

UC Irvine

Open Access Policy Deposits

Title

Trefftzs Award: The Capital Growth Model: An Empirical Investigation

Permalink

<https://escholarship.org/uc/item/8xm84960>

Journal

The Journal of Financial and Quantitative Analysis, 8(2)

ISSN

0022-1090

Authors

Bicksler, James L

Thorp, Edward O

Publication Date

1973-03-01

DOI

10.2307/2330023

Copyright Information

This work is made available under the terms of a Creative Commons Attribution License, available at

<https://creativecommons.org/licenses/by/4.0/>

Peer reviewed

THE CAPITAL GROWTH MODEL:
AN EMPIRICAL INVESTIGATION

James L. Bicksler and Edward O. Thorp*

I. Introduction

Modern micro-capital theory offers three major alternative choice theoretic approaches from which a set of market equilibrium prices can be derived. These approaches are:

1. Time-state preference theory of Arrow [1] and Debreu [6],
2. The capital asset pricing model (hereafter CAPM) of Sharpe [34], Lintner [23], and Fama [7],
3. The capital growth model of Kelly [16], Breiman [5], and Latané [17].¹

Of the three approaches, time-state preference theory is probably the most eloquent and robust theoretical approach to normative portfolio and capital market theory inasmuch as it is "capable of being shifted comfortably between individual and market levels of analysis, between perfect and imperfect markets, and between conditions of certainty and uncertainty with the fundamental theoretical elements exhibiting a continuity and a clarity otherwise unattainable."² Also, this theory is intuitively appealing because of its consistency with the multi-period consumption-investment utility maximization problem of the individual. Its views of securities as time-dated state-dependent claims are, of course, easily integrated into the dynamic utility problem of the investor.

Unfortunately, operational attempts to apply the theory require a useful partitioning of the mutually exclusive and exhaustive requirements of the states of the world over time. Present-day attempts to *simplify* the assessment process have met with little success.³ Consequently, "for some time to come, this should preclude any hope of practical application of the model in its strict form as well as complicated empirical testing."⁴ Further, there is not a complete set of Arrow-Debreu certificates and, additionally, there are more states of nature than firms. Under these conditions, thought to be

*Rutgers University and the University of California, Irvine, respectively. This research was supported in part by the Air Force Office of Scientific Research through Grant AF-AFOSR 70-1870C.

¹Also see Thorp [37] and Thorp and Whitley [38]. For an extension of the capital growth criterion under conditions whereby the investment opportunity set is stochastically temporally dependent, see Hakansson and Liu [14].

²Ricks and Rubinstein [30, p. 6].

³Witness Fama-Miller's comment that "the complex problem of defining what is meant by a state has not been adequately treated in the literature." See Fama and Miller [9, p. 158].

⁴Ricks and Rubinstein [30, p. 94].

descriptive of "real world" capital markets, time-state preference theory simply does not provide a general framework for determining a set of equilibrium prices and yields.⁵

The second micro-capital theory approach, that of the CAPM, rests on the conceptual foundations of the mean-variance portfolio selection models of Markowitz [26] and Tobin [39]. There is a mixture of intuitively appealing and unappealing properties emerging from the two-parameter CAPM. The basic query of a model's empirical robustness is not tested, however, by a direct examination of its underlying assumptions. Instead, the hypothesis test is whether its implications regarding capital market phenomena are consistent with observed behavior. The present state of the evidence regarding the CAPM's empirical validity is far from definitive. At a minimum the studies of Blume-Friend ([3] and [4]) and Black-Jensen-Scholes [2] suggest that the CAPM will need some modification.⁶

Indeed, current work in this arena vividly illustrates the interaction of theory and empiricism within the context of enriching the reformulation of the foundations of microeconomics.

An alternative micro-capital theory approach is the capital growth model. Despite its almost complete neglect, until recently, in the finance literature, it has numerous appealing economic properties. Hakansson [13, pp. 873-874] postulates the following properties:

1. Implies stationary risk aversion,
2. Consistent with the von Neumann-Morgenstern axioms,
3. Consistent with the notion of absolute preference,
4. Implies decreasing absolute risk aversion as the familiar Arrow-Pratt measure,
5. Implies that optimal portfolio strategy is myopic,
6. Implies that the optimal strategy will never risk ruin,
7. Derives the only efficient portfolio sequence in the long run,
8. Can be used as a basis for mutual fund, etc., portfolio strategy.

While a theory of equilibrium prices is just beginning to emerge from the foundations of the capital growth model, and despite some scathing criticism of its

⁵Stiglitz [36, pp. 25-60].

⁶Also, note, Merton's comment [27, p. 1] that "those proponents who agree with the theoretical objections, but who argue that the capital market operates 'as if' these assumptions were satisfied are not without criticism either. While the model predicts that the expected excess return from holding an asset is proportional to the covariance of its return with the market portfolio (its 'beta'), the careful empirical work of Black, Jensen, and Scholes has demonstrated that this is not the case. In particular, they found that 'low beta' assets return on average more than is forecasted by the model."

inapplicability to investors having isoelastic utility functions, it seems reasonable for it to be judged, like other theories of economics, within the methodology of positivism.⁷

In this regard, the recent Roll study [31, p. 21] compared the CAPM and the capital growth model and concluded that the two models showed "a close empirical correspondence . . . for common stock returns." Indeed, "the two models could not be distinguished on an empirical basis. In every period, estimated corresponding coefficients of the two models were nearly equal." Roll [31, p. 15] further argues that the result of "approximately equivalent empirical implications" should not be surprising "if certain restrictions are placed on the ranges of individual rates of return." As to the properties of the two alternative frameworks, Roll states that "a close correspondence was demonstrated between their qualitative implications. For example, both models imply that an asset's expected return will equal the risk-free interest rate if the covariance between the asset's return and the average return on all assets, (R_j, R_m) , is zero. For most cases, the growth-optimum model also shares the Sharpe-Lintner implication that an asset's expected return will exceed the risk-free rate if and only if $\text{Cov}(R_j, R_m) = 0$ " [31, p. 20].

⁷For the beginning of an equilibrium framework of equity prices in a world where investors have either lognormal or Bernoulli utility functions, see Litzenberger and Budd [24, pp. 1277-1282]. In this context, while Samuelson's comment [32, p. 245] that "the geometric mean criterion is definitely too conservative to maximize an isoelastic utility function" is formally correct, it neglects questions of the robustness of geometric mean derived market models and their confrontation with "real world" market phenomena. Further, the implied risk aversion properties of the logarithmic utility function are far more appealing than those of the quadratic utility functions, which may be used as a rationale for mean-variance portfolio models. Also, see Samuelson [33, pp. 2493-2496] and Mossin [28].

Markowitz [25, pp. 3-4] in a rebuttal to the Samuelson and Mossin argument that for a wide category of utility functions the expected log rule is nonoptimal sets forth a new proof of the log rule as a rational investment strategy in the long run. The essence of Markowitz argument is that "comparing the theorem 1 with the Mossin-Samuelson argument, one might be tempted to conclude that the expected log rule is asymptotically desirable if constant $V(g)$ is assumed, but generally not asymptotically desirable if constant $U(W_T)$ is assumed. But theorems 2 and 3 show, under fairly general conditions, that the expected log rule is asymptotically optimal even when constant $U(W_T)$ is bounded from above and below. I.e., the expected log rule is asymptotically optimal, under the conditions of theorems 2 and 3, provided that U does not approach $+\infty$ as W_T approaches ∞ , and does not approach $-\infty$ as W_T approaches 0. Essential to the Mossin-Samuelson result, then, was the fact that every utility function, in the class considered by them, was unbounded either from above or below."

On the latter point, Markowitz maintains that "along the lines of Menger (1934), that given any $U(W_T)$ which is unbounded (either above or below) a St. Petersburg type of game can be constructed which shows that the particular unbounded $U(W_T)$ is absurd. From this it is argued that $U(W_T)$ should be assumed to be bounded, hence theorems 2 and 3 are applicable.

In brief, the Markowitz view is that "it is must more plausible to analyze behavior for the long run by assuming constant $V(g)$ than constant $U(W_T)$ as T increases. In this case theorem 1 applies, but even if constant $U(W_T)$ were assumed, it must be bounded to avoid the absurd behavior shown in section 14." In this case theorems 2 and 3 apply. Theorems 1, 2, and 3 show general conditions under which the expected log rule provides asymptotically optimum, or almost asymptotically optimum, expected utility."

The reader should note that the Markowitz argument does not depend in any way upon the law of large numbers and that the strategy is asymptotically optimal. This asymptotic property will undoubtedly be noted by Samuelson.

II. The Capital Growth Model as a Normative Investment Rule:

Some Empirical Questions

There are several germane empirical questions of interest to investors regarding the capital growth model other than its robustness as an explanatory framework of equilibrium prices. One such query involves an extension of Hakansson's work determining "whether the growth-optimal portfolio has any desirable properties in the short and/or intermediate run."⁸ Alternatively stated, given the asymptotically appealing properties of the capital growth model, what is the nature of the convergency properties in the short and intermediate run? This query along with an investigation of other empirical properties of the capital growth model is the focus of this study.

III. Research Design and Empirical Results

To investigate the empirical properties of the capital growth model, the cumulative distributions of the logs of terminal wealth were estimated under a variety of postulated conditions. These conditions were three different investor time horizons (10, 20, and 40 periods), two different distributions of returns (one generated from the CRISPE file plus an assumed uniform distribution), and ten different lambda values (indicating the magnitude of borrowing-lending [levering-unlevering] of the portfolio.)⁹ This results in 60 (3 investor time horizons x 2 different distributions x 10 lambda values) different cumulative distributions of the logs of the terminal wealth relatives. This generated data indicates the payoffs (geometric returns) and associated probabilities under a wide variety, though obviously far from all, of sets of conditions.

Space does not permit a numerical and visual examination of all of the distributions, but a number of preliminary findings will be indicated.¹⁰ The findings are based on a comparison of the cumulatives via FDSD (first-degree stochastic dominance) and SDSD (second-degree stochastic dominance.)¹¹ This provides a vehicle for stating positively which portfolios are surely not growth optimal but does not permit one to state which specific portfolio is the growth optimal portfolio.

Among the conclusions are:

1. The optimal growth portfolio is not the portfolio with the highest lambda.
2. There is *little spread* in the distribution functions for the optimal growth portfolio and the portfolios closely neighboring the optimal growth portfolio.
3. Similarly, a corollary of (2) quite likely is that, in a world of transaction costs, *suboptimal* portfolios closely bordering the optimal portfolio and purchased under a buy-and-hold strategy would not be stochastically dominated (indeed, they might be preferred if a different preference ordering rule were stipulated) by the period-by-period rebalancing scheme designed to obtain the optimal growth portfolio within each of the periods of the individual's investment horizon.

⁸Hakansson [12, p. 540]. Also, see Latané and Tuttle [19, p. 633].

⁹Fifty runs of the specific characteristics were utilized to generate the simulated cumulative distributions of the logs of the terminal wealth relatives.

¹⁰Illustrative sample distributions are presented in Figures A to F of the Appendix.

¹¹The meaning of these concepts and the conditions under which they lead to the same preference ordering as the expected utility maximization rule is contained in Hadar and Russell [11, pp. 25-34], Quirk and Saposnik [29, pp. 140-146], and Hanoch and Levy [15, pp. 335-346].

4. Another corollary of (2) quite likely is that, to obtain a significant spread in the distribution functions of the growth optimal portfolio and the neighboring portfolios, a *far longer* sequence of investment periods is required.
5. The optimal growth portfolio is *highly* levered.

The relationship of these findings to other studies and to on-going financial research should be of interest. Inference (1) appears to be consistent with the Friend-Blume finding that "for the 20 percent of stock with the highest beta, the rate of return declines."¹²

Inference (5) is compatible with the Latané-Young finding that "optimum leverage leads to major gains. The proper use of debt can increase returns greatly."¹³

The recent Fama-MacBeth manuscript is consistent with inferences (1), (2), and (5) and quite likely with corollaries (3) and (4). For example, the Fama-MacBeth evidence suggests that the optimal growth portfolio has a far higher beta than does the optimal risky portfolio (where the beta equals one) for a Tobin maximizer (e.g., the particular efficient Markowitz portfolio that is tangent to the capital market line). Indeed, the value of beta for the growth optimal portfolio for 1935-6/68, the longest test period utilized, was 2.46.^{14, 15} Further, a visual inspection of the Fama-MacBeth plots clearly shows that the growth optimal portfolio's geometric return does not vary

¹²Friend and Blume [10, p. 9]. The Friend-Blume finding was derived from a comparison of geometric returns computed under (1) a buy-and-hold policy and (2) a yearly revision policy for deciles of firms ranked on the magnitude of their beta.

¹³Latané and Young [10, p. 611]. Indeed, the present study is in some ways a variation of the Latané-Young theme. However, the Latané-Young investigation utilized as the rule of dominance the highest geometric return for the sample testing period. In this respect, our results are at the same time both stronger and less general inasmuch as our dominance rule has more restrictions incorporated within the sufficiency conditions. For an elaboration of the concept that the broader the rule of dominance, the larger the efficient set, see Levy and Paroush [22].

¹⁴The betas of the other periods for the optimal growth portfolio were:

<u>Period</u>	<u>Beta_G</u>
1946-6/68	2.82
1935-45	2.34
1946-55	1.70
1956-6/68	3.54
1935-40	1.15
1941-45	>5.00
1946-50	1.35
1951-55	2.36
1956-60	<i>ex post</i> efficient slope was negative
1961-6/68	>5.00

See Fama and MacBeth [8, Table 7].

¹⁵The difference regarding inference (5) between this study and the Fama-MacBeth study is that we utilize lambda, instead of beta, values. However, this distinction is trivial in a world of borrowing-lending. Borrowing (levering) the optimal risky portfolio until one obtains the optimal growth portfolio which has a *high* beta leads directly to (or is consistent with) inference (1).

substantially from its neighboring portfolios' geometric returns. Indeed, in most plots it is amazing to see the width of the range over which it is nearly horizontal.¹⁶

Inferences (3) and (4) are postulated to be corollaries, under certain reasonable assumptions or conditions, of inference (2). Specifically, it is quite likely, given the fee schedule prevailing for security transactions, that portfolio strategies involving readjustment of securities to obtain the optimal growth portfolio will not dominate portfolios closely neighboring the optimal growth portfolio which are purchased and maintained under a buy-and-hold rule.¹⁷ Ergo, inference (3).

Lastly, regarding inference (4), Hakansson [13, p. 877] showed that the growth optimal portfolio converges "very rapidly to the efficient N-Period portfolio even when the 'starting point' (for N=1) is very poor." This would seem to suggest that the asymptotically appealing properties of the capital growth model are obtained even in the short, or certainly in the intermediate, run. The evidence of this study as well as the results of the Fama-MacBeth study seem, however, to indicate that Hakansson's view is suspect.^{18, 19, 20}

To conclude, Roll [31, p. 1] has noted that "in the past ten years, however, little work on growth maximization of portfolio value has appeared in the academic literature of finance or economics." Since the situation is rapidly changing, it is hoped that this study adds a strand of empirical evidence to this arena. Much, of course, remains to be done using more "realistic" return distributions, etc. In this sense, the results of this study are quite preliminary.

¹⁶See Fama-MacBeth for these plots. At least two exceptions are noted. They are for the periods 1941/45 and 1956/60. However, the plots through the range of $\beta > 5$ are not available and hence no germane statements relevant to these periods can be made.

¹⁷For insights into the structure of the dynamic investor portfolio problem, the nature of the turnpike theorem and its implications, and the importance of transaction costs, see Leland [21] and Smith [35, pp. 425-439].

¹⁸Hakansson [12, p. 515] shows, via a simplified example, using an investment opportunity set of three securities that "it is remarkable how close" the capital growth portfolio "comes to dominating each of the other step functions after six periods."

¹⁹The question of convergency properties is indeterminably linked to the time interval required for the optimal growth portfolio to dominate alternative portfolios. The Fama-MacBeth evidence indicates that the plots of the short intervals (e.g., five years) are more horizontal than the slightly humped plots of the longer time periods. The Thorp-Bicksler evidence shows under more stringent conditions of dominance that even for periods of, for example, 30 years it is not always clear that growth optimal portfolios dominate.

²⁰In this regard, Latané and Tuttle [18, p. 361] comment that "the rapidity of convergence and the effects of convergence depend on two factors (1) the time it takes for *ex ante* probability beliefs about returns to become reflected in *ex post* frequency distributions and (2) the differences among the strategies being considered for adoption." Regarding our query about the convergency properties of the capital growth model, it is difficult, if not impossible, operationally to derive meaningful and well-defined hypotheses about (1). Concerning (2), Latané and Tuttle imply that the answer will change depending upon the set of strategies considered. A more important query is whether the *ex ante* growth optimal portfolio dominates the *ex ante* nongrowth optimal portfolios.

APPENDIX

Cumulative Distribution of Terminal Wealth Relative

Figures A through F illustrate the variation of the cumulative distribution of the terminal wealth relative with the fraction λ of capital reinvested. These figures are part of a run in which we assumed the return per unit in each time period was uniformly distributed between 0.75 and 1.50, for an expected return of 12.5 percent.

We assumed 10 time periods, and let λ range from approximately 0.4 to 4.0 in steps of approximately 0.4. There were fifty simulations for each λ , leading to empirical distributions such as those of the figures.

FIGURE A
CUMULATION DISTRIBUTION OF TERMINAL WEALTH RELATIVE
 $\lambda = 0.403$
CUMULATIVE PROBABILITY

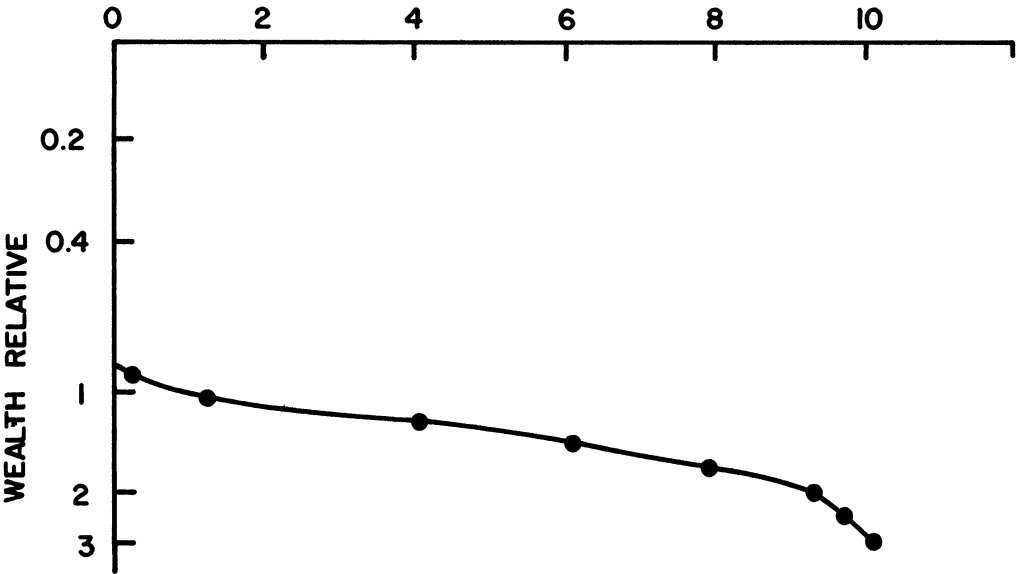


FIGURE B
CUMULATION DISTRIBUTION OF TERMINAL WEALTH RELATIVE
LAMBDA=1.208
CUMULATIVE PROBABILITY

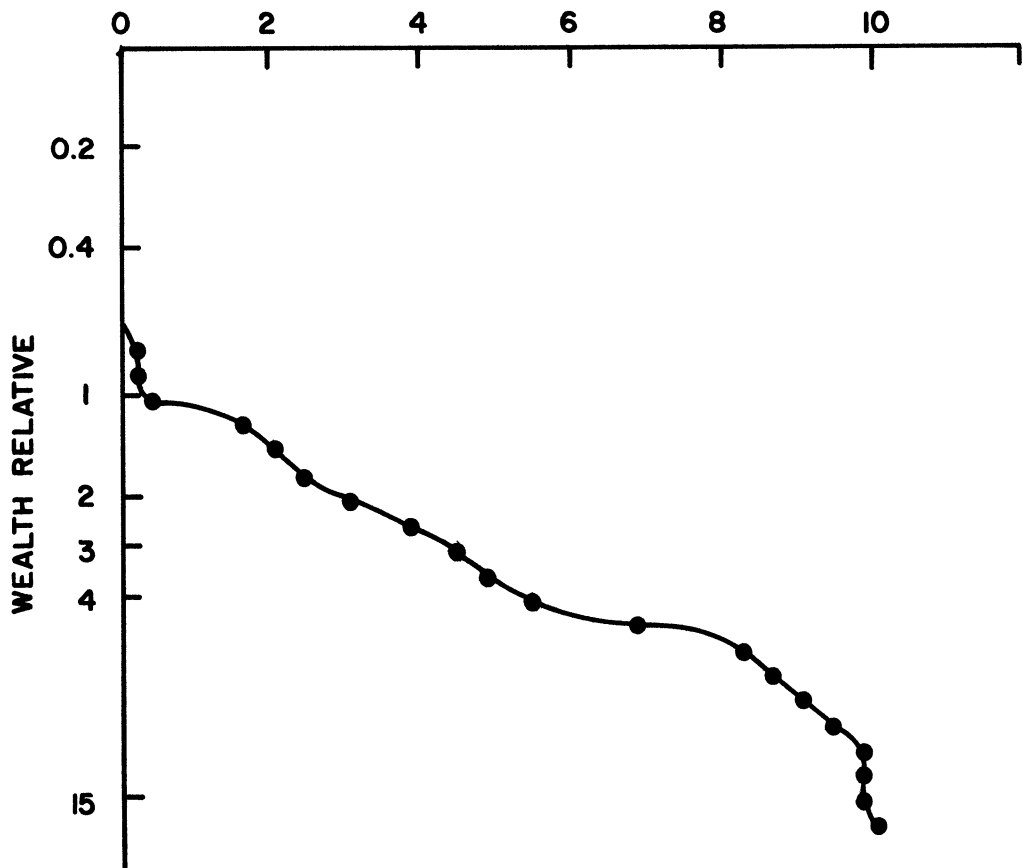


FIGURE C
CUMULATION DISTRIBUTION OF TERMINAL WEALTH RELATIVE
 $\lambda = 1.611$
CUMULATIVE PROBABILITY

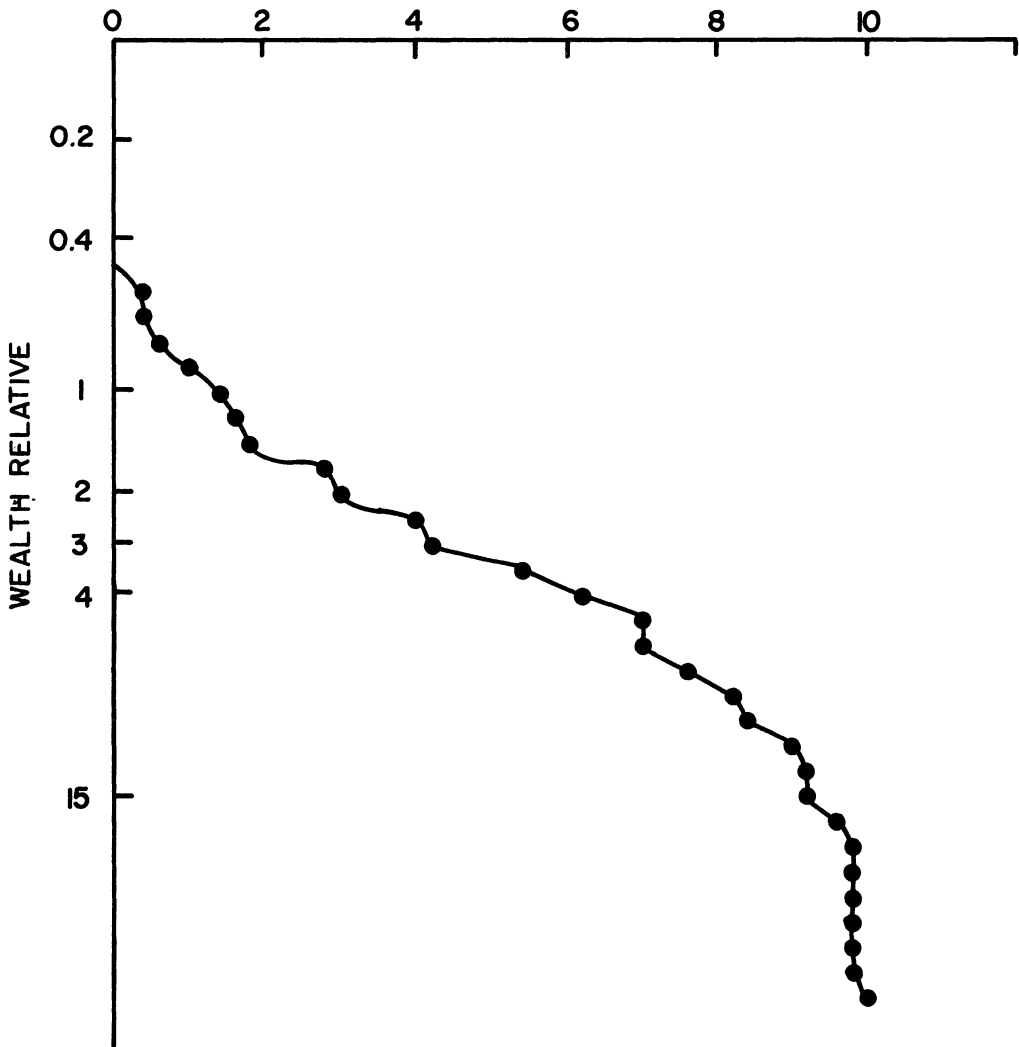


FIGURE D
CUMULATION DISTRIBUTION OF TERMINAL WEALTH RELATIVE
 $\lambda = 2.417$
CUMULATIVE PROBABILITY

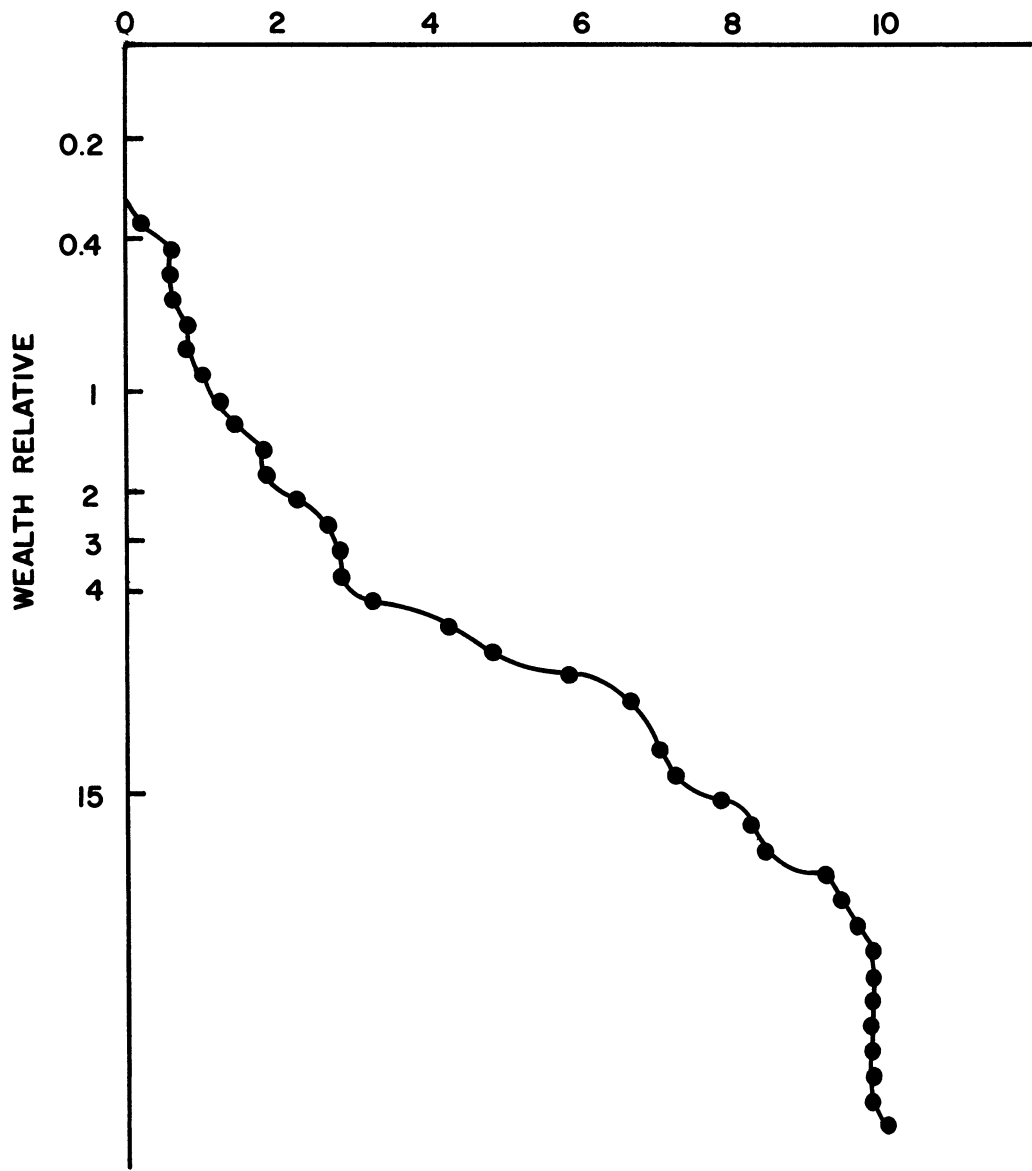


FIGURE E
CUMULATION DISTRIBUTION OF TERMINAL WEALTH RELATIVE
 $\lambda = 2.819$
CUMULATIVE PROBABILITY

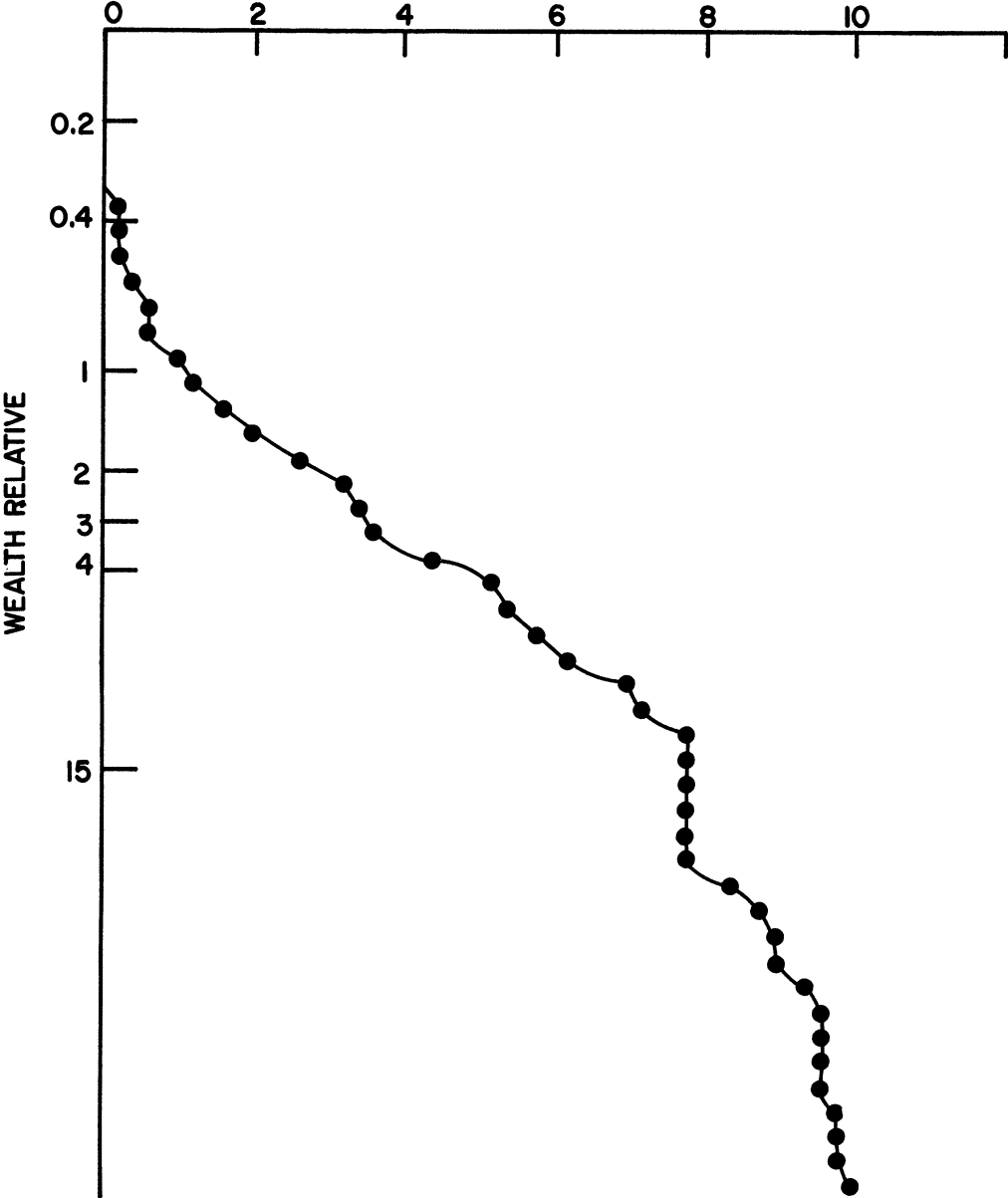
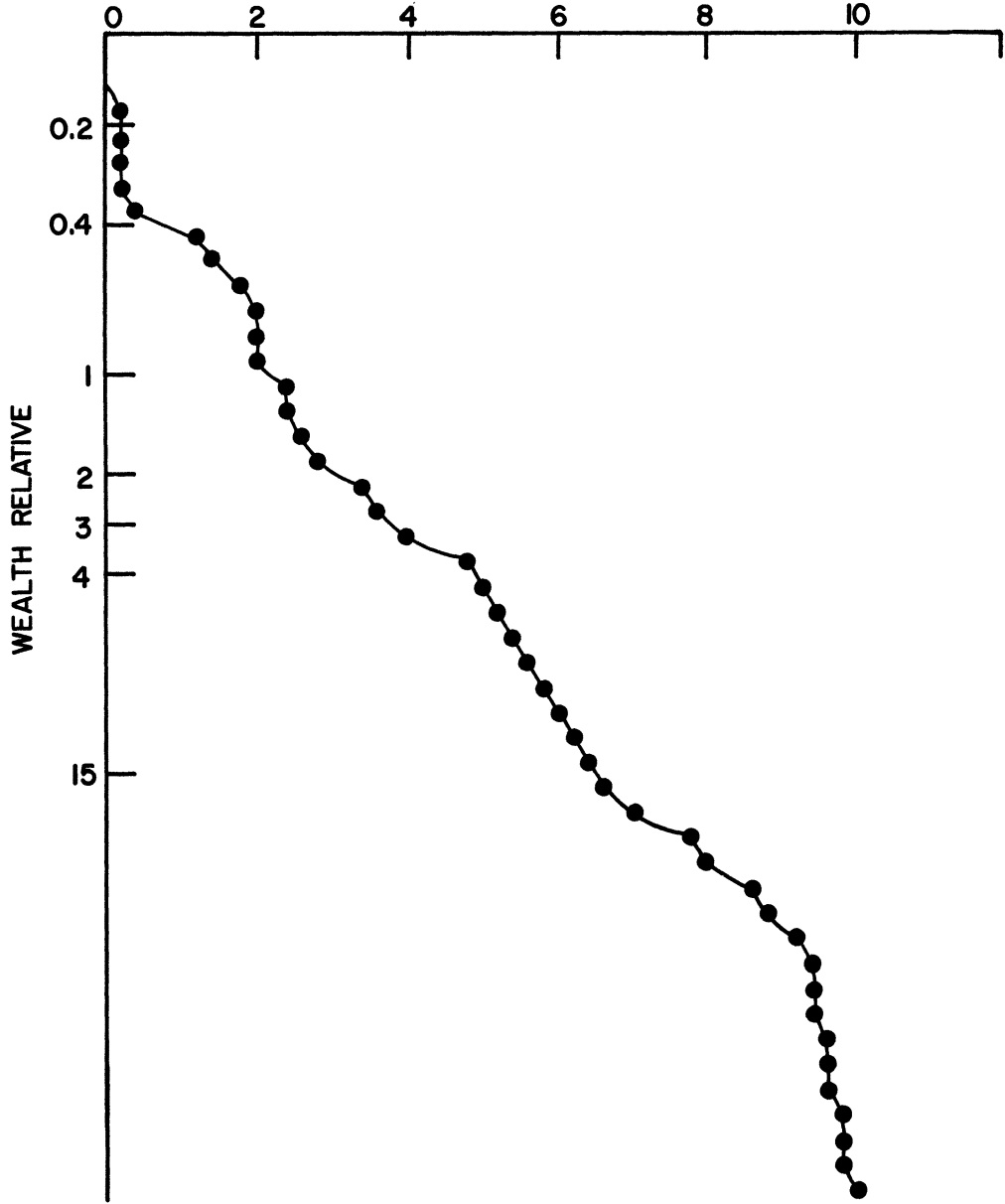


FIGURE F
CUMULATION DISTRIBUTION OF TERMINAL WEALTH RELATIVE
 $\lambda = 3.222$
CUMULATIVE PROBABILITY



REFERENCES

- [1] Arrow, K. J. "The Role of Securities in the Optimal Allocation of Risk Bearing." *Review of Economic Studies*, Vol. 31 (April 1964), pp. 91-96.
- [2] Black, F.; M. C. Jensen, and M. Scholes. "The Capital Asset Pricing Model: Some Empirical Tests." In *Studies in the Theory of Capital Markets*, edited by M. C. Jensen. New York, N. Y.: Praeger Press, 1972, pp. 79-121.
- [3] Blume, M., and I. Friend. "Measurement of Portfolio Performance Under Uncertainty." *American Economic Review*, September 1970.
- [4] _____. "A New Look at the Capital Asset Pricing Model." *Journal of Finance*, forthcoming.
- [5] Breiman, L. "Investment Policies for Expanding Business in a Long-Run Sense." *Naval Research Logistics Quarterly*, December 1960, pp. 647-651.
- [6] Debreu, G. *The Theory of Value*. New York, N. Y.: John Wiley and Sons, Inc., 1959.
- [7] Fama, E. "Risk, Returns, and Equilibrium: Some Clarifying Comments." *Journal of Finance*, March 1968, pp. 29-40.
- [8] Fama, E., and J. MacBeth. "Long-Run Growth in a Short-Run Market." Unpublished manuscript, August 1972.
- [9] Fama, E., and M. H. Miller. *The Theory of Finance*. New York, N. Y.: Holt, Reinhart and Winston, 1972.
- [10] Friend, I., and M. Blume. "Risk and the Long-Run Rates of Return on NYSE Common Stock." Working Paper No. 18-72, Rodney L. White Center for Financial Research, University of Pennsylvania, 1972.
- [11] Hadar, J., and W. R. Russell. "Rules for Ordering Uncertain Prospects." *American Economic Review*, January 1969, pp. 25-34.
- [12] Hakansson, N. H. "Capital Growth and the Mean-Variance Approach to Portfolio Selection." *Journal of Financial and Quantitative Analysis*, January 1971.
- [13] _____. "Multi-Period Mean-Variance Analysis: Towards a General Theory of Portfolio Choice." *Journal of Finance*, September 1971.
- [14] Hakansson, N. H., and Tien-Ching Liu. "Optimal Growth Portfolios when Yields are Serially Correlated." *Review of Economics and Statistics*, November 1970, pp. 385-394.
- [15] Hanoch, G., and H. Levy. "The Efficiency Analysis of Choices Involving Risk." *Review of Economic Studies*, 1969, pp. 335-346.
- [16] Kelly, J. "A New Interpretation of Information Rate." *Bell System Technical Journal*, August 1956.
- [17] Latané, H. "Criteria for Choice Among Risky Ventures." *Journal of Political Economy*, April 1959.
- [18] Latané, H., and D. Tuttle. "Criteria for Portfolio Building." *Journal of Finance*, September 1967.
- [19] _____. *Security Analysis and Portfolio Management*. New York, N. Y.: Ronald Press, 1970.
- [20] Latané, H., and W. Young. "Test of Portfolio Building Rules." *Journal of Finance*, September 1969.
- [21] Leland, H. E. *Dynamic Portfolio Theory*, Ph.D. dissertation, Harvard University, 1968.

- [22] Levy, H., and J. Paroush. "Multi-Period Stochastic Dominance." Unpublished paper, Israel, June 1972.
- [23] Lintner, J. "Security Prices, Risk, and Maximal Gains from Diversification." *Journal of Finance*, March 1968, pp. 29-40.
- [24] Litzenberger, R., and A. Budd. "A Note on Geometric Mean Portfolio Selection and the Market Price of Equities." *Journal of Financial and Quantitative Analysis*, December 1971, pp. 1277-1282.
- [25] Markowitz, H. "Investment for the Long Run." Working Paper No. 20-72, Rodney L. White Center for Financial Research, 1972.
- [26] _____. *Portfolio Selection: Efficient Diversification of Investments*. New York, N. Y.: John Wiley and Sons, Inc., 1959.
- [27] Merton, R. "An Intertemporal Capital Asset Pricing Model." Working Paper 588-72, Alfred P. Sloan School of Management, M.I.T., February 1972.
- [28] Mossin, J. "Optimal Multiperiod Portfolio Policies." *Journal of Business*, April 1968.
- [29] Quirk, J. P., and R. Saposnik. "Admissibility and Measurable Utility Functions." *Review of Economic Studies*, 1962, pp. 140-146.
- [30] Ricks, B., and M. Rubinstein. "Portfolio Theory in Perspective." Unpublished manuscript, n.d.
- [31] Roll, R. "Evidence on the Growth Optimal Model." Unpublished manuscript, April 1972.
- [32] Samuelson, P. A. "Lifetime Portfolio Selection by Dynamic Programming." *Review of Economics and Statistics*, May 1969.
- [33] _____. "The 'Fallacy' of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling." *Proceedings National Academy of Sciences*, vol. 68, No. 10, October 1971, pp. 2493-2496.
- [34] Sharpe, W. F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance*, vol. 19 (September 1964), pp. 425-442.
- [35] Smith, K. V. "A Transition Model for Portfolio Revision." *Journal of Business*, September 1967, pp. 425-439.
- [36] Stiglitz, J. "Optimality of Stock Market Allocation." *Quarterly Journal of Economics*, February 1972, pp. 25-60.
- [37] Thorp, E. "Portfolio Choice and the Kelly Criterion." *Business and Economics Statistics Proceedings, American Statistical Association*, 1971, pp. 215-224.
- [38] Thorp, E., and R. Whitley. "Concave Utilities are Distinguished by Their Optimal Strategies." Unpublished manuscript, 1972.
- [39] Tobin, J. "Liquidity Preference as Behavior Towards Risk." *Review of Economic Studies*, vol. 25 (February 1958), pp. 65-86.