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ABSTRACT

The significance of inertial frames and the 4-space-dependent Lorentz transformations among them in the mathematics of general relativity is expounded here in textbook style. The laws governing the components of affine connections on a frame are presented, and the Christoffel symbols are deduced from a viewpoint congenial to the notation of frame components. The parallel displacement of spinors is discussed, and is compared with the analogous notion for vectors.

It is owing to the recent interest in generalized gauge-invariance notions, in the sense of, e. g., Yang and Mills, that these notes, reproduced from notes circulated privately by the author last year, are now distributed. Indeed, such notions of gauge invariance, which stem historically from Einstein's general theory, are most evidently related to that theory in the notation which emphasizes the rôle of inertial frames.

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CONTENTS

Frames and Lorentz Invariance in General Relativity

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I. Introduction

One is usually introduced to general relativity by the remark that the 6-parameter family of inertial frames at a point should be determined dynamically. One soon introduces curvilinear coördinates in 4-space, and replaces the concept of 6-parameter family of inertial frames by a symmetric metric tensor, $g_{\mu\nu}$. By means of any of a 6-parameter family of linear coördinate transformations, one can bring $g_{\mu\nu}(x)$ to the form

$$\eta_{(\mu\nu)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{1}$$

the Lorentz metric,[1] at any given point x, in which case the vectors $f^{(\alpha)}_{\mu}(x)$ which constitute an inertial frame at x may be chosen to be δ^{α}_{μ}. The symbol $f^{(\alpha)}_{\mu}(x)$ designates the μth component of the covariant vector $f^{(\alpha)}(x)$, so that the parenthesized index (α) is a label which plays no role in coördinate transformations. Thus, the inertial frames can be recovered from the metric tensor.

It is only when half-odd-integral spin appeared in physics that the frames were discussed at length, under the name of vierbeine ("fourlegs"), or anholonomic reference systems.[2] Nevertheless, the whole of general relativity may profitably be discussed from the standpoint of local inertial frames. The main advantage of the frames in discussions not involving half-odd-integral spin is the sharp separation of the physical requirement of Lorentz covariance from the purely conventional concomitant of the use of curvilinear coördinates: to wit, general covariance.

The development of the theory of parallel displacement without half-odd-integral spin, involving a derivation of the Christoffel symbols from ideas referent directly to the frames, occupies Section II. The results may be found here and there in the references (1); here they are presented in a less abbreviated form, and apart from the problems of unified field theory.

Section III presents the theory of parallel displacement of a 2-spinor. Since the close historical relation between spinors and inertial frames may appear puzzling, it may be appropriate to remark on this here.

Local fields are required to transform as representations of the proper homogeneous Lorentz group--the group of transitions between inertial frames at a point. It happens that the tensors defined with respect to a totally different group: the group of general linear transformations induced on the space of vectors at a point by the group of transitions between different curvilinear coördinate systems, may be pressed into service as quantities under integral-spin Lorentz transformations. Different tensors correspond to essentially the same Lorentz-transformation quantity (the same, if Minkowski notation is used), but this awkwardness is resolved through the use of $g_{\mu\nu}$ and its inverse, $g^{\mu\nu}$, which by lowering and raising indices interconvert different tensors belonging to essentially the same Lorentz-group quantity. But there are no general linear group tensors to correspond to the half-odd-integral spin representations. In fact, any continuous function defined on the general linear group must be single-valued because that group is simply connected; all representations are therefore single-valued, and therefore restrict to single-valued representations of the Lorentz group. When these are reduced, none of the double-valued half-odd-integral spin representations may appear. Spinors, then, essentially require a quadratic form, a set of orthogonal frames, or other notion of orthogonality, for their very definition; they belong more directly to the Lorentz group than the integral-spin representation, which may be obtained first from a different group, with the notion of orthogonality brought in afterwards only to relate covariant and contravariant quantities.[3]

II. Skew Frames, Inertial Frames, and Parallel Displacement of Tensors

1. Skew Frames

The usual tensor calculus is adopted. A skew frame at each point of a manifold is specified in a curvilinear coördinate system by the functions

$$f_{\mu}^{(\alpha)}(x), \tag{2}$$

restricted only by the usual regularity conditions, and that the $f^{(\alpha)}$ be a complete system of linearly independent vectors at x. The skew frame is then given in all coördinate systems by the condition that the $f^{(\alpha)}$ be covariant vectors:

$$\bar{f}_{\mu}^{(\alpha)}(\bar{x}) = \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}} f_{\nu}^{(\alpha)}(x).$$

The field of reciprocal skew frames $f_{(\alpha)}^{\mu}(x)$ is defined as the matrix inverse of (2). Thus,

$$f_{\mu}^{(\alpha)}(x)\ f_{(\beta)}^{\mu}(x) = \delta_{\beta}^{\alpha}, \tag{3a}$$

$$f_{\mu}^{(\alpha)}(x)\ f_{(\alpha)}^{\nu}(x) = \delta_{\mu}^{\nu}. \tag{3b}$$

Equation (3a) implies that the $f_{(\alpha)}^{\mu}(x)$, for fixed (α), transform as the components of a contravariant vector field.

The frames allow us to reduce an arbitrary tensor to an equivalent collection of scalars:

$$\begin{aligned} T_{\mu\nu}{}^{\varpi}(x) \rightarrow T_{(\alpha\beta)}{}^{(\gamma)}(x) &= f_{(\alpha)}^{\mu}(x)\ f_{(\beta)}^{\nu}(x)\ f_{\varpi}^{(\gamma)}(x)\ T_{\mu\nu}{}^{\varpi}(x), \\ T_{\mu\nu}{}^{\varpi}(x) &= f_{\mu}^{(\alpha)}(x)\ f_{\nu}^{(\beta)}(x)\ f_{(\gamma)}^{\varpi}(x)\ T_{(\alpha\beta)}{}^{(\gamma)}(x). \end{aligned} \tag{4}$$

These parenthesized components have the geometrical meaning of components on a frame, or in the case of vectors, of the dot products with the frame vectors.[4]

2. Frame Transformations

The introduction of new skew frames $\bar{f}$ to replace old skew frames f defines a linear transformation at each point x:

$$\bar{f}^{(\alpha)}_{\mu}(x) = L^{(\alpha)}{}_{(\beta)}(x)\ f^{(\beta)}_{\mu}(x), \tag{5a}$$

$$\bar{f}^{\mu}_{(\alpha)}(x) = L_{(\alpha)}{}^{(\beta)}(x)\ f^{\mu}_{(\beta)}(x). \tag{5b}$$

The requirement that $\bar{f}^{\mu}_{(\alpha)}$ be reciprocal to $\bar{f}^{(\alpha)}_{\mu}$ is equivalent to the requirement that the two L's of (5a), (5b) be reciprocal matrices in the sense:

$$L^{(\alpha)}{}_{(\beta)}(x)\ L_{(\gamma)}{}^{(\beta)}(x) = \delta^{\alpha}_{\gamma}, \tag{6a}$$

$$L^{(\alpha)}{}_{(\beta)}(x)\ L_{(\alpha)}{}^{(\gamma)}(x) = \delta^{\gamma}_{\beta}. \tag{6b}$$

Note that no change of curvilinear coördinates is involved in a frame transformation. The same frame transformation in different coördinates x' is given by $L'^{(\alpha)}{}_{(\beta)}(x') = L^{(\alpha)}{}_{(\beta)}(x)$, in the sense that $f \rightarrow f' \rightarrow \overline{f'}$ and $f \rightarrow \bar{f} \rightarrow \bar{f}'$ then agree; $\bar{f}' = \overline{f'}$.

If a tensor has all or some ordinary indices converted to parenthesized indices, then it will frame-transform via factors of L, as in (5a) or (5b), owing to the factors f contained in its expression (4) in terms of unparenthesized components.

The trivial derivative (7) corresponds to (12) with $c = 0$, which is not frame-invariant; see (10).

How do the ordinary components of a covariant vector parallelly displace?

$$v_{\|\mu}(x, \delta) \doteq f_{\mu}^{(\alpha)}(x + \delta)\ v_{\|(\alpha)}(x, \delta); \tag{13}$$

$$v_{\|\mu}(x, \delta) = v_{\mu}(x) + c^{\nu}_{\mu\varpi}(x)\ v_{\nu}(x)\ \delta^{\varpi} + \frac{\partial f_{\mu}^{(\alpha)}}{\partial x^{\varpi}}(x)\ f^{\nu}_{(\alpha)}(x)\ v_{\nu}(x)\ \delta^{\varpi};$$

or

$$v_{\|\mu}(x, \delta) = v_{\mu}(x) + \Gamma^{\nu}_{\mu\varpi}(x)\ v_{\nu}(x)\ \delta^{\varpi}, \tag{14a}$$

with

$$\Gamma^{\nu}_{\mu\varpi} = c^{\nu}_{\mu\varpi} + f^{\nu}_{(\alpha)}\ \partial f_{\mu}^{(\alpha)}/\partial x^{\varpi}. \tag{14b}$$

The trivial non-frame-invariant derivative (7) corresponds, as we have already noted, to $c = 0$; the ordinary affine connection corresponding to this trivial derivative is the second term of (14b), which may also be seen directly by writing out (7) in terms of the v_{ν}. The first term $c^{\nu}_{\mu\varpi} \equiv f^{\nu}_{(\beta)}\ f_{\mu}^{(\alpha)}\ f_{\varpi}^{(\gamma)}\ c^{(\beta)}_{(\alpha\gamma)}$ of (14b), in general, transforms as an ordinary tensor under coördinate transformations. Thus, in general, the curvilinear-coördinate affine connection character resides in the second term of (14b), whereas the peculiar frame-transformation character (10) of c which corresponds to frame-invariant differentiations is cancelled in (14b) by the frame dependence of the second term. By transposing that term, the peculiar coördinate-transformation character appears cancelled on the left-hand side, and the peculiar frame-transformation character, instead, comes into evidence.

From (13) it follows that

$$v_{\mu}(x + \delta) - v_{\|\mu}(x, \delta) \doteq f_{\mu}^{(\alpha)}(x + \delta)(v_{(\alpha)}(x + \delta) - v_{\|(\alpha)}(x, \delta)),$$

or

$$v_{\mu,\nu}\ \delta^{\nu} \doteq f_{\mu}^{(\alpha)}(x+\delta)\ v_{(\alpha,\beta)}\ \delta^{(\beta)} \doteq f_{\mu}^{(\alpha)}(x)\ v_{(\alpha,\beta)}\ \delta^{(\beta)};$$
$$v_{\mu,\nu}\ \delta^{\nu} = f_{\mu}^{(\alpha)}\ v_{(\alpha,\beta)}\ f_{\nu}^{(\beta)}\ \delta^{\nu},$$

so that we have simply

$$v_{\mu,\nu} = f_{\mu}^{(\alpha)}\ f_{\nu}^{(\beta)}\ v_{(\alpha,\beta)},$$
$$v_{(\alpha,\beta)} = f_{(\alpha)}^{\mu}\ f_{(\beta)}^{\nu}\ v_{\mu,\nu}; \tag{15}$$

in words, the requirement that the two covariant derivatives be the parenthesized and the ordinary components of the same quantity coincides with the natural correspondence (13) of parallel displacements, or, equivalently, with the relation (14b).

In a similar way, one can discuss parallel displacement of a contravariant vector, starting with the formula

$$v_{\parallel}^{(\alpha)}(x,\ \delta) = v^{(\alpha)}(x) - \hat{c}^{(\alpha)}{}_{(\beta\gamma)}(x)\ v^{(\beta)}(x)\ \delta^{(\gamma)}, \tag{16}$$

which leads to the same conditions on the $\hat{c}^{(\alpha)}{}_{(\beta\gamma)}$ that were derived for the c: namely, that they be invariant to coördinate transformations and that they transform according to the law (10) under change of skew frames.

If one now applies the usual rule for displacing a product to the scalar $v_{\mu}(x)\ w^{\mu}(x) = v_{(\alpha)}(x)\ w^{(\alpha)}(x)$, one finds that

$$(v_{(\alpha)}\ w^{(\alpha)})_{\parallel}(x,\ \delta) \doteq v_{\parallel(\alpha)}(x,\ \delta)\ w_{\parallel}^{(\alpha)}(x,\ \delta) \doteq v_{(\alpha)}(x)\ w^{(\alpha)}(x)$$
$$+\ v_{(\beta)}(x)\ w^{(\alpha)}(x)\ (c^{(\beta)}{}_{(\alpha\gamma)}(x) - \hat{c}^{(\beta)}{}_{(\alpha\gamma)}(x))\ \delta^{(\gamma)}.$$

If the parallelly displaced scalar is to be computable from the original scalar, without reference to the individual components of v and w, then clearly $c^{(\beta)}{}_{(\alpha\gamma)}(x) - \hat{c}^{(\beta)}{}_{(\alpha\gamma)}(x) = \delta_{\alpha}^{\beta}\ u_{(\gamma)}(x)$, under which circumstance the covariant derivative of a scalar s would be given by $s_{,(\alpha)} = \partial_{(\alpha)}s - u_{(\alpha)}$, where, as in (11),

$$\partial_{(\alpha)} \equiv f_{(\alpha)}^{\mu}(x)\ \frac{\partial}{\partial x^{\mu}}, \tag{17}$$

with $u_{(\alpha)}$ the parenthesized components of a covariant vector field, which corresponds to H. Weyl's pre-spinor gauge invariance.

If one, however, wishes to get along with only one connection, and not two—c and $\hat{c}$—or more, one can put the connections for differentiation of covariant and contravariant vectors equal; $c = \hat{c}$, and have a scalar be unchanged by parallel displacement; $u_{(\alpha)} = 0$. Thus, the scalar product of coveriant and contravariant vectors is defined independently of any notion of orthogonality or metric, and the most economical use of an arbitrary connection $c^{(\alpha)}{}_{(\beta\gamma)}(x)$ leads to invariance of this scalar product under parallel displacement.

In the same spirit, one can use c and the usual product rule to displace and differentiate arbitrary tensors which are products of vectors, and then transfer the derived laws to arbitrary tensors.

The coefficients c are easily annulled at a point, by choosing the skew frames in the neighborhood of the point appropriately. In fact, set up new skew frames in the neighborhood of x by parallelly displacing the frame at x to the new points -- either along curves linear in the x^{μ}, or, e. g., along continuously self-parallel curves, or "geodesics." If a vector be parallelly displaced with the frame, the inner products of the vector with the frame vectors of opposite variance remain invariant. But these inner products are precisely the parenthesized components of the parallelly displaced vector, whence from (8), we see that, on these particular frames, $c^{(\beta)}{}_{(\alpha\gamma)}(x) = 0$, if x is the initial point of the construction.

We see this more formally as follows. Displace the frame vector $f^{(\delta)}$ parallelly, where now (δ) is an inactive label:

$$f_{\parallel(\alpha)}^{(\delta)}(x,\ \delta) = f_{(\alpha)}^{(\delta)}(x) + c^{(\beta)}{}_{(\alpha\gamma)}(x)\ f_{(\beta)}^{(\delta)}(x)\ \delta^{(\gamma)};$$

$$f_{\parallel(\alpha)}^{(\delta)}(x,\ \delta) = \delta_{\alpha}^{\delta} + c^{(\delta)}{}_{(\alpha\gamma)}(x)\ \delta^{(\gamma)}, \tag{18a}$$

since $f_{(\alpha)}^{(\delta)}(x) = f_{(\alpha)}^{\mu}(x)\ f_{\mu}^{(\delta)}(x) = \delta_{\alpha}^{\delta}$. If the result, $f_{\parallel}$, is to be our frame at $x + \delta$, to first order in δ, then it also must reduce to δ_{α}^{δ}, so that $c^{(\delta)}{}_{(\alpha\gamma)}(x) = 0$. On the other hand, if we keep the a priori frames, so that c does not necessarily vanish at x, and $f_{\parallel}$ is not regarded as the frame at $x + \delta$, we may replace δ_{α}^{δ} in (18a) by $f_{(\alpha)}^{(\delta)}(x + \delta)$, and write (18a) in the form

$$f_{(\alpha)}^{(\delta)}(x + \delta) - f_{\parallel(\alpha)}^{(\delta)}(x,\ \delta) = -c^{(\delta)}{}_{(\alpha\gamma)}(x)\ \delta^{(\gamma)}, \tag{18b}$$

in which form (18) represents the c as yielding the change of frame referred to its own notion of parallel displacement, where (α) on the left-hand side indicates the computation of components on the a priori frame at $x + \delta$, a situation which may be clarified by introducing some ordinary indices:

$$f_{\mu}^{(\delta)}(x + \delta) - f_{\parallel\mu}^{(\delta)}(x,\ \delta) = -c^{(\delta)}{}_{\mu\nu}(x)\ \delta^{\nu}. \tag{18c}$$

4. Lorentz Transformations and Inertial Frames

The Lorentz metric, $\eta_{(\alpha\beta)} = \eta^{(\alpha\beta)}$ of eq. (1) is now introduced to form inner products of vectors of the same variance, and to raise and lower parenthesized indices. Thus, one can pass from ordinary covariant components v_μ of a vector to contravariant components by first going to parenthesized indices, then raising with $\eta^{(\alpha\beta)}$, and finally passing back to ordinary indices: $v_\mu \rightarrow v_{(\alpha)} = f^{\mu}_{(\alpha)} v_\mu$; $v_{(\alpha)} \rightarrow v^{(\beta)} = \eta^{(\beta\alpha)} v_{(\alpha)}$; $v^{(\beta)} \rightarrow v^\nu = f^{\nu}_{(\beta)} v^{(\beta)}$; or

$$v^\nu(x) = f^{\nu}_{(\beta)}(x)\, \eta^{(\beta\alpha)}\, f^{\mu}_{(\alpha)}(x)\, v_\mu(x). \qquad (19a)$$

Inversely,

$$v_\mu(x) = f^{(\alpha)}_{\mu}(x)\, \eta_{(\alpha\beta)}\, f^{(\beta)}_{\nu}(x)\, v^\nu(x), \qquad (19b)$$

which follows from

$$\eta_{(\alpha\beta)}\eta^{(\beta\gamma)} = \delta^{\gamma}_{\alpha} \qquad (20)$$

and from (3).

In general, the v^ν deduced from v_μ according to (19a) will depend on the $f^\mu_{(\alpha)}(x)$, as well as on the $v_\mu(x)$, and will vary with an arbitrary change of skew frames. Those frame transformations $L_{(\alpha)}{}^{(\beta)}(x)$ which leave the complete raising operator

$$g^{\nu\mu}(x) \equiv f^\nu_{(\beta)}(x)\,\eta^{(\beta\alpha)}\;f^\mu_{(\alpha)}(x) \tag{21}$$

invariant are known as <u>Lorentz transformations</u>. As usual, restrictions of continuity and differentiability will also be imposed.[5]

Note that

$$g^{\mu\nu}(x) = g^{\nu\mu}(x),$$

and that

$$g_{\mu\nu}(x) \equiv f_\mu^{(\alpha)}(x)\,\eta_{(\alpha\beta)}\;f_\nu^{(\beta)}(x)$$

is the matrix reciprocal to $g^{\mu\nu}(x)$. Thus, the formalism of frames and parenthesized indices extends the usual formalism.

By writing out the condition

$$\bar{f}^\mu_{(\alpha)}(x)\,\eta^{(\alpha\beta)}\;\bar{f}^\nu_{(\beta)}(x) = f^\mu_{(\alpha)}(x)\,\eta^{(\alpha\beta)}\;f^\nu_{(\beta)}(x),$$

with

$$\bar{f}^\mu_{(\alpha)}(x) = L_{(\alpha)}{}^{(\beta)}(x)\;f^\mu_{(\beta)}(x),$$

the condition that $L_{(\alpha)}{}^{(\beta)}(x)$ be a Lorentz transformation is easily seen to be equivalent to

$$L^{(\alpha)}{}_{(\gamma)}(x)\,\eta_{(\alpha\beta)}\;L^{(\beta)}{}_{(\delta)}(x) = \eta_{(\gamma\delta)}. \tag{22a}$$

This may be written

$$L^{(\alpha)}{}_{(\gamma)}(x)\;\left(\eta_{(\alpha\beta)}\,L^{(\beta)}{}_{(\delta)}(x)\,\eta^{(\delta\epsilon)}\right) = \delta^\epsilon_\gamma,$$

which may be expressed with the aid of (6) as

$$L_{(\alpha)}{}^{(\epsilon)}(x) = \eta_{(\alpha\beta)}\;L^{(\beta)}{}_{(\delta)}(x)\,\eta^{(\delta\epsilon)}, \tag{22b}$$

a form which possesses the following happy verbal interpretation: The use of η's to raise and lower parenthesized indices may be extended to Lorentz transformations, without conflicting with the notation (6) for the reciprocal matrix. It will be convenient to adopt the new symbols $L_{(\alpha\delta)}(x) = \eta_{(\alpha\beta)}\, L^{(\beta)}{}_{(\delta)}(x)$, and $L^{(\beta\epsilon)}(x) = L^{(\beta)}{}_{(\delta)}(x)\, \eta^{(\delta\epsilon)}$, in conformity with such free extension of the use of η's.

Except for the extra variables x^{μ}, which have not yet entered essentially into the discussion, the Lorentz transformations here are precisely those of special relativity.

A notion of orthogonality, in the sense of the metric η, is specified by declaring an arbitrary system of frames, $f_{\mu}^{(\alpha)}(x)$, to be an <u>inertial system of frames</u>, for clearly, once one has chosen a complete independent set of vectors at each point to be generalized unit vectors in the sense of a metric η, and to be mutually orthogonal, one has defined the inner product of arbitrary vectors of similar variance. The expression of this notion of orthogonality in the language of unparenthesized indices involves the use of (21) as a metric tensor.

The <u>inertial frames</u> are all the frames equivalent to the original arbitrarily chosen inertial system of frames, in the sense that they yield the same notion of orthogonality; i. e., the same $g^{\mu\nu}(x)$; they are thus the frames which may be obtained from an arbitrary inertial system of frames by Lorentz transformations.

5. Antisymmetric Connections

We have already seen that the inner product of a covariant and a contravariant vector is invariant to parallel displacement, if we use the same connection for displacement of the two vectors. However, with the aid of η, we may write inner products of vectors of the same variance, which inner products are invariant to Lorentz frame transformations. By requiring that such inner products remain invariant to parallel displacements, or, equivalently, that $\eta_{(\alpha\beta)}$ or $\eta^{(\alpha\beta)}$ displace trivially, we obtain a restriction on the coefficients c, which involves the $\eta^{(\alpha\beta)}$, and which may be written as an antisymmetry property, if advantage is taken of the notation of raising indices. Thus,

$$\eta^{(\alpha\beta)} v_{\parallel(\alpha)} w_{\parallel(\beta)} = \eta^{(\alpha\beta)} v_{(\alpha)} w_{(\beta)} \iff$$
$$\eta^{(\alpha\beta)} (v_{(\alpha)} c^{(\epsilon)}{}_{(\beta\xi)} w_{(\epsilon)} \delta^{(\xi)} + c^{(\gamma)}{}_{(\alpha\delta)} v_{(\gamma)} \delta^{(\delta)} w_{(\beta)}) = 0 \iff$$
$$\eta^{(\alpha\beta)} c^{(\epsilon)}{}_{(\beta\xi)} + \eta^{(\beta\epsilon)} c^{(\alpha)}{}_{(\beta\xi)} = 0 \iff$$
$$c^{(\epsilon\alpha)}{}_{(\xi)} + c^{(\alpha\epsilon)}{}_{(\xi)} = 0. \tag{23}$$

Connections satisfying (23) will henceforth be termed antisymmetric for the sake of brevity; no restriction of symmetry relating to the third index is to be inferred from this designation.

In the case of an antisymmetric connection, the construction of skew frames, which in general led to the vanishing of the $c^{(\beta)}{}_{(\alpha\gamma)}(x)$ at the point x, will lead to inertial frames, provided only that the original frame at x be chosen inertial. Consequently, the frame transformation leading from an initial system of inertial frames to the frames of the construction is a Lorentz transformation.

The eq. (23)-antisymmetric and -symmetric parts, c_A and c_S, of an arbitrary connection, are respectively a connection and a tensor under Lorentz transformations -- the $\frac{\partial L}{\partial x}$ terms from (10) cancel in τ_S by virtue of (22). Thus, in the form $c_S = 0$, (23) is seen to be a Lorentz-invariant property.

6. Christoffel Symbols

Since an antisymmetric connection preserves all inner products in parallel displacement, or, equivalently, is permutable with the raising and lowering of parenthesized indices, and since such a connection can be reduced to zero at an arbitrary point by a suitable Lorentz transformation from its original expression on an inertial system of frames, it may be suspected that the antisymmetry property (23) alone uniquely determines the analogue in frame language of the usual Christoffel symbols, which indeed are determined uniquely from the $g_{\mu\nu}(x)$ in the language of unparenthesized indices by the permutability of parallel displacement and all inner products, and by the possibility of rendering $\Gamma^{\nu}{}_{\mu\varpi}(x) = 0$ at an arbitrary point x by a coördinate transformation, which possibility is equivalent to the symmetry property,

$$\Gamma^{\nu}{}_{\mu\varpi}(x) = \Gamma^{\nu}{}_{\varpi\mu}(x). \tag{24}$$

But this is not so: That c satisfy (23) does not determine c uniquely, and hence the associated Γ given by (14b) is not determined uniquely. This is obvious directly: add to $c_{\nu\mu\varpi}$ a nonzero arbitrary ordinary tensor $\hat{c}_{\nu\mu\varpi}$ antisymmetric in its first pair of indices; then $c' = c + \hat{c}$ has correctly coördinate-transformation invariant parenthesized components, it satisfies the frame-transformation law (10), and the antisymmetry property (23); further, such antisymmetric $\hat{c}_{\nu\mu\varpi}$ exist: e. g., let $\hat{c}_{(\alpha\beta\gamma)}(x) = \epsilon_{0\alpha\beta\gamma}\phi(x)$, where ϵ is the completely alternating symbol, and ϕ is an arbitrary nonzero scalar function, e. g., a nonzero constant.[6]

What part of the usual ingredients which allow the computation of Γ is missing? The equation

$$g_{\mu\nu;\varpi} = 0 \tag{25}$$

is equivalent, by (15), to $g_{(\alpha\beta,\gamma)} = 0$; since $g_{(\alpha\beta)} \equiv \eta_{(\alpha\beta)}$, to $\eta_{(\alpha\beta,\gamma)} = 0$, which is none other than the condition of permutability of η and c expressed by (23). We therefore have (25), and it is (24) that is missing. Conversely, (25) implies $\eta_{(\alpha\beta,\gamma)} = 0$, which implies the antisymmetry property (23), so that the lack of uniqueness of antisymmetric c is in one-to-one correspondence with the lack of uniqueness of the Γ consistent with (25), but not necessarily with (24), in the language of unparenthesized indices.

Of course, we could assume (24), derive the Christoffel symbols Γ directly from (25) in the usual way, and then find c from (14b); uniqueness is obvious,[6] so that the usual computations are needed only for existence and explicit form, although uniqueness appears again as a byproduct.

Since the usual Γ is linear homogeneous in the first derivatives of the components of g, we have by (21) and (14b) that the corresponding c is linear homogeneous in the first derivatives of the frame components.

From the standpoint of frames, however, assumptions about behavior under curvilinear coördinate transformations seem highly arbitrary; (24), which is effectively the assumption that it be possible to coördinate-transform Γ to zero at a point, seems an awkward hypothesis, especially since it is independent of the apparently similar but effectively empty additional condition that it be possible to render c = 0 at a point by a Lorentz frame transformation.

We will instead start from another condition, which also, of necessity, relates c, the frames, and the coördinates, to derive the

Christoffel symbols. Namely, we assume in addition to the proper transformation laws and (23) that c be linear homogeneous in the $\partial f/\partial x$, and have an expression in terms of the f and $\partial f/\partial x$ which preserves its form under Lorentz frame transformations, and thereupon directly compute a unique c. This may seem like the natural formalization of our original introduction of the $c^{(\beta)}{}_{(\alpha\gamma)}(x)$ as compensators for the rotation of the reference frame, in the definition of parallel displacement; it may seem natural that the rotation of the frame be proportional to the explicit first derivatives $\partial f/\partial x$. Indeed, the definition of parallel displacement also accepts the importance of the coördinates in computing explicit derivatives in its naïve term, and is therefore a kind of precedent for such a condition.

Theorem. A unique curvilinear-coördinate transformation invariant c is determined by the Lorentz-transformation property (10), the antisymmetry property (23), and the requirement that it be linear homogeneous in the $\partial f/\partial x$. Its expression (38) is determined in the proof.

Proof. By differentiating (3), we obtain $\frac{\partial f^{(\alpha)}_{\mu}}{\partial x^{\varpi}} f^{\mu}_{(\beta)} = -f^{(\alpha)}_{\mu} \frac{\partial f^{\mu}_{(\beta)}}{\partial x^{\varpi}}$ and $\frac{\partial f^{(\alpha)}_{\mu}}{\partial x^{\varpi}} f^{\nu}_{(\alpha)} = -f^{(\alpha)}_{\mu} \frac{\partial f^{\nu}_{(\alpha)}}{\partial x^{\varpi}}$, which with the remark that $\eta_{(\alpha\beta)}$, $\eta^{(\alpha\beta)}$ are constants allows us to freely transform derivatives, and thereby to reduce the general linear homogeneous expression bearing precisely three free lower parenthesized indices to

$$\begin{aligned} c_{(\alpha\beta\gamma)} = {} & c_1 \frac{\partial f_{(\alpha)\mu}}{\partial x^{\nu}} f^{\mu}_{(\beta)} f^{\nu}_{(\gamma)} + c_2 \frac{\partial f_{(\alpha)\mu}}{\partial x^{\nu}} f^{\mu}_{(\gamma)} f^{\nu}_{(\beta)} \\ & + c_3 \frac{\partial f_{(\beta)\mu}}{\partial x^{\nu}} f^{\mu}_{(\gamma)} f^{\nu}_{(\alpha)} + c_4 \frac{\partial f_{(\beta)\mu}}{\partial x^{\nu}} f^{\mu}_{(\alpha)} f^{\nu}_{(\gamma)} \\ & + c_5 \frac{\partial f_{(\gamma)\mu}}{\partial x^{\nu}} f^{\mu}_{(\alpha)} f^{\nu}_{(\beta)} + c_6 \frac{\partial f_{(\beta)\mu}}{\partial x^{\nu}} f^{\mu}_{(\beta)} f^{\nu}_{(\alpha)} . \end{aligned} \qquad (26)$$

That the three parenthesized indices appear below is equivalent to (10) for constant L. The antisymmetry property (23) further specializes (26) to

$$c_{(\alpha\beta\gamma)} = c_1 \left(\frac{\partial f_{(\alpha)\mu}}{\partial x^\nu} f^\mu_{(\beta)} f^\nu_{(\gamma)} - \frac{\partial f_{(\beta)\mu}}{\partial x^\nu} f^\mu_{(\alpha)} f^\nu_{(\gamma)}\right)$$
$$+ c_3 \left(\frac{\partial f_{(\beta)\mu}}{\partial x^\nu} f^\mu_{(\gamma)} f^\nu_{(\alpha)} - \frac{\partial f_{(\alpha)\mu}}{\partial x^\nu} f^\mu_{(\gamma)} f^\nu_{(\beta)}\right) \qquad (27)$$
$$+ c_5 \left(\frac{\partial f_{(\gamma)\mu}}{\partial x^\nu} f^\mu_{(\alpha)} f^\nu_{(\beta)} - \frac{\partial f_{(\gamma)\mu}}{\partial x^\nu} f^\mu_{(\beta)} f^\nu_{(\alpha)}\right).$$

The coefficient of c_5 is already invariant under curvilinear coördinate transformations, for it is the contraction of the covariant tensor $s_{\mu\nu} = \frac{\partial f_{(\gamma)\mu}}{\partial x^\nu} - \frac{\partial f_{(\gamma)\nu}}{\partial x^\mu}$ with the contravariant tensor $t^{\mu\nu} = f^\mu_{(\alpha)} f^\nu_{(\beta)}$. If we put $c_1 = c_3 + (c_1 - c_3)$ and leave aside the $c_1 - c_3$ part, the first two lines, now bearing the common coefficient c_3, form scalars in a similar way: combine the first part of the first line with the second part of the second line, and the second part of the first line with the first part of the second.

Thus, (27) is invariant to curvilinear coördinate transformations if and only if the remainder,

$$(c_1 - c_3)\left(\frac{\partial f_{(\alpha)\mu}}{\partial x^\nu} f^\mu_{(\beta)} f^\nu_{(\gamma)} - \frac{\partial f_{(\beta)\mu}}{\partial x^\nu} f^\mu_{(\alpha)} f^\nu_{(\gamma)}\right), \qquad (28)$$

is. It is not, unless $c_1 = c_3$; see Appendix 1.

Therefore, putting $c_1 = c_3$, we have

$$c_{(\alpha\beta\gamma)} = c_1 \left(\frac{\partial f_{(\alpha)\mu}}{\partial x^\nu} - \frac{\partial f_{(\alpha)\nu}}{\partial x^\mu}\right) f^\mu_{(\beta)} f^\nu_{(\gamma)}$$
$$+ c_1 \left(\frac{\partial f_{(\beta)\mu}}{\partial x^\nu} - \frac{\partial f_{(\beta)\nu}}{\partial x^\mu}\right) f^\mu_{(\gamma)} f^\nu_{(\alpha)} \qquad (29)$$
$$+ c_5 \left(\frac{\partial f_{(\gamma)\mu}}{\partial x^\nu} - \frac{\partial f_{(\gamma)\nu}}{\partial x^\mu}\right) f^\mu_{(\alpha)} f^\nu_{(\beta)}.$$

We now impose Lorentz invariance. The original expression was taken to have precisely three lower parenthesized indices to assure

Lorentz invariance under constant Lorentz transformations; hence, when (29) is transformed, the terms not involving the $\partial L/\partial x$ will already balance. It is therefore necessary only that the terms involving $\partial L/\partial x$ sum to the proper derivative term in the expression for $\bar{c}_{(\alpha\beta\gamma)}$, namely, that obtained from (10) by lowering (β) and interchanging α and β:

$$L_{(\alpha\epsilon)}\, L_{(\gamma)}{}^{(\xi)}\, f^{\mu}_{(\xi)}\, \frac{\partial L_{(\beta)}{}^{(\epsilon)}}{\partial x^{\mu}} = L_{(\alpha\epsilon)}\, \frac{\partial L_{(\beta)}{}^{(\epsilon)}}{\partial x^{\mu}}\, \bar{f}^{\mu}_{(\gamma)}. \tag{30}$$

A typical term from (29) transforms as follows:

$$\frac{\partial \bar{f}_{(\alpha)\mu}}{\partial x^{\nu}}\, \bar{f}^{\mu}_{(\beta)}\, \bar{f}^{\nu}_{(\gamma)} = \text{term without } \frac{\partial L}{\partial x} + f_{(\delta)\mu}\, f^{\mu}_{(\epsilon)}\, f^{\nu}_{(\xi)}\, \frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\nu}}\, L_{(\beta)}{}^{(\epsilon)} L_{(\gamma)}{}^{(\xi)},$$

of which the $\frac{\partial L}{\partial x}$ term reduces to

$$\eta_{(\delta\epsilon)}\, \frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\nu}}\, L_{(\beta)}{}^{(\epsilon)}\, \bar{f}^{\nu}_{(\gamma)} = \frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\mu}}\, L_{(\beta\delta)}\, \bar{f}^{\mu}_{(\gamma)}. \tag{31}$$

The equation of the sum of all six $\partial L/\partial x$ terms from (29) to expression (30) reads

$$\begin{aligned} c_1 \Big(&\frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\beta\delta)}\, \bar{f}^{\mu}_{(\gamma)} - \frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\gamma\delta)}\, \bar{f}^{\mu}_{(\beta)} \\ &+ \frac{\partial L_{(\beta)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\gamma\delta)}\, \bar{f}^{\mu}_{(\alpha)} - \frac{\partial L_{(\beta)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\alpha\delta)}\, \bar{f}^{\mu}_{(\gamma)}\Big) \\ + c_5 \Big(&\frac{\partial L_{(\gamma)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\alpha\delta)}\, \bar{f}^{\mu}_{(\beta)} - \frac{\partial L_{(\gamma)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\beta\delta)}\, \bar{f}^{\mu}_{(\alpha)}\Big) \\ = {}& L_{(\alpha\delta)}\, \frac{\partial L_{(\beta)}{}^{(\delta)}}{\partial x^{\mu}}\, \bar{f}^{\mu}_{(\gamma)}. \end{aligned} \tag{32}$$

Since $L_{(\alpha)}{}^{(\delta)}\, L_{(\beta\delta)} = \eta_{(\alpha\beta)}$, we have $\frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\beta\delta)} = -L_{(\alpha)}{}^{(\delta)}\, \frac{\partial L_{(\beta\delta)}}{\partial x^{\mu}} = -L_{(\alpha\delta)}\, \frac{\partial L_{(\beta)}{}^{(\delta)}}{\partial x^{\mu}}$; the first and fourth terms are equal, and, except for the coefficient, coincide with the right-hand side. The fifth term is similarly related to the second, the sixth to the third, so that (32) reduces to

$$\begin{aligned} (c_1 + c_5)\Big(- \frac{\partial L_{(\alpha)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\gamma\delta)}\, \bar{f}^{\mu}_{(\beta)} + \frac{\partial L_{(\beta)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\gamma\delta)}\, \bar{f}^{\mu}_{(\alpha)}\Big) \\ = (1 + 2c_1)\frac{\partial L_{(\beta)}{}^{(\delta)}}{\partial x^{\mu}} L_{(\gamma\delta)}\, \bar{f}^{\mu}_{(\alpha)}. \end{aligned} \tag{33}$$

By putting $\bar{f}^{\mu}_{(\gamma)} = \delta^{\mu}_{\gamma}$, $L_{(\alpha)}{}^{(\beta)} = \delta^{\beta}_{\alpha}$, eq. (33) is simplified without restricting the $\partial L/\partial x$: By putting

$$L_{(\alpha)}{}^{(\beta)}(x+\delta) = \delta^{\beta}_{\alpha} + \frac{\partial L_{(\alpha)}{}^{(\beta)}}{\partial x^{\mu}}(x)\,\delta^{\mu} \tag{34a}$$

into (22a), it is easy to verify that the antisymmetry conditions

$$\frac{\partial L_{(\alpha\beta)}}{\partial x^{\mu}} + \frac{\partial L_{(\beta\alpha)}}{\partial x^{\mu}} = 0 \tag{34b}$$

are necessary and sufficient for $L_{(\alpha)}{}^{(\beta)}(x+\delta)$ to be a Lorentz transformation, to first order in δ. Equation (33) thereupon simplifies to

$$(c_1 + c_5)\left(\frac{\partial L_{(\beta\gamma)}}{\partial x^{\alpha}} - \frac{\partial L_{(\alpha\gamma)}}{\partial x^{\beta}}\right) = (1 + 2c_1)\,\frac{\partial L_{(\beta\alpha)}}{\partial x^{\gamma}}. \tag{35}$$

Put $\frac{\partial L_{(12)}}{\partial x^3} = 1 = -\frac{\partial L_{(21)}}{ox^3}$, all other $\frac{\partial L_{(\alpha\beta)}}{\partial x^{\gamma}} = 0$. With $\alpha = 1$, $\beta = 2$, $\gamma = 3$, (35) thereupon reduces to

$$0 = 1 + 2c_1; \tag{36}$$

$\beta = 1$, $\gamma = 2$, $\alpha = 3$ yields

$$c_1 + c_5 = 0, \tag{37}$$

so that $c_1 = -\frac{1}{2}$, $c_5 = \frac{1}{2}$, and

$$\begin{aligned} 2c_{(\alpha\beta\gamma)} = {} & \left(\frac{\partial f_{(\alpha)\nu}}{\partial x^{\mu}} - \frac{\partial f_{(\alpha)\mu}}{\partial x^{\nu}}\right) f^{\mu}_{(\beta)}\, f^{\nu}_{(\gamma)} \\ & + \left(\frac{\partial f_{(\beta)\nu}}{\partial x^{\mu}} - \frac{\partial f_{(\beta)\mu}}{\partial x^{\nu}}\right) f^{\mu}_{(\gamma)}\, f^{\nu}_{(\alpha)} \\ & + \left(\frac{\partial f_{(\gamma)\nu}}{\partial x^{\mu}} - \frac{\partial f_{(\gamma)\mu}}{\partial x^{\nu}}\right) f^{\mu}_{(\beta)}\, f^{\nu}_{(\alpha)}, \end{aligned} \tag{38a}$$

or

$$\begin{aligned} 2c_{\rho\sigma\tau} = {} & f^{(\alpha)}_{\rho}\left(\frac{\partial f_{(\alpha)\tau}}{\partial x^{\sigma}} - \frac{\partial f_{(\alpha)\sigma}}{\partial x^{\tau}}\right) \\ & + f^{(\alpha)}_{\sigma}\left(\frac{\partial f_{(\alpha)\rho}}{\partial x^{\tau}} - \frac{\partial f_{(\alpha)\tau}}{\partial x^{\rho}}\right) \\ & + f^{(\alpha)}_{\tau}\left(\frac{\partial f_{(\alpha)\rho}}{\partial x^{\sigma}} - \frac{\partial f_{(\alpha)\sigma}}{\partial x^{\rho}}\right), \end{aligned} \tag{38b}$$

which is the unique connection determined by our conditions.

The part of c antisymmetric in the final indices is a curl:

$$c^{(\alpha)}{}_{\sigma\tau} - c^{(\alpha)}{}_{\tau\sigma} = \frac{\partial f^{(\alpha)}_{\tau}}{\partial x^{\sigma}} - \frac{\partial f^{(\alpha)}_{\sigma}}{\partial x^{\tau}}. \tag{39}$$

From (14b), the corresponding Γ is seen to be

$$\Gamma_{\rho\sigma\tau} = c_{\rho\sigma\tau} + f_{(\alpha)\rho} \frac{\partial f_{\sigma}^{(\alpha)}}{\partial x^{\tau}} = c_{\rho\sigma\tau} + f_{\rho}^{(\alpha)} \frac{\partial f_{(\alpha)\sigma}}{\partial x^{\tau}};$$

$$\begin{aligned} 2\Gamma_{\rho\sigma\tau} = {} & f_{\rho}^{(\alpha)} \frac{\partial f_{(\alpha)\tau}}{\partial x^{\sigma}} + f_{\tau}^{(\alpha)} \frac{\partial f_{(\alpha)\rho}}{\partial x^{\sigma}} \\ & + f_{\sigma}^{(\alpha)} \frac{\partial f_{(\alpha)\rho}}{\partial x^{\tau}} + f_{\rho}^{(\alpha)} \frac{\partial f_{(\alpha)\sigma}}{\partial x^{\tau}} \\ & - f_{\sigma}^{(\alpha)} \frac{\partial f_{(\alpha)\tau}}{\partial x^{\rho}} - f_{\tau}^{(\alpha)} \frac{\partial f_{(\alpha)\sigma}}{\partial x^{\rho}}. \end{aligned}$$

Since $f_{\rho}^{(\alpha)} \frac{\partial f_{(\alpha)\tau}}{\partial x^{\sigma}} + f_{\tau}^{(\alpha)} \frac{\partial f_{(\alpha)\rho}}{\partial x^{\sigma}} = \frac{\partial}{\partial x^{\sigma}}(f_{\rho}^{(\alpha)} \eta_{(\alpha\beta)} f_{\tau}^{(\beta)}) = \frac{\partial g_{\rho\tau}}{\partial x^{\sigma}}$,

we have indeed the conventional Christoffel symbols,

$$\Gamma_{\rho\sigma\tau} = \tfrac{1}{2}\left(\frac{\partial g_{\rho\tau}}{\partial x^{\sigma}} + \frac{\partial g_{\rho\sigma}}{\partial x^{\tau}} - \frac{\partial g_{\sigma\tau}}{\partial x^{\rho}}\right). \tag{38c}$$

The requirement that the c be given in terms of the f by a Lorentz-invariant expression is the mathematical concretization of the notion of a completely equivalent 6-parameter family of inertial frames at a point, of special relativity. It renders the $f_{\mu}^{(\alpha)}(x)$ "less powerful" than a physical field. However, the $f_{\mu}^{(\alpha)}(x)$ may be used more strongly; e.g., a special system of frames may be used to define a connection Γ according to (7). In such a case, the $f_{\mu}^{(\alpha)}(x)$ introduce a non-Lorentz-invariant feature in the space, and in this way resemble a physical field, which has a special expression on certain inertial frames at a point that is not assumed on a general inertial frame at the point. It is not, then, surprising that frames and connections other than that of the Christoffel symbols appear together in some papers on unified field theory.

7. Curvature and Torsion

In this section we return to the arbitrary connections of Section 3; the antisymmetry property (23), in particular, is dropped; inasmuch as curvature and torsion are general attributes which involve no reference to the Lorentz metric.

Equation (15) is easily generalized to an arbitrary tensor; e. g., $t_{(\alpha\beta,\gamma)} = f^{\mu}_{(\alpha)} f^{\nu}_{(\beta)} f^{\varpi}_{(\gamma)} t_{\mu\nu,\varpi}$. Therefore, the tensors of curvature, R, and torsion, T, defined by

$$v_{(\alpha,\beta,\gamma)} - v_{(\alpha,\gamma,\beta)} = v_{(\delta)} R^{(\delta)}{}_{(\alpha\beta\gamma)} + v_{(\alpha,\delta)} T^{(\delta)}{}_{(\beta\gamma)}, \tag{40a}$$

and those defined by

$$v_{\mu,\nu,\varpi} - v_{\mu,\varpi,\nu} = v_{\rho} R^{\rho}{}_{\mu\nu\varpi} + v_{\mu,\rho} T^{\rho}{}_{\nu\varpi}, \tag{40b}$$

coincide, or correspond, in the sense that they are respectively the parenthesized and unparenthesized components of common tensors -- at least, if the separations into a term proportional to v and one proportional to its absolute derivative <u>are not made differently</u> in the two languages -- for the right-hand sums correspond, and if the splittings into v and derivative terms correspond, then the R and T must correspond, since their coefficients, v and its derivative, respectively correspond. By carrying out differentiations and subtractions, the forms (40a, b) may be verified, with the following explicit forms for R and T:[7]

$$R^{(\epsilon)}{}_{(\alpha\beta\gamma)} = c^{(\epsilon)}{}_{(\alpha\delta)}\, f^{\mu}_{(\beta)}\, f^{\nu}_{(\gamma)} \left(\frac{\partial f^{(\delta)}_{\nu}}{\partial x^{\mu}} - \frac{\partial f^{(\delta)}_{\mu}}{\partial x^{\nu}}\right)$$
$$- f^{\mu}_{(\gamma)} \frac{\partial c^{(\epsilon)}{}_{(\alpha\beta)}}{\partial x^{\mu}} + f^{\mu}_{(\beta)} \frac{\partial c^{(\epsilon)}{}_{(\alpha\gamma)}}{\partial x^{\mu}}$$
$$+ c^{(\epsilon)}{}_{(\delta\beta)}\, c^{(\delta)}{}_{(\alpha\gamma)} - c^{(\epsilon)}{}_{(\delta\gamma)}\, c^{(\delta)}{}_{(\alpha\beta)} \tag{41a}$$
$$= c^{(\epsilon)}{}_{(\alpha\delta)} \left(f^{\mu}_{(\gamma)}\, \partial_{(\beta)} f^{(\delta)}_{\mu} - f^{\mu}_{(\beta)}\, \partial_{(\gamma)} f^{(\delta)}_{\mu}\right)$$
$$- \partial_{(\gamma)} c^{(\epsilon)}{}_{(\alpha\beta)} + \partial_{(\beta)} c^{(\epsilon)}{}_{(\alpha\gamma)}$$
$$+ c^{(\epsilon)}{}_{(\delta\beta)}\, c^{(\delta)}{}_{(\alpha\gamma)} - c^{(\epsilon)}{}_{(\delta\gamma)}\, c^{(\delta)}{}_{(\alpha\beta)};$$

$$T^{(\epsilon)}{}_{(\beta\gamma)} = c^{(\epsilon)}{}_{(\gamma\beta)} - c^{(\epsilon)}{}_{(\beta\gamma)} + f^{\mu}_{(\beta)}\, f^{\nu}_{(\gamma)} \left(\frac{\partial f^{(\epsilon)}_{\nu}}{\partial x^{\mu}} - \frac{\partial f^{(\epsilon)}_{\mu}}{\partial x^{\nu}}\right) \tag{41b}$$
$$= c^{(\epsilon)}{}_{(\gamma\beta)} - c^{(\epsilon)}{}_{(\beta\gamma)} + f^{\mu}_{(\gamma)}\, \partial_{(\beta)} f^{(\epsilon)}_{\mu} - f^{\mu}_{(\beta)}\, \partial_{(\gamma)} f^{(\epsilon)}_{\mu};$$

$$R^{\sigma}_{\mu\nu\varpi} = -\frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\varpi}} + \frac{\partial \Gamma^{\sigma}_{\mu\varpi}}{\partial x^{\nu}} + \Gamma^{\sigma}_{\rho\nu}\Gamma^{\rho}_{\mu\varpi} - \Gamma^{\sigma}_{\rho\varpi}\Gamma^{\rho}_{\mu\nu}; \tag{41c}$$

$$T^{\sigma}_{\nu\varpi} = -\Gamma^{\sigma}_{\nu\varpi} + \Gamma^{\sigma}_{\varpi\nu}. \tag{41d}$$

From (14b), it is clear that the T <u>do correspond</u> directly, whence the splittings of (40a) and (40b) do correspond, so that the correspondence of the R follows as above. The R correspondence may also be verified directly by tedious calculation from the explicit forms (41a, c) via (14b).

Since T is minus twice the antisymmetric part of Γ, property (24) finds a geometrical meaning as

$$T^{(\delta)}{}_{(\beta\gamma)} = 0, \tag{42}$$

in (40a). The objection made to the use of (24) as a condition for the determination of the Christoffel symbols in conjunction with (23), in that either in its simple but formal statement, or in the form that Γ be reducible to zero by a curvilinear coördinate transformation, its meaning in the language of frame components was unclear, is partially removed. In this connection, it may be interesting to also note that only the curvature tensor R emerges from the discussion of the change in a vector produced by parallel displacement around an infinitesimal loop, even when $T \neq 0$ -- which may indeed seem reasonable directly from (40), inasmuch as loop displacement of a single vector will not easily be pressed into definition of vector field sufficient for the covariant derivative to be defined. That T vanish therefore corresponds to the condition that the process of antisymmetric double differentiation of a vector field be no richer than that of loop displacement of a single vector.

The simplifications of (41a, b) attained at one point x by choosing the frame $f^{(\alpha)}_{\mu}(x) = \delta^{\alpha}_{\mu}$ and the frames at neighboring points by parallel displacement, so that $c^{(\beta)}{}_{(\alpha\gamma)}(x) = 0$, are obvious. Similar simplification of (41c) by special coördinate transformations will not be discussed; at least for $T = 0$, they are thoroughly treated under the title of <u>normal coördinates</u> in textbooks;[8] the familiar unparenthesized-component formulae appear here only for the purpose of comparison to the frame-component formulae.

III. Spin $\frac{1}{2}$

1. Spinor Algebra

The clear isolation of a Lorentz group at each point x afforded by the language of inertial frames leads to a discussion of spin $\frac{1}{2}$ which, except for the extra parameters x^μ, does not differ from the usual discussion in purely special relativity.[9]

There are four representations $L^{(\mu)}{}_{(\nu)} \to S$ of the group of Lorentz transformations at a point by complex (2 by 2) matrices. Given any one representation $L \to S$, we obtain another, the contragredient representation, as usual, by use of the adjoint matrices, $L \to (S^{-1})^\sim$; that the representations are complex allows us to generate two more representations by taking complex conjugates of these: $L \to S^*$, and $L \to ((S^{-1})^\sim)^* = (S^{-1})^\dagger$.

A conventional notation for the action of S on the spinors is

$$\overline{\psi}^a(x) = S^a{}_b(x)\ \psi^b(x). \tag{43}$$

If the notation for the inverse matrix be written in analogy to the notation of (6), namely,

$$S^a{}_b(x)\ S_c{}^b(x) = \delta^a_c, \tag{44a}$$

$$S^a{}_b(x)\ S_a{}^c(x) = \delta^c_b, \tag{44b}$$

one obtains the usual simple notation for the contragredient representation,

$$\overline{\chi}_a(x) = S_a{}^b(x)\ \chi_b(x),$$

and ψ^a may be termed a contravariant spinor, χ_a a covariant spinor. The indices belonging to complex-conjugate representations are conventionally denoted by dots, and one speaks of dotted contravariant and of dotted covariant spinors. Thus, $(S^a{}_b(x))^* = S^{\dot{a}}{}_{\dot{b}}(x)$, and the complex conjugate of (43) reads

$$\overline{\psi}^{\mathring{a}}(x) = S^{\mathring{a}}{}_{\mathring{b}}(x)\ \psi^{\mathring{b}}(x).$$

If the $\psi^{a}(x)$ are q-numbers, then $\psi^{\dot{a}}(x)$ designates the hermitean conjugate, for a fixed value of the index a.

Clearly, one may pass from ordinary to dotted components of a spinor by complex conjugation. One may also raise and lower indices by the matrix $\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, if the Pauli matrices

$$\sigma^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \sigma^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma^{(2)} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma^{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{45}$$

are used.[8] The Pauli matrices are defined, more generally, as twice the coefficient of i in the infinitesimal parts of the respective rotation operators in the representation S, for $\mu = 1, 2, 3$; $\sigma^{(0)} = 1$ is a convention which ties a given representation to one of the two inequivalent Lorentz vector representations via eq. (46), and the choice of plus sign in $\sigma^{(1)}\ \sigma^{(2)} = \pm i\sigma^{(3)}$ specifies S as either of two of the four inequivalent spinor representations; general Pauli matrices subject only to these conventions are related to the special Pauli matrices (45) by a similarity transformation.

The famous tie between spinor and vector representations is

$$S^{\dagger}(x)\ \sigma^{(\mu)}\ S(x) = L^{(\mu)}{}_{(\nu)}(x)\ \sigma^{(\nu)}. \tag{46a}$$

The constant matrices $\sigma^{(\mu)}$ are written $\sigma^{(\mu)}_{\dot{a}b}$ because then (46a) reads

$$S^{\dot{d}}{}_{\dot{a}}(x)\ \sigma^{(\mu)}_{\dot{d}c}\ S^{c}{}_{b}(x) = L^{(\mu)}{}_{(\nu)}(x)\ \sigma^{(\nu)}_{\dot{a}b}, \tag{46b}$$

or

$$L_{(\mu)}{}^{(\varpi)}(x)\ S^{\dot{d}}{}_{\dot{a}}(x)\ S^{c}{}_{b}(x)\ \sigma^{(\mu)}_{\dot{d}c} = \sigma^{(\varpi)}_{\dot{a}b}, \tag{46c}$$

which is precisely the assertion that if the $\sigma^{(\mu)}$, defined in terms of the constant infinitesimal parts of the S representation, are "erroneously" transformed according to all their indices, then nevertheless no real error is committed, with this assignment of index positions. Thus, an otherwise complicated expression, e. g., $\sigma^{(\mu)}_{\dot{a}b}\ \psi^b(x)\ v_{(\mu)}(x)$, has its Lorentz-transformation rendered obvious -- in this case, we have a dotted covariant spinor. Further flexibility is attained by noting that

$$\sigma^{(\varpi)}_{\dot{a}d} = \eta_{(\varpi\mu)}\ \epsilon^{ab}\ \epsilon^{\dot{d}\dot{c}}\ \sigma^{(\mu)}_{b\dot{c}} = \sigma^{a\dot{d}}_{(\varpi)}\,, \tag{47}$$

where the left-hand equality is a directly computable identity with $\sigma^{(\mu)}_{b\dot{c}} = (\sigma^{(\mu)}_{\dot{b}c})^*$, by virtue of which the numbers $\sigma^{(\varpi)}_{\dot{a}d}$ may also be taken to transform on all indices according to the alternate notation on the right.

The Lorentz-transformation character for spinors is thus given in precisely the same way as for vectors: by the action of a representation of the Lorentz group, varying with x, acting on a field, expressed in curvilinear-coördinate-transformation invariant components.

In consequence of (46b),

$$\chi^{\dot{a}}(x)\ \sigma^{(\mu)}_{\dot{a}b}\,\psi^b(x) \equiv v^{(\mu)}(x)\,, \tag{48}$$

invariant to curvilinear coördinate transformations because all quantities on the left-hand side are thus invariant, undergoes Lorentz transformation in the manner of the upper parenthesized or frame components of a vector field.

The ordinary contravariant components of v are

$$v^{\nu}(x) = \chi^{\dot{a}}(x)\ \sigma^{\nu}_{\dot{a}b}(x)\,\psi^b(x)\,,$$
$$\sigma^{\nu}_{\dot{a}b}(x) = f^{\nu}_{(\mu)}(x)\ \sigma^{(\mu)}_{\dot{a}b}\,,$$

but whether one designates vectors by parenthesized, frame indices or by ordinary indices, the <u>spinor</u> components ψ^b refer to inertial frames, and do not transform under change of curvilinear coördinates; the difference between the two vector notations is absorbed by the Pauli matrices. This curvilinear-coördinate invariance of the spinors cannot be "remedied," because there are no spin representations of the full linear group, as has been noted in the introduction. Thus, when spinors are included in a discussion, the language of inertial frames appears unified, in that <u>all</u> Lorentz-group quantities remain invariant under curvilinear coördinate transformations, whereas the usual notation, which draws exclusive attention to the unparenthesized components of tensors, appears disjointed. The $\sigma^{\mu}(x)$ can be used as a foundation for the geometry in place of the $f^{\mu}_{(\nu)}$; in fact, as $\sigma^{\mu}(x) = f^{\mu}_{(\nu)}(x)\ \sigma^{(\nu)}$, and $\frac{1}{2}\ \mathrm{Tr}\ \sigma^{(\mu)}\sigma^{(\nu)} = \delta^{\nu}_{\mu}$, or, more explicitly,

$$\tfrac{1}{2}\ \sigma^{(\mu)}_{a\dot{b}}\sigma^{\dot{}(\nu)}_{\dot{b}a} = \tfrac{1}{2}\ \sigma^{a\dot{b}}_{(\mu)}\ \sigma^{\dot{}(\nu)}_{\dot{b}a} = \delta^{\nu}_{\mu}, \tag{49}$$

we have $f^{\mu}_{(\nu)} = \frac{1}{2}\mathrm{Tr}\ \sigma^{(\nu)}\ \sigma^{\mu}$, or, more explicitly,

$$f^{\mu}_{(\nu)}(x) = \tfrac{1}{2}\ \sigma^{a\dot{b}}_{(\nu)}\ \sigma^{\dot{}\mu}_{\dot{b}a}(x) \tag{50}$$

-- so the way to do it is to prefix the discussion of frames by this definition of the frame vectors, which is, however, extremely formal, and may be misleading if the development covers up the presence of the equivalent frames (50).[3]

2. The Parallel Displacement and Covariant Derivative of a Spinor

The spinor obtained from $\psi^a(x)$ by infinitesimal parallel displacement is defined in terms of coefficients of connection, in precise analogy to the case of vectors:

$$\psi^a_{\parallel}(x, \delta) = \psi^a(x) - k^a{}_{b(\alpha)}(x)\, \psi^b(x)\, \delta^{(\alpha)}. \tag{51}$$

The requirement that $\psi^a_{\parallel}(x, \delta)$ behave like a contravariant spinor at $x + \delta$, to first order in δ, with respect to both curvilinear coördinate transformations and Lorentz frame transformations, gives us the transformation properties of k. Thus, all quantities in (51), save perhaps k, are invariant to curvilinear coördinate transformations; whence, however, k must be invariant, too. For the behavior of k under Lorentz transformations, (51) is written for different frames, the known forms of all the quantities except $\bar{k}^a{}_{b(\alpha)}(x)$ are inserted, and then the result is solved for $\bar{k}$:

$$S^a{}_c(x + \delta)\ (\psi^c(x) - k^c{}_{b(\alpha)}(x)\, \psi^b(x)\, \delta^{(\alpha)})$$
$$\doteq S^a{}_c(x)\, \psi^c(x) - \bar{k}^a{}_{b(\alpha)}(x)\ S^b{}_c(x)\, \psi^c(x)\ L^{(\alpha)}{}_{(\beta)}(x)\ \delta^{(\beta)}.$$

The coefficient of ψ^c in the first-order part is

$$-\frac{\partial S^a{}_c}{\partial x^\mu}\, \delta^\mu + S^a{}_d\, k^d{}_{c(\alpha)}\, \delta^{(\alpha)} = \bar{k}^a{}_{b(\alpha)}\ S^b{}_c\ L^{(\alpha)}{}_{(\beta)}\ \delta^{(\beta)}.$$

The coefficient, now, of $\delta^{(\beta)}$, with, of course, $\delta^\mu = f^\mu_{(\beta)}\ \delta^{(\beta)}$, is

$$-\frac{\partial S^a{}_c}{\partial x^\mu}\, f^\mu_{(\beta)} + S^a{}_d\, k^d{}_{c(\beta)} = \bar{k}^a{}_{b(\alpha)}\ S^b{}_c\ L^{(\alpha)}{}_{(\beta)},$$

and solving by applying $S_e{}^c\, L_{(\gamma)}{}^{(\beta)}$ yields

$$\bar{k}^a{}_{e(\gamma)} = S^a{}_d\, S_e{}^c\, L_{(\gamma)}{}^{(\beta)}\, k^d{}_{c(\beta)} - S_e{}^c\, \frac{\partial S^a{}_c}{\partial x^\mu}\, f^\mu_{(\beta)}\, L_{(\gamma)}{}^{(\beta)}; \tag{52}$$

as usual, the connection has a "naïve" term, and a term depending on the nonuniformity of the transformation.

The covariant derivative is, as usual, defined as the coefficients in the absolute differential, obtained by subtracting the parallel displacement from the actual first-order extrapolated spinor field:

$$\psi^{a}{}_{,(\alpha)}(x)\ \delta^{(\alpha)} \doteq \psi^{a}(x+\delta) - \psi^{a}_{\parallel}(x,\ \delta);$$

$$\psi^{a}{}_{,(\alpha)}(x) = \partial_{(\alpha)}\psi^{a}(x) + k^{a}{}_{b(\alpha)}(x)\ \psi^{b}(x). \tag{53}$$

The three other kinds of spinors can be transformed to contravariant spinors by ϵ and complex conjugation, then parallelly displaced, then transformed back, so that the k above already defines parallel displacements for all spinors. In particular,

$$\psi^{\dot{a}}_{\parallel}(x,\ \delta) = \psi^{\dot{a}}(x) - k^{\dot{a}}{}_{\dot{b}(\alpha)}(x)\ \psi^{\dot{b}}(x)\ \delta^{(\alpha)}, \tag{54}$$

obtained by taking the complex conjugate of (51), where, of course, $k^{\dot{a}}{}_{\dot{b}(\alpha)}(x) = (k^{a}{}_{b(\alpha)}(x))^{*}$.

3. Relation to the Parallel Displacement of a Vector

The quantity $v^{(\mu)}(x)$ of eq. (48) is a vector, with determinate parallel displacement c when a k is given to parallelly displace spinors. Thus,

$$v_{\|}^{(\mu)}(x,\ \delta) \doteq (\chi^{\dot a} - k^{\dot a}{}_{\dot c(\alpha)}\, \chi^{\dot c}\, \delta^{(\alpha)})\ \sigma^{(\mu)}_{\dot a b}\ (\psi^b - k^b{}_{d(\alpha)}\,\psi^d\,\delta^{(\alpha)})$$
$$\doteq v^{(\mu)}(x) - \chi^{\dot a}\ (\sigma^{(\mu)}_{\dot a b}\ k^b{}_{d(\alpha)} + \sigma^{(\mu)}_{\dot c d}\ k^{\dot c}{}_{\dot a(\alpha)})\ \psi^d\ \delta^{(\alpha)},$$

whereas, also,

$$v_{\|}^{(\mu)}(x,\ \delta) = v^{(\mu)}(x) - c^{(\mu)}{}_{(\beta\alpha)}(x)\ v^{(\beta)}(x)\ \delta^{(\alpha)}$$
$$= v^{(\mu)}(x) - c^{(\mu)}{}_{(\beta\alpha)}\ \chi^{\dot a}\ \sigma^{(\beta)}_{\dot a d}\ \psi^d\ \delta^{(\alpha)},$$

so that, on equating the two forms, one obtains

$$c^{(\mu)}{}_{(\beta\alpha)}\ \sigma^{(\beta)}_{\dot a c} = \sigma^{(\mu)}_{\dot a b}\ k^b{}_{c(\alpha)} + \sigma^{(\mu)}_{\dot b c}\ k^{\dot b}{}_{\dot a(\alpha)}. \tag{55}$$

This may obviously be solved for the c, most directly, by traces. Thus, by applying $\frac{1}{2}\ \sigma^{(\gamma)}_{\dot c a} = \frac{1}{2}\ \sigma^{c\dot a}_{(\gamma)}$, one obtains

$$c^{(\mu)}{}_{(\gamma\alpha)} = \tfrac{1}{2}(\sigma^{c\dot a}_{(\gamma)}\ \sigma^{(\mu)}_{\dot a b}\ k^b{}_{c(\alpha)} + \sigma^{(\mu)}_{\dot b c}\ \sigma^{c\dot a}_{(\gamma)}\ k^{\dot b}{}_{\dot a(\alpha)}). \tag{56}$$

What are the properties of such a c, derived from a k? Since the proposed parallelly displaced vector $v_{\|}(x,\ \delta)$ already is known to transform to first order in δ as a vector at $x + \delta$, the c must necessarily have the correct transformation properties of frame-oriented coefficients of connection. By writing eq. (55) in the abbreviated notation (58) below, one sees immediately that $c^{(\mu)}{}_{(\beta\alpha)}\ \sigma^{(\beta)}$ is hermitean, whence from the hermiticity and linear independence of the $\sigma^{(\beta)}$, it follows that the $c^{(\mu)}{}_{(\beta\alpha)}$ are real.

The c derived from a k are therefore ordinary coefficients of connection, and we address ourselves to the determination of any special properties of such a connection c that may follow from its compatibility with, or derivability from, a connection for spinors, k. We will do this in the process of determining what is the class of k's consistent with a given c; for c's incompatible with a k, the class of k's will be found to be empty.

For the purpose of this computation, we will go to a fixed inertial system of frames,[10] and we will drop the obviously Lorentz-covariant notation of positional and dotted indices. In this system of frames, $k^{b}{}_{c(\alpha)}(x)$ is, for each α and x, a (2 by 2) matrix, and may therefore be expressed as a linear combination of the Pauli matrices $\sigma^{(\beta)}_{\dot{b}c}$. Thus,

$$k^{b}{}_{c(\alpha)}(x) = k_{(\alpha)}(x) = k_{(\alpha\beta)}(x)\ \sigma^{(\beta)}_{\dot{b}c}, \tag{57}$$

where $k_{(\alpha)}(x)$ is a (2 by 2) matrix, for fixed α and x, and the $k_{(\alpha\beta)}(x)$ are complex numbers. Equation (55) then reads

$$c^{(\mu)}{}_{(\beta\alpha)}\ \sigma^{(\beta)} = \sigma^{(\mu)}\ k_{(\alpha)} + k_{(\alpha)}^{\dagger}\ \sigma^{(\mu)}, \tag{58}$$

or

$$c^{(\mu)}{}_{(\beta\alpha)}\ \sigma^{(\beta)} = \sigma^{(\mu)}\ \sigma^{(\beta)}\ k_{(\alpha\beta)} + k_{(\alpha\beta)}^{*}\ \sigma^{(\beta)}\ \sigma^{(\mu)}. \tag{59}$$

It remains to compute the dependence of the $k_{(\alpha\beta)}$ on the $c^{(\mu)}{}_{(\beta\alpha)}$, and to verify that the resulting k possess the necessary properties of a spinor connection. A fast start is obtained by specializing (59) to $\mu = 0$:

$$c^{(0)}{}_{(\beta\alpha)}\ \sigma^{(\beta)} = (k_{(\alpha\beta)} + k_{(\alpha\beta)}^{*})\ \sigma^{(\beta)},$$

or

$$c^{(0)}{}_{(\beta\alpha)} = 2\ \mathrm{Re}\ k_{(\alpha\beta)}. \tag{60}$$

The relation $\sigma^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which has been applied, is invariant to a similarity transformation, so that the special representation (45) has not been invoked.

We continue by specializing (59) to $\mu = i$, where $1 \leq i \leq 3$:

is a fixed index:

$$c^{(i)}{}_{(0\alpha)}\,\sigma^{(0)} + c^{(i)}{}_{(i\alpha)}\,\sigma^{(i)} + c^{(i)}{}_{(i+1,\alpha)}\,\sigma^{(i+1)} + c^{(i)}{}_{(i+2,\alpha)}\,\sigma^{(i+2)}$$

$$= \sigma^{(i)}\,k_{(\alpha 0)} + \sigma^{(0)}\,k_{(\alpha i)} + i\sigma^{(i+2)}\,k_{(\alpha,\ i+1)} - i\sigma^{(i+1)}\,k_{(\alpha,\ i+2)}$$
$$+ \sigma^{(i)}\,k_{(\alpha 0)}{}^{*} + \sigma^{(0)}\,k_{(\alpha i)}{}^{*} - i\sigma^{(i+2)}\,k_{(\alpha,\ i+1)}{}^{*} + i\sigma^{(i+1)}\,k_{(\alpha,\ i+2)}{}^{*}, \tag{61}$$

where indices distinct from 0 are given in an obvious modulo-3 notation. The relations applied here were $\sigma^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\sigma^{(i)}\,\sigma^{(i+1)} = i\sigma^{(i+2)} = -i\sigma^{(i+1)}\,\sigma^{(i)}$, which are invariant to similarity transformations, and are therefore also independent of the special form (45). Equating the coefficients of the various $\sigma^{(\mu)}$ to zero gives

$$c^{(i)}{}_{(0\alpha)} = k_{(\alpha i)} + k_{(\alpha i)}{}^{*}, \tag{62a}$$

$$c^{(i)}{}_{(i\alpha)} = k_{(\alpha 0)} + k_{(\alpha 0)}{}^{*}, \tag{62b}$$

$$c^{(i)}{}_{(i+1,\alpha)} = -ik_{(\alpha,\ i+2)} + ik_{(\alpha,\ i+2)}{}^{*}, \tag{62c}$$

$$c^{(i)}{}_{(i+2,\alpha)} = ik_{(\alpha,\ i+1)} - ik_{(\alpha,\ i+1)}{}^{*}. \tag{62d}$$

From (60) and (62a), we find, by lowering the upper index in each case,

$$c_{(0i\alpha)} = -\,c_{(i0\alpha)}. \tag{63}$$

Since this has been proved in an arbitrary inertial system[10] of frames, it must be a Lorentz-invariant relation. In Appendix 2, this requirement is shown to be equivalent to the form

$$c_{(\alpha\beta\gamma)} = \hat{c}_{(\alpha\beta\gamma)} + \eta_{(\alpha\beta)}\,v_{(\gamma)}, \tag{64}$$

where $\hat{c}$ is a connection satisfying the antisymmetry property (23), and $v_{(\gamma)}$ is an arbitrary vector. This vector is real, as we have already noted that the coefficients c will all be real, or, directly, from (62b);

$$c^{(i)}{}_{(i\alpha)} = v_{(\alpha)} = 2\ \mathrm{Re}\ k_{(\alpha 0)},\ \text{no sum on } i. \tag{65}$$

Equation (62c) may be written

$$c_{(i,\ i+1,\alpha)} = -2\ \mathrm{Im}\ k_{(\alpha,\ i+2)}, \tag{66a}$$

which together with the antisymmetry property on the first two indices when these are distinct determines $c_{(i+1,\ i,\ \alpha)}$, and hence $c_{(i,\ i+2,\alpha)}$, in a manner redundant with (62d).

Therefore, the class of c's compatible with some k is, tentatively, determined to be those c's of form (64); the necessity of that form has already been proven. For a c of that form, the k is partially fixed in that the real parts of all its coefficients are given by (60),

$$2 \text{ Re } k_{(\alpha,\beta)} = c_{(0,\beta,\alpha)}, \tag{60}$$

the imaginary parts of all the coefficients save $k_{(\alpha,\ 0)}$ by

$$2 \text{ Im } k_{(\alpha,\ i)} = c_{(i-1,\ i+1,\alpha)}. \tag{66b}$$

$k_{(\alpha,\ i)}$ is thus completely determined by the antisymmetric part $\hat{c}$ of c. If we restrict ourselves to vector connections that preserve arbitrary inner products, i. e., to antisymmetric connections, then $v_{(\alpha)} = 0 = 2 \text{ Re } k_{(\alpha,\ 0)}$. In any case, Im $k_{(\alpha,\ 0)}$ is not determined by the c.

In Appendix 3, it is verified that the k obtained from an antisymmetric connection $\hat{c}$ by setting $k_{(\alpha,\ 0)} = 0$, namely,

$$k_{(\alpha,\ i)} = \tfrac{1}{2}(\hat{c}_{(0,\ i,\alpha)} + i\hat{c}_{(i-1,\ i+1,\alpha)}), \tag{67}$$

does, in fact, define a spinor connection through (57), which may be termed the distinguished spinor connection determined by an antisymmetric vector connection. In Appendix 4, it is shown that the coefficients $k_{(\alpha,\ 0)}$ of any spinor connection transform like a (complex) vector, which may be written

$$k_{(\alpha,\ 0)} = \tfrac{1}{2}\, v_{(\alpha)} + i\, a_{(\alpha)} \tag{68}$$

in terms of real vectors v and <u>a</u>, which finally establishes that the necessary and <u>sufficient</u> condition for a connection c to be compatible with some spinor connection is that it be of form (64), and in fact that the class of k's compatible with such a c is given by (67), (68), and

(57), in terms of the parts $\hat{c}$ and v determined by c, and an arbitrary real vector, $a_{(\alpha)}$.

That the condition that c be compatible with a k imposes a restriction which tends strongly in the direction of (23), namely, the restriction (64), should not be too surprising, inasmuch as (23) corresponds to preservation of the Lorentz inner product, whereas the completely general spinor connection k is also linked with the Lorentz group, inasmuch as it refers implicitly to the meaningfulness of spinors, and involves the spin representation of Lorentz transformations in its frame transformation law, and these are Lorentz-group, and not affine, concepts.

4. Dirac Equations

The use of covariant differentiation and Pauli matrices to write a Dirac-Weyl-Pauli-Majorana-Lee-Yang-Case equation is obvious:

$$\sigma^{(\alpha)}_{\dot{a}b}\, \psi^{b}{}_{,(\alpha)}(x) = 0,$$

or

$$\sigma^{(\alpha)}_{\dot{a}b}\, \psi^{b}{}_{,(\alpha)} + im\psi_{\dot{a}} = 0,$$

for nonvanishing mass.

By dissecting the 4-component-spinor Dirac equation into its 2-component pieces in the usual way, it also can be directly written:

$$\sigma^{(\alpha)}_{\dot{a}b}\, \psi^{b}{}_{,(\alpha)} + im\chi_{\dot{a}} = 0,$$
$$\sigma^{a\dot{b}}_{(\alpha)}\, \chi_{\dot{b}}{}^{,(\alpha)} + im\psi^{a} = 0.$$

The term $a_{(\alpha)}(x) = \mathrm{Im}\, k_{(\alpha,\, 0)}(x)$ of last Section, will be seen to act in the usual manner of a vector potential if the covariant derivative is written out.[11] This term $a_{(\alpha)}(x)$ has appeared, without being specifically sought, because we have asked for the <u>most general</u> k consistent with a c. Obviously, neither $a_{(\alpha)}$ nor $v_{(\alpha)} = 2\,\mathrm{Re}\, k_{(\alpha,\, 0)}$ distinct from zero appear if one imposes the "natural" condition that k be linear homogeneous in the $\partial f/\partial x$: one then obtains the distinguished k determined through (67) from the Christoffel-symbol c's and $k_{(\alpha,\, 0)} = 0$.

Appendix 1

Non-Invariance of a Quantity in the Christoffel Symbol Derivation

This is a direct verification that (28) is not invariant to curvilinear coördinate transformations unless $c_1 = c_3$.

$$\frac{\partial \bar{f}_{(\alpha)\mu}}{\partial \bar{x}^\nu} \bar{f}^\mu_{(\beta)} \bar{f}^\nu_{(\gamma)} = \frac{\partial\left(\frac{\partial x^\varpi}{\partial \bar{x}^\mu} f_{(\alpha)\varpi}\right)}{\partial \bar{x}^\nu} \frac{\partial \bar{x}^\mu}{\partial x^\rho} f^\rho_{(\beta)} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} f^\sigma_{(\gamma)}$$

$$= \frac{\partial f_{(\alpha)\mu}}{\partial x^\nu} f^\mu_{(\beta)} f^\nu_{(\gamma)} + f_{(\alpha)\varpi} f^\rho_{(\beta)} f^\sigma_{(\gamma)} \frac{\partial^2 x^\varpi}{\partial \bar{x}^\nu \partial \bar{x}^\mu} \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial \bar{x}^\nu}{\partial x^\sigma},$$

so that the second term on the right-hand side is the increment consequent on transformation of $\frac{\partial f_{(\alpha)\mu}}{\partial x^\nu} f^\mu_{(\beta)} f^\nu_{(\gamma)}$. That (28) be invariant with $c_1 \neq c_3$ requires that this increment, antisymmetrized on α, β, vanish;

$$(f_{(\alpha)\varpi} f^\rho_{(\beta)} - f_{(\beta)\varpi} f^\rho_{(\alpha)}) f^\sigma_{(\gamma)} \frac{\partial^2 x^\varpi}{\partial \bar{x}^\nu \partial \bar{x}^\mu} \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} \stackrel{?}{=} 0.$$

Cancel $f^\sigma_{(\gamma)}$, then $\frac{\partial \bar{x}^\nu}{\partial x^\sigma}$, to obtain

$$(f_{(\alpha)\varpi} f^\rho_{(\beta)} - f_{(\beta)\varpi} f^\rho_{(\alpha)}) \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial^2 x^\varpi}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \stackrel{?}{=} 0.$$

By suitable choice of coördinates x, we may put $f^\rho_{(\beta)} = \delta^\rho_\beta$, $f_{(\alpha)\varpi} = \eta_{(\alpha\varpi)}$, without prejudice to the generality of the transformation $x \rightarrow \bar{x}$, thereby obtaining

$$\eta_{(\alpha\varpi)} \frac{\partial^2 x^\varpi}{\partial x^\beta \partial \bar{x}^\nu} \stackrel{?}{=} \eta_{(\beta\varpi)} \frac{\partial^2 x^\varpi}{\partial x^\alpha \partial \bar{x}^\nu}.$$

By putting $\alpha = 0$ and $\beta = 1$ we obtain

$$\frac{\partial^2 x^0}{\partial x^1 \partial \bar{x}^\nu} \stackrel{?}{=} -\frac{\partial^2 x^1}{\partial x^0 \partial \bar{x}^\nu},$$

but by putting $\alpha = 1$ and $\beta = 2$, we obtain

$$-\frac{\partial^2 x^1}{\partial x^2 \partial \bar{x}^\nu} \stackrel{?}{=} -\frac{\partial^2 x^2}{\partial x^1 \partial \bar{x}^\nu}.$$

In particular, for invariance of (28) with $c_1 \neq c_3$, these must both be generally valid laws for a transformation of two variables, but as such they are incompatible -- a sample nonlinear transformation will show that neither is valid -- so that the condition of invariance is not fulfilled

Appendix 2

Form of c Compatible With Some k

To show that

$$c_{(0,i,\alpha)} + c_{(i,0,\alpha)} = 0, \tag{63}$$

for all $i \neq 0$, is satisfied for an arbitrary inertial system of frames if and only if $c_{(\mu,\nu,\alpha)}$ is of form (64):

$$\bar{c}_{(0,i,\alpha)} + \bar{c}_{(i,0,\alpha)} = 0,$$

which for a constant Lorentz transformation becomes

$$L_{(0)}{}^{(\mu)} L_{(i)}{}^{(\nu)} L_{(\alpha)}{}^{(\varpi)} c_{(\mu\nu\varpi)} + L_{(i)}{}^{(\nu)} L_{(0)}{}^{(\mu)} L_{(\alpha)}{}^{(\varpi)} c_{(\nu\mu\varpi)} = 0;$$

by cancelling $L_{(\alpha)}{}^{(\varpi)}$, we obtain

$$L^{(0\mu)} L^{(i\nu)} (c_{(\mu\nu\varpi)} + c_{(\nu\mu\varpi)}) = 0.$$

By putting $L_{(\alpha)}{}^{(\beta)} = \delta_\alpha^\beta + \epsilon\cdot\lambda_{(\alpha)}{}^{(\beta)}$ and computing to first order in ϵ, one obtains

$$(\eta^{(0\mu)} \lambda^{(i\nu)} + \lambda^{(0\mu)} \eta^{(i\nu)})(c_{(\mu\nu\varpi)} + c_{(\nu\mu\varpi)}) = 0,$$

or

$$\lambda^{(i\nu)} (c_{(0\nu\varpi)} + c_{(\nu 0\varpi)}) - \lambda^{(0\mu)} (c_{(\mu i\varpi)} + c_{(i\mu\varpi)}) = 0,$$

which simplifies, by virtue of (63), to

$$2\lambda^{(i0)} c_{(00\varpi)} - \lambda^{(0j)} (c_{(ji\varpi)} + c_{(ij\varpi)}) = 0, \tag{69}$$

where j is summed, but only from 1 to 3, and the $\lambda^{(\mu\nu)}$ need satisfy only the relations

$$\lambda^{(\mu\nu)} + \lambda^{(\nu\mu)} = 0, \tag{70}$$

for an infinitesimal Lorentz transformation; this is essentially the same condition as (34b), derived in the same way.

Put $\lambda^{(i0)} = 1 = -\lambda^{(0i)}$ for a fixed $i \neq 0$, and $\lambda^{(\alpha\beta)} = 0$ for all other cases, in conformity with (70), into (69), to obtain

$$2c_{(00\varpi)} + 2c_{(ii\varpi)} = 0, \text{ no sum on } i. \tag{71}$$

On the other hand, put $\lambda^{(k0)} = 1 = -\lambda^{(0k)}$, $0 \neq k \neq i$, all other $\lambda^{(\alpha\beta)} = 0$, to obtain

$$c_{(ki\varpi)} + c_{(ik\varpi)} = 0, \quad k \neq i; \tag{72}$$

this has been just proved for $k \neq 0$, but the original condition (63) validates (72) for $k = 0$.

In words, (72), supported by (63), affirms that the part of $c_{(\alpha\beta\gamma)}$ off-diagonal in the sense $\alpha \neq \beta$ is antisymmetric, whereas from (71), the corresponding diagonal elements may be obtained thus:

$$c_{(\mu\mu\varpi)} = \eta_{(\mu\mu)}\, c_{(00\varpi)}, \quad \text{no sum on } \mu. \tag{73}$$

Let $v_{(\varpi)} = c_{(00\varpi)}$ be a vector which coincides with $c_{(00\varpi)}$ on the original frames. By the remark at the end of Section II.5, $\eta_{(\mu\nu)}\, v_{(\varpi)}$ Lorentz-transforms in the same way as $c_{(\mu\nu\varpi)} - \hat{c}_{(\mu\nu\varpi)}$, where $\hat{c}$ is the antisymmetric part of c, so that the relation

$$c_{(\mu\nu\varpi)} = \hat{c}_{(\mu\nu\varpi)} + \eta_{(\mu\nu)}\, v_{(\varpi)}, \tag{64}$$

which by (73) holds on the original frames, actually obtains generally.

Thus, c is reduced to the sum of an arbitrary antisymmetric connection and the outer product of η and an arbitrary vector, which form does in fact satisfy (63) in all inertial systems of frames, so that the form (64) is equivalent to the condition (63) in all inertial systems of frames.

Appendix 3

Existence of a k Compatible with an Antisymmetric c

The problem is to verify that (67), where c is a connection satisfying the antisymmetry property (23), $k_{(\alpha,\,0)} = 0$, and (57) define a spinor connection k.

That the expression "k" given by the formulae is invariant under curvilinear coördinate transformations is immediate, as the c and σ are so, by definition.

The explicit verification that the Lorentz frame-transformation property (52) is satisfied will be circumvented by logical argument. It has already been argued that an arbitrary properly transforming k generates a properly transforming c: the brief remark that the $v_{\parallel}^{(\alpha)}(x,\,\delta)$ defined by spinors and k transforms as a proper parallelly displaced vector constituted a sufficient argument, inasmuch as this condition on $v_{\parallel(\alpha)}(x,\,\delta)$ served to determine the transformation law for a proper c -- formal completeness requires the additional and obvious remark that spinors exist to represent an arbitrary vector field as $v^{(\alpha)}(x) = \psi^{\dot{a}}(x)\ \sigma_{\dot{a}b}^{(\alpha)}\ \chi^{b}(x)$.

Now, <u>define</u> k in one frame by (67) and $k_{(\alpha,0)} = 0$, in terms of the components in that frame of a properly transforming c satisfying (23), and extend the definition of k by the proper k-transformation law (52) to all Lorentz-related frames. This extended k generates a properly transforming c, as has been noted in the last paragraph, which must agree with the original c, inasmuch as agreement obtains in the original frame, and the law for c transformation is unique; that antisymmetric c related to k by (67) and $k_{(\alpha,0)} = 0$ properly relate k and c in the original frames is the import of the work in the text.

Conversely, by the work in the text, the expression of this extended k in terms of c follows the law (67), supplemented by $\mathrm{Re}\ k_{(\alpha,0)} = 0$, for antisymmetric c. All is therefore proved, except for the statement that $\mathrm{Im}\ k_{(\alpha,0)} = 0$ in all inertial systems of frames, given that it is true in the original inertial system.

If the argument is now repeated for c of form (64), then we gain the information that $2\ \mathrm{Re}\ k_{(\alpha,0)} = v_{(\alpha)}$, with $v_{(\alpha)}$ a vector, is Lorentz-invariant. This suggests that $\mathrm{Im}\ k_{(\alpha,0)} = a_{(\alpha)}$, with $a_{(\alpha)}$ a vector, is also Lorentz-invariant , which if true, would complete the proof.

A direct verification that $k_{(\alpha,0)}$ transforms like the lower parenthesized components of a vector is given, instead, in Appendix 4.

Appendix 4

That $k_{(\alpha,0)}$ Is a Vector

It is verified here that the $k_{(\alpha,0)}$ Lorentz-transform like the lower parenthesized components of a vector.

By substituting the definition (57) of the $k_{(\alpha,\beta)}$ into the transformation law (52) of a spinor connection $k^{a}{}_{e(\gamma)}$, one obtains

$$\bar{k}_{(\gamma\alpha)}\ \sigma^{(\alpha)}_{\dot{a}e} = S^{a}{}_{d}\ S_{e}{}^{c}\ L_{(\gamma)}{}^{(\beta)}\ k_{(\beta\alpha)}\ \sigma^{(\alpha)}_{\dot{d}c} - S_{e}{}^{c}\ \frac{\partial S^{a}{}_{e}}{\partial x^{\mu}}\ f^{\mu}_{(\beta)}\ L_{(\gamma)}{}^{(\beta)},$$

which yields

$$\bar{k}_{(\gamma\delta)} = \tfrac{1}{2}\ L_{(\gamma)}{}^{(\beta)}\left(k_{(\beta\alpha)}\ \sigma^{(\alpha)}_{\dot{d}c}\ S_{e}{}^{c}\ \sigma^{e\dot{a}}_{(\delta)}\ S^{a}{}_{d} - \sigma^{e\dot{a}}_{(\delta)}\ S_{e}{}^{c}\ \frac{\partial S^{a}{}_{c}}{\partial x^{\mu}}\ f^{\mu}_{(\beta)}\right) \tag{74}$$

as the Lorentz-transformation law for the $k_{(\alpha\beta)}$, on applying the trace operation, $\frac{1}{2}\ \sigma^{e\dot{a}}_{(\delta)}$. It is easier to read (74) in ordinary matrix notation for spin indices: $\sigma^{(\alpha)}_{\dot{d}c} = \sigma^{\dot{d}c}_{(\alpha)} \rightarrow \sigma^{(\alpha)}_{dc}$; $S_{e}{}^{c} \rightarrow s_{ec}$, whence $S^{a}{}_{d}\ S_{b}{}^{d} = \delta^{a}_{b}$ and $(s^{-1})_{da}\ s_{bd} = 1_{ba}$ give $S^{a}{}_{d} = (s^{-1})_{da}$.

In this notation, (74) becomes

$$\bar{k}_{(\gamma\delta)} = \tfrac{1}{2}\ L_{(\gamma)}{}^{(\beta)}\left[k_{(\beta\alpha)}\ \mathrm{Tr}(\sigma^{(\alpha)}\ s^{\sim}\ \sigma^{(\delta)}\ s^{-1\sim}) - \mathrm{Tr}(s\ \frac{\partial(s^{-1})}{\partial x^{\mu}}\ \sigma^{(\delta)\sim})\ f^{\mu}_{(\beta)}\right]. \tag{75}$$

Since $\sigma^{(0)} = 1$, we have, for $\delta = 0$,

$$\bar{k}_{(\gamma,0)} = L_{(\gamma)}{}^{(\beta)}\left[k_{(\beta,\alpha)}\ \tfrac{1}{2}\ \mathrm{Tr}\ \sigma^{(\alpha)} - f^{\mu}_{(\beta)}\ \tfrac{1}{2}\ \mathrm{Tr}\ s\ \frac{\partial(s^{-1})}{\partial x^{\mu}}\right].$$

Since $\frac{1}{2}\ \mathrm{Tr}\ \sigma^{(\alpha)} = \delta^{\alpha}_{0}$, we have the desired result,

$$\bar{k}_{(\gamma,0)} = L_{(\gamma)}{}^{(\beta)}\ k_{(\beta,0)}, \tag{76}$$

provided that

$$\mathrm{Tr}\ s\ \frac{\partial(s^{-1})}{\partial x^{\mu}} = 0. \tag{77}$$

Since $L \rightarrow s$ is a representation of the Lorentz group,[5] $L \rightarrow \det s$ is a one-dimensional representation, of which the only one is the identical representation; $\det s = 1$. Therefore, if

$$s = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix},$$

then

$$s^{-1} = \begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix},$$

and

$$\begin{aligned} \mathrm{Tr}\ s \frac{\partial(s^{-1})}{\partial x^{\mu}} &= s_{11} \frac{\partial(s^{-1})_{11}}{\partial x^{\mu}} + s_{12} \frac{\partial(s^{-1})_{21}}{\partial x^{\mu}} + s_{21} \frac{\partial(s^{-1})_{12}}{\partial x^{\mu}} + s_{22} \frac{\partial(s^{-1})_{22}}{\partial x^{\mu}} \\ &= s_{11} \frac{\partial s_{22}}{\partial x^{\mu}} - s_{12} \frac{\partial s_{21}}{\partial x^{\mu}} - s_{21} \frac{\partial s_{12}}{\partial x^{\mu}} + s_{22} \frac{\partial s_{11}}{\partial x^{\mu}} \\ &= \frac{\partial}{\partial x^{\mu}} (s_{11} s_{22} - s_{12} s_{21}) \\ &= \frac{\partial}{\partial x^{\mu}} (\det s) \\ &= 0, \end{aligned}$$

so that (77), and hence the transformation law (76), is verified.

Footnotes

1. Neither the 4-dimensionality of the basic manifold nor the detailed distribution of 1's and -1's in the η metric is essential to the arguments; as long as $\eta^2 = 1$, they will apply, although the appellations "inertial frame" and "Lorentz transformation" to be introduced later would in more general cases be unconventional terminology. The section on spin $\frac{1}{2}$ is to be excepted, but only because it was not thought worthwhile to work out the spinor algebra in the greatest generality.

2. T. Levi-Civita, Sitzungsberichte preuss. Akad. Wiss., Phys.-math. Klasse, 346 (1932); H. Weyl, Z. Phys. 56, 330 (1929); J. A. Schouten and D. van Dantzig, Ann. Math. 34, 271 (1933); F. J. Belinfante, Physica 7, 305 (1940); more recent references are D. Brill and J. A. Wheeler, Revs. Modern Phys. 29, 465 (1957), Bade and Jehle, Revs. Modern Phys. 25, 714 (1953); the latter contains an extensive bibliography. Also T. W. B. Kibble, J. Math. Phys. 2, 212 (1961).

The term "anholonomic reference system" derives from the fact that if sixteen functions $f^{(\alpha)}_{\mu}(x)$ specify four vectors $f^{(0)}_{\mu}$, $f^{(1)}_{\mu}$, $f^{(2)}_{\mu}$, $f^{(3)}_{\mu}$ at all points x at once, subject to only continuity and differentiability conditions, then there will not in general be four functions $\phi^{(\alpha)}(x)$ such that $f^{(\alpha)}_{\mu}(x) = \frac{\partial\phi^{(\alpha)}}{\partial x^{\mu}}(x)$; i. e., the vectors considered as differential forms are not in general exact; neither will $f^{(\alpha)}_{\mu}(x) = \lambda^{(\alpha)}(x)\,\frac{\partial\phi^{(\alpha)}}{\partial x^{\mu}}(x)$, in general, with integrating factors $\lambda^{(\alpha)}(x)$ and no sum on α, so that the frames are not even integrable, or "holonomic". The matter is of interest, because the differentials of the four curvilinear coördinates are exact, and would be the first sort of frames to appear in a discussion emphasizing the coördinates. The restriction of exactness, or even that of integrability, is, however, an intolerable restriction, because the inertial frames, which are the frames of interest in the common expression of local physics, are usually anholonomic. One should not think of the anholonomic frames as peculiar -- rather, it is the coördinates that are distorted or "curvilinear" due to curvature, so that a condition of simplicity based too directly on them, like exactness, is likely to be worthless.

3. For similar remarks, see footnote 7 of H. Weyl's Z. Phys. 56 article, page 320 of F. J. Belinfante's article in Physica 7, and the end of Section III.5 here. That one can feel strongly about this matter, even without being faced with the spin representations, is illustrated by the remark on page 136 of H. Weyl's "Classical Groups", 2d Ed. (Princeton University Press, 1946).

4. $f^{(\alpha)}_{\mu}(x)\ f^{\mu}_{(\beta)}(x) = f^{(\alpha)}_{(\beta)}(x)$ according to the notation of (4), so that (3a) may be written $f^{(\alpha)}_{(\beta)}(x) = \delta^{\alpha}_{\beta}$. For the sake of clarity, we will not exploit this identity to eliminate either of the symbols "f" or "δ".

5. The group involved at one point x will, as a concomitant of the notion of continuity of the variation of frames, which will remain implicit, be the group without any of the reflection operations. In special arguments and special cases, invariance properties may extend to one or more reflections.

Note that by the use of our normal terminology, expression (21) would be written $\eta^{\nu\mu}(x)$; just as in the case discussed in footnote 4, two letters, here g and η, could be replaced by one. As in that case, however, both letters will be retained, to accentuate the difference between the constant and universal components of the parenthesized metric components, and the coördinate-dependent ordinary tensor components. Further, $\eta^{(\beta\alpha)} f^{\mu}_{(\alpha)} = f^{(\beta)\mu}$, and $g^{\nu\mu} = f^{\nu}_{(\beta)} f^{(\beta)\mu} = f^{\nu\mu}$, so that, in principle, all the four letters in this and the previous footnote-- g, f, η, and δ--are redundant.

6. If we are given a c which satisfies (23) and (24), it is unique, by a similar argument: The difference $\hat{c}$ between two such c's is antisymmetric in its first pair of indices, and, because the f $\partial f/\partial x$ discrepancies between the c and their Γ of eq. (14b) drop out in the subtraction, $\hat{c}$ is symmetric, like the Γ, in its last pair of indices; whereas any 3-index symbol with these simultaneous antisymmetry and symmetry properties is easily shown to vanish.

7. The order of indices is chosen to have (40) read smoothly, and is not guaranteed to conform with other notations. $R^{\rho}{}_{\mu\nu\sigma}$ happens to agree with the notation in Einstein's "Meaning of Relativity", Princeton University Press -- if the difference in sign of the metric tensor is eliminated by comparing precisely this form, rather than the one with all indices below; the R here also differs only in sign from that of reference 8.

8. E. g., O. Veblen, "Invariants of Quadratic Differential Forms", No. 24 of Cambridge Tracts in Mathematics and Mathematical Physics, (Cambridge University Press, 1952).

9. See, e. g., E. M. Corson, "Introduction to Tensors, Spinors, and Relativistic Wave-Equations" (Blackie & Son, 1953). For the use of arbitrary Pauli matrices, see K. M. Case, Phys. Rev. 107, 307 (1957). The use of special Pauli matrices in the sequel is tied only to use of the special matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for the raising and lowering operator ϵ, which is discussed more generally in Case's paper as a (2 by 2) charge-conjugation matrix. I have not deemed it important to adopt such general notation, because the important point about the Pauli matrices is that they are constants under both curvilinear-coördinate and Lorentz transformations; that other Pauli matrices, or even an abstract algebra of Pauli objects, may function in place of particular Pauli matrices is here an incidental point.

10. Since spinors are specifically Lorentz-group quantities, the system of frames which may come into consideration at once are necessarily related by Lorentz transformations. By terming any one of these systems of frames inertial, in the manner of Section II.4, our terminology designates all the systems as inertial. Thus, the Lorentz-transformation law (52) for k does not define $\bar{k}$ for arbitrary linear transformations, inasmuch as S has been defined only for Lorentz transformations, and, as has been noted in the Introduction, the definition of S as a representation cannot be extended to the class of all nonsingular linear transformations.

11. The single 2-component equation with mass does not remain invariant when ψ is transformed by a phase factor, and a is augmented by the phase gradient, for the mass term transforms with the complex

conjugate phase factor, but the coupled pair of 2-component equations does remain invariant under a similar scheme of transformations, if ψ and χ are taken to transform with opposite phases.