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# Convergence of Perturbation Series in the Microcanonical Formulation of Quantum Field Theories

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# Convergence of Perturbation Series in the Microcanonical Formulation of Quantum Field Theories

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Abstract

It is shown that the perturbation series in the microcanonical formulation of scalor $\lambda\phi^4$ and gauge theories on the lattice is absolutely convergent for small enough expansion parameter, $\lambda$. These series can be either the weak ($\lambda$) or the strong ($\lambda^{-1}$) coupling expansion series.

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Recently, the microcanonical quantization method[1] has received much attention as a means of performing numerical calculations. Notably, it has been used in lattice gauge theories. Since the method is quite new to particle physicists, it is important to examine in detail the basis[2] of this quantum formulation.

We shall show in this letter that for both the strong and weak coupling expansion of scalar field theories or gauge theories, the perturbation series are convergent if we quantize in a finite volume with the microcanonical ensemble density.[1] It is possible, in principle, to sum these perturbation series.

As is well known,[3] the perturbation series in the standard formulations (Feynman's path integral or canonical quantization) is not convergent, even if we use an ultraviolet cut off and a finite volume. Although the series in the standard perturbation expansion can be summed up in some cases with appropriate methods of summation (Borel summation, etc.), the final results may still depend on the methods used. Furthermore, so called non-perturbative effects can not be recovered unambiguously from the standard perturbation expansion. On the other hand, the convergence of the microcanonical perturbation series implies that we can obtain all the effects. The underlying reason for the convergence of the series is that the microcanonical density is expanded with respect to an interaction Hamiltonian, but not with respect to the total energy, which in general depends itself on an expansion parameter and should be determined self-consistently (see below).

First, let us recapitulate briefly the microcanonical quantization method.[1,2] Consider a scalar theory on a d-dimensional lattice. The action is

$$S \equiv S_0 + \lambda S_I \quad , \quad S_0 \equiv \frac{V}{N}\left\{\sum_{\chi,\mu} \frac{(\phi_{\chi+\mu} - \phi_\chi)^2}{2a^2} + \sum_{\mu} \frac{m^2}{2}\phi_\chi^2\right\}$$

and

$$S_I = \sum_{\chi} \frac{V}{N}\phi_\chi^4 \quad , \tag{1}$$

where a, m and $\lambda$ are the lattice spacing, bare mass and bare coupling constant, respectively, and where $\chi$ is the vector belonging to a lattice point and $\mu$ is a unit vector in the $\mu$ direction. The volume $V = Na^d$ is taken to be finite. According to the microcanonical method, we compute the reguralized Green's function as follows

$$G(\chi_1 \ldots \chi_n) = \lim_{N\to\infty} \frac{1}{Z} \int d\mu\ \phi(\chi_1) \ldots \phi(\chi_n)\delta(E - H)$$

$$Z \equiv \int d\mu\ \delta(E - H) \quad , \tag{2}$$

with $H = \sum_{\chi=1}^{N} \frac{P_\chi^2}{2} + S$ and $E = \frac{N}{2} + \langle S\rangle$ .

The measured used in (2) is given by

$$d\mu \equiv \prod_{\chi=1}^{N} dP_\chi d\phi_\chi \tag{3}$$

and the average $\langle S\rangle$ in E is

$$\langle S\rangle = \frac{\int \prod_{\chi=1}^{N} d\phi_\chi e^{-S} S}{\int \prod_{\chi=1}^{N} d\phi_\chi e^{-S}} \quad . \tag{4}$$

We shall show below that the Green's function in (2) is identical to the one given in the standard formulation. In a previous paper,[2] we have shown this equivalence perturbatively. It is easy to see that the formula in (2) is the same as the one used in the numerical calculations of Ref. 4. We note that the quantity <S> may also be obtained as

$$\lim_{N\to\infty} \langle S\rangle/N = \lim_{N\to\infty} \frac{1}{Z} \int d\mu \; S\delta(E - H)/N \quad . \tag{5}$$

This is a consistency condition, by use of which the total energy E (or <S>) is to be evaluated.

Now we turn to the microcanonical perturbation theory and show the convergence of its series. The perturbation series we consider is defined by expanding the microcanonical density with respect to $\lambda$,

$$\delta(E - H) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dE^n} \delta(E - H_0)(\lambda S_I)^n \tag{6}$$

with $H = \sum_{x=1}^{N} \frac{P_x^2}{2} + S_0 \quad .$

To demonstrate the convergence of the perturbation series of the Green's function, it is sufficient to show the convergence of the following series

$$Z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dE^n} \int d\mu \; \delta(E - H_0)(\lambda S_I)^n \quad . \tag{7}$$

It is worthwhile to stress that the limit $N \to \infty$ is taken after summing the perturbation series. In our previous paper,[2] we took the limit order by order in the series together with the extra condition $E = N$ and obtained the usual

Feynman rules. In general, the infinite volume limit and the summation over the order n of the expansion do not commute. This situation is analogous to that in the standard functional formulation, where an exchange of the functional integration and the summation $e^{-S} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_I^n e^{-S_0}$ is also impossible.

To prove the convergence of the series (7), we scale $P_x$ and $\phi_x$ as $P_x \to \sqrt{E}\, P_x$ and $\phi_x \to \sqrt{E}\, \phi_x$. The n-th order of the series becomes then

$$A_n \equiv \frac{(-1)^n}{n!} \frac{d^n}{dE^n} E^{N+2n-1} \int d\mu\, \delta(1 - H_0)(\lambda S_I)^n$$

$$= \frac{(-1)^n}{n!} \frac{(N + 2n - 1)!}{(N + n - 1)!} E^{N+n-1} \int d\mu\, \delta(1 - H_0)(\lambda S_I)^n$$

$$< \frac{1}{n!} \frac{(N + 2n - 1)!}{(N + n - 1)!} E^{N+n-1} \int d\mu\, \delta(1 - H_0')(\lambda S_I)^n \quad , \qquad (8)$$

where $H_0' \equiv \Delta V \sum_{x=1}^{N} \left( \frac{P_x^2}{2} + \frac{m^2 \phi_x^2}{2} \right)$ and $\Delta V \equiv \frac{V}{N}$ .

Using the equality

$$\int d\mu\, \delta(1 - H_0')(\lambda S_I)^n < (\lambda S_I(\phi_x = c))^n \int d\mu\, \delta(1 - H_0') \quad , \qquad (9)$$

where $c = \left( \frac{2}{\Delta V\, m^2} \right)^{1/2}$ , we find that $A_n$ is less than

$$\frac{(N+2n-1)!}{n!(N+n-1)!} E^{N+n-1} (\lambda S_I(\phi_x = c)^n \int d\mu\, \delta(1 - H_0')$$

$$\xrightarrow[n\to\infty]{} O\left(\frac{1}{\sqrt{n}} (4\lambda\, S_I E)^n\right) \qquad . \tag{10}$$

Thus, we have discovered that the perturbation series (7) for given energy E converges absolutely for small enough $\lambda$. Summing the series for small $\lambda$, we obtain the Green's function by taking the limit $N \to \infty$ afterwards. Before taking this limit, it is necessary to insert for E an explicit value which depends on $\lambda$ and to make an appropriate analytic continuation in $\lambda$. The reason for the former operation is that when we expanded the microcanonical density as in (6), we did not expand the total energy E, which has to be determined from the self-consistency equation (5), but only the interaction part $S_I$. This is the crux for obtaining a convergent perturbation series.

Next, let us expand the ensemble density with respect to $\lambda^{-1/2}$ (i.e. a strong coupling expansion)

$$\begin{aligned} \int d\mu\, \delta(E - H) &= \int d\mu\, \delta(E - H_0 - \lambda\, S_I) \\ &= \lambda^{-\frac{N}{4}} \int d\mu\, \delta(E - H_I - \lambda^{-\frac{1}{2}} S_0) \\ &= \lambda^{-\frac{N}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dE^n} \int d\mu\, \delta(E - H_I)(\lambda^{-\frac{1}{2}} S_0)^n \end{aligned} \tag{11}$$

with $H_I \equiv \Delta V \sum_x \left(\frac{p_x^2}{2} + \phi_x^4\right)$ , where we have scaled $\phi_x$ as $\phi_x \to \lambda^{-\frac{1}{4}}\phi_x$. The n-th order (=$A_n$) of the series can be written as

$$\lambda^{\frac{N}{4}}A_n = \frac{(-1)^n}{n!}\frac{d^n}{dE^n}\int d\mu\ \delta(E - H_I)(\lambda^{-\frac{1}{2}}S_0)^n$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dE^n}\left(E^{\frac{3}{4}N+\frac{n}{2}-1}\right)\lambda^{-\frac{n}{2}}\int d\mu\ \delta(1 - H_I)S_0^n \quad . \tag{12}$$

If we note the following equality,

$$S_0 \leqq \Delta V\left\{\sum_{x,\mu}\frac{\phi_{x,\mu}^2 + \phi_x^2}{a^2} + \sum_x \frac{m^2}{2}\phi_x^2\right\} \equiv S_0' \tag{13}$$

it is easy to show that

$$A_n \leqq |A_n| \leqq \frac{\lambda^{\frac{N}{4}}}{n!}\left|\frac{d^n}{dE^n}\left(E^{\frac{3N}{4}+\frac{n}{2}-1}\right)\right|\lambda^{-\frac{n}{2}}\int d\mu\ \delta(1 - H_I)(S_0')^n$$

$$< \frac{\lambda^{\frac{N}{4}}}{n!}\left|\frac{d^n}{dE^n}\left(E^{\frac{3N}{4}+\frac{n}{2}-1}\right)\right|\lambda^{-\frac{n}{2}}\ S_0'(\phi_x = c')^n\int d\mu\ \delta(1 - H_I) \tag{14}$$

where $C' \equiv (\Delta V)^{-\frac{1}{4}} = (\frac{V}{N})^{-\frac{1}{4}}$ and where we have replaced $\phi_x$ with a possible maximal value of C'. Therefore, we conclude that the series (11) in the strong coupling expansion for given E also converges absolutely for large enough $\lambda$.

In the previous discussion we have demonstrated the convergence of the perturbation series for the scalar field theory. For lattice gauge theories, it is also possible to prove the convergence of the strong coupling expansion. In the case of an even number of degrees of freedom (N = even), we obtain a finite series.

Let us write the action of alattice gauge theory as $\beta S$, where $\beta$ is the inverse of the coupling constant squared. The microcanonical partition function is then

$$\int d\mu\, \delta\left(E - \sum_{i=1}^{N} \frac{P_i^2}{2} - \beta S\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dE^n} \left(E^{\frac{N}{2}-1}\right) \int d\mu\, \delta\left(1 - \sum_{i=1}^{N} \frac{P_i^2}{2}\right)(\beta S)^n$$

$$\text{with}\quad d\mu \equiv \prod_{i=1}^{N} dP_i \times \prod_{\ell} dU_\ell \quad\text{and}\quad E = \frac{n}{2} + \langle\beta S\rangle \quad . \qquad (15)$$

In (15) $dU_\ell$ is the Haar measure of a link variable and N is the total number of degrees of freedom, which equals (number of lattice points) x $\frac{d(d-1)}{2}$ x (number of gauge degrees of freedom). It is straightforward to show the convergence of the series (15). We find that the effective expansion parameter is $\beta N/2E$, which is less than $\beta$, so that a series like (15) in $\beta N/2E$ converges faster than one in $\beta$ of the standard formulation.

It is interesting that the series in (15) terminates at order $n_c(= \frac{N}{2} - 1)$ if N is even. This implies that the series of the weak coupling expansion, which is obtained by multiplying the series (15) by $\beta^{-(\frac{N}{2}-1)}$, is convergent also.

Finally, we show that under some reasonable assumptions the microcanonical expectation value (2) equals the one derived in the standard functional formulation, which is given by

$$\lim_{N\to\infty} \frac{1}{Z_c} \int \prod_{x=1}^{N} d\phi_x\, \phi(x_1)\ldots\phi(x_n) e^{-S} = G(x_1\ldots x_n) \qquad (16)$$

$$\text{with}\quad Z_c \equiv \int \prod_{x=1}^{N} d\phi_x\, e^{-S} \quad .$$

Using the formula, $\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda(E-H)} = \delta(E - H)$ and performing the integrations over $P_i$, we obtain from eq. (2)

$$\frac{1}{Z_c}\int d\mu\, \delta(E - H)f(\phi) = \frac{1}{Z_c'}\int_{-\infty}^{\infty} d\lambda \prod_{x=1}^{N} d\phi_x f(\phi)\lambda^{-\frac{N}{2}} e^{i\lambda(E-S)}$$

$$= \frac{1}{Z_c'}\int_{-\infty}^{\infty} d\lambda\, e^{-\frac{N}{2}\log\lambda + i\lambda E + W} G_\lambda(f) \qquad (17)$$

with $e^W \equiv \int \prod_{x=1}^{N} d\phi_x e^{-i\lambda S}$ , $G_\lambda(f) \equiv \dfrac{\int \prod_{x=1}^{N} d\phi_x e^{-i\lambda S} f(\phi)}{e^W}$

and $f(\phi) \equiv \phi(x_1)\ldots\phi(x_n)$ .

In (17) we have not specified an irrelevant normalization factor. In order to evaluate the integral over $\lambda$ as $N \to \infty$, we assume that both of the following limits exist

$$\lim_{N\to\infty} G_\lambda(f) \quad , \quad \lim_{N\to\infty} \frac{\partial}{\partial\lambda} W/N \quad \text{for} \quad \text{Im}\,\lambda \leqq 0 \quad . \qquad (18)$$

Then, by using the steepest descent method as $N \to \infty$ and the relation $E = \frac{N}{2} + \langle S \rangle$, it is easy to derive the standard formula (16). The stationary point required can be found by solving the equation,

$$-\frac{1}{2\lambda} + \frac{iE}{N} + \frac{1}{N}\frac{\partial W}{\partial\lambda} = 0 \quad , \quad E = \frac{N}{2} + \langle S \rangle \quad , \qquad (19)$$

where $\langle S \rangle$ is given by (4).

To summarize, we have shown that for a sufficiently small expansion parameter, the perturbation series converges in the microcanonical formulation of the scalar theory and of the gauge theories with an even total number of degrees of freedom. The convergence of the series is only guaranteed for a finite volume. However, if we are able to sum the series and make appropriate analytic continuation in the expansion parameter of the result, we may construct, in principle, all the regularized Green's functions in the infinite volume limit. The reason for being able to obtain non-perturbative effects from perturbation theory is eq. (5), in which we need to estimate non-perturbatively the quantity $\langle S \rangle$ and then develop the perturbation theory as discussed above.

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