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Hamiltonian Structure of Particle Motion in an Ideal
Helical Wiggler with Guide Field*

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Abstract

The ideal helical wiggler with guide field is shown to possess an integrable Hamiltonian. Explicit generating functions are presented for the canonical transformation to action/angle variables.

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The following is a reduction of the motion of a particle in the magnetic field:

$$\vec{B} = B_0 \hat{z} + B_w (\hat{x} \cos kz + \hat{y} \sin kz),$$

to integrable form, including a partial reduction to action/angle variables. Essentially the same problem was solved by Roberts and Buchsbaum [1], but without revealing the Hamiltonian structure. The practical use of the calculation presented here is as a basis for a perturbative treatment of more difficult and/or realistic magnetic field models [2].

Choose units so that $e=m=c=1$, and the following gauge:

$$\vec{A} = B_0 xy + \frac{B_w}{k} (-\hat{x} \cos kz - \hat{y} \sin kz) .$$

Then the relativistic Hamiltonian (squared) is

$$H^2(\vec{x}, \vec{p}) = (p_x + \frac{B_w}{k} \cos kz)^2 + (p_y - B_0 x + \frac{B_w}{k} \sin kz)^2 + p_z^2 + 1.$$

Now introduce the following canonical transformation:

$(\vec{x}, \vec{p}) \rightarrow (Q_1, Q_2, z=Q_3; P_1, P_2, P_3)$, given by the generating function:

$$\begin{aligned} F(x, y, z; P_1, P_2, P_3) = & (P_1 + B_0 y) \left(x - \frac{B_w}{B_0 k} \sin kz \right) + P_2 \left(\frac{P_1}{B_0} + y + \frac{B_w}{B_0 k} \cos kz \right) \\ & + P_3 z - \frac{B_w^2}{4B_0 k^2} \sin 2kz - \frac{B_w^2}{2B_0 k} z. \end{aligned}$$

This yields the following transformation equations:

$$p_x = p_1 + B_0 y ,$$

$$p_y = p_2 + B_0 x - \frac{B_w}{k} \sin kz ,$$

$$p_z = p_3 - \frac{B_w}{B_0} (\cos kz) (p_1 + B_0 y) - \frac{B_w}{B_0} (\sin kz) p_2 - \frac{B_w^2}{B_0 k} \cos^2 kz ,$$

$$Q_1 = p_2/B_0 + x - \frac{B_w}{B_0 k} \sin kz ,$$

$$Q_2 = p_1/B_0 + y - \frac{B_w}{B_0 k} \cos kz ,$$

$$Q_3 = z .$$

Untangling this, we obtain

$$x = Q_1 - p_2/B_0 + \frac{B_w}{B_0 k} \sin kz ,$$

$$y = Q_2 - p_1/B_0 - \frac{B_w}{B_0 k} \cos kz ,$$

$$z = Q_3 ,$$

$$p_x = B_0 Q_2 - \frac{B_w}{k} \cos kz ,$$

$$p_y = B_0 Q_1 ,$$

$$p_z = p_3 - \frac{B_w}{B_0} (B_0 Q_2 \cos kz + p_2 \sin kz) .$$

In the new variables, we have

$$H^2 = p_2^2 + B_0^2 Q_2^2 + [p_3 - \frac{B_w}{B_0} (B_0 Q_2 \cos kz + p_2 \sin kz)]^2 + 1 .$$

Note that this H is ignorable in both Q_1 and P_1 , so both of these are invariants. In fact, Q_1 and P_1 are closely related to a kind of generalized guiding center position, denoted by X and Y :

$$X = Q_1, \quad Y = -P_1/B_0$$

The Hamiltonian now has only 2 degrees of freedom. It is still not clear that it is integrable, since it depends on all 4 canonical variables: $Q_2, P_2; z, P_3$. Notice that it has a harmonic oscillator term in Q_2 and P_2 , with frequency B_0 . The physical significance of the variables Q_2 and P_2 is that they are essentially the x and y components of the four-velocity:

$$\dot{x} = B_0 Q_2, \quad \dot{y} = P_2$$

However, the harmonic oscillator term is coupled to the longitudinal motion through the term $[P_3 - \dots]^2$.

Introduce action/angle variables for the Q_2, P_2 variables, i.e. set

$$\begin{aligned} Q_2 &= (2J/B_0)^{1/2} \sin \theta, \\ P_2 &= (2JB_0)^{1/2} \cos \theta, \end{aligned}$$

so that the new canonical variables are $(\theta, J; z, P_3)$. The Hamiltonian becomes

$$H^2 = 2B_0 J + [P_3 - B_w (2J/B_0)^{1/2} \sin(\theta + kz)]^2 + 1.$$

Now introduce a new canonical transformation: $(\theta, J; z, P_3) \rightarrow (\phi, J; z, P'_3)$, generated by

$$F'(\theta, z; J, P'_3) = J(\theta + kz) + z P'_3,$$

yielding

$$\phi = \theta + kz, \quad P_3 = P'_3 + kJ.$$

Then the Hamiltonian becomes

$$H^2 = 2B_0 J + [P'_3 + kJ - B_w (2J/B_0)^{1/2} \sin \phi]^2 + 1.$$

This is now ignorable in z , so that P_3' is an invariant.

The Hamiltonian now has only one essential degree of freedom, and so is integrable. The integration can be carried out via elliptic functions [3].

Acknowledgments

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2. H. Freund and A. Ganguly, IEEE J. Qu. Electronics 21 (1985) 1073.
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