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Miksis Jr., Joseph John

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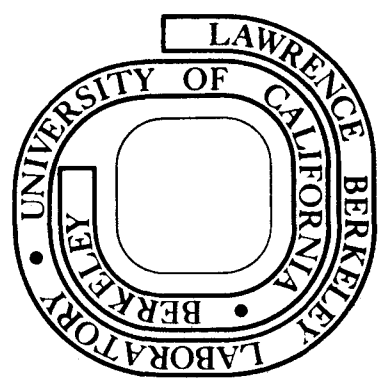
PRIMARY RESISTANCES FOR RING-DISK ELECTRODES

Joseph John Miksis, Jr.
(M. S. thesis)

November 1975

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Primary Resistances for Ring-Disk Electrodes

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Primary Resistances for Ring-Disk Electrodes

Joseph John Miksis, Jr.

Materials and Molecular Research Division, Lawrence Berkeley Laboratory
and Department of Chemical Engineering, University of California,
Berkeley, California 94720

November, 1975

Abstract

A system consisting of a disk electrode, a concentric ring electrode, and a large counterelectrode at infinity has three independent resistance values describing the primary potential difference between any two electrodes when current is passed between any two electrodes. These resistance values are calculated and presented as dimensionless correlations as functions of the ratios of radii of the disk and ring.

Introduction

A common electrode geometry in electroanalytical and research applications involves a disk electrode and a concentric ring electrode both embedded in an insulating plane and rotated about the axis of the disk. Species produced by an electrochemical reaction at the disk can frequently be detected quantitatively by electrochemical reaction at the ring.^{1,2,3} In some of these applications it is desirable to assess the ohmic potential drop in the solution. For example, to have a controlled electrode potential for the reaction at the disk one needs to know how a current to the disk and a current to the ring separately influence the potential in the solution in the neighborhood of the disk.^{4,5} To ensure that a limiting current is maintained on the ring involves a similar question.⁶

Experimental efforts to answer these questions involve abrupt changes in the current to either the ring or the disk followed by a measurement of the change in potential of both the ring and the disk as shortly thereafter as possible.^{4,5,7} Such rapid changes in potential and current are associated with the primary distributions of potential and current.⁸

Consequently, we can define a mathematical problem in which the potential obeys Laplace's equation,

$$\nabla^2 \phi = 0, \quad (1)$$

the potential is zero at infinity, and has a uniform value in the solution adjacent to each electrode. Corresponding to a zero current

density, the normal component of the potential gradient is zero on the insulating annulus between the disk and the ring and on the plane surrounding the ring. This problem excludes consideration of the variation of conductivity within the thin diffusion layer adjacent to the electrodes and effectively regards the change in potential drop to be determined by the bulk of the solution. Also excluded from consideration is the effect of electrode kinetics, it being assumed that the double-layer capacity is sufficiently large that the potential difference across it does not change during the time of the measurement.⁸ (The course of events involving the change of the charge of the double-layer capacity has been examined by Nişancıoğlu and Newman.^{9,10,11})

The problem thus defined is limited in scope since it involves only the geometry of the system, the conductivity of the solution, and the potentials and currents themselves. The principal result of the model is the expression of the disk and ring potentials in terms of the disk and ring currents:

$$V_d = R_{dd}I_d + R_{dr}I_r, \quad (2)$$

$$V_r = R_{rd}I_d + R_{rr}I_r, \quad (3)$$

where I_d and I_r are the total currents to the disk and ring electrodes, respectively, and V_d and V_r are the potentials, presumed uniform, in the solution adjacent to the two electrodes. In the absence of concentration and surface overpotentials, V_d and V_r can

be regarded to be the potentials of the electrodes themselves, and this is the usual manner of speaking when discussing primary-distribution problems. Bear in mind that in the applications discussed above these quantities I_d , I_r , V_d , and V_r probably represent instantaneous changes in the electrode currents and the corresponding instantaneous changes in the electrode potentials.

R_{dd} , R_{dr} , R_{rd} , and R_{rr} are the primary resistances defined by equations 2 and 3 for this ring-disk system. We can attach a physical meaning to them by the following considerations. When there is no ring current, $I_r = 0$, we see that R_{dd} represents the resistance between the disk electrode and a counterelectrode at infinity. This resistance will be lower in the presence of the ring than for the disk alone because current can find a path through the ring electrode to the disk, bypassing some of the resistance of the solution. This is true even though there is no net current to the ring. Under these circumstances, the potential of the ring will take on a definite value to satisfy the condition of no net current to the ring. This value is determined by R_{rd} in equation 3. Thus, R_{rd} is a quantity having the dimensions of a resistance but which yields the potential on the ring due to a current on the disk.

In a similar manner, we see that when there is no disk current, R_{rr} is the resistance between the ring and a counterelectrode at infinity while R_{dr} reproduces the potential on the disk due to a current on the ring. As shown below, $R_{dr} = R_{rd}$.

The geometry of the ring-disk system is defined adequately by the ratio r_0/r_1 of the disk radius to the inner radius of the ring

and the ratio r_1/r_2 of the inner and outer radii of the ring. The resistances can be made dimensionless with the conductivity κ of the solution and a characteristic length, which we choose to be the outer radius r_2 of the ring. Therefore, the results of this study can be presented simply by correlating three dimensionless resistances ($R_D^D = \kappa r_2 R_{dd}$, $R_D^R = R_R^D = \kappa r_2 R_{dr}$, and $R_R^R = \kappa r_2 R_{rr}$) as functions of two geometric ratios (r_0/r_1 and r_1/r_2). This simplicity and generality is a further justification for restricting the problem to the primary resistances.

In a subsequent paper from this laboratory,¹² we shall discuss some more complicated behavior of the ring-disk system in which concentration variations and electrode kinetics are considered in order to assess the current distribution on a sectioned electrode (composed of the ring and disk at the same potential) below the limiting current, the collection efficiency of the system when the current distribution on the disk is nonuniform due to the ohmic potential drop in the solution, and the anomalous diffusion coefficient for a redox couple measured by means of the limiting current to a ring electrode with zero current to the disk.

Symmetry of Resistances

Let us consider two cases: case 1 where $I_d = 0$ and case 2 where $I_r = 0$. For any two functions ϕ_1 and ϕ_2 , Green's theorem says¹³

$$\int (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) dV_o = \oint (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \underline{dS} . \quad (4)$$

The integral over the volume V_o is zero here because both ϕ_1 and ϕ_2 obey Laplace's equation. The surface integral is over the entire area enclosing the volume V_o , which we shall take to be the entire half-space between the plane of the disk and the counterelectrode at infinity. The integral over the insulating surfaces is zero because the normal component of the potential gradient is zero there. The integral over a hemisphere at infinity is zero because each potential is inversely proportional to the radius, the potential gradient is inversely proportional to the square of the radius, and \underline{dS} is proportional to the square of the radius.

This leaves us with integrals over only the surfaces of the electrodes:

$$\int_d (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \underline{dS} = - \int_r (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \underline{dS} . \quad (5)$$

Now, by the definition of the primary distributions, the potential adjacent to each electrode is uniform and can be removed from the integral, with the result

$$V_{d1} \int_d \nabla \phi_2 \cdot dS - V_{d2} \int \nabla \phi_1 \cdot dS = -V_{r1} \int_r \nabla \phi_2 \cdot dS + V_{r2} \int_r \nabla \phi_1 \cdot dS \quad (6)$$

Furthermore, the integral of the normal component of the potential gradient over the surface of an electrode is proportional to the total current to the electrode. Equation 6 becomes

$$V_{d1} I_{d2} - V_{d2} I_{d1} = -V_{r1} I_{r2} + V_{r2} I_{r1} \quad (7)$$

For the cases chosen here, $I_{d1} = 0$ and $I_{r2} = 0$, and this reduces to

$$V_{d1} I_{d2} = V_{r2} I_{r1} \quad (8)$$

Substitution of equations 2 and 3 for the electrode potentials, with $I_{d1} = I_{r2} = 0$, yields

$$R_{dr} I_{r1} I_{d2} = R_{rd} I_{d2} I_{r1} \quad (9)$$

or

$$R_{dr} = R_{rd} \quad (10)$$

Gabrielli et al.⁷ state this result and provide supporting experimental results. Equation 10 could be considered to be an example of the Onsager reciprocal relation.

Analysis

Newman¹⁴ reviews methods of calculating current and potential distributions in ring or disk geometries. At first we thought that we could treat the ring-disk system as a composite disk of radius r_2 and use the method of separation of variables in rotational elliptic coordinates. Then the current density would be zero on the insulating annulus while the potentials would be specified on the ring and disk, and the coefficients of the series would be determined by trial and error or by matrix inversion so as to satisfy these boundary conditions. However, such a series is inadequate to represent the distributions of potential and current in this system because the current density approaches infinity at the inner edge of the ring and at the edge of the disk. (The coordinate system does allow treatment in a natural way of the infinite current density near the outer edge of the ring, just as it does for the primary distribution near the edge of a disk without a ring.¹⁵)

As an alternative, the currents due to the ring and the disk were treated separately by different methods. First a series of ten cases was defined with prescribed current distributions on the ring. For cases 1 and 3, these current distributions were

$$i_{r1} = \frac{2}{\sqrt{1-x^2}} \quad (11)$$

and

$$i_{r3} = \sqrt{\frac{2}{1+x}}, \quad (12)$$

where

$$x = \frac{2r - r_1 - r_2}{r_2 - r_1}. \quad (13)$$

Case 2 has a zero current density everywhere on the ring but will have a current assigned to the disk as described below. Cases 4 through 10 were assigned the following current distributions on the ring:

$$i_{r,k} = P_{k-4}(x), \quad (14)$$

where $P_k(x)$ is the Legendre polynomial.

It was felt that these cases would represent a complete set which could be superposed to reproduce any primary current distribution on the ring electrode. In particular, case 1 has an infinite current density at both the inner and the outer edge of the ring, and the current density approaches infinity in the manner required when an electrode is embedded in an insulating plane, namely, by being inversely proportional to the square root of the distance from the edge. Case 3 involves an infinite current density only at the inner edge of the ring. A superposition of cases 1 and 3 should be able to match the way in which any primary current distribution goes to infinity at the inner and outer edges of the ring. The residual current distribution should be finite over the ring and adequately

represented by a superposition of the remaining cases 4 through 10. For some values of r_0/r_1 and r_1/r_2 where the accuracy of the results was questionable, the number of cases was extended from 10 to 20.

The next step in the procedure is to evaluate the potential distribution on both the disk and the ring due to the current distribution on the ring for each of the cases described above. For this purpose, we use the formula for the potential in the plane of the disk¹⁴

$$\Phi_0(r) = \frac{2}{\pi K} \int_{r_1}^{r_2} \frac{i(r')K'(m)r'dr'}{r+r'}, \quad (15)$$

where

$$m = \frac{4rr'}{(r+r')^2} \quad (16)$$

and $K(m)$ is the complete elliptic integral of the first kind. The evaluation of this integral for the potential distribution on the ring requires care, first of all, because the elliptic integral approaches infinity when $r' = r$. Additional difficulties are introduced for cases 1 and 3 where the current distribution approaches infinity at the inner or outer edge of the ring.

The potential distributions obtained above will be nonuniform on both the ring and the disk. For each case, the potential can be made uniform on the disk by superposing the potential distribution due

to a current distribution introduced on the disk. Here we use rotational elliptic coordinates η and ξ based on the radius r_0 of the disk. The coordinate transformation reads

$$z = r_0 \xi \eta \quad \text{and} \quad r = r_0 \sqrt{(1 + \xi^2)(1 - \eta^2)}, \quad (17)$$

and the solution of Laplace's equation by separation of variables in this coordinate system is^{14,16}

$$\Phi = \sum_{n=0}^{\infty} B_n P_{2n}(\eta) M_{2n}(\xi), \quad (18)$$

where B_n represents arbitrary coefficients, P_{2n} is again the Legendre polynomial, and $M_{2n}(\xi)$ (called $M_n(\xi)$ in reference 14) is a Legendre function of imaginary argument having properties described earlier. Selection of even Legendre polynomials in equation 18 ensures that the corresponding current distribution is zero in the plane outside the disk; hence, the current distribution is not modified on the ring by superposing a potential distribution of the type in equation 18.

In practice, equation 18 is truncated after a finite number of terms, say 20. For each case, the B values are now chosen so that the potential (including that due to the ring current) will be zero on the surface of the disk. Up to this point, case 2 has not been defined or modified. We now require that the potential Φ_0 be equal to unity on the surface of the disk, for case 2, which is equivalent to setting $B_0 = 1$. The superposition of the disk potential function in equation 18 will generate a nonzero net current

and a uniform potential for the disk for each case.

Next, for each case, we should calculate the potential distribution on the ring due to the current distribution on the disk, and we should add this to the potential distribution previously obtained from the current distribution on the ring. This step involves the use of equation 18 with values of ξ greater than zero since

$$\Phi_o = \sum_{n=0}^{\infty} B_n P_{2n}(0) M_{2n}(\xi) \quad (19)$$

in the plane for r greater than r_o . The evaluation of $M_{2n}(\xi)$ has been necessary in earlier work,⁶ and we have introduced refinements here to permit accurate calculation for large values of ξ and n (see appendix A).

The several cases that have been treated now each have prescribed current distributions on the ring and disk, known total currents, a uniform potential on the disk, and a nonuniform but finite potential distribution on the ring. The final step of the procedure is to superpose cases 3 through 10 onto cases 1 and 2, in turn, in such a way that the potential distribution on the ring is made uniform. More cases can be used to attain a higher degree of uniformity.

Cases 1 and 2 now satisfy all the requirements of a primary distribution -- they have uniform potentials on the ring and the disk, and they satisfy Laplace's equation and all the other boundary conditions. Analysis of cases 1 and 2 according to equations 2 and 3 yields values of the resistances R_{dd} , R_{dr} , R_{rd} , and R_{rr} .

This solution for the primary potential and current distributions by superposition may seem involved and complicated, but it is economical and accurate, and it avoids any trial-and-error calculations. The functions chosen for superposition make special allowance for the geometry of the system and can treat the infinite current densities at the edges of the electrodes even when the insulating annulus is quite thin.

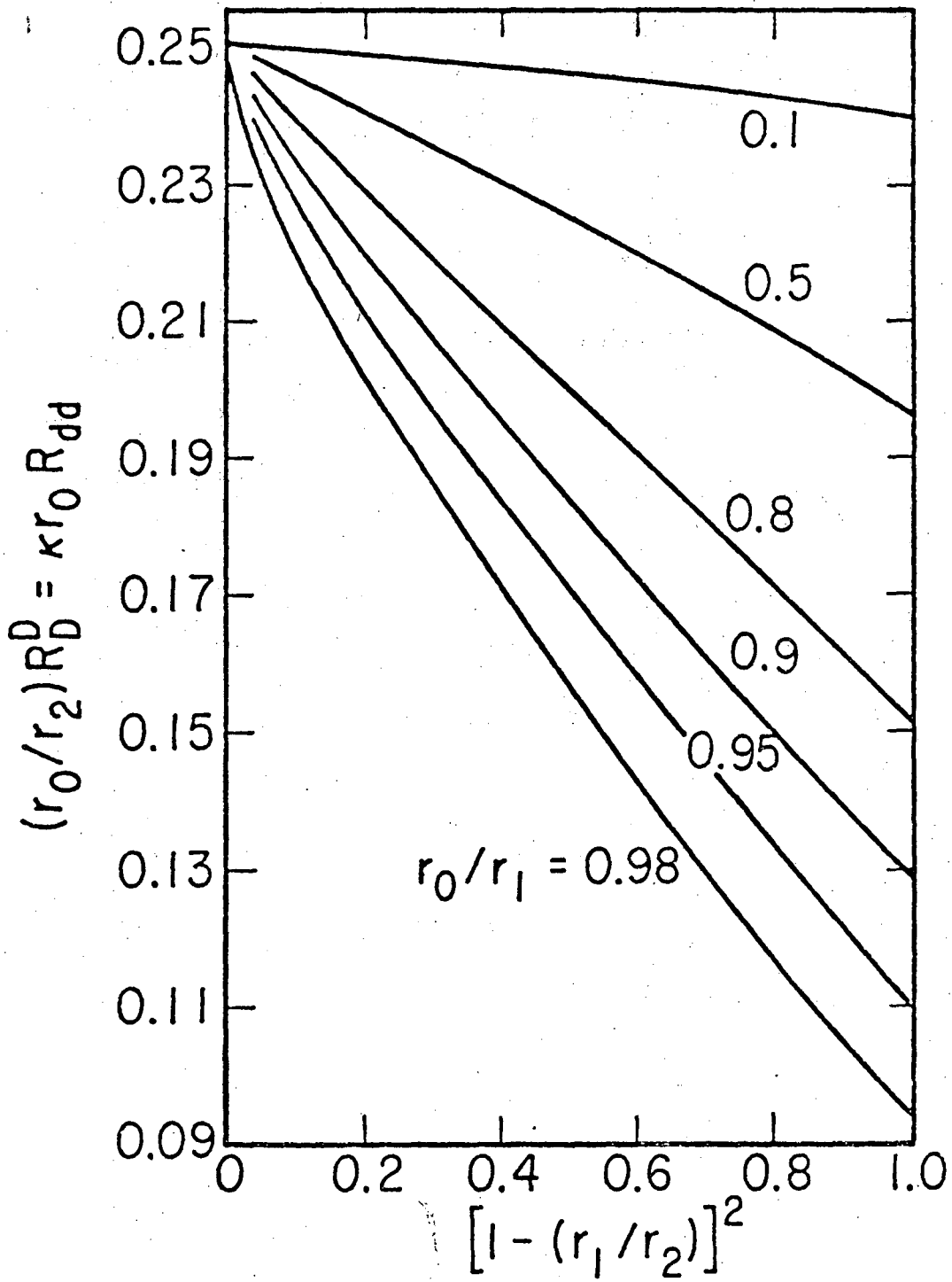
Results

In the computed results, R_{dr} and R_{rd} usually agreed to within 0.01 percent. Certain limiting situations could also be checked to ensure the validity of the results.

Figures 1, 2, and 3 show the values of the three independent resistances as functions of the geometric ratios r_0/r_1 and r_1/r_2 . For a very thin ring, R_{rr} becomes infinite. Consequently, on figure 3 we have added a term which compensates for this and produces a finite limit as r_1 approaches r_2 . An exception is the (unrealistic) limit of a zero gap distance. As r_0 approaches r_1 , the value of $\kappa r_2 R_{rr}$ approaches 0.25, independent of the value of r_1/r_2 .

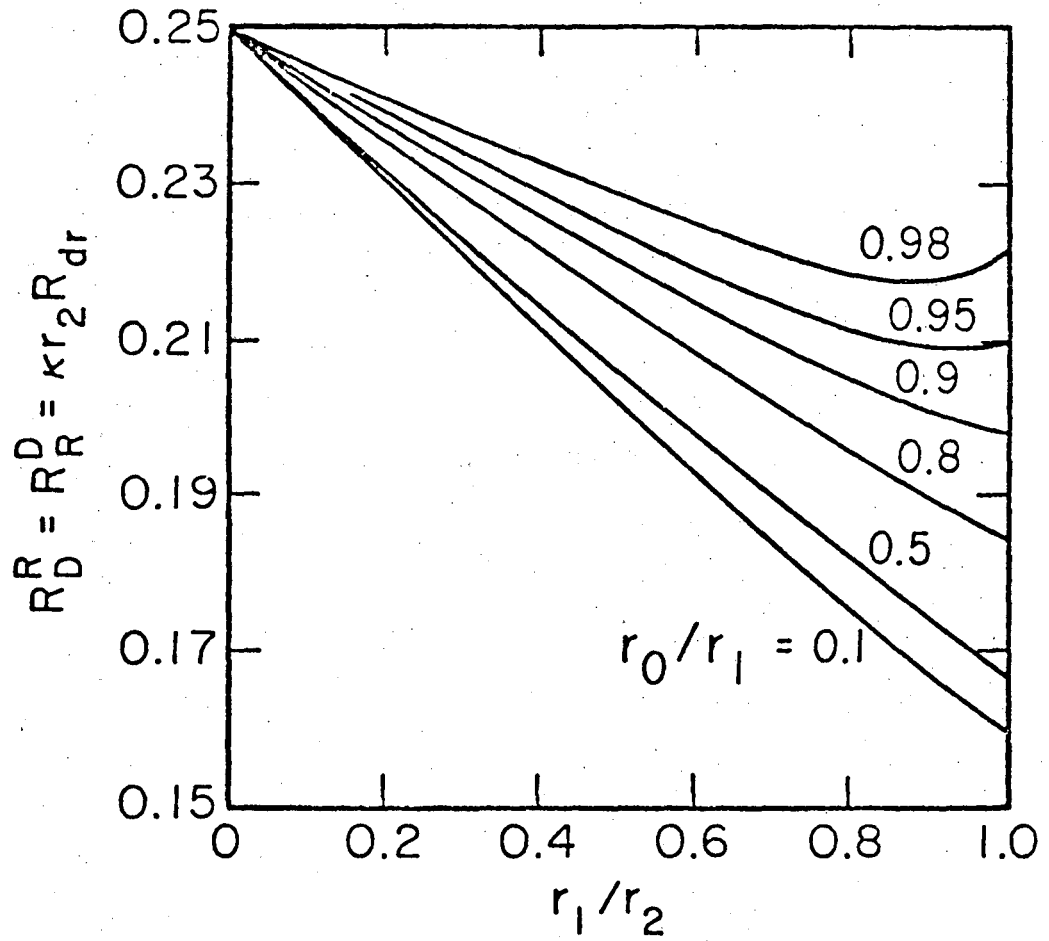
Discussion

The results for R_{dd} can be comprehended in relation to the value $1/4\kappa r_0$ for the primary resistance¹⁵ for a single disk in an insulating plane. The values for the disk resistance, as plotted in



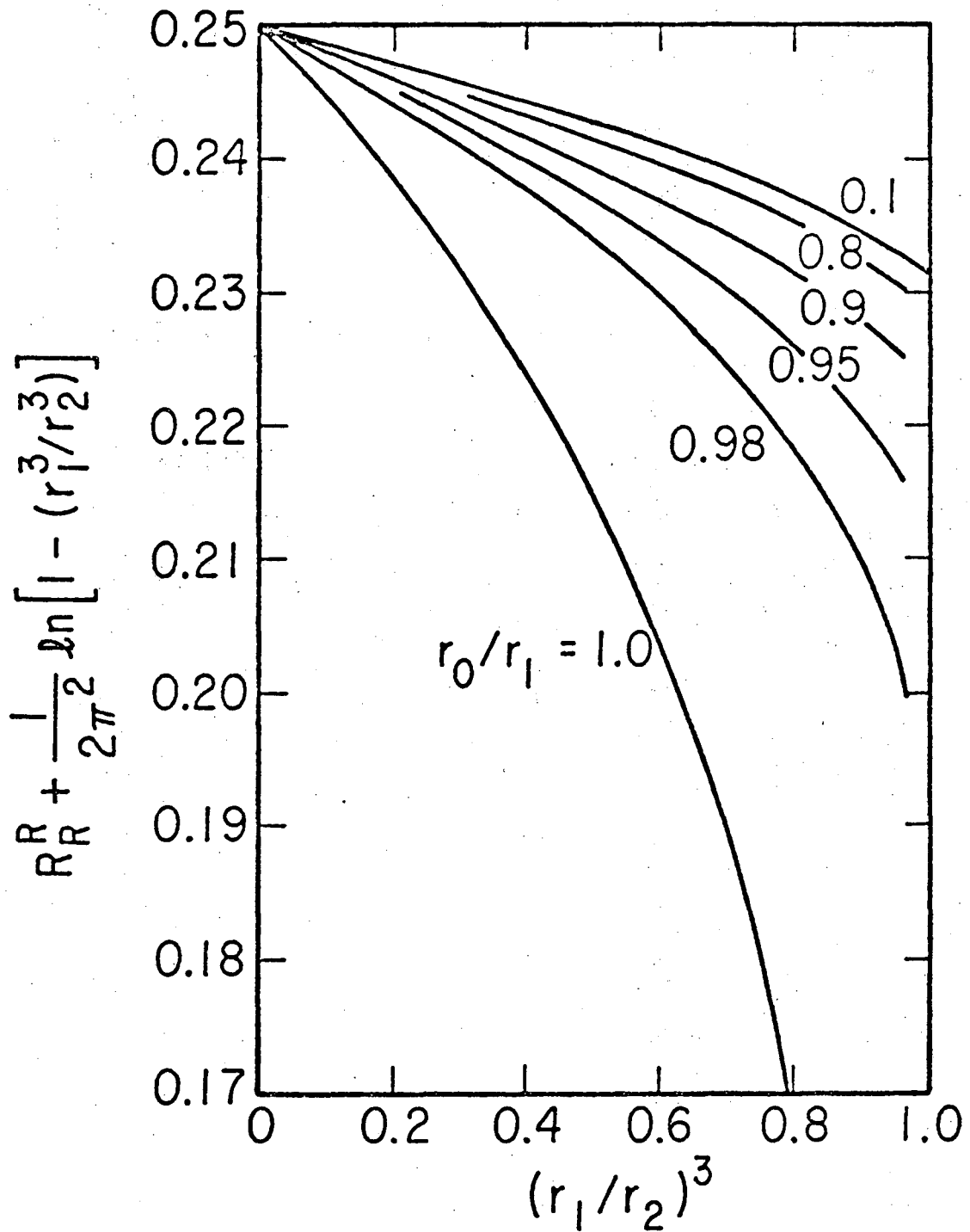
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Figure 1. Correlation of the disk resistance.



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Figure 2. Correlation of the interaction resistance.



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Figure 3. Correlation of the ring resistance.

figure 1, therefore approach the value 0.25 as the influence of the ring becomes negligible -- either for thin rings ($r_1 \rightarrow r_2$) or for wide gaps between the ring and the disk ($r_0 \ll r_1$). The influence of the ring is always to lower the resistance value $\kappa r_0 R_{dd}$ below the value 0.25 because the ring provides an alternative current path which can help the current get from infinity to the neighborhood of the disk. Figure 1 shows how this effect becomes more pronounced for wide rings and narrow gaps.

There are several ways of thinking about the coupling resistances $R_{dr} = R_{rd}$. First imagine a current to the disk with no current to the ring. Then the potential distribution will bear some resemblance to that for a single disk in an insulating plane, and the similarity will become exact in the limit of a thin ring. The ring, in addition to distorting this potential field, will acquire a potential corresponding to the single disk at some radial position r_* which lies between r_1 and r_2 . Since

$$(r/r_0)^2 = 1 + \xi^2 \quad (20)$$

on the ring and since

$$M_0(\xi) = \frac{2}{\pi} \text{ctn}^{-1}(\xi) , \quad (21)$$

the potential in the plane at a radial position r_* due to the primary distribution on a single disk is

$$V_r = V_d \operatorname{ctn}^{-1}(\xi) = \frac{I_d}{2\pi\kappa r_o} \sin^{-1}\left(\frac{r_o}{r_*}\right). \quad (22)$$

This leads to the resistance value

$$\kappa r_2 R_{rd} = \frac{r_2}{2\pi r_o} \sin^{-1}\left(\frac{r_o}{r_*}\right). \quad (23)$$

This formula becomes rigorous for thin rings when we set r_* equal to r_2 . Thus, the intercept on the right side of figure 2 is known with certainty. The limit for the ordinate is 0.25 for narrow gaps ($r_o \rightarrow r_1$) and $1/2\pi = 0.1592$ for wide gaps ($r_o \ll r_1$).

For thick rings, it is convenient to think of a zero current on the disk. Then the ring itself will look like a disk, with a small imperfection at the center, and the potential distribution will be nearly that for a disk of radius r_2 in an insulating plane. The small disk of radius r_o can then sense only one potential, that approximately equal to the potential of the ring $V_r = I_r/4\kappa r_2$.

This leads to the limit

$$\kappa r_2 R_{dr} \rightarrow 0.25 \text{ as } r_2/r_1 \rightarrow \infty, \quad (24)$$

independent of the value of r_o/r_1 .

By an analysis of the current deflected from the insulating region for $r < r_1$, one can find a correction to equation 24 for small disks:

$$\kappa r_2 R_{dr} = \frac{1}{4} - \frac{1}{\pi^2} \frac{r_1}{r_2} \quad \text{for } r_1 \ll r_2 \quad \text{and } r_o \ll r_1. \quad (25)$$

This limiting slope is verified in figure 2.

For rings which are neither thick nor thin, we can use the results in figure 2 to calculate the value of r_* according to equation 23. It turns out that r_* varies from the arithmetic average of r_1 and r_2 for thin rings to a value of $2r_2/\pi$ for thick rings (in order to reproduce the limit in equation 24). This suggests the method of correlation of R_{dr} shown in figure 4. Here a value of r_* is calculated a priori, and the ratio of the left and right sides of equation 23 represents a deviation function which is close to unity. The only advantage of figure 4 over figure 2 is that the scale can be expanded because the minimum and maximum values now differ by a factor of 1.05 instead of a factor of 1.57.

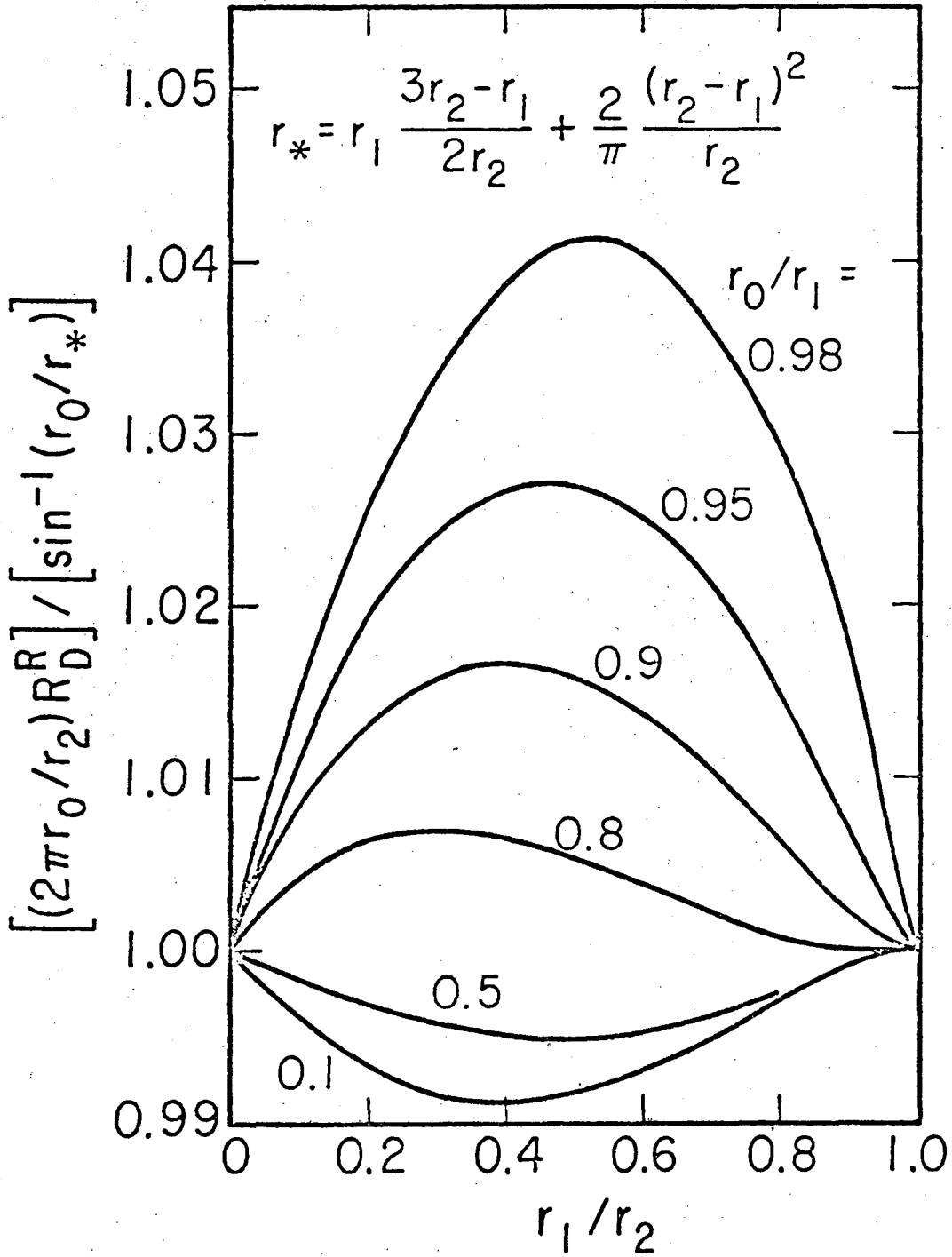
Let us next turn our attention to the ring resistance R_{rr} . For wide rings, it is clear that the resistance value is given by

$$\kappa r_2 R_{rr} = 0.25, \quad (26)$$

the value for a single disk of radius r_2 . In the other extreme,

$$\kappa r_2 R_{rr} + \frac{1}{2\pi^2} \ln \left(1 - \frac{r_1^3}{r_2^3} \right) = \frac{\ln 96}{2\pi^2} = 0.2312 \quad (27)$$

for thin rings ($r_1 \rightarrow r_2$) and small disks ($r_o \ll r_1$).



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Figure 4. Correlation of interaction resistance.

Figure 3 was plotted so that the small disk case ($r_o \ll r_1$) would show clearly these limits. According to this figure, the effect of a nonzero disk is always to lower the ring resistance, because an alternative path is provided between the counterelectrode at infinity and the ring electrode. The correction to equation 27 for small disks is very small, $-(r_o/r_2)^5/45\pi^2$. Thus, we see that the curve for $r_o/r_1 = 0.8$ is already very close to the curve for $r_o/r_1 = 0.1$.

Gabrielli et al.⁷ have measured resistances for four ring-disk geometries. They verified the coupling relationship between R_{rd} and R_{dr} . A comparison between their measurements and our calculated values is made in table 1. For this purpose, l/κ was given the value 2.25 ohm-cm for a 2 N sulfuric acid solution. The comparison cannot be regarded as satisfactory. Two experimental values of $\kappa r_o R_{dd}$ are greater than 0.25, which should not be possible. The other two values of $\kappa r_o R_{dd}$ show good agreement. Measured values of the coupling resistance are consistently lower than those calculated. One value of $\kappa r_2 R_{rr}$ is lower than 0.25, which should not be possible. The other measured values of $\kappa r_2 R_{rr}$ are significantly higher than the calculated values.

Shabrang and Bruckenstein⁵ analyze their results in terms of equations of the form

$$V_d - V_T = R_D I_d + (I_d + I_r) R_C \quad (28)$$

and

Table 1. Comparison of calculated resistances with those measured by Gabrielli et al.⁷ for four ring-disk geometries.

r_o/r_1	r_1/r_2	$\kappa r_2 R_{rr}$		$\kappa r_2 R_{dr}$		$\kappa r_o R_{dd}$	
		meas.	calc.	meas.	calc.	meas.	calc.
0.952	0.42	0.244	0.252	0.211	0.228	0.307	0.192
0.968	0.62	0.272	0.261	0.194	0.22	0.217	0.216
0.976	0.82	0.311	0.273	0.189	0.218	0.231	0.238
0.976	0.976	1.213	0.342	0.177	0.219	0.262	0.2495

$$V_r - V_T = R_R I_r + (I_d + I_r) R'_C, \quad (29)$$

where R_D , R_R , R_C , and R'_C are resistances and V_T is the potential of the reference electrode and can be expressed as

$$V_T = R_{Aux} I_d + R'_{Aux} I_r. \quad (30)$$

Comparison with equations 2 and 3 shows that we can make the associations

$$R_{dd} = R_D + R_C + R_{Aux}, \quad (31)$$

$$R_{dr} = R_C + R'_{Aux}, \quad (32)$$

$$R_{rd} = R'_C + R_{Aux}, \quad (33)$$

and

$$R_{rr} = R_R + R'_C + R'_{Aux}. \quad (34)$$

In view of equation 10, we can write

$$R_C - R'_C = R_{Aux} - R'_{Aux}. \quad (35)$$

Shabrang and Bruckenstein take these differences to be zero. Indeed, if the counterelectrode is far away and the reference electrode is moderately far away from the ring-disk system, we can estimate¹⁵

$$R_{Aux} = R'_{Aux} = \frac{1}{2\pi\kappa\rho}, \quad (36)$$

where ρ is the radial position of the reference electrode in spherical coordinates. However, the currents I_d and I_r do not, in general, need to have the same influence on the potential V_T in equation 30; the difference will become accentuated the closer the reference electrode probe is to the ring-disk system.

From figures 1, 2, and 3, we find $\kappa r_o R_{dd} = 0.249$, $\kappa r_2 R_{rd} = 0.209$, and $\kappa r_2 R_{rr} = 0.3238$ for the geometry of Shabrang and Bruckenstein ($r_o/r_1 = 0.95$ and $r_1/r_2 = 8/8.4$). R_D corresponds approximately to $R_{dd} - R_{dr}$, and R_R corresponds approximately to $R_{rr} - R_{rd}$. (Shabrang and Bruckenstein come to a different conclusion.) For the ratio $R_D/(R_D + R_R)$, they find values of 0.37, 0.35, 0.34, 0.39, 0.36, 0.34, and 0.31, whereas we calculate 0.366 for the corresponding ratio. (Here, we assume that the labels V_o/V_D and V_o/V_R are interchanged in their table III.)

Because of uncertainties in the position of the reference electrode and the conductivity of the solution, we refrain from further comparisons with their data.

From the results of Miller and Bellavance⁴ we deduce an experimental value of $\kappa r_2 R_{rd} = 0.192$. The corresponding value from figure 2 is $\kappa r_2 R_{rd} = 0.206$ for $r_o/r_1 = 0.909$ and $r_1/r_2 = 0.812$.

Conclusion

Computed values of the primary resistances for a ring-disk system, as presented here, should permit estimation of the uncompensated resistances when an attempt is made to control the potentials of the

electrodes. There are few geometries for which this information is available.

Discrepancies between calculated and experimental values may lead to refined experiments or to considerations beyond the scope of the primary resistances.

Acknowledgment

This work was supported by the United States Energy Research and Development Administration.

List of Symbols

B_n	coefficients in series 18 for potential
I_d	disk current, A
I_r	ring current, A
K	complete elliptic integral of the first kind
m	see equation 16
M_{2n}	Legendre function of imaginary argument
P_k	Legendre polynomial
r	radial position in cylindrical coordinates, cm
r_0	radius of disk, cm
r_1	inner radius of ring, cm
r_2	outer radius of ring, cm
r_*	position on ring electrode, cm
$R_{dd}, R_{dr}, R_{rd}, R_{rr}$	resistances defined by equations 2 and 3, ohm
R_D, R_R, R_C, R'_C	resistances defined by equations 28 and 29, ohm

R_{Aux}, R'_{Aux} resistances defined by equation 30, ohm

$R_D^D, R_D^R, R_R^D, R_R^R$ dimensionless resistances

S surface area, cm^2

V_d disk potential, V

V_r ring potential, V

V_T potential at reference electrode, V

V_o volume, cm^3

x see equation 13

z distance from the plane of the disk, cm

η rotational elliptic coördinate

κ conductivity of the solution, $ohm^{-1} - cm^{-1}$

ξ rotational elliptic coördinate

ρ radial position in spherical coördinates, cm

Φ potential in the solution, V

APPENDIX A: Calculation of the M Functions

Of fundamental importance in the calculation of potential distributions in disk systems are the functions M_n defined by¹⁶

$$\frac{d}{d\xi} \left[(1 + \xi^2) \frac{dM_n}{d\xi} \right] = n(n + 1)M_n, \quad (\text{A-1})$$

subject to the boundary conditions

$$M_n = 1 \quad \text{at} \quad \xi = 0, \quad (\text{A-2})$$

$$M_n = 0 \quad \text{at} \quad \xi = \infty. \quad (\text{A-3})$$

We describe here several methods which have been used for evaluating $M_n(\xi)$ and its derivative.

Power series

Equation A-1 is Legendre's equation of imaginary argument.

Development in a power series

$$M_n = \sum_{k=0}^{\infty} C_k \xi^k \quad (\text{A-4})$$

is assured to converge for $|\xi| < 1$. From the boundary condition A-2 we have

$$C_0 = 1. \quad (\text{A-5})$$

The recursion relation is

$$C_k = -C_{k-2} \frac{(k-n-2)(k+n-1)}{k(k-1)} \quad \text{for } k > 1 \quad (\text{A-6})$$

It has been shown elsewhere¹⁶ that

$$M'_{2n}(0) = -\frac{2}{\pi} \frac{(2^n n!)^4}{[(2n)!]^2}, \quad (\text{A-7})$$

the result being restricted to even values of $2n$. We generalize by noting that

$$C_1 = -2/\pi \quad \text{for } n = 0, \quad (\text{A-8})$$

$$C_1 = -\pi/2 \quad \text{for } n = 1, \quad (\text{A-9})$$

and a recursion relation for higher values of n is

$$C_{1,n} = \left(\frac{n}{n-1}\right)^2 C_{1,n-2}. \quad (\text{A-10})$$

The function subprogram FUNM(N,XI) calculates $M_n(\xi)$ according to this method beginning at statement 7. The function subprogram FUNMP(N,XI) calculates the derivative $M'_n(\xi)$ by the same technique.

Series in $1/\xi$

On the basis of equations 8.1.3 and 15.1.1 of reference 17, one can develop the expression, convergent for $|\xi| > 1$,


```

FUNCTION FUNM(N,XI) $ XI2=XI**2 $ ADD=1.0 $ SUM=1.0
PI2=1.5707963267948966192
IF(ABS(XI).LT.1.0) GO TO 7 $ DO 1 K=1,500
ADD=-ADD*FLOAT(2*K+N)*FLOAT(2*K+N-1)/4.0/FLOAT(K)/(FLOAT(K+N)+0.5)
1/XI2
SUM=SUM+ADD
1 IF(ABS(ADD).LT.1.0E-9*ABS(SUM)) GO TO 2
2 FUNM=SUM/XI**(1+N)/PI2 $ IF(N.EQ.0) RETURN
DO 3 NN=1,N
3 FUNM=FUNM*FLOAT(NN)/FLOAT(2*NN+1) $ N2=N/2
IF(N.NE.2*N2) GO TO 5 $ DO 4 NN=1,N2
4 FUNM=FUNM*FLOAT(NN)/(FLOAT(NN)-0.5) $ RETURN
5 FUNM=FUNM*PI2 $ IF(N2.EQ.0) RETURN $ DO 6 NN=1,N2
6 FUNM=FUNM*(FLOAT(NN)+0.5)/FLOAT(NN) $ RETURN
7 A2=1.0 $ N2=N/2 $ IF(2*N2.EQ.N) GO TO 9 $ A1=-PI2
IF(N.EQ.1) GO TO 11 $ DO 8 NN=3,N,2
8 A1=A1/(1.0-1.0/FLOAT(NN))**2 $ GO TO 11
9 A1=-1.0/PI2 $ IF(N.EQ.0) GO TO 11 $ DO 10 NN=2,N,2
10 A1=A1/(1.0-1.0/FLOAT(NN))**2
11 A1=A1*XI $ FUNM=A2+A1 $ DO 12 K=2,500,2
A2=-A2*XI2*FLOAT(K-N-2)*FLOAT(K+N-1)/FLOAT(K*K-K)
A1=-A1*XI2*FLOAT(K-N-1)*FLOAT(K+N)/FLOAT(K*K+K) $ ADD=A2+A1
FUNM=FUNM+ADD
12 IF(ABS(ADD).LT.1.E-9*ABS(FUNM)) RETURN $ RETURN $ END

```

```

FUNCTION FUNMP(N,XI) $ XI2=XI**2 $ ADD=1.0 $ SUM=1.0
PI2=1.5707963267948966192
IF(ABS(XI).LT.1.0) GO TO 7 $ DO 1 K=1,500
ADD=-ADD*FLOAT(2*K+N)*FLOAT(2*K+N-1)/4.0/FLOAT(K)/(FLOAT(K+N)+0.5)
1/XI2
SUM=SUM+ADD
1 IF(ABS(ADD).LT.1.0E-9*ABS(SUM)) GO TO 2
2 FUNMP=-SUM/XI**(2+N)/PI2*FLOAT(N+1)
IF(N.EQ.0) RETURN $ DO 3 NN=1,N
3 FUNMP=FUNMP*FLOAT(NN)/FLOAT(2*NN+1) $ N2=N/2
IF(N.NE.2*N2) GO TO 5 $ DO 4 NN=1,N2
4 FUNMP=FUNMP*FLOAT(NN)/(FLOAT(NN)-0.5) $ RETURN
5 FUNMP=FUNMP*PI2 $ IF(N2.EQ.0) RETURN $ DO 6 NN=1,N2
6 FUNMP=FUNMP*(FLOAT(NN)+0.5)/FLOAT(NN) $ RETURN
7 A2=1.0/XI $ N2=N/2 $ IF(2*N2.EQ.N) GO TO 9 $ A1=-PI2
IF(N.EQ.1) GO TO 11 $ DO 8 NN=3,N,2
8 A1=A1/(1.0-1.0/FLOAT(NN))**2 $ GO TO 11
9 A1=-1.0/PI2 $ IF(N.EQ.0) GO TO 11 $ DO 10 NN=2,N,2
10 A1=A1/(1.0-1.0/FLOAT(NN))**2
11 FUNMP=A1 $ DO 12 K=2,500,2
A2=-A2*XI2*FLOAT(K-N-2)*FLOAT(K+N-1)/FLOAT(K*K-K)
A1=-A1*XI2*FLOAT(K-N-1)*FLOAT(K+N)/FLOAT(K*K+K)
ADD=A2*FLOAT(K)+A1*FLOAT(K+1) $ FUNMP=FUNMP+ADD
12 IF(ABS(ADD).LT.1.E-9*ABS(FUNMP)) RETURN $ RETURN $ END

```

$$M_n(\xi) = \frac{\Gamma(n+1)}{\left[\Gamma\left(\frac{1}{2} + \frac{n}{2}\right)\right]^2} \frac{1}{2^n \xi^{n+1}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(1 + \frac{n}{2} + k\right) \Gamma\left(\frac{1}{2} + \frac{n}{2} + k\right)}{\Gamma\left(n + \frac{3}{2} + k\right) k! \xi^{2k}}$$

$$= \frac{Q_n(i\xi)}{Q_n(0)}, \quad (\text{A-11})$$

where $Q_n(i\xi)$ is a Legendre function of the second kind of imaginary argument whose value at the origin is determined to be

$$Q_n(0) = \frac{\Gamma\left(\frac{1}{2} + \frac{n}{2}\right)}{2\Gamma\left(1 + \frac{n}{2}\right)} \frac{\sqrt{\pi}}{i^{n+1}}. \quad (\text{A-12})$$

One can simplify the expressions somewhat by noting that

$$M_0(\xi) \rightarrow \frac{2}{\pi\xi} \quad \text{as } \xi \rightarrow \infty, \quad (\text{A-13})$$

$$M_1(\xi) \rightarrow \frac{1}{2\xi^2} \quad \text{as } \xi \rightarrow \infty, \quad (\text{A-14})$$

and

$$M_{n+2}(\xi) \rightarrow \frac{(n+2)^2}{(2n+5)(2n+3)} \frac{M_n(\xi)}{\xi^2} \quad \text{as } \xi \rightarrow \infty. \quad (\text{A-15})$$

Then, if we write equation A-11 as

$$M_n(\xi) = \sum_{k=0}^{\infty} \frac{D_k}{\xi^{2k+n+1}}, \quad (\text{A-16})$$

then the values of D_0 can be inferred from equations A-13 to A-15 and the higher coefficients are determined by the recursion relation

$$D_k = -D_{k-1} \frac{(2k+n)(2k+n-1)}{4k\left(k+n+\frac{1}{2}\right)}. \quad (\text{A-17})$$

The function subprogram FUNM(N,XI) calculates $M_n(\xi)$ according to this method for $\xi > 1$. The function subprogram FUNMP(N,XI) calculates the derivative $M'_n(\xi)$ by the same technique.

Series expansion in $\text{ctn}^{-1}(\xi)$

Introduction of the variable

$$y = \text{ctn}^{-1}(\xi) \quad (\text{A-18})$$

transforms the differential equation A-1 to

$$\sin^2 y \frac{d^2 M_n}{dy^2} = n(n+1)M_n \quad (\text{A-19})$$

with boundary conditions

$$M_n = 0 \quad \text{at} \quad y = 0. \quad (\text{A-20})$$

$$M_n = 1 \quad \text{at} \quad y = \pi/2. \quad (\text{A-21})$$

The asymptotic forms A-13 to A-15 lead to

$$M_0 \rightarrow \frac{2}{\pi} y \quad \text{as} \quad y \rightarrow 0, \quad (\text{A-22})$$

$$M_1 \rightarrow \frac{1}{2} y^2 \text{ as } y \rightarrow 0, \quad (\text{A-23})$$

and

$$M_{n+2}(y) \rightarrow \frac{(n+2)^2 y^2}{(2n+5)(2n+3)} M_n(y) \text{ as } y \rightarrow 0. \quad (\text{A-24})$$

The solution can be expressed as a power series in y :

$$M_n = \sum_{k=0}^{\infty} A_k y^{2k+n+1}. \quad (\text{A-25})$$

This series is observed to converge for all the values of interest, that is, y between 0 and $\pi/2$ or ξ between 0 and ∞ . The values of A_0 can be inferred from equations A-22 to A-24. A recursion relation can be derived for the higher coefficients. This involves the expansion of $\sin^2 y$ in a power series in y .

The subroutine COEF(NMAX,KMAX,A) establishes the coefficients for $M_{2n}(\xi)$, that is, for even values of $2n$. The subroutine DISK(R,PE,A2N,NM) calculates the values of $M_{2n}(\xi)$ for a given value of ξ (related to R) and stores the result in the array PE(N). These programs are specialized for the ring-disk geometry treated in this thesis and are intended to calculate the potential on the ring due to a current on the disk, in conjunction with the computer programs discussed in appendix B. It was not necessary for this purpose to develop the derivative $M'_{2n}(\xi)$. However, the subroutine DISKP(R,PPE,A2N,NM) calculates the values of $M'_{2n}(\xi)$ and stores them in the array PPE(N).

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```

SUBROUTINE COEF(NMAX,KMAX,A) $ DIMENSION A(20,40)
A(1,1)=1./1.5707963267948956192 $ DO 1 K=2,KMAX $ DO 1 N=1,NMAX
1 A(N,K)=0.0 $ DO 3 N=2,NMAX $ ON=N-1
A(N,1)=4.0*ON**2*A(N-1,1)/(4.0*ON+1.)/(4.0*ON-1.0)
DO 3 KK=2,KMAX $ OK=KK-1 $ SUM=0.0 $ PFAC=1.0
SIGNP=-1.0 $ DO 2 LP=1,OK $ P=LP-1 $ SIGNP=-SIGNP $ FACM=1.0
IF(LP.GT.1) PFAC=PFAC*(2.0*P+1.0)*2.0*P $ SIGN=SIGNP
DO 2 MM=LP,OK $ OM=MM-1
IF(MM.GT.LP) FACM=FACM*2.0*(OM-P)*(2.0*OM-2.0*P+1.0)
SUM=SUM+A(N,OK-MM+1)*SIGN*4.0*(OK-OM+ON+0.5)*(OK-OM+ON)/PFAC/FACM
2 SIGN=-SIGN
3 A(N,OK)=-SUM/4.0/OK/(2.0*ON+OK+0.5) $ RETURN $ END

```

```

SUBROUTINE DISK(R,PE,A2N,NM) $ DIMENSION PE(20),A2N(20,40),YM(60)
CALCULATES M(XI), THE LEGENDRE FUNCTION OF IMAGINARY ARGUMENT
C Y=PI/2.-ATAN(XI), WHERE XI=SQRT(R**2-1.0)
C Y=ASIN(1.0/R) $ YM(1)=Y $ Y2=Y*Y $ MAX=NM+38 $ DO 1 M=1,MAX
1 YM(M+1)=YM(M)*Y2 $ DO 3 N=1,NM $ PE(N)=0.0 $ DO 2 M=1,40
ADD=A2N(N,M)*YM(M+N-1) $ PE(N)=PE(N)+ADD
2 IF(ABS(ADD).LT.1.E-7*ABS(PE(N))) GO TO 3
3 CONTINUE $ RETURN $ END

```

```

SUBROUTINE DISKP(R,PPE,A2N,NM) $ DIMENSION PPE(20),A2N(20,40),YM(60)
CALCULATES THE DERIVATIVE OF M(XI), THE LEGENDRE FUNCTION OF
C IMAGINARY ARGUMENT Y=PI/2.-ATAN(XI)
C Y=ASIN(1.0/R) $ YM(1)=1.0 $ Y2=Y*Y $ MAX=NM+38 $ DO 1 M=1,MAX
1 YM(M+1)=YM(M)*Y2 $ DO 3 N=1,NM $ PPE(N)=0.0 $ DO 2 M=1,40
ADD=A2N(N,M)*YM(M+N-1)*FLOAT(2*M+2*N-3) $ PPE(N)=PPE(N)+ADD
2 IF(ABS(ADD).LT.1.E-7*ABS(PPE(N))) GO TO 3
3 PPE(N)=-PPE(N)/R**2 $ RETURN $ END

```

Recursion in order n

Let

$$Q_0(i\xi) = -i \operatorname{ctn}^{-1}(\xi) , \quad (\text{A-26})$$

$$Q_1(i\xi) = \xi \operatorname{ctn}^{-1}(\xi) - 1 , \quad (\text{A-27})$$

and

$$(n + 1)Q_{n+1}(i\xi) = (2n + 1)i\xi Q_n(i\xi) - nQ_{n-1}(i\xi) . \quad (\text{A-28})$$

A recursion relation for the derivative is

$$(1 + \xi^2) \frac{dQ_n(i\xi)}{d\xi} = n\xi Q_n(i\xi) + niQ_{n-1}(i\xi) . \quad (\text{A-29})$$

Now $M_n(\xi)$ is related to Q_n by

$$M_n(\xi) = \frac{Q_n(i\xi)}{Q_n(0)} , \quad (\text{A-30})$$

where $Q_n(0)$ is given by equation A-12.

To avoid complex arithmetic, define

$$q_n(\xi) = i^{n+1} Q_n(i\xi) . \quad (\text{A-31})$$

Equations A-28 and A-29 become

$$(n + 1)q_{n+1}(\xi) = -(2n + 1)\xi q_n(\xi) + nq_{n-1}(\xi) \quad (\text{A-32})$$

and

$$(1 + \xi^2) \frac{dq_n(\xi)}{d\xi} = n\xi q_n(\xi) - nq_{n-1}(\xi) . \quad (A-33)$$

The function subprogram XI(R,Z,ETA,INDEX,ANS) calculates $M_{2n}(\xi)$ and $M'_{2n}(\xi)$ by means of the recursion relations developed above. It also calculates a conversion from cylindrical coordinates to elliptic coordinates since it was originally designed to be used in a program to calculate the primary resistance in a cylindrical cell with a small disk centered in one end and with the counterelectrode comprising the other end of the cell. For this reason, the function XI also calculates $P_{2n}(\eta)M'_{2n}(\xi)$ and $P'_{2n}(\eta)M_{2n}(\xi)$ as well as the r derivative of $P_{2n}(\eta)M_{2n}(\xi)$.

The recursion method loses accuracy for large values of n , and this becomes more severe for values of ξ greater than 1. An error in the sixth significant figure is detectable at $n = 24$ for $\xi = 0.5$, at $n = 14$ for $\xi = 0.9$ and 1.0 , at $n = 8$ for $\xi = 2$, and at $n = 6$ for $\xi = 3$ and 4 .

Finite differences

A finite-difference solution of equation A-1 or equation A-19 has been developed using the variable $y = \text{ctn}^{-1}(\xi)$ as the independent variable. While this method gave generally satisfactory results, its accuracy can be exceeded by using one of the methods described here.

```

FUNCTION XI(P,Z,ETA,INDEX,ANS,0)
DIMENSION O(20,5),PHI(20),PHIETA(20),PHIXI(20),ANS(20)
JMAX=20
C CALCULATES CONVERSION FROM CYLINDRICAL COORDINATES R,Z TO
C ELLIPTIC COORDINATES ETA,XI.
Z2=Z**2 $ RHO2=P**2+Z2-1.0 $ IF(RHO2.GE.0.0) GO TO 1
X2=3.0*Z2/(-RHO2+SQRT(RHO2**2+4.0*Z2)) $ GO TO 2
1 X2=0.5*(RHO2+SQRT(RHO2**2+4.0*Z2))
2 XI=SQRT(X2) $ IF(XI.EQ.0.0) GO TO 3
IF(7.LT.0.0) XI=-XI $ ETA=Z/XI $ GO TO 4
3 ETA=SQRT(1.0-P**2)
C CALCULATES LEGENDRE POLYNOMIALS P(ETA) AND LEGENDRE FUNCTIONS
C M(XI), AS WELL AS THE DERIVATIVES. RESULTS ARE FOR EVEN ORDER.
4 XI=1.0+XI**2 $ XF=ETA**2+XI**2
PI2=1.5707963267948966192 $ O(1,2)=PI2-ATAN(XI)
OO=1.0-XI*O(1,2) $ O(1,1)=1.0 $ PO=ETA $ O(1,3)=0.0
O(1,4)=-1./XI $ PPO=1.0 $ O(1,5)=PI2 $ N=0 1 DO 5 J=2,JMAX
N=N+1 $ O(J,1)=((2*N+1)*ETA*PO-N*O(J-1,1))/(N+1)
O(J,2)=((2*N+1)*XI*OO-N*O(J-1,2))/(N+1)
O(J,3)=((2*N+1)*(OO+ETA*PPO)-N*O(J-1,3))/(N+1)
O(J,5)=-N*O(J-1,5)/(N+1) $ N=N+1
O(J,4)=N*(XI*O(J,2)+OO)/XI $ PO=((2*N+1)*ETA*O(J,1)-N*PO)/(N+1)
OO=(-(2*N+1)*XI*O(J,2)-N*OO)/(N+1)
5 PPO=((2*N+1)*O(J,1)+ETA*O(J,3)-N*PPO)/(N+1) $ DO 5 J=1,JMAX
PHI(J)=O(J,1)*O(J,2)/O(J,5) $ PHIETA(J)=O(J,3)*O(J,2)/O(J,5)
6 PHIXI(J)=O(J,1)*O(J,4)/O(J,5) $ DO 7 J=1,JMAX
C DERIVATIVE OF POTENTIAL, INDEX=1
GO TO (11,12,13,14),INDEX
11 ANS(J)=R/XF*(XI*PHIXI(J)-ETA*PHIETA(J)) $ GO TO 7
C POTENTIAL, INDEX=2
12 ANS(J)=PHI(J) $ GO TO 7
C FOR HEMISPHERICAL PIT, INDEX=4
14 ANS(J)=(1.0-ETA**2)*PHIETA(J)+(1.0+ETA**2)*PHIXI(J) $ GO TO 7
C M2N(XI) INDEX 3
13 ANS(J)=O(J,2)/O(J,5)
7 CONTINUE $ RETURN $ FND

```


Summary

The series expansion in $y = \text{ctn}^{-1}(\xi)$ should converge for $y < \pi$ or $\xi > -\infty$, which covers every conceivable value of interest. The accuracy of this method is always excellent, but the calculation of the coefficients in the series is expensive.

The series expansion in $1/\xi$ is inexpensive and generally accurate except for values of ξ quite close to 1. The series expansion in ξ is also inexpensive. Its accuracy is deficient for values of ξ quite close to 1, and it also fails for large order n . An error in the sixth significant figure is detectable at $n = 20$ for $\xi = 0.5$ and at $n = 12$ for $\xi = 0.9$.

The recursion relation fails badly for large ξ and large n , but it is better than the power series in ξ for $\xi \leq 1$. An inexpensive procedure for moderate n should probably combine recursion for $|\xi| \leq 1.01$ with the series in $1/\xi$ for $\xi > 1.01$.

APPENDIX B: Computer Programs

The programs for the calculation of the primary resistances for the ring-disk system are recorded here. These include the main program RIDISK, the subroutines RING and ALLPOL, and the function EK. In addition, one needs the subroutines COEF and DISK, discussed and reproduced in appendix A.

The function EK calculates the complete elliptic integral of the first kind, $K(m)$.

Subroutine ALLPOL calculates both odd and even Legendre polynomials of argument X and stores them in the array P .

The subroutine RING calculates the values of the potential due to a ring current by evaluating the integral involving the elliptic integral:

$$\phi_o(r) = \frac{2}{\pi K} \int_{r_1}^{r_2} \frac{i(r')K(m)r' dr'}{r + r'}, \quad (B-1)$$

where

$$m = \frac{4rr'}{(r + r')^2}. \quad (B-2)$$

This is done for each case of ring current discussed in the text and is carried out for each value of r in the array RR (either on the ring or on the disk).

The main program must carry out the manipulations referred to in the text. First one reads two sets of abscissae and weight factors for Gaussian integration. The first set (X and W) establishes values of r at which potentials are calculated on both the ring and the disk. The second set (U and V in the main program and AP and V in subroutine RING) establishes the values of r' at which the current density on the ring is specified and the basis for the integration in equation B-1.

We used the values $IM = 32$ and $JM = 64$. More values are required for the accurate integration of equation B-1 than for the subsequent straightening out of the potential distributions on the ring and the disk.

For KM and NM we used values of 10 and 20. However, for some of the calculations, KM was increased to 20, as mentioned in the text.

For each problem, the computer reads the values of r_0/r_1 and r_1/r_2 . In the output, ZDD, ZDR, ZRD, and ZRR refer to values of $\kappa r_2 R_{dd}$, $\kappa r_2 R_{dr}$, $\kappa r_2 R_{rd}$, and $\kappa r_2 R_{rr}$. GAP refers to the resistance between the ring and the disk when no current flows to the counterelectrode at infinity, and ELECT refers to the resistance between a composite electrode (consisting of the ring and disk at the same potential) and a counterelectrode at infinity. Both of these resistances are multiplied by κr_2 to make them dimensionless. ZDRRD and ZRRR are auxiliary values related to the development of improved ways of presenting the dependence of R_{dr} and R_{rr} on r_0/r_1 and r_1/r_2 . See also appendix C.

```
PROGRAM RIDISK (INPUT,OUTPUT,TAPE1=OUTPUT,PUNCH)
DIMENSION X(32),W(32),R1(32),RR(32),PC(32,22),FR(32,22),
1AMPO(22),AMPR(22),PO(2),PRI(2),BD(20,22),PE(20),A2N(20,40),P( 79)
2,PO(20),C(20)
COMMON A0,CP(64,22),CR( 32,24),IM,JM,KM, U(64),V(64),PI,R1,R2,OO
100 FORMAT(5I5)
101 FCKMAT(4F15.1)
102 FORMAT (2F9.5,8F13.7)
103 FCKMAT(3F10.1)
104 FCKMAT(2F30.1)
105 FCKMAT (1M , 3X,*R0/R1*, 4X,*R1/R2*, 8X,*ZDC*,10X,*ZCR*,10X,*ZR
10*,10X,*ZKR*,10X,*GAP*,9X,*ELECT*/)
106 FCKMAT (6F12.3)
PI2=1.5707963267949 $ PI=2.0*PI2 $ A0=1.38629436112
READ 100,IM,IM,JM,KM $ NM=20
IHP1=IM+1 $ READ 104,(X(I),I=IHP1,IM) $ READ 104,(W(I),I=IHP1,IM)
JHP1=JM+1 $ READ 101,(U(J),J=JHP1,JM) $ READ 101,(V(J),J=JHP1,JM)
DO 3 I=1,IM $ IF(I.GT.IM) GO TO 1
IL=IM-1+1 $ X(I)=0.5-0.5*X(IL) $ W(I)=W(IL) $ GO TO 2
1 X(I)=0.5+0.5*X(I)
2 CALL ALLPOL(KM-2,2.0*X(I)-1.0,P) $ DO 3 K=2,KM
3 CR(I,K+2)=P(K-1) $ DO 6 J=1,JM $ IF(J.GT.JM) GO TO 4
JL=JM-J+1 $ U(J)=0.5-0.5*U(JL) $ V(J)=V(JL) $ GO TO 5
4 U(J)=0.5+0.5*U(J)
5 CALL ALLPOL(KM-4,2.0*U(J)-1.0,P) $ DO 6 K=4,KM
6 CP(J,K)=P(K-3) $ CALL ALLPOL(2*NM-2,0.0,P) $ DO 7 N=1,NM
7 PO (N)=P(2*N-1) $ CALL COEF(NM,40,A2N) $ PRINT 105 $ R2=1.0
99 READ 103, R01,R1 $ IF(R01.EQ.0.0) STOP $ R0=R01*R1 $ DO=1.0-R1
C****CALCULATION OF RADII AND ETA VALUES****
DO 8 I=1,IM $ RO(I)=R0*(1.0-X(I)**2)**0.5
8 RR(I)=R1+(R2-R1)*X(I)
C*****ASSIGN CURRENT VALUES TO THE RING*****
AMPR(4)=(R2-R1)*(R2+R1)*PI $ AMPR(2)=0.0 $ DO 9 K=6,KM
9 AMPR(K)=0.0 $ AMPR(5)=(R2-R1)**2*PI/3.0
AMPR(3)=(R2-R1)*(R2+2.0*R1)*PI/0.75 $ AMPR(1)=PI*AMPR(4)
C****CALCULATE POTENTIAL ON THE DISK AND RING*****
CALL RING(PO,RO) $ CALL RING(PR,RR)
C****CALCULATE THE B VALUES FOR DISK ONLY PROBLEM*****
DO 10 N=1,NM $ DO 10 K=1,KM
10 BD(N,K)=0.0 $ DO 11 I=1,IM $ CALL ALLPOL(2*NM-2,X(I),P)
DO 11 N=1,NM $ DO 11 K=1,KM
11 BU(N,K)=BU(N,K)-PO(I,K)*P(2*N-1)*W(I) $ DO 12 N=1,NM $ DO 12K=1,KM
12 BD(N,K)=BU(N,K)*(4*N-3)*0.5 $ BD(1,2)=1.0
C*****EVALUATE DISK CURRENTS*****
DO 13 K=1,KM
13 AMPO(K)=BD(1,K)*4.0*R0
C*****EVALUATE NEW RING POTENTIALS*****
DO 14 I=1,IM $ CALL DISK(PR(I)/R0,PE,A2N,NM)
DO 14 K=1,KM $ DO 14 N=1,NM
14 PR(I,K)=PR(I,K)+UD(N,K)*PE(N)*PO(N)
C****STRAIGHTEN THE RING POTENTIALS*****
DO 17 N=3,KM $ DO 15 K=1,KM $ C(K)=0.0 $ DO 15 I=1,IM
15 C(K)=C(K)+CR(I,N+2)*FR(I,K)*W(I)
DO 17 K=1,KM $ IF(K.EQ.N) GO TO 17 $ F=C(K)/C(N) $ DO 16 I=1,IM
16 PR(I,K)=PR(I,K)-F*PR(I,N) $ AMPO(K)=AMPO(K)-F*AMPO(N)
AMPR(K)=AMPR(K)-F*AMPR(N)
```

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```

17 CONTINUE
C*****CALCULATE AVERAGE POTENTIALS*****
DO 18 K=1,2 $ PRI(K)=0.0 $ PUI(K)=0.0 $ DO 18 I=1,IM
18 PRI(K)=FRI(K)+PR(I,K)*W(I)
PRI(1)=0.5*PRI(1) $ FRI(2)=0.5*PRI(2) $ POI(2)=1.0
C*****CALCULATE THE RESISTANCE RELATIONSHIPS*****
DET=AMFR(1)*AMPD(2)-AMFR(2)*AMPD(1)
ZRR=(PRI(1)*AMPD(2)-PRI(2)*AMPD(1))/DET
ZRD=(AMFR(1)*RI(2)-AMFR(2)*RI(1))/DET
ZDR=(POI(1)*AMPD(2)-POI(2)*AMPD(1))/DET
ZDC=(AMFR(1)*PI(2)-AMFR(2)*PI(1))/DET
DET2=PRI(1)*POI(2)-PRI(2)*POI(1)
GAP=DET2*2.0/(ZDR+ZRC)/DET $ ELECT=DET2/(ZRR+ZCC-ZDR-ZRC)/DET
RSTAR=R1*(3.0-R1)/2.0+DD**2/PI2
ZDRRC=(ZDR+ZRC)*PI*R0/R2/ASIN(R0/RSTAR)
ZRRR =ZRR+0.5*(R1/PI)**2*ALOG(DD*(139.-107.*SQRT(1.-R1**2))/32.)
R01=R0/R1 $ PRINT 102, R01,R1,ZDD,ZDR,ZRD,ZRR,GAP,ELECT,ZDRRC,ZRRR
PUNCH 106, R01,R1,ZDD,ZDR,ZRD,ZRR
GO TO 99 $ END

```

```

SUBROUTINE ALLPOL(NMAX,X,P) $ DIMENSION P(79)
C * * * CALCULATION OF LEGENDRE POLYNOMIALS * * * * *
P(1)=1.0 $ P(2)=X $ IF(NMAX.LE.1) RETURN $ DO 1 N=2,NMAX
1 P(N+1)=((2*N-1)*X*P(N)-(N-1)*P(N-1))/N $ RETURN $ END

```

```

FUNCTION EK(EM)
EM1=1.0-EM $ EK=-50.0 $ IF(EM1.LE.0.0) RETURN
EK=1.38629436112+EM1*(0.09666344259+EM1*(0.03590092383+EM1*(
10.03742563713+EM1*(0.01451196212))))-(0.5+EM1*(0.12498593597+EM1*(
20.06880248576+EM1*(0.03328355346+EM1*(0.00441787012)))))*ALOG(EM1)
RETURN $ END

```

```

SUBROUTINE RING(PCT,RR) & DIMENSION PCT(32,22),RR(32),S2(64)
C****SUBROUTINE CALCULATES VALUES OF THE POTENTIAL DUE TO A RING****
C****CURRENT BY EVALUATING THE INTEGRAL INVOLVING THE ELLIPTIC INTEGRAL
COMMON AO,CP(64,22),CR( 32,24),IM,JM,KM,AP(64),V(64),PI,R1,R2,CD
DO 38 I=1,IM & A=(RR(I)-R1)/(R2-R1) & G=A+R1/DC & DO 1 K=1,KM
1 POT(I,K)=0.0 & IF(A.GE.0.0 .AND. A.LE.1.0) GO TO 7
EL1=EK(1.0-(A/(2.*G-A))**2)*R1/(DD*A+2.*R1)
EL2=EK(1.0-((1.0-A)/(2.*G+1.0-A))**2)*R2/DD/(2.*G+1.-A)
DO 5 J=1,JM & Y=AP(J)-A & H=(Y/(Y+2.*G))**2 & EL =EK(1.0-H)
IF(AP(J).EQ.0.0) GO TO 2
POT(I,3)=POT(I,3)+V(J)*((Y+G)*EL /((Y+2.*G)-EL1)/AP(J)**0.5
IF(AP(J).NE.1.0) GO TO 3 & S3=-EL1 & GO TO 4
2 S3=-EL2 & GO TO 4
3 S3=(Y+G)*EL /((Y+2.*G)/(AP(J)*(1.0-AP(J))))**0.5
1-EL1/ AP(J)**0.5-EL2/(1.0-AP(J))**0.5
4 POT(I,1)=POT(I,1)+S3*V(J) & DO 5 K=4,KM
5 POT(I,K)=POT(I,K)+(Y+G)*EL *CP(J,K)/(Y+2.*G)*V(J) & DC 6 K=4,KM
6 POT(I,K)=POT(I,K)*DD/PI & POT(I,1)=DD*(POT(I,1)+4.*(EL1+EL2))/PI
POT(I,3)=DD*(POT(I,3)+4.*EL1)/PI & GO TO 38
7 SUM=(AO+ALOG(2.*G))/2. & S3=0.0 & S5=0.0 & H=(A/(2.*G-A))**2
IF(A.NE.0.0) S5=(G-A)*EK(1.0-H)/(2.0*G-A)+ALCG(A**2)/4.-SUM
S4=0.0 & H=((1.0-A)/(1.0-A+2.*G))**2
IF(A.NE.1.0) S4=((1.0-A+G)*EK(1.0-H)/(1.0-A+2.*G)+ALOG((1.-A)**2)
1/4.-SUM
DO 9 J=1,JM & Y=AP(J)-A & H=(Y/(Y+2.*G))**2 & S2(J)=0.0
IF(Y.EQ.0.0) GO TO 9 & EL =EK(1.0-H) & DO 8 K=4,KM
8 POT(I,K)=POT(I,K)+(Y+G)*EL *CP(J,K)/(Y+2.*G)*V(J)
S2(J)=(Y+G)*EL /((Y+2.*G)+ALOG(Y**2))/4.-SUM
S3=S3+(G/(Y+2.*G))**2*ALOG(H)*V(J)
9 CONTINUE & S3=S3*CD/PI & ADD=0.0 & AS=A/(A+2.*R1/DC)
IF(AS.GT.0.0) ADD=AS*(ALOG(AS)-1.0) & AT=(1.-A)/(1.+A+2.*R1/DC)
IF(AT.GT.0.0) ADD=ADD+AT*(ALOG(AT)-1.)
S3=S3-ADD*2.*(DC*A+R1)/PI & DO 10 K=4,KM
10 POT(I,K)=POT(I,K)*DD/PI +S3*CR(I,K)
DO 13 J=1,JM & IF(AP(J).EQ.0.0) GO TO 11
POT(I,3)=POT(I,3)+V(J)*(S2(J)-S5)/AP(J)**0.5
IF(AP(J).NE.1.0) GO TO 12 & S1=-S5 & GO TO 13
11 S1=-S4 & GO TO 13
12 S1=S2(J)/(AP(J)*(1.-AP(J)))*0.5-S5/AP(J)**0.5-S4/(1.-AP(J))**0.5
13 POT(I,1)=POT(I,1)+S1*V(J)
POT(I,1)=DD*(POT(I,1)+4.*(S5+S4))/PI+DD*(AO+3.*ALOG(2.)+ALCG(G))
PL=FC(I,3)
VA=A**0.5 & PL=DD*(PL+4.*(S5+SUM))/PI
IF(A.EC.0.0) GO TO 35 & IF(A.EQ.1.0) GO TO 36
PL=PL-2.0*DD*(ALOG(1.0-A)-2.0+VA*ALOG((1.0+VA)/(1.0-VA)))/PI
GO TO 37
35 PL=PL+ 4.0*DD/PI & GO TO 37
36 PL=PL-2.0*DD*(ALOG(2.)-2.0)/PI
37 POT(I,3)=PL
38 CONTINUE & RETURN & END

```

APPENDIX C: Computed Results

Computed results were punched on cards to permit correlation without repeating the expense of the initial calculation. These results are printed in the following table. Rather than print ZRD, we have given the percent deviation DEV from the value of ZDR. Values indicated by a check were obtained with $KM = 20$.

Following the table of results is a simple program for testing correlation formulas. Some of these correlations have been discussed in the text.

R01	R1	ZDD	ZDR	ZPR	DEV
.10	.99	2.52524612	.16021369	.41066874	-.0000
.10	.98	2.55099439	.16100423	.37699795	-.0000
.10	.95	2.63140872	.16337595	.33430296	-.0000
.10	.90	2.77704159	.16735141	.30449557	-.0000
.10	.80	3.12152515	.17545410	.27851527	-.0000
.10	.60	4.14628138	.19246175	.25904760	.0001
.10	.40	6.17430763	.21065415	.25234139	-.0000
.10	.20	12.20380746	.22994821	.25027332	-.0105
.10	.10	24.21275001	.23990154	.25003120	-.0333
.10	.05	48.20770664	.24494691	.25000338	-.0113 ✓
.50	.99	.50504207	.16741149	.41057039	-.0000
.50	.98	.51016995	.16815326	.37690203	-.0000
.50	.95	.52609461	.17037284	.33421420	-.0000
.50	.90	.55461332	.17407914	.30441827	-.0000
.50	.80	.62067639	.18159675	.27845851	0.
.50	.60	.80920122	.19728337	.25402189	.0002
.50	.40	1.16411683	.21398755	.25233381	-.0001
.50	.20	2.17595689	.23165690	.25027227	-.0116
.50	.10	4.15232216	.24076067	.25003106	-.0131
.50	.05	8.08442934	.24537808	.25000337	-.0133 ✓
.80	.99	.31563925	.18499500	.40835158	-.0000
.80	.98	.31880840	.18551244	.37476850	-.0000
.80	.95	.32851615	.18707605	.33231781	-.0000
.80	.90	.34548642	.18974933	.30286191	-.0000
.80	.80	.38336965	.19536460	.27742238	.0001
.80	.60	.48554675	.20757427	.25860997	.0004
.80	.40	.66768596	.22091291	.25221444	-.0005
.80	.20	1.16640456	.23516019	.25025734	-.0074
.80	.10	2.12520187	.24251361	.25002901	.0504
.80	.05	4.02249594	.24625902	.25000313	-.0123 ✓
.90	.99	.28055157	.19818490	.40310695	-.0000
.90	.98	.28332150	.19837673	.36984528	-.0000
.90	.95	.29165884	.19908958	.32820587	-.0000
.90	.90	.30586956	.20062551	.29974717	.0000
.90	.80	.33668813	.20452458	.27556338	.0002
.90	.60	.41720082	.21417307	.25795673	.0006
.90	.40	.55719472	.22529443	.25203701	-.0000
.90	.20	.93454220	.23736223	.25023564	.0002
.90	.10	1.65899944	.24360448	.25002588	.1406
.90	.05	3.07935040	.24681233	.25000278	.0009 ✓
.95	.99	.26575637	.20957384	.39406724	-.0000 ✓
.95	.98	.26830384	.20930569	.36166491	-.0000 ✓
.95	.95	.27578912	.20897303	.32190950	0. ✓
.95	.90	.28920875	.20934505	.29536748	.0000 ✓
.95	.80	.31454085	.21174759	.27318007	.0001 ✓
.95	.60	.38200135	.21936287	.25718642	.0003 ✓
.95	.40	.49770481	.22974950	.25187612	.0004 ✓
.95	.20	.80683911	.23911740	.25021335	-.0008 ✓
.95	.10	1.39477209	.24452091	.25002557	-.0003 ✓
.95	.05	2.55721873	.24724652	.25000236	.0268 ✓
.98	.99	.25754844	.22112912	.37728800	-.0000 ✓
.98	.98	.25985586	.22014364	.34739798	-.0000 ✓
.98	.95	.26640597	.21850522	.31205857	.0000 ✓
.98	.90	.27698941	.21770326	.28907230	.0001 ✓
.98	.80	.29908594	.21874622	.26999061	.0002 ✓
.98	.60	.35502573	.22449037	.25620834	.0003 ✓
.98	.40	.45002018	.23219805	.25158612	.0001 ✓
.98	.20	.70214304	.24086449	.25018408	.0005 ✓
.98	.10	1.18232443	.24538951	.25002187	.0130 ✓
.98	.05	2.14466423	.24766471	.25000182	.0717 ✓

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```

PROGRAM DATA(INPUT,OUTPUT,TAPE1=OUTPUT)
100 FORMAT (6X,*R01*,4X,*R1*,6X,*ZDD*,9X,*ZDR*,9X,*ZRR*,7X,*DEV*,13X,*
1ZD*,8X,*ZDRPD*,9X,*ZD*)
101 FORMAT (3X,2F6.2,3F12.8,F8.4,5X,3F12.8)
102 FORMAT (1H )
106 FORMAT (6F12.8)
PRINT 100
PI=3.141592654
98 PRINT 102
99 READ 105, R01,R1,ZDD,ZDR,ZRD,ZRR $ IF(R01.EQ.0.0) GO TO 99
IF(R01.LT.0.0) STOP
R0=R01*R1
DEV=100.0*(ZRD/ZDR-1.0)
ZD =1.0/ZDD/R0+2.0*(1.0-R1)**2*ALOG(1.0-R01)
RSTAR=R1*(3.-R1)/2.+(1.-R1)**2/PI*2.
ZDRPD=2.0*ZDR*PI*R0/ASIN(R0/RSTAR)
ZR =ZRR+0.5/PI**2*ALOG(1.0-R1**3)
PRINT 101, R01,R1,ZDD,ZDR,ZRR,DEV,ZD,ZDRPD,ZR
GO TO 99 $ END

```

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