.2.22)$$

$$\Pi_H^{\text{bd}}, \check{\Pi}_H^{\text{bd}}, \widehat{\Pi}_H^{\text{bd}}, \widetilde{\Pi}_H^{\text{bd}}, \quad (6.2.23)$$

$$\Pi_H, \check{\Pi}_H, \widetilde{\Pi}_H. \quad (6.2.24)$$

Then the corresponding rational finite projective modules over these rings carrying the corresponding (φ, Γ) -structures. For the deformed we look at the corresponding rings in the following:

$$\Pi_{H,A}, \check{\Pi}_{H,A}, \widetilde{\Pi}_{H,A}. \quad (6.2.25)$$

Then we have the corresponding parallel results as well.

Proof. See [KL2, Theorem 5.7.4]. □

Remark 6.2.21. One can certainly compare the corresponding (φ, Γ) -modules above with the corresponding Frobenius sheaves over the Kummer pro-étale topology in setting 6.2.18 as in [KL2] by adding additional effective conditions on the objects.

6.3 Hodge-Iwasawa Modules

6.3.1 The Categories of Hodge-Iwasawa Modules

We recall the corresponding Hodge-Iwasawa modules we defined in the previous papers [XT1] and [XT2]. We use our notations in [XT2] for the corresponding tower and the corresponding period rings.

Setting 6.3.1. We use the notation (H, H^+) to denote the corresponding tower in our situation. Here we assume that (H, H^+) is finite étale, noetherian, weakly-decompleting, decompleting, Galois with the Galois topological group Γ . We also then follow the corresponding setting of [KL2, Hypothesis 3.1.1]. And fix some $a > 0$ positive integer.

Setting 6.3.2. Recall that we have the corresponding period rings in the following, defined in [KL1] and [KL2]:

$$\tilde{\Omega}_H^{\text{int}}, \tilde{\Omega}_H, \tilde{\Pi}_H^{\text{int},r}, \tilde{\Pi}_H^{\text{int}}, \tilde{\Pi}_H^{\text{bd},r}, \tilde{\Pi}_H^{\text{bd}}, \tilde{\Pi}_H^r, \tilde{\Pi}_H^I, \tilde{\Pi}_H^\infty, \tilde{\Pi}_H$$

form the corresponding group of perfect period rings along the tower (H, H^+) , here $r > 0$ is some radius while I corresponds to some closed interval. Then as in [KL2, Definition 5.2.1] we have various imperfection of the corresponding rings listed above. We mainly need the corresponding Robba rings in the following fashion:

$$\check{\Pi}_H^r, \check{\Pi}_H^I, \check{\Pi}_H^\infty, \check{\Pi}_H,$$

and

$$\Pi_H^r, \Pi_H^I, \Pi_H^\infty, \Pi_H.$$

Setting 6.3.3. The Hodge-Iwasawa modules we considered and studied are basically over the corresponding deformed version of the corresponding period rings we recalled above over some affinoid algebra A over the base field E as in [XT2]:

$$\check{\Pi}_{H,A}^r, \check{\Pi}_{H,A}^I, \check{\Pi}_{H,A}^\infty, \check{\Pi}_{H,A},$$

and

$$\Pi_{H,A}^r, \Pi_{H,A}^I, \Pi_{H,A}^\infty, \Pi_{H,A},$$

and

$$\tilde{\Pi}_{H,A}^r, \tilde{\Pi}_{H,A}^I, \tilde{\Pi}_{H,A}^\infty, \tilde{\Pi}_{H,A}.$$

Recall that we have the following corresponding key categories involved in our study ([XT2]). First we consider the corresponding finite projective objects for any suitable s, r such that $0 \leq s \leq r/p^{ah}$ (for any A such that the corresponding rings building the corresponding Fargues-Fontaine curve namely the perfect Robba rings with respect to some intervals are sheafy as adic rings):

A. Look at the corresponding space $\mathrm{Spa}(H_0, H_0^+)$ which is the corresponding base of the tower, then we have the corresponding category of all the Frobenius sheaves over the pro-étale site of this space over the sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}$. Here the Frobenius will be chosen in our situation to be φ^a ;

B. We look at the same space as in the previous item, and we look at the corresponding category of the Frobenius bundles over the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}$;

C. We look at the same base space and then the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}^{[s, r]}$ for $s \leq r/p^{ah}$, we then have the category of the corresponding finite projective Frobenius modules over the period rings in this setting;

D. We then have the corresponding finite projective modules over the period ring $\widetilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

E. We then have the corresponding finite projective bundles over the period ring $\widetilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

F. We then have the corresponding finite projective modules over the period ring $\widetilde{\Pi}_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;

G. We then have the corresponding finite projective modules over the period ring $\Pi_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

H. We then have the corresponding finite projective bundles over the period ring $\Pi_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

I. We then have the corresponding finite projective modules over the period ring $\Pi_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;

J. We then have the corresponding finite projective modules over the period ring $\check{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

K. We then have the corresponding finite projective bundles over the period ring $\check{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

L. We then have the corresponding finite projective modules over the period ring $\check{\Pi}_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;

M. We have the corresponding adic version of the corresponding Fargues-Fontaine curve $FF_{\overline{H}_\infty, A}$. And we can consider the corresponding category of Γ -equivariant quasi-coherent locally free sheaves over this adic version of Fargues-Fontaine curve, and we assume the corresponding Γ -equivariant action to be semilinear and continuous over each Γ -equivariant affinoid subspace.

Remark 6.3.4. One has to be careful when one would like to define the objects in A, B, C which could be compared to the other objects. One needs to consider the corresponding locally finite projective objects instead of locally finite free objects.

Then we have:

Proposition 6.3.5.

(See [XT2, Proposition 5.44], after Kedlaya-Liu [KL2, Theorem 5.7.4]) *The categories described above are equivalent to each other.*

Then we consider the corresponding pseudocoherent objects, where we further assume that the corresponding ring R_H is now F -(finite projective) (for any A such that the corresponding rings building the corresponding Fargues-Fontaine curve namely the perfect Robba rings with respect to some intervals are sheafy as adic rings).

A. Look at the corresponding space $\mathrm{Spa}(H_0, H_0^+)$ which is the corresponding base of the tower, then we have the corresponding category of all the Frobenius sheaves over the pro-étale site of this space over the sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}$. Here the Frobenius will be chosen in our situation to be φ^a ;

B. We look at the same space as in the previous item, and we look at the corresponding category of the Frobenius bundles over the corresponding sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}$;

C. We look at the same base space and then the corresponding sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}^{[s, r]}$ for $s \leq r/p^{ah}$, we then have the category of the corresponding pseudocoherent Frobenius modules over the period rings in this setting;

D. We then have the corresponding pseudocoherent modules over the period ring $\tilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

E. We then have the corresponding pseudocoherent bundles over the period ring $\tilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

F. We then have the corresponding pseudocoherent modules over the period ring $\tilde{\Pi}_{H, A}^{[s, r]}$,

carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;

G. We then have the corresponding pseudocoherent modules over the period ring $\Pi_{H,A}$, carrying the corresponding action coming from (φ^a, Γ) ;

H. We then have the corresponding pseudocoherent bundles over the period ring $\Pi_{H,A}$, carrying the corresponding action coming from (φ^a, Γ) ;

I. We then have the corresponding pseudocoherent modules over the period ring $\Pi_{H,A}^{[s,r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;

J. We then have the corresponding pseudocoherent modules over the period ring $\check{\Pi}_{H,A}$, carrying the corresponding action coming from (φ^a, Γ) ;

K. We then have the corresponding pseudocoherent bundles over the period ring $\check{\Pi}_{H,A}$, carrying the corresponding action coming from (φ^a, Γ) ;

L. We then have the corresponding pseudocoherent modules over the period ring $\check{\Pi}_{H,A}^{[s,r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;

M. We have the corresponding adic version of the corresponding Fargues-Fontaine curve $FF_{\overline{H}_\infty, A}$. And we can consider the corresponding category of pseudocoherent sheaves over this adic version of Fargues-Fontaine curve.

Remark 6.3.6. One has to be careful when one would like to define the objects in A, B, C which could be compared to the other objects. One needs to consider the corresponding locally pseudocoherent objects instead of locally pseudocoherent objects. Moreover in this current setting locally we have to impose p -adic functional analytic conditions on the pseudocoherent objects when one would like to do the localization in different topologies.

Then we have:

Proposition 6.3.7.

(See [XT2, Proposition 5.51], after Kedlaya-Liu [KL2, Theorem 5.9.4]) *The categories described above are equivalent to each other.*

Proposition 6.3.8. (After Kedlaya-Liu [KL2, Theorem 5.9.4]) *The categories above are abelian.*

Proof. By our assumption the corresponding ring $\check{\Pi}_{H,A}^{[s,r]}$ is coherent. □

At least in the corresponding finite projective situation, we also have some comparison on the corresponding objects in the logarithmic setting:

Proposition 6.3.9. (After Kedlaya-Liu [KL2, Lemma 5.4.11]) *Let (H, H^+) be the one of the corresponding logarithmic towers we encountered in the previous section, we then have the following categories are basically equivalent:*

A. *Look at the corresponding space $\mathrm{Spa}(H_0, H_0^+)$ which is the corresponding base of the tower, then we have the corresponding category of all the Frobenius sheaves over the pro-étale site of this space over the sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{prokét}, A}}$. Here the Frobenius will be chosen in our situation to be φ^a ;*

B. *We look at the same space as in the previous item, and we look at the corresponding category of the Frobenius bundles over the corresponding sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{prokét}, A}}$;*

C. *We look at the same base space and then the corresponding sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{prokét}, A}}^{[s, r]}$ for $s \leq r/p^{ah}$, we then have the category of the corresponding finite projective Frobenius modules over the period rings in this setting.*

Proof. This is because the corresponding log perfectoid affinoids form a basis of the corresponding pro-Kummer étale site (see the corresponding development in [DLLZ2, Chapter 5.3]). □

Proposition 6.3.10. (After Kedlaya-Liu [KL2, Theorem 5.7.4]) *Let (H, H^+) be the one of the corresponding logarithmic towers we encountered in the previous section. Then we have the following categories of Hodge-Iwasawa modules are equivalent:*

D. *We then have the corresponding finite projective modules over the period ring $\tilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;*

E. *We then have the corresponding finite projective bundles over the period ring $\tilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;*

F. *We then have the corresponding finite projective modules over the period ring $\tilde{\Pi}_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;*

G. *We then have the corresponding finite projective modules over the period ring $\Pi_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;*

H. *We then have the corresponding finite projective bundles over the period ring $\Pi_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;*

I. *We then have the corresponding finite projective modules over the period ring $\Pi_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$;*

- J. We then have the corresponding finite projective modules over the period ring $\check{\Pi}_{H,A}$, carrying the corresponding action coming from (φ^a, Γ) ;
- K. We then have the corresponding finite projective bundles over the period ring $\check{\Pi}_{H,A}$, carrying the corresponding action coming from (φ^a, Γ) ;
- L. We then have the corresponding finite projective modules over the period ring $\check{\Pi}_{H,A}^{[s,r]}$, carrying the corresponding action coming from (φ^a, Γ) , $0 \leq s \leq r/p^{ah}$.

Proof. The vertical comparison along changing the corresponding ring with respect to the accents was established in proposition 6.2.20. Then to compare the corresponding objects in the remaining cases see [KL2, Theorem 5.7.4] and [XT2, Proposition 5.44]. \square

In Iwasawa theory, the corresponding aspects of Hodge-Iwasawa theory will usually happen over some Fréchet-Stein algebras. Here we will use some geometric language which is different from the situation we considered in our previous work. We now first consider the corresponding commutative setting. We use the notation X to denote a general Stein space:

$$X := \bigcup_n X_n$$

regarded as a general quasi-Stein adic space in the corresponding context of [KL2, Chapter 2.6]. Then we have the corresponding sheaf will be organized in the following sense:

$$\mathcal{O}_X := \varprojlim_n \mathcal{O}_{X_n}. \quad (6.3.1)$$

Therefore we will correspondingly consider the period rings over the corresponding spaces taking the form of X as above. We assume each \mathcal{O}_{X_n} satisfies the corresponding assumption as that for the ring A as discussed above.

Definition 6.3.11. For any ring R (commutative) which could be written as the corresponding inverse system $\varprojlim_n R_n$. We call the projective system of modules over R as a projective system $\{\mathcal{M}_n\}_n$ of all modules M_n over R_n for each $n \geq 0$.

A. Look at the corresponding space $\mathrm{Spa}(H_0, H_0^+)$ which is the corresponding base of the tower, then we have the corresponding category of all the projective systems of the Frobenius sheaves over the pro-étale site of this space over the sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t.}}} \varprojlim_n \mathcal{O}_{X_n}$.

Here the Frobenius will be chosen in our situation to be φ^a ;

B. We look at the same space as in the previous item, and we look at the corresponding category of all the projective systems of the Frobenius bundles over the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}}, \varprojlim_n \mathcal{O}_{X_n}}$;

C. We look at the same base space and then the corresponding sheaf of ring

$\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}}, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$ for $s \leq r/p^{ah}$, we then have the category of the projective systems of corresponding finite projective Frobenius modules over the period rings in this setting;

D. We then have the corresponding projective systems of finite projective modules over the period ring $\widetilde{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;

E. We then have the corresponding projective systems of finite projective bundles over the period ring $\widetilde{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;

F. We then have the corresponding projective systems of finite projective modules over the period ring $\widetilde{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;

G. We then have the corresponding projective systems of finite projective modules over the period ring $\Pi_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;

H. We then have the corresponding projective systems of finite projective bundles over the period ring $\Pi_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;

I. We then have the corresponding projective systems of finite projective modules over the period ring $\Pi_{H, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;

J. We then have the corresponding projective systems of finite projective modules over the period ring $\check{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;

K. We then have the corresponding projective systems of finite projective bundles over the period ring $\check{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;

L. We then have the corresponding projective systems of finite projective modules over the period ring $\check{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;

M. We have the corresponding adic version of the corresponding Fargues-Fontaine curve $FF_{\overline{H}_\infty, \varprojlim_n \mathcal{O}_{X_n}}$. And we can consider the corresponding category of the projective systems of Γ -equivariant quasi-coherent locally free sheaves over this adic version of Fargues-Fontaine curve, and we assume the corresponding Γ -equivariant action to be semilinear and continuous over each Γ -equivariant affinoid subspace.

Then we have:

Proposition 6.3.12.

(See [XT2, Theorem 4.11], after Kedlaya-Liu [KL2, Theorem 5.7.4]) *The categories described above are equivalent to each other.*

For the pseudocoherent setting we have the following categories:

- A. Look at the corresponding space $\mathrm{Spa}(H_0, H_0^+)$ which is the corresponding base of the tower, then we have the corresponding category of all the projective systems of the pseudocoherent Frobenius sheaves over the pro-étale site of this space over the sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}}, \varprojlim_n \mathcal{O}_{X_n}}$. Here the Frobenius will be chosen in our situation to be φ^a ;
- B. We look at the same space as in the previous item, and we look at the corresponding category of all the projective systems of the pseudocoherent Frobenius bundles over the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}}, \varprojlim_n \mathcal{O}_{X_n}}$;
- C. We look at the same base space and then the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}}, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$ for $s \leq r/p^{ah}$, we then have the category of the projective systems of corresponding pseudocoherent Frobenius modules over the period rings in this setting;
- D. We then have the corresponding projective systems of pseudocoherent modules over the period ring $\widetilde{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;
- E. We then have the corresponding projective systems of pseudocoherent bundles over the period ring $\widetilde{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;
- F. We then have the corresponding projective systems of pseudocoherent modules over the period ring $\widetilde{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;
- G. We then have the corresponding projective systems of pseudocoherent modules over the period ring $\Pi_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;
- H. We then have the corresponding projective systems of pseudocoherent bundles over the period ring $\Pi_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;
- I. We then have the corresponding projective systems of pseudocoherent modules over the period ring $\Pi_{H, \varprojlim_n \mathcal{O}_{X_n}}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;
- J. We then have the corresponding projective systems of pseudocoherent modules over the period ring $\check{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;
- K. We then have the corresponding projective systems of pseudocoherent bundles over the period ring $\check{\Pi}_{H, \varprojlim_n \mathcal{O}_{X_n}}$, carrying the corresponding action coming from (φ^a, Γ) ;
- L. We then have the corresponding projective systems of pseudocoherent modules over

the period ring $\check{\Pi}_{H, \varprojlim_n}^{[s,r]} \mathcal{O}_{X_n}$, carrying the corresponding action coming from (φ^a, Γ) ;

M. We have the corresponding adic version of the corresponding Fargues-Fontaine curve $FF_{\overline{H}_\infty, \varprojlim_n} \mathcal{O}_{X_n}$. And we can consider the corresponding category of the projective systems of Γ -equivariant pseudocoherent sheaves over this adic version of Fargues-Fontaine curve, and we assume the corresponding Γ -equivariant action to be semilinear and continuous over each Γ -equivariant affinoid subspace.

Then we have:

Proposition 6.3.13.

(See [XT2, Theorem 4.11], after Kedlaya-Liu [KL2, Theorem 5.9.6]) *The categories described above are equivalent to each other. And they are actually abelian.*

6.3.2 The Cohomologies of Hodge-Iwasawa Modules

We then define the corresponding Herr style cohomology in our current deformed setting for the corresponding. Note that the corresponding cohomology over the corresponding Fréchet-Stein algebras are very important in the study of Iwasawa conjectures and the corresponding p -adic Tamagawa number conjectures.

First keep the following setting:

Setting 6.3.14. We use the notation (H, H^+) to denote the corresponding tower in our situation. Here we assume that (H, H^+) is finite étale, noetherian, weakly-decompleting, decompleting, Galois with the Galois topological group Γ . We also then follow the corresponding setting of [KL2, Hypothesis 3.1.1]. And fix some $a > 0$ positive integer.

A. Look at the corresponding space $\mathrm{Spa}(H_0, H_0^+)$ which is the corresponding base of the tower, then we have the corresponding category of all the Frobenius sheaves over the pro-étale site of this space over the sheaf of ring $\tilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}$. Here the Frobenius will be chosen in our situation to be φ^a ;

Definition 6.3.15. (After Kedlaya-Liu, [KL2, Definition 4.4.4]) For the category in the above (and the corresponding setting in pseudocoherent situation) we define the

corresponding complex $C_\varphi^\bullet(M)$ of any object \mathcal{M} as the corresponding hypercomplex of the following complex for just the Frobenius operator:

$$0 \longrightarrow \mathcal{M} \xrightarrow{\varphi^a - 1} \mathcal{M} \longrightarrow 0.$$

B. We look at the same space as in the previous item, and we look at the corresponding category of the Frobenius bundles over the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}$;

C. We look at the same base space and then the corresponding sheaf of ring $\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}^{[s, r]}$ for $s \leq r/p^{ah}$, we then have the category of the corresponding finite projective Frobenius modules over the period rings in this setting;

Definition 6.3.16. (After Kedlaya-Liu, [KL2, Definition 4.4.4]) For the three categories in the above (and the corresponding setting in pseudocoherent situation) we define the corresponding complex $C_\varphi^\bullet(M)$ of any object \mathcal{M} as the corresponding hypercomplex of the following complex for just the Frobenius operator:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}^{[s, r]}} \widetilde{\Pi}_{\mathrm{Spa}(H_0, H_0^+)_{\mathrm{pro\acute{e}t}, A}}^{[s, r/p^{ah}]} \longrightarrow 0.$$

D. We then have the corresponding finite projective modules over the period ring $\widetilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

E. We then have the corresponding finite projective bundles over the period ring $\widetilde{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

F. We then have the corresponding finite projective modules over the period ring $\widetilde{\Pi}_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;

G. We then have the corresponding finite projective modules over the period ring $\Pi_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

H. We then have the corresponding finite projective bundles over the period ring $\Pi_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

I. We then have the corresponding finite projective modules over the period ring $\Pi_{H, A}^{[s, r]}$, carrying the corresponding action coming from (φ^a, Γ) ;

J. We then have the corresponding finite projective modules over the period ring $\check{\Pi}_{H, A}$, carrying the corresponding action coming from (φ^a, Γ) ;

K. We then have the corresponding finite projective bundles over the period ring $\check{\Pi}_{H, A}$,

carrying the corresponding action coming from (φ^a, Γ) ;

L. We then have the corresponding finite projective modules over the period ring $\check{\Pi}_{H,A}^{[s,r]}$, carrying the corresponding action coming from (φ^a, Γ) ;

Definition 6.3.17. (After Kedlaya-Liu, [KL2, Definition 5.7.9]) For the categories (except for the corresponding bundles) in the above (and the corresponding setting in pseudocoherent situation) we define the corresponding complex $C_{\varphi, \Gamma}^{\bullet}(\mathcal{M})$ of any object \mathcal{M} as the corresponding totalization of the following complex:

$$0 \longrightarrow C_{\Gamma}^{\bullet}(\mathcal{M}) \xrightarrow{\varphi^a - 1} C_{\Gamma}^{\bullet}(\mathcal{M}) \longrightarrow 0.$$

or

$$0 \longrightarrow C_{\Gamma}^{\bullet}(\mathcal{M}) \xrightarrow{\varphi^a - 1} C_{\Gamma}^{\bullet}(\mathcal{M}) \otimes \Delta \longrightarrow 0,$$

respectively for different situations in the above where $\Delta \in \{\check{\Pi}_{H,A}^{[s,r/p^{ah}]}, \Pi_{H,A}^{[s,r/p^{ah}]}, \tilde{\Pi}_{H,A}^{[s,r/p^{ah}]}\}$.

M. We have the corresponding adic version of the corresponding Fargues-Fontaine curve $FF_{\overline{H}_{\infty}, A}$. And we can consider the corresponding category of Γ -equivariant quasi-coherent locally free sheaves over this adic version of Fargues-Fontaine curve (with the deformation from the algebra A), and we assume the corresponding Γ -equivariant action to be semilinear and continuous over each Γ -equivariant affinoid subspace.

Definition 6.3.18. (After Kedlaya-Liu, [KL2]) For the categories in the above we define the corresponding complex $C_{\Gamma}^{\bullet}(\mathcal{M})$ of any object \mathcal{M} as the corresponding hypercomplex of the complex $C_{\Gamma}^{\bullet}(\mathcal{M})$. In pseudocoherent situation, we have the parallel definition.

Remark 6.3.19. Although we will not basically repeat the corresponding construction of the cohomology as above, but all the definitions on the corresponding cohomologies could be translated to the corresponding construction for the corresponding logarithmic setting and the corresponding Fréchet-Stein context we considered seriously above.

Based on the corresponding comparison theorem above we can actually consider the corresponding comparison between the corresponding cohomologies. The corresponding statements will take the following forms:

Proposition 6.3.20. (After Kedlaya-Liu, [KL2, Theorem 5.7.10]) *Suppose we have a corresponding finite projective module \mathcal{M} over $\Pi_{H,A}$ carrying the corresponding action of (φ^a, Γ) , and we use the notation $\check{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ to be the corresponding base change of \mathcal{M} from the original base ring to the ring being one of $\check{\Pi}_{H,A}$ and $\widetilde{\Pi}_{H,A}$ respectively. Then we have the following three complexes are mutually quasi-isomorphic:*

$$C_{\varphi, \Gamma}^{\bullet}(\mathcal{M}), C_{\varphi, \Gamma}^{\bullet}(\check{\mathcal{M}}), C_{\varphi, \Gamma}^{\bullet}(\widetilde{\mathcal{M}}). \quad (6.3.2)$$

Proposition 6.3.21. (After Kedlaya-Liu, [KL2, Theorem 5.7.10]) *Suppose we have a corresponding finite projective module \mathcal{M} over $\Pi_{H, \varprojlim_n} \mathcal{O}_{X_n}$ carrying the corresponding action of (φ^a, Γ) , and we use the notation $\check{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ to be the corresponding base change of \mathcal{M} from the original base ring to the ring being one of $\check{\Pi}_{H, \varprojlim_n} \mathcal{O}_{X_n}$ and $\widetilde{\Pi}_{H, \varprojlim_n} \mathcal{O}_{X_n}$ respectively. Then we have the following three complexes are mutually quasi-isomorphic:*

$$C_{\varphi, \Gamma}^{\bullet}(\mathcal{M}), C_{\varphi, \Gamma}^{\bullet}(\check{\mathcal{M}}), C_{\varphi, \Gamma}^{\bullet}(\widetilde{\mathcal{M}}). \quad (6.3.3)$$

As in [KL2] starting from a corresponding finite projective module carrying the corresponding (φ^a, Γ) -structure, one can compare the corresponding cohomologies of modules over the full Robba rings with those of those modules over some Robba rings with respect to some intervals.

Proposition 6.3.22. (After Kedlaya-Liu, [KL2, Theorem 5.7.11]) *Descend \mathcal{M} which is a finite projective module over $\Pi_{H,A}$ carrying the action of (φ^a, Γ) to the corresponding Robba ring $\Pi_{H,A}^{r_0}$ for some specific positive radius $r_0 > 0$. Then as in the previous proposition we consider the corresponding base change of the module \mathcal{M} to the corresponding Robba rings $\check{\Pi}_{H,A}^{[s,r]}$ for $0 < s \leq r \leq r_0$ with that $s \leq r/p^{ah}$, and to the corresponding Robba rings $\widetilde{\Pi}_{H,A}^{[s,r]}$. We denote these two modules by $\check{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$. Then for sufficiently large integer $m \geq 0$ suppose we consider the following three complexes:*

$$\begin{aligned} 0 &\longrightarrow C_{\Gamma}^{\bullet}(\check{\mathcal{M}}_{[s,r]}) \xrightarrow{\varphi^a-1} C_{\Gamma}^{\bullet}(\check{\mathcal{M}}_{[s,r/p^{ah}]}) \longrightarrow 0, \\ 0 &\longrightarrow C_{\Gamma}^{\bullet}(\widetilde{\mathcal{M}}_{[s,r]}) \xrightarrow{\varphi^a-1} C_{\Gamma}^{\bullet}(\widetilde{\mathcal{M}}_{[s,r/p^{ah}]}) \longrightarrow 0, \\ 0 &\longrightarrow \varphi^{-am} C_{\Gamma}^{\bullet}(\mathcal{M}_{[s/p^{ahm}, r/p^{ahm}]}) \xrightarrow{\varphi^a-1} \varphi^{-am} C_{\Gamma}^{\bullet}(\mathcal{M}_{[s/p^{ahm}, r/p^{ah(m+1)}]}) \longrightarrow 0. \end{aligned}$$

Then we have that these three complexes and the corresponding complex $C_{\varphi, \Gamma}^{\bullet}(\mathcal{M})$ are quasi-isomorphic to each other.

Proof. See [KL2, Theorem 5.7.11]. □

Remark 6.3.23. We want to mention here that actually the corresponding context should work in general for the tower which is finite étale but not actually Galois. Although in our previous work [XT2] we only considered the corresponding towers which are required to be Galois with respect to some Galois topological group Γ .

6.4 Applications to General Analytic Spaces

6.4.1 Contact with Rigid Analytic Spaces

We now consider the corresponding context of more global geometric objects, over smooth proper rigid analytic spaces. We certainly will consider very general coefficients namely the corresponding affinoid algebras in the same framework. The two considerations happen to carry the same context coming from the corresponding rigid analytic spaces, but more strictly speaking they will interact in certain situations such as when we look at some nonabelian quotient of the profinite fundamental groups.

We basically consider the following context, where X will be a corresponding smooth proper rigid analytic space over some analytic field K . Here we assume that the corresponding analytic field K is containing the p -adic number field \mathbb{Q}_p . The affinoid algebra A over \mathbb{Q}_p in the following discussion is assumed to be sousperfectoid as those considered in [KH]. Then we can consider the following category.

Definition 6.4.1. (After Kedlaya-Liu, [KL2, Definition 8.10.1]) We use the corresponding notation $D_{\text{pseudo}, \tilde{\Pi}_{X,A}^{[s,r]}}$ ($0 < s \leq r/p^a$) to denote the corresponding category of all the pseudocoherent $\tilde{\Pi}_{X,A}^{[s,r]}$ sheaves, which is regarded as a subcategory of the corresponding category of all the corresponding sheaves over $\tilde{\Pi}_{X,A}^{[s,r]}$, in the corresponding pro-étale topology.

Also we have the following category of all the (φ^a, Γ) over the space X as in the above:

Definition 6.4.2. (After Kedlaya-Liu, [KL2, Definition 8.10.5]) We use the corresponding notation $D_{\text{pseudo}, \varphi^a, \tilde{\Pi}_{X,A}}$ to denote the corresponding category of all the pseudocoherent $\tilde{\Pi}_{X,A}$ sheaves carrying the corresponding Frobenius operator φ^a , which is regarded as a subcategory of the corresponding category of all the corresponding sheaves over $\tilde{\Pi}_{X,A}$, in the corresponding pro-étale topology.

Proposition 6.4.3. (After Kedlaya-Liu, [KL2, Theorem 8.10.6]) *The corresponding category $D_{\text{pseudo}, \varphi^a, \tilde{\Pi}_{X,A}}$ is abelian, where the corresponding kernels and cokernels are compatible with the corresponding category of all the sheaves of $\tilde{\Pi}_{X,A}$ -modules over the corresponding pro-étale site.*

Proof. Locally the corresponding object in this category will be φ^a -sheaf in the corresponding category $D_{\text{pseudo}, \varphi^a, \tilde{\Pi}_{X_0, A}}$ where X_0 is a smooth affinoid subspace. We then deduce the corresponding Γ -action will be realized by the corresponding Galois group of the corresponding toric tower. Then by the comparison we established in the above, we can have the chance to directly compare this to the corresponding (φ^a, Γ) -module over the ring $\tilde{\Pi}_{H, A}$ for some restricted tower H , which implies the corresponding result since this ring is coherent. \square

6.4.2 Contact with k_Δ -Analytic Spaces

In this section we are going to consider the corresponding k_Δ -analytic space in the sense of [DFN, Part I Chapter I]. The corresponding Δ was denoted by H in [DFN, Part I Chapter I], namely a commutative multiplication group satisfying the corresponding relation:

$$|k^\times| \subset H \subset \mathbb{R}_+^\times.$$

Here for k we assume that this is an analytic field of characteristic 0. Then we are going to use the notation Δ to denote a corresponding commutative multiplicative group such that we have the following:

$$|k^\times| \subset \Delta \subset \mathbb{R}_+^\times.$$

Recall from [DFN, Part I Chapter I.3] we have the corresponding notation of k_Δ -affinoid algebras and k_Δ -affinoid spaces. These are basically the corresponding non-strictly generalization of the corresponding strictly affinoid algebras and the corresponding spectrum. For instance the corresponding k_Δ -affinoid algebras are defined to be those taking the corresponding form coming from the admissible quotient from some algebra taking the form of:

$$k\{T_1/x_1, \dots, T_d/x_d\}$$

for $x_i \in \Delta$ and $i = 1, \dots, d$. Then we have the corresponding notion from this of the affinoid spaces from some suitable geometric point of view, we will use the language of adic spaces as in [KL2]. So we define:

Definition 6.4.4. (After Temkin, [DFN, Part I Chapter I.3]) We define the corresponding k_Δ -affinoid spaces as the corresponding adic space associated some k_Δ -affinoid algebra R . Note that this is endowed with the structure sheaf $\mathcal{O}_{\text{Spa}(R, R^+)}$.

Then as in [DFN, Part I Chapter I] we can define the corresponding general k_Δ -analytic spaces as in the following:

Definition 6.4.5. (After Temkin, [DFN, Part I Chapter I.3]) We define a general k_Δ -analytic space to be an adic space X which could be covered by an atlas forming by the corresponding k_Δ -analytic affinoid spaces.

We would like to study the corresponding properties of the corresponding sheaves over a general k_Δ -analytic space X as in [KL2]. And certainly eventually we contact with the corresponding Hodge-Iwasawa theory. But we would like to further make some further more detailed discussion around the current generalized context.

Remark 6.4.6. We should mention that we reproduce many key arguments following [KL2, Chapter 8] in the following presentation of the parallel results to [KL2, Chapter 8]. However, if one really wants to work with some larger base field (by taking the valuation group even up to \mathbb{R}_+^\times), then one should be able to simplified many arguments below.

Remark 6.4.7. The corresponding desingularization result from Temkin (as in the context of [KL2, Remark 8.1.2]) in our situation namely we can desingularize a corresponding k_Δ -affinoid algebra A by the corresponding excellence of this algebra as in [DFN, Part I Chapter I Fact 3.1.2.1]. Namely in the category of schemes we can choose a map with smooth domain $Y \rightarrow \text{Spec}A$ which will have the corresponding analytification in the categories of all the k_Δ -affinoid algebras, again with smooth domain.

Then we consider the corresponding correspondence with the Ax-Sen-Tate style result in our context by using the corresponding pro-étale site. So we use the notation $f_{\text{proét}}$ to denote the corresponding morphism $f_{\text{proét}} : X_{\text{proét}} \rightarrow X$.

Proposition 6.4.8. (After Kedlaya-Liu, [KL2, Theorem 8.2.3]) *Suppose we have that the k_Δ -analytic space X is seminormal. Then we have the corresponding isomorphism $\mathcal{O}_X \xrightarrow{\sim} f_{\text{proét},*} \widehat{\mathcal{O}}_X$. On the other hand this isomorphism will also imply that the corresponding space X is essentially seminormal.*

Proof. We follow the proof of [KL2, Theorem 8.2.3], which proves the smooth situation first and then proves the corresponding general cases by considering the corresponding desingularization. If k is perfectoid and the space X is smooth and affinoid, which as in

[KL2, Theorem 8.2.3] could be directly proved by relating to the corresponding restricted toric towers, then the corresponding result follows from [KL2, Theorem 8.2.3]. Then to remove the corresponding current assumption on the perfectoidness we just consider the corresponding perfectoidization k_1 of k and the corresponding uniform completion of $k_2 := k_1 \otimes_k k_1$, and we consider the base changes of X (affinoid) to the corresponding k_1 and k_2 to form the following short exact sequence:

$$0 \longrightarrow X \longrightarrow X \otimes k_1 \longrightarrow X \otimes k_2.$$

Then as in [KL2, Theorem 8.2.3] one further reduces to the corresponding situation around the fields. Finally to consider the general space X we consider the corresponding desingularization as in remark 6.4.7 to consider some smooth space Y which maps under h say to X , then by considering the corresponding pullbacks of perfectoid subdomain one can basically up to such desingularization compare to have the desired isomorphism by the smooth case as above. After this one considers the fact that seminormality implies that we have the isomorphism $\mathcal{O}_X \xrightarrow{\sim} h_* \mathcal{O}_Y$, which finishes the proof in this case. Then conversely if we have the corresponding isomorphism in the statement of the proposition, then we choose the corresponding seminormalization X^{seminor} of X which then finishes the corresponding proof as in [KL2, Theorem 8.2.3].

□

Proposition 6.4.9. (After Kedlaya-Liu, [KL2, Corollary 8.2.4]) *When we consider the corresponding vector bundle V over X which is assumed to be seminormal, we will have then the corresponding isomorphism:*

$$V \xrightarrow{\sim} f_{\text{proét},*} f_{\text{proét}}^* V.$$

Proof. From the previous proposition. □

Proposition 6.4.10. (After Kedlaya-Liu, [KL2, Lemma 8.2.7]) *Let X be seminormal, and consider any coherent subsheaf of any locally finite free \mathcal{O}_X -module V , which is denoted by W . We then have the following identity:*

$$W = f_{\text{proét},*} \text{Image}(f_{\text{proét}}^* W \rightarrow f_{\text{proét}}^* V).$$

Proof. As in [KL2, Lemma 8.2.7], we basically realize the corresponding coherent subsheaf as the kernel of a map taking the form of $V \rightarrow V'$. Then we have that:

$$0 \longrightarrow \text{Image}(f_{\text{proét}}^* W \rightarrow f_{\text{proét}}^* V) \longrightarrow f_{\text{proét}}^* V \longrightarrow f_{\text{proét}}^* V'.$$

is exact sequence. Then the previous proposition directly show the corresponding identity after taking the corresponding pushforward along the canonical morphism for the corresponding pro-étale site of X . \square

Proposition 6.4.11. (After Kedlaya-Liu, [KL2, Lemma 8.2.8]) *Now we consider the following correspondence. Consider a corresponding locally finite free \mathcal{O}_X -module V over X which is assumed to be seminormal, and also consider the corresponding locally finite free \mathcal{O}_X -sheaf $f_{\text{proét}}^* V$. Then the corresponding functor $W \rightarrow \text{Image}(f_{\text{proét}}^* W \rightarrow f_{\text{proét}}^* V)$ will establish an equivalence between the coherent subsheaves of V and the coherent subsheaves of its pullbacks.*

Proof. One can finish the proof as in the proof of [KL2, Lemma 8.2.8]. To recall the argument, we briefly present the proof for the readers. The corresponding fully faithfulness comes from the previous proposition. For the essential surjectivity as in [KL2, Lemma 8.2.8] we first consider the corresponding smooth case, which could be then reduced to the [KL2, Lemma 8.2.6] by relating to the corresponding restricted toric tower. In general the argument is by considering any decreasing sequence of ideal of \mathcal{O}_X whose pullback will basically contain some given ideal B in $\widehat{\mathcal{O}}_X$. We denote this sequence by $\{A_i\}_i$. Then for each A_i choose some finite free C_i which maps to A_i , and one can basically consider the corresponding inverse image of B in the pullback of C_i which could be denoted by B_i , which corresponds to by the previous smooth case a coherent subsheaf of the C_i . Now considering further pushforwarding this into $\widehat{\mathcal{O}}_X$ we will have B_{i+1} . Now we have if $A_{i+1} = A_i$ for some $i \geq 1$ then we have that $B = B_i$. However this will eventually be the case by the corresponding noetherian property. \square

Proposition 6.4.12. (After Kedlaya-Liu)

([KL2, Proposition 8.2.9, Corollary 8.2.10, Corollary 8.2.11]) *A. When we have that X is seminormal, we will have that the corresponding functor $f_{\text{proét}}^*$ establishes embedding from the category of coherent sheaves over \mathcal{O}_X to the category of pseudocoherent sheaves over $\widehat{\mathcal{O}}_X$, which is fully faithful and exact. B. And for any coherent sheaf F*

over any analytic k_Δ -space, then the tensoring with F will then establish a functor on the category of all the pseudocoherent sheaves over $\widehat{\mathcal{O}}_X$ which is an endofunctor, here X could be allowed to be more general by considering the corresponding seminormalization. *C.* Over X which is again assumed to be seminormal, then we have the injective map $f_{\text{proét},*}^* V \hookrightarrow V$ from any locally finite free sheaf V over $\widehat{\mathcal{O}}_X$.

Proof. See [KL2, Proposition 8.2.9, Corollary 8.2.10, Corollary 8.2.11]. □

Then building on the basic facts from [KL2, Lemma 8.2.12, Corollary 8.2.15] on the corresponding restricted toric towers, we can now study the corresponding categories of the corresponding \mathcal{O}_X -sheaves.

Setting 6.4.13. We now assume we are working over $X = \text{Spa}(L, L^+)$ which is seminormal and affinoid. And consider (H_\bullet, H_\bullet^+) which is assumed to be a perfectoid and finite étale tower over the base X . Take any ideal $l \subset L$ we can form the corresponding quotient $\overline{L} := L/JL$ and $\overline{\Omega} := \overline{H}_\infty/J\overline{H}_\infty$.

We first have the following:

Proposition 6.4.14. (After Kedlaya-Liu, [KL2, Lemma 8.3.3]) *The map $L \rightarrow \overline{H}_\infty$ is faithfully flat.*

Proof. See [KL2, Lemma 8.3.3]. □

Proposition 6.4.15. (After Kedlaya-Liu, [KL2, Lemma 8.3.4]) *Γ -modules over the quotient $\overline{\Omega}$ satisfy the following property. The finitely generatedness of any M of such corresponding modules as such could be promoted to be finite projective one if one works over $\overline{\Omega}_e$ with M_e , where e exists as an element in L which is not mapped to zero divisors in the quotients \overline{L} and $\overline{\Omega}$.*

Proof. The corresponding proof is basically parallel to [KL2]. To achieve so we consider the corresponding construction as in the following. First we consider the corresponding minimal ideals of L containing the prescribed ideal l . Then for each of such ideal \mathfrak{p} (finitely many) we look at the Fitting ideal Fitt_k no contained in this prime (for minimized such k), choose any element in $\text{Fitt}_k - \mathfrak{p}$, then this will do the job that the corresponding base change of M to $(L/l)_{e_{\mathfrak{p}}}$ will be finite projective the reasonable quotient. Then to integrate all the information along all such minimal primes we consider for each pair $\mathfrak{p} \neq \mathfrak{q}$ and choose any element $e_{\mathfrak{p} \neq \mathfrak{q}}$. Finally just put e to be $\sum_{\mathfrak{p}} e_{\mathfrak{p}} \prod_{\mathfrak{p} \neq \mathfrak{q}} e_{\mathfrak{p} \neq \mathfrak{q}}$. □

The following will be crucial in the understanding of the corresponding category of all the $\widehat{\mathcal{O}}_X$ -sheaves over some k_Δ -analytic space X .

Proposition 6.4.16. (After Kedlaya-Liu, [KL2, Proposition 8.3.5]) *Suppose over the ring \overline{H}_∞ , we have a finite module which is carrying additionally the structure of Γ -module. Denote this by M . Then this is automatically pseudocoherent and more importantly étale-stably pseudocoherent.*

Proof. This is a nontrivial result around the Γ -modules. We recall the argument in [KL2, Proposition 8.3.5] in the following, which will prove our result in our current generalized context. First the ideal is using the corresponding noetherian reduction along some prescribed ideal $l \subset L$. As in [KL2, Proposition 8.3.5] we reduce ourselves to the situation where \overline{L} is reduced and connected. To perform the argument, one first chooses some element $e \in L$ as in the previous proposition such that the statement of the previous proposition hold. Then consider all the corresponding e -torsions over L which is denoted by C . After applying the corresponding induction hypothesis we have for M/C that the corresponding $M/C/eM/C$ is then pseudocoherent (not known to be strictly or so on). And consider the fact that $(M/C)_e = M_e$ and $M/C[e]$ is trivial, we have then as in [KL2, Proposition 8.3.5, Lemma 1.5.8] that M_C is then pseudocoherent (again not known to be strictly or so on). But then this implies the torsion part is then finitely generated, which is killed by some power of e up to some power h for instance. Then repeat considering using the corresponding induction hypothesis we can show that the whole module M is then pseudocoherent (again not known to be strictly or so on) by considering the quotient $e^{x-1}C/e^xC$ for $x = 0, \dots, h$. Then to consider the corresponding strictly pseudocoherence, we play with then the kernel along some finite free cover of M , which is assumed to be linear over the ring L . Then consider kernel and complete the kernel, then apply the corresponding previous paragraph to show that the quotient of M by this kernel is pseudocoherent, which by [KL2, Corollary 1.2.11] shows that this kernel is actually complete initially, which shows that the corresponding strictly pseudocoherent (not known to be étale-stably pseudocoherent). To finish relate the étale map out of the period ring to some step in the tower, which shows by the previous argument above we are done. \square

We now choose to consider the corresponding parallel discussion as above but for the Robba ring taking the form of $\widetilde{\Pi}_H$ and the corresponding related Robba rings. The

order of our discussion is different from [KL2], where we will study the corresponding categories of pseudocoherent sheaves over $\widehat{\mathcal{O}}_X$ and $\widetilde{\Pi}_X$ together after we finish more concrete local study. Consider at this moment X which is affinoid defined over a perfectoid field k , and consider (H_\bullet, H_\bullet^+) a finite étale perfectoid tower as in [KL2].

Proposition 6.4.17. (After Kedlaya-Liu, [KL2, Lemma 8.7.3]) *Consider the ring $\widetilde{\Pi}_H^{[s,r]}$ and consider any Γ -stable nonzero ideal of this ring. Then we have that this is basically containing some nonzero element in $\widetilde{\Pi}_H$.*

Proof. Since over a corresponding restricted toric tower this is just the same as in the proof of [KL2, Lemma 8.7.3] we will not repeat it. And then one can follow the strategy to tackle more general situation by looking at the corresponding $\widetilde{\Pi}_X^{[s,r]}$ -sheaf of module associated to the quotient of $\widetilde{\Pi}_H^{[s,r]}$ by the ideal given. Then by applying the previous case one can finish the proof. \square

Proposition 6.4.18. (After Kedlaya-Liu, [KL2, Corollary 8.7.4]) *Consider the ring $\widetilde{\Pi}_H^{[s,r]}$ and consider any Γ -module which is now assumed to be finite generated over this ring. Then we can find an element such that inverting this element over the base ring the module will be finite projective.*

Proof. See [KL2, Corollary 8.7.4]. \square

Proposition 6.4.19. (After Kedlaya-Liu, [KL2, Proposition 8.8.9]) *Under the same condition as in [KL2, Proposition 8.8.9], consider the ring $\widetilde{\Pi}_H^{[s,r]}$ and consider an element t which is coprime to all the images of t_θ (see [KL2, Proposition 8.8.9] for the definition of this element) under all the powers of the Frobenius. By modulo this element any nontrivial ideal of this ring, suppose we have that this ideal will be same under the two corresponding base change to $\widetilde{\Pi}_{H^1}^{[s,r]}$. Then we have that this is just a trivial ideal.*

Proof. See [KL2, Proposition 8.8.9]. \square

Proposition 6.4.20. (After Kedlaya-Liu, [KL2, Corollary 8.8.10]) *Under the same condition as in [KL2, Corollary 8.8.10], consider the ring $\widetilde{\Pi}_H^{[s,r]}$ and consider an element t which is coprime to all the images of t_θ (see [KL2, Corollary 8.8.10] for the definition of this element) under all the powers of the Frobenius. Then by considering taking the*

corresponding quotient by this element we will have the chance to promote finitely generatedness to finite projectivity. Now to be more precise suppose over the quotient over $\tilde{\Pi}_H^{[s,r]}$ by this element we have a module which is contained in a finitely generated one and which carries an action from Γ , then we have that actually this is projective.

Proof. See [KL2, Corollary 8.8.10]. □

Proposition 6.4.21. (After Kedlaya-Liu, [KL2, Proposition 8.9.2]) *Suppose over the ring $\tilde{\Pi}_H^{[s,r]}$ for some $0 < s \leq r$, we have a finite module which is carrying additionally the structure of Γ -module. Denote this by M . Then this is automatically pseudocoherent and more importantly étale-stably pseudocoherent.*

Proof. See [KL2, Proposition 8.9.2]. This is actually just parallel to proposition 6.4.16 by using decomposition of t to supercede the corresponding role of the element e . □

After these technical discussion we can now consider the corresponding categories of the corresponding pseudocoherent sheaves over the ring $\widehat{\mathcal{O}}_X$ and $\tilde{\Pi}_X^{[s,r]}$ for $0 < s \leq r$. Here now X will be then a general k_Δ -analytic space. The affinoid algebra A over \mathbb{Q}_p in the following discussion is assumed to be sousperfectoid as those considered in [KH].

Definition 6.4.22. (After Kedlaya-Liu, [KL2, Definition 8.4.1]) We use the corresponding notation $D_{\text{pseudo}, \widehat{\mathcal{O}}_X}$ to denote the corresponding category of all the pseudocoherent $\widehat{\mathcal{O}}_X$ sheaves over X (certainly under the corresponding pro-étale site), which is regarded as a subcategory of the corresponding category of all the corresponding sheaves over $\widehat{\mathcal{O}}_X$, in the corresponding pro-étale topology.

Proposition 6.4.23. (After Kedlaya-Liu, [KL2, Theorem 8.4.3]) *The corresponding category $D_{\text{pseudo}, \widehat{\mathcal{O}}_X}$ is abelian, where the corresponding kernels and cokernels are compatible with the corresponding category of all the sheaves of $\widehat{\mathcal{O}}_X$ -modules over the corresponding pro-étale site.*

Proof. This is a direct consequence of the corresponding properties in proposition 6.4.16. □

Definition 6.4.24. (After Kedlaya-Liu, [KL2, Definition 8.10.1]) We use the corresponding notation $D_{\text{pseudo}, \tilde{\Pi}_{X,A}^{[s,r]}}$ ($0 < s \leq r/p^a$) to denote the corresponding category

of all the pseudocoherent $\tilde{\Pi}_{X,A}^{[s,r]}$ sheaves, which is regarded as a subcategory of the corresponding category of all the corresponding sheaves over $\tilde{\Pi}_{X,A}^{[s,r]}$, in the corresponding pro-étale topology.

Also we have the following category of all the (φ^a, Γ) over the space X as in the above:

Definition 6.4.25. (After Kedlaya-Liu, [KL2, Definition 8.10.5]) We use the corresponding notation $D_{\text{pseudo}, \varphi^a, \tilde{\Pi}_{X,A}}$ to denote the corresponding category of all the pseudocoherent $\tilde{\Pi}_{X,A}$ sheaves carrying the corresponding Frobenius operator φ^a , which is regarded as a subcategory of the corresponding category of all the corresponding sheaves over $\tilde{\Pi}_{X,A}$, in the corresponding pro-étale topology.

Proposition 6.4.26. (After Kedlaya-Liu, [KL2, Theorem 8.10.2]) *The corresponding category $D_{\text{pseudo}, \tilde{\Pi}_{X,A}^{[s,r]}}$ is abelian, where the corresponding kernels and cokernels are compatible with the corresponding category of all the sheaves of $\tilde{\Pi}_{X,A}^{[s,r]}$ -modules over the corresponding pro-étale site. Here A is just \mathbb{Q}_p .*

Proof. This is a direct consequence of the corresponding properties in proposition 6.4.21. □

Proposition 6.4.27. (After Kedlaya-Liu, [KL2, Theorem 8.10.6]) *The corresponding category $D_{\text{pseudo}, \varphi^a, \tilde{\Pi}_{X,A}}$ is abelian, where the corresponding kernels and cokernels are compatible with the corresponding category of all the sheaves of $\tilde{\Pi}_{X,A}$ -modules over the corresponding pro-étale site. Here A is just \mathbb{Q}_p .*

Proof. See the previous proposition. □

Proposition 6.4.28. (After Kedlaya-Liu, [KL2, Theorem 8.10.6])

When X is smooth. The corresponding category $D_{\text{pseudo}, \varphi^a, \tilde{\Pi}_{X,A}}$ is abelian, where the corresponding kernels and cokernels are compatible with the corresponding category of all the sheaves of $\tilde{\Pi}_{X,A}$ -modules over the corresponding pro-étale site.

Proof. See proposition 6.4.3. □

6.4.3 Contact with Arithmetic Riemann-Hilbert Correspondence

We now consider the corresponding setting where the base field is finite extension of $\mathbb{F}_p((t))$ or \mathbb{Q}_p . The affinoid algebra A over $\mathbb{F}_p((t))$ or \mathbb{Q}_p in the following discussion is assumed to be sousperfectoid as those considered in [KH]. And we consider the corresponding setting above on the corresponding Robba rings.

We would like to take this chance to develop a little bit on the corresponding Hodge-Iwasawa B -pairs in our context. This was considered by [KP] with some restriction in the representation theoretic context. Since we would like to discuss the corresponding some tower where the internal structure is a little bit complicated to handle (although this is definitely doable). Note that one might want also to have the chance to consider more general tower contacting with the corresponding Iwasawa theory such as the Lubin-Tate as proposed in [KP].

Definition 6.4.29. (After Kedlaya-Pottharst [KP, Definition 2.17]) Now we first consider more general rigid analytic spaces. Let X be a rigid analytic space over a finite extension E of \mathbb{Q}_p or $\mathbb{F}_p((t))$. Now we consider the corresponding sheaves $\mathbb{B}_{\mathrm{dR},X,A}$, $\mathbb{B}_{e,X,A}$ and $\mathbb{B}_{\mathrm{dR},X,A}^+$ which are defined in the following way. For each of the corresponding perfectoid subdomain Y of the space X in our situation the corresponding rings $\mathbb{B}_{\mathrm{dR},X,A}(Y)$, $\mathbb{B}_{e,X,A}(Y)$ and $\mathbb{B}_{\mathrm{dR},X,A}^+(Y)$ are defined as in the following. So the corresponding map $Y_A \rightarrow \mathrm{Proj}P_{X,A}(Y)$ will by considering the corresponding zero locus define a closed subscheme in the corresponding deformed schematic Fargues-Fontaine curve by A . Here in more detail this is just the corresponding map $Y_A \rightarrow \mathrm{Proj}P_{Y^b,A}$. Then we consider the corresponding open complement subspace by removing this subscheme, whose coordinate ring will be the ring $\mathbb{B}_{e,X,A}(Y)$, then we take the corresponding completion of the corresponding deformed schematic Fargues-Fontaine curve along the corresponding closed subscheme to define the corresponding space whose coordinate ring is $\mathbb{B}_{\mathrm{dR},X,A}^+(Y)$, finally taking the corresponding space corresponding to the fiber product of the open affine subscheme and this completion we can take the corresponding coordinate ring which is denoted by $\mathbb{B}_{\mathrm{dR},X,A}(Y)$. Then we take the corresponding graded rings we can define the sheaves $\mathbb{B}_{\mathrm{HT},X,A}^+$ and $\mathbb{B}_{\mathrm{HT},X,A}$ as in [KL2, Definition 8.6.5].

Definition 6.4.30.

(After Tan-Tong, Shimizu [TT, Section 2.1], [Shi, Section 2]) In the very parallel way one can consider the corresponding cristalline period sheaves (by taking the corresponding completion in the corresponding de Rham periods above) $\mathbb{B}_{\text{cris},X,A}^+$ and $\mathbb{B}_{\text{cris},X,A}$ as in [TT, Section 2.1]. In the very parallel way one can consider the corresponding semistable period sheaves $\mathbb{B}_{\text{st},X,A}^+$ and $\mathbb{B}_{\text{st},X,A}$ as in [Shi, Section 2].

Then we have the following generalization of Bloch-Kato [BK1] and Nakamura's [Nakamura1] fundamental sequences in rigid family:

Lemma 6.4.31. *We then have the corresponding Bloch-Kato fundamental sequence which is defined by the following short exact sequence:*

$$0 \longrightarrow \mathbb{B}_{e,X,A} \cap \mathbb{B}_{\text{dR},X,A}^+ \longrightarrow \mathbb{B}_{e,X,A} \oplus \mathbb{B}_{\text{dR},X,A}^+ \longrightarrow \mathbb{B}_{e,X,A} \cup \mathbb{B}_{\text{dR},X,A}^+ \longrightarrow 0.$$

Proof. This comes from the corresponding construction directly, for instance see [KL2, Definition 4.8.2]. \square

Definition 6.4.32.

(After Scholze, Kedlaya-Liu [Sch1, Definition 6.8], [KL2, Definition 8.6.5])

Now we first consider more general rigid analytic spaces. Let X be a rigid analytic space over a finite extension E of \mathbb{Q}_p or $\mathbb{F}_p((t))$. Now we consider the corresponding sheaves $\mathcal{O}\mathbb{B}_{\text{dR},X,A}$, $\mathcal{O}\mathbb{B}_{e,X,A}$ and $\mathcal{O}\mathbb{B}_{\text{dR},X,A}^+$ which are defined in the following way. $\mathcal{O}\mathbb{B}_{\text{dR},X,A}^+$ is the sheafification of the following presheaf. For any perfectoid subdomain taking the form of $(P_i)_i$ as rings we consider the corresponding product $P_i^+ \otimes_{W(\kappa_E)} W(P^{b+})$, then take the corresponding ϖ -adic completion and invert the corresponding φ . Then we consider the corresponding completion along the ring P_i and the kernel of the map θ . Then taking the limit throughout all i we can define the corresponding ring $\mathcal{O}\mathbb{B}_{\text{dR},X,A}^+((P_i)_i)$. Then by considering the base change to $\mathbb{B}_{\text{dR},X,A}^+$, one can define the corresponding ring $\mathcal{O}\mathbb{B}_{\text{dR},X,A}^+$. Then by considering the base change to $\mathbb{B}_{\text{dR},X,A}$, one can define the corresponding ring $\mathcal{O}\mathbb{B}_{\text{dR},X,A}$. Then similarly one can define $\mathcal{O}\mathbb{B}_{e,X,A}$. Then taking the corresponding graded pieces we have $\mathcal{O}\mathbb{B}_{\text{HT},X,A}$ and $\mathcal{O}\mathbb{B}_{\text{HT},X,A}^+$ as in [KL2, Definition 8.6.5].

Definition 6.4.33.

(After Tan-Tong, Shimizu [TT, Section 2.1], [Shi, Section 2]) In the very parallel way one can consider the corresponding cristalline period sheaves (by taking the

corresponding completion in the corresponding de Rham periods above) $\mathcal{O}\mathbb{B}_{\text{cris},X,A}^+$ and $\mathcal{O}\mathbb{B}_{\text{cris},X,A}$ as in [TT, Section 2.1]. In the very parallel way one can consider the corresponding semistable period sheaves $\mathcal{O}\mathbb{B}_{\text{st},X,A}^+$ and $\mathcal{O}\mathbb{B}_{\text{st},X,A}$ as in [Shi, Section 2].

Then we have the following generalization of Bloch-Kato [BK1] and Nakamura's [Nakamura1] fundamental sequences in rigid family:

Lemma 6.4.34. *For X smooth, we then have the corresponding Bloch-Kato fundamental sequence which is defined by the following short exact sequence:*

$$0 \longrightarrow \mathcal{O}\mathbb{B}_{e,X,A} \cap \mathcal{O}\mathbb{B}_{\text{dR},X,A}^+ \longrightarrow \mathcal{O}\mathbb{B}_{e,X,A} \oplus \mathcal{O}\mathbb{B}_{\text{dR},X,A}^+ \longrightarrow \mathcal{O}\mathbb{B}_{e,X,A} \cup \mathcal{O}\mathbb{B}_{\text{dR},X,A}^+ \longrightarrow 0.$$

Proof. In the situation where X is smooth, we do have the corresponding explicit expression for the pro-étale sheaves, for instance for $\mathcal{O}\mathbb{B}_{\text{dR},X,A}^+$ and $\mathcal{O}\mathbb{B}_{\text{dR},X,A}$ one could follow [Sch1, Proposition 6.10] to have the chance to find the corresponding expressions which represents these rings as formal power series over the rings $\mathbb{B}_{\text{dR},X,A}^+$ and $\mathbb{B}_{\text{dR},X,A}$. For the ring $\mathcal{O}\mathbb{B}_{e,X,A}$ one could make the parallel argument we will not repeat this again. \square

Then we can discuss the corresponding B -pairs:

Definition 6.4.35. (After Kedlaya-Liu [KL1, Definition 9.3.11])

We now define a B -pair over the corresponding triplet $(\mathbb{B}_{\text{dR},X,A}, \mathbb{B}_{e,X,A}, \mathbb{B}_{\text{dR},X,A}^+)$ is a triplet $(\mathbb{M}_{\text{dR},X,A}, \mathbb{M}_{e,X,A}, \mathbb{M}_{\text{dR},X,A}^+)$ which are the corresponding finite projective objects over the corresponding rings involved above, such that that we have the following isomorphisms:

$$\mathbb{M}_{\text{dR},X,A} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A}^+ \otimes_{\mathbb{B}_{\text{dR},X,A}^+} \mathbb{B}_{\text{dR},X,A} \xrightarrow{\sim} \mathbb{M}_{e,X,A}^+ \otimes_{\mathbb{B}_{e,X,A}} \mathbb{B}_{\text{dR},X,A}.$$

Similarly as in [KL1, Definition 9.4.4] we can define the corresponding cohomology of the B -pairs by considering the hypercohomology of the following sequence:

$$0 \longrightarrow \mathbb{M}_{e,X,A} \oplus \mathbb{M}_{\text{dR},X,A}^+ \longrightarrow \mathbb{M}_{\text{dR},X,A} \longrightarrow 0,$$

which is given by subtraction from the first coordinate by the second coordinate in the first non-zero term in this sequence.

Remark 6.4.36. The finite projective objects mean those ones which are locally finite projective. Since we are carrying the corresponding coefficient A which is essentially some

nontrivial deformation, therefore locally on the space X we have to further glue along A in order to really consider the corresponding finite projective objects which are interesting enough in our development.

Then we can discuss the corresponding $\mathcal{O}B$ -pairs:

Definition 6.4.37. (After Kedlaya-Liu [KL1, Definition 9.3.11])

We now define a $\mathcal{O}B$ -pair over the corresponding triplet $(\mathcal{O}\mathbb{B}_{\mathrm{dR},X,A}, \mathcal{O}\mathbb{B}_{e,X,A}, \mathcal{O}\mathbb{B}_{\mathrm{dR},X,A}^+)$ is a triplet $(\mathbb{M}_{\mathrm{dR},X,A}, \mathbb{M}_{e,X,A}, \mathbb{M}_{\mathrm{dR},X,A}^+)$ which are the corresponding finite projective objects over the corresponding rings involved above, such that that we have the following isomorphisms:

$$\mathbb{M}_{\mathrm{dR},X,A} \xrightarrow{\sim} \mathbb{M}_{\mathrm{dR},X,A}^+ \otimes_{\mathcal{O}\mathbb{B}_{\mathrm{dR},X,A}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR},X,A} \xrightarrow{\sim} \mathbb{M}_{e,X,A}^+ \otimes_{\mathcal{O}\mathbb{B}_{e,X,A}} \mathcal{O}\mathbb{B}_{\mathrm{dR},X,A}.$$

Similarly as in [KL1, Definition 9.4.4] we can define the corresponding cohomology of the B -pairs by considering the hypercohomology of the following sequence:

$$0 \longrightarrow \mathbb{M}_{e,X,A} \oplus \mathbb{M}_{\mathrm{dR},X,A}^+ \longrightarrow \mathbb{M}_{\mathrm{dR},X,A} \longrightarrow 0,$$

which is given by subtraction from the first coordinate by the second coordinate in the first non-zero term in this sequence.

We now construct the corresponding deformed geometrization of the corresponding Bloch-Kato exponentials. To do that we consider the following derived functors which are inspired by [LZ, Theorem 1.5] which could be also dated back to Nakamura's work [Nakamura2, 2.1] in the nonderived situation:

Definition 6.4.38. For any B -pair we set the corresponding derived de Rham functor D_{dR}^\bullet and derived Hodge-Tate functor D_{HT}^\bullet on the B -pairs as in the following. For any B -pair we set the corresponding derived cristalline functor $D_{\mathrm{cris}}^\bullet$ and derived semistable functor D_{st}^\bullet on the B -pairs as in the following. Consider a B -pair $(\mathbb{M}_{\mathrm{dR},X,A}, \mathbb{M}_{e,X,A}, \mathbb{M}_{\mathrm{dR},X,A}^+)$ we have the following functors:

$$D_{\mathrm{dR}}^\bullet(\mathbb{M}_{\mathrm{dR},X,A}) := R^\bullet f_{\mathrm{pro\acute{e}t},*} \mathbb{M}_{\mathrm{dR},X,A} \otimes_{\mathbb{B}_{\mathrm{dR},X,A}} \mathcal{O}\mathbb{B}_{\mathrm{dR},X,A},$$

and

$$D_{\mathrm{HT}}^\bullet(\mathbb{M}_{\mathrm{dR},X,A}) := R^\bullet f_{\mathrm{pro\acute{e}t},*} \mathbb{M}_{\mathrm{dR},X,A} \otimes_{\mathbb{B}_{\mathrm{dR},X,A}} \mathcal{O}\mathbb{B}_{\mathrm{HT},X,A}$$

and

$$D_{\text{cris}}^{\bullet}(\mathbb{M}_{\text{dR},X,A}) := R^{\bullet}f_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}}$$

and

$$D_{\text{st}}^{\bullet}(\mathbb{M}_{\text{dR},X,A}) := R^{\bullet}f_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}.$$

We call that the B -pair de Rham if we have the following isomorphism:

$$f_{\text{proét}}^*(f_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}},$$

and we call this Hodge-Tate if we have the corresponding similar isomorphism. We will use the corresponding notation \mathbb{D} to denote the corresponding functor on the derived category level. And we call that the B -pair cristalline if we have the corresponding isomorphism:

$$f_{\text{proét}}^*(f_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}}.$$

And we call that the B -pair semi-stable if we have the corresponding isomorphism:

$$f_{\text{proét}}^*(f_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{st},X,A}} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}.$$

Now we address some issues around the corresponding finiteness of the construction getting involved as in the above. First when the space X is smooth and proper over the corresponding base fields, in the situation where A is just \mathbb{Q}_p we have in this case $f_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}$ of any B -pair is coherent sheaf over \mathcal{O}_X , which is the consequence of [KL2, Theorem 8.6.2 (a)]. And moreover in this case since the corresponding space is smooth as noted in [KL3, Definition 10.10] we have that the corresponding existence of the connection ∇ which will ensure the corresponding coherent sheaves are also projective modules. We now first consider the situation where we have the ring A not just \mathbb{Q}_p in more general situation. Certainly to study the corresponding B -pairs it will be basically convenient to relate to the corresponding (φ, Γ) -modules as in the earlier work from Kedlaya-Liu and Nakamura. To do so we can first consider the following higher dimensional generalization of [KP, Theorem 2.18]:

Remark 6.4.39. Note that at this moment here we assume that the corresponding base fields are of 0 characteristic.

Proposition 6.4.40. (After Kedlaya-Pottharst [KP, Theorem 2.18]) *We look at the two groups of objects. In the first consideration we look at the corresponding finite*

locally free sheaves over the corresponding A -deformed version of the schematic Fargues-Fontaine curves attache to any local perfectoid chart for some ring R . In the other second consideration we look at the corresponding A -deformed version of the corresponding B -pairs defined above but after taking the corresponding section over some perfectoid domain associated to some ring R . Then we have the corresponding equivalence between the corresponding categories respecting the two groups of objects.

Proof. As in [KP, Theorem 2.18] this is by Beauville-Laszlo descent. □

Remark 6.4.41. Then we could basically translate the corresponding results on the cohomology of B -pairs to those on the corresponding cohomology of the vector bundles over the Fargues-Fontaine curves and further more to those on that of the Frobenius modules over the pro-étale site. Then we have that this will be further related to (φ, Γ) -modules over the imperfect Robba rings with respect to the corresponding base spaces. So we move on to consider the corresponding (φ, Γ) -modules over the corresponding pro-étale site.

Following Nakamura one can further generalize the corresponding Bloch-Kato long exact sequence which provides the corresponding exponentials:

Definition 6.4.42. (After Bloch-Kato-Nakamura) For X smooth, consider any (φ, Γ) -module M over the pro-étale sheaf $\tilde{\Pi}_{X,A}^\infty$ we then take the corresponding pro-étale invariance, namely we consider the following two functor with the one D_{dR} :

$$D_{\mathcal{O}}^\bullet(M) := R^\bullet f_{\text{proét},*}(M \otimes_{\tilde{\Pi}_{X,A}^\infty} (\mathcal{O}_{\mathbb{B}_{e,X,A}} \cap \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}^+)),$$

$$D_{\oplus}^\bullet(M) := R^\bullet f_{\text{proét},*}(M \otimes_{\tilde{\Pi}_{X,A}^\infty} (\mathcal{O}_{\mathbb{B}_{e,X,A}} \bigoplus \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}^+)),$$

as well as the corresponding definitions on the derived category level:

$$\mathbb{D}_{\mathcal{O}}^\bullet(M) := R^\bullet f_{\text{proét},*}(M \otimes_{\tilde{\Pi}_{X,A}^\infty} (\mathcal{O}_{\mathbb{B}_{e,X,A}} \cap \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}^+)),$$

$$\mathbb{D}_{\oplus}^\bullet(M) := R^\bullet f_{\text{proét},*}(M \otimes_{\tilde{\Pi}_{X,A}^\infty} (\mathcal{O}_{\mathbb{B}_{e,X,A}} \bigoplus \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}^+)),$$

Taking the corresponding complexes induced and considering the corresponding mapping cone we have the following distinguished triangle in $D(\mathcal{O}_X \widehat{\otimes}_{\mathbb{Q}_p} A)$ (the derived category of all the $\mathcal{O}_X \widehat{\otimes}_{\mathbb{Q}_p} A$ -modules):

$$C_{\mathcal{O}}^\bullet(M) \longrightarrow C_{\oplus}^\bullet(M) \longrightarrow C_{\text{dR}}^\bullet(M) \longrightarrow C_{\mathcal{O}}^\bullet(M)[1].$$

Consider the corresponding degree 0 and 1 we will have geometric and arithmetic family version of the corresponding Bloch-Kato-Nakamura exponential map:

$$h^0(C_{\mathrm{dR}}^\bullet(M)) \rightarrow h^1(C_{\mathcal{O}}^\bullet(M)),$$

as the map of sheaves over $\mathcal{O}_X \widehat{\otimes}_{\mathbb{Q}_p} A$. Here we assume that the corresponding B -pairs getting involved are de Rham.

Definition 6.4.43. (After Kedlaya-Liu [KL2, Definition 8.6.5]) For any φ -module M over the sheaf of ring $\widetilde{\Pi}_{X,A}^\infty$ we set the corresponding derived de Rham functor D_{dR}^\bullet and derived Hodge-Tate functor D_{HT}^\bullet on the φ -modules M over the sheaf of ring $\widetilde{\Pi}_{X,A}^\infty$ as in the following. For any φ -module M over the sheaf of ring $\widetilde{\Pi}_{X,A}^\infty$ we set the corresponding derived cristalline functor $D_{\mathrm{cris}}^\bullet$ and derived semistable functor D_{st}^\bullet on the φ -modules M over the sheaf of ring $\widetilde{\Pi}_{X,A}^\infty$ as in the following. Consider a φ -module M over the sheaf of ring $\widetilde{\Pi}_{X,A}^\infty$ we have the following functors:

$$D_{\mathrm{dR}}^\bullet(M) := R^\bullet f_{\mathrm{pro\acute{e}t},*} M \otimes_{\widetilde{\Pi}_{X,A}^\infty} \mathcal{O}_{\mathbb{B}_{\mathrm{dR},X,A}},$$

and

$$D_{\mathrm{HT}}^\bullet(M) := R^\bullet f_{\mathrm{pro\acute{e}t},*} M \otimes_{\widetilde{\Pi}_{X,A}^\infty} \mathcal{O}_{\mathbb{B}_{\mathrm{HT},X,A}}$$

and

$$D_{\mathrm{cris}}^\bullet(M) := R^\bullet f_{\mathrm{pro\acute{e}t},*} M \otimes_{\widetilde{\Pi}_{X,A}^\infty} \mathcal{O}_{\mathbb{B}_{\mathrm{cris},X,A}},$$

and

$$D_{\mathrm{st}}^\bullet(M) := R^\bullet f_{\mathrm{pro\acute{e}t},*} M \otimes_{\widetilde{\Pi}_{X,A}^\infty} \mathcal{O}_{\mathbb{B}_{\mathrm{st},X,A}}.$$

We call that the B -pair de Rham if we have the following isomorphism:

$$f_{\mathrm{pro\acute{e}t}}^*(f_{\mathrm{pro\acute{e}t},*} M \otimes_{\widetilde{\Pi}_{X,A}^\infty} \mathcal{O}_{\mathbb{B}_{\mathrm{dR},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\mathrm{dR},X,A}} \xrightarrow{\sim} M \otimes_{\widetilde{\Pi}_{X,A}^\infty} \mathcal{O}_{\mathbb{B}_{\mathrm{dR},X,A}},$$

and we call this Hodge-Tate if we have the corresponding similar isomorphism. We will use the corresponding notation \mathbb{D} to denote the corresponding functor on the derived category level. And we define the corresponding properties of being cristalline and semistable in the same parallel way.

Proposition 6.4.44. *For general rigid analytic space X (note that at this moment we are in the 0 characteristic situation), the corresponding cohomology $D_{\mathrm{dR}}^i(M)$ is coherent sheaf over X for each i . In the situation where the space is smooth we have the further projectivity.*

Proof. This is a relative version of the result of [KL3, below Definition 10.10]. Namely we consider the corresponding resolution of singularities and reduce to the corresponding smooth case. In the corresponding smooth case we further reduce to the corresponding case for the base of a corresponding toric tower in the restricted sense. Then we relate to pseudocoherent Γ -modules over $\overline{H}_\infty \widehat{\otimes} A$ in our context, which further relates to the the corresponding pseudocoherent Γ -modules over $\widetilde{\Pi}_A$ and then over $H_\infty \widehat{\otimes} A$. Then we descend to some finite level ring $H_k \widehat{\otimes} A$ for some k . Then we need to consider as in [KL2, Corollary 5.9.5] that the corresponding p.f. descent will allow us to realize the corresponding Γ -cohomology as the abstract one by using the higher period rings. Then this will realize the corresponding finiteness. The projectivity follows from the existence of the connection. \square

Proposition 6.4.45. *For general rigid analytic space X (note that at this moment we are in the 0 characteristic situation), the corresponding cohomology $D_{\text{cris}}^i(M), D_{\text{st}}^i(M)$ are coherent sheaves over X for each i . In the situation where the space is smooth we have the further projectivity.*

Proof. See the proof in the previous proposition. \square

Corollary 6.4.46. *The corresponding sheaf*

$$f_{\text{proét},*} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}$$

over $\mathcal{O}_X \widehat{\otimes} A$ is basically a coherent sheaf. When we have that the corresponding space X is smooth we could further have the corresponding projectivity in our context.

Remark 6.4.47. The construction above is generalization of [KL2, Definition 8.6.5], which is different slightly from [LZ, Section 3.2]. But we can also push things down to the étale site along $g_{\text{proét}} : X_{\text{proét}} \rightarrow X_{\text{ét}}$.

Definition 6.4.48. For any B -pair we set the corresponding derived de Rham functor E_{dR}^\bullet and derived Hodge-Tate functor E_{HT}^\bullet on the B -pairs as in the following. For any B -pair we set the corresponding derived crystalline functor E_{cris}^\bullet and derived semistable functor E_{st}^\bullet on the B -pairs as in the following. Consider a B -pair $(\mathbb{M}_{\text{dR},X,A}, \mathbb{M}_{e,X,A}, \mathbb{M}_{\text{dR},X,A}^+)$ we have the following functors:

$$E_{\text{dR}}^\bullet(\mathbb{M}_{\text{dR},X,A}) := R^\bullet g_{\text{proét},*} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}},$$

and

$$E_{\text{HT}}^{\bullet}(\mathbb{M}_{\text{dR},X,A}) := R^{\bullet}g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{HT},X,A}}$$

and

$$E_{\text{cris}}^{\bullet}(\mathbb{M}_{\text{dR},X,A}) := R^{\bullet}g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}}$$

and

$$E_{\text{st}}^{\bullet}(\mathbb{M}_{\text{dR},X,A}) := R^{\bullet}g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}.$$

We call that the B -pair de Rham if we have the following isomorphism:

$$g_{\text{proét}}^*(g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}},$$

and we call this Hodge-Tate if we have the corresponding similar isomorphism. We will use the corresponding notation \mathbb{E} to denote the corresponding functor on the derived category level. And we call that the B -pair cristalline if we have the corresponding isomorphism:

$$g_{\text{proét}}^*(g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}}.$$

And we call that the B -pair semi-stable if we have the corresponding isomorphism:

$$g_{\text{proét}}^*(g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{st},X,A}} \xrightarrow{\sim} \mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}.$$

Definition 6.4.49. (After Kedlaya-Liu [KL2, Definition 8.6.5]) For any φ -module M over the sheaf of ring $\tilde{\Pi}_{X,A}^{\infty}$ we set the corresponding derived de Rham functor E_{dR}^{\bullet} and derived Hodge-Tate functor E_{HT}^{\bullet} on the φ -modules M over the sheaf of ring $\tilde{\Pi}_{X,A}^{\infty}$ as in the following. For any φ -module M over the sheaf of ring $\tilde{\Pi}_{X,A}^{\infty}$ we set the corresponding derived cristalline functor $E_{\text{cris}}^{\bullet}$ and derived semistable functor E_{st}^{\bullet} on the φ -modules M over the sheaf of ring $\tilde{\Pi}_{X,A}^{\infty}$ as in the following. Consider a φ -module M over the sheaf of ring $\tilde{\Pi}_{X,A}^{\infty}$ we have the following functors:

$$E_{\text{dR}}^{\bullet}(M) := R^{\bullet}g_{\text{proét},*}M \otimes_{\tilde{\Pi}_{X,A}^{\infty}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}},$$

and

$$E_{\text{HT}}^{\bullet}(M) := R^{\bullet}g_{\text{proét},*}M \otimes_{\tilde{\Pi}_{X,A}^{\infty}} \mathcal{O}_{\mathbb{B}_{\text{HT},X,A}}$$

and

$$E_{\text{cris}}^{\bullet}(M) := R^{\bullet}g_{\text{proét},*}M \otimes_{\tilde{\Pi}_{X,A}^{\infty}} \mathcal{O}_{\mathbb{B}_{\text{cris},X,A}},$$

and

$$E_{\text{st}}^{\bullet}(M) := R^{\bullet}g_{\text{proét},*}M \otimes_{\tilde{\Pi}_{X,A}^{\infty}} \mathcal{O}_{\mathbb{B}_{\text{st},X,A}}.$$

We call that the B -pair de Rham if we have the following isomorphism:

$$g_{\text{proét}}^*(g_{\text{proét},*}M \otimes_{\tilde{\Pi}_{X,A}^{\infty}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}) \otimes \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}} \xrightarrow{\sim} M \otimes_{\tilde{\Pi}_{X,A}^{\infty}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}},$$

and we call this Hodge-Tate if we have the corresponding similar isomorphism. We will use the corresponding notation \mathbb{E} to denote the corresponding functor on the derived category level. And we define the corresponding properties of being cristalline and semistable in the same parallel way.

Proposition 6.4.50. *For general rigid analytic space X (note that at this moment we are in the 0 characteristic situation), the corresponding cohomology $E_{\text{dR}}^i(M)$ for each i is a coherent sheaf over X . In the situation where the space is smooth we have the further projectivity.*

Proof. One looks at the following composition:

$$X_{\text{proét}} \rightarrow X_{\text{ét}} \rightarrow X_{\text{an}}. \quad (6.4.1)$$

We call the second functor $f_{\text{ét},\text{an}}$. In effect, the second one realizes the corresponding equivalence between the category of all the coherent $\mathcal{O}_{X_{\text{ét}}}$ -modules and the category of all the coherent $\mathcal{O}_{X_{\text{an}}}$ -modules. We have that $f_{\text{proét},*} = f_{\text{ét},\text{an},*}g_{\text{proét},*}$. Then the result follows from the corresponding proposition 6.4.44. \square

Proposition 6.4.51. *For general rigid analytic space X (note that at this moment we are in the 0 characteristic situation), the corresponding cohomology $E_{\text{cris}}^i(M), E_{\text{st}}^i(M)$ for each i are coherent sheaves over X . In the situation where the space is smooth we have the further projectivity.*

Proof. See the previous proposition. \square

Corollary 6.4.52. *The corresponding sheaf*

$$g_{\text{proét},*}\mathbb{M}_{\text{dR},X,A} \otimes_{\mathbb{B}_{\text{dR},X,A}} \mathcal{O}_{\mathbb{B}_{\text{dR},X,A}}$$

over $\mathcal{O}_X \widehat{\otimes} A$ is basically a coherent sheaf. When we have that the corresponding space X is smooth we could further have the corresponding projectivity in our context.

Now we consider the corresponding reconstruction process as in the philosophy of [KP].

Setting 6.4.53. Namely we take the space X to be smooth, and we take the corresponding A to be some local chart of some abelian Fréchet-Stein algebra A_∞ (but the local quasi-compacts are assumed to be within our sousperfectoid requirement). Take a B -pair M now over $\mathcal{O}\mathbb{B}_{\mathrm{dR},X,A_\infty}$ namely let us require the corresponding finiteness and projectivity at the infinite level, as a family of B -pair over $\mathcal{O}\mathbb{B}_{\mathrm{dR},X,A_n}$ for all n .

Proposition 6.4.54. *Let X be the space attached to $E\{T_1, \dots, T_n\}$, and define the functor $\mathbb{D}_{\mathrm{dR}}^\bullet$ as above for a family M in the previous setting over $\mathcal{O}\mathbb{B}_{\mathrm{dR},X,A_\infty}$. Then we have that $\mathbb{D}_{\mathrm{dR}}^\bullet$ consists of coadmissible and finite projective modules over $\mathcal{O}_X \widehat{\otimes} A_\infty$ where A_∞ is assumed to come from a p -adic Lie group as in [Z1, Theorem 3.10] (but the local quasi-compacts are assumed to be within our sousperfectoid requirement).*

Proof. Our result above for A_n for each $n = 0, 1, \dots$ shows the coadmissibility. The finite projectivity follows from the existence of the connection ∇_X and a theorem due to Zábřadi [Z1, Theorem 3.10]. \square

Conjecture 6.4.55. *If the space X is seminormal, then we have the same result.*

Remark 6.4.56. This would reconstruct, generalize and geometrize the corresponding Berger and Fourquaux's Lubin-Tate Iwasawa construction [BerFou]. And in the corresponding cyclotomic situation this also provides the corresponding generalization and geometrization of the corresponding construction after Nakamura [Nakamura1].

6.5 Applications to Equivariant Iwasawa Theory

6.5.1 Equivariant Big Perrin-Riou-Nakamura Exponential Maps

Now we establish the corresponding equivariant version of Nakamura's de Rham Iwasawa theory [Nakamura1] of (φ, Γ) -modules over the Robba ring in the corresponding cyclotomic situation. The idea and basic philosophy come from the corresponding reconstruction process observed by the work of [KPX] and [KP], namely the corresponding Fontaine's original construction using ψ -operator of the corresponding Iwasawa cohomology of (φ, Γ) -modules could be reconstructed by using the corresponding Iwasawa analytic deformation and even Frobenius sheaves in the style of Kedlaya-Liu [KL1] and [KL2] over the pro-étale sites after Scholze.

In our previous work in [T1], we partially looked at some sort of generalized version of the picture above (although [T1] is really and essentially aimed at the corresponding direction after [PZ1] and [CKZ]), namely instead of just deforming the corresponding (φ, Γ) -modules by some Iwasawa deformation over $\Pi^\infty(\Gamma)$ which is the corresponding Fréchet-Stein algebra attached to the group Γ , we look at more general Iwasawa deformation over $\Pi^\infty(\Gamma) \widehat{\otimes} \Pi^\infty(\Gamma')$ namely we could look at a suitable Fréchet-Stein algebra attached to the group $\Gamma \times \Gamma'$ such that we have that it could be written as family of affinoid algebras in rigid analytic geometry. The corresponding (φ, Γ) cohomology of such modules could be regarded as the corresponding genuine Γ' -equivariant Iwasawa cohomology.

Definitely we work in the philosophy of the corresponding Burns-Flach-Fukaya-Kato's grand picture on the so-called noncommutative Tamagawa number conjectures as in [BF1], [BF2] and [FK]. Nakamura's original Iwasawa theory on the (φ, Γ) -modules happens over the tower of Iwasawa tower. Namely the corresponding big Perrin-Riou-Nakamura exponential maps should be related to the corresponding characteristic ideals of the (φ, Γ) -cohomology of the corresponding Iwasawa deformation of any de Rham (φ, Γ) -module in some very precise way. One should regard our generalization as climbing even higher in some equivariant way. Namely the corresponding big Perrin-Riou-Nakamura exponential maps should be related to the corresponding characteristic ideals

of the (φ, Γ) -cohomology of the corresponding higher dimensional equivariant deformation of any de Rham (φ, Γ) -module in some very precise way.

We now follow [Nakamura1] to establish the equivariant version of corresponding big (dual-)exponential maps after Perrin-Riou-Nakamura. This happens at the infinite level of the corresponding generalized admissible abelian tower in our setting, which generalizes the corresponding setting considered in [Nakamura1].

Setting 6.5.1. We consider an abelian p -adic Lie group Γ' such that the Fréchet-Stein algebra $\Pi^\infty(\Gamma')$ in the sense of [ST1] could be written as a family of affinoid algebras $\Pi^\infty(\Gamma')_k, k = 0, 1, \dots$ in the sense of rigid analytic geometry:

$$\Pi^\infty(\Gamma') \xrightarrow{\sim} \varprojlim_k \Pi^\infty(\Gamma')_k.$$

If we use the notation X to denote the Fréchet-Stein space attached to the algebra $\Pi^\infty(\Gamma')$ which could be written as:

$$X \xrightarrow{\sim} \varinjlim_k X_k :$$

with:

$$\mathcal{O}_X \xrightarrow{\sim} \varprojlim_k \mathcal{O}_{X_k} :$$

Now we start to consider the corresponding relative Robba rings where our Iwasawa theory happens:

Definition 6.5.2. Many definitions of the Robba rings are actually compatible with [Nakamura1], the only difference is that we will deform over some general algebras. We use the notations $\Pi_{\text{an,con},K}^r$ and $\Pi_{\text{an,con},K}$ to denote the corresponding Robba rings as in [Nakamura1, Section 2.1]. Now we consider the corresponding Robba rings

$\Pi_{\text{an,con},K,\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')}^r$ and $\Pi_{\text{an,con},K,\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')}$ in the relative sense:

$$\Pi_{\text{an,con},K,\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')}^r := \Pi_{\text{an,con},K}^r \widehat{\otimes} \Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma'), \quad (6.5.1)$$

$$\Pi_{\text{an,con},K,\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')} := \Pi_{\text{an,con},K} \widehat{\otimes} \Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma'). \quad (6.5.2)$$

$$(6.5.3)$$

One also has the corresponding definitions for Robba rings with respect to some interval $[s, r]$. And be very careful that in our current development the corresponding Robba ring

$\Pi_{\text{an,con},K}^?$ is the ring $B_{\text{rig},K}^?$ where $? =, r, [s, r]$, instead of the ring $\Pi_{\text{an,con}}^?$ (π_K) considered in [KPX, Definition 2.2.2].

Setting 6.5.3. Our motivic objects are still usual (φ, Γ) -modules over the Robba ring $\Pi_{\text{an,con},K}$. Namely we consider those modules finite locally free over this big non-noetherian ring carrying commuting semilinear action from the φ and Γ .

Remark 6.5.4. The ideas we presented here actually are initiated in [KP]. One replaces the corresponding p -adic Lie group Γ by a pro-étale site, and considers the deformation of the sheaves (both the corresponding motivic objects and the corresponding Berger's p -adic differential equations attached in whatever reasonable context like Lubin-Tate setting or some else) by the corresponding Iwasawa deformations, and then considers the corresponding big exponential maps by using the connections and their duals.

To study Iwasawa theory we look at those geometric (φ, Γ) -modules, namely those geometric ones in the sense of p -adic Hodge theory after Berger's celebrated thesis. We do not go beyond [Nakamura1] on this, but we would like to recall the basic construction on the corresponding p -adic differential equation attached to a de Rham (φ, Γ) -module M .

Setting 6.5.5. Recall the basic definition of the corresponding p -adic differential equation attached to a de Rham (φ, Γ) -module M after Berger. As in [Nakamura1, Section 3.2] (we will use some different notations from [Nakamura1, Section 3.2]), we consider the following constructions. Let M be a (φ, Γ) -module M over $\Pi_{\text{an,con},K}$. In our situation we work with at least de Rham ones in the sense of Berger's celebrated paper [Ber1]. Then we have the corresponding relation for any integer $k \geq l'$ for some $l'(K, M) \geq 0$ fixed which is assumed to be sufficiently large depending on K, M :

$$F_{\text{dif},k}(M) \xrightarrow{\sim} F_{\text{dR},K}(M) \otimes K_k((t)) \tag{6.5.4}$$

with the corresponding lattice $F_{\text{dR},K}(M) \otimes K_k[[t]]$ inside with reasonable rank. Then we have the corresponding continuous differential:

$$\widehat{\Omega}_{\Pi_{\text{an,con},K}/K_0^{\text{ur}}} = \widehat{\Omega}_{\Pi_{\text{an,con},K}} dT.$$

The corresponding differential equation in the sense of Berger is the following construction which is denoted by $N_{\text{dif}}(M)$, we will denote this by $\Theta_{\text{Ber,dif}}(M)$ which is

actually defined in the following way. Recall as in [Nakamura1, Theorem 3.5] by [Ber1, 5.10] and [Ber2, 3.2.3], first we have $\Theta_{\text{Ber,dif},k}(M)$ which is defined to be the subset of all the elements in $M^{(k)}[1/t]$ such that the corresponding image of them under each map ι'_k for all $k' \geq k$ lives in $F_{\text{dR},K}(M) \otimes K'_k[[t]]$. Recall from [Nakamura1, Section 3.2] we have then the corresponding set $\Theta_{\text{Ber,dif}}(M) = \varinjlim_k \Theta_{\text{Ber,dif},k}(M)$ which could be endowed with furthermore the structure of (φ, Γ) -module over the Robba ring $\Pi_{\text{an,con},K}$. And furthermore we have the corresponding properties that $d_0(\Theta_{\text{Ber,dif}}(M)) \subset t\Theta_{\text{Ber,dif}}(M)$ where the differential operator d_0 is defined to be $\frac{\log(\gamma)}{\log(\chi(\gamma))}$ living in in our notation $\Pi^\infty(\Gamma_K)$.

Therefore recall from [Nakamura1, Section 3.2] then we can define the connection $\partial := \frac{1}{t}d_0$ mapping the corresponding differential (φ, Γ) -module considered above to itself. The differential structures and the corresponding Frobenius structures and the corresponding Lie group action structures are compatible in the following sense:

$$\partial\varphi - p\varphi\partial = 0, \tag{6.5.5}$$

$$\partial\gamma - \chi(\gamma)\gamma\partial = 0, \tag{6.5.6}$$

and:

$$\varphi(dT) - pdT = 0, \tag{6.5.7}$$

$$\gamma(dT) - \chi(\gamma)dT = 0. \tag{6.5.8}$$

Lemma 6.5.6. *Now we set $d_i := d_0 - i$ for $i \geq 0$ integer, we have for any m -th ($m \geq 0$) Hodge filtration of M (which is assumed to be de Rham) which gives the full module:*

$$d_{m-1}d_{m-2}d_{m-3}\dots d_1d_0(\Theta_{\text{Ber,dif}}(M)) \subset M.$$

Proof. This is just [Nakamura1, Lemma 3.6]. □

We now define the corresponding equivariant big exponential map after Perrin-Riou-Nakamura.

Assumption 6.5.7. Now we assume that there is a quotient $G_K \rightarrow \Gamma'$ in our situation among all the other requirements as above.

Definition 6.5.8. Take any sufficiently large $m \geq 0$ as in the previous lemma. First we consider the differential equation $\Theta_{\text{Ber,dif}}(M)$ attached to M which is assumed to

be de Rham. We consider the Γ' -deformation of M over the algebra $\Pi^\infty(\Gamma')$. This happens by considering the corresponding quotient $G_K \rightarrow \Gamma'$ which defines a corresponding Galois representation with value in $\Pi^\infty(\Gamma')$, which by corresponding embedding fully faithful gives rise to a corresponding (φ, Γ) -module over the relative Robba ring $\Pi_{\text{an,con},K} \widehat{\otimes} \Pi^\infty(\Gamma')$. Denote this by $\text{Dfm}_{\Gamma'}$ which could be written as the projective limit of the families $\{\text{Dfm}_{\Gamma',p}\}_{p \geq 0}$ over $\Pi_{\text{an,con},K} \widehat{\otimes} \Pi^\infty(\Gamma')_p$ for all $p \geq 0$. Then we consider the product $\text{Dfm}_{\Gamma',p} \widehat{\otimes} M$ then take the corresponding projective limit over all $p \geq 0$. Then we consider the following definition which is an equivariant version of [Nakamura1, Definition 3.7]:

$$\text{Exp}_{M,m}^{\Pi^\infty(\Gamma')} : (\varprojlim \text{Dfm}_{\Gamma',p} \widehat{\otimes} \Theta_{\text{Ber,dif}}(M))^{\psi=1} \rightarrow (\varprojlim \text{Dfm}_{\Gamma',p} \widehat{\otimes} M)^{\psi=1}$$

which is just defined by extension of $d_{m-1}d_{m-2}d_{m-3}\dots d_1d_0$. This is now $\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')$ -linear instead of just $\Pi^\infty(\Gamma_K)$. And the corresponding (φ, Γ) -modules here could be basically geometrized by the corresponding Hodge-Iwasawa construction above. We consider the corresponding family of vector bundle $\varprojlim \mathcal{F}(M)_p$ and $\varprojlim \mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p$ over the corresponding adic version of Fargues-Fontaine curve deformed by $\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')$. We have now that the geometrized big exponential map by taking the corresponding sheaf cohomology:

$$\text{Exp}_{\mathcal{F},m}^{\Pi^\infty(\Gamma')} : R^1 \Gamma_{\text{sheaf}}(\varprojlim \mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p) \rightarrow R^1 \Gamma_{\text{sheaf}}(\varprojlim \mathcal{F}(M)_p),$$

and the corresponding derived version:

$$\text{Exp}_{\mathcal{F},m}^{\Pi^\infty(\Gamma'), \bullet} : R^\bullet \Gamma_{\text{sheaf}}(\varprojlim \mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p) \rightarrow R^\bullet \Gamma_{\text{sheaf}}(\varprojlim \mathcal{F}(M)_p).$$

6.5.2 Characteristic Ideal Sheaves

Now we follow the corresponding idea in [Nakamura1] to consider the corresponding characteristic ideal and the corresponding determinant of a given derived maps with respect to the corresponding equivariant Iwasawa cohomology. For this we consider the corresponding map defined above, and we look at the corresponding equivariant Iwasawa cohomology, i.e. the corresponding more general commutative deformation in our current

context.

First in our situation we make the following assumption:

Assumption 6.5.9. We now assume that the corresponding deformation happens over Krull noetherian rings. This could basically allow us to define the corresponding characteristic ideal over the sheaf $\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}$ of any torsion module M , which we will denote it by $\text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(M)$ which is again a sheaf of module over $\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}$.

In the current generalized context, following Nakamura [Nakamura1, Section 3.4] we define the corresponding determinant of a general map $f : M \rightarrow N$ of coadmissible modules over the sheaf $\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}$. First as above we have the corresponding characteristic ideals $\text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(M_{\text{torsion}})$ and $\text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(N_{\text{torsion}})$. Then quotienting out the corresponding torsion part we have:

$$\bar{f} : M/M_{\text{torsion}} \rightarrow N/N_{\text{torsion}}. \quad (6.5.9)$$

We will regard this as a corresponding map on the sheaves over the the space $\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}$, then for any $Y = \text{Sp}(\Pi^\infty(\Gamma')_p)$ quasicompact of $X_{\Pi^\infty(\Gamma')}$ we consider the corresponding ring $\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')_p$, then we have the notion of determinant of \bar{f} which we will denote it by $\det_{\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')_p}(\bar{f})$. We then organize this construction to get the determinant sheaf $\det_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(\bar{f})$. Then now we define the determinant of f as:

$$\det_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(f) \quad (6.5.10)$$

$$:= \det_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(\bar{f}) \text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(M_{\text{torsion}}) \text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} X_{\Pi^\infty(\Gamma')}}(N_{\text{torsion}})^{-1}. \quad (6.5.11)$$

On the corresponding derived level we consider after [Nakamura1, Section 3.4] the following construction for the big exponential map:

$$\text{Exp}_{\mathcal{F},m}^{\Pi^\infty(\Gamma'),\bullet} : R^\bullet \Gamma_{\text{sheaf}}(\varinjlim \mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p) \rightarrow R^\bullet \Gamma_{\text{sheaf}}(\varinjlim \mathcal{F}(M)_p).$$

For each p we consider the localized map:

$$\text{Exp}_{\mathcal{F},m}^{\Pi^\infty(\Gamma')_p,\bullet} : R^\bullet \Gamma_{\text{sheaf}}(\mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p) \rightarrow R^\bullet \Gamma_{\text{sheaf}}(\mathcal{F}(M)_p).$$

Then what we can do is to define the corresponding determinant of the this by:

$$\det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(\text{Exp}_{\mathcal{F},m}^{\Pi^\infty(\Gamma')_p,\bullet}) := \quad (6.5.12)$$

$$\det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^1\Gamma_{\text{sheaf}}(\mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p) \rightarrow R^1\Gamma_{\text{sheaf}}(\mathcal{F}(M)_p)). \quad (6.5.13)$$

$$\det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^2\Gamma_{\text{sheaf}}(\mathcal{F}(\Theta_{\text{Ber,dif}}(M))_p) \rightarrow R^2\Gamma_{\text{sheaf}}(\mathcal{F}(M)_p))^{-1}. \quad (6.5.14)$$

Then we can organize this to be the corresponding determinant sheaf:

$$\det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(\text{Exp}_{\mathcal{F},m}^{X_{\Pi^\infty(\Gamma')_p,\bullet}}) \quad (6.5.15)$$

$$= \det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^1\Gamma_{\text{sheaf}}(\mathcal{F}(\{\Theta_{\text{Ber,dif}}(M)\}_p)_p) \rightarrow R^1\Gamma_{\text{sheaf}}(\{\mathcal{F}(M)_p\}_p)). \quad (6.5.16)$$

$$\det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^2\Gamma_{\text{sheaf}}(\mathcal{F}(\{\Theta_{\text{Ber,dif}}(M)\}_p)_p) \rightarrow R^2\Gamma_{\text{sheaf}}(\{\mathcal{F}(M)_p\}_p))^{-1} \quad (6.5.17)$$

$$= \det_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^1\Gamma_{\text{sheaf}}(\mathcal{F}(\{\Theta_{\text{Ber,dif}}(M)\}_p)_p) \rightarrow R^1\Gamma_{\text{sheaf}}(\{\mathcal{F}(M)_p\}_p)). \quad (6.5.18)$$

$$\text{char}_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^2\Gamma_{\text{sheaf}}(\mathcal{F}(\{\Theta_{\text{Ber,dif}}(M)\}_p)_p))^{-1}. \quad (6.5.19)$$

$$\text{char}_{\Pi^\infty(\Gamma_K)\widehat{\otimes}\Pi^\infty(\Gamma')_p}(R^2\Gamma_{\text{sheaf}}(\{\mathcal{F}(M)_p\}_p)). \quad (6.5.20)$$

The following conjecture reads that in our current situation we have the corresponding equivariant Iwasawa main conjecture. But we will only consider the situation where $K = \mathbb{Q}_p$ and the corresponding full admissible p -adic Lie extension after [LZ1] namely we consider $K_\infty = \mathbb{Q}_p^{\text{ur}}(\mu_{p^\infty})$. We first recall Nakamura's result in [Nakamura1]:

Theorem 6.5.10. (Nakamura's Iwasawa Main Conjecture) *In our current notations, let Γ' be trivial. Consider a de Rham (φ, Γ) module M over the usual Robba ring. We assume that this module has rank g with g Hodge-Tate weights a_1, \dots, a_g . Then we have the following formula linking the determinant of the Perrin-Riou-Nakamura big exponen-*

tial map and the characteristic ideals in the equivariant way:

$$\prod_i \prod_{b_i}^{g-a_i-1} \nabla_{a_i+b_i} \tag{6.5.21}$$

$$= \det_{\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')} (R^1 \Gamma_{\text{sheaf}}(\mathcal{F}(\{\Theta_{\text{Ber,dif}}(M)\}_p)_p) \rightarrow R^1 \Gamma_{\text{sheaf}}(\{\mathcal{F}(M)_p\}_p)). \tag{6.5.22}$$

$$\text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')} (R^2 \Gamma_{\text{sheaf}}(\mathcal{F}(\{\Theta_{\text{Ber,dif}}(M)\}_p)_p))^{-1}. \tag{6.5.23}$$

$$\text{char}_{\Pi^\infty(\Gamma_K) \widehat{\otimes} \Pi^\infty(\Gamma')} (R^2 \Gamma_{\text{sheaf}}(\{\mathcal{F}(M)_p\}_p)). \tag{6.5.24}$$

Then in this situation:

Conjecture 6.5.11. (Equivariant Iwasawa Main Conjecture, after Nakamura)

Consider a de Rham (φ, Γ) module M over the usual Robba ring. We assume that this module has rank g with g Hodge-Tate weights a_1, \dots, a_g . Then we conjecture we have the corresponding analog of the formula linking the determinant of the Perrin-Riou-Nakamura big exponential map and the characteristic ideals in the equivariant way.

Remark 6.5.12. This is also inspired by the work of [LVZ] and [LZ1], while we work in the generality after [Nakamura1]. Although we have not mentioned explicitly but we definitely conjecture that one has the parallel statement for Lubin-Tate differential equations after [BerFou] in the equivariant setting.

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- Tong, Xin. "Category and Cohomology of Hodge-Iwasawa Modules." arXiv preprint arXiv:2012.07308 (2020).

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Chapter 7

Period Rings with Big Coefficients and Applications I

7.1 Introduction

7.1.1 Some Motivation of Big Coefficients

We initiate our development on the corresponding period rings with big coefficients and application, with the corresponding motivation partially coming from our Hodge-Iwasawa theory (see [T1], [T2]), although we do not want to restrict ourselves to Hodge-Iwasawa theory. In our previous work [T1], [T2] and actually also [T3], we developed very carefully the corresponding deformation of various type of relative p -adic Hodge structures over rigid analytic spaces. It is not necessary to just consider things over rigid analytic spaces namely those analytic spaces which admit atlas made up of strictly affinoids. We first want to mention that one could have the chance to generalize this.

One of our target here is the corresponding adic space coefficient consideration, which is natural if one looks at the corresponding context of [CKZ] or [PZ], where one has some multivariate Robba rings which admit multi partial Frobenius actions. For instance when we have two variables taking quotient by one partial Frobenius will basically give us some adic space coefficients. The remaining Frobenius gives rise to some relative Frobenius Hodge structures.

The corresponding [CKZ] actually proposed that one can also even consider the product version of the corresponding context of [KL1] and [KL2], which is our second goal here. We want to systematically consider deforming the Frobenius modules structures of [KL1] and [KL2] after our work [T1] and [T2]. We focus on the translating the corresponding results in [T1] and [T2] to the corresponding context in the current adic space consideration, again after [KL1], [KL2] as well. We hope to work in full generality, namely over perfectoid uniform adic Banach coefficients and those adic space coefficients forming from the quotient of the perfectoid uniform adic Banach ones.

In the situation where we have the corresponding Kedlaya-Liu's perfect Robba rings with general coefficients in some Banach rings, one might have to impose some of the corresponding sheafiness condition. But we would like to use the ∞ -Huber spectrum given by [BK] to tackle this from some other derived perspective where we do have the

corresponding derived sheafness. Definitely this is very complicated to manipulate, but we initiate some of the discussion though not complete at all. We hope we could come back to this systematically later at some point.

Finally we revisit the corresponding noncommutative Hodge-Iwasawa theory we initiated in the corresponding papers [T1] and [T2]. We use some observation coming from Kedlaya to tackle the corresponding noncommutative descent for finite projective datum.

7.1.2 Results

In this section we make some summary on our main results covered in the main body of our current paper:

1. The first scope of the discussion is around some perfectoid or preperfectoid deformation of essentially the Frobenius modules and the corresponding quasi-coherent sheaves over the deformed version of the adic Fargues-Fontaine curves by perfectoids or preperfectoids. In theorem 7.3.16, we consider the comparison between the following objects (with the corresponding notations in theorem 7.3.16):

- A. The pseudocoherent sheaves over the adic relative Fargues-Fontaine curve $Y_{\text{FF},R,A}$ where A is a perfectoid adic Banach uniform algebra over E , here we consider étale topology;
- B. The pseudocoherent sheaves over the adic relative Fargues-Fontaine curve $Y_{\text{FF},R,A}$ where A is a perfectoid adic Banach uniform algebra over E , here we consider pro-étale topology;
- C. The pseudocoherent modules over the relative A -coefficient Robba ring $\tilde{\Pi}_{R,A}$, carrying the Frobenius action, where A is a perfectoid adic Banach uniform algebra over E ;
- D. The pseudocoherent modules over the relative A -coefficient Robba ring $\tilde{\Pi}_{R,A}^{\infty}$, carrying the Frobenius action, where A is a perfectoid adic Banach uniform algebra over E ;
- E. The pseudocoherent bundles over the relative A -coefficient Robba ring $\tilde{\Pi}_{R,A}$, carrying the Frobenius action, where A is a perfectoid adic Banach uniform algebra over E ;
- F. The pseudocoherent bundles over the relative A -coefficient Robba ring $\tilde{\Pi}_{R,A}^{[s,r]}$, carrying

the Frobenius action, where A is a perfectoid adic Banach uniform algebra over E , $0 < s \leq r/p^{ah}$.

Theorem 7.1.1. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *The categories mentioned above are equivalent to each other. (See theorem 7.3.16.)*

Now for an adic space D in the pro-étale topology which could be covered by perfectoid subdomains over E , we have the following categories (objects defined over D are actually compatible families):

- A. The pseudocoherent sheaves over the adic relative Fargues-Fontaine curve $Y_{\text{FF},R,D}$, here we consider étale topology;
- B. The pseudocoherent sheaves over the adic relative Fargues-Fontaine curve $Y_{\text{FF},R,D}$, here we consider pro-étale topology;
- C. The pseudocoherent modules over the relative D -coefficient Robba ring $\tilde{\Pi}_{R,D}$, carrying the Frobenius action;
- D. The pseudocoherent modules over the relative D -coefficient Robba ring $\tilde{\Pi}_{R,D}^{\infty}$, carrying the Frobenius action;
- E. The pseudocoherent bundles over the relative D -coefficient Robba ring $\tilde{\Pi}_{R,D}$, carrying the Frobenius action;
- F. The pseudocoherent bundles over the relative D -coefficient Robba ring $\tilde{\Pi}_{R,D}^{[s,r]}$, carrying the Frobenius action, $0 < s \leq r/p^{ah}$.

Theorem 7.1.2. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *The categories mentioned above are equivalent to each other. (See theorem 7.3.15.)*

We then apply to the corresponding construction to the corresponding Fargues-Fontaine curve $Y_{\text{FF},R}$ we have the following categories:

- A. The pseudocoherent sheaves over the adic relative Fargues-Fontaine curve $Y_{\text{FF},R,Y_{\text{FF},R}}$, here we consider étale topology;
- B. The pseudocoherent sheaves over the adic relative Fargues-Fontaine curve $Y_{\text{FF},R,Y_{\text{FF},R}}$, here we consider pro-étale topology;
- C. The pseudocoherent modules over the relative $Y_{\text{FF},R}$ -coefficient Robba ring $\tilde{\Pi}_{R,Y_{\text{FF},R}}$,

carring the Frobenius action;

D. The pseudocoherent modules over the relative $Y_{\text{FF},R}$ -coefficient Robba ring $\tilde{\Pi}_{R,Y_{\text{FF},R}}^{\infty}$, carring the Frobenius action;

E. The pseudocoherent bundles over the relative $Y_{\text{FF},R}$ -coefficient Robba ring $\tilde{\Pi}_{R,Y_{\text{FF},R}}$, carring the Frobenius action;

F. The pseudocoherent bundles over the relative $Y_{\text{FF},R}$ -coefficient Robba ring $\tilde{\Pi}_{R,Y_{\text{FF},R}}^{[s,r]}$, carring the Frobenius action, $0 < s \leq r/p^{ah}$.

Theorem 7.1.3. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *The categories mentioned above are equivalent to each other. (See corollary 7.3.17.)*

2. For general Banach objects, we believe the corresponding picture will still be very interesting and have more potential applications. At least the relative analytic geometry (over the period rings) will be very interesting. That being said, the corresponding study of such relative geometry is really not that easy. The first difficulty will be essentially the sheafiness. Bambozzi-Kremnizer established some derived sheafiness when we replace the Huber's spectrum with the derived ones in [BK]. For any Banach ring S we have (with the notation in the section 7.4) the ∞ -analytic space $\text{Spa}^h(S)$. We then apply to our period rings with general Banach coefficient A we have the corresponding ∞ -period rings. And we then look at the following categories (with the notation in the section 7.4):

A. The corresponding f -projective Frobenius φ^a -bundles over the ∞ -Robba ring $\tilde{\Pi}_{R,A}^h$, here A is arbitrary Banach adic uniform algebra over \mathbb{Q}_p ;

B. The corresponding f -projective Frobenius φ^a -modules over the ∞ -Robba ring $\tilde{\Pi}_{R,A}^{[s,r],h}$, here $0 < s \leq r/p^{ah}$.

Theorem 7.1.4. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *The categories mentioned above are equivalent to each other. (See theorem 7.4.12.)*

3. We revisit the corresponding noncommutative descent which was initiated in [T2]. We will noncommutativate some argument due to Kedlaya on the corresponding Kiehl's glueing properties without the noetherian assumption. This means that we could actually

descent the corresponding finite projective modules in the corresponding noncommutative setting.

Proposition 7.1.5. *The descent for finite projective bimodules over noncommutative Banach rings holds under the conditions in [KL1, Definition 2.7.3 (a),(b)]. (See proposition 7.5.12.)*

Then we apply this to p -adic Hodge theory, we look at the following categories:

- A. The corresponding finite projective Frobenius φ^a -bundles over the Robba ring $\tilde{\Pi}_{R,A}$, here A is arbitrary Banach algebra over \mathbb{Q}_p ;
- B. The corresponding finite projective Frobenius φ^a -modules over the Robba ring $\tilde{\Pi}_{R,A}^{[s,r]}$, here $0 < s \leq r/p^{ah}$.

Proposition 7.1.6. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *The categories mentioned above are equivalent to each other. (See proposition 7.5.13.)*

7.1.3 Convention and Notation

We make some convention here. When we say an ∞ -analytic stack, we will mean a ∞ -sheaf fibered over the corresponding category of all the seminormed commutative monoids satisfying the condition and framework in [BBBK]. Please note the difference between a derived analytic stack, an ∞ -analytic stack and a stack, fibered over category of seminormed commutative monoids, as in [BBBK].

7.1.4 Remarks on Future Work

We will consider more general ∞ -glueing in the style of Kedlaya-Liu [KL2] after the foundations we have applied from [BK] (or possibly equivalently work of Clausen-Scholze [CS]) whenever one would study more general quasi-coherent sheaves over some sheaves of Banach \mathcal{O} -algebras over reasonable sites essentially encoded in our current work.

We have not considered the corresponding imperfect setting of the corresponding period rings with really big coefficients along the corresponding style and fashions we established in [T1] and [T2] in the context of reasonable towers in [KL1] and [KL2]. We would like to study the corresponding commutative and noncommutative setting in future work along these towers going upside and down. There will be some consequences for some specific analytic spaces. Also we could basically contact the corresponding ∞ -context as well by considering the corresponding sheaves of ∞ -Banach algebras over some interesting pro-étale site over adic spaces.

7.2 Period Rings with Coefficients in General Adic Spaces

7.2.1 Period Rings with Coefficients in General Banach Adic Rings

Setting 7.2.1. We now consider the corresponding Kedlaya-Liu's period sheaves with the big coefficients in general adic spaces. We will consider those adic spaces which could be written as the corresponding quotients of the corresponding affinoid (pre-)perfectoid spaces. To be more precise we look at the following rings, we first consider some finite multi-intervals $[s_I, r_I]$ (with real bounds) with $0 \leq s_\alpha \leq r_\alpha < \infty$ for each $\alpha \in I$ and we consider the corresponding base field E which is complete nonarchimedean discrete valued carrying some Banach norm, taking the form of \mathbb{Q}_p or $\mathbb{F}_p((\bar{t}))$ with the uniform notation π for chosen uniformizer. We then consider over E the corresponding E -strictly affinoids:

$$E\{s_1/T_1, s_2/T_2, \dots, s_n/T_n, T_1/r_1, \dots, T_n/r_n\} \quad (7.2.1)$$

and we have the corresponding Fréchet rings:

$$\varprojlim_{r_\alpha \rightarrow 1, \forall \alpha \in I} E\{s_1/T_1, s_2/T_2, \dots, s_n/T_n, T_1/r_1, \dots, T_n/r_n\} \quad (7.2.2)$$

with the corresponding ind-Fréchet rings:

$$\varinjlim_{s_\alpha \rightarrow 0^+, \forall \alpha \in I} \varprojlim_{r_\alpha \rightarrow 1, \forall \alpha \in I} E\{s_1/T_1, s_2/T_2, \dots, s_n/T_n, T_1/r_1, \dots, T_n/r_n\}. \quad (7.2.3)$$

We then consider the following:

$$A_{[s_I, r_I]} \quad (7.2.4)$$

$$= E(\pi^{1/p^\infty})\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge, \quad (7.2.5)$$

$$A_{[s_I, r_I]}^+ \quad (7.2.6)$$

$$= \mathcal{O}[\pi^{1/p^\infty}]\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge. \quad (7.2.7)$$

And we also consider the corresponding Fréchet ones:

$$A_{s_I} = \tag{7.2.8}$$

$$\varprojlim_{r_\alpha \rightarrow 1, \forall \alpha \in I} E(\pi^{1/p^\infty})\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge, \tag{7.2.9}$$

$$A_{r_I}^+ = \tag{7.2.10}$$

$$\varprojlim_{r_\alpha \rightarrow 1, \forall \alpha \in I} \mathcal{O}[\pi^{1/p^\infty}]\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge. \tag{7.2.11}$$

Setting 7.2.2. We use the corresponding notations as in the following to denote the corresponding preperfectoid ones:

$$A'_{[s_I, r_I]} = E\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge, \tag{7.2.12}$$

$$A'^+_{[s_I, r_I]} = \mathcal{O}\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge. \tag{7.2.13}$$

And we also consider the corresponding Fréchet ones:

$$A'_{s_I} = \varprojlim_{r_\alpha \rightarrow 1, \forall \alpha \in I} E\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge, \tag{7.2.14}$$

$$A'^+_{r_I} = \varprojlim_{r_\alpha \rightarrow 1, \forall \alpha \in I} \mathcal{O}\{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge. \tag{7.2.15}$$

Then we consider the corresponding context of [T2, Chapter 1], where with trivial coefficients we just consider the same context after [KL2]. First we have the corresponding consideration, let F be the corresponding field which is a complete discrete valued nonarchimedean field with uniformizer π with normalized Banach norm $\|\cdot\|_F$ such the corresponding evaluation on π is just p^{-1} , and we assume the residue field of F is just \mathbb{F}_{p^h} for some integer $h > 0$. Now we define the following big Robba rings:

Definition 7.2.3. (After Kedlaya-Liu [KL2, Definition 4.1.1]) We let A be a perfectoid adic uniform Banach algebra over E with integral subring \mathcal{O}_A over \mathcal{O}_E . Recall

from the corresponding context in [KL1] we have the corresponding period rings in the relative setting. We follow the corresponding notations we used in [T2, Section 2.1] for those corresponding period rings. We first have for a pair (R, R^+) where R is a uniform perfect adic Banach ring over the assumed base \mathcal{O}_F . Then we take the corresponding generalized Witt vectors taking the form of $W_{\mathcal{O}_E}(R)$, which is just the ring $\tilde{\Omega}_R^{\text{int}}$, and by inverting the corresponding uniformizer we have the ring $\tilde{\Omega}_R$, then by taking the completed product with A we have $\tilde{\Omega}_{R,A}$. Now for some $r > 0$ we consider the ring $\tilde{\Pi}_R^{\text{int},r}$ which is the completion of $W_{\mathcal{O}_E}(R^+)[[R]]$ (note that this is actually just $W_{\mathcal{O}_E}(R^+)[[x], x \in R]$) by the norm $\|\cdot\|_{\alpha^r}$ defined by:

$$\|\cdot\|_{\alpha^r} \left(\sum_{n \geq 0} \pi^n [\bar{r}_n] \right) = \sup_{n \geq 0} \{ p^{-n} \alpha(\bar{r}_n)^r \}. \quad (7.2.16)$$

Then we have the product $\tilde{\Pi}_{R,A}^{\text{bd},r}$ defined as the completion under the product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_A$ of the corresponding ring $\tilde{\Pi}_R^{\text{bd},r} \otimes_E A$, which could be also defined from the corresponding integral Robba rings defined above. Then we define the corresponding Robba ring for some interval $I \subset (0, \infty)$ with coefficient in the perfectoid ring A denoted by $\tilde{\Pi}_{R,A}^I$ as the following complete tensor product:

$$\tilde{\Pi}_R^I \hat{\otimes}_E A$$

under the the corresponding tensor product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_A$. Then we set $\tilde{\Pi}_{R,A}^r$ as $\varprojlim_{s \rightarrow 0} \tilde{\Pi}_{R,A}^{[s,r]}$, and then we define $\tilde{\Pi}_{R,A}$ as $\varinjlim_{r \rightarrow \infty} \tilde{\Pi}_{R,A}^{[s,r]}$, and we define $\tilde{\Pi}_{R,A}^\infty$ as $\varprojlim_{r \rightarrow \infty} \tilde{\Pi}_{R,A}^r$. And we also have the corresponding full integral Robba ring and the corresponding full bounded Robba ring by taking the corresponding union through all $r > 0$.

One can even consider the corresponding coefficients in the corresponding Banach adic uniform algebra over the base field.

Definition 7.2.4. (After Kedlaya-Liu [KL2, Definition 4.1.1]) We let B be an adic uniform Banach algebra over E with integral subring \mathcal{O}_B over \mathcal{O}_E . Recall from the corresponding context in [KL1] we have the corresponding period rings in the relative setting. We follow the corresponding notations we used in [T2, Section 2.1] for those corresponding period rings. We first have for a pair (R, R^+) where R is a uniform perfect adic Banach ring over the assumed base \mathcal{O}_F . Then we take the corresponding generalized Witt vectors taking the form of $W_{\mathcal{O}_E}(R)$, which is just the ring $\tilde{\Omega}_R^{\text{int}}$, and by inverting the

corresponding uniformizer we have the ring $\tilde{\Omega}_R$, then by taking the completed product with B we have $\tilde{\Omega}_{R,B}$. Now for some $r > 0$ we consider the ring $\tilde{\Pi}_R^{\text{int},r}$ which is the completion of $W_{\mathcal{O}_E}(R^+)[[r] : r \in R]$ by the norm $\|\cdot\|_{\alpha^r}$ defined by:

$$\|\cdot\|_{\alpha^r}(\sum_{n \geq 0} \pi^n [\bar{r}_n]) = \sup_{n \geq 0} \{p^{-n} \alpha(\bar{r}_n)^r\}. \quad (7.2.17)$$

Then we have the product $\tilde{\Pi}_{R,B}^{\text{bd},r}$ defined as the completion under the product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_B$ of the corresponding ring $\tilde{\Pi}_R^{\text{bd},r} \otimes_E B$, which could be also defined from the corresponding integral Robba rings defined above. Then we define the corresponding Robba ring for some interval $I \subset (0, \infty)$ with coefficient in the perfectoid ring B denoted by $\tilde{\Pi}_{R,B}^I$ as the following complete tensor product:

$$\tilde{\Pi}_R^I \widehat{\otimes}_E B$$

under the the corresponding tensor product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_B$. Then we set $\tilde{\Pi}_{R,B}^r$ as $\varprojlim_{s \rightarrow 0} \tilde{\Pi}_{R,B}^{[s,r]}$, and then we define $\tilde{\Pi}_{R,B}$ as $\varinjlim_{r \rightarrow \infty} \tilde{\Pi}_{R,B}^{[s,r]}$, and we define $\tilde{\Pi}_{R,B}^\infty$ as $\varprojlim_{r \rightarrow \infty} \tilde{\Pi}_{R,B}^r$. And we also have the corresponding full integral Robba ring and the corresponding full bounded Robba ring by taking the corresponding union through all $r > 0$.

In some situation we also need the corresponding preperfectoid coefficients for instance the local chart coming from the adic Fargues-Fontaine curves:

Definition 7.2.5. (After Kedlaya-Liu [KL2, Definition 4.1.1]) We let A be a preperfectoid adic uniform Banach algebra over E with integral subring \mathcal{O}_A over \mathcal{O}_E . Recall from the corresponding context in [KL1] we have the corresponding period rings in the relative setting. We follow the corresponding notations we used in [T2, Section 2.1] for those corresponding period rings. We first have for a pair (R, R^+) where R is a uniform perfect adic Banach ring over the assumed base \mathcal{O}_F . Then we take the corresponding generalized Witt vectors taking the form of $W_{\mathcal{O}_E}(R)$, which is just the ring $\tilde{\Omega}_R^{\text{int}}$, and by inverting the corresponding uniformizer we have the ring $\tilde{\Omega}_R$, then by taking the completed product with A we have $\tilde{\Omega}_{R,A}$. Now for some $r > 0$ we consider the ring $\tilde{\Pi}_R^{\text{int},r}$ which is the completion of $W_{\mathcal{O}_E}(R^+)[[r] : r \in R]$ by the norm $\|\cdot\|_{\alpha^r}$ defined by:

$$\|\cdot\|_{\alpha^r}(\sum_{n \geq 0} \pi^n [\bar{r}_n]) = \sup_{n \geq 0} \{p^{-n} \alpha(\bar{r}_n)^r\}. \quad (7.2.18)$$

Then we have the product $\tilde{\Pi}_{R,A}^{\text{bd},r}$ defined as the completion under the product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_A$ of the corresponding ring $\tilde{\Pi}_R^{\text{bd},r} \otimes_E A$, which could be also defined from the corresponding integral Robba rings defined above. Then we define the corresponding Robba ring for some interval $I \subset (0, \infty)$ with coefficient in the perfectoid ring A denoted by $\tilde{\Pi}_{R,A}^I$ as the following complete tensor product:

$$\tilde{\Pi}_R^I \widehat{\otimes}_E A$$

under the the corresponding tensor product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_A$. Then we set $\tilde{\Pi}_{R,A}^r$ as $\varprojlim_{s \rightarrow 0} \tilde{\Pi}_{R,A}^{[s,r]}$, and then we define $\tilde{\Pi}_{R,A}$ as $\varinjlim_{r \rightarrow \infty} \tilde{\Pi}_{R,A}^{[s,r]}$, and we define $\tilde{\Pi}_{R,A}^\infty$ as $\varprojlim_{r \rightarrow \infty} \tilde{\Pi}_{R,A}^r$. And we also have the corresponding full integral Robba ring and the corresponding full bounded Robba ring by taking the corresponding union through all $r > 0$.

In the previous situation where we have the more explicit representation of the corresponding perfectoid coefficients we can basically consider more explicit construction:

Definition 7.2.6. (After Kedlaya-Liu [KL2, Definition 4.1.1]) We let A be a perfectoid adic uniform Banach algebra over E with integral subring \mathcal{O}_A over \mathcal{O}_E taking the form of one of the rings $A_{[s_I, r_I]}$. Recall from the corresponding context in [KL1] we have the corresponding period rings in the relative setting. We follow the corresponding notations we used in [T2, Section 2.1] for those corresponding period rings. We first have for a pair (R, R^+) where R is a uniform perfect adic Banach ring over the assumed base \mathcal{O}_F . Then we take the corresponding generalized Witt vectors taking the form of $W_{\mathcal{O}_E}(R)$, which is just the ring $\tilde{\Omega}_R^{\text{int}}$, and by inverting the corresponding uniformizer we have the ring $\tilde{\Omega}_R$, then by taking the completed product with A we have $\tilde{\Omega}_{R,A}$. Now for some $r > 0$ we consider the ring $\tilde{\Pi}_{R,A}^{\text{int},r}$ which is the completion of $W_{\mathcal{O}_E}(R^+)[[r] : r \in R] \otimes \mathcal{O}_A$ by the norm $\|\cdot\|_{\alpha^r, A}$ defined by:

$$\|\cdot\|_{\alpha^r} \left(\sum_{n, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{r}_n] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k} \right) \quad (7.2.19)$$

$$= \sup_{n \geq 0} \{p^{-n} \alpha(\bar{r}_n)^r\}. \quad (7.2.20)$$

Then we have the product $\tilde{\Pi}_{R,A}^{\text{bd},r}$ defined as the completion under the product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_A$ of the corresponding ring $\tilde{\Pi}_R^{\text{bd},r} \otimes_E A$, which could be also defined from the corresponding integral Robba rings defined above. Then we define the corresponding

Robba ring for some interval $I \subset (0, \infty)$ with coefficient in the perfectoid ring A denoted by $\tilde{\Pi}_{R,A}^I$ as the following complete tensor product:

$$\tilde{\Pi}_R^I \widehat{\otimes}_E E(\pi^{1/p^\infty}) \{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge$$

under the the corresponding tensor product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_A$. Then we set $\tilde{\Pi}_{R,A}^r$ as $\varprojlim_{s \rightarrow 0} \tilde{\Pi}_{R,A}^{[s,r]}$, and then we define $\tilde{\Pi}_{R,A}$ as $\varinjlim_{r \rightarrow \infty} \tilde{\Pi}_{R,A}^{[s,r]}$, and we define $\tilde{\Pi}_{R,A}^\infty$ as $\varprojlim_{r \rightarrow \infty} \tilde{\Pi}_{R,A}^r$. And we also have the corresponding full integral Robba ring and the corresponding full bounded Robba ring by taking the corresponding union through all $r > 0$.

7.2.2 Properties

As in our work [T2] and [KL2], we can discuss some of the key properties of the corresponding period rings defined above.

Setting 7.2.7. We will mainly consider the corresponding coefficients in

$$E(\pi^{1/p^\infty}) \{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge$$

and their Banach uniform adic strict quotient through the following strict morphism:

$$A_{[s_I, r_I]} \rightarrow \overline{A_{[s_I, r_I]}} \rightarrow 0.$$

Also we consider the corresponding preperfectoid setting:

$$E \{(s_1/T_1)^{1/p^\infty}, (s_2/T_2)^{1/p^\infty}, \dots, (s_n/T_n)^{1/p^\infty}, (T_1/r_1)^{1/p^\infty}, \dots, (T_n/r_n)^{1/p^\infty}\}^\wedge$$

and their Banach uniform adic strict quotient through the following strict morphism:

$$A'_{[s_I, r_I]} \rightarrow \overline{A'_{[s_I, r_I]}} \rightarrow 0.$$

Proposition 7.2.8. (After Kedlaya-Liu [KL1, Lemma 5.2.6])

Under the corresponding definitions and the corresponding notations in our context we have the following identification:

$$\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, s} \cap \tilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r]} = \tilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, r} \quad (7.2.21)$$

$$\overline{\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, s}} \cap \overline{\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r]}} = \overline{\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, r}}. \quad (7.2.22)$$

Proof. For the first corresponding identity above we perform the corresponding argument as in [KL2, Lemma 5.2.6] and [T2]. First consider now arbitrary element h in the corresponding intersection on the left. Then we use a corresponding sequence of elements in the bounded Robba ring to approximate the corresponding element h above, say h_1, h_2, \dots . This will mean that for any k we could find some sufficiently large number P_k such that the following holds for any $t \in [s, r]$ and for any $i \geq P_k$:

$$\|\cdot\|_{\alpha^t, A_{[s_I, r_I]}}(h - h_i) \leq p^{-k}. \quad (7.2.23)$$

Then we are going to consider the corresponding decomposition of each element h_i in the following way:

$$h_i = \sum_{n \in \mathbb{Z}[1/p], i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.24)$$

Then we consider the corresponding integral part of this decomposition namely we consider:

$$g_i = \sum_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.25)$$

Now since we have that the elements involved are also living in the integral rings we have then that in the expression of h_i we have:

$$\|\cdot\|_{\alpha^s, A_{[s_I, r_I]}}(\pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}) \leq p^{-k}, \forall n < 0. \quad (7.2.26)$$

Then we have:

$$\alpha(\bar{h}_{i,n}) \leq p^{-(k-n)/s}. \quad (7.2.27)$$

Then we have:

$$\|\cdot\|_{\alpha^r, A_{[s_I, r_I]}}(h - g_i) \leq p^{-n} p^{-(i-n)r/s}.$$

This will prove the corresponding results since we can now use the corresponding elements in the integral Robba rings to approximate.

For the corresponding second corresponding identity we consider the corresponding strictness of the quotient in the original formation of the rings. Now take any element \bar{h} in the

quotient, then lift this to h . For the first corresponding identity above we perform the corresponding argument as in [KL2, Lemma 5.2.6] and [T2]. First consider now arbitrary element h in the corresponding intersection on the left. Then we use a corresponding sequence of elements in the bounded Robba ring to approximate the corresponding element h above, say h_1, h_2, \dots . This will mean that for any k we could find some sufficiently large number P_k such that the following holds for any $t \in [s, r]$ and for any $i \geq P_k$:

$$\|\cdot\|_{\alpha^t, A_{[s_I, r_I]}}(h - h_i) \leq p^{-k}. \quad (7.2.28)$$

Then we are going to consider the corresponding decomposition of each element h_i in the following way:

$$h_i = \sum_{n \in \mathbb{Z}[1/p], i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.29)$$

Then we consider the corresponding integral part of this decomposition namely we consider:

$$g_i = \sum_{n \in \mathbb{Z}[1/p]_{\geq 0}, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.30)$$

Now since we have that the elements involved are also living in the integral rings we have then that in the expression of h_i we have:

$$\|\cdot\|_{\alpha^s, A_{[s_I, r_I]}}(\pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}) \leq p^{-k}, \forall n < 0. \quad (7.2.31)$$

Then we have:

$$\alpha(\bar{h}_{i,n}) \leq p^{-(k-n)/s}. \quad (7.2.32)$$

Then we have:

$$\|\cdot\|_{\alpha^r, A_{[s_I, r_I]}}(h - g_i) \leq p^{-n} p^{-(i-n)r/s}.$$

This will prove the corresponding results since we can now use the corresponding elements in the integral Robba rings to approximate. Then we have that:

$$\|\cdot\|_{\alpha^r, \overline{A_{[s_I, r_I]}}}(\bar{h} - \bar{g}_i) \leq p^{-n} p^{-(i-n)r/s}, \quad (7.2.33)$$

which shows the corresponding approximating process in the strict quotient. □

Proposition 7.2.9. (After Kedlaya-Liu [KL1, Lemma 5.2.6]) *Under the corresponding definitions and the corresponding notations in our context we have the following identification:*

$$\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, s} \cap \tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r]} = \tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, r} \quad (7.2.34)$$

$$\frac{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, s}}{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}} \cap \frac{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r]}}{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}} = \frac{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, r}}{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}}. \quad (7.2.35)$$

Proof. For the first corresponding identity above we perform the corresponding argument as in [KL2, Lemma 5.2.6] and [T2]. First consider now arbitrary element h in the corresponding intersection on the left. Then we use a corresponding sequence of elements in the bounded Robba ring to approximate the corresponding element h above, say h_1, h_2, \dots . This will mean that for any k we could find some sufficiently large number P_k such that the following holds for any $t \in [s, r]$ and for any $i \geq P_k$:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(h - h_i) \leq p^{-k}. \quad (7.2.36)$$

Then we are going to consider the corresponding decomposition of each element h_i in the following way:

$$h_i = \sum_{n \in \mathbb{Z}, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.37)$$

Then we consider the corresponding integral part of this decomposition namely we consider:

$$g_i = \sum_{n \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.38)$$

Now since we have that the elements involved are also living in the integral rings we have then that in the expression of h_i we have:

$$\|\cdot\|_{\alpha^s, A'_{[s_I, r_I]}}(\pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}) \leq p^{-k}, \forall n < 0. \quad (7.2.39)$$

Then we have:

$$\alpha(\bar{h}_{i,n}) \leq p^{-(k-n)/s}. \quad (7.2.40)$$

Then we have:

$$\|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(h - g_i) \leq p^{-n} p^{-(i-n)r/s}.$$

This will prove the corresponding results since we can now use the corresponding elements in the integral Robba rings to approximate.

For the corresponding second corresponding identity we consider the corresponding strictness of the quotient in the original formation of the rings. Now take any element \bar{h} in the quotient, then lift this to h . For the first corresponding identity above we perform the corresponding argument as in [KL2, Lemma 5.2.6] and [T2]. First consider now arbitrary element h in the corresponding intersection on the left. Then we use a corresponding sequence of elements in the bounded Robba ring to approximate the corresponding element h above, say h_1, h_2, \dots . This will mean that for any k we could find some sufficiently large number P_k such that the following holds for any $t \in [s, r]$ and for any $i \geq P_k$:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(h - h_i) \leq p^{-k}. \quad (7.2.41)$$

Then we are going to consider the corresponding decomposition of each element h_i in the following way:

$$h_i = \sum_{n \in \mathbb{Z}, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.42)$$

Then we consider the corresponding integral part of this decomposition namely we consider:

$$g_i = \sum_{n \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}. \quad (7.2.43)$$

Now since we have that the elements involved are also living in the integral rings we have then that in the expression of h_i we have:

$$\|\cdot\|_{\alpha^s, A'_{[s_I, r_I]}}(\pi^n [\bar{h}_{i,n}] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}) \leq p^{-k}, \forall n < 0. \quad (7.2.44)$$

Then we have:

$$\alpha(\bar{h}_{i,n}) \leq p^{-(k-n)/s}. \quad (7.2.45)$$

Then we have:

$$\|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(h - g_i) \leq p^{-n} p^{-(i-n)r/s}.$$

This will prove the corresponding results since we can now use the corresponding elements in the integral Robba rings to approximate. Then we have that:

$$\|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(\bar{h} - \bar{g}_i) \leq p^{-n} p^{-(i-n)r/s}, \quad (7.2.46)$$

which shows the corresponding approximating process in the strict quotient. \square

Proposition 7.2.10. (After Kedlaya-Liu [KL1, Lemma 5.2.7])

We consider the Robba ring over some closed interval $\widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r]}$ we have the following decomposition for each element x in this ring into the following way:

$$x = y + z$$

where $y \in \pi^n \widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, r}$ and $z \in \widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r']}$ for each $r' \geq r$, and here $n > 0$ lives in $\mathbb{Z}[1/p]$.

With this decomposition we have the following estimate:

$$\|\cdot\|_{\alpha^t, A_{[s_I, r_I]}}(z) \leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A_{[s_I, r_I]}}(x)^{t/r}. \quad (7.2.47)$$

And we consider the Robba ring over some closed interval $\widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r]}$ we have the following decomposition for each element x in this ring into the following way:

$$x = y + z$$

where $y \in p^n \widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, r}$ and $z \in \widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r']}$ for each $r' \geq r$, and here $n > 0$ lives in $\mathbb{Z}[1/p]$.

With this decomposition we have the following estimate:

$$\|\cdot\|_{\alpha^t, \overline{A_{[s_I, r_I]}}} (z) \leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, \overline{A_{[s_I, r_I]}}} (x)^{t/r}. \quad (7.2.48)$$

Proof. For the first part, we first consider any element r in the bounded Robba ring. Each such element admits the corresponding expansion by the linear combination of the Teichmüller elements, each term takes the corresponding form as $\pi^l[r_l]$ summing up from some negative integer. Then the corresponding decomposition just reads:

$$y = \sum_{l \geq n, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^l[\bar{r}_l] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}, \quad (7.2.49)$$

$$z = r - y. \quad (7.2.50)$$

Then in this case we have the corresponding desired estimate:

$$\|\cdot\|_{\alpha^t, A_{[s_I, r_I]}} (\pi^l[\bar{r}_l] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}) \quad (7.2.51)$$

$$\leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A_{[s_I, r_I]}} (\pi^l[\bar{r}_l] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k})^{t/r}. \quad (7.2.52)$$

Then we choose a corresponding sequence of the corresponding elements in the bounded Robba ring to approximate any chosen arbitrary element in $\widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r]}$ such that we have:

$$\|\cdot\|_{\alpha^t, A_{[s_I, r_I]}}(r - r_0 - \dots - r_i) \leq p^{-1-i} \|\cdot\|_{\alpha^t, A_{[s_I, r_I]}}(r), \forall t \in [s, r].$$

Then apply the corresponding construction in the previous situation to each r_i we have that there is a corresponding decomposition:

$$r_i = y_i + z_i$$

with the corresponding desired extracted sum $\sum_i y_i$ which converges to the desired element y . While for z_i we consider the corresponding estimate as in the following:

$$\|\cdot\|_{\alpha^t, A_{[s_I, r_I]}}(z_i) \leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A_{[s_I, r_I]}}(r_i)^{t/r} \quad (7.2.53)$$

$$\leq p^{-i} p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A_{[s_I, r_I]}}(r_i)^{t/r}, \quad (7.2.54)$$

which shows the corresponding sum $\sum_i z_i$ converges to desired element z in the corresponding union of all the Robba rings $\widetilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s, r']}$ for all $r' \geq r$. Then for the second statement we just perform as before to look at the corresponding lifting of elements through the corresponding strict quotient morphism. □

Proposition 7.2.11. (After Kedlaya-Liu [KL1, Lemma 5.2.7])

We consider the Robba ring over some closed interval $\widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r]}$ we have the following decomposition for each element x in this ring into the following way:

$$x = y + z$$

where $y \in \pi^n \widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, r}$ and $z \in \widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r']}$ for each $r' \geq r$, and here $n > 0$ is some arbitrary given integer. With this decomposition we have the following estimate:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(z) \leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(x)^{t/r}. \quad (7.2.55)$$

And we consider the Robba ring over some closed interval $\widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r]}$ we have the following decomposition for each element x in this ring into the following way:

$$x = y + z$$

where $y \in p^n \widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, r}$ and $z \in \widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r']}$ for each $r' \geq r$, and here $n > 0$ lives in \mathbb{Z} . With this decomposition we have the following estimate:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(z) \leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(x)^{t/r}. \quad (7.2.56)$$

Proof. For the first part, we first consider any element r in the bounded Robba ring. Each such element admits the corresponding expansion by the linear combination of the Teichmüller elements, each term takes the corresponding form as $\pi^l[r_l]$ summing up from some negative integer. Then the corresponding decomposition just reads:

$$y = \sum_{l \geq n, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{Z}[1/p]_{\geq 0}} \pi^l[\bar{r}_l] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}, \quad (7.2.57)$$

$$z = r - y. \quad (7.2.58)$$

Then in this case we have the corresponding desired estimate:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(\pi^l[\bar{r}_l] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k}) \quad (7.2.59)$$

$$\leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(\pi^l[\bar{r}_l] (s_1/T_1)^{i_1} \dots (s_k/T_k)^{i_k} (T_1/r_1)^{j_1} \dots (T_k/r_k)^{j_k})^{t/r}. \quad (7.2.60)$$

Then we choose a corresponding sequence of the corresponding elements in the bounded Robba ring to approximate any chosen arbitrary element in $\widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r]}$ such that we have:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(r - r_0 - \dots - r_i) \leq p^{-1-i} \|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(r), \forall t \in [s, r].$$

Then apply the corresponding construction in the previous situation to each r_i we have that there is a corresponding decomposition:

$$r_i = y_i + z_i$$

with the corresponding desired extracted sum $\sum_i y_i$ which converges to the desired element y . While for z_i we consider the corresponding estimate as in the following:

$$\|\cdot\|_{\alpha^t, A'_{[s_I, r_I]}}(z_i) \leq p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(r_i)^{t/r} \quad (7.2.61)$$

$$\leq p^{-i} p^{(1-n)(1-t/r)} \|\cdot\|_{\alpha^r, A'_{[s_I, r_I]}}(r_i)^{t/r}, \quad (7.2.62)$$

which shows the corresponding sum $\sum_i z_i$ converges to desired element z in the corresponding union of all the Robba rings $\widetilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s, r']}$ for all $r' \geq r$. Then for the second statement we just perform as before to look at the corresponding lifting of elements through the corresponding strict quotient morphism. \square

Proposition 7.2.12. (After Kedlaya-Liu [KL1, Lemma 5.2.10]) *We could have the following result in our new context around the corresponding period rings with respect to two intervals $[s_1, r_1], [s_2, r_2]$ such that we have:*

$$0 < s_1 \leq s_2 \leq r_1 \leq r_2. \quad (7.2.63)$$

To be more precise in our current situation, we have:

$$\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s_1, r_1]} \cap \tilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s_2, r_2]} = \tilde{\Pi}_{R, A_{[s_1, r_2]}}^{\text{int}, r} \quad (7.2.64)$$

and

$$\frac{\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s_1, r_1]}}{\tilde{\Pi}_{R, A_{[s_I, r_I]}}} \cap \frac{\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{[s_2, r_2]}}{\tilde{\Pi}_{R, A_{[s_I, r_I]}}} = \frac{\tilde{\Pi}_{R, A_{[s_I, r_I]}}^{\text{int}, r}}{\tilde{\Pi}_{R, A_{[s_I, r_I]}}}. \quad (7.2.65)$$

Proof. See the proof of [KL1, Lemma 5.2.10]. \square

Proposition 7.2.13. (After Kedlaya-Liu [KL1, Lemma 5.2.10]) *We could have the following result in our new context around the corresponding period rings with respect to two intervals $[s_1, r_1], [s_2, r_2]$ such that we have:*

$$0 < s_1 \leq s_2 \leq r_1 \leq r_2. \quad (7.2.66)$$

To be more precise in our current situation, we have:

$$\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s_1, r_1]} \cap \tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s_2, r_2]} = \tilde{\Pi}_{R, A'_{[s_1, r_2]}}^{\text{int}, r} \quad (7.2.67)$$

and

$$\frac{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s_1, r_1]}}{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}} \cap \frac{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{[s_2, r_2]}}{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}} = \frac{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}^{\text{int}, r}}{\tilde{\Pi}_{R, A'_{[s_I, r_I]}}}. \quad (7.2.68)$$

Proof. See the proof of the previous proposition. \square

7.3 Period Rings and Sheaves with coefficient in Adic Spaces

7.3.1 Fundamental Settings

Scholze's diamonds [Sch1] are very general adic stacks where the category is big and convenient enough to include the corresponding perfectoid spaces and the corresponding seminormal rigid analytic spaces. The two spaces happen to be our main interests at the same time. However, we do not actually use the language of diamonds.

Now we use the notation D to denote a general adic space over E in the sense of [KL1] and [KL2]. Then we define the following period rings with coefficient in D by using the corresponding Banach algebra structures over E .

Definition 7.3.1. We now consider the following sheaves in adic topology, étale topology and pro-étale topology (which are indeed not just presheaves by using the corresponding Schauder basis):

$$\tilde{\Pi}_{R,D}, \tilde{\Pi}_{R,D}^{\infty}, \tilde{\Pi}_{R,D}^I, \tilde{\Pi}_{R,D}^r \quad (7.3.1)$$

which assign each affinoid subdomain or perfectoid subdomain $\mathrm{Spa}(A, A^+)$ of D to the following rings:

$$\tilde{\Pi}_{R,A}, \tilde{\Pi}_{R,A}^{\infty}, \tilde{\Pi}_{R,A}^I, \tilde{\Pi}_{R,A}^r. \quad (7.3.2)$$

To define Frobenius modules over D we first define the following local picture:

Definition 7.3.2. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding φ^a -modules over the corresponding period rings above, which we could use some uniform notation $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,B}$ to denote these. We define any φ^a -module over the period rings with index \emptyset above to be a finitely generated projective module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M$. We define any φ^a -module over the period rings with index r above to be a finitely generated projective module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M \otimes \Delta^{rp^{-ah}}$. We define any φ^a -module over the period rings with index $[s, r]$ above to be a finitely generated projective module over $\Delta^?$ carrying semilinear action

from the Frobenius operator φ^a such that $\varphi^{a*}M \otimes_{\Delta^{[sp-ha, rp-ha]}} \Delta^{[s, rp^{-ha}]} \xrightarrow{\sim} M \otimes \Delta^{[s, rp^{-ha}]}$. For a corresponding object over Δ^\emptyset , we assume the module is the base change from some module over Δ^{r_0} for some $r_0 > 0$.

Definition 7.3.3. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding pseudocoherent φ^a -modules over the corresponding period rings above, which we could use some uniform notation $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,B}$ to denote these. We define any pseudocoherent φ^a -module over the period rings with index above to be a pseudocoherent module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M$. We define any pseudocoherent φ^a -module over the period rings with index r above to be a pseudocoherent module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M \otimes \Delta^{rp^{-ah}}$. We define any pseudocoherent φ^a -module over the period rings with index $[s, r]$ above to be a pseudocoherent module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \otimes_{\Delta^{[sp-ha, rp-ha]}} \Delta^{[s, rp^{-ha}]} \xrightarrow{\sim} M \otimes \Delta^{[s, rp^{-ha}]}$. For a corresponding object over Δ^\emptyset , we assume the module is the base change from some module over Δ^{r_0} for some $r_0 > 0$. As in [KL2, Definition 4.4.4] we impose the corresponding topological condition on the corresponding modules by imposing that all the modules are complete with respect to the corresponding natural topology, and over the Robba rings with respect to some specific interval we assume that the corresponding modules are étale-stably pseudocoherent. And we assume the corresponding stability with respect to the base change as in [KL2, Theorem 4.6.1] for modules over $\Delta^{r_0}, \Delta^\infty$.

Definition 7.3.4. (After Kedlaya-Liu [KL2, Definition 4.4.6]) We now define the corresponding φ^a -bundles over the corresponding period rings above, which we could use some uniform notation $\Delta^?_{*,B}, ? = \emptyset, r, * = R$ to denote these. We define a φ^a -bundle over $\Delta^?_{*,B}, ? = \emptyset, r$ to be a compatible family of finitely generated projective φ^a -modules over $\Delta^?_{*,B}, ? = [s', r']$ for suitable $[s', r']$ (namely contained in $(0, r]$ if we are looking at the ring $\Delta^r_{*,B}$) satisfying the corresponding restriction compatibility and cocycle condition.

Definition 7.3.5. (After Kedlaya-Liu [KL2, Definition 4.4.6]) We now define the corresponding pseudocoherent φ^a -bundles over the corresponding period rings above, which we could use some uniform notation $\Delta^?_{*,B}, ? = \emptyset, r, * = R$ to denote these. We define a pseudocoherent φ^a -bundle over $\Delta^?_{*,B}, ? = \emptyset, r$ to be a compatible family of étale-

stably pseudocoherent φ^a -modules over $\Delta_{*,B}^?, ? = [s', r']$ for suitable $[s', r']$ (namely contained in $(0, r]$ if we are looking at the ring $\Delta_{*,B}^r$) satisfying the corresponding restriction compatibility and cocycle condition.

Remark 7.3.6. The p -adic functional analytic property we imposed on the algebraic pseudocoherent modules could be also translated to adic topology, which is the corresponding stably pseudocoherent one. This is very important whenever one works with the corresponding analytic topology instead of the corresponding pro-étale topology.

Then we define things over the adic spaces.

Definition 7.3.7. In adic topology, we define the φ^a -module over any

$\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of φ^a -modules over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each affinoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.8. In adic topology, we define a pseudocoherent φ^a -module over any

$\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of pseudocoherent φ^a -modules over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each affinoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.9. In adic topology, we define the φ^a -bundle over any

$\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of φ^a -bundles over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each affinoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.10. In adic topology, we define a pseudocoherent φ^a -bundle over any

$\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of pseudocoherent φ^a -bundles over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each affinoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.11. In pro-étale topology, we define the φ^a -module over any $\Delta^?, ? =$

$\emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of φ^a -modules over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each perfectoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.12. In pro-étale topology, we define a pseudocoherent φ^a -module over

any $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of pseudocoherent φ^a -modules over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each perfectoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.13. In pro-étale topology, we define the φ^a -bundle over any $\Delta^?, ? =$

$\emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of φ^a -bundles over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each perfectoid subdomain $\text{Spa}(A, A^+)$ of D .

Definition 7.3.14. In pro-étale topology, we define a pseudocoherent φ^a -bundle over any $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,D}$ to be a compatible family of pseudocoherent φ^a -bundles over $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,A}$ for each perfectoid subdomain $\text{Spa}(A, A^+)$ of D .

7.3.2 Fundamental Comparisons

Theorem 7.3.15. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories:*

1. *The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,D}$ in the corresponding étale topology;*
2. *The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{pro-étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,D}$ in the corresponding pro-étale topology;*
3. *The corresponding category of all the pseudocoherent bundles over the period ring $\tilde{\Pi}_{R,D}$, carrying the corresponding Frobenius action from the operator φ^a ;*
4. *The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,D}^\infty$, carrying the corresponding Frobenius action from the operator φ^a ;*
5. *The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,D}$, carrying the corresponding Frobenius action from the operator φ^a ;*
6. *The corresponding category of all the finitely generated projective modules over the period ring $\tilde{\Pi}_{R,D}^{[s,r]}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$.*

Then we have that they are equivalent.

This could be proved by the following theorem which is also considered in [T2] in the corresponding Hodge-Iwasawa theory:

Theorem 7.3.16. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories (A is a perfectoid algebra corresponding to some local chart of D in the previous theorem):*

1. *The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,A}$ in the corresponding étale topology;*
2. *The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{pro-étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,A}$ in the corresponding pro-étale topology;*
3. *The corresponding category of all the pseudocoherent bundles over the period ring $\tilde{\Pi}_{R,A}$,*

carrying the corresponding Frobenius action from the operator φ^a ;

4. The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,A}^\infty$, carrying the corresponding Frobenius action from the operator φ^a ;

5. The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,A}$, carrying the corresponding Frobenius action from the operator φ^a (note that here we assume that the module descends to some module carrying Frobenius over $\tilde{\Pi}_{R,A}^r$ for some radius $r_0 > 0$, which further base changes to some module carrying Frobenius over $\tilde{\Pi}_{R,A}^{[s,r]}$ for some interval $[s, r]$ which is assumed to be étale-stably pseudocoherent);

6. The corresponding category of all the finitely generated projective modules over the period ring $\tilde{\Pi}_{R,A}^{[s,r]}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$.

Then we have that they are equivalent.

Proof. The proof is parallel to [KL2, Theorem 4.6.1]. From 1,2 to 3, the corresponding results follow from [KL2, Theorem 4.4.3] due to the natural sheafiness since we are working over perfectoid spaces. Then to glue a bundle to get a module over the Robba ring over the corresponding Robba ring $\tilde{\Pi}_{R,A}^\infty$, we use the corresponding reified adic space $\text{Spra}(\tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}]}, \tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}], \text{Gr}})$ to cover the corresponding whole spaces, and use the corresponding Frobenius pullback to basically control the corresponding finiteness of the corresponding sections over each member in this covering, then we can derive the result from [KL2, Proposition 2.6.17]. The same strategy allows us to go from 3 to 4,5. The corresponding equivalence between 3 and 6 could be proved by the following argument. First from 3 to 6 we just take the corresponding projection. Then back we consider the corresponding Frobenius to control finiteness over each $\text{Spra}(\tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}]}, \tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}], \text{Gr}})$ for any $k \geq 0$, then glueing over any arbitrary interval by using the sheafiness of the period rings. One also checks the corresponding pseudocoherence as in [KL2, Theorem 4.6.1]. \square

Corollary 7.3.17. *We have the following categories are equivalent predicted in [CKZ] partially and implicitly (here R' is a perfect algebra with the same type as that of R but needs not to be the same as R):*

1. The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF}, R, Y_{\text{FF}, R'}}$ in the corresponding étale topology;

2. The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{pro-étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,Y_{\text{FF},R'}}$ in the corresponding pro-étale topology;
3. The corresponding category of all the pseudocoherent bundles over the period ring $\tilde{\Pi}_{R,Y_{\text{FF},R'}}$, carrying the corresponding Frobenius action from the operator φ^a ;
4. The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,Y_{\text{FF},R'}}^\infty$, carrying the corresponding Frobenius action from the operator φ^a ;
5. The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,Y_{\text{FF},R'}}$, carrying the corresponding Frobenius action from the operator φ^a ;
6. The corresponding category of all the finitely generated projective modules over the period ring $\tilde{\Pi}_{R,Y_{\text{FF},R'}}^{[s,r]}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$.

For some application one could consider the corresponding preperfectoid coefficients:

Theorem 7.3.18. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories (A is a preperfectoid algebra over E):*

1. The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,A}$ in the corresponding étale topology;
2. The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{pro-étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF},R,A}$ in the corresponding pro-étale topology;
3. The corresponding category of all the pseudocoherent bundles over the period ring $\tilde{\Pi}_{R,A}$, carrying the corresponding Frobenius action from the operator φ^a ;
4. The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,A}^\infty$, carrying the corresponding Frobenius action from the operator φ^a ;
5. The corresponding category of all the pseudocoherent modules over the period ring $\tilde{\Pi}_{R,A}$, carrying the corresponding Frobenius action from the operator φ^a (note that here we assume that the module descends to some module carrying Frobenius over $\tilde{\Pi}_{R,A}^r$ for some radius $r > 0$, which further base changes to some module carrying Frobenius over $\tilde{\Pi}_{R,A}^{[s,r]}$ for some interval $[s,r]$ which is assumed to be étale-stably pseudocoherent);
6. The corresponding category of all the finitely generated projective modules over the period ring $\tilde{\Pi}_{R,A}^{[s,r]}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$.

Then we have that they are equivalent.

Proof. The proof is parallel to [KL2, Theorem 4.6.1]. From 1,2 to 3, the corresponding results follow from [KL2, Theorem 4.4.3] due to the natural sheafiness since we are working over preperfectoid spaces. Then to glue a bundle to get a module over the Robba ring over the corresponding Robba ring $\tilde{\Pi}_{R,A}^\infty$, we use the corresponding reified adic space $\text{Spra}(\tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}]}, \tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}], \text{Gr}})$ to cover the corresponding whole spaces, and use the corresponding Frobenius pullback to basically control the corresponding finiteness of the corresponding sections over each member in this covering, then we can derive the result from [KL2, Proposition 2.6.17]. The same strategy allows us to go from 3 to 4,5. The corresponding equivalence between 3 and 6 could be proved by the following argument. First from 3 to 6 we just take the corresponding projection. Then back we consider the corresponding Frobenius to control finiteness over each $\text{Spra}(\tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}]}, \tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}], \text{Gr}})$ for any $k \geq 0$, then glueing over any arbitrary interval by using the sheafiness of the period rings. One also checks the corresponding pseudocoherence as in [KL2, Theorem 4.6.1]. □

Definition 7.3.19. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding φ^a - φ'^a -modules over the corresponding period rings, which we could use some uniform notation $\Delta^? \widehat{\otimes} \Delta^{?'} , ? , ?' = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_R$ to denote these. We define any φ^a - φ'^a -module over the period rings with index \emptyset above to be a finitely generated projective module over $\Delta^? \widehat{\otimes} \Delta^{?}'$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M$ where φ^a comes from the first factor, and carrying semilinear action from the Frobenius operator φ'^a such that $\varphi'^{a*} M \xrightarrow{\sim} M$ where φ'^a comes from the second factor, and we assume that this module descends to some $\Delta^r \widehat{\otimes} \Delta^{r'}$ for $0 < r, r' < \infty$. We define any φ^a - φ'^a -module over the period rings with index r_1, r_2 above to be a finitely generated projective module over $\Delta^? \widehat{\otimes} \Delta^{?}'$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M \otimes \Delta^{r_1 p^{-ah}} \widehat{\otimes} \Delta^{?}'$, carrying semilinear action from the Frobenius operator φ'^a such that $\varphi'^{a*} M \xrightarrow{\sim} M \otimes \Delta^? \widehat{\otimes} \Delta^{r_2 p^{-ah}}$. We define any φ^a - φ'^a -module over the period rings with index $[s_1, r_1], [s_2, r_2]$ above to be a finitely generated projective module over $\Delta^? \widehat{\otimes} \Delta^{?}'$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \otimes_{\Delta^{[s_1 p^{-ha}, r_1 p^{-ha}]}} \Delta^{[s_1, r_1 p^{-ha}]} \widehat{\otimes} \Delta^{?}' \xrightarrow{\sim} M \otimes \Delta^{[s_1, r_1 p^{-ha}]} \widehat{\otimes} \Delta^{?}'$, carrying semilinear action from the Frobenius operator φ'^a such that $\varphi'^{a*} M \otimes_{\Delta^? \widehat{\otimes} \Delta^{[s_2 p^{-ha}, r_2 p^{-ha}]}} \Delta^? \widehat{\otimes} \Delta^{[s_2, r_2 p^{-ha}]} \xrightarrow{\sim} M \otimes \Delta^? \widehat{\otimes} \Delta^{[s_2, r_2 p^{-ha}]}$. For a corresponding object over $\Delta^\emptyset \widehat{\otimes} \Delta^\emptyset$, we assume the module

is the base change from some module over $\Delta^{r_0} \widehat{\otimes} \Delta^{r'_0}$ for some $r_0, r'_0 > 0$, as mentioned above.

Definition 7.3.20. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding pseudocoherent φ^a - φ'^a -modules over the corresponding period rings above, which we could use some uniform notation $\Delta^? \widehat{\otimes} \Delta^{?'}$, $?, ?' = \emptyset, I, r, \infty, \Delta = \widetilde{\Pi}_R$ to denote these. We define any pseudocoherent φ^a - φ'^a -module over the period rings with index \emptyset above to be a pseudocoherent module over $\Delta^? \widehat{\otimes} \Delta^{?'}$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M$ where φ^a comes from the first factor, and carrying semilinear action from the Frobenius operator φ'^a such that $\varphi'^{a*} M \xrightarrow{\sim} M$ where φ'^a comes from the second factor, and we assume that this module descends to some $\Delta^r \widehat{\otimes} \Delta^{r'}$ for $0 < r, r' < \infty$. We define any pseudocoherent φ^a - φ'^a -module over the period rings with index r_1, r_2 above to be a pseudocoherent module over $\Delta^? \widehat{\otimes} \Delta^{?'}$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M \otimes \Delta^{r_1 p^{-ah}} \widehat{\otimes} \Delta^{?'}$, carrying semilinear action from the Frobenius operator φ'^a such that $\varphi'^{a*} M \xrightarrow{\sim} M \otimes \Delta^? \widehat{\otimes} \Delta^{r_2 p^{-ah}}$. We define any pseudocoherent φ^a - φ'^a -module over the period rings with index $[s_1, r_1], [s_2, r_2]$ above to be a pseudocoherent module over $\Delta^? \widehat{\otimes} \Delta^{?'}$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \otimes_{\Delta^{[s_1 p^{-ha}, r_1 p^{-ha}]} \widehat{\otimes} \Delta^{?'}} \Delta^{[s_1, r_1 p^{-ha}]} \widehat{\otimes} \Delta^{?'} \xrightarrow{\sim} M \otimes \Delta^{[s_1, r_1 p^{-ha}]} \widehat{\otimes} \Delta^{?'}$, carrying semilinear action from the Frobenius operator φ'^a such that $\varphi'^{a*} M \otimes_{\Delta^? \widehat{\otimes} \Delta^{[s_2 p^{-ha}, r_2 p^{-ha}]}} \Delta^? \widehat{\otimes} \Delta^{[s_2, r_2 p^{-ha}]} \xrightarrow{\sim} M \otimes \Delta^? \widehat{\otimes} \Delta^{[s_2, r_2 p^{-ha}]}$. For a corresponding object over $\Delta^\emptyset \widehat{\otimes} \Delta^\emptyset$, we assume the module is the base change from some module over $\Delta^{r_0} \widehat{\otimes} \Delta^{r'_0}$ for some $r_0, r'_0 > 0$. As in [KL2, Definition 4.4.4] we impose the corresponding topological condition on the corresponding modules by imposing that all the modules are complete with respect to the corresponding natural topology, and over the Robba rings with respect to some specific multi-interval we assume that the corresponding modules are étale-stably pseudocoherent.

Theorem 7.3.21. *We have the following categories are equivalent predicted in [CKZ]:*

1. *The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF}, R, Y_{\text{FF}, R}}$ in the corresponding étale topology;*
2. *The corresponding category of all the sheaves of pseudocoherent $\mathcal{O}_{\text{pro-étale}}$ -modules over the adic Fargues-Fontaine curve $Y_{\text{FF}, R, Y_{\text{FF}, R}}$ in the corresponding pro-étale topology;*
3. *The corresponding category of all the pseudocoherent modules over the period ring*

$\tilde{\Pi}_R^{[s,r]} \widehat{\otimes} \tilde{\Pi}_R^{[s',r']}$, carrying the corresponding partial Frobenius action from the operators φ^a and φ'^a from the first R and the second R respectively, here we assume that $0 < s \leq r/p^{ah} < \infty$ and $0 < s' \leq r'/p^{ah} < \infty$ (which are certainly assumed to be étale-stably pseudocoherent).

Proof. From 1,2 to 3, we consider the corresponding projection to the subspace with respect to the first factor namely:

$$\mathrm{Spa}(\tilde{\Pi}_{R, Y_{\mathrm{FF}, R}}^{[s,r]}, \tilde{\Pi}_{R, Y_{\mathrm{FF}, R}}^{[s,r],+}) / \varphi^{\mathbb{Z}}. \quad (7.3.3)$$

By the quasi-compactness of $Y_{\mathrm{FF}, R}$ we have that for some fixed nice $[s, r]$ the pseudocoherent \mathcal{O} -modules could be regarded in some equivalent way as pseudocoherent $\tilde{\Pi}_{R, Y_{\mathrm{FF}, R}}^{[s,r]}$ -modules carrying the partial Frobenius action from φ^a (namely the corresponding finiteness is preserved during this functor). Then consider the preperfectoid coefficients in $\tilde{\Pi}_R^{[s,r]}$ we have that this category could be further related to the one of all the pseudocoherent $\tilde{\Pi}_R^{[s,r]} \widehat{\otimes} \tilde{\Pi}_R^{[s',r']}$ -modules carrying the partial Frobenius action from φ^a and the partial Frobenius action from φ'^a . We then start from any object in 3, we then use the corresponding reified adic space:

$$\mathrm{Spr}(\tilde{\Pi}_R^{[sp^{-kha}, rp^{-kha}]} \widehat{\otimes} \tilde{\Pi}_R^{[s'p^{-k'ha}, r'p^{-k'ha}]}, (\tilde{\Pi}_R^{[sp^{-kha}, rp^{-kha}]} \widehat{\otimes} \tilde{\Pi}_R^{[s'p^{-k'ha}, r'p^{-k'ha}]})^{\mathrm{Gr}}) \quad (7.3.4)$$

to cover the corresponding spaces showing the corresponding essential surjectivity of the functor from 1 to 3. Again we need to apply [KL2, Proposition 2.6.17] to obtain the finiteness around the global section.

□

7.4 Sheafiness and ∞ -Period Rings and Sheaves

7.4.1 The ∞ -Period Rings and Sheaves

Here we would like to mention something related to the corresponding sheafiness of the Robba rings in our context. Namely when we discuss the corresponding modules over period rings with big coefficients we might come across the issues around the sheafiness. However recent work [BK] could let us get around the issue by just considering the ∞ -adic space $(\mathrm{Spa}^h(S), \mathcal{O}_{\mathrm{Spa}^h(S)}^{\mathrm{der}})$ attached a Banach algebra S , which is just an ∞ -analytic stack.

Following [BK, Introduction] we define the following ∞ -analytic space which are the corresponding coverings of the ∞ -version of Fargues-Fontaine curves as those considered in [KL1], [KL2] and [T2]. Here we drop the corresponding sheafiness condition on the ring $\tilde{\Pi}_{R,A}^{[s,r]}$ for any closed subinterval of $(0, \infty)$. Then we define:

$$X_{\mathrm{FF},R,A}^{\infty} := \varinjlim_{0 < s \leq r < \infty} \mathrm{Spa}^h(\tilde{\Pi}_{R,A}^{[s,r]})$$

which carries a corresponding derived sheaves of ∞ -rings, namely we have ringed ∞ -adic space:

$$\left(\varinjlim_{0 < s \leq r < \infty} \mathrm{Spa}^h(\tilde{\Pi}_{R,A}^{[s,r]}), \varinjlim_{0 < s \leq r < \infty} \mathcal{O}_{\mathrm{Spa}^h(\tilde{\Pi}_{R,A}^{[s,r]})}^{\mathrm{der}} \right). \quad (7.4.1)$$

Remark 7.4.1. In [BK], there are two different notions of the corresponding ∞ -Huber spectra attached to Banach algebras. We choose to focus on $(\mathrm{Spa}_{\mathrm{Rat}}^h(S), \mathcal{O}_{\mathrm{Spa}_{\mathrm{Rat}}^h(S)}^{\mathrm{der}})$ as our $(\mathrm{Spa}^h(S), \mathcal{O}_{\mathrm{Spa}^h(S)}^{\mathrm{der}})$ in this section. Recall the corresponding Tate acyclicity in this case still holds for derived standard rational localization.

Setting 7.4.2. Note (!) that in our current section we are going to use the notation A to denote any commutative Banach uniform adic algebra over \mathbb{Q}_p .

Definition 7.4.3. We define the corresponding ∞ deformed Fargues-Fontaine curves in general notation $Y_{\mathrm{FF},R,A}^{\infty}$ to be the corresponding quotient of the space by the \mathbb{Z} -power of the Frobenius operator which acts through the corresponding affinoids on the algebraic level.

The presheaf of ∞ -rings $\varprojlim_{0 < s \leq r < \infty} \mathcal{O}_{\mathrm{Spa}^h(\tilde{\Pi}_{R,A}^{[s,r]})}^{\mathrm{der}}$ are general limit of the corresponding sheaves $\mathcal{O}_{\mathrm{Spa}^h(\tilde{\Pi}_{R,A}^{[s,r]})}^{\mathrm{der}}$, which provides upgrading of the discussion in [T2]. So now we consider similarly as above the following ∞ -adic spaces:

$$\mathrm{Spa}^h \tilde{\Pi}_{R,A}, \mathrm{Spa}^h \tilde{\Pi}_{R,A}^\infty, \mathrm{Spa}^h \tilde{\Pi}_{R,A}^I, \mathrm{Spa}^h \tilde{\Pi}_{R,A}^r. \quad (7.4.2)$$

Then take the corresponding global section we have the following ∞ -period rings:

$$\tilde{\Pi}_{R,A}^h, \tilde{\Pi}_{R,A}^{\infty,h}, \tilde{\Pi}_{R,A}^{I,h}, \tilde{\Pi}_{R,A}^{r,h}. \quad (7.4.3)$$

Definition 7.4.4. We define the corresponding Frobenius (on the ∞ -rings) as in the corresponding non-derived situation through the corresponding affinoids on the algebraic level.

Definition 7.4.5. (After Lurie [Lu1, Corollary 7.2.2.9, Definition 7.2.2.10]) For general projective with finiteness condition we look at the following definition following [Lu1, Corollary 7.2.2.9]. We will use the notion a *f-projective* module over a ∞ -algebra S to mean a projective object M in the corresponding ∞ -category of all the S -modules such that we have $\pi_0 M$ is finite projective over $\pi_0 S$.

Definition 7.4.6. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding φ^a -modules over the corresponding ∞ -period rings above, which we could use some uniform notation $\Delta^{?,h}, ? = \emptyset, I, r, \infty$ to denote these. We define any φ^a -module over the ∞ -period rings with index \emptyset above to be a *f-projective* module over $\Delta^{?,h}$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M$. We define any φ^a -module over the derived period rings with index r above to be a *f-projective* module over $\Delta^{?,h}$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M \otimes \Delta^{rp^{-ah},h}$. We define any φ^a -module over the ∞ -period rings with index $[s, r]$ above to be a *f-projective* module over $\Delta^{?,h}$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \otimes \Delta^{[s, rp^{-ha}],h} \xrightarrow{\sim} M \otimes_{\Delta^{[sp^{-ha}, rp^{-ha}],h}} \Delta^{[s, rp^{-ha}],h}$. Here $\Delta = \tilde{\Pi}_{R,A}$. For a corresponding object over $\Delta^{\emptyset,h}$, we assume the module is the base changes from some module over $\Delta^{r_0,h}$ for some $r_0 > 0$.

Remark 7.4.7. We believe that one can discuss more general objects such as the corresponding coherent sheaves after [Lu1] and [Lu2], but at this moment let us just focus on the derived vector bundles.

Definition 7.4.8. (After Kedlaya-Liu [KL2, Definition 4.4.6]) We now define the corresponding φ^a -bundles over the corresponding ∞ -period rings above, which we could use some uniform notation $\Delta_{*,A}^{?,h}, ? = \emptyset, I, r, \infty, * = R$ to denote these. We define a φ^a -bundle over $\Delta_{*,A}^{?,h}, ? = \emptyset, r$ to be a compatible family of f -projective φ^a -modules over $\Delta_{*,A}^{?,h}, ? = [s', r']$ for suitable $[s', r']$ (namely contained in $(0, r]$) satisfying the corresponding restriction compatibility and cocycle condition.

7.4.2 Some Results on the ∞ -Descent

Conjecture 7.4.9. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories:*

1. *The corresponding category of all the bundles over the homotopical period ring $\tilde{\Pi}_{R,A}^h$, carrying the corresponding Frobenius action from the operator φ^a ;*
2. *The corresponding category of all the f -projective modules over the homotopical period ring $\tilde{\Pi}_{R,A}^{\infty,h}$, carrying the corresponding Frobenius action from the operator φ^a ;*
3. *The corresponding category of all the f -projective modules over the homotopical period ring $\tilde{\Pi}_{R,A}^{[s,r],h}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$.*

Then we have that they are equivalent.

In the corresponding sheafy case, we do have some good comparison:

Theorem 7.4.10. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *In the situation where the Robba ring $\tilde{\Pi}_{R,A}^{[s,r]}$ for any closed interval $[s, r]$ is stably uniform. Consider the following categories:*

1. *The corresponding category of all the bundles over the homotopical period ring $\tilde{\Pi}_{R,A}^h$, carrying the corresponding Frobenius action from the operator φ^a ;*
2. *The corresponding category of all the f -projective modules over the homotopical period ring $\tilde{\Pi}_{R,A}^{\infty,h}$, carrying the corresponding Frobenius action from the operator φ^a ;*
3. *The corresponding category of all the f -projective modules over the homotopical period ring $\tilde{\Pi}_{R,A}^{[s,r],h}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$. Then we have that they are equivalent.*

Proof. This amounts to the corresponding statement for modules over classical rings. The proof is parallel to [KL2, Theorem 4.6.1]. Then to glue a bundle to get a module over the

Robba ring over the corresponding Robba ring $\tilde{\Pi}_{R,A}^\infty$, we use the corresponding reified adic space $\text{Spra}(\tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}]}, \tilde{\Pi}_{R,A}^{[sp^{-kah}, rp^{-kah}], \text{Gr}})$ to cover the corresponding whole spaces, and use the corresponding Frobenius pullback to basically control the corresponding finiteness of the corresponding sections over each member in this covering, then we can derive the result from [KL2, Proposition 2.6.17, Corollary 2.6.10]. This will show the equivalence between 1 and 2. Then the functor from 1 to 3 is just the corresponding projection. On the other hand the functor from 3 to 1, we use the corresponding Frobenius to reach any interval taking the form of $[sp^{-nha}, rp^{-nha}]$ for any $n \in \mathbb{Z}$, then we still have to consider the corresponding extraction of a single module from two over some overlapped such specific intervals coming from the Frobenius. However this is achievable since for π_0 of the corresponding two modules we are done through [KL1, Theorem 1.3.9]. \square

Remark 7.4.11. The stably-uniform condition here relates directly to the construction of [BK].

Theorem 7.4.12. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories:*

1. *The corresponding category of all the bundles over the homotopical period ring $\tilde{\Pi}_{R,A}^h$, carrying the corresponding Frobenius action from the operator φ^a , such that the corresponding underlying modules are plat (here we assume that the modules are f -projective);*
2. *The corresponding category of all the f -projective modules over the homotopical period ring $\tilde{\Pi}_{R,A}^{[s,r],h}$, carrying the corresponding Frobenius action from the operator φ^a , where $0 < s \leq r/p^{ah}$, such that the corresponding underlying modules are plat.*

Then we have that they are equivalent.

Proof. Due to the fact that the ∞ -presheaf $\mathcal{O}_{\text{Spa}^h(\tilde{\Pi}_{R,A}^{I,h})}^{\text{der}}$ is a ∞ -sheaf, the proof reduces to [KL2, Theorem 4.6.1] which we also performed in [T2, Section 4.2]. To compare, the functor from 1 to 2 is just the corresponding projection. On the other hand the functor from 2 to 1, we use the corresponding Frobenius to reach any interval taking the form of $[sp^{-nha}, rp^{-nha}]$ for any $n \in \mathbb{Z}$, then we still have to consider the corresponding extraction of a single module from two over some overlapped such specific intervals coming from the Frobenius. However this is achievable since for π_0 of the corresponding two modules we are done through [KL1, Theorem 1.3.9]. \square

Remark 7.4.13. Professor Kedlaya has informed us that the condition on the spectrum in [KL1, Theorem 1.3.9 (b)] could be removed (see our amplification in section 7.5 below namely proposition 7.5.11). This made us believe above holds. We want to mention here that since as mentioned in [BK] the authors of [BK] believe there is deep relationship between [BK] and Clausen-Scholze's work [CS]. Therefore we believe there is deep relationship between ours and the possible ones after Clausen-Scholze [CS].

7.5 Noncommutative Descent Revisit

7.5.1 Big Noncommutative Coefficients

We now also consider the corresponding more general noncommutative Banach coefficients as in our previous work [T2], which is targeted at the corresponding K -theory of the general Robba rings.

Setting 7.5.1. Note (!) that in our current section we are going to use the notation B to denote any Banach algebra over E .

Definition 7.5.2. (After Kedlaya-Liu [KL2, Definition 4.1.1]) We let B be a Banach algebra over E with integral subring \mathcal{O}_B over \mathcal{O}_E . Recall from the corresponding context in [KL1] we have the corresponding period rings in the relative setting. We follow the corresponding notations we used in [T2, Section 2.1] for those corresponding period rings. We first have for a pair (R, R^+) where R is a uniform perfect adic Banach ring over the assumed base \mathcal{O}_F . Then we take the corresponding generalized Witt vectors taking the form of $W_{\mathcal{O}_E}(R)$, which is just the ring $\tilde{\Omega}_R^{\text{int}}$, and by inverting the corresponding uniformizer we have the ring $\tilde{\Omega}_R$, then by taking the completed product with B we have $\tilde{\Omega}_{R,B}$. Now for some $r > 0$ we consider the ring $\tilde{\Pi}_R^{\text{int},r}$ which is the completion of $W_{\mathcal{O}_E}(R^+)[[r] : r \in R]$ by the norm $\|\cdot\|_{\alpha^r}$ defined by:

$$\|\cdot\|_{\alpha^r} \left(\sum_{n \geq 0} \pi^n [\bar{r}_n] \right) = \sup_{n \geq 0} \{ p^{-n} \alpha(\bar{r}_n)^r \}. \quad (7.5.1)$$

Then we have the product $\tilde{\Pi}_{R,B}^{\text{bd},r}$ defined as the completion under the product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_B$ of the corresponding ring $\tilde{\Pi}_R^{\text{bd},r} \otimes_{\mathbb{Q}_p} B$, which could be also defined from the corresponding integral Robba rings defined above. Then we define the corresponding Robba ring for some interval $I \subset (0, \infty)$ with coefficient in the perfectoid ring B denoted by $\tilde{\Pi}_{R,B}^I$ as the following complete tensor product:

$$\tilde{\Pi}_R^I \hat{\otimes}_{\mathbb{Q}_p} B$$

under the the corresponding tensor product norm $\|\cdot\|_{\alpha^r} \otimes \|\cdot\|_B$. Then we set $\tilde{\Pi}_{R,B}^r$ as $\varprojlim_{s \rightarrow 0} \tilde{\Pi}_{R,B}^{[s,r]}$, and then we define $\tilde{\Pi}_{R,B}$ as $\varinjlim_{r \rightarrow \infty} \tilde{\Pi}_{R,B}^{[s,r]}$, and we define $\tilde{\Pi}_{R,B}^\infty$ as $\varprojlim_{r \rightarrow \infty} \tilde{\Pi}_{R,B}^r$. And we also have the corresponding full integral Robba ring and the corresponding full bounded Robba ring by taking the corresponding union through all $r > 0$.

We first define the corresponding right Frobenius modules and bundles:

Definition 7.5.3. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding right φ^a -modules over the corresponding period rings above, which we could use some uniform notation $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \widetilde{\Pi}_{R,B}$ to denote these. We define any right φ^a -module over the period rings with index above to be a finitely generated projective right module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M$. We define any right φ^a -module over the period rings with index r above to be a finitely generated projective right module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M \otimes \Delta^{rp^{-ah}}$. We define any right φ^a -module over the period rings with index $[s, r]$ above to be a finitely generated projective right module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \otimes_{\Delta^{[sp^{-ha}, rp^{-ha}]}} \Delta^{[s, rp^{-ha}]} \xrightarrow{\sim} M \otimes \Delta^{[s, rp^{-ha}]}$. For a corresponding object over Δ^\emptyset , we assume the module is the base changes from some module over Δ^{r_0} for some r_0 .

Definition 7.5.4. (After Kedlaya-Liu [KL2, Definition 4.4.6]) We now define the corresponding right φ^a -bundles over the corresponding period rings above, which we could use some uniform notation $\Delta_{*,B}^?, ? = \emptyset, I, r, \infty, * = R$ to denote these. We define a right φ^a -bundle over $\Delta_{*,B}^?, ? = \emptyset, r$ to be a compatible family of finitely generated projective right φ^a -modules over $\Delta_{*,B}^?, ? = [s', r']$ for suitable $[s', r']$ (namely contained in $(0, r]$) satisfying the corresponding restriction compatibility and cocycle condition.

We first define the corresponding left Frobenius modules and bundles:

Definition 7.5.5. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding left φ^a -modules over the corresponding period rings above, which we could use some uniform notation $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \widetilde{\Pi}_{R,B}$ to denote these. We define any left φ^a -module over the period rings with index above to be a finitely generated projective left module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} M$. We define any left φ^a -module over the period rings with index r above to be a finitely generated projective left module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*}M \xrightarrow{\sim} \Delta^{rp^{-ah}} \otimes M$. We define any left φ^a -module over the period rings with index $[s, r]$ above to be a finitely generated projective

left module over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\Delta^{[s, rp^{-ha}]} \otimes_{\Delta^{[sp^{-ha}, rp^{-ha}]}} \varphi^{a*} M \xrightarrow{\sim} \Delta^{[s, rp^{-ha}]} \otimes M$. For a corresponding object over Δ^\emptyset , we assume the module is the base changes from some module over Δ^{r_0} for some $r_0 > 0$.

Definition 7.5.6. (After Kedlaya-Liu [KL2, Definition 4.4.6]) We now define the corresponding left φ^a -bundles over the corresponding period rings above, which we could use some uniform notation $\Delta_{*,B}^?, ? = \emptyset, I, r, \infty, * = R$ to denote these. We define a left φ^a -bundle over $\Delta_{*,B}^?, ? = \emptyset, r$ to be a compatible family of finitely generated projective left φ^a -modules over $\Delta_{*,B}^?, ? = [s', r']$ for suitable $[s', r']$ (namely contained in $(0, r)$) satisfying the corresponding restriction compatibility and cocycle condition.

We first define the corresponding Frobenius bimodules and bibundles:

Definition 7.5.7. (After Kedlaya-Liu [KL2, Definition 4.4.4]) We define the corresponding φ^a -bimodules over the corresponding period rings above, which we could use some uniform notation $\Delta^?, ? = \emptyset, I, r, \infty, \Delta = \tilde{\Pi}_{R,B}$ to denote these. We define any φ^a -bimodule over the period rings with index above to be a finitely generated projective bimodule over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} M$. We define any φ^a -bimodule over the period rings with index r above to be a finitely generated projective bimodule over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\varphi^{a*} M \xrightarrow{\sim} \Delta^{rp^{-ah}} \otimes M$ and $\varphi^{a*} M \xrightarrow{\sim} M \otimes \Delta^{rp^{-ah}}$. We define any φ^a -bimodule over the period rings with index $[s, r]$ above to be a finitely generated projective bimodule over $\Delta^?$ carrying semilinear action from the Frobenius operator φ^a such that $\Delta^{[s, rp^{-ha}]} \otimes_{\Delta^{[sp^{-ha}, rp^{-ha}]}} \varphi^{a*} M \xrightarrow{\sim} \Delta^{[s, rp^{-ha}]} \otimes M$ and $\varphi^{a*} M \otimes_{\Delta^{[sp^{-ha}, rp^{-ha}]}} \Delta^{[s, rp^{-ha}]} \xrightarrow{\sim} M \otimes \Delta^{[s, rp^{-ha}]}$. For a corresponding object over Δ^\emptyset , we assume the module is the base changes from some module over Δ^{r_0} for some $r_0 > 0$.

Definition 7.5.8. (After Kedlaya-Liu [KL2, Definition 4.4.6]) We now define the corresponding φ^a -bibundles over the corresponding period rings above, which we could use some uniform notation $\Delta_{*,B}^?, ? = \emptyset, I, r, \infty, * = R$ to denote these. We define a φ^a -bibundle over $\Delta_{*,B}^?, ? = \emptyset, r$ to be a compatible family of finitely generated projective φ^a -bimodules over $\Delta_{*,B}^?, ? = [s', r']$ for suitable $[s', r']$ (namely contained in $(0, r)$) satisfying the corresponding restriction compatibility and cocycle condition.

7.5.2 Glueing Noncommutative Vector Bundles

The following is what we achieved in the corresponding paper [T2, Lemma 6.82]:

Proposition 7.5.9. *Consider the following exact sequence of Banach algebras satisfying the corresponding conditions in [KL1, Definition 2.7.3 (a),(b)]:*

$$0 \rightarrow \Pi \rightarrow \Pi_1 \bigoplus \Pi_2 \rightarrow \Pi_{12} \rightarrow 0. \quad (7.5.2)$$

And consider the corresponding right glueing datum (M_1, M_2, M_{12}) over the corresponding rings above, and we assume that the glueing datum is right finite projective. Then we have that the corresponding kernel M of

$$M_1 \bigoplus M_2 \rightarrow M_{12} \quad (7.5.3)$$

as right module over Π is finitely presented, with the corresponding isomorphisms:

$$M \otimes \Pi_1 \xrightarrow{\sim} M_1, \quad (7.5.4)$$

$$M \otimes \Pi_2 \xrightarrow{\sim} M_2. \quad (7.5.5)$$

Proof. This is essentially proved in [T2, Lemma 6.82]. □

By considering the parallel argument one has the following:

Proposition 7.5.10. *Consider the following exact sequence of Banach algebras satisfying the corresponding conditions in [KL1, Definition 2.7.3 (a),(b)]:*

$$0 \rightarrow \Pi \rightarrow \Pi_1 \bigoplus \Pi_2 \rightarrow \Pi_{12} \rightarrow 0. \quad (7.5.6)$$

And consider the corresponding left glueing datum (M_1, M_2, M_{12}) over the corresponding rings above, and we assume that the glueing datum is left finite projective. Then we have that the corresponding kernel M of

$$M_1 \bigoplus M_2 \rightarrow M_{12} \quad (7.5.7)$$

as left module over Π is finitely presented, with the corresponding isomorphisms:

$$M \otimes \Pi_1 \xrightarrow{\sim} M_1, \quad (7.5.8)$$

$$M \otimes \Pi_2 \xrightarrow{\sim} M_2. \quad (7.5.9)$$

Then we consider the corresponding bimodules, and by relying on some argument essentially due to Kedlaya we have the following:

Proposition 7.5.11. (Kedlaya) *Assume now the Banach algebras are commutative. Consider the following exact sequence of Banach algebras satisfying the corresponding conditions in [KL1, Definition 2.7.3 (a),(b)]:*

$$0 \rightarrow \Pi \rightarrow \Pi_1 \oplus \Pi_2 \rightarrow \Pi_{12} \rightarrow 0. \quad (7.5.10)$$

And consider the corresponding glueing datum (M_1, M_2, M_{12}) over the corresponding rings above, and we assume that the glueing datum is finite projective. Then we have that the corresponding kernel M of

$$M_1 \oplus M_2 \rightarrow M_{12} \quad (7.5.11)$$

as module over Π is finite projective, with the corresponding isomorphisms:

$$M \otimes \Pi_1 \xrightarrow{\sim} M_1, \quad (7.5.12)$$

$$M \otimes \Pi_2 \xrightarrow{\sim} M_2. \quad (7.5.13)$$

Proof. This proof is due to Kedlaya. The corresponding finitely presentedness is given in [KL1, Theorem 1.3.9(a)], note here that we are under the conditions (a), (b) in [KL1, Definition 2.7.3 (a),(b)]. To promote the corresponding property to that being finite projective. We look at the corresponding diagram coming from the desired presentation:

$$\begin{array}{ccccc}
 B & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & B_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & A_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12}.
 \end{array}$$

Now we look at the sets:

$$\mathrm{Hom}(M_1, B_1), \mathrm{Hom}(M_2, B_2) \quad (7.5.14)$$

which actually are also modules over the corresponding base rings. By [KL1, Theorem 1.3.9(a)] we can basically apply the construction to

$$\mathrm{Hom}(M_1, B_1), \mathrm{Hom}(M_2, B_2), \mathrm{Hom}(M_{12}, B_{12}) \quad (7.5.15)$$

which relates directly to a glueing square, which gives rise to the surjective morphism:

$$\mathrm{Hom}(M_1, B_1) \oplus \mathrm{Hom}(M_2, B_2) \rightarrow \mathrm{Hom}(M_{12}, B_{12}). \quad (7.5.16)$$

Then we choose the corresponding splittings s_1, s_2 for M_1 and M_2 respectively. Then map them to $\mathrm{Hom}(M_{12}, B_{12})$. Then we have there is an element $s \in \mathrm{Hom}(M_{12}, B_{12})$ such that $f_{1,12}(s_1) - f_{2,12}(s_2) = s$. Then choose the corresponding preimage $s'_1 + s'_2$ of s in $\mathrm{Hom}(M_1, B_1) \oplus \mathrm{Hom}(M_2, B_2)$ through:

$$\mathrm{Hom}(M_1, B_1) \oplus \mathrm{Hom}(M_2, B_2) \rightarrow \mathrm{Hom}(M_{12}, B_{12}). \quad (7.5.17)$$

This makes the corresponding correct modification to the corresponding splitting $s_1 + s_2$ to make sure that $s''_1 + s''_2$ lives in the kernel of

$$\mathrm{Hom}(M_1, B_1) \oplus \mathrm{Hom}(M_2, B_2) \rightarrow \mathrm{Hom}(M_{12}, B_{12}). \quad (7.5.18)$$

Then this gives rise to a corresponding splitting for M . □

Proposition 7.5.12. (After Kedlaya) *Consider the following exact sequence of Banach algebras satisfying the corresponding conditions in [KL1, Definition 2.7.3 (a),(b)]:*

$$0 \rightarrow \Pi \rightarrow \Pi_1 \bigoplus \Pi_2 \rightarrow \Pi_{12} \rightarrow 0. \quad (7.5.19)$$

And consider the corresponding glueing datum (M_1, M_2, M_{12}) over the corresponding rings above, and we assume that the glueing datum is finite projective. Then we have that the corresponding kernel M of

$$M_1 \bigoplus M_2 \rightarrow M_{12} \quad (7.5.20)$$

as bimodule over Π is finite projective, with the corresponding isomorphisms:

$$M \otimes \Pi_1 \xrightarrow{\sim} M_1, \quad (7.5.21)$$

$$M \otimes \Pi_2 \xrightarrow{\sim} M_2. \quad (7.5.22)$$

Proof. The left module and the right module structures have already given us the corresponding finitely presentedness. To promote the corresponding property to that being finite projective. We look at the corresponding diagram coming from the desired presentation:

$$\begin{array}{ccccc}
 B & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & B_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & A_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12}.
 \end{array}$$

Now we look at the sets:

$$\text{Hom}(M_1, B_1), \text{Hom}(M_2, B_2) \tag{7.5.23}$$

which actually are also left modules. By the left glueing process we consider before we can basically apply the construction to

$$\text{Hom}(M_1, B_1), \text{Hom}(M_2, B_2), \text{Hom}(M_{12}, B_{12}) \tag{7.5.24}$$

which relates directly to a glueing square, which gives rise to the surjective morphism:

$$\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2) \rightarrow \text{Hom}(M_{12}, B_{12}). \tag{7.5.25}$$

This is again as left module consideration. Then we choose the corresponding splittings s_1, s_2 for M_1 and M_2 respectively. Then map them to $\text{Hom}(M_{12}, B_{12})$. Then we have there is an element $s \in \text{Hom}(M_{12}, B_{12})$ such that $f_{1,12}(s_1) - f_{2,12}(s_2) = s$. Then choose the corresponding preimage $s'_1 + s'_2$ of s in $\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2)$ through:

$$\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2) \rightarrow \text{Hom}(M_{12}, B_{12}). \tag{7.5.26}$$

This makes the corresponding correct modification to the corresponding splitting $s_1 + s_2$ to make sure that $s''_1 + s''_2$ lives in the kernel of

$$\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2) \rightarrow \text{Hom}(M_{12}, B_{12}). \tag{7.5.27}$$

Then this gives rise to a corresponding splitting for M as left module. Then on the other hand, we look at the corresponding diagram coming from the desired presentation:

$$\begin{array}{ccccc}
 B & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & B_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & A_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12}.
 \end{array}$$

Now we look at the sets:

$$\text{Hom}(M_1, B_1), \text{Hom}(M_2, B_2) \tag{7.5.28}$$

which actually are also right modules. By the right glueing process we consider before we can basically apply the construction to

$$\text{Hom}(M_1, B_1), \text{Hom}(M_2, B_2), \text{Hom}(M_{12}, B_{12}) \tag{7.5.29}$$

which relates directly to a glueing square, which gives rise to the surjective morphism:

$$\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2) \rightarrow \text{Hom}(M_{12}, B_{12}). \tag{7.5.30}$$

This is again as right module consideration. Then we choose the corresponding splittings s_1, s_2 for M_1 and M_2 respectively. Then map them to $\text{Hom}(M_{12}, B_{12})$. Then we have there is an element $s \in \text{Hom}(M_{12}, B_{12})$ such that $f_{1,12}(s_1) - f_{2,12}(s_2) = s$. Then choose the corresponding preimage $s'_1 + s'_2$ of s in $\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2)$ through:

$$\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2) \rightarrow \text{Hom}(M_{12}, B_{12}). \tag{7.5.31}$$

This makes the corresponding correct modification to the corresponding splitting $s_1 + s_2$ to make sure that $s''_1 + s''_2$ lives in the kernel of

$$\text{Hom}(M_1, B_1) \oplus \text{Hom}(M_2, B_2) \rightarrow \text{Hom}(M_{12}, B_{12}). \tag{7.5.32}$$

Then this gives rise to a corresponding splitting for M as right module as well.

□

Apply the corresponding general results developed above we have the following result for glueing finite projective Frobenius modules.

Proposition 7.5.13. *Consider the following two categories. The first category is the corresponding finite projective φ^a -bibundles over the corresponding period ring $\widetilde{\Pi}_{R,B}$. And the corresponding second finite projective φ^a -bimodules over the period ring $\widetilde{\Pi}_{R,B}^{[s,r]}$ such that we have $0 < s \leq rp^{-ha}$. Then we have that the corresponding involved categories are actually equivalent to each other.*

Proof. The functor from the first category to the second one realizing this equivalence is just the corresponding projection. To show that for any bimodule over $\widetilde{\Pi}_{R,B}^{[s,r]}$ we have the corresponding essential surjectivity we lift the corresponding bimodule up to achieve a family of bimodules with the same rank by applying the corresponding Frobenius with respect to different intervals taking the form of $[sp^{-akh}, rp^{-akh}]$. Then for general interval we are free to glue finite projective modules by applying proposition 7.5.12.

□

This chapter is based on the following paper which is being revised for the journal, where the author of this dissertation is the main author:

- Tong, Xin. "Period Rings with Big Coefficients and Applications I." arXiv preprint arXiv:2012.07338 (2020).

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Chapter 8

Period Rings with Big Coefficients and Applications II

8.1 Introduction

8.1.1 Noncommutative Topological Pseudocoherence after Illusie-Kedlaya-Liu

In our previous work on noncommutative deformation of period rings and application [T1], [T2] and [T3], we considered many useful deformation of the sheaves from many deep constructions coming from Kedlaya-Liu. The corresponding local pictures in both commutative setting and noncommutative setting were carefully developed.

The globalization of the picture will happen both in the corresponding coefficients and in the corresponding underlying spaces. Both of them require very deep descent results, although in the commutative setting one do not have to modify much from [KL1] and [KL2]. However in the noncommutative setting, things will be very complicated. Since if one would like to really consider the corresponding localization with respect to the noncommutative coefficient Banach algebra, one has to study the corresponding noncommutative toposes. This will be really complicated since the corresponding topological issues are really hard to manipulate on the level of toposes. That being said, with respect to some of the corresponding application in mind, we could actually do some extension of Kedlaya-Liu glueing to noncommutative setting by keeping track of the noncommutative spatial information in the following sense.

Over \mathbb{Q}_p or $\mathbb{F}_p((t))$, in our application in mind we consider the corresponding deformation of period rings in various context and the corresponding period sheaves over certain sites. One could deal with such very big Banach sheaves by considering the corresponding localization concentrated at the corresponding commutative part but release the corresponding deformation components. This will basically be slightly different from Kedlaya-Liu's original extension to Banach context [KL2] from Illusie's original notion of pseudocoherence in [SGAVI]. What is happening is that we will have some stability notion which is only sensible to the commutative spatial part. To be more precise when we consider the product of a commutative Banach adic algebra A with a noncommutative Banach algebra B , we will consider just the corresponding stability with respect to the

corresponding rational localization from A (not at all for B). Therefore when we sheafify the modules with such stability we will have a sheaf locally attached to modules with such stability in the same fashion.

In some of the applications one might want to consider the corresponding deformations by more general coefficients, for instance in noncommutative Iwasawa theory one might want to consider the corresponding noncommutative Iwasawa algebras. Therefore in such situation one could deform the structure sheaves by some pro-Banach algebras, motivated by Burns-Flach-Fukaya-Kato [BF1], [BF2] and [FK1]. On the other hand suppose one considers deformations by some limit of Fréchet rings such as the corresponding ring B_e in the theory of Bergers B -pairs in [Ber1] and the corresponding various types of Robba rings with respect to some radius $r > 0$ or nothing (namely the full Robba rings) in [KL1] and [KL2]. Carrying these LF algebras one can consider some types of mixed-type of Hodge-structures along the work of [CKZ], [PZ] and [Ked1] in more general sense or in the original context.

In this paper, we will also consider the following discussion. First we would like to tackle the corresponding integral aspects of the corresponding story, especially when we have the adic spaces which are not Tate but just analytic (since we are dealing with spaces over \mathbb{Z}_p) in the sense of [Ked2]. Certainly we will carry some large coefficients. We will expect that this will have the potential application to integral p -adic Hodge theory such as in [BMS] and [BS]. That being said, the corresponding descent in this paper will definitely have its own interests. We work in detail out the corresponding descent of the corresponding pseudocoherent sheaves with some stability carrying the corresponding large coefficients in the corresponding uniform analytic Huber pair situation, while we also translated several results to the corresponding uniform analytic adic Banach rings after [KL2].

8.1.2 Main Results

The corresponding notions of stably-pseudocoherence is generalized to noncommutative deformed setting from [KL2]. Over affinoids we realized the following theorems around taking global section of pseudocoherent sheaves. Namely with the corresponding notations in theorem 8.2.15:

Theorem 8.1.1. (After Kedlaya-Liu [KL2, Theorem 2.5.5]) *In analytic topology, taking the global section will realize an equivalence between the category of pseudocoherent $\mathcal{O}\widehat{\otimes}B$ -sheaves and the category of B -stably-pseudocoherent $\mathcal{O}(X)\widehat{\otimes}B$ modules. Here X is an adic affinoid $\mathrm{Spa}(A, A^+)$.*

In further application, one can also establish the desired parallel results in the étale topology. Namely with the corresponding notations in theorem 8.2.23:

Theorem 8.1.2. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *In étale topology, taking the global section will realize an equivalence between the category of pseudocoherent $\mathcal{O}_{\text{ét}}\widehat{\otimes}B$ -sheaves and the category of B -étale-stably-pseudocoherent $\mathcal{O}_{\text{ét}}(X)\widehat{\otimes}B$ modules. Here X is an adic affinoid $\mathrm{Spa}(A, A^+)$.*

As we mentioned one could also consider more general deformation such as the corresponding. The first scope of the consideration could be made in order to include the context of pro-Banach families associated to some p -adic rational Iwasawa algebra with Banach coefficients $B[[G]]$ attached to some p -adic Lie group G . Namely with the corresponding notations in theorem 8.3.15:

Theorem 8.1.3. (After Kedlaya-Liu [KL2, Theorem 2.5.5]) *In analytic topology, taking the global section will realize an equivalence between the category of pseudocoherent $\mathcal{O}\widehat{\otimes}B[[G]]$ -sheaves and the category of $B[[G]]$ -stably-pseudocoherent $\mathcal{O}(X)\widehat{\otimes}B[[G]]$ modules. Here X is an adic affinoid $\mathrm{Spa}(A, A^+)$, and here all the modules are pro-systems over $\mathbb{Q}_p[[G]]$.*

In further application, one can also establish the desired parallel results in the étale topology. Namely with the corresponding notations in theorem 8.3.23:

Theorem 8.1.4. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *In étale topology, taking the global section will realize an equivalence between the category of pseudocoherent*

$\mathcal{O}_{\text{ét}} \widehat{\otimes} B[[G]]$ -sheaves and the category of $B[[G]]$ -étale-stably-pseudocoherent $\mathcal{O}_{\text{ét}}(X) \widehat{\otimes} B[[G]]$ modules. Here X is an adic affinoid $\text{Spa}(A, A^+)$, and here all the modules are pro-systems over $\mathbb{Q}_p[[G]]$.

One can also even consider the limit of Fréchet coefficients with some further application to some equivariant sheaves over adic Fargues-Fontaine curves. Namely with the corresponding notations in theorem 8.4.15:

Theorem 8.1.5. (After Kedlaya-Liu [KL2, Theorem 2.5.5]) *In analytic topology, taking the global section will realize an equivalence between the category of pseudocoherent $\mathcal{O} \widehat{\otimes} B$ -sheaves and the category of B -stably-pseudocoherent $\mathcal{O}(X) \widehat{\otimes} B$ modules. Here X is an adic affinoid $\text{Spa}(A, A^+)$, and here B is injective limit of Banach algebras $\varinjlim_h B_h$.*

In further application, one can also establish the desired parallel results in the étale topology. Namely with the corresponding notations in theorem 8.4.23:

Theorem 8.1.6. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *In étale topology, taking the global section will realize an equivalence between the category of pseudocoherent $\mathcal{O}_{\text{ét}} \widehat{\otimes} B$ -sheaves and the category of B -étale-stably-pseudocoherent $\mathcal{O}_{\text{ét}}(X) \widehat{\otimes} B$ modules. Here X is an adic affinoid $\text{Spa}(A, A^+)$, and here B is injective limit of Banach algebras $\varinjlim_h B_h$.*

One can also consider the corresponding families of affinoids in some coherent way for the commutative parts, namely this is the notion of quasi-Stein spaces from [KL2, Chapter 2.6]. Then with the corresponding notations in theorem 8.2.32, theorem 8.3.32 and theorem 8.4.32:

Theorem 8.1.7. (After Kedlaya-Liu [KL2, Proposition 2.6.17]) *Carrying Banach, pro-Banach or limit of Fréchet coefficient B one could have the finiteness of the global sections of B -stably-pseudocoherent sheaves over some quasi-Stein space X as long as one has m -uniform covering as in [KL2, Proposition 2.6.17].*

The application will be encoded in our last chapter, where we focused on finding relationship between the corresponding Frobenius equivariant sheaves over Fargues-Fontaine space:

$$\bigcup \mathrm{Spa}(\tilde{\Pi}_R^{[r_1, r_2]}, \tilde{\Pi}_R^{[r_1, r_2], +}) \quad (8.1.1)$$

and the corresponding Frobenius equivariant modules over the global sections:

$$\tilde{\Pi}_R^{[r_1, r_2]}, \forall 0 < s < r. \quad (8.1.2)$$

in glueing fashion. The sheaves and modules here carrying some big Banach coefficients (noncommutative). And we also considered some very interesting pro-Banach situation and LF situation, see theorem 8.7.29 and theorem 8.7.30.

8.1.3 Further Consideration

One thing we have observed (although have not written so) is that suppose we apply our results to some locally noetherian spaces, then we will have no stability issue, and we will have not just pseudoflatness but even more. Instead, this will immediately mean that we are going to be more algebraic throughout. The corresponding story is very interesting in the situation of rigid analytic spaces.

The discussion in the previous paragraph in fact implies that in the noetherian situation, one should be able to extend the discussion around ∞ -glueing along the style of Kedlaya-Liu by applying Bambozzi-Kremnizer spectrum [BK1] and Clausen-Scholze space [CS] to the corresponding coherent sheaves which are interesting to us where the corresponding highly nontrivial stability and nonflatness issues in [KL2] disappear. Again note that we are still carrying some large coefficients.

8.2 Foundations on Noncommutative Descent for Adic Spectra in Banach Case

8.2.1 Noncommutative Pseudocoherence in Analytic Topology

We now establish some foundations in the noncommutative setting on glueing noncommutative pseudocoherent modules after [KL2, Chapter 2]. But we remind the readers that we will fix the base space:

Setting 8.2.1. Consider a corresponding sheafy Banach adic uniform algebra (A, A^+) over \mathbb{Q}_p or $\mathbb{F}_p((t))$, we consider the base space $\mathrm{Spa}(A, A^+)$ as in [KL2, Chapter 2]. And now we will consider a further noncommutative Banach algebra (B, B^+) over \mathbb{Q}_p or $\mathbb{F}_p((t))$.

Definition 8.2.2. (After Kedlaya-Liu [KL2, Definition 2.4.1]) For any left $A\widehat{\otimes}B$ -module M , we call it m - B -stably pseudocoherent if we have that it is m - B -pseudocoherent, complete for the natural topology and for any morphism $A \rightarrow A'$ which is the corresponding rational localization, the base change of M to $A'\widehat{\otimes}B$ is complete for the natural topology as the corresponding left $A'\widehat{\otimes}B$ -module. As in [KL2, Definition 2.4.1] we call that the corresponding left $A\widehat{\otimes}B$ -module M just B -stably pseudocoherent if we have that it is simply just ∞ - B -stably pseudocoherent.

Remark 8.2.3. Throughout the whole paper, it is obvious we need to use the corresponding noncommutative version of the corresponding pseudocoherent modules on the algebraic level due to Illusie along the development of Grothendieck's SGAVI [SGAVI], but the main difference is just that we consider the corresponding modules with action happening on one side.

Since our spaces are commutative, so the rational localization is not flat which is exactly the same situation as in [KL1] and [KL2], therefore in our situation we need to define something weaker than the corresponding flatness which will be sufficient for us to apply in the following development:

Definition 8.2.4. (After Kedlaya-Liu [KL2, Definition 2.4.4]) For any Banach left $A\widehat{\otimes}B$ -module M , we call it is m - B -pseudoflat if for any right $A\widehat{\otimes}B$ -module M' m - B -stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B}(M', M) = 0$. For any Banach right $A\widehat{\otimes}B$ -module M , we

call it is m - B -pseudoflat if for any left $A\widehat{\otimes}B$ -module M' m - B -stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B}(M, M') = 0$.

Definition 8.2.5. (After Kedlaya-Liu [KL2, Definition 2.4.6]) We consider the corresponding notion of the corresponding pro-projective module. We define over A the corresponding B -pro-projective module M to be a corresponding left $A\widehat{\otimes}B$ -module such that one could find a filtered sequence of projectors such that in the sense of taking the pro-limit we have the corresponding projector will converge to any chosen element in M . Here we assume the module is complete with respect to natural topology, and we assume the projectors are $A\widehat{\otimes}B$ -linear and we assume that the corresponding image of the projectors are modules which are also finitely generated and projective.

Lemma 8.2.6. (After Kedlaya-Liu [KL2, Lemma 2.4.7]) *Suppose we have over the space $\mathrm{Spa}(A, A^+)$ a corresponding B -pro-projective left module M . And suppose that we have a 2- B -pseudocoherent right module C over A . And we assume that C is complete with respect to the natural topology. Then we have that the corresponding product $C \otimes_{A\widehat{\otimes}B} M$ is then complete under the corresponding natural topology. And moreover we have that in our situation:*

$$\mathrm{Tor}_1^{A\widehat{\otimes}B}(C, M) = 0.$$

Proof. This will be just a noncommutative version of the corresponding [KL2, Lemma 2.4.7]. We adapt the corresponding argument in [KL2, Lemma 2.4.7] to our situation. What we are going to consider in this case is then first consider the following presentation:

$$M' \longrightarrow (A\widehat{\otimes}B)^k \longrightarrow M \longrightarrow 0,$$

where the left module M' is finitely presented. Then we consider the following commutative diagram:

$$\begin{array}{ccccccc} C \otimes_{A\widehat{\otimes}B} M' & \longrightarrow & C \otimes_{A\widehat{\otimes}B} (A\widehat{\otimes}B)^k & \longrightarrow & C \otimes_{A\widehat{\otimes}B} M & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C \widehat{\otimes}_{A\widehat{\otimes}B} M' & \longrightarrow & C \widehat{\otimes}_{A\widehat{\otimes}B} (A\widehat{\otimes}B)^k & \longrightarrow & C \widehat{\otimes}_{A\widehat{\otimes}B} M \longrightarrow 0. \end{array}$$

The first row is exact as in [KL2, Lemma 2.4.7], and we have the corresponding second

row is also exact since we have that by hypothesis the corresponding module M is B -pro-projective. Then as in [KL2, Lemma 2.4.7] the results follow. \square

Proposition 8.2.7. (After Kedlaya-Liu [KL2, Corollary 2.4.8]) *Over the space $\mathrm{Spa}(A, A^+)$, we have any B -pro-projective left module is 2- B -pseudoflat.*

Proof. This is a direct consequence of the previous lemma. \square

Lemma 8.2.8. (After Kedlaya-Liu [KL2, Corollary 2.4.9]) *Keep the notation above, suppose we are working over the adic space $\mathrm{Spa}(A, A^+)$. Suppose now the left $A\widehat{\otimes}B$ -module is finitely generated, then we have that the following natural maps:*

$$A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T\}, \quad (8.2.1)$$

$$A\{T, T^{-1}\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T, T^{-1}\} \quad (8.2.2)$$

are surjective. Suppose now the left $A\widehat{\otimes}B$ -module is finitely presented, then we have that the following natural maps:

$$A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T\}, \quad (8.2.3)$$

$$A\{T, T^{-1}\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T, T^{-1}\} \quad (8.2.4)$$

are bijective. Here all the modules are assumed to be complete with respect to the natural topology in our context.

Proof. This is the corresponding consequence of the previous lemma. \square

Proposition 8.2.9. (After Kedlaya-Liu [KL2, Lemma 2.4.12]) *We keep the corresponding notations in [KL2, Lemma 2.4.10]. Then in our current sense we have that the corresponding morphism $A \rightarrow B_2$ is then 2- B -pseudoflat.*

Proof. The corresponding proof could be made parallel to the corresponding proof of [KL2, Lemma 2.4.12]. We need to keep track the corresponding coefficient B in our current context. To be more precise we consider the following commutative diagram from the corresponding chosen exact sequence (which is just the analog of the corresponding

one in [KL2, Lemma 2.4.12], namely as below we choose arbitrary short exact sequence with P as a left $A\widehat{\otimes}B$ -module which is 2- B -stably pseudocoherent):

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

which then induces the following corresponding big commutative diagram:

$$\begin{array}{ccccccc}
& & & & 0 & & 0 & , \\
& & & & \downarrow & & \downarrow & \\
0 & \longrightarrow & A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M & \longrightarrow & A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} N & \longrightarrow & A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} P & \longrightarrow 0, \\
& & \downarrow 1-fT & & \downarrow 1-fT & & \downarrow 1-fT & \\
0 & \longrightarrow & A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M & \longrightarrow & A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} N & \longrightarrow & A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} P & \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & ?B_2\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M & \longrightarrow & B_2\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} N & \longrightarrow & B_2\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} P & \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow & \\
& & 0 & & 0 & & 0 &
\end{array}$$

with the notations in [KL2, Lemma 2.4.10] where we actually have the corresponding exactness along the horizontal direction at the corner around $B_2\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M$ marked with ?, by diagram chasing. \square

Proposition 8.2.10. (After Kedlaya-Liu [KL2, Lemma 2.4.13]) *We keep the corresponding notations in [KL2, Lemma 2.4.10]. Then in our current sense we have that the corresponding morphism $A \rightarrow B_1$ is then 2- B -pseudoflat. And in our current sense we have that the corresponding morphism $A \rightarrow B_{12}$ is then 2- B -pseudoflat.*

Proof. The statement for $A \rightarrow B_{12}$ could be proved by consider the composition $A \rightarrow B_2 \rightarrow B_{12}$ where the 2- B -pseudoflatness could be proved as in [KL2, Lemma 2.4.13] by inverting the corresponding variables. For $A \rightarrow B_1$, the corresponding proof could be made parallel to the corresponding proof of [KL2, Lemma 2.4.13]. To be more precise we consider the following commutative diagram from the corresponding chosen exact sequence (which is just the analog of the corresponding one in [KL2, Remark 2.4.5], namely as below we choose arbitrary short exact sequence with P as a left $A \widehat{\otimes} B$ -module which is 2- B -stably pseudocoherent):

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

which then induces the following corresponding big commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (B_1 \widehat{\otimes} B \oplus B_2 \widehat{\otimes} B) \otimes_{A \widehat{\otimes} B} M & \longrightarrow & (B_1 \widehat{\otimes} B \oplus B_2 \widehat{\otimes} B) \otimes_{A \widehat{\otimes} B} N & \longrightarrow & (B_1 \widehat{\otimes} B \oplus B_2 \widehat{\otimes} B) \otimes_{A \widehat{\otimes} B} P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & B_{12} \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M & \longrightarrow & B_{12} \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} N & \longrightarrow & B_{12} \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with the notations in [KL2, Lemma 2.4.10] where we actually have the corresponding

exactness along the horizontal direction at the corner around $M \otimes_{A \widehat{\otimes} B} (B_1 \widehat{\otimes} B \oplus B_2 \widehat{\otimes} B)$ marked with $?$, by diagram chasing. \square

After these foundational results, as in [KL2] we have the following proposition which is the corresponding noncommutative generalization of the corresponding result established in [KL2, Theorem 2.4.15].

Proposition 8.2.11. (After Kedlaya-Liu [KL2, Theorem 2.4.15]) *In our current context we have that for any rational localization $\mathrm{Spa}(A', A'^+) \rightarrow \mathrm{Spa}(A, A^+)$ we have that along this base change the corresponding B -stably pseudocoherence is preserved.*

Then we have the corresponding Tate's acyclicity in the noncommutative deformed setting:

Theorem 8.2.12. (After Kedlaya-Liu [KL2, Theorem 2.5.1]) *Now suppose we have in our corresponding setting 8.2.1 a corresponding B -stably pseudocoherent module M . Then we consider the corresponding assignment such that for any $U \subset \mathrm{Spa}(A, A^+)$ we define $\widetilde{M}(U)$ as in the following:*

$$\widetilde{M}(U) := \varprojlim_{\mathrm{Spa}(S, S^+) \subset U, \text{rational}} S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.2.5)$$

Then we have that for any \mathfrak{B} which is a rational covering of $U = \mathrm{Spa}(S, S^+) \subset \mathrm{Spa}(A, A^+)$ (certainly this U is also assumed to be rational) we have that the vanishing of the following two cohomology groups:

$$H^i(U, \widetilde{M}), \check{H}^i(U : \mathfrak{B}, \widetilde{M}) \quad (8.2.6)$$

for any $i > 0$. When concentrating at the degree zero we have:

$$H^0(U, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M, \check{H}^0(U : \mathfrak{B}, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.2.7)$$

Proof. By [KL1, Propositions 2.4.20-2.4.21], we could then finish proof as in [KL2, Theorem 2.5.1] by using our previous proposition 8.2.11 and proposition 8.2.10 as above. \square

We now consider the corresponding noncommutative deformed version of Kiehl's glueing properties for stably pseudocoherent modules after [KL2]:

Definition 8.2.13. (After Kedlaya-Liu [KL2, Definition 2.5.3]) Consider in our context (over $\mathrm{Spa}(A, A^+)$) the corresponding sheaves $\mathcal{O} \widehat{\otimes} B$, we will then define the corresponding pseudocoherent sheaves over $\mathrm{Spa}(A, A^+)$ to be those locally defined by attaching stably-pseudocoherent modules over the section.

Lemma 8.2.14. (After Kedlaya-Liu [KL2, Lemma 2.5.4]) Consider the corresponding notations in [KL2, Lemma 2.4.10], we have the corresponding morphism $A \rightarrow B_1 \oplus B_2$. Then we have that this morphism is an descent morphism effective for the corresponding B -stably pseudocoherent Banach modules.

Proof. This is a B -relative version of the [KL2, Lemma 2.5.4]. We adapt the corresponding proof to our situation. First consider the corresponding notations on the commutative adic space $\mathrm{Spa}(A, A^+)$ as in [KL2, Lemma 2.4.10]. And we have a covering $\{U_1, U_2\}$. Then consider a descent datum for this cover of B -stably pseudocoherent sheaves. By theorem 8.2.12 we could realize this as a corresponding pseudocoherent sheaf over $\mathcal{O} \widehat{\otimes} B$. We denote this by \mathcal{M} . Then by (the proof of) [T2, Lemma 6.82], we have that this is finitely generated and we have the corresponding vanishing of $\check{H}^i(\mathrm{Spa}(A, A^+), \mathfrak{B}; \mathcal{M})$ for $i > 0$. Then by theorem 8.2.12 again we have that the vanishing of $H^i(U_j; \mathcal{M})$ for $j \in \{1, 2\}, i > 0$. Then as in [KL2, Lemma 2.5.4] we have the corresponding vanishing of $H^i(\mathrm{Spa}(A, A^+); \mathcal{M})$ for $i > 0$. Then as in [KL2, Lemma 2.5.4] we choose a corresponding covering of some finite free object \mathcal{F} and take its kernel \mathcal{K} which is actually pseudocoherent by the corresponding proposition 8.2.11. Then forming the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (B_p \widehat{\otimes} B) \otimes \mathcal{K}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (B_p \widehat{\otimes} B) \otimes \mathcal{F}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (B_p \widehat{\otimes} B) \otimes \mathcal{M}(\mathrm{Spa}(A, A^+)) \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K}(U_p) & \longrightarrow & \mathcal{F}(U_p) & \longrightarrow & \mathcal{M}(U_p) \longrightarrow 0,
\end{array}$$

where we have $p = 1, 2$. The first row is exact by B -pseudoflatness as in proposition 8.2.11 and the second is row is exact as well by theorem 8.2.12. The middle vertical arrow is an isomorphism while the rightmost vertical arrow is surjective and the leftmost vertical arrow is surjective. The surjectivity just here comes from actually [T2, Lemma 6.82].

This will imply that the rightmost vertical arrow is injective as well. Then we have the corresponding isomorphism $\mathcal{M} \xrightarrow{\sim} \widetilde{\mathcal{M}(\mathrm{Spa}(A, A^+)})$. Then we consider the following diagram:

$$\begin{array}{ccccccc}
(C \widehat{\otimes} B) \otimes \mathcal{K}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C \widehat{\otimes} B) \otimes \mathcal{F}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C \widehat{\otimes} B) \otimes \mathcal{M}(\mathrm{Spa}(A, A^+)) & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{F}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{M}(\mathrm{Spa}(C, C^+)) & \longrightarrow & 0,
\end{array}$$

C is any rational localization of A . Here the first row is exact by B -pseudoflatness and the second row is exact as well. The middle vertical arrow is an isomorphism while the rightmost vertical arrow is surjective as in [KL2, Lemma 2.5.4]. Then we apply the same argument to the leftmost vertical arrow (with a new diagram), we will have the corresponding surjectivity of this leftmost vertical arrow. This will imply that the rightmost vertical arrow is injective as well. Then we have the corresponding isomorphism for the rightmost vertical arrow. \square

Theorem 8.2.15. (After Kedlaya-Liu [KL2, Theorem 2.5.5]) *Taking global section will realize the equivalence between the following two categories: A. The category of all the pseudocoherent $\mathcal{O} \widehat{\otimes} B$ -sheaves; B. The category of all the B -stably pseudocoherent modules over $A \widehat{\otimes} B$.*

Proof. See [KL2, Theorem 2.5.5]. We need to still apply [KL1, Proposition 2.4.20], as long as one considers instead in our situation proposition 8.5.13 and theorem 8.2.12. \square

8.2.2 Noncommutative Pseudocoherence in Étale Topology

We consider the extension of the corresponding discussion in the previous subsection to the étale topology. At this moment we consider the following same context on the geometric level as in [KL2]:

Setting 8.2.16. (After Kedlaya-Liu [KL2, Hypothesis 2.5.8]) As in the previous subsection we consider now the corresponding geometric setting namely an adic space $\mathrm{Spa}(A, A^+)$ where A is assumed to be sheafy. Then we consider B as above. And we consider the corresponding étale site $\mathrm{Spa}(A, A^+)_{\mathrm{ét}}$. And by keeping the corresponding setting in the geometry as in [KL2, Hypothesis 2.5.8], we assume that there is a stable basis \mathfrak{B} containing the space $\mathrm{Spa}(A, A^+)$ itself.

Definition 8.2.17. (After Kedlaya-Liu [KL2, Definition 2.5.9]) For any left $A\widehat{\otimes}B$ -module M , we call it m - B -étale stably pseudocoherent with respect to \mathfrak{B} if we have that it is m - B -pseudocoherent, complete for the natural topology and for any rational localization $A \rightarrow A'$, the base change of M to $A'\widehat{\otimes}B$ is complete for the natural topology as the corresponding left $A'\widehat{\otimes}B$ -module. As in [KL2, Definition 2.4.1] we call that the corresponding left $A\widehat{\otimes}B$ -module M just B -stably pseudocoherent if we have that it is simply just ∞ - B -stably pseudocoherent.

Definition 8.2.18. (After Kedlaya-Liu [KL2, Below Definition 2.5.9]) For any Banach left $A\widehat{\otimes}B$ -module M , we call it m - B -étale-pseudoflat if for any right $A\widehat{\otimes}B$ -module M' m - B -étale-stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B}(M', M) = 0$. For any Banach right $A\widehat{\otimes}B$ -module M , we call it is m - B -étale-pseudoflat if for any left $A\widehat{\otimes}B$ -module M' m - B -étale-stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B}(M, M') = 0$.

The following proposition holds in our current setting.

Proposition 8.2.19. (After Kedlaya-Liu [KL2, Lemma 2.5.10]) *One can actually find another basis \mathfrak{C} in \mathfrak{B} such that any morphism in \mathfrak{C} could be 2- B -étale-pseudoflat with respect to either \mathfrak{C} or \mathfrak{B} .*

Proof. We follow the proof of [KL2, Lemma 2.5.10] in our current B -relative situation, however not that much needs to be adapted by relying on the proof of [KL2, Lemma 2.5.10]. To be more precise the corresponding selection of the new basis \mathfrak{C} comes from including all the morphisms made up of some composition of rational localization and the corresponding finite étale ones. For these we have shown above the corresponding 2- B -étale pseudoflatness. Then the rest will be pure geometric for the analytic spaces which are just the same as the situation of [KL2, Lemma 2.5.10], therefore we just omit the corresponding argument, see [KL2, Lemma 2.5.10]. \square

Then we have the corresponding Tate's acyclicity in the noncommutative deformed setting in étale topology (here fix \mathfrak{C} as above):

Theorem 8.2.20. (After Kedlaya-Liu [KL2, Theorem 2.5.11]) *Now suppose we have in our corresponding setting 8.2.1 a corresponding B -étale-stably pseudocoherent module M . Then we consider the corresponding assignment such that for any $U \subset \mathrm{Spa}(A, A^+)$ we define $\widetilde{M}(U)$ as in the following:*

$$\widetilde{M}(U) := \varprojlim_{\mathrm{Spa}(S, S^+) \subset U, \mathfrak{C}} S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.2.8)$$

Then we have that for any \mathfrak{B} which is a covering of $U = \mathrm{Spa}(S, S^+) \subset \mathrm{Spa}(A, A^+)$ (certainly this U is also assumed to be in \mathfrak{C} , and we assume this covering is formed by using the corresponding members in \mathfrak{C}) we have that the vanishing of the following two cohomology groups:

$$H^i(U, \widetilde{M}), \check{H}^i(U, \mathfrak{B}, \widetilde{M}) \quad (8.2.9)$$

for any $i > 0$. When concentrating at the degree zero we have:

$$H^0(U, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M, \check{H}^0(U, \mathfrak{B}, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.2.10)$$

Proof. By [KL1, Proposition 8.2.21], we could then finish proof as in [KL2, Theorem 2.5.11] by using our previous proposition 8.2.19 and the corresponding faithfully flat descent as in the situation of [KL2, Theorem 2.5.11]. \square

We now consider the corresponding noncommutative deformed version of Kiehl's glueing property for stably pseudocoherent modules after [KL2, Definition 2.5.12]:

Definition 8.2.21. (After Kedlaya-Liu [KL2, Definition 2.5.12]) Consider in our context (over $\mathrm{Spa}(A, A^+)_{\text{ét}}$) the corresponding sheaves $\mathcal{O}_{\mathrm{Spa}(A, A^+)_{\text{ét}}} \widehat{\otimes} B$, we will then define the corresponding pseudocoherent sheaves over $\mathrm{Spa}(A, A^+)_{\text{ét}}$ to be those locally defined by attaching étale-stably-pseudocoherent modules over the section.

Lemma 8.2.22. (After Kedlaya-Liu [KL2, Lemma 2.5.13]) *Consider the corresponding notations in [KL2, Lemma 2.4.10], we have the corresponding morphism $A \rightarrow B_1 \oplus B_2$. Then we have that this morphism is an descent morphism effective for the corresponding B -étale-stably pseudocoherent Banach modules.*

Proof. This is a B -relative version of the [KL2, Lemma 2.5.13]. We adapt the corresponding proof to our situation. First consider the corresponding notations on the commutative adic space $\mathrm{Spa}(A, A^+)$ as in [KL2, Lemma 2.4.10]. And we have a covering $\{U_1, U_2\}$. Then consider a descent datum for this cover of B -étale-stably pseudocoherent sheaves. By theorem 8.2.20 we could realize this as a corresponding pseudocoherent sheaf over $\mathcal{O}_{\text{ét}} \widehat{\otimes} B$. We denote this by \mathcal{M} . Then by consider the corresponding results in the rational localization situation above we have that the vanishing of $H^i(U_j; \mathcal{M})$ for $j \in \{1, 2\}, i > 0$. And note that we have the corresponding equality on the degree zero namely $H_{\text{ét}}^i(U_j; \mathcal{M}) = H^i(U_j; \mathcal{M})$ for just i which is zero. Then as in [KL2, Lemma 2.5.13] we have the corresponding vanishing of $H^i(\mathrm{Spa}(A, A^+); \mathcal{M})$ for $i > 0$. Namely this is due to the fact that the covering is already one in the analytic topology. Then as in [KL2, Lemma 2.5.13] we choose a corresponding covering of some finite free object \mathcal{F} and take its kernel \mathcal{K} which gives rise to the corresponding exact sequence:

$$0 \longrightarrow \mathcal{K}(\mathrm{Spa}(A, A^+)) \longrightarrow \mathcal{F}(\mathrm{Spa}(A, A^+)) \longrightarrow \mathcal{M}(\mathrm{Spa}(A, A^+)) \longrightarrow 0,$$

where we have $p = 1, 2$. We have already the corresponding stably-pseudocoherence of the global section. As in [KL2, Lemma 2.5.13] we only need to show that it is stable under étale morphism. Then we consider the following diagram:

$$\begin{array}{ccccccc} (C \widehat{\otimes} B) \otimes \mathcal{K}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C \widehat{\otimes} B) \otimes \mathcal{F}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C \widehat{\otimes} B) \otimes \mathcal{M}(\mathrm{Spa}(A, A^+)) & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \mathcal{K}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{F}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{M}(\mathrm{Spa}(C, C^+)) & \longrightarrow 0, \end{array}$$

$A \rightarrow C$ is some member in \mathfrak{C} . Here the first row is exact by B -étale-pseudoflatness and the second is row is exact as well. The middle vertical arrow is an isomorphism while the rightmost vertical arrow is surjective as in [KL2, Lemma 2.5.13]. Then we apply the same argument to the leftmost vertical arrow (with a new diagram), we will have the corresponding surjectivity of this leftmost vertical arrow. This will imply that the rightmost vertical arrow is injective as well. Then we have the corresponding isomorphism for the rightmost vertical arrow. \square

Theorem 8.2.23. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *Taking global section will realize the equivalence between the following two categories: A. The category of all the pseudocoherent $\mathcal{O}_{\mathrm{Spa}(A, A^+)_{\text{ét}}}\widehat{\otimes}B$ -sheaves; B. The category of all the B -étale-stably pseudocoherent modules over $A\widehat{\otimes}B$.*

Proof. See [KL2, Theorem 2.5.14]. We need to still apply [KL1, Theorem 8.2.22], as long as one considers instead in our situation, and [SP, Tag 03OD].

□

8.2.3 Noncommutative Deformation over Quasi-Stein Spaces

In this subsection we consider the corresponding noncommutative deformation over the corresponding context in [KL2, Chapter 2.6].

Setting 8.2.24. Let X be a corresponding quasi-Stein adic space over \mathbb{Q}_p or $\mathbb{F}_p((t))$ in the sense of [KL2, Definition 2.6.2]. Recall that what is happening is that X could be written as the corresponding direct limit of affinoids $X := \varinjlim_i X_i$.

Lemma 8.2.25. (After Kedlaya-Liu [KL2, Lemma 2.6.3]) *We now consider the corresponding rings $A_i := \mathcal{O}_{X_i}$ for all $i = 0, 1, \dots$, and in our current situation we consider the corresponding rings $A_i\widehat{\otimes}B$ (over \mathbb{Q}_p or $\mathbb{F}_p((t))$) for all $i = 0, 1, \dots$. And in our situation we consider the corresponding modules M_i over $A_i\widehat{\otimes}B$ for all $i = 0, 1, \dots$ with the same requirement as in [KL2, Lemma 2.6.3] (namely those complete with respect to the natural topology). Suppose that we have bounded surjective map from $f_i : A_i\widehat{\otimes}B \otimes_{A_{i+1}\widehat{\otimes}B} M_{i+1} \rightarrow M_i, i = 0, 1, \dots$. Then we have first the density of the corresponding image of $\varprojlim_i M_i$ in each M_i for any $i = 0, 1, 2, \dots$. And we have as well the corresponding vanishing of $R^1\varprojlim_i M_i$.*

Proof. This is the B -relative version of the result in [KL2, Lemma 2.6.3]. For the first statement we just choose sequence of Banach norms on all the corresponding modules for all $i = 0, 1, \dots$ such that we have $\|f_i(x_{i+1})\|_i \leq 1/2\|x_{i+1}\|_{i+1}$ for any $x_{i+1} \in M_{i+1}$. Then for any $x_i \in M_i$ and any $\delta > 0$, we now consider for any $j \geq 1$ the corresponding x_{i+j} such that we have $\|x_{i+j} - f_{i+j+1}(x_{i+j+1})\|_{i+j} \leq \delta$. Then the sequence $x_{i+j+k}, k = 0, 1, \dots$ will converge to some well-defined y_{i+j} with in our situation the corresponding $y_i = f_i(y_{i+1})$.

We then have $\|x_i - y_i\|_i \leq \delta$. This will prove the first statement. For the second statement as in [KL2, Lemma 2.6.3] we form the product $M_0 \times M_1 \times M_2 \times \dots$ and consider the induced map F from $M_{i+1} \rightarrow M_i$, and consider the corresponding cokernel of the map $1 - F$ since this is just the corresponding limit we are considering. Then to show that the cokernel is zero we just look at the corresponding cokernel of the corresponding map on the corresponding completed direct summand which will project to the original one. But then we will have $\|f_i(v)\|_i \leq 1/2\|v\|_i$, which produces an inverse to $1 - F$ which will basically finish the proof for the second statement. \square

Proposition 8.2.26. (After Kedlaya-Liu [KL2, Lemma 2.6.4]) *In the same situation as above, suppose we have that the corresponding modules M_i are basically B -stably pseudocoherent over the rings $A_i \widehat{\otimes} B$ for all $i = 0, 1, \dots$. Now we consider the situation where $f_i : A_i \widehat{\otimes} B \otimes_{A_{i+1} \widehat{\otimes} B} M_{i+1} \rightarrow M_i$ is an isomorphism. Then the conclusion in our situation is then that the corresponding projection from $\varprojlim M_i$ to M_i for each $i = 0, 1, 2, \dots$ is an isomorphism.*

Proof. This is a B -relative version of the [KL2, Lemma 2.6.4]. We adapt the argument to our situation as in the following. First we choose some finite free covering T of the limit M such that we have for each i the corresponding map $T_i \rightarrow M_i$ is surjective. Then we consider the index $j \geq i$ and set the kernel of the map from T_j to M_j to be S_j . By the direct analog of [KL2, Lemma 2.5.6] we have that $A_i \widehat{\otimes} B \otimes S_j \rightarrow S_i$ realizes the isomorphism, and we have that the corresponding surjectivity of the corresponding map from $\varprojlim_i S_i$ projecting to the S_i . Then one could finish the proof by 5-lemma to the following commutative diagram as in [KL2, Lemma 2.6.4]:

$$\begin{array}{ccccccc}
 (A_i \widehat{\otimes} B) \otimes \varprojlim_i S_i & \longrightarrow & (A_i \widehat{\otimes} B) \otimes \varprojlim_i F_i & \longrightarrow & (A_i \widehat{\otimes} B) \otimes \varprojlim_i M_i & \longrightarrow & 0, \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & S_i & \longrightarrow & F_i & \longrightarrow & M_i \longrightarrow 0.
 \end{array}$$

\square

Proposition 8.2.27. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the map*

$\mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ is surjective for any B -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B$.

Proof. This is just the corresponding corollary of the previous proposition. \square

Proposition 8.2.28. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *We have that the stalk \mathcal{M}_x is generated over the stalk $\mathcal{O}_{X,x}$ for any $x \in X$ by $\mathcal{M}(X)$, for any B -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B$.*

Proof. This is just the corresponding corollary of the proposition before the previous proposition. \square

Proposition 8.2.29. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the corresponding vanishing of the corresponding sheaf cohomology groups $H^k(X, \mathcal{M})$ of \mathcal{M} for higher $k > 0$, for any B -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B$.*

Proof. We follow the idea of the proof of [KL2, Theorem 2.6.5] by comparing this to the corresponding Čech cohomology with some covering $\mathfrak{X} = \{X_1, \dots, X_N, \dots\}$:

$$\check{H}^k(X, \mathfrak{X} = \{X_1, \dots, X_N, \dots\}; \mathcal{M}) = H^k(X, \mathcal{M}), \quad (8.2.11)$$

$$\check{H}^k(X_i, \mathfrak{X} = \{X_1, \dots, X_i\}; \mathcal{M}) = H^k(X_i, \mathcal{M}) = 0, k \geq 1. \quad (8.2.12)$$

Here we have applied the corresponding [SP, Tag 01EW]. Now we consider the situation where $k > 1$:

$$\check{H}^{k-1}(X_{j+1}, \mathfrak{X} = \{X_1, \dots, X_{j+1}\}; \mathcal{M}) \rightarrow \check{H}^{k-1}(X_j, \mathfrak{X} = \{X_1, \dots, X_j\}; \mathcal{M}) \rightarrow 0, \quad (8.2.13)$$

which induces the following isomorphism by [Kie1, 2.6 Hilfssatz]:

$$\varprojlim_{j \rightarrow \infty} \check{H}^{k-1}(X_j, \mathfrak{X} = \{X_1, \dots, X_j\}; \mathcal{M}) \xrightarrow{\sim} \check{H}^k(X, \mathfrak{X} = \{X_1, \dots, X_N, \dots\}; \mathcal{M}). \quad (8.2.14)$$

Then we have the corresponding results for the index $k > 1$. For $k = 1$, one can as in [KL2, Theorem 2.6.5] relate this to the corresponding $R^1 \varprojlim_i$, which will finishes the proof in the same fashion.

\square

Corollary 8.2.30. (After Kedlaya-Liu [KL2, Corollary 2.6.6]) *The corresponding functor from the corresponding B -deformed pseudocoherent sheaves over X to the corresponding B -stably by taking the corresponding global section is an exact functor.*

Corollary 8.2.31. (After Kedlaya-Liu [KL2, Corollary 2.6.8]) *Consider a particular $\mathcal{O}_X \widehat{\otimes} B$ -pseudocoherent sheaf \mathcal{M} which is finite locally free throughout the whole space X . Then we have that the global section $\mathcal{M}(X)$ as $\mathcal{O}_X(X) \widehat{\otimes} B$ left module admits the corresponding structures of finite projective structure if and only if we have the corresponding global section is finitely generated.*

Proof. As in [KL2, Corollary 2.6.8] one could find some global splitting through the local splittings. □

We now consider the following B -relative analog of [KL2, Proposition 2.6.17]:

Theorem 8.2.32. (After Kedlaya-Liu [KL2, Proposition 2.6.17]) *Consider the following two statements for a particular $\mathcal{O}_X \widehat{\otimes} B$ -pseudocoherent sheaf \mathcal{M} . First is that one can find finite many generators (the number is up to some uniform fixed integer $n \geq 0$) for each section of $\mathcal{M}(X_i)$ for each $i = 1, 2, \dots$. The second statement is that the global section $\mathcal{M}(X)$ is just finitely generated. Then in our situation the two statement is equivalent if we have that the corresponding space X admits an m -uniform covering in the exact same sense of [KL2, Proposition 2.6.17].*

Proof. One direction is obvious, the other direct could be proved in the same way as in [KL2, Proposition 2.6.17] where essentially the information on X does not change at all. Namely as in [KL2, Proposition 2.6.17] we could basically consider one single subcovering $\{Y_u\}_{u \in U}$ indexed by U of the covering which is m -uniform. For any $u \in U$ we consider the smallest i such that $X_i \cap Y_u$, and we denote this by $i(u) \geq 0$. Then we form:

$$V_u := X_{i(u)} \bigcup_{v \in U, v \leq u} Y_v. \tag{8.2.15}$$

Then as in [KL2, Proposition 2.6.17] we can find some $x_u, u \in U$ (which are denoted by the general form of x_i in [KL2, Proposition 2.6.17]) in the global section of $\mathcal{O}_X(X)$ such that it restricts to some topologically nilpotent onto the space V_u and it restricts to some inverse of some topologically nilpotent to Y_u . Then we can build up a corresponding

generating set for the global section by the following approximation process just as in [KL2, Proposition 2.6.17]. To be more precise what we have consider is to modify the corresponding generator for each Y_u (for instance let them be denoted by y_1, \dots, y_u) to be $y_{u,1}, \dots, y_{u,n}$ (n is the corresponding uniform integer in the statement of the theorem) for all $u \in U$. It is achieved through induction, when we have $y_{u,1}, \dots, y_{u,n}$ then we will set that to be 0 otherwise in the situation where there exists some predecessor u' of u in the corresponding set U we then set this to be $y_{u',1}, \dots, y_{u',n}$. And we set as in [KL2, Proposition 2.6.17]:

$$y_{u,j} := y_{u-1,j} + x_u^c y_u$$

by lifting the corresponding power c to be as large as possible. This will guarantee the convergence of:

$$\lim_{u \rightarrow \infty} \{y_{u,1}, \dots, y_{u,n}\}, \tag{8.2.16}$$

which gives the set of the corresponding global generators desired. □

8.3 Foundations on Noncommutative Descent for Adic Spectra in Pro-Banach Case

8.3.1 Noncommutative Pseudocoherence in Analytic Topology

We now establish some foundations in the noncommutative setting on glueing noncommutative pseudocoherent modules after [KL2, Chapter 2]. However the corresponding difference we will make here from the previous section is that we will consider some further Iwasawa deformation similar to the sense of Fukaya-Kato's adic rings. We first consider the simpler situation where we will deform over $B[[G]]$ where B as before comes from the following beginning setting up and G is a given p -adic Lie group:

Setting 8.3.1. Consider a corresponding sheafy Banach adic uniform algebra (A, A^+) over \mathbb{Q}_p , we consider the base space $\mathrm{Spa}(A, A^+)$. And now we will consider a further noncommutative Banach algebra (B, B^+) over \mathbb{Q}_p .

Setting 8.3.2. In what follow when we mean a module M over $A \widehat{\otimes} B[[G]]$ we will mean a pro-system $\{M_E\}_{E \subset G}$ of left module over $A \widehat{\otimes} B[[G]]$. And the corresponding definition of pseudocoherence is also defined on the adic system $\{M_E\}_{E \subset G}$ namely for each member in the system. We abuse the notation a bit when we write the notation $[[G]]/E$ which will really mean that $\mathbb{Q}_p[[G/E]]$. Here we have considered a corresponding representation of the algebra $B[[G]]$ by $\varprojlim_E B[[G/E]]$.

Definition 8.3.3. (After Kedlaya-Liu [KL2, Definition 2.4.1]) For any left $A \widehat{\otimes} B[[G]]$ -module M , we call it m - $B[[G]]$ -stably pseudocoherent if we have that it is m - $B[[G]]$ -pseudocoherent, complete for the natural topology and for any morphism $A \rightarrow A'$ which is the corresponding rational localization, the base change of M to $A' \widehat{\otimes} B[[G]]$ is complete for the natural topology as the corresponding left $A' \widehat{\otimes} B[[G]]$ -module (namely each member in the family is still remained to be complete after the corresponding rational localization). As in [KL2, Definition 2.4.1] we call that the corresponding left $A \widehat{\otimes} B[[G]]$ -module M just $B[[G]]$ -stably pseudocoherent if we have that it is simply just ∞ - $B[[G]]$ -stably pseudocoherent.

Since our spaces are commutative, so the rational localization is not flat, therefore in our situation we need to define something weaker than the corresponding flatness which

will be sufficient for us to apply in the following development:

Definition 8.3.4. (After Kedlaya-Liu [KL2, Definition 2.4.4]) For any Banach left $A\widehat{\otimes}B[[G]]$ -module M , we call it is m - $B[[G]]$ -pseudoflat if for any right $A\widehat{\otimes}B[[G]]$ -module M' m - $B[[G]]$ -stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B[[G]]}(M', M) = 0$ (namely $\{\mathrm{Tor}_1^{A\widehat{\otimes}B[[G]]/E}(M'_E, M_E)\}_{E \subset G} = \{0\}_{E \subset G}$). For any Banach right $A\widehat{\otimes}B[[G]]$ -module M , we call it is m - $B[[G]]$ -pseudoflat if for any left $A\widehat{\otimes}B[[G]]$ -module M' m - $B[[G]]$ -stably pseudocoherent we have

$$\mathrm{Tor}_1^{A\widehat{\otimes}B[[G]]}(M, M') = 0.$$

Definition 8.3.5. (After Kedlaya-Liu [KL2, Definition 2.4.6]) We consider the corresponding notion of the corresponding pro-projective modules. We define over A the corresponding $B[[G]]$ -pro-projective module M to be a corresponding left $A\widehat{\otimes}B[[G]]$ -module such that one could find a filtered sequence of projectors such that in the sense of taking the prolimit we have the corresponding projector will converge to any chosen element in M . Here we assume the module is complete with respect to natural topology, and we assume the projectors are $A\widehat{\otimes}B[[G]]$ -linear and we assume that the corresponding image of the projectors are modules which are also finitely generated and projective.

Lemma 8.3.6. (After Kedlaya-Liu [KL2, Lemma 2.4.7]) *Suppose we have over the space $\mathrm{Spa}(A, A^+)$ a corresponding $B[[G]]$ -pro-projective left module M . And suppose that we have a 2 - $B[[G]]$ -pseudoflat right module C over A . And here we assume that C is complete. Then we have that the corresponding product $C \otimes_{A\widehat{\otimes}B[[G]]} M$ is then complete under the corresponding natural topology. And moreover we have that in our situation:*

$$\mathrm{Tor}_1^{A\widehat{\otimes}B[[G]]}(C, M) = 0.$$

Proof. This will be just a noncommutative version of the corresponding [KL2, Lemma 2.4.7]. We adapt the corresponding argument in [KL2, Lemma 2.4.7] to our situation. We just consider each member in the families for the adic modules. What we are going to consider in this case is then first to consider the following presentation:

$$M' \longrightarrow (A\widehat{\otimes}B[[G]]/E)^k \longrightarrow M \longrightarrow 0,$$

where the left module M' is finitely presented. Then we consider the following commutative diagram:

$$\begin{array}{ccccccc}
C \otimes_{A \widehat{\otimes} B[[G]]/E} M' & \longrightarrow & C \otimes_{A \widehat{\otimes} B[[G]]/E} (A \widehat{\otimes} B[[G]]/E)^k & \longrightarrow & C \otimes_{A \widehat{\otimes} B[[G]]/E} M & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & C \widehat{\otimes}_{A \widehat{\otimes} B[[G]]/E} M' & \longrightarrow & C \widehat{\otimes}_{A \widehat{\otimes} B[[G]]/E} (A \widehat{\otimes} B[[G]]/E)^k & \longrightarrow & C \widehat{\otimes}_{A \widehat{\otimes} B[[G]]/E} M & \longrightarrow & 0.
\end{array}$$

The first row is exact as in [KL2, Lemma 2.4.7], and we have the corresponding second row is also exact since we have that by hypothesis the corresponding module M is $[[G]]/E$ -pro-projective. Then as in [KL2, Lemma 2.4.7] the results follow. \square

Proposition 8.3.7. (After Kedlaya-Liu [KL2, Corollary 2.4.8]) *Over the space $\mathrm{Spa}(A, A^+)$, we have any $B[[G]]$ -pro-projective left module is 2- $B[[G]]$ -pseudoflat.*

Proof. This is a direct consequence of the previous lemma. \square

Lemma 8.3.8. (After Kedlaya-Liu [KL2, Corollary 2.4.9]) *Keep the notation above, suppose we are working over the adic space $\mathrm{Spa}(A, A^+)$. Suppose now the left $A \widehat{\otimes} B[[G]]$ -module is finitely generated (again note that we are talking about a family of objects over the corresponding quotients), then we have that the following natural maps:*

$$A\{T\} \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M \rightarrow M\{T\}, \quad (8.3.1)$$

$$A\{T, T^{-1}\} \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M \rightarrow M\{T, T^{-1}\} \quad (8.3.2)$$

are surjective. Suppose now the left $A \widehat{\otimes} B$ -module is finitely presented, then we have that the following natural maps:

$$A\{T\} \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M \rightarrow M\{T\}, \quad (8.3.3)$$

$$A\{T, T^{-1}\} \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M \rightarrow M\{T, T^{-1}\} \quad (8.3.4)$$

are bijective. All modules here are assumed to be complete.

Proof. This is the corresponding consequence of the previous lemma. \square

Proposition 8.3.9. (After Kedlaya-Liu [KL2, Lemma 2.4.12]) *We keep the corresponding notations in [KL2, Lemma 2.4.10]. Then in our current sense we have that the corresponding morphism $A \rightarrow B_2$ is then 2 - $B[[G]]$ -pseudoflat.*

Proof. The corresponding proof could be made parallel to the corresponding proof of [KL2, Lemma 2.4.12]. To be more precise we consider the following commutative diagram from the corresponding chosen exact sequence (which is just the analog of the corresponding one in [KL2, Lemma 2.4.12], namely as below we choose arbitrary short exact sequence with P as a left $A \widehat{\otimes} B[[G]]/E$ -module which is 2 - $B[[G]]/E$ -stably pseudocoherent):

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

which then induces the following corresponding big commutative diagram:

$$\begin{array}{ccccccc}
& & & 0 & & & 0 \\
& & & \downarrow & & & \downarrow \\
0 & \longrightarrow & A\{T\} \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} M & \longrightarrow & A\{T\} \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} N & \longrightarrow & A\{T\} \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} P \longrightarrow 0, \\
& & \downarrow 1-fT & & \downarrow 1-fT & & \downarrow 1-fT \\
0 & \longrightarrow & A\{T\} \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} M & \longrightarrow & A\{T\} \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} N & \longrightarrow & A\{T\} \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} P \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B_2 \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} M & \longrightarrow & B_2 \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} N & \longrightarrow & B_2 \widehat{\otimes} B[[G]]/E \otimes_{A \widehat{\otimes} B[[G]]/E} P \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

with the notations in [KL2, Lemma 2.4.10] where we actually have the corresponding exactness along the horizontal direction at the corner around $M \otimes_{A \widehat{\otimes} B[[G]]/E} B_2 \widehat{\otimes} B[[G]]/E$ marked with ?, by diagram chasing. \square

Proposition 8.3.10. (After Kedlaya-Liu [KL2, Lemma 2.4.13]) *We keep the corresponding notations in [KL2, Lemma 2.4.10]. Then in our current sense we have that the corresponding morphism $A \rightarrow B_1$ is then 2 - $B[[G]]$ -pseudoflat. And in our current sense we have that the corresponding morphism $A \rightarrow B_{1,2}$ is then 2 - $B[[G]]$ -pseudoflat.*

Proof. The statement for $A \rightarrow B_{1,2}$ could be proved by consider the composition $A \rightarrow B_2 \rightarrow B_{12}$ where the $2\text{-}B[[G]]$ -pseudoflatness could be proved as in [KL2, Lemma 2.4.13] by inverting the corresponding variable. For $A \rightarrow B_1$, the corresponding proof could be made parallel to the corresponding proof of [KL2, Lemma 2.4.13]. To be more precise we consider the following commutative diagram from the corresponding chosen exact sequence (which is just the analog of the corresponding one in [KL2, Remark 2.4.5], namely as below we choose arbitrary short exact sequence with P as a left $A \widehat{\otimes} B[[G]]/E$ -module which is $2\text{-}B[[G]]/E$ -stably pseudocoherent):

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

which then induces the following corresponding big commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \downarrow & & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (B_1 \widehat{\otimes}_E^{?} B[[G]] \oplus B_2 \widehat{\otimes}_E B[[G]]) \otimes M & \longrightarrow & (B_1 \widehat{\otimes}_E B[[G]] \oplus B_2 \widehat{\otimes}_E B[[G]]) \otimes N & \longrightarrow & (B_1 \widehat{\otimes}_E B[[G]] \oplus B_2 \widehat{\otimes}_E B[[G]]) \otimes P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & B_{12} \widehat{\otimes}_E B[[G]] \otimes M & \longrightarrow & B_{12} \widehat{\otimes}_E B[[G]] \otimes N & \longrightarrow & B_{12} \widehat{\otimes}_E B[[G]] \otimes P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with the notations in [KL2, Lemma 2.4.10] where we actually have the corresponding exactness along the horizontal direction at the corner around

$$M \otimes_{A \widehat{\otimes} B[[G]]/E} (B_1 \widehat{\otimes} B[[G]]/E \oplus B_2 \widehat{\otimes} B[[G]]/E)$$

marked with ?, by diagram chasing. □

After these foundational results as in [KL2] we have the following proposition which is the corresponding noncommutative generalization of the corresponding result established in [KL2, Theorem 2.4.15].

Proposition 8.3.11. (After Kedlaya-Liu [KL2, Theorem 2.4.15]) *In our current context we have that for any rational localization $\mathrm{Spa}(A', A'^+) \rightarrow \mathrm{Spa}(A, A^+)$ we have that along this base change the corresponding $B[[G]]$ -stably pseudocoherence is preserved.*

Then we have the corresponding Tate's acyclicity in the noncommutative deformed setting:

Theorem 8.3.12. (After Kedlaya-Liu [KL2, Theorem 2.5.1]) *Now suppose we have in our corresponding setting 8.2.1 a corresponding $B[[G]]$ -stably pseudocoherent module M . Then we consider the corresponding assignment such that for any $U \subset \mathrm{Spa}(A, A^+)$ we define $\widetilde{M}(U)$ as in the following:*

$$\widetilde{M}(U) := \varprojlim_{\mathrm{Spa}(S, S^+) \subset U, \text{rational}} S \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M. \quad (8.3.5)$$

Then we have that for any \mathfrak{B} which is a rational covering of $U = \mathrm{Spa}(S, S^+) \subset \mathrm{Spa}(A, A^+)$ (certainly this U is also assumed to be rational) we have that the vanishing of the following two cohomology groups:

$$H^i(U, \widetilde{M}), \check{H}^i(U : \mathfrak{B}, \widetilde{M}) \quad (8.3.6)$$

for any $i > 0$. When concentrating at the degree zero we have:

$$H^0(U, \widetilde{M}) = S \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M, \check{H}^0(U : \mathfrak{B}, \widetilde{M}) = S \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M. \quad (8.3.7)$$

Proof. By [KL1, Propositions 2.4.20-2.4.21], we could then finish proof as in [KL2, Theorem 2.5.1] by using our previous proposition 8.3.11 and proposition 8.3.10 as above. \square

We now consider the corresponding noncommutative deformed version of Kiehl's glueing properties for stably pseudocoherent modules after [KL2]:

Definition 8.3.13. (After Kedlaya-Liu [KL2, Definition 2.5.3]) *Consider in our context (over $\mathrm{Spa}(A, A^+)$) the corresponding sheaves $\mathcal{O} \widehat{\otimes} B[[G]]$, we will then define the corresponding pseudocoherent sheaves over $\mathrm{Spa}(A, A^+)$ to be those locally defined by attaching stably-pseudocoherent modules over the section.*

Lemma 8.3.14. (After Kedlaya-Liu [KL2, Lemma 2.5.4]) *Consider the corresponding notations in [KL2, Lemma 2.4.10], we have the corresponding morphism $A \rightarrow B_1 \oplus B_2$. Then we have that this morphism is an descent morphism effective for the corresponding $B[[G]]$ -stably pseudocoherent Banach modules.*

Proof. This is a $B[[G]]$ -relative version of the [KL2, Lemma 2.5.4]. We adapt the corresponding proof to our situation. First consider the corresponding notations on the commutative adic space $\mathrm{Spa}(A, A^+)$ as in [KL2, Lemma 2.4.10]. And we have a covering $\{U_1, U_2\}$. Then consider a descent datum for this cover of $B[[G]]$ -stably pseudocoherent sheaves. By theorem 8.3.12 we could realize this as a corresponding pseudocoherent sheaf over $\mathcal{O} \widehat{\otimes} B[[G]]$. We denote this by \mathcal{M} . Then by (the proof of) [T2, Lemma 6.82], we have that this is finitely generated and we have the corresponding vanishing of $\check{H}^i(\mathrm{Spa}(A, A^+), \mathfrak{B}; \mathcal{M})$ for $i > 0$. Then by theorem 8.3.12 again we have that the vanishing of $H^i(U_j; \mathcal{M})$ for $j \in \{1, 2\}, i > 0$. Then as in [KL2, Lemma 2.5.4] we have the corresponding vanishing of $H^i(\mathrm{Spa}(A, A^+); \mathcal{M})$ for $i > 0$. Then as in [KL2, Lemma 2.5.4] we choose a corresponding covering of some finite free object \mathcal{F} and take its kernel \mathcal{K} which is actually pseudocoherent by the corresponding proposition 8.3.11. Then forming the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (B_p \widehat{\otimes} B[[G]]/E) \otimes \mathcal{K}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (B_p \widehat{\otimes} B[[G]]/E) \otimes \mathcal{F}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (B_p \widehat{\otimes} B[[G]]/E) \otimes \mathcal{M}(\mathrm{Spa}(A, A^+)) \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K}(U_p) & \longrightarrow & \mathcal{F}(U_p) & \longrightarrow & \mathcal{M}(U_p) \longrightarrow 0,
\end{array}$$

where we have $p = 1, 2$. The first row is exact by $B[[G]]/E$ -pseudoflatness as in proposition 8.2.11 and the second row is exact as well by theorem 8.2.12. The middle vertical arrow is an isomorphism while the rightmost vertical arrow is surjective and the leftmost vertical arrow is surjective. The surjectivity just here comes from actually [T2, Lemma 6.82]. This will imply that the rightmost vertical arrow is injective as well. Then we have the corresponding isomorphism $\mathcal{M} \xrightarrow{\sim} \widetilde{\mathcal{M}(\mathrm{Spa}(A, A^+)})$. Then we consider the following diagram:

$$\begin{array}{ccccccc}
(C \widehat{\otimes} B[[G]]/E) \otimes \mathcal{K}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C \widehat{\otimes} B[[G]]/E) \otimes \mathcal{F}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C \widehat{\otimes} B[[G]]/E) \otimes \mathcal{M}(\mathrm{Spa}(A, A^+)) & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{F}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{M}(\mathrm{Spa}(C, C^+)) \longrightarrow 0,
\end{array}$$

C is any rational localization of A . Here the first row is exact by $B[[G]]/E$ -pseudoflatness and the second is row is exact as well. The middle vertical arrow is an isomorphism while the rightmost vertical arrow is surjective as in [KL2, Lemma 2.5.4]. Then we apply the same argument to the leftmost vertical arrow (with a new diagram), we will have the corresponding surjectivity of this leftmost vertical arrow. This will imply that the rightmost vertical arrow is injective as well. Then we have the corresponding isomorphism for the rightmost vertical arrow. \square

Theorem 8.3.15. (After Kedlaya-Liu [KL2, Theorem 2.5.5]) *Taking global section will realize the equivalence between the following two categories: A. The category of all the pseudocoherent $\mathcal{O} \widehat{\otimes} B[[G]]$ -sheaves; B. The category of all the $B[[G]]$ -stably pseudocoherent modules over $A \widehat{\otimes} B[[G]]$.*

Proof. See [KL2, Theorem 2.5.5]. We need to still apply [KL1, Proposition 2.4.20], as long as one considers instead in our situation lemma 8.3.14 and theorem 8.3.12. \square

8.3.2 Noncommutative Pseudocoherence in Étale Topology

We consider the extension of the corresponding discussion in the previous subsection to the étale topology. At this moment we consider the following same context on the geometric level as in [KL2]:

Setting 8.3.16. (After Kedlaya-Liu [KL2, Hypothesis 2.5.8]) As in the previous subsection we consider now the corresponding geometric setting namely an adic space $\mathrm{Spa}(A, A^+)$ where A is assumed to be sheafy. Then we consider $B[[G]]$ as above. And we consider the corresponding étale site $\mathrm{Spa}(A, A^+)_{\text{ét}}$. And by keeping the corresponding

setting in the geometry as in [KL2, Hypothesis 2.5.8], we assume that there is a stable basis \mathfrak{B} containing the space $\text{Spa}(A, A^+)$ itself.

Definition 8.3.17. (After Kedlaya-Liu [KL2, Definition 2.5.9]) For any left $A \widehat{\otimes} B[[G]]$ -module M , we call it m - $B[[G]]$ -étale stably pseudocoherent with respect to \mathfrak{B} if we have that it is m - $B[[G]]$ -pseudocoherent, complete for the natural topology and for any rational localization $A \rightarrow A'$, the base change of M to $A' \widehat{\otimes} B[[G]]$ is complete for the natural topology as the corresponding left $A' \widehat{\otimes} B[[G]]$ -module. As in [KL2, Definition 2.4.1] we call that the corresponding left $A \widehat{\otimes} B[[G]]$ -module M just $B[[G]]$ -stably pseudocoherent if we have that it is simply just ∞ - $B[[G]]$ -stably pseudocoherent.

Definition 8.3.18. (After Kedlaya-Liu [KL2, Below Definition 2.5.9]) For any Banach left $A \widehat{\otimes} B[[G]]$ -module M , we call it is m - $B[[G]]$ -étale-pseudoflat if for any right $A \widehat{\otimes} B[[G]]$ -module M' m - $B[[G]]$ -étale-stably pseudocoherent we have $\text{Tor}_1^{A \widehat{\otimes} B[[G]]}(M', M) = 0$. For any Banach right $A \widehat{\otimes} B[[G]]$ -module M , we call it is m - $B[[G]]$ -étale-pseudoflat if for any left $A \widehat{\otimes} B[[G]]$ -module M' m - $B[[G]]$ -étale-stably pseudocoherent we have

$$\text{Tor}_1^{A \widehat{\otimes} B[[G]]}(M, M') = 0.$$

The following proposition holds in our current setting.

Proposition 8.3.19. (After Kedlaya-Liu [KL2, Lemma 2.5.10]) *One can actually find another basis \mathfrak{C} in \mathfrak{B} such that any morphism in \mathfrak{C} could be 2 - $B[[G]]$ -étale-pseudoflat with respect to either \mathfrak{C} or \mathfrak{B} .*

Proof. We follow the proof of [KL2, Lemma 2.5.10] in our current $B[[G]]$ -relative situation, however not that much needs to proof by relying on the proof of [KL2, Lemma 2.5.10]. To be more precise the corresponding selection of the new basis \mathfrak{C} comes from including all the morphism made up of some composition of rational localization and the corresponding finite étale ones. For these we have shown above the corresponding 2 - $B[[G]]$ -étale pseudoflatness. Then the rest will be pure geometric for the analytic spaces which are just the same as the situation of [KL2, Lemma 2.5.10], therefore we just omit the corresponding argument, see [KL2, Lemma 2.5.10]. \square

Then we have the corresponding Tate's acyclicity in the noncommutative deformed setting in étale topology (here fix \mathfrak{C} as above):

Theorem 8.3.20. (After Kedlaya-Liu [KL2, Theorem 2.5.11]) *Now suppose we have in our corresponding setting 8.3.1 a corresponding $B[[G]]$ -étale-stably pseudocoherent module M . Then we consider the corresponding assignment such that for any $U \subset \mathrm{Spa}(A, A^+)$ we define $\widetilde{M}(U)$ as in the following:*

$$\widetilde{M}(U) := \varprojlim_{\mathrm{Spa}(S, S^+) \subset U, \in \mathfrak{C}} S \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M. \quad (8.3.8)$$

Then we have that for any \mathfrak{B} which is a covering of $U = \mathrm{Spa}(S, S^+) \subset \mathrm{Spa}(A, A^+)$ (certainly this U is also assumed to be in \mathfrak{C} , and we assume this covering is formed by using the corresponding members in \mathfrak{C}) we have that the vanishing of the following two cohomology groups:

$$H^i(U, \widetilde{M}), \check{H}^i(U : \mathfrak{B}, \widetilde{M}) \quad (8.3.9)$$

for any $i > 0$. When concentrating at the degree zero we have:

$$H^0(U, \widetilde{M}) = S \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M, \check{H}^0(U : \mathfrak{B}, \widetilde{M}) = S \widehat{\otimes} B[[G]] \otimes_{A \widehat{\otimes} B[[G]]} M. \quad (8.3.10)$$

Proof. By [KL1, Proposition 8.2.21], we could then finish proof as in [KL2, Theorem 2.5.11] by using our previous proposition 8.3.19 and the corresponding faithfully flat descent as in the situation of [KL2, Theorem 2.5.11]. \square

We now consider the corresponding noncommutative deformed version of Kiehl's glueing properties for stably pseudocoherent modules after [KL2]:

Definition 8.3.21. (After Kedlaya-Liu [KL2, Definition 2.5.12]) *Consider in our context (over $\mathrm{Spa}(A, A^+)_{\text{ét}}$) the corresponding sheaves $\mathcal{O}_{\mathrm{Spa}(A, A^+)_{\text{ét}}} \widehat{\otimes} B[[G]]$, we will then define the corresponding pseudocoherent sheaves over $\mathrm{Spa}(A, A^+)_{\text{ét}}$ to be those locally defined by attaching étale-stably-pseudocoherent modules over the section.*

Lemma 8.3.22. (After Kedlaya-Liu [KL2, Lemma 2.5.13]) *Consider the corresponding notations in [KL2, Lemma 2.4.10], we have the corresponding morphism $A \rightarrow B_1 \oplus B_2$. Then we have that this morphism is an descent morphism effective for the corresponding $B[[G]]$ -étale-stably pseudocoherent Banach modules.*

Proof. This is a $B[[G]]$ -relative version of the [KL2, Lemma 2.5.13]. We adapt the corresponding proof to our situation. First consider the corresponding notations on the

commutative adic space $\mathrm{Spa}(A, A^+)$ as in [KL2, Lemma 2.4.10]. And we have a covering $\{U_1, U_2\}$. Then consider a descent datum for this cover of $B[[G]]$ -étale-stably pseudocoherent sheaves. By theorem 8.2.20 we could realize this as a corresponding pseudocoherent sheaf over $\mathcal{O}\widehat{\otimes}B[[G]]$. We denote this by \mathcal{M} . Then by consider the corresponding results in the rational localization situation above we have that the vanishing of $H^i(U_j; \mathcal{M})$ for $j \in \{1, 2\}, i > 0$. Then as in [KL2, Lemma 2.5.13] we have the corresponding vanishing of $H^i(\mathrm{Spa}(A, A^+); \mathcal{M})$ for $i > 0$. Then as in [KL2, Lemma 2.5.13] we choose a corresponding covering of some finite free object \mathcal{F} and take its kernel \mathcal{K} which gives rise to the corresponding exact sequence:

$$0 \longrightarrow \mathcal{K}(\mathrm{Spa}(A, A^+)) \longrightarrow \mathcal{F}(\mathrm{Spa}(A, A^+)) \longrightarrow \mathcal{M}(\mathrm{Spa}(A, A^+)) \longrightarrow 0,$$

where we have $p = 1, 2$. We have already the corresponding stably-pseudocoherence of the global section. As in [KL2, Lemma 2.5.13] we only need to show that it is stable under étale morphism. Then we consider the following diagram:

$$\begin{array}{ccccccc} (C\widehat{\otimes}B[[G]]/E) \otimes \mathcal{K}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C\widehat{\otimes}B[[G]]/E) \otimes \mathcal{F}(\mathrm{Spa}(A, A^+)) & \longrightarrow & (C\widehat{\otimes}B[[G]]/E) \otimes \mathcal{M}(\mathrm{Spa}(A, A^+)) & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \mathcal{K}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{F}(\mathrm{Spa}(C, C^+)) & \longrightarrow & \mathcal{M}(\mathrm{Spa}(C, C^+)) & \longrightarrow 0, \end{array}$$

$A \rightarrow C$ is some member in \mathfrak{C} . Here the first row is exact by $B[[G]]/E$ -étale-pseudoflatness and the second is row is exact as well. The middle vertical arrow is an isomorphism while the rightmost vertical arrow is surjective as in [KL2, Lemma 2.5.13]. Then we apply the same argument to the leftmost vertical arrow (with a new diagram), we will have the corresponding surjectivity of this leftmost vertical arrow. This will imply that the rightmost vertical arrow is injective as well. Then we have the corresponding isomorphism for the rightmost vertical arrow. \square

Theorem 8.3.23. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *Taking global section will realize the equivalence between the following two categories: A. The category of all the pseudocoherent $\mathcal{O}_{\mathrm{Spa}(A, A^+)_{\text{ét}}}\widehat{\otimes}B[[G]]$ -sheaves; B. The category of all the $B[[G]]$ -étale-stably pseudocoherent modules over $A\widehat{\otimes}B[[G]]$.*

Proof. See [KL2, Theorem 2.5.14]. We need to still apply [KL1, Theorem 8.2.22], as long as one considers instead in our situation, and [SP, Tag 03OD].

□

8.3.3 Noncommutative Deformation over Quasi-Stein Spaces

In this subsection we consider the corresponding noncommutative deformation over the corresponding context in [KL2, Chapter 2.6].

Setting 8.3.24. Let X be a corresponding quasi-Stein adic space over \mathbb{Q}_p in the sense of [KL2, Definition 2.6.2]. Recall that what is happening is that X could be written as the corresponding direct limit of affinoids $X := \varinjlim_i X_i$.

Lemma 8.3.25. (After Kedlaya-Liu [KL2, Lemma 2.6.3]) *We now consider the corresponding rings $A_i := \mathcal{O}_{X_i}$ for all $i = 0, 1, \dots$, and in our current situation we consider the corresponding rings $A_i \widehat{\otimes} B[[G]]$ (over \mathbb{Q}_p) for all $i = 0, 1, \dots$. And in our situation we consider the corresponding modules M_i over $A_i \widehat{\otimes} B[[G]]$ for all $i = 0, 1, \dots$ with the same requirement as in [KL2, Lemma 2.6.3] (namely those complete with respect to the natural topology). Suppose that we have bounded surjective map from $f_i : A_i \widehat{\otimes} B[[G]] \otimes_{A_{i+1} \widehat{\otimes} B[[G]]} M_{i+1} \rightarrow M_i, i = 0, 1, \dots$. Then we have first the density of the corresponding image of $\varprojlim_i M_i$ in each M_i for any $i = 0, 1, 2, \dots$. And we have as well the corresponding vanishing of $R^1 \varprojlim_i M_i$.*

Proof. This is the $B[[G]]$ -relative version of the result in [KL2, Lemma 2.6.3]. One needs to consider the corresponding truncation through some specific open subgroup E . Then see lemma 8.2.25. □

Proposition 8.3.26. (After Kedlaya-Liu [KL2, Lemma 2.6.4]) *In the same situation as above, suppose we have that the corresponding modules M_i are basically $B[[G]]$ -stably pseudocoherent over the rings A_i for all $i = 0, 1, \dots$. Now we consider the situation where $f_i : A_i \widehat{\otimes} B[[G]] \otimes_{A_{i+1} \widehat{\otimes} B[[G]]} M_{i+1} \rightarrow M_i$ is an isomorphism. Then the conclusion in our situation is then that the corresponding projection from $\varprojlim M_i$ to M_i for each $i = 0, 1, 2, \dots$ is an isomorphism.*

Proof. This is a $B[[G]]$ -relative version of the [KL2, Lemma 2.6.4]. We adapt the argument to our situation as in the following. First we choose some finite free covering T of the

limit M such that we have for each i the corresponding map $T_i \rightarrow M_i$ is surjective. Then we consider the index $j \geq i$ and set the kernel of the map from T_j to M_j to be S_j . By the direct analog of [KL2, Lemma 2.5.6] we have that $A_i \widehat{\otimes} B[[G]]/E \otimes S_j \rightarrow S_i$ realizes the isomorphism, and we have that the corresponding surjectivity of the corresponding map from $\varprojlim_i S_i$ projecting to the S_i . Then one could finish the proof by 5-lemma to the following commutative diagram as in [KL2, Lemma 2.6.4]:

$$\begin{array}{ccccccc}
(A_i \widehat{\otimes} B[[G]]/E) \otimes \varprojlim_i S_i & \longrightarrow & (A_i \widehat{\otimes} B[[G]]/E) \otimes \varprojlim_i F_i & \longrightarrow & (A_i \widehat{\otimes} B[[G]]/E) \otimes \varprojlim_i M_i & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_i & \longrightarrow & F_i & \longrightarrow & M_i & \longrightarrow & 0.
\end{array}$$

□

Proposition 8.3.27. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the map $\mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ is surjective for any $B[[G]]$ -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B[[G]]$.*

Proof. This is just the corresponding corollary of the previous proposition. □

Proposition 8.3.28. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *We have that the stalk \mathcal{M}_x is finitely generated over the stalk $\mathcal{O}_{X,x}$ for any $x \in X$ by $\mathcal{M}(X)$, for any $B[[G]]$ -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B[[G]]$.*

Proof. This is just the corresponding corollary of the proposition before the previous proposition. □

Proposition 8.3.29. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the corresponding vanishing of the corresponding sheaf cohomology groups $H^k(X, \mathcal{M})$ of \mathcal{M} for higher $k > 0$, for any $B[[G]]$ -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B[[G]]$.*

Proof. See the proof of proposition 8.2.29. Again in our situation we only have to work over some quotient. □

Corollary 8.3.30. (After Kedlaya-Liu [KL2, Corollary 2.6.6]) *The corresponding functor from the corresponding $B[[G]]$ -deformed pseudocoherent sheaves over X to the corresponding $B[[G]]$ -stably by taking the corresponding global section is an exact functor.*

Corollary 8.3.31. (After Kedlaya-Liu [KL2, Corollary 2.6.8]) *Consider a particular $\mathcal{O}_X \widehat{\otimes} B[[G]]$ -pseudocoherent sheaf \mathcal{M} which is finite locally free throughout the whole space X . Then we have that the global section $\mathcal{M}(X)$ as $\mathcal{O}_X(X) \widehat{\otimes} B[[G]]$ left module admits the corresponding structures of finite projective structure if and only if we have the corresponding global section is finitely generated.*

Proof. As in [KL2, Corollary 2.6.8] one could find some global splitting through the local splittings. □

We now consider the following $B[[G]]$ -relative analog of [KL2, Proposition 2.6.17]:

Theorem 8.3.32. (After Kedlaya-Liu [KL2, Proposition 2.6.17]) *Consider the following two statements for a particular $\mathcal{O}_X \widehat{\otimes} B[[G]]$ -pseudocoherent sheaf \mathcal{M} . First is that one can find finite many generators (the number is up to some uniform fixed integer $n \geq 0$) for each section of $\mathcal{M}(X_i)$ for each $i = 1, 2, \dots$. The second statement is that the global section $\mathcal{M}(X)$ is just finitely generated. Then in our situation the two statement is equivalent if we have that the corresponding space X admits a m -uniformed covering in the exact same sense of [KL2, Proposition 2.6.17].*

Proof. See the proof of theorem 8.2.32. □

8.4 Foundations on Noncommutative Descent for Adic Spectra in Limit of Fréchet Case

8.4.1 Noncommutative Pseudocoherence in Analytic Topology

We now establish some foundations in the noncommutative setting on glueing noncommutative pseudocoherent modules after [KL2, Chapter 2]. But we remind the readers that we will fix the base space:

Setting 8.4.1. Consider a corresponding sheafy Banach adic uniform algebra (A, A^+) over \mathbb{Q}_p or $\mathbb{F}_p((t))$, we consider the base space $\mathrm{Spa}(A, A^+)$. And now we will consider a further noncommutative Banach algebra (B, B^+) over \mathbb{Q}_p or $\mathbb{F}_p((t))$ which could be written as the following injective limit:

$$B = \varinjlim_h B_h,$$

where each B_h is Banach (or more general Fréchet although we do not consider such generality).

Example 8.4.2. There are many interesting models in our current context, for instance the period ring B_e as considered by Berger in [Ber1] and the corresponding Robba rings in the full setting as in [KL2] (note that the latter is really indeed ind-Fréchet).

Definition 8.4.3. (After Kedlaya-Liu [KL2, Definition 2.4.1]) For any left $A \widehat{\otimes} B$ -module M , we call it m - B -stably pseudocoherent if we have that it is m - B -pseudocoherent, complete for the natural topology and for any morphism $A \rightarrow A'$ which is the corresponding rational localization, the base change of M to $A' \widehat{\otimes} B$ is complete for the natural LF topology as the corresponding left $A' \widehat{\otimes} B$ -module. As in [KL2, Definition 2.4.1] we call that the corresponding left $A \widehat{\otimes} B$ -module M just B -stably pseudocoherent if we have that it is simply just ∞ - B -stably pseudocoherent.

Definition 8.4.4. (After Kedlaya-Liu [KL2, Definition 2.4.4]) For any Banach left $A \widehat{\otimes} B$ -module M , we call it is m - B -pseudoflat if for any right $A \widehat{\otimes} B$ -module M' m - B -stably pseudocoherent we have $\mathrm{Tor}_1^{A \widehat{\otimes} B}(M', M) = 0$. For any Banach right $A \widehat{\otimes} B$ -module M , we call it is m - B -pseudoflat if for any left $A \widehat{\otimes} B$ -module M' m - B -stably pseudocoherent we have $\mathrm{Tor}_1^{A \widehat{\otimes} B}(M, M') = 0$.

Definition 8.4.5. (After Kedlaya-Liu [KL2, Definition 2.4.6]) We consider the corresponding notion of the corresponding pro-projective module. We define over A the corresponding B -pro-projective module M to be a corresponding left $A\widehat{\otimes}B$ -module such that one could find a filtered sequence of projectors such that in the sense of taking the prolimit we have the corresponding projector will converge to any chosen element in M . Here we assume the module is complete with respect to natural topology, and we assume the projectors are $A\widehat{\otimes}\varinjlim_h B_h$ -linear and we assume that the corresponding image of the projectors are modules which are also finitely generated and projective.

Lemma 8.4.6. (After Kedlaya-Liu [KL2, Lemma 2.4.7]) *Suppose we have over the space $\mathrm{Spa}(A, A^+)$ a corresponding B -pro-projective left module M . And suppose that we have a 2- B -pseudoflat right module C over A . And we assume that C is complete for the LF topology. Then we have that the corresponding product $C\otimes_{A\widehat{\otimes}B}M$ is then complete under the corresponding natural topology. And moreover we have that in our situation:*

$$\mathrm{Tor}_1^{A\widehat{\otimes}B}(C, M) = 0.$$

Proof. This will be just a noncommutative and LF version of the corresponding [KL2, Lemma 2.4.7]. We adapt the corresponding argument in [KL2, Lemma 2.4.7] to our situation. What we are going to consider in this case is then first consider the following presentation:

$$M' \longrightarrow (A\widehat{\otimes}\varinjlim_h B_h)^k \longrightarrow M \longrightarrow 0,$$

where the left module M' is finitely presented. Then we consider the following commutative diagram:

$$\begin{array}{ccccccc} C \otimes_{A\widehat{\otimes}\varinjlim_h B_h} M' & \longrightarrow & C \otimes_{A\widehat{\otimes}\varinjlim_h B_h} (A\widehat{\otimes}\varinjlim_h B_h)^k & \longrightarrow & C \otimes_{A\widehat{\otimes}\varinjlim_h B_h} M & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & C\widehat{\otimes}_{A\widehat{\otimes}\varinjlim_h B_h} M' & \longrightarrow & C\widehat{\otimes}_{A\widehat{\otimes}B} (A\widehat{\otimes}\varinjlim_h B_h)^k & \longrightarrow & C\widehat{\otimes}_{A\widehat{\otimes}\varinjlim_h B_h} M & \longrightarrow 0. \end{array}$$

The first row is exact as in [KL2, Lemma 2.4.7], and we have the corresponding second row is also exact since we have that by hypothesis the corresponding module M is B -pro-projective. Then as in [KL2, Lemma 2.4.7] the results follow. \square

Proposition 8.4.7. (After Kedlaya-Liu [KL2, Corollary 2.4.8]) *Over the space $\mathrm{Spa}(A, A^+)$, we have any B -pro-projective left module is 2- B -pseudoflat.*

Proof. This is a direct consequence of the previous lemma. \square

Lemma 8.4.8. (After Kedlaya-Liu [KL2, Corollary 2.4.9]) *Keep the notation above, suppose we are working over the adic space $\mathrm{Spa}(A, A^+)$. Suppose now the left $A\widehat{\otimes}B$ -module is finitely generated, then we have that the following natural maps:*

$$A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T\}, \quad (8.4.1)$$

$$A\{T, T^{-1}\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T, T^{-1}\} \quad (8.4.2)$$

are surjective. Suppose now the left $A\widehat{\otimes}B$ -module is finitely presented, then we have that the following natural maps:

$$A\{T\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T\}, \quad (8.4.3)$$

$$A\{T, T^{-1}\}\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M \rightarrow M\{T, T^{-1}\} \quad (8.4.4)$$

are bijective. Here all the modules are assumed to be complete for the LF topology.

Proof. This is the corresponding consequence of the previous lemma. \square

Proposition 8.4.9. (After Kedlaya-Liu [KL2, Lemma 2.4.12]) *We keep the corresponding notations in [KL2, Lemma 2.4.10]. Then in our current sense we have that the corresponding morphism $A \rightarrow B_2$ is then 2- B -pseudoflat.*

Proof. The corresponding proof could be made parallel to the corresponding proof of [KL2, Lemma 2.4.12]. To be more precise we consider the following commutative diagram from the corresponding chosen exact sequence (which is just the analog of the corresponding one in [KL2, Lemma 2.4.12], namely as below we choose arbitrary short exact sequence with P as a left $A\widehat{\otimes}\varinjlim_h B_h$ -module which is $2\text{-}\varinjlim_h B_h$ -stably pseudocoherent):

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

which then induces the following corresponding big commutative diagram:

$$\begin{array}{ccccccc}
& & & 0 & & & 0 \\
& & & \downarrow & & & \downarrow \\
0 & \longrightarrow & A\{T\} \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} M & \longrightarrow & A\{T\} \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} N & \longrightarrow & A\{T\} \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} P \longrightarrow 0, \\
& & \downarrow 1-fT & & \downarrow 1-fT & & \downarrow 1-fT \\
0 & \longrightarrow & A\{T\} \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} M & \longrightarrow & A\{T\} \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} N & \longrightarrow & A\{T\} \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} P \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \overset{?}{B_2 \widehat{\otimes} \varinjlim_h B_h} \otimes_{A \widehat{\otimes} \varinjlim_h B_h} M & \longrightarrow & B_2 \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} N & \longrightarrow & B_2 \widehat{\otimes} \varinjlim_h B_h \otimes_{A \widehat{\otimes} \varinjlim_h B_h} P \longrightarrow 0, \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

with the notations in [KL2, Lemma 2.4.10] where we actually have the corresponding exactness along the horizontal direction at the corner around $M \otimes_{A \widehat{\otimes} \varinjlim_h B_h} B_2 \widehat{\otimes} \varinjlim_h B_h$ marked with ?, by diagram chasing. \square

Proposition 8.4.10. (After Kedlaya-Liu [KL2, Lemma 2.4.13]) *We keep the corresponding notations in [KL2, Lemma 2.4.10]. Then in our current sense we have that the corresponding morphism $A \rightarrow B_1$ is then 2- B -pseudoflat. And in our current sense we have that the corresponding morphism $A \rightarrow B_{1,2}$ is then 2- B -pseudoflat.*

Proof. The statement for $A \rightarrow B_{1,2}$ could be proved by consider the composition $A \rightarrow B_2 \rightarrow B_{1,2}$ where the 2- B -pseudoflatness could be proved as in [KL2, Lemma 2.4.13] by inverting the corresponding variable. For $A \rightarrow B_1$, the corresponding proof could be made parallel to the corresponding proof of [KL2, Lemma 2.4.13]. To be more precise we consider the following commutative diagram from the corresponding chosen exact

sequence (which is just the analog of the corresponding one in [KL2, Remark 2.4.5], namely as below we choose arbitrary short exact sequence with P as a left $A \widehat{\otimes} B$ -module which is 2- B -stably pseudocoherent):

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

which then induces the following corresponding big commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (B_1 \widehat{\otimes} B_\infty \oplus B_2 \widehat{\otimes} B_\infty) \otimes_{A \widehat{\otimes} B_\infty} M & \longrightarrow & (B_1 \widehat{\otimes} B_\infty \oplus B_2 \widehat{\otimes} B_\infty) \otimes_{A \widehat{\otimes} B_\infty} N & \longrightarrow & (B_1 \widehat{\otimes} B_\infty \oplus B_2 \widehat{\otimes} B_\infty) \otimes_{A \widehat{\otimes} B_\infty} P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & B_{12} \widehat{\otimes} B_\infty \otimes_{A \widehat{\otimes} B_\infty} M & \longrightarrow & B_{12} \widehat{\otimes} B_\infty \otimes_{A \widehat{\otimes} B_\infty} N & \longrightarrow & B_{12} \widehat{\otimes} B_\infty \otimes_{A \widehat{\otimes} B_\infty} P \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with the notations in [KL2, Lemma 2.4.10] where we actually have the corresponding exactness along the horizontal direction at the corner around

$$M \otimes_{A \widehat{\lim}_h B_h} (B_1 \widehat{\lim}_h B_h \oplus B_2 \widehat{\lim}_h B_h)$$

marked with ?, by diagram chasing. Here $B_\infty := \varinjlim_h B_h$. □

After these foundational results as in [KL2] we have the following proposition which is the corresponding noncommutative LF generalization of the corresponding result established in [KL2, Theorem 2.4.15].

Proposition 8.4.11. (After Kedlaya-Liu [KL2, Theorem 2.4.15]) *In our current context we have that for any rational localization $\mathrm{Spa}(A', A'^+) \rightarrow \mathrm{Spa}(A, A^+)$ we have that along this base change the corresponding B -stably pseudocoherence is preserved.*

Then we have the corresponding Tate's acyclicity in the noncommutative deformed setting:

Theorem 8.4.12. (After Kedlaya-Liu [KL2, Theorem 2.5.1]) *Now suppose we have in our corresponding setting 8.4.1 a corresponding B -stably pseudocoherent module M . Then we consider the corresponding assignment such that for any $U \subset \mathrm{Spa}(A, A^+)$ we define $\widetilde{M}(U)$ as in the following:*

$$\widetilde{M}(U) := \varprojlim_{\mathrm{Spa}(S, S^+) \subset U, \text{rational}} S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.4.5)$$

Then we have that for any \mathfrak{B} which is a rational covering of $U = \mathrm{Spa}(S, S^+) \subset \mathrm{Spa}(A, A^+)$ (certainly this U is also assumed to be rational) we have that the vanishing of the following two cohomology groups:

$$H^i(U, \widetilde{M}), \check{H}^i(U : \mathfrak{B}, \widetilde{M}) \quad (8.4.6)$$

for any $i > 0$. When concentrating at the degree zero we have:

$$H^0(U, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M, \check{H}^0(U : \mathfrak{B}, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.4.7)$$

Proof. By [KL1, Propositions 2.4.20-2.4.21], we could then finish proof as in [KL2, Theorem 2.5.1] by using our previous proposition 8.4.11 and proposition 8.4.10 as above. \square

We now consider the corresponding noncommutative LF deformed version of Kiehl's glueing properties for stably pseudocoherent modules after [KL2]:

Definition 8.4.13. (After Kedlaya-Liu [KL2, Definition 2.5.3]) Consider in our context (over $\mathrm{Spa}(A, A^+)$) the corresponding sheaves $\mathcal{O} \widehat{\otimes} B$, we will then define the corresponding pseudocoherent sheaves over $\mathrm{Spa}(A, A^+)$ to be those locally defined by attaching stably-pseudocoherent modules over the section.

Lemma 8.4.14. (After Kedlaya-Liu [KL2, Lemma 2.5.4]) *Consider the corresponding notations in [KL2, Lemma 2.4.10], we have the corresponding morphism $A \rightarrow B_1 \oplus B_2$. Then we have that this morphism is an descent morphism effective for the corresponding B -stably pseudocoherent Banach modules.*

Proof. See the proof of proposition 8.5.13. □

Theorem 8.4.15. (After Kedlaya-Liu [KL2, Theorem 2.5.5]) *Taking global section will realize the equivalence between the following two categories: A. The category of all the pseudocoherent $\mathcal{O} \widehat{\otimes} B$ -sheaves; B. The category of all the B -stably pseudocoherent modules over $A \widehat{\otimes} B$.*

Proof. See [KL2, Theorem 2.5.5]. We need to still apply [KL1, Proposition 2.4.20], as long as one considers instead in our situation lemma 8.4.14 and theorem 8.4.12. □

8.4.2 Noncommutative Pseudocoherence in Étale Topology

We consider the extension of the corresponding discussion in the previous subsection to the étale topology. At this moment we consider the following same context on the geometric level as in [KL2]:

Setting 8.4.16. (After Kedlaya-Liu [KL2, Hypothesis 2.5.8]) As in the previous subsection we consider now the corresponding geometric setting namely an adic space $\mathrm{Spa}(A, A^+)$ where A is assumed to be sheafy. Then we consider $B := \varinjlim_h B_h$ as above. And we consider the corresponding étale site $\mathrm{Spa}(A, A^+)_{\text{ét}}$. And by keeping the corresponding setting in the geometry as in [KL2, Hypothesis 2.5.8], we assume that there is a stable basis \mathfrak{B} containing the space $\mathrm{Spa}(A, A^+)$ itself.

Definition 8.4.17. (After Kedlaya-Liu [KL2, Definition 2.5.9]) For any left $A \widehat{\otimes} B$ -module M , we call it m - B -étale stably pseudocoherent with respect to \mathfrak{B} if we have that it is m - B -pseudocoherent, complete for the natural topology and for any rational localization $A \rightarrow A'$, the base change of M to $A' \widehat{\otimes} B$ is complete for the natural LF topology as the corresponding left $A' \widehat{\otimes} B$ -module. As in [KL2, Definition 2.4.1] we call that the corresponding left $A \widehat{\otimes} B$ -module M just B -stably pseudocoherent if we have that it is simply just ∞ - B -stably pseudocoherent.

Definition 8.4.18. (After Kedlaya-Liu [KL2, Below Definition 2.5.9]) For any Banach left $A\widehat{\otimes}B$ -module M , we call it is m - B -étale-pseudoflat if for any right $A\widehat{\otimes}B$ -module M' m - B -étale-stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B}(M', M) = 0$. For any Banach right $A\widehat{\otimes}B$ -module M , we call it is m - B -étale-pseudoflat if for any left $A\widehat{\otimes}B$ -module M' m - B -étale-stably pseudocoherent we have $\mathrm{Tor}_1^{A\widehat{\otimes}B}(M, M') = 0$.

The following proposition holds in our current setting.

Proposition 8.4.19. (After Kedlaya-Liu [KL2, Lemma 2.5.10]) *One can actually find another basis \mathfrak{C} in \mathfrak{B} such that any morphism in \mathfrak{C} could be 2- B -étale-pseudoflat with respect to either \mathfrak{C} or \mathfrak{B} .*

Proof. We follow the proof of [KL2, Lemma 2.5.10] in our current B -relative situation, however not that much needs to proof by relying on the proof of [KL2, Lemma 2.5.10]. To be more precise the corresponding selection of the new basis \mathfrak{C} comes from including all the morphism made up of some composition of rational localization and the corresponding finite étale ones. For these we have shown above the corresponding 2- B -étale pseudoflatness. Then the rest will be pure geometric for the analytic spaces which are just the same as the situation of [KL2, Lemma 2.5.10], therefore we just omit the corresponding argument, see [KL2, Lemma 2.5.10]. \square

Then we have the corresponding Tate's acyclicity in the noncommutative deformed setting in étale topology (here fix \mathfrak{C} as above):

Theorem 8.4.20. (After Kedlaya-Liu [KL2, Theorem 2.5.11]) *Now suppose we have in our corresponding setting 8.4.1 a corresponding B -étale-stably pseudocoherent module M . Then we consider the corresponding assignment such that for any $U \subset \mathrm{Spa}(A, A^+)$ we define $\widetilde{M}(U)$ as in the following:*

$$\widetilde{M}(U) := \varprojlim_{\mathrm{Spa}(S, S^+) \subset U, \mathfrak{C}} S\widehat{\otimes}B \otimes_{A\widehat{\otimes}B} M. \quad (8.4.8)$$

Then we have that for any \mathfrak{B} which is a covering of $U = \mathrm{Spa}(S, S^+) \subset \mathrm{Spa}(A, A^+)$ (certainly this U is also assumed to be in \mathfrak{C} , and we assume this covering is formed by using the corresponding members in \mathfrak{C}) we have that the vanishing of the following two cohomology groups:

$$H^i(U, \widetilde{M}), \check{H}^i(U : \mathfrak{B}, \widetilde{M}) \quad (8.4.9)$$

for any $i > 0$. When concentrating at the degree zero we have:

$$H^0(U, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M, \check{H}^0(U : \mathfrak{B}, \widetilde{M}) = S \widehat{\otimes} B \otimes_{A \widehat{\otimes} B} M. \quad (8.4.10)$$

Proof. By [KL1, Proposition 8.2.21], we could then finish proof as in [KL2, Theorem 2.5.11] by using our previous proposition 8.4.19 and the corresponding faithfully flat descent as in the situation of [KL2, Theorem 2.5.11]. \square

We now consider the corresponding noncommutative LF deformed version of Kiehl's glueing properties for stably pseudocoherent modules after [KL2]:

Definition 8.4.21. (After Kedlaya-Liu [KL2, Definition 2.5.12]) Consider in our context (over $\mathrm{Spa}(A, A^+)_{\acute{e}t}$) the corresponding sheaves $\mathcal{O}_{\mathrm{Spa}(A, A^+)_{\acute{e}t}} \widehat{\otimes} B$, we will then define the corresponding pseudocoherent sheaves over $\mathrm{Spa}(A, A^+)_{\acute{e}t}$ to be those locally defined by attaching étale-stably-pseudocoherent modules over the section.

Lemma 8.4.22. (After Kedlaya-Liu [KL2, Lemma 2.5.13]) *Consider the corresponding notations in [KL2, Lemma 2.4.10], we have the corresponding morphism $A \rightarrow B_1 \oplus B_2$. Then we have that this morphism is an descent morphism effective for the corresponding B -étale-stably pseudocoherent Banach modules.*

Proof. See the proof of lemma 8.2.22. \square

Theorem 8.4.23. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *Taking global section will realize the equivalence between the following two categories: A. The category of all the pseudocoherent $\mathcal{O}_{\mathrm{Spa}(A, A^+)_{\acute{e}t}} \widehat{\otimes} B$ -sheaves; B. The category of all the B -étale-stably pseudocoherent modules over $A \widehat{\otimes} B$.*

Proof. See [KL2, Theorem 2.5.14]. We need to still apply [KL1, Theorem 8.2.22], as long as one considers instead in our situation, and [SP, Tag 03OD]. \square

8.4.3 Noncommutative Deformation over Quasi-Stein Spaces

In this subsection we consider the corresponding noncommutative deformation over the corresponding context in [KL2, Chapter 2.6].

Setting 8.4.24. Let X be a corresponding quasi-Stein adic space over \mathbb{Q}_p or $\mathbb{F}_p((t))$ in the sense of [KL2, Definition 2.6.2]. Recall that what is happening is that X could be written as the corresponding direct limit of affinoids $X := \varinjlim_i X_i$.

Lemma 8.4.25. (After Kedlaya-Liu [KL2, Lemma 2.6.3]) *We now consider the corresponding rings $A_i := \mathcal{O}_{X_i}$ for all $i = 0, 1, \dots$, and in our current situation we consider the corresponding rings $A_i \widehat{\otimes} B$ (over \mathbb{Q}_p or $\mathbb{F}_p((t))$) for all $i = 0, 1, \dots$. And in our situation we consider the corresponding modules M_i over $A_i \widehat{\otimes} B$ for all $i = 0, 1, \dots$ with the same requirement as in [KL2, Lemma 2.6.3] (namely those complete with respect to the natural topology which is LF topology in our current situation). Suppose that we have bounded surjective map from $f_i : A_i \widehat{\otimes} B \otimes_{A_{i+1} \widehat{\otimes} B} M_{i+1} \rightarrow M_i, i = 0, 1, \dots$. Then we have first the density of the corresponding image of $\varprojlim_i M_i$ in each M_i for any $i = 0, 1, 2, \dots$. And we have as well the corresponding vanishing of $R^1 \varprojlim_i M_i$.*

Proof. This is the $\varinjlim_h B_h$ -relative version of the result in [KL2, Lemma 2.6.3]. We need to then use the corresponding family of Banach norms parametrized by the index h . For the first statement we just choose sequence of Banach norms on all the corresponding modules for all $i = 0, 1, \dots$ and h such that we have $\|f_i(x_{i+1})\|_i^h \leq 1/2 \|x_{i+1}\|_{i+1}^h$ for any $x_{i+1} \in M_{i+1}$. Then for any $x_i \in M_i$ and any $\delta > 0$, we now consider for any $j \geq 1$ and $\forall h$ the corresponding x_{i+j} such that we have $\|x_{i+j} - f_{i+j+1}(x_{i+j+1})\|_{i+j}^h \leq \delta$. Then the sequence $x_{i+j+k}, k = 0, 1, \dots$ will converge to some well-defined y_{i+j} with in our situation the corresponding $y_i = f_i(y_{i+1})$. We then have $\|x_i - y_i\|_i^h \leq \delta$. This will prove the first statement. For the second statement as in [KL2, Lemma 2.6.3] we form the product $M_0 \times M_1 \times M_2 \times \dots$ and then consider the induced map F from $M_{i+1} \rightarrow M_i$, and consider the corresponding cokernel of the map $1 - F$ since this is just the corresponding limit we are considering. Then to show that the cokernel is zero we just look at the corresponding cokernel of the corresponding map on the corresponding completed direct summand which will project to the original one. But then we will have $\|f_i(v)\|_i^h \leq 1/2 \|v\|_i^h$, which produces an inverse to $1 - F$ which will basically finish the proof for the second statement. \square

Proposition 8.4.26. (After Kedlaya-Liu [KL2, Lemma 2.6.4]) *In the same situation as above, suppose we have that the corresponding modules M_i are basically B -stably pseudocoherent over the rings A_i for all $i = 0, 1, \dots$. Now we consider the situation where $f_i : A_i \widehat{\otimes} B \otimes_{A_{i+1} \widehat{\otimes} B} M_{i+1} \rightarrow M_i$ is an isomorphism. Then the conclusion in our situation*

is then that the corresponding projection from $\varprojlim M_i$ to M_i for each $i = 0, 1, 2, \dots$

Proof. This is a $\varinjlim_h B_h$ -relative version of the [KL2, Lemma 2.6.4]. We adapt the argument to our situation as in the following. First we choose some finite free covering T of the limit M such that we have for each i the corresponding map $T_i \rightarrow M_i$ is surjective. Then we consider the index $j \geq i$ and set the kernel of the map from T_j to M_j to be S_j . By the direct analog of [KL2, Lemma 2.5.6] we have that $A_i \widehat{\otimes} \varinjlim_h B_h \otimes S_j \rightarrow S_i$ realizes the isomorphism, and we have that the corresponding surjectivity of the corresponding map from $\varprojlim_i S_i$ projecting to the S_i . Then one could finish the proof by 5-lemma to the following commutative diagram as in [KL2, Lemma 2.6.4]:

$$\begin{array}{ccccccc}
 (A_i \widehat{\otimes} \varinjlim_h B_h) \otimes \varprojlim_i S_i & \longrightarrow & (A_i \widehat{\otimes} \varinjlim_h B_h) \otimes \varprojlim_i F_i & \longrightarrow & (A_i \widehat{\otimes} \varinjlim_h B_h) \otimes \varprojlim_i M_i & \longrightarrow & 0, \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & S_i & \longrightarrow & F_i & \longrightarrow & M_i \longrightarrow 0.
 \end{array}$$

□

Proposition 8.4.27. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the map $\mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ is surjective for any B -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B$.*

Proof. This is just the corresponding corollary of the previous proposition. □

Proposition 8.4.28. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *We have that the stalk \mathcal{M}_x is finitely generated over the stalk $\mathcal{O}_{X,x}$ for any $x \in X$ by $\mathcal{M}(X)$, for any B -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B$.*

Proof. This is just the corresponding corollary of the proposition before the previous proposition. □

Proposition 8.4.29. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the corresponding vanishing of the corresponding sheaf cohomology groups $H^k(X, \mathcal{M})$ of \mathcal{M} for higher $k > 0$, for any B -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} B$.*

Proof. See the proof of proposition 8.2.29. □

Corollary 8.4.30. (After Kedlaya-Liu [KL2, Corollary 2.6.6]) *The corresponding functor from the corresponding B -deformed pseudocoherent sheaves over X to the corresponding B -stably by taking the corresponding global section is an exact functor.*

Corollary 8.4.31. (After Kedlaya-Liu [KL2, Corollary 2.6.8]) *Consider a particular $\mathcal{O}_X \widehat{\otimes} B$ -pseudocoherent sheaf \mathcal{M} which is finite locally free throughout the whole space X . Then we have that the global section $\mathcal{M}(X)$ as $\mathcal{O}_X(X) \widehat{\otimes} B$ left module admits the corresponding structures of finite projective structure if and only if we have the corresponding global section is finitely generated.*

Proof. As in [KL2, Corollary 2.6.8] one could find some global splitting through the local splittings. □

We now consider the following B -relative analog of [KL2, Proposition 2.6.17]:

Theorem 8.4.32. (After Kedlaya-Liu [KL2, Proposition 2.6.17]) *Consider the following two statements for a particular $\mathcal{O}_X \widehat{\otimes} B$ -pseudocoherent sheaf \mathcal{M} . First is that one can find finite many generators (the number is up to some uniform fixed integer $n \geq 0$) for each section of $\mathcal{M}(X_i)$ for each $i = 1, 2, \dots$. The second statement is that the global section $\mathcal{M}(X)$ is just finitely generated. Then in our situation the two statement is equivalent if we have that the corresponding space X admits a m -uniformed covering in the exact same sense of [KL2, Proposition 2.6.17].*

Proof. See theorem 8.2.32. □

8.5 Descent over Analytic Huber Pairs over \mathbb{Z}_p

8.5.1 Noncommutative Deformation over Analytic Huber Pairs in the Analytic Topology

We now study the corresponding glueing of stably pseudocoherent sheaves after our previous work [T3]. However since we currently are working in the corresponding framework of [Ked2], we will consider the corresponding analytic Huber pair in [Ked2].

Setting 8.5.1. Now we consider the corresponding Huber pair taking the general form of (A, A^+) as in [Ked2, Definition 1.1.2], namely it is uniform analytic. We now assume we are going to work over some base $(V, V^+) = (\mathbb{Z}_p, \mathbb{Z}_p)$. And we will assume that we have another topological ring Z which is assumed to satisfy the following condition: for any exact sequence of uniform analytic Huber rings

$$0 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2 \longrightarrow \Gamma_3 \longrightarrow 0,$$

we have that the following is always exact:

$$0 \longrightarrow \Gamma_1 \widehat{\otimes} Z \longrightarrow \Gamma_2 \widehat{\otimes} Z \longrightarrow \Gamma_3 \widehat{\otimes} Z \longrightarrow 0.$$

This is achieved for instance when we have that the map $C \rightarrow Z$ splits in the category of all the topological modules.

Setting 8.5.2. We will maintain the corresponding assumption in [Ked2, Hypothesis 1.7.1]. To be more precise we will have a map $(A, A^+) \rightarrow (B, B^+)$ which is a corresponding rational localization. And recall the corresponding complex in [Ked2]:

$$0 \longrightarrow B \longrightarrow B \left\{ \begin{matrix} f \\ g \end{matrix} \right\} \oplus B \left\{ \begin{matrix} g \\ f \end{matrix} \right\} \longrightarrow B \left\{ \begin{matrix} f, g \\ g, f \end{matrix} \right\} \longrightarrow 0.$$

Setting 8.5.3. We will need to consider the corresponding nice rational localizations in the sense of [Ked2, Definition 1.9.1] where such localizations are defined to be those composites of the corresponding rational localizations in the Laurent or the balanced situation. For the topological modules, we will always assume that the modules are left modules.

Definition 8.5.4. (After Kedlaya [Ked2, Definition 1.9.1]) Over A , we define a corresponding Z -stably-pseudocoherent module M to be a pseudocoherent module M over $A\widehat{\otimes}Z$ which is complete with respect to the natural topology. And for any rational localization $A \rightarrow A'$ with respect to A , the completeness still holds. Similar we can define the corresponding m - Z -stably-pseudocoherent modules in the similar way where we just consider the corresponding m -pseudocoherent modules on the algebraic level. Over A , we define a corresponding Z -nice-stably-pseudocoherent module M to be a pseudocoherent module M over $A\widehat{\otimes}Z$ which is complete with respect to the natural topology. And for any nice rational localization $A \rightarrow A'$ with respect to A , the completeness still holds. Similar we can define the corresponding m - Z -nice-stably-pseudocoherent modules in the similar way where we just consider the corresponding m -pseudocoherent modules on the algebraic level.

Lemma 8.5.5. (After Kedlaya [Ked2, Lemma 1.9.3]) *Now assume that we are in the corresponding assumption of [Ked2, 1.7.1]. Also consider the corresponding complex in [Ked2, 1.6.15.1]:*

$$0 \longrightarrow B \longrightarrow B \left\{ \frac{f}{g} \right\} \oplus B \left\{ \frac{g}{f} \right\} \longrightarrow B \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

And we assume now that this is exact. Now take g to be $1-f$ or 1 . Here the corresponding elements f, g come from B . Now over $B\widehat{\otimes}Z$ suppose we have a module M which is assumed now to be finitely presented and complete with respect to the natural topology over $B\widehat{\otimes}Z$. Then we have that the group $\mathrm{Tor}_1(B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\}, M)$ is zero.

Proof. This is a Z -relative version of the corresponding [Ked2, Lemma 1.9.3]. We briefly recall the argument being adapted in our situation. As in [Ked2, Lemma 1.9.3] we can reduce ourselves to the situation where g is just 1 . Then regard the corresponding ring $B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\}$ as $B\widehat{\otimes}Z \{T\} / (1 - fT)$. Then we consider the corresponding in the ring $B\widehat{\otimes}Z[[T]]$ the formal inverse of $1 - fT$ namely the series $1 + fT + f^2T^2 + \dots$. This will force multiplication by this element to be injective. This just finishes the proof as in [Ked2, Lemma 1.9.3]. \square

Lemma 8.5.6. (After Kedlaya [Ked2, Lemma 1.9.4]) *Now assume that we are in the corresponding assumption of [Ked2, 1.7.1]. Also consider the corresponding complex*

in [Ked2, 1.6.15.1]:

$$0 \longrightarrow B \longrightarrow B \left\{ \frac{f}{g} \right\} \oplus B \left\{ \frac{g}{f} \right\} \longrightarrow B \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

And we assume now that this is exact. Now take g to be $1-f$ or 1 . Here the corresponding element f, g come from B . Now over $B\widehat{\otimes}Z$ suppose we have a module M which is assumed now to be finitely presented and complete with respect to the natural topology over $B\widehat{\otimes}Z$, and furthermore the corresponding completeness is stable under any corresponding nice rational localization $B \rightarrow C$. Then we have that the group $\mathrm{Tor}_1(B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\}, M)$ is zero and the group $\mathrm{Tor}_1(B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}, M)$ is zero as well.

Proof. Choose a corresponding presentation for M taking the form of:

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow M \longrightarrow 0.$$

Here the module X_2 is a finite free left module. Then we have by the previous lemma the following exact sequence:

$$0 \longrightarrow B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \otimes_{B\widehat{\otimes}Z} X_1 \longrightarrow B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \otimes_{B\widehat{\otimes}Z} X_2 \longrightarrow B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \otimes_{B\widehat{\otimes}Z} M \longrightarrow 0.$$

Now apply the previous lemma again as in the proof of [Ked2, Lemma 1.9.4] we have the following exact sequence:

$$0 \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \otimes_{B\widehat{\otimes}Z} X_1 \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \otimes_{B\widehat{\otimes}Z} X_2 \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \otimes_{B\widehat{\otimes}Z} M \longrightarrow 0.$$

Then to tackle the situation for $\frac{f}{g}$, we just further take the tensor product with the corresponding sequence as in the following:

$$0 \longrightarrow B\widehat{\otimes}Z \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\} \oplus B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

□

Then we have the following corollary which is the corresponding analog of [Ked2, Corollary 1.9.5]:

Corollary 8.5.7. (After Kedlaya [Ked2, Corollary 1.9.5]) *Working over a uniform analytic Huber pair (A, A^+) which is sheafy, suppose we consider a finitely generated $A\widehat{\otimes}Z$ module complete with respect to the natural topology. Then we have that for any nice rational localization $A \rightarrow B$ in the sense of [Ked2, Definition 1.9.1] then tensoring with the corresponding Z we have the vanishing of $\mathrm{Tor}_1(B, M)$.*

Proof. By the previous lemmas. □

Corollary 8.5.8. (After Kedlaya [Ked2, Corollary 1.9.6]) *Working over a uniform analytic Huber pair (A, A^+) which is sheafy, suppose we consider a nice Z -stably pseudo-coherent left $A\widehat{\otimes}Z$ module M . Then we have that for any rational localization $A \rightarrow B$ and any rational localization from B to C in the nice setting as in [Ked2, Definition 1.9.1]. We have that base change along the map $B \rightarrow C$ (note that this is assumed to be nice) will preserve the corresponding stably pseudocoherence for M .*

Lemma 8.5.9. (After Kedlaya [Ked2, Lemma 1.9.7]) *Now assume that we are in the corresponding assumption of [Ked2, 1.7.1]. Also consider the corresponding complex in [Ked2, 1.6.15.1]:*

$$0 \longrightarrow B \longrightarrow B \left\{ \frac{f}{g} \right\} \oplus B \left\{ \frac{g}{f} \right\} \longrightarrow B \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

And we assume now that this is exact. Now take g to be $1-f$ or 1 . Here the corresponding element f, g come from B . Now over $B\widehat{\otimes}Z$ suppose we have a module M which is assumed now to be finitely presented and complete with respect to the natural topology over $B\widehat{\otimes}Z$, and furthermore the corresponding completeness is stable under any corresponding nice rational localization $B \rightarrow C$. Then we have that over then $B\widehat{\otimes}Z$ in our situation tensoring with M will preserve the corresponding exactness of the following:

$$0 \longrightarrow B\widehat{\otimes}Z \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\} \oplus B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

Proof. See [Ked2, Lemma 1.9.7]. □

Theorem 8.5.10. (After Kedlaya [Ked2, Corollary 1.9.8]) *Working over a uniform analytic Huber pair (A, A^+) which is sheafy, suppose we consider a Z -nice-stably pseudo-coherent left $A\widehat{\otimes}Z$ module M . Then we have that the presheaf \widetilde{M} associated to the M sheafified along the ring A only is acyclic.*

Proof. See [Ked2, Corollary 1.9.8]. □

Now we establish the corresponding analogs of the corresponding results in [Ked2] which are needed in the corresponding descent of pseudocoherent sheaves.

Lemma 8.5.11. (After Kedlaya [Ked2, Lemma 1.9.9]) *Again as in [Ked2, Lemma 1.9.9] we suppose we are in the situation of 1.7.1 of [Ked2] for the adic rings. Now one can actually find a neighbourhood around 0 of the corresponding ring $B\widehat{\otimes}Z\left\{\frac{f}{g}, \frac{g}{f}\right\}$. Then one can find a corresponding decomposition for any invertible matrix M where $M - 1$ takes coefficients in this neighbourhood into M_1M_2 where invertible M_1 takes coefficients in $B\widehat{\otimes}Z\left\{\frac{f}{g}\right\}$ and invertible M_2 takes coefficients in $B\widehat{\otimes}Z\left\{\frac{g}{f}\right\}$.*

Proof. See [Ked2, Lemma 1.9.9]. □

Lemma 8.5.12. (After Kedlaya [Ked2, Lemma 1.9.10]) *Again as in [Ked2, Lemma 1.9.10] we suppose we are in the situation of 1.7.1 of [Ked2] for the adic rings. Now for a glueing datum M_1, M_2, M_{12} over the rings respectively $B\widehat{\otimes}Z\left\{\frac{f}{g}\right\}, B\widehat{\otimes}Z\left\{\frac{g}{f}\right\}, B\widehat{\otimes}Z\left\{\frac{f}{g}, \frac{g}{f}\right\}$. We assume that they are finitely generated. We recall this means that we have:*

$$f_1 : B\widehat{\otimes}Z\left\{\frac{f}{g}, \frac{g}{f}\right\} \otimes_{B\widehat{\otimes}Z\left\{\frac{f}{g}\right\}} M_1 \xrightarrow{\sim} M_{12}, \quad (8.5.1)$$

$$f_2 : B\widehat{\otimes}Z\left\{\frac{f}{g}, \frac{g}{f}\right\} \otimes_{B\widehat{\otimes}Z\left\{\frac{g}{f}\right\}} M_2 \xrightarrow{\sim} M_{12}. \quad (8.5.2)$$

Then we have that the corresponding strictly surjectivity of the map $M_1 \oplus M_2 \rightarrow M_{12}$ given by $f_1(x_1) - f_2(x_2)$. And we have that the corresponding equalizer M will satisfy that $B\widehat{\otimes}Z\left\{\frac{f}{g}\right\} \otimes_{B\widehat{\otimes}Z} M \rightarrow M_1, B\widehat{\otimes}Z\left\{\frac{g}{f}\right\} \otimes_{B\widehat{\otimes}Z} M \rightarrow M_2$ are strictly surjective.

Proof. This is a Z -relative version of the corresponding result in [Ked2, Lemma 1.9.10]. We adapt the corresponding argument there to our situation. Choose basis of M_1 namely e_1, \dots, e_n , and choose basis of M_2 namely h_1, \dots, h_n . Now we set the corresponding matrix of the corresponding $f_2(w_i)$ under the basis e_1, \dots, e_n to be P while we set the corresponding matrix of the corresponding $f_1(v_i)$ under the basis h_1, \dots, h_n to be Q . Then what is happening is that we can apply the previous lemma to our current situation to extract a neighbourhood such that $P(f^{-k}Q' - Q)$ (for some sufficiently k) takes the value in this neighbourhood and we have that the matrix $P(f^{-k}Q' - Q) + 1$ will be decomposed into $K_1K_2^{-1}$ with the corresponding conditions on the entries as in the previous lemma. Then what we could do is as in [Ked2, Lemma 1.9.10] to consider the setting $(x_1, x_2) := (\sum f^k X_{1ij} e_i, \sum Q' X_{2ij} h_i)$ which gives rise to that $f_1(x_1) - f_2(x_2) = 0$ by our construction. Then as in [Ked2, Lemma 1.9.10] the corresponding result in the first statement will follow once one considers as well [Ked2, Lemma 1.8.1]. Then for the second statement,

we consider application of the first statement as in [Ked2, Lemma 1.9.10]. Briefly recalling, we consider the second part of statement (while the first part could be derived in the same fashion). To be more precise we consider any element $m_2 \in M_2$ then send by f_2 to M_{12} , we will have by the first statement some $y_1 \in M_1, y_2 \in M_2$ such that $f_1(y_1) - f_2(y_2) = f_2(m_2)$. This will gives $(y_1, y_2 + m_2)$ lives in the equalizer, which will proves the result as in [Ked2, Lemma 1.9.10].

□

Proposition 8.5.13. (After Kedlaya [Ked2, Lemma 1.9.11]) *Consider the same situation of the previous lemma. And assume that the corresponding [Ked2, 1.6.15.1] is exact. And we assume that the corresponding modules M_1, M_2, M_{12} are Z -stably pseudo-coherent over the rings respectively $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}, B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}, B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}$. Then we have that actually the corresponding equalizer M is just pseudocoherent in our current situation.*

Proof. This is a Z -relative version of [Ked2, Lemma 1.9.11]. We adapt the corresponding argument to our situation. First we consider corresponding modules M_1, M_2, M_{12} , and choose finite free modules L_1, L_2, L_{12} as in [Ked2, Lemma 1.9.11] over $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}$,

$B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}$, $B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}$, then we can form the corresponding commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F_{12} \xrightarrow{?} 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & L_1 \oplus L_2 & \longrightarrow & L_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where two arrows marked by ? are not a priori known to exist to make the corresponding parts of the sequences exact. Here the left vertical arrows come from taking snake lemma (recall that one first chooses a corresponding finite free module L mapping to M which will base change to the corresponding module L_1, L_2, L_{12} , then F will be taken as in [Ked2, Lemma 1.9.11] to be the corresponding equalizer of the corresponding horizontal map, the idea is that the map $L \rightarrow M$ is just a map which is not a priori known to be surjective while eventually this is indeed the case). The corresponding modules F_1, F_2, F_{12} are the corresponding kernels of the corresponding coverings by finite free modules. Then what is happening is that the corresponding first horizontal complex is actually within the same situation we are considering, therefore the previous lemma implies that the ?-marked arrows exist and make the sequences exact. We then consider the following commutative

diagram:

$$\begin{array}{ccccccc}
B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\} \otimes F & \longrightarrow & B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\} \otimes L & \longrightarrow & B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\} \otimes M & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & F_1 & \longrightarrow & L_1 & \longrightarrow & M_1 & \longrightarrow 0.
\end{array}$$

What we know is that the corresponding middle vertical arrow is isomorphism, while the left and right ones are surjective, which implies the right most one is an isomorphism. We then consider the following commutative diagram:

$$\begin{array}{ccccccc}
B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \otimes F & \longrightarrow & B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \otimes L & \longrightarrow & B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \otimes M & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & F_2 & \longrightarrow & L_2 & \longrightarrow & M_2 & \longrightarrow 0.
\end{array}$$

What we know is that the corresponding middle vertical arrow is isomorphism, while the left and right ones are surjective, which implies the right most one is an isomorphism. Finally to show that the module M is pseudocoherent we consider the corresponding induction on the number of the finite projective modules participated in the resolution. $k = 1$ will be obvious, while in general the induction step will be achieved from k to $k + 1$ by considering as in [Ked2, Lemma 1.9.11] the module F and apply the corresponding construction in the same fashion. \square

Proposition 8.5.14. (After Kedlaya [Ked2, Lemma 1.9.13]) *Take the corresponding assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of A . Suppose we have that all the modules considered are Z -stably complete with respect to nice rational localization. Then consider C_1 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}$. Then consider C_2 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\}$. Then consider C_3 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\}$. Then consider C_4 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z$. Then we have a corresponding equalizer functor:*

$$C_2 \times_{C_1} C_3 \rightarrow C \tag{8.5.3}$$

which is exact and fully faithful, and we have well-established inverse.

Proof. See [Ked2, Lemma 1.9.13]. □

Proposition 8.5.15. (After Kedlaya [Ked2, Lemma 1.9.16]) *Take the corresponding assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of A . Suppose we have that all the modules considered are Z -stably complete with respect to nice rational localization. Let C be the category of such Z -nice-stably pseudocoherent modules over A and let C' be the category of such Z -nice-stably pseudocoherent sheaves over X . If we consider the corresponding functor taking the form of the corresponding sheafification of \widetilde{M} with respect to the corresponding nice rational subspaces. Then we have that in fact this realizes an exact equivalence between C and C' .*

Proof. See [Ked2, Lemma 1.9.16]. □

Proposition 8.5.16. (After Kedlaya [Ked2, Lemma 1.9.17]) *Take the corresponding assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of A . Suppose we have that all the modules considered are Z -stably complete with respect to nice rational localization. Let C be the category of such Z -nice-stably pseudocoherent modules over A and let C' be the category of such Z -nice-stably pseudocoherent modules over B . Then we have that in fact this realizes an exact functor from C to C' .*

Proof. See [Ked2, Lemma 1.9.17]. □

Proposition 8.5.17. (After Kedlaya [Ked2, Corollary 1.9.18]) *Take the corresponding assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of A . Let C be the category of such Z -stably pseudocoherent modules over A and let C' be the category of such Z -nice-stably pseudocoherent modules over A . Then we have that in fact this realizes an equality from $C \hookrightarrow C'$.*

Proof. See [Ked2, Corollary 1.9.18]. □

Theorem 8.5.18.

(After Kedlaya [Ked2, Theorem 1.4.14, Theorem 1.4.16, Theorem 1.4.18])

Take the corresponding assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of A . Then: I. We have the corresponding stability under rational localization for Z -stably pseudocoherent modules. II. We have the corresponding acyclicity of the corresponding presheaf \widetilde{M} attached to any Z -stably pseudocoherent module. III. We have the

corresponding glueing of the corresponding Z -stably pseudocoherent modules as in [Ked2, Theorem 1.4.18].

Proof. This is a very parallel Z -relative analog of [Ked2, Theorem 1.4.14, Theorem 1.4.16, Theorem 1.4.18], we refer the readers to the proof of [Ked2, Theorem 1.4.14, Theorem 1.4.16, Theorem 1.4.18]. See [Ked2, After Corollary 1.9.18, the proof of Theorem 1.4.14, the proof of Theorem 1.4.16, the proof of Theorem 1.4.18]. \square

8.6 Descent over Adic Banach Rings over \mathbb{Z}_p

8.6.1 Noncommutative Deformation over Adic Banach Rings in the Analytic Topology

We now study the corresponding glueing of stably pseudocoherent sheaves after our previous work [T3]. We now translate the corresponding discussion to the context which is more related to [KL2] over adic Banach rings. One can see that these are really parallel to the discussion over analytic Huber pair.

Setting 8.6.1. Now we consider the corresponding adic Banach ring taking the general form of (A, A^+) , namely it is uniform analytic. We now assume we are going to work over some base $(V, V^+) = (\mathbb{Z}_p, \mathbb{Z}_p)$. And we will assume that we have another Banach ring Z over V which is assumed to satisfy the following condition: for any exact sequence of uniform analytic adic Banach rings

$$0 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2 \longrightarrow \Gamma_3 \longrightarrow 0,$$

we have that the following is always exact:

$$0 \longrightarrow \Gamma_1 \widehat{\otimes} Z \longrightarrow \Gamma_2 \widehat{\otimes} Z \longrightarrow \Gamma_3 \widehat{\otimes} Z \longrightarrow 0.$$

This is achieved for instance when we have that the map $C \rightarrow Z$ splits in the category of all the Banach modules.

Setting 8.6.2. We will maintain the corresponding assumption in [Ked2, Hypothesis 1.7.1]. To be more precise we will have a map $(A, A^+) \rightarrow (B, B^+)$ which is a corresponding rational localization. And recall the corresponding complex in [Ked2]:

$$0 \longrightarrow B \longrightarrow B \left\{ \begin{array}{c} f \\ g \end{array} \right\} \oplus B \left\{ \begin{array}{c} g \\ f \end{array} \right\} \longrightarrow B \left\{ \begin{array}{c} f, g \\ g, f \end{array} \right\} \longrightarrow 0.$$

Setting 8.6.3. We will need to consider the corresponding nice rational localizations in the sense of [Ked2, Definition 1.9.1] where such localizations are defined to be those composites of the corresponding rational localizations in the Laurent or the balanced situation. For the topological modules, we will always assume that the modules are left modules.

Definition 8.6.4. (After Kedlaya [Ked2, Definition 1.9.1]) Over A , we define a corresponding Z -stably-pseudocoherent module M to be a pseudocoherent module M over $A\widehat{\otimes}Z$ which is complete with respect to the natural topology. And for any rational localization $A \rightarrow A'$ with respect to A , the completeness still holds. Similar we can define the corresponding m - Z -stably-pseudocoherent modules in the similar way where we just consider the corresponding m -pseudocoherent modules on the algebraic level. Over A , we define a corresponding Z -nice-stably-pseudocoherent module M to be a pseudocoherent module M over $A\widehat{\otimes}Z$ which is complete with respect to the natural topology. And for any nice rational localization $A \rightarrow A'$ with respect to A , the completeness still holds. Similar we can define the corresponding m - Z -nice-stably-pseudocoherent modules in the similar way where we just consider the corresponding m -pseudocoherent modules on the algebraic level.

Lemma 8.6.5. (After Kedlaya [Ked2, Lemma 1.9.3]) *Now assume that we are in the corresponding assumption of [Ked2, 1.7.1]. Also consider the corresponding complex in [Ked2, 1.6.15.1]:*

$$0 \longrightarrow B \longrightarrow B \left\{ \begin{matrix} f \\ g \end{matrix} \right\} \oplus B \left\{ \begin{matrix} g \\ f \end{matrix} \right\} \longrightarrow B \left\{ \begin{matrix} f, g \\ g, f \end{matrix} \right\} \longrightarrow 0.$$

And we assume now that this is exact. Now take g to be $1-f$ or 1 . Here the corresponding element f, g come from B . Now over $B\widehat{\otimes}Z$ suppose we have a module M which is assumed now to be finitely presented and complete with respect to the natural topology over $B\widehat{\otimes}Z$. Then we have that the group $\mathrm{Tor}_1(B\widehat{\otimes}Z \left\{ \begin{matrix} g \\ f \end{matrix} \right\}, M)$ is zero.

Proof. See lemma 8.5.5. □

Lemma 8.6.6. (After Kedlaya [Ked2, Lemma 1.9.4]) *Now assume that we are in the corresponding parallel assumption of [Ked2, 1.7.1]. Also consider the corresponding complex in [Ked2, 1.6.15.1]:*

$$0 \longrightarrow B \longrightarrow B \left\{ \begin{matrix} f \\ g \end{matrix} \right\} \oplus B \left\{ \begin{matrix} g \\ f \end{matrix} \right\} \longrightarrow B \left\{ \begin{matrix} f, g \\ g, f \end{matrix} \right\} \longrightarrow 0.$$

And we assume now that this is exact. Now take g to be $1-f$ or 1 . Here the corresponding element f, g come from B . Now over $B\widehat{\otimes}Z$ suppose we have a module M which is assumed now to be finitely presented and complete with respect to the natural topology over $B\widehat{\otimes}Z$, and furthermore the corresponding completeness is stable under any corresponding nice

rational localization $B \rightarrow C$. Then we have that the group $\mathrm{Tor}_1(B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\}, M)$ is zero and the group $\mathrm{Tor}_1(B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}, M)$ is zero as well.

Proof. See lemma 8.5.6. □

Then we have the following corollary which is the corresponding analog of [Ked2, Corollary 1.9.5]:

Corollary 8.6.7. (After Kedlaya [Ked2, Corollary 1.9.5]) *Working over a uniform analytic adic Banach ring (A, A^+) which is sheafy, suppose we consider a finitely generated $A\widehat{\otimes}Z$ module complete with respect to the natural topology. Then we have that for any nice rational localization $A \rightarrow B$ in the sense of [Ked2, Definition 1.9.1] then tensoring with the corresponding Z we have the vanishing of $\mathrm{Tor}_1(B, M)$.*

Proof. By the previous lemmas. □

Corollary 8.6.8. (After Kedlaya [Ked2, Corollary 1.9.6]) *Working over a uniform analytic adic Banach ring (A, A^+) which is sheafy, suppose we consider a nice Z -stably pseudocoherent left $A\widehat{\otimes}Z$ module M . Then we have that for any rational localization $A \rightarrow B$ and any rational localization from B to C in the nice setting as in [Ked2, Definition 1.9.1]. We have that base change along the map $B \rightarrow C$ (note that this is assumed to be nice) will preserve the corresponding stably pseudocoherence for M .*

Lemma 8.6.9. (After Kedlaya [Ked2, Lemma 1.9.7]) *Now assume that we are in the corresponding parallel assumption of [Ked2, 1.7.1]. Also consider the corresponding complex in [Ked2, 1.6.15.1]:*

$$0 \longrightarrow B \longrightarrow B \left\{ \frac{f}{g} \right\} \oplus B \left\{ \frac{g}{f} \right\} \longrightarrow B \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

And we assume now that this is exact. Now take g to be $1-f$ or 1 . Here the corresponding element f, g come from B . Now over $B\widehat{\otimes}Z$ suppose we have a module M which is assumed now to be finitely presented and complete with respect to the natural topology over $B\widehat{\otimes}Z$, and furthermore the corresponding completeness is stable under any corresponding nice rational localization $B \rightarrow C$. Then we have that over then $B\widehat{\otimes}Z$ in our situation tensoring with M will preserve the corresponding exactness of the following:

$$0 \longrightarrow B\widehat{\otimes}Z \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g} \right\} \oplus B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\} \longrightarrow B\widehat{\otimes}Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \longrightarrow 0.$$

Proof. See [Ked2, Lemma 1.9.7]. □

Theorem 8.6.10. (After Kedlaya [Ked2, Corollary 1.9.8]) *Working over a uniform analytic adic Banach ring (A, A^+) which is sheafy, suppose we consider a Z -nice-stably pseudocoherent left $A \widehat{\otimes} Z$ module M . Then we have that the presheaf \widetilde{M} associated to the M sheafified along the ring A only is acyclic.*

Proof. See [Ked2, Corollary 1.9.8]. □

Now we establish the corresponding analogs of the corresponding results in [Ked2] which are needed in the corresponding descent of pseudocoherent sheaves.

Lemma 8.6.11. (After Kedlaya [Ked2, Lemma 1.9.9]) *Again as in [Ked2, Lemma 1.9.9] we suppose we are in the parallel situation of 1.7.1 of [Ked2, Lemma 1.9.9] for the adic Banach rings. Now take a sufficiently small constant $\delta > 0$. Then one can find a corresponding decomposition for any invertible matrix M where $M - 1$ takes norm $\|M - 1\| \leq \delta$ into $M_1 M_2$ where invertible M_1 takes coefficients in $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}$ and invertible M_2 takes coefficients in $B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}$.*

Proof. See [Ked2, Lemma 1.9.9]. □

Lemma 8.6.12. (After Kedlaya [Ked2, Lemma 1.9.10]) *Again as in [Ked2, Lemma 1.9.10] we suppose we are in the parallel situation of 1.7.1 of [Ked2, Lemma 1.9.10] for the adic Banach rings. Now for a glueing datum M_1, M_2, M_{12} over the rings respectively $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}, B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}, B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}$. We assume that they are finitely generated. We recall this means that we have:*

$$f_1 : B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \otimes_{B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}} M_1 \xrightarrow{\sim} M_{12}, \quad (8.6.1)$$

$$f_2 : B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\} \otimes_{B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}} M_2 \xrightarrow{\sim} M_{12}. \quad (8.6.2)$$

Then we have that the corresponding strictly surjectivity of the map $M_1 \oplus M_2 \rightarrow M_{12}$ given by $f_1(x_1) - f_2(x_2)$. And we have that the corresponding equalizer M will satisfy that $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\} \otimes_{B \widehat{\otimes} Z} M \rightarrow M_1, B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\} \otimes_{B \widehat{\otimes} Z} M \rightarrow M_2$ are strictly surjective.

Proof. See lemma 8.5.12. □

Proposition 8.6.13. (After Kedlaya [Ked2, Lemma 1.9.11]) *Consider the same situation of the previous lemma. And assume that the corresponding [Ked2, 1.6.15.1] is exact. And we assume that the corresponding modules M_1, M_2, M_{12} are Z -stably pseudocoherent over the rings respectively $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}, B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}, B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}$. Then we have that actually the corresponding equalizer M is just pseudocoherent in our current situation.*

Proof. This is a Z -relative version of [Ked2, Lemma 1.9.11]. We adapt the corresponding argument to our situation. First we consider corresponding modules M_1, M_2, M_{12} , and choose finite free modules L_1, L_2, L_{12} as in [Ked2, Lemma 1.9.11] over $B \widehat{\otimes} Z \left\{ \frac{f}{g} \right\}, B \widehat{\otimes} Z \left\{ \frac{g}{f} \right\}, B \widehat{\otimes} Z \left\{ \frac{f}{g}, \frac{g}{f} \right\}$, then we can form the corresponding commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F_{12} \xrightarrow{?} 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & L_1 \oplus L_2 & \longrightarrow & L_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where two arrows marked by ? are not a priori known to exist to make the corresponding parts of the sequences exact. Here the left vertical arrows come from taking snake lemma (recall that one first chooses a corresponding finite free module L mapping to M which will base change to the corresponding module L_1, L_2, L_{12} , then F will be taken as in [Ked2, Lemma 1.9.11] to be the corresponding equalizer of the corresponding horizontal map, the idea is that the map $L \rightarrow M$ is just a map which is not a priori known to be surjective

while eventually this is indeed the case). The corresponding modules F_1, F_2, F_{12} are the corresponding kernels of the corresponding coverings by finite free modules. Then what is happening is that the corresponding first horizontal complex is actually within the same situation we are considering, therefore the previous lemma implies that the $?$ -marked arrows exists and make the sequences exact. We then consider the following commutative diagram:

$$\begin{array}{ccccccc}
B\widehat{\otimes}Z \left\{ \begin{smallmatrix} f \\ g \end{smallmatrix} \right\} \otimes F & \longrightarrow & B\widehat{\otimes}Z \left\{ \begin{smallmatrix} f \\ g \end{smallmatrix} \right\} \otimes L & \longrightarrow & B\widehat{\otimes}Z \left\{ \begin{smallmatrix} f \\ g \end{smallmatrix} \right\} \otimes M & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & F_1 & \longrightarrow & L_1 & \longrightarrow & M_1 & \longrightarrow 0.
\end{array}$$

What we know is that the corresponding middle vertical arrow is isomorphism, while the left and right ones are surjective, which implies the right most one is an isomorphism. We then consider the following commutative diagram:

$$\begin{array}{ccccccc}
B\widehat{\otimes}Z \left\{ \begin{smallmatrix} g \\ f \end{smallmatrix} \right\} \otimes F & \longrightarrow & B\widehat{\otimes}Z \left\{ \begin{smallmatrix} g \\ f \end{smallmatrix} \right\} \otimes L & \longrightarrow & B\widehat{\otimes}Z \left\{ \begin{smallmatrix} g \\ f \end{smallmatrix} \right\} \otimes M & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & F_2 & \longrightarrow & L_2 & \longrightarrow & M_2 & \longrightarrow 0.
\end{array}$$

What we know is that the corresponding middle vertical arrow is isomorphism, while the left and right ones are surjective, which implies the right most one is an isomorphism. Finally to show that the module M is pseudocoherent we consider the corresponding induction on the number of the finite projective modules participated in the resolution. $k = 1$ will be obvious, while in general the induction step will achieved from k to $k + 1$ by considering as in [Ked2, Lemma 1.9.11] the module F and apply the corresponding construction in the same fashion. \square

Proposition 8.6.14. (After Kedlaya [Ked2, Lemma 1.9.13]) *Take the corresponding parallel assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of A . Suppose we have that all the modules considered are Z -stably complete with respect to nice rational localization. Then consider C_1 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z \left\{ \begin{smallmatrix} f \\ g, \frac{g}{f} \end{smallmatrix} \right\}$. Then consider C_2 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z \left\{ \begin{smallmatrix} f \\ g \end{smallmatrix} \right\}$. Then consider C_3*

the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z \left\{ \frac{g}{f} \right\}$. Then consider C_4 the category of all such modules which are assumed to be pseudocoherent over $B\widehat{\otimes}Z$. Then we have a corresponding equalizer functor:

$$C_2 \times_{C_1} C_3 \rightarrow C \tag{8.6.3}$$

which is exact and fully faithful, and we have well-established inverse.

Proof. See [Ked2, Lemma 1.9.13]. □

Proposition 8.6.15. (After Kedlaya [Ked2, Lemma 1.9.16]) *Take the corresponding parallel assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of the adic Banach A . Suppose we have that all the modules considered are Z -stably complete with respect to nice rational localization. Let C be the category of such Z -nice-stably pseudocoherent modules over A and let C' be the category of such Z -nice-stably pseudocoherent sheaves over X . If we consider the corresponding functor taking the form of the corresponding sheafification of \widetilde{M} with respect to the corresponding nice rational subspaces. Then we have that in fact this realizes an exact equivalence between C and C' .*

Proof. See [Ked2, Lemma 1.9.16]. □

Proposition 8.6.16. (After Kedlaya [Ked2, Lemma 1.9.17]) *Take the corresponding parallel assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of the adic Banach ring A . Suppose we have that all the modules considered are Z -stably complete with respect to nice rational localization. Let C be the category of such Z -nice-stably pseudocoherent modules over A and let C' be the category of such Z -nice-stably pseudocoherent modules over B . Then we have that in fact this realizes an exact functor from C to C' .*

Proof. See [Ked2, Lemma 1.9.17]. □

Proposition 8.6.17. (After Kedlaya [Ked2, Corollary 1.9.18]) *Take the corresponding parallel assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of the adic Banach ring A . Let C be the category of such Z -stably pseudocoherent modules over A and let C' be the category of such Z -nice-stably pseudocoherent modules over A . Then we have that in fact this realizes an equality from $C \subset C'$.*

Proof. See [Ked2, Corollary 1.9.18]. □

Theorem 8.6.18.**(After Kedlaya [Ked2, Theorem 1.4.14, Theorem 1.4.16, Theorem 1.4.18])**

Take the corresponding parallel assumption in [Ked2, 1.7.1] and furthermore assume the sheafiness of the adic Banach ring A . Then: I. we have the corresponding stability under rational localization for Z -stably pseudocoherent modules. II. We have the corresponding acyclicity of the corresponding presheaf \widetilde{M} attached to any Z -stably pseudocoherent module. III. We have the corresponding glueing result as in [Ked2, Theorem 1.4.18].

Proof. This is a very parallel Z -relative analog of [Ked2, Theorem 1.4.14, Theorem 1.4.16, Theorem 1.4.18], we refer the readers to the proof of [Ked2, Theorem 1.4.14, Theorem 1.4.16, Theorem 1.4.18]. See [Ked2, After Corollary 1.9.18, the proof of Theorem 1.4.14, the proof of Theorem 1.4.16, the proof of Theorem 1.4.18]. \square

8.6.2 Noncommutative Deformation over Quasi-Stein Spaces

We now consider the corresponding analytic quasi-Stein spaces over \mathbb{Z}_p , Z will be an adic Banach ring as above:

Setting 8.6.19. (After Kedlaya-Liu [KL2, Definition 2.6.2]) Fix an adic Banach space X over \mathbb{Z}_p which is assumed to be an injective limit of analytic adic affinoids $X = \varinjlim_i X_i$, where we assume that the corresponding transition map $\mathcal{O}_{X_{i+1}} \rightarrow \mathcal{O}_{X_i}$ takes dense images for all $i = 1, 2, \dots$

Lemma 8.6.20. (After Kedlaya-Liu [KL2, Lemma 2.6.3]) *We now consider the corresponding rings $H_i := \mathcal{O}_{X_i}$ for all $i = 0, 1, \dots$, and in our current situation we consider the corresponding rings $H_i \widehat{\otimes} Z$ (over \mathbb{Z}_p) for all $i = 0, 1, \dots$. And in our situation we consider the corresponding modules M_i over $H_i \widehat{\otimes} Z$ for all $i = 0, 1, \dots$ with the same requirement as in [KL2, Lemma 2.6.3] (namely those complete with respect to the natural topology). Suppose that we have bounded surjective map from $f_i : H_i \widehat{\otimes} Z \otimes_{H_{i+1} \widehat{\otimes} Z} M_{i+1} \rightarrow M_i$, $i = 0, 1, \dots$. Then we have first the density of the corresponding image of $\varprojlim_i M_i$ in each M_i for any $i = 0, 1, 2, \dots$. And we have as well the corresponding vanishing of $R^1 \varprojlim_i M_i$.*

Proof. This is the Z -relative version of the result in [KL2, Lemma 2.6.3]. For the first statement we just choose sequence of Banach norms on all the corresponding modules for

all $i = 0, 1, \dots$ such that we have $\|f_i(x_{i+1})\|_i \leq 1/2\|x_{i+1}\|_{i+1}$ for any $x_{i+1} \in M_{i+1}$. Then for any $x_i \in M_i$ and any $\delta > 0$, we now consider for any $j \geq 1$ the corresponding x_{i+j} such that we have $\|x_{i+j} - f_{i+j+1}(x_{i+j+1})\|_{i+j} \leq \delta$. Then the sequence $x_{i+j+k}, k = 0, 1, \dots$ will converge to some well-defined y_{i+j} with in our situation the corresponding $y_i = f_i(y_{i+1})$. We then have $\|x_i - y_i\|_i \leq \delta$. This will prove the first statement. For the second statement as in [KL2, Lemma 2.6.3] we form the product $M_0 \times M_1 \times M_2 \times \dots$ and then consider the induced map F from $M_{i+1} \rightarrow M_i$, and consider the corresponding cokernel of the map $1 - F$ since this is just the corresponding limit we are considering. Then to show that the cokernel is zero we just look at the corresponding cokernel of the corresponding map on the corresponding completed direct summand which will project to the original one. But then we will have $\|f_i(v)\|_i \leq 1/2\|v\|_i$, which produces an inverse to $1 - F$ which will basically finish the proof for the second statement. \square

Proposition 8.6.21. (After Kedlaya-Liu [KL2, Lemma 2.6.4]) *In the same situation as above, suppose we have that the corresponding modules M_i are basically Z -stably pseudocoherent over the rings $H_i \widehat{\otimes} Z$ for all $i = 0, 1, \dots$. Now we consider the situation where $f_i : H_i \widehat{\otimes} Z \otimes_{H_{i+1} \widehat{\otimes} Z} M_{i+1} \rightarrow M_i$ is an isomorphism. Then the conclusion in our situation is then that the corresponding projection from $\varprojlim M_i$ to M_i for each $i = 0, 1, 2, \dots$ is an isomorphism.*

Proof. This is a Z -relative and analytic version of the [KL2, Lemma 2.6.4]. We adapt the argument to our situation as in the following. First we choose some finite free covering T of the limit M such that we have for each i the corresponding map $T_i \rightarrow M_i$ is surjective. Then we consider the index $j \geq i$ and set the kernel of the map from T_j to M_j to be S_j . By the direct analog of [KL2, Lemma 2.5.6] we have that $H_i \widehat{\otimes} Z \otimes S_j \rightarrow S_i$ realizes the isomorphism, and we have that the corresponding surjectivity of the corresponding map from $\varprojlim_i S_i$ projecting to the S_i . Then one could finish the proof by 5-lemma to the

following commutative diagram as in [KL2, Lemma 2.6.4]:

$$\begin{array}{ccccccc}
(H_i \widehat{\otimes} Z) \otimes \varprojlim_i S_i & \longrightarrow & (H_i \widehat{\otimes} Z) \otimes \varprojlim_i F_i & \longrightarrow & (H_i \widehat{\otimes} Z) \otimes \varprojlim_i M_i & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_i & \longrightarrow & F_i & \longrightarrow & M_i & \longrightarrow & 0.
\end{array}$$

□

Proposition 8.6.22. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the map $\mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ is surjective for any Z -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} Z$.*

Proof. This is just the corresponding corollary of the previous proposition. □

Proposition 8.6.23. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *We have that the stalk \mathcal{M}_x is generated over the stalk $\mathcal{O}_{X,x}$ for any $x \in X$ by $\mathcal{M}(X)$, for any Z -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} Z$.*

Proof. This is just the corresponding corollary of the proposition before the previous proposition. □

Proposition 8.6.24. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *For any quasi-compact adic affinoid space of X which is denoted by Y , we have that the corresponding vanishing of the corresponding sheaf cohomology groups $H^k(X, \mathcal{M})$ of \mathcal{M} for higher $k > 0$, for any Z -stably pseudocoherent sheaf \mathcal{M} over the sheaf $\mathcal{O}_X \widehat{\otimes} Z$.*

Proof. We follow the idea of the proof of [KL2, Theorem 2.6.5] by comparing this to the corresponding Čech cohomology with some covering $\mathfrak{X} = \{X_1, \dots, X_N, \dots\}$:

$$\check{H}^k(X, \mathfrak{X} = \{X_1, \dots, X_N, \dots\}; \mathcal{M}) = H^k(X, \mathcal{M}), \quad (8.6.4)$$

$$\check{H}^k(X_i, \mathfrak{X} = \{X_1, \dots, X_i\}; \mathcal{M}) = H^k(X_i, \mathcal{M}) = 0, k \geq 1. \quad (8.6.5)$$

Here we have applied the corresponding [SP, Tag 01EW]. Now we consider the situation where $k > 1$:

$$\check{H}^{k-1}(X_{j+1}, \mathfrak{X} = \{X_1, \dots, X_{j+1}\}; \mathcal{M}) \rightarrow \check{H}^{k-1}(X_j, \mathfrak{X} = \{X_1, \dots, X_j\}; \mathcal{M}) \rightarrow 0, \quad (8.6.6)$$

which induces the following isomorphism by [Kie1, 2.6 Hilfssatz]:

$$\varprojlim_{j \rightarrow \infty} \check{H}^{k-1}(X_j, \mathfrak{X} = \{X_1, \dots, X_j\}; \mathcal{M}) \xrightarrow{\sim} \check{H}^k(X, \mathfrak{X} = \{X_1, \dots, X_N, \dots\}; \mathcal{M}). \quad (8.6.7)$$

Then we have the corresponding results for the index $k > 1$. For $k = 1$, one can as in [KL2, Theorem 2.6.5] relate this to the corresponding $R^1 \varprojlim_i$, which will finish the proof in the same fashion. \square

Corollary 8.6.25. (After Kedlaya-Liu [KL2, Corollary 2.6.6]) *The corresponding functor from the corresponding Z -deformed pseudocoherent sheaves over X to the corresponding Z -stably by taking the corresponding global section is an exact functor.*

Corollary 8.6.26. (After Kedlaya-Liu [KL2, Corollary 2.6.8]) *Consider a particular $\mathcal{O}_X \widehat{\otimes} Z$ -pseudocoherent sheaf \mathcal{M} which is finite locally free throughout the whole space X . Then we have that the global section $\mathcal{M}(X)$ as $\mathcal{O}_X(X) \widehat{\otimes} Z$ left module admits the corresponding structures of finite projective structure if and only if we have the corresponding global section is finitely generated.*

Proof. As in [KL2, Corollary 2.6.8] one could find some global splitting through the local splittings. \square

We now consider the following Z -relative and analytic analog of [KL2, Proposition 2.6.17]:

Theorem 8.6.27. (After Kedlaya-Liu [KL2, Proposition 2.6.17]) *Consider the following two statements for a particular $\mathcal{O}_X \widehat{\otimes} Z$ -pseudocoherent sheaf \mathcal{M} . First is that one can find finite many generators (the number is up to some uniform fixed integer $n \geq 0$) for each section of $\mathcal{M}(X_i)$ for each $i = 1, 2, \dots$. The second statement is that the global section $\mathcal{M}(X)$ is just finitely generated. Then in our situation the two statement is equivalent if we have that the corresponding space X admits an m -uniform covering in the exact same sense of [KL2, Proposition 2.6.17].*

Proof. One direction is obvious, the other direct could be proved in the same way as in [KL2, Proposition 2.6.17] where essentially the information on X does not change at all. Namely as in [KL2, Proposition 2.6.17] we could basically consider one single subcovering $\{Y_u\}_{u \in U}$ indexed by U of the covering which is m -uniform. For any $u \in U$ we consider the smallest i such that $X_i \cap Y_u$, and we denote this by $i(u) \geq 0$. Then we form:

$$V_u := X_{i(u)} \bigcup_{v \in U, v \leq u} Y_v. \quad (8.6.8)$$

Then as in [KL2, Proposition 2.6.17] we can find some $x_u, u \in U$ (which are denoted by the general form of x_i in [KL2, Proposition 2.6.17]) in the section $\mathcal{O}_X(V_u)$ such that it restricts to some 0 onto the space V_u and it restricts to some element a_1 such that we have:

$$a_1 b_1 + \dots + a_k b_k = 1 \quad (8.6.9)$$

where b_1, \dots, b_k are the corresponding topologically nilpotent elements to Y_u . By the corresponding density of the corresponding global section $\mathcal{O}_X(X)$ in $\mathcal{O}_X(V_u)$ we could find some x_i (with the same notation) in the global section $\mathcal{O}_X(X)$ such that it restricts to some topologically nilpotent element onto the space V_u and it restricts to some element a'_1 such that we have:

$$a'_1 b'_1 + \dots + a'_k b'_k = 1 \quad (8.6.10)$$

where b_1, \dots, b_k are the corresponding topologically nilpotent elements to Y_u , by simultaneously approximating the corresponding the coefficient a_1, \dots, a_k and b_1, \dots, b_k to achieve a'_1, \dots, a'_k and b'_1, \dots, b'_k such that we have:

$$a'_1 b'_1 + \dots + a'_k b'_k = 1. \quad (8.6.11)$$

Then we can build up a corresponding generating set for the global section by the following approximation process just as in [KL2, Proposition 2.6.17]. To be more precise what we have consider is to modify the corresponding generator for each Y_u (for instance let them be denoted by y_1, \dots, y_u) to be $y_{u,1}, \dots, y_{u,n}$ (n is the corresponding uniform integer in the statement of the theorem) for all $u \in U$. It is achieved through induction, when we have $y_{u,1}, \dots, y_{u,n}$ then we will set that to be 0 otherwise in the situation where there exists

some predecessor u' of u in the corresponding set U we then set this to be $y_{u',1}, \dots, y_{u',n}$. And we set as in [KL2, Proposition 2.6.17]:

$$y_{u,j} := y_{u-1,j} + x_u^c y_u$$

by lifting the corresponding power c to be as large as possible. This will guarantee the convergence of:

$$\lim_{u \rightarrow \infty} \{y_{u,1}, \dots, y_{u,n}\}, \tag{8.6.12}$$

which gives the set of the corresponding global generators desired. □

8.7 Extensions and Applications

8.7.1 Deforming the $(\infty, 1)$ -Sheaves over Bambozzi-Kremnizer ∞ -Analytic Spaces

Here we follow [BK1] to extend our discussion above partially to the corresponding ∞ -analytic space attached to a Banach adic uniform algebra A (required to be commutative) over $\mathbb{F}_p((t))$ or \mathbb{Q}_p and we consider furthermore a Banach algebra B over $\mathbb{F}_p((t))$ or \mathbb{Q}_p .

Remark 8.7.1. We should definitely mention that one can apply the corresponding foundation from Clausen-Scholze [CS] to do this, namely the corresponding animated objects. But we will only apply the corresponding foundation in [BK1].

First we have the following ∞ -version of Tate-Čech acyclicity in our context:

Proposition 8.7.2. (After Bambozzi-Kremnizer [BK1, Theorem 4.15]) *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow A^h \widehat{\otimes} B \longrightarrow \prod_i A_i \widehat{\otimes} B \longrightarrow \prod_{i,j} A_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} A_j \widehat{\otimes} B \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec} A_i \rightarrow \mathrm{Spec} A^h\}_{i \in I}$ in the sense of [BK1, Theorem 4.15]. Here we have already attached the corresponding homotopical ring A^h to A .

Proof. This is a direct consequence of the corresponding result in [BK1, Theorem 4.15]. □

Along this idea one can then build up a sheaf of ∞ -Banach ring over Bambozzi-Kremnizer's spectrum $\mathrm{Spa}^h(A)$, which we will denote it by $\mathcal{O}_{\mathrm{Spa}^h(A), B}$. Namely we have defined a new ∞ -ringed space $(\mathrm{Spa}^h(A), \mathcal{O}_{\mathrm{Spa}^h(A), B})$.

Definition 8.7.3. (After Lurie [Lu1, Definition 2.9.1.1]) We will call a finite projective module spectrum M over the ring $A^h \widehat{\otimes} B$ finite locally free, namely it will be basically sit in some finite coproduct of copies of $A^h \widehat{\otimes} B$ as a direct summand. Note that one can obviously construct a retract from some finite free module spectrum to such object, which is surjective namely with respect to the homotopy group π_0 .

As in [Lu2, Proposition 7.2.4.20], we have that finite locally free sheaves are actually flat and pseudocoherent in the strong ∞ -sense. This will cause the following to be true:

Proposition 8.7.4. *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow M \longrightarrow \prod_i A_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} M \longrightarrow \prod_{i,j} (A_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} A_j \widehat{\otimes} B) \widehat{\otimes}^{\mathbb{L}} M \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec} A_i \rightarrow \mathrm{Spec} A^h\}_{i \in I}$ in the sense of [BK1, Theorem 2.15, Theorem 4.15].

Theorem 8.7.5. (After Kedlaya-Liu [KL1, Theorem 2.7.7]) *Carrying the corresponding coefficient in noncommutative Banach ring B , we have that ∞ -descent for locally free finitely generated module spectra with respect to simple Laurent covering.*

Proof. Certainly along the idea of [KL1, Theorem 2.7.7] one can actually try to consider the corresponding glueing of finite projective objects, but this will basically involve a derived or ∞ version of [KL1, Proposition 2.4.20]. Instead we consider the following restricted situation which is also considered restrictively in [T3, Section 5]. First we consider the corresponding glueing datum along A , namely suppose we have the corresponding short exact sequence of Banach adic uniform algebra over \mathbb{Q}_p :

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \longrightarrow \Pi_{12} \longrightarrow 0,$$

which is assumed to satisfy conditions (a) and (b) of [KL1, Definition 2.7.3], namely this is strictly exact and the map $\Pi_1 \rightarrow \Pi_{12}$ is of image that is dense. Now we consider the Bambozzi-Kremnizer spaces and rings:

$$\mathrm{Spa}^h(\Pi), \mathrm{Spa}^h(\Pi_1), \mathrm{Spa}^h(\Pi_2), \mathrm{Spa}^h(\Pi_{12}), \tag{8.7.1}$$

$$\Pi^h, \Pi_1^h, \Pi_2^h, \Pi_{12}^h. \tag{8.7.2}$$

Now consider the following finite locally free bimodule spectra forming the corresponding glueing datum:

$$M_1, M_2, M_{12} \tag{8.7.3}$$

over:

$$\Pi_1^h \widehat{\otimes} B, \Pi_2^h \widehat{\otimes} B, \Pi_{12}^h \widehat{\otimes} B. \tag{8.7.4}$$

Then we can regard these as certain sheaves of bimodules:

$$\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_{12} \tag{8.7.5}$$

over the ∞ -sheaves:

$$\mathcal{O}_{\Pi_1^h} \widehat{\otimes} B, \mathcal{O}_{\Pi_2^h} \widehat{\otimes} B, \mathcal{O}_{\Pi_{12}^h} \widehat{\otimes} B. \tag{8.7.6}$$

Then we can have the chance to take the global section along the space $\mathrm{Spa}^h(\Pi)$ which gives rise to a certain module spectrum M over Π^h . However the corresponding $\pi_0(M)$ will then be finite projective over $\pi_0(\Pi)$ by [T3, Proposition 5.12]. This finishes the corresponding glueing of vector bundle process in this specific situation. \square

Remark 8.7.6. Again one can apply the whole machinery from [CS] to achieve this, as long as one would like to work with noncommutative analogs of the animated rings.

Now we consider $\mathbb{Q}_p[[G]]$ -adic objects in the situation over \mathbb{Q}_p . First we have the following ∞ -version of Tate-Čech acyclicity in our context:

Proposition 8.7.7. (After Bambozzi-Kremnizer [BK1, Theorem 4.15]) *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow A^h \widehat{\otimes}_{B[[G]]} E \longrightarrow \prod_i A_i \widehat{\otimes}_{B[[G]]} E \longrightarrow \prod_{i,j} A_i \widehat{\otimes}_{B[[G]]} E \widehat{\otimes}_{A_j} E \widehat{\otimes}_{B[[G]]} E \longrightarrow \dots),$$

for any $E \subset G$, and for any homotopy Zariski covering $\{\mathrm{Spec} A_i \rightarrow \mathrm{Spec} A^h\}_{i \in I}$ in the sense of [BK1, Theorem 2.15, Theorem 4.15]. Here we have already attached the corresponding homotopical ring A^h to A .

Proof. This is a direct consequence of the corresponding result in [BK1, Theorem 4.15]. \square

Along this idea one can then build up a sheaf of ∞ -Banach ring over Bambozzi-Kremnizer's spectrum $\mathrm{Spa}^h(A)$, which we will denote it by $\mathcal{O}_{\mathrm{Spa}^h(A), B[[G]]}$. Namely we have defined a new ∞ -ringed space $(\mathrm{Spa}^h(A), \mathcal{O}_{\mathrm{Spa}^h(A), B[[G]]})$.

Definition 8.7.8. (After Lurie [Lu1, Definition 2.9.1.1]) We will call a finite projective module spectrum M over the ring $A^h \widehat{\otimes} B[[G]]$ finite locally free, namely it will be basically sit in some finite coproduct of copies of $A^h \widehat{\otimes} B[[G]]$ as a direct summand. Note that one can obviously construct a retract from some finite free module spectrum to such object.

As in [Lu2, Proposition 7.2.4.20], we have that finite locally free sheaves are actually flat and pseudocoherent in the strong ∞ -sense. This will cause the following to be true:

Proposition 8.7.9. *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow M \longrightarrow \prod_i A_i \widehat{\otimes}_{B[[G]]/E} M \longrightarrow \prod_{i,j} (A_i \widehat{\otimes}_{B[[G]]/E} \otimes^{\mathbb{L}} A_j \widehat{\otimes}_{B[[G]]/E} \otimes^{\mathbb{L}} M \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec} A_i \rightarrow \mathrm{Spec} A^h\}_{i \in I}$ in the sense of [BK1, Theorem 4.15, Theorem 2.15].

Now we consider the LF deformation, namely here B could be written as $\varinjlim_h B_h$. First we have the following ∞ -version of Tate-Čech acyclicity in our context:

Proposition 8.7.10. (After Bambozzi-Kremnizer [BK1, Theorem 4.15]) *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow A^h \widehat{\otimes} B \longrightarrow \prod_i A_i \widehat{\otimes} B \longrightarrow \prod_{i,j} A_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} A_j \widehat{\otimes} B \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec} A_i \rightarrow \mathrm{Spec} A^h\}_{i \in I}$ in the sense of [BK1, Theorem 4.15]. Here we have already attached the corresponding homotopical ring A^h to A .

Proof. This is a direct consequence of the corresponding result in [BK1, Theorem 4.15]. □

Along this idea one can then build up a sheaf of ∞ -Banach ring over Bambozzi-Kremnizer's spectrum $\mathrm{Spa}^h(A)$, which we will denote it by $\mathcal{O}_{\mathrm{Spa}^h(A),B}$. Namely we have defined a new ∞ -ringed space $(\mathrm{Spa}^h(A), \mathcal{O}_{\mathrm{Spa}^h(A),B})$.

Definition 8.7.11. (After Lurie [Lu1, Definition 2.9.1.1]) We will call a finite projective module spectrum M over the ring $A^h \widehat{\otimes} B$ finite locally free, namely it will be basically sit in some finite coproduct of copies of $A^h \widehat{\otimes} B$ as a direct summand. Note that one can obviously construct a retract from some finite free module spectrum to such object.

As in [Lu2, Proposition 7.2.4.20], we have that finite locally free sheaves are actually flat and pseudocoherent in the strong ∞ -sense. This will cause the following to be true:

Proposition 8.7.12. *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow M \longrightarrow \prod_i A_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} M \longrightarrow \prod_{i,j} (A_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} A_j \widehat{\otimes} B) \widehat{\otimes}^{\mathbb{L}} M \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec} A_i \rightarrow \mathrm{Spec} A^h\}_{i \in I}$ in the sense of [BK1, Theorem 2.15, Theorem 4.15].

Theorem 8.7.13. (After Kedlaya-Liu [KL1, Theorem 2.7.7]) *Carrying the corresponding coefficient in the noncommutative Limit of Fréchet B , we have that ∞ -descent for locally free finitely generated module spectra with respect to simple Laurent covering.*

Proof. Along the discussion in the Banach situation, we can also make the parallel discussion in the corresponding limit of Fréchet situation (LF). Certainly along the idea of [KL1, Theorem 2.7.7] one can actually try to consider the corresponding glueing of finite projective objects, but this will basically involve a derived or ∞ version of [KL1, Proposition 2.4.20]. Instead we consider the following restricted situation which is also considered restrictively in [T3, Section 5]. First we consider the corresponding glueing datum along A , namely suppose we have the corresponding short exact sequence of Banach adic uniform algebra over \mathbb{Q}_p :

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \longrightarrow \Pi_{12} \longrightarrow 0,$$

which is assumed to satisfy conditions (a) and (b) of [KL1, Definition 2.7.3], namely this is strictly exact and the map $\Pi_1 \rightarrow \Pi_{12}$ is of image that is dense. Now we consider the Bambozzi-Kremnizer spaces and rings:

$$\mathrm{Spa}^h(\Pi), \mathrm{Spa}^h(\Pi_1), \mathrm{Spa}^h(\Pi_2), \mathrm{Spa}^h(\Pi_{12}), \quad (8.7.7)$$

$$\Pi^h, \Pi_1^h, \Pi_2^h, \Pi_{12}^h. \quad (8.7.8)$$

Now consider the following finite locally free bimodule spectra forming the corresponding glueing datum:

$$M_1, M_2, M_{12} \quad (8.7.9)$$

over:

$$\Pi_1^h \widehat{\lim}_h B_h, \Pi_2^h \widehat{\lim}_h B_h, \Pi_{12}^h \widehat{\lim}_h B_h. \quad (8.7.10)$$

Then we can regard these as certain sheaves of bimodules:

$$\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_{12} \tag{8.7.11}$$

over the ∞ -sheaves:

$$\mathcal{O}_{\Pi_1^h} \widehat{\otimes} \lim_h B_h, \mathcal{O}_{\Pi_2^h} \widehat{\otimes} \lim_h B_h, \mathcal{O}_{\Pi_{12}^h} \widehat{\otimes} \lim_h B_h. \tag{8.7.12}$$

Then we can have the chance to take the global section along the space $\mathrm{Spa}^h(\Pi)$ which gives rise to a certain module spectrum M over Π^h . However the corresponding $\pi_0(M)$ will then be finite projective over $\pi_0(\Pi)$ by [T3, Proposition 5.12]. This finishes the corresponding glueing of vector bundle process in this specific situation. \square

Remark 8.7.14. Again one can apply the whole machinery from [CS] to achieve this, as long as one would like to work with noncommutative analogs of the animated rings.

The above discussion could be made to be more general by looking at some derived rational localization $A \rightarrow D$ of A as in [BK1] which we will denote by $D \in \infty - \mathbf{Comm}$. For instance one can just take D to be some derived rational localization taking the form such as $A \left\langle \frac{a_0}{a_0}, \dots, \frac{a_k}{a_0} \right\rangle^h$ (and consider the glueing over $\mathrm{Spec}A \left\langle \frac{a_0}{a_0}, \dots, \frac{a_k}{a_0} \right\rangle^h \in \infty - \mathbf{Comm}^{\mathrm{op}}$). We now assume the following assumption:

Assumption 8.7.15. Without any sheafiness condition on the ring A in the corresponding classical sense, we assume the following totalization is strictly exact:

$$\mathrm{Tot}(0 \longrightarrow D \widehat{\otimes} B \longrightarrow \prod_i D_i \widehat{\otimes} B \longrightarrow \prod_{i,j} D_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} D_j \widehat{\otimes} B \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec}D_i \rightarrow \mathrm{Spec}D\}_{i \in I}$ in the sense of [BK1, Theorem 4.15]. Here recall the notation $\mathrm{Spec}D$ means the corresponding object in $\infty - \mathbf{Comm}^{\mathrm{op}}$. Here B is just assumed to be Banach.

Remark 8.7.16. By the corresponding Dold-Kan correspondence we could regard D as an $(\infty, 1)$ -commutative object in the corresponding category of all the simplicial semi-normed monoids, namely an E_∞ -ring which carries the corresponding simplicial Banach structures. However then the corresponding ring $D \widehat{\otimes} B$ will then be a product of an E_∞ -ring with a discrete (in the sense of [Lu2, Chapter 7.2]) E_1 -ring (or if you wish a discrete A_∞ -ring).

Definition 8.7.17. (After Lurie [Lu1, Definition 2.9.1.1]) We will call a finite projective module spectrum M over the ring $D \widehat{\otimes} B$ finite locally free, namely it will be basically sit in some finite coproduct of copies of $D \widehat{\otimes} B$ as a direct summand. Note that one can obviously construct a retract from some finite free module spectrum to such object, which is surjective namely with respect to the homotopy group π_0 .

Definition 8.7.18. (After Lurie [Lu1, Definition 2.9.1.1]) We will call a sheaf M over the ∞ -ringed Grothendieck site:

$$(\mathrm{Spec} D, \mathcal{O}_{\mathrm{Spec} D \widehat{\otimes} B}, \{\mathrm{Spec} D \left\langle \frac{f_0}{f_i}, \dots, \frac{f_d}{f_i} \right\rangle^h\}_{i=0, \dots, d}) \quad (8.7.13)$$

locally finite free if it is locally (over some derived rational subset $\mathrm{Spec} D_i \in \infty - \mathbf{Comm}^{\mathrm{op}}$ and $D_i := D \left\langle \frac{f_0}{f_i}, \dots, \frac{f_d}{f_i} \right\rangle^h, i = 1, \dots, d$) sitting in some coproduct of finite many copies of $D_i \widehat{\otimes} B$ as a direct summand.

As in [Lu2, Proposition 7.2.4.20], we have that finite locally free sheaves are actually flat and pseudocoherent in the strong ∞ -sense. This will cause the following to be true:

Proposition 8.7.19. *The following totalization is strictly exact:*

$$\mathrm{Tot}(0 \longrightarrow M \longrightarrow \prod_i D_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} M \longrightarrow \prod_{i,j} (D_i \widehat{\otimes} B \widehat{\otimes}^{\mathbb{L}} D_j \widehat{\otimes} B) \widehat{\otimes}^{\mathbb{L}} M \longrightarrow \dots)$$

for any homotopy Zariski covering $\{\mathrm{Spec} D_i \rightarrow \mathrm{Spec} D\}_{i \in I}$ in the sense of [BK1, Theorem 2.15, Theorem 4.15].

Now we take a corresponding fiber sequence (under the corresponding Dold-Kan correspondence one can also consider the corresponding distinguished triangles

$$D \longrightarrow \prod_{i=1,2} D_i \longrightarrow D_1 \widehat{\otimes}^{\mathbb{L}} D_2 \longrightarrow D[1],$$

for the corresponding associated complexes):

$$D \longrightarrow \prod_{i=1,2} D_i \longrightarrow D_{12},$$

and assume that the following:

$$0 \longrightarrow \pi_0 D \longrightarrow \pi_0 D_1 \oplus \pi_0 D_2 \longrightarrow \pi_0 D_{12} \longrightarrow 0$$

to satisfy [KL1, Definition 2.7.3, condition (a), (b)]. Then we have the effective ∞ -descentness of the corresponding morphism $D \widehat{\otimes} B \rightarrow D_1 \widehat{\otimes} B \oplus D_2 \widehat{\otimes} B$.

Theorem 8.7.20. (After Kedlaya-Liu [KL1, Theorem 2.7.7]) *Carrying the corresponding coefficient in noncommutative Banach ring B , we have that ∞ -descent for locally free finitely generated module spectra with respect to binary covering on the corresponding π_0 in the situation described above.*

Proof. Certainly along the idea of [KL1, Theorem 2.7.7] one can actually try to consider the corresponding glueing of finite projective objects in more general setting, but this will basically involve a derived or ∞ version of [KL1, Proposition 2.4.20]. Instead we consider the following restricted situation which is also considered restrictively in [T3, Section 5]. First we consider the corresponding glueing datum along $\pi_0 D$, namely suppose we have the corresponding short exact sequence of Banach adic uniform algebra over \mathbb{Q}_p :

$$0 \longrightarrow \pi_0 D \longrightarrow \pi_0 D_1 \oplus \pi_0 D_2 \longrightarrow \pi_0 D_{12} \longrightarrow 0$$

which is assumed to satisfy conditions (a) and (b) of [KL1, Definition 2.7.3], namely this is strictly exact and the map $\Pi_1 \rightarrow \Pi_{12}$ is of image that is dense. Now consider the following finite locally free bimodule spectra forming the corresponding glueing datum:

$$M_1, M_2, M_{12} \tag{8.7.14}$$

over:

$$D_1 \widehat{\otimes} B, D_2 \widehat{\otimes} B, D_{12} \widehat{\otimes} B. \tag{8.7.15}$$

Then we can regard these as certain sheaves of bimodules:

$$\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_{12} \tag{8.7.16}$$

over the ∞ -sheaves:

$$\mathcal{O}_{\mathrm{Spec} D_1} \widehat{\otimes} B, \mathcal{O}_{\mathrm{Spec} D_2} \widehat{\otimes} B, \mathcal{O}_{\mathrm{Spec} D_{12}} \widehat{\otimes} B. \tag{8.7.17}$$

Then we can have the chance to take the global section along the space $\mathrm{Spec} D$ which gives rise to a certain module spectrum M over D . However the corresponding $\pi_0(M)$ will then be finite projective over $\pi_0(\Pi)$ by [T3, Proposition 5.12]. This finishes the corresponding glueing of vector bundle process in this specific situation. \square

8.7.2 Application to Equivariant Sheaves over Adic Fargues-Fontaine Curves

We now apply our study to the first specific situation around the corresponding mixed-type Hodge structures inspired essentially by the work [CKZ], [PZ] and [Ked1], which is also considered in [T3]. Let us first describe the corresponding situation:

Setting 8.7.21. We will use the notations in [T1], [T2] and [T3]. We now consider the corresponding Fargues-Fontaine curves as our target to relativization. We will use the corresponding FF_R to denote the adic Fargues-Fontaine curve in our current situation (as in [KL2, Theorem 4.6.1]) which is defined to be the corresponding suitable quotient of the following adic space:

$$\bigcup_{0 < r_1 < r_2} \mathrm{Spa}(\tilde{\Pi}_R^{[r_1, r_2]}, \tilde{\Pi}_R^{[r_1, r_2], +}). \quad (8.7.18)$$

Here R is specified perfectoid Banach uniform algebra over \mathbb{F}_{p^h} .

Assumption 8.7.22. In this section we assume that the corresponding base field will be just of mixed-characteristic.

In our current framework we look at the following deformation:

Setting 8.7.23. Over \mathbb{Q}_p we consider the any Galois perfectoid tower (H_\bullet, H_\bullet^+) (in the sense of [KL2, Definition 5.1.1]) with Galois group $\mathrm{Gal}_H := \mathrm{Gal}(\overline{H}_0/H_0)$, and we use the notation \tilde{H}_∞ to denote the corresponding perfectoid top level of the tower in mixed-characteristic situation, and we consider the corresponding ring \overline{H}_∞ as its tilt. Now we consider the period rings $B_{\mathrm{dR}, \overline{H}_\infty}^+$, $B_{\mathrm{dR}, \overline{H}_\infty}$ and $B_{e, \overline{H}_\infty}$ (as constructed in [KL2, Definition 8.6.5]).

We now try to be more specific, where we just consider the situation where the ring \tilde{H}_∞ is just the ring \mathbb{C}_p . We now consider some Banach algebra B over \mathbb{Q}_p as before. Now what is happening is then we consider the following ringed space:

$$(\mathrm{FF}_R, \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+, \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}, \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}). \quad (8.7.19)$$

Now we encode Berger's B -pair in some relativization after Kedlaya-Liu ([KL1, Definition 9.3.11] and [KL2, Definition 4.8.2]) and Kedlaya-Pottharst ([KP, Definition 2.17]):

Definition 8.7.24. We now define a pseudocoherent FF_R - B - Gal_H sheaf to be a triplet $(M_{\mathrm{dR}}^+, M_{\mathrm{dR}}, M_e)$ of $B \widehat{\otimes} ?$ -stably pseudocoherent sheaves such that the form a corresponding descent triplet with respect to the following diagram by natural base change:

$$\mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+ \longrightarrow \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty} \longleftarrow \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}, \quad (8.7.20)$$

where $? = B_{\mathrm{dR}, \overline{H}_\infty}^+, B_{\mathrm{dR}, \overline{H}_\infty}, B_{e, \overline{H}_\infty}$. For the rings:

$$\mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty} \quad (8.7.21)$$

the corresponding definition of pseudocoherent sheaves are defined to be locally associated to the $B \widehat{\otimes} B_{e, \overline{H}_\infty}$ -stably pseudocoherent modules in the ind-Fréchet topology, namely the completeness with respect to the ind-Fréchet topology is then stable under the corresponding rational localization concentrated at the part for FF_R . And we assume that the corresponding sheaves admit Gal_H -equivariant action in the semilinear fashion.

In the étale topology we consider the following ringed space:

$$(\mathrm{FF}_R, \mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+, \mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}, \mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}). \quad (8.7.22)$$

Now we encode Berger's B -pair in some relativization after Kedlaya-Liu ([KL1, Definition 9.3.11] and [KL2, Definition 4.8.2]) and Kedlaya-Pottharst ([KP, Definition 2.17]):

Definition 8.7.25. We now define a pseudocoherent FF_R - B - Gal_H sheaf to be a triplet $(M_{\mathrm{dR}}^+, M_{\mathrm{dR}}, M_e)$ of $B \widehat{\otimes} ?$ -étale-stably pseudocoherent sheaves such that these form a corresponding descent triplet with respect to the following diagram by natural base change:

$$\mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+ \longrightarrow \mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty} \longleftarrow \mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}, \quad (8.7.23)$$

where $? = B_{\mathrm{dR}, \overline{H}_\infty}^+, B_{\mathrm{dR}, \overline{H}_\infty}, B_{e, \overline{H}_\infty}$. For the rings:

$$\mathcal{O}_{\mathrm{FF}_R, \acute{\mathrm{e}}\mathrm{t}} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty} \quad (8.7.24)$$

the corresponding definition of pseudocoherent sheaves are defined to be locally associated to the $B \widehat{\otimes} B_{e, \overline{H}_\infty}$ -étale-stably pseudocoherent modules in the ind-Fréchet topology, namely the completeness with respect to the ind-Fréchet topology is then stable under the corresponding étale morphisms concentrated at the part for FF_R . And we assume that the corresponding sheaves admit Gal_H -equivariant action in the semilinear fashion.

Definition 8.7.26. We now define a projective FF_R - B - Gal_H sheaf to be a triplet $(M_{\mathrm{dR}}^+, M_{\mathrm{dR}}, M_e)$ of locally finite projective sheaves such that these form a corresponding descent triplet with respect to the following diagram by natural base change:

$$\mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+ \longrightarrow \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty} \longleftarrow \mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}, \quad (8.7.25)$$

where $? = B_{\mathrm{dR}, \overline{H}_\infty}^+, B_{\mathrm{dR}, \overline{H}_\infty}, B_{e, \overline{H}_\infty}$. For the rings:

$$\mathcal{O}_{\mathrm{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty} \quad (8.7.26)$$

the corresponding definition of projective sheaves are defined to be locally associated to the finite projective modules in the ind-Fréchet topology, namely the completeness with respect to the ind-Fréchet topology is then stable under the corresponding rational localization concentrated at the part for FF_R . And we assume that the corresponding sheaves admit Gal_H -equivariant action in the semilinear fashion.

In the étale topology we consider the following ringed space:

$$(\mathrm{FF}_R, \mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+, \mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}, \mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}). \quad (8.7.27)$$

Now we encode Berger's B -pair in some relativization after Kedlaya-Liu ([KL1, Definition 9.3.11] and [KL2, Definition 4.8.2]) and Kedlaya-Pottharst ([KP, Definition 2.17]):

Definition 8.7.27. We now define a projective FF_R - B - Gal_H sheaf to be a triplet $(M_{\mathrm{dR}}^+, M_{\mathrm{dR}}, M_e)$ of locally finite projective sheaves such that these form a corresponding descent triplet with respect to the following diagram by natural base change:

$$\mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty}^+ \longrightarrow \mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{\mathrm{dR}, \overline{H}_\infty} \longleftarrow \mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}, \quad (8.7.28)$$

where $? = B_{\mathrm{dR}, \overline{H}_\infty}^+, B_{\mathrm{dR}, \overline{H}_\infty}, B_{e, \overline{H}_\infty}$. For the rings:

$$\mathcal{O}_{\mathrm{FF}_R, \text{ét}} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty} \quad (8.7.29)$$

the corresponding definition of projective sheaves are defined to be locally associated to the finite projective modules in the ind-Fréchet topology, namely the completeness with respect to the ind-Fréchet topology is then stable under the corresponding étale morphisms concentrated at the part for FF_R . And we assume that the corresponding sheaves admit Gal_H -equivariant action in the semilinear fashion.

Remark 8.7.28. One can basically define the same B - Gal_H -sheaves over any adic Banach affinoid algebra or space in the same fashion which we will not repeat again.

We now prove the following theorems:

Theorem 8.7.29. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories:*

I. *The category of all the projective FF_R - B - Gal_H sheaves over the site:*

$$(\text{FF}_R, \mathcal{O}_{\text{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}^+, \mathcal{O}_{\text{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}, \mathcal{O}_{\text{FF}_R} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}). \quad (8.7.30)$$

II. *The category of all the compatible families (with locally glueability and local cocycle condition) of all the projective $\{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})\}_{[s,r]}$ - B - Gal_H sheaves over the sites:*

$$(\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+}), \mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}^+, \quad (8.7.31)$$

$$\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}, \mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty}), \quad (8.7.32)$$

for any $[s, r] \subset (0, \infty)$, carrying the corresponding semilinear action of the Frobenius coming from just the Robba ring part.

III. *The category of all the compatible families (with locally glueability and local cocycle condition) of all the projective $\{\widetilde{\Pi}_R^{[s,r]}\}_{[s,r]}$ - B - Gal_H modules over the rings:*

$$(\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}^+, \quad (8.7.33)$$

$$\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}, \quad (8.7.34)$$

$$\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty})|_{\text{Spa}(\widetilde{\Pi}_R^{[s,r]}, \widetilde{\Pi}_R^{[s,r],+})}, \quad (8.7.35)$$

for any $[s, r] \subset (0, \infty)$, carrying the corresponding semilinear action of the Frobenius coming from just the Robba ring part.

IV. *The category of all the projective $\widetilde{\Pi}_R^\infty$ - B - Gal_H modules over the rings:*

$$(\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^\infty, \widetilde{\Pi}_R^{\infty,+})} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}^+, \quad (8.7.36)$$

$$\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^\infty, \widetilde{\Pi}_R^{\infty,+})} \widehat{\otimes} B \widehat{\otimes} B_{\text{dR}, \overline{H}_\infty}, \quad (8.7.37)$$

$$\mathcal{O}_{\text{Spa}(\widetilde{\Pi}_R^\infty, \widetilde{\Pi}_R^{\infty,+})} \widehat{\otimes} B \widehat{\otimes} B_{e, \overline{H}_\infty})|_{\text{Spa}(\widetilde{\Pi}_R^\infty, \widetilde{\Pi}_R^{\infty,+})}, \quad (8.7.38)$$

carrying the corresponding semilinear action of the Frobenius coming from just the Robba ring part.

Then we have the corresponding categories above are equivalent.

Proof. The corresponding families of sheaves with respect to the intervals, the families of modules with respect to the intervals and the sheaves over FF_R could be related equivalently through the functor taking the corresponding local global section, by applying theorem 8.4.15 and theorem 8.3.15. The corresponding families of sheaves with respect to the intervals, the families of modules with respect to the intervals and the modules over $\tilde{\Pi}_R^\infty$ could be related equivalently through the functor taking the corresponding local projective limit global section, by applying theorem 8.4.32 and theorem 8.3.32 and by considering the corresponding Frobenius pullback to control the rank throughout. \square

Theorem 8.7.30. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the following categories:*

I. *The category of all the pseudocoherent sheaves over the site:*

$$(\mathrm{FF}_R, \mathcal{O}_{\mathrm{FF}_R, \acute{e}t} \widehat{\otimes} B). \quad (8.7.39)$$

II. *The category of all the compatible families (with locally glueability and local cocycle condition) of all the pseudocoherent $\{\mathrm{Spa}(\tilde{\Pi}_R^{[s,r]}, \tilde{\Pi}_R^{[s,r],+})\}_{\{[s,r]\}}$ sheaves over the sites:*

$$(\mathrm{Spa}(\tilde{\Pi}_R^{[s,r]}, \tilde{\Pi}_R^{[s,r],+}), \mathcal{O}_{\mathrm{Spa}(\tilde{\Pi}_R^{[s,r]}, \tilde{\Pi}_R^{[s,r],+}), \acute{e}t} \widehat{\otimes} B), \quad (8.7.40)$$

for any $[s, r] \subset (0, \infty)$, carrying the corresponding semilinear action of the Frobenius coming from just the Robba ring part.

III. *The category of all the compatible families (with locally glueability and local cocycle condition) of all the stably-pseudocoherent $\{\tilde{\Pi}_R^{[s,r]}\}_{\{[s,r]\}}$ modules over the rings:*

$$\mathcal{O}_{\mathrm{Spa}(\tilde{\Pi}_R^{[s,r]}, \tilde{\Pi}_R^{[s,r],+}), \acute{e}t} \widehat{\otimes} B|_{\mathrm{Spa}(\tilde{\Pi}_R^{[s,r]}, \tilde{\Pi}_R^{[s,r],+})}, \quad (8.7.41)$$

for any $[s, r] \subset (0, \infty)$, carrying the corresponding semilinear action of the Frobenius coming from just the Robba ring part.

IV. *The category of all the stably-pseudocoherent $\tilde{\Pi}_R^\infty$ modules over the rings:*

$$\mathcal{O}_{\mathrm{Spa}(\tilde{\Pi}_R^\infty, \tilde{\Pi}_R^{\infty,+}), \acute{e}t} \widehat{\otimes} B|_{\mathrm{Spa}(\tilde{\Pi}_R^\infty, \tilde{\Pi}_R^{\infty,+})}, \quad (8.7.42)$$

carrying the corresponding semilinear action of the Frobenius coming from just the Robba ring part.

Then we have the corresponding categories above are equivalent.

Proof. The corresponding families of sheaves with respect to the intervals, the families of modules with respect to the intervals and the sheaves over FF_R could be related equivalently through the functor taking the corresponding local global section, by applying theorem 8.2.23. The corresponding families of sheaves with respect to the intervals, the families of modules with respect to the intervals and the modules over $\tilde{\Pi}_R^\infty$ could be related equivalently through the functor taking the corresponding local projective limit global section, by applying theorem 8.2.32 and by considering the corresponding Frobenius pullback to control the rank throughout. \square

This chapter is based on the following paper which is being revised for the journal, where the author of this dissertation is the main author:

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Chapter 9

Period Rings with Big Coefficients and Applications III

9.1 Introduction

9.1.1 Introduction and Main Results

In our previous work on the corresponding period rings with big coefficients and applications, we essentially studied many very general deformations of the corresponding pseudocoherent sheaves over adic Banach rings in the sense of [KL1] and [KL2]. We studied the corresponding glueing in the fashion of [KL1] and [KL2] carrying sufficiently large coefficients. Although we have some conditions on both the corresponding adic spaces we are considering and the corresponding coefficients, but the descent results on their own are already general enough to tackle some specific situations in the Hodge-Iwasawa theoretic consideration and noncommutative analytic geometry such as in [TX3] and [TX4].

First in our current scope of discussion we will first consider the corresponding extension of our paper [TX2] to the corresponding pro-étale topology setting. In the Tate setting, this will follow the corresponding unrelative situation in [KL2] while in the analytic situation the corresponding will following the corresponding unrelative situation in [Ked1] (in the context of section 9.2 and section 9.3):

Theorem 9.1.1. (See proposition 9.2.10, proposition 9.3.17) *Over sheafy Tate adic Banach rings or analytic Huber rings, the corresponding descent of stably pseudocoherent modules (with respect to the underlying spaces) carrying the corresponding noncommutative coefficients holds over étale sites and pro-étale sites. Here for pro-étale sites we will consider only perfectoid rings.*

We then consider some application to the corresponding 'complete' noetherian situation (this will mean that we consider the noetherian deformation of noetherian rings) in the context of section 9.6:

Theorem 9.1.2. (See proposition 9.6.9) *Over noetherian sheafy Tate adic Banach rings, the corresponding descent of finitely presented modules (with respect to the underlying spaces) carrying the corresponding noncommutative coefficients (such that the products of rings are noetherian as well) holds over étale sites.*

We also discussed the corresponding possible application to the descent in certain cases over the corresponding ∞ -analytic stacks after Bambozzi-Ben-Bassat-Kremnizer

[BBBK] and Bambozzi-Kremnizer [BK], where the latter is really more related to the adic geometry we considered after [KL1] and [KL2]. Our approach certainly is definitely parallel to [KL1] and [KL2], which is not actually parallel to more ∞ -method considered by Ben-Bassat-Kremnizer in [BBK].

9.1.2 Future Work

We expect more interesting applications. For instance partial noetherian spaces could admit some desired descent carrying general noncommutative coefficients (certainly not need to be noetherian coefficients), such as different types of eigenvarieties or Shimura varieties.

9.2 Descent over Analytic Adic Banach Rings in the Rational Setting

9.2.1 Noncommutative Functional-Analytic Pseudocoherence over Pro-Étale Topology

Now we consider the corresponding discussion of the corresponding glueing deformed pseudocoherent sheaves over étale topology which generalize the corresponding discussion in [Ked1].

Setting 9.2.1. Let (W, W^+) be Tate adic Banach uniform ring which is defined over \mathbb{Q}_p . And we assume that the ring V over \mathbb{Q}_p is a Banach ring over the \mathbb{Q}_p . Assume that the ring W is sheafy.

Remark 9.2.2. The corresponding glueing of the corresponding pseudocoherent sheaves over the corresponding pro-étale site does not need the corresponding notions on the corresponding stability beyond the corresponding étale-stably pseudocoherence.

Lemma 9.2.3. (After Kedlaya-Liu [KL2, Proposition 3.4.3]) *Suppose that we are taking (W, W^+) to be Fontaine perfectoid (note that we are considering the corresponding context of Tate adic Banach situation). Then we have that over the corresponding étale site, the corresponding group $H^i(\mathrm{Spa}(W, W^+)_{\mathrm{pét}}, \mathcal{O} \widehat{\otimes} V)$ vanishes for each $i > 0$, while we have that the corresponding group $H^0(\mathrm{Spa}(W, W^+)_{\mathrm{pét}}, \mathcal{O} \widehat{\otimes} V)$ is just isomorphic $W \widehat{\otimes} V$.*

Proof. See [KL2, Proposition 3.4.3], where the corresponding [KL2, Corollary 3.3.20] applies. □

Now we proceed to consider the corresponding pro-étale topology. First we recall the following result from [KL2, Lemma 3.4.4] which does hold in our situation since we did not change much on the corresponding underlying adic spaces.

Lemma 9.2.4. (Kedlaya-Liu [KL2, Lemma 3.4.4]) *Consider the ring W as above which is furthermore assumed to be Fontaine perfectoid in the sense of [KL2, Definition 3.3.1], and consider any direct system of faithfully finite étale morphisms as:*

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \dots,$$

where A_0 is just the corresponding base A . Then the corresponding completion of the infinite level could be decomposed as:

$$W_0 \oplus \widehat{\bigoplus_{k=0}^{\infty} W_k/W_{k-1}}. \quad (9.2.1)$$

As in [KL2, Lemma 3.4.4] we endow the corresponding completion mentioned above with the corresponding seminorm spectral. And for the corresponding quotient we endow with the corresponding quotient norm.

Then we consider the corresponding deformation by the ring V over \mathbb{Q}_p :

Lemma 9.2.5. (After Kedlaya-Liu [KL2, Lemma 3.4.4]) Consider the ring A as above which is furthermore assumed to be Fontaine perfectoid in the sense of [KL2, Definition 3.3.1], and consider any direct system of faithfully finite étale morphisms as:

$$W_{0,V} \longrightarrow W_{1,V} \longrightarrow W_{2,V} \longrightarrow \dots,$$

where W_0 is just the corresponding base W . Then the corresponding completion of the infinite level could be decomposed as:

$$W_{0,V} \oplus \widehat{\bigoplus_{k=0}^{\infty} W_{k,V}/W_{k-1,V}}. \quad (9.2.2)$$

Corollary 9.2.6. (After Kedlaya-Liu [KL2, Corollary 3.4.5]) Starting with an adic Banach ring which is as in [KL2, Corollary 3.4.5] assumed to be Fontaine perfectoid in the sense of [KL2, Definition 3.3.1]. And as in the previous two lemmas we consider an admissible infinite direct system:

$$W_{0,V} \longrightarrow W_{1,V} \longrightarrow W_{2,V} \longrightarrow \dots,$$

whose infinite level will be assumed to take the corresponding spectral seminorm as in the [KL2, Corollary 3.4.5]. Then carrying the corresponding coefficient V , we have in such situation the corresponding 2-pseudoflatness of the corresponding embedding map:

$$W_{0,V} \rightarrow W_{\infty,V}. \quad (9.2.3)$$

Here $W_{\infty,V}$ denotes the corresponding completion of the limit $\varinjlim_{k \rightarrow \infty} W_{k,V}$.

Proof. See [KL2, Corollary 3.4.5]. □

Now recall that from [KL2] since we do not have to modify the corresponding underlying spatial context, so we will also only have to consider the corresponding stability with respect to étale topology even although we are considering the corresponding profinite étale site in our current section. We first generalize the corresponding Tate acyclicity in [KL2] to the corresponding V -relative situation in our current situation:

Remark 9.2.7. In the following we only consider the corresponding perfectoid ring W .

Proposition 9.2.8. (After Kedlaya-Liu [KL2, Theorem 3.4.6]) *Now we consider the corresponding pro-étale site of X which we denote it by $X_{\text{pét}}$. We assume we have a stable basis \mathbb{H} of the corresponding perfectoid subdomains for this profinite étale site where each morphism therein will be étale pseudoflat. We consider as in [KL2, Theorem 3.4.6] a corresponding module over $W \widehat{\otimes} V$ which is assumed to be étale-stably pseudocoherent. Then consider the corresponding presheaf \widetilde{G} attached to this étale-stably pseudocoherent module with respect to in our situation the corresponding chosen well-defined basis \mathbb{H} , we have that the corresponding sheaf over some element $Y \in \mathbb{H}$ (that is to say over $\mathcal{O}_{Y_{\text{pét}}} \widehat{\otimes} V$) is acyclic and is acyclic with respect to some Čech covering coming from elements in \mathbb{H} .*

Proof. The corresponding proof could be made as in the corresponding nonrelative setting as in [KL2, Theorem 3.4.6]. As in [KL2, Theorem 3.4.6] the corresponding first step is to check the statement is true with respect to some specific basis consisting of all the perfectoid subdomains which are faithfully finite étale over some specific perfectoid subdomain Y (along towers). Then the corresponding statement could be reduced to those with respect to covering which is simple Laurent rational and the corresponding covering in the sense of the previous corollary. Then actually in this first scope of consideration, the statement holds by previous corollary and [TX2, Theorem 2.12]. Then in general for some general Y we consider the corresponding covering of Y consisting of elements satisfying the previous situation, then apply the previous argument to the corresponding basis in current situation satisfying the previous situation Then we consider the corresponding sheafification of the following chosen short exact sequence:

$$0 \longrightarrow G'' \longrightarrow G' \longrightarrow G \longrightarrow 0,$$

with G' finite projective. Then the idea is to consider the corresponding diagram factoring through the corresponding pro-étale sheafification of this short exact sequence down to the corresponding étale site. Then the five lemma will finish the corresponding proof as one considers the following commutative diagram:

$$\begin{array}{ccccccc}
G''_{V_Y} & \longrightarrow & G'_{V_Y} & \longrightarrow & G_{V_Y} & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(Y_{\text{ét}}, \widetilde{G}'') & \longrightarrow & \Gamma(Y_{\text{ét}}, \widetilde{G}') & \longrightarrow & \Gamma(Y_{\text{ét}}, \widetilde{G}) \longrightarrow 0.
\end{array}$$

Here the corresponding ring V_Y is the adic Banach ring such that $Y = \text{Spa}(V_Y, V_Y^+)$. \square

Definition 9.2.9. (After Kedlaya-Liu [KL2, Definition 3.4.7]) Consider the pro-étale site of X attached to the adic Banach ring (W, W^+) . We will define a sheaf of module G over $\widehat{\mathcal{O}}_{\text{pét}} \widehat{\otimes} V$ to be V -pseudocoherent if locally we can define this as a sheaf attached to a V -étale-stably pseudocoherent module. As in [KL2, Definition 3.4.7], we do not have to consider the corresponding notion of V -profinite-étale-stably pseudocoherent module.

Proposition 9.2.10. (After Kedlaya-Liu [KL2, Theorem 3.4.8]) *Taking the corresponding global section will realize the corresponding equivalence between the following two categories. The first one is the corresponding one of all the V -pseudocoherent sheaves over $\widehat{\mathcal{O}}_{\text{pét}} \widehat{\otimes} V$, while the second one is the corresponding one of all the V -étale-stably pseudocoherent modules over $W \widehat{\otimes} V$.*

Proof. As in [KL2, Theorem 3.4.8], the corresponding [KL1, Proposition 9.2.6] applies in the way that the corresponding conditions of [KL1, Proposition 9.2.6] hold in our current situation. \square

Obviously we have the following analog of [KL2, Corollary 3.4.9]:

Corollary 9.2.11. (After Kedlaya-Liu [KL2, Corollary 3.4.9]) *The following two categories are equivalent. The first is the corresponding category of all V -pseudocoherent sheaves over $\widehat{\mathcal{O}}_{\text{pét}} \widehat{\otimes} V$. The second is the corresponding category of all V -pseudocoherent sheaves over $\mathcal{O}_{\text{ét}} \widehat{\otimes} V$. The corresponding functor is the corresponding pullback along the corresponding morphism of sites $X_{\text{pét}} \rightarrow X_{\text{ét}}$.*

9.3 Descent over Analytic Huber Rings in the Integral Setting

9.3.1 Noncommutative Topological Pseudocoherence over Étale Topology

Now we consider the corresponding discussion of the corresponding glueing deformed pseudocoherent sheaves over étale topology which generalize the corresponding discussion in [Ked1]. And we will consider the corresponding situation as assumed in the following setting:

Setting 9.3.1. Let (W, W^+) be analytic Huber uniform pair which is defined over \mathbb{Z}_p . And we assume that the ring V over \mathbb{Z}_p is a topological ring (complete) and splitting over the \mathbb{Z}_p . We now fix a corresponding stable basis \mathbb{H} for the corresponding étale site of the adic space $\mathrm{Spa}(W, W^+)$, locally consisting of the corresponding compositions of rational localizations and the corresponding finite étale morphisms. And as in [Ked1, Hypothesis 1.10.3] we need to assume that the corresponding basis is made up of the corresponding adic spectrum of sheafy rings. Assume the corresponding sheafiness of the Huber ring W .

Definition 9.3.2. (After Kedlaya-Liu [KL2, Definition 2.5.9]) We define a V -étale stably pseudocoherent module over the corresponding Huber ring W with respect to the corresponding basis \mathbb{H} chosen above for the corresponding étale site of the analytic adic space X . We define a module over $W \widehat{\otimes} V$ is a V -étale stably pseudocoherent module if it is algebraically pseudocoherent (namely formed by the corresponding possibly infinite length resolution of finitely generated and projective modules) and at least complete with respect to the corresponding natural topology and also required to be also complete with respect to the natural topology along some base change with respect to any morphism in \mathbb{H} .

Definition 9.3.3. (After Kedlaya-Liu [KL2, Definition 2.5.9]) Along the previous definition we have the corresponding notion of V -étale-pseudoflat left (or right respectively) modules with respect to the corresponding chosen basis \mathbb{H} . A such module is defined to be a topological module G over $W \widehat{\otimes} V$ complete with respect to the natural

topology and for any right (or left respectively) V -étale stably pseudocoherent module, they will jointly give the Tor_1 vanishing.

Lemma 9.3.4. (After Kedlaya-Liu [KL2, Lemma 2.5.10]) *One could find another basis \mathbb{H}' which is in our situation contained in the corresponding original basis \mathbb{H} such that any morphism in \mathbb{H} could be V -étale-pseudoflat with respect \mathbb{H}' or just \mathbb{H} itself.*

Proof. The derivation of such new basis is actually along the same way as in [KL2, Lemma 2.5.10] since the corresponding compositions of rational localizations and the finite étales are actually satisfying the corresponding conditions in the statement by [TX2, Theorem 2.12] over the analytic topology. In general one just show any general morphism will also decompose in the same way, which one can prove certainly as in [KL2, Lemma 2.5.10], where one instead consider in the current context the corresponding basis spreading result in [Ked1, Lemma 1.10.4]. \square

Proposition 9.3.5. (After Kedlaya-Liu [KL2, Theorem 2.5.11]) *Consider the site $X_{\text{ét}}$ and consider the basis \mathbb{H} . Take any V -étale stably pseudocoherent module M over $W \widehat{\otimes} V$. Consider the corresponding presheaf by taking the inverse limit throughout all the corresponding base change along morphisms in \mathbb{H} . Then we have the corresponding acyclicity of the presheaf over any element in \mathbb{H} and any covering of this element by elements in \mathbb{H} .*

Proof. As in [KL2, Theorem 2.5.11] apply the corresponding [KL1, Proposition 8.2.21]. \square

Proposition 9.3.6. (After Kedlaya-Liu [KL2, Lemma 2.5.13]) *The corresponding glueing of V -étale-stably pseudocoherent modules holds in this current situation over W along binary morphisms (namely along binary rational decomposition).*

Proof. We adapt the argument of [KL2, Lemma 2.5.13] to our current situation. Take the corresponding map to be $W \rightarrow W_1 \oplus W_2$, and we denote the two spaces associated W_1 and W_2 by Y_1 and Y_2 . Establish a corresponding descent datum of V -étale-stably pseudocoherent modules along this decomposition of $\mathrm{Spa}(W, W^+)$. One can realize this by acyclicity a sheaf G over the whole space $\mathrm{Spa}(W, W^+)$. Then we have for each $k = 1, 2$ that the corresponding identification of $H^0(Y_{k, \text{ét}}, G)$ with $H^0(Y_k, G)$ and the vanishing of the i -th

cohomology for higher i . By the corresponding results we have in the corresponding rational localization situation we vanishing of the corresponding $H^i(\mathrm{Spa}(W, W^+), G), i \geq 1$. Now consider a finite free covering C of G and take the kernel K to form the corresponding exact sequence:

$$0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0,$$

which will give rise to the following short exact sequence:

$$0 \longrightarrow K(\mathrm{Spa}(W, W^+)) \longrightarrow C(\mathrm{Spa}(W, W^+)) \longrightarrow G(\mathrm{Spa}(W, W^+)) \longrightarrow 0.$$

To finish we have also to show the corresponding resulting global section is actually V -étale-stably pseudocoherent. For this, we refer the readers to [TX2, Lemma 2.22].

□

Proposition 9.3.7. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *Taking the corresponding global section will realize the equivalence between the following two categories. The first is the one of all the corresponding V -pseudocoherent sheaves over $\mathcal{O}_{\text{ét}} \widehat{\otimes} V$. The second is the one of all the corresponding V -étale-stably pseudocoherent modules.*

Proof. This is by considering and applying the corresponding [Ked1, Lemma 1.10.4] in our current context. The refinement comes from the acyclicity, the transitivity is straightforward, the corresponding binary rational localization situation is the previous proposition, while the corresponding last condition will basically comes from the corresponding f.p. descent [SGAI, Chapitre VIII].

□

9.3.2 Noncommutative Topological Pseudocoherence over Pro-Étale Topology

Setting 9.3.8. Let (W, W^+) be analytic Huber ring which is defined over \mathbb{Z}_p . And we assume that the ring V over \mathbb{Z}_p is a complete topological ring over the \mathbb{Z}_p which is completely exact. Assume that the ring W is sheafy. We assume p is a topologically nilpotent element.

Remark 9.3.9. The corresponding glueing of the corresponding pseudocoherent sheaves over the corresponding pro-étale site does not need the corresponding notions on the corresponding stability beyond the corresponding étale-stably pseudocoherence.

Lemma 9.3.10. (After Kedlaya-Liu [KL2, Proposition 3.4.3]) *Suppose that we are taking (W, W^+) to be perfectoid now in the sense of [Ked1, Definition 2.1.1] (note that we are considering the corresponding context of analytic Huber pair situation). Then we have that over the corresponding étale site, the corresponding group $H^i(\mathrm{Spa}(W, W^+)_{\text{pét}}, \mathcal{O} \widehat{\otimes} V)$ vanishes for each $i > 0$, while we have that the corresponding group $H^0(\mathrm{Spa}(W, W^+)_{\text{pét}}, \mathcal{O} \widehat{\otimes} V)$ is just isomorphic $W \widehat{\otimes} V$.*

Proof. See [KL2, Proposition 3.4.3]. □

Now we proceed to consider the corresponding pro-étale topology. First we recall the following result from [KL2, Lemma 3.4.4] which does hold in our situation since we did not change much on the corresponding underlying adic spaces.

Lemma 9.3.11. (Kedlaya-Liu [KL2, Lemma 3.4.4]) *Consider the ring W as above which is furthermore assumed to be perfectoid in the sense of instead [Ked1, Definition 2.1.1], and consider any direct system of faithfully finite étale morphisms as:*

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \dots,$$

where A_0 is just the corresponding base A . Then the corresponding completion of the infinite level could be decomposed as:

$$W_0 \oplus \widehat{\bigoplus}_{k=0}^{\infty} W_k/W_{k-1}. \tag{9.3.1}$$

As in [KL2, Lemma 3.4.4] we endow the corresponding completion mentioned above with the corresponding seminorm spectral. And for the corresponding quotient we endow with the corresponding quotient norm.

Then we consider the corresponding deformation by the ring V over \mathbb{Z}_p :

Lemma 9.3.12. (After Kedlaya-Liu [KL2, Lemma 3.4.4]) *Consider the ring A as above which is furthermore assumed to be perfectoid in the sense of [Ked1, Definition 2.1.1], and consider any direct system of faithfully finite étale morphisms as:*

$$W_{0,V} \longrightarrow W_{1,V} \longrightarrow W_{2,V} \longrightarrow \dots,$$

where W_0 is just the corresponding base W . Then the corresponding completion of the infinite level could be decomposed as:

$$W_{0,V} \oplus \widehat{\bigoplus_{k=0}^{\infty} W_{k,V}/W_{k-1,V}}. \quad (9.3.2)$$

Corollary 9.3.13. (After Kedlaya-Liu [KL2, Corollary 3.4.5]) *Starting with an analytic Huber ring which is as in [KL2, Corollary 3.4.5] assumed to be perfectoid in the sense of [Ked1, Definition 2.1.1]. And as in the previous two lemmas we consider an admissible infinite direct system:*

$$W_{0,V} \longrightarrow W_{1,V} \longrightarrow W_{2,V} \longrightarrow \dots,$$

whose infinite level will be assumed to take the corresponding spectral seminorm as in the [KL2, Corollary 3.4.5]. Then carrying the corresponding coefficient V , we have in such situation the corresponding 2-pseudoflatness of the corresponding embedding map:

$$W_{0,V} \rightarrow W_{\infty,V}. \quad (9.3.3)$$

Here $W_{\infty,V}$ denotes the corresponding completion of the limit $\varinjlim_{k \rightarrow \infty} W_{k,V}$.

Proof. See [KL2, Corollary 3.4.5]. □

Now recall that from [KL2] since we do not have to modify the corresponding underlying spatial context, so we will also only have to consider the corresponding stability with respect to étale topology even although we are considering the corresponding profinite étale site in our current section. We first generalize the corresponding Tate acyclicity in [KL2] to the corresponding V -relative situation in our current situation:

Remark 9.3.14. In the following we only consider the corresponding perfectoid ring W .

Proposition 9.3.15. (After Kedlaya-Liu [KL2, Theorem 3.4.6]) *Now we consider the corresponding pro-étale site of X which we denote it by $X_{\text{pét}}$. Then we assume we have a stable basis \mathbb{H} of the corresponding perfectoid subdomains for this profinite étale site where each morphism therein will be étale pseudoflat. Then we consider as in [KL2, Theorem 3.4.6] a corresponding module over $W \widehat{\otimes} V$ which is assumed to be étale-stably*

pseudocoherent. Then consider the corresponding presheaf \tilde{G} attached to this étale-stably pseudocoherent with respect to in our situation the corresponding chosen well-defined basis \mathbb{H} , we have that the corresponding sheaf over some element $Y \in \mathbb{H}$ (that is to say over $\mathcal{O}_{Y_{\text{pét}}}\widehat{\otimes}V$) is acyclic and is acyclic with respect to some Čech covering coming from elements in \mathbb{H} .

Proof. See proposition 9.2.8. □

Definition 9.3.16. (After Kedlaya-Liu [KL2, Definition 3.4.7]) Consider the pro-étale site of X attached to the adic Banach ring (W, W^+) . We will define a sheaf of module G over $\widehat{\mathcal{O}}_{\text{pét}}\widehat{\otimes}V$ to be V -pseudocoherent if locally we can define this as a sheaf attached to a V -étale-stably pseudocoherent module. As in [KL2, Definition 3.4.7], we do not have to consider the corresponding notion of V -profinite-étale-stably pseudocoherent module.

Proposition 9.3.17. (After Kedlaya-Liu [KL2, Theorem 3.4.8]) *Taking the corresponding global section will realize the corresponding equivalence between the following two categories. The first one is the corresponding one of all the V -pseudocoherent sheaves over $\widehat{\mathcal{O}}_{\text{pét}}\widehat{\otimes}V$, while the second one is the corresponding one of all the V -étale-stably pseudocoherent module over $W\widehat{\otimes}V$.*

Proof. As in [KL2, Theorem 3.4.8], the corresponding analog of [KL1, Proposition 9.2.6] applies in the way that the corresponding conditions of the analog of [KL1, Proposition 9.2.6] hold in our current situation. Also see [Ked1, Theorem 2.9.9, Remark 2.9.10 and Lemma 1.10.4]. □

Obviously we have the following analog of [KL2, Corollary 3.4.9]:

Corollary 9.3.18. (After Kedlaya-Liu [KL2, Corollary 3.4.9]) *The following two categories are equivalent. The first is the corresponding category of all V -pseudocoherent sheaves over $\widehat{\mathcal{O}}_{\text{pét}}\widehat{\otimes}V$. The second is the corresponding category of all V -pseudocoherent sheaves over $\mathcal{O}_{\text{ét}}\widehat{\otimes}V$. The corresponding functor is the corresponding pullback along the corresponding morphism of sites $X_{\text{pét}} \rightarrow X_{\text{ét}}$.*

9.4 Descent in General Setting for Analytic Huber Pairs

We now consider the situation where there is no condition on the base (except that we will assume certainly the corresponding sheafiness). We will discuss along [Ked1] the corresponding descent in analytic and étale topology, as well as the corresponding quasi-Stein spaces. The current situation will not assume we are working over \mathbb{Z}_p .

Remark 9.4.1. The corresponding discussion in this section and in the following section is actually an exercise proposed by Kedlaya included in [Ked1], where the corresponding étale situation essentially is proposed by Kedlaya in [Ked1, Discussion after Lemma 1.10.4] while we give the corresponding exposition in both étale and pro-étale situations closely after [KL1] and [KL2].

9.4.1 Étale Topology

Setting 9.4.2. Let (W, W^+) be analytic Huber uniform pair. We now fix a corresponding stable basis \mathbb{H} for the corresponding étale site of the adic space $\mathrm{Spa}(W, W^+)$, locally consisting of the corresponding compositions of rational localizations and the corresponding finite étale morphisms. And as in [Ked1, Hypothesis 1.10.3] we need to assume that the corresponding basis is made up of the corresponding adic spectrum of sheafy rings. Assume the corresponding sheafiness of the Huber ring W .

Remark 9.4.3. One should treat our discussion in this section as essentially some detailization of some exercise predicted in [Ked1].

Definition 9.4.4. (After Kedlaya-Liu [KL2, Definition 2.5.9]) We define the étale stably pseudocoherent module over the corresponding Huber ring W with respect to the corresponding basis \mathbb{H} chosen above for the corresponding étale site of the analytic adic space X . We define a module over W to be an étale stably pseudocoherent module if it is algebraically pseudocoherent (namely formed by the corresponding possibly infinite length resolution of finitely generated and projective modules) and at least complete with respect to the corresponding natural topology and also required to be also complete with respect to the natural topology along some base change with respect to any morphism in \mathbb{H} .

Definition 9.4.5. (After Kedlaya-Liu [KL2, Definition 2.5.9]) Along the previous definition we have the corresponding notion of étale-pseudoflat left (or right respectively) modules with respect to the corresponding chosen basis \mathbb{H} . A such module is defined to be a topological module G over W complete with respect to the natural topology and for any right (or left respectively) étale stably pseudocoherent module, they will jointly give the Tor_1 vanishing.

Remark 9.4.6. Although in this definition we considered the corresponding left or right modules, but essentially speaking this is no different from the usual situation since we are in the corresponding essential site theoretic situation.

Lemma 9.4.7. (After Kedlaya-Liu [KL2, Lemma 2.5.10]) *One could find another basis \mathbb{H}' which is in our situation contained in the corresponding original basis \mathbb{H} such that any morphism in \mathbb{H} could be étale-pseudoflat with respect \mathbb{H}' or just \mathbb{H} itself.*

Proof. The derivation of such new basis is actually along the same way as in [KL2, Lemma 2.5.10] since the corresponding compositions of rational localizations and the finite étales are actually satisfying the corresponding conditions in the statement by [TX2, Theorem 2.12] over the analytic topology. In general one just show any general morphism will also decompose in the same way, which one can prove certainly as in [KL2, Lemma 2.5.10], where one instead consider in the current context the corresponding basis spreading result in [Ked1, Lemma 1.10.4]. \square

Proposition 9.4.8. (After Kedlaya-Liu [KL2, Theorem 2.5.11]) *Consider the site $X_{\text{ét}}$ and consider the basis \mathbb{H} . Take any étale stably pseudocoherent module M over W . Consider the corresponding presheaf by taking the inverse limit throughout all the corresponding base change along morphisms in \mathbb{H} . Then we have the corresponding acyclicity of the presheaf over any element in \mathbb{H} and any covering of this element by elements in \mathbb{H} .*

Proof. As in [KL2, Theorem 2.5.11] apply the corresponding [KL1, Proposition 8.2.21]. \square

Proposition 9.4.9. (After Kedlaya-Liu [KL2, Lemma 2.5.13]) *The corresponding glueing of étale-stably pseudocoherent modules holds in this current situation over W along binary morphisms (namely along binary rational decomposition).*

Proof. We adapt the argument of [KL2, Lemma 2.5.13] to our current situation. Take the corresponding map to be $W \rightarrow W_1 \oplus W_2$, and we denote the two spaces associated W_1 and W_2 by Y_1 and Y_2 . Establish a corresponding descent datum of V -étale-stably pseudocoherent modules along this decomposition of $\mathrm{Spa}(W, W^+)$. One can realize this by acyclicity a sheaf G over the whole space $\mathrm{Spa}(W, W^+)$. Then we have for each $k = 1, 2$ that the corresponding identification of $H^0(Y_{k,\mathrm{ét}}, G)$ with $H^0(Y_k, G)$ and the vanishing of the i -th cohomology for higher i . By the corresponding results we have in the corresponding rational localization situation we have vanishing of the corresponding $H^i(\mathrm{Spa}(W, W^+), G), i \geq 1$. Now consider a finite free covering C of G and take the kernel K to form the corresponding exact sequence:

$$0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0,$$

which will give rise to the following short exact sequence:

$$0 \longrightarrow K(\mathrm{Spa}(W, W^+)) \longrightarrow C(\mathrm{Spa}(W, W^+)) \longrightarrow G(\mathrm{Spa}(W, W^+)) \longrightarrow 0.$$

To finish we have also to show the corresponding resulting global section is actually étale-stably pseudocoherent. For this, we refer the readers to [TX2, Lemma 2.22].

□

Proposition 9.4.10.

(Kedlaya [Ked1, Below Lemma 1.10.4])

(After Kedlaya-Liu [KL2, Theorem 2.5.14]) *Taking the corresponding global section will realize the equivalence between the following two categories. The first is the one of all the corresponding pseudocoherent sheaves over $\mathcal{O}_{\mathrm{ét}}$. The second is the one of all the corresponding étale-stably pseudocoherent modules.*

Proof. This is by considering and applying the corresponding [Ked1, Lemma 1.10.4] in our current context. The refinement comes from the acyclicity, the transitivity is straightforward, the corresponding binary rational localization situation is the previous proposition, while the corresponding last condition will basically come from the corresponding f.p. descent [SGAI, Chapitre VIII].

□

9.4.2 Pro-étale Topology

Setting 9.4.11. Let (W, W^+) be an analytic Huber ring. Assume that the ring W is sheafy. We assume p is a topologically nilpotent element.

Remark 9.4.12. Although it seems that the current discussion is basically outside the corresponding deformed setting, but the foundation here may be generalized to the deformed setting by taking the corresponding completed tensor product over \mathbb{F}_1 as in [BBBK].

Remark 9.4.13. The corresponding glueing of the corresponding pseudocoherent sheaves over the corresponding pro-étale site does not need the corresponding notions on the corresponding stability beyond the corresponding étale-stably pseudocoherence.

Lemma 9.4.14. (After Kedlaya-Liu [KL2, Proposition 3.4.3]) *Suppose that we are taking (W, W^+) to be perfectoid now in the sense of [Ked1, Definition 2.1.1] (note that we are considering the corresponding context of analytic Huber pair situation). Then we have that over the corresponding étale site, the corresponding group $H^i(\mathrm{Spa}(W, W^+)_{\mathrm{pét}}, \mathcal{O})$ vanishes for each $i > 0$, while we have that the corresponding group $H^0(\mathrm{Spa}(W, W^+)_{\mathrm{pét}}, \mathcal{O})$ is just isomorphic to W .*

Proof. See [KL2, Proposition 3.4.3]. □

Now we proceed to consider the corresponding pro-étale topology. First we recall the following result from [KL2, Lemma 3.4.4] which does hold in our situation since we did not change much on the corresponding underlying adic spaces.

Lemma 9.4.15. (Kedlaya-Liu [KL2, Lemma 3.4.4]) *Consider the ring W as above which is furthermore assumed to be perfectoid in the sense of instead [Ked1, Definition 2.1.1], and consider any direct system of faithfully finite étale morphisms as:*

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \dots,$$

where W_0 is just the corresponding base W . Then the corresponding completion of the infinite level could be decomposed as:

$$W_0 \oplus \widehat{\bigoplus_{k=0}^{\infty} W_k/W_{k-1}}. \tag{9.4.1}$$

As in [KL2, Lemma 3.4.4] we endow the corresponding completion mentioned above with the corresponding seminorm spectral. And for the corresponding quotient we endow with the corresponding quotient norm.

Corollary 9.4.16. (After Kedlaya-Liu [KL2, Corollary 3.4.5]) *Starting with an analytic Huber ring which is as in [KL2, Corollary 3.4.5] assumed to be perfectoid in the sense of [Ked1, Definition 2.1.1]. And as in the previous two lemmas we consider an admissible infinite direct system:*

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \dots,$$

whose infinite level will be assumed to take the corresponding spectral seminorm as in the [KL2, Corollary 3.4.5]. Then we have in such situation the corresponding 2-pseudoflatness of the corresponding embedding map:

$$W_0 \rightarrow W_\infty. \tag{9.4.2}$$

Here W_∞ denotes the corresponding completion of the limit $\varinjlim_{k \rightarrow \infty} W_k$.

Proof. See [KL2, Corollary 3.4.5]. □

Now recall that from [KL2] since we do not have to modify the corresponding underlying spatial context, so we will also only have to consider the corresponding stability with respect to étale topology even although we are considering the corresponding profinite étale site in our current section. We first generalize the corresponding Tate acyclicity in [KL2] to the corresponding our current situation.

Remark 9.4.17. In the following we only consider the base spaces which are perfectoid.

Proposition 9.4.18. (After Kedlaya-Liu [KL2, Theorem 3.4.6]) *Now we consider the corresponding pro-étale site of X which we denote it by $X_{\text{pét}}$. Then we assume we have a stable basis \mathbb{H} of the corresponding perfectoid subdomains for this profinite étale site where each morphism therein will be étale pseudoflat. Then we consider as in [KL2,*

Theorem 3.4.6] a corresponding module over W which is assumed to be étale-stably pseudocoherent. Then consider the corresponding presheaf \tilde{G} attached to this étale-stably pseudocoherent with respect to in our situation the corresponding chosen well-defined basis \mathbb{H} , we have that the corresponding sheaf over some element $Y \in \mathbb{H}$ (that is to say over $\mathcal{O}_{Y_{\text{pét}}}$) is acyclic and is acyclic with respect to some Čech covering coming from elements in \mathbb{H} .

Proof. See proposition 9.2.8. □

Definition 9.4.19. (After Kedlaya-Liu [KL2, Definition 3.4.7]) Consider the pro-étale site of X attached to the analytic Huber pair (W, W^+) . We will define a sheaf of module G over $\hat{\mathcal{O}}_{\text{pét}}$ to be pseudocoherent if locally we can define this as a sheaf attached to an étale-stably pseudocoherent module. As in [KL2, Definition 3.4.7], we do not have to consider the corresponding notion of profinite-étale-stably pseudocoherent modules.

Proposition 9.4.20.

(Kedlaya [Ked1, Section 3.8], after Kedlaya-Liu [KL2, Theorem 3.4.8]) Taking the corresponding global section will realize the corresponding equivalence between the following two categories. The first one is the corresponding one of all the pseudocoherent sheaves over $\hat{\mathcal{O}}_{\text{pét}}$, while the second one is the corresponding one of all the étale-stably pseudocoherent modules over W .

Proof. As in [KL2, Theorem 3.4.8], the corresponding analog of [KL1, Proposition 9.2.6] applies in the way that the corresponding conditions of the analog of [KL1, Proposition 9.2.6] hold in our current situation. Also see [Ked1, Theorem 2.9.9, Remark 2.9.10 and Lemma 1.10.4]. □

Obviously we have the following analog of [KL2, Corollary 3.4.9]:

Corollary 9.4.21. (After Kedlaya-Liu [KL2, Corollary 3.4.9]) The following two categories are equivalent. The first is the corresponding category of all pseudocoherent sheaves over $\hat{\mathcal{O}}_{\text{pét}}$. The second is the corresponding category of all pseudocoherent sheaves over $\mathcal{O}_{\text{ét}}$. The corresponding functor is the corresponding pullback along the corresponding morphism of sites $X_{\text{pét}} \rightarrow X_{\text{ét}}$.

9.5 Descent in General Setting for Analytic Adic Banach Rings

We now translate the results in previous section to the corresponding analytic adic Banach context. Again everything is essentially as proposed in [Ked1, Below Lemma 1.10.4, also see Section 3.8].

9.5.1 Étale Topology

Setting 9.5.1. Let (W, W^+) be analytic adic Banach ring. We now fix a corresponding stable basis \mathbb{H} for the corresponding étale site of the adic space $\mathrm{Spa}(W, W^+)$, locally consisting of the corresponding compositions of rational localizations and the corresponding finite étale morphisms. And as in [Ked1, Hypothesis 1.10.3] we need to assume that the corresponding basis is made up of the corresponding adic spectrum of sheafy rings. Assume the corresponding sheafiness of the Huber ring W .

Definition 9.5.2. (After Kedlaya-Liu [KL2, Definition 2.5.9]) We define the étale-stably pseudocoherent module over the corresponding adic Banach W with respect to the corresponding basis \mathbb{H} chosen above for the corresponding étale site of the analytic adic space X . We define a module over W to be an étale stably pseudocoherent module if it is algebraically pseudocoherent (namely formed by the corresponding possibly infinite length resolution of finitely generated and projective modules) and at least complete with respect to the corresponding natural topology and also required to be also complete with respect to the natural topology along some base change with respect to any morphism in \mathbb{H} .

Definition 9.5.3. (After Kedlaya-Liu [KL2, Definition 2.5.9]) Along the previous definition we have the corresponding notion of étale-pseudoflat left (or right respectively) modules with respect to the corresponding chosen basis \mathbb{H} . A such module is defined to be a topological module G over W complete with respect to the natural topology and for any right (or left respectively) étale stably pseudocoherent module, they will jointly give the Tor_1 vanishing.

Lemma 9.5.4. (After Kedlaya-Liu [KL2, Lemma 2.5.10]) *One could find another*

basis \mathbb{H}' which is in our situation contained in the corresponding original basis \mathbb{H} such that any morphism in \mathbb{H} could be étale-pseudoflat with respect \mathbb{H}' or just \mathbb{H} itself.

Proof. See lemma 9.4.7. □

Proposition 9.5.5. (After Kedlaya-Liu [KL2, Theorem 2.5.11]) *In our current analytic adic Banach situation, consider the site $X_{\text{ét}}$ and consider the basis \mathbb{H} . Take any étale stably pseudocoherent module M over W . Consider the corresponding presheaf by taking the inverse limit throughout all the corresponding base change along morphisms in \mathbb{H} . Then we have the corresponding acyclicity of the presheaf over any element in \mathbb{H} and any covering of this element by elements in \mathbb{H} .*

Proof. As in [KL2, Theorem 2.5.11] apply the corresponding [KL1, Proposition 8.2.21]. □

Proposition 9.5.6. (After Kedlaya-Liu [KL2, Lemma 2.5.13]) *In our current analytic adic Banach situation, the corresponding glueing of étale-stably pseudocoherent modules holds in this current situation over W along binary morphisms (namely along binary rational decomposition).*

Proof. We adapt the argument of [KL2, Lemma 2.5.13] to our current situation over analytic adic Banach rings. Take the corresponding map to be $W \rightarrow W_1 \oplus W_2$, and we denote the two spaces associated W_1 and W_2 by Y_1 and Y_2 . Establish a corresponding descent datum of V -étale-stably pseudocoherent modules along this decomposition of $\text{Spa}(W, W^+)$. One can realize this by acyclicity a sheaf G over the whole space $\text{Spa}(W, W^+)$ Then we have for each $k = 1, 2$ that the corresponding identification of $H^0(Y_{k,\text{ét}}, G)$ with $H^0(Y_k, G)$ and the vanishing of the i -th cohomology for higher i . By the corresponding results we have in the corresponding rational localization situation we vanishing of the corresponding $H^i(\text{Spa}(W, W^+), G), i \geq 1$. Now consider a finite free covering C of G and take the kernel K to form the corresponding exact sequence:

$$0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0,$$

which will give rise to the following short exact sequence:

$$0 \longrightarrow K(\text{Spa}(W, W^+)) \longrightarrow C(\text{Spa}(W, W^+)) \longrightarrow G(\text{Spa}(W, W^+)) \longrightarrow 0.$$

To finish we have also to show the corresponding resulting global section is actually étale-stably pseudocoherent. For this, we refer the readers to [TX2, Lemma 2.22].

□

Proposition 9.5.7. (After Kedlaya-Liu [KL2, Theorem 2.5.14]) *In our current analytic adic Banach situation, taking the corresponding global section will realize the equivalence between the following two categories. The first is the one of all the corresponding pseudocoherent sheaves over $\mathcal{O}_{\text{ét}}$. The second is the one of all the corresponding étale-stably pseudocoherent modules.*

Proof. This is by considering and applying the corresponding [Ked1, Lemma 1.10.4] in our current context. The refinement comes from the acyclicity, the transitivity is straightforward, the corresponding binary rational localization situation is the previous proposition, while the corresponding last condition will basically comes from the corresponding f.p. descent [SGAI, Chapitre VIII].

□

9.5.2 Pro-étale Topology

Setting 9.5.8. Let (W, W^+) be an analytic perfectoid adic Banach ring. Assume that the ring W is sheafy. We assume p is a topologically nilpotent element.

Remark 9.5.9. As in the corresponding analytic Huber ring situation, the corresponding glueing of the corresponding pseudocoherent sheaves over the corresponding pro-étale site does not need the corresponding notions on the corresponding stability beyond the corresponding étale-stably pseudocoherence.

We translate the corresponding notions of perfectoid rings to the current context from [Ked1, Definition 2.1.1]:

Definition 9.5.10. (Kedlaya [Ked1, Definition 2.1.1]) Consider a general analytic adic Banach ring (W, W^+) , we will call it perfectoid if there exists a definition ideal $d \subset W^+$ such that d^p contains p and we have the corresponding Frobenius map from W^+/d to W^+/d^p is required to be surjective.

Lemma 9.5.11. (After Kedlaya-Liu [KL2, Proposition 3.4.3]) *Suppose that we are taking (W, W^+) to be perfectoid now in the sense of definition 9.5.10 (note that we are considering the corresponding context of analytic adic Banach situation). Then we have that over the corresponding étale site, the corresponding group $H^i(\mathrm{Spa}(W, W^+)_{\mathrm{pét}}, \mathcal{O})$ vanishes for each $i > 0$, while we have that the corresponding group $H^0(\mathrm{Spa}(W, W^+)_{\mathrm{pét}}, \mathcal{O})$ is just isomorphic W .*

Proof. See [KL2, Proposition 3.4.3]. □

Now we proceed to consider the corresponding pro-étale topology. First we recall the following result from [KL2, Lemma 3.4.4].

Lemma 9.5.12. (Kedlaya-Liu [KL2, Lemma 3.4.4]) *Consider the ring W as above which is furthermore assumed to be perfectoid in the sense of instead definition 9.5.10, and consider any direct system of faithfully finite étale morphisms as:*

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \dots,$$

where W_0 is just the corresponding base W . Then the corresponding completion of the infinite level could be decomposed as:

$$W_0 \oplus \widehat{\bigoplus_{k=0}^{\infty} W_k/W_{k-1}}. \tag{9.5.1}$$

As in [KL2, Lemma 3.4.4] we endow the corresponding completion mentioned above with the corresponding seminorm spectral. And for the corresponding quotient we endow with the corresponding quotient norm.

Corollary 9.5.13. (After Kedlaya-Liu [KL2, Corollary 3.4.5]) *Starting with an analytic adic Banach ring which is as in [KL2, Corollary 3.4.5] assumed to be perfectoid in the sense of definition 9.5.10. And as in the previous two lemmas we consider an admissible infinite direct system:*

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \dots,$$

whose infinite level will be assumed to take the corresponding spectral seminorm as in the [KL2, Corollary 3.4.5]. Then we have in such situation the corresponding 2-pseudoflatness

of the corresponding embedding map:

$$W_0 \rightarrow W_\infty. \tag{9.5.2}$$

Here W_∞ denotes the corresponding completion of the limit $\varinjlim_{k \rightarrow \infty} W_k$.

Proof. See [KL2, Corollary 3.4.5]. □

Now recall that from [KL2] since we do not have modified the corresponding underlying spatial context, so we will also only have to consider the corresponding stability with respect to étale topology even although we are considering the corresponding profinite étale site in our current section. We first generalize the corresponding Tate acyclicity in [KL2] to the corresponding our current situation:

Proposition 9.5.14. (After Kedlaya-Liu [KL2, Theorem 3.4.6]) *Now we consider the corresponding pro-étale site of X which we denote it by $X_{\text{pét}}$. Then we assume we have a stable basis \mathbb{H} of the corresponding perfectoid subdomains for this profinite étale site where each morphism therein will be étale pseudoflat. Then we consider as in [KL2, Theorem 3.4.6] a corresponding module over W which is assumed to be étale-stably pseudocoherent. Then consider the corresponding presheaf \tilde{G} attached to this étale-stably pseudocoherent with respect to in our situation the corresponding chosen well-defined basis \mathbb{H} , we have that the corresponding sheaf over some element $Y \in \mathbb{H}$ (that is to say over $\mathcal{O}_{Y_{\text{pét}}}$) is acyclic and is acyclic with respect to some Čech covering coming from elements in \mathbb{H} .*

Proof. See proposition 9.2.8. □

Definition 9.5.15. (After Kedlaya-Liu [KL2, Definition 3.4.7]) Consider the pro-étale site of X attached to the adic Banach ring (W, W^+) . We will define a sheaf of module G over $\widehat{\mathcal{O}}_{\text{pét}}$ to be pseudocoherent if locally we can define this as a sheaf attached to an étale-stably pseudocoherent module. As in [KL2, Definition 3.4.7], we do not have to consider the corresponding notion of profinite-étale-stably pseudocoherent module.

Proposition 9.5.16. (After Kedlaya-Liu [KL2, Theorem 3.4.8]) *Taking the corresponding global section will realize the corresponding equivalence between the following two categories. The first one is the corresponding one of all the pseudocoherent sheaves over $\widehat{\mathcal{O}}_{\text{pét}}$, while the second one is the corresponding one of all the étale-stably pseudocoherent modules over W .*

Proof. As in [KL2, Theorem 3.4.8], the corresponding analog of [KL1, Proposition 9.2.6] applies in the way that the corresponding conditions of the analog of [KL1, Proposition 9.2.6] hold in our current situation. Also see [Ked1, Theorem 2.9.9, Remark 2.9.10 and Lemma 1.10.4]. Certainly in this situation we need to translate the corresponding statements in [Ked1, Theorem 2.9.9, Remark 2.9.10 and Lemma 1.10.4] to the corresponding adic Banach setting. \square

Obviously we have the following analog of [KL2, Corollary 3.4.9]:

Corollary 9.5.17. (After Kedlaya-Liu [KL2, Corollary 3.4.9]) *The following two categories are equivalent. The first is the corresponding category of all pseudocoherent sheaves over $\widehat{\mathcal{O}}_{\text{pét}}$. The second is the corresponding category of all pseudocoherent sheaves over $\mathcal{O}_{\text{ét}}$. The corresponding functor is the corresponding pullback along the corresponding morphism of sites $X_{\text{pét}} \rightarrow X_{\text{ét}}$.*

9.5.3 Quasi-Stein Spaces

We now generalize our discussion in [TX2] around pseudocoherent sheaves over quasi-Stein spaces to the general base situation without the corresponding restrictive assumption that we are working over \mathbb{Z}_p .

Setting 9.5.18. Assume that we are working over some quasi-Stein space X which could be written as the corresponding direct limit of analytic adic Banach affinoids: $X := \varinjlim_i X_i$ such that X_i could be written as the corresponding analytic affinoid $\text{Spa}(W_i, W_i^+)$.

Lemma 9.5.19. (After Kedlaya-Liu [KL2, Lemma 2.6.3]) *Consider a compatible family of Banach modules over the projective system $\mathcal{O}_{X_i}(X_i)$. Suppose that the corresponding transition map $p_i : \mathcal{O}_{X_i}(X_i) \widehat{\otimes}_{\mathcal{O}_{X_{i+1}}(X_{i+1})} M_{i+1} \rightarrow M_i$ is surjective. Then we have: I. The corresponding global section of M throughout the limit $\mathcal{O}(X) = \varprojlim_i \mathcal{O}_{X_i}(X_i)$ is dense in each section M_i for any $i \geq 0$; II. The corresponding vanishing of $R^1 \varprojlim_{i \rightarrow \infty}$ holds in our situation. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. This is basically the general version of the corresponding result in [KL2, Lemma 2.6.3]. One could prove this in the same way. \square

Lemma 9.5.20. (After Kedlaya-Liu [KL2, Corollary 2.6.4]) *Consider a compatible family of Banach modules over the projective system $\mathcal{O}_{X_i}(X_i)$. Suppose that each member in the family is stably-pseudocoherent. And suppose that the corresponding transition map $p_i : \mathcal{O}_{X_i}(X_i) \widehat{\otimes}_{\mathcal{O}_{X_{i+1}}(X_{i+1})} M_{i+1} \rightarrow M_i$ is strictly isomorphism. Then we have in our situation the corresponding projection from the global section $M = \varprojlim_i M_i$ to each M_i (for each $i \geq 0$) is then isomorphism in our current situation as in [KL2, Corollary 2.6.4]. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. This is basically the general version of the corresponding result in [KL2, Corollary 2.6.4]. One could prove this in the same way. \square

Proposition 9.5.21. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *With the corresponding notations as above, we have that the global section of any pseudocoherent sheaf M over X will be dense in the section over any quasicompact. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. See [KL2, Theorem 2.6.5]. \square

Proposition 9.5.22. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *With the corresponding notations as above, we have that the global section of any pseudocoherent sheaf M over X will finitely generate each fiber M_x over $\mathcal{O}_{X,x}$ for any $x \in X$. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. See [KL2, Theorem 2.6.5]. \square

Proposition 9.5.23. (After Kedlaya-Liu [KL2, Theorem 2.6.5]) *With the corresponding notations as above, we have that the corresponding sheaf cohomology of any pseudocoherent sheaf M over X admits vanishing for degree bigger than 0. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. See [KL2, Theorem 2.6.5]. \square

Proposition 9.5.24. (After Kedlaya-Liu [KL2, Corollary 2.6.6]) *For any pseudocoherent sheaf M over the space X , we have in our current situation that the exact functor from the corresponding category of all the pseudocoherent sheaves to the corresponding one of all the corresponding $\mathcal{O}_X(X)$ -modules through the corresponding taking the global section. (Parallel statement in analytic Huber ring situation holds as well.)*

Proposition 9.5.25. (After Kedlaya-Liu [KL2, Corollary 2.6.8]) *For any pseudo-coherent sheaf M admitting a structure of vector bundles over the space X , we have in our current situation that the corresponding sufficient and necessary condition for the global section to be finite projective is exactly that the global section of M is finitely generated. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. See [KL2, Corollary 2.6.8], it is straightforward to exact a global splitting from the local ones over quasi-compacts. □

Proposition 9.5.26. (After Kedlaya-Liu [KL2, Proposition 2.6.17]) *Assume that the space X is basically m -uniform in the sense of [KL2]. Then we have the corresponding finiteness of the global section is equivalent to to the corresponding uniform finiteness through out all $X_i, i = 0, 1, 2, \dots$. (Parallel statement in analytic Huber ring situation holds as well.)*

Proof. See the proof given in [TX2, Theorem 6.27]. □

9.6 Applications

9.6.1 Application to Noetherian Cases

The corresponding noetherian situation is expected to be in some sense better than just being pseudoflat with respect to the corresponding rational localization.

Setting 9.6.1. We now work with the corresponding analytic adic Banach rings over \mathbb{Q}_p or $\mathbb{F}_p((t))$ as the corresponding base spaces. The deformation will happen along some Banach ring over \mathbb{Q}_p or $\mathbb{F}_p((t))$. And moreover we have to assume that the corresponding noetherianness preserves under the corresponding deformation over some ring V . Namely we have to assume that $W \widehat{\otimes}_* V$ is always noetherian, for instance this could be achieved when we have that V over \mathbb{Q}_p comes from truncations of distribution algebra over some p -adic Lie groups by considering some further p -adic microlocal analysis.

Lemma 9.6.2. *With the notations in the [KL2, Lemma 2.4.10], we have that the corresponding morphisms $W \widehat{\otimes} V \rightarrow B_1 \widehat{\otimes} V$, $W \widehat{\otimes} V \rightarrow B_2 \widehat{\otimes} V$ and $W \widehat{\otimes} V \rightarrow B_{12} \widehat{\otimes} V$ are 2-pseudoflat. And furthermore in our current context they are actually flat.*

Proof. In our current situation, the corresponding 2-pseudoflatness is achieved since this is already true in the corresponding more general situation. However for finite presented modules (we do not have to consider the topological issues since we are in the noetherian situation) this is already flat which implies for finitely generated modules (we do not have to consider the topological issues since we are in the noetherian situation) this is already flat as well. Then for any module which could be written as injective limit of finite ones, the result holds. \square

Corollary 9.6.3. *The corresponding rational localization with respect to the adic Banach ring (W, W^+) is flat (namely for all the corresponding finitely presented module over $W \widehat{\otimes} V$).*

Proposition 9.6.4. *Then we consider the corresponding presheafification of any finitely generated module M over $W \widehat{\otimes} V$, to be more precise over $\mathrm{Spa}(W, W^+)$ we will define the corresponding presheaf \widetilde{M} by taking the inverse limit of the base changes of M throughout all the rational localizations of W . Then we have that the Tate glueing property holds in our current situation for such presheaf \widetilde{M} .*

Proof. This reduces to the previous lemma by using [KL1, Proposition 2.4.20, Proposition 2.4.21]. \square

The corresponding glueing finitely generated modules is also achieved in current noetherian situation. First we consider the corresponding result:

Proposition 9.6.5. *The descent along the morphism $W \rightarrow B_1 \oplus B_2$ is effective as long as one focuses on the category of all the finitely presented modules over $W \widehat{\otimes} V$.*

Proof. This is the corresponding noetherian implication of the corresponding result in [TX2, Lemma 2.14]. \square

Proposition 9.6.6. *The corresponding functor of taking the corresponding global section will give rise to the corresponding equivalence between the corresponding V -coherent sheaves and the V finitely presented modules.*

Proof. This is also the corresponding noetherian implication of the corresponding result in [TX2, Theorem 2.15]. \square

Then fixing a stable basis \mathbb{H} for the étale site of the space $\text{Spa}(W, W^+)$, then we have the results over the étale site as well:

Proposition 9.6.7. *We consider the corresponding presheafification of any finitely generated module M over $W \widehat{\otimes} V$, to be more precise over $\text{Spa}(W, W^+)$ we will define the corresponding presheaf \widetilde{M} by taking the inverse limit of the base changes of M throughout all the member in the basis \mathbb{H} . Then we have that the Tate glueing property holds in our current situation for such presheaf \widetilde{M} .*

Proposition 9.6.8. *The descent (in the étale topology) along the morphism $W \rightarrow B_1 \oplus B_2$ is effective as long as one focuses on the category of all the finitely presented modules over $W \widehat{\otimes} V$.*

Proof. This is the corresponding noetherian implication of the corresponding result in [TX2, Lemma 2.22]. \square

Proposition 9.6.9. *The corresponding functor (in the étale topology) of taking the corresponding global section will give rise to the corresponding equivalence between the corresponding V -coherent sheaves and the V finitely presented modules.*

Proof. This is also the corresponding noetherian implication of the corresponding result in [TX2, Theorem 2.23]. \square

9.6.2 Application to Descent over Noncommutative ∞ -Analytic Prestacks after Bambozzi-Ben-Bassat-Kremnizer

We now contact with the corresponding derived analytic spaces from [BBBK]. Recall that we have the categories $\text{Simp}(\text{Ind}^m(\text{BanSets}_E))$ and $\text{Simp}(\text{Ind}(\text{BanSets}_E))$ of the corresponding simplicial sets in the corresponding inductive category of Banach sets over $E = \mathbb{Q}_p, \mathbb{F}_p((t))$.

Theorem 9.6.10. (Bambozzi-Ben-Bassat-Kremnizer) *The categories $\text{Simp}(\text{Ind}^m(\text{BanSets}_E))$ and $\text{Simp}(\text{Ind}(\text{BanSets}_E))$ admit symmetric monoidal model categorical structure.*

Therefore based on this nice structure [BBBK] defined the corresponding ∞ -categories of \mathbb{E}_∞ -rings:

$$\text{sComm}(\text{Simp}(\text{Ind}^m(\text{BanSets}_E))) \tag{9.6.1}$$

$$\text{sComm}(\text{Simp}(\text{Ind}(\text{BanSets}_E))). \tag{9.6.2}$$

We now consider the corresponding some object A which is a corresponding ∞ -locally convex ring in the categories above and we consider the corresponding object in the corresponding opposite category we call that $\text{Spec}A$. And we consider the corresponding homotopy Zariski topology, which allows us to talk about the corresponding ∞ -analytic stacks.

Now recall that a connective \mathbb{E}_1 -ring is called noetherian if we have that $\pi_0(A)$ is noetherian and $\pi_n(A)$ is basically finitely generated over $\pi_0(A)$.

Remark 9.6.11. The corresponding Koszul simplicial Banach ring considered in [BK] is actually concentrated in the nonpositive degrees¹, in the cohomological convention,

¹Thanks for Federico Bambozzi for reminding us of this.

namely $H^{-n}(\cdot) = 0$ for $n < 0$. But as in any abstract homotopy theory we consider the corresponding conventional transformation:

$$\pi_n := H^{-n}, n \geq 0. \tag{9.6.3}$$

Now we consider an ∞ -analytic stack X as considered in [BBBK]. And we consider the corresponding ringed site attached to X under the corresponding homotopy Zariski topology, denoted by (X, \mathcal{O}_X) .

Example 9.6.12. The examples which are very interesting to us are the corresponding ∞ -Huber spectra constructed from any Banach rings over E from [BK] by using Koszul complex. The corresponding classical sheafness issue could be really forgotten.

Now we consider the corresponding \mathbb{E}_1 -ring A in the corresponding categories $\text{Simp}(\text{Ind}^m(\text{BanSets}_E))$ and $\text{Simp}(\text{Ind}(\text{BanSets}_E))$. We make the following assumption:

Assumption 9.6.13. We assume that all the ring objects below are noetherian, as \mathbb{E}_1 -rings.

Suppose now we have four \mathbb{E}_1 Banach rings A, A_1, A_2, A_{12}^2 in $\text{Simp}(\text{Ind}^m(\text{BanSets}_E))$ and $\text{Simp}(\text{Ind}(\text{BanSets}_E))$ respectively. And we assume that we have the following short strictly exact sequence in the sense of a glueing square:

$$0 \longrightarrow \pi_0(A) \longrightarrow \pi_0(A_1) \oplus \pi_0(A_2) \longrightarrow \pi_0(A_{12}) \longrightarrow 0,$$

and we assume that the image of $\pi_0(A_i)$ in $\pi_0(A_{12})$ is dense for each $i = 1, 2$. And we assume the corresponding morphism $\pi_0(A_i) \rightarrow \pi_0(A_{12})$ is flat for $i = 1, 2$. And we assume that A, A_1, A_2, A_{12} form the corresponding derived glueing sequence.

Example 9.6.14. One can construct the following example. First consider any glueing sequence of Banach rings (not required to be sheafy or commutative) over \mathbb{Q}_p :

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \longrightarrow \Pi_{12} \longrightarrow 0.$$

This is very natural in the corresponding situation for example where we deform from a corresponding nice short exact sequence of commutative sheafy rings of the same type,

²Namely all the homotopy groups have to be Banach instead of Bornological or ind-Banach.

but in any rate we do not require the corresponding commutativity or the corresponding sheafyness. Then take any derived global section of some $\mathrm{Spa}^h(R)$ from [BK], denoted by R^h . Over \mathbb{Q}_p , we have the following short strictly exact sequence:

$$0 \longrightarrow \Pi \widehat{\otimes}_{\mathbb{Q}_p} \pi_0(R^h) \longrightarrow \Pi_1 \widehat{\otimes}_{\mathbb{Q}_p} \pi_0(R^h) \oplus \Pi_2 \widehat{\otimes}_{\mathbb{Q}_p} \pi_0(R^h) \longrightarrow \Pi_{12} \widehat{\otimes}_{\mathbb{Q}_p} \pi_0(R^h) \longrightarrow 0,$$

but since the corresponding ring $\Pi, \Pi_1, \Pi_2, \Pi_{12}$ will actually realize the situation where $\Pi_* \widehat{\otimes} \pi_0(R^h) \xrightarrow{\sim} \pi_0(\Pi_* \widehat{\otimes}^{\mathbb{L}} R^h)$, we will have the desired situation as long as restrict to now the noetherian situation.

Conjecture 9.6.15. *The map $A \rightarrow \prod_{i=1,2} A_i$ is an effective descent morphism with respect to the finitely presented left module spectra (meaning we have finitely presented π_0).*

Proposition 9.6.16. *The operation of taking equalizer along the corresponding map $\prod_{i=1,2} A_i \rightarrow A_{12}$ preserves the property of being finitely presented.*

Proof. The situation where we deform from the corresponding \mathbb{E}_∞ by some noncommutative deformation could be implied by [TX2, Lemma 2.14]. While in general this follows from [TX4, Lemma 6.83]. Note that we achieve the finiteness for each $\pi_n(M)$ just by application of the results we know in the classical deformed situation, which is because the connecting homomorphism vanishes on each level. \square

What could happen in the commutative setting is actually also interesting. We have the following example in mind:

Example 9.6.17. Consider the context of the previous example, but assume that all the rings involved are Banach commutative. First we consider over \mathbb{Q}_p :

$$0 \longrightarrow \Pi \longrightarrow \Pi_1 \oplus \Pi_2 \longrightarrow \Pi_{12} \longrightarrow 0,$$

a strictly exact sequence of sheafy rings. We now deform this along some spectrum $\mathrm{Spa}^h(R)$ in [BK] which will produce a very interesting situation where we can actually obtain desired situation for the descent of finitely presented module spectra. And note that in commutative setting we could also have more geometric contexts to rely on as in [BK] and [BBBK]. This may have contact with the corresponding derived Galois deformation theory of [GV] for instance by considering the corresponding simplicial pro-Artin rings.

Remark 9.6.18. The corresponding machinery from [CS] should definitely reflect similar things here even in the \mathbb{E}_1 -ring context. Also we would like to mention in the commutative setting that actually the descent for certain quasi-coherent modules are also considered extensively in [BBK]. Strikingly the ideas in [BBK] (although developed in a quite commutative setting) come from partially work from Kontsevich-Rosenberg [KR] namely essentially the noncommutative descent. Our feeling is that definitely the descent for certain quasi-coherent modules in [BBBK] could be established to noncommutative setting in some form both in the archimedean and nonarchimedean situations.

This chapter is based on the following paper, where the author of this dissertation is the main author:

- Tong, Xin. "Period Rings with Big Coefficients and Applications III." arXiv preprint arXiv:arXiv:2102.10692(2021).

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Chapter 10

Topics on Geometric and Representation Theoretic Aspects of Period Rings I

10.1 Introduction

10.1.1 General Perfectoids and Preperfectoids

Our previous work on Hodge-Iwasawa theory aimed at the corresponding deformation of the corresponding Hodge structure after [KL1] and [KL2]. The corresponding application in our mind targets the corresponding noncommutative Iwasawa theory and the corresponding p -adic local systems with general Banach coefficients. The corresponding background of the foundations on the analytic geometry comes from essentially [KL1] where Banach analog of Fontaine's perfectoid spaces in the Tate case was defined and studied extensively.

The corresponding analytic Huber analog of the main results of [KL1] was initiated in [Ked1] where the analytic Huber analog of perfectoids are defined, and the analytic Huber analog of the corresponding perfectoid correspondence is established and the corresponding descent for vector bundles and stably-pseudocoherent sheaves is established in the analytic topology.

In our current development of the corresponding Hodge-Iwasawa theory we consider the corresponding generality at least parallel to [Ked1]. We now consider the corresponding analytic analog of our previous consideration which generalized the corresponding [KL1] and [KL2] to the corresponding big coefficient situation. Without deformation in our sense [KL1] and [KL2] have already considered the corresponding free of trivial norm cases in certain context and considerations.

Certainly inverting p takes back us to the Tate situation, but what we know is that this is at least not included in our previous consideration, since the relative Robba rings are not the same (even in our current situation many are still Tate as their owns). So we have to basically establish the corresponding parallel theory, in order to treat the situation where the base spaces or rings are not Tate. And more importantly the integral Robba rings are not Tate, which needs to be discussed in order to apply to the situation where the base spaces or rings are not Tate.

With the corresponding notations in Chapter 5, we have:

Theorem 10.1.1. *Over A -relative preperfectoid Robba rings $\tilde{\Pi}_{?,R,A}$ ($? = , r, \infty, [r_1, r_2], \{[s, r]\}$, $0 < r_1 \leq r_2/p^{hk}$) constructed over analytic base (R, R^+) , we have the equivalence among the categories of the Frobenius étale-stably-pseudocoherent modules, which could be further compared in equivalence to the the pseudocoherent sheaves over adic Fargues-Fontaine curves in both étale and pro-étale topology.*

This is basically the corresponding commutative version where we encode the corresponding ring A into the corresponding Huber spectrum in [KL1] and [KL2]. The corresponding noncommutative version is also true but we have to do this slightly different by deforming the structure sheaves. With the corresponding notations again in Chapter 5, we have:

Theorem 10.1.2. *Over B -relative preperfectoid Robba rings $\tilde{\Pi}_{?,R,B}$ ($? = , r, \infty, [r_1, r_2], \{[s, r]\}$, $0 < r_1 \leq r_2/p^{hk}$) constructed over analytic base (R, R^+) , we have the equivalence among the categories of the Frobenius B -étale-stably-pseudocoherent modules, which could be further compared in equivalence to the the B -pseudocoherent sheaves over adic Fargues-Fontaine curves in étale topology.*

Our ultimate consideration will after [GR], where (R, R^+) is not necessarily always assumed to be analytic, especially in the integral setting this is quite general. With the corresponding notations again in Chapter 6, we have:

Theorem 10.1.3. *Over A -relative preperfectoid Robba rings $\tilde{\Pi}_{?,R,A}$ ($? = [r_1, r_2], \{[s, r]\}$, $0 < r_1 \leq r_2/p^{hk}$) constructed over analytic base (R, R^+) , we have the equivalence of the categories of the Frobenius étale-stably-pseudocoherent modules, which could be further compared in equivalence to the the pseudocoherent sheaves over adic Fargues-Fontaine curves in the étale or pro-étale topology.*

Theorem 10.1.4. *Over B -relative preperfectoid Robba rings $\tilde{\Pi}_{?,R,B}$ ($? = [r_1, r_2], \{[s, r]\}$, $0 < r_1 \leq r_2/p^{hk}$) constructed over analytic base (R, R^+) , we have the equivalence of the categories of the Frobenius B -stably-pseudocoherent modules.*

10.1.2 Some Consideration in the Future

More thorough applications of [Lu2] will be expected in our study on the corresponding derived algebraic geometry of the corresponding period rings¹. We only touched this to some very transparent extent which will be somehow reflecting some further consideration along the corresponding applications in our mind.

New robust developments from Clausen-Scholze [CS], Bambozzi-Kremnizer [BK] and Bambozzi-Ben-Bassat-Kremnizer [BBBK] may shed lights on the derived Galois (possibly in more general setting for fundamental groups) deformation theory (and some unobstructed crystalline substacks) such as in [GV] and possibly (we conjecture) the corresponding possible theory of some derived eigenvarieties and derived Selmer varieties in Kim's nonabelian Chabauty. We hope to give more hard thorough study and understanding around this. Especially in our situation we will come across many technical issues where these new developments could help.

¹The original motivation comes from the study of the algebraic pseudocoherent sheaves over algebraic Fargues-Fontaine curves originally in [KL2], and also in [XT1] and [XT2] where the corresponding functional analytic information around locally convex vector spaces (after Bourbaki) is not considered.

10.2 Period Rings in General Setting with General Coefficients

In this section we start by defining Kedlaya-Liu's style period rings with coefficients in Tate adic Banach rings which are of finite type.

Setting 10.2.1. We consider the corresponding setting up which is as in the following. First we consider a field E which is analytic nonarchimedean with normalized norm such that the corresponding uniformizer π_E is of norm $1/p$. And we assume that the corresponding residue field of the field E takes the form of \mathbb{F}_{p^h} . Then we fix a corresponding uniform adic Banach algebra (R, R^+) over \mathbb{F}_{p^h} which is not required to be contain topological nilpotent unit. However we need to assume that that this is analytic in the sense of [KL1] and [Ked1, Definition 1.1.2]. We recall that this means the set of the corresponding topological nilpotents generates the unit ideal. And let A be a general rigid analytic affinoid over \mathbb{Q}_p or over $\mathbb{F}_p((t))$.

Then in this generality we define our large coefficient Robba rings following Kedlaya-Liu in the Tate algebra situation namely we have $A = \mathbb{Q}_p\{X_1, \dots, X_d\}$ or $A = \mathbb{F}_p((t))\{X_1, \dots, X_d\}$:

Definition 10.2.2. (After Kedlaya-Liu [KL2, Definition 4.1.1]) Now consider the following constructions. First we consider the corresponding Witt vectors coming from the corresponding adic ring (R, R^+) . First we consider the corresponding generalized Witt vectors with respect to (R, R^+) with the corresponding coefficients in the Tate algebra with the general notation $W(R^+)[[R]]$. The general form of any element in such deformed ring could be written as $\sum_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^i [\bar{y}_i] X_1^{i_1} \dots X_d^{i_d}$. Then we take the corresponding completion with respect to the following norm for some radius $t > 0$:

$$\|\cdot\|_{t,A} \left(\sum_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^i [\bar{y}_i] X_1^{i_1} \dots X_d^{i_d} \right) := \max_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} p^{-i} \|\cdot\|_R(\bar{y}_i) \quad (10.2.1)$$

which will give us the corresponding ring $\tilde{\Pi}_{\text{int},t,R,A}$ such that we could put furthermore that:

$$\tilde{\Pi}_{\text{int},R,A} := \bigcup_{t>0} \tilde{\Pi}_{\text{int},t,R,A}. \quad (10.2.2)$$

Then as in [KL2, Definition 4.1.1], we now put the ring $\tilde{\Pi}_{\text{bd},t,R,A} := \tilde{\Pi}_{\text{int},t,R,A}[1/\pi]$ and we set:

$$\tilde{\Pi}_{\text{bd},R,A} := \bigcup_{t>0} \tilde{\Pi}_{\text{bd},t,R,A}. \quad (10.2.3)$$

The corresponding Robba rings with respect to some intervals and some radius could be defined in the same way as in [KL2, Definition 4.1.1]. To be more precise we consider the completion of the corresponding ring $W(R^+)[[R]][1/\pi]$ with respect to the following norm for some $t > 0$ where t lives in some prescribed interval $I = [s, r]$:

$$\|\cdot\|_{t,A} \left(\sum_{i,i_1 \geq 0, \dots, i_d \geq 0} \pi^i [\bar{y}_i] X_1^{i_1} \dots X_d^{i_d} \right) := \max_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} p^{-i} \|\cdot\|_R(\bar{y}_i). \quad (10.2.4)$$

This process will produce the corresponding Robba rings with respect to the given interval $I = [s, r]$. Now for particular sorts of intervals $(0, r]$ we will have the corresponding Robba ring $\tilde{\Pi}_{r,R,A}$ and we will have the corresponding Robba ring $\tilde{\Pi}_{\infty,R,A}$ if the corresponding interval is taken to be $(0, \infty)$. Then in our situation we could just take the corresponding union throughout all the radius $r > 0$ to define the corresponding full Robba ring taking the notation of $\tilde{\Pi}_{R,A}$.

Remark 10.2.3. The corresponding Robba rings $\tilde{\Pi}_{\text{bd},R,A}$, $\tilde{\Pi}_{R,A}$, $\tilde{\Pi}_{I,R,A}$, $\tilde{\Pi}_{r,R,A}$, $\tilde{\Pi}_{\infty,R,A}$ are actually themselves Tate adic Banach rings. However in many further application the non-Tateness of the ring R will cause some reason for us to do the corresponding modification, which is considered on this level in fact in [KL1] in the context therein.

Definition 10.2.4. Then for any general affinoid algebra A over the corresponding base analytic field, we just take the corresponding quotients of the corresponding rings defined in the previous definition over some Tate algebras in rigid analytic geometry, with the same notations though A now is more general. Note that one can actually show that the definition does not depend on the corresponding choice of the corresponding presentations over A .

Remark 10.2.5. Again in this situation more generally, the corresponding Robba rings $\tilde{\Pi}_{\text{bd},R,A}$, $\tilde{\Pi}_{R,A}$, $\tilde{\Pi}_{I,R,A}$, $\tilde{\Pi}_{r,R,A}$, $\tilde{\Pi}_{\infty,R,A}$ are actually themselves Tate adic Banach rings.

Note that we can also as in the situation of [KL2] and [XT2] consider the corresponding the corresponding property checking of the corresponding period rings defined

above. We collect the corresponding statements here while the the proof could be found in [XT2]:

Lemma 10.2.6. (After Kedlaya-Liu [KL2, Lemma 5.2.6]) *For any two radii $0 < r_1 < r_2$ we have the corresponding equality:*

$$\tilde{\Pi}_{\text{int},r_2,R,\mathbb{Q}_p\{T_1,\dots,T_d\}} \cap \tilde{\Pi}_{[r_1,r_2],R,\mathbb{Q}_p\{T_1,\dots,T_d\}} = \tilde{\Pi}_{\text{int},r_1,R,\mathbb{Q}_p\{T_1,\dots,T_d\}}. \quad (10.2.5)$$

Proof. See [KL2, Lemma 5.2.6] and [XT2, Proposition 2.13]. □

Lemma 10.2.7. (After Kedlaya-Liu [KL2, Lemma 5.2.6]) *For any two radii $0 < r_1 < r_2$ we have the corresponding equality:*

$$\tilde{\Pi}_{\text{int},r_2,R,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} \cap \tilde{\Pi}_{[r_1,r_2],R,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} = \tilde{\Pi}_{\text{int},r_1,R,\mathbb{F}_p((t))\{T_1,\dots,T_d\}}. \quad (10.2.6)$$

Proof. See [KL2, Lemma 5.2.6] and [XT2, Proposition 2.13]. □

Lemma 10.2.8. (After Kedlaya-Liu [KL2, Lemma 5.2.6]) *For general affinoid A as above (over \mathbb{Q}_p or $\mathbb{F}_p((t))$) and for any two radii $0 < r_1 < r_2$ we have the corresponding equality:*

$$\tilde{\Pi}_{\text{int},r_2,R,A} \cap \tilde{\Pi}_{[r_1,r_2],R,A} = \tilde{\Pi}_{\text{int},r_1,R,A}. \quad (10.2.7)$$

Proof. See [KL2, Lemma 5.2.6] and [XT2, Proposition 2.14]. □

Lemma 10.2.9. (After Kedlaya-Liu [KL2, Lemma 5.2.10]) *For any four radii $0 < r_1 < r_2 < r_3 < r_4$ we have the corresponding equality:*

$$\tilde{\Pi}_{[r_1,r_3],R,\mathbb{Q}_p\{T_1,\dots,T_d\}} \cap \tilde{\Pi}_{[r_2,r_4],R,\mathbb{Q}_p\{T_1,\dots,T_d\}} = \tilde{\Pi}_{[r_1,r_4],R,\mathbb{Q}_p\{T_1,\dots,T_d\}}. \quad (10.2.8)$$

Proof. See [KL2, Lemma 5.2.10] and [XT2, Proposition 2.16]. □

Lemma 10.2.10. (After Kedlaya-Liu [KL2, Lemma 5.2.10]) *For any four radii $0 < r_1 < r_2 < r_3 < r_4$ we have the corresponding equality:*

$$\tilde{\Pi}_{[r_1,r_3],R,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} \cap \tilde{\Pi}_{[r_2,r_4],R,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} = \tilde{\Pi}_{[r_1,r_4],R,\mathbb{F}_p((t))\{T_1,\dots,T_d\}}. \quad (10.2.9)$$

Proof. See [KL2, Lemma 5.2.10] and [XT2, Proposition 2.16]. □

Lemma 10.2.11. (After Kedlaya-Liu [KL2, Lemma 5.2.10]) *For any four radii $0 < r_1 < r_2 < r_3 < r_4$ we have the corresponding equality:*

$$\tilde{\Pi}_{[r_1, r_3], R, A} \cap \tilde{\Pi}_{[r_2, r_4], R, A} = \tilde{\Pi}_{[r_1, r_4], R, A}. \quad (10.2.10)$$

Proof. See [KL2, Lemma 5.2.10] and [XT2, Proposition 2.17]. □

10.3 The Frobenius Modules and Frobenius Sheaves over the Period Rings

We now consider the corresponding Frobenius actions and the corresponding action reflected on the corresponding Hodge-Iwasawa modules over general analytic R . As in the Tate situation we consider the corresponding 'arithmetic' Frobenius as well.

Setting 10.3.1. We are going to use the corresponding notation F to denote the corresponding relative Frobenius induced from the corresponding p^h -power absolute Frobenius from R . This would mean that when we consider the corresponding Frobenius up to higher order.

Furthermore in our situation we have the corresponding sheaves of period rings (carrying the coefficient A) over $\mathrm{Spa}(R, R^+)$ in the corresponding pro-étale site:

$$\tilde{\Pi}_{\mathrm{bd}, \mathrm{Spa}(R, R^+)_{\mathrm{pro\acute{e}t}, A}}, \tilde{\Pi}_{\mathrm{Spa}(R, R^+)_{\mathrm{pro\acute{e}t}, A}}, \tilde{\Pi}_{I, \mathrm{Spa}(R, R^+)_{\mathrm{pro\acute{e}t}, A}}, \tilde{\Pi}_{r, \mathrm{Spa}(R, R^+)_{\mathrm{pro\acute{e}t}, A}}, \tilde{\Pi}_{\infty, \mathrm{Spa}(R, R^+)_{\mathrm{pro\acute{e}t}, A}} \quad (10.3.1)$$

which we will also briefly write as:

$$\tilde{\Pi}_{\mathrm{bd}, *, A}, \tilde{\Pi}_{*, A}, \tilde{\Pi}_{I, *, A}, \tilde{\Pi}_{r, *, A}, \tilde{\Pi}_{\infty, *, A} \quad (10.3.2)$$

when the corresponding topology is basically clear to the readers (one can also consider other kinds of topology in the same fashion such as the corresponding étale situation in our current context).

Definition 10.3.2. (After Kedlaya-Liu [KL2, Definition 4.4.4]) The corresponding Frobenius modules over the corresponding period sheaves will be defined to be finite projective modules carrying the semilinear action from F^k (where we allow some power $k > 0$) with some pullback isomorphic requirement which could be defined in the following way. Over the rings without intervals or radius this will mean that we have $F^{k*}M \xrightarrow{\sim} M$. While over the corresponding Robba rings with radius $r > 0$ this will mean that we have $F^{k*}M \otimes_{*_{r/p^{hk}}} *_{r/p^{hk}} \xrightarrow{\sim} M \otimes_{*_{r/p^{hk}}} *_{r/p^{hk}}$. While over the corresponding Robba rings with interval $[r_1, r_2]$ this will mean that we have $F^{k*}M \otimes_{*_{[r_1/p^{hk}, r_2/p^{hk}]}} *_{[r_1, r_2/p^{hk}]} \xrightarrow{\sim} M \otimes_{*_{[r_1, r_2]}} *_{[r_1, r_2/p^{hk}]}$. Over the corresponding Robba rings without any intervals or radii we assume that they are base change from some modules over the corresponding Robba rings with some radii.

Definition 10.3.3. (After Kedlaya-Liu [KL2, Definition 4.4.4]) The corresponding pseudocoherent or fpd Frobenius modules over the corresponding period sheaves will be defined to be pseudocoherent or fpd modules carrying the semilinear action from F^k (where we allow some power $k > 0$) with some pullback isomorphic requirement which could be defined in the following way. Over the rings without intervals or radius this will mean that we have $F^{k*}M \xrightarrow{\sim} M$. While over the corresponding Robba rings with radius $r > 0$ this will mean that we have $F^{k*}M \otimes_{*_{r/p^{hk}}} *_{r/p^{hk}} \xrightarrow{\sim} M \otimes_{*_r} *_{r/p^{hk}}$. While over the corresponding Robba rings with interval $[r_1, r_2]$ this will mean that we have $F^{k*}M \otimes_{*_{[r_1/p^{hk}, r_2/p^{hk}]}} *_{[r_1, r_2/p^{hk}]} \xrightarrow{\sim} M \otimes_{*_{[r_1, r_2]}} *_{[r_1, r_2/p^{hk}]}$. Over the corresponding Robba rings without any intervals or radii we assume that they are base changes from some modules over the corresponding Robba rings with some radii. And then all the modules are required to be complete with respect to the natural topology over any perfectoid subdomain within the corresponding pro-étale topology, and the corresponding modules over the corresponding Robba rings with respect to some radius $r > 0$ will basically be required to be base change to any étale-stably pseudocoherent (over any specific chosen perfectoid subdomain) modules (note that this will include the corresponding glueing along the space A with respect to the corresponding implicit étale topology induced over A).

Then as in [KL2, Definition 4.4.4] we consider the corresponding Frobenius modules over the period rings instead of period sheaves (and we do not either put topological conditions, which will be added later in specific consideration):

Definition 10.3.4. (After Kedlaya-Liu [KL2, Definition 4.4.4]) The corresponding Frobenius modules over the corresponding period rings will be defined to be finite projective modules carrying the semilinear action from F^k (where we allow some power $k > 0$) with some pullback isomorphic requirement which could be defined in the following way. Over the rings without intervals or radius this will mean that we have $F^{k*}M \xrightarrow{\sim} M$. While over the corresponding Robba rings with radius $r > 0$ this will mean that we have $F^{k*}M \otimes_{*_{r/p^{hk}}} *_{r/p^{hk}} \xrightarrow{\sim} M \otimes_{*_r} *_{r/p^{hk}}$. While over the corresponding Robba rings with interval $[r_1, r_2]$ this will mean that we have $F^{k*}M \otimes_{*_{[r_1/p^{hk}, r_2/p^{hk}]}} *_{[r_1, r_2/p^{hk}]} \xrightarrow{\sim} M \otimes_{*_{[r_1, r_2]}} *_{[r_1, r_2/p^{hk}]}$. Over the corresponding Robba rings without any intervals or radii

we assume that they are base change from some modules over the corresponding Robba rings with some radii.

Remark 10.3.5. As in [KL2, Definition 4.4.4] we did not put any topological conditions on the ring theoretic and algebraic representation theoretic objects defined above, but this will be definitely more precise in later development. Certainly this is more relevant with respect to the following definition.

Definition 10.3.6. (After Kedlaya-Liu [KL2, Definition 4.4.4]) The corresponding pseudocoherent or fpd Frobenius modules over the corresponding period rings will be defined to be pseudocoherent or fpd modules carrying the semilinear action from F^k (where we allow some power $k > 0$) with some pullback isomorphic requirement which could be defined in the following way. Over the rings without intervals or radius this will mean that we have $F^{k*}M \xrightarrow{\sim} M$. While over the corresponding Robba rings with radius $r > 0$ this will mean that we have $F^{k*}M \otimes_{*_{r/p^{hk}}} *_{r/p^{hk}} \xrightarrow{\sim} M \otimes_{*_r} *_{r/p^{hk}}$. While over the corresponding Robba rings with interval $[r_1, r_2]$ this will mean that we have $F^{k*}M \otimes_{*_{[r_1/p^{hk}, r_2/p^{hk}]}} *_{[r_1, r_2/p^{hk}]} \xrightarrow{\sim} M \otimes_{*_{[r_1, r_2]}} *_{[r_1, r_2/p^{hk}]}$. Over the corresponding Robba rings without any intervals or radii we assume that they are base change from some modules over the corresponding Robba rings with some radii.

Then we define over our current analytic base the corresponding pseudocoherent and fpd sheaves carrying the corresponding Frobenius modules.

Definition 10.3.7. (After Kedlaya-Liu [KL2, Definition 4.4.6]) Over the Robba rings with respect to the corresponding all finite intervals contained in $(0, \infty)$, we consider a family $(M_{[s,r]})_{[s,r]}$ of finite projective modules with respect to the corresponding intervals. Then we call this family a finite projective F^k -bundle over $\tilde{\Pi}_{R,A}$ if each member in the family is assumed to be a corresponding finite projective F^k -modules.

Definition 10.3.8. (After Kedlaya-Liu [KL2, Definition 4.4.6]) Over the Robba rings with respect to the corresponding all finite intervals contained in $(0, \infty)$, we consider a family $(M_{[s,r]})_{[s,r]}$ of fpd or pseudocoherent modules with respect to the corresponding intervals. Then we call this family a fpd or pseudocoherent F^k -sheaf over $\tilde{\Pi}_{R,A}$ if each member in the family is assumed to be a corresponding stably or étale-stably fpd or pseudocoherent F^k -modules.

Proposition 10.3.9. (After Kedlaya-Liu [KL2, Lemma 4.4.8]) *Any pseudocoherent F^k -modules over the corresponding period rings above admits a surjective morphism from some finite projective F^k -modules over the same types of period rings.*

Proof. This is the corresponding relative and analytic version of the corresponding [KL2, Lemma 4.4.8]. The proof is parallel, see [KL2, Lemma 4.4.8]. \square

10.4 Comparison in the Étale Setting

We now consider the corresponding parallel consideration to [KL2, Section 4.5]. Essentially in our current context, this is really the corresponding purely analytic setting especially when we do not have the chance to invert π . Certainly we will also consider the corresponding deformed setting. The corresponding space we are going to work on is just the corresponding perfectoid spaces (in the corresponding analytic situation) taking the corresponding general form of $\mathrm{Spa}(R, R^+)$, and consider any topological ring Z and the corresponding sheaf \underline{Z} . We will consider the corresponding finite projective, pseudocoherent and finite projective dimension local systems of \underline{Z} -modules. Note that again in this analytic setting the corresponding definitions of such modules are locally of the corresponding same types. This means in particular in the situation of finite projective situation the corresponding definition is locally the finite projective sheaves of modules over \underline{Z} instead of the corresponding naive locally finite free ones especially in our large coefficient situation.

Lemma 10.4.1. (After Kedlaya-Liu [KL2, Lemma 4.5.3]) *The corresponding taking element to the corresponding element after the action of the corresponding operator $F - 1$ will be to realize the following exact sequences as in [KL2, Lemma 4.5.3]:*

$$0 \longrightarrow \underline{\mathcal{O}}_{E_k} \longrightarrow \tilde{\Omega}_{\mathrm{int},R} \longrightarrow \tilde{\Omega}_{\mathrm{int},R} \longrightarrow 0,$$

$$0 \longrightarrow \underline{E}_k \longrightarrow \tilde{\Omega}_R \longrightarrow \tilde{\Omega}_R \longrightarrow 0,$$

$$0 \longrightarrow \underline{\mathcal{O}}_{E_k} \longrightarrow \tilde{\Pi}_{\mathrm{int},R} \longrightarrow \tilde{\Pi}_{\mathrm{int},R} \longrightarrow 0,$$

$$0 \longrightarrow \underline{E}_k \longrightarrow \tilde{\Pi}_R \longrightarrow \tilde{\Pi}_R \longrightarrow 0.$$

Proof. See [KL2, Lemma 4.5.3]. □

Proposition 10.4.2. (After Kedlaya-Liu [KL2, Lemma 4.5.4]) *We have the corresponding strictly pseudocoherence of any corresponding finitely presented modules over $\tilde{\Omega}_{\mathrm{int},R}$ or $\tilde{\Pi}_{\mathrm{int},R}$. Therefore we do not have to consider the corresponding topological issue and the stability issue further in this certain special situation.*

Proof. See [KL2, Lemma 4.5.4]. □

The following result is then achievable:

Proposition 10.4.3. (After Kedlaya-Liu [KL2, Theorem 4.5.7]) *Consider the following categories. The first one is the corresponding category of all the finite projective or pseudocoherent \mathcal{O}_{E_k} -local systems. The second one is the corresponding category of all the finite projective or stably-pseudocoherent F^k - $\tilde{\Omega}_{\text{int},R}$ -modules. The third one is the corresponding category of all the finite projective or stably-pseudocoherent F^k - $\tilde{\Pi}_{\text{int},R}$ -modules. The fourth one is the corresponding category of all the finite projective or stably-pseudocoherent F^k - $\tilde{\Omega}_{\text{int},*}$ -sheaves. The last one is the corresponding category of all the finite projective or stably-pseudocoherent F^k - $\tilde{\Pi}_{\text{int},*}$ -sheaves. Here we consider the corresponding analytic topology.*

Then we have that these categories are actually equivalent. Similar parallel statement could then be made to fpd objects.

Remark 10.4.4. The corresponding stably-pseudocoherent indication in the previous proposition is actually not serious as in [KL2, Lemma 4.5.7], the reason is that by the previous proposition, everything is already complete with respect to the natural topology.

Proof. This is just the corresponding analytic analog of the corresponding result in [KL2, Lemma 4.5.7]. We just briefly mention the corresponding functors which realize the corresponding equivalence. The corresponding one from the first category to the corresponding sheaves is naturally just the corresponding base change. The corresponding functor from the sheaves to the corresponding modules is naturally the corresponding global section functor after the corresponding foundation in [Ked1, Theorem 1.4.2, Theorem 1.4.18]. □

Remark 10.4.5. Since we touched the corresponding foundations in [Ked1, Theorem 1.4.2, Theorem 1.4.18] on the corresponding glueing of the vector bundles and the corresponding stably-pseudocoherent modules in the situation of analytic topology. However the corresponding descent in the étale, pro-étale and v -topology situations are actually parallel to the corresponding [KL2, Theorem 2.5.5, Theorem 2.5.14, Theorem 3.4.8, Theorem 3.5.8], which will basically upgrade naturally this proposition to the étale, pro-étale and v -topology situations, although we have not established the chance to present.

Now carrying the corresponding coefficient A we could have at least the corresponding fully faithfulness.

Proposition 10.4.6. (After Kedlaya-Liu [KL2, Theorem 4.5.7]) *Consider the following categories. The first one is the corresponding category of all the finite projective or pseudocoherent $\underline{\mathcal{O}}_{E_k, A}$ -local systems. The second one is the corresponding category of all the finite projective or stably-pseudocoherent $F^k\text{-}\tilde{\Omega}_{\text{int}, R, A}$ -modules. The third one is the corresponding category of all the finite projective or stably-pseudocoherent $F^k\text{-}\tilde{\Pi}_{\text{int}, R, A}$ -modules. The fourth one is the corresponding category of all the finite projective or stably-pseudocoherent $F^k\text{-}\tilde{\Omega}_{\text{int}, *, A}$ -sheaves. The last one is the corresponding category of all the finite projective or stably-pseudocoherent $F^k\text{-}\tilde{\Pi}_{\text{int}, *, A}$ -sheaves. Here we consider the corresponding analytic topology.*

Then we have that the first category could be embedded fully faithfully into the rest four categories and the corresponding modules could be compared to the sheaves under equivalences of the corresponding categories as long as we assume the corresponding sousperfectoidness of the ring A which preserves the corresponding exactness of the exact sequences:

$$0 \longrightarrow \underline{\mathcal{O}}_{E_k} \longrightarrow \tilde{\Omega}_{\text{int}, R} \longrightarrow \tilde{\Omega}_{\text{int}, R} \longrightarrow 0,$$

$$0 \longrightarrow \underline{E}_k \longrightarrow \tilde{\Omega}_R \longrightarrow \tilde{\Omega}_R \longrightarrow 0,$$

$$0 \longrightarrow \underline{\mathcal{O}}_{E_k} \longrightarrow \tilde{\Pi}_{\text{int}, R} \longrightarrow \tilde{\Pi}_{\text{int}, R} \longrightarrow 0,$$

$$0 \longrightarrow \underline{E}_k \longrightarrow \tilde{\Pi}_R \longrightarrow \tilde{\Pi}_R \longrightarrow 0.$$

Similar statement could be made to the corresponding fpd objects.

Proof. See [KL2, Theorem 4.5.7]. □

10.5 Comparison in the Non-Étale Setting

We now consider the corresponding analog of the corresponding [KL2, Chapter 4.6] over $\mathrm{Spa}(R, R^+)$ as above which is just assumed to be analytic.

Remark 10.5.1. As we mentioned before, the corresponding analyticity of the ring R will still not deduce from the analyticity of the corresponding rational Robba rings and sheaves.

Theorem 10.5.2. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Assume the corresponding Robba rings with respect to closed intervals are sheafy. Consider the corresponding categories in the following:*

A. The corresponding category of all the corresponding étale-stably-pseudocoherent sheaves over the adic Fargues-Fontaine curve (associated to $\{\tilde{\Pi}_{I,R,A}\}_{I \subset (0,\infty)}$) in the corresponding étale topology;

B. The corresponding category of all the corresponding étale-stably-pseudocoherent F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,R,A}\}_{I \subset (0,\infty)}$;

C. The corresponding category of all the corresponding étale-stably-pseudocoherent modules over any Robba rings $\tilde{\Pi}_{I,R,A}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$;

D. The corresponding category of all the corresponding pseudocoherent modules over any Robba rings $\tilde{\Pi}_{R,A}$, but admitting models of strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{r,R,A}$ for some radius $r > 0$ whose base changes to some $\tilde{\Pi}_{[r_1,r_2],R,A}$ will produce corresponding F^k -modules which are étale-stably-pseudocoherent;

E. The corresponding category of all the corresponding strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{\infty,R,A}$ whose base changes to some $\tilde{\Pi}_{[r_1,r_2],R,A}$ will provide corresponding F^k -modules which are étale-stably-pseudocoherent;

F. The corresponding category of all the corresponding étale-stably-pseudocoherent sheaves over the adic Fargues-Fontaine curve (associated to $\{\tilde{\Pi}_{I,R,A}\}_{I \subset (0,\infty)}$) in the corresponding pro-étale topology.

Then we have that they are actually equivalent.

Proof. This is just parallel [KL2, Theorem 4.6.1], see [XT2, Theorem 4.11]. □

Theorem 10.5.3. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Assume A is sous-perfectoid. Consider the corresponding categories of sheaves over $\mathrm{Spa}(R, R^+)_{\mathrm{pét}}$ in the following:*

A. The corresponding category of all the corresponding étale-stably-pseudocoherent modules over any Robba rings $\tilde{\Pi}_{I,,A}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$;*

B. The corresponding category of all the corresponding pseudocoherent modules over any Robba rings $\tilde{\Pi}_{,A}$, but admit models of strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{r,*,A}$ for some radius $r > 0$ whose base changes to some $\tilde{\Pi}_{[r_1, r_2], R, A}$ will provide corresponding F^k -modules which are étale-stably-pseudocoherent;*

C. The corresponding category of all the corresponding étale-strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{\infty,,A}$ whose base changes to some $\tilde{\Pi}_{[r_1, r_2],*,A}$ will provide corresponding F^k -modules which are étale-stably-pseudocoherent.*

Then we have that they are actually equivalent.

Proof. This is parallel to [KL2, Theorem 4.6.1] The proof is the same as the one where $*$ is just a ring R . See [XT2, Theorem 4.11]. □

Theorem 10.5.4. (After Kedlaya-Liu [KL2, Corollary 4.6.2]) *The corresponding analogs of the previous two theorems hold in the corresponding finite projective setting. Here we assume that A is in the same hypotheses as above.*

Proof. See [KL2, Corollary 4.6.2]. □

Now we drop the corresponding condition on the sheafiness on A by using the corresponding derived spectrum $\mathrm{Spa}^h(A)$ from [BK] where we have the corresponding ∞ -sheaf $\mathcal{O}_{\mathrm{Spa}^h(A)}$, we now apply the corresponding construction to the Robba rings (in the rational setting over \mathbb{Q}_p) $*_{R,A}$ and denote the corresponding derived version by $*_{R,A}^h$. Here A is assumed to be just Banach over \mathbb{Q}_p or $\mathbb{F}_p((t))$, which is certainly in a very general situation.

Remark 10.5.5. One can apply the corresponding results of [CS] to achieve so as well, which we believe will be robust as well.

Theorem 10.5.6. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Take a derived rational localization of $\tilde{\Pi}_{I_1 \cup I_2, R, A}$, which we denote it by $\tilde{\Pi}_{I_1 \cup I_2, R, A}^{h,*}$, where we choose two*

overlapped closed intervals I_1 and I_2 . Now take the corresponding base changes of this localization along:

$$\tilde{\Pi}_{I_1 \cup I_2, R, A} \rightarrow \tilde{\Pi}_{I_1, R, A}, \quad (10.5.1)$$

$$\tilde{\Pi}_{I_1 \cup I_2, R, A} \rightarrow \tilde{\Pi}_{I_2, R, A}, \quad (10.5.2)$$

which we will denote by $\tilde{\Pi}_{I_1, R, A}^{h,*}$ and $\tilde{\Pi}_{I_2, R, A}^{h,*}$. Consider the corresponding categories in the following:

A. The corresponding product of the category of all the corresponding finite projective F^k -modules over the Robba ring $\tilde{\Pi}_{I_1, R, A}^{h,*}$ and the the category of all the corresponding finite projective F^k -modules over the Robba ring $\tilde{\Pi}_{I_2, R, A}^{h,*}$, over the category of all the corresponding finite projective F^k -modules over $\tilde{\Pi}_{I_1, R, A}^{h,*} \hat{\otimes}^{\mathbb{L}} \tilde{\Pi}_{I_2, R, A}^{h,*}$;

B. The corresponding category of all the corresponding finite projective modules over the Robba ring $\tilde{\Pi}_{I_1 \cup I_2, R, A}^{h,*}$.

Then we have that they are actually equivalent.

Proof. We consider a pair of overlapped intervals $I_1 = [r_1, r_2], I_2 = [s_1, s_2]$, and we have the corresponding ∞ -Robba rings which form the desired situation, where we have the sequence which is basically exact up to higher homotopy:

$$\tilde{\Pi}_{I_1 \cup I_2, R, A}^{h,*} \longrightarrow \tilde{\Pi}_{I_1, R, A}^{h,*} \oplus \tilde{\Pi}_{I_2, R, A}^{h,*} \longrightarrow \tilde{\Pi}_{I_1, R, A}^{h,*} \hat{\otimes}^{\mathbb{L}} \tilde{\Pi}_{I_2, R, A}^{h,*}.$$

Now we consider the corresponding two finite projective module spectra (which admit retracts from finite free module spectra as in [Lu1, Proposition 7.2.2.7]) M_1, M_2, M_{12} over the rings in the middle and the rightmost positions, and assume that they form a corresponding glueing datum. Then we consider the corresponding presentation diagram

for the module spectra:

$$\begin{array}{ccccc}
 F & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \longrightarrow & G_1 \oplus G_2 & \longrightarrow & G_{12} \\
 \downarrow & & \downarrow r_1 \oplus r_2 & & \downarrow r_{12} \\
 M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12}
 \end{array}$$

by just taking the corresponding fibers, here G, G_1, G_2, G_{12} are the finite free modules and we have the corresponding retracts r_1, r_2, r_{12} , but note that the corresponding map $r_1 \oplus r_2$ might not be a priori a retract. However consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 \pi_0 F & \longrightarrow & \pi_0 F_1 \oplus \pi_0 F_2 & \longrightarrow & \pi_0 F_{12} & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \pi_0 G & \longrightarrow & \pi_0 G_1 \oplus \pi_0 G_2 & \longrightarrow & \pi_0 G_{12} & \longrightarrow & 0 \\
 \downarrow & & \downarrow \pi_0(r_1) \oplus \pi_0(r_2) & & \downarrow r_{12} & & \\
 \pi_0 M & \longrightarrow & \pi_0 M_1 \oplus \pi_0 M_2 & \longrightarrow & \pi_0 M_{12} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

We then consider the corresponding the argument of Kedlaya as in [XT3, Proposition 5.11] which gives us modification on sections $\pi_0 s_1$ and $\pi_0 s_2$ of $\pi_0 r_1$ and $\pi_0 r_2$ respectively

such that we have (with the same notations) the new $\pi_0 s_1$ and $\pi_0 s_2$ give the corresponding new lifts s_1 and s_2 whose coproduct actually restricts to M which produce a section for M for the map $G \rightarrow M$, which proves the desired finite projectivity on M . From A to B we just take the corresponding projection while going back we consider the corresponding binary glueing to tackle the corresponding interval which cannot be reached by taking Frobenius translations but could be covered by two ones reachable by taking Frobenius translations. The lifts could be also modified by directly on the derived level.

□

In [GV], Galatius and Venkatesh considered some derived Galois deformation theory. Now we make some discussion around this point by using the corresponding simplicial Banach rings (over \mathbb{Q}_p or over $\mathbb{F}_p((t))$) in [BK], namely we use the notation A^h to denote any local charts ² coming from the corresponding Bambozzi-Kremnizer spectrum in [BK] attached to some Banach algebra over (over \mathbb{Q}_p or over $\mathbb{F}_p((t))$). Over the analytic field specified we take the corresponding completed tensor products the period rings involved with A^h , which we will denote by $*_{R,A^h}$. Then we have the following similar results as above:

Conjecture 10.5.7. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the corresponding categories in the following:*

A. The corresponding category of all the corresponding locally finite free sheaves over the adic Fargues-Fontaine curve (associated to $\{\tilde{\Pi}_{I,R,A^h}\}_{I \subset (0,\infty)}$) in the corresponding homotopy Zariski topology from [BK] and [BBBK] induced by Koszul derived rational localization;

B. The corresponding category of all the corresponding finite projective F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,R,A^h}\}_{I \subset (0,\infty)}$;

C. The corresponding category of all the corresponding finite projective modules over any Robba rings $\tilde{\Pi}_{I,R,A^h}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$.

Then we have that they are actually equivalent.

Remark 10.5.8. This conjecture and the one we made in the same fashion later namely

²Even one can take the more general simplicial Banach rings as in [BBBK], especially one would like to focus on the corresponding analytification of derived Galois deformation rings as in [GV], at least our feeling is that [BBBK] will allow one to take the corresponding derived adic generic fiber to produce some desired simplicial Banach rings to tackle derived Galois deformation problems in [GV].

conjecture 10.6.9 could be formulated by using Clausen-Scholze analytic spaces in [CS], which is in a different geometric setting, but we will believe the picture is quite similar. In fact once one replaces the geometric context we considered in order to use the context of [CS], then this will be a direct consequence of the corresponding general descent result ([CS, Theorem 14.9, Remark 14.10]) in the corresponding nonderived (but possibly highly nonsheafy) situation, which includes the corresponding descending of the corresponding general perfect complex as well.

Remark 10.5.9. One can apply the corresponding results of [CS] to achieve so as well, which we believe will be robust as well.

Theorem 10.5.10. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the corresponding categories in the following:*

A. *The corresponding category of all the corresponding finite projective F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,R,A^h}\}_{I \subset (0,\infty)}$;*

B. *The corresponding category of all the corresponding finite projective modules over any Robba rings $\tilde{\Pi}_{I,R,A^h}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$.*

Then we have that they are actually equivalent.

Proof. Note that in our situation we do have the following nice short exact sequences:

$$0 \longrightarrow \pi_k \tilde{\Pi}_{I_1 \cup I_2, R, A^h} \longrightarrow \pi_k \tilde{\Pi}_{I_1, R, A^h} \oplus \pi_k \tilde{\Pi}_{I_2, R, A^h} \longrightarrow \pi_k \tilde{\Pi}_{I_1 \cap I_2, R, A^h} \longrightarrow 0, k = 0, 1, \dots$$

We consider a pair of overlapped intervals $I_1 = [r_1, r_2], I_2 = [s_1, s_2]$, and we have the corresponding ∞ -Robba rings which form a corresponding glueing sequence in the sense [KL1, Definition 2.7.3]:

$$0 \longrightarrow \pi_0 \tilde{\Pi}_{I_1 \cup I_2, R, A^h} \longrightarrow \pi_0 \tilde{\Pi}_{I_1, R, A^h} \oplus \pi_0 \tilde{\Pi}_{I_2, R, A^h} \longrightarrow \pi_0 \tilde{\Pi}_{I_1 \cap I_2, R, A^h} \longrightarrow 0.$$

Now we consider the corresponding two finite projective module spectra (which admit retracts from finite free module spectra as in [Lu1, Proposition 7.2.2.7]) M_1, M_2, M_{12} over the rings in the middle and the rightmost position, and assume that they form a corresponding glueing datum. Then we consider the corresponding presentation diagram

for the module spectra:

$$\begin{array}{ccccc}
 F & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & F_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \longrightarrow & G_1 \oplus G_2 & \longrightarrow & G_{12} \\
 \downarrow & & \downarrow r_1 \oplus r_2 & & \downarrow r_{12} \\
 M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12}
 \end{array}$$

by just taking the corresponding fibers, here G, G_1, G_2, G_{12} are the finite free modules and we have the corresponding retracts r_1, r_2, r_{12} , but note that the corresponding map $r_1 \oplus r_2$ might not be a priori a retract. However consider the following commutative diagram (which admits more exactness than in the previous theorem):

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \downarrow & & & \downarrow \\
 \pi_0 F & \longrightarrow & \pi_0 F_1 \oplus \pi_0 F_2 & \longrightarrow & \pi_0 F_{12} & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 \longrightarrow & \pi_0 G & \longrightarrow & \pi_0 G_1 \oplus \pi_0 G_2 & \longrightarrow & \pi_0 G_{12} & \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow r_{12} & & \\
 0 \longrightarrow & \pi_0 M & \longrightarrow & \pi_0 M_1 \oplus \pi_0 M_2 & \longrightarrow & \pi_0 M_{12} & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

We then consider the corresponding the argument of Kedlaya as in [XT3, Proposition 5.11] which gives us modification on sections $\pi_0 s_1$ and $\pi_0 s_2$ of $\pi_0 r_1$ and $\pi_0 r_2$ respectively such that we have (with the same notations) the new $\pi_0 s_1$ and $\pi_0 s_2$ give the corresponding new lifts s_1 and s_2 whose coproduct actually restricts to M which produce a section for M for the map $G \rightarrow M$, which proves the desired finite projectivity on M . From A to B we just take the corresponding projection while going back we consider the corresponding binary glueing to tackle the corresponding interval which cannot be reached by taking Frobenius translations but could be covered by two ones reachable by taking Frobenius translations. The lifts could be also modified by directly on the derived level. Or one can just show the flatness on the derived level directly by using derived Tor, which will directly proves that M is finite projective as a module spectrum. \square

Now we even drop the hypothesis (by using the corresponding techniques in [XT3] and [XT4]) on the commutativity on A , namely now A could be allowed to be strictly quotient coming from the free Tate algebras $\mathbb{Q}_p \langle Z_1, \dots, Z_d \rangle$ and $\mathbb{F}_p((t)) \langle Z_1, \dots, Z_d \rangle$, which we will use some different notation B to denote this. Note certainly that A is some special case of such B , therefore now we are going to indeed generalize the discussion above to the noncommutative setting.

Remark 10.5.11. One can definitely consider more general coefficients in the Banach setting, but for Hodge-Iwasawa theory, we prefer to locate our discussion in the area where we could get the interesting rings by taking admissible and strictly quotient from free Tate algebras over analytic field, which certainly includes the corresponding situation of rigid analytic affinoids.

Setting 10.5.12. By the work [XT3] and [XT4] we can freely translate between the language of B -stably-pseudocoherent sheaves over analytic or étale topology and étale setting and the corresponding B -stably pseudocoherent modules or B -étale stably pseudocoherent modules, which makes the discussion in our context possible. What was happening indeed when we are in the corresponding sheafy tensor product situation is that one can direct read off such result in the next theorem by using the descent for the pseudocoherent modules with desired stability in [KL2] and [Ked1].

Theorem 10.5.13. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Assume the corresponding Robba rings with respect to closed intervals are sheafy. Consider the correspond-*

ing categories in the following (all modules are left):

A. The corresponding category of all the corresponding B -étale-stably-pseudocoherent sheaves over the adic Fargues-Fontaine curve (associated to $\{\tilde{\Pi}_{I,R,B}\}_{I \subset (0,\infty)}$) in the corresponding étale topology;

B. The corresponding category of all the corresponding B -étale-stably-pseudocoherent F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,R,B}\}_{I \subset (0,\infty)}$;

C. The corresponding category of all the corresponding B -étale-stably-pseudocoherent modules over any Robba rings $\tilde{\Pi}_{I,R,B}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$;

D. The corresponding category of all the corresponding B -pseudocoherent modules over any Robba rings $\tilde{\Pi}_{R,B}$, but admit models of B -strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{r,R,B}$ for some radius $r > 0$ whose base changes to some $\tilde{\Pi}_{[r_1,r_2],R,B}$ will provide corresponding F^k -modules which are B -étale-stably-pseudocoherent;

E. The corresponding category of all the corresponding B -strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{\infty,R,B}$ whose base changes to some $\tilde{\Pi}_{[r_1,r_2],R,B}$ will provide corresponding F^k -modules which are B -étale-stably-pseudocoherent.

Then we have that they are actually equivalent.

Proof. This is just parallel [KL2, Theorem 4.6.1], see [XT2, Theorem 4.11]. The corresponding modules could be regarded as sheaves over the deformed sites as in our previous work [XT3] and [XT4]. \square

Theorem 10.5.14. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) Consider the corresponding categories of sheaves over $\mathrm{Spa}(R, R^+)_{\text{ét}}$ in the following (all the modules are left over the corresponding rings):

A. The corresponding category of all the corresponding B -étale-stably-pseudocoherent modules over any Robba rings $\tilde{\Pi}_{I,*,B}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$;

B. The corresponding category of all the corresponding B -pseudocoherent modules over any Robba rings $\tilde{\Pi}_{*,B}$, but admit models of B -strictly-pseudocoherent F^k -modules over some $\tilde{\Pi}_{r,*,B}$ for some radius $r > 0$ whose base changes to some $\tilde{\Pi}_{[r_1,r_2],R,B}$ will provide corresponding F^k -modules which are B -étale-stably-pseudocoherent;

C. The corresponding category of all the corresponding B -strictly-pseudocoherent F^k -

modules over some $\tilde{\Pi}_{\infty,*,B}$ whose base changes to some $\tilde{\Pi}_{[r_1,r_2],*,B}$ will provide corresponding F^k -modules which are B -étale-stably-pseudocoherent.

Then we have that they are actually equivalent.

Proof. This is parallel to [KL2, Theorem 4.6.1] The proof is the same as the one where $*$ is just a ring R . See [XT2, Theorem 4.11]. \square

Theorem 10.5.15. (After Kedlaya-Liu [KL2, Corollary 4.6.2]) *The corresponding analogs of the previous two theorems hold in the corresponding finite projective setting (one has to consider the bimodules). Here we assume that B is in the same hypotheses as above.*

Proof. See [KL2, Corollary 4.6.2] and [XT3, Proposition 5.12]. \square

10.6 Discussion on the Generality of Gabber-Ramero

10.6.1 General Period Rings

We can encode now the corresponding discussion in the previous section actually, but we choose to separately discuss the corresponding results in detail here. Certainly many results in [KL1] and [KL2], and [XT1], [XT2], [XT3] and [XT4] rely on the corresponding topologically nilpotent units and systems of topologically nilpotents. Therefore we will discuss the corresponding admissible and reasonable generalization after [KL1], [KL2], [XT1], [XT2], [XT3], [XT4] and [GR].

Setting 10.6.1. Drop the condition on the analyticity on (R, R^+) and we switch to the corresponding notation (S, S^+) .

After Kedlaya-Liu [KL2, Definition 4.1.1], we consider the following constructions. First we consider the corresponding Witt vectors coming from the corresponding adic ring (S, S^+) . First we consider the corresponding generalized Witt vectors with respect to (S, S^+) with the corresponding coefficients in the Tate algebra with the general notation $W(S^+)[[S]]$. The general form of any element in such deformed ring could be written as $\sum_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^i [\bar{y}_i] X_1^{i_1} \dots X_d^{i_d}$. Then we take the corresponding completion with respect to the following norm for some radius $t > 0$:

$$\|\cdot\|_{t,A} \left(\sum_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} \pi^i [\bar{y}_i] X_1^{i_1} \dots X_d^{i_d} \right) := \max_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} p^{-i} \|\cdot\|_S(\bar{y}_i) \quad (10.6.1)$$

which will give us the corresponding ring $\tilde{\Pi}_{\text{int},t,S,A}$ such that we could put furthermore that:

$$\tilde{\Pi}_{\text{int},S,A} := \bigcup_{t>0} \tilde{\Pi}_{\text{int},t,S,A}. \quad (10.6.2)$$

Then as in [KL2, Definition 4.1.1], we now put the ring $\tilde{\Pi}_{\text{bd},t,S,A} := \tilde{\Pi}_{\text{int},t,S,A}[1/\pi]$ and we set:

$$\tilde{\Pi}_{\text{bd},S,A} := \bigcup_{t>0} \tilde{\Pi}_{\text{bd},t,S,A}. \quad (10.6.3)$$

The corresponding Robba rings with respect to some intervals and some radius could be defined in the same way as in [KL2, Definition 4.1.1]. To be more precise we consider the

completion of the corresponding ring $W(S^+)[[S]][1/\pi]$ with respect to the following norm for some $t > 0$ where t lives in some prescribed interval $I = [s, r]$:

$$\|\cdot\|_{t,A} \left(\sum_{i, i_1 \geq 0, \dots, i_d \geq 0} \pi^i [\bar{y}_i] X_1^{i_1} \dots X_d^{i_d} \right) := \max_{i \geq 0, i_1 \geq 0, \dots, i_d \geq 0} p^{-i} \|\cdot\|_S(\bar{y}_i). \quad (10.6.4)$$

This process will produce the corresponding Robba rings with respect to the given interval $I = [s, r]$. Now for particular sorts of intervals $(0, r]$ we will have the corresponding Robba ring $\tilde{\Pi}_{r,S,A}$ and we will have the corresponding Robba ring $\tilde{\Pi}_{\infty,S,A}$ if the corresponding interval is taken to be $(0, \infty)$. Then in our situation we could just take the corresponding union throughout all the radius $r > 0$ to define the corresponding full Robba ring taking the notation of $\tilde{\Pi}_{S,A}$.

The corresponding Robba rings $\tilde{\Pi}_{\text{bd},S,A}$, $\tilde{\Pi}_{S,A}$, $\tilde{\Pi}_{I,S,A}$, $\tilde{\Pi}_{r,S,A}$, $\tilde{\Pi}_{\infty,S,A}$ are actually themselves Tate adic Banach rings. However in many further application the non-Tateness of the ring S will cause some reason for us to do the corresponding modification.

Then for any general affinoid algebra A over the corresponding base analytic field, we just take the corresponding quotients of the corresponding rings defined in the previous definition over some Tate algebras in rigid analytic geometry, with the same notations though A now is more general. Note that one can actually show that the definition does not depend on the corresponding choice of the corresponding presentations over A .

Again in this situation more generally, the corresponding Robba rings $\tilde{\Pi}_{\text{bd},S,A}$, $\tilde{\Pi}_{S,A}$, $\tilde{\Pi}_{I,S,A}$, $\tilde{\Pi}_{r,S,A}$, $\tilde{\Pi}_{\infty,S,A}$ are actually themselves Tate adic Banach rings.

Lemma 10.6.2. (After Kedlaya-Liu [KL2, Lemma 5.2.6]) *For any two radii $0 < r_1 < r_2$ we have the corresponding equality:*

$$\tilde{\Pi}_{\text{int},r_2,S,\mathbb{Q}_p\{T_1,\dots,T_d\}} \bigcap \tilde{\Pi}_{[r_1,r_2],S,\mathbb{Q}_p\{T_1,\dots,T_d\}} = \tilde{\Pi}_{\text{int},r_1,S,\mathbb{Q}_p\{T_1,\dots,T_d\}}. \quad (10.6.5)$$

Proof. See [KL2, Lemma 5.2.6] and [XT2, Proposition 2.13]. □

Lemma 10.6.3. (After Kedlaya-Liu [KL2, Lemma 5.2.6]) *For any two radii $0 < r_1 < r_2$ we have the corresponding equality:*

$$\tilde{\Pi}_{\text{int},r_2,S,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} \bigcap \tilde{\Pi}_{[r_1,r_2],S,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} = \tilde{\Pi}_{\text{int},r_1,S,\mathbb{F}_p((t))\{T_1,\dots,T_d\}}. \quad (10.6.6)$$

Proof. See [KL2, Lemma 5.2.6] and [XT2, Proposition 2.13]. □

Lemma 10.6.4. (After Kedlaya-Liu [KL2, Lemma 5.2.6]) *For general affinoid A as above (over \mathbb{Q}_p or $\mathbb{F}_p((t))$) and for any two radii $0 < r_1 < r_2$ we have the corresponding equality:*

$$\tilde{\Pi}_{\text{int},r_2,S,A} \cap \tilde{\Pi}_{[r_1,r_2],S,A} = \tilde{\Pi}_{\text{int},r_1,S,A}. \quad (10.6.7)$$

Proof. See [KL2, Lemma 5.2.6] and [XT2, Proposition 2.14]. □

Lemma 10.6.5. (After Kedlaya-Liu [KL2, Lemma 5.2.10]) *For any four radii $0 < r_1 < r_2 < r_3 < r_4$ we have the corresponding equality:*

$$\tilde{\Pi}_{[r_1,r_3],S,\mathbb{Q}_p\{T_1,\dots,T_d\}} \cap \tilde{\Pi}_{[r_2,r_4],S,\mathbb{Q}_p\{T_1,\dots,T_d\}} = \tilde{\Pi}_{[r_1,r_4],S,\mathbb{Q}_p\{T_1,\dots,T_d\}}. \quad (10.6.8)$$

Proof. See [KL2, Lemma 5.2.10] and [XT2, Proposition 2.16]. □

Lemma 10.6.6. (After Kedlaya-Liu [KL2, Lemma 5.2.10]) *For any four radii $0 < r_1 < r_2 < r_3 < r_4$ we have the corresponding equality:*

$$\tilde{\Pi}_{[r_1,r_3],S,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} \cap \tilde{\Pi}_{[r_2,r_4],S,\mathbb{F}_p((t))\{T_1,\dots,T_d\}} = \tilde{\Pi}_{[r_1,r_4],S,\mathbb{F}_p((t))\{T_1,\dots,T_d\}}. \quad (10.6.9)$$

Proof. See [KL2, Lemma 5.2.10] and [XT2, Proposition 2.16]. □

Lemma 10.6.7. (After Kedlaya-Liu [KL2, Lemma 5.2.10]) *For any four radii $0 < r_1 < r_2 < r_3 < r_4$ we have the corresponding equality:*

$$\tilde{\Pi}_{[r_1,r_3],S,A} \cap \tilde{\Pi}_{[r_2,r_4],S,A} = \tilde{\Pi}_{[r_1,r_4],S,A}. \quad (10.6.10)$$

Proof. See [KL2, Lemma 5.2.10] and [XT2, Proposition 2.17]. □

Definition 10.6.8. All the Frobenius finite projective, pseudocoherent and fpd modules under F^k could be defined in the exact same way as in section 10.3. We will not repeat the corresponding definition, but note that we change the notation for R to be just S .

Conjecture 10.6.9. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the corresponding categories in the following:*

A. The corresponding category of all the corresponding locally finite free sheaves over the adic Fargues-Fontaine curve (associated to $\{\tilde{\Pi}_{I,S,A^h}\}_{I \subset (0,\infty)}$) in the corresponding homotopy Zariski topology from [BK] and [BBBK] induced by Koszul derived rational localization;

B. The corresponding category of all the corresponding finite projective F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,S,A^h}\}_{I \subset (0,\infty)}$;

C. The corresponding category of all the corresponding finite projective modules over any Robba rings $\tilde{\Pi}_{I,S,A^h}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$.

Then we have that they are actually equivalent.

Remark 10.6.10. One can apply the corresponding results of [CS] to achieve so as well, which we believe will be robust as well.

Theorem 10.6.11. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) Consider the corresponding categories in the following:

A. The corresponding category of all the corresponding finite projective F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,S,A^h}\}_{I \subset (0,\infty)}$;

B. The corresponding category of all the corresponding finite projective modules over any Robba rings $\tilde{\Pi}_{I,S,A^h}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$.

Then we have that they are actually equivalent.

Proof. As in the situation in the analytic setting we consider the following argument which will show the desired result. In our situation we do have the following nice short exact sequences:

$$0 \longrightarrow \pi_k \tilde{\Pi}_{I_1 \cup I_2, S, A^h} \longrightarrow \pi_k \tilde{\Pi}_{I_1, S, A^h} \oplus \pi_k \tilde{\Pi}_{I_2, S, A^h} \longrightarrow \pi_k \tilde{\Pi}_{I_1 \cap I_2, S, A^h} \longrightarrow 0, k = 0, 1, \dots$$

We consider a pair of overlapped intervals $I_1 = [r_1, r_2], I_2 = [s_1, s_2]$, and we have the corresponding ∞ -Robba rings which form a corresponding glueing sequence in the sense [KL1, Definition 2.7.3]:

$$0 \longrightarrow \pi_0 \tilde{\Pi}_{I_1 \cup I_2, S, A^h} \longrightarrow \pi_0 \tilde{\Pi}_{I_1, S, A^h} \oplus \pi_0 \tilde{\Pi}_{I_2, S, A^h} \longrightarrow \pi_0 \tilde{\Pi}_{I_1 \cap I_2, S, A^h} \longrightarrow 0.$$

Now we consider the corresponding two finite projective module spectra (which admits retracts from finite free module spectra as in [Lu1, Proposition 7.2.2.7]) M_1, M_2, M_{12} over the rings in the middle and the rightmost position, and assume that they form a corresponding glueing datum. Then we consider the corresponding presentation diagram

for the module spectra:

$$\begin{array}{ccccc}
 L & \longrightarrow & L_1 \oplus L_2 & \longrightarrow & L_{12} \\
 \downarrow & & \downarrow & & \downarrow \\
 N & \longrightarrow & N_1 \oplus N_2 & \longrightarrow & N_{12} \\
 \downarrow & & \downarrow r_1 \oplus r_2 & & \downarrow r_{12} \\
 M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_{12}
 \end{array}$$

by just taking the corresponding fibers, here N, N_1, N_2, N_{12} are the finite free modules and we have the corresponding retracts r_1, r_2, r_{12} , but note that the corresponding map $r_1 \oplus r_2$ might not be a priori a retract. However consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \downarrow & & & \downarrow \\
 \pi_0 F & \longrightarrow & \pi_0 F_1 \oplus \pi_0 F_2 & \longrightarrow & \pi_0 F_{12} & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 \longrightarrow & \pi_0 G & \longrightarrow & \pi_0 G_1 \oplus \pi_0 G_2 & \longrightarrow & \pi_0 G_{12} & \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow r_{12} & & \\
 0 \longrightarrow & \pi_0 M & \longrightarrow & \pi_0 M_1 \oplus \pi_0 M_2 & \longrightarrow & \pi_0 M_{12} & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

We then consider the corresponding the argument of Kedlaya as in [XT3, Proposition 5.11] which gives us modification on sections $\pi_0 s_1$ and $\pi_0 s_2$ of $\pi_0 r_1$ and $\pi_0 r_2$ respectively such

that we have (with the same notations) the new $\pi_0 s_1$ and $\pi_0 s_2$ give the corresponding new lifts s_1 and s_2 whose coproduct actually restricts to M which produce a section for M for the map $N \rightarrow M$, which proves the desired finite projectivity on M . From A to B we just take the corresponding projection while going back we consider the corresponding binary glueing to tackle the corresponding interval which cannot be reached by taking Frobenius translations but could be covered by two ones reachable by taking Frobenius translations. One can actually derive this on the derived level directly by taking the corresponding lift of the difference of the base changes of s_1 and s_2 to the right in $\text{Ext}^0(M_{12}, L_{12})$ to $\text{Ext}^0(M_1 \oplus M_2, L_1 \oplus L_2)$ to modify the corresponding sections s_1 and s_2 in order to restrict to M , which proves the result by the argument of Kedlaya as in [XT3, Proposition 5.11]. \square

Now again we even drop the hypothesis (by using the corresponding techniques in [XT3] and [XT4]) on the commutativity on A , namely now A could be allowed to be strictly quotient coming from the free Tate algebras $\mathbb{Q}_p \langle Z_1, \dots, Z_d \rangle$ and $\mathbb{F}_p((t)) \langle Z_1, \dots, Z_d \rangle$, which we will use some different notation B to denote this. Note certainly that A is some special case of such B , therefore now we are going to indeed generalize the discussion above to the noncommutative setting.

Theorem 10.6.12. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Assume the corresponding Robba rings with respect to closed intervals are sheafy. Consider the corresponding categories in the following (all modules are left):*

A. The corresponding category of all the corresponding finite locally projective sheaves over the adic Fargues-Fontaine curve (associated to $\{\tilde{\Pi}_{I,S,B}\}_{I \subset (0,\infty)}$) in the corresponding étale topology;

B. The corresponding category of all the corresponding finite projective F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,S,B}\}_{I \subset (0,\infty)}$;

C. The corresponding category of all the corresponding finite projective modules over any Robba rings $\tilde{\Pi}_{I,S,B}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$.

Then we have that they are actually equivalent.

Proof. This is just parallel [KL2, Theorem 4.6.1], see [XT2, Theorem 4.11]. The corresponding modules could be regarded as sheaves over the deformed sites as in our previous work [XT3] and [XT4]. \square

Theorem 10.6.13. (After Kedlaya-Liu [KL2, Theorem 4.6.1]) *Consider the corresponding categories of sheaves over $\mathrm{Spa}(S, S^+)_{\text{ét}}$ in the following (all the modules are bimodules over the corresponding rings):*

A. *The corresponding category of all the corresponding finite projective F^k -sheaves over families of Robba rings $\{\tilde{\Pi}_{I,*,B}\}_{I \subset (0, \infty)}$;*

B. *The corresponding category of all the corresponding finite projective modules over any Robba rings $\tilde{\Pi}_{I,*,B}$ such that the interval $I = [r_1, r_2]$ satisfies that $0 \leq r_1 \leq r_2/p^{hk}$;*

Then we have that they are actually equivalent.

Proof. Note that in this situation the corresponding space is not even analytic adic space, but the Robba rings over some certain perfectoid domains are analytic. This is parallel to [KL2, Theorem 4.6.1] The proof is the same as the one where $*$ is just a ring R . See [XT2, Theorem 4.11]. \square

10.6.2 Derived Algebraic Geometry of ∞ -Period Rings

We now consider the contact with the corresponding algebraic sheaves over the corresponding schemes associated to the corresponding period rings in [KL1], [KL2] as in [XT1], [XT2], [XT3], [XT4] along [KL1].

Remark 10.6.14. Certainly we are now dropping the corresponding topology and functional analyticity. This is motivated by the corresponding discussion on the algebraic quasi-coherent sheaves over schematic Fargues-Fontaine curves in [KL2].

Setting 10.6.15. A is assumed to be now general Banach (commutative at this moment) over the base analytic field, and keep R as in the situation before in this section (in the most general setting we considered so far). Now we apply the whole machinery in [Lu1] to the corresponding ∞ -Bambozzi-Kremnizer rings $*_{R,A}^h$ attached to the period rings $*_{R,A}$ carrying A . Then we regard $*_{R,A}^h$ as a corresponding \mathbb{E}_1 -ring in [Lu1]³.

Proposition 10.6.16. *The corresponding category of all the perfect, almost perfect (after [Lu1, Section 7.2.4], namely pseudocoherent) F^k -equivariant sheaves over the families of ∞ -Robba étale ∞ -Deligne-Mumford toposes $(\mathrm{Spec} \tilde{\Pi}_{I,R,A}^h, \mathcal{O})_{\{I \subset (0, \infty)\}}$ (as in [Lu2, Chapter 1.4]) is equivalent to the category of all the perfect, almost perfect (after [Lu1,*

³Note that the original convention of [BK] is cohomological and vanishing in positive degree.

Section 7.2.4], namely pseudocoherent) F^k -equivariant sheaves over some ∞ -Robba étale ∞ -Deligne-Mumford topos $(\mathrm{Spec}\tilde{\Pi}_{I,R,A}^h, \mathcal{O})$ (as in [Lu2, Chapter 1.4]) where $I = [r_1, r_2]$ ($0 < r_1 \leq r_2/p^{hk}$). Here we need to assume the corresponding conjecture 10.6.21 holds.

Proof. This is actually quite transparent that we just take the corresponding Frobenius translation to compare the corresponding sheaves. \square

Proposition 10.6.17. *The corresponding category of all the locally finite free (after [Lu1, Section 7.2.2, 7.2.4], namely pseudocoherent and flat) F^k -equivariant sheaves over the families of ∞ -Robba étale ∞ -Deligne-Mumford toposes $(\mathrm{Spec}\tilde{\Pi}_{I,R,A}^h, \mathcal{O})_{\{I \subset (0, \infty)\}}$ (as in [Lu2, Chapter 1.4]) is equivalent to the category of all the locally finite free (after [Lu1, Section 7.2.2, 7.2.4], namely pseudocoherent and flat) F^k -equivariant sheaves over some ∞ -Robba étale ∞ -Deligne-Mumford topos $(\mathrm{Spec}\tilde{\Pi}_{I,R,A}^h, \mathcal{O})$ (as in [Lu2, Chapter 1.4]) where $I = [r_1, r_2]$ ($0 < r_1 \leq r_2/p^{hk}$).*

Proof. This is again actually quite transparent that we just take the corresponding Frobenius translation to compare the corresponding sheaves. \square

Example 10.6.18. Everything will be certainly more interesting when we maintain in the corresponding noetherian situation, where we do have the corresponding derived analytic consideration. But since our current goal is to study the corresponding derived algebraic geometry carrying relative p -adic Hodge structure we will not explicitly mention the corresponding analytic context.

Similarly as what we did in the corresponding derived analytic geometry above, we consider the corresponding contact with [GV] in the derived algebraic geometric setting. We keep the notation as above. Then we have the following results with the same arguments as in the above on derived deformation of Hodge structures (note that this is more related to the non-derived situation):

Proposition 10.6.19. *The corresponding category of all the perfect, almost perfect (after [Lu1, Section 7.2.4], namely pseudocoherent) F^k -equivariant sheaves over the families of ∞ -Robba étale ∞ -Deligne-Mumford toposes $(\mathrm{Spec}\tilde{\Pi}_{I,R,A^h}^h, \mathcal{O})_{\{I \subset (0, \infty)\}}$ (as in [Lu2, Chapter 1.4]) is equivalent to the category of all the perfect, almost perfect (after [Lu1, Section 7.2.4], namely pseudocoherent) F^k -equivariant sheaves over some ∞ -Robba étale*

∞ -Deligne-Mumford topos $(\mathrm{Spec}\tilde{\Pi}_{I,R,A^h}, \mathcal{O})$ (as in [Lu2, Chapter 1.4]) where $I = [r_1, r_2]$ ($0 < r_1 \leq r_2/p^{hk}$). Here we need to assume the corresponding conjecture 10.6.21 holds.

Proposition 10.6.20. *The corresponding category of all the locally finite free (after [Lu1, Section 7.2.2, 7.2.4], namely pseudocoherent and flat) F^k -equivariant sheaves over the families of ∞ -Robba étale ∞ -Deligne-Mumford toposes $(\mathrm{Spec}\tilde{\Pi}_{I,R,A^h}, \mathcal{O})_{\{I \subset (0, \infty)\}}$ (as in [Lu2, Chapter 1.4]) is equivalent to the category of all the locally finite free (after [Lu1, Section 7.2.2, 7.2.4], namely pseudocoherent and flat) F^k -equivariant sheaves over some ∞ -Robba étale ∞ -Deligne-Mumford topos $(\mathrm{Spec}\tilde{\Pi}_{I,R,A^h}, \mathcal{O})$ (as in [Lu2, Chapter 1.4]) where $I = [r_1, r_2]$ ($0 < r_1 \leq r_2/p^{hk}$).*

The following is expected to hold in full generality.

Conjecture 10.6.21. *In the two settings in this subsection above, we conjecture that the corresponding descent holds (along binary rational coverings) for perfect, almost perfect and finite projective modules (which will allow us to compare sheaves and the global sections) in the noetherian situation.*

Remark 10.6.22. In fact we believe that this is a direct consequence (in the Huber ring setting) after we apply the results of [CS, Theorem 14.9, Remark 14.10] since we are in the noetherian setting.

Remark 10.6.23. The corresponding descent could happen when we have the noetherianness, where the corresponding rational localization is actually automatically flat, which will imply that one can certainly glue coherent sheaves in our situation, see [KL1, Theorem 1.3.9]. Beyond the corresponding noetherianness it might be not safe to conjecture so. One can also use [CS] to tackle this in the corresponding context of even general ∞ -quasi-coherent sheaves, and beyond the corresponding noetherian situation purely algebraic definition will be very hard to control (where we think we have to definitely replace the purely algebraic consideration with the corresponding consideration in [CS] or [BK]).

This chapter is based on the following paper, where the author of this dissertation is the main author:

- Tong, Xin. "Topics on Geometric and Representation Theoretic Aspects of Period Rings I." arXiv preprint arXiv:2102.10693 (2021).

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Chapter 11

Topologization and Functional Analytification I: Intrinsic Morphisms of Commutative Algebras

11.1 Introduction

11.1.1 Main Consideration

Scholze's diamond [Sch2] is actually very general notion beyond the corresponding perfectoid spaces, partially because it contains the corresponding diamantine spaces after Hansen-Kedlaya [HK]. This point of view certainly gives the motivation for this notion from Hansen-Kedlaya. Therefore one could regard to some extent diamantine spaces as giving some (maybe better to say more ring theoretic) analogs of the corresponding Scholze's diamonds [Sch2] [FS], moreover they behave as if they are perfectoid spaces [Sch3], [KL1], [KL2], [Ked1] and [GR]. Similar discussion could be made to sousperfectoid space, which should be more 'perfectoid' generalization.

Hansen-Kedlaya [HK] have given the definition of naive étale morphisms among any Tate Huber pairs namely these are locally composites of the rational localizations and finite étale morphisms (very importantly with strongly sheafy domains and targets). This is because one definitely believes that correct notion of étale morphism should admit such admissible decomposition and factorization. However what should be the correct intrinsic one has not been given in full detail yet. In the significant strongly sheafy situation, we are going to try to answer this question as proposed in [Ked1, Appendix 5]. In this paper, we try to study the corresponding properties of the corresponding naive étale morphisms along the ideas of [Ked1, Appendix 5] and [HK]. The goal is to accurately characterize the corresponding naive étale morphisms in some intrinsic way.

Sheafiness plays a very crucial role in the discussion above. However suppose we do not have to worry about the sheafiness at all (in fact in some sense we really do not have to worry about this at all by the work of Clausen-Scholze [CS] and Bambozzi-Kremnizer [BK]), then one might want to believe that the robust definitions could be made even more robust. Therefore we investigate the corresponding morphisms of the corresponding ring objects where sheafiness could be replaced by ∞ -sheafiness (namely sheafiness up to higher homotopy) after [BBBK] and [BK]. One should be able to consider Clausen-Scholze's foundation [CS] as well, however we will mainly focus on the ∞ -locally convex objects in [BBBK] and [BK], as in the corresponding schematic situation in [Lu1],

[Lu2], [TV1] and [TV2]. We consider the corresponding interesting approaches through the corresponding formal and PD completions just as in the ∞ -schematic situation in [R] which is very related to the corresponding Drinfeld's stacky construction [Dr1] and [Dr2] by using the Čech-Alexander complex on the corresponding crystalline cohomology and prismatic cohomology.

The current list of definitions will be established for discrete \mathbb{E}_∞ -ring objects and \mathbb{E}_∞ -ring objects in suitable locally convex ∞ -categories after after Bambozzi-Ben-Bassat-Kremnizer [BBBK]:

- D1. Localized intrinsic étale morphisms of open mapping Huber rings;
- D2. Localized intrinsic étale morphisms of open mapping adic Banach rings;
- D3. Localized intrinsic lisse morphisms of open mapping Huber rings;
- D4. Localized intrinsic lisse morphisms of open mapping adic Banach rings;
- D5. Localized intrinsic non-ramifié morphisms of open mapping Huber rings;
- D6. Localized intrinsic non-ramifié morphisms of open mapping adic Banach rings;
- ∞ 1. De Rham intrinsic étale-like morphisms of ∞ -analytic functors;
- ∞ 2. De Rham intrinsic lisse-like morphisms of ∞ -analytic functors;
- ∞ 3. De Rham intrinsic non-ramifié-like morphisms of ∞ -analytic functors;
- ∞ 4. PD (crystalline) intrinsic étale-like morphisms of ∞ -analytic functors;
- ∞ 5. PD (crystalline) intrinsic lisse-like morphisms of ∞ -analytic functors;
- ∞ 6. PD (crystalline) intrinsic non-ramifié-like morphisms of ∞ -analytic functors.

Certainly for general locally convex spaces producing nice ring structures we really have to be very precise and accurate in any sorts of characterization. However we have not unfortunately achieve this due to some very subtle issues, mainly coming from the corresponding issues in very general functional analytification. That being otherwise all said, we still actually could literally talk about the desired definitions for simplicial noetherian Banach rings in certain situations.

11.1.2 Further Consideration

Our ultimate goal is certainly to study the corresponding geometric sites (étale, pro-étale, crystalline and prismatic [SGAIV],[Gro1],[Sch1],[Sch2],[Sch3],[KL1],[KL2],[Ked1],[BS],[Dr1],[Dr2]) and the corresponding cohomologies (étale, pro-étale, crystalline and prismatic) for really general ∞ -analytic spaces (possibly also noncommutative analogs of those in [KR1]) over \mathbb{F}_1 and try to apply to the locally noetherian situations, the strongly noetherian situations (such as in [G1], [GL]), the strongly sheafy situations under the foundation of ∞ -locally convex spaces (as in [HK], [KL1] and more general situations), although our very beginning corresponding motivation for this article is an attempt to answer some questions in [Ked1, Appendix A5].

11.2 Affinoid Morphisms of Huber Rings

We start with the discussion on the corresponding intrinsic definition of étale morphisms.

Setting 11.2.1. We start with an analytic uniform Huber pair (A, A^+) . And we will consider the category of all such rings. We assume the corresponding completeness for the Huber pairs.

Definition 11.2.2. (Hansen-Kedlaya [HK, Definition 5.1]) We call a map of Huber rings $(A, A^+) \rightarrow (B, B^+)$ naive étale after [HK, Definition 5.1] if it admit a factorization into rational localizations and finite étale morphisms. Here we assume (A, A^+) is strong sheafy and we assume that (B, B^+) is strongly sheafy.¹

Definition 11.2.3. (Kedlaya [Ked1, Definition A5.2]) Recall from [Ked1, Definition A5.2], we have the corresponding affinoid morphism from any strongly sheafy Huber ring A , namely a morphism $A \rightarrow B$, such that B admits some surjective covering from $A \langle T_1, \dots, T_d \rangle$ and through this map we have that B is a stably-pseudocoherent sheaf over $A \langle T_1, \dots, T_d \rangle$ and the corresponding ring B is assumed to be sheafy².

The belief (as proposed in [Ked1, Problem A5.3, Problem A5.4]) is that somehow the corresponding affinoid morphisms in the definition should be directly used in the corresponding definitions of lisse morphisms and unramified morphisms, as well as certainly the étale morphisms. To investigate this kind of idea, we are going to first investigate the corresponding naive étale morphisms along this idea.

Lemma 11.2.4. *Let $f_1 : \Gamma_1 \rightarrow \Gamma_2$ and $f_2 : \Gamma_2 \rightarrow \Gamma_3$ be two affinoid morphisms, then the composition $f_2 \circ f_1$ is also affinoid.*

Proof. Straightforward. □

¹Certainly one needs to be careful since we are now considering more general context than [HK] without assuming the corresponding Tateness.

²Certainly one needs to be more careful since this is also slightly different from the original definition [Ked1, Definition A5.2], thanks Professor Kedlaya for telling me this should be better. We want to mention that this is a quite subtle point around the sheafiness (see [Ked1, Theorem 1.4.20]), the point here is that we do not know the kernel of an affinoid morphism is closed or not, if it is closed then we could keep the knowledge that B being stably-pseudocoherent is equivalent to B being sheafy. However if this is not closed, then we do not have this sort of equivalence to our knowledge.

Lemma 11.2.5. (Kedlaya) *For any standard binary rational localization of A with respect to $f, g \in A$, suppose we know that there are two surjective morphisms:*

$$s_1 : A \left\langle \frac{f}{g} \right\rangle \langle T_1, \dots, T_n \rangle \rightarrow B \left\langle \frac{f}{g} \right\rangle, \quad (11.2.1)$$

$$s_2 : A \left\langle \frac{g}{f} \right\rangle \langle T_1, \dots, T_n \rangle \rightarrow B \left\langle \frac{g}{f} \right\rangle. \quad (11.2.2)$$

Then we have that there is a surjective morphism:

$$s : A \langle T_1, \dots, T_n \rangle \rightarrow B. \quad (11.2.3)$$

Proof. The following argument is due to Kedlaya ¹, we work out it for the convenience of the readers. First, we have the following short exact sequence:

$$0 \longrightarrow B \longrightarrow B \left\langle \frac{f}{g} \right\rangle \oplus B \left\langle \frac{g}{f} \right\rangle \longrightarrow B \left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \longrightarrow 0.$$

Take any $b \in B$, and use the notation (b_1, b_2) for the image in the middle. By the surjectivity of the maps s_1, s_2 we have that there exist some element $a_1 \in A \left\langle \frac{f}{g} \right\rangle \langle T_1, \dots, T_n \rangle$ and some element $a_2 \in A \left\langle \frac{g}{f} \right\rangle \langle T_1, \dots, T_n \rangle$ such that we have:

$$s_1(a_1) = b_1, \quad (11.2.4)$$

$$s_2(a_2) = b_2. \quad (11.2.5)$$

With more explicit expression we have the following:

$$s_1 \left(\sum_{i_1, \dots, i_n} \sum_i a_1^{i, i_1, \dots, i_n} u^i T_1^{i_1} \dots T_n^{i_n} \right) = \sum_i b_1^i u^i, \quad (11.2.6)$$

$$s_2 \left(\sum_{i_1, \dots, i_n} \sum_i a_2^{i, i_1, \dots, i_n} v^i T_1^{i_1} \dots T_n^{i_n} \right) = \sum_i b_2^i v^i, \quad (11.2.7)$$

under the corresponding presentations up to liftings:

$$B \left\langle \frac{f}{g} \right\rangle = B \langle u \rangle / (gu - f), \quad (11.2.8)$$

$$B \left\langle \frac{g}{f} \right\rangle = B \langle v \rangle / (fv - g). \quad (11.2.9)$$

$$(11.2.10)$$

¹Thanks Professor Kedlaya for mentioning the similarity of this to the corresponding locality of morphisms of finite type as in Grothendieck's EGA I and II.

Then to finish we only have to take some finite sum in the summation to make approximation. We first claim that such finite sum approximation and modification will not change the corresponding surjectivity of the map s_1 and s_2 . Namely for each $k = 1, 2$ the map s_k will maintain surjective once we modify the image of T_1, \dots, T_n infinitesimally around some neighbourhood U of 0, in other words it will maintain to be surjective even if we set $s_k(T_1), \dots, s_k(T_n)$ to be x_1, \dots, x_n whenever $x_1 - s_k(T_1), \dots, x_n - s_k(T_n)$ lives in the neighbourhood U , and moreover we have that the corresponding modification could be assumed to take T_i to x_i with $i = 1, \dots, n$. By open mapping, we have that the corresponding lifts of the corresponding differences $x_1 - s_k(T_1), \dots, x_n - s_k(T_n)$ could be made to be living in some arbitrarily chosen neighbourhood V of 0. Then we only have to consider the following map factoring through the corresponding map s_k :

$$h : A_k \langle T_1, \dots, T_n \rangle \rightarrow A_k \langle T_1, \dots, T_n \rangle \quad (11.2.11)$$

$$T_i \mapsto T_i + \text{lifts of } x_i - s_k(T_k) \quad (11.2.12)$$

where A_1 is the ring $A \langle \frac{f}{g} \rangle$ while we have A_2 is the ring $A \langle \frac{g}{f} \rangle$, which basically proves the claim. Then this will indicate that one can find some joint finite subset $T := \{T_1, \dots, T_{n'}\}$ for $B \langle \frac{f}{g} \rangle$ and $B \langle \frac{g}{f} \rangle$ such that the modified

$$s_1 : A \langle \frac{f}{g} \rangle \langle T_1, \dots, T_{n'} \rangle \rightarrow B \langle \frac{f}{g} \rangle, \quad (11.2.13)$$

$$s_2 : A \langle \frac{g}{f} \rangle \langle T_1, \dots, T_{n'} \rangle \rightarrow B \langle \frac{g}{f} \rangle, \quad (11.2.14)$$

are basically surjective and they fit into the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \langle \frac{f}{g} \rangle \langle T_1, \dots, T_{n'} \rangle \oplus A \langle \frac{g}{f} \rangle \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \langle \frac{f}{g}, \frac{g}{f} \rangle \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & B \langle \frac{f}{g} \rangle \oplus B \langle \frac{g}{f} \rangle & \longrightarrow & B \langle \frac{f}{g}, \frac{g}{f} \rangle & \longrightarrow & 0,
\end{array}$$

where the middle and the rightmost vertical arrows are surjective. Then claim is then that the left vertical one is also surjective. The kernels $K_1 \oplus K_2$ in the middle is mapped surjectively to the kernel K_{12} of the rightmost vertical map. So the snake lemma will force the cokernel of the left vertical arrow to be zero which shows the corresponding exactness

at the corresponding location ? in the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & K_1 \oplus K_2 & \longrightarrow & K_{12} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \langle \frac{f}{g} \rangle \langle T_1, \dots, T_{n'} \rangle \oplus A \langle \frac{q}{f} \rangle \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \langle \frac{f}{g}, \frac{q}{f} \rangle \langle T_1, \dots, T_{n'} \rangle \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B & \longrightarrow & B \langle \frac{f}{g} \rangle \oplus B \langle \frac{q}{f} \rangle & \longrightarrow & B \langle \frac{f}{g}, \frac{q}{f} \rangle \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where K_1, K_2, K_{12} are pseudocoherent, which implies that the corresponding module K is also pseudocoherent.

□

Proposition 11.2.6. *Any naive étale morphism is affinoid.*

Proof. First let $f : (A, A^+) \rightarrow (B, B^+)$ be any naive étale morphism. Then locally this is basically composition of the corresponding rational localizations and finite étale maps. Locally rational localizations involved are actually affinoid, and locally the corresponding finite étale maps from strongly sheafy rings will have strongly sheafy target, which will imply that locally finite étale maps are affinoid. Then this could be globalized to force the global map f to be affinoid. The properties of factoring through a surjection globally could be proved by glueing local ones through lemma 11.3.6, by considering [KL1, Proposition 2.4.20]. And globally the corresponding ring B is stably-pseudocoherent over $A \langle T_1, \dots, T_n \rangle$ for some n since this is a local property. □

Therefore we have proved that the corresponding étale maps in the corresponding naive sense is actually affinoid in the above sense. Therefore it is now natural to try to

find the corresponding properties which may completely characterize the corresponding naive étale morphisms which are affinoid.

Certainly we may have the corresponding conjectures that all the naive étale morphisms will satisfy the corresponding properties of algebraically étale ones (such as in [EGAIV4, Chapitre 17], [SP, Tag 00U1]). We now discuss the corresponding completed cotangent complex after Huber [Hu1, 1.6.2]. Recall for our current B the corresponding completed differential $\Omega_{B/A, \text{topo}}^1$ (see [Hu1, 1.6.2] for the construction for any f -adic rings in the noetherian setting). Therefore we consider the corresponding topological naive cotangent complex:

$$\tau_{\leq 1} \mathbb{L}_{B/A, \text{topo}} \quad (11.2.15)$$

for any naive étale map $f : (A, A^+) \rightarrow (B, B^+)$. We now discuss the construction without the corresponding strongly noetherian requirement in our current situation. First we know that B is of topologically finite type over A :

$$B = A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} / I. \quad (11.2.16)$$

Then we could first define the topological free differentials:

$$\Omega^1 := A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_1 + \dots + A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_n. \quad (11.2.17)$$

Then we have:

$$\Omega_{B/A, \text{topo}}^1 := \Omega^1 / (I \cup d(I)) \Omega^1. \quad (11.2.18)$$

Here everything is assumed to be basically complete with respect to the corresponding natural topology. Namely we need to take the corresponding completion always with respect to the corresponding induced topology. Certainly here Ω^1 is already complete due to the fact that it is finitely projective. Recall that a map $f : \Gamma_1 \rightarrow \Gamma_2$ is called étale in the scheme theory if the naive cotangent complex (truncated and could be regarded as an ∞ -module spectrum) is quasi-isomorphic to zero. The corresponding underlying complex reads:

$$[I/I^2 \longrightarrow \Omega_{\Gamma_2/\Gamma_1, \text{topo}}^1].$$

In the situation where we consider $A \rightarrow B$ is affinoid, the corresponding ideal I is actually stably-pseudocoherent over $A \langle X_1, \dots, X_n \rangle$. It is pseudocoherent by the corresponding two out of three property. The stability holds locally, so we have the case. And if

the morphism if furthermore naive étale then we have I/I^2 is also stably-pseudocoherent, see lemma 11.4.8.

Remark 11.2.7. Note that we are considering the very general and complicated non-noetherian situation, modules will need to be endowed with the natural topology and complete, although finite projective modules are complete automatically. This will have nontrivial things to do with the corresponding definition of $\Omega_{B/A, \text{topo}}^1$.

One can actually generalize the corresponding full cotangent complexes and derived de Rham complexes to this topological context following [III1], [III2] and [B1]. First for the corresponding topological cotangent complex we consider the following definition (note that we have to assume the corresponding topologically finite type condition). We start with the corresponding algebraic ones for $B^h = A[X_1, \dots, X_n]_{T_1, \dots, T_n}/I$, under the topologization we have the corresponding derived cotangent complex:

$$\mathbb{L}_{B^h/A, \text{alg}}, \quad (11.2.19)$$

by taking the usual algebraic one. Then we take the corresponding completion with respect to the corresponding topologization which gives rise to the following topological one:

$$\mathbb{L}_{B/A, \text{topo}}. \quad (11.2.20)$$

We define the corresponding de Rham complex in the following parallel way. What is happen is that consider the presentation $B^h = A[X_1, \dots, X_n]_{T_1, \dots, T_n}/I$ which gives rise to the corresponding algebraic de Rham complex:

$$0 \longrightarrow B^h \longrightarrow \Omega_{B^h/A, \text{alg}}^1 \longrightarrow \Omega_{B^h/A, \text{alg}}^2 \longrightarrow \dots \longrightarrow \Omega_{B^h/A, \text{alg}}^\bullet \longrightarrow \dots,$$

which will give rise to the corresponding topological one if we take the corresponding completion induced from the subset T_1, \dots, T_n :

$$0 \longrightarrow B \longrightarrow \Omega_{B^h/A, \text{topo}}^1 \longrightarrow \Omega_{B^h/A, \text{topo}}^2 \longrightarrow \dots \longrightarrow \Omega_{B^h/A, \text{topo}}^\bullet \longrightarrow \dots$$

From our construction for $\Omega_{B^h/A, \text{topo}}^1$, one can actually define:

$$\Omega_{B^h/A, \text{topo}}^{\bullet, \text{f}} := \bigoplus_{i_1, \dots, i_\bullet \in \{1, \dots, n\}} A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_{i_1} \wedge dX_{i_2} \wedge \dots \wedge dX_{i_\bullet}. \quad (11.2.21)$$

and then define:

$$\Omega_{B/A, \text{topo}}^\bullet := \left(\bigoplus_{i_1, \dots, i_\bullet \in \{1, \dots, n\}} A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_{i_1} \wedge dX_{i_2} \wedge \dots \wedge dX_{i_\bullet} \right) / \quad (11.2.22)$$

$$\left((I \cup dI \cup d^\bullet I) \bigoplus_{i_1, \dots, i_\bullet \in \{1, \dots, n\}} A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_{i_1} \wedge dX_{i_2} \wedge \dots \wedge dX_{i_\bullet} \right), \quad (11.2.23)$$

after taking suitable completion when needed ¹.

¹11As in [B1] and [GL] where one takes the corresponding derived p -completion out from the algebraic cotangent complex and the corresponding derived algebraic de Rham complex.

11.3 Affinoid Morphisms of Banach Rings

We now consider the parallel situation of Banach rings.

Setting 11.3.1. We start with a uniform adic Banach ring (A, A^+) in the general sense of [KL1] and [KL2] (without assumption on the topologically nilpotent units being existing, but we assume this is open mapping). And we will consider the category of all such rings. We assume the corresponding completeness as well.

Definition 11.3.2. (Hansen-Kedlaya [HK, Definition 5.1]) We call a map of adic Banach rings $(A, A^+) \rightarrow (B, B^+)$ naive étale after [HK, Definition 5.1] if it admit a factorization into rational localizations and finite étale morphisms. Here we assume (A, A^+) is strong sheafy and we assume that (B, B^+) is sheafy.

Remark 11.3.3. Certainly this is in more general situation than the corresponding context of [HK].

Definition 11.3.4. (Kedlaya [Ked1, Definition A5.2]) Recall from [Ked1, Definition A5.2], we have the corresponding affinoid morphism from any strongly sheafy adic Banach ring A , namely a morphism $A \rightarrow B$, such that B admits some surjective covering from $A \langle T_1, \dots, T_d \rangle$ and through this map we have that B is a stably-pseudocoherent sheaf over $A \langle T_1, \dots, T_d \rangle$ and we assume that (B, B^+) is strongly sheafy.

Remark 11.3.5. Of course the corresponding foundation is not the same but parallel to such situation we are considering now, however it is definitely reasonable and parallel to follow [Ked1, Appendix A5] to give the definition here.

The belief (as proposed in [Ked1, Problem A5.3, Problem A5.4]) is that somehow the corresponding affinoid morphisms in the definition should be directly used in the corresponding definitions of lisse morphisms and unramified morphisms, as well as certainly the étale morphisms. To investigate this kind of idea, we are going to first investigate the corresponding naive étale morphisms along this idea.

Lemma 11.3.6. (Kedlaya) *For any standard binary rational localization of A with re-*

spect to $f, g \in A$, suppose we know that there are two surjective morphisms:

$$s_1 : A \left\langle \frac{f}{g} \right\rangle \langle T_1, \dots, T_n \rangle \rightarrow B \left\langle \frac{f}{g} \right\rangle, \quad (11.3.1)$$

$$s_2 : A \left\langle \frac{g}{f} \right\rangle \langle T_1, \dots, T_n \rangle \rightarrow B \left\langle \frac{g}{f} \right\rangle. \quad (11.3.2)$$

Then we have that there is a surjective morphism:

$$s : A \langle T_1, \dots, T_n \rangle \rightarrow B. \quad (11.3.3)$$

Proof. The following argument is due to Kedlaya, we work out it for the convenience of the readers. And this is the corresponding Banach analog of corresponding parallel result in the Huber ring situation. First, we have the following short exact sequence:

$$0 \longrightarrow B \longrightarrow B \left\langle \frac{f}{g} \right\rangle \oplus B \left\langle \frac{g}{f} \right\rangle \longrightarrow B \left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \longrightarrow 0.$$

Take any $b \in B$, and use the notation (b_1, b_2) for the image in the middle. By the surjectivity of the maps s_1, s_2 we have that there exist some element $a_1 \in A \left\langle \frac{f}{g} \right\rangle \langle T_1, \dots, T_n \rangle$ and some element $a_2 \in A \left\langle \frac{g}{f} \right\rangle \langle T_1, \dots, T_n \rangle$ such that we have:

$$s_1(a_1) = b_1, \quad (11.3.4)$$

$$s_2(a_2) = b_2. \quad (11.3.5)$$

With more explicit expression we have the following:

$$s_1 \left(\sum_{i_1, \dots, i_n} \sum_i a_1^{i, i_1, \dots, i_n} u^i T_1^{i_1} \dots T_n^{i_n} \right) = \sum_i b_1^i u^i, \quad (11.3.6)$$

$$s_2 \left(\sum_{i_1, \dots, i_n} \sum_i a_2^{i, i_1, \dots, i_n} v^i T_1^{i_1} \dots T_n^{i_n} \right) = \sum_i b_2^i v^i, \quad (11.3.7)$$

under the corresponding presentations up to liftings:

$$B \left\langle \frac{f}{g} \right\rangle = B \langle u \rangle / (gu - f), \quad (11.3.8)$$

$$B \left\langle \frac{g}{f} \right\rangle = B \langle v \rangle / (fv - g). \quad (11.3.9)$$

$$(11.3.10)$$

Then to finish we only have to take some finite sum in the summation to make approximation. We first claim that such finite sum approximation and modification will not change

the corresponding surjectivity of the map s_1 and s_2 . Namely for each $k = 1, 2$ the map s_k will maintain surjective once we modify the image of T_1, \dots, T_n infinitesimally around some neighbourhood U of 0, in other words it will maintain to be surjective even if we set $s_k(T_1), \dots, s_k(T_n)$ to be x_1, \dots, x_n whenever $\|x_1 - s_k(T_1)\| \leq \delta, \dots, \|x_n - s_k(T_n)\| \leq \delta$ for some prescribed constant $\delta < 1$ and moreover we have that the corresponding modification could be assumed to take T_i to x_i with $i = 1, \dots, n$. By open mapping in this current context, we have that the corresponding lifts of the corresponding differences $x_1 - s_k(T_1), \dots, x_n - s_k(T_n)$ could be made to be living in some arbitrarily chosen neighbourhood V of 0 namely, we can find lifts y_1, \dots, y_n of these differences such that

$$\|y_1\| < 1, \dots, \|y_n\| < 1. \quad (11.3.11)$$

Then we only have to consider the following map factors through the corresponding map s_k :

$$h : A_k \langle T_1, \dots, T_n \rangle \rightarrow A_k \langle T_1, \dots, T_n \rangle \quad (11.3.12)$$

$$T_i \mapsto T_i + \text{lifts of } x_i - s_k(T_k) \quad (11.3.13)$$

where A_1 is the ring $A \left\langle \frac{f}{g} \right\rangle$ while we have A_2 is the ring $A \left\langle \frac{g}{f} \right\rangle$, which basically proves the claim. Then this will indicate that one can find some joint finite subset $T := \{T_1, \dots, T_{n'}\}$ for $B \left\langle \frac{f}{g} \right\rangle$ and $B \left\langle \frac{g}{f} \right\rangle$ such that the modified

$$s_1 : A \left\langle \frac{f}{g} \right\rangle \langle T_1, \dots, T_{n'} \rangle \rightarrow B \left\langle \frac{f}{g} \right\rangle, \quad (11.3.14)$$

$$s_2 : A \left\langle \frac{g}{f} \right\rangle \langle T_1, \dots, T_{n'} \rangle \rightarrow B \left\langle \frac{g}{f} \right\rangle, \quad (11.3.15)$$

are basically surjective and they fit into the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \left\langle \frac{f}{g} \right\rangle \langle T_1, \dots, T_{n'} \rangle \oplus A \left\langle \frac{g}{f} \right\rangle \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \langle T_1, \dots, T_{n'} \rangle \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & B \left\langle \frac{f}{g} \right\rangle \oplus B \left\langle \frac{g}{f} \right\rangle & \longrightarrow & B \left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \longrightarrow 0 \end{array}$$

where the middle and the rightmost vertical arrows are surjective. Then claim is then that the left vertical one is also surjective. The kernels $K_1 \oplus K_2$ in the middle is mapped

surjectively to the kernel K_{12} of the rightmost vertical map. So the snake lemma will force the cokernel of the left vertical arrow to be zero which shows the corresponding exactness at the corresponding location ? in the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & K_1 \oplus K_2 & \longrightarrow & K_{12} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \langle \frac{f}{g} \rangle \langle T_1, \dots, T_{n'} \rangle \oplus A \langle \frac{q}{f} \rangle \langle T_1, \dots, T_{n'} \rangle & \longrightarrow & A \langle \frac{f}{g}, \frac{q}{f} \rangle \langle T_1, \dots, T_{n'} \rangle \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & B & \longrightarrow & B \langle \frac{f}{g} \rangle \oplus B \langle \frac{q}{f} \rangle & \longrightarrow & B \langle \frac{f}{g}, \frac{q}{f} \rangle \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where K_1, K_2, K_{12} are pseudocoherent, which implies that the corresponding module K is also pseudocoherent.

□

Lemma 11.3.7. *Let $f_1 : \Gamma_1 \rightarrow \Gamma_2$ and $f_2 : \Gamma_2 \rightarrow \Gamma_3$ be two affinoid morphisms, then the composition $f_2 \circ f_1$ is also affinoid.*

Proof. Straightforward.

□

Proposition 11.3.8. *Any naive étale morphism is affinoid.*

Proof. See proposition 11.3.8.

□

Therefore as in the corresponding Huber pair situation, we have proved that the corresponding étale maps in the corresponding naive sense is actually affinoid in the above

sense. Therefore it is now natural to try to find the corresponding properties which may completely characterize the corresponding naive étale morphisms which are affinoid.

Certainly we may have the corresponding conjectures that all the naive étale morphisms will satisfy the corresponding properties of algebraically étale ones (such as in [EGAIV4, Chapitre 17], [SP, Tag 00U1]). We now discuss the corresponding Banach completed cotangent complex after Huber [Hu1, 1.6.2]. Recall for our current B the corresponding completed differential $\Omega_{B/A, \text{topo}}^1$ (see [Hu1, 1.6.2] for the construction for any f -adic rings). Certainly in the context of adic Banach rings we have the parallel completed version of the differentials by taking Banach completion, which is in some trivial way the current situation. Therefore we consider the corresponding topological naive cotangent complex:

$$\tau_{\leq 1} \mathbb{L}_{B/A, \text{topo}} \tag{11.3.16}$$

for any naive étale map $f : (A, A^+) \rightarrow (B, B^+)$. We consider the construction without the corresponding strongly noetherian requirement in our current situation. First we know that B is of topologically finite type over A :

$$B = A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} / I. \tag{11.3.17}$$

Then we could first define the Banach free differentials:

$$\Omega^1 := A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_1 + \dots + A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_n. \tag{11.3.18}$$

Then we have:

$$\Omega_{B/A, \text{topo}}^1 := \Omega^1 / (I \bigcup d(I)) \Omega^1. \tag{11.3.19}$$

Here everything is assumed to be basically complete with respect to the corresponding natural topology. Namely we need to take the corresponding completion always with respect to the corresponding induced norms. Certainly here Ω^1 is already complete due to the fact that it is finitely projective. Recall that a map $f : \Gamma_1 \rightarrow \Gamma_2$ is called étale in the scheme theory if the naive cotangent complex (truncated and could be regarded as an ∞ -module spectrum) is quasi-isomorphic to zero. The corresponding underlying complex reads:

$$[I/I^2 \longrightarrow \Omega_{\Gamma_2/\Gamma_1, \text{topo}}^1].$$

In the situation where we consider $A \rightarrow B$ is affinoid, the corresponding ideal I is actually stably-pseudocoherent over $A \langle T_1, \dots, T_n \rangle$. It is pseudocoherent by the corresponding two out of three property. The stability holds locally, so we have the case. And if the morphism is furthermore naive étale then we have I/I^2 is also stably-pseudocoherent, see lemma 11.4.8.

Remark 11.3.9. Note that we are considering the very general and complicated non-noetherian situation, modules will need to be endowed with the natural topology coming from the Banach structures on the Banach rings and complete, although in this situation as well finite projective modules are complete automatically. This will have nontrivial things to do with the corresponding definition of $\Omega_{B/A, \text{topo}}^1$.

In the Banach world, one can actually generalize the corresponding full cotangent complexes and de Rham complex to this context. First for the corresponding topological cotangent complex we consider the following definition (note that we have to assume the corresponding topologically finite type condition). We start with the corresponding algebraic ones for $B^h = A[X_1, \dots, X_n]_{T_1, \dots, T_n}/I$, under the topologization we have the corresponding derived cotangent complex:

$$\mathbb{L}_{B^h/A, \text{alg}}, \tag{11.3.20}$$

by taking the usual algebraic one. Then we take the corresponding completion with respect to the corresponding topologization which gives rise to the following topological one:

$$\mathbb{L}_{B/A, \text{topo}}. \tag{11.3.21}$$

We define the corresponding de Rham complex in the following way parallelly. What is happen is that consider the presentation $B^h = A[X_1, \dots, X_n]_{T_1, \dots, T_n}/I$ which gives rise to the corresponding algebraic de Rham complex:

$$0 \longrightarrow B^h \longrightarrow \Omega_{B^h/A, \text{alg}}^1 \longrightarrow \Omega_{B^h/A, \text{alg}}^2 \longrightarrow \dots \longrightarrow \Omega_{B^h/A, \text{alg}}^\bullet \longrightarrow \dots,$$

which will give rise to the corresponding topological one if we take the corresponding

completion under $(\|\cdot\|, \text{Ban})$ induced from the subset T_1, \dots, T_n :

$$0 \longrightarrow B_{\|\cdot\|, \text{Ban}}^h \longrightarrow \Omega_{B^h/A, \text{alg}, \|\cdot\|, \text{Ban}}^1 \longrightarrow \Omega_{B^h/A, \text{alg}, \|\cdot\|, \text{Ban}}^2 \longrightarrow \dots \longrightarrow \Omega_{B^h/A, \text{alg}, \|\cdot\|, \text{Ban}}^\bullet \longrightarrow ,$$

$$0 \longrightarrow B \longrightarrow \Omega_{B^h/A, \text{topo}}^1 \longrightarrow \Omega_{B^h/A, \text{topo}}^2 \longrightarrow \dots \longrightarrow \Omega_{B^h/A, \text{topo}}^\bullet \longrightarrow .$$

From our construction for $\Omega_{B^h/A, \text{topo}}^1$, one can actually define:

$$\Omega_{B^h/A, \text{topo}}^{\bullet, \text{f}} := \bigoplus_{i_1, \dots, i_\bullet \in \{1, \dots, n\}} A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_{i_1} \wedge dX_{i_2} \wedge \dots \wedge dX_{i_\bullet} \quad (11.3.22)$$

and then define:

$$\Omega_{B^h/A, \text{topo}}^\bullet := \left(\bigoplus_{i_1, \dots, i_\bullet \in \{1, \dots, n\}} A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_{i_1} \wedge dX_{i_2} \wedge \dots \wedge dX_{i_\bullet} \right) / \quad (11.3.23)$$

$$\left((I \cup dI \cup d^\bullet I) \bigoplus_{i_1, \dots, i_\bullet \in \{1, \dots, n\}} A \langle X_1, \dots, X_n \rangle_{T_1, \dots, T_n} dX_{i_1} \wedge dX_{i_2} \wedge \dots \wedge dX_{i_\bullet} \right), \quad (11.3.24)$$

after taking suitable completion under Banach norms when needed. One can also follow the construction in [III1], [III2], [B1] and [GL] to first consider the corresponding polynomial resolution P_\bullet for B , then consider the corresponding algebraic derived de Rham complex $\Omega_{P_\bullet/A, \text{alg}}^*$, then take the corresponding Banach completion to produce the corresponding topological one $\Omega_{P_\bullet/A, \text{topo}}^*$. Then as in [III1], [III2], [GL] and [B1] around analytic derived p -adic de Rham complex we can take the corresponding suitable derived filtered completion to get the complex $\widehat{\Omega}_{B/A, \text{topo}}^\bullet$ (certainly we need to consider Banach version of some filtered derived category of simplicial Banach rings). In the pro-étale site theoretic setting for rigid spaces (namely the corresponding p -complete context) this recovers the corresponding construction in [GL]. Recall in [GL] the corresponding analytic derived p -adic de Rham complex is constructed by first define the integral version $\Omega_{B_0/A_0, \text{topo}}^\bullet$, and then take the colimit throughout all such rings of definition, and invert p , and then take the filtered completion.

Remark 11.3.10. Certainly after this previous discussion we can construct the corresponding Banach derived de Rham complex, Banach cotangent complex and Banach André-Quillen homology for any morphism $A \rightarrow B$ of Banach rings admissible in our situation. Recall from [III1], [III2], [B1] we have the corresponding algebraic p -adic derived

cotangent complex:

$$\Omega_{A[B]^\bullet/A}^1 \otimes_{A[B]^\bullet} B \quad (11.3.25)$$

where $A[B]^\bullet$ is just the corresponding standard cofibrant replacement (in the topology theoretic language) of B/A . Then we take the corresponding derived completion ³ under the induced norm to achieve the corresponding topological one:

$$\mathbb{L}_{B/A,\text{topo}} := (\Omega_{A[B]^\bullet/A}^1 \otimes_{A[B]^\bullet} B)_{\|\cdot\|}^\wedge. \quad (11.3.26)$$

We have the corresponding algebraic p -adic derived de Rham complex:

$$\Omega_{A[B]^\bullet/A}^\bullet \quad (11.3.27)$$

where $A[B]^\bullet$ is just the corresponding standard cofibrant replacement (in the topology theoretic language) of B/A . Then we take the corresponding completion under the induced norm to achieve the corresponding topological one:

$$\text{dR}_{B/A,\text{topo}} := (\Omega_{A[B]^\bullet/A}^\bullet)_{\|\cdot\|}^\wedge, \quad (11.3.28)$$

carrying certain filtration $\text{Fil}_{\text{dR}_{B/A,\text{topo}}}^*$, which allows one take the corresponding filtered completion to achieve the corresponding final object:

$$\text{dR}_{B/A,\text{topo}}^\wedge := (\Omega_{A[B]^\bullet/A}^\bullet)_{\|\cdot\|, \text{Fil}_{\text{dR}_{B/A,\text{topo}}}^*}^\wedge. \quad (11.3.29)$$

³The derived completion in our Banach situation is the derived Banach completion which for instance could happen by using the 'completion' functor from $\text{Simp}(\text{Ind}(\text{NormSets}))$ to $\text{Simp}(\text{Ind}(\text{BanachSets}))$ literally in [BBBK].

11.4 Naive Étale Morphisms and Intrinsic Étale Morphisms

As discussed above we now study the corresponding properties of naive étale morphisms aiming at the corresponding characterization of the corresponding correct definitions of intrinsic étale morphisms. We now assume the corresponding analyticity of the adic rings.

Lemma 11.4.1. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any rational localization map. Then we have that B is of topologically finite type.*

Proof. Straightforward. □

Lemma 11.4.2. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any rational localization map. Then we have that B is affinoid over A in the sense definition 11.3.4.*

Proof. See the proof of proposition 11.3.8. □

Lemma 11.4.3. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any rational localization map. Then we have*

$$\tau_{\leq 1} \mathbb{L}_{B/A, \text{topo}} \tag{11.4.1}$$

is quasi-isomorphic to zero.

Proof. It suffices to reduce to standard binary localization such as simple Laurent or balanced localization, where we give a proof in the case of simple Laurent one:

$$A \rightarrow A\{T\}/(T - f) \tag{11.4.2}$$

for some $f \in A$. Then we have that actually the corresponding topological cotangent complex will be the corresponding completion of the corresponding algebraic ones (see [Hu1, Proposition 1.6.3]). The corresponding quasi-isomorphism could be defined directly. One just considers the following algebraic differential map:

$$I = (T - f) \rightarrow A[T]/(T - f)dT \tag{11.4.3}$$

$$\tag{11.4.4}$$

which is actually surjective since for any:

$$\sum_{i \geq 0} a_i T^i + g(T)(T - f)dT \in A[T]/(T - f)dT \quad (11.4.5)$$

one takes the corresponding integration of:

$$\int_f^T \sum_{i \geq 0} a_i T^i + g(T)(T - f)dT \quad (11.4.6)$$

$$= \int_f^T \sum_{i \geq 0} a_i T^i + \left(\sum_{i \geq 0} g_i T^i \right) (T - f)dT \quad (11.4.7)$$

$$= \int_f^T \sum_{i \geq 0} a_i T^i + \int_f^T \left(\sum_{i \geq 0} g_i T^i \right) (T - f)dT \quad (11.4.8)$$

$$= \int_f^T \sum_{i \geq 0} a_i T^i + \int_f^T \sum_{i \geq 0} g_i T^{i+1} - f \int_f^T \sum_{i \geq 0} g_i T^i \quad (11.4.9)$$

$$= \sum_{i \geq 0} a_i \frac{1}{i+1} T^{i+1} \Big|_f^T + \sum_{i \geq 0} g_i \frac{1}{i+2} T^{i+2} \Big|_f^T - f \sum_{i \geq 0} g_i \frac{1}{i+1} T^{i+1} \Big|_f^T \quad (11.4.10)$$

$$= (*) (T - f), \quad (11.4.11)$$

where we only have finite sums here since we are considering the corresponding topologized polynomial. For this algebraic map, we have that kernel is $(T - f)^2$, for instance consider:

$$d((T - f)h(T)) = 0, \quad (11.4.12)$$

we will have:

$$h(T) + (T - f)h'(T)dT = 0, \quad (11.4.13)$$

which implies that the image of $(T - f)h(T)$ lives in the corresponding quotient $A[T]/(T - f)dT$. Therefore we have that the topologized (not complete yet) cotangent complex:

$$[(T - f)/(T - f)^2 \longrightarrow A[T]/(T - f)dT],$$

which is quasi-isomorphic to zero. Then we take the corresponding completion with respect to the corresponding topology induced from A we have the desired result. \square

Lemma 11.4.4. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any finite étale map. Then we have that B is of topologically finite type.*

Proof. Since we have that B is affinoid over A by proposition 11.3.8. □

Lemma 11.4.5. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any finite étale map. Then we have that B is affinoid over A in the sense definition 11.3.4.*

Proof. See the proof of proposition 11.3.8. □

Lemma 11.4.6. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any finite étale map. Then we have*

$$\tau_{\leq 1} \mathbb{L}_{B/A, \text{topo}} \tag{11.4.14}$$

is quasi-isomorphic to zero.

Proof. This is basically nontrivial due to the fact that we are discussing the corresponding topological cotangent complex. However, one takes the corresponding finite presentation $B = A[T_1, \dots, T_n]/(f_1, \dots, f_p)$ (note that in fact that we have that B is finite over A) since we are considering a finite étale map, which will realize the corresponding desired algebraic cotangent complex:

$$[I/I^2 \longrightarrow \Omega_{B/A}^1].$$

For the topological situation we have that $B = A\{T_1, \dots, T_n\}/(f_1, \dots, f_p)$ by taking the corresponding completion. Again note that this means that actually the corresponding A -algebra B is still finite over A (since that is the very assumption). Therefore we could have the chance to right B as just $\bigoplus_i Ae_i$ this is basically inducing the same differential module $\bigoplus_j (\bigoplus_i Ae_i) dT_j$ in both the corresponding topological setting and the algebraic setting. Then one could get the corresponding desired topological cotangent complex which is quasi-isomorphic to zero. □

Then we consider the corresponding local composition:

Lemma 11.4.7. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any naive étale map. Then we have that B is of topologically finite type.*

Proof. Since we have that B is affinoid over A by proposition 11.3.8. □

Lemma 11.4.8. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any naive étale morphism. Then we have*

$$\tau_{\leq 1} \mathbb{L}_{B/A, \text{topo}} \tag{11.4.15}$$

is quasi-isomorphic to zero locally with respect to the corresponding rational localization.

Proof. Locally we have that any naive étale morphism takes the corresponding truncated cotangent complex to be trivialized, by the corresponding composition properties of the cotangent complex [SP, Tag 08PN]. \square

In order to globalize the picture one has to work harder. First we have the following:

Proposition 11.4.9. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be any naive étale morphism. Then f is affinoid, of topologically finite type.*

Proof. This is by proposition 11.3.8 for the affinoidness, which implies the corresponding second property. \square

Definition 11.4.10. We now define localized ⁴ intrinsic étale morphism to be a morphism $f : (A, A^+) \rightarrow (B, B^+)$ which is affinoid with strongly sheafy target, and locally (with respect to the rational localization) the corresponding truncated topological cotangent complex is quasi-isomorphic to zero.

We now consider the corresponding intrinsic étale morphisms of the corresponding special adic spaces after [HK, Section 4, Section 7 and Section 11]:

Setting 11.4.11. We now consider the three special adic spaces after [HK, Section 4, Section 7 and Section 11], they are the corresponding strongly sheafy adic spaces, the corresponding sousperfectoid adic spaces and the corresponding diamantine adic spaces. We will use the notations T, S, D to denote them in general respectively.

The corresponding categories of strongly sheafy adic spaces, sousperfectoid spaces and diamantine adic spaces are nice enough since at least we have well-defined notion of naive étale morphisms (which is certainly the correct one) and furthermore well-defined étale sites.

Definition 11.4.12. For strongly sheafy adic spaces, a morphism $T_1 \rightarrow T_2$ is called localized intrinsic étale if locally on T_1 this is localized intrinsic étale, namely for any neighbourhood $U \subset T_1$ we have that the morphism $(\mathcal{O}_{T_2}(U), \mathcal{O}_{T_2}^+(U)) \rightarrow (\mathcal{O}_{T_1}(U), \mathcal{O}_{T_1}^+(U))$ is localized intrinsic étale.

⁴However this could actually be globalized easily.

Definition 11.4.13. For sousperfectoid adic spaces, a morphism $S_1 \rightarrow S_2$ is called localized intrinsic étale if locally on S_1 this is localized intrinsic étale, namely for any neighbourhood $U \subset S_1$ we have that the morphism $(\mathcal{O}_{S_2}(U'), \mathcal{O}_{S_2}^+(U')) \rightarrow (\mathcal{O}_{S_1}(U), \mathcal{O}_{S_1}^+(U))$ is localized intrinsic étale.

Definition 11.4.14. For diamantine adic spaces, a morphism $D_1 \rightarrow D_2$ is called localized intrinsic étale if locally on D_1 this is localized intrinsic étale, namely for any neighbourhood $U \subset D_1$ we have that the morphism $(\mathcal{O}_{D_2}(U'), \mathcal{O}_{D_2}^+(U')) \rightarrow (\mathcal{O}_{D_1}(U), \mathcal{O}_{D_1}^+(U))$ is localized intrinsic étale.

11.5 Properties

We now study the corresponding properties of the corresponding localized intrinsic étale morphisms, following [EGAIV4, Chapitre 17] and [Hu1, Especially Section 1.6, Section 1.7]. We now assume the corresponding analyticity of the adic rings.

Conjecture 11.5.1. *Any localized intrinsic étale morphism of strongly sheafy adic spaces is locally a composition of rational localization and finite étale morphism.*

Here is the special situation.

Proposition 11.5.2. *As in [Hu1, Section 1.6, Section 1.7], namely in the strongly noetherian situation we have the conjecture holds.*

Proof. This is because in that setting our definition in the intrinsic setting coincides with the more algebraic one in [Hu1, Section 1.6, Section 1.7]. And note that in this setting the affinoidness of the morphism reduces to just being admitting surjections from Tate algebra over the source. \square

If this is true then we have:

Corollary 11.5.3. *Any localized intrinsic étale morphism of sousperfectoid adic spaces is locally a composition of rational localization and finite étale morphism. Any localized intrinsic étale morphism of diamantine adic spaces is locally a composition of rational localization and finite étale morphism.*

Proposition 11.5.4. *Compositions of localized intrinsic étale morphisms of strongly sheafy adic spaces are again localized intrinsic étale morphism.*

Proof. Locally it is the corresponding compositions of topologically finite type morphism, and locally it is the corresponding compositions of the corresponding affinoid morphisms, and locally it is the corresponding compositions of morphisms giving rise to the quasi-isomorphic to zero truncated cotangent complex. \square

Corollary 11.5.5. *Compositions of localized intrinsic étale morphisms of sousperfectoid adic spaces are again localized intrinsic étale morphism. Compositions of localized intrinsic étale morphisms of diamantine adic spaces are again localized intrinsic étale morphism.*

Proposition 11.5.6. *The localized intrinsic étaleness of any morphism $T_1 \rightarrow T_2$ of strongly sheafy adic rings is preserved under the base change along any morphism of $T_3 \rightarrow T_2$.*

Proof. The base change of any morphism of topologically finite type is again of topologically finite type. The affinoidness of morphism is also preserved under any base change morphism. Finally for the cotangent complex locally, we definitely have the corresponding result as well. \square

Proposition 11.5.7. *The étale property of a morphism between strongly sheafy rings could be detected locally at each point.*

Proof. Straightforward. \square

We now consider some functoriality issue in our current situation. Now we consider the corresponding localized intrinsic étale morphisms under the construction of Witt vectors. Now let:

$$A \rightarrow B \tag{11.5.1}$$

be a general morphism in positive characteristic. Therefore we can take the corresponding Witt vector construction:

$$W(A^b) \rightarrow W(B^b), \tag{11.5.2}$$

where we assume that $A^b \rightarrow B^b$ is localized intrinsic étale. Here we take the completion if needed along the corresponding Fontainisation.

Proposition 11.5.8. *The map*

$$W(A^b) \rightarrow W(B^b), \tag{11.5.3}$$

*is affinoid if the kernel is closed*¹.

Proof. We only need to check this locally, locally we have that there is a lifting:

$$W(A^b)\{T_1, \dots\} \rightarrow W(B^b) \rightarrow 0 \tag{11.5.4}$$

¹This is again due to the very subtle point around the sheafiness such as in [Ked1, Theorem 1.4.20].

from the corresponding surjection:

$$A^{\flat}\{\overline{T}_1, \dots\} \rightarrow B^{\flat} \rightarrow 0. \quad (11.5.5)$$

And what we have is that this map on the Witt vector level is also realizing the target as a stably-pseudocoherent module over the source since we have that the target is sheafy ([Ked1, Theorem 1.4.20]). \square

Proposition 11.5.9. *Same holds for the construction of integral Robba ring $\widetilde{\mathcal{R}}_*^r$ and Robba ring $\widetilde{\mathcal{R}}_*^{[s,r]}$ with respect to closed intervals as in [KL1, Definition 5.1.1] and [KL2, Definition 4.1.1].*

11.6 Étale-Like Morphisms of ∞ -Banach Rings and the ∞ -Analytic Stacks

11.6.1 Approach through De Rham Stacks

We now extend the corresponding discussion to the E_∞ objects in [BBBK, Remark 3.16] by using the ideas as in [R]. Recall from [BBBK, Theorem 3.14] we have the corresponding categories $\text{SimpInd}^m(\text{BanSets}_H)$ and $\text{SimpInd}(\text{BanSets}_H)$ which are the corresponding categories of the corresponding simplicial sets over the corresponding inductive categories of the corresponding Banach sets over some Banach ring H ⁵.

Theorem 11.6.1. (Bambozzi-Ben-Bassat-Kremnizer) *The corresponding categories $\text{SimpInd}^m(\text{BanSets}_H)$ and $\text{SimpInd}(\text{BanSets}_H)$ admit symmetric monoidal model categorical structure. Same holds for $\text{SimpInd}^m(\text{NrSets}_H)$ and $\text{SimpInd}(\text{NrSets}_H)$.*

Corollary 11.6.2.

The corresponding categories $\text{SimpInd}^m(\text{BanSets}_H)$ and $\text{SimpInd}(\text{BanSets}_H)$ admit presentations as $(\infty, 1)$ -categories. Same holds for

$$\text{SimpInd}^m(\text{NrSets}_H)$$

and

$$\text{SimpInd}(\text{NrSets}_H).$$

Then recall from [BBBK, Remark 3.16] we have the corresponding ring objects in the ∞ -categories above:

$$\text{sComm}(\text{SimpInd}^m(\text{BanSets}_H)), \tag{11.6.1}$$

$$\text{sComm}(\text{SimpInd}(\text{BanSets}_H)). \tag{11.6.2}$$

and

$$\text{sComm}(\text{SimpInd}^m(\text{NrSets}_H)), \tag{11.6.3}$$

$$\text{sComm}(\text{SimpInd}(\text{NrSets}_H)). \tag{11.6.4}$$

⁵It is safer to assume the open mapping properties on homotopy groups.

Now we use general notation A to denote any object in these categories, regarding as a general E_∞ -ring. We consider the general morphism $A \rightarrow B$ in the first two categories in the following discussion.

Definition 11.6.3. For any general morphism $A \rightarrow B$, we call this affinoid if we have that that $\pi_0(B)$ is affinoid over $\pi_0(A)$, namely we have that there is a surjection map $\pi_0(A) \langle X_1, \dots, X_d \rangle \rightarrow \pi_0(B)$. And moreover we assume that $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \xrightarrow{\sim} \pi_n(B)$, for any n .

Remark 11.6.4. Kedlaya's theorem [Ked1, Theorem 1.4.20] is actually expected to hold in more general setting, at least in the situation where the definition of the affinoid morphisms could be made independent from the corresponding stably-pseudocoherence for open mapping rings (note that we are working over analytic fields). However in the previous definition, we have been not really exact. To be really accurate in the characterization of some desired notion of the affinoidness we think that one has to add certain ∞ -sheafiness (which certainly holds in [BK]). To be more precise for any general morphism $A \rightarrow B$ in [BK] (namely in current situation one considers the corresponding Banach algebras over the analytic fields in our situation), we call this affinoid if we have that that $\pi_0(B)$ is affinoid over $\pi_0(A)$, namely we have that there is a surjection map $\pi_0(A) \langle X_1, \dots, X_d \rangle \rightarrow \pi_0(B)$. And moreover we assume that $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \xrightarrow{\sim} \pi_n(B)$, for any n . In this situation we have the nice sheafiness (up to higher homotopy). Again similar discussion could be made in the context of [CS]. Note that [Ked1, Theorem 1.4.20] literally says that the sheafiness is equivalent (in some nice sense but in more flexible derived sense) to the stably-pseudocoherence.

Definition 11.6.5. For any general morphism $A \rightarrow B$, we call this localized intrinsic étale if we have that that $\pi_0(B)$ is localized intrinsic étale over $\pi_0(A)$, namely we have that there is a surjection map $\pi_0(A) \langle X_1, \dots, X_d \rangle \rightarrow \pi_0(B)$ and we have that locally the corresponding truncated topological cotangent complex is basically quasi-isomorphic to zero. And moreover we assume that $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \xrightarrow{\sim} \pi_n(B)$, for any n .

We now use the corresponding $X = \text{Spec}A$ to denote the corresponding ∞ -stack in the opposite categories with respect to the ring A . We now define the corresponding de Rham stack attached to X as in [R, Remark 1.2]:

Definition 11.6.6. We now define:

$$X_{\text{dR}}(R) := \varinjlim_I X(\pi_0(R)/I) \quad (11.6.5)$$

for any R in

$$\text{sComm}(\text{SimpInd}^m(\text{BanSets}_H)), \quad (11.6.6)$$

$$\text{sComm}(\text{SimpInd}(\text{BanSets}_H)). \quad (11.6.7)$$

And the injective limit is taking throughout all nilpotent ideals of $\pi_0(R)$.

Remark 11.6.7. As in [R, Definition 1.1, Remark 1.2], one can actually define the corresponding de Rham and crystalline spaces for any functor from $\text{sComm}(\text{SimpInd}^m(\text{BanSets}_H))$ and $\text{sComm}(\text{SimpInd}(\text{BanSets}_H))$ to $\underline{\text{sSets}}$. This means we do not have to consider $(\infty, 1)$ -sheaves satisfying certain ∞ -descent conditions.

Definition 11.6.8. The corresponding formal completion of any morphism $X = \text{Spec}B \rightarrow Y = \text{Spec}A$:

$$Y_{X,\text{dR}} \quad (11.6.8)$$

is defined to be:

$$Y_{X,\text{dR}}(R) := \varinjlim_I X(\pi_0(R)/I) \times_{Y(\pi_0(R)/I)} Y(R), \quad (11.6.9)$$

for any R in

$$\text{sComm}(\text{SimpInd}^m(\text{BanSets}_{\mathbb{Q}_p})), \quad (11.6.10)$$

$$\text{sComm}(\text{SimpInd}(\text{BanSets}_{\mathbb{Q}_p})). \quad (11.6.11)$$

And the injective limit is taking throughout all nilpotent ideals of $\pi_0(R)$.

Remark 11.6.9. Certainly it is actually not clear how really we should deal with the corresponding ideals here, namely we are not for sure if we need to consider closed ideals. But for simplicial noetherian rings we really have some nice definitions, which will certainly be tangential to the corresponding Huber's original consideration.

Definition 11.6.10. We now define the corresponding de Rham intrinsic étale morphism to be an affinoid morphism $X = \text{Spec}B \rightarrow Y = \text{Spec}A$ which satisfies the condition:

$$\pi_0(X(R)) \xrightarrow{\sim} \pi_0(Y_{X,\text{dR}}(R)), \quad (11.6.12)$$

for any E_∞ -object R .

Proposition 11.6.11. *Compositions of de Rham intrinsic étale morphisms are again PD intrinsic étale morphisms.*

Proof. This is formal. □

11.6.2 Approach through Crystalline Stack and PD-morphisms

We now use the corresponding $X = \text{Spec}A$ to denote the corresponding ∞ -stack in the opposite categories with respect to the ring A . We now define the corresponding crystalline stack attached to X as in [R, Definition 1.1]:

Definition 11.6.12. We now define:

$$X_{\text{crys}}(R) := \varinjlim_{I,\gamma} X(\pi_0(R)/I) \quad (11.6.13)$$

for any R in

$$\text{sComm}(\text{SimpInd}^m(\text{BanSets}_H)), \quad (11.6.14)$$

$$\text{sComm}(\text{SimpInd}(\text{BanSets}_H)). \quad (11.6.15)$$

And the injective limit is taking throughout all nilpotent ideals of $\pi_0(R)$ and the corresponding PD-structures.

Definition 11.6.13. The corresponding PD completion of any morphism $X = \text{Spec}B \rightarrow Y = \text{Spec}A$:

$$Y_{X,\text{crys}} \quad (11.6.16)$$

is defined to be:

$$Y_{X,\text{crys}}(R) := \varinjlim_{I,\gamma} X(\pi_0(R)/I) \otimes_{Y(\pi_0(R)/I)} Y(R), \quad (11.6.17)$$

for any R in

$$\text{sComm}(\text{SimpInd}^m(\text{BanSets}_H)), \quad (11.6.18)$$

$$\text{sComm}(\text{SimpInd}(\text{BanSets}_H)). \quad (11.6.19)$$

And the injective limit is taking throughout all nilpotent ideals of $\pi_0(R)$ and all the corresponding PD structures.

Remark 11.6.14. Certainly it is actually not clear how really we should deal with the corresponding ideals here and the corresponding PD structures, namely we are not for sure if we need to consider closed ideals. But for simplicial noetherian rings we really have some nice definitions, which will certainly be tangential to the corresponding Huber's original consideration.

Definition 11.6.15. We now define the corresponding PD intrinsic étale morphism to be an affinoid morphism $X = \text{Spec}B \rightarrow Y = \text{Spec}A$ which satisfies the condition:

$$\pi_0(X(R)) \xrightarrow{\sim} \pi_0(Y_{X,\text{crys}}(R)), \quad (11.6.20)$$

for any E_∞ -object R .

Proposition 11.6.16. *We have that any de Rham intrinsic étale morphism is a PD intrinsic étale morphism.*

Proof. This is formal. □

Proposition 11.6.17. *Compositions of PD intrinsic étale morphisms are again PD intrinsic étale morphisms.*

Proof. This is formal. □

11.7 Lisse-Like and Non-Ramifié-Like Morphisms of ∞ -Banach Rings and the ∞ -Analytic Stacks

11.7.1 Approach through De Rham Stacks

We now define the corresponding lisse-like morphisms along the idea in the previous section:

Definition 11.7.1. For any general morphism $A \rightarrow B$, we call this localized intrinsic lisse if we have that that $\pi_0(B)$ is localized intrinsic lisse over $\pi_0(A)$, namely we have that there is a surjection map $\pi_0(A) \langle X_1, \dots, X_d \rangle \rightarrow \pi_0(B)$ and we have that locally the corresponding truncated topological cotangent complex is basically quasi-isomorphic to $\Omega_{\pi_0(B)/\pi_0(A)}[0]$. And moreover we assume that $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \xrightarrow{\sim} \pi_n(B)$, for any n .

Definition 11.7.2. We now define the corresponding de Rham intrinsic lisse morphism to be an affinoid morphism $X = \text{Spec}B \rightarrow Y = \text{Spec}A$ which satisfies the condition:

$$\pi_0(X(R)) \rightarrow \pi_0(Y_{X,\text{dR}}(R)) \tag{11.7.1}$$

being surjective, for any E_∞ -object R .

Definition 11.7.3. For any general morphism $A \rightarrow B$, we call this localized intrinsic non-ramifié if we have that that $\pi_0(B)$ is localized intrinsic non-ramifié over $\pi_0(A)$, namely we have that there is a surjection map $\pi_0(A) \langle X_1, \dots, X_d \rangle \rightarrow \pi_0(B)$ and we have that locally the corresponding truncated topological cotangent complex is basically quasi-isomorphic to $\Omega_{\pi_0(B)/\pi_0(A)}[0]$ which vanishes as well. And moreover we assume that $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \xrightarrow{\sim} \pi_n(B)$, for any n .

11.7.2 Approach through Crystalline Stack and PD-morphisms

Definition 11.7.4. We now define the corresponding PD intrinsic lisse morphism to be an affinoid morphism $X = \text{Spec}B \rightarrow Y = \text{Spec}A$ which satisfies the condition:

$$\pi_0(X(R)) \rightarrow \pi_0(Y_{X,\text{crys}}(R)) \tag{11.7.2}$$

being surjective, for any E_∞ -object R ⁶.

⁶For non-ramifié situation one considers injectivity.

Proposition 11.7.5. *We have that any de Rham intrinsic lisse morphism is a PD intrinsic lisse morphism.*

11.8 Perfectization and Fontainisation of ∞ -Analytic Stacks Situation

11.8.1 Perfectization, Fontainisation and Crystalline Stacks

Now we consider the corresponding perfectoidization of ∞ -analytic stacks after [R] and [Dr1] in the situation where H is assumed to be of characteristic p .

Definition 11.8.1. For any object in $\infty - \text{Fun}(\text{sCommSimpInd}(\text{Ban}_H), \underline{\text{sSets}})$, denoted by X , we define the corresponding *perfectization* X^{1/p^∞} of X to be the corresponding functor such that for any $R \in \text{sCommSimpInd}(\text{Ban}_H)$ we have that $X^{1/p^\infty}(R) := X(R^b)$ where we define the corresponding tilting *Fontainisation* R^b of R to be the corresponding derived completion of:

$$\varprojlim \{ \dots \xrightarrow{\text{Fro}} R \xrightarrow{\text{Fro}} R \xrightarrow{\text{Fro}} R \}. \quad (11.8.1)$$

In the situation of the corresponding monomorphically inductive Banach sets, we have the parallel definition:

Definition 11.8.2. For any object in $\infty - \text{Fun}(\text{sCommSimpInd}^m(\text{BanSets}_H), \underline{\text{sSets}})$, denoted by X , we define the corresponding *perfectization* X^{1/p^∞} of X to be the corresponding functor such that for any ring $R \in \text{sCommSimpInd}^m(\text{BanSets}_H)$ we have that $X^{1/p^\infty}(R) := X(R^b)$ where we define the corresponding tilting *Fontainisation* R^b of R to be the corresponding derived completion of:

$$\varprojlim \{ \dots \xrightarrow{\text{Fro}} R \xrightarrow{\text{Fro}} R \xrightarrow{\text{Fro}} R \}. \quad (11.8.2)$$

Remark 11.8.3. This is very general notion beyond the corresponding $(\infty, 1)$ -sheaves satisfying certain descent with respect to the derived rational localizations or more general homotopy Zariski topology as in [BK] and [BBBK].

Now we follow [R, Proposition 5.3] and [Dr1, Section 1.1] to give the following discussion around the corresponding Witt crystalline Stack:

Definition 11.8.4. We define the corresponding *Witt Crystalline Stack* X_W of any X over H/\mathbb{F}_p in

$$\infty - \text{Fun}(\text{sCommSimpInd}(\text{Ban}_H), \underline{\text{sSets}})$$

or

$$\infty - \text{Fun}(\text{sCommSimpInd}^m(\text{Ban}_H), \underline{\text{sSets}})$$

to be the functor $(W(\pi_0(X)^b)_{\pi_0(X), \text{crys}})_p^\wedge$. And we define the corresponding pre-crystals to be sheaves of \mathcal{O} -modules over this functors when we have that X is an ∞ -analytic stack with reasonable topology.

Example 11.8.5. For instance if we have that $X = \text{Spa}^h(R)$ coming from the corresponding Bambozzi-Kremnizer spectrum of any Banach ring over $\mathbb{F}_p((t))$ as constructed in [BK]. Then we have that the corresponding functor is $(W(\pi_0(X)^b)_{\pi_0(X), \text{crys}})_p^\wedge$ is now admitting structures coming from the corresponding homotopy Zariski topology from X .

Example 11.8.6. For instance if we have that $X = \text{Spec}(\mathbb{F}_p[[t]])$ coming from the corresponding object in the opposite category of $\mathbb{F}_p[[t]]$. Then we have that the corresponding functor is $(W(\pi_0(X)^b)_{\pi_0(X), \text{crys}})_p^\wedge$ is now admitting structures coming from the corresponding homotopy Zariski topology from X , is just the same as the corresponding one in the algebraic setting constructed in [R, Proposition 5.3] and [Dr1, Section 1.1].

11.8.2 Perfectization, Fontainisation and Robba Stacks

Now we contact [KL1] and [KL2] to look at the corresponding Robba Stacks. Now take any X to be any ∞ -analytic stack which admits structures of simplicial complete Bornological rings or ind-Fréchet structures, namely we have the corresponding complete bornological topology or ind-Fréchet topology on $\pi_0(X)$. We work over H/\mathbb{F}_p as well.

Example 11.8.7. For instance we take that $X = \text{Spa}^h(R)$ coming from the corresponding Bambozzi-Kremnizer spectrum of any Banach ring over $\mathbb{F}_p((t))$ as constructed in [BK].

Definition 11.8.8. For any ∞ -analytic stack X as above, we consider the corresponding Witt vector functor $W_n(\pi_0(X)^b)$ and then consider

$$\varinjlim_{n \rightarrow \infty} W_n(\pi_0(X)^b),$$

namely $W(\pi_0(X)^b)$, then we consider the ring $W(\pi_0(X)^b)[1/p]$. Then we can take the corresponding completion with respect to the Gauss norm $\|\cdot\|_{\pi_0(X),[s,r]}$ coming from the corresponding norm on $\pi_0(X)$ with respect to some interval $[s, r] \in (0, \infty)$ as in [KL2, Definition 4.1.1]:

$$\tilde{\Pi}(X)_{[s,r]} := (W(\pi_0(X)^b)[1/p])^{\wedge}_{\|\cdot\|_{\pi_0(X),[s,r]},\text{Fré}} \quad (11.8.3)$$

Then following [KL2, Definition 4.1.1] we consider the following:

$$\tilde{\Pi}(X)_r := \varprojlim_{s>0} (W(\pi_0(X)^b)[1/p])^{\wedge}_{\|\cdot\|_{\pi_0(X),[s,r]},\text{Fré}} \quad (11.8.4)$$

and

$$\tilde{\Pi}(X) := \varinjlim_{r>0} \varprojlim_{s>0} (W(\pi_0(X)^b)[1/p])^{\wedge}_{\|\cdot\|_{\pi_0(X),[s,r]},\text{Fré}}. \quad (11.8.5)$$

We call these *Robba stacks*.

Example 11.8.9. In the situation where X is some ∞ -analytic stack carrying the corresponding sheaves of simplicial Banach rings (namely not in general bornological or ind-Fréchet) we have that the finite projective modules over the three Robba stacks (for by enough intervals) carrying semilinear Frobenius action which realizes the corresponding isomorphisms by Frobenius pullbacks are equivalent. In the noetherian setting we have the same holds for locally finite presented sheaves as well. This is the main results of [KL2, Theorem 4.6.1].

This chapter is based on the following paper, where the author of this dissertation is the main author:

- Tong, Xin. "Topologization and Functional Analytification I: Intrinsic Morphisms of Commutative Algebras." arXiv preprint arXiv:2102.10766 (2021).

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Chapter 12

Topologization and Functional Analytification II: Motivic Constructions

12.1 Introduction

In our previous paper on the intrinsic morphisms of the corresponding topological, bornological and ind-Fréchet rings, we discussed some generalization of the corresponding contexts of [Huber1] and [Huber2], along two directions. One is along some answer to a question of Kedlaya in [Ked1, Appendix 5] namely the corresponding affinoid morphisms, and the other is along some stacky understanding of these such as in [Dr1], [Dr2], [R].

Under some certain foundation, [Pau1] actually studied many parallel considerations of our project. The parallel considerations are the corresponding ind-Banach analytic spaces, the corresponding étale and pro-étale sites of ind-Banach analytic spaces, the corresponding de Rham stacks of such analytic spaces, global de Rham cohomology and comparison over ind-Banach spaces, and algebraic derived de Rham complex over the corresponding ind-Banach spaces. And the corresponding descent over the corresponding derived ind-Banach spaces. We will consider the corresponding topologization and functional analytification of the cohomology theories (topological derived de Rham after [B1], [Bei], [GL], [III1], [III2] topological logarithmic derived de Rham after Gabber such as in [B1] and [O], certainly functional analytification construction after [KL1] and [KL2] and etc). We would like to mention that our project is also largely inspired by the corresponding development in [CS1], [CS2], [FS], [Sch2] where the corresponding v -stacky consideration is extensively developed. Note that v -stacks are very significant analytic spaces, especially in the corresponding geometrization of local Langlands correspondence in [FS], where complicated derived categories are constructed through deep foundations in [Sch2] and [CS1], [CS2] on condensed sets. One certainly believes that the corresponding foundations by using normed sets in [BBBK], [BBK] and [BK] will also have potential applications to certainly motivic and functorial constructions in analytic geometry. Note that the programs in [B1], [BMS] and [BS1] have already indicated that working with the corresponding correct simplicial spaces in the analytic setting is not only a pure generalization but also a very significant point around even the non simplicial analytic spaces especially in the corresponding singular situations.

Then we follow [An1], [An2], [B1], [Bei], [BS1], [G1], [GL], [III1], [III2], [Qui]

to extend the corresponding discussion to certainly spaces, we mainly have focused on the corresponding rigid analytic spaces, pseudorigid spaces in some explicit way. And we then extend the corresponding discussion to the corresponding derived setting. The corresponding derived setting is literally following [An1], [An2], [B1], [B2], [Bei], [BMS], [BS1], [G1], [GL], [III1], [III2], [Qui] and the corresponding derived logarithmic setting is literally following [B1], [Ko1] and [O]¹.

We first consider the following list of motivic constructions in this paper which we hope to establish in very general topological and Banach setting, where one can actually believe that the constructions are somehow parallel and in some sense equivalent even on the ∞ -categorical level²:

Setting 12.1.1.

- A. Derived Topological de Rham complexes and Derived Topological Logarithmic de Rham complexes of simplicial derived I -complete rings and over pro-étale site, after [B1], [B2], [Bei], [BMS], [BS1], [G1], [GL], [III1], [III2], [O];
- B. $\mathcal{O}B_{\text{dR}}$ -sheaves and $\mathcal{O}B_{\text{dR},\log}$ -sheaves over general analytic adic spaces³ after [DLLZ2], [Sch2];
- C. $\varphi\text{-}\tilde{C}_X$ -sheaves, relative B -pairs over general analytic adic spaces after [KL1] and [KL2].

Remark 12.1.2. One could also consider the corresponding derived I -complete version⁴ of the corresponding left Kan extended THH and HH \mathbb{E}_1 -ring spectra of derived I -topological \mathbb{E}_1 -ring spectra as in [NS]. In certain situation, this should be able to be compared to the constructions above.

¹The corresponding [Ko1] has already mentioned the corresponding derived logarithmic prismatic cohomology.

²We are actually talking about the spaces which might not be smooth.

³These are not necessarily p -adic Tate. But as long as p is topological nilpotent we do have a basis consisting of perfectoid subdomains as in [Ked1, Theorem 2.9.9, Remark 2.9.10].

⁴In fact we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3].

12.2 Preliminary

12.2.1 The Rings in Algebraic Topology

We consider some very general topological rings. We will consider the corresponding Huber's adic rings, namely we allow the corresponding rings to contain some adic open subrings which have then the corresponding linear topology. And we consider more general simplicial topological rings which are more general adic in some derived sense.

Setting 12.2.1. We consider the corresponding category $C_{\infty, \mathbb{E}_{\infty}}$ of all the \mathbb{E}_{∞} rings⁵. We then consider the corresponding category $C_{\infty, \mathbb{E}_{\infty}, \text{rat}, I}$ of all the \mathbb{E}_{∞} rings which have subrings being derived I -adically complete⁶, where I is finite generating set coming from elements in π_0 . And we have the smaller category $C_{\infty, \mathbb{E}_{\infty}, \text{int}, I}$ of all the all the \mathbb{E}_{∞} rings being derived I -adically complete.

Example 12.2.2. We have many interesting ring spectra from classical algebraic topology. One can further takes the corresponding derived p -completion to achieve many interesting objects⁷. For more discussion see [MP, Chapter 5-13] and [N].

Example 12.2.3. As also discussed in [MP, Chapter 14-19] on the level of just model categories or more general construction in [Lu1, Chapter 1.3] one considers the stable homotopy category $D(R)$ of some ring spectra, namely the corresponding ∞ -enhancement of the corresponding classical triangulated categories. One considers the corresponding derived complete objects in this ∞ -category, then one will have some interesting \mathbb{E}_{∞} -spectra.

Example 12.2.4. We then have the example that is the corresponding rigid analytic affinoids in [Ta, Definition 4.1]. They have open subrings which are actually derived p -adically complete.

Example 12.2.5. We then have the example that is the corresponding pseudorigid analytic affinoids as in [Bel1, Definition 3.1], [Bel2] and [L, Definition 4.1]. They have

⁵These are the corresponding ring objects in the algebraic topology, see [Lu1, Chapter 7]. So we want to consider the corresponding interesting objects in classical algebraic topology such as in [MP] and [N].

⁶In fact we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3].

⁷In fact we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3]. This should be different in more general setting from the consideration in algebraic topology and more general ∞ -category theory.

open subrings which are actually derived t -adically complete for some specific pseudouniformizer t .

Example 12.2.6. We now fix a bounded morphism of simplicial adic rings $A \rightarrow B$ over A^* where A^* contains a corresponding ring of definition A_0^* which is complete with respect to the (p, I) -topology and we assume that A is adic and we assume that (A_0^*, I) is a prism in [BS1] namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For A or B respectively we assume this contains a subring A_0 or B_0 (over A_0^*) respectively such that we have A_0 or B_0 respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A). All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* .

We also consider the corresponding simplicial prelog rings as in [B1, Chapter 6], and also consider the corresponding application of such algebraic derived constructions to some derived I -adic rings carrying the corresponding prelog structures such as in [DLLZ1, Chapter 2].

Setting 12.2.7. We consider the corresponding category $C_{\infty, \mathbb{E}_{\infty}, \text{prelog}}$ of all the \mathbb{E}_{∞} rings carrying the corresponding prelog structures. We then consider the corresponding category $C_{\infty, \mathbb{E}_{\infty}, \text{rat}, I, \text{prelog}}$ of all the \mathbb{E}_{∞} rings carrying prelog structures which have subrings being derived I -adically complete⁸, where I is finite generating set coming from elements in π_0 . And we have the smaller category $C_{\infty, \mathbb{E}_{\infty}, \text{int}, I, \text{prelog}}$ of all the all the \mathbb{E}_{∞} rings carrying prelog structures being derived I -adically complete.

Example 12.2.8. The first example is the corresponding rigid analytic affinoids in [Ta, Definition 4.1]. They have open subrings which are actually derived p -adically complete. Then one adds the corresponding prelog structures from [DLLZ1, Chapter 2].

Example 12.2.9. The second example is the corresponding pseudorigid analytic affinoids as in [Bel1, Definition 3.1], [Bel2] and [L, Definition 4.1]. They have open subrings which are actually derived t -adically complete for some specific pseudouniformizer t . Then one adds the corresponding prelog structures from [DLLZ1, Chapter 2].

⁸In fact again we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3].

Example 12.2.10. We now fix a bounded morphism of logarithmic simplicial adic rings $(A, M) \rightarrow (B, N)$ over A^* where A^* contains a corresponding ring of definition A_0^* which is complete with respect to the (p, I) -topology and we assume that A is adic and we assume that (A_0^*, I) is a prism namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For (A, M) or (B, N_0) respectively we assume this contains a subring (A_0, M_0) or (B_0, N_0) (over A_0^*) respectively such that we have (A_0, M_0) or (B_0, N_0) respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A). All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* .

12.3 Topological André-Quillen Homology and Topological Derived de Rham Complexes

12.3.1 Derived p -Complete Derived de Rham Complex

We now first discuss the corresponding Banach version of André-Quillen Homology and the corresponding Banach version of Derived de Rham complex parallel to [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [Qui, Section 4]. We would like to start from the corresponding context of [GL, Chapter 3, Chapter 4], and represent the construction for the convenience of the readers. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$ of p -complete rings. This is the corresponding derived differential complex attached to the polynomial resolution of B :

$$A[A[B]]\dots, \quad (12.3.1)$$

which is now denoted by $K_L\Omega_{B/A,\text{alg}}^\bullet := K_L\Omega_{-/A,\text{alg}}^\bullet(B)$ after taking the corresponding left Kan extension which will be the same for all the following constructions ⁹. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B/A,\text{alg}} := \Omega_{A[B]^\bullet/A,\text{alg}}^1 \otimes_{A[B]^\bullet} B. \quad (12.3.2)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet,\text{AQ,alg}} := \pi_\bullet(\mathbb{L}_{B/A,\text{alg}}). \quad (12.3.3)$$

The corresponding topological André-Quillen complex is actually the completed version of the corresponding algebraic ones above by considering the corresponding certain p -completion over the simplicial module structure.

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{B/A,\text{alg}}^\bullet, K_L\text{Fil}_{\Omega_{B/A,\text{alg}}^\bullet}^*. \quad (12.3.4)$$

⁹We have already considered the corresponding left Kan extension to all the rings which are not concentrated at degree zero after [BMS, Example 5.11, Example 5.12], which is also discussed in [B2, Lecture 7].

We then take the corresponding Banach completion and we denote that by:

$$K_L \Omega_{B/A, \text{topo}}^\bullet, K_L \text{Fil}_{\Omega_{B/A, \text{topo}}^\bullet}^* . \quad (12.3.5)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above:

$$K_L \widehat{\Omega}_{B/A, \text{topo}}^\bullet, K_L \widehat{\text{Fil}}_{\widehat{\Omega}_{B/A, \text{topo}}^\bullet}^* . \quad (12.3.6)$$

This is basically the corresponding analytic and complete version the corresponding algebraic de Rham complex. Furthermore we allow large coefficients with rigid affinoid algebra Z over \mathbb{Q}_p . Therefore we take the corresponding completed tensor product in the following.

Definition 12.3.1. We define the following Z deformed version of the corresponding complete version of the corresponding André-Quillen homology and the corresponding complete version of the corresponding derived de Rham complex. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding derived differential complex attached to the polynomial resolution of B_0 :

$$A_0[A_0[B_0]] \dots, \quad (12.3.7)$$

which is now denoted by $K_L \Omega_{B_0/A_0, \text{alg}}^\bullet$. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0, \text{alg}} := \Omega_{A_0[B_0]^\bullet/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^\bullet} B_0. \quad (12.3.8)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ, alg}} := \pi_\bullet(\mathbb{L}_{B_0/A_0, \text{alg}}). \quad (12.3.9)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived p -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R \varprojlim_k \text{Kos}_{p^k} \left((\Omega_{A_0[B_0]^\bullet/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^\bullet} B_0) \right). \quad (12.3.10)$$

Taking the product with \mathcal{O}_Z we have the corresponding integral version of the topological André-Quillen complex:

$$\mathbb{L}_{B_0/A_0, \text{topo}, Z} := \mathbb{L}_{B_0/A_0, \text{topo}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_Z \quad (12.3.11)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L \Omega_{B_0/A_0, \text{alg}}^\bullet, K_L \text{Fil}_{\Omega_{B_0/A_0, \text{alg}}^1}^*. \quad (12.3.12)$$

We then take the corresponding derived p -completion and we denote that by:

$$K_L \Omega_{B_0/A_0, \text{topo}}^\bullet := R \varprojlim_k \text{Kos}_{p^k} \left(K_L \Omega_{B_0/A_0, \text{alg}}^\bullet, K_L \text{Fil}_{\Omega_{B_0/A_0, \text{alg}}^*}^\bullet \right), \quad (12.3.13)$$

$$K_L \text{Fil}_{\Omega_{B_0/A_0, \text{topo}}^*}^\bullet := R \varprojlim_k \text{Kos}_{p^k} \left(K_L \text{Fil}_{\Omega_{B_0/A_0, \text{alg}}^*}^\bullet \right). \quad (12.3.14)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with \mathcal{O}_Z to get:

$$K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet := K_L \Omega_{B_0/A_0, \text{topo}}^\bullet \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} \mathcal{O}_Z, \quad (12.3.15)$$

$$K_L \text{Fil}_{\Omega_{B_0/A_0, \text{topo}, Z}^*}^\bullet := K_L \text{Fil}_{\Omega_{B_0/A_0, \text{topo}}^*}^\bullet \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} \mathcal{O}_Z. \quad (12.3.16)$$

Then we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B_0/A_0, \text{topo}, Z} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}, Z}[1/p], \quad (12.3.17)$$

$$H_{\bullet, \text{AQ}, \text{topo}, Z} := \pi_\bullet(\mathbb{L}_{B/A, \text{topo}, Z}), \quad (12.3.18)$$

$$K_L \Omega_{B/A, \text{topo}, Z}^\bullet := \text{Colim}_{A_0 \rightarrow B_0} K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet[1/p], \quad (12.3.19)$$

$$K_L \text{Fil}_{\Omega_{B/A, \text{topo}, Z}^*}^\bullet := \text{Colim}_{A_0 \rightarrow B_0} K_L \text{Fil}_{\Omega_{B_0/A_0, \text{topo}, Z}^*}^\bullet[1/p]. \quad (12.3.20)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{B/A, \text{topo}, \text{Hodge}, Z}^\bullet, K_L \text{Fil}_{\Omega_{B/A, \text{topo}, \text{Hodge}, Z}^*}^\bullet. \quad (12.3.21)$$

Definition 12.3.2. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $K_L \Omega_{B/A, \text{topo}, Z}^\bullet$ with the corresponding induced filtrations.

Definition 12.3.3. We define the corresponding almost perfect ¹⁰ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $K_L \Omega_{B/A, \text{topo}, Z}^\bullet$ with the corresponding induced filtrations.

The following is derived from the main Poincaré Lemma from [GL, Theorem 1.2] in the non-deformed situation. Consider a corresponding smooth rigid analytic space X over k/\mathbb{Q}_p (where k is a corresponding unramified analytic field which is discretely-valued and the corresponding residue field is finite). Then we have the following:

Proposition 12.3.4. *Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Then we have the following two strictly exact long exact sequences ¹¹:*

$$0 \longrightarrow K_L \Omega_{k[\hat{\mathcal{O}}]^\bullet/k, \text{topo}, Z}^\bullet \xrightarrow{\partial} K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \xrightarrow{\partial} K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X, \text{topo}}^1 \xrightarrow{\partial} \dots,$$

and

$$0 \longrightarrow K_L \Omega_{k[\hat{\mathcal{O}}]^\bullet/k, \text{topo}, Z}^\bullet \xrightarrow{\partial} K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X_{\text{ét}}, \text{topo}, Z}^\bullet \xrightarrow{\partial} K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X_{\text{ét}}, \text{topo}}^1 \xrightarrow{\partial} \dots$$

Proof. This is actually a direct consequence [GL, Theorem 1.2]. □

Proposition 12.3.5. *Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Let M be a corresponding Z -projective differential crystal spectrum. Then we have the following two strictly exact long exact sequences ¹²:*

$$M^{\otimes^L}(0) \longrightarrow \Omega_{k[\hat{\mathcal{O}}]^\bullet/k, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X, \text{topo}}^1 \xrightarrow{\partial} \dots,$$

and

$$M^{\otimes^L}(0) \longrightarrow \Omega_{k[\hat{\mathcal{O}}]^\bullet/k, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet/X_{\text{ét}}, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X_{\text{ét}}, \text{topo}}^1 \xrightarrow{\partial} \dots.$$

¹⁰This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

¹¹We should have the corresponding naturality taking into the following form:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_L \Omega_{k[\hat{\mathcal{O}}]^\bullet/k, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X, \text{topo}}^1 \xrightarrow{\partial} \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_L \Omega_{k[\hat{\mathcal{O}}]^\bullet/k, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X_{\text{ét}}, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & K_L \Omega_{X[\hat{\mathcal{O}}]^\bullet/X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X_{\text{ét}}, \text{topo}}^1 \xrightarrow{\partial} \longrightarrow \end{array}$$

¹²Here we drop the notation K_L .

Proof. This is actually a direct consequence of the previous proposition due to the corresponding flatness. \square

12.3.2 Analytic André-Quillen Homology and Analytic Derived de Rham complex of Pseudorigid Spaces

We now consider the analytic André-Quillen Homology and analytic Derived de Rham complex of pseudorigid space, which are very crucial in some development in [Bel1] in the arithmetic family. Therefore we just investigate the corresponding picture in the corresponding geometric family.

Setting 12.3.6. We consider now a corresponding morphism taking the corresponding form of $A \rightarrow B$ where A is going to be a pseudorigid affinoid algebra over \mathbb{Z}_p , and B is going to be a perfectoid chart of A in the corresponding pro-étale site of the pseudorigid affinoid space attached to A . As in [Bel1, Definition 3.1, and below Definition 3.1] in our situation A is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle [1/t]$, where K is a discrete valued field containing \mathbb{Q}_p and $(a, b) = 1$ ¹³.

From [Bel1, Definition 3.1, and below Definition 3.1] we have the following:

Lemma 12.3.7. *We have the following statements:*

- A. *A as above is Tate, complete over \mathcal{O}_K ;*
- B. *The ring A has a ring of definition A_0 which is of \mathcal{O}_K -formally finite type;*
- C. *The ring A_0 is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle$.*

Proof. It is very easy to see that the corresponding results hold in our current situation over \mathcal{O}_K . □

As in the corresponding construction in the rigid situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [Qui, Section 4] we give the following definitions. First we consider the following new setting:

Setting 12.3.8. Now as in the above notion we will consider a general map of rings $A \rightarrow B$ over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle [1/t]$ where we have the corresponding map of the associated ring of definitions $A_0 \rightarrow B_0$ over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle$ such that A_0, B_0 are basically I -adic (where $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle$ is I -adic).

¹³Certainly one can also consider the characteristic p situation.

Definition 12.3.9. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding derived differential complex attached to the polynomial resolution of B_0 :

$$A_0[A_0[B_0]]\dots, \quad (12.3.22)$$

which is now denoted by $K_L\Omega_{B_0/A_0, \text{alg}}^\bullet$ after taking the corresponding left Kan extension. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0, \text{alg}} := \Omega_{A_0[B_0]^\bullet/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^\bullet} B_0. \quad (12.3.23)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ, alg}} := \pi_\bullet(\mathbb{L}_{B_0/A_0, \text{alg}}). \quad (12.3.24)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R\varprojlim_I \text{Kos}_I \left((\Omega_{A_0[B_0]^\bullet/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^\bullet} B_0) \right). \quad (12.3.25)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{B_0/A_0, \text{alg}}^\bullet, K_L\text{Fil}_{K_L\Omega_{B_0/A_0, \text{alg}}^\bullet}^*. \quad (12.3.26)$$

We then take the corresponding derived I -completion and we denote that by:

$$K_L\Omega_{B_0/A_0, \text{topo}}^\bullet := R\varprojlim_I \text{Kos}_I \left(K_L\Omega_{B_0/A_0, \text{alg}}^\bullet, K_L\text{Fil}_{K_L\Omega_{B_0/A_0, \text{alg}}^\bullet}^* \right), \quad (12.3.27)$$

$$\text{Fil}_{K_L\Omega_{B_0/A_0, \text{topo}}^\bullet}^* := R\varprojlim_I \text{Kos}_I \left(K_L\text{Fil}_{K_L\Omega_{B_0/A_0, \text{alg}}^\bullet}^* \right). \quad (12.3.28)$$

Then we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}}[1/t], \quad (12.3.29)$$

$$H_{\bullet, \text{AQ, topo}} := \pi_\bullet(\mathbb{L}_{B_0/A_0, \text{topo}}), \quad (12.3.30)$$

$$K_L\Omega_{B_0/A_0, \text{topo}}^\bullet := \text{Colim}_{A_0 \rightarrow B_0} K_L\Omega_{B_0/A_0, \text{topo}}^\bullet[1/t], \quad (12.3.31)$$

$$K_L\text{Fil}_{K_L\Omega_{B_0/A_0, \text{topo}}^\bullet}^* := \text{Colim}_{A_0 \rightarrow B_0} \text{Fil}_{K_L\Omega_{B_0/A_0, \text{topo}}^\bullet}^*[1/t]. \quad (12.3.32)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L\Omega_{B/A,\text{topo,Hodge}}^\bullet, \text{Fil}_{K_L\Omega_{B/A,\text{topo,Hodge}}^\bullet}^* . \quad (12.3.33)$$

Definition 12.3.10. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $K_L\Omega_{B/A,\text{topo}}^\bullet$ with the corresponding induced filtrations.

Definition 12.3.11. We define the corresponding almost perfect ¹⁴ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $K_L\Omega_{B/A,\text{topo}}^\bullet$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over pseudorigid spaces where Z now is a topological algebra over \mathbb{Z}_p ¹⁵.

Definition 12.3.12. We define the following Z deformed version of the corresponding complete version of the corresponding André-Quillen homology and the corresponding complete version of the corresponding derived de Rham complex. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding derived differential complex attached to the polynomial resolution of B_0 :

$$A_0[A_0[B_0]]\dots, \quad (12.3.34)$$

which is now denoted by $\Omega_{A_0[B_0]^\bullet/A_0,\text{alg}}^\bullet$. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0,\text{alg}} := \Omega_{A_0[B_0]^\bullet/A_0,\text{alg}}^1 \otimes_{A_0[B_0]^\bullet} B_0. \quad (12.3.35)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet,\text{AQ,alg}} := \pi_\bullet(\mathbb{L}_{B_0/A_0,\text{alg}}). \quad (12.3.36)$$

¹⁴This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

¹⁵It is better to assume that it is basically completely flat.

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R \varprojlim_k \text{Kos}_I \left((\Omega_{A_0[B_0]^\bullet/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^\bullet} B_0) \right). \quad (12.3.37)$$

Taking the product with Z we have the corresponding integral version of the topological André-Quillen complex¹⁶:

$$\mathbb{L}_{B_0/A_0, \text{topo}, Z} := \mathbb{L}_{B_0/A_0, \text{topo}} \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z \quad (12.3.38)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L \Omega_{B_0/A_0, \text{alg}}^\bullet, \text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^\bullet}^* \quad (12.3.39)$$

We then take the corresponding derived I -completion and we denote that by:

$$K_L \Omega_{B_0/A_0, \text{topo}}^\bullet := R \varprojlim_k \text{Kos}_I \left(K_L \Omega_{B_0/A_0, \text{alg}}^\bullet, \text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^\bullet}^* \right), \quad (12.3.40)$$

$$K_L \text{Fil}_{\Omega_{B_0/A_0, \text{topo}}^\bullet}^* := R \varprojlim_k \text{Kos}_I \left(K_L \text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^\bullet}^* \right). \quad (12.3.41)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get¹⁷:

$$K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet := K_L \Omega_{B_0/A_0, \text{topo}}^\bullet \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (12.3.42)$$

$$K_L \text{Fil}_{\Omega_{B_0/A_0, \text{topo}, Z}^\bullet}^* := \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}}^\bullet}^* \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (12.3.43)$$

Then we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B_0/A_0, \text{topo}, Z} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}, Z}[1/t], \quad (12.3.44)$$

$$H_{\bullet, \text{AQ}, \text{topo}, Z} := \pi_\bullet(\mathbb{L}_{B/A, \text{topo}, Z}), \quad (12.3.45)$$

$$K_L \Omega_{B/A, \text{topo}, Z}^\bullet := \text{Colim}_{A_0 \rightarrow B_0} K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet[1/t], \quad (12.3.46)$$

$$K_L \text{Fil}_{K_L \Omega_{B/A, \text{topo}, Z}^\bullet}^* := \text{Colim}_{A_0 \rightarrow B_0} K_L \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet}^*[1/t]. \quad (12.3.47)$$

¹⁶Here we did not take the corresponding derived completion, but in some situation this is achievable.

¹⁷Here again we did not take the corresponding derived completion, but in some situation this is achievable, for instance when the ring Z is endowed with derived J -complete topology.

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{B/A, \text{topo}, \text{Hodge}, Z}^\bullet, K_L \text{Fil}_{K_L \Omega_{B/A, \text{topo}, \text{Hodge}, Z}^1}^* \quad (12.3.48)$$

Example 12.3.13. Now we construct the corresponding pseudorigid analog of the corresponding example in [GL, Example 4.7]. Now we consider the corresponding the map:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm \rangle \longrightarrow \mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm/p^\infty}, X_2^{\pm/p^\infty}, \dots, X_d^{\pm/p^\infty} \rangle \quad (12.3.49)$$

which could be written as:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm \rangle \longrightarrow \quad (12.3.50)$$

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm \rangle \langle Y_1^{\pm/p^\infty}, Y_2^{\pm/p^\infty}, \dots, Y_d^{\pm/p^\infty} \rangle / (X_i - Y_i, i = 1, \dots, d). \quad (12.3.51)$$

So in our situation the corresponding Hodge complete topological derived de Rham complex will be just:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle Y_1^{\pm/p^\infty}, Y_2^{\pm/p^\infty}, \dots, Y_d^{\pm/p^\infty} \rangle [[Z_1, \dots, Z_d, Z_i = X_i - Y_i, i = 1, \dots, d]]. \quad (12.3.52)$$

We now follow the corresponding previous works [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1], [III2, Chapitre VIII] to study the corresponding functoriality for some triple $A \rightarrow B \rightarrow C$ where we have first in the algebraic setting the following result. Here we first consider more general setting where we will consider those rings over $\mathcal{O}_K[[t]] \left\langle \frac{\pi^a}{t^b} \right\rangle$ which is t -adically complete.

Proposition 12.3.14. *The corresponding functoriality for the corresponding algebraic derived de Rham complex holds in our situation for the corresponding integral adic rings A, B, C in the context of this section, namely we have the following corresponding com-*

mutative diagram:

$$\begin{array}{ccc}
 K_L\Omega_{B/A,\text{alg}}^\bullet & \longrightarrow & K_L\Omega_{B/A,\text{alg}}^\bullet \\
 \downarrow & & \downarrow \\
 B & \longrightarrow & K_L\Omega_{B/A,\text{alg}}^\bullet
 \end{array}$$

Proof. See [GL, Lemma 3.3]. □

Proposition 12.3.15. *The corresponding functoriality for the corresponding algebraic derived de Rham complex holds in our situation for the corresponding integral adic rings A, B, C in the context of this section, namely we have the following corresponding commutative diagram:*

$$\begin{array}{ccc}
 K_L\Omega_{B/A,\text{topo,Hodge}}^\bullet & \longrightarrow & K_L\Omega_{B/A,\text{topo,Hodge}}^\bullet \\
 \downarrow & & \downarrow \\
 B & \longrightarrow & K_L\Omega_{B/A,\text{topo,Hodge}}^\bullet
 \end{array}$$

Proof. Just take the corresponding derived I -completion of the diagram in the previous proposition. □

The following conjectures are literally inspired by the rigid analytic situation in [GL, Theorem 1.2]. We use the notation $*$ to denote the ring $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle [1/t]$. We assume the spaces in the following two conjectures are smooth pseudo-rigid analytic spaces over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle [1/t]$.

Conjecture 12.3.16. *The construction in this section could be carried over the pro-étale sites ¹⁸. Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Then we have the following two strictly exact long*

¹⁸For more on the foundations here for the pro-étale sites see [Ked1, Theorem 2.9.9, Remark 2.9.10]

exact sequences ¹⁹:

$$0 \longrightarrow \Omega_{*[\hat{\mathcal{O}}]^\bullet / *, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X, \text{topo}}^1 \xrightarrow{\partial} \dots,$$

and

$$0 \longrightarrow \Omega_{*[\hat{\mathcal{O}}]^\bullet / *, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X_{\acute{e}t}, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X_{\acute{e}t}, \text{topo}}^1 \xrightarrow{\partial} \dots$$

Conjecture 12.3.17. Consider the corresponding projective map $g : X_{\text{pro-}\acute{e}tale} \rightarrow X_{\acute{e}t}$ and the the projective map $f : X_{\text{pro-}\acute{e}tale} \rightarrow X$. Let M be a corresponding Z -projective differential crystal spectrum. Then we have the following two strictly exact long exact sequences:

$$M_{\otimes^{\mathbb{L}}}^{\mathbb{L}}(0 \longrightarrow \Omega_{\mathcal{O}_k[\hat{\mathcal{O}}]^\bullet / \mathcal{O}_k, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X, \text{topo}}^1 \xrightarrow{\partial} \dots),$$

and

$$M_{\otimes^{\mathbb{L}}}^{\mathbb{L}}(0 \longrightarrow \Omega_{\mathcal{O}_k[\hat{\mathcal{O}}]^\bullet / \mathcal{O}_k, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X_{\acute{e}t}, \text{topo}, Z}^\bullet \xrightarrow{\partial} \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X_{\acute{e}t}, \text{topo}}^1 \xrightarrow{\partial} \dots).$$

¹⁹We should have the corresponding naturality taking into the following form:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{*[\hat{\mathcal{O}}]^\bullet / *, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X, \text{topo}}^1 \xrightarrow{\partial} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{*[\hat{\mathcal{O}}]^\bullet / *, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & \Omega_{X[\hat{\mathcal{O}}]^\bullet / X_{\acute{e}t}, \text{topo}, Z}^\bullet & \xrightarrow{\partial} & \Omega_{X[\hat{\mathcal{O}}]^\bullet / X, \text{topo}, Z}^\bullet \otimes f^{-1} \Omega_{X_{\acute{e}t}, \text{topo}}^1 \xrightarrow{\partial} \dots \end{array}$$

12.3.3 Derived (p, I) -Complete Derived de Rham complex of Derived Adic Rings

The corresponding construction of the derived de Rham complex could be defined for general derived spaces. Along our discussion in the situations of rigid analytic spaces and the corresponding pseudorigid spaces we now focus on the corresponding derived adic rings.

Setting 12.3.18. We now fix a bounded morphism of simplicial adic rings $A \rightarrow B$ over A^* where A^* contains a corresponding ring of definition A_0^* which is derived complete with respect to the (p, I) -topology and we assume that A is adic and we assume that (A_0^*, I) is a prism in [BS1] namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For A or B respectively we assume this contains a subring A_0 or B_0 (over A_0^*) respectively such that we have A_0 or B_0 respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A). All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* . We are going to assume that p is a topologically nilpotent element.

As in the corresponding construction in the rigid and pseudocoherent situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [Qui, Section 4], we have the following:

Definition 12.3.19. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding left Kan extended derived differential complex attached to B_0/A_0 which is now denoted by $K_L\Omega_{B_0/A_0, \text{alg}}^\bullet$ ²⁰ from the corresponding (p, I) -complete commutative rings. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0, \text{alg}} := \Omega_{P_{B_0}^\bullet/A_0, \text{alg}}^1 \otimes_{P_{B_0}^\bullet} B_0. \quad (12.3.53)$$

²⁰Namely over A_0 we take the suitable corresponding left Kan extension, then apply to B_0 .

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ, alg}} := \pi_{\bullet}(\mathbb{L}_{B_0/A_0, \text{alg}}). \quad (12.3.54)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R\varprojlim \text{Kos}_{(p, I)} \left((\Omega_{P_{B_0}^{\bullet}/A_0, \text{alg}}^1 \otimes_{P_{B_0}^{\bullet}} B_0) \right). \quad (12.3.55)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L \Omega_{-/A_0, \text{alg}}^{\bullet}(B_0), \text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^{\bullet}}^*. \quad (12.3.56)$$

We then take the corresponding derived (p, I) -completion and we denote that by:

$$K_L \Omega_{B_0/A_0, \text{topo}}^{\bullet} := R\varprojlim \text{Kos}_{(p, I)} \left(K_L \Omega_{B_0/A_0, \text{alg}}^{\bullet}, \text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^{\bullet}}^* \right), \quad (12.3.57)$$

$$K_L \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}}^{\bullet}}^* := R\varprojlim \text{Kos}_{(p, I)} \left(\text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^{\bullet}}^* \right). \quad (12.3.58)$$

Then in the situation that all the rings are classical adic rings we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B/A, \text{topo}} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}}[1/(d)], \quad (12.3.59)$$

$$H_{\bullet, \text{AQ, topo}} := \pi_{\bullet}(\mathbb{L}_{B/A, \text{topo}}), \quad (12.3.60)$$

$$K_L \Omega_{B/A, \text{topo}}^{\bullet} := \text{Colim}_{A_0 \rightarrow B_0} K_L \Omega_{B_0/A_0, \text{topo}}^{\bullet}[1/(d)], \quad (12.3.61)$$

$$K_L \text{Fil}_{K_L \Omega_{B/A, \text{topo}}^{\bullet}}^* := \text{Colim}_{A_0 \rightarrow B_0} \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}}^{\bullet}}^*[1/(d)]. \quad (12.3.62)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{B/A, \text{topo, Hodge}}^{\bullet}, \text{Fil}_{K_L \Omega_{B/A, \text{topo, Hodge}}^{\bullet}}^*. \quad (12.3.63)$$

Definition 12.3.20. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_{∞} -ring $K_L \Omega_{B/A, \text{topo}}^{\bullet}$ with the corresponding induced filtrations.

Definition 12.3.21. We define the corresponding almost perfect ²¹ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $K_L\Omega_{B/A,\text{topo}}^\bullet$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over pseudorigid spaces where Z now is a simplicial topological algebra over \mathbb{Z}_p ²².

Definition 12.3.22. We define the following Z deformed version of the corresponding complete version of the corresponding André-Quillen homology and the corresponding complete version of the corresponding derived de Rham complex. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding left Kan extended derived differential complex attached to B_0 from the corresponding derived (p, I) -complete commutative rings which is now denoted by $K_L\Omega_{B_0/A_0,\text{alg}}^\bullet$. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0,\text{alg}} := \Omega_{P_{B_0}^\bullet/A_0,\text{alg}}^1 \otimes_{P_{B_0}^\bullet} B_0. \quad (12.3.64)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet,\text{AQ,alg}} := \pi_\bullet(\mathbb{L}_{B_0/A_0,\text{alg}}). \quad (12.3.65)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0,\text{topo}} := R\varprojlim_{\leftarrow} \text{Kos}_{(p,I)} \left((\Omega_{P_{B_0}^\bullet/A_0,\text{alg}}^1 \otimes_{P_{B_0}^\bullet} B_0) \right). \quad (12.3.66)$$

Taking the product with Z we have the corresponding integral version of the topological André-Quillen complex:

$$\mathbb{L}_{B_0/A_0,\text{topo},Z} := \mathbb{L}_{B_0/A_0,\text{topo}} \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z \quad (12.3.67)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{B_0/A_0,\text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{B_0/A_0,\text{alg}}^\bullet}^*. \quad (12.3.68)$$

²¹This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

²²It is better to assume that it is basically derived completely flat.

We then take the corresponding derived (p, I) -completion and we denote that by:

$$K_L \Omega_{B_0/A_0, \text{topo}}^\bullet := R \lim_{\leftarrow k} \text{Kos}_{(p, I)} \left(K_L \Omega_{B_0/A_0, \text{alg}}^\bullet, \text{Fil}_{\Omega_{B_0/A_0, \text{alg}}^\bullet}^* \right), \quad (12.3.69)$$

$$\text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}}^\bullet}^* := R \lim_{\leftarrow k} \left(\text{Fil}_{K_L \Omega_{B_0/A_0, \text{alg}}^\bullet}^* \right). \quad (12.3.70)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get:

$$K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet := K_L \Omega_{B_0/A_0, \text{topo}}^\bullet \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (12.3.71)$$

$$\text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet}^* := \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}}^\bullet}^* \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (12.3.72)$$

When we have that the corresponding ring Z is also derived (p, I) -topologized and commutative, then we can further take the corresponding derived (p, I) -completion to achieve the corresponding derived completed version:

$$K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet := K_L \Omega_{B_0/A_0, \text{topo}}^\bullet \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (12.3.73)$$

$$\text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet}^* := \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}}^\bullet}^* \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (12.3.74)$$

Then in the situation that all the rings are classical adic rings we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B/A, \text{topo}, Z} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}, Z}[1/(d)], \quad (12.3.75)$$

$$H_{\bullet, \text{AQ}, \text{topo}, Z} := \pi_\bullet(\mathbb{L}_{B/A, \text{topo}, Z}), \quad (12.3.76)$$

$$K_L \Omega_{B/A, \text{topo}, Z}^\bullet := \text{Colim}_{A_0 \rightarrow B_0} K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet[1/(d)], \quad (12.3.77)$$

$$\text{Fil}_{K_L \Omega_{B/A, \text{topo}, Z}^\bullet}^* := \text{Colim}_{A_0 \rightarrow B_0} \text{Fil}_{K_L \Omega_{B_0/A_0, \text{topo}, Z}^\bullet}^*[1/(d)]. \quad (12.3.78)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{B/A, \text{topo}, \text{Hodge}, Z}^\bullet, \text{Fil}_{K_L \Omega_{B/A, \text{topo}, \text{Hodge}, Z}^\bullet}^*. \quad (12.3.79)$$

12.4 Topological Logarithmic Derived De Rham Complexes

12.4.1 Logarithmic Setting for Rigid Analytic Spaces

In this section we are now going to follow Gabber's construction as in [O, Chapter 8] and the extension by [B1, Chapter 5, Chapter 6, Chapter 7] to consider the corresponding construction of the topological complete logarithmic cotangent complexes and the corresponding topological logarithmic derived de Rham complex following the corresponding construction for rigid spaces. Certainly the corresponding construction will also follow closely the corresponding [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [Qui, Section 4] as if we do not have the corresponding log structures, in particular the geometry context underlying is just as in [GL, Chapter 3, Chapter 4]. For the geometric foundation of log adic spaces, see [DLLZ1]. We start from the corresponding construction of the algebraic p -adic logarithmic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$ of p -complete rings carrying the corresponding log structures, here we use the corresponding notation $(*, ?)$, $* = A, B$ to denote the corresponding admissible log rings where $?$ represents the corresponding monoids in the consideration. This is the corresponding derived differential complex attached to the canonical resolution (which we will denote it by the same notation as in the non logarithmic setting) of (B, N) :

$$(A, M)[(B, N)]^\bullet, \quad (12.4.1)$$

which is now denoted by $K_L\Omega_{(B,N)/(A,M),\text{alg}}^\bullet := K_L\Omega_{-(A,M),\text{alg}}^\bullet((B, N))$ after taking the left Kan extension as in [B1, Chapter 6] and applying to the ring (B, N) . The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B,N)/(A,M),\text{alg}} := K_L\Omega_{(A,M)[(B,N)]^\bullet/(A,M),\text{alg}}^1 \otimes_{(A,M)[(B,N)]^\bullet} (B, N). \quad (12.4.2)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\bullet,\text{AQ,alg}} := \pi_\bullet(\mathbb{L}_{(B,N)/(A,M),\text{alg}}). \quad (12.4.3)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding certain p -completion over the simplicial module structure.

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L \Omega_{(B,N)/(A,M),\text{alg}}^\bullet, \text{Fil}_{K_L \Omega_{(B,N)/(A,M),\text{alg}}^1}^* \cdot \quad (12.4.4)$$

We then take the corresponding Banach completion and we denote that by:

$$K_L \Omega_{(B,N)/(A,M),\text{topo}}^\bullet, \text{Fil}_{K_L \Omega_{(B,N)/(A,M),\text{topo}}^1}^* \cdot \quad (12.4.5)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above:

$$K_L \widehat{\Omega}_{(B,N)/(A,M),\text{topo}}^\bullet, \text{Fil}_{K_L \widehat{\Omega}_{(B,N)/(A,M),\text{topo}}^1}^* \cdot \quad (12.4.6)$$

This is basically the corresponding analytic and complete version the corresponding algebraic log de Rham complex. Furthermore we allow large coefficients with rigid affinoid algebra Z over \mathbb{Q}_p . Therefore we take the corresponding completed tensor product in the following.

Definition 12.4.1. We define the following Z deformed version of the corresponding complete version of the corresponding logarithmic André-Quillen homology and the corresponding complete version of the corresponding logarithmic derived de Rham complex. We start from the corresponding construction of the algebraic p -adic logarithmic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^\bullet, \quad (12.4.7)$$

which is now denoted by $K_L \Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^\bullet$ after taking the corresponding left Kan extension as in [B1, Chapter 6]. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0 / (A_0, M_0), \text{alg}} := K_L \Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0). \quad (12.4.8)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ}, \text{alg}, \log} := \pi_\bullet(\mathbb{L}_{(B_0, N_0) / (A_0, M_0), \text{alg}}). \quad (12.4.9)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived p -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (12.4.10)$$

$$:= R \varprojlim_k \text{Kos}_{p^k} \left((K_L \Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, N_0)[(B_0, N_0)]^\bullet} (B_0, N_0)) \right). \quad (12.4.11)$$

Taking the product with \mathcal{O}_Z we have the corresponding integral version of the topological logarithmic André-Quillen complex:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_Z \quad (12.4.12)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet}^* \quad (12.4.13)$$

We then take the corresponding derived p -completion and we denote that by:

$$K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet \quad (12.4.14)$$

$$:= R \varprojlim_k \text{Kos}_{p^k} \left(K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet}^* \right), \quad (12.4.15)$$

$$\text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* := R \varprojlim_k \text{Kos}_{p^k} \left(\text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet}^* \right). \quad (12.4.16)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with \mathcal{O}_Z to get:

$$K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet := K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_Z, \quad (12.4.17)$$

$$\text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet}^* := \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_Z. \quad (12.4.18)$$

Then we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} [1/p], \quad (12.4.19)$$

$$H_{\bullet, \text{AQ}, \text{topo}, Z} := \pi_\bullet(\mathbb{L}_{(B, N)/(A, M), \text{topo}, Z}), \quad (12.4.20)$$

$$K_L \Omega_{(B, N)/(A, M), \text{topo}, Z}^\bullet := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet [1/p], \quad (12.4.21)$$

$$\text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}, Z}^\bullet}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet}^* [1/p]. \quad (12.4.22)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L\Omega_{(B,N)/(A,M),\text{topo,Hodge},Z}^\bullet, \text{Fil}_{K_L\Omega_{(B,N)/(A,M),\text{topo,Hodge},Z}^1}^* \quad (12.4.23)$$

Definition 12.4.2. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $K_L\Omega_{(B,N)/(A,M),\text{topo},Z}^\bullet$ with the corresponding induced filtrations.

Definition 12.4.3. We define the corresponding almost perfect ²³ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $K_L\Omega_{(B,N)/(A,N),\text{topo},Z}^\bullet$ with the corresponding induced filtrations.

The following is inspired by the main Poincaré Lemma from [GL, Theorem 1.2] in the non-deformed situation. Consider a corresponding smooth log rigid analytic space X over k/\mathbb{Q}_p (where k is a corresponding unramified analytic field which is discretely-valued and the corresponding residue field is finite). Then we have the following:

Conjecture 12.4.4. *Consider the corresponding projective map $g : X_{\text{Kummer-pro-étale}} \rightarrow X_{\text{Kummer-ét}}$ and the the projective map $f : X_{\text{Kummer-pro-étale}} \rightarrow X$. Then we have the logarithmic versions of the strictly exact long exact sequences as in the Poincaré lemma in the rigid analytic situation.*

²³This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

12.4.2 Logarithmic Setting for Pseudorigid Spaces

We now consider the analytic logarithmic André-Quillen Homology and analytic logarithmic Derived de Rham complex of pseudorigid space, which are very crucial in some development in [Bel1] in the arithmetic family. Therefore we just investigate the corresponding picture in the corresponding geometric family.

Setting 12.4.5. We consider now a corresponding morphism taking the corresponding form of $(A, M) \rightarrow (B, N)$ where (A, M) is going to be a log pseudorigid affinoid algebra over \mathbb{Z}_p (where we consider the corresponding foundation in [DLLZ1, Chapter 2] for the corresponding log adic rings), and (B, N) is going to be a Kummer perfectoid chart of A in the corresponding Kummer pro-étale site of the log pseudorigid affinoid space attached to A (where we consider the corresponding foundation in [DLLZ1, Chapter 5] for the corresponding Kummer pro-étale sites). As in [Bel1, Definition 3.1, and below Definition 3.1] in our situation A is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle [1/t]$, where K is a discrete valued field containing \mathbb{Q}_p and $(a, b) = 1$ ²⁴.

From [Bel1, Definition 3.1, and below Definition 3.1] we have the following:

Lemma 12.4.6. *We have the following statements:*

- A. *A as above is Tate, complete over \mathcal{O}_K ;*
- B. *The ring A has a ring of definition A_0 which is of \mathcal{O}_K -formally finite type;*
- C. *The ring A_0 is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle$.*

Proof. Since we did not change the corresponding assumption on the corresponding ring theoretic consideration. □

As in the corresponding construction in the rigid situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 5, Chapter 6, Chapter 7], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [O, Chapter 8], [Qui, Section 4]. We keep now the following setting:

Setting 12.4.7. Now as in the above notion we will consider a general map of logarithmic rings $(A, M) \rightarrow (B, N)$ over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle [1/t]$ where we have the corresponding map of the associated ring of definitions $(A_0, M_0) \rightarrow (B_0, N_0)$ over $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle$ such that A_0, B_0 are basically I -adic (where $\mathcal{O}_K[[t]] \langle \pi^a/t^b \rangle$ is I -adic).

²⁴Certainly one can also consider the characteristic p situation.

Definition 12.4.8. We start from the corresponding construction of the algebraic p -adic log derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^\bullet, \quad (12.4.24)$$

which is now denoted by $K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet$. This is after taking the left Kan extension as in [B1, Chapter 6]. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0). \quad (12.4.25)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ}, \text{alg}} := \pi_\bullet(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (12.4.26)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (12.4.27)$$

$$:= R\lim_{\leftarrow I} \text{Kos}_I \left((K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0)) \right). \quad (12.4.28)$$

Then we consider the corresponding logarithmic derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^*. \quad (12.4.29)$$

We then take the corresponding derived I -completion and we denote that by:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet := \quad (12.4.30)$$

$$R\lim_{\leftarrow I} \text{Kos}_I \left(K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \right), \quad (12.4.31)$$

$$\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* := R\lim_{\leftarrow I} \text{Kos}_I \left(\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \right). \quad (12.4.32)$$

Then we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}}[1/t], \quad (12.4.33)$$

$$H_{\bullet, \text{AQ}, \text{topo}} := \pi_{\bullet}(\mathbb{L}_{(B, M)/(A, M), \text{topo}}), \quad (12.4.34)$$

$$K_L \Omega_{(B, N)/(A, M), \text{topo}}^{\bullet} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\bullet}[1/t], \quad (12.4.35)$$

$$\text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}}^{\bullet}}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\bullet}}^*[1/t]. \quad (12.4.36)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^{\bullet}, \text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^{\bullet}}^*. \quad (12.4.37)$$

Definition 12.4.9. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_{∞} -ring $K_L \Omega_{(B, N)/(A, M), \text{topo}}^{\bullet}$ with the corresponding induced filtrations.

Definition 12.4.10. We define the corresponding almost perfect ²⁵ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_{∞} -ring $K_L \Omega_{(B, N)/(A, M), \text{topo}}^{\bullet}$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over logarithmic pseudorigid spaces where Z now is a topological algebra over \mathbb{Z}_p ²⁶.

Definition 12.4.11. We define the following Z deformed version of the corresponding complete version of the corresponding logarithmic André-Quillen homology and the corresponding complete version of the corresponding logarithmic derived de Rham complex. We start from the corresponding construction of the logarithmic algebraic p -adic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^{\bullet}, \quad (12.4.38)$$

²⁵This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

²⁶It is better to assume that it is basically completely flat.

which is now denoted by $K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet$. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0). \quad (12.4.39)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ}, \text{alg}} := \pi_\bullet(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (12.4.40)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (12.4.41)$$

$$:= R\lim_{\leftarrow k} \text{Kos}_I \left((K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0)) \right). \quad (12.4.42)$$

Taking the product with Z we have the corresponding integral version of the topological logarithmic version of the corresponding André-Quillen complex:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \otimes_{\mathbb{Z}_p} \mathcal{O}_Z \quad (12.4.43)$$

Then we consider the corresponding logarithmic derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \quad (12.4.44)$$

We then take the corresponding derived I -completion and we denote that by:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet := \quad (12.4.45)$$

$$R\lim_{\leftarrow k} \text{Kos}_I \left(K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \right), \quad (12.4.46)$$

$$\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* := R\lim_{\leftarrow k} \text{Kos}_I \left(\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \right). \quad (12.4.47)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet := K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (12.4.48)$$

$$\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet}^* := \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (12.4.49)$$

Then we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}[1/t], \quad (12.4.50)$$

$$H_{\bullet, \text{AQ}, \text{topo}, Z} := \pi_{\bullet}(\mathbb{L}_{(B, N)/(A, M), \text{topo}, Z}), \quad (12.4.51)$$

$$K_L \Omega_{(B, N)/(A, M), \text{topo}, Z}^{\bullet} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^{\bullet}[1/t], \quad (12.4.52)$$

$$\text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}, Z}^{\bullet}}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^{\bullet}}^*[1/t]. \quad (12.4.53)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}, Z}^{\bullet}, \text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}, Z}^{\bullet}}^*. \quad (12.4.54)$$

Example 12.4.12. Now we construct the corresponding pseudorigid analog of the corresponding example in [GL, Example 4.7]. Now we consider the corresponding the map:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm}, X_2^{\pm}, \dots, X_d^{\pm}, X_{d+1}, \dots, X_e \rangle \quad (12.4.55)$$

$$\longrightarrow \mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm/p^{\infty}}, X_2^{\pm/p^{\infty}}, \dots, X_d^{\pm/p^{\infty}}, X_{d+1}^{1/p^{\infty}}, \dots, X_e^{1/p^{\infty}} \rangle \quad (12.4.56)$$

which could be written as:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm}, X_2^{\pm}, \dots, X_d^{\pm}, X_{d+1}, \dots, X_e \rangle \longrightarrow \quad (12.4.57)$$

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm}, X_2^{\pm}, \dots, X_d^{\pm}, X_{d+1}, \dots, X_e \rangle \left\langle Y_1^{\pm/p^{\infty}}, \dots, Y_d^{\pm/p^{\infty}}, Y_{d+1}^{1/p^{\infty}}, \dots, Y_e^{1/p^{\infty}} \right\rangle \quad (12.4.58)$$

$$/(X_i - Y_i, i = 1, \dots, e). \quad (12.4.59)$$

So in our situation the corresponding Hodge complete topological derived de Rham complex will be just:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \left\langle Y_1^{\pm/p^{\infty}}, Y_2^{\pm/p^{\infty}}, \dots, Y_d^{\pm/p^{\infty}}, Y_{d+1}^{1/p^{\infty}}, \dots, Y_e^{1/p^{\infty}} \right\rangle \quad (12.4.60)$$

$$[[Z_1, \dots, Z_e, Z_i = X_i - Y_i, i = 1, \dots, e]]. \quad (12.4.61)$$

The following conjectures are literally inspired by the rigid analytic situation in [GL, Theorem 1.2]. We assume the spaces are smooth.

Conjecture 12.4.13. *After the whole foundation of [DLLZ1] and [Ked1, Theorem 2.9.9, Remark 2.9.10], the construction in this section could be carried over the corresponding certain Kummer-pro-étale sites where logarithmic perfectoid subdomains form a corresponding basis of neighbourhood of the topology.*

Based on this conjecture one can conjecture the following:

Conjecture 12.4.14. *Consider the corresponding projective map $g : X_{\text{Kummer-pro-étale}} \rightarrow X_{\text{Kummer-ét}}$ and the the projective map $f : X_{\text{Kummer-pro-étale}} \rightarrow X$. Then we have the log versions of the strictly exact Poincaré long exact sequences as in the pseudorigid analytic situation.*

and

Conjecture 12.4.15. *Consider the corresponding projective map $g : X_{\text{Kummer-pro-étale}} \rightarrow X_{\text{Kummer-ét}}$ and the the projective map $f : X_{\text{Kummer-pro-étale}} \rightarrow X$. Let M be a corresponding Z -projective differential crystal spectrum. Then we have the logarithmic versions of the strictly exact Poincaré long exact sequences for M as in the pseudorigid analytic situation.*

12.4.3 Derived (p, I) -Complete Logarithmic Derived de Rham complex

The corresponding construction of the derived de Rham complex could be defined for general derived spaces. Along our discussion in the situations of rigid analytic spaces and the corresponding pseudorigid spaces we now focus on the corresponding derived rings. And we consider the corresponding logarithmic setting.

Setting 12.4.16. We now fix a bounded morphism of log simplicial topological rings $(A, M) \rightarrow (B, N)$ over A^* where A^* contains a corresponding ring of definition A_0^* which is derived complete with respect to the (p, I) -topology and we assume that (A, M) is adic in the sense to be defined just below and we assume that (A_0^*, I) is a prism namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For (A, M) or (B, N) respectively we assume this contains a subring (A_0, M) or (B_0, N_0) (over A_0^*) respectively such that we have A_0 or B_0 respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A)²⁷. All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* . We are going to assume that p is a topologically nilpotent element.

As in the corresponding non-logarithmic situation following [B1, Chapter 6]²⁸ we first consider the category $\text{Alg}_{\text{prelog, smooth}, A_0^*}$ of all the smooth prelog algebras over A_0^* and take the corresponding left Kan extension to the corresponding ∞ -category $\text{Alg}_{\infty, \text{prelog}, A_0^*}$ of all the prelog simplicial A_0^* -algebras. Then we may take the corresponding derived (p, I) -completion to get completed objects. Finally we could basically apply this to the relatively more specific rings in the previous setting. These are basically parallel to the situations where we work over \mathbb{Z}_p and the pseudorigid disc²⁹.

Remark 12.4.17. The corresponding idea behind this is certainly inspired by [LL] for instance where the authors compared the corresponding prismatic cohomology of some

²⁷One can also invert p to consider the rationality with respect to p as in the rigid analytic situation.

²⁸Here the corresponding left Kan extension happens along the embedding of the free prelog rings to the ∞ -category of simplicial prelog rings. See [B1, Chapter 6] for the construction.

²⁹Certainly we could work in more general setting over any derived J -complete ring, such as in the situation where we do not require that the ring A_0^* is a prism.

suitable ring R over A_0^*/I for instance and the corresponding derived de Rham cohomology canonically attached to the corresponding morphism, and the corresponding Hodge filtration is compared to the Nygaard filtration.

As in the corresponding construction in the rigid and pseudocoherent situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 5, Chapter 6, Chapter 7], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [O, Chapter 8], [Qui, Section 4] we give the following definition.

Definition 12.4.18. We start from the corresponding construction of the algebraic p -adic logarithmic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the cofibrant replacement of (B_0, N_0) :

$$P_{(B_0, N_0)}^\bullet, \quad (12.4.62)$$

which is now denoted by $K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet$ after taking the corresponding left Kan extension as in [B1, Chapter 6]. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := K_L\Omega_{P_{(B_0, N_0)}^\bullet/(A_0, M_0), \text{alg}}^1 \otimes_{P_{(B_0, N_0)}^\bullet} (B_0, N_0). \quad (12.4.63)$$

The corresponding logarithmic algebraic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ}, \text{alg}} := \pi_\bullet(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (12.4.64)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} := R\varprojlim_{\leftarrow} \text{Kos}_{(p, I)} \left((K_L\Omega_{P_{(B_0, N_0)}^\bullet/(A_0, M_0), \text{alg}}^1 \otimes_{P_{(B_0, N_0)}^\bullet} (B_0, N_0)) \right). \quad (12.4.65)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^*. \quad (12.4.66)$$

We then take the corresponding derived (p, I) -completion and we denote that by:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet := R\varprojlim \text{Kos}_{(p, I)} \left(K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet}^* \right), \quad (12.4.67)$$

$$\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* := R\varprojlim \text{Kos}_{(p, I)} \left(\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet}^* \right). \quad (12.4.68)$$

Then in the situation that all the rings are classical logarithmic adic rings we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{(B, N)/(A, M), \text{topo}} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} [1/(d)], \quad (12.4.69)$$

$$H_{\bullet, \text{AQ}, \text{topo}} := \pi_\bullet(\mathbb{L}_{(B, N)/(A, M), \text{topo}}), \quad (12.4.70)$$

$$K_L\Omega_{(B, N)/(A, M), \text{topo}}^\bullet := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet [1/(d)], \quad (12.4.71)$$

$$\text{Fil}_{K_L\Omega_{(B, N)/(A, M), \text{topo}}^\bullet}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* [1/(d)]. \quad (12.4.72)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L\Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^\bullet, \text{Fil}_{K_L\Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^\bullet}^*. \quad (12.4.73)$$

Definition 12.4.19. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $K_L\Omega_{(B, N)/(A, M), \text{topo}}^\bullet$ with the corresponding induced filtrations.

Definition 12.4.20. We define the corresponding almost perfect ³⁰ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $K_L\Omega_{(B, N)/(A, M), \text{topo}}^\bullet$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over pseudorigid spaces where Z now is a simplicial topological algebra over \mathbb{Z}_p ³¹.

Definition 12.4.21. We define the following Z deformed version of the corresponding complete version of the corresponding logarithmic André-Quillen homology and the cor-

³⁰This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

³¹It is better to assume that it is basically derived completely flat.

responding complete version of the corresponding logarithmic derived de Rham complex. We start from the corresponding construction of the logarithmic algebraic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^\bullet, \quad (12.4.74)$$

which is now denoted by $K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet$. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet/(A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0). \quad (12.4.75)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\bullet, \text{AQ}, \text{alg}} := \pi_\bullet(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (12.4.76)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (12.4.77)$$

$$:= R\varprojlim_k \text{Kos}_{(p, I)} \left((K_L\Omega_{(A_0, M_0)[(B_0, N_0)]^\bullet/(A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^\bullet} (B_0, N_0)) \right). \quad (12.4.78)$$

Taking the product with Z we have the corresponding integral version of the topological logarithmic version of the corresponding André-Quillen complex:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \otimes_{\mathbb{Z}_p} Z \quad (12.4.79)$$

Then we consider the corresponding logarithmic derived algebraic de Rham complex which is just defined to be:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^*. \quad (12.4.80)$$

We then take the corresponding derived (p, I) -completion and we denote that by:

$$K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet := \quad (12.4.81)$$

$$R\varprojlim_k \text{Kos}_{(p, I)} \left(K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^\bullet, \text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \right), \quad (12.4.82)$$

$$\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* := R\varprojlim_k \text{Kos}_{(p, I)} \left(\text{Fil}_{K_L\Omega_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^* \right). \quad (12.4.83)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get:

$$K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet := K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (12.4.84)$$

$$\text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}}^* := \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (12.4.85)$$

When we have that the corresponding ring Z is also derived (p, I) -topologized and commutative, then we can further take the corresponding derived (p, I) -completion to achieve the corresponding derived completed version:

$$K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}^\bullet := K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (12.4.86)$$

$$\text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}}^* := \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}}^\bullet}^* \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (12.4.87)$$

Then in the situation that all the rings are classical logarithmic adic rings we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} [1/(p, I)], \quad (12.4.88)$$

$$H_{\bullet, \text{AQ}, \text{topo}, Z} := \pi_\bullet(\mathbb{L}_{(B, N)/(A, M), \text{topo}, Z}), \quad (12.4.89)$$

$$K_L \Omega_{(B, N)/(A, M), \text{topo}, Z}^\bullet := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} K_L \Omega_{(B, N)/(A_0, M_0), \text{topo}, Z}^\bullet [1/(p, I)], \quad (12.4.90)$$

$$\text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}, Z}}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{K_L \Omega_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z}}^* [1/(p, I)]. \quad (12.4.91)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$K_L \Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}, Z}^\bullet, \text{Fil}_{K_L \Omega_{(B, N)/(A, M), \text{topo}, \text{Hodge}, Z}}^*. \quad (12.4.92)$$

12.5 Robba Sheaves and Frobenius Sheaves

12.5.1 Pseudorigid Relative Toric Tower

In this section we discuss the corresponding Robba sheaves of [KL1] and [KL2] over general spaces. Here we consider the corresponding discussion for the pseudorigid situation which is certainly following the corresponding treatment in [KL2, Chapter 8]. But we do not really understand if the corresponding generality could be achieved at this moment. So one definitely has to be very careful in the corresponding analysis ³². These kinds of spaces are actually different from the rigid analytic space, although the theories are really related to each other, such as in [Bel1], [Bel2] and [L]. Following the ideas in the rigid situation in [KL2], we consider the towers in the smooth situation in the following:

Setting 12.5.1. We consider the following towers as in [KL2, Chapter 5]. First we let A_0 be the ring $\mathcal{O}_K[[u]] \langle \frac{\pi^a}{u^b} \rangle [1/u] \langle T_1^\pm, \dots, T_d^\pm \rangle$ and we put A_0^+ to be the ring $\mathcal{O}_K[[u]] \langle \frac{\pi^a}{u^b} \rangle \cdot \langle T_1^\pm, \dots, T_d^\pm \rangle$. And for the higher level we have the following rings:

$$(A_n, A_n^+) \tag{12.5.1}$$

$$:= (\mathcal{O}_K(\pi^{1/p^n})[[u]] \langle \frac{\pi^a}{u^b} \rangle [1/u] (u^{\pm 1/p^n}) \langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n} \rangle), \tag{12.5.2}$$

$$\mathcal{O}_K(\pi^{1/p^n})[[u]] \langle \frac{\pi^a}{u^b} \rangle [u^{1/p^n}] \langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n} \rangle), \tag{12.5.3}$$

which implies that we have:

$$A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_n \longrightarrow \dots, \forall n \geq 0.$$

Proposition 12.5.2. *The tower above is finite étale.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \langle V_1^\pm, \dots, V_e^\pm \rangle.$$

³²Pseudorigid spaces are actually along our generalization in our mind, which are the first kinds of spaces we would like to study in our project before considering more general topological or functional analytic spaces, namely those general spaces whose structure sheaves of simplicial rings carry topologies or norms such as in [BBBK] and [CS2].

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \tag{12.5.4}$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle, \tag{12.5.5}$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle). \tag{12.5.6}$$

This is certainly a finite étale tower. \square

Proposition 12.5.3. *The tower above is perfectoid.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \langle V_1^{\pm}, \dots, V_e^{\pm} \rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \tag{12.5.7}$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle, \tag{12.5.8}$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle). \tag{12.5.9}$$

The corresponding ∞ -level is just:

$$A_\infty \tag{12.5.10}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] [1/u^{1/p^\infty}] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge, \tag{12.5.11}$$

$$A_\infty^+ \tag{12.5.12}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge. \tag{12.5.13}$$

These are certainly the corresponding perfectoid rings in the sense of [KL2, Definition 3.3.1], also see [KL2, Lemma 3.3.28]³³. \square

Remark 12.5.4. Here the corresponding rings are regarded as the corresponding topological rings instead of Banach rings, but we can certainly consider the corresponding Banach ring structure induced from the linear topology namely really the ∞ -level of this tower is Fontaine perfectoid adic Banach ring which is Tate in the sense of [KL2, Definition 3.1.1].

³³As in [KL2, Lemma 3.3.28] one takes the original pseudouniformizer, and goes to a suitably high level to achieve some root u' of the function $X^{p^n} - Xu - u$, and eventually the corresponding element u'^p will divide p .

Proposition 12.5.5. *The tower is weakly decompleting.*

Proof. We give the proof where $K = \mathbb{Q}_p$, which is certainly carried over to more general situation. The corresponding tower is actually weakly decompleting once one considers the corresponding presentation as above for the corresponding ring of definition:

$$A_\infty \tag{12.5.14}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}][[1/u^{1/p^\infty}] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge, \tag{12.5.15}$$

$$A_\infty^+ \tag{12.5.16}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge. \tag{12.5.17}$$

This transfers to the positive characteristic situation under tilting:

$$R_\infty \tag{12.5.18}$$

$$:= k_K[\bar{\pi}^{1/p^\infty}][[\bar{u}^{1/p^\infty}, \bar{S}_1^{1/p^\infty}, \dots, \bar{S}_d^{1/p^\infty}][[1/\bar{u}^{1/p^\infty}] \langle \bar{V}_1^{\pm 1/p^\infty}, \dots, \bar{V}_e^{\pm 1/p^\infty} \rangle^\wedge, \tag{12.5.19}$$

$$R_\infty^+ \tag{12.5.20}$$

$$:= k_K[\bar{\pi}^{1/p^\infty}][[\bar{u}^{1/p^\infty}, \bar{S}_1^{1/p^\infty}, \dots, \bar{S}_d^{1/p^\infty}] \langle \bar{V}_1^{\pm 1/p^\infty}, \dots, \bar{V}_e^{\pm 1/p^\infty} \rangle^\wedge. \tag{12.5.21}$$

Then one can finish as in [KL2, Lemma 7.1.2] by comparing this with $R_{(H_\bullet, H_\bullet^+)}$. \square

12.5.2 Pseudorigid Logarithmic Relative Toric Tower

Setting 12.5.6. We consider the following towers as in [KL2, Chapter 5]. First we let A_0 be the ring $\mathcal{O}_K[[u]] \langle \frac{\pi^a}{u^b} \rangle [1/u] \langle T_1^\pm, \dots, T_d^\pm, T_{d+1}^-, \dots, T_f^- \rangle$ and we put A_0^+ to be the ring

$$\mathcal{O}_K[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle \cdot \langle T_1^\pm, \dots, T_d^\pm, T_{d+1}^-, \dots, T_f^- \rangle. \tag{12.5.22}$$

And for the higher level we have the following rings:

$$(A_n, A_n^+) \tag{12.5.23}$$

$$:= (\mathcal{O}_K(\pi^{1/p^n})[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [1/u] (u^{\pm 1/p^n}) \langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n}, T_{d+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \rangle, \tag{12.5.24}$$

$$\mathcal{O}_K(\pi^{1/p^n})[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [u^{1/p^n}] \langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n}, T_{d+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \rangle, \tag{12.5.25}$$

which implies that we have:

$$A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_n \longrightarrow \dots, \forall n \geq 0.$$

Proposition 12.5.7. *The tower above is finite étale.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \langle V_1^\pm, \dots, V_e^\pm \rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \tag{12.5.26}$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \rangle), \tag{12.5.27}$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{d+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \rangle). \tag{12.5.28}$$

This is certainly a finite étale tower. □

Proposition 12.5.8. *The tower above is perfectoid.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \langle V_1^\pm, \dots, V_e^\pm \rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \tag{12.5.29}$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \rangle), \tag{12.5.30}$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \rangle). \tag{12.5.31}$$

The corresponding ∞ -level is just:

$$A_\infty := \tag{12.5.32}$$

$$\mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}][[1/u^{1/p^\infty}]] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge, \tag{12.5.33}$$

$$A_\infty^+ \tag{12.5.34}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge. \tag{12.5.35}$$

These are certainly the corresponding perfectoid rings in the sense of [KL2, Definition 3.3.1], also see [KL2, Lemma 3.3.28]³⁴. \square

Remark 12.5.9. Here the corresponding rings are regarded as the corresponding topological rings instead of Banach rings, but we can certainly consider the corresponding Banach ring structure induced from the linear topology namely really the ∞ -level of this tower is Fontaine perfectoid adic Banach ring which is Tate in the sense of [KL2, Definition 3.1.1].

Proposition 12.5.10. *The tower is weakly decompleting.*

Proof. We give the proof where $K = \mathbb{Q}_p$, which is certainly carried over to more general situation. The corresponding tower is actually weakly decompleting once one considers the corresponding presentation as above for the corresponding ring of definition:

$$A_\infty := \tag{12.5.36}$$

$$\mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}][[1/u^{1/p^\infty}]] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge, \tag{12.5.37}$$

$$A_\infty^+ \tag{12.5.38}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge. \tag{12.5.39}$$

³⁴As in [KL2, Lemma 3.3.28] one takes the original pseudouniformizer, and goes to a suitably high level to achieve some root u' of the function $X^{p^n} - Xu - u$, and eventually the corresponding element u'^p will divide p .

This transfers to the positive characteristic situation under tilting:

$$R_\infty := \tag{12.5.40}$$

$$k_K[\bar{\pi}^{1/p^\infty}][[\bar{u}^{1/p^\infty}, \bar{S}_1^{1/p^\infty}, \dots, \bar{S}_d^{1/p^\infty}][[1/\bar{u}^{1/p^\infty}]] \left\langle \bar{V}_1^{\pm 1/p^\infty}, \dots, \bar{V}_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge, \tag{12.5.41}$$

$$R_\infty^+ \tag{12.5.42}$$

$$:= k_K[\bar{\pi}^{1/p^\infty}][[\bar{u}^{1/p^\infty}, \bar{S}_1^{1/p^\infty}, \dots, \bar{S}_d^{1/p^\infty}]] \left\langle \bar{V}_1^{\pm 1/p^\infty}, \dots, \bar{V}_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge. \tag{12.5.43}$$

Then one can finish as in [KL2, Lemma 7.1.2] by comparing this with $R_{(H_\bullet, H_\bullet^+)}$. \square

12.5.3 Robba Sheaves over Pseudorigid Spaces

Now we consider Kedlaya-Liu's Robba sheaves, which are also some motivic and functorial construction. Our space is actually regarded over \mathbb{Z}_p , but note that the corresponding pseudorigid affinoid could be defined over arbitrary integral ring \mathcal{O}_K of some analytic field K . Let X be a pseudorigid space over \mathcal{O}_K , but we regard this as a corresponding Tate adic space over \mathbb{Z}_p . Now we consider the corresponding pro-étale site of X which we denote it as $X_{\text{proét}}$.

Definition 12.5.11. We now apply the corresponding definitions of the Robba rings in [KL2, Definition 4.1.1] with E therein being just \mathbb{Q}_p to any perfectoid subdomain (P_∞, P_∞^+) of $X_{\text{proét}}$. For each such perfectoid, we promote it to be a Banach ring $(P_\infty, P_\infty^+, \|\cdot\|_\infty)$, then we have the corresponding construction of the Robba rings in [KL2, Definition 4.1.1] (we use the notation Π to represent the notation \mathcal{R}):

$$\tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s, r]} \tag{12.5.44}$$

$$\varprojlim_s \tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s, r]} \tag{12.5.45}$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s, r]} \tag{12.5.46}$$

Then one just organizes these to be certain presheaves over the site $X_{\text{proét}}$:

$$\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \quad (12.5.47)$$

$$\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \quad (12.5.48)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}. \quad (12.5.49)$$

However the corresponding construction is not canonical since the promotion to Banach rings locally is not functorial. But we do have the situation that they are actually sheaves due to the fact that we can regard them as sheaves over some preperfectoid spaces under the corresponding tilting of the total spaces, where we use the same notation to denote the sheaves.

$$\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \quad (12.5.50)$$

$$\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \quad (12.5.51)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}. \quad (12.5.52)$$

The corresponding total spaces are defined as:

$$\text{Spa}_{\text{total}}(\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}), \quad (12.5.53)$$

$$\text{Spa}_{\text{total}}(\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}), \quad (12.5.54)$$

$$\text{Spa}_{\text{total}}(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}). \quad (12.5.55)$$

Definition 12.5.12. As in [KL2, Chapter 4.3 and Chapter 8], consider the corresponding total space:

$$\text{Spa}_{\text{total}}(\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}). \quad (12.5.56)$$

We define the corresponding φ -sheaves (where φ is Frobenius lifting from p -th power Frobenius coming from the characteristic p rings encoded in the construction recalled above from [KL2]) to be sheaves locally attached to étale-stably pseudocoherent sheaves carrying the corresponding semilinear Frobenius morphisms realizing the isomorphisms under pullback. Then over

$$\text{Spa}_{\text{total}}(\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}) \quad (12.5.57)$$

we define the similar pseudocoherent sheaves (complete with respect to the natural topology) locally base change to some sheaves in the previous kind. Finally we define the similar pseudocoherent sheaves (complete with respect to the natural topology) over

$$\mathrm{Spa}_{\mathrm{total}}\left(\varinjlim_r \varprojlim_s \widetilde{\Pi}_{X,\mathrm{pro\acute{e}t}}^{[s,r]}, \varinjlim_r \varprojlim_s \widetilde{\Pi}_{X,\mathrm{pro\acute{e}t}}^{[s,r],+}\right), \quad (12.5.58)$$

locally base change to some sheaves in the previous kind³⁵.

12.5.4 Robba Sheaves over (p, I) -Adic Spaces

Now we consider Kedlaya-Liu's Robba sheaves over more general adic spaces, which is also some motivic and functorial construction. Let X be a Tate adic space over \mathbb{Z}_p where p is assumed to be topologically nilpotent. Now we consider the corresponding pro-étale site of X which we denote it as $X_{\mathrm{pro\acute{e}t}}$.

Definition 12.5.13. We now apply the corresponding definitions of the Robba rings in [KL2, Definition 4.1.1] with E therein being just \mathbb{Q}_p to any perfectoid subdomain (P_∞, P_∞^+) of $X_{\mathrm{pro\acute{e}t}}$. For each such perfectoid, we promote it to be a Banach ring $(P_\infty, P_\infty^+, \|\cdot\|_\infty)$, then we have the corresponding construction of the Robba rings in [KL2, Definition 4.1.1] (we use the notation Π to represent the notation \mathcal{R}):

$$\widetilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]}, \quad (12.5.59)$$

$$\varprojlim_s \widetilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]}, \quad (12.5.60)$$

$$\varinjlim_r \varprojlim_s \widetilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]}. \quad (12.5.61)$$

Then one just organizes these to be certain presheaves over the site $X_{\mathrm{pro\acute{e}t}}$:

$$\widetilde{\Pi}_{X,\mathrm{pro\acute{e}t}}^{[s,r]}, \quad (12.5.62)$$

$$\varprojlim_s \widetilde{\Pi}_{X,\mathrm{pro\acute{e}t}}^{[s,r]}, \quad (12.5.63)$$

$$\varinjlim_r \varprojlim_s \widetilde{\Pi}_{X,\mathrm{pro\acute{e}t}}^{[s,r]}. \quad (12.5.64)$$

However the corresponding construction is not canonical since the promotion to Banach rings locally is not functorial. But we do have the situation that they are actually sheaves

³⁵Here it is certainly not expect to be the case where we could have uniform radius $r > 0$.

due to the fact that we can regard them as sheaves over some preperfectoid spaces under the corresponding tilting of the total spaces, where we use the same notation to denote the sheaves.

$$\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \quad (12.5.65)$$

$$\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \quad (12.5.66)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}. \quad (12.5.67)$$

The corresponding total spaces are defined as:

$$\text{Spa}_{\text{total}}(\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}), \quad (12.5.68)$$

$$\text{Spa}_{\text{total}}(\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}), \quad (12.5.69)$$

$$\text{Spa}_{\text{total}}(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}). \quad (12.5.70)$$

Definition 12.5.14. As in [KL2, Chapter 4.3 and Chapter 8], consider the corresponding total space:

$$\text{Spa}_{\text{total}}(\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}). \quad (12.5.71)$$

We define the corresponding φ -sheaves (where φ is Frobenius lifting from p -th power Frobenius coming from the characteristic p rings encoded in the construction recalled above from [KL2]) to be sheaves locally attached to étale-stably pseudocoherent sheaves carrying the corresponding semilinear Frobenius morphism realizing the isomorphism under pullback. Then over

$$\text{Spa}_{\text{total}}(\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}) \quad (12.5.72)$$

we define the similar pseudocoherent sheaves (complete with respect to the natural topology) locally base change to some sheaves in the previous kind. Finally we define the similar pseudocoherent sheaves (complete with respect to the natural topology) over

$$\text{Spa}_{\text{total}}(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}), \quad (12.5.73)$$

locally base change to some sheaves in the previous kind³⁶.

³⁶Here it is certainly not expected to be the case where we could have uniform radius $r > 0$.

Remark 12.5.15. The corresponding sheaves of the full Robba rings are well-defined since one could regard them as structure sheaves of some total spaces.

12.6 Derived I -Complete THH and HH of Derived I -Complete Rings

12.6.1 Derived I -Complete Objects

The corresponding THH and HH of derived I -complete rings are topological constructions which are very closely related to the corresponding relative p -adic motives. And certain derived I -complete versions are also relevant in some highly nontrivial way. We make some discussion closely after [BMS, Section 2.2, Section 2.3], [BS1] and [NS, Chapter 3]. We now consider the following rings:

Setting 12.6.1. We consider the derived I -complete \mathbb{E}_1 -rings. For instance one can consider some derived I -complete \mathbb{E}_1 -rings relative to the corresponding integral pseudorigid disc. In the following I will be some two sided ideal in the π_0 of the base spectrum R . We regard all the ring spectra as being in the ∞ -category of all ring spectra over R and in the derived ∞ -category $\mathbb{D}(R)$ of all the module spectra over R . We regard all the derived I -complete ring spectra as being in the ∞ -category of all derived I -complete ring spectra over R and in the derived ∞ -category $\mathbb{D}_I(R)$ of all the derived I -complete module spectra over R . We assume that I is central in the noncommutative setting.

Definition 12.6.2. For any such ring R which is assumed to be \mathbb{E}_∞ , we will use the corresponding notations $L\mathrm{THH}(R)$ and $L\mathrm{HH}(R)$ to denote the corresponding left Kan extended THH or the corresponding HH functor from [NS, Chapter 3]. And we will use the corresponding notations $L\mathrm{THH}(R)_I$ and $L\mathrm{HH}(R)_I$ to denote the corresponding left Kan extended THH or the corresponding HH functor after taking the corresponding derived I -completion as in [NS, Chapter 3].

Definition 12.6.3. For any such ring R which is assumed to be \mathbb{E}_1 , we will use the corresponding notations $\mathrm{THH}(R)$ and $\mathrm{HH}(R)$ to denote the corresponding THH or the corresponding HH functor from [NS, Chapter 3]. And we will use the corresponding notations $\mathrm{THH}(R)_I$ and $\mathrm{HH}(R)_I$ to denote the corresponding THH or the corresponding HH functor after taking the corresponding derived I -completion as in [NS, Chapter 3] and [BS1, Chapter 1 Notation].

Example 12.6.4. For instance one considers the adic rings in [FK, Section 1.4], one takes such a ring R which is complete with respect to a corresponding two sided finitely generated ideal I , then the corresponding construction above namely $LTHH(R)_I$ and $LHH(R)_I$ will also be basically topological versions of the spectra in [ELS, Definition 1.3, Definition 1.4] in the corresponding derived noncommutative deformation theory.

As in [ELS, Chapter 5] and more generally, one has the corresponding noncommutative version of the corresponding relative Kähler differential complexes $\Omega_{\text{noncom}, B/A}^\bullet$. For instance if we have the corresponding map $A \rightarrow B$ of adic rings in [FK, Section 1.4] we could then take the corresponding derived I -completion from $\Omega_{\text{noncom}, B/A}^\bullet$ to achieve the corresponding topological version $\Omega_{\text{noncom}, B/A, \text{topo}}^\bullet$.

Remark 12.6.5. As mentioned in the introduction, the corresponding topological derived I -adic version of the corresponding $LTHH$ and LHH spectra should be interesting to study, as over a quasisyntomic site of a quasisyntomic ring Q , [BMS, Proposition 5.15] shows that the corresponding sheaf $\pi_0 \text{HC}^-(-/Q)_p$ will be closely related to the corresponding p -adic complete and Hodge-complete derived de Rham sheaf over the same site after taking the corresponding unfolding through the quasiregular semiperfectoids. We do not know if this relation could hold in some sense if we consider derived I -adic rings, but one might want to believe that in the situations we considered before the story should be in some sense easier to be established.

12.7 Bibliography

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