Title
Nontrivial tori in spaces of symplectic embeddings

Author
Munteanu, Cristian Mihai

Publication Date
2019

Peer reviewed|Thesis/dissertation

Nontrivial tori in spaces of symplectic embeddings

by

Cristian Mihai Munteanu

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Michael Hutchings, Chair
Professor Fraydoun Rezakhanlou
Professor Surjeet Rajendran

Spring 2019

Nontrivial tori in spaces of symplectic embeddings

Abstract

Nontrivial tori in spaces of symplectic embeddings

by

Cristian Mihai Munteanu

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Michael Hutchings, Chair

This dissertation is comprised of two papers studying the topology of certain spaces of symplectic embeddings.

The first paper shows how given two 4–dimensional ellipsoids whose symplectic sizes satisfy a specified inequality, a certain loop of symplectic embeddings between the two ellipsoids is noncontractible. The statement about symplectic ellipsoids is a particular case of a more general result which claims that given two convex toric domains whose first and second ECH capacities satisfy a specified inequality, one can prove that a certain loop of symplectic embeddings between the two convex toric domains is noncontractible.

The second paper proves how given two $2n$–dimensional symplectic ellipsoids whose symplectic sizes satisfy certain inequalities, a certain map from the n–torus to the space of symplectic embeddings from one ellipsoid to the other induces an injective map at the level of homology with mod 2 coefficients.

To my parents,
Gabriel and Olivia,

and to my brother,
Adrian.

Tell the truth or at least don't lie.
Jordan Peterson

Contents

Acknowledgments

In *Le petit prince*, Antoine de Saint–Exupéry wrote

Tu deviens responsable pour toujours de ce que tu as apprivoisé.[1]

and I believe that now is the most appropriate moment to let this piece of wisdom sink in. As I am now responsible for the few pieces of mathematics I have "tamed", I owe who I have become in the process to the many people who share the responsibility for "taming" me. I would like to express my gratitude towards them.

First and foremost, I would like to thank my advisor, Michael Hutchings, without whose help and advice this would not have been possible. Michael has taught me the importance of clarity and conciseness in arguments while trusting me to be able to figure things out by myself. While thinking about the math discussed in this thesis, I have benefited from conversations with many people, including Dan Cristofaro–Gardiner, Dusa McDuff, Felix Schlenk, and Chris Wendl, who I would especially like to thank for sharing their knowledge with me. I would like to thank Klaus Mohnke and Chris Wendl for hosting me during the Spring of 2017 at Humbold–Universität zu Berlin where I did part of the work on this thesis and, moreover, to thank Klaus for offering me the opportunity to continue my career as a mathematician there starting in 2019. I would also like to thank Fraydoun Rezakhanlou for helping me become a better teacher.

I would like to thank my friend Julian Chaidez for coauthoring the second paper in this thesis while being patient enough to teach me a lot of things I did not understand. I would like to thank my friend Chris Gerig for convincing me to choose Michael as an advisor and also for explaining lots of mathematics to me. I would like to thank my friends Ben Fillipenko, Andrew Hanlon, and Jeff Hicks for sharing an office and discussing math with me. I would especially like to thank Barb, Jennifer, Judie, Marsha, and Vicky for making my life in the department a lot more pleasant.

I began doing mathematics when I was 8 and during all the years that have passed since then, I was lucky to encounter teachers and mentors who are all to blame for my passion for mathematics. I would like to mention a few in particular. My father, Gabriel, discovered my inclination for mathematics and helped me start on this path; Florin Crețu and Florin Slujitoru taught me the basics when I was very young; Enache Pãtrașcu, whose love and wisdom I still cherish, taught me invaluable things about life, while giving me a strong foundation for mathematics; Sorin Bodaș taught me how mathematics is the language of physics; Tase (and his 15 dogs) taught me how to focus my mind, write clearly, and see beauty in mathematics; Keivan Mallahi–Karai, Stefan Maubach, Daniel Meyer, and Ivan Penkov gave me a great mathematical education that made a great difference in graduate school; Sergei Tabachnikov guided me through my first mathematical research and encouraged me to choose Berkeley; Alan Huckleberry inspired me to do research in geometry and taught

[1] You become forever responsible for that which you have tamed.

me to distinguish good mathematics from Mickey Mouse mathematics; Michael Hutchings helped me mature as a mathematician and gain clarity in thought.

All the work I have done would not have been possible without the support and love of my close friends. Chris, Philipp, and Sevan became my family in Berkeley and shared their love in more ways than I can count, and I will forever be grateful for what they have given me; SC was my first and only friend in the beginning times of confusion and loneliness in Berkeley; Lawrence played tennis and shared his passion for Federer with me; Julian and Chris shared more than just mathematics with me; Carlo taught me how to cook genuine Italian food and complain like a true Italian; Henrik watched a lot of movies with me and listened to a lot more rants than a friend should be required to; Jenny shared a home with me and showed me all the delicious food places in Berkeley; Andra and Dan hosted me on the gorgeous island of Tenerife where part of this thesis was written; Pavel and Stuti listened to me while growing up and becoming an adult; Mihaela believed in me that I will discover who I truly am; Andrei, Ciprian, Dorel, George, Gică, Gîț, Livache, Maria, Mili, Pană, and Toader made life a lot more fun to live; Andrei always reminded me why I love mathematics, while entertaining me with the best political discussions; Dan, Dragoș, Mircea, and Theo remained as close as family regardless the distance and the very few times we met in the past few years.

Last and definitely not least, I would like to thank my family, even though words cannot contain how much they have given and sacrificed for me to get here. This work is dedicated to them. My father has always seen more in me than I have managed to see myself. I would like to thank him for constantly inspiring me to become a better human. My mother, who probably suffered the most from the distance between us, is the one who taught me how to help others selflessly. I would like to thank her for bringing me into this world and offering me the freedom to choose my own path. I wish I will be able to do for my children what my mother and my father have done for me. My brother is the one who taught me what unconditional love is and I am forever indebted to him for that. I wish the life I live inspires him as much as his life inspires me. I will close by thanking Clara who even though left this world too soon, has kept on inspiring me through the love and wisdom she shared with me in times of difficulty.

Format

This thesis consists of the papers:

1. *Noncontractible loops of symplectic embeddings between convex toric domains*

   and

2. *Essential tori in spaces of symplectic embeddings* (with Julian Chaidez),

which were written while the author was a graduate student at UC Berkeley. Each chapter in this dissertation contains one paper and is completely self–contained.

Both papers are studying the topology of the space of symplectic embeddings between certain starshaped subdomains in $\mathbb{C}^n$.

Basic definitions

Recall that a *symplectic manifold* is a smooth, even–dimensional manifold M equipped with a *symplectic form*, i.e. a closed nondegenerate 2–form ω on M. The classical example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dy_i)$, where $(x_1, y_1, \ldots, x_n, y_n)$ are linear coordinates on $\mathbb{R}^{2n}$.

Given two symplectic manifolds (M, ω_M) and (N, ω_N), a *symplectic embedding* of M into N is a smooth embedding $\varphi : M \to N$ such that $\varphi^* \omega_N = \omega_M$.

Recall also that a *contact manifold* is a smooth, $(2n-1)$–dimensional manifold Y equipped with a *contact form*, i.e. a 1–form λ satisfying $\lambda \wedge (d\lambda)^{n-1} > 0$. The *Reeb vector field* associated to λ, R_λ, is the vector field determined uniquely by $d\lambda(R_\lambda, \cdot) = 0$ and $\lambda(R_\lambda) = 1$. A closed curve $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ such that $\gamma'(t) = R_\lambda(\gamma(t))$ is called a *Reeb orbit*.

Given (M, ω) a symplectic manifold with boundary, one can prove that if there exists a vector field ρ defined in a neighborhood of ∂M that is transverse to ∂M satisfying $\mathcal{L}_\rho \omega = \omega$ then the boundary ∂M is a contact manifold. Such a vector field ρ is called a *Liouville vector field* and it induces the contact form $\lambda = \iota_\rho \omega|_{\partial M}$ on ∂M.

A basic example that provides nonetheless nontrivial results is the study of symplectic embeddings involving the *symplectic ellipsoid*

$$E(a_1, a_2, \ldots, a_n) := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi |z_1|^2}{a_1} + \ldots \frac{\pi |z_n|^2}{a_n} \leq 1 \right\},$$

which together with the restriction of the standard symplectic form ω_{std} is a symplectic manifold with boundary.

For the symplectic ellipsoid $E(a_1, a_2, \ldots, a_n)$, the radial vector field $\rho = \sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i})$ is a Liouville vector field which induces the standard contact form $\lambda_{\text{std}} = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$ on the boundary $\partial E(a_1, a_2, \ldots, a_n)$. This holds true in more generality for any subdomain $U \subset \mathbb{R}^{2n}$ that contains the origin and is transverse to the radial vector field, i.e. a *starshaped* domain.

It turns out one can study symplectic geometry of a starshaped domain U by studying the contact geometry of ∂U and we elaborate in the papers that follow how one can take advantage of this nice fact. More specifically for the study of symplectic embeddings, given two starshaped domains U and V and a symplectic embeddings $\varphi : U \to V$, we will define the *completed symplectic cobordism*

$$\widehat{W}_\varphi = (-\infty, 0] \times \varphi(\partial U) \cup V \setminus \mathrm{int}(\varphi(U)) \cup [0, \infty) \times \partial V$$

and consider moduli spaces J–holomorphic curves

$$u : (\dot{\Sigma}, j) \to (\widehat{W}_\varphi, J)$$

with asymptotical cylindrical ends over Reeb orbits. The study of these moduli spaces will provide obstructions to the existence of certain families of symplectic embeddings $U \to V$ and thus say something about the topology of the space of symplectic embeddings $U \to V$.

Chapter 1

Noncontractible loops of symplectic embeddings between convex toric domains

Mihai Munteanu

1.1 Introduction

1.1.1 Previous results and a new result about ellipsoids

Questions about symplectic embeddings of one symplectic manifold into another have always been one of the main study directions in symplectic geometry. The pioneering work of Gromov in [16] introduced new methods that made it possible to answer many open questions about symplectic embeddings that had been until then unanswered. The survey by Schlenk, [41], presents in detail the type of results one can prove about symplectic embeddings together with the tools used to prove such results.

Most of the questions that have been answered (in the positive or the negative) concern the existence of symplectic embeddings of one symplectic manifold into another. For example, see [30], [31], [33], and [35] for symplectic embeddings involving 4–dimensional ellipsoids, see [7], [9], [10], and [24] for symplectic embeddings involving more general 4–dimensional symplectic manifolds, and also see [17], [19], and [21] for results in higher dimensions.

Another direction where significant progress has been made is the study of the connectivity of certain spaces of symplectic embeddings. In [31], McDuff shows the connectivity of spaces of symplectic embeddings between 4–dimensional ellipsoids, while in [9], Cristofaro–Gardiner extends this result to symplectic embeddings from concave toric domains to convex toric domains, both of which are subdomains of $\mathbb{R}^4$ whose definition we recall below in §1.1.2. In [21], Hind proves the non-triviality of π_0 for spaces of symplectic embeddings involving certain 4–dimensional polydisks, extending a result that was initially proved in [15]. In

[20], the authors prove that certain spaces of symplectic embeddings involving more general 4–dimensional symplectic manifolds are disconnected, while in [36], the authors study the connectivity of symplectic embeddings into generalized "camel" spaces in higher dimensions, extending results in [14].

Following yet another direction, in this paper we study the fundamental group of certain spaces of symplectic embeddings in 4 dimensions. Let us first clarify the notation we will be using. For real numbers a and b with $0 < a \leq b$, the set

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}$$

together with the restriction of the standard symplectic form from $\mathbb{R}^4$ is called a *closed symplectic ellipsoid*, or more simply an *ellipsoid*. Moreover, we define the symplectic ball $B^4(a) := E(a,a)$. Also, if M and N are symplectic manifolds, let $\mathrm{SympEmb}(M,N)$ denote the space of symplectic embeddings of M into N.

Here are a few results about the fundamental group of spaces of symplectic embeddings that motivated our work. The first result in this direction is an immediate consequence of the methods that Gromov introduced in [16] in order to prove the nonsqueezing theorem.

Theorem 1.1.1 ([14])**.** *Let S be an embedded unknotted 2–sphere in $(\mathbb{R}^4, \omega_{\mathrm{std}})$. Write $X_S = \mathbb{R}^4 \setminus S$ and let $e : \mathrm{SympEmb}(B^4(r), X_S) \to X_S$ be the evaluation map $f \mapsto f(0)$. Then the induced homomorphism $e_* : \pi_1(\mathrm{SympEmb}(B^4(r), X_S)) \to \pi_1(X_S)$ is surjective for $2\pi r^2 < \int_S \omega$ and trivial otherwise.*

Another situation where the fundamental group of a space of symplectic embeddings can be computed is the following.

Theorem 1.1.2 ([22])**.** *If $\epsilon < 1$ the space $\mathrm{SympEmb}(B^4(\epsilon), B^4(1))$ deformation retracts to $U(2)$.*

A more recent result that is closer in spirit to the results of this paper can be found in [5], where the author constructs a loop $\{\phi_\mu\}_{\mu \in [0,1]}$ in $\mathrm{SympEmb}(E(a,b) \sqcup E(a,b), B^4(R))$ and shows that if the positive real numbers a, b, and R satisfy $\frac{a}{b} \notin \mathbb{Q}$, $2a < R < a + b$, and $b < 2a$, then the constructed loop is noncontractible in $\mathrm{SympEmb}(E(a,b) \sqcup E(a,b), B^4(R))$. Moreover, the loop becomes contractible if $R > a + b$.

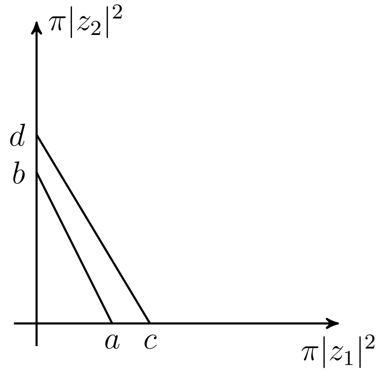
By contrast to [5], we study symplectic embeddings whose domain is connected. More specifically, this paper is concerned with the study of restrictions of the loop of symplectic linear maps defined in (1.1.1) below to certain domains in $\mathbb{R}^4$.

Definition 1.1.3. Let $\{\Phi_t\}_{t \in [0,1]} \subset \mathrm{Sp}(4, \mathbb{R})$ denote the loop of symplectic linear maps

$$\Phi_t(z_1, z_2) = \begin{cases} (e^{4\pi i t} z_1, z_2), & t \in \left[0, \frac{1}{2}\right] \\ (z_1, e^{-4\pi i t} z_2), & t \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{1.1.1}$$

The loop Φ_t is a concatenation of the 2π counterclockwise rotation in the z_1–plane followed by the 2π clockwise rotation in the z_2–plane. The loop $\{\Phi_t\}_{t\in[0,1]}$ is contractible in $\mathrm{Sp}(4,\mathbb{R})$, but it restricts to give some noncontractible loops of symplectic embeddings. For example:

Theorem 1.1.4. *Assume that $a < c < b < d$ and $c < 2a$. Then, for Φ_t defined as in (1.1.1), the loop of symplectic embeddings $\{\varphi_t = \Phi_t|_{E(a,b)}\}_{t\in[0,1]}$ is noncontractible in $\mathrm{SympEmb}(E(a,b), E(c,d))$.*



If $\max(a,b) \leq \min(c,d)$, then one can fit a ball between $E(a,b)$ and $E(c,d)$, meaning there exists $r > 0$ such that $E(a,b) \subset B(r) \subset E(c,d)$, see Figure 1.1. Under this assumption, the loop $\{\varphi_t\}_{t\in[0,1]}$ is contractible. For a more general statement, see Proposition 1.1.10 below.

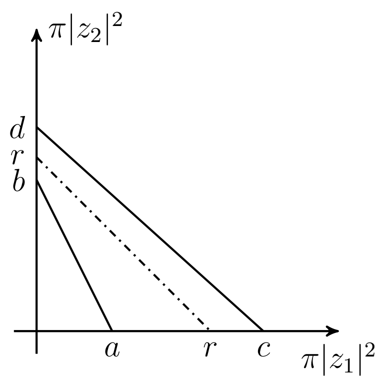


Figure 1.1: The loop $\{\varphi_t\}_{t\in[0,1]}$ is contractible if $\max(a,b) \leq \min(c,d)$.

The method of proof we present in §1.4 does not answer whether the loop $\{\varphi_t\}_{t\in[0,1]}$ is contractible or not under the following assumption.

Open question 1.1.5. *Assume $2a < c < b < d$. Is the loop $\{\varphi_t = \Phi_t|_{E(a,b)}\}_{t\in[0,1]}$ contractible in $\mathrm{SympEmb}(E(a,b), E(c,d))$?*

1.1.2 Main theorem

We begin by recalling an important example of 4–dimensional symplectic manifolds with boundary, in order to prepare for the statement of the main theorem. Given a domain $\Omega \subset \mathbb{R}^2_{\geq 0}$, we define the *toric domain*

$$X_\Omega = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \pi(|z_1|^2, |z_2|^2) \in \Omega \right\} \tag{1.1.2}$$

which, together with the restriction of the standard symplectic form $\omega_{\text{std}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on $\mathbb{C}^2$, is a symplectic manifold with boundary. For example, if Ω is the triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$, then X_Ω is the ellipsoid $E(a,b)$ defined above, while if Ω is the rectangle with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b), then X_Ω is the polydisk $P(a,b) = B^2(a) \times B^2(b)$. Note that we allow domains that have non-smooth boundary. The toric domains we work with in this paper have the following particular property.

Definition 1.1.6. A *convex toric domain* is a toric domain X_Ω defined by

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} \;\middle|\; 0 \leq x \leq a,\ 0 \leq y \leq f(x) \right\} \tag{1.1.3}$$

such that its defining function $f : [0,a] \to \mathbb{R}_{\geq 0}$ is nonincreasing and concave.

Even though we will not work with this type of domains in this paper, let us also recall that a *concave toric domain* is a toric domain defined also by (1.1.3) such that its defining function $f : [0,a] \to \mathbb{R}_{\geq 0}$ is nonincreasing, convex, and $f(a) = 0$. For example, ellipsoids are the only toric domains that are both convex and concave, and polydisks are convex toric domains. We next explain how to compute the first few embedded contact homology (ECH) capacities of convex toric domains in order to state the main result of this paper.

Given a 4–dimensional symplectic manifold (X, ω) with contact boundary $\partial X = Y$, its ECH capacities are a sequence of real numbers

$$0 = c_0^{\text{ECH}}(X,\omega) < c_1^{\text{ECH}}(X,\omega) \leq \cdots \leq \infty$$

constructed using a filtration by action of the ECH chain complex. The ECH capacities obstruct symplectic embeddings, meaning that if there exists a symplectic embedding $(X,\omega) \to (X',\omega')$ then $c_k(X,\omega) \leq c_k(X',\omega')$ for all $k \geq 0$. In particular, for the first and second ECH capacities of a convex toric domain, we can use the following explicit formulas, see [24, Proposition 5.6] for details.

Proposition 1.1.7. *For a convex toric domain X_Ω with nice defining function $f : [0,a] \to \mathbb{R}_{\geq 0}$,*

$$c_1^{\text{ECH}}(X_\Omega) = \min(a, f(0)) \text{ and}$$
$$c_2^{\text{ECH}}(X_\Omega) = \min(2a, x + f(x), 2f(0)),$$

where $x \in (0,a)$ is the unique point where $f'(x) = -1$.

For the definition of a *nice* defining function, see §1.2.4. Every defining function can be perturbed to be nice. Having introduced all the ingredients, we are ready to state the main result of this paper.

Theorem 1.1.8. *Let X_{Ω_1} and X_{Ω_2} be convex toric domains with defining functions $f_1 : [0, a] \to \mathbb{R}_{\geq 0}$ and $f_2 : [0, c] \to \mathbb{R}_{\geq 0}$, respectively. Assume that $X_{\Omega_1} \subset X_{\Omega_2}$, $a < c < f_1(0) < f_2(0)$, and $c_1^{\mathrm{ECH}}(X_{\Omega_2}) < c_2^{\mathrm{ECH}}(X_{\Omega_1})$. Then, for Φ_t defined as in (1.1.1), the loop of symplectic embeddings $\{\varphi_t = \Phi_t|_{X_{\Omega_1}}\}_{t\in[0,1]}$ is noncontractible in* $\mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$.

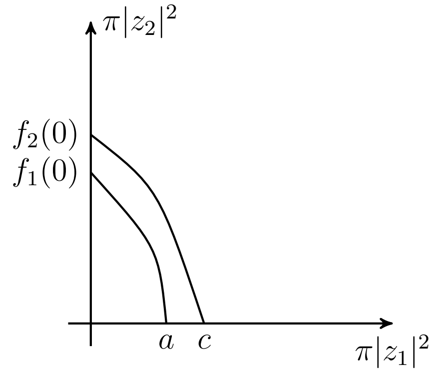


Figure 1.2: The loop $\{\varphi_t\}_{t\in[0,1]}$ is noncontractible if $X_{\Omega_1} \subset X_{\Omega_2}$, $a < c < f_1(0) < f_2(0)$, and $c_1^{\mathrm{ECH}}(X_{\Omega_2}) < c_2^{\mathrm{ECH}}(X_{\Omega_1})$.

Remark 1.1.9.

i. By symmetry, Theorem 1.1.8 also holds if we assume $f_1(0) < f_2(0) < a < c$ instead of $a < c < f_1(0) < f_2(0)$. See Figure 1.2 for an example where the bounds in the hypothesis of Theorem 1.1.8 hold.

ii. For $X_{\Omega_1} = E(a, b)$ and $X_{\Omega_2} = E(c, d)$ satisfying $a < c < b < d$, as in the hypothesis of Theorem 1.1.4, we compute $c_1^{\mathrm{ECH}}(E(c, d)) = \min(c, d) = c$ and $c_2^{\mathrm{ECH}}(E(a, b)) = \min(2a, b)$. Hence, Theorem 1.1.4 is a special case of Theorem 1.1.8.

If the target X_{Ω_2} is large enough, the loop $\{\varphi_t\}_{t\in[0,1]}$ becomes contractible, see Figure 1.3.

Proposition 1.1.10. *Assume there exists $r > 0$ such that $X_{\Omega_1} \subset B^4(r) \subset X_{\Omega_2}$. Then the loop $\{\varphi_t = \Phi_t|_{X_{\Omega_1}}\}_{t\in[0,1]}$ is contractible in* $\mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$.

Proof. Since the loop $\{\Phi_t\}_{t\in[0,1]}$ is contractible in $U(2)$, there exists a homotopy of unitary maps $\{\Phi_z\}_{z\in\mathbb{D}}$ contracting it, where $\mathbb{D}$ denotes the closed unit disk. For each $z \in \mathbb{D}$, the operator norm of $\Phi_z \in U(2)$ is $||\Phi_z|| = 1$, and hence $\mathrm{im}\left(\Phi_z|_{X_{\Omega_1}}\right) \subset B(r) \subset X_{\Omega_2}$. So the 2–parameter family of restrictions $\{\Phi_z|_{X_{\Omega_1}}\}_{z\in\mathbb{D}}$ is contained in $\mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$ and provides a homotopy from $\{\varphi_t\}_{t\in[0,1]}$ to the constant loop. $\square$

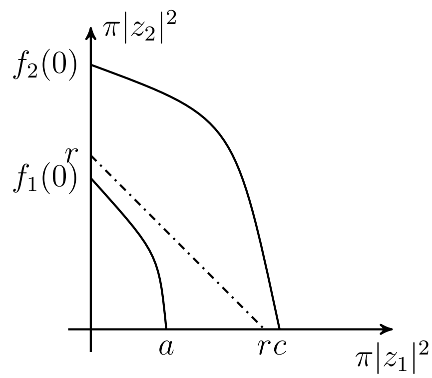


Figure 1.3: If $X_{\Omega_1} \subset B(r) \subset X_{\Omega_2}$, the loop $\{\varphi_t\}_{t\in[0,1]}$ is contractible.

1.1.3 Strategy of proof and the organization of the paper

We use the following strategy to prove Theorem 1.1.8. For each symplectic embedding $\varphi : X_{\Omega_1} \to X_{\Omega_2}$, we add to the compact symplectic cobordism $(X_{\Omega_2} \setminus \mathrm{int}(\varphi(X_{\Omega_1})), \omega_{\mathrm{std}})$, a positive cylindrical end at ∂X_{Ω_2} and a negative cylindrical end at ∂X_{Ω_1}, in order to construct the completed symplectic cobordism $\widehat{W}_\varphi = (-\infty, 0] \times \partial X_{\Omega_1} \cup (X_{\Omega_2} \setminus \mathrm{int}\ \varphi(X_{\Omega_1})) \cup [0,\infty) \times \partial X_{\Omega_2}$. After choosing an almost complex structure J that is compatible with the cobordism structure on $\widehat{W}_\varphi$, we define the moduli space $\mathcal{M}_J(\varphi)$ which consists of J–holomorphic cylinders in $\widehat{W}_\varphi$ that have a positive end at the shortest Reeb orbit on ∂X_{Ω_2} and a negative end at the shortest Reeb orbit on ∂X_{Ω_1}.

Using automatic transversality together with a compactness argument which works under the hypothesis of Theorem 1.1.8, we show that for each $\varphi \in \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$ and for each compatible almost complex structure J, the moduli space $\mathcal{M}_J(\varphi)$ is a finite set. We directly construct an almost complex structure $\widehat{J}$ and a $\widehat{J}$–holomorphic cylinder with the right asymptotics, to show that $\mathcal{M}_{\widehat{J}}(\varphi_0)$ is nonempty for the restriction of the inclusion map φ_0 and the particular choice of $\widehat{J}$. We describe the cylinders near their asymptotic ends to prove that, whenever nonempty, $\mathcal{M}_J(\varphi)$ contains a unique J–holomorphic cylinder.

We complete the proof using an argument by contradiction. We assume the loop $\{\varphi_t\}_{t\in[0,1]}$ is contractible by the homotopy $\{\varphi_z\}_{z\in\mathbb{D}}$, $\varphi_z \in \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$ for each $z \in \mathbb{D}$. We choose a 2–parameter family of almost complex structures $\mathfrak{J} = \{J_z\}_{z\in\mathbb{D}}$ so that J_z is compatible with the cobordism structure on $\widehat{W}_{\varphi_z}$ and $J_z = \widehat{J}$ for all $z \in \partial\mathbb{D}$. We define the universal moduli space $\mathcal{M}_{\mathfrak{J}} = \sqcup_{z\in\mathbb{D}} \mathcal{M}_{J_z}(\varphi_z)$ and, using parametric transversality for generic families of almost complex structures, we show that, for a generic choice of $\mathfrak{J}$ as above, the moduli space $\mathcal{M}_{\mathfrak{J}}$ is a 2–dimensional manifold. Assuming the bounds in the hypothesis of Theorem 1.1.8, we conclude using SFT compactness and the description of each $\mathcal{M}_{J_z}(\varphi_z)$ that $\mathcal{M}_{\mathfrak{J}}$ is homeomorphic to the closed disk $\mathbb{D}$.

For the final details, we fix a parametrization of the shortest Reeb orbit on ∂X_{Ω_2} together with a point p on the same Reeb orbit. For each φ_z, we trace, on the unique cylinder

$[u_z] \in \mathcal{M}_{J_z}(\varphi)$, the vertical ray that is asymptotic to p at ∞ and record the point p_z where it lands at $-\infty$ on the shortest Reeb orbit on ∂X_{Ω_1}. We then study the composition of maps

$$\begin{array}{ccccccc} S^1 & \to & \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2}) & \to & \mathcal{M}_{\mathfrak{J}} & \to & S^1 \\ t & \mapsto & \varphi_t = \varphi_z & \mapsto & (z, [u_z]) & \mapsto & p_z. \end{array}$$

and show that this circle map has degree -1. This provides the contradiction we are looking for, since we previously showed that $\mathcal{M}_{\mathfrak{J}}$ is homeomorphic to the closed disk $\mathbb{D}$.

The paper is divided in sections as follows. In §1.2, we classify the embedded Reeb orbits on the boundary of a convex toric domain. We make use of this classification, together with an automatic transversality argument, to prove the compactness of the moduli space $\mathcal{M}_J(\varphi)$ in §1.3. We also use the classification in §1.2 to show the compactness of the moduli space $\mathcal{M}_{\mathfrak{J}}$ in §1.4.3. Finally, §1.4.1 contains the argument for the existence of J–holomorphic cylinders with the right asymptotics, §1.4.2 contains the argument for the uniqueness of J–holomorphic cylinders in $\mathcal{M}_J(\varphi)$, and §1.4.3 presents the details behind the construction of the circle map above, in order to complete the proof.

Acknowledgements. I would like to thank my advisor, Michael Hutchings, for all the help and ideas he shared with me. I would also like to thank Chris Wendl for clarifying some of my mathematical confusions during my visit at Humboldt–Universität zu Berlin and Felix Schlenk for the helpful comments on the first draft. Finally, I would like to thank my friends, Julian Chaidez and Chris Gerig, for the many helpful conversations we had. The author was partially supported by NSF Grant No. DMS–1708899.

1.2 Reeb dynamics and the ECH index

1.2.1 Geometric setup

Let (Y, ξ) be a closed 3–dimensional contact manifold with contact form λ, i.e. $\xi = \ker \lambda$. The *Reeb vector field* R corresponding to λ is uniquely defined as the vector field satisfying $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 0$. A *Reeb orbit* is a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$, modulo translations of the domain, such that $\gamma'(t) = R(\gamma(t))$. The *action* of a Reeb orbit γ is defined by $\mathcal{A}(\gamma) = \int_{S^1} \gamma^* \lambda$ and is also equal to the period of γ.

For a fixed Reeb orbit γ, the linearization of the Reeb flow of R induces a symplectic linear map $P_\gamma : (\xi_{\gamma(0)}, d\lambda) \to (\xi_{\gamma(0)}, d\lambda)$, called the *linearized return map.* A Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z}$ is called *nondegenerate* if its linearized return map P_γ does not have 1 as an eigenvalue. We call γ *elliptic* if the eigenvalues of P_γ are complex conjugate on the unit circle, *positive hyperbolic* if the eigenvalues of P_γ are real and positive, and *negative hyperbolic* if the eigenvalues of P_γ are real and negative. A contact form λ is called *nondegenerate* if all its Reeb orbits are nondegenerate.

1.2.2 Reeb dynamics on ∂X_Ω

In this section we compute the Reeb dynamics on the boundary of convex toric domains. Recall that a convex toric domain $X_\Omega \subset \mathbb{R}^4$ is defined by (1.1.2), with defining set Ω given by (1.1.3). Similarly to the computations in [25, §4.3], we choose scaled polar coordinates $(z_1, z_2) = (\sqrt{r_1/\pi}e^{i\theta_1}, \sqrt{r_2/\pi}e^{i\theta_2})$ on $\mathbb{C}^2$ to obtain

$$\omega_{\text{std}} = \frac{1}{2\pi}\left(dr_1 \wedge d\theta_1 + dr_2 \wedge d\theta_2\right).$$

The radial vector field

$$\rho = r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2}$$

is a Liouville vector field for ω_{std} defined on all $\mathbb{R}^4$. The boundary of the toric domain ∂X_Ω is transverse to ρ and so

$$\lambda_{\text{std}} = \iota_\rho \omega_{\text{std}} = \frac{1}{2\pi}\left(r_1 d\theta_1 + r_2 d\theta_2\right)$$

restricts to a contact form on ∂X_Ω. The Reeb vector field R corresponding to λ_{std} has the following expression. In the two coordinate planes, R is given by

$$R = \begin{cases} \frac{2\pi}{a}\frac{\partial}{\partial \theta_1} & \text{if } z_2 = 0 \\ \frac{2\pi}{f(0)}\frac{\partial}{\partial \theta_2} & \text{if } z_1 = 0. \end{cases}$$

While if $\pi(|z_1|^2, |z_2|^2) = (r_1, r_2) = (x, f(x))$ for some $x \in (0, a)$ with $f'(x) = \tan\phi$, $\phi \in [-\pi/2, 0]$, then

$$R = \frac{2\pi}{-x\sin\phi + f(x)\cos\phi}\left(-\sin\phi \frac{\partial}{\partial \theta_1} + \cos\phi \frac{\partial}{\partial \theta_2}\right).$$

The embedded Reeb orbits of $\lambda_{\text{std}}|_{\partial X_\Omega}$ are classified as follows:

- The circle $e_{0,1} = \partial X_\Omega \cap \{z_2 = 0\}$ is an embedded elliptic Reeb orbit with action $\mathcal{A}(e_{0,1}) = a$.

- The circle $e_{1,0} = \partial X_\Omega \cap \{z_1 = 0\}$ is an embedded elliptic Reeb orbit with action $\mathcal{A}(e_{1,0}) = f(0)$.

- For each $x \in (0, a)$ with $f'(x) \in \mathbb{Q}$ and $f''(x) \neq 0$, the torus

  $$\{z \in \partial X_\Omega | \pi(|z_1|^2, |z_2|^2) = (x, f(x))\}$$

  is foliated by a Morse-Bott circle of Reeb orbits. If $f'(x) = -\frac{p}{q}$ with p, q relatively prime positive integers, then we call this torus $T_{p,q}$ and we compute that each orbit in this family has action $\mathcal{A} = qx + pf(x)$.

Remark 1.2.1. The existence of Morse-Bott circles of Reeb orbits implies that the contact form $\lambda_{\mathrm{std}}|_{\partial X_\Omega}$ is degenerate. We need to perturb it in order to make it nondegenerate since the nondegeneracy allows the study of J–holomorphic curves with cylindrical ends asymptotic to Reeb orbits.

For each $\epsilon > 0$, we can perturb $\lambda_{\mathrm{std}}|_{\partial X_\Omega}$ to a nondegenerate $\lambda = h\lambda_{\mathrm{std}}|_{\partial X_\Omega}$, where $||h - 1||_{C^0} < \epsilon$, so that each Morse-Bott family $T_{p,q}$ that has action $\mathcal{A} < 1/\epsilon$ becomes two embedded Reeb orbits of approximately the same action, more specifically an elliptic orbit $e_{p,q}$ and a hyperbolic orbit $h_{p,q}$. Moreover, no Reeb orbits of action $\mathcal{A} < 1/\epsilon$ are created and the Reeb orbits $e_{0,1}$ and $e_{1,0}$ are unaffected.

Such a perturbation of the contact form is equivalent to a perturbation of the hypersurface ∂X_Ω on which the restriction of λ_{std} becomes nondegenerate.

1.2.3 ECH index

Embedded contact homology (ECH) is an invariant for 3–dimensional contact manifolds due to Hutchings. See [25] for a detailed account of history, motivation, construction, and applications of ECH. We give a brief overview of the definition of ECH following the notation from [26].

Let (Y, λ) be a contact 3–dimensional manifold with nondegenerate contact form λ. Given a convex toric domain X_Ω, the boundary ∂X_Ω together with a perturbation of $\lambda_{\mathrm{std}}|_{\partial X_\Omega}$, as in Remark 1.2.1, is such a contact manifold.

An *orbit set* is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$, where α_i are distinct embedded Reeb orbits and m_i are positive integers. We will also use the multiplicative notation $\alpha = \prod \alpha_i^{m_i}$ for an orbit set $\alpha = \{(\alpha_i, m_i)\}$. Denote by $[\alpha]$ the sum $\sum_i m_i[\alpha_i] \in H_1(Y)$ and define the action of α by $\mathcal{A}(\alpha) = \sum_i m_i \mathcal{A}(\alpha_i)$. If $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ are two orbit sets with $[\alpha] = [\beta] \in H_1(Y)$, then define $H_2(Y, \alpha, \beta)$ to be the set of relative homology classes of 2–chains A such that $\partial A = \sum m_i\alpha_i - \sum n_j\beta_j$. Note that $H_2(Y, \alpha, \beta)$ is an affine space over $H_2(Y)$.

Given a $Z \in H_2(Y, \alpha, \beta)$, define the *ECH index* of Z by the formula

$$I(\alpha, \beta, Z) = c_\tau(Z) + Q_\tau(Z) + CZ_\tau^I(\alpha) - CZ_\tau^I(\beta) \tag{1.2.1}$$

where τ is a choice of symplectic trivializations of ξ over the Reeb orbits α_i and β_j, $c_\tau(Z) = c_1(\xi|_Z, \tau)$ denotes the relative first Chern class (see [26, §2.5]), $Q_\tau(Z)$ denotes the relative self-intersection number (see [26, §2.7]), and

$$CZ_\tau^I(\alpha) = \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k),$$

where $CZ_\tau(\gamma)$ is the Conley–Zehnder index with respect to τ of the orbit γ (see [26, §2.3]).

The ECH index does not depend on the choice of symplectic trivialization. The definition of the ECH index I can be extended to symplectic cobordisms by generalizing the definitions of the relative first Chern class and of the self intersection number (see [26, §4.2]).

If $Z \in H_2(Y, \alpha, \beta)$ and $W \in H_2(Y, \beta, \gamma)$, then $I(Z+W) = I(Z)+I(W)$. In the particular case of starshaped hypersurfaces in $\mathbb{R}^4$, this implies there is an absolute $\mathbb{Z}$ grading on orbit sets as follows. Since $H_2(Y) = H_2(S^3) = 0$, for every pair of orbit sets α and β there is an unique class $Z \in H_2(Y, \alpha, \beta)$. Define $I(\emptyset) = 0$ for the empty orbit set and set

$$I(\alpha) := I(\alpha, \emptyset, Z) \in \mathbb{Z},$$

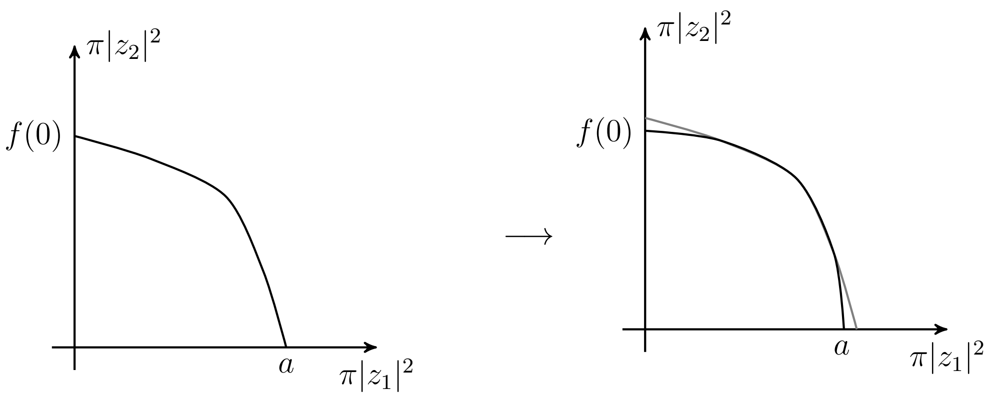
where Z is the unique element of $H_2(Y, \alpha, \emptyset)$. Also, let $c_\tau(\alpha) := c_\tau(Z)$ and $Q_\tau(\alpha) := Q_\tau(Z)$.

1.2.4 Absolute grading on ∂X_Ω

Following the details in [24, §5], we recall the classification of the orbit sets on the boundary of a convex toric domain X_Ω that have ECH index $I \leq 4$.

Similarly to [24, Lemma 5.4], we first perform a perturbation of the geometry of ∂X_Ω (see Figure 1.4). This means we can assume, without loss of generality, that the function $f : [0, a] \to \mathbb{R}_{\geq 0}$ defining Ω is *nice*, meaning that f satisfies the following properties:

- f is smooth,
- $f'(0)$ is irrational and is approximately 0,
- $f'(a)$ is irrational and is very large, close to $-\infty$,
- $f''(x) < 0$ except for x in small connected neighborhoods of 0 and a.



(a) Original toric domain

(b) Defining function perturbed to be nice

Figure 1.4: Perturbating X_Ω to a nice position

Lemma 1.2.2 ([24, Example 1.12]). *Let X_Ω be a convex toric domain defined by a nice function f. Let λ be a nondegenerate contact structure obtained by perturbing $\lambda_{\mathrm{std}}|_{\partial X_\Omega}$ up to sufficiently large action. Then the orbit sets with ECH index $I \leq 4$ are classified as follows.*

- $I = 0$: $\emptyset$.
- $I = 1$: *no orbit sets.*
- $I = 2$: $e_{0,1}$ *and* $e_{1,0}$.
- $I = 3$: $h_{1,1}$.
- $I = 4$: $e_{0,1}^2$, $e_{1,1}$, *and* $e_{1,0}^2$.

In general, the classification of orbit set generators, up to larger ECH index and action, provides a combinatorial model to compute the sequence of ECH capacities of a convex toric domain using the following formula.

Lemma 1.2.3 ([24, Lemma 5.6])**.** *For a convex toric domain X_Ω and a nonnegative integer k,*

$$c_k^{\mathrm{ECH}}(X_\Omega) = \min\{\mathcal{A}(\alpha) \mid I(\alpha) = 2k\}.$$

In particular, the equalities claimed in Proposition 1.1.7 hold. Moreover, one can deduce the following lemma which we will use to rule out breaking.

Lemma 1.2.4. *For a convex toric domain X_Ω, orbit sets α with ECH index $I(\alpha) \geq 5$ have action $\mathcal{A}(\alpha) \geq c_2^{\mathrm{ECH}}(X_\Omega)$.*

1.3 Ruling out breaking

1.3.1 Completed symplectic cobordisms

Let $(Y_\pm, \lambda_\pm)$ be closed contact 3–dimensional manifolds. A *compact symplectic cobordism* from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic manifold (W, ω) with boundary $\partial W = -Y_- \sqcup Y_+$ such that $\omega|_{Y_\pm} = d\lambda_\pm$.

Given a compact symplectic cobordism (W, ω), one can find neighborhoods N_- of Y_- and N_+ of Y_+ in W, and symplectomorphisms

$$(N_-, \omega) \to ([0, \epsilon) \times Y_-, d(e^s \lambda_-))$$

and

$$(N_+, \omega) \to ((-\epsilon, 0] \times Y_+, d(e^s \lambda_+)),$$

where s denotes the coordinate on $[0, \epsilon)$ and $(-\epsilon, 0]$. Using these identifications, we can complete the compact symplectic cobordism (W, ω) by adding cylindrical ends $(-\infty, 0] \times Y_-$ and $[0, \infty) \times Y_+$ to obtain the *completed symplectic cobordism*

$$\widehat{W} = [0, \infty) \times Y_+ \cup_{Y_+} W \cup_{Y_-} (-\infty, 0] \times Y_-.$$

In accordance with [2], we restrict the class of almost complex structures on a completed cobordism $\widehat{W}$ as follows. An almost complex structure J on a completed symplectic cobordism $\widehat{W}$ as above is called *compatible* (in [2], the authors use the term *adjusted*) if:

- On $[0,\infty)\times Y_+$ and $(-\infty,0]\times Y_-$, the almost complex structure J is $\mathbb{R}$–invariant, maps ∂_s (the $\mathbb{R}$ direction) to $R_{\lambda_\pm}$, and maps $\xi_\pm$ to itself compatibly with $d\lambda_\pm$.
- On the compact symplectic cobordism W, the almost complex structure J is tamed by ω.

Call $\mathcal{J}(\widehat{W})$ the set of all such compatible almost complex structures on $\widehat{W}$.

Choose a compatible almost complex structure $J \in \mathcal{J}(\widehat{W})$ on $\widehat{W}$ and let (Σ, j) be a compact Riemann surface. We will consider curves

$$u : (\dot{\Sigma} = \Sigma \setminus \{x_1, \dots, x_k, y_1, \dots, y_l\}, j) \to (\widehat{W}, J)$$

that are J–holomorphic, i.e. $du \circ j = J \circ du$, and have k positive ends at $\Gamma^+ = (\gamma_1^+, \dots, \gamma_k^+)$ corresponding to the punctures $(x_1, \dots, x_k)$, and l negative ends at $\Gamma^- = (\gamma_1^-, \dots, \gamma_l^-)$ corresponding to the punctures $(y_1, \dots, y_l)$. Denote by $\mathcal{M}_J(\Gamma^+, \Gamma^-)$ the space of such J–holomorphic curves u modulo reparametrizations of the domain $\dot{\Sigma}$.

Recall that a *positive end* of u at γ means a puncture, near which u is asymptotic to $\mathbb{R} \times \gamma$. More specifically, that means there is a choice of coordinates $(s,t) \in [0,\infty) \times \mathbb{R}/T\mathbb{Z}$ on a neighborhood of the puncture, with $j(\partial_s) = \partial_t$ and such that $\lim_{s\to\infty} \pi_{\mathbb{R}}(u(s,t)) = \infty$ and $\lim_{s\to\infty} \pi_{Y_+}(u(s,\cdot)) = \gamma$. Similarly, at a *negative end* there is a choice of coordinates $(s,t) \in (-\infty, 0] \times \mathbb{R}/T\mathbb{Z}$ on a neighborhood of the puncture, with $j(\partial_s) = \partial_t$ and such that $\lim_{s\to-\infty} \pi_{\mathbb{R}}(u(s,t)) = \infty$ and $\lim_{s\to-\infty} \pi_{Y_-}(u(s,\cdot)) = \gamma$.

Given a J–holomorphic curve u as above, define the *Fredholm index* of u by

$$\operatorname{ind}(u) = -\chi(u) + 2c_\tau(u) + \sum_{i=1}^{k} CZ_\tau(\gamma_i^+) - \sum_{j=1}^{l} CZ_\tau(\gamma_j^-), \tag{1.3.1}$$

where τ is a trivialization of ξ over $\gamma_i^\pm$ that is symplectic with respect to $d\lambda$, $\chi(u)$ is the Euler characteristic of $\dot{\Sigma}$, $c_\tau(u) := c_1(u^*\xi, \tau)$ denotes the relative first Chern class, and $CZ_\tau(\gamma_i^\pm)$ is the Conley–Zehnder index with respect to τ, as before. The significance of the Fredholm index is that for a generic choice of compatible almost complex structure J and for a somewhere–injective J–holomorphic curve u, the moduli space $\mathcal{M}_J(\Gamma^+, \Gamma^-)$ is a manifold of dimension $\operatorname{ind}(u)$ near u. See [44, §6] for more details.

1.3.2 Moduli spaces

Let X_{Ω_1} and X_{Ω_2} be two convex toric domains defined by nice functions $f_1 : [0,a] \to \mathbb{R}_{\geq 0}$ and $f_2 : [0,c] \to \mathbb{R}_{\geq 0}$, respectively. Also, let $\varphi : X_{\Omega_1} \to X_{\Omega_2}$ be a symplectic embedding. The manifold $W_\varphi := X_{\Omega_2} \setminus \operatorname{int}(X_{\Omega_1})$ is a compact symplectic cobordism from $(\partial X_{\Omega_2}, \lambda_{\text{std}}|_{\partial X_{\Omega_2}})$ to $(\partial X_{\Omega_1}, \lambda_{\text{std}}|_{\partial X_{\Omega_1}})$, where λ_{std} denotes the standard Liouville form on $\mathbb{R}^4$.

Following the explanation in Remark 1.2.1, perturb the boundary components ∂X_{Ω_1} and ∂X_{Ω_2} of W_φ in such a way that the Liouville form λ_{std} restricts to nondegenerate contact forms λ_1 and λ_2 on ∂X_{Ω_1} and ∂X_{Ω_2}, respectively. Add cylindrical ends to W_φ and call $\widehat{W_\varphi}$ the completed symplectic cobordism.

To clean up notation, call γ_a the $e_{0,1}$ embedded Reeb orbit on ∂X_{Ω_1}, and call γ_c the $e_{0,1}$ embedded Reeb orbit on ∂X_{Ω_2}. Recall that $\mathcal{A}(\gamma_a) = a$ and $\mathcal{A}(\gamma_c) = c$.

For a given almost complex structure $J \in \mathcal{J}(\widehat{W_\varphi})$, define $\mathcal{M}_J(\varphi)$ to be the moduli space of J–holomorphic cylinders $u : (\mathbb{R} \times S^1, j) \to (\widehat{W_\varphi}, J)$ such that u has a positive end at γ_c and a negative end at γ_a, modulo translation and rotations of the domain $\mathbb{R} \times S^1$.

All such J–holomorphic cylinders have Fredholm index $\operatorname{ind}(u) = 0$ and the automatic transversality result in Lemma 1.3.1 below implies that $\mathcal{M}_J(\varphi)$ is a 0–dimensional manifold for any choice of J. Moreover, $\mathcal{M}_J(\varphi)$ can be compactified with broken holomorphic curves using the SFT compactness theorem, [2, Theorem 10.2], since all the J–holomorphic cylinders in $\mathcal{M}_J(\varphi)$ have the same asymptotics.

1.3.3 Automatic transversality

A much more general automatic transversality result than the one we need to use is proven by Wendl in [42]. In the language employed in this paper, the particular case that we need to use is stated as follows. See also [27, Lemma 4.1] for a very similar statement and proof in the case of symplectizations.

Lemma 1.3.1. *Let $\widehat{W}$ be a completed symplectic cobordism and let $u : \dot{\Sigma} \to \widehat{W}$ be an immersed J–holomorphic curve that has asymptotic ends to Reeb orbits. Let N denote the normal bundle to u in $\widehat{W}$ and*

$$D_u : L^2(\Sigma, N) \to L^2(\Sigma, T^{0,1}\mathbb{C} \otimes N)$$

denote the normal linearized operator of u. Also let $h^+(u)$ denote the number of ends of u at positive hyperbolic orbits. If

$$2g(\Sigma) - 2 + h^+(u) < \operatorname{ind}(u),$$

then D_u is surjective, i.e. the moduli space of J–holomorphic curves near u is a manifold that is cut out transversely and has dimension $\operatorname{ind}(u)$.

Note that there are no genericity assumptions on the almost complex structure J in Lemma 1.3.1. Also, the result applies to the J–holomorphic cylinders in $\mathcal{M}_J(\varphi)$ since they have ends only at elliptic Reeb orbits and the adjunction formula introduced below in (1.4.3) implies that they are embedded. Hence $\mathcal{M}_J(\varphi)$ is cut out transversely, for any choice of compatible almost complex structure J.

1.3.4 Ruling out breaking

In this section, we study the possible boundary of the union $\sqcup_{J\in\mathfrak{J}}\mathcal{M}_J(\varphi)$, where $\mathfrak{J}$ is a smooth parametrized family of compatible almost complex structures. We prove that, assuming the bounds in the hypothesis of Theorem 1.1.8, a sequence of cylinders in $\sqcup_{J\in\mathfrak{J}}\mathcal{M}_J(\varphi)$ cannot converge to a broken holomorphic building with multiple levels.

Proposition 1.3.2. *Assume X_{Ω_1} and X_{Ω_2} are convex toric domains satisfying the bounds in the hypothesis of Theorem 1.1.8. Let $\{\varphi_i \in \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})\}_{i\geq 1}$ be a sequence of symplectic embeddings, C^0–converging to $\varphi_0 \in \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})\}$. Let $\{J_i \in \mathcal{J}(\widehat{W}_{\varphi_i})\}_{i\geq 1}$ be a sequence of compatible almost complex structures converging to $J_0 \in \mathcal{J}(\widehat{W}_{\varphi_0})$. Let $u_i \in \mathcal{M}_{J_i}(\varphi_i)$. Then the sequence $\{u_i\}_{i\geq 1}$ cannot converge in the sense of [2] to a J_0–holomorphic building with more than one level.*

Proof. In general, if there exists a J–holomorphic curve from the orbit set α to the orbit set β, then $\mathcal{A}(\alpha) \geq \mathcal{A}(\beta)$. Assume that, in the limit, the cylinders u_i break into a J_0–holomorphic building $u_0 = (v_1, v_2, \ldots, v_l)$, where v_1 denotes the top level. Assume that α_j is the orbit set at which the level v_j has negative ends. Then $\mathcal{A}(\alpha_j) \in [a, c]$. Note first that c is the lowest action of an orbit set in ∂X_{Ω_2}. This means that v_1 lives in the cobordism level. Secondly, the assumption $c_1^{\mathrm{ECH}}(X_{\Omega_2}) < c_2^{\mathrm{ECH}}(X_{\Omega_1})$ translates to

$$c < \min(2a, \mathcal{A}(e_{1,1}), 2f_1(0)) = \min(\mathcal{A}(\gamma_a^2), \mathcal{A}(e_{1,1}), \mathcal{A}(e_{1,0}^2)),$$

where $\gamma_a = e_{0,1}$, $e_{1,1}$, and $e_{1,0}$ are the Reeb orbits on ∂X_{Ω_1}. Thirdly, for a small enough perturbation of ∂X_{Ω_1}, we also have $c < \mathcal{A}(h_{1,1})$ since $\mathcal{A}(h_{1,1})$ is approximately $\mathcal{A}(e_{1,1})$. Lastly, Lemma 1.2.4 implies that all orbit sets α on ∂X_{Ω_1} with $I(\alpha) \geq 5$ satisfy $c < \mathcal{A}(\alpha)$.

Using the classification by ECH index in Lemma 1.2.2, together with the action inequalities above, we conclude that the only orbit set through which the cylinders u_i could hypothetically break is $\alpha = e_{0,1}$. This means that the only broken building we still have to rule out is $u_0 = (v_1, v_2)$, where v_1 is a Fredholm index 0 cylinder from γ_c to $e_{1,0}$ in the cobordism level and v_2 is a Fredholm index 0 cylinder from $e_{1,0}$ to $\gamma_a = e_{0,1}$ in the lower symplectization level. The nontrivial cylinder v_2 is a Fredholm index 0 J_0–holomorphic cylinder in a symplectization, and so, by automatic transversality, it cannot appear.

$\square$

Proposition 1.3.2 together with the automatic transversality from Lemma 1.3.1, and SFT compactness, [2, Theorem 10.2], imply that $\mathcal{M}_J(\varphi)$ is a compact 0–dimensional manifold, i.e. a finite set of points.

1.4 Proof of main theorem

1.4.1 Nonemptiness of moduli spaces

First, we prove the nonemptiness of $\mathcal{M}_{\widehat{J}}(\varphi_0)$ for the inclusion map $\varphi_0 : X_{\Omega_1} \to X_{\Omega_2}$ and a certain compatible almost complex structure $\widehat{J}$.

Proposition 1.4.1. *There exists $\widehat{J} \in \mathcal{J}(\widehat{W}_{\varphi_0})$ such that the moduli space $\mathcal{M}_{\widehat{J}}(\varphi_0)$ is nonempty.*

Proof. We will construct a compatible almost complex structure $\widehat{J}$ that is invariant under the S^1–action by rotations in the z_2–plane and prove that an appropriate restriction of the z_1–plane is the $\widehat{J}$–holomorphic cylinder we are looking for. Our construction is similar to [5, §5.2]. Whenever we say "S^1–equivariant", we mean invariant under the S^1–action by rotations in the z_2–plane.

Recall that ∂X_{Ω_1} and ∂X_{Ω_2} are contact hypersurfaces in the compact symplectic cobordism $(W_{\varphi_0}, \omega_{\text{std}} = d\lambda_{\text{std}})$. Moreover, notice that they are S^1–equivariant. Using an S^1–equivariant version of the Moser trick, one can prove that there exist S^1–equivariant neighborhoods N_1 of ∂X_{Ω_1} and N_2 of ∂X_{Ω_2} in W_{φ_0}, and S^1–equivariant symplectomorphisms

$$\psi_1 : (N_1, \omega) \to ([0, \epsilon) \times \partial X_{\Omega_1}, d(e^s \lambda_1))$$

and

$$\psi_2 : (N_2, \omega) \to ((-\epsilon, 0] \times \partial X_{\Omega_2}, d(e^s \lambda_2)),$$

where $\lambda_i = \lambda_{\text{std}}|_{\partial X_{\Omega_i}}$, and s denotes the coordinate on $[0, \epsilon)$ and $(-\epsilon, 0]$.

Choose almost complex structures J_1 on $((0, \frac{\epsilon}{3}) \cup (\frac{2\epsilon}{3}, \epsilon)) \times \partial X_{\Omega_1}$ and J_2 on $((-\epsilon, -\frac{2\epsilon}{3}) \cup (-\frac{\epsilon}{3}, 0)) \times \partial X_{\Omega_2}$, that are S^1–equivariant and compatible with the cylindrical ends near the boundary of W_{φ_0}, and that pull back under ψ_i to the standard complex structure on $\mathbb{C}^2$ near the interior of W_{φ_0}, i.e. $\psi_1^*(J_1|_{(\frac{2\epsilon}{3}, \epsilon) \times \partial X_{\Omega_1}}) = i$ and $\psi_2^*(J_2|_{(-\epsilon, -\frac{2\epsilon}{3}) \times \partial X_{\Omega_2}}) = i$. Define

$$\widehat{J}(p) := \begin{cases} \psi_1^*(J_1(\psi_1(p))), & p \in \psi_1^{-1}((0, \frac{\epsilon}{3}) \cup (\frac{2\epsilon}{3}, \epsilon)) \times \partial X_{\Omega_1}) \\ i, & p \in W_{\varphi_0} \setminus (N_1 \cup N_2) \\ \psi_2^*(J_2(\psi_2(p))), & p \in \psi_2^{-1}((-\epsilon, -\frac{2\epsilon}{3}) \cup (-\frac{\epsilon}{3}, 0) \times \partial X_{\Omega_2}). \end{cases} \tag{1.4.1}$$

The compatibility of $\widehat{J}$ with the cylindrical ends near the boundary of the compact symplectic cobordism W_{φ_0} makes it possible to extend $\widehat{J}$ to a compatible S^1–equivariant almost complex structure on the cylindrical ends of the completed symplectic cobordism $\widehat{W}_{\varphi_0}$. We still need to interpolate between the standard complex structure in the interior of W_{φ_0} and the almost complex structure on the cylindrical ends.

Let $g(\cdot, \cdot) := \omega(\cdot, \widehat{J}\cdot)$ be the positive definite Riemannian metric defined by the compatibility of ω and $\widehat{J}$ and note that g is S^1–equivariant. Extend the Riemannian metric g to W_{φ_0} and average the obtained extension over the S^1–action to obtain an S^1–equivariant Riemannian metric $\widehat{g}$ on W_{φ_0}. Note that $\widehat{g} = g$ wherever g is defined since g is S^1–equivariant. Define $\widehat{J}$ to be the unique compatible almost complex structure that satisfies $\widehat{g}(\cdot, \cdot) = \omega(\cdot, \widehat{J}\cdot)$ and note that this definition extends the definition in (1.4.1), since $\widehat{g} = g$ wherever g is defined. Note that since $\widehat{g}$ and ω_{std} are S^1–equivariant, then $\widehat{J}$ is also S^1–equivariant.

Let $S := W_{\varphi_0} \cap \{z_1 = 0\}$. Note that S is a closed annulus which we can complete by adding cylindrical ends to get

$$\widehat{S} := (-\infty, 0] \times \gamma_a \cup S \cup [0, \infty) \times \gamma_c.$$

We will now show that $\widehat{J}$ being invariant under the S^1–action in the z_2–plane implies that $\widehat{J}$ preserves the tangent space of $\widehat{S}$. Let $h_\theta(z_1, z_2) := (z_1, e^{i\theta} z_2)$, for $\theta \in [0, 2\pi]$. Knowing $\widehat{J}$ is invariant under the S^1–action in the z_2–plane implies that

$$\widehat{J}_{h_\theta(p)} \circ d_p h_\theta = d_p h_\theta \circ \widehat{J}_p,$$

for any $p \in W_{\varphi_0}$ and any $\theta \in [0, 2\pi]$. In the basis $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right\}$, this equality can be written in 2×2 block matrix notation as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{h_\theta(p)} \begin{pmatrix} I & 0 \\ 0 & R_\theta \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & R_\theta \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}_p, \tag{1.4.2}$$

for any $p \in W_{\varphi_0}$ and any $\theta \in [0, 2\pi]$, and where $\widehat{J}_p = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_p$ is the almost complex structure in coordinates and $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a rotation matrix. After carrying out the multiplications in (1.4.2), we see that

$$\begin{pmatrix} A_{h_\theta(p)} & B_{h_\theta(p)} R_\theta \\ C_{h_\theta(p)} & D_{h_\theta(p)} R_\theta \end{pmatrix} = \begin{pmatrix} A_p & B_p \\ R_\theta C_p & R_\theta D_p \end{pmatrix}.$$

Note that for $p = (z_1, 0)$, $h_\theta(p) = p$, and so the above equality implies $B_p R_\theta = B_p$ for any $p \in S$ and $\theta \in [0, 2\pi]$. This implies $B_p = 0$ and hence, $\widehat{J}$ preserves the tangent bundle of S. Moreover, by construction, $\widehat{J}$ preserves the tangent spaces on the cylindrical ends of $\widehat{S}$ and so $\widehat{J}$ preserves the tangent bundle of $\widehat{S}$.

Hence, $(\widehat{S}, \widehat{J})$ is a Riemann surface which is diffeomorphic to a punctured plane. By the Uniformization theorem, $(\widehat{S}, \widehat{J})$ is biholomorphically equivalent to either the punctured plane, the punctured disk, or an open annulus. Since $\widehat{J}$ is compatible with the infinite cylindrical ends of $\widehat{W}_{\varphi_0}$, $(\widehat{S}, \widehat{J})$ must be biholomorphic to a punctured plane, and hence also biholomorphic to a cylinder. We conclude that there exists a $\widehat{J}$–holomorphic map $u : (\mathbb{R} \times S^1, j) \to (\widehat{W}_{\varphi_0}, \widehat{J})$ with image $\widehat{S}$, and hence, $[u] \in \mathcal{M}_{\widehat{J}}(\varphi_0)$.

Finally, note that the perturbation of the hypersurfaces ∂X_{Ω_i}, for $i = 1, 2$, needed to make $\lambda_{\text{std}}|_{\partial X_{\Omega_i}}$ nondegenerate, happens away from the z_1–plane and so the curve $[u]$ persists after the perturbation. $\square$

Remark 1.4.2. All the symplectic embeddings that form the loop considered in Theorem 1.1.8 have the same image in X_{Ω_2}, so $\widehat{W}_{\varphi_t} = \widehat{W}_{\varphi_0}$, for any $t \in [0, 1]$. Hence the moduli space $\mathcal{M}_{\widehat{J}}(\varphi_t)$ contains the same $\widehat{J}$–holomorphic cylinders as $\mathcal{M}_{\widehat{J}}(\varphi_0)$.

1.4.2 Counting the cylinders

We next prove the uniqueness of the J–holomorphic cylinders using asymptotic analysis estimates. Let us begin by recalling the *adjunction formula*:

Lemma 1.4.3. *Let $u : \dot{\Sigma} \to X$ be a somewhere–injective J–holomorphic curve. Then u has finitely many singularities, and*

$$c_\tau(u) = \chi(u) + Q_\tau(u) + w_\tau(u) - 2\delta(u) \tag{1.4.3}$$

where $c_\tau(u)$ is the relative first Chern class as before (see [26, §4.2]), $\chi(u)$ is the Euler characteristic of the domain of u, $Q_\tau(u)$ is the relative self intersection number as before (see [26, §4.2]), $w_\tau(u)$ is the asymptotic writhe defined in [26, §2.6], and $\delta(u)$ is a count of singularities of u with positive integer weights.

For a proof of this statement, see [23, §3]. Following the details in [26, §2.6], we give an overview of the definition of writhe, linking number, and winding number in this context, as they will become useful in the proof of Proposition 1.4.6 below.

Let γ be a simple Reeb orbit and let k be a positive integer. A *braid* with k strands around γ is an oriented link ζ contained in a tubular neighborhood N of γ, such that the tubular neighborhood projection $\zeta \to \gamma$ is an orientation–preserving degree k submersion.

Choose a symplectic trivialization τ over γ and extend it to the tubular neighborhood N of γ to identify N with $S^1 \times \mathbb{D}$, such that the projection of $\zeta \subset N$ to the S^1 factor is a submersion. Identify further $S^1 \times \mathbb{D}$ with a solid torus in $\mathbb{R}^3$ by applying an orientation preserving diffeomorphism. We thus obtain an embedding $\phi_\tau : N \to \mathbb{R}^3$. We set up the identifications in such a way that $\phi_\tau(\zeta)$ is an oriented link in $\mathbb{R}^3$ with no vertical tangents. Hence, it has a well defined writhe by counting signed self–crossings in the projection to $\mathbb{R}^2 \times \{0\}$. We use the sign convention where counterclockwise twists contribute positively to the writhe.

We define the *writhe* of a braid ζ around γ, $w_\tau(\zeta) \in \mathbb{Z}$, to be the writhe of the oriented link $\phi_\tau(\zeta)$ in $\mathbb{R}^3$. Also if ζ and ζ' are two disjoint braids around γ, define the *linking number* of ζ and ζ', $l_\tau(\zeta, \zeta') \in \mathbb{Z}$, to be the linking number of the oriented links $\phi_\tau(\zeta)$ and $\phi_\tau(\zeta')$ in $\mathbb{R}^3$. This latter quantity is defined as one half the signed count of crossings of the projections of the two links to $\mathbb{R}^2 \times \{0\}$. Note that, if ζ and ζ' are two disjoint braids around γ then

$$w_\tau(\zeta \cup \zeta') = w_\tau(\zeta) + w_\tau(\zeta') + 2l_\tau(\zeta, \zeta').$$

For a braid ζ around γ that is disjoint from γ we define the *winding number* of ζ around γ to be $\mathrm{wind}_\tau(\zeta) := l_\tau(\zeta, \gamma)$.

The following two lemmas explain how to bound the writhe and the winding number in terms of the Conley–Zehnder index. The formulation is adapted from [27]. For more details, see also [23].

Lemma 1.4.4 ([27, Lemma 3.2])**.** *Let γ be an embedded Reeb orbit and let N be a tubular neighborhood around γ. Let $u : \dot{\Sigma} \to \mathbb{R} \times Y$ be a J–holomorphic curve with a positive end at γ^d which is not part of a trivial cylinder or a multiply covered component and let ζ denote the intersection of this end with $\{s\} \times Y$. If $s >> 0$, then the following hold:*

a. ζ is the graph in N of a nonvanishing section of ξ_{γ^d} and has well defined winding number $\mathrm{wind}_\tau(\zeta)$*.*

b. $\mathrm{wind}_\tau(\zeta) \leq \left\lfloor \frac{CZ_\tau(\gamma^d)}{2} \right\rfloor$.

c. If J is generic, $CZ_\tau(\gamma^d)$ is odd, and if $\mathrm{ind}(u) \leq 2$ *then equality holds in (b).*

d. $w_\tau(\zeta) \leq (d-1)\mathrm{wind}_\tau(\zeta)$.

An equivalent statement holds for the asymptotic winding number and writhe at a negative cylindrical end of a J–holomorphic curve.

Lemma 1.4.5 ([27, Lemma 3.4])**.** *Let γ be an embedded Reeb orbit and let N be a tubular neighborhood around γ. Let $u : \dot{\Sigma} \to \mathbb{R} \times Y$ be a J–holomorphic curve with a negative end at γ^d which is not part of a trivial cylinder or a multiply covered component and let ζ denote the intersection of this end with $\{s\} \times Y$. If $s << 0$, then the following hold:*

a. ζ is the graph in N of a nonvanishing section of ξ_{γ^d} and has well defined winding number $\mathrm{wind}_\tau(\zeta)$.

b. $\mathrm{wind}_\tau(\zeta) \geq \left\lceil \frac{CZ_\tau(\gamma^d)}{2} \right\rceil$.

c. If J is generic, $CZ_\tau(\gamma^d)$ is odd, and if $\mathrm{ind}(u) \leq 2$ *then equality holds in (b).*

d. $w_\tau(\zeta) \geq (d-1)\mathrm{wind}_\tau(\zeta)$.

Fix a symplectic embedding $\varphi \in \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$ and fix an almost complex structure $J \in \mathcal{J}(\widehat{W}_\varphi)$.

Proposition 1.4.6. *If the moduli space $\mathcal{M}_J(\varphi)$ is nonempty, then it contains exactly one index zero cylinder.*

Proof. Assume there are two different cylinders, u_1 and u_2, in $\mathcal{M}_J(\varphi)$. For $s << 0$, $\zeta_a = (u_1 \cup u_2) \cap (\{s\} \times \partial X_{\Omega_1})$ is a braid around γ_a with two components, ζ_1^a and ζ_2^a, each having one strand. For $s >> 0$, $\zeta_c = (u_1 \cup u_2) \cap (\{s\} \times \partial X_{\Omega_2})$ is a braid around γ_c with two components, ζ_1^c and ζ_2^c, each with one strand. Lemma 1.4.4 implies

$$\mathrm{wind}_\tau(\zeta_i^c) \leq \left\lfloor \frac{CZ_\tau(\gamma_c)}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0.$$

Similarly, Lemma 1.4.5 implies

$$\mathrm{wind}_\tau(\zeta_i^a) \geq \left\lceil \frac{CZ_\tau(\gamma_a)}{2} \right\rceil = \left\lceil \frac{1}{2} \right\rceil = 1.$$

The linking numbers of the different strands of the two braids are given by $l_\tau(\zeta_1^a, \zeta_2^a) = \mathrm{wind}(\zeta_2^a)$ and $l_\tau(\zeta_1^c, \zeta_2^c) = \mathrm{wind}(\zeta_2^c)$. See [26, Lemma 4.17] for details. This means

$$\begin{aligned} w_\tau(\zeta_a) &= w_\tau(\zeta_1^a \cup \zeta_2^a) = w_\tau(\zeta_1^a) + w_\tau(\zeta_2^a) + 2 \cdot l_\tau(\zeta_1^a, \zeta_2^a) \\ &= 0 + 0 + 2 \cdot \mathrm{wind}(\zeta_2^a) \geq 2 \end{aligned}$$

and

$$\begin{aligned} w_\tau(\zeta_c) &= w_\tau(\zeta_1^c \cup \zeta_2^c) = w_\tau(\zeta_1^c) + w_\tau(\zeta_2^c) + 2 \cdot l_\tau(\zeta_1^c, \zeta_2^c) \\ &= 0 + 0 + 2 \cdot \mathrm{wind}(\zeta_2^c) \leq 0. \end{aligned}$$

Hence

$$w_\tau(u_1 \cup u_2) = w_\tau(\zeta_c) - w_\tau(\zeta_a) \leq -2.$$

Since $c_\tau(u_1 \cup u_2) = Q_\tau(u_1 \cup u_2) = 0$, the relative adjunction formula recalled in (1.4.3) applied to $u_1 \cup u_2$ gives

$$0 = 0 + 0 + w_\tau(u_1 \cup u_2) - 2\delta(u_1 \cup u_2).$$

This is a contradiction since $w_\tau(u_1 \cup u_2) \leq -2$ and $\delta(u_1 \cup u_2) \geq 0$. □

1.4.3 Final steps of the proof

We have all the details needed to complete the proof of Theorem 1.1.8. Assume that the loop $\{\varphi_t\}_{t\in[0,1]}$ is contractible in $\mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$. This means there exists a 2–parameter family $\{\varphi_z\}_{z\in\mathbb{D}} \subset \mathrm{SympEmb}(X_{\Omega_1}, X_{\Omega_2})$, parametrized by the unit disk $\mathbb{D}$, such that $\{\varphi_z\}_{z\in\partial\mathbb{D}} = \{\varphi_t\}_{t\in[0,1]}$. The family of embeddings $\{\varphi_z\}_{z\in\mathbb{D}}$ generates a 2–parameter family of completed symplectic cobordisms $\{\widehat{W}_{\varphi_z}\}_{z\in\mathbb{D}}$. Let $\mathfrak{J} = \{J_z\}_{z\in\mathbb{D}}$ be a generic 2–parameter family of compatible almost complex structures such that $J_z \in \mathcal{J}(\widehat{W}_{\varphi_z})$ for every $z \in \mathbb{D}$ and $J_z = \widehat{J}$ for every $z \in \partial\mathbb{D}$, where $\widehat{J}$ is the almost complex structure constructed in Proposition 1.4.1. Remark 1.4.2 provides an explanation as to why we can choose the same almost complex structure $\widehat{J}$ for all $z \in \partial\mathbb{D}$.

Consider the moduli space

$$\mathcal{M}_{\mathfrak{J}} := \{(z, u_z) \mid z \in \mathbb{D},\ u_z \in \mathcal{M}_{J_z}(\varphi_z)\}.$$

Claim 1.4.7. *$\mathcal{M}_{\mathfrak{J}}$ is homeomorphic to the closed disk $\mathbb{D}$.*

Proof. By the parametric regularity theorem, [44, Theorem. 7.2 & Remark 7.4], for a generic choice of 2–parameter family of compatible almost complex structures $\mathfrak{J}$, the moduli space $\mathcal{M}_{\mathfrak{J}}$ is a 2–dimensional manifold that is cut out transversely. The holomorphic curves in $\mathcal{M}_{\mathfrak{J}}$ have fixed asymptotics and so, by the SFT compactness result presented in [2, Theorem 10.2], there exists a compactification of $\mathcal{M}_{\mathfrak{J}}$ with broken holomorphic buildings. Proposition 1.3.2 implies that, under the assumptions made in the hypothesis of Theorem 1.1.8, no such breaking is possible and so, $\mathcal{M}_{\mathfrak{J}}$ is already compact.

The automatic transversality result presented in Lemma 1.3.1, together with the nonemptiness result proved in Proposition 1.4.1 and the uniqueness result proved in Proposition 1.4.6, implies that $\mathcal{M}_{\mathfrak{J}}$ contains exactly one cylinder above each parameter $z \in \partial\mathbb{D}$ and at most one cylinder above each parameter $z \in \mathrm{int}\mathbb{D}$. Given that the moduli space $\mathcal{M}_{\mathfrak{J}}$ is compact, it must contain exactly one cylinder above every parameter $z \in \mathbb{D}$ and so we can conclude that $\mathcal{M}_{\mathfrak{J}}$ is homeomorphic to the disk $\mathbb{D}$. □

Let $\gamma_c : \mathbb{R}/c\mathbb{Z} \to \partial X_{\Omega_2}$ be the parametrization of γ_c such that $p = \gamma_c(0) = \left(\sqrt{\frac{c}{\pi}}, 0\right) \in \mathbb{C}^2$. There exists a unique representative $u_z : \mathbb{R} \times S^1 \to \widehat{W}_{\varphi_z}$ of the unique class in $\mathcal{M}_{J_z}(\varphi_z)$ such that $\lim_{s\to\infty} u_z(s,0) = p$. Define $p_z := \lim_{s\to-\infty} u_z(s,0)$. This construction induces a well defined composition of maps

$$\begin{array}{ccccccc} S^1 & \to & \operatorname{SympEmb}(X_{\Omega_1}, X_{\Omega_2}) & \to & \mathcal{M}_{\mathfrak{J}} & \to & \gamma_a \simeq S^1 \\ t & \mapsto & \varphi_t = \varphi_z & \mapsto & (z, [u_z]) & \mapsto & p_z. \end{array}$$

Claim 1.4.8. *The above composition is a degree -1 circle map.*

Proof. Remark 1.4.2 explains why for any two parameters $z, w \in \partial\mathbb{D}$, the moduli spaces $\mathcal{M}_{J_z}(\varphi_z)$ and $\mathcal{M}_{J_w}(\varphi_w)$ are the same. Moreover, note that the choice of fixed asymptotics, $\lim_{s\to\infty} u_z(s,0) = p = \lim_{s\to\infty} u_w(s,0)$, implies that the representatives u_z and u_w are also the same. Hence, we can easily trace the movement of the point p_z on the orbit γ_a as z goes around the boundary of the parameter space.

Recall that the image of X_{Ω_1} under the loop of symplectic embeddings $\{\varphi_t\}_{t\in[0,1]}$ does a counterclockwise 2π rotation in the z_1–plane, which rotates the orbit γ_a, followed by a clockwise 2π rotation in the z_2–plane, which does not rotate the orbit γ_a. Let $q := p_1$ be the point on γ_a corresponding to the parameter $1 \in \mathbb{D}$. Then

$$p_{e^{2\pi i t}} = \begin{cases} e^{-4\pi i t} q, & t \in \left[0, \frac{1}{2}\right] \\ q, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

and so the above composition is a degree -1 circle map. $\square$

This last claim provides us with a contradiction, given that a degree -1 circle map cannot factor through the disk $\mathcal{M}_{\mathfrak{J}} \simeq \mathbb{D}$.

Chapter 2

Essential tori in spaces of symplectic embeddings

Julian Chaidez, Mihai Munteanu

2.1 Introduction

The study of symplectic embeddings is a major area of focus in symplectic geometry. Remarkably, the space of such embeddings can have a rich and complex structure, even when the domain and target manifolds are relatively simple.

Symplectic embeddings between ellipsoids are a well–studied instance of this phenomenon. For a nondecreasing sequence of positive real numbers $a = (a_1, a_2, \ldots, a_n)$ define the *symplectic ellipsoid* $E(a)$ by

$$E(a) = E(a_1, a_2, \ldots, a_n) := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \leq 1 \right\}. \tag{2.1.1}$$

The space $E(a)$ carries the structure of an exact symplectic manifold with boundary endowed with the restriction of the standard Liouville form λ on $\mathbb{C}^n$, given by

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} (x_i dy_i - y_i dx_i). \tag{2.1.2}$$

A special case is the *symplectic ball* $B^{2n}(r)$, which is simply $E(a)$ for $a = (r, \ldots, r)$.

The types of results that one can prove about symplectic embeddings, together with the tools used to do so, are surveyed at length by Schlenk in [41]. Most research has thus far sought to address the existence problem. Let us recall some of the more striking progress in this direction. The first nontrivial result was Gromov's eponymous *nonsqueezing theorem*, proven in the seminal paper [16].

Theorem 2.1.1 ([16])**.** *There exists a symplectic embedding*

$$B^{2n}(r) \to B^2(R) \times \mathbb{C}^{2n-2}$$

if and only if $r \leq R$*.*

This result demonstrated that there are obstructions to symplectic embeddings beyond the volume and initiated the study of quantitative symplectic geometry. Note that Theorem 2.1.1 can be seen as a result about ellipsoid embeddings, since $B^2(R) \times \mathbb{C}^{2n-2}$ can be viewed as the degenerate ellipsoid $E(R, \infty, \ldots, \infty)$.

In dimension 4, the question of when the ellipsoid $E(a, b)$ symplectically embeds into the ellipsoid $E(a', b')$ was answered by McDuff in [32]. Let $\{N_k(a, b)\}_{k \geq 0}$ denote the sequence of nonnegative integer linear combinations of a and b, ordered nondecreasingly with repetitions.

Theorem 2.1.2 ([32])**.** *There exists a symplectic embedding*

$$\operatorname{int}(E(a, b)) \to E(a', b')$$

if and only if $N_k(a, b) \leq N_k(a', b')$ *for every nonnegative integer* k*.*

A special case of this embedding problem, where the target ellipsoid is the ball $B^4(\lambda)$, was studied by McDuff and Schlenk in an earlier paper [35] using methods different from [32]. In that paper, McDuff and Schlenk give a remarkable calculation of the function $c_0 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$c_0(a) := \inf \left\{ \lambda \;\middle|\; E(1, a) \text{ symplectically embeds into } B^4(\lambda) \right\}.$$

In particular, they show that for $a \in [1, (\frac{1+\sqrt{5}}{2})^4]$, the function c_0 is given by a piecewise linear function involving the Fibonacci numbers, which they call the *Fibonacci staircase.* Some higher dimensional cases of the existence problem for symplectic embeddings have been studied in a similar manner. For instance, a family of stabilized analogues of the function c_0, which are defined as

$$c_n(a) := \inf \left\{ \lambda \;\middle|\; E(1, a) \times \mathbb{C}^n \text{ symplectically embeds into } B^4(\lambda) \times \mathbb{C}^n \right\},$$

are studied in the more recent papers [11] and [12].

Beyond problems of existence, one can ask about the algebraic topology of the space of symplectic embeddings $\mathrm{SympEmb}(U, V)$ between two symplectic manifolds U and V, with respect to the C^∞ topology. Again, most results have been proven in dimensions 2 and 4. For instance, in [31], McDuff demonstrated that the space of embeddings between 4–dimensional symplectic ellipsoids is connected whenever it is nonempty. Other results in dimension 4 can be found in [1] and [22].

More recently, in [37], the second author developed methods to show that the contractibility of certain loops of symplectic embeddings of ellipsoids depends on the relative sizes of the two ellipsoids.

2.1.1 Main result

In this paper, we build upon the methods developed in [37] to tackle the question of describing the higher homology groups of spaces of symplectic embeddings between ellipsoids in any dimension.

More precisely, we will be studying families of symplectic embeddings that are restrictions of the following unitary maps. For $\theta = (\theta_1, \ldots, \theta_n) \in T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, let U_θ denote the unitary transformation

$$U_\theta(z_1, \ldots, z_n) := (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n). \tag{2.1.3}$$

Given symplectic ellipsoids $E(a)$ and $E(b)$ such that $a_i < b_i$ for every $i \in \{1 \ldots, n\}$, we may define the family of ellipsoid embeddings

$$\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b)), \qquad \Phi(\theta) = U_\theta|_{E(a)} \tag{2.1.4}$$

by restricting the domain of the maps U_θ. The following theorem about the family Φ is the main result of this paper.

Theorem 2.1.3 (Main theorem)**.** *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of real numbers satisfying*

$$a_i < b_i < a_{i+1} \quad \text{for all} \quad i \in \{1, \ldots, n-1\} \quad \text{and} \quad a_n < b_n < 2a_1.$$

Furthermore, let $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ be the family of symplectic embeddings (2.1.4). Then the induced map

$$\Phi_* : H_*(T^n; \mathbb{Z}/2) \to H_*(\mathrm{SympEmb}(E(a), E(b)); \mathbb{Z}/2)$$

on homology with $\mathbb{Z}/2$–coefficients is injective.

In order to demonstrate the nontriviality of Theorem 2.1.3, we note that the map induced by $\Phi : T^n \to \mathrm{Symp}(E(a), E(b))$ on $\mathbb{Z}/2$–homology has a sizeable kernel when $E(a)$ is very small relative to $E(b)$. More precisely, we have the following.

Proposition 2.1.4. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two nondecreasing sequences of real numbers satisfying $a_n < b_1$. Furthermore, let $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ be as in (2.1.4). Then the induced map Φ_* on $\mathbb{Z}/2$–homology has rank* 1 *in degree* ≤ 1 *and rank* 0 *otherwise.*

Unlike the proof of Theorem 2.1.3, the proof of Proposition 2.1.4 is an elementary calculation in algebraic topology which we defer to §2.2.

Remark 2.1.5 (Comparison to [37])**.** In dimension 4, the fact that Φ_* is injective in degree 1 was proven by the second author, Munteanu, in [37]. Specifically, this is equivalent to [37, Theorem 1.4] which states that the loop

$$\Psi : S^1 \to \mathrm{Symp}(E(a), E(b))$$

defined by

$$\Psi(t)(z_1, z_2) := \begin{cases} (e^{4\pi i t} z_1, z_2) & t \in \left[0, \frac{1}{2}\right] \\ (e^{-4\pi i t} z_1, z_2) & t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

is noncontractible. In fact, [37] actually addresses the more general 4–dimensional case where $E(a)$ and $E(b)$ are replaced with convex toric domains in $\mathbb{C}^2$. We expect Theorem 2.1.3 to hold at this level of generality, and we hope to address this in future work using somewhat different methods (see Remark 2.1.7).

Remark 2.1.6 ($\mathbb{Z}$ vs $\mathbb{Z}/2$ coefficients)**.** Our use of $\mathbb{Z}/2$ coefficients, instead of $\mathbb{Z}$ coefficients, allows us to use the methods of §2.4 to work entirely with smooth manifolds with boundary as opposed to cochains. While the contents of §2.4 provide a nice technical work around, we expect Theorem 2.1.3 to hold at the level of $\mathbb{Z}$ coefficients as well. We plan to develop the methods needed to work over $\mathbb{Z}$ in forthcoming work (see Remark 2.1.7).

Remark 2.1.7 (Lagrangian analogues)**.** In forthcoming work, we hope to demonstrate results analogous to Theorem 2.1.3 for families of Lagrangian torus embeddings in toric domains. We anticipate that these results will be useful for demonstrating the various generalizations of Theorem 2.1.3 discussed in Remarks 2.1.5 and 2.1.6 above.

Organization. The rest of the paper is organized as so. In §2.2, we give the proof of Theorem 2.1.3. The final two section are dedicated to demonstrating some technical results needed to deduce the steps of the proof. Namely, in §2.3 we recall the definition of the contact dg–algebra (as constructed in full generality by Pardon in [39]) together with the computations for symplectic ellipsoids that are relevant to Theorem 2.1.3. In §2.4, we prove some useful technical results about the topology of all symplectic embeddings spaces.

Acknowledgements. We would like to thank our advisor, Michael Hutchings for all the helpful discussions. JC was supported by the NSF Graduate Research Fellowship under Grant No. 1752814. MM was partially supported by NSF Grant No. DMS–1708899.

2.2 Proof of the main result

In this section, we prove Theorem 2.1.3 assuming a small number of technical results discussed in §2.3–2.4. Here is a brief overview of the proof to help guide the reader.

We assume by contradiction that the map Φ_* induced by the family Φ of (2.1.4) is not injective in degree k. Using this assumption and the results in §2.4, we find a certain family of symplectic embeddings, parametrized by a union of an odd number of k–tori $\sqcup_1^m T^k$ and built from Φ, which is null–bordant in the space $\mathrm{SympEmp}(E(a), E(b))$. This means that the family extends to a smooth $(k+1)$–dimensional family of symplectic embeddings $\Psi : P \to \mathrm{SympEmp}(E(a), E(b))$ where P is a smooth, compact, $(k+1)$–dimensional manifold with boundary $\partial P \simeq \sqcup_1^m T^k$.

Using Ψ, we construct a moduli space of holomorphic curves $\mathcal{M}_I(\mathfrak{J})$ in completed symplectic cobordisms parametrized by P. Moreover, we construct an associated evaluation map $\mathrm{ev}_I : \mathcal{M}_I(\mathfrak{J}) \to T^k$ to a k–torus T^k. We then show that the degree of this evaluation map is 1 mod 2 when restricted to any of the torus components of $\partial\mathcal{M}_I(\mathfrak{J})$. This is the contradiction, since the evaluation map extends to the bounding manifold $\mathcal{M}_I(\mathfrak{J})$ and so must have degree 0.

2.2.1 Review of contact geometry

We now provide a quick review of basic contact geometry, and in the process establish notation for §2.2 and §2.3. We also discuss the Reeb dynamics on the boundary of a symplectic ellipsoid with rationally independent parameters.

Review 2.2.1 (Contact manifolds)**.** Recall that a *contact manifold* (Y, ξ) is a smooth $(2n-1)$–manifold Y together with a rank $2n-2$ sub–bundle $\xi \subset TY$ that is given fiberwise by the kernel $\xi = \ker(\alpha)$ of a contact 1–form $\alpha \in \Omega^1(Y)$. A *contact form* α is a 1–form on Y satisfying $\alpha \wedge d\alpha^{n-1} \neq 0$ everywhere.

Every contact form α on Y has a naturally associated *Reeb vector field* R_α defined implicitly from α via the equations

$$\iota_{R_\alpha}\alpha = 1, \qquad \iota_{R_\alpha} d\alpha = 0. \tag{2.2.1}$$

The *Reeb flow* $\Phi_\alpha : Y \times \mathbb{R} \to Y$ is the flow of the vector field R_α, i.e. the family of diffeomorphisms satisfying

$$\left.\frac{d\Phi^t_\alpha(y)}{dt}\right|_{t=s} = R_\alpha \circ \Phi^s_\alpha(y). \tag{2.2.2}$$

A *Reeb orbit* is a closed orbit of the flow Φ_α, i.e. a curve $\gamma : S^1 = \mathbb{R}/L\mathbb{Z} \to Y$ satisfying $\frac{d\gamma}{dt} = R_\alpha \circ \gamma$ for some positive number L which is called the *period.* Note that L coincides with the *action* $\mathcal{A}_\alpha(\gamma)$ of γ, which is defined as

$$\mathcal{A}_\alpha(\gamma) = \int_{S^1} \gamma^*\alpha. \tag{2.2.3}$$

A Reeb orbit γ is called *nondegenerate* if the differential $T\Phi^L_{\gamma(0)}$ of the time L flow satisfies

$$\det(T\Phi^L_{\gamma(0)}|_\xi - \mathrm{Id}_\xi) \neq 0. \tag{2.2.4}$$

A contact form α is called *nondegenerate* if every Reeb orbit of α is nondegenerate.

Review 2.2.2 (Conley–Zehnder indices)**.** Any nondegenerate Reeb orbit γ posseses a fundamental numerical invariant called the *Conley–Zehnder index* $\mathrm{CZ}(\gamma, \tau)$, whose definition and computation we now review.

The Conley–Zehnder index $\mathrm{CZ}(\gamma, \tau)$ depends on a choice of symplectic trivialization $\tau : \gamma^*\xi \simeq S^1 \times \mathbb{C}^{n-1}$. The invariant is defined by $\mathrm{CZ}(\gamma, \tau) := \mu_{\mathrm{RS}}(\phi)$ were μ_{RS} denotes the Robbin–Salamon index (see [40]) and ϕ is the path of symplectic matrices defined as

$$\phi : [0, L] \to \mathrm{Sp}(2n-2), \qquad \phi(t) := \tau_{\gamma(t)} \circ T\Phi^t_{\gamma(0)}|_\xi \circ \tau^{-1}_{\gamma(0)}.$$

In the case where $c_1(\xi) = 0 \in H^2(Y;\mathbb{Z})$ and $[\gamma] = 0 \in H_1(Y;\mathbb{Z})$, a canonical Conley–Zehnder index $\mathrm{CZ}(\gamma)$ (which does not depend on a choice of trivialization) can be associated to γ via the following procedure. Extend γ to a map $u : \Sigma \to Y$ from an oriented surface Σ with boundary $\partial\Sigma = S^1$ satisfying $u|_{\partial\Sigma} = \gamma$. Pick a symplectic trivialization $\sigma : u^*\xi \simeq \Sigma \times \mathbb{C}^{n-1}$ and define $\mathrm{CZ}(\gamma)$ by the formula

$$\mathrm{CZ}(\gamma) := \mathrm{CZ}(\gamma, \sigma|_{\partial\Sigma}). \tag{2.2.5}$$

The fact that $\mathrm{CZ}(\gamma)$ is independent of Σ and σ follows from the vanishing of the first Chern class. The index $\mathrm{CZ}(\gamma)$ can be related to the index $\mathrm{CZ}(\gamma, \tau)$ with respect to a trivialization τ by the formula

$$\mathrm{CZ}(\gamma) = \mathrm{CZ}(\gamma, \tau) + 2c_1(\gamma, \tau). \tag{2.2.6}$$

Here $c_1(\gamma, \tau)$ is the relative first Chern number with respect to τ of the pullback $u^*\xi$ of ξ to a capping surface u of γ.

For the purposes of this paper, we are interested in a specific family of examples of contact manifolds, namely boundaries $(\partial E(a), \alpha)$ of irrational symplectic ellipsoids.

Example 2.2.3 (Ellipsoids)**.** Let $E(a)$ be a symplectic ellipsoid with parameters $a = (a_1, \dots, a_n) \in (0, \infty)^n$. Consider the boundary of the ellipsoid $(\partial E(a), \alpha)$ as a contact manifold with contact form $\alpha = \lambda|_{\partial E(a)}$, induced by the standard Liouville form λ on $\mathbb{C}^n$ defined by (2.1.2). Assume that the parameters a_i satisfy $a_i/a_j \notin \mathbb{Q}$ for each $i \neq j$. The Reeb vector field R_α is given by

$$R_\alpha = 2\pi \sum_i a_i^{-1} \frac{\partial}{\partial\theta_i}. \tag{2.2.7}$$

Here θ_i is the angular coordinate in the ith $\mathbb{C}$ factor of $\mathbb{C}^n$, which we denote by $\mathbb{C}_i$. The Reeb flow Φ_α on $\partial E(a)$ is given by:

$$\Phi_\alpha : \partial E(a) \times \mathbb{R} \to Y, \qquad \Phi^t_\alpha(z_1, \dots, z_n) = (e^{2\pi t/a_1} z_1, \dots, e^{2\pi t/a_n}). \tag{2.2.8}$$

Due to our assumption that $a_i/a_j \notin \mathbb{Q}$ for each $i \neq j$, there are precisely n simple orbits γ_i for $1 \le i \le n$. Each curve γ_i is a parametrization of the curve of points in Y with $z_j = 0$ for all $j \neq i$. The iterates γ_i^m (for any $m \ge 1$ and $1 \le i \le n$) are all nondegenerate, as we will show below by computing the linearized flow. The action of γ_i^m is given by $\mathcal{A}_\alpha(\gamma_i^m) = ma_i$ by (2.2.8).

To compute the Conley–Zehnder indices of the Reeb orbits γ_i^m, we proceed as follows. Note that along γ_i^m the fiber $\xi_{\gamma_i^m(t)}$ agrees at each t with the orthogonal complex subspace to the ith component $\mathbb{C}_i^\perp = \oplus_{j\neq i}\mathbb{C}_j \subset \mathbb{C}^n \simeq T\mathbb{C}^n_{\gamma_i^m(t)}$. The linearized flow in this trivialization is a direct sum of loops $t \mapsto e^{2\pi i m a_i t/a_j}$ for $j \neq i$ for $t \in [0,1]$. Thus, in this trivialization, the Conley–Zehnder index is given by

$$\mathrm{CZ}(\gamma_i^m, \tau) = \sum_{j\neq i}\left(2\left\lfloor \frac{ma_i}{a_j}\right\rfloor + 1\right).$$

On the other hand, the relative Chern number $c_1(\gamma_i^m, \tau)$ with respect to τ is

$$c_1(\gamma_i^m, \tau) = m.$$

Thus we have the following formula for the canonical CZ index of γ_i^m.

$$\mathrm{CZ}(\gamma_i^m) = \sum_{j\neq i}\left(2\left\lfloor \frac{ma_i}{a_j}\right\rfloor + 1\right) + 2m, \tag{2.2.9}$$

which after some smart rewriting becomes

$$\mathrm{CZ}(\gamma_i^m) = n - 1 + 2\left|\{L \in \mathrm{Spec}(Y,\alpha) \mid L \leq ma_i\}\right|. \tag{2.2.10}$$

Next, we review the basic terminology of exact symplectic cobordisms and associated structures. Throughout the discussion for the rest of the section, let $(Y_\pm, \alpha_\pm)$ be closed contact $(2n-1)$–manifolds with contact forms $\alpha_\pm$.

Review 2.2.4 (Exact symplectic cobordisms)**.** Recall that an *exact symplectic cobordism* (W, λ, ι) from (Y_+, α_+) to (Y_-, α_-) consists of the following data.

- A compact, exact symplectic manifold (W, λ) with boundary ∂W such that the *Liouville vector field* Z (defined by the equation $d\lambda(Z, \cdot) = \lambda$) is transverse to ∂W everywhere. In this situation, $\partial W = \partial_+ W \sqcup \partial_- W$ where Z points outward along $\partial_+ W$ and inward along $\partial_- W$.
- A pair of boundary inclusion maps ι_+ and ι_-, which are strict contactomorphisms of the form

$$\iota_+ : (Y_+, \alpha_+) \simeq (\partial_+ W, \lambda|_{\partial_+ W}) \qquad \iota_- : (Y_-, \alpha_-) \simeq (\partial_- W, \lambda|_{\partial_- W}) \tag{2.2.11}$$

We will generally suppress the inclusions in the notation, using ι_+ and ι_- when needed. The maps ι_+ and ι_- extend, via flow along Z or $-Z$, to collar coordinates

$$([0, \epsilon) \times Y_-, e^s\lambda_-) \simeq (N_-, \lambda|_{N_-}), \qquad ((-\epsilon, 0] \times Y_+, e^s\lambda_+) \simeq (N_+, \lambda|_{N_+}). \tag{2.2.12}$$

Here N_- and N_+ are collar neighborhoods of Y_- and Y_+ respectively, the maps preserve the 1–forms above and s denotes the coordinate on $[0,\epsilon)$ and $(-\epsilon, 0]$.

Given exact symplectic cobordisms (W, λ, ι) from (Y_0, α_0) to (Y_1, α_1) and (W', λ', ι') from (Y_1, α_1) to (Y_2, α_2), we can form the *composition* $(W\#W', \lambda\#\lambda', \iota\#\iota')$ by gluing W and W' via the identification $(\iota'_+)^{-1} \circ \iota_-$ of $\partial_- W$ and $\partial_+ W'$. The Liouville forms and inclusions extend in the obvious way to the glued manifold.

Using these identifications (2.2.12), we can complete the exact symplectic cobordism (W, λ) by adding cylindrical ends $(-\infty, 0] \times Y_-$ and $[0, \infty) \times Y_+$ to obtain the *completed exact symplectic cobordism* $(\widehat{W}, \widehat{\lambda})$, given by

$$\widehat{W} = (-\infty, 0] \times Y_- \sqcup_{\iota_-} W \sqcup_{\iota_+} [0, \infty) \times Y_+. \tag{2.2.13}$$

The Liouville 1–forms λ, $e^s\alpha_-$ and $e^s\alpha_+$ glue together to a Liouville form $\widehat{\lambda}$ on $\widehat{W}$. An important special caase of completed cobordisms is given by the *symplectization* of a contact manifold $(\mathbb{R} \times Y, e^s\alpha)$, which is denoted by $\widehat{Y}$.

Given a manifold P (with or without boundary), a *P–parametrized family of exact symplectic cobordisms* $(W_p, \lambda_p)_{p\in P}$ from Y_+ to Y_- is a fiber bundle $W \to P$ over P with a 1–form λ on W and a bundle map $\iota^\pm : P \times Y^\pm \to W$ such that $(W_p, \lambda_p, \iota_p)$ is an exact symplectic cobordism for each $p \in P$. A pair of exact symplectic cobordisms $(V, \eta, \jmath)$ and $(V, \eta', \jmath')$ are called *deformation equivalent* if there is a $[0, 1]$–parametrized family of exact symplectic cobordisms such that $(V, \eta, \jmath) \simeq (W_0, \lambda_0, \iota_0)$ and $(V', \eta', \iota') \simeq (W_1, \lambda_1, \iota_1)$.

Review 2.2.5. (Almost complex structures) Recall that a compatible almost complex structure J on the symplectic vector bundle ξ gives rise to an $\mathbb{R}$–invariant compatible almost complex structure $\widehat{J}$ on the symplectization $\widehat{Y} = \mathbb{R} \times Y$, defined by

$$\widehat{J}(\partial_s) = R_\alpha, \qquad \widehat{J}(R_\alpha) = -\partial_s, \qquad \widehat{J}|_\xi = J.$$

We denote the set of compatible almost complex structures on Y by $\mathcal{J}(Y)$, and the $\mathbb{R}$-invariant almost complex structures arising from these as $\mathcal{J}(\widehat{Y})$.

An almost complex structure J on a completed exact symplectic cobordism $\widehat{W}$ as above is called *compatible* if it has the following properties.

- On the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, J restricts to $\mathbb{R}$-invariant complex structures arising from $J_+ \in \mathcal{J}(Y_+)$ and $J_- \in \mathcal{J}(Y_-)$, respectively.
- The almost complex structure J is compatible with the symplectic form $d\lambda$.

Such an almost complex structure extends to an almost complex structure on $\widehat{W}$ in the obvious way. We let $\mathcal{J}(W)$ denote the set of all such compatible almost complex structures on a given exact symplectic cobordism W.

As with contact manifolds, we are interested in a particular family of examples of exact symplectic cobordisms related to ellipsoid embeddings.

Notation 2.2.6 (Cobordisms of embeddings)**.** Let $E(a)$ and $E(b)$ be irrational ellipsoids. Given a symplectic embedding $\varphi : E(a) \to \mathrm{int}(E(b))$, we denote by W_φ the exact symplectic cobordism given by

$$W_\varphi := E(b) \setminus \mathrm{int}(\varphi(E(a))), \qquad \iota_+ := \mathrm{Id}\,|_{\partial E(b)}, \qquad \iota_- := \varphi|_{\partial E(a)}. \tag{2.2.14}$$

In this context, we label the simple Reeb orbits of $\partial E(b)$ by γ_i^+ and the simple Reeb orbits of $\partial E(a)$ by γ_i^-. The simple Reeb orbits of the negative boundary of W_φ are, of course, the images $\varphi(\gamma_i^-)$ and will be denoted as such.

More generally, let P ve a compact manifold with boundary and $\Psi : P \times E(a) \to \mathrm{int}(E(b))$ be a P–parametrized family of symplectic embeddings such that $\mathrm{Im}(\Psi_p)$ is independent of p for p near ∂P. We then acquire a family of cobordisms $(W_{\Psi_p}, \lambda_{\Psi_p})$ with fiber given by (2.2.14). We let $W_{\partial P} = E(b) \setminus \Psi_p(E(a))$ for $p \in \partial P$ and $\lambda_{\partial P}$ be the Liouville form. Note that in this case, the cobordisms $(W_{\Psi_p}, \lambda_{\Psi_p}, \iota_{\Psi_p})$ for $p \in \partial P$ differ only by the boundary inclusion ι_{Ψ_p}. In situations where ι_{Ψ_p} plays no role, we will often not distinguish between $(W_{\Psi_p}, \lambda_{\Psi_p}, \iota_{\Psi_p})$ for different $p \in \partial P$.

In this setting, we let $\mathcal{J}(\Psi)$ denote the set of P–parametrized families $\mathfrak{J} = \{J_p \mid p \in P\}$ of almost complex structures with the following properties:

- J_p is compatible with $\widehat{W}_{\Psi_p}$ for each $p \in P$, i.e. $J_p \in \mathcal{J}(\widehat{W}_{\Psi_p})$,
- J_p is equal to some p-independent $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ for p near ∂P.

We note that $\mathcal{J}(\widehat{W})$ is contractible for any W (see for instance [34, Proposition 4.11]). This implies that the space of families $\mathcal{J}(\Psi)$ is also contractible, and that any family $\{J_p \mid p \in \partial P\}$ over ∂P extends to a family $\{J_p \mid p \in P\}$ over all of P.

2.2.2 Moduli spaces in cobordisms

We now introduce the spaces of holomorphic curves that are relevant to our proof and we derive the salient properties of these spaces, namely generic transversality (Lemma 2.2.10) and compactness (Lemma 2.2.12). We also state a point count result for one of the moduli spaces of interest, whose proof we defer to §2.3.

Notation 2.2.7 (Curve domains)**.** Fix a subset $I \subset \{1, \dots, n\}$ and denote by $|I|$ the size of I. For the remainder of §2.2, we adopt the following notation.

For each $i \in I$, let Σ_i denote a copy of the twice punctured Riemann sphere $\mathbb{R} \times S^1 \simeq \mathbb{CP}^1 \setminus \{0, \infty\}$ with the usual complex structure $j_{\mathbb{CP}^1}$ and let $\overline{\Sigma}_i$ denote the corresponding copy of $\mathbb{C}P^1$ itself. Let p_i^+ and p_i^- denote the points ∞ and 0 in the copy $\overline{\Sigma}_i$ of $\mathbb{C}P^1$. We refer to p_i^+ and p_i^- as the positive and negative punctures of Σ_i, respectively. Denote by Σ_I the disjoint union $\sqcup_{i \in I} \Sigma_i$.

Definition 2.2.8 (Unparametrized moduli space)**.** Given any symplectic embedding $\varphi \in \mathrm{Symp}(E(a), E(b))$ and any admissible almost complex structure $J_\varphi \in \mathcal{J}(\varphi)$ on $\widehat{W}_\varphi$ as above, we denote by $\mathcal{M}_I(\widehat{W}_\varphi; J_\varphi)$ the moduli space defined as

$$\mathcal{M}_I(\widehat{W}_\varphi; J_\varphi) := \left\{ u : \Sigma_I \to \widehat{W}_\varphi \;\middle|\; \begin{array}{c} (du)^{0,1}_{J_\varphi} = 0 \\ u \to \gamma_i^\pm \text{ at } p_i^\pm \end{array} \right\} / (\mathbb{C}^\times)^{|I|}. \tag{2.2.15}$$

That is, $u : \Sigma_I \to \widehat{W}_\varphi$ is a J_φ–holomorphic curve such that u is asymptotic to the trivial cylinder over γ_i^+ in $[0, \infty) \times \partial_+ W_\varphi \simeq [0, \infty) \times \partial E(b)$ at the puncture p_i^+ and u is asymptotic to the trivial cylinder over $\varphi(\gamma_i^-)$ in $(-\infty, 0] \times \partial_- W_\varphi \simeq (-\infty, 0] \times \varphi(\partial E(a))$ at the puncture p_i^-, for each $i \in I$. We quotient the space of such maps by the group of domain reparametrizations, which is the product $(\mathbb{C}^\times)^{|I|}$ of the biholomorphism groups $\mathbb{C}^\times$ of each component cylinder $\Sigma_i \simeq \mathbb{R} \times S^1$.

Definition 2.2.9 (Parametrized moduli space over P)**.** Given a compact manifold with boundary P, a P–parametrized family of symplectic embeddings $\Psi : P \times E(a) \to E(b)$ and a P–parametrized family of complex structure $\mathfrak{J} \in \mathcal{J}(\Psi)$, let $\mathcal{M}_I(\mathfrak{J})$ denote the moduli space of pairs

$$\mathcal{M}_I(\mathfrak{J}) := \left\{ (p, u) \;\middle|\; p \in P,\ u \in \mathcal{M}_I(\widehat{W}_{\Psi_p}; J_p) \right\}. \tag{2.2.16}$$

Lemma 2.2.10 (Transversality)**.** *Let $E(a)$ and $E(b)$ be irrational symplectic ellipsoids with parameters $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ satisfying*

$$a_i < b_i, \qquad a_i < 2a_1, \quad \text{and} \quad b_i < 2b_1 \qquad \text{for all } i \text{ with } 1 \le i \le n. \tag{2.2.17}$$

Then there exists a comeager $\mathcal{J}^{\mathrm{reg}}(\Psi) \subset \mathcal{J}(\Psi)$ with $J|_{\partial P}$

(a) Every $u \in \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is Fredholm regular (see [44, Definition 7.14]), and thus the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is a 0–dimensional manifold.

(b) Every $(p, u) \in \mathcal{M}_I(\mathfrak{J})$ is parametrically Fredholm regular (see [44, Remark 7.4] and [43, Definition 4.5.5]) and thus $\mathcal{M}_I(\mathfrak{J})$ is a $(|I| + 1)$–dimensional manifold with boundary $\partial\mathcal{M}_I(\mathfrak{J}) \simeq \partial P \times \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$.

Proof. This essentially follows from the general transversality results of [43] and [44, §7], which we now discuss in some detail.

First, observe that every curve $u \in \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ must be somewhere injective (see [44, p. 123]) for any choice of $J_{\partial P}$. Indeed, note that all of the orbits γ_i^- and γ_i^+ are simple. This means that none of them can be factored as $\eta \circ \varphi$ where η is a closed Reeb orbit and $\varphi : S^1 \to S^1$ is a k–fold cover with $k \ge 2$. This implies that u is simple as well, i.e. that u cannot factor as $v \circ \phi$ where $v : \Sigma' \to \widehat{W}_{\partial P}$ is a J-holomorphic curve and $\phi : \Sigma_I \to \Sigma'$ is a holomorphic branched cover. Simple curves are somewhere injective. In fact, these conditions are equivalent in our setting, see [44, Theorem 6.19]. The same reasoning shows that any curve u appearing as a factor in a point $(p,u) \in \mathcal{M}_I(\mathfrak{J})$ is somewhere injective for any choice of $\mathfrak{J}$.

To see (a), we now note that by [44, Theorems 7.1–7.2] there exists a comeager subset $\mathcal{J}^{\mathrm{reg}}(\widehat{W}_{\partial P}) \subset \mathcal{J}(\widehat{W}_{\partial P})$ with the property that for any $J_{\partial P} \in \mathcal{J}^{\mathrm{reg}}(\widehat{W}_{\partial P})$, every somewhere injective curve $u \in \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is Fredholm regular. Furthermore, [44, Theorems 7.1] states that the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is a manifold near these points with dimension given by the index formula

$$\mathrm{ind}(u) = \sum_{i \in I} \left((n-3)\chi(\Sigma_i) + 2c_1(u|_{\Sigma_i}, \tau) + \mathrm{CZ}(\gamma_i^+, \tau) - \mathrm{CZ}(\gamma_i^-, \tau) \right). \tag{2.2.18}$$

Here the Conley–Zehnder indices $\mathrm{CZ}(\gamma^\pm, \tau)$ and relative Chern numbers $c_1(u|_{\Sigma_i}, \tau)$ are as in Review 2.2.2, and τ denotes a trivialization of ξ over $\sqcup_i(\gamma_i^+ \sqcup \gamma_i^-)$.

Note that $\partial E(a)$ and $\partial E(b)$ are simply connected and $W_{\partial P}$ is diffeomorphic to a product. Thus we may choose τ by taking capping disks D_i for γ_i^-, thus inducing trivializations of ξ along γ_i^-, and then extending τ to a trivialization along Σ_i to induce trivializations of ξ along γ_i^+. The resulting trivialization has $c_1(u|_{\Sigma_i}, \tau) = 0$, $\mathrm{CZ}(\gamma_i^+, \tau) = \mathrm{CZ}(\gamma_i^+)$ and $\mathrm{CZ}(\gamma_i^-, \tau) = \mathrm{CZ}(\gamma_i^-)$. Here $\mathrm{CZ}(\gamma_i^+)$ and $\mathrm{CZ}(\gamma_i^-)$ denote the canonical indices described in Review 2.2.2. Thus, using this special choice of τ and noting that $\chi(\Sigma_i) = 0$, the formula (2.2.18) simplifies to

$$\dim(\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})) = \sum_{i \in I} \left(\mathrm{CZ}(\gamma_i^+) - \mathrm{CZ}(\gamma_i^-) \right) \tag{2.2.19}$$

Finally, we observe that the hypotheses (2.2.17) and the Conley–Zehnder index formula (2.2.10) imply that $\mathrm{CZ}(\gamma_i^+) = \mathrm{CZ}(\gamma_i^-) = n-1+2i$. Therefore, the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is 0–dimensional, and we have proven (a).

To see (b), we apply the appropriate parametric version of transversality (see [44, Remark 7.4] and [43, §4.5]), which states that there exists a family $\mathfrak{J} \in \mathcal{J}(\Psi)$, such that $\mathfrak{J}|_{\partial P} \equiv J_{\partial P}$ and $\mathcal{M}_I(\mathfrak{J})$ is a manifold with boundary. Since $\mathfrak{J}$ is independent of $p \in \partial P$ on the boundary, the boundary of the moduli space is simply the product $\partial \mathcal{M}_I(\mathfrak{J}) = \partial P \times \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$. The dimension is given by

$$\dim(\mathcal{M}_I(\mathfrak{J})) = \dim(P) + \dim(\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})) = k+1.$$

This concludes the proof of (b), and also the whole proof of Lemma 2.2.10. □

Before continuing on to the proof of compactness in Lemma 2.2.12, let us give a brief, very simplified review of a version of SFT compactness. We refer the reader to [2, §10] for the original proof and to [43, §9.4] for a detailed overview.

Review 2.2.11 (SFT Compactness)**.** Let P be a compact manifold with boundary, and let (Y_*, α_*) for $* \in \{+,-\}$ be closed, nondegenerate contact manifolds. Let (W_p, λ_p, J_p) be a

P–paramaterized family of exact symplectic cobordisms from Y_+ to Y_- equipped with a P–parametrized family of compatible almost complex structures on $\widehat{W}_p$ such that $J_p|_{[0,\infty)\times Y_+} = J_+$ and $J_p|_{(-\infty,0]\times Y_-} = J_-$ for some fixed almost complex structures $J_\pm$. Fix a surface Σ, acquired by taking a closed surface $\overline{\Sigma}$ and removing a finite set of punctures. Finally, consider a sequence $p_i \in P$ and $u^i : \Sigma \to (\widehat{W}_{p_i}, J_{p_i})$ of J_{p_i}–holomorphic curves asymptoting to collections of Reeb orbits Γ^+ (at the positive end of $\widehat{W}_{p_i}$) and Γ^- (at the negative end of $\widehat{W}_{p_i}$) independent of i.

The SFT compactness theorem states that, after passing to a subsequence, $p_i \to p \in P$ and u^i converges to a *J_p–holomorphic building*, which is a tuple of the form

$$v = (u_1^+, \ldots, u_M^+, u^W, u_1^-, \ldots, u_N^-). \tag{2.2.20}$$

Here $M, N \in \mathbb{Z}^{\geq 0}$ are integers and the elements of the tuple (called *levels*) are holomorphic maps from punctured surfaces of the form

$$u_j^* : S_j^* \to (\mathbb{R} \times Y_*, J_*) \text{ for } * \in \{+,-\} \quad \text{and} \quad u^W : S^W \to (\widehat{W}_p, J_p).$$

The maps u_j^* and the map u^W are considered modulo domain reparametrization, and modulo translation when the target manifold is a symplectization. The surfaces S_j can be glued together along the boundary punctures asymptotic to matching Reeb orbits, and this glued surface $\#_j S_j$ is homeomorphic to Σ.

All of the curves u_j^* and u^W must be asymptotic to a Reeb orbit at each positive and negative puncture. We denote the collections of positive and negative limit Reeb orbits of u^W (with multiplicity) by $\Gamma^+(u^W)$ and $\Gamma^-(u^W)$, respectively, and we adopt similar notation for u_j^*. The asymptotics of the u_j^* and u^W must be compatible, in the sense that the negative ends of u_j^* and the positive ends of u_{j+1}^* must agree (and likewise for u_M^+ and u^W, etc.). Furthermore, we must have $\Gamma^+(u_1^+) = \Gamma^+$ and $\Gamma^-(u_N^-) = \Gamma^-$. Finally, every symplectization level u_j^* must have at least one component that is not a trivial cylinder $\mathbb{R} \times \gamma$.

Since (W_p, λ_p) is an exact symplectic cobordism, one may apply Stoke's theorem to derive the following expression for the energies of the levels of v:

$$\mathcal{E}(u^W) := \int_{S^W} [u^W]^* d\lambda_p = \sum_{\eta^+ \in \Gamma^+(u^W)} \mathcal{A}(\eta^+) - \sum_{\eta^- \in \Gamma^-(u^W)} \mathcal{A}(\eta^-) \tag{2.2.21}$$

and

$$\mathcal{E}(u_j^\pm) := \int_{S_j} [u_j^\pm]^* d(e^t \alpha_\pm) = \sum_{\eta^+ \in \Gamma^+(u_j^\pm)} \mathcal{A}(\eta^+) - \sum_{\eta^- \in \Gamma^-(u_j^\pm)} \mathcal{A}(\eta^-). \tag{2.2.22}$$

The positivity of the energy of any holomorphic curve implies that the right hand sides of (2.2.21) and (2.2.22) are nonnegative. More generally, if we let $\mathcal{A}[\Gamma]$ denote the total action of a collection of Reeb orbits, then we have the string of inequalities

$$\begin{gathered} \mathcal{A}[\Gamma^-] = \mathcal{A}[\Gamma(u_N^-)] \leq \cdots \leq \mathcal{A}[\Gamma(u_1^-)] \leq \mathcal{A}[\Gamma(u^W)] \leq \\ \leq \mathcal{A}[\Gamma(u_M^+)] \leq \cdots \leq \mathcal{A}[\Gamma(u_1^+)] = \mathcal{A}[\Gamma^+]. \end{gathered} \tag{2.2.23}$$

There is some of additional data, beyond the holomorphic curves themselves, associated to a holomorphic building. However, we suppress this data since it will play no role in any of our arguments below.

With the above review of SFT compactness finished, we are ready to move on to the statement and proof of Lemma 2.2.12.

Lemma 2.2.12 (Compactness)**.** *Let $E(a)$ and $E(b)$ be irrational symplectic ellipsoids with parameters $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ satisfying*

$$a_i < b_i < a_{i+1} \text{ for all } i \in \{1, \dots, n-1\} \quad \text{and} \quad b_n < 2a_1. \tag{2.2.24}$$

Choose an almost complex structure $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ and a family $\mathfrak{J} \in \mathcal{J}(\Psi)$ as in Lemma 2.2.10. Then the moduli spaces $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ and $\mathcal{M}_I(\mathfrak{J})$ are compact.

Proof. Let (p_i, u^i) be a sequence in $\mathcal{M}_I(\mathfrak{J})$. By SFT compactness, after passing to a subsequence u^i converges to a limit building v. We use the notation of Review 2.2.11 for this building. We must show that v has no symplectization levels. By considering components of Σ_I and $\#_j S_j$, we can assume that $|I| = 1$, i.e. that Σ_I has one component and each u^i is positively asymptotic to a single $\gamma_{l_i}^+$ where $1 \le l_i \le n$.

Now consider a positive symplectization level u_j^+ of v. Due to action monotonicity (2.2.23), the collections $\Gamma^+(u_j^+)$ and $\Gamma^-(u_j^+)$ of positive and negative limit Reeb orbits of $(Y_+, \alpha_+) \simeq (\partial E(b), \lambda|_{\partial E(b)})$ must satisfy

$$a_{l_i} = \mathcal{A}(\gamma_{l_i}^-) \le \mathcal{A}[\Gamma^-(u_j^+)] \le \mathcal{A}[\Gamma^+(u_j^+)] \le \mathcal{A}(\gamma_{l_i}^+) = b_{l_i}.$$

Consider $\Gamma^+(u_j^+)$ only. Due to the hypotheses (2.2.24), $\Gamma^+(u_j^+)$ cannot contain either a copy of γ_r^+ for $r > l_i$ or a copy of an iterate $(\gamma_r^+)^m$ for any $m \ge 2$ and any r. Otherwise, we would have $\mathcal{A}[\Gamma^+(u_j^+)] > b_{l_i}$. This implies that $\Gamma^+(u_j^+)$ can only contain Reeb orbits γ_r^+ for $r \le l_i$. Moreover, since $\mathcal{A}[\Gamma^+(u_j^+)] \ge a_{l_i}$ and $\mathcal{A}(\gamma_r^+) = b_r < a_{l_i}$ for $r < l_i$ (again by (2.2.24)), we must have $\Gamma^+(u_j^+) = \{\gamma_{l_i}^+\}$. The same reasoning shows that $\Gamma^-(u_j^+) = \{\gamma_{l_i}^+\}$. Thus the energy $\mathcal{E}(u_j^+)$ of the level u_j^+ is 0 by (2.2.22) and the level u_j^+ must be a branched cover of a trivial cylinder (see [43, Lemma 9.9]). Since the ends are embedded, u_j^+ must be simple and thus a trivial cylinder. This is disallowed by the SFT compactness statement, so u_j^+ cannot exist.

The same reasoning implies that negative levels u_j^- of v cannot exist. Thus the building v consists of a single level u^W, whose domain is a cylinder and which is asymptotic to $\gamma_{l_i}^+$ and $\gamma_{l_i}^-$ at the positive and negative ends. We have found a limit curve $(p, u^W) \in \mathcal{M}_I(\mathfrak{J})$ for a subsequence of (p_i, u^i) and thus we have proven the compactness of $\mathcal{M}_I(\mathfrak{J})$. The compactness of $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ follows from that of $\mathcal{M}_I(\mathfrak{J})$. $\square$

Finally, we state the following curve count lemma. The proof is an application of the (full) contact homology, and we defer it to §2.3.2.

Lemma 2.2.13 (Curve count)**.** *Let $E(a)$ and $E(b)$ be irrational symplectic ellipsoids with parameters $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ satisfying*

$$a_i < b_i < a_{i+1} \text{ for all } i \in \{1, \dots, n-1\} \quad \text{and} \quad b_n < 2a_1. \tag{2.2.25}$$

Then there is a comeager $\mathcal{J}^{\mathrm{reg}} \subset \mathcal{J}(\widehat{W}_{\partial P})$ such that, for any $J_{\partial P} \in \mathcal{J}^{\mathrm{reg}}$, the compact 0–dimensional manifold $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ has an odd number of points.

2.2.3 Proofs of Theorem 2.1.3 and Proposition 2.1.4

In this section, we use the moduli spaces constructed in §2.2.2 to prove our main result, Theorem 2.1.3. We also provide a proof of Proposition 2.1.4. The following small piece of notation will be helpful for both proofs.

Notation 2.2.14. For any $n \in \mathbb{Z}^+$ and any $I \subset \{1, \dots, n\}$, define the $|I|$–torus by

$$T_I = \{(\theta_1, \dots, \theta_n) \in T^n \mid \theta_j = 0, \ \forall j \notin I\}. \tag{2.2.26}$$

Note that the kth homology group $H_k(T^n; \mathbb{Z}/2)$ of the n–torus T^n is generated by the fundamental classes $[T_I]$, where I runs over all subsets of size $|I| = k$.

For Theorem 2.1.3, we also require the following result, which is proven in §2.4.

Lemma 2.2.15. *Let U and V be compact symplectic manifolds with boundary. Let Z be a closed manifold with total Stieffel–Whitney class $w(Z) = 1 \in H^*(Z; \mathbb{Z}/2)$ and let Φ be a smooth family of symplectic embeddings*

$$\Phi : Z \to \mathrm{SympEmb}(U, V) \quad \textit{with} \quad \Phi_*[Z] = 0.$$

Then there exists a compact manifold P with boundary Z and an extension of Φ to a smooth family Ψ of symplectic embeddings

$$\Psi : P \to \mathrm{SympEmb}(U, V) \quad \textit{with} \quad \Psi|_{\partial P} = \Phi.$$

Given the above preparation, we are now ready for the proof of Theorem 2.1.3.

Proof. (Theorem 2.1.3) We pursue the argument by contradiction outlined at the begining of §2.2. Fix an integer k with $1 \le k \le n$ and suppose that there were a nonzero $\mathbb{Z}/2$–homology class of the n–torus of the form

$$[A] = \sum_L c_L [T_L] \quad \text{with} \quad \Phi_*[A] = 0 \in H_k(\mathrm{SympEmb}(E(a), E(b)); \mathbb{Z}/2). \tag{2.2.27}$$

Let $Z = \sqcup_{c_L \neq 0} T_L$. Then Lemma 2.2.15 states that there exists a smooth $(k+1)$–dimensional manifold P with boundary $\partial P = Z$ and a smooth family of embeddings

$$\Psi : P \to \mathrm{SympEmb}(E(a), E(b)) \quad \text{with} \quad \Psi|_{T_L} = \Phi|_{T_L} \text{ for each } T_L \subset Z. \tag{2.2.28}$$

By passing to subellipsoids, we may assume that $E(a)$ and $E(b)$ are irrational. In this setting, Lemmas 2.2.10 and 2.2.12 state that there exist choices of $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ and $\mathfrak{J} \in \mathcal{J}(\Psi)$ such that the parametrized moduli space $\mathcal{M}_I(\mathfrak{J})$ is a compact $(k+1)$–dimensional manifold with boundary $\partial\mathcal{M}(\mathfrak{J}) \simeq Z \times \mathcal{M}(\widehat{W}_{\partial P}; J_{\partial P})$. On the parametrized moduli space $\mathcal{M}_I(\mathfrak{J})$, we can define an evaluation map

$$\mathrm{Ev}_I : \prod_{i \in I} \gamma_i^+ \times \mathcal{M}_I(\mathfrak{J}) \to \prod_{i \in I} \gamma_i^- \simeq T^k \tag{2.2.29}$$

via the following procedure. Let $q = (q_i)_{i \in I}$ be a point in $\times_{i \in I} \gamma_i^+$ and let $(p, u) \in \mathcal{M}_I(\mathfrak{J})$. According to (2.2.16), (p, u) is a pair of a point $p \in P$ and an equivalence class of holomorphic maps $u : \Sigma_I \to \widehat{W}_{\Psi_p}$ up to reparametrization. Pick a representative holomorphic curve $\widetilde{u}$ of u, which consists of k maps $\widetilde{u}_i : \Sigma_i \to \widehat{W}_{\Psi_p}$ for each $i \in I$. We have limit parametrizations of γ_i^+ and γ_i^- induced by $\widetilde{u}_i$, defined by

$$\lim_{+} \widetilde{u}_i : S^1 \to \gamma_i^+ \subset \partial E(b), \qquad \lim_{+} \widetilde{u}_i(t) := \lim_{s \to +\infty} \pi_{\partial_+ W_{\Psi_p}} \widetilde{u}_i(s, t)$$

$$\lim_{-} \widetilde{u}_i : S^1 \to \Psi_p(\gamma_i^-) \subset \Psi_p(\partial E(a)), \qquad \lim_{-} \widetilde{u}_i(t) := \lim_{s \to -\infty} \pi_{\partial_- W_{\Psi_p}} \widetilde{u}_i(s, t)$$

Here $\pi_{\partial_+ W_{\Psi_p}}$ and $\pi_{\partial_- W_{\Psi_p}}$ denote projection to the positive and negative boundaries of W_{Ψ_p}. Note that these projections are only defined in the limit as $s \to \pm\infty$. In terms of these parametrizations, we define the evaluation map Ev_I by the formula

$$\mathrm{Ev}_I(q; p, u) := \left([\Psi_p^{-1} \circ \lim_{-} \widetilde{u}_i \circ (\lim_{+} \widetilde{u}_i)^{-1}](q_i) \right)_{i \in I} \in \prod_{i \in I} \gamma_i^-. \tag{2.2.30}$$

This definition is independent of the choice of representative $\widetilde{u}$. Finally, fix an arbitrary $\underline{q}$ in the product $\prod_{i \in I} \gamma_i^+$ and define

$$\mathrm{ev}_I : \mathcal{M}_I(\mathfrak{J}) \to \prod_{i \in I} \gamma_i^-, \qquad \mathrm{ev}_I(p, u) := \mathrm{Ev}_I(\underline{q}; p, u). \tag{2.2.31}$$

Now consider the restriction of ev_I to each component $T_L \times \{u\}$ of the boundary $Z \times \mathcal{M}(\widehat{W}_{\partial P}, J_{\partial P})$ of $\mathcal{M}(\mathfrak{J})$. Since the equivalence class of curve u is independent of $\theta \in T_L \subset T^n$, we can use (2.2.30) and (2.2.31) to write

$$\mathrm{ev}_I(\theta, u) = (\Psi_\theta^{-1}(\underline{r}_i))_{i \in I} \qquad \text{with} \quad \underline{r} := \left([\lim_{-} \widetilde{u}_i \circ (\lim_{+} \widetilde{u}_i)^{-1}](\underline{q}_i) \right)_{i \in I} \in \prod_{i \in I} \gamma_i^-.$$

Here $\underline{r}$ is independent of θ. Using the fact that $\Psi^{-1}|_Z = \Phi^{-1}$ and the formula (2.1.4) for the family of embeddings Φ, we have the formula

$$\mathrm{ev}_I(\theta, u) = (\Phi_\theta^{-1}(\underline{r}_i))_{i \in I} = (e^{-2\pi i \theta_i} \cdot \underline{r}_i)_{i \in I}. \tag{2.2.32}$$

In the right–most expression of (2.2.32), we identify $\underline{r}_i$ with an element of $\mathbb{C}^n$ via the inclusion $\gamma_i^- \subset E(a) \subset \mathbb{C}^n$.

The expression (2.2.32) allows us to compute the degree of ev_I on each component $T_L \times \{u\}$. There are two cases. If $L = I$, then (2.2.32) shows that $\mathrm{ev}_I|_{T_L\times\{u\}}$ is degree 1. If $I \neq L$, then for any $j \in L \setminus (L \cap I)$, θ_j is constant for every $\theta \in T_L$ and it follows from (2.2.32) that the degree of $\mathrm{ev}_I|_{T_L\times\{u\}}$ is 0. To derive our final contradictiction, we now observe that the total degree mod 2 of ev_I restricted to the boundary is

$$\begin{aligned}\deg(\mathrm{ev}_I|_{\partial\mathcal{M}_I(\mathfrak{J})}) &= \sum_{T_L\times\{u\}\subset\partial\mathcal{M}_I(\mathfrak{J})} \deg(\mathrm{ev}_I|_{T_L\times\{u\}}) = \\ &= |\mathcal{M}_I(\widehat{W}_{\partial P}, J_{\partial P})| \equiv 1 \mod 2.\end{aligned} \tag{2.2.33}$$

The right–most equality in (2.2.33) crucially uses the point count of Lemma 2.2.13. The equality (2.2.33) also provides the contradiction, since the degree of the restricton of a map to a boundary must be 0 mod 2. This concludes the proof. □

Having concluded the proof of Theorem 2.1.3, we now move on to Proposition 2.1.4. The proof is much less involved than that of Theorem 2.1.3, and does not use any of the machinery from §2.2.1–2.2.2. We begin with a lemma about the homology groups of the unitary group $U(n)$.

Lemma 2.2.16. *Consider the map $U : T^n \to U(n)$ given by $\theta \mapsto U_\theta$. Then the induced map $U_* : H_*(T^n;\mathbb{Z}/2) \to H_*(U(n);\mathbb{Z}/2)$ on $\mathbb{Z}/2$–homology is:*

(a) surjective if $* = 0$ or $* = 1$.

(b) identically 0 if $* \geq 2$.

Proof. To show (a), we first note that T^n and $U(n)$ are connected so $U_*|_{H_0} = \mathrm{Id}$. Furthermore, if we consider the loop $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to T^n$ given by $\theta \mapsto (\theta, 0, \dots, 0)$, we see that the composition

$$\det_{\mathbb{C}} \circ U \circ \gamma : \mathbb{R}/2\pi\mathbb{Z} \to U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$$

is the identity. Since $\det_{\mathbb{C}} : U(n) \to U(1)$ induces an isomorphism on H_1, the induced map of U must be surjective on H_1.

To show (b) we proceed as follows. It suffices to show that $U_*[T_L] = 0$ for all L with $|L| \geq 2$. We can factorize $T_L = T_J \times T_K$ for $J \sqcup K = L$ and $|J| = 2$, and

$$T_L = T_J \times T_K \xrightarrow{\iota_J\times\iota_K} U(2) \times U(n-2) \xrightarrow{j} U(n).$$

Here j is the inclusion of a product of unitary subgroups, and ι_J and ι_K are inclusions of the tori into these unitary subgroups. It suffices to show that $(\iota_J \times \iota_K)_*[T_L] = [\iota_J]_*[T_J] \otimes [\iota_K]_*[T_K] = 0$, or simply that $[\iota_J]_*[T_J] = 0 \in H_2(U(2);\mathbb{Z}/2)$.

Now we simply note that $\dim(U(2)) = 4$ and $H^*(U(2);\mathbb{Z}/2) \simeq \mathbb{Z}/2[c_1, c_3]$ where c_i is a generator of index i. In particular, $H_2(U(2);\mathbb{Z}/2) \simeq H^2(U(2);\mathbb{Z}/2) = 0$. □

Using Lemma 2.2.16, we can now prove Propositon 2.1.4. The point is that the entire unitary group $U(n)$ embeds into $\mathrm{SympEmb}(E(a), E(b))$ via domain restriction when $a_i < b_j$ for all i and j (which is equivalent to $a_n < b_1$ by our ordering convention).

Proof. (Proposition 2.1.4) Let $D : \mathrm{SympEmb}(E(a), E(b)) \to U(n)$ denote the map $\varphi \mapsto r(d\varphi|_0)$, given by taking derivatives $d\varphi|_0 \in \mathrm{Sp}(2n)$ at the origin and composing with a retraction $r : \mathrm{Sp}(2n) \to U(n)$. Under the hypotheses on a and b, we can factor the identity $\mathrm{Id} : U(n) \to U(n)$ and $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ as

$$\mathrm{Id} : U(n) \xrightarrow{\mathrm{res}} \mathrm{Symp}(E(a), E(b)) \xrightarrow{D} U(n),$$

$$\Phi : T^n \xrightarrow{U} U(n) \xrightarrow{\mathrm{res}} \mathrm{SympEmb}(E(a), E(b)).$$

Here $\mathrm{res}(\varphi) := \varphi|_{E(a)}$ denotes restriction of domain. In particular, $\mathrm{res} : U(n) \to \mathrm{Symp}(E(a), E(b))$ is injective on homology and $\mathrm{Im}(\Phi_*) \simeq \mathrm{Im}(U_*)$ as $\mathbb{Z}$–graded $\mathbb{Z}/2$–vector spaces. The result thus follows from Lemma 2.2.16. ☐

2.3 Contact homology

In this section, we discuss the main Floer–theoretic tool in this paper, the full contact homology $CH(Y, \xi)$ of a closed contact manifold (Y, ξ). The goal is to extract the point count result, Lemma 2.2.13 in §2.2.2, from the basic properties of this invariant.

2.3.1 Contact dg–algebra

We first review the contact dg–algebra of a contact manifold and the cobordism dg–algebra maps induced by exact symplectic cobordisms. This invariant package was originally introduced by Eliashberg–Givental–Hofer [13] without foundations. Here we use the construction by Pardon [39] using virtual fundamental cycles (VFC).

Remark 2.3.1 (Assumptions)**.** We restrict our discussion to contact manifolds Y and exact symplectic cobordisms W satisfying the following assumptions:

$$H_1(Y; \mathbb{Z}) = H_2(Y; \mathbb{Z}) = 0 \qquad \text{and} \qquad c_1(\xi) = 0, \tag{2.3.1}$$

$$H_1(W; \mathbb{Z}) = H_2(W; \mathbb{Z}) = 0 \qquad \text{and} \qquad c_1(TW) = 0. \tag{2.3.2}$$

The hypotheses (2.3.1) and (2.3.2) are sufficient for our applications. Furthermore, they allow us to simplify various definitions, formulas, and notations from [39] by suppressing the homology classes of holomorphic curves and using $\mathbb{Z}$–gradings.

We begin by fixing notation for the choices of Floer data that are needed to define the relevant chain groups and cobordism maps.

Setup 2.3.2 (Contact manifold data)**.** Fix the following setup and notation.

(a) (Y, ξ) is a closed contact $(2n-1)$–manifold satisfying (2.3.1), with nondegenerate contact form α and associated Reeb vector field R_α.

(b) $b(\underline{\gamma})$ is a basepoint for each simple orbit $\underline{\gamma}$ of (Y, α). Denote the set of orbits of (Y, α) by $\mathcal{P}(Y, \alpha)$. Given any orbit $\gamma \in \mathcal{P}(Y, \alpha)$, denote the underlying simple orbit by $\underline{\gamma}$ and the degree of the covering map $\gamma \to \underline{\gamma}$ by $d(\gamma)$.

(c) J is a $d\alpha$–compatible complex structure on ξ and $\widehat{J}$ is the associated $\mathbb{R}$–invariant almost complex structure on the symplectization $\widehat{Y} := \mathbb{R} \times Y$.

(d) $\theta \in \Theta(Y, \alpha, J)$ is a choice of virtual perturbation data (in the sense of [39] §1.1) for the compactified moduli spaces of holomorphic curves $\overline{\mathcal{M}}_\partial(\widehat{Y}; \gamma^+, \Gamma^-)$ (see Definition 2.3.5) for all good orbits γ^+ and collections of good orbits Γ^-.

We use $\mathsf{Data}(Y, \xi)$ to denote the set of choices of data associated to a fixed contact manifold (Y, ξ). Note that $\mathsf{Data}(Y, \xi)$ is natural: any contactomorphism $\Phi : (Y, \xi) \to (Y', \xi')$ induces an obvious map $\Phi_* : \mathsf{Data}(Y, \xi) \to \mathsf{Data}(Y', \xi')$ acquired by pushing forward the contact forms, markers, complex structures and VFC data.

Setup 2.3.3 (Symplectic cobordism data)**.** Fix the following setup and notation.

(a) For each $* \in \{+, -\}$, let Y_*, ξ_* and α_* be contact data, J_* and $\widehat{J}_*$ be complex data, and θ_* be virtual perturbation data, all as in Setup 2.3.2.

Furthermore, fix the following setup and notation for corresponding cobordism data.

(b) (W, λ) is an exact symplectic $2n$–cobordism from $\partial_+ W \simeq Y_+$ to $\partial_- W \simeq Y_-$ satisfying (2.3.2), with completion $(\widehat{W}, \widehat{\lambda})$ as in Review 2.2.4.

(c) J is a $d\lambda$–compatible complex structure on W agreeing with J_* on symplectic collar neighborhoods of Y_*, and $\widehat{J}$ is the associated complex structure on $\widehat{W}$.

(d) $\theta \in \Theta(W, \lambda, \widehat{J})$ is a choice of virtual perturbation data (in the sense of [39] §1.3) for the compactified moduli spaces of holomorphic curves $\overline{\mathcal{M}}_{\mathrm{c}}(\widehat{W}; \gamma^+, \Gamma^-)$ (see Definition 2.3.5) for all good orbits γ^+ and collections of good orbits Γ^-.

We use $\mathsf{Data}[W, \lambda]$ to denote the set of choices of data as above for a fixed deformation class $[W, \lambda]$ of exact symplectic cobordism. For each $* \in \{+, -\}$, there is a projection map of Floer data

$$\pi_* : \mathsf{Data}[W, \lambda] \to \mathsf{Data}(Y_*, \xi_*), \qquad (\lambda, J, \theta) \mapsto (\alpha_*, J_*, \pi_* \theta).$$

Here we use the projection map of VFC data $\pi_* : \Theta(W, \lambda, J) \to \Theta(Y_*, \alpha_*, J_*)$ described in [39], §1.3. Note that the product map

$$\pi_+ \times \pi_- : \Theta(W, \lambda, J) \to \Theta(Y_+, \alpha_+, J_+) \times \Theta(Y_-, \alpha_-, J_-)$$

is surjective.

Before discussing the moduli spaces involved in contact homology, we recall the definition of asymptotic markers.

Definition 2.3.4 (Asymptotic markers)**.** Let (Σ, j) be a closed Riemann surface with punctures. Let $\overline{\Sigma}$ denote the surface Σ with punctures added back in. An *asymptotic marker* $m(p)$ at a puncture p of Σ is a point in the projectivized tangent space $S_p\overline{\Sigma}$.

Let $u : (\Sigma, j) \to (\widehat{W}, \widehat{J})$ asymptotic at the positive or negative end to an orbit γ at p, where W and J are as in Setup 2.3.3(a)–(c). A natural map $Su_p : S_p\overline{\Sigma} \to \underline{\gamma}$ may be defined in this setting as the limit

$$S_p u(v) = \lim_{\epsilon\to 0} \pi_{Y_+} u(\eta(\epsilon)) \quad \text{ or } \quad S_p u(v) = \lim_{\epsilon\to 0} \pi_{Y_-} u(\eta(\epsilon)),$$

for any arc $\eta : [0,1] \to \overline{\Sigma}$ with $\eta(0) = p$ and $\frac{d\eta}{d\epsilon} = v$. In either of these cases, we say that $m(p)$ is asymptotic $x \in \underline{\gamma}$ under u if $S_p u(m(p)) = x$.

Here are precise descriptions of the moduli spaces referenced in Setup 2.3.2 and Setup 2.3.3 above. The reader should reference [39, §2.3] for Pardon's definitions.

Definition 2.3.5 (Moduli spaces for CH)**.** Let $(\widehat{W}, \widehat{\lambda})$ and $\widehat{J}$ be as in Setup 2.3.3(a)–(b). Let γ^+ be a Reeb orbit of (Y_+, α_+) and let Γ^- be a collection of k Reeb orbits in (Y_-, α_-) (allowing repitition of orbits). Consider holomorphic curves $\mathbf{u}$ consisting of the following data.

(a) A punctured, genus 0 Riemann surface (Σ, j) with a positive puncture p^+ and k negative punctures P^-.

(b) A holomorphic map $u : (\Sigma, j) \to (\widehat{W}, \widehat{J})$ which is asymptotic to γ^+ at the puncture p^+ and the orbits Γ^- at the punctures P^-.

(c) Asymptotic markers $m(p^+)$ at p^+ and $m(p^-)$ at each $p^- \in P^-$. These markers must have the property that $m(p^+)$ is asymptotic to $b(\underline{\gamma})$ under u and p^- is asymptotic to $b(\underline{\gamma}^-)$, for any $p^- \in P^-$ and corresponding $\gamma^- \in \Gamma^-$.

A pair of holomorphic curves $\mathbf{u}_* = (\Sigma_*, j_*, u_*, p_*^+, P_*^-, m_*)$ for $* \in \{0, 1\}$ are equivalent, denoted by $\mathbf{u}_0 \sim \mathbf{u}_1$, if there is a smooth map $\varphi : \Sigma_0 \to \Sigma_1$ such that

(d) φ is holomorphic, i.e. $T\varphi \circ j_0 = j_1 \circ T\varphi$.

(e) φ sends punctures to punctures, i.e. $\varphi(p_0^+) = p_1^-$ and $\varphi(P_0^-) = P_1^-$.

(f) φ sends markers to markers, i.e. the induced map $S_{p^+}\varphi : S_{p_0^+}\Sigma_0 \to S_{p_1^+}\Sigma_1$ satisfies $S_{p^+}\varphi(m_0(p_0^+)) = m_1(p_1^+)$ and similarly for negative punctures.

Let $\mathcal{M}(\widehat{W};\gamma^+,\Gamma^-)$ denote the moduli space of such holomorphic curves $\mathbf{u}$ modulo equivalence. When the points of this moduli space are Fredholm regular (in the sense of [39, Definition 2.39]), the moduli space is a smooth manifold of dimension

$$\dim(\mathcal{M}(\widehat{W};\gamma^+,\Gamma^-)) = |\gamma^+| - |\Gamma^-| \in \mathbb{Z}. \tag{2.3.3}$$

If the completed cobordism $(\widehat{W},\widehat{\lambda})$ and $\widehat{J}$ arise via a symplectization $(\widehat{Y}, e^t\alpha)$ as in Setup 2.3.2(a)–(b), then this moduli space poseses a natural $\mathbb{R}$ action given by $\mathbb{R}$–translation on $\widehat{Y}$. We adopt the notation

$$\mathcal{M}_\partial(\widehat{Y};\gamma^+,\Gamma^-) := \mathcal{M}(\widehat{Y};\gamma^+,\Gamma^-)/\mathbb{R}. \tag{2.3.4}$$

for the quotient. In the general case of a completed cobordism $(\widehat{W},\widehat{\lambda})$ as above, we write

$$\mathcal{M}_c(\widehat{W};\gamma^+,\Gamma^-) := \mathcal{M}(\widehat{W};\gamma^+,\Gamma^-). \tag{2.3.5}$$

We let $\overline{\mathcal{M}}_\partial(\widehat{Y};\gamma^+,\Gamma^-)$ and $\overline{\mathcal{M}}_c(\widehat{W};\gamma^+,\Gamma^-)$ denote the compactifications of (2.3.4) and (2.3.5) defined in [39], respectively. A detailed understanding of this compactification is unnecessary for this paper, although it is similar to the SFT compactification described in [2].

Given the setup discussed above, we now give an overview of the construction of the contact dg–algebra and cobordism maps.

Construction 2.3.6 (Contact dg–algebra)**.** Consider a contact manifold (Y,ξ) along with a choice of data $\mathcal{D} \in \mathsf{Data}(Y,\xi)$, all as in Setup 2.3.2.

We now give the construction of the $\mathbb{Z}$–graded differential algebra $CC(Y,\xi)_\mathcal{D}$ with differential $\partial_\mathcal{D}$ of degree -1, called the *contact dg–algebra* or the *full contact homology algebra.* We denote the homology of this dg–algebra by

$$CH(Y,\xi)_\mathcal{D} := H(CC(Y,\xi)_\mathcal{D}, \partial_\mathcal{D}).$$

(Algebra) To define the algebra $CC(Y,\xi)_\mathcal{D}$, consider the set $\mathcal{P}(Y,\alpha)$ of unparametrized Reeb orbits of (Y,α). We can divide $\mathcal{P}(Y,\alpha)$ into good orbits $\mathcal{P}_g(Y,\alpha)$ and bad orbits $\mathcal{P}_b(Y,\alpha)$. An orbit γ is *bad* if it is a cover of an orbit γ' with $\mathrm{CZ}(\gamma) - \mathrm{CZ}(\gamma') \equiv 1 \mod 2$. An orbit is *good* if it is not bad. To each good orbit we can associate an $\mathbb{Z}$–graded orientation line $\mathfrak{o}_\gamma$ supported in grading $|\gamma| = CZ(\gamma) + n - 3$ (see for instance [3]). For a finite set of orbits $\Gamma \subset \mathcal{P}_g(Y,\alpha)$ of good orbits, we define $\mathfrak{o}_\Gamma := \otimes_{\gamma\in\Gamma}\mathfrak{o}_\gamma$ and $|\Gamma| := \sum_{\gamma\in\Gamma}|\gamma|$. We then define the contact dg–algebra as

$$CC(Y,\xi)_\mathcal{D} := \bigwedge\Big(\bigoplus_{\gamma\in\mathcal{P}_g(Y,\alpha)} \mathfrak{o}_\gamma \otimes_\mathbb{Z} \mathbb{Q}\Big). \tag{2.3.6}$$

That is, $CC(Y,\xi)_\mathcal{D}$ is the free, graded–commutative, unital $\mathbb{Z}$–graded $\mathbb{Q}$–algebra generated by the orientation lines $\mathfrak{o}_\gamma$ for $\gamma \in \mathcal{P}_g(Y,\alpha)$.

(Differential) We define the differential $\partial_{\mathcal{D}} : CC(Y,\xi)_{\mathcal{D}} \to CC(Y,\xi)_{\mathcal{D}}$ on any pure element $x \in \mathfrak{o}_{\gamma_+}$ of the algebra by the formula

$$\partial_{\mathcal{D}} x := \sum_{|\gamma^+|=|\Gamma^-|+1} \frac{\#_\theta \overline{\mathcal{M}}_\partial(\widehat{Y};\gamma^+,\Gamma^-)}{|\mathrm{Aut}(\gamma^+,\Gamma^-)|} \cdot x. \tag{2.3.7}$$

The symbol $\#_\theta$ denotes taking a virtual point count (with respect to the VFC data θ coming with the data $\mathcal{D}$) valued in $\mathfrak{o}_{\Gamma^-} \otimes \mathfrak{o}^\vee_{\gamma^+}$. See [39, §1.2, 2.3] for a full discussion. The differential is extended to the entire algebra by imposing the graded Leibniz rule $\partial_{\mathcal{D}}(xy) = \partial_{\mathcal{D}}(x)y + (-1)^{|x|}x\partial_{\mathcal{D}}(y)$ for any $x, y \in CC(Y,\xi)_{\mathcal{D}}$.

Construction 2.3.7 (Contactomorphism Maps)**.** Consider a contactomorphism $\Phi : (Y,\xi) \to (Y',\xi')$ between contact manifolds (Y,ξ) and (Y',ξ'), along with a choice of data $\mathcal{D} \in \mathsf{Data}(Y,\xi)$, all as in Setup 2.3.2. Then we can construct a morphism of $\mathbb{Z}$–graded dg–algebras

$$CC(\Phi)_{\mathcal{D}} : CC(Y,\xi)_{\mathcal{D}} \to CC(Y',\xi')_{\Phi_*\mathcal{D}}.$$

To define this map, consider the bijection $\mathcal{P}_{\mathrm{g}}(Y,\alpha) \to \mathcal{P}_{\mathrm{g}}(Y',(\Phi^{-1})^*\alpha)$ given by $\gamma \mapsto \Phi \circ \gamma$. This map comes with isomorphisms of $\mathbb{Z}$–graded $\mathbb{Z}$–modules $\mathfrak{o}_\gamma \to \mathfrak{o}_{\Phi\circ\gamma}$ for each $\gamma \in \mathcal{P}(Y,\alpha)$ (see [3]). We therefore define $CH(\Phi)$ to be the unique algebra map where the restriction $CH(\Phi)|_{\mathfrak{o}_\gamma\otimes\mathbb{Q}}$ to $\mathfrak{o}_\gamma \otimes \mathbb{Q}$ is the induced map of orientation lines (tensored up to $\mathbb{Q}$).

Construction 2.3.8 (Cobordism Maps)**.** Consider an exact symplectic cobordism (W,λ) between contact manifolds (Y_+,ξ_+) and (Y_-,ξ_-), along with a choice of data $\mathcal{D} \in \mathsf{Data}(W,V,\lambda)$ as in Setup 2.3.3, where $\mathcal{A} = \pi_+\mathcal{D}$ and $\mathcal{B} = \pi_-\mathcal{D}$. Then we can construct a morphism of $\mathbb{Z}$–graded dg–algebras

$$CC(W,\lambda)_{\mathcal{D}} : CC(Y_+,\xi_+)_{\mathcal{A}} \to CC(Y_-,\xi_-)_{\mathcal{B}}.$$

To define this cobordism map, we define its value on generators and extend it to an algebra map. On a generator $x \in \mathfrak{o}_{\gamma^+}$ for $\gamma^+ \in \mathcal{P}(Y_+,\alpha_+)$, it is given by the sum

$$CC(W,\lambda)_{\mathcal{D}}(x) = \sum_{|\gamma^+|=|\Gamma^-|+1} \frac{\#_\theta \overline{\mathcal{M}}_{\mathrm{c}}(\widehat{W};\gamma^+,\Gamma^-)}{|\,\mathrm{Aut}(\gamma^+,\Gamma^-)|} \cdot x. \tag{2.3.8}$$

The symbol $\#_\theta$ denotes taking a virtual moduli count in $\mathfrak{o}_{\Gamma^-} \otimes \mathfrak{o}^\vee_{\gamma^+}$ with respect to the VFC data θ, similarly to the differential. See [39, §1.3, 2.3] for a full discussion.

The salient features of Constructions 2.3.6 and 2.3.8 can be summarized in the following Theorem, various parts of which are covered in [39, §1.3–1.8].

Theorem 2.3.9 ([39] Contact homology)**.** *The dg–algebra and maps described in Constructions 2.3.6 and 2.3.8 have the following properties:*

(a) *(Homology) The map $\partial_{\mathcal{D}}$ of Construction 2.3.6 is a differential and the homology algebra $CH(Y,\xi) \simeq CH(Y,\xi)_{\mathcal{D}}$ is independent of the choice of data $\mathcal{D} \in \mathsf{Data}(Y,\xi)$ up to canonical isomorphism.*

(b) *(Contactomorphism) The contactomorphism map $CC(W,\lambda)_{\mathcal{D}}$ of Construction 2.3.7 induces a well–defined isomorphism*

$$CH(\Phi) : CH(Y,\xi) \simeq CH(Y',\xi').$$

(c) *(Cobordism) The cobordism map $CC(W,\lambda)_{\mathcal{D}}$ of Construction 2.3.8 is a map of dg–algebras, and induces a well–defined map*

$$CH[W,\lambda] : CH(Y_+,\xi_+) \to CH(Y_-,\xi_-).$$

(d) *(Deformation/Composition) The cobordism map $CH[W,\lambda]$ depends only on the deformation class of (W,λ) (see Review 2.2.4). Furthermore, if (W_{01},λ_{01}) and (W_{12},λ_{12}) are cobordisms from Y_0 to Y_1 and Y_1 to Y_2, respectively, then*

$$CH[W_{12}\#_{Y_1}W_{01}, \lambda_{12}\#_{Y_1}\lambda_{01}] = CH[W_{12},\lambda_{12}] \circ CH[W_{01},\lambda_{01}].$$

(e) *(Transversality) Suppose that a 0–dimensional moduli space used in either Constructions 2.3.6 and 2.3.8, i.e. one of the spaces*

$$\mathcal{M}_{\partial}(\widehat{Y};\gamma^+,\Gamma^-) \quad or \quad \mathcal{M}_{\mathrm{c}}(\widehat{W};\gamma^+,\Gamma^-),$$

is Fredholm regular (see [39, §2.11]) and SFT compact. Then the corresponding virtual count, which is either

$$\#_{\theta}\mathcal{M}_{\partial}(\widehat{Y};\gamma^+,\Gamma^-) \quad or \quad \#_{\theta}\mathcal{M}_{\mathrm{c}}(\widehat{W};\gamma^+,\Gamma^-),$$

is given by a signed point count (according to coherent orientations, see [44, §11]), after one identifies the orientation lines $\mathfrak{o}_\gamma$ of all good orbits γ with $\mathbb{Z}$.

2.3.2 Proof of point count

We now compute the examples of contact homology and cobordism maps that are relevant to this paper, and use these computations to prove Lemma 2.2.13.

Definition 2.3.10. A contact form α on a closed contact manifold (Y,ξ) is *lacunary* if for every good orbit $\gamma \in \mathcal{P}_{\mathrm{g}}(Y,\alpha)$ the grading $|\gamma| \in \mathbb{Z}$ is even.

The contact dg–algebra of a contact manifold with lacunary contact form has vanishing differential for grading reasons. Therefore we have the following lemma.

Lemma 2.3.11. *Let (Y, ξ) be a contact manifold with lacunary contact form α. Then*

$$CH(Y,\xi) \simeq \bigwedge (\mathbb{Q}[\mathcal{P}_{\mathrm{g}}(Y,\alpha)])\,. \tag{2.3.9}$$

Example 2.3.12 (Ellipsoid CH)**.** Consider the boundary $(\partial E(a), \alpha)$ of an irrational ellipsoid $E(a)$ equipped with the standard contact structure ξ and contact form α, as in Example 2.2.3. Note that $(\partial E(a), \xi)$ satisfies the hypotheses (2.3.1).

The Conley–Zehnder index formula (2.2.10) implies that every orbit of $(\partial E(a), \alpha)$ is good, and that we have the grading formula

$$|\gamma_i^m| = 2n - 4 + 2|\{L \in \mathrm{Spec}(Y,\alpha) | L \le m a_i\}| \in \mathbb{Z}. \tag{2.3.10}$$

Therefore the grading of $\mathfrak{o}_{\gamma_i^m}$ is even and the contact form is lacunary. The contact homology is thus computed by (2.3.9).

Next, we use the axioms of Theorem 2.3.9 to demonstrate some properties of the cobordism maps on contact homology induced by the cobordisms of Example 2.2.6.

Lemma 2.3.13. *Let $E(a)$ and $E(b)$ be irrational ellipsoids with $a_i < b_i$ for all i, and let $c \in (0,1)$ be a constant such that $c \cdot b_i < a_i$ for all i. Let $\iota : E(a) \to \mathrm{int}(E(b))$ and $\jmath : E(c \cdot b) \to E(a)$ be the standard inclusions, and let $\mathrm{sc} : \partial E(b) \to \partial E(c \cdot b)$ be the contactomorphism given by scaling, i.e. $\mathrm{sc}(x) := c \cdot x$.*

Then the cobordism maps $CH[W_\iota, \lambda_\iota]$ and $CH[W_\jmath, \lambda_\jmath]$ induced by the cobordism (W_ι, λ_ι) and $(W_\jmath, \lambda_\jmath)$ (see Example 2.2.6) are inverses to each other, i.e.

$$CH[W_\jmath, \lambda_\jmath] \circ CH[W_\iota, \lambda_\iota] = CH(\mathrm{sc}). \tag{2.3.11}$$

Proof. Notice that we have the isomorphisms

$$(W_\jmath, \lambda_\jmath) \#_{\partial E(a)} (W_\iota, \lambda_\iota) \simeq (W_{\iota\circ\jmath}, \lambda_{\iota\circ\jmath}) \simeq (E(a) \times [\log(c), 0], e^s \lambda|_{\partial E(a)}) \tag{2.3.12}$$

of exact symplectic cobordisms from $(\partial E(a), \lambda|_{\partial E(a)})$ to $(\partial E(c \cdot a), \lambda|_{\partial E(c\cdot a)})$. Therefore by Theorem 2.3.9(c), the identity of cobordism maps

$$CH[W_\jmath, \lambda_\jmath] \circ CH[W_\iota, \lambda_\iota] = CH[E(a) \times [\log(c), 0], e^s \lambda|_{\partial E(a)}] \tag{2.3.13}$$

holds. The completion of the right–hand side of (2.3.11) is simply the symplectization of $\partial E(a)$. By [39, Lemma 1.2], the induced map is equal to $CH(\mathrm{sc})$. $\square$

With the above computations in hand, we now begin the proof of Lemma 2.2.13 from §2.2.2. We first verify that the moduli spaces of Definition 2.2.8 are, in our case of interest, given by a product of those provided by the cobordism maps of contact homology, described in Definition 2.3.5 and [39].

Lemma 2.3.14. *Let* $(W_{\partial P}, \lambda_{\partial P})$ *and* $J_{\partial P}$ *be as in Lemma 2.2.13. Pick basepoints* $b(\gamma_i^+)$ *and* $b(\gamma_i^-)$ *for all* i*, as in Setup 2.3.2(b). Then there is a natural bijection*

$$\mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P}) \simeq \prod_{i \in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-). \tag{2.3.14}$$

Proof. To prove this, we construct natural maps between the two moduli spaces that are inverses of each other.

($\rightarrow$) Let u be a point in $\mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P})$. By Definition 2.2.8, u is a tuple $(u_i)_{i \in I}$ of reparametrization classes u_i of maps $\Sigma_i \to \widehat{W}_{\partial P}$ where $\Sigma_i := \mathbb{C}P^1 \setminus \{0, \infty\}$. Let $\widetilde{u}_i$ be a representative for each $i \in I$. Because the orbits γ_i^+ and γ_i^- are embedded, there is a unique choice of asymptotic markers $m_i(p_i^+)$ and $m_i(p_i^-)$ at the punctures p_i^+ and p_i^- of Σ_i which satisfy Definition 2.3.5(c). We thus define the map

$$\mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P}) \to \prod_{i \in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-), \quad u \mapsto \prod_{i \in I} [\Sigma_i, j_{\mathbb{C}P^1}, \widetilde{u}_i, p_i^+, p_i^-, m_i]. \tag{2.3.15}$$

The bracket $[-]$ within the product means that we have taken the equivalence class of the curve up to the relation $\sim$ described in Definition 2.3.5(d)–(f). Any two choices of representative produce curves equivalent under $\sim$.

($\leftarrow$) In the other direction, let $\mathbf{u}$ be a point in the moduli space of Definition 2.3.5, which can be written as

$$\mathbf{u} = (\mathbf{u}_i)_{i \in I} \in \prod_{i \in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-), \qquad \mathbf{u}_i = [S_i, j_i, \widetilde{u}_i, p_i^+, p_i^-, m_i].$$

As above, $\widetilde{u}_i$ is a holomorphic cylinder and u_i is the corresponding reparametrization class. Due to uniformization, for each $i \in I$ we can pick a biholomorphism $\varphi_i : (\Sigma_i, j_{\mathbb{C}P^1}) \simeq (S_i, j_i)$ preserving the marked points. We define the map

$$\prod_{i \in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-) \to \mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P}), \qquad \mathbf{u} \mapsto (\widetilde{u}_i \circ \varphi_i)_{i \in I}. \tag{2.3.16}$$

By the right–hand side we mean the reprametrization class of $\widetilde{u}_i \circ \varphi_i$. Any two choices of φ_i evidently produce reparametrization equivalent tuples.

Verifying that the maps (2.3.15) and (2.3.16) are inverses of each other, modulo the equivalence relations involved, is straightforward. $\square$

Next, we need a regularity and compactness result for the cobordism moduli space $\mathcal{M}_{\mathrm{c}}(\gamma_i^+; \gamma_i^-)$ under weaker hypotheses than Lemmas 2.2.12 and 2.2.12. The proof is very similar to the proofs of those Lemmas, so we will be terse in our exposition.

Lemma 2.3.15 (Compactness/transversality for $\mathcal{M}_{\mathrm{c}}$)**.** *Let* $E(c)$ *and* $E(d)$ *be irrational symplectic ellipsoids with parameters* $c = (c_1, \ldots, c_n)$ *and* $d = (d_1, \ldots, d_n)$ *satisfying*

$$c_i < d_i \quad \text{for all } i \in \{1, \ldots, n-1\}, \qquad d_n < 2d_1, \quad \text{and} \quad c_n < 2c_1. \tag{2.3.17}$$

Let $\jmath : E(c) \to E(d)$ be the inclusion and let $(W_\jmath, \lambda_\jmath)$ be the associated embedding.

Then there exists a comeager $\mathcal{J}^{\mathrm{reg}}(\jmath) \subset \mathcal{J}(W_\jmath)$ such that the space $\mathcal{M}_c(\widehat{W}; \gamma_i^+, \gamma_i^-)$ of Definition 2.3.5 is Fredholm regular and SFT compact for any $J_\jmath \in \mathcal{J}^{\mathrm{reg}}(\jmath)$.

Proof. Let $Y_+ = \partial E(d)$ and $Y_- = \partial E(d)$, and consider the moduli spaces

$$\mathcal{M}_\partial(\widehat{Y}_+; \gamma_i^+, \gamma_j^+), \qquad \mathcal{M}_\partial(\widehat{Y}_-; \gamma_i^-, \gamma_j^-), \quad \text{and} \quad \mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \Gamma^-). \tag{2.3.18}$$

Here $\gamma_*^\pm$ can be any orbit in $Y_\pm$ and Γ^- is any finite collection of orbits on $\partial_- W_\jmath \simeq \partial E(a)$. By energy monotonicity, only finitely many such moduli spaces can be nonempty. These spaces will be needed in the compactness argument.

(Regularity) Note that the orbits $\gamma_i^\pm$ are all embedded. Therefore, by the same discussion as in the proof of Lemma 2.2.10, every curve in $\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \Gamma^-)$ is somewhere injective, and we can invoke [44, Theorems 7.1–7.2] to see that there is a comeager subset $\mathcal{J}^{\mathrm{reg}}(W_\jmath; \gamma_i^+, \Gamma^-) \subset \mathcal{J}(W_\jmath)$ such that the moduli space $\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \Gamma^-)$ is transverse. An analogous argument (using [44, Theorem 8.1]) shows transversality for any of the $\mathcal{M}_\partial$ moduli spaces for $J_\pm$ in a comeager $\mathcal{J}^{\mathrm{reg}}(Y_\pm; \gamma_i^\pm, \gamma_j^\pm)$. Intersecting these (countably many) comeager sets yields the desired $\mathcal{J}^{\mathrm{reg}}(\jmath)$.

(Compactness) Pick a $J_\jmath \in \mathcal{J}^{\mathrm{reg}}(\jmath)$ as above and let (p_i, u^i) be a sequence in $\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \gamma_i^-)$ converging to a building v. We use the notation of Review 2.2.11 for this building. As in Lemma 2.2.12, we show that v has no symplectization levels.

First, consider a positive symplectization level u_k^+. By action monotonicity, we know that $\mathcal{A}[\Gamma^\pm(u_k^+)] \le \mathcal{A}(\gamma_i^+)$. It follows from (2.3.17) that there is a sequence $\{a_k\}_0^M$ with $a_0 = i$ such that

$$\Gamma^-(u_k^+) = \{\gamma_{a_k}^+\} \quad \text{and} \quad u_k^+ \in \mathcal{M}(\widehat{Y}_+; \gamma_{a_{k-1}}^+, \gamma_{a_k}^+) \quad \text{for all } k \in \{1, \dots, M\}.$$

Next, consider the cobordism level u^W. Since $\Gamma^+(u^W) = \Gamma^-(u_M^+) = \{\gamma_{a_M}^+\}$, we know by the above transversality argument above that the moduli space of u^W is Fredholm regular, with dimension given by

$$\dim(\mathcal{M}_c(\widehat{W}_\jmath; \gamma_{a_M}^+, \Gamma^-(u^W)) = |\gamma_{a_M}^+| - |\Gamma^-(u^W)| \ge 0.$$

Using the grading formula (2.3.10), we see that if $\Gamma^-(u^W)$ contains an iterate γ_i^m for $m \ge 2$ or more than one orbit, then $|\Gamma^-(u^W)| \ge 4n - 2$. On the other hand, $|\gamma_{a_M}^+| \le 4n - 4$. Therefore, $\Gamma^-[u^W] = \{\gamma_{b_0}^-\}$ with $1 \le b_0 \le n$. Finally, we can argue analogously to the positive symplectization case to show that there is a sequence $\{b_k\}_0^N$ with $b_N = i$ such that

$$\Gamma^-(u_k^-) = \{\gamma_{b_k}^-\} \quad \text{and} \quad u_k^+ \in \mathcal{M}(\widehat{Y}_+; \gamma_{b_{k-1}}^+, \gamma_{b_k}^+) \quad \text{for all } k \in \{1, \dots, N\}.$$

We have thus shown that every level of v is Fredholm regular, and therefore

$$|\gamma_{a_k}^+| - |\gamma_{a_{k+1}}^+| \ge 1, \qquad |\gamma_{b_k}^-| - |\gamma_{b_{k+1}}^-| \ge 1, \qquad |\gamma_{a_M}^+| - |\gamma_{b_0}^-| \ge 0.$$

The above equations contradict the fact that $|\gamma_i^+| - |\gamma_i^-| = 0$ unless $M = N = 0$, i.e. unless there are no symplectization levels. $\square$

Finally, we proceed with the actual proof of Lemma 2.2.13.

Proof. (Lemma 2.2.13) Since $(W_{\partial P}, \lambda_{\partial P}) = (W_\iota, \lambda_\iota)$ (without considering the boundary inclusion maps), we know that the following equality of moduli spaces

$$\mathcal{M}_c(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-) = \mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)$$

holds. This equality, along with Lemma 2.3.14, then implies that it suffices for us to show that $\mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)$ has an odd number of points for J in a comeager set.

To show this, we argue as so. By (2.3.10) in Example 2.3.12, we know that on any of the contact manifolds $\partial E(c \cdot a)$, $\partial E(a)$ or $\partial E(b)$, a non–simple orbit η satisfies $|\eta| \geq 4n - 2$ and any set of 2 or more orbits Γ satisfies $|\Gamma| \geq 4n - 2$. In particular, we have the following isomorphisms for $1 \leq i \leq n$:

$$CH_{2(n+i-2)}(\partial E(a)) \simeq \mathfrak{o}_{\gamma_i} \otimes \mathbb{Q}, \qquad CH_{2(n+i-2)}(\partial E(c \cdot a)) \simeq \mathfrak{o}_{\mathrm{sc}(\gamma_i)} \otimes \mathbb{Q}.$$

By applying Lemma 2.3.13, along with the definitions of $CH(W_\iota)$ and $CH(W_\jmath)$ in terms of the virtual moduli count, we find that

$$CH(\mathrm{sc}) = CH[W_\iota, \lambda_\iota] \circ CH[W_\jmath, \lambda_\jmath] = \frac{\#_\theta \mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^-, \mathrm{sc}(\gamma_i^+))}{|\operatorname{Aut}(\gamma_i^-, \mathrm{sc}(\gamma_i^+))|} \circ \frac{\#_\theta \mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)}{|\operatorname{Aut}(\gamma_i^+, \gamma_i^-)|}.$$

Now note that $\operatorname{Aut}(\gamma_i^-, \mathrm{sc}(\gamma_i^+))$ and $\operatorname{Aut}(\gamma_i^+, \gamma_i^-)$ are both trivial groups since the orbits γ_i^+, γ_i^- and $\mathrm{sc}(\gamma_i^+)$ are all embedded. Furthermore, by Lemma 2.3.15 there are comeager sets of almost complex structures such that the moduli spaces above are both Fredholm regular and compact. Thus, by applying Theorem 2.3.9(e) and choosing identifications of $\mathfrak{o}_{\gamma_i^+}, \mathfrak{o}_{\gamma_i^-}$ and $\mathfrak{o}_{\mathrm{sc}(\gamma_i^+)}$ with $\mathbb{Z}$ such that the map $\mathbb{Z} \to \mathbb{Z}$ induced by $CH(\mathrm{sc})$ is the identity, we acquire the formula

$$1 = \#\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^-, \mathrm{sc}(\gamma_i^+)) \cdot \#\mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-) \in \mathbb{Z}$$

Here # denotes taking a $\mathbb{Z}$–valued, signed point count and so we can conclude that

$$\#\mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-) = \pm 1 \quad \text{and} \quad |\mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)| \equiv 1 \mod 2.$$

This proves the point count for the cobordism moduli-spaces, and ends the proof. $\square$

2.4 Spaces of symplectic embeddings

In this section, we discuss some basic results about the Fréchet manifold of symplectic embeddings $\mathrm{SympEmb}(U, V)$ between symplectic manifolds with boundary. In §2.4.1, we construct the Fréchet manifold structure on $\mathrm{SympEmb}(U, V)$. In §2.4.2, we discuss the relationship between the bordism groups and homology groups of a Fréchet manifold. Last, we prove a version of the Weinstein neighborhood with boundary as Proposotion 2.4.13 in §2.4.3.

2.4.1 Fréchet manifold structure

Let (U, ω_U) and (V, ω_V) be $2n$–dimensional compact symplectic manifolds with nonempty contact boundaries. We now give a proof of the folklore result that the space of symplectic embeddings from U to V is a Fréchet manifold.

Proposition 2.4.1. *The space* $\mathrm{SympEmb}(U, V)$ *of symplectic embeddings* $\varphi : U \to \mathrm{int}(V)$ *with the* C^∞ *compact open topology is a metrizable Fréchet manifold.*

Proof. Let $(U \times V, \omega_{U\times V})$, with $\omega_{U\times V} = \pi_U^*\omega_U - \pi_V^*\omega_V$, denote the product symplectic manifold with corners. Given a symplectic embedding $\varphi : U \to \mathrm{int}(V)$, we may associate the graph $\Gamma(\varphi) \subset U \times V$ given by

$$\Gamma(\varphi) := \{(u, \varphi(u)) \in U \times V\}.$$

The graph is a Lagrangian submanifold with boundary transverse to the characteristic foliation $T(\partial U)^\omega$ on the contact hypersurface $\partial U \times \mathrm{int}(V)$. By the Weinstein neighborhood theorem with boundary, Proposition 2.4.13, there is a neighborhood A of U, a neighborhood B of $\Gamma(\varphi)$ and a symplectomorphism $\psi : A \simeq B$ with $\psi|_U : U \to \Gamma(\varphi)$ given by $u \mapsto (u, \varphi(u))$ and $\psi^*\omega_{U\times V} = \omega_{\mathrm{std}}$.

Let $\mathcal{A}(\varphi, \psi) \subset \ker(d : \Omega^1(L) \to \Omega^2(L))$ and $\mathcal{B}(\varphi, \psi) \subset \mathrm{SympEmb}(U, V)$ denote the open subsets given by

$$\mathcal{A}(\varphi, \psi) := \left\{\alpha \in \Omega^1(L) \;\middle|\; d\alpha = 0 \text{ and } \mathrm{Im}(\alpha) \subset A\right\},$$

$$\mathcal{B}(\varphi, \psi) := \{\phi \in \mathrm{SympEmb}(U, V) \mid \mathrm{Im}(\varphi) \subset B\}.$$

Then we have maps $\Phi : \mathcal{A}(\varphi, \psi) \to \mathcal{B}(\varphi, \psi)$ and $\Psi : \mathcal{A}(\varphi, \psi) \to \mathcal{B}(\varphi, \psi)$ given by

$$\alpha \mapsto \Phi[\alpha] := (\pi_V \circ \psi \circ \alpha) \circ (\pi_U \circ \psi \circ \alpha)^{-1},$$

$$\phi \mapsto \Psi[\phi] := (\psi^{-1} \circ (\mathrm{Id} \times \phi)) \circ (\pi_L \circ \psi^{-1} \circ (\mathrm{Id} \times \phi))^{-1}.$$

It is a tedious but straightforward calculation to check that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$. The fact that Φ and Ψ are continuous in the C^∞ compact open topologies on the domain and images follows from the fact that function composition defines a continuous map $C^\infty(M, N) \times C^\infty(N, O) \to C^\infty(M, O)$ for any compact manifolds M, N, and O (in fact, smooth; see [29, Theorem 42.13]).

Since $C^\infty(U, V)$ is metrizable under the compact open C^∞–topology (see [29, Corollary 41.12]), the subspace $\mathrm{SympEmb}(U, V)$ is also metrizable. □

Lemma 2.4.2. *Let* L *be a compact manifold with boundary and let* $\sigma : L \to T^*L$ *be a section. Then* $\sigma(L)$ *is Lagrangian if and only if* σ *is closed.*

Proof. The same as the closed case, see [34, Proposition 3.4.2]. □

2.4.2 Bordism groups of Fréchet manifolds

We now discuss (unoriented) bordism groups and their structure in the case of Fréchet maifolds. We begin by defining the relevant notions of (continuous and smooth) bordism.

Definition 2.4.3 (Bordisms)**.** Let X be a topological space and $f : Z \to X$ be a map from a closed manifold. We say that the pair (Z, f) is *null–bordant* if there exists a pair (Y, g) of a compact manifold with boundary Y and a continuous map $g : Y \to X$ such that $\partial Y = Z$ and $g|_{\partial Y} = f$. Given a pair of manifold/map pairs (Z_i, f_i) for $i \in \{0, 1\}$, we say that (Z_0, f_0) and (Z_1, f_1) are *bordant* if $(Z_0 \sqcup Z_1, f_0 \sqcup f_1)$ is null–bordant.

Definition 2.4.4 (Smooth bordism)**.** Let X be a Fréchet manifold and $f : Z \to X$ be a smooth map from a smooth closed manifold. Then (Z, f) is *smoothly null–bordant* if it is null–bordant via a pair (Y, g) where $g : Y \to X$ be a smooth map of Banach manifolds with boundary. Similarly, a pair (Z_i, f_i) for $i \in \{0, 1\}$ is *smoothly bordant* if $(Z_0 \sqcup Z_1, f_0 \sqcup Z_1)$ is smoothly null–bordant.

The above notions come with accompanying versions of the bordism group.

Definition 2.4.5 (Bordism group of X)**.** The *n–th bordism group* $\Omega_n(X; \mathbb{Z}/2)$ of a topological space X is group generated by equivalence classes $[Z, f]$ of pairs (Z, f), where Z is a closed n–dimensional manifold and $f : Z \to X$ is a continuous map, modulo the relation that $(Z_0, f_0) \sim (Z_1, f_1)$ if the pair is bordant. Addition is defined by disjoint union

$$[Z_0, f_0] + [Z_1, f_1] := [Z_0 \sqcup Z_1, f_0 \sqcup f_1].$$

Definition 2.4.6 (Smooth bordism group of X)**.** The *n–th smooth bordism group* $\Omega_n^\infty(X; \mathbb{Z}/2)$ of a Fréchet manifold X is group generated by equivalence classes $[Z, f]$ of pairs (Z, f), where Z is a closed n–dimensional manifold and $f : Z \to X$ is a smooth map, modulo the relation that $(Z_0, f_0) \sim (Z_1, f_1)$ if the pair is smoothly bordant. Addition in the group $\Omega_*^\infty(X; \mathbb{Z}/2)$ is defined by disjoint union as before.

Lemma 2.4.7. *The natural map $\Omega_*^\infty(X; \mathbb{Z}_2) \to \Omega_*(X; \mathbb{Z}_2)$ is an isomorphism.*

Proof. The argument uses smooth approximation and is identical to the case where X is a finite dimensional smooth manifold, which can be found in [8, Section I.9]. □

Given the above terminology, we can now prove the main result of this subsection, Proposition 2.4.8. It provides a class of submanifolds for which being null–bordant and being null–homologous are equivalent.

Proposition 2.4.8. *Let X be a metrizable Fréchet manifold, and let $f : Z \to X$ be a smooth map from a closed manifold Z with Stieffel–Whitney class $w(Z) = 1 \in H^*(Z; \mathbb{Z}/2)$. Then $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$ if and only $[Z, f] = 0 \in \Omega_*^\infty(X; \mathbb{Z}_2)$.*

Proof. Proposition 2.4.8 will follow immediately from the following results. First, by Lemma 2.4.7, it suffices to show $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$ if and only $[Z, f] = 0 \in \Omega_*(X; \mathbb{Z}_2)$. By Proposition 2.4.9, we can replace X with a CW complex. Lemma 2.4.10 proves the result in this context. $\square$

Proposition 2.4.9 ([38, Theorem 14])**.** *A metrizable Fréchet manifold is homotopy equivalent to a CW complex.*

Lemma 2.4.10. *Let X homotopy equivalent to a CW complex, and let $f : Z \to X$ be a continuous map from a closed manifold Z with Stieffel–Whitney class $w(Z) = 1 \in H^*(Z; \mathbb{Z}/2)$. Then $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$ if and only $[Z, f] = 0 \in \Omega_*(X; \mathbb{Z}_2)$.*

Remark 2.4.11. Crucially, we make no finiteness assumptions on the CW structure.

Proof. ($\Rightarrow$) Suppose that $f_*[Z] = 0 \in H_2(Z; \mathbb{Z}/2)$. Pick a homotopy equivalence $\varphi : X \simeq X'$ with a CW complex X'. Such an equivalence induces an isomorphism of unoriented bordism groups $\Omega_*(X; \mathbb{Z}_2) \simeq \Omega_*(X'; \mathbb{Z}_2)$, so it suffices to show that the pair $(Z, \varphi \circ f)$ is null–bordant, or equivalently to assume that X is a CW complex to begin with.

So assume that X is a CW complex. By Lemma 2.4.12, we can find a finite sub–complex $A \subset X$ such that $f(Z) \subset A$ and $f_*[Z] = 0 \in H_*(A; \mathbb{Z}/2)$. By Theorem 17.2 of [8], $[Z, f] = 0 \in \Omega_*(A; \mathbb{Z}_2)$ if and only if the Stieffel–Whitney numbers $\mathrm{sw}_{\alpha,I}[Z, f]$ are identically 0. Recall that the Stieffel–Whitney number $\mathrm{sw}_{\alpha,I}[Z, f]$ associated to $[Z, f]$, a cohomology class $\alpha \in H_k(A; \mathbb{Z}_2)$ and a partition $I = (i_1, \ldots, i_k)$ of $\dim(Z) - k$ is defined to be

$$\mathrm{sw}_{\alpha,I}[Z, f] = \langle w_{i_1}(Z) w_{i_2}(Z) \ldots w_{i_k}(Z) f^*\alpha, [Z] \rangle \in \mathbb{Z}_2.$$

Here $w_j(Z) \in H^j(Z; \mathbb{Z}_2)$ denotes the j–th Stieffel–Whitney class of Z. By assumption, $w(Z) = 1$ and so $w_j(Z) = 0$ for all $j \neq 0$. In particular, the only possible nonzero Stieffel–Whitney numbers have $I = (0)$. But we see that

$$\mathrm{sw}_{\alpha,(0)}[Z, f] = \langle f^*\alpha, [Z] \rangle = \langle \alpha, f_*[Z] \rangle = 0.$$

Therefore, $\mathrm{sw}_{\alpha,I}[Z, f] \equiv 0$ and $[Z, f]$ must be null–bordant.

($\Leftarrow$) This direction is completely obvious, since the map $\Omega_*(X) \to H_*(X; \mathbb{Z}/2)$ given by $[Z, f] \mapsto f_*[Z]$ is well defined. $\square$

Lemma 2.4.12. *Let X be a CW complex, and let $f : Z \to X$ be a map from a closed manifold Z with $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$. Then there exists a finite sub–complex $A \subset X$ with $f(Z) \subset A$ and $f_*[Z] = 0 \in H_*(A; \mathbb{Z}/2)$.*

Proof. A very convenient tool for this is the stratifold homology theory of [28], which we now review briefly.

Given a space M, the n–th stratifold group $sH_n(M; \mathbb{Z}/2)$ with $\mathbb{Z}/2$–coefficients (see Proposition 4.4 in [28]) is generated by equivalence classes of pairs (S, g) of a compact, regular stratifold S and a continuous map $g : S \to M$. Two pairs (S_i, g_i) for $i \in \{0, 1\}$ are equivalent if they are bordant by a c–stratifold, i.e. if there is a pair (T, h) of a compact, regular c–stratifold and a continuous map $g : T \to M$ such that $(\partial T, h|_{\partial T}) = (S_0 \sqcup S_1, g_0 \sqcup g_1)$ (see Chapter 3 and Section 4.4 of [28]). Given a map $\varphi : M \to N$ of spaces, the pushforward map $\varphi_* : sH(M;\mathbb{Z}_2) \to sH(M;\mathbb{Z}_2)$ on stratifold homology is given (on generators) by $[S, g] \mapsto [S, \varphi \circ g] = \varphi_*[\Sigma, g]$.

Stratifold homology satisfies the Eilenberg–Steenrod axioms (see Chapter 20 of [28]), and thus if M is a CW complex then there is a natural isomorphism $sH_*(M;\mathbb{Z}_2) \simeq H_*(M;\mathbb{Z}_2)$. If M is a manifold of dimension n, the fundamental class $[M] \in sH_n(M;\mathbb{Z}_2)$ is given by the tautological equivalence class $[M] = [M, \mathrm{Id}]$.

The proof of the lemma is simple with the above machinery in place. Since $f_*[Z] = 0$, the pair (Z, f) must be null–bordant via some compact c–stratifold (Y, g). Since Y and its image $g(Y)$ are both compact, we can choose a sub–complex $A \subset X$ such that $g(T) \subset A \subset X$. Then the pair (Z, f) are null–bordant by (Y, g) in A as well, so that $[Z, f] = 0 \in sH_*(A;\mathbb{Z}_2)$ and thus $f_*[Z] = 0 \in H_*(A;\mathbb{Z}_2)$ via the isomorphism $sH_*(A;\mathbb{Z}_2) \simeq H_*(A;\mathbb{Z}_2)$. □

2.4.3 Weinstein neighborhood theorem with boundary

In this section, we prove the analogue of the Weinstein neighborhood theorem for a Lagrangian L with boundary, within a symplectic manifold X with boundary. We could find no reference for this fact in the literature.

Proposition 2.4.13 (Weinstein neighborhood theorem with boundary)**.** *Let (X, ω) be a symplectic manifold with boundary ∂X and let $L \subset X$ be a properly embedded, Lagrangain submanifold with boundary $\partial L \subset \partial X$ transverse to $T(\partial X)^\omega$.*

Then there exists a neighborhood $U \subset T^*L$ of L (as the zero section), a neighborhood $V \subset X$ of L and a diffeomorphism $f : U \simeq V$ such that $\varphi^*(\omega|_V) = \omega_{\mathrm{std}}|_U$.

Proof. The proof has two steps. First, we construct neighborhoods $U \subset T^*L$ and $V \subset X$ of L, and a diffeomorphism $\varphi : U \simeq V$ such that

$$\varphi|_L = \mathrm{Id}, \qquad \varphi^*(\omega|_V)|_L = \omega_{\mathrm{std}}|_L, \qquad T(\partial U)^{\omega_{\mathrm{std}}} = T(\partial U)^{\varphi^*\omega}. \tag{2.4.1}$$

Here $T(\partial U)^{\omega_{\mathrm{std}}} \subset T(\partial U)$ is the symplectic perpendicular to $T(\partial U)$ with respect to ω_{std} (and similarly for $T(\partial U)^{\varphi^*\omega}$. Second, we apply Lemma 2.4.14 and a Moser type argument to conclude the result.

(Step 1) Let J be a compatible almost complex structure on X and g be the induced metric on L. Recall that the normal bundle $\nu_g L$ with respect to g is a bundle over L with Lagrangian fiber, and that $J : TL \to \nu_g L$ gives a natural isomorphism. Let $\Phi^g : T^*L \to TL$ denote the bundle isomorphism induced by the metric g and let $\exp^g$ denote the exponential map with respect to g.

Since L is compact, we can choose a tubular neighborhood U' of νL such that $\exp^g : U \to X$ is a diffeomorphism onto its image V. We then let

$$U := [J \circ \Phi^g]^{-1}(U') \subset T^*L$$

and also

$$\phi^g : U \simeq V, \qquad (x,v) \mapsto \exp^g_x(J \circ \Phi_g(v)).$$

Note that $\phi^g|_L = \mathrm{Id}$ and $[\phi^g]^*\omega|_L = \omega_{\mathrm{std}}|_L$ by the same calculations as in [34, Theorem 3.4.13]. We now must modify U, V, and ϕ^g to satisfy the last condition of (2.4.1).

To this end, we apply Lemma 2.4.15. Taking $\kappa_0 = T(\partial U)^{\omega_{\mathrm{std}}}$ and $\kappa_1 = T(\partial U)^{[\phi^g]^*\omega}$, we acquire a neighborhood $N \subset \partial(T^*L)$ of ∂L and a family of embeddings $\psi : N \times I \to \partial(T^*L)$ with the following four properties:

$$\psi_t|_{\partial L} = \mathrm{Id}, \qquad d(\psi_t)_u = \mathrm{Id} \text{ for } u \in \partial L, \qquad \psi_0 = \mathrm{Id},$$

$$[\psi_1]_*(T(\partial U)^{\omega_{\mathrm{std}}}) = T(\partial U)^{[\phi^g]^*\omega}.$$

Note here that we are using the fact that $T(\partial U)^{\omega_{\mathrm{std}}}|_L = T(\partial U)^{[\phi^g]^*\omega}|_L$ already by the construction of ϕ^g. By shrinking N and U, we can simply assume that $N = \partial U$. Let $\mathrm{tc} : [0,1) \times \partial U \simeq T \subset U$ be tubular neighborhood coordinates near boundary. By choosing the tubular neighborhood coordinates $\mathrm{tc} : [0,1) \times \partial U \simeq T$ appropriately, we can also assume that $\mathrm{tc}([0,1) \times \partial L) = L \cap T$. We define a map $\Phi : U \to T^*L$ by

$$\Phi(u) = \begin{cases} (s, \psi_{1-s}(v)) & \text{if } u = (s,v) \in [0,1) \times \partial U \text{ via tc}, \\ u & \text{otherwise.} \end{cases}$$

The map Φ has the following properties which are analogous to those of ψ_s:

$$\Phi|_L = \mathrm{Id}, \qquad d(\Phi)_u = \mathrm{Id} \text{ for } u \in L, \qquad \Phi_*(T(\partial L)^{\omega_{\mathrm{std}}}) = T(\partial L)^{[\phi^g]^*\omega}.$$

Also note that Φ is smooth since ψ_t is constant for t near 0 and 1. We thus define f as the composition $\varphi = \phi^g \circ \Phi$. It is immediate that f has the properties in (2.4.1).

(Step 2) We closely follows the Moser type argument of [34, Lemma 3.2.1]. By shrinking U, we may assume that it is an open disk bundle. Let $\omega_t = (1-t)\omega_{\mathrm{std}} + tf^*\omega$ and $\tau = \frac{d}{dt}(\omega_t) = f^*\omega - \omega_{\mathrm{std}}$. Let $\kappa = T(\partial U)^{\omega_t}$ (by the previous work, it does not depend on t). Note that τ satisfies all of the assumptions of Lemma 2.4.14(2.4.3). We prove that κ is invariant under the scaling map $\phi_t(x,u) = (x,tu)$ in Lemma 2.4.16. We can thus find a σ satisfying the properties listed in (2.4.2).

Let Z_t be the unique family of vector fields satisfying $\sigma = \iota(Z_t)\omega_t$. Due to the properties of σ, Z_t satisfies the following properties for each t.

$$Z_t|_L = 0, \qquad Z_t|_{\partial U} \in T(\partial U) \text{ for all } t.$$

The first property is immediate, while the latter is a consequence of the fact that

$$\omega_t(Z_t, \cdot)|_\kappa = \sigma|_\kappa = 0$$

implies

$$Z_t \in (\kappa)^{\omega_t} = T(\partial U).$$

These two properties imply that Z_t generates a map $\Psi : U' \times [0,1] \to U$ for some smaller tubular neighborhood $U' \subset U$ with the property that $\Psi_t|_L = \mathrm{Id}$ and $\Psi_t^* \omega_t = \omega_0$ (see [34, §3.2], as the reasoning is identical to the closed case). In particular, we get a map $\Psi_1 : U' \to U$ with $\Psi_1|_L = \mathrm{Id}$ and $\Psi_1^* f^* \omega$. By shrinking U, taking $\varphi = f \circ \Psi_1$ and taking $V = \varphi(U)$, we at last acquire the desired result. □

The remainder of this section is devoted to proving the various lemmas that we used in the proof above.

Lemma 2.4.14 (Fiber integration with boundary)**.** *Let X be a compact manifold with boundary, $\pi : E \to X$ be a rank k vector bundle with metric and $\pi : U \to X$ be the (open) disk bundle of E with closure $\overline{U}$. Let $\kappa \subset T(\partial U)$ be a distribution on ∂U such that $d\phi_t(\kappa_u) = \kappa_{\phi_t(u)}$ for all $u \in U$, where $\phi : U \times I \to U$ denote the family of smooth maps given by $\phi_t(x, u) := (x, tu)$.*

Finally, suppose that $\tau \in \Omega^{k+1}(\overline{U})$ is a $(k+1)$–form such that

$$d\tau = 0, \qquad \tau|_X = 0, \qquad (\iota_{\partial X}^* \tau)|_\kappa = 0. \tag{2.4.2}$$

Then there exists a k–form $\sigma \in \Omega^k(\overline{U})$ with

$$d\sigma = \tau, \qquad \sigma|_X = 0, \qquad (\iota_{\partial X}^* \sigma)|_\kappa = 0. \tag{2.4.3}$$

Proof. We use integration over the fiber, as in [34, p. 109]. Note that the maps $\phi_t : U \to \phi_t(U) \subset U$ are diffeomorphisms for each $t > 0$, $\phi_0 = \pi$, $\phi_1 = \mathrm{Id}$ and $\phi_t|_X = \mathrm{Id}$. Therefore we have

$$\phi_0^* \tau = 0, \qquad \phi_1^* \tau = \tau.$$

We may define a vector field Z_t for all $t > 0$ and a k–form σ_t for all $t \geq 0$ by

$$Z_t := (\frac{d}{dt}\phi_t) \circ \phi_t^{-1} \text{ for } t > 0, \qquad \sigma_t := \phi_t^*(\iota(Z_t)\tau) \text{ for } t \geq 0.$$

Although Z_t is singular at $t = 0$, as in [34] one can verify in local coordinates that σ_t is smooth at $t = 0$. Since $Z_t|_X = 0$, the k–form σ_t satisfies $\sigma_t|_X = 0$. Furthermore, for any vector field $K \in \Gamma(\kappa)$ on ∂X which is parallel to κ, we have $\iota(K)\sigma_t = \phi_t^*(\iota(Z_t)\iota(d\phi_t(K))\tau) = 0$ on the boundary, so that $\iota_{\partial X}^*(\sigma_t)|_\kappa = 0$. Finally, σ_t satisfies the equation

$$\begin{aligned} \tau = \phi_1^* \tau - \phi_0^* \tau &= \int_0^1 \frac{d}{dt}(\phi_t^* \tau) dt = \int_0^1 \phi_t^*(\mathcal{L}_{X_t} \tau) dt \\ &= \int_0^1 d(\phi_t^*(\iota(X_t)\tau)) dt = \int_0^1 d\sigma_t dt = d(\int_0^1 \sigma_t dt). \end{aligned}$$

Therefore, if we define $\sigma := \int_0^1 \sigma_t dt$, it is simple to verify the desired properties using the corresponding properties for σ_t. □

Lemma 2.4.15. *Let U be a manifold and $L \subset U$ be a closed submanifold. Let κ_0, κ_1 be rank 1 orientable distributions in TU such that $\kappa_i|_L \cap TL = \{0\}$ and $\kappa_0|_L = \kappa_1|_L$.*

Then there exists a neighborhood $U' \subset U$ of L and a family of smooth embeddings $\psi : U's \times I \to U$ with the following four properties:

$$\psi_t|_{\partial L} = \mathrm{Id}, \qquad d(\psi_t)_u = \mathrm{Id} \ \textit{for } u \in L, \qquad \psi_0 = \mathrm{Id}, \qquad [\psi_1]_*(\kappa_0) = \kappa_1.$$

Furthermore, we can take ψ_t to be t–independent for t near 0 and 1.

Proof. Since κ_0 and κ_1 are orientable, we can pick nonvanishing sections Z_0 and Z_1 We may assume that $Z_0 = Z_1$ along L. We let Z_t denote the family of vector fields $Z_t := (1-t)Z_0 + tZ_1$. Since $Z_0 = Z_1$ along L, we can pick a neighborhood N of L such that Z_t is nowhere vanishing for all t. We also select a submanifold $\Sigma \subset N$ with $\dim(\Sigma) = \dim(U) - 1$ and such that

$$\Sigma \pitchfork Z_t \text{ for all } t \quad \text{and} \quad L \subset \Sigma.$$

We can find such a Σ by, say, picking a metric and using the exponential map on a neighborhood of L in the sub–bundle $\nu L \cap \kappa_0^{\perp}$ of TL. By shrinking Σ and scaling Z_t to λZ_t, $0 < \lambda < 1$, we can define a smooth family of embeddings

$$\Psi : (-1,1)_s \times \Sigma \times [0,1]_t \to N, \qquad \Psi_t(s,x) = \exp[Z_t]_s(x).$$

Here $\exp[Z_t]$ denotes the flow generated by Z_t. We let $\psi_t = \Psi_t \circ \Psi_0^{-1}$. To see the properties of (2.4.1), note that $\Psi_t(0,l) = l$ for all $l \in L$ and $d(\Psi_t)_{0,l}(s,u) = sZ_t + u$. This implies the first two properties. The third is trivial, while the fourth is immediate from $[\Phi_t]_*(\partial_t) = Z_t$. We can make ψ_t constant near 0 and 1 by simply reparametrizing with respect to t. □

Lemma 2.4.16. *Let L be a manifold with boundary and let (T^*L, ω) be the cotangent bundle with the standard symplectic form. Let $\kappa = T(\partial T^*L)^{\omega}$ denote the characteristic foliation of the boundary ∂T^*L and let $\phi : T^*L \times (0,1] \to T^*L$ denote the family of maps $\phi_t(x,v) = (x, tv)$. Then $[\phi_t]_*(\kappa) = \kappa$.*

Proof. By passing to a chart, we may assume that $L \subset \mathbb{R}^+_{x_1} \times \mathbb{R}^{n-1}_x$ and $T^*L \subset \mathbb{R}^+_{x_1} \times \mathbb{R}^{n-1}_x \times \mathbb{R}^n_p$. Then κ is simply given on $\partial T^*L \subset \{0\} \times \times \mathbb{R}^{n-1}_x \times \mathbb{R}^n_p$ by

$$\kappa = \mathrm{span}(\partial_{p_1}) = \mathrm{span}(\partial_{x_1})^{\omega} \subset T(\partial T^*L).$$

Under the scaling map, we have $[\phi_t]_*(\partial_{p_1}) = t \cdot \partial_{p_1}$. This implies that $[\phi_t]_*(\kappa) = \kappa$. □

Bibliography

[1] S. Anjos, F. Lalonde, M. Pinsonnault, *The homotopy type of the space of symplectic balls in rational ruled 4–manifolds*, Geometry and Topology 13 (2009): 1177-1227.

[2] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, E. Zehnder, *Compactness results in symplectic field theory*, Geometry and Topology 7.2 (2003): 799-888.

[3] F. Bourgeois, K. Mohnke, *Coherent orientations in symplectic field theory*, Mathematische Zeitschrift 248.1 (2004): 123-146.

[4] F. Bourgeois, A. Oancea, *Symplectic homology, autonomous Hamiltonians and Morse–Bott moduli spaces*, Duke mathematical journal 146.1 (2009): 71-174.

[5] E. Burkard, *First steps in homotopy results for symplectic embeddings of ellipsoids*, PhD thesis (2016).

[6] O. Buse, R. Hind, *Ellipsoid embeddings and symplectic packing stability*, Compositio Mathematica 149.5 (2013): 889-902.

[7] K. Choi, D. Cristofaro–Gardiner, D. Frenkel, M. Hutchings, V. G. B. Ramos, *Symplectic embeddings into four-dimensional concave toric domains*, Journal of Topology 7.4 (2014): 1054-1076.

[8] P. E. Conner, E. E. Floyd, *Differentiable periodic maps*, Bulletin of the American Mathematical Society 68.2 (1962): 76-86.

[9] D. Cristofaro–Gardiner, *Symplectic embeddings from concave toric domains into convex ones*, arXiv:1409.4378 (2014).

[10] D. Cristofaro–Gardiner, D. Frenkel, F. Schlenk, *Symplectic embeddings of four-dimensional ellipsoids into integral polydiscs* Algebraic & Geometric Topology 17.2 (2017): 1189-1260.

[11] D. Cristofaro–Gardiner, R. Hind, *Symplectic embeddings of products*, arXiv:1508.02659 (2015).

[12] D. Cristofaro–Gardiner, R. Hind, D. McDuff, *The ghost stairs stabilize to sharp symplectic embedding obstructions*, Journal of Topology 11.2 (2018): 309-378.

[13] Y. Eliashberg, A. Givental, H. Hofer *Introduction to symplectic field theory*, Visions in Mathematics, Birkhuser Basel (2000): 560-673.

[14] Y. Eliashberg, M. Gromov, *Convex symplectic manifolds*, Several complex variables and complex geometry, Part 2 (1991): 135-162.

[15] A. Floer, H. Hofer, K. Wysocki, *Applications of symplectic homology I*, Mathematische Zeitschrift 217.1 (1994): 577-606.

[16] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Inventiones mathematicae 82.2 (1985): 307-347.

[17] L. Guth, *Symplectic embeddings of polydisks*, Inventiones mathematicae 172.3 (2008): 477-489.

[18] J. Gutt, *The positive equivariant symplectic homology as an invariant of some contact manifolds*, arXiv:1503.01443 (2015).

[19] J. Gutt, M. Hutchings, *Symplectic capacities from positive S^1–equivariant symplectic homology*, Algebraic & Geometric Topology 18.6 (2018): 3537-3600.

[20] J. Gutt, M. Usher, *Symplectically knotted codimension–zero embeddings of domains in* $\mathbb{R}^4$, arXiv:1708.01574 (2017).

[21] R. Hind, *Symplectic folding and nonisotopic polydisks*, Algebraic & Geometric Topology 13.4 (2013): 2171-2192.

[22] R. Hind, M. Pinsonnault, W. Wu, *Symplectomorphism groups of non–compact manifolds, orbifold balls, and a space of Lagrangians*, arXiv:1305.7291 (2013).

[23] M. Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, Journal of the European Mathematical Society 4.4 (2002): 313-361.

[24] M. Hutchings, *Beyond ECH capacities*, Geometry & Topology 20.2 (2016): 1085-1126.

[25] M. Hutchings, *Lecture notes on embedded contact homology*, Contact and symplectic topology. Springer, Cham, 2014. 389-484.

[26] M. Hutchings, *The embedded contact homology index revisited*, New perspectives and challenges in symplectic field theory 49 (2009): 263-297.

[27] M. Hutchings, J. Nelson, *Cylindrical contact homology for dynamically convex contact forms in three dimensions*, arXiv:1407.2898 (2014).

[28] M. Kreck, *Differential algebraic topology*, Graduate Studies In Mathematics Volume 110, AMS.

[29] A. Kriegl, P. W. Michor, *The convenient setting of global analysis*, Volume 53, AMS (1997).

[30] D. McDuff, *Blow ups and symplectic embeddings in dimension 4*, Topology 30.3 (1991): 409-421.

[31] D. McDuff, *Symplectic embeddings of 4–dimensional ellipsoids*, Journal of Topology 2.1 (2009): 1-22.

[32] D. McDuff, *The Hofer conjecture on embedding symplectic ellipsoids*, Journal Of Differential Geometry 88 (2001): 519-532.

[33] D. McDuff, L. Polterovich, *Symplectic packings and algebraic geometry*, Inventiones mathematicae 115.1 (1994): 405-429.

[34] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford University Press (2017).

[35] D. McDuff, F. Schlenk, *The embedding capacity of 4–dimensional symplectic ellipsoids*, Annals of Mathematics (2012): 1191-1282.

[36] D. McDuff, L. Traynor, *The 4-dimensional symplectic camel and related results*, London Math. Soc. Lecture Note Ser 192 (1993): 169182.

[37] M. Munteanu, *Noncontractible loops of symplectic embeddings between convex toric domains*, arXiv:1809.04147 (2018).

[38] R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5.1 (1966): 1-16.

[39] J. Pardon, *Contact homology and virtual fundamental cycles*, arXiv:1508.03873 (2015).

[40] J. Robbin, D. Salamon, *The Maslov index for paths*, Topology 32.4 (1993): 827-844.

[41] F. Schlenk, *Symplectic embedding problems, old and new*, Bulletin of the American Mathematical Society 55.2 (2018): 139-182.

[42] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, Commentarii Mathematici Helvetici 85 (2008).

[43] C. Wendl, *Lectures on holomorphic curves in symplectic and contact geometry*, arXiv:1011.1690 (2010).

[44] C. Wendl, *Lectures on symplectic field theory*, arXiv:1612.01009 (2016).