UNIVERSITY OF CALIFORNIA, SAN DIEGO

A tale of scale, conformal, and superconformal invariance

A dissertation submitted in partial satisfaction of the

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in

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by

Andreas Stergiou

Committee in charge:

Professor Kenneth Intriligator, Chair
Professor Julius Kuti, Co-Chair
Professor Mark Gross
Professor Justin Roberts
Professor Frank Wuerthwein

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The dissertation of Andreas Stergiou is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

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VITA

2005	B.S. in Physics, National and Capodistrian University of Athens
2007	M.S. in Theoretical Physics, University of Amsterdam
2007–2013	Teaching and Research Assistant, University of California, San Diego
2013	Ph.D. in Physics, University of California, San Diego

PUBLICATIONS

Jean-François Fortin and A.S., *"Field-theoretic Methods in Strongly-Coupled Models of General Gauge Mediation"*, Nucl.Phys. **B873**, 92 (2013), `arXiv:1212.2202`.

Jean-François Fortin, Benjamín Grinstein, Christopher W. Murphy and A.S., *"On limit cycles in supersymmetric theories"*, Phys.Lett. **B719**, 170 (2013), `arXiv:1210.2718`.

Jean-François Fortin, Benjamín Grinstein and A.S., *"Limit cycles and conformal invariance"*, JHEP **1301**, 184 (2013), `arXiv:1208.3674`.

Jean-François Fortin, Benjamín Grinstein and A.S., *"Limit cycles in four dimensions"*, JHEP **1212**, 112 (2012), `arXiv:1206.2921`.

Jean-François Fortin, Benjamín Grinstein and A.S., *"Scale without conformal invariance at three loops"*, JHEP **0812**, 85 (2012), `arXiv:1202.4757`.

Jean-François Fortin, Benjamín Grinstein and A.S., *"Cyclic unparticle physics"*, Phys. Lett. **B709**, 408 (2012), `arXiv:1110.1634`.

Jean-François Fortin, Kenneth Intriligator and A.S., *"Superconformally Covariant OPE and General Gauge Mediation"*, JHEP **1112**, 64 (2011), `arXiv:1109.4940`.

Jean-François Fortin, Benjamín Grinstein and A.S., *"Scale without Conformal Invariance: Theoretical Foundations"*, JHEP **1207**, 25 (2012), `arXiv:1107.3840`.

Jean-François Fortin, Kenneth Intriligator and A.S., *"Current OPEs in Superconformal Theories"*, JHEP **1109**, 71 (2011), `arXiv:1107.1721`.

A.S., *"The `chet` package"*, `arXiv:1106.2809`.

Jean-François Fortin, Benjamín Grinstein and A.S., *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`.

A.S., *"Precession of Pericenter: A More Accurate Approach"*, `arXiv:gr-qc/0610158`.

ABSTRACT OF THE DISSERTATION

A tale of scale, conformal, and superconformal invariance

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This dissertation consists of two parts. In the first, we study the possibility of recurrent trajectories in renormalization-group flows of unitary four-dimensional gauge theories, and the relation of scale and conformal invariance. We carry out three-loop computations of the beta function in dimensional regularization, and we establish that the beta-function vector field of four-dimensional gauge theories admits recurrent trajectories. It is then demonstrated that theories that live on these trajectories are conformal. Along the way, we construct a perturbative proof that scale implies conformal invariance in relativistic unitary quantum field theories in four spacetime dimensions. We also point out that the beta function of supersymmetric theories does not admit limit cycles in perturbation theory.

The second part of this dissertation pertains to theories that are superconformal, or approximately superconformal. We use the constraints of superconformal symmetry to illustrate features of two- and three-point correlators involving conserved-current insertions. This is motivated by "general gauge mediation" where two-point current-correlators parametrize the soft masses of the minimal supersymmetric standard model. We show that the superconformal symmetry and current conservation are enough to fix the operator products of descendants in terms of those of the primaries. Subsequently we consider soltly broken superconformal symmetry and we study analyticity properties of these correlators, e.g. their discontinuities. We then use the optical theorem to relate them to total scattering cross sections from visible to hidden sector states. We also discuss how the current-current OPE can be truncated to the first few terms to get a good approximation to the soft masses. Finally, we demonstrate our techniques in several examples, both at weak and strong coupling. Among them, we introduce a new framework where supersymmetry-breaking arises both from a hidden sector and dynamically.

Chapter 1

Introduction

In this first, introductory chapter we present a summary of the results presented in the following chapters, and we attempt to elucidate the motivation for this work.

1.1. The framework

The most successful language for the description of the elementary constituents of matter is that of quantum field theory (QFT). It arises when quantum mechanics and the special theory of relativity are combined, and it is so powerful and rich that it has helped us reveal profound results about the nature of the subatomic world for several decades. Despite its long study, QFT keeps hiding a lot of secrets, and active research is being undertaken to uncover more features of the fundamental particles and their interactions.

The success of QFT can be summarized in the discovery of the Higgs boson at the Large Hadron Collider in Geneva, Switzerland, in 2012. It was theorized in the mid-60's, based on thinking in the context of QFT, that a spin-zero particle should exist, its role being the generation of the property we call mass. A huge experimental effort, of a scale never before seen in history, and technological advances that would have not been made without the motivation of finding out if the Higgs particle exists, led to its discovery almost fifty years after its first appearance in the mathematical formulas of QFT. The discovery of the Higgs boson is only one example of the guidance

QFT provides to experiment. Without it, it would have been nearly impossible to experimentally approach the question of the origin of mass in the universe.

Although the study of QFT is well-motivated by its early success, it has become clear that the best formulation of QFT is currently out of reach. This is a rather unsatisfactory state of affairs, but we are willing to put up with it because of QFT's unique ability to describe the subatomic world. However, very complicated and impractical calculations are required nowadays to make progress, so it has become common to analyze systems with a lot of symmetry as a way to simplify the seemingly intractable calculations. Luckily, even in the case where higher symmetry is involved, QFT still contains a wealth of striking phenomena.

1.2. Symmetries in physics

Probably the most important lesson one learns in early physics education is that it is much easier to solve problems with a high degree of symmetry. As an example, it is very easy to find the electric field outside charge distributed continuously on a sphere, like the one in Figure 1.1a. However, if the charge was distributed on an egg like the one in Figure 1.1b, then the calculation would be a lot more complicated, although an egg is not that asymmetric.

(a) A sphere [1]　　　　(b) An egg [2]
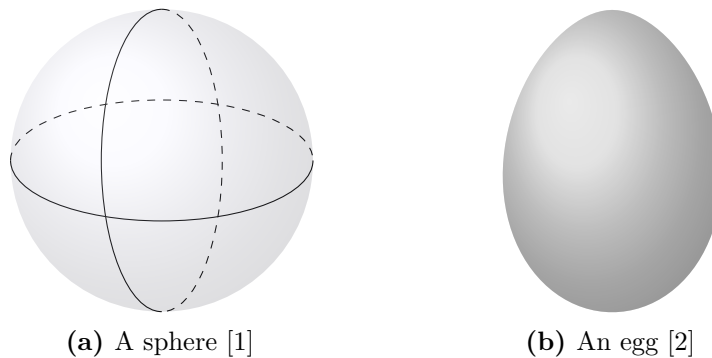
Figure 1.1: Electric charge can be distributed continuously on the surface of a sphere or that of an egg. The electric field outside the sphere is much easier to compute than that outside the egg.

The above example illustrates clearly that, when possible, one should study the most symmetric systems, for those capture the essential physics without introducing computational complications. Of course one should not be misguided: the most symmetric examples may not exhibit the most general behavior.

In high-energy physics there is a spacetime symmetry we always impose on our theories: Poincaré invariance. This is the sanity requirement that experiments performed in various locations in space and at various times should give the same answers, as well as the requirement of Lorentz invariance, i.e. that rotations and boosts of the experimental apparatus do not affect the essential results of the experiment. The Standard Model of particle physics, the most successful theory at our disposal, is Poincaré invariant.

Although Poincaré invariance imposes some restrictions on the form of the theories we can consider, it is natural to ask if we can impose further symmetries in a consistent way. The motivation is of course that the more symmetry we have, the easier it becomes to analyze the theory under consideration. Now, it turns out that aside from spacetime symmetries QFTs can also have global symmetries that act on the fields of QFT but not on spacetime. At this point an example is useful. A free complex scalar field ϕ of mass m is described by the Klein–Gordon Lagrangian,

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{1.2.1}$$

The scalar field ϕ inherits a transformation property from the fact that it is a function on spacetime: under a Lorentz transformation L, ϕ obeys $U^{-1}(L)\phi(x)U(L) = \phi(L^{-1}x)$, where L^{-1} is the inverse Lorentz transformation, and $U(L)$ is a unitary operator representing the Lorentz transformation L. In other words, the Lorentz transformed field at the Lorentz transformed point has a value equal to the value of the untransformed field at the original point. This is the statement that ϕ is a scalar field. It is not hard to see that the Lagrangian (1.2.1) is also a scalar under Lorentz transformations.

On top of the Lorentz symmetry, we can quickly see that we can multiply

the field ϕ in (1.2.1) by a complex constant phase, $\phi \to e^{i\alpha}\phi$, $\alpha^* = \alpha$, and leave $\mathscr{L}$ invariant. This is a new type of symmetry of $\mathscr{L}$, called a global symmetry because the parameter α is not a function of spacetime.

Now, in the quest for higher symmetry, one may wonder if the Poincaré group can be combined with a global symmetry group in a non-trivial way. The answer was given in [3] and it is negative. The result of [3], known as the Coleman–Mandula theorem, is that a QFT's symmetry group is a direct product of the Poincaré group and the global symmetry group; no non-trivial mixing between the two is allowed.

Like every theorem, the Coleman–Mandula theorem relies on certain assumptions. One is free to relax these assumptions and, if that is done in a meaningful way, it results in novel ideas. It turns out that relaxing key assumptions of the Coleman–Mandula theorem leads to supersymmetry, scale, and conformal symmetry. The details are rather technical, but the point that should be clear is that the two possible loopholes of the Coleman–Mandula theorem result in the extension of the spacetime symmetry group of QFT. Supersymmetric theories, as well as conformal and even superconformal theories, enjoy a high degree of symmetry which makes them far more tractable than ordinary QFTs. It is not a surprise, then, that the study of such theories has dominated the QFT research for the past forty years. In this dissertation we study conformal, supersymmetric and superconformal theories.

1.3. Scale and conformal invariance

In the real world phenomena we observe and measurements we make are fully dressed with all the effects of quantum mechanics and the various interactions. In our theoretical studies, however, we simply introduce a set of parameters and we perform calculations with these parameters trying to reproduce our observations. Therefore, in order to make contact with experiment, we have to build a bridge between parameters of our theoretical model and parameters with which we describe our observations.

This bridge is called renormalization, and it is necessary in order to make sense of computations in QFT. A great deal of confusion resulted during the development of QFT from the fact that renormalization was not included in the calculations, which thus gave divergent results. With the advent of renormalization it was soon realized that the divergences, which were there in intermediate steps of the computations, were actually simply an artifact of the way the calculations were performed.

At the technical level, in order to implement the idea of renormalization one introduces an arbitrary energy scale μ in the theory. The requirement is then that physical quantities do not depend on μ, and it is encoded in the renormalization-group equation (RGE). The RGE is describing the evolution of the theory's parameters as we vary the renormalization scale μ. If the parameters of the theory are collectively denoted by g^i, then the RGE that describes the evolution of the coupling with μ is a first-order differential equation:

$$\mu \frac{dg^i}{d\mu} = \beta^i(g), \tag{1.3.1}$$

where $\beta^i(g)$ is the so-called beta function. If we think of the couplings as coordinates and of $t = -\ln(\mu/\mu_0)$ as time, then the beta function is the velocity of evolution of the system. The solution to the RGE is called a flow.

The content of (1.3.1) is that as we vary the energy at which we study the theory, the renormalized parameters of our theory change. We can think of this as a flow in the space of theories, since a theory is defined by the values of its parameters. Now suppose that we follow the flow all the way to its end, i.e. to the limit $t \to \infty$. What can that end be? This question was first considered in its generality in [4], where the possibility of a fixed point was considered along with that of a limit cycle and other more exotic final states.

The interpretation of the physics of the fixed point is well-known: the theory is conformally-invariant. Let us review this fact. A conformal transformation is a change

of coordinates that results in the metric transforming as

$$\gamma_{\mu\nu}(x) \rightarrow e^{-2\sigma(x)}\gamma_{\mu\nu}(x).$$

Infinitesimally this becomes $\delta\gamma_{\mu\nu} = -2\gamma_{\mu\nu}\delta\sigma$. The variation of the action is then

$$\delta S = \int d^d x \, \frac{\delta S}{\delta\gamma^{\mu\nu}}\delta\gamma^{\mu\nu} = -2\int d^d x \sqrt{\gamma}\, \frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\mu\nu}}\gamma^{\mu\nu}\,\delta\sigma,$$

where γ is the determinant of the metric, and we work in Euclidean spacetime with dimension d. The stress-energy tensor is defined as

$$T_{\mu\nu}(x) = -\frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\mu\nu}(x)},$$

and it is by construction a symmetric tensor. With this definition we can write

$$\delta S = -\tfrac{1}{2}\int d^d x \sqrt{\gamma}\, T_{\mu\nu}\delta\gamma^{\mu\nu} = \int d^d x \sqrt{\gamma}\, T^{\mu}_{\mu}\delta\sigma, \qquad (1.3.2)$$

where $T^{\mu}_{\mu} \equiv \gamma^{\mu\nu}T_{\mu\nu}$ is the trace of the stress-energy tensor.

The infinitesimal variation $\delta\sigma$ can be x-dependent as well as x-independent. In the former case the transformation is called a scale transformation, while in the latter a special conformal transformation. Now, if we are performing a scale transformation, then the action will remain invariant if the trace of the stress-energy tensor is a total derivative, as is clear from (1.3.2). If $\delta\sigma$ is x-dependent, then the theory enjoys invariance under conformal transformations if the trace of the stress-energy tensor is zero.

But what is the relation between this and the fact that a theory is conformal at a fixed point of the RG running? It turns out that the trace of the stress-energy tensor in a QFT is also given by $T^{\mu}_{\mu} = \beta^i \mathcal{O}_i$, where $\mathcal{O}_i$ is a complete set of scale-dimension-d

operators in the QFT and β^i are the beta functions.[1] Consequently, if the beta functions are zero, i.e. if we are at a fixed point of the renormalization-group flow, then the theory is conformal.

Let us summarize what we have found. If we take a QFT in flat-space, then the theory is conformal if $T^\mu_\mu = 0$, and it is only scale-invariant if $T^\mu_\mu = \partial_\mu V^\mu$ for some local operator $V^\mu(x)$ (without explicit x-dependence). For technical reasons, the operator V^μ cannot be equal to a linear combination of a conserved current and the divergence of a two-index tensor. There are several questions that arise at this point. Are there theories that are invariant under scale transformations but not under the special conformal ones? If $T^\mu_\mu = 0$ corresponds to fixed points, what does $T^\mu_\mu = \partial_\mu V^\mu$ correspond to?

To answer these questions certain assumptions have to be made. We are interested in relativistic QFTs that are unitary, renormalizable, and have a well-defined stress-energy tensor. We will call these theories well-behaved from now on.

1.3.1. Two-dimensional QFT

In two-dimensions the question of scale without conformal invariance was answered a long time ago. The basis for the answer was given by the results of Zamolodchikov [6] on the so called *c*-theorem. More specifically, Zamolodchikov showed that in any two-dimensional QFT one can define a function of the couplings that is monotonically decreasing along the RG flow. This result corroborates the intuition that the "number of degrees of freedom" of a QFT decreases in the flow from high to low energies, an intuition based on the idea that one can excite more degrees of freedom using high-energy probes.

Regarding the relation of scale and conformal invariance, Polchinski showed they are actually equivalent in two dimensions [7]. In other words, there is no well-behaved two-dimensional QFT that is scale-invariant without being conformal, or, to connect

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[1]For a derivation using dimensional regularization the reader is referred to [5].

with the previous section, there is no appropriate operator V^μ such that $T^\mu_\mu = \partial_\mu V^\mu$.

The results of Zamolodchikov and Polchinski are based on very general principles, and so it was suggested soon after their discovery that even QFTs in higher spacetime dimensions should display similar properties. Although a wealth of evidence supported this expectation, it was not until very recently that significant progress was made.

Let us note here that the results we describe below are based on methods that are available only for QFTs defined in even dimensions. Similar questions can be asked about QFTs in odd spacetime dimensions, and there is a very strong interest in the answers. Nevertheless, we will concentrate in the case of even dimensions in this dissertation.

1.3.2. Four-dimensional QFT

The results we reviewed in the previous section imply that there are no exotic flows in the RG running of two-dimensional QFTs. More specifically, the endpoints of any RG flow in two-dimensional QFTs are conformal field theories, where the beta functions vanish. There is no room for limit cycles or other more exotic flows as envisioned by Wilson. Nevertheless, the situation in higher dimensions is more rich, as we now explain. Explicit calculations and further comments can be found in chapter 2.

The usual calculation of beta functions in QFTs is done in perturbation theory and proceeds order by order in the loop expansion. The beta function is then a function of the couplings, and CFTs are positions in coupling space with vanishing beta functions. Now suppose that, contrary to the case of two-dimensional QFTs, an appropriate operator V^μ exists such that $T^\mu_\mu = \partial_\mu V^\mu \neq 0$. To make the discussion more concrete, let us work with multi-flavor ϕ^4 theory,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \tfrac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d. \tag{1.3.3}$$

This example was considered by Polchinski [7], who proceeded as follows. We know that $T^\mu_\mu = \beta^i\mathcal{O}_i = \beta_{abcd}\phi_a\phi_b\phi_c\phi_d$, and we can also verify that the most general candidate

V^μ is $V^\mu = Q_{ab}\phi_a\partial^\mu\phi_b$, with Q_{ab} anti-symmetric. Then, $\partial_\mu V^\mu = Q_{ab}\phi_a\partial^2\phi_b$, and after we use the equations of motion we find that the theory under consideration will be scale-invariant without being conformal at positions in coupling space for which

$$\beta_{abcd} = Q_{ae}\lambda_{ebcd} + \text{permutations.} \qquad (1.3.4)$$

The aim is now to find values for the couplings and the entries of Q for which (1.3.4) is true. Polchinski then worked in $d = 4 - \epsilon$ spacetime dimensions, a necessary trade-off that allows computational control, and used the one-loop beta function to prove that there are no solutions to (1.3.4) that do not make both sides zero. Although his result is valid at one loop and in $d = 4 - \epsilon$, it is nevertheless rather interesting that there are no non-trivial one-loop solutions in the space of couplings of the form of (1.3.4).

Prompted by Polchinski's result, Dorigoni and Rychkov [8] considered a more general theory, a theory of real scalars ϕ_a and Weyl spinors ψ_i, with the most general scale-invariant couplings allowed:

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi_a\,\partial^\mu\phi_a + i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i - \tfrac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d - \tfrac{1}{2}y_{a|ij}\phi_a\psi_i\psi_j - \tfrac{1}{2}y^*_{a|ij}\phi_a\bar{\psi}_i\bar{\psi}_j.$$

The candidate virial current is now

$$V_\mu = Q_{ab}\phi_a\partial_\mu\phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}_\mu\psi_j,$$

where Q is anti-symmetric and P anti-Hermitian. At one loop, they showed a result similar to Polchinski's, i.e. that there are no non-trivial solutions to

$$\beta_{abcd} - \mathcal{Q}_{abcd} = 0, \qquad \text{and} \qquad \beta_{a|ij} - \mathcal{P}_{a|ij} = 0, \qquad (1.3.5)$$

where

$$\mathcal{Q}_{abcd} = Q_{ae}\lambda_{ebcd} + 3 \text{ permutations,}$$

$$\mathcal{P}_{a|ij} = Q_{ab}y_{b|ij} + (P_{ki}y_{a|jk} + i \leftrightarrow j).$$

(1.3.6)

The first examination of (1.3.6) in higher orders in perturbation theory was considered in [9–11]. It came as a surprise that there are solutions to (1.3.5) when higher loop orders are included. In order to find the solutions, one has to work at three loops.

The next step was to move away from $d = 4 - \epsilon$ and go to the interesting case of $d = 4$. As explained in chapter 2 below, one can still find nontrivial solutions of (1.3.5) in a gauge theory with scalars and Weyl spinors, whose gauge coupling sits at a fixed point.

1.3.3. Interpretation of the solutions

According to our arguments above, when the beta functions are zero the theory is conformal and sits at a fixed point of the RG. But what about beta functions of the form of (1.3.5)? In that case, one can verify that the evolution of the couplings is described by

$$\bar{\lambda}_{abcd}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'} ,$$

$$\bar{y}_{a|ij}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'} ,$$

where the $\widehat{Z}(t)$ matrices are given by

$$\widehat{Z}_{aa'}(t) = \left(e^{Qt}\right)_{aa'} , \qquad \widehat{Z}_{ii'}(t) = \left(e^{Pt}\right)_{ii'} .$$

Then any point $(\bar{\lambda}_{abcd}(t, \lambda, y), \bar{y}_{a|ij}(t, \lambda, y))$ lies on a trajectory that satisfies (1.3.5), since the couplings and also the beta functions transform homogeneously along the

trajectory:

$$\bar{\beta}_{abcd}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\beta_{a'b'c'd'}\ ,$$

$$\bar{\beta}_{a|ij}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)\beta_{a'|i'j'}\ .$$

(1.3.7)

Here, unbarred parameters are evaluated at $(\lambda_{abcd}, y_{a|ij})$, i.e., at a solution of (1.3.5). The behavior (1.3.7) ensures that Q_{ab} and P_{ij} are constant along the scale-invariant trajectory.

Therefore, since the eigenvalues of Q and P are purely imaginary, we have found that solutions of (1.3.5) are recurrent trajectories of the beta function, i.e. limit cycles and ergodic trajectories! But what is the physics of these trajectories? It turns out that even if the beta functions are nonzero and of the specific form of (1.3.5), then the theory is still conformal! This stems from the work of Jack and Osborn [5], which we now review. More details can be found in chapter 3.

The crucial observation of Jack and Osborn is the fact that the trace of the stress-energy tensor can get contributions from divergence terms, i.e.

$$T^{\mu}_{\ \mu} = \beta^i \mathcal{O}_i + \partial_\mu J^\mu,$$

where J^μ is a dimension-three operator. Note that if the theory is conformal and the operator J^μ has no anomalous dimension, then $\partial_\mu J^\mu = 0$. Therefore, we consider the case where J^μ has an anomalous dimension, but of course $T^{\mu}_{\ \mu}$ does not. Now, one can express $\partial_\mu J^\mu$ in the basis of the dimension-four operators $\mathcal{O}_i$, and that will induce a shift of the beta functions. So the expectation is that the beta function is not the quantity that can tell us if a theory is conformal.

To be more explicit, let us take the example (1.3.3). Then, the most general candidate J^μ is $J^\mu = S_{ab}\phi_a \partial^\mu \phi_b$, where S_{ab} is an anti-symmetric matrix with entries

that are functions of the couplings. After we use the equations of motion, we then find

$$T^\mu_\mu = (\beta_I - (S\lambda)_I)\mathcal{O}_I$$

where the index I denotes the collection of indices $(abcd)$, and $(S\lambda)_I = S_{ae}\lambda_{ebcd} +$ permutations.

Thus, we arrive at the conclusion that a theory with couplings g_I is conformal if

$$B_I = \beta_I - (Sg)_I = 0. \tag{1.3.8}$$

One can actually show that if $\beta_I = 0$ then $B_I = 0$, but the converse does not hold. As it turns out, the solutions found in [9–11] have $B_I = 0$, and the same is true for the solutions in chapter 2. This is shown in chapter 3, with an explicit calculation of S at three loops. Consequently, we have found new conformal theories.

One can further argue that, at least in perturbation theory, whenever solutions of $\beta_I = (Qg)_I$ are found, then $(Qg)_I = (Sg)_I$, and thus the theory is conformal by (1.3.8). This follows from consistency conditions stemming from the Abelian nature of the Weyl group, and is shown explicitly in chapter 3. If there were solutions to $\beta_I = (Qg)_I$ with $(Qg)_I \neq (Sg)_I$, then the theory would be scale-invariant without being conformal. With our arguments we have thus shown that scale implies conformal invariance in perturbation theory in well-behaved theories. This is the main result of chapter 3.

1.4. Supersymmetry and its breaking

As we have already remarked, the Coleman–Mandula theorem constrains the nature of the spacetime symmetry group of a QFT. Nevertheless, Coleman and Mandula only considered bosonic operators to construct their proof, and so one can relax that assumption to obtain theories with an enlarged symmetry group, i.e. supersymmetric

theories. Global supersymmetry transformations are generated by the fermionic quantum operators Q, called supercharges, which transform fermionic states into bosonic states and vice-versa:

$$Q|\text{fermion}\rangle \propto |\text{boson}\rangle \qquad \text{and} \qquad Q|\text{boson}\rangle \propto |\text{fermion}\rangle.$$

This immediately implies that the number of fermionic and bosonic degrees of freedom in a supersymmetric theory are equal.

1.4.1. A digression

One can ask if scale and conformal invariance are equivalent in supersymmetric theories. This is the topic explored in chapter 4 and the answer is of course positive as follows from the previous section. Nevertheless, the question is still interesting in its own right, since in supersymmetric theories the extra symmetry results in simplifications.

More specifically, if we calculate the S of (1.3.8) in the most general renormalizable and classically scale-invariant supersymmetric theory, then we find $S = 0$ to all orders in perturbation theory. This is yet another example of the simplicity enforced by symmetry. As a result, the beta functions of supersymmetric theories do not have limit cycles or ergodic trajectories, and the only conformal theories are the ones associated with fixed points of the RG flows.

1.4.2. Supersymmetry breaking

From our brief description of supersymmetry (SUSY), it is clear that SUSY is not part of our world, for we do not observe an equality of fermionic and bosonic degrees of freedom. Therefore, if SUSY was once a symmetry of the universe, it must have been broken somewhere along the evolution of the universe. However, SUSY could not have been broken arbitrarily, since, although it is certainly broken today, the symmetry still manifests itself in certain ways, mainly through properties of the superpartner particles. These are particles that are partnered with the observed particles to give equal number

of bosonic and fermionic degrees of freedom at some point in the past. As an example, SUSY predicts the existence of fermionic superpartners of the gauge bosons, called gauginos, and bosonic partners of quarks called squarks. If SUSY were not broken, these would have the same mass with the corresponding gauge bosons and quarks, but since we do not observe them the symmetry is broken and they have acquired large masses. That's how they avoid detection in experiments.

Various attempts to keep SUSY-breaking consistent with the observed phenomenology have led to the idea of mediation of SUSY-breaking, whereby SUSY is broken in a hidden sector and the breaking is communicated to our visible sector through interactions. The interactions will certainly be gravitational, but one could in addition construct models where the gauge interactions of the Standard Model play an essential role. Indeed, gauge mediation requires that SUSY be broken in a hidden sector with the breaking communicated to the visible sector through the familiar gauge interactions. All soft SUSY-breaking terms in the visible sector are generated via loop effects, and desired phenomenology is obtained very naturally.

In the minimal version of gauge mediation one assumes the existence of a hidden sector that contains a gauge singlet chiral superfield S, as well as a messenger sector with fields $\Phi, \widetilde{\Phi}$ that are charged under the gauge interactions of the standard model. Through interactions in the hidden sector S develops a vacuum expectation value both in its first and its last component, $\langle S \rangle = \langle S \rangle + \theta^2 \langle F_S \rangle$. The superpotential that couples the hidden sector with the messenger sector is $W_{\mathrm{h \otimes m}} \propto S \, \mathrm{Tr}(\widetilde{\Phi}\Phi)$, such that the SUSY-breaking of the hidden sector is transmitted to the messenger sector. The usual gauge interactions then communicate the SUSY-breaking to the supersymmetric extension of the standard model generating the appropriate soft SUSY-breaking terms.

1.4.3. General gauge mediation

A powerful framework for the study of gauge mediation, dubbed general gauge mediation (GGM), was introduced in [12]. In GGM soft terms are written in terms

of one- and two-point correlators of components of a current (linear) superfield of the hidden sector,

$$\mathcal{J}(z) = J(x) + i\theta j(x) - i\bar{\theta}\bar{\jmath}(x) - \theta\sigma^\mu\bar{\theta}j_\mu(x) + \cdots, \tag{1.4.1}$$

where the ellipsis stands for derivative terms, following from the conservation equations $D^2\mathcal{J} = \bar{D}^2\mathcal{J} = 0$, where D and $\bar{D}$ are appropriate covariant derivatives. The linear superfield is the SUSY generalization of a conserved current. Among the virtues of GGM is its ability to disentangle genuine characteristics of gauge mediation from possible model-dependent features. GGM also leads to phenomenological superpartner-mass sum rules, that could be verified by experiments.

The correlators one considers in GGM are

$$
\begin{aligned}
\langle J(x)J(0)\rangle &= C_0(x) \xrightarrow{\text{F.T.}} \widetilde{C}_0(p), \\
\langle j_\alpha(x)\bar{\jmath}_{\dot\alpha}(0)\rangle &= -i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu C_{1/2}(x) \xrightarrow{\text{F.T.}} \sigma^\mu_{\alpha\dot\alpha}p_\mu\widetilde{C}_{1/2}(p), \\
\langle j_\mu(x)j_\nu(0)\rangle &= (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)C_1(x) \xrightarrow{\text{F.T.}} -(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\widetilde{C}_1(p), \\
\langle j_\alpha(x)j_\beta(0)\rangle &= \epsilon_{\alpha\beta}B_{1/2}(x) \xrightarrow{\text{F.T.}} \epsilon_{\alpha\beta}\widetilde{B}_{1/2}(p),
\end{aligned}
\tag{1.4.2}
$$

where F.T. stands for Fourier-transforming, $\text{F.T.} \equiv i\int d^4x\, e^{-ip\cdot x}$.

Motivated by the theoretical appeal of gauge mediation, in chapter 5 we study constraints of superconformal symmetry on correlation functions involving the linear superfield $\mathcal{J}$. The foundation for our work was laid down by Osborn [13], who worked out the general form of two- and three-point correlation functions of superconformal primary operators. Using the results of [13] we find the general form of three-point functions with two current insertions, and we show that, within a superconformal field theory, the superconformal algebra and current conservation are powerful enough to relate all possible two-operator products of components of the current superfield (1.4.1) to the operator product $J(x)J(0)$. Consequently, only the correlator $\langle J(x)J(0)\rangle$ is necessary, while all other correlators in (1.4.2) can be expressed in terms of $\langle J(x)J(0)\rangle$ with the help of the superconformal group. We also point out that in superconformal

theories the spacetime symmetry is not enough to guarantee that only the lowest-components' operator product is necessary. The Ward identity of current conservation is essential in deriving our result for the operator product $\mathcal{J}(z)\mathcal{J}(0)$.

Here we also rely on the operator product expansion (OPE). The idea behind the OPE is that local physics can be captured by local operators. Consider the situation where one wants to calculate the correlation function $\langle J(x)J(0)\rangle$, in the limit $x \to 0$, where J is some operator. The product operator $J(x)J(0)$ is not a local operator, yet it is reasonable to expect that in the limit $x \to 0$ there is an expansion of $J(x)J(0)$ which can approximately describe the same physics as the full operator, i.e. such that it can substitute $J(x)J(0)$ in any correlation function. So we write

$$J(x)J(0) \sim \sum c_{\mathcal{O}}(x)\mathcal{O}(0)$$

where the operator $\mathcal{O}$ is of course local and the x-dependence of the left-hand side is captured by the coefficient c-functions $c_{\mathcal{O}}(x)$, called the Wilson coefficients. The Wilson coefficients are universal, i.e. they don't depend on the correlation function in which the substitution is made.

In practical applications of the OPE one splits all momentum integrals in two regions. If a vacuum expectation value of an operator is considered, then only the low-energy behavior is captured, while if a Wilson coefficient is considered, then the calculation is only taking care of the high-energy effects. This splitting of scales inherent in the OPE makes it a very useful tool for the study of theories even at strong coupling.

In chapter 6 we study the $J(x)J(0)$ OPE and we find an approximation to the soft MSSM SUSY-breaking masses even for strongly-coupled hidden sectors. The expansion relies on several approximations (e.g. cuts at supersymmetric threshold, uniform convergence of the OPE) but, at least in the simple case of minimal gauge mediation, a complete knowledge of the OPE leads to an exact evaluation of the soft SUSY-breaking masses. These are the masses of the squarks and gauginos that SUSY predicts, and so they are interesting to theorists and experimentalists alike.

It seems contradictory that we can use the power of the symmetry, which we took advantage of in chapter 5, even when the symmetry is broken, but this is typical of spontaneous breaking of symmetries. The theory still respects the symmetry, but the vacuum does not. So in order to carry out our computations we promote all symmetry-breaking parameters to fields with suitable transformation properties. The symmetry is then restored, and the expectation value of the fields gives rise to spontaneous symmetry breaking.

To avoid complications such as lengthy OPE computations and analytic continuations, a further approximation is introduced in chapter 6, for which one only needs to identify the lowest-dimension operators that have non-zero vacuum expectation values after acted upon with the SUSY operators Q^2 and $\bar{Q}^2Q^2$. In the example of minimal gauge mediation there is only one such operator, namely $S^\dagger S$, and one finds that our approximation to the soft masses is actually only a factor of two smaller than the exact answers at this order.

Finally, in chapter 7 we use the results of chapter 6 to understand the generation of soft masses even when the sector responsible for SUSY-breaking is strongly-coupled. At strong coupling direct computational control is lost, but the OPE can still be used since the calculation of the Wilson coefficients is done at high energies where the theories we consider are weakly-coupled due to asymptotic freedom.

The hidden-sector theory we study is the supersymmetric extension of QCD, which is known even at strong coupling to break SUSY in interesting ways [14]. In this context, we derive explicit expressions for the approximate values of gaugino and squark masses. Chapter 7 serves as a useful illustration of the ideas presented in chapter 6, but it also studies a new model of SUSY breaking in supersymmetric QCD, where two sources of SUSY-breaking exist. This gives a deformation of the SUSY-breaking pattern of [14], and allows for non-zero gaugino masses as required by phenomenology.

References

[1] T. Bombadil, *"3D bodies in TikZ"*, `http://goo.gl/i7qMl`.

[2] P. Gaborit, *"How can I draw an egg using TikZ?"*, `http://goo.gl/bUPGR`.

[3] S.R. Coleman & J. Mandula, *"All possible symmetries of the S matrix"*, Phys.Rev. **159**, 1251 (1967).

[4] K.G. Wilson, *"The Renormalization Group and Strong Interactions"*, Phys.Rev. **D3**, 1818 (1971).

[5] I. Jack & H. Osborn, *"Analogs for the c theorem for four-dimensional renormalizable field theories"*, Nucl.Phys. **B343**, 647 (1990).

[6] A. Zamolodchikov, *"Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory"*, JETP Lett. **43**, 730 (1986).

[7] J. Polchinski, *"Scale and conformal invariance in quantum field theory"*, Nucl.Phys. **B303**, 226 (1988).

[8] D. Dorigoni & V.S. Rychkov, *"Scale Invariance + Unitarity $\Rightarrow$ Conformal Invariance?"*, `arXiv:0910.1087`.

[9] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`.

[10] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: Theoretical Foundations"*, JHEP **1207**, 025 (2012), `arXiv:1107.3840`.

[11] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance at Three Loops"*, JHEP **1208**, 085 (2012), `arXiv:1202.4757`.

[12] P. Meade, N. Seiberg & D. Shih, *"General Gauge Mediation"*, Prog.Theor.Phys.Suppl. **177**, 143 (2009), `arXiv:0801.3278`.

[13] H. Osborn, *"N=1 superconformal symmetry in four-dimensional quantum field theory"*, Annals Phys. **272**, 243 (1999), `hep-th/9808041`.

[14] K.A. Intriligator, N. Seiberg & D. Shih, *"Dynamical SUSY breaking in meta-stable vacua"*, JHEP **0604**, 021 (2006), `hep-th/0602239`.

Part I

Scale and Conformal Invariance

Chapter 2

Limit Cycles in Four Dimensions

2.1. Introduction

A necessary prerequisite for the complete understanding of quantum field theory (QFT) is the appreciation of its possible phases. In some cases a phase may be out of direct computational reach, e.g., the confining phase of QCD, while in others one may be able to use perturbation theory to gain an understanding of the dynamics of the theory. For a long time the only perturbatively accessible phase of QFT has been presumed to be that of a theory at a conformal fixed point, where, e.g., correlators exhibit power-law scaling.

Recently, the existence of renormalization-group (RG) limit cycles was established by us in $d = 4 - \epsilon$ spacetime dimensions with a three-loop calculation in a unitary theory of scalars and fermions [1–3]. Theories in $d = 4 - \epsilon$ are of course unphysical, but working with them has always been useful in the study of properties of the RG [4], in the sense that RG effects found in such theories have invariably been shown to have counterparts in more physical cases. It was therefore suggested by our results that limit cycles should also occur in integer spacetime dimensions. In the present note we show that this is indeed the case in a four-dimensional unitary gauge theory.

This new feature of the RG gives rise to an obvious question: "what phase of QFT is described by a limit cycle?" It follows from the work of Jack and Osborn [5] that theories that live on limit cycles may be CFTs. As we show in [6] this is indeed

the case for the limit cycle we present below. Thus, although beta functions admit limit cycles, theories that live on these cycles are conformal.

The existence of recurrent trajectories in the RG has implications for the c-theorem. This theorem reflects the intuition that coarse-graining reduces the number of massless degrees of freedom of a QFT, and it comes in different versions, as explained, e.g., in [7]. The strong version, i.e., that there exists a scalar function of the couplings c, along any RG flow, that obeys $dc/dt \leq 0$, with t the RG time and the inequality saturated only at fixed points, was proved long ago for QFTs in $d = 2$ [8], and has been elaborated on heavily in the literature. Soon thereafter it was suggested that a strong c-theorem should be true in $d = 4$ as well [9], and that was indeed shown to be the case at weak coupling [5, 10], at least when renormalization effects of certain composite operators are not of relevance [3, 11]. A proof of the four-dimensional version of the weak version of the c-theorem was recently claimed [12] (see also [13]), i.e., that there is a c-function such that if two four-dimensional CFTs are connected by an RG flow, then $c_{\mathrm{UV}} > c_{\mathrm{IR}}$. Similar ideas were used in an attempt for a proof of the weak version of the c-theorem in $d = 6$ [14].

We hasten to remark that the existence of limit cycles in the beta-function vector field does not contradict intuition derived from the c-theorem. In particular, the quantity c that satisfies a c-theorem is constant even on limit cycles, and is expected to have the same monotonic behavior when it flows from a UV fixed point or limit cycle to an IR fixed point or limit cycle. However, the existence of RG limit cycles obviously demonstrates that beta-function flows are not gradient flows.

The outline of the paper is as follows. In the next section we present our example. We describe in detail the three-loop calculation that establishes the limit cycle, and we show that the dilatation current of the theory on the limit cycle is well-defined and free of anomalies. In the last section we conclude and mention a few open questions.

2.2. The 4d example

In this section we describe in detail the first example of a limit cycle in $d = 4$.

2.2.1. The theory

Our theory has an $SU(3)$ gauge group with two singlet real scalars, ϕ_1 and ϕ_2, two pairs of fundamental and antifundamental active Weyl fermions, $(\psi_{1,2}, \tilde{\psi}_{1,2})$, as well as $\frac{1}{2}(29 - 3\varepsilon)$ pairs of fundamental and antifundamental sterile Weyl fermions. The kinetic terms are canonical and the interactions are given by[1]

$$V = \tfrac{1}{24}\lambda_1\phi_1^4 + \tfrac{1}{24}\lambda_2\phi_2^4 + \tfrac{1}{4}\lambda_3\phi_1^2\phi_2^2 + \tfrac{1}{6}\lambda_4\phi_1^3\phi_2 + \tfrac{1}{6}\lambda_5\phi_1\phi_2^3$$

$$+ \left(\phi_1 \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} + \phi_2 \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} \begin{pmatrix} y_5 & y_6 \\ y_7 & y_8 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} + \text{h.c.} \right).$$

In contrast with the active Weyl spinors, the sterile ones do not interact with the scalars, but they do interact with the gluons through their kinetic terms. One needs sterile fermions in order to get a perturbative fixed point for the gauge coupling, *à la* Banks–Zaks [2, 15]. The smallest value of ε for which our theory is physical is $\varepsilon = \frac{1}{3}$, but we will treat ε as an expansion parameter and take $\varepsilon \to \frac{1}{3}$ at the end. As we will see, our perturbative results can be trusted in this limit.

The most general virial current in our theory is[2]

$$V^\mu = Q_{ab}\phi_a\partial^\mu\phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}^\mu\psi_j \,, \tag{2.2.1}$$

where Q_{ab} is antisymmetric and P_{ij} anti-Hermitian, i.e., $Q_{ba} = -Q_{ab}$ and $P_{ji}^* = -P_{ij}$. For compactness we have denoted by $\psi_{3,4}$ the two antifundamentals $\tilde{\psi}_{1,2}$. By gauge

---

[1] The beta functions for all couplings in this theory can be found at `http://het.ucsd.edu/misc/4D_betas2s12f.m`.

[2] Lower case indices from the beginning of the roman alphabet are indices in flavor space for scalar fields, while lower case indices from the middle are indices in flavor and gauge space for Weyl spinors.

invariance $P_{ij} = P_{ji} = 0$ for $i = 1, 2$ and $j = 3, 4$. All these constraints are satisfied by

$$Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} ip_1 & p_5 + ip_6 & 0 & 0 \\ -p_5 + ip_6 & ip_2 & 0 & 0 \\ 0 & 0 & ip_3 & p_7 + ip_8 \\ 0 & 0 & -p_7 + ip_8 & ip_4 \end{pmatrix}.$$

The virial current (2.2.1) contains a fermionic part, something that can lead to an ABJ-like anomaly [16, 17] for the dilatation current. In this case, the virial current would have an extra contribution to its anomalous dimension,[3] beyond the one calculated from its two-point function. This is not allowed by conformal invariance [6]. Therefore, we expect that a limit cycle solution should have the property that the virial current be not anomalous. Consequently, a consistent limit cycle in a gauge theory with an $SU(n \geq 3)$ gauge group and fundamental and antifundamental fermions should have

$$\text{Tr}\, P = 0.$$

This condition provides a powerful check on our computations.

2.2.2. The three-loop computation

It is convenient to rewrite compactly the interactions in V as

$$V = \tfrac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d + (\tfrac{1}{2} y_{a|ij} \phi_a \psi_i \psi_j + \text{h.c.}).$$

---

[3]As is also the case, e.g., for the axial current [18, Appendix C].

Here, again, we are using the compact notation for the Weyl spinors, with $\psi_{3,4}$ standing for $\tilde{\psi}_{1,2}$. To find a limit cycle we must exhibit solutions to

$$\beta^g(g, y, \lambda) = 0\,,$$

$$\beta_{a|ij}(g, y, \lambda) = -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'}\,, \tag{2.2.2}$$

$$\beta_{abcd}(g, y, \lambda) = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}\,,$$

that do not require zero $\beta_{a|ij}$ and/or β_{abcd}. This requires both determining the values of the coupling constants and of the matrices Q and P for which the equations are satisfied. It would appear, naively, that the system of equations (2.2.2) has more unknowns than equations, due to the presence of the unknowns Q_{ab} and P_{ij}, and is thus ill-defined. However, in searching for particular solutions, one is free to set some coupling constants to zero. This is accomplished by using the freedom to redefine the scalar fields by an $O(2)$ transformation and the active Weyl spinors by a $U(2) \times U(2)$ transformation, with the concomitant redefinition of coupling constants. Note that a coupling may become zero without its beta function becoming zero, since the couplings are not exclusively multiplicatively renormalized. Hence, the number of unknowns in (2.2.2) is reduced and we obtain a well-defined system with equal numbers of equations and unknowns.

As in Ref. [3] we can calculate the entries of Q and P on a limit cycle in an expansion in ε. To that end, we expand in the small parameter ε the couplings,

$$g = \sum_{n \geq 1} g^{(n)}\varepsilon^{n-\frac{1}{2}}, \qquad y_{a|ij} = \sum_{n \geq 1} y_{a|ij}^{(n)}\varepsilon^{n-\frac{1}{2}}, \qquad \lambda_{abcd} = \sum_{n \geq 1} \lambda_{abcd}^{(n)}\varepsilon^n\,,$$

and the unknown parameters in the virial current,

$$Q_{ab} = \sum_{n \geq 3} Q_{ab}^{(n)}\varepsilon^n, \qquad P_{ij} = \sum_{n \geq 3} P_{ij}^{(n)}\varepsilon^n\,,$$

and we solve Eqs. (2.2.2) order by order in ε. The lowest order entries in Q and P

are of order ε^3, for at lower orders in ε, corresponding to one- and two-loop orders in perturbation theory, the beta functions produce a gradient flow [5].

To establish a limit cycle we have to compute the ε^3-order terms in the ε expansion of the parameters of the virial current. For a complete calculation of these we need the two-loop beta function for the quartic coupling, the three-loop beta function for the Yukawa coupling, and the four-loop beta function for the gauge coupling. To see why, let us explain how the ε expansion works.

The ε expansion of the beta functions can be written schematically as

$$
\frac{\beta^g}{g^2} = \sum_{n \geq 1} f_g^{(n)} \varepsilon^{n+\frac{1}{2}} = f_g^{(1)}(g^{(1)}, y^{(1)}) \varepsilon^{3/2} + f_g^{(2)}(g^{(1)}, y^{(1)}, \lambda^{(1)}; g^{(2)}, y^{(2)}) \varepsilon^{5/2}
$$
$$
+ f_g^{(3)}(g^{(1)}, y^{(1)}, \lambda^{(1)}, g^{(2)}, y^{(2)}, \lambda^{(2)}; g^{(3)}, y^{(3)}) \varepsilon^{7/2} + \cdots ,
$$

$$
\beta^y = \sum_{n \geq 1} f_y^{(n)} \varepsilon^{n+\frac{1}{2}} = f_y^{(1)}(g^{(1)}, y^{(1)}) \varepsilon^{3/2} + f_y^{(2)}(g^{(1)}, y^{(1)}, \lambda^{(1)}; g^{(2)}, y^{(2)}) \varepsilon^{5/2}
$$
$$
+ f_y^{(3)}(g^{(1)}, y^{(1)}, \lambda^{(1)}, g^{(2)}, y^{(2)}, \lambda^{(2)}; g^{(3)}, y^{(3)}) \varepsilon^{7/2} + \cdots ,
$$

$$
\beta^\lambda = \sum_{n \geq 1} f_\lambda^{(n)} \varepsilon^{n+1} = f_\lambda^{(1)}(g^{(1)}, y^{(1)}; \lambda^{(1)}) \varepsilon^2 + f_\lambda^{(2)}(g^{(1)}, y^{(1)}, \lambda^{(1)}, g^{(2)}, y^{(2)}; \lambda^{(2)}) \varepsilon^3 + \cdots .
$$

Note that the gauge-coupling beta function is divided by g^2. This way systems of equations obtained at a specific ε order contain the same coefficients in the ε expansion of the couplings and can thus be solved simultaneously. All couplings, f's, and beta functions carry flavor indices which we omit for brevity. It is important to realize that both the one- and the two-loop order of β^g contribute to $f_g^{(1)}$, for we are fixing the gauge coupling to a point *à la* Banks–Zaks. The first step is to simultaneously solve $f_g^{(1)} = 0$ and $f_y^{(1)} = 0$, a system of nonlinear equations from which we get a set of solutions $\{(g^{(1)}, y^{(1)})\}$. Each solution in this set is then used to solve $f_\lambda^{(1)} = 0$, another system of nonlinear equations, which also gives a set of solutions $\{\lambda^{(1)}\}$. At this point we can discard solutions with complex $\lambda^{(1)}$'s—those correspond to nonunitary theories—and construct the set of solutions $\mathcal{S} = \{(g^{(1)}, y^{(1)}, \lambda^{(1)})\}$. The determination of the unknowns in $f_x^{(n \geq 2)}$ requires solving simultaneous linear equations, and so we

have a unique solution for each element of $\mathcal{S}$. At the next step we use solutions in $\mathcal{S}$ to simultaneously solve $f_g^{(2)} = 0$ and $f_y^{(2)} = 0$ for the unknowns $g^{(2)}$ and $y^{(2)}$, which are thus uniquely determined. These are used in $f_\lambda^{(2)}$ from which $\lambda^{(2)}$ is determined, and then we consider $f_g^{(3)}$ and $f_y^{(3)}$. These two functions receive contributions from the $(n \leq 4)$-loop orders of β^g and the $(n \leq 3)$-loop orders of β^y. At this level we also have to take Q and P into account, i.e., we have to see if there are solutions in the set $\mathcal{S}$ that can lead to solutions of the linear equations $f_g^{(3)} = 0$ and $f_y^{(3)} = Qy + Py$ with Q and/or P nonzero. An indication of which solutions in $\mathcal{S}$ may lead to non-vanishing Q or P is that, already at the previous order, the beta functions for the coupling constants that were set to zero do not vanish.

Now, the two-loop Yukawa and scalar coupling beta functions and the three-loop gauge beta function can be found in the literature [19, 20]. To establish the non-vanishing of Q or P at lowest order, ε^3, the three-loop Yukawa beta function and the four-loop gauge beta function are required. Fortunately, a complete calculation of these beta functions is not needed. We parametrize the beta functions at these orders by summing all possible monomials of coupling constants of appropriate order with arbitrary coefficients c_n. Then, by solving the set of linear equations $f_g^{(3)} = 0$ and $f_y^{(3)} = Qy + Py$, we determine which of these coefficients are involved in the determination of Q and P. There is a one-to-one correspondence between each monomial in the beta functions and a three- or four-loop Feynman diagram. Thus, rather than computing some 1200 three-loop diagrams for the Yukawa beta function and a larger number of four-loop diagrams for the gauge beta function, we find that only a small number of diagrams needs to be computed.

For the present model, following the procedure outlined in the previous para-

graph, we find, to lowest order in ε, that the point

$$y_1 = \frac{219\sqrt{92\,534(11\,430\,301 - 30\,212\sqrt{19\,370})}}{27\,828\,258\,757}\pi\varepsilon^{3/2} + i\frac{24\sqrt{3559}}{3559}\pi\sqrt{\varepsilon} + \cdots,$$

$$y_4 = \frac{8\sqrt{17\,795}}{3559}\pi\sqrt{\varepsilon} + \cdots, \qquad y_5 = \frac{16\sqrt{10\,677}}{3559}\pi\sqrt{\varepsilon} + \cdots,$$

$$\lambda_1 = \frac{-3(4\,177\,004 + 11\,781\sqrt{19\,370})}{7\,819\,123}\pi^2\varepsilon + \cdots, \qquad \lambda_2 = \frac{-75(93\,964 + 1245\sqrt{19\,370})}{7\,819\,123}\pi^2\varepsilon + \cdots,$$

$$\lambda_3 = \frac{1743(9\sqrt{19\,370} - 676)}{7\,819\,123}\pi^2\varepsilon + \cdots, \qquad \lambda_4 = \frac{-249\sqrt{78(11\,430\,301 - 30\,212\sqrt{19\,370})}}{7\,819\,123}\pi^2\varepsilon + \cdots,$$

$$\lambda_5 = \frac{-63\sqrt{78(11\,430\,301 - 30\,212\sqrt{19\,370})}}{7\,819\,123}\pi^2\varepsilon + \cdots, \qquad g = \frac{6\sqrt{78\,298}}{3559}\pi\sqrt{\varepsilon} + \cdots,$$

$$(2.2.3)$$

where we omit couplings that are zero at this point, lies on a limit cycle. Among the zero couplings only the imaginary part of y_5 and the real part of y_8 have nonzero beta functions and are thus generated on the limit cycle. Since not all imaginary parts of $y_{1,\ldots,8}$ can be rotated away, the theory violates CP. For the entries of Q and P we find

$$q^{(3)} = \frac{3\sqrt{891\,563\,478 - 2\,356\,536\sqrt{19\,370}}}{3\,763\,549\,370\,814\,194}(2\,061\,664 + 143\,986c_1 + 127\,268c_2$$

$$- 735\,868c_3 + 63\,634c_4 - 735\,868c_5 - 1\,117\,968c_6 - 1\,593\,120c_7$$

$$+ 654\,696c_8 + 1\,309\,392c_9 + 1\,726\,320c_{10} + 2\,146\,752c_{11} - 25\,316\,928c_{12} \qquad (2.2.4)$$

$$+ 24\,431\,904c_{13} - 863\,136c_{14} + 4\,779\,648c_{15} + 106\,491c_{16}$$

$$- 212\,982c_{17} + 212\,982c_{18} + 106\,491c_{19} - 212\,982c_{20}),$$

and

$$p_1^{(3)} = -\frac{18\sqrt{297\,187\,826 - 785\,512\sqrt{19\,370}}}{1\,881\,774\,685\,407\,097}(389\,632 + 4300c_1 + 50\,720c_2 - 105\,124c_3$$

$$+ 25\,360c_4 - 105\,124c_5 - 94\,632c_6 - 357\,744c_7 + 93\,528c_8 + 187\,056c_9$$

$$+ 276\,648c_{10} + 276\,648c_{11} - 3\,616\,704c_{12} + 3\,490\,272c_{13} - 155\,844c_{14}$$

$$+ 862\,992c_{15} + 15\,213c_{16} - 30\,426c_{17} + 30\,426c_{18} + 15\,213c_{19} - 30\,426c_{20}) - p_3^{(3)},$$

$$(2.2.5)$$

where the coefficients $c_{1,\ldots,20}$ are given by the contributions of the three-loop diagrams of Fig. 2.1 to the Yukawa beta function. None of the three- or four-loop contributions to β^g appears in $q^{(3)}$ or $p_1^{(3)}$. For the other entries of P we find $p_{5,6,7,8}^{(3)} = 0$, and that $p_{2,3,4}^{(3)}$ are undetermined with $p_4^{(3)} = -p_2^{(3)}$.[4] The condition for absence of anomalies of the dilatation current, $\mathrm{Tr}\,P = 0$, is thus $p_1^{(3)} + p_3^{(3)} = 0$. We remark that $q^{(3)}$ and $p_1^{(3)}$ can be determined simply because the running couplings $\mathrm{Im}\,y_5$ and $\mathrm{Re}\,y_8$ run through zero at the point (2.2.3).

$D_1^{(3)}$ $D_2^{(3)}$ (and its symmetric) $D_3^{(3)}$ (and its symmetric)

$D_4^{(3)}$ $D_5^{(3)}$ (and its symmetric) $D_6^{(3)}$

$D_7^{(3)}$ (and its symmetric) $D_8^{(3)}$ $D_9^{(3)}$ (and its symmetric)

---
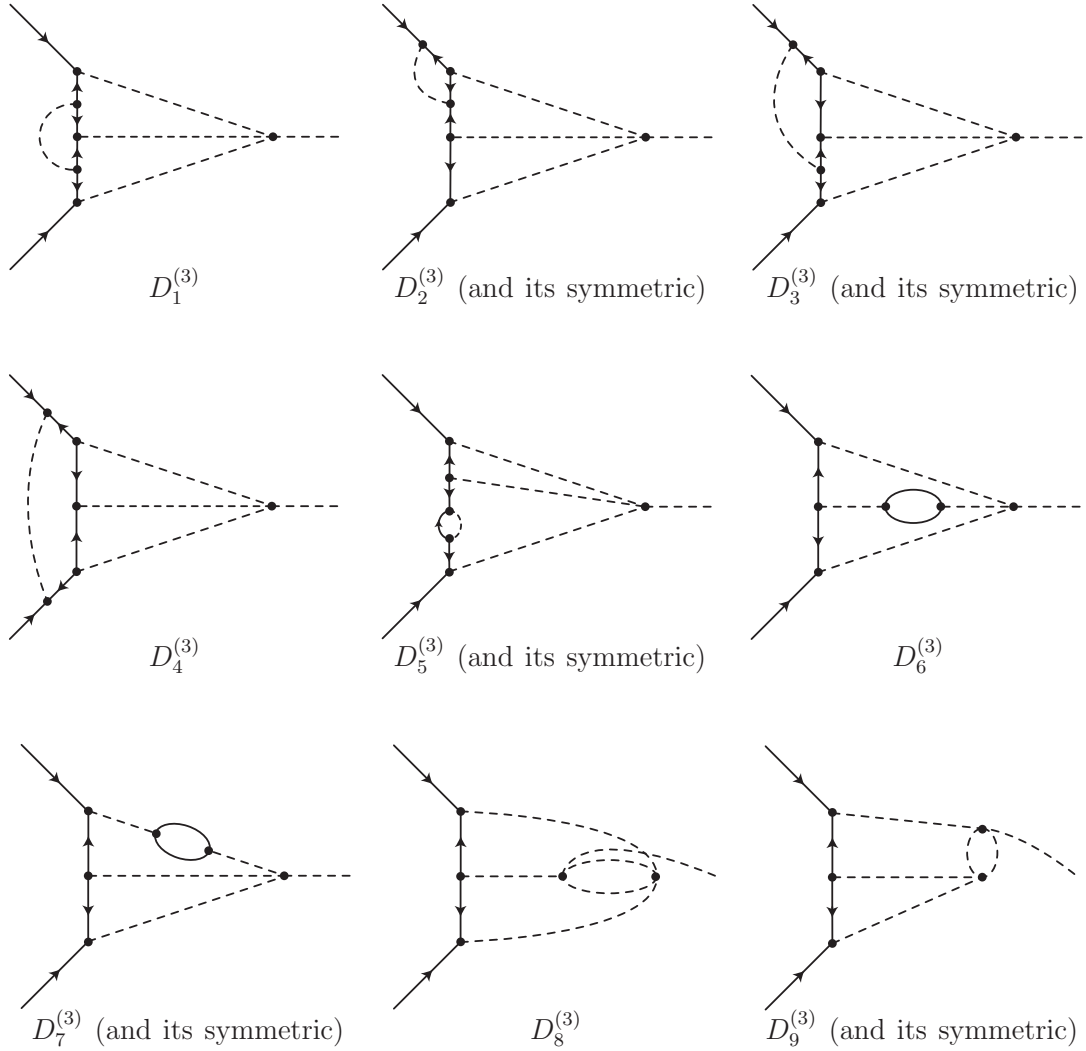
[4]Undetermined entries of P multiply operators that are conserved, i.e., they correspond to global symmetries in the fermionic sector of the theory.

$D_{10}^{(3)}$

$D_{11}^{(3)}$

$D_{12}^{(3)}$

$D_{13}^{(3)}$ (and its symmetric)

$D_{14}^{(3)}$

$D_{15}^{(3)}$

$D_{16}^{(3)}$

$D_{17}^{(3)}$ (and its symmetric)

$D_{18}^{(3)}$ (and its symmetric)

$D_{19}^{(3)}$

$D_{20}^{(3)}$ (and its symmetric)
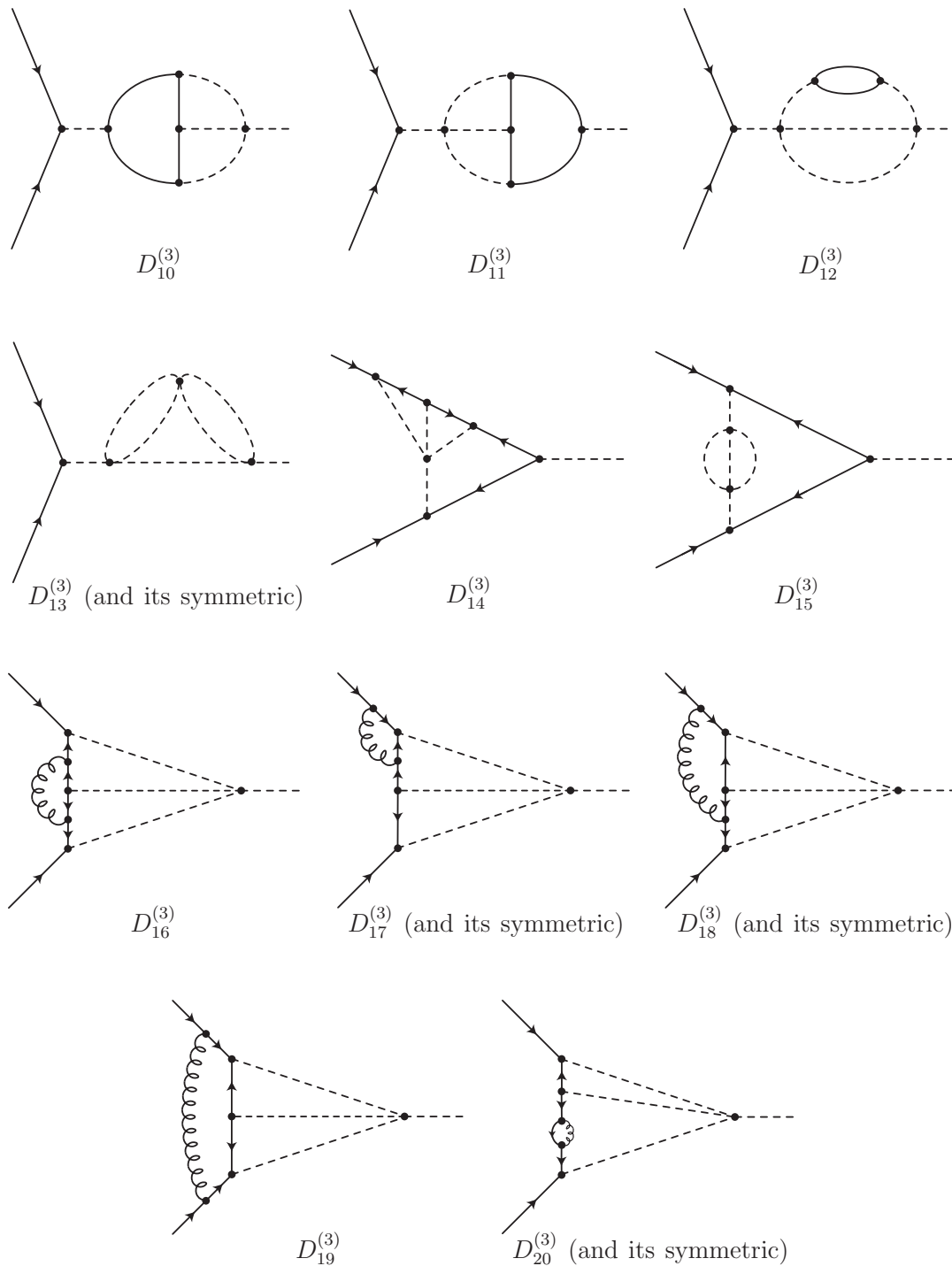
Figure 2.1: Three-loop diagrams that contribute to $q^{(3)}$ and $p^{(3)}$.

Note that both $q^{(3)}$ and $p_1^{(3)}$ receive contributions from exactly the same diagrams, although with different weights, and that twelve of these diagrams ($D_1^{(3)}$–$D_{10}^{(3)}$,

$D_{12}^{(3)}$, and $D_{13}^{(3)}$) are exactly the diagrams that contributed to the frequency of the cycle of Ref. [3]. All diagrams involve at least two types of couplings, as expected from the "interference" arguments of Wallace and Zia [21], as was also seen in our three-loop calculations in $d = 4 - \epsilon$ spacetime dimensions [3].

In dimensional regularization with $d = 4 - \epsilon$ the three-loop diagrams of Fig. 2.1 have simple ϵ-poles and thus they contribute to the Yukawa beta function. The residues of the simple ϵ-poles of $D_{1-20}^{(3)}$ lead to the coefficients $c_{1,\dots,20}$ in

$$(16\pi^2)^3 \beta_{a|ij} \supset c_1 (y_b y_c^* y_d y_c^* y_e)_{ij} \lambda_{abde} + \cdots + c_{20} g^2 [(y_b y_c^* t^{*A} t^{*A} y_d)_{ij} + \{i \leftrightarrow j\}] \lambda_{abcd},$$

as explained, for example, in [22]. We performed the three-loop computation with the method developed in Ref. [23] and the results of Ref. [24]. Since $q^{(3)}$ and $p_1^{(3)}$ are gauge-invariant, we can easily incorporate a quick check in our calculation by using the full gluon propagator, with the gauge parameter ξ. We find[5]

$$c_1 = 3, \quad c_2 = -1, \quad c_3 = 2, \quad c_4 = 5, \quad c_5 = \tfrac{1}{2}, \quad c_6 = \tfrac{3}{2},$$

$$c_7 = \tfrac{1}{2}, \quad c_8 = \tfrac{3}{2}, \quad c_9 = \tfrac{1}{2}, \quad c_{10} = \tfrac{5}{8}, \quad c_{11} = \tfrac{5}{8}, \quad c_{12} = -\tfrac{5}{32}$$

$$c_{13} = -\tfrac{1}{16}, \quad c_{14} = 3, \quad c_{15} = -\tfrac{3}{8}, \quad c_{16} = -7 + 3\xi, \quad c_{17} = 4\xi, \quad c_{18} = -7 - \xi,$$

$$c_{19} = 19 + 5\xi, \quad c_{20} = -\xi.$$

Inserting these into the expressions (2.2.4) and (2.2.5) we obtain

$$q^{(3)} = \frac{20\,745 \sqrt{891\,563\,478 - 2\,356\,536\sqrt{19\,370}}}{99\,040\,772\,916\,163} \approx 5 \times 10^{-6},$$

and

$$p_1^{(3)} = -p_3^{(3)}.$$

---

[5]The symmetry factors are included in the c's. Diagrams $D_{6-11}^{(3)}$ have symmetry factor $s = \tfrac{1}{2}$, diagrams $D_{12,13}^{(3)}$ have $s = \tfrac{1}{4}$, and diagram $D_{15}^{(3)}$ has $s = \tfrac{1}{6}$. All other diagrams have $s = 1$.

That $q^{(3)} \neq 0$ indicates that we have a limit cycle in the RG running of a four-dimensional unitary, renormalizable, well-defined gauge theory. This is the first example ever exhibited of such behavior. As expected, there is no ξ-dependence in the final answer. As expected, the dilatation current is automatically non-anomalous. These are nontrivial checks and lend credibility to our calculation. We have found in the same theory a distinct second limit cycle, in another position in coupling space, with exactly the same properties as the one we presented above.

We have verified that our results can be trusted in the $\varepsilon \to \frac{1}{3}$ limit. More specifically, the expansion parameters are bounded on the cycle: $|\lambda|/16\pi^2 < 5\%$, $|y|^2/16\pi^2 < 1\%$, and $g^2/16\pi^2 = 0.46\%$. Hence, they remain perturbative along the whole cycle.

The only unsatisfactory feature of our example is the fact that, as can be seen from Eqs. (2.2.3), the tree-level scalar potential is unbounded from below. Still the model can be studied in perturbation theory, since the vacuum state $\phi = 0$ is perturbatively stable and its non-perturbative lifetime τ is exponentially long, $\ln(\tau) \sim 1/\max(\lambda_a)$. This is similar in spirit to perturbative studies of renormalization for ϕ^3 models in six dimensions. However, we expect that four-dimensional limit cycles with bounded scalar potential also exist. Our expectation is based on our results in $d = 4 - \epsilon$, where by progressing from the simplest examples, which displayed unbounded tree-level potentials, to more involved examples, we found limit cycles with bounded tree-level scalar potentials [1, 3]. In any case, the behavior of the effective potential in any of these theories remains an open question.

2.3. Conclusion

The existence of limit cycles brings to light a new facet of unitary four-dimensional QFT. Many new questions arise:

- What is the nature of RG flows away from limit cycles? Are there flows to or

from fixed points from or to cycles or ergodic trajectories?

- Are there limit cycles in supersymmetric theories?

- Are there limit cycles in $d = 3$ and $d > 4$? Are there strongly-coupled limit cycles in $d = 3$ that correspond to the $\epsilon \to 1$ limit of the $d = 4 - \epsilon$ perturbative models?

- Are there limit cycles one can be establish in more indirect ways, i.e., without the need of three-loop computations?

- Are there new possibilities for beyond the standard model physics associated with limit cycles [25]?

- What is the holographic description of limit cycles? (This question has been considered in Refs. [26, 27].)

- Are there applications for condensed matter systems?

Answers to these questions will allow a more complete understanding of QFT, and may lead to a new class of phenomena with unique characteristics. It should already be clear, though, that RG flows display behavior that is much richer than previously thought.

Acknowledgments

References

[1] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`.

[2] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: Theoretical Foundations"*, JHEP **1207**, 025 (2012), `arXiv:1107.3840`.

[3] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance at Three Loops"*, JHEP **1208**, 085 (2012), `arXiv:1202.4757`.

[4] K. Wilson & J.B. Kogut, *"The Renormalization group and the epsilon expansion"*, Phys.Rept. **12**, 75 (1974).

[5] I. Jack & H. Osborn, *"Analogs for the c theorem for four-dimensional renormalizable field theories"*, Nucl.Phys. **B343**, 647 (1990).

[6] J.F. Fortin, B. Grinstein & A. Stergiou, *"Limit Cycles and Conformal Invariance"*, `arXiv:1208.3674v2`.

[7] E. Barnes, K.A. Intriligator, B. Wecht & J. Wright, *"Evidence for the strongest version of the 4d a-theorem, via a-maximization along RG flows"*, Nucl.Phys. **B702**, 131 (2004), `hep-th/0408156`.

[8] A. Zamolodchikov, *"Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory"*, JETP Lett. **43**, 730 (1986).

[9] J.L. Cardy, *"Is There a c Theorem in Four-Dimensions?"*, Phys.Lett. **B215**, 749 (1988).

[10] H. Osborn, *"Derivation of a four-dimensional c theorem"*, Phys.Lett. **B222**, 97 (1989).

[11] H. Osborn, *"Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories"*, Nucl.Phys. **B363**, 486 (1991).

[12] Z. Komargodski & A. Schwimmer, *"On Renormalization Group Flows in Four Dimensions"*, JHEP **1112**, 099 (2011), `arXiv:1107.3987`.

[13] Z. Komargodski, *"The Constraints of Conformal Symmetry on RG Flows"*, `arXiv:1112.4538`.

[14] H. Elvang, D.Z. Freedman, L.Y. Hung, M. Kiermaier, R.C. Myers et al., *"On renormalization group flows and the a-theorem in 6d"*, `arXiv:1205.3994`.

[15] T. Banks & A. Zaks, *"On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions"*, Nucl.Phys. **B196**, 189 (1982).

[16] S.L. Adler, *"Axial vector vertex in spinor electrodynamics"*, Phys.Rev. **177**, 2426 (1969).

[17] J. Bell & R. Jackiw, *"A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ-model"*, Nuovo Cim. **A60**, 47 (1969).

[18] D.B. Kaplan & A. Manohar, *"Strange Matrix Elements in the Proton from Neutral Current Experiments"*, Nucl.Phys. **B310**, 527 (1988).

[19] I. Jack & H. Osborn, *"General background field calculations with fermion fields"*, Nucl.Phys. **B249**, 472 (1985).

[20] A. Pickering, J. Gracey & D. Jones, *"Three loop gauge beta function for the most general single gauge coupling theory"*, Phys.Lett. **B510**, 347 (2001), `hep-ph/0104247`.

[21] D. Wallace & R. Zia, *"Gradient Properties of the Renormalization Group Equations in Multicomponent Systems"*, Annals Phys. **92**, 142 (1975).

[22] M.E. Machacek & M.T. Vaughn, *"Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization"*, Nucl.Phys. **B222**, 83 (1983).

[23] K.G. Chetyrkin, M. Misiak & M. Münz, *"Beta functions and anomalous dimensions up to three loops"*, Nucl.Phys. **B518**, 473 (1998), `hep-ph/9711266`.

[24] H.K. Dreiner, H.E. Haber & S.P. Martin, *"Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry"*, Phys.Rept. **494**, 1 (2010), `arXiv:0812.1594`.

[25] J.F. Fortin, B. Grinstein & A. Stergiou, *"Cyclic unparticle physics"*, Phys.Lett. **B709**, 408 (2011), `arXiv:1110.1634`.

[26] Y. Nakayama, *"Gravity Dual for Cyclic Renormalization Group Flow without Conformal Invariance"*, `arXiv:1107.2928`.

[27] Y. Nakayama, *"Holographic Renormalization of Foliation Preserving Gravity and Trace Anomaly"*, `arXiv:1203.1068`.

[28] J. Kuipers, T. Ueda, J. Vermaseren & J. Vollinga, *"FORM version 4.0"*, `arXiv:1203.6543`.

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Chapter 3

Limit Cycles and Conformal Invariance

3.1. Overview in lieu of Introduction

Two recent reported results can potentially greatly enrich our understanding of quantum field theory (QFT). On the one hand, Komargodski and Schwimmer (KS) [1], following earlier work by Cappelli, D'Appollonio, Guida and Magnoli (CDGM) [2, 3], have delineated a nonperturbative proof of an inequality satisfied when a four-dimensional QFT flows between two fixed points of the renormalization group (RG). On the other hand, we have discovered closed RG trajectories[1] in theories in $d = 4 - \epsilon$ [4–6] and $d = 4$ [7] spacetime dimensions, in a regime where perturbation theory is applicable. While the former result can impose restrictions on the possible realizations of long distance (IR) phases of QFTs, the latter exhibits explicitly a novel feature of QFTs. A question naturally arises as to whether these results are compatible.

In this work we will show perturbatively that unitary, interacting, scale-invariant cycles[2] in $d = 4$ correspond to conformal field theories (CFTs), that is, theories with invariance under the full conformal group, not just Poincaré plus dilatations. This follows from the work of Jack and Osborn (JO) [8]. Compatibility of this type of

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[1]Meaning closed flow-lines of the familiar dim-reg beta-function vector field, in conventions where the anomalous-dimension matrix is symmetric. For a word on conventions and their effects on RG functions see Appendix 3.A.

[2]More precisely "limit recursive flows" of the dim-reg beta-function vector field. In what follows we refer to both limit cycles and limiting ergodic behavior simply as "cycles."

cycles[3] with the aforementioned inequality is then not surprising since the inequality still compares a quantity defined on CFTs, be it a CFT at an endpoint of an RG flow or a CFT corresponding to a limit cycle of the RG flow.

Luty, Polchinski and Rattazzi (LPR) [11] argued that limit cycles cannot exist in $d = 4$ unitary QFT, and hence that scale without conformal invariance is excluded. As we shall see, limit cycles do occur, but QFTs on them are fully conformal, not just scale-invariant. LPR have informed us that their manuscript is being replaced with one that contains a corrected version of their argument, with their conclusion regarding the absence of scale without conformal invariance unchanged.

The work of KS is not sensitive to the presence of cycles. Indeed, KS assume the existence of a flow from a short distance (UV) CFT to an IR CFT, and argue that the coefficient a of the Euler density in the curved-space trace anomaly,

$$T^{\mu}_{\ \mu} = \text{operator terms} + c(\text{Weyl tensor})^2 - a(\text{Euler term}),$$

is larger at the UV than the IR fixed point: $a_{\text{UV}} > a_{\text{IR}}$. This, then, is a proof of the "weak version" of the c-theorem. The KS argument incorporates putative flows from a fixed point or cycle to another fixed point or cycle, since in both cases the theories encountered are CFTs.

In $d = 2$ a stronger result holds: there exists a quantity c, local in the RG scale, that is monotonically decreasing along any RG flow [12]. This is referred to as the "strong version" of the c-theorem, and it was first argued to also be true in $d = 4$ by Cardy [13]. A proof was later found by JO (see also [14]), albeit only in perturbation theory. Away from fixed points the quantity that plays the role of c in the arguments of JO is not exactly equal to a (the coefficient of the Euler term in the curved-space trace anomaly). However, it agrees with a at endpoints/limit cycles of the RG trajectories.

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[3]The condition for scale invariance, $\mu \, dg^i/d\mu = Q^i_{\ j} g^j$, $Q^T = -Q = \text{constant}$ [9], gives recursive flows [10]. Our study of cycles here is concerned with this type of closed trajectories, given by a rotation of the coupling constants by a compact Abelian group generated by Q. Whether recursive flows that are not of this type exist is an open question.

This is in agreement with the result of KS that the weak version of the c-theorem is valid for a. In this paper we extend the perturbative proof of JO to include RG cycles.

Of course it is well-known that a may *increase* away from trivial UV fixed points: for example, for pure Yang–Mills (YM) theory with beta function $\beta^g = -\beta_0 g^3/16\pi^2 - \beta_1 g^5/(16\pi^2)^2 - \cdots$ one has [8]

$$a = a_0 + \frac{n_V \beta_1}{8(16\pi^2)^3} g^4 + \mathcal{O}(g^6).$$

Here a_0 is the free field theory (one-loop) value of a and $n_V = \dim(\mathrm{Adj})$ is the number of vector fields.[4] Nevertheless, even in this case JO showed that there exists a quantity, $\tilde{\beta}_b$, which flows monotonically (to all orders in perturbation theory). The quantity $\tilde{\beta}_b$ is related to a, which in JO is denoted by β_b, by

$$\tilde{\beta}_b = \beta_b + \tfrac{1}{8} w \beta^g, \qquad \beta_b \equiv a.$$

Here w is a function of the coupling g, and $\beta^g = -dg/dt$ is the beta function. While $\tilde{\beta}_b$ and β_b agree on fixed points, the difference is parametrically large away from fixed points. In Section 3.2 we explain this in detail.

The result of JO follows from careful inspection of how the theory responds to Weyl rescaling. The KS method, or an elaboration on it by LPR [11], extensively uses Weyl rescaling and takes advantage of the particularly simple form this takes on fixed points. However, in trying to extend the KS arguments to produce a proof of the strong version of the c-theorem, LPR use Weyl rescaling away from fixed points. We explain how consistency requires introducing spacetime-dependent coupling constants and then in addition new counterterms that involve derivatives of the couplings. We use the very rescaling in LPR to derive JO's consistency conditions anew, of which the monotonic flow of $\tilde{\beta}_b$ is but one example.

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[4]We thank K. Intriligator for discussions on this point.

For models which display cycles the state of affairs is significantly more complex. In all these models the kinetic terms of the Lagrangian are invariant under a "flavor" symmetry group G_F (that commutes with the gauge group). Scalar self-interactions and Yukawa couplings of scalars with fermions break G_F. The dependence of counterterms on the coupling constants characterizing these interactions is restricted by the pattern of breaking of G_F. There is a well-known, simple method of accounting for this. The coupling constants are treated as spurions, that is, as non-dynamical fields, and allowed to transform under G_F precisely so as to render the Lagrangian invariant under these symmetry transformations. Then, if the regulator respects the symmetry, so will the counterterms. It follows that the entries in the trace anomaly respect the symmetry too. As a is the coefficient of the G_F-invariant Euler density, it is itself G_F-invariant as well. And since the flow on a cycle corresponds to a G_F-transformation of the couplings, a remains constant on the cycle.

This raises the following question: "how is the monotonic flow of $\tilde{\beta}_b$ consistent with the constancy of a?" Actually, $\tilde{\beta}_b$ is also G_F-invariant, and is thus also guaranteed to be constant along the cyclic flow. The answer is found in the flow equation for $\tilde{\beta}_b$ given by Osborn in [14]. His equation is a generalization, applicable to these more complex theories, of that found by JO. This flow equation is not guaranteed to give monotonic flows, but can and does give constancy of $\tilde{\beta}_b$ on cycles. We review the work of JO concerning these more complex theories in Section 3.3, and show that a quantity $\widetilde{B}_b$ decreases monotonically along RG flows, at least in perturbation theory, and agrees with β_b on fixed points and cycles. This is a result essentially contained in [8, 14], although it is not explicitly mentioned there.

To obtain this result an understanding of the modifications to the trace-anomaly equation in theories with cycles is required. It is a little-known fact that in theories with many fermions and scalars there generically appears a term in the trace anomaly of the form of the divergence of a current. The current generates transformations that correspond to a particular element S of the Lie algebra of G_F, that is a function of

the coupling constants. JO showed by direct computation that S vanishes to three loops if the field content of the QFT consists solely of scalars, and to two loops if of scalars and Dirac spinors. On the other hand, the element Q of the Lie algebra of G_F that generates the flow along the cycle is found in our computations to arise at three-loop order in gauge theories that include both scalars and spinors [7]. Could it be, then, that S is non-zero at three loops in these theories? And if so, what is the relation between Q and S? In Section 3.4.1 we take on the task of computing the lowest-order contribution to JO's S for the most general four-dimensional QFT, compare with Q and demonstrate that S agrees with Q on cycles and vanishes on fixed points. A corollary of this result is that scale implies conformal invariance in relativistic unitary perturbative four-dimensional QFTs.

That S agrees with Q on cycles suggests that the two terms in the flat-space trace anomaly may cancel. That is, the well-known $\beta \, \partial \mathscr{L} / \partial g$ term may cancel against the little-known divergence of the S-current term, since the β-term is determined by Q on cycles. This is indeed what happens: the trace of the stress-energy tensor vanishes for unitary, scale-invariant cycles, and hence these models display invariance under the full conformal group. In the rest of Section 3.4.2 we prove this and explore a few of its consequences. Armed with these results, we return to the proof of the c-theorem in Section 3.5. There we give a slightly streamlined version of the LPR version of the KS argument, with care to address the possible differences that may arise when the CFTs at the ends of the RG flow correspond to cycles.

Let us specify here that we follow closely the notation of JO [8], with some notable exceptions. From here on, following JO, in the anomaly equation we use β_a and β_b for $-c$ and a respectively, although we still use the terminology c-theorem instead of the more accurate β_b-theorem. Also, we call $\lambda_{a,b,c}$ rather than a, b, c the infinite counterterms that give rise to $\beta_{a,b,c}$ (having infinite counterterms labeled by a and c can certainly produce confusion with the corresponding "beta functions" that appear in the Weyl anomaly and that are commonly referred to as a and c). Throughout this

paper RG time is defined by $t = -\ln(\mu/\mu_0)$, so that it increases as we flow to the IR.

Many of our results are extracted from the work of JO. So it is perhaps necessary to remark that, besides parsing the results of JO to hopefully make them slightly more accessible to the general reader, we have made several novel contributions:

- We have discovered where in the argument of LPR the quantity β_b is replaced by $\tilde{\beta}_b$ (or, in more generality, by $\widetilde{B}_b$).

- We have extended JO's calculation of S to third loop order, which is the leading non-vanishing contribution to S in a Yang–Mills theory with scalars and spinors.

- We have shown that S vanishes on fixed points and agrees with the generator Q of limit cycles on them.

- We have demonstrated in perturbation theory that unitary, scale and Poincaré invariant, interacting QFTs in $d = 4$ have vanishing trace of the stress-energy tensor and hence are invariant under the full conformal group.

- We have used the above to

  ○ find, using arguments of JO, a perturbative proof of an extension of the strong version of the c-theorem, i.e., that there exists a quantity that monotonically decreases in flows out of UV fixed points and cycles, and

  ○ clarify that the arguments of KS apply even in the presence of cycles, i.e., that $(\beta_b)_{\text{UV}} > (\beta_b)_{\text{IR}}$ for presumed RG flows that can now originate or terminate on cycles as well as fixed points, valid even outside perturbation theory (provided the implicit assumptions in KS do not invalidate their result).

3.2. Weyl consistency conditions

In this section we review the derivation of the Weyl consistency conditions of JO. The method uses as a starting point the expressions of Weyl invariance used by

KS and by LPR. The presentation is formulated so that it becomes clear that the assumptions in those works already implicitly lead to the JO consistency conditions. Hence, although the derivation presented here may seem novel, it actually follows closely JO. We have included it here for completeness, for pedagogy and because it makes clear that neither the results of KS nor those of LPR should be in conflict with those of JO.

Let us briefly review Osborn's argument for the consistency conditions [14]. These are analogous to the well-known Wess–Zumino consistency conditions [15]. Let $\Delta_\sigma \widetilde{W}$ denote the action of a Weyl transformation on $\widetilde{W}$, the generating functional for connected renormalized Green functions. Because of the Abelian nature of the Weyl group, the Weyl consistency conditions follow:

$$[\Delta_\sigma, \Delta_{\sigma'}]\widetilde{W} = 0.$$

In JO the same consistency conditions are obtained by requiring finiteness of the trace of the stress-energy tensor in curved background and with spacetime-dependent couplings. One can also obtain the Weyl consistency conditions based on the arguments of LPR.

LPR start from a quantum action S_0 which is a function of a conformally flat metric, $\gamma_{\mu\nu} = e^{-2\tau(x)}\eta_{\mu\nu}$ and coupling constants $g^i(\mu)$ (in $d = 4 - \epsilon$ regularization, with, say, minimal subtraction (MS)). By rescaling the fields, which are dummy variables of integration anyway, by $\phi \to (e^\tau)^\delta \phi$, where δ is the canonical dimension of the field (as in $\delta = (d-2)/2$ for scalars), and using the μ-independence of the bare couplings, one sees that the τ-dependence in S_0 arises only due to the scale dependence of renormalized coupling constants, $g^i(e^\tau \mu)$. Effectively, the regularized generating functional W satisfies

$$W[e^{-2\tau(x)}\eta_{\mu\nu}, g^i(\mu)] = W[\eta_{\mu\nu}, g^i(e^{\tau(x)}\mu)]. \tag{3.2.1}$$

Alternatively, Komargodski [16] argues that the functional is made invariant under

Weyl transformations by adding a conformal compensator $\tau(x)$. One can write

$$W[e^{-2\tau(x)}\eta_{\mu\nu}, g^i(e^{-\tau(x)}\mu)] = W[\eta_{\mu\nu}, g^i(\mu)],$$

or, equivalently, that the left-hand side is invariant under $\tau \to \tau + \sigma$. For finiteness it is also necessary to include in W all possible counterterms that are functions of spacetime-dependent background and coupling constants, $\gamma_{\mu\nu}(x)$ and $g^i(x)$. It is from counterterms that do not vanish for spacetime-independent coupling constants that the $\beta_{a,b,c}$-anomalies arise. It is convenient, in order to keep track of curvature-dependent terms, to do this in a more general background metric,

$$W[e^{-2\tau(x)}\gamma_{\mu\nu}(x), g^i(\mu)] = W[\gamma_{\mu\nu}(x), g^i(e^{\tau(x)}\mu)], \tag{3.2.2a}$$

$$W[e^{-2\tau(x)}\gamma_{\mu\nu}(x), g^i(e^{-\tau(x)}\mu)] = W[\gamma_{\mu\nu}(x), g^i(\mu)]. \tag{3.2.2b}$$

At the risk of restating the trivial, let us emphasize that it is not consistent to neglect the spacetime dependence of couplings when studying Weyl anomalies, since the Weyl transformation involves spacetime-dependent couplings. The counterterms associated with spacetime derivatives of these couplings will lead to additional anomalies. Some of these may—and as we will see, do—contribute to the dilaton/compensator scattering amplitude even after one takes the limit of flat background and spacetime-independent coupling constants.

The approach of LPR allows one to compute quantities associated with a model in a curved background with spacetime-independent coupling constants in terms of corresponding quantities for the same model but in a flat background with, however, spacetime-dependent coupling constants. This observation is not new. For example, in JO the same observation is used precisely for the same purpose, namely, to compute the anomalies associated with scale transformations using only computations in flat space. Similarly, the approach of Komargodski allows for an explicit nonlinear realization of scale invariance, at the price of introducing spacetime-dependent coupling constants.

In either case it is important to realize that new counterterms are required to render the model finite, much like counterterms involving derivatives of the metric need to be introduced to render finite the model in a curved background. These new counterterms must involve derivatives of the coupling constants and lead to new Weyl anomalies. At the end of this section we study how these new anomalies contribute to the Wess–Zumino action for the conformal compensator $\tau(x)$ even after the couplings and the metric are taken to be spacetime-independent. For the remainder of this section we take a closer look at these counterterms, the anomalies they produce and the relations between them, that is, the JO consistency conditions, that follow from (3.2.2).

Consider the theory in the background of an arbitrary metric $\gamma_{\mu\nu}(x)$ and arbitrary spacetime-dependent coupling constants $g^i(x)$ corresponding to interaction terms $g^i(x)\mathcal{O}_i(x)$ in the Lagrangian. The arbitrary spacetime dependence of the couplings allows one to use them as sources for operators in the interaction part of the Lagrangian, by taking functional derivatives of the generating functional with respect to $g^i(x)$. If the quantum action is renormalized, then this procedure automatically gives finite correlations functions for the insertions of these operators. Let $\widetilde{W}$ stand for the renormalized generating functional. It is convenient to separate the counterterms that are independent of quantum fields from the rest of the action. They can be taken out of the functional integral and contribute directly to $\widetilde{W}$:

$$\widetilde{W} = W + W_{\text{c.t.}}.$$

The generating functional W results from performing the functional integral over quantum fields in the absence of the quantum-field-independent counterterms. The counterterms required to render the theory finite were first classified in JO. They consist of all possible diff-invariant dimension-four operators constructed out of the metric and couplings and their derivatives:

$$W_{\text{c.t.}} = -\int \sqrt{-\gamma}\,\mu^{-\epsilon}\lambda \cdot \mathscr{R},$$

where dimensional regularization is used with $d = 4 - \epsilon$ and

$$\lambda \cdot \mathscr{R} = \lambda_a F + \lambda_b G + \lambda_c H^2 + \mathscr{E}_i \partial_\mu g^i \partial^\mu H + \tfrac{1}{2}\mathscr{F}_{ij} \partial_\mu g^i \partial^\mu g^j H + \tfrac{1}{2}\mathscr{G}_{ij} \partial_\mu g^i \partial_\nu g^j G^{\mu\nu}$$

$$+ \tfrac{1}{2}\mathscr{A}_{ij} \nabla^2 g^i \nabla^2 g^j + \tfrac{1}{2}\mathscr{B}_{ijk} \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + \tfrac{1}{4}\mathscr{C}_{ijkl} \partial_\mu g^i \partial^\mu g^j \partial_\nu g^k \partial^\nu g^l.$$

$$(3.2.3)$$

Here F is the Weyl tensor squared, G is the Euler density, H is proportional to the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor:

$$F = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - \frac{4}{d-2} R^{\mu\nu} R_{\mu\nu} + \frac{2}{(d-2)(d-1)} R^2,$$

$$G = \frac{2}{(d-3)(d-2)} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2),$$

$$H = \frac{1}{d-1} R, \qquad G_{\mu\nu} = \frac{2}{d-2} (R_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu} R).$$

The quantities above are defined as in JO, with inessential d-dependent factors for later convenience. Each of the counterterms in $\lambda \cdot \mathscr{R}$ is an expansion in $1/\epsilon$ chosen to render $\widetilde{W}$ finite—for this one must in addition introduce wave-function and coupling constant counterterms, as usual. The coefficients $\lambda = (\lambda_a, \lambda_b, \dots, \mathscr{C}_{ijkl})$ are in general functions of the couplings $g^i(x)$.

The anomalous variation of the generating functional is dictated by these counterterms. While W satisfies (3.2.1) and (3.2.2), this is not true of $W_{\text{c.t.}}$, as can be seen by explicit computation. The anomaly is precisely the statement that the infinitesimal transformation $\tau \to \tau + \sigma$ in (3.2.2b),

$$\Delta_\sigma W_{\text{c.t.}} = W_{\text{c.t.}}[(1 - 2\sigma)\gamma_{\mu\nu}, g^i - \sigma\mu \, dg^i/d\mu] - W_{\text{c.t.}}[\gamma_{\mu\nu}, g^i],$$

fails to vanish. The anomalous variation can be split into a term that would occur even if σ were spacetime-independent plus a term proportional to the derivative of σ:

$$\Delta W_{\text{anomaly}} = \Delta_\sigma W_{\text{c.t.}} = -\int \sqrt{-\gamma} \, (\sigma \beta_\lambda \cdot \mathscr{R} + \partial_\mu \sigma \, \mathscr{Z}^\mu). \qquad (3.2.4)$$

These terms again can be expanded using dimensional analysis and diff-invariance:

$$\beta_\lambda \cdot \mathscr{R} = \beta_a F + \beta_b G + \beta_c H^2 + \chi_i^e \partial_\mu g^i \partial^\mu H + \tfrac{1}{2}\chi_{ij}^f \partial_\mu g^i \partial^\mu g^j H + \tfrac{1}{2}\chi_{ij}^g \partial_\mu g^i \partial_\nu g^j G^{\mu\nu}$$

$$+ \tfrac{1}{2}\chi_{ij}^a \nabla^2 g^i \nabla^2 g^j + \tfrac{1}{2}\chi_{ijk}^b \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + \tfrac{1}{4}\chi_{ijkl}^c \partial_\mu g^i \partial^\mu g^j \partial_\nu g^k \partial^\nu g^l,$$

$$(3.2.5)$$

and[5]

$$\mathscr{Z}_\mu = G_{\mu\nu} w_i \partial^\nu g^i + \partial_\mu(Hd) + H Y_i \partial_\mu g^i$$

$$+ \partial_\mu(U_i \nabla^2 g^i + \tfrac{1}{2}V_{ij}\partial_\nu g^i \partial^\nu g^j) + S_{ij}\partial_\mu g^i \nabla^2 g^j + \tfrac{1}{2}T_{ijk}\partial_\nu g^i \partial^\nu g^j \partial_\mu g^k,$$

$$(3.2.6)$$

up to terms with vanishing divergence. Since $\widetilde{W}$ is finite and the σ-variation of W vanishes, it must be that the variation of $W_{\text{c.t.}}$ is finite by itself.

Calculations of the coefficients in $\beta_\lambda \cdot \mathscr{R}$ and $\mathscr{Z}_\mu$ can be done using standard techniques of dimensional regularization with a mass-independent renormalization scheme, say MS. For now, let us concentrate on the relatively straightforward computation of $\beta_\lambda \cdot \mathscr{R}$. Since for constant σ the transformation $\delta\gamma_{\mu\nu} = -2\sigma\gamma_{\mu\nu}$ just counts dimensions, and the dimension of the volume element is d while that of the operators in $W_{\text{c.t.}}$ is four, we obtain

$$(\epsilon - \hat{\beta}^i \hat{\partial}_i)\lambda \cdot \mathscr{R} = \beta_\lambda \cdot \mathscr{R}, \qquad (3.2.7)$$

where the beta function is

$$\mu \frac{dg^i}{d\mu} = \hat{\beta}^i = -\epsilon k^i g^i + \beta^i(g) \qquad \text{(no sum over } i\text{)}.$$

Here the derivative is taken holding the bare parameters fixed. k^i is defined by requiring that the Lagrangian scales appropriately: for $\phi' = \mu^{\delta\epsilon}\phi$ and $g'^i = \mu^{k^i\epsilon}g^i$, then $\mathscr{L}(\phi', g') = \mu^{-\epsilon}\mathscr{L}(\phi, g)$. Note that $\hat{\beta}^i \hat{\partial}_i \equiv \hat{\beta}^i \hat{\partial}/\hat{\partial}g^i$ denotes substitution of g^i by β^i

---

[5]The second term involves the function of coupling constants d, which is not to be confused with $d = 4 - \epsilon$. We follow Osborn in this unfortunate choice of notation, hoping that with this warning no confusion will arise in what follows.

wherever g^i may be found, e.g., $\hat{\beta}^i\hat{\partial}_i(\partial_\mu g^j) \equiv \partial_\mu\beta^j = \partial_i\beta^j\partial_\mu g^i$, and of course respects the standard rules of differentiation. Using (3.2.7), it is straightforward to show that, e.g.,

$$\chi^a_{ij} = (\epsilon - \hat{\beta}^k\partial_k)\mathscr{A}_{ij} - \mathscr{A}_{ik}\partial_j\hat{\beta}^k - \mathscr{A}_{jk}\partial_i\hat{\beta}^k.$$

The consistency conditions of JO can be understood as following from requiring that (3.2.2) applied to the complete renormalized generating function $\widetilde{W}$ fails only up to the finite, anomaly terms. The left-hand side of (3.2.2a) does not involve any counterterms from spacetime-dependent couplings, while the right-hand side does not involve any from a curved background. Hence, the counterterms in one and the other case must be related. Consider on the right-hand side of (3.2.2a), for example, the counterterm

$$\tfrac{1}{2}\mathscr{A}_{ij}\nabla^2 g^i\nabla^2 g^j = \tfrac{1}{2}\mathscr{A}_{ij}\hat{\beta}^i\hat{\beta}^j(\nabla^2\tau)^2 + \cdots , \tag{3.2.8}$$

where we have expanded to lowest order in $\tau(x)$. Comparing with the counterterms on the left-hand side of (3.2.2a), that arise solely from a curved background, we have,

$$\sqrt{-\gamma}\,(\lambda_a F + \lambda_b G + \lambda_c H^2) = 8\lambda_b\left[(\nabla^2\tau)^2 - (\partial_\mu\partial_\nu\tau)^2 + \cdots\right] + 4\lambda_c\left[(\nabla^2\tau)^2 + \cdots\right]. \tag{3.2.9}$$

The λ_b term is a total derivative so for localized $\tau(x)$ it does not contribute (recall there is an implicit spacetime integration). Matching the terms in (3.2.8) and (3.2.9) we find that the counterterms are related,

$$4\lambda_c \sim \tfrac{1}{2}\mathscr{A}_{ij}\hat{\beta}^i\hat{\beta}^j, \tag{3.2.10}$$

where the symbol $\sim$ denotes equality up to finite terms, that is, the difference is finite as $\epsilon \to 0$. This precisely corresponds to Eq. (3.12) of JO. Applying $\mu\,d/d\mu$ on the bare couplings to derive RGEs and the corresponding beta functions, one then derives from

this JO's consistency condition (3.21a),

$$8\beta_c = \chi_{ij}^a \beta^i \beta^j - \beta^i \partial_i X, \tag{3.2.11}$$

where X arises from the finite difference between the left- and right-hand sides of (3.2.10), and β_c and χ_{ij}^a are beta functions for λ_c and $\mathscr{A}_{ij}$, respectively. The remaining consistency conditions in JO can be obtained in a similar fashion. We only quote here one other consistency condition that plays an important role in what follows. Using (3.2.2a) the lowest order terms in $\tau(x)$ that are linear in the Einstein tensor give

$$8\partial_i \lambda_b \sim \mathscr{G}_{ij} \hat{\beta}^j. \tag{3.2.12}$$

With the finite difference between the two sides of (3.2.12) denoted by w_i one obtains

$$8\partial_i \beta_b = \chi_{ij}^g \beta^j - \beta^j \partial_j w_i - \partial_i \beta^j w_j.$$

This consistency condition is the origin of the proposal in JO for a c-function,

$$\tilde{\beta}_b \equiv \beta_b + \tfrac{1}{8}\beta^i w_i,$$

which satisfies

$$\partial_i \tilde{\beta}_b = \tfrac{1}{8}(\chi_{ij}^g + \partial_{[i}w_{j]})\beta^j, \tag{3.2.13}$$

where $\partial_{[i}w_{j]} = \partial_i w_j - \partial_j w_i$. Then its RG flow is monotonic provided the "metric" χ_{ij}^g is positive-definite, for

$$-\frac{d\tilde{\beta}_b}{dt} = \beta^i \partial_i \tilde{\beta}_b = \tfrac{1}{8}\chi_{ij}^g \beta^i \beta^j.$$

To summarize, the extension (3.2.2) of the invariance requirement of LPR in (3.2.1), when applied to *the complete set of counterterms required for finiteness when coupling constants have spacetime dependence*, leads to the consistency conditions of JO.

3.2.1. The trace anomaly and the computation of $\nabla_\mu \mathscr{Z}^\mu$

As formulated, the renormalized generating functional $\widetilde{W}$ is a finite function of the background metric and of renormalized spacetime-dependent coupling constants. As such we can obtain finite insertions of composite operators in Green functions by functional differentiation,

$$\langle T_{\mu\nu}(x)\rangle = \frac{2}{\sqrt{-\gamma}}\,\frac{\delta\widetilde{W}}{\delta\gamma^{\mu\nu}(x)} \qquad \text{and} \qquad \langle[\mathcal{O}_i(x)]\rangle = \frac{1}{\sqrt{-\gamma}}\,\frac{\delta\widetilde{W}}{\delta g^i(x)}\ . \tag{3.2.14}$$

Note that $[\mathcal{O}_i(x)]$ stands for the fully renormalized insertion of the composite operator $\mathcal{O}_i(x)$, which may differ from the operator monomial in an expectation value. Following JO, we make this distinction explicit by introducing the notation $[\ldots]$.

Using (3.2.14) in (3.2.2) and (3.2.4) one obtains

$$T^\mu_\mu = \hat{\beta}^i[\mathcal{O}_i] - \mu^{-\epsilon}\beta_\lambda \cdot \mathscr{R} + \mu^{-\epsilon}\nabla_\mu\mathscr{Z}^\mu. \tag{3.2.15}$$

This is the well-known trace anomaly, accounting for the effects of curved background and spacetime-dependent coupling constants. However, this equation is not quite correct in the most generality: there are two terms missing on the right-hand side. The first is an operator that vanishes by the equations of motion times the anomalous dimension of the corresponding quantum field. We have lost track of this term because the relation (3.2.1) is only correct up to terms that vanish by the equations of motion. The second missing term is more subtle: we have missed counterterms that may be needed to render some theories finite. When the kinetic terms of the Lagrangian exhibit a continuous symmetry the current associated with this symmetry is a dimension-three operator and a new type of counterterm is required in the presence of spacetime-dependent couplings, that is, a counterterm proportional to the product of the current and the derivative of a coupling. This will be discussed extensively, and the anomaly equation will be fixed, in Section 3.3.

Let us turn to the computation of $\mathscr{Z}^\mu$ in (3.2.4). It follows, of course, straightfor-

wardly from the definition (3.2.4). Slightly less trivial is the fact that the computation must give a finite current $\mathscr{Z}^\mu$. That this must be so can be seen from the trace anomaly in (3.2.15), in which all other terms are already finite. This means that there must be cancellations among infinite terms that contribute to $\mathscr{Z}^\mu$. In fact, these cancellations are nothing but the consistency conditions, e.g., (3.2.10) and (3.2.12). For example, the terms in (3.2.4) involving the Einstein tensor (modulo terms that do not vanish for spacetime-independent σ) are

$$
\int \sqrt{-\gamma}\, G^{\mu\nu}(-8\lambda_b \nabla_\mu \partial_\nu \sigma - \mathscr{G}_{ij}\partial_\mu g^i \hat{\beta}^j \partial_\nu \sigma) = \int \sqrt{-\gamma}\, \partial_\nu \sigma\, G^{\mu\nu} \partial_\mu g^i (8\partial_i \lambda_b - \mathscr{G}_{ij}\hat{\beta}^j)
$$
$$
= \int \sqrt{-\gamma}\, \partial_\nu \sigma\, G^{\mu\nu}\partial_\mu g^i\, w_i.
$$

Here, in going from the first to the second line we used the finiteness condition (3.2.12) and the definition that the finite difference is w_i. Thus we have reproduced the first term in $\mathscr{Z}_\mu$ of (3.2.6). The remaining terms in (3.2.6) can be similarly found.

3.2.2. Wess–Zumino action

The Wess–Zumino action, W_{WZ}, is a function of $\tau(x)$ that will give $-\Delta W_{\mathrm{anomaly}}$ upon a Weyl transformation, $\tau(x) \to \tau(x)+\sigma(x)$. Focusing on the β_b-term in $\Delta W_{\mathrm{anomaly}}$,

$$
-\int \sqrt{-\gamma}\, \sigma \beta_b G, \tag{3.2.16}
$$

KS write the corresponding WZ term as[6]

$$
\int \sqrt{-\gamma}\, \left\{ \tau \beta_b G - 4\beta_b \left[G^{\mu\nu}\tau_{,\mu}\tau_{,\nu} + \tau_{,\mu}\tau^{,\mu}\nabla^2 \tau + \tfrac{1}{2}(\tau_{,\mu}\tau^{,\mu})^2 \right] \right\},
$$

where we have introduced the shorthand $\tau_{,\mu} = \partial_\mu \tau$. However, this computation is incomplete. The problem with this is that we have ignored the effect of the new

---

[6]The sign in the term cubic in τ is opposite to that of KS because we use the opposite sign convention for the conformal compensator, which gives Δ_σ as in JO.

counterterms arising from spacetime dependence of the couplings. Since we will not need a Wess–Zumino action for our generalization of the KS argument to theories with cycles, we will not aim at being complete and only point out one interesting consequence here. Consider, for example, the term in $\mathscr{Z}^\mu$

$$-\int \sqrt{-\gamma}\, \partial_\mu \sigma \, w_i G^{\mu\nu} \partial_\nu g^i. \tag{3.2.17}$$

Now with $\partial_\mu g^i = -\beta^i \tau_{,\mu}$, one has the following generalization of the Wess–Zumino dilaton action:

$$\int \sqrt{-\gamma}\,\left\{ \beta_b \tau G - 4(\beta_b + \tfrac{1}{8} w_i \beta^i)\left[G^{\mu\nu}\tau_{,\mu}\tau_{,\nu} + \tau_{,\mu}\tau^{,\mu}\nabla^2\tau + \tfrac{1}{2}(\tau_{,\mu}\tau^{,\mu})^2 \right] \right\}. \tag{3.2.18}$$

The Weyl variation of (3.2.18) gives the sum of (3.2.16) and (3.2.17) (if $\partial_\mu g^i = -\beta^i \tau_{,\mu}$ there). The correction that takes β_b into $\tilde{\beta}_b = \beta_b + \tfrac{1}{8} w_i \beta^i$ is generally of lower order than the running of β_b. That is, $\tfrac{1}{8} w_i \beta^i$ is of lower order than $\beta_b - \beta_{b0}$, where β_{b0} stands for the free field theory value of β_b.

Let us be more explicit. Consider, for example, the perturbative result of JO for a pure YM theory with gauge coupling g,

$$\beta_b = \beta_{b0} + \frac{n_V \beta_1}{8(16\pi^2)^3} g^4 + \mathcal{O}(g^6), \tag{3.2.19}$$

from which β_b is seen to increase in the flow out of the trivial UV fixed point. JO also give $gw = 2n_V/16\pi^2 + \cdots$, and therefore

$$\tilde{\beta}_b = \beta_{b0} - \frac{n_V \beta_0}{4(16\pi^2)^2} g^2 + \mathcal{O}(g^4),$$

which shows that the leading-order running of $\tilde{\beta}_b$ is modified by the $\tfrac{1}{8} w \beta^g$ term. Note that $\tilde{\beta}_b$ decreases in the flow out of the trivial UV fixed point, as opposed to β_b which, as seen from (3.2.19), increases. Therefore, a strong c-theorem in four dimensions should involve $\tilde{\beta}_b$, not β_b. Of course $\tilde{\beta}_b$ and β_b agree at fixed points.

There is another subtle point we would like to address. The coefficient appearing in the two-point function of the trace of the stress-energy tensor appears to play the role of the "metric" χ_{ij}^g in the consistency condition (3.2.13). In Appendix 3.B we point out, following JO, that this is actually related to $-2\chi_{ij}^a$, see (3.2.11). Explicit computations show that $-2\chi_{ij}^a$ agrees with χ_{ij}^g to second order in perturbation theory in any four-dimensional theory. As we show in Appendix 3.B this agreement fails, for example, at third order in a YM theory with a single gauge coupling.

3.3. Flavor symmetries, dimension-three operators and the corrected trace anomaly

As we have mentioned above, in deriving the trace anomaly we have missed counterterms that may be needed to render some theories finite. When the kinetic terms of the Lagrangian exhibit a continuous symmetry the current associated with this symmetry is a dimension-three operator and a new type of counterterm is required in the presence of spacetime-dependent couplings, that is, a counterterm proportional to the product of the current and the derivative of a coupling.

Consider a theory with n_S real scalar fields interacting through the usual quartic interaction. The kinetic part of the bare Lagrangian,

$$\mathscr{L}_{0K} = \tfrac{1}{2}\partial^\mu \phi_{0a}\partial_\mu \phi_{0a},$$

exhibits a continuous symmetry under transformations of the fields $\delta\phi_{0a} = -\omega_{ab}\phi_{0b}$, where ω is in the algebra of the flavor group $G_F = SO(n_S)$. In the process of renormalization we introduce a renormalization matrix Z and write

$$\mathscr{L}_{0K} = \tfrac{1}{2}\partial^\mu \phi^T Z \partial_\mu \phi,$$

where renormalized fields, ϕ, are related to bare fields by $\phi_0 = Z^{1/2}\phi$.[7] In the presence of spacetime-dependent coupling constants new divergences arise and thus new counterterms are needed. For example, one must introduce a new counterterm of the form

$$\mathscr{L}_{\text{c.t.}} = (\partial^\mu g^i)(N_i)_{ab}\phi_{0b}\partial_\mu\phi_{0a}, \tag{3.3.1}$$

with $(N_i)_{ab} = -(N_i)_{ba}$, that is, in the algebra of G_F. Note that this new counterterm is not accounted for in $W_{\text{c.t.}}$ which by construction is independent of quantum fields. Note also that additional counterterms, symmetric under $a \leftrightarrow b$, must also be introduced. One may integrate by parts to write these as terms with no derivatives acting on the quantum fields. While necessary, they do not play a central role in what follows.

To be more explicit, we consider a theory of real scalars and write for the bare Lagrangian

$$\mathscr{L}_0 = \tfrac{1}{2}\gamma^{\mu\nu} D_{0\mu}\phi_{0a} D_{0\nu}\phi_{0a} + \tfrac{1}{8}(d-2)\phi_{0a}\phi_{0a}H - \tfrac{1}{4!}g^0_{abcd}\phi_{0a}\phi_{0b}\phi_{0c}\phi_{0d}. \tag{3.3.2}$$

This is written in term of bare fields ϕ_0. The second term is introduced to ensure conformal invariance of the classical action. In the potential term, the bare couplings g^0_{abcd} are completely symmetric under exchange of the indices a, b, c and d. The covariant derivative,

$$D_{0\mu}\phi_0 = (\partial_\mu + A_{0\mu})\phi_0,$$

is introduced with an eye towards including the counterterm (3.3.1), since

$$A_{0\mu} = A_\mu + N_I(D_\mu g)_I, \qquad D_\mu = \partial_\mu + A_\mu. \tag{3.3.3}$$

Here, following JO, we use the compact notation $I = (abcd)$ and we have left implicit

---

[7]Note that in this step we have the freedom to introduce an orthogonal matrix O and define $\phi_0 = \tilde{Z}^{1/2}\phi$, where $\tilde{Z}^{1/2} = OZ^{1/2}$. This does not affect $Z = Z^{T/2}Z^{1/2}$. Nevertheless, such a freedom leads to an ambiguity in the definition of beta functions and anomalous dimensions as we explain in Appendix 3.A.

the Lie-algebra indices (so that $N_I^T = -N_I$ and $A_\mu^T = -A_\mu$). Note that N_I is a function of the renormalized couplings that has an ϵ-expansion starting at order $1/\epsilon$. If the theory contains gauge fields and some of the scalars are charged under the gauge group $G_g \subseteq G_F$, it is straightforward to include an additional quantum gauge field in addition to the background field A_μ.

The Lagrangian (3.3.2) is explicitly locally G_F-symmetric if we agree to transform the couplings and the gauge fields:

$$\delta g_{abcd}^0 = -\omega_{ae} g_{ebcd}^0 + \text{permutations} \qquad (\delta g_I^0 = -(\omega g^0)_I \text{ for short}),$$

$$\delta A_\mu = D_\mu \omega.$$

The first of these is already used in defining the covariant derivative $(D_\mu g)_I$ in (3.3.3). It is very important to note at this point that if this explicit local invariance is non-anomalous it can (and will) be used to constrain the counterterms and the generating functional $\widetilde{W}$,

$$\widetilde{W}[\gamma^{\mu\nu}(x), (\Omega g)_I(x), \Omega D_\mu \Omega^{-1}] = \widetilde{W}[\gamma^{\mu\nu}(x), g_I(x), A_\mu], \tag{3.3.4}$$

where $\Omega(x) = \exp(\omega(x)) \in G_F$. Of course, in theories without spinors the symmetry is trivially non-anomalous. Furthermore, derivatives of the generating functional with respect to the background field now give insertions of the scalar current.

It is not our intention to repeat the calculations of JO in their entirety here. We will instead describe the main ingredients and results. We have already described the two main new ingredients, namely, the need for new counterterms and the introduction of a background field to ensure invariance under G_F in (3.3.4). As before, additional quantum-field-independent counterterms are required. These are as in (3.2.3) but with the replacement $\partial_\mu \to D_\mu$ to ensure G_F invariance. Additional counterterms involving

the field strength $F_{\mu\nu} = [D_\mu, D_\nu]$ are also required,

$$\tilde{\lambda} \cdot \mathscr{R} = \lambda \cdot \mathscr{R} + \tfrac{1}{4}\text{Tr}(\mathscr{K}F^{\mu\nu}F_{\mu\nu}) + \tfrac{1}{2}\text{Tr}(\mathscr{P}_{IJ}F^{\mu\nu})(D_\mu g)_I(D_\nu g)_J. \tag{3.3.5}$$

Moreover, as advertised, new field-dependent counterterms must be included,

$$\mathscr{Q} = \eta_{ab}\phi_a\phi_b H + (\delta_I)_{ab}\phi_a\phi_b(D^2 g)_I + \tfrac{1}{2}(\epsilon_{IJ})_{ab}\phi_a\phi_b(D^\mu g)_I(D_\mu g)_J. \tag{3.3.6}$$

Proceeding much as before, JO find[8] [8, Eq. (6.15)]

$$T^\mu_\mu = \hat{\beta}_I[\mathcal{O}_I] + [\beta^\mathscr{Q}] + [(D^\mu\phi)^T\beta^A_\mu\phi] - \mu^{-\epsilon}\beta_{\tilde{\lambda}} \cdot \mathscr{R} + \nabla_\mu(J^\mu + J^\mu_\Theta + \widetilde{\mathscr{Z}^\mu}) - ((1+\hat{\gamma})\phi) \cdot \frac{\delta}{\delta\phi}\tilde{S}_0, \tag{3.3.7}$$

which, using the underlying gauge invariance, they rewrite as [8, Eq. (6.23)]

$$T^\mu_\mu = \widehat{B}_I[\mathcal{O}_I] + [\beta^\mathscr{Q}] + [(D_\mu\phi)^T B^A_\mu\phi] - \mu^{-\epsilon}\beta_{\tilde{\lambda}} \cdot \mathscr{R} + \nabla_\mu(J^\mu_\Theta + \widetilde{\mathscr{Z}^\mu}) - ((1+\hat{\gamma}+S)\phi) \cdot \frac{\delta}{\delta\phi}\tilde{S}_0. \tag{3.3.8}$$

Many comments are in order. The last term, involving the derivative of the full action integral $\tilde{S}_0$, vanishes by the equations of motion. We have included it here for completeness. We have already commented that a similar term is missing from (3.2.15). The operator $[\mathcal{O}_I]$ corresponds to the interaction term in the Lagrangian, $\mathcal{O}_I = \tfrac{1}{4!}\phi_a\phi_b\phi_c\phi_d$, but differs from it, $[\mathcal{O}_I] = \mathcal{O}_I - \nabla_\mu J^\mu_I$ where $J^\mu_I = (D^\mu_0\phi_0)^T N_I\phi_0$. Its coefficient in (3.3.8) is given by

$$\widehat{B}_I \equiv \hat{\beta}_I - (Sg)_I,$$

---

[8]Note that in Jack and Osborn the first term contains $\hat{\beta}^V = \mu\, dV/d\mu$, where V is the renormalized potential, and with the derivative taken by holding the bare fields, ϕ_0, and the bare potential, V_0, constant (independent of RG time). With a potential of the form $V = g_I\mathcal{O}_I$ and following Jack and Osborn's definitions we then obtain $[\hat{\beta}^V] = \hat{\beta}_I[\mathcal{O}_I]$ as expected.

where $\hat{\beta}_I = \mu \, dg_I/d\mu = -\epsilon g_I + \beta_I$. The current J^μ in (3.3.7) is defined as

$$J^\mu = (D_0^\mu \phi_0)^T N_I \hat{\beta}_I \phi_0 \qquad (3.3.9)$$

and it is finite as required from consistency of (3.3.7). Note that the combination $[\mathcal{O}_I] + \nabla_\mu J_I^\mu$ appearing in (3.3.7) is just $\mathcal{O}_I$. However, $\mathcal{O}_I$ is not by itself a finite operator. While $\hat{\beta}_I \mathcal{O}_I$ is finite, since it is the sum of two finite operators, replacing $\hat{\beta}_I$ by its $\epsilon \to 0$ limit, $\beta_I \mathcal{O}_I$, is not by itself finite. Finiteness of J^μ implies that it can be brought to the form

$$J^\mu = [(D^\mu \phi)^T S \phi].$$

The Lie-algebra element S is then defined by $\widehat{B}_I N_I = S$. Since S is finite it is required that the infinite pieces of $\widehat{B}_I N_I$ cancel automatically, i.e.,

$$\widehat{B}_I N_I = S \ \Rightarrow \ S = -N_I^1 g_I, \qquad (3.3.10)$$

where $N_I = \sum_{n=1}^\infty N_I^n/\epsilon^n$, so that N_I^1 is the residue of the simple ϵ-pole in N_I. Cancellation of the infinite pieces requires that $B_I N_I^n - g_I N_I^{n+1} = 0$ for $n \geq 1$. The beta functions for the field-dependent quadratic counterterms are

$$\beta^{\mathscr{Q}} \equiv \beta^\eta_{ab} \phi_a \phi_b H + (\beta^\delta_I)_{ab} \phi_a \phi_b (D^2 g)_I + \tfrac{1}{2}(\beta^\epsilon_{IJ})_{ab} \phi_a \phi_b (D_\mu g)_I (D_\mu g)_J \, .$$

The term $\beta_{\tilde{\lambda}} \cdot \mathscr{R}$ is the obvious generalization of (3.2.5) while the current $\widetilde{\mathscr{Z}^\mu}$ is defined as in (3.2.6) but rendered G_F-invariant by replacing derivatives by covariant derivatives. In addition, $\widetilde{\mathscr{Z}^\mu}$ has contributions from the new counterterms in (3.3.5), and there are additional contributions to the terms with the $\mathscr{A}$ and $\mathscr{B}$ of (3.2.3). The third term in (3.3.8) involves

$$B^A_\mu \equiv \beta^A_\mu + D_\mu S \equiv \rho_I (D_\mu g)_I + D_\mu S \equiv P_I (D_\mu g)_I, \qquad \rho_I = g_J \partial_J N_I^1 + N_I^1,$$

where $\beta_\mu^A \equiv \mu \, dA_\mu/d\mu$ is the beta function for the background gauge field A_μ. Finally, the current J_Θ^μ arises from the counterterms in (3.3.6) and has a complicated expression in terms of the simple ϵ-poles in δ_I and ϵ_{IJ} (see JO for details [8, Eqs. (6.21–22)]).

At this point we can take the limit of flat spacetime, spacetime-independent couplings and no background gauge field in (3.3.8). This gives

$$T^\mu_\mu = \widehat{B}_I[\mathcal{O}_I] - ((1 + \hat{\gamma} + S)\phi) \cdot \frac{\delta}{\delta\phi} S_0. \tag{3.3.11}$$

Since $[\mathcal{O}_I]$ is finite we can now safely conclude that a theory is conformal if and only if $B_I = 0$. This does not require that $\beta_I = 0$.

In the general case considered here the JO consistency conditions are modified relative to what has been presented in Section 3.2. On the one hand the conditions have to be covariant under transformations by the symmetry group G_F. On the other, there are additional terms that arise from the additional counterterms required to render the theory finite. Osborn gives the form of these most general consistency conditions [14]. Two conditions play a role in our discussion:

$$8\partial_I \beta_b = \chi_{IJ}^g B_J - B_J \partial_J w_I - (\partial_I B_J) w_J - (P_I g)_J w_J \tag{3.3.12}$$
$$= \chi_{IJ}^g B_J - \beta_J \partial_J w_I - (\partial_I \beta_J) w_J - (\rho_I g)_J w_J,$$

and

$$B_I P_I = 0. \tag{3.3.13}$$

In addition, covariance under G_F gives, e.g.,

$$(\omega g)_I \partial_I \beta_b = 0 \qquad \text{and} \qquad (\omega g)_I \partial_I S = [\omega, S]. \tag{3.3.14}$$

Of course, the first of these applies to any G_F-invariant while the second to any antisymmetric tensor (for example, any Lie-algebra valued function). Using the first of

(3.3.14) in (3.3.12) gives a nontrivial relation among several beta functions:

$$(\omega g)_I \left[\chi^g_{IJ} B_J - B_J \partial_J w_I - (\partial_I B_J) w_J - (P_I g)_J w_J \right] = 0, \qquad (3.3.15)$$

or, equivalently,

$$(\omega g)_I \left[\chi^g_{IJ} B_J - \beta_J \partial_J w_I - (\partial_I \beta_J) w_J - (\rho_I g)_J w_J \right] = 0. \qquad (3.3.16)$$

These conditions can be used to understand aspects of the flow of β_b. Consider the flow defined by some arbitrary function $f_I(g)$,

$$\frac{d\bar{g}_I}{d\eta} = -f_I(\bar{g}(\eta)).$$

If one takes $f_I = \beta_I$ then the flow can be identified with the RG flow, with $\eta = t = -\ln(\mu/\mu_0)$. From (3.3.12) we have

$$-8\frac{d\widetilde{B}_b}{d\eta} = \chi^g_{IJ} f_I B_J + f_I B_J \partial_{[I} w_{J]} - (P_I g)_J f_I w_J, \qquad (3.3.17)$$

where

$$\widetilde{B}_b = \beta_b + \tfrac{1}{8} B_I w_I, \qquad (3.3.18)$$

and

$$-8\frac{d\tilde{\beta}_b}{d\eta} = \chi^g_{IJ} f_I B_J + f_I \beta_J \partial_{[I} w_{J]} - (\rho_I g)_J f_I w_J. \qquad (3.3.19)$$

Three special cases are of most interest. Consider first $f_I(g) = -(\omega g)_I$. From the second equation in (3.3.14) we see that on this flow ω is constant. This is a recursive flow (cycle or ergodic). It follows from the G_F-invariance of $\widetilde{B}_b$ and $\tilde{\beta}_b$ that these remain constant on the flow. This is a consequence of the detailed cancellations that must be satisfied by the beta functions in (3.3.15) and (3.3.16). This general result can be applied to limit cycles, $\beta_I = (Qg)_I$, for which $\omega = Q$. We thus see that counterterms that ensure G_F-covariance guarantee constancy of β_b (and $\tilde{\beta}_b$) on recursive flows.

The second and third special cases correspond to $f_I = B_I$ and $f_I = \beta_I$. While the B_I and β_I flows are generally distinguishable, one may use (3.3.14) to show the two flows are identical for G_F-invariants.[9] Using (3.3.17) with $f_I = B_I$ and the consistency condition (3.3.13) we see that

$$-8\frac{d\widetilde{B}_b}{dt} = \chi^g_{IJ} B_I B_J. \tag{3.3.20}$$

This shows that $\widetilde{B}_b$ decreases monotonically along both flows and is a good candidate for the c function of the c-theorem. Indeed this shows a strong version of the c-theorem in perturbation theory. To two loops $\chi^g_{IJ} = -2\chi^a_{IJ} > 0$, where unitarity is required for the inequality, so the right-hand side of (3.3.20) is positive-definite along a perturbative flow.

The relation between the B_I and β_I flows can be made more explicit, hopefully clarifying their relation. Consider the flows

$$-\frac{dg_I}{dt} = \beta_I(g(t)) \qquad \text{and} \qquad -\frac{d\bar{g}_I}{d\eta} = B_I(\bar{g}(\eta)).$$

The solution to the η-flow is given in terms of the one for the RG flow by

$$\bar{g}(\eta) = F(\eta)g(\eta) \qquad \text{where} \qquad F(\eta) = T\left(\exp\left[-\int_{-\infty}^{\eta} d\eta' S(\eta')\right]\right).$$

Here T is the η-ordered product and $F \in G_F$. As such, $\beta_I(\bar{g}) = \beta_I(Fg) = (F\beta)_I(g)$ and similarly for B_I and indeed for any tensor function of the couplings. Of special interest are G_F-invariants, like $\widetilde{B}_b$, for which $\widetilde{B}_b(\bar{g}) = \widetilde{B}_b(Fg) = \widetilde{B}_b(g)$. So we see again that the monotonic η-flow of $\widetilde{B}_b$ gives a monotonic RG flow of $\widetilde{B}_b$.

The quantity $\tilde{\beta}_b$ does not appear to be a good candidate for the c function of the c-theorem. Using (3.3.19) to study its flow, so the term $f_I \beta_J \partial_{[I} w_{J]}$ automatically

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[9]This was pointed out to us by J. Polchinski.

vanishes, we obtain

$$-8\frac{d\tilde{\beta}_b}{dt} = \chi^g_{IJ}\beta_I B_J - (\rho_I g)_J \beta_I w_J. \tag{3.3.21}$$

Were we to ignore the last term on the right-hand side we would be able to establish a perturbative c-theorem for $\tilde{\beta}_b$. Indeed, to two loops $B_I = \beta_I$ and $\chi^g_{IJ} = -2\chi^a_{IJ} > 0$ so the right-hand side of (3.3.21) would be positive-definite along a perturbative flow. However, the last term is parametrically of the same order as the first on the right-hand side of (3.3.21) so this does not give a perturbative c-theorem for $\tilde{\beta}_b$.

3.4. Scale implies conformal invariance

3.4.1. S is Q (on cycles)

In this subsection we elucidate the relation between Q and S. Our treatment is focused on theories in $d = 4$. We remind the reader that Q is defined as the solution to the equations $\beta^g = 0$ and $\beta_I = (Qg)_I$, defining an RG cycle on which Q remains a constant while S is defined as a function of couplings that makes explicit the finiteness of the current J^μ in (3.3.9). There is no a priori reason they should be related.

What is known about S? JO have shown, by direct calculation, and we have verified, that in a scalar field theory S vanishes up to third order in the loop expansion. The result holds even if gauge fields are included and the scalars are charged under the gauge group. For theories with scalars and fermions, JO have shown, and we have verified, that S remains zero to two loops. However, this is consistent with a possible equality of S and Q on cycles. Indeed, we have obtained previously that Q is of third order in the loop expansion in Yang–Mills theories with scalars and fermions, while in purely-scalar field theories a non-vanishing Q, if it exists, must be at least of fifth order in the loop expansion.

As might be expected from the discussion above, we will show that (up to conserved current)

1. S is Q on cycles,

2. S vanishes at fixed points.

In light of these results the computation of Q can be tremendously simplified given an explicit expression for S. Presently, the procedure to determine Q involves determining first the beta functions for the coupling constants to second order in the loop expansion for scalar self-couplings, to third order in the loop expansion for Yukawa couplings and to fourth order in the loop expansion for Yang–Mills couplings, and then solving the system of nonlinear coupled equations $\beta^g = 0$ and $\beta_I = (Qg)_I$ (we implicitly use here that g_I can also stand for Yukawa couplings). Since S must have a perturbative expansion that starts at third order in the loop expansion, to determine Q from S it suffices to evaluate it with coupling constants on the cycle computed to lowest order in the loop expansion. So Q is obtained from S by determining the zeroes of the one-loop beta functions (two-loop for gauge couplings): if $S = 0$ on the zero of the beta functions, the zero is a fixed point of the RGE, but if $S \neq 0$ on the zero, then the zero is a point on a cycle and $Q = S$ there.

To this end an explicit, three-loop expression for S is required. But as pointed out above, there has been no computation of S to the order where one would expect it to be non-vanishing if S were to equal Q on cycles. We have endeavored to compute S to third order in the loop expansion for a general theory containing n_S real scalars and n_f Weyl spinors, possibly charged under a gauge group. The potential in the Lagrangian is

$$V = \tfrac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d + (\tfrac{1}{2}y_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}).$$

The details of the computation are spelled out in Appendix 3.C. The surprisingly simple result is

$$(16\pi^2)^3 S_{ab} = \tfrac{5}{8}\,\text{tr}(y_a y_c^* y_d y_e^*)\lambda_{bcde} + \tfrac{3}{8}\,\text{tr}(y_a y_c^* y_d y_d^* y_b y_c^*) - \{a \leftrightarrow b\} + \text{h.c..}$$

We have evaluated this expression on the fixed points and cycles of the theories we

explored in [4, 6, 7] and found that in each case, even in examples in $d = 4 - \epsilon$, S vanishes at all fixed points and equals our previous determination of Q on all cycles.

Now for the (perturbative) proof of the propositions above. First we show that $S = Q$ on cycles. Consider the η-flow with $f_I = B_I$, with boundary condition that at $\eta = 0$ the point $\bar{g}_I(0)$ is on the cycle. Then $B_I(0) = \beta_I(\bar{g}(0)) - (Sg(0))_I = ([Q - S]g(0))_I$, with $Q - S$ in the Lie algebra of G_F and the left-hand side of (3.3.20) vanishes. Since χ^g_{IJ} is positive-definite to second order in the loop expansion, (3.3.20) gives $B_I(0) = 0$. This implies $S = Q + \Delta Q$ on cycles, where $(\Delta Q \, g)_I = 0$. But if $\Delta Q \neq 0$ this corresponds to a conserved current, $\nabla_\mu [(D^\mu \phi)^T \Delta Q \, \phi] = 0$, and we are free to redefine the scale current by a conserved current by $Q \to Q + \Delta Q$. Hence, $S = Q$ on cycles.[10]

For theories with two scalars there is an alternative, perhaps simpler proof that S equals Q when evaluated on a cycle. Consider (3.3.20) evaluated on a point on the cycle. It is easy to show that S is a constant on the cycle: $-dS/dt = \beta_I \partial_I S = (Qg)_I \partial_I S = [Q, S] = 0$, where the last two steps follow from (3.3.14) and the fact that, for two flavors, the flavor group, $SO(2)$, is Abelian. Now, as before, we consider the η-flow defined by the B_I function starting from a point on the RG-cycle (we make the distinction of the actual RG-cycle and a η-cycle explicit, to avoid confusion). The flow is defined by $-d\bar{g}_I/d\eta = B_I = \beta_I - (Sg)_I = ([Q - S]g)_I$, where the last step follows from assuming the initial point is on the RG-cycle and then noting that the solution corresponds to a trajectory that traverses the same cycle but at a different angular speed (the angular speeds are Q_{12} and $Q_{12} - S_{12}$ for the RG- and η-cycles, respectively). Therefore the η-cycle is generated by a trajectory in G_F and it follows that, just as for an RG-cycle, any G_F-invariant remains constant on the η-cycle. But the consistency condition (3.3.20) then implies that $B_I = ([Q - S]g)_I = 0$ on the cycle. Since Q and S are each characterized by a single number the only solution is $S_{12} = Q_{12}$ (on the cycle).

It is easy to show that $(Sg)_I = 0$ at a fixed point, and this is consistent with

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[10]In unitary theories with $\mathcal{N} = 1$ supersymmetry we recently showed, without relying on perturbation theory, that $S = 0$ [17]. It thus follows that RG limit cycles do not arise in such theories.

the notion that a fixed point corresponds to the case $(Qg)_I = 0$. To see this, notice that at a fixed point $B_I = -(Sg)_I$ so at that point the flow corresponds to a first-order G_F-transformation. That is, the first derivative with respect to η of G_F-invariants vanishes at the fixed point. Hence, (3.3.20) gives that $\chi^g_{IJ}(Sg)_I(Sg)_J = 0$ and hence $(Sg)_I = 0$ at the fixed point. The solution is that either $S = 0$ at the fixed point, or there is an emergent symmetry at the fixed point, and J^μ is the corresponding conserved current. This completes the proof of the two propositions above.

3.4.2. Cyclic CFTs

A perturbative proof that scale imples conformal invariance

The condition for a theory in $d > 2$ to be scale-invariant is that the trace of its stress-energy tensor be a total derivative [9],

$$T^\mu_{\ \mu} = \partial_\mu V^\mu,$$

where $V^\mu \neq j^\mu + \partial_\nu L^{\mu\nu}$ with $\partial_\mu j^\mu = 0$ and, without loss of generality, $L^{\mu\nu} = L^{\nu\mu}$. A candidate for V^μ is $V^\mu = \partial^\mu \phi^T P \phi$. If the theory includes spinors an additional current can be added to V^μ but the argument below is easily generalized by trivial extensions, e.g., by interpreting the index I as including all couplings. Using the equations of motion, or alternatively a G_F-transformation, this can be cast as an algebraic condition,

$$B_I = (Pg)_I. \tag{3.4.1}$$

It is easy to see now that in $d = 4$ the B_I-flow of $\widetilde{B}_b$ requires $(Pg)_I = 0$. Indeed, using (3.4.1) in (3.3.20) the left-hand side vanishes on account of B_I being of the form $(\omega g)_I$, and then perturbative positivity of χ^g_{IJ} implies $B_I = 0$. While P may not vanish, the current V^μ can at most be a symmetry of the theory, $V^\mu = j^\mu$. This concludes the proof that scale implies conformal invariance in perturbation theory.

Some properties of cyclic CFTs

Our result that scale implies conformal invariance implies that the non-trivial cycle found in [7] actually corresponds to a CFT. We dub such CFTs cyclic CFTs. It is quite surprising that CFTs can be found at points where the beta functions do not vanish. It is unclear what, if anything, distinguishes these theories from fixed-point CFTs. Presumably the special current J^μ plays a crucial role. We hope to address these questions in the future, but at present have no progress to report.

Since the stress-energy tensor is not renormalized, and since the divergence of the special current J^μ appears in the trace-anomaly equation, one may suspect its anomalous dimension vanishes. If so this would correspond to a non-conserved vector operator of dimension exactly three (no anomalous dimension), which is impossible in a unitary CFT. However, the operator actually mixes under renormalization. A simple computation gives

$$\mu\frac{d}{d\mu}[\mathcal{O}_I] = -\partial_I\hat{\beta}_J[\mathcal{O}_J] + \partial_\mu[\partial^\mu\phi^T\rho_I\phi],$$

$$\mu\frac{d}{d\mu}[\partial^\mu\phi^T\omega\phi] = -[\partial^\mu\phi^T\rho_I(\omega g)_I\phi],$$

which allows one to readily verify that (i) the combination $\hat{\beta}_I[\mathcal{O}_I] + \partial_\mu J^\mu$ is RG-invariant, (ii) a symmetry current is RG-invariant, and (iii) J^μ is not RG-invariant, $\mu\frac{d}{d\mu}J^\mu = -[\partial^\mu\phi^T\hat{\beta}_I\rho_I\phi]$.

Even if the beta function is non-vanishing, properties that follow directly from the conformal symmetry apply to these cyclic CFTs. Consider for example the well-known fact that two point correlators of primary operators can be diagonalized and

$$\langle\mathcal{O}(x)^\dagger\mathcal{O}(0)\rangle = (x^2)^{-\Delta_\mathcal{O}}.$$

Now contrast this with the two point function of the elementary real scalars ϕ_a in a

cyclic CFT. Scale and Poincaré invariance alone give [5]

$$\langle \phi(x)\phi^T(0) \rangle = (x^2)^{-\frac{1}{2}\Delta} G (x^2)^{-\frac{1}{2}\Delta^T}, \tag{3.4.2}$$

where G is a fixed real, positive, symmetric matrix and $\Delta = 1 + \gamma + Q$, with $\gamma^T = \gamma$ the anomalous dimension matrix of the elementary fields ϕ and $Q^T = -Q$ defining the cycle through $\beta_I = (Qg)_I$. Now one can redefine the field by $\phi \to M^{-1}\phi$ with M chosen so that $MGM^T = 1$, which is always possible with real M for a real, positive, symmetric matrix. This effectively redefines $\Delta \to M\Delta M^{-1}$. The condition for invariance under special conformal transformations then gives[11] $\Delta^T = \Delta$. A further field redefinition by an appropriate orthogonal transformation R finally brings Δ into diagonal form, $\Delta \to R\Delta R^T$. The entries of this diagonal form of Δ correspond to the roots of the characteristic polynomial of $1 + \gamma + Q$ which must be real. It is interesting that this puts restrictions on the possible values of Q: given a fixed value of γ, for large enough Q some roots will be complex. To put it differently, from our proof that these theories are conformally invariant we infer that if a matrix XAX^{-1} is diagonal for a real matrix A and a real, symmetric, invertible matrix X, then all the roots of the characteristic polynomial of A are real.

This unfortunately means that the large-Q scenario of [5], which leads to interesting oscillatory behavior in unparticle physics, is excluded by conformal invariance. More generally, the constraints that unitarity and scale invariance alone place on the scaling dimensions of operators are weaker than those that follow if conformal invariance is also imposed [18]. These weaker conditions are popular in unparticle phenomenology as they amplify the putative effects of unparticles. Of course our proof does not rule out theories that are scale-invariant but not conformal outside the realm of perturbation theory, leaving a smidgen of hope for unparticle enthusiasts.

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[11]Alternatively, special conformal transformations on (3.4.2) require that $\Delta G = G\Delta^T$.

3.5. The *c*-theorem in the presence of cycles

As we have seen, the consistency relations of JO lead to the *c*-theorem in perturbation theory,

$$-\frac{d\widetilde{B}_b}{dt} = \tfrac{1}{8}\chi^g_{IJ}B_I B_J \geq 0,$$

with $\widetilde{B}_b$ defined in (3.3.18). Only the last step in this sequence of relations invokes perturbation theory, for the positivity of the metric χ^g_{ij} is established perturbatively. For a non-perturbative proof we turn to the method of KS.

Let us review the argument of KS. Our presentation is closer in spirit to that of LPR. We will try to note explicitly when implicit assumptions in that argument are made. While plausible, these assumptions should be justified for the theorem to be established. We deviate from both presentations in that we do not derive nor use a Wess–Zumino dilaton action for, as we will see, this is not necessary for the computation.

Consider the four point function of the operator $\frac{1}{2}\partial_\mu(x_\nu T^{\mu\nu})$ in an arbitrary four-dimensional theory which is classically scale-invariant. Furthermore, we will consider kinematics such that $p_i^2 = 0$, $i = 1, \ldots, 4$, for the momenta p_i of the four insertions, so that the Mandelstam variables satisfy $s + t + u = 0$. Equivalently, for the theory on a conformally flat background, $g_{\mu\nu} = e^{-2\tau(x)}\eta_{\mu\nu}$, one may compute the $\tau(x)$ scattering amplitude $A(s,t)$ with the on-shell condition $\nabla^2\tau = 0$.

Now, we will assume that the forward scattering amplitude $\mathcal{A}_{\text{fwd}}(s) = \mathcal{A}(s,0)$ exists, that is that the limit $t \to 0$ of $\mathcal{A}(s,t)$ exists. This could fail if $\mathcal{A}(s,t)$ had terms of the form, e.g., $\sim s^2\ln(t)$. Now, $\mathcal{A}_{\text{fwd}}(s)$ can be computed by taking four $\tau(x)$-derivatives of the generating functional and then taking the metric as flat, the coupling constants to be spacetime-independent and the background field A_μ and the conformal compensator to vanish. Alternatively, and more straightforwardly, one can work with a conformally flat metric and having the only spacetime dependence in couplings and A_μ arise through the dependence on the conformal factor $\tau(x)$, so that

one merely needs to take $\tau(x) = 0$ after four times differentiating $\widetilde{W}$. Now the first derivative simply gives the conformal anomaly equation

$$T^\mu_\mu = \widehat{B}_I[\mathcal{O}_I] + [\beta^{\mathscr{2}}] + [(D^\mu\phi)^T B^A_\mu \phi] - \mu^{-\epsilon}\beta_{\tilde{\lambda}} \cdot \mathscr{R} + \nabla_\mu(J^\mu_\Theta + \widetilde{\mathscr{Z}^\mu}) - ((1 + \hat{\gamma} + S)\phi) \cdot \frac{\delta}{\delta\phi}\tilde{S}_0,$$

One need only thrice differentiate this equation to obtain the four-point amplitude of T^μ_μ. Note that on fixed points and cycles, where we will need this, the first term vanishes since $\widehat{B}_I = 0$. Also, the last term, which vanishes by the equations of motion, can be ignored for the computation of the amplitude. Most of the remaining terms vanish once the couplings are taken to be spacetime-independent (and the metric flat and $A_\mu = 0$). The remaining terms arise from the G and H^2 terms in $\beta_\lambda \cdot \mathscr{R}$. For a conformally flat metric, $\gamma_{\mu\nu} = \exp(-2\tau(x))\eta_{\mu\nu}$, one has (in d spacetime dimensions)

$$e^{-4\tau}G = 8(\partial^2\tau)^2 - 8\tau_{,\mu\nu}\tau^{,\mu\nu} - 16\tau_{,\mu}\tau_{,\nu}\tau^{,\mu\nu}$$
$$- 8(d-3)\tau_{,\mu}\tau^{,\mu}\partial^2\tau + 2(d-1)(d-4)(\tau_{,\mu}\tau^{,\mu})^2,$$
$$e^{-4\tau}H^2 = 4(\partial^2\tau)^2 - 4(d-2)\tau_{,\mu}\tau^{,\mu}\partial^2\tau + (d-2)^2(\tau_{,\mu}\tau^{,\mu})^2.$$

The cubic term in H^2 vanishes for an "on-shell" conformal factor $\partial^2\tau = 0$ and so the only contribution to the "on-shell" forward scattering amplitude is from G:

$$\mathcal{A}_{\text{fwd}}(s)|_{\text{FP or cycle}} = -\beta_b(s^2 + t^2 + u^2)|_{t=0} = -2\beta_b s^2.$$

Let's assume that there exists an RG trajectory from a UV fixed point or cycle to an IR fixed point or cycle. On this trajectory this equation no longer holds. However, we can inspect limiting behavior. Since $\mathcal{A}_{\text{fwd}}/s^2$ depends on s only through the dimensionless ratio μ^2/s, its behavior is dictated by the renormalization group. Hence,

$$\lim_{s\to\infty}\frac{\mathcal{A}_{\text{fwd}}(s)}{s^2} = \lim_{s\to\infty}\frac{\mathcal{A}_{\text{fwd}}(s)|_{\text{FP or cycle}}}{s^2} = -2(\beta_b)_{\text{UV}}$$

and

$$\lim_{s \to 0} \frac{\mathcal{A}_{\text{fwd}}(s)}{s^2} = \lim_{s \to 0} \frac{\mathcal{A}_{\text{fwd}}(s)|_{\text{FP or cycle}}}{s^2} = -2(\beta_b)_{\text{IR}},$$

where $(\beta_b)_{\text{UV}}$ and $(\beta_b)_{\text{IR}}$ are the limiting UV and IR values of β_b along the trajectory and correspond to those on the fixed point or cycle. LPR study the approach to these limiting values using conformal perturbation theory.

Following LPR we next consider the integral of $\mathcal{A}_{\text{fwd}}(s)/s^3$ over the contour in Fig. 3.1. The integral over the semicircle I_1 cannot be easily computed, but in the
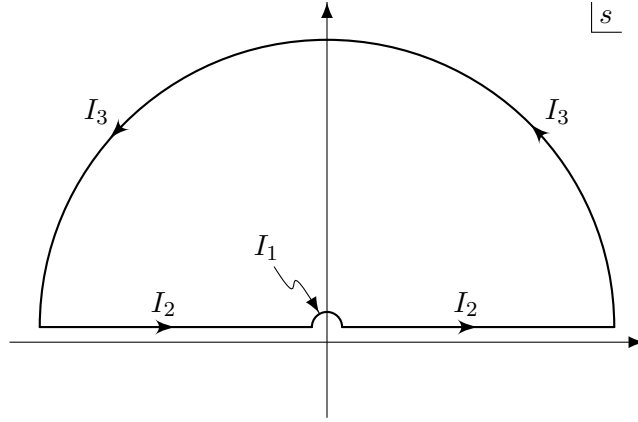
Figure 3.1: The contour of integration for $\int ds \, \mathcal{A}_{\text{fwd}}(s)/s^3$.

limit that the radius of the semicircle vanishes it is reasonable that one can use the limiting value,

$$\int_{I_1} \frac{ds}{s^3} \mathcal{A}_{\text{fwd}}(s) \approx \int_{I_1} \frac{ds}{s} 2(\beta_b)_{\text{IR}} = 2\pi i (\beta_b)_{\text{IR}}, \qquad (3.5.1)$$

where the last step corresponds to taking the vanishing limit of the radius of the semicircle I_1. Similarly, the large circle I_3 gives

$$\int_{I_3} \frac{ds}{s^3} \mathcal{A}_{\text{fwd}}(s) \approx \int_{I_3} \frac{ds}{s} 2(\beta_b)_{\text{UV}} = -2\pi i (\beta_b)_{\text{UV}}. \qquad (3.5.2)$$

It follows from Cauchy's theorem that

$$(\beta_b)_{\text{UV}} - (\beta_b)_{\text{IR}} = \frac{1}{2\pi i} \int_{I_2} \frac{ds}{s^3} \mathcal{A}_{\text{fwd}}(s)$$
$$= \frac{1}{\pi} \int_0^\infty \frac{ds}{s^3} \operatorname{Im}(\mathcal{A}_{\text{fwd}}(s + i0)),$$

where in the last line LPR assume crossing symmetry to write $\mathcal{A}_{\text{fwd}}(-s + i0) = \mathcal{A}^*_{\text{fwd}}(s + i0)$. Finally, the KS argument invokes the optical theorem that relates the imaginary part of the forward scattering amplitude to a positive-definite cross section to conclude that

$$(\beta_b)_{\text{UV}} - (\beta_b)_{\text{IR}} > 0.$$

We note in passing that the optical theorem is known to apply for forward scattering amplitudes of (on-shell) physical particles. It is not clear a priori that it applies to Green functions of composite operators at $p_i^2 = 0$, even if it corresponds to the scattering amplitude of would-be dilaton scattering. We think the assumption of positivity is reasonable, so we press on.

What steps in the argument above require special attention when the theory admits dimension-three currents? As we have pointed out, the trace of the stress-energy tensor now has an additional $\partial_\mu J^\mu$ term, but we have already accounted for this in the presentation above: the current can be eliminated by replacing B_I for β_I in the expression for the trace of the stress-energy tensor. Throughout the flow this makes no difference to the argument above, since the positivity of the integral over the segments I_2 of the contour follows from the optical theorem. For cycles one is not free to ignore the $\tau(x)$ dependence of the couplings or the background vector field in the anomaly equation. But on the cycle the couplings are covariantly constant. Hence, the terms that vanish at fixed points because of the constancy of couplings also vanish for cycles, but now because they are covariantly constant. Finally, the validity of the limits in (3.5.1) and (3.5.2) needs to be established anew for limit cycles. However, the same method of conformal perturbation theory may be applied to establish the result. Since

it is only scaling that is used in this step of the argument by LPR, the proof goes through as presented there.

3.6. Summary and concluding remarks

We have shown that the Komargodski–Schwimmer proof of the weak version of the c-theorem includes the more general case that a renormalization group flow goes from a fixed point or cycle to another fixed point or cycle. Regarding the strong version of the c-theorem, proven in perturbation theory by Jack and Osborn, we pointed out that the quantity that plays the role of c is $\widetilde{B}_b$ (defined in (3.3.18)) which is closely related to the a-anomaly (β_b in the notation of Jack and Osborn); these quantities agree at fixed points and on cycles, but are not generally the same.

We presented a calculation of the Lie-algebra function of coupling constants S introduced by Jack and Osborn. This is the first calculation of S to an order (third) in the loop expansion where it does not vanish. We then proved that $S = 0$ on fixed points and that S precisely corresponds to the generator Q of limit cycles when evaluated at any point on the limit cycle. This gives a major improvement on the method of searching for limit cycles: one merely needs to find zeroes of the beta functions to the first order in the loop expansion (second order for Yang–Mills couplings) and evaluate S there. If $S = 0$ the zero corresponds to a fixed point, while if $S = Q \neq 0$ the zero corresponds to a limit cycle with Q the generator of the cycle.

We used these results to show that the trace of the stress-energy tensor vanishes on cycles, and hence that scale implies conformal invariance (perturbatively in unitary relativistic $d = 4$ QFT). If "theory space" is understood as the space of couplings of a model modulo the action of G_F on these couplings (with G_F the group of symmetries of the free Lagrangian), then cycles and fixed points are mapped to single points. It is remarkable that all such points describe in fact CFTs.

Some questions remain which we intend to turn to in the future. Among them

are:

- Are there renormalization group flows between fixed points and cycles?

- Are there limit cycles in four dimensions with bounded tree-level scalar potential?

- Are there any properties of cyclic CFTs that generically distinguish them from fixed point CFTs? In particular, does the current associated with the generator S play a special role?

- Can a non-perturbative proof of the strong version of the c-theorem be given by extending the perturbative proof, say, by showing positivity of the metric χ^g_{IJ} using dispersion relations?

- Do relativistic, unitary QFTs admit recursive RG flows that do not correspond to motions by generators in G_F?

We look forward to addressing these questions.

3.A. Ambiguities in RG functions

It is well-known that anomalous-dimension matrices and beta functions are dependent on the renormalization scheme. Nevertheless, physical quantities obtained

from the anomalous-dimension matrices and the beta functions which are relevant to the study of scale-invariant theories are, as expected, independent of the scheme [6].

It is however usually not appreciated that anomalous-dimension matrices and beta functions exhibit another freedom, mentioned briefly in the beginning of Section 3.3, which we review here. For simplicity consider a theory of real multi-component scalars with bare Lagrangian

$$\mathscr{L}_0 = \tfrac{1}{2}\partial_\mu\phi_{0a}\partial^\mu\phi_{0a} - \tfrac{1}{4!}g^0_{abcd}\phi_{0a}\phi_{0b}\phi_{0c}\phi_{0d}.$$

There is an ambiguity in the definition of the wavefunction renormalization matrix $Z^{1/2}$, corresponding to the freedom of choosing $\tilde{Z}^{1/2} = OZ^{1/2}$ where $O^TO = \mathbb{1}$ [8]. In this appendix we study the effect of this ambiguity in the definition of RG functions. For simplicity we present this analysis in the flat background limit. Dimensional regularization is used throughout.

Bare couplings and fields are related to the corresponding renormalized quantities by

$$g^0_I = \mu^{k_I\epsilon}(g_I + L_I(g)), \qquad \phi_0 = \mu^{\delta\epsilon}\check{Z}(g)\phi,$$

where $\check{Z} = Z^{1/2}$, and $\check{Z} - 1$ and L_I have expansions in ϵ-poles starting at $1/\epsilon$. The anomalous-dimension matrices and the beta functions, as well as the antisymmetric matrix S of (3.3.10), are given by

$$\hat{\gamma} = \delta\epsilon - k_I g_I \partial_I \check{Z}^1, \qquad \hat{\beta}_I = -k_I g_I \epsilon - k_I L_I^1 + k_J g_J \partial_J L_I^1, \qquad S = -k_I g_I N_I^1,$$

where the superscript denotes residues of simple poles. The index carried by k is exempt from the summation convention. In the present example $k_I = 1$, but we keep it for generality. Since we are interested in ambiguities that arise because of different choices in the subtraction of infinite quantities, we assume that O has an expansion in ϵ-poles, $O = 1 + O^1/\epsilon + \cdots$, where O^1 is antisymmetric as required by $O^TO = 1$. Then, under

the freedom mentioned above, it is easy to verify that the relevant quantities change as

$$\check{Z}^1 \to \check{Z}^1 + O^1, \qquad L_I^1 \to L_I^1 + (O^1 g)_I, \qquad N_I^1 \to N_I^1 - \partial_I O^1,$$

This induces a change in the anomalous-dimension matrix, the beta functions, and the antisymmetric matrix S:

$$\hat{\gamma} \to \hat{\gamma} - \omega, \qquad \hat{\beta}_I \to \hat{\beta}_I + (\omega g)_I, \qquad S \to S + \omega,$$

where $\omega = k_I g_I \partial_I O^1$. This ambiguity, or "gauge" freedom, in the definition of anomalous dimensions and beta functions is usually resolved by requiring that the anomalous-dimension matrix be symmetric. Note, however, that the trace of the stress-energy tensor, being a physical quantity, has to be invariant under this unphysical freedom. Indeed, this is obviously the case in (3.3.11). As we see $\hat{\beta}$, $\hat{\gamma}$ and S are gauge-covariant, but $\widehat{B}_I = \hat{\beta}_I - (Sg)_I$ and $\widehat{\Gamma} = \hat{\gamma} + S$ are gauge-invariant. Although RG flows are specified by $\hat{\beta}$, there is a gauge, defined by $\omega = -S$ so that $S = 0$, in which $\widehat{B} = \hat{\beta}$ and $\widehat{\Gamma} = \hat{\gamma}$.

Finally, it is worth pointing out that $\widehat{B}_I$ could be seen as the proper vector field whose RG flows one should consider, and whose fixed points describe CFTs. This vector field does not admit cycles in perturbation theory.

3.B. The relation between the metrics χ_{ij}^a and χ_{ij}^g

The coefficient c_g of LPR appears to play the role of the "metric" χ_{ij}^g in the consistency condition (3.2.13). As we mention in the end of Section 3.2.2 and elaborate on further here, this is not the case. To see the connection with the work of JO, following LPR we write

$$\Delta W_{\text{anomaly}} = \frac{1}{2} \int d^4x \, d^4y \, \tau(x) \tau(y) \langle \Theta(x) \Theta(y) \rangle,$$

where $\Theta = \beta^i \mathcal{O}_i$, and therefore

$$\frac{d}{dt}\Delta W_{\text{anomaly}} = \frac{1}{2}\int d^4x\, d^4y\, \tau(x)\tau(y)\frac{d}{dt}\langle\Theta(x)\Theta(y)\rangle. \qquad (3.\text{B}.1)$$

In Ref. [14, Eq. (3.18b)] Osborn finds the RGE for the product of two local renormalized operators,

$$-\frac{d}{dt}\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle + \partial_i\beta^j\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle + \partial_j\beta^k\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = -\chi_{ij}^a\partial^2\partial^2\delta^{(4)}(x).$$

The quantity χ_{ij}^a can be thought of as the beta function associated with the counterterm needed in order to renormalize the correlator $\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle$. Now since $-d\beta^i/dt = \beta^j\partial_j\beta^i$, it is easy to see that

$$\frac{d}{dt}\langle\Theta(x)\Theta(0)\rangle = \chi_{ij}^a\beta^i\beta^j\partial^2\partial^2\delta^{(4)}(x).$$

Using this in (3.B.1) we see that the metric of LPR is $-2\chi^a$, which is always positive. This suggests the question "is there a relation between χ^g and χ^a?"

In the specific example of a gauge theory with a simple gauge group G and charged Dirac fermions in some representation, JO give [8, Eqs. (5.12)], at two loops,

$$\chi^{a(2)} = -\frac{1}{2}\chi^{g(2)} = -\frac{n_V}{8\pi^2 g^2}\left[1 + \left(17C - \frac{20}{3}R\right)h\right], \qquad h \equiv \frac{g^2}{16\pi^2}, \qquad (3.\text{B}.2)$$

where $\text{tr}(t_{\text{adj}}^a t_{\text{adj}}^b) = C\delta^{ab}$, R is similarly defined for the representation of the Dirac fermions, and $n_V = \dim(\text{Adj})$ is the number of vectors. However, the relation $\chi^g = -2\chi^a$ of (3.B.2) does not hold in general, and so the task of computing χ^g is complicated. Nevertheless, Weyl consistency conditions give the general relation between χ^a and χ^g [8, Eq. (3.23)]:

$$\chi_{ij}^g + 2\chi_{ij}^a - \bar{\chi}_{ijk}^a\beta^k = -\beta^k\partial_k V_{ij} - \partial_i\beta^j\, V_{kj} - \partial_j\beta^k\, V_{ik}, \qquad (3.\text{B}.3)$$

where $\zeta V_{ij} = \bar{\chi}^a_{ijk} k^k g^k$ (no sum over the index k), and $\bar{\chi}^a_{ijk} = \partial_k \chi^a_{ij} - \frac{1}{2}(\chi^b_{ikj} + \chi^b_{jki})$, with χ^b_{ijk} necessary to regulate infinities in three-point functions, and ζ defined as an operator counting the number of loops, whose form can be read off from $\mathcal{O}(\epsilon)$ terms of the finiteness condition (3.9e) of JO:

$$\zeta V_{ij} = (1 + k^k g^k \partial_k) V_{ij} + 2k^i V_{ij} \qquad \text{(no sum over the index carried by } k\text{)}$$

(cf. JO's (3.16b)).

In our gauge-theory example (3.B.3) becomes

$$\chi^g + 2\chi^a - \bar{\chi}^a \beta^g = -\beta^g \frac{\partial V}{\partial g} - 2\frac{\partial \beta^g}{\partial g} V, \qquad \zeta V = \tfrac{1}{2}\bar{\chi}^a g, \qquad (3.B.4)$$

where $\zeta V = (2 + \frac{1}{2} g\, \partial/\partial g) V = (2 + h\, \partial/\partial h) V$, the beta function for the gauge coupling is

$$\frac{1}{g}\beta^g = -\beta_0 h - \beta_1 h^2 + \mathcal{O}(h^3), \qquad \beta_0 = \frac{1}{3}(11C - 4R), \qquad \beta_1 = \frac{2}{3}C(17C - 10R),$$

and $\bar{\chi}^a = \partial \chi^a/\partial g - \chi^b$, where χ^b is given at two loops by $\chi^{b(2)} = \frac{n_V}{4\pi^2 g^3}(1 + 4\beta_0 h)$. It follows that

$$\bar{\chi}^a = -\frac{n_V}{\pi^2 g^3}[\beta_0 h + \mathcal{O}(h^2)].$$

Expanding $V = v_0 + v_1 h + \cdots$ gives $\zeta V = 2v_0 + 3v_1 h + \cdots = \frac{1}{2} g \bar{\chi}^a$, or

$$V = -\frac{n_V \beta_0}{64\pi^4} + \mathcal{O}(h).$$

With these results (3.B.4) gives

$$\chi^g + 2\chi^a = -\frac{n_V \beta_0^2}{32\pi^4} h + \mathcal{O}(h^2),$$

and, therefore, beyond two-loop order, $\chi^g \neq -2\chi^a$.

To summarize, the results of LPR correspond to using JO's $-2\chi_{ij}^a$ as a metric, which however is not in general equal to JO's metric χ_{ij}^g. Indeed, $\chi_{ij}^g + 2\chi_{ij}^a$ fails to vanish beyond the first few orders in the loop expansion. The positivity of χ_{ij}^g may also fail non-perturbatively (for example, if its perturbative expression has finite radius of convergence).

3.C. How to calculate N_I and S

The calculation of JO's N_I proceeds order by order in perturbation theory. In this appendix we calculate contributions to N_I in a quantum field theory with real scalars and Weyl spinors up to two loops, and we also perform a three-loop calculation of the part of N_I that is needed in order to compute S.

As can be seen from (3.3.1), in order to calculate N_I we need to compute self-energies of scalars but with coupling constants as spectator fields. Equivalently, the calculation can be done by considering scalar self-energy diagrams and letting momentum come in from external legs and go out through couplings. From these diagrams we can then pick up the contribution linear in the momentum of the field and linear in the momentum of the coupling. After we antisymmetrize, we have a contribution to N_I.

It is perhaps helpful to remind the reader here that in a theory with scalars and fermions the I index can be either $(abcd)$ or $(a|ij)$. Let us also remark that S appears first at three loops in a theory with scalars and spinors. The reason is easily seen from (3.3.10): a diagram that contributes to N will only contribute to S if it is not symmetric under $a \leftrightarrow b$. As it turns out there are no such diagrams in scalar self-energies at one and two loops, but there are four such diagrams at three loops. Consequently, even if the theory contains gauge fields, diagrams with gauge fields will not contribute to S at three loops, but certainly will do so at higher order. Therefore, even in a gauge theory we don't need to include gauge fields in our leading-order

calculation of S.

3.C.1. One loop

At one loop the calculation proceeds with no subtleties since renormalization is trivial, i.e., there are no subdivergences to be subtracted. The two diagrams that contribute to N_I and their corresponding counterterms are shown in Fig. 3.2.
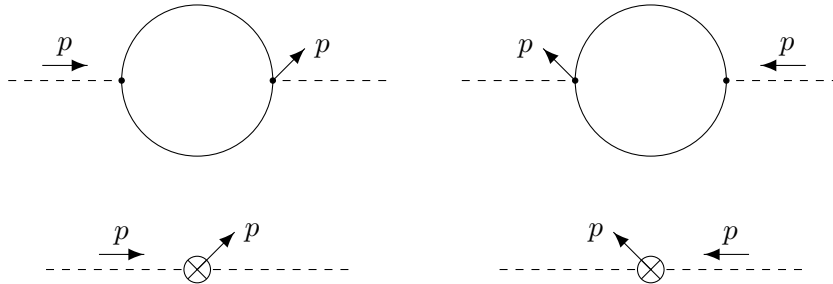
Figure 3.2: Diagrams that contribute to $N_{a|ij}$ at one loop and their corresponding counterterms.

A straightforward calculation gives

$$(N_{c|ij})_{ab} = -\frac{1}{16\pi^2\epsilon}\frac{1}{2}(y^*_{a|ij}\delta_{bc} - y^*_{b|ij}\delta_{ac}) + \text{finite},$$

and there is of course a complex conjugate $(N^*_{c|ij})_{ab}$.

In order to simplify the notation we write the result for the residue of the simple ϵ-pole in N_I in the form

$$16\pi^2(N^1_I)_{ab}\partial^\mu g_I = -\tfrac{1}{2}[\text{tr}(y_a\partial^\mu y^*_b) + \text{h.c.} - \{a \leftrightarrow b\}],$$

where g_I on the left-hand side stands here for $y_{c|ij}$ or $y^*_{c|ij}$. Selecting the appropriate derivatives one easily reads off the corresponding N^1_I. Our result reproduces JO's equation (7.16) for ρ_I when we use Dirac spinors.

3.C.2. Two loops

At two loops there are three Feynman diagrams that contribute to N_I, listed in Fig. 3.3. The calculation of the residues of the simple ϵ-poles of N_I requires now
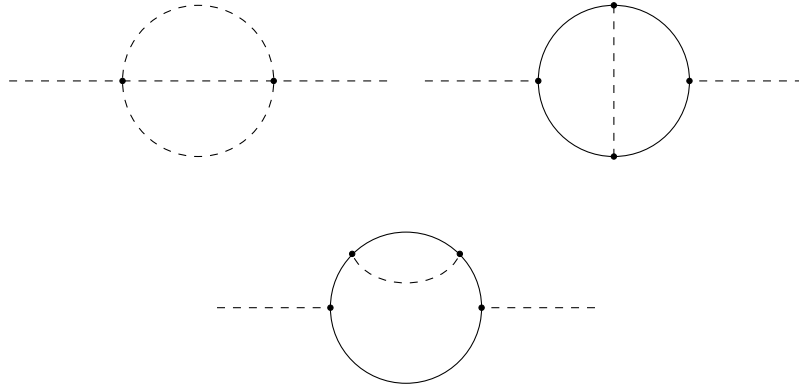
Figure 3.3: Feynman diagrams that contribute to N_I at two loops.

a subtraction of subdivergences, something that proceeds, for the most part, in the usual way. However, there is a small subtlety, not seen in the usual treatments of renormalization, that we would like to point out. Clearly, the two right-most diagrams of Fig. 3.3 have subdivergences so we have to add to them the diagrams with the insertions of the corresponding counterterms. For the right-most diagram the graph with the insertion of the counterterm is
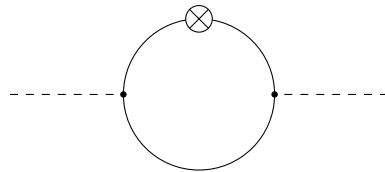
Now, when the momentum that comes in from, say the left external leg, flows out through the counterterm, then there are two diagrams that contribute, namely
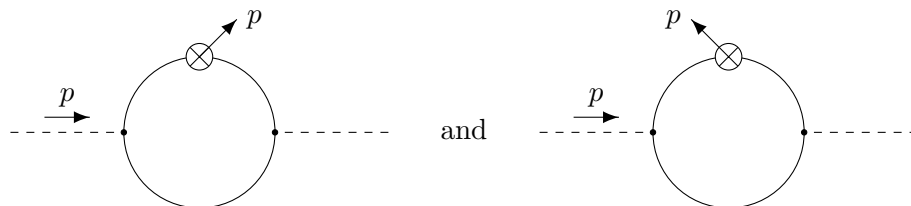
where the momentum exits to the north-east or to the north-west depending on which vertex it flows out of in the original diagram in Fig. 3.3. In both cases the counterterm is the same, but the diagram with the insertion of the counterterm is different as a result of the difference in the momentum of the internal leg that the counterterm picks up. That is, had we retained different momenta for the various vertices, there would be two momenta associated with the counterterm.

The two-loop result for N_I^1, previously unpublished, is

$$(16\pi^2)^2(N_I^1)_{ab}\partial^\mu g_I = -\tfrac{1}{24}\lambda_{acde}\,\partial^\mu\lambda_{bcde} + [\tfrac{1}{4}\operatorname{tr}(y_a^* y_c \partial^\mu y_b^* y_c)$$

$$+ \tfrac{1}{8}\operatorname{tr}(y_a^*\partial^\mu y_c y_c^* y_b) + \tfrac{3}{8}\operatorname{tr}(y_a^* y_c y_c^* \partial^\mu y_b) + \text{h.c.}] - \{a \leftrightarrow b\}.$$

It follows that S vanishes at this order. This can be seen, term by term (when anti-symmetrized in a and b) by replacing g_I for $\partial^\mu g_I$.

3.C.3. Three loops

At three loops there are many diagrams that contribute to N_I, but only four are not symmetric under $a \leftrightarrow b$ and thus end up contributing to S. These diagrams are shown in Fig. 3.4, and we here only compute their contributions to N_I^1. From these
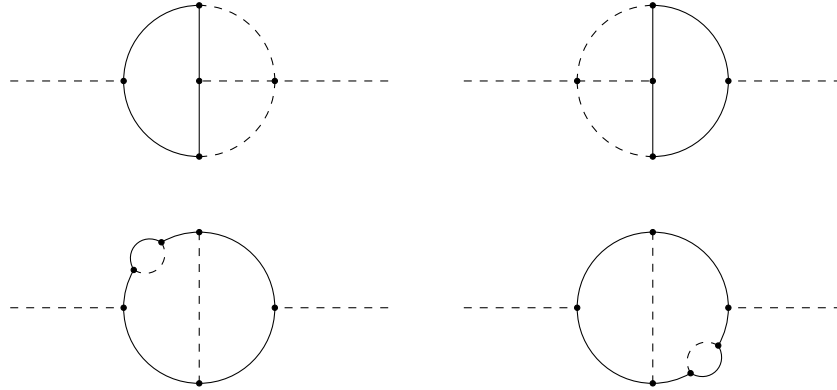
Figure 3.4: Three-loop diagrams that contribute to N_I not symmetric under $a \leftrightarrow b$, and thus leading to contributions to S at three loops.

diagrams (and the corresponding counterterms), using the methods for the calculation

of pole parts of three-loop diagrams given in [19], we find

$$(16\pi^2)^3(N_I^1)_{ab}\partial^\mu g_I \supset -\tfrac{1}{2}\operatorname{tr}(y_a\partial^\mu y_c^* y_d y_e^*)\lambda_{bcde} - \tfrac{1}{3}\operatorname{tr}(y_a y_c^* \partial^\mu y_d y_e^*)\lambda_{bcde}$$

$$-\tfrac{1}{2}\operatorname{tr}(y_a y_c^* y_d \partial^\mu y_e^*)\lambda_{bcde} - \tfrac{5}{24}\operatorname{tr}(y_a y_c^* y_d y_e^*)\partial^\mu\lambda_{bcde}$$

$$-\tfrac{1}{24}\operatorname{tr}(y_b\partial^\mu y_c^* y_d y_e^*)\lambda_{acde} - \tfrac{5}{24}\operatorname{tr}(y_b y_c^* \partial^\mu y_d y_e^*)\lambda_{acde}$$

$$-\tfrac{1}{24}\operatorname{tr}(y_b y_c^* y_d \partial^\mu y_e^*)\lambda_{acde} - \tfrac{5}{24}\operatorname{tr}(\partial^\mu y_b y_c^* y_d y_e^*)\lambda_{acde}$$

$$-\tfrac{7}{32}\operatorname{tr}(y_a\partial^\mu y_c^* y_d y_d^* y_b y_c^*) - \tfrac{7}{96}\operatorname{tr}(y_a y_c^* \partial^\mu y_d y_d^* y_b y_c^*)$$

$$-\tfrac{23}{96}\operatorname{tr}(y_a y_c^* y_d \partial^\mu y_d^* y_b y_c^*) - \tfrac{7}{96}\operatorname{tr}(y_a y_c^* y_d y_d^* \partial^\mu y_b y_c^*)$$

$$-\tfrac{7}{32}\operatorname{tr}(y_a y_c^* y_d y_d^* y_b \partial^\mu y_c^*) + \tfrac{1}{16}\operatorname{tr}(y_a\partial^\mu y_c^* y_c y_d^* y_b y_d^*)$$

$$-\tfrac{5}{48}\operatorname{tr}(y_a y_c^* \partial^\mu y_c y_d^* y_b y_d^*) - \tfrac{1}{48}\operatorname{tr}(y_a y_c^* y_c \partial^\mu y_d^* y_b y_d^*)$$

$$-\tfrac{7}{96}\operatorname{tr}(y_a y_c^* y_c y_d^* \partial^\mu y_b y_d^*) + \tfrac{1}{16}\operatorname{tr}(y_a y_c^* y_c y_d^* y_b \partial^\mu y_d^*)$$

$$+ \text{h.c.} - \{a\leftrightarrow b\},$$

and since

$$S \equiv -k_I N_I^1 g_I = -N_{abcd}^1\lambda_{abcd} - (\tfrac{1}{2}N_{a|ij}^1 y_{a|ij} + \text{h.c.})$$

we finally obtain

$$(16\pi^2)^3 S_{ab} = \tfrac{5}{8}\operatorname{tr}(y_a y_c^* y_d y_e^*)\lambda_{bcde} + \tfrac{3}{8}\operatorname{tr}(y_a y_c^* y_d y_d^* y_b y_c^*) + \text{h.c.} - \{a\leftrightarrow b\}.$$

As already remarked in the main body, evaluating this on points in coupling space where we have found fixed points and cycles in Refs. [4, 6, 7], we find that S vanishes at all fixed points and equals Q on all cycles.

References

[1] Z. Komargodski & A. Schwimmer, *"On Renormalization Group Flows in Four Dimensions"*, JHEP **1112**, 099 (2011), arXiv:1107.3987.

[2] A. Cappelli, G. D'Appollonio, R. Guida & N. Magnoli, *"On the c theorem in more*

than two-dimensions", JHEP , (2000), `hep-th/0009119`.

[3] A. Cappelli, R. Guida & N. Magnoli, *"Exact consequences of the trace anomaly in four-dimensions"*, Nucl.Phys. **B618**, 371 (2001), `hep-th/0103237`.

[4] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`.

[5] J.F. Fortin, B. Grinstein & A. Stergiou, *"Cyclic unparticle physics"*, Phys.Lett. **B709**, 408 (2012), `arXiv:1110.1634`.

[6] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance at Three Loops"*, `arXiv:1202.4757`.

[7] J.F. Fortin, B. Grinstein & A. Stergiou, *"Limit Cycles in Four Dimensions"*, `arXiv:1206.2921`.

[8] I. Jack & H. Osborn, *"Analogs for the c theorem for four-dimensional renormalizable field theories"*, Nucl.Phys. **B343**, 647 (1990).

[9] J. Polchinski, *"Scale and conformal invariance in quantum field theory"*, Nucl.Phys. **B303**, 226 (1988).

[10] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: Theoretical Foundations"*, JHEP **1207**, 025 (2012), `arXiv:1107.3840`.

[11] M.A. Luty, J. Polchinski & R. Rattazzi, *"The a-theorem and the Asymptotics of 4D Quantum Field Theory"*, `arXiv:1204.5221v1`.

[12] A. Zamolodchikov, *"Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory"*, JETP Lett. **43**, 730 (1986).

[13] J.L. Cardy, *"Is There a c Theorem in Four-Dimensions?"*, Phys.Lett. **B215**, 749 (1988).

[14] H. Osborn, *"Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories"*, Nucl.Phys. **B363**, 486 (1991).

[15] J. Wess & B. Zumino, *"Consequences of anomalous Ward identities"*, Phys.Lett. **B37**, 95 (1971).

[16] Z. Komargodski, *"The Constraints of Conformal Symmetry on RG Flows"*, `arXiv:1112.4538`.

[17] J.F. Fortin, B. Grinstein, C.W. Murphy & A. Stergiou, *"On Limit Cycles in Supersymmetric Theories"*, `arXiv:1210.2718`.

[18] B. Grinstein, K.A. Intriligator & I.Z. Rothstein, *"Comments on Unparticles"*, Phys.Lett. **B662**, 367 (2008), `arXiv:0801.1140`.

[19] K.G. Chetyrkin, M. Misiak & M. Münz, *"Beta functions and anomalous dimensions up to three loops"*, Nucl.Phys. **B518**, 473 (1998), `hep-ph/9711266`.

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Chapter 4

On Limit Cycles in Supersymmetric Theories

4.1. Introduction

In recent papers by some of us two independent methods were used to claim the existence of unitary four-dimensional quantum field theories that are scale but not conformally invariant (SFTs) [1–3]. A natural interpretation of the renormalization-group (RG) behavior of such theories is that they live on RG limit cycles with a constant "number of degrees of freedom." Nevertheless, the work of Jack and Osborn [4] (see also [5]), which we think is widely unappreciated in the literature, has lead us to a new understanding of the conditions for conformal invariance.[1] More specifically, it has become clear that a theory does not need to have zero beta functions in order for it to be conformal, and that the claimed examples of non-conformal scale-invariant field theories [1–3] are actually conformal.

We will not have much to say here about this new understanding—more details will be given in a forthcoming publication [6]. Our aim in the present note is to show that unitary $\mathcal{N} = 1$ supersymmetric theories in four dimensions cannot flow to a superconformal phase with nonzero beta functions. In other words, we will show that the beta-function vector field of supersymmetric theories does not admit limit cycles, in contrast to that of non-supersymmetric theories. (Let us remark here that we use

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[1]We acknowledge helpful discussions on this point with Markus Luty, Joseph Polchinski and Riccardo Rattazzi, as well as informative correspondence with Hugh Osborn.

"limit cycles" loosely to mean recursive flows in the beta-function vector field of a theory, that is, flows that may be cyclic or ergodic.) A corollary of this result is that there are no unitary $\mathcal{N} = 1$ supersymmetric SFTs in four dimensions.

The subject of scale without conformal invariance in unitary $\mathcal{N} = 1$ supersymmetric theories with an R-symmetry was investigated recently by Antoniadis and Buican [7]. Their treatment relies on carefully analyzing constraints in the operator content of such theories, and relies on various well motivated assumptions. A criterion is then given for a unitary supersymmetric theory to contain a superscale-invariant phase: it has to contain at least two real nonconserved dimension-two scalar singlet operators [7]. The most constraining assumption in the analysis of [7] is perhaps that an R-symmetry is required along the RG flow.

The operator content of possible supersymmetric SFTs was also studied by Nakayama [8], without the requirement of an R-symmetry. The so-called virial multiplet was constructed and its implications for scale without conformal invariance in supersymmetric theories were explored. In concrete examples difficulties were found in constructing a nontrivial virial multiplet in perturbation theory. However, relaxing the constraint of unitarity produced non-conformal scale-invariant field theories in a simple Wess–Zumino model.

With the recent work mentioned in the last two paragraphs in mind,[2] it seems unlikely that supersymmetric theories can host a superscale-invariant phase that is not superconformal. Still, we think it is interesting to ponder the existence of supersymmetric limit cycles. Examples of limit cycles in non-supersymmetric theories are more generic than previously thought: in addition to a four-dimensional example, limit cycles in $4 - \epsilon$ dimensions have also been found [1, 2, 11]. Thus, it is worthwhile to analyze the constraints supersymmetry imposes on such RG behavior.

The conclusion of our present note is that supersymmetry does not allow for limit cycles, and thus it does not allow for SFTs. Our method of proof, as will become

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[2]For other studies of superscale and superconformal invariance see [9, 10].

clear below, is very different in spirit from that employed by Antoniadis and Buican, and by Nakayama. More specifically, in order to reach our conclusion we analyze supersymmetric theories with superspace-dependent couplings, and show that a quantity corresponding to the S of [4] (see also [6]) is constrained to be zero by supersymmetry. The quantity S is related to the frequency with which a theory traverses its putative limit cycle, and thus the fact that $S = 0$ in supersymmetry immediately shows that supersymmetric limit cycles cannot occur.

Note Added: As this work was being finalized, Nakayama added an appendix to [8] where he also showed that S must vanish to all orders in perturbation theory in $\mathcal{N} = 1$ supersymmetric field theories.

4.2. Preliminaries

In this section we give a brief review of material that is necessary for our arguments.

We are interested in four-dimensional theories that are classically scale-invariant. They are parametrized by coupling constants g_i. Following Jack and Osborn we promote these to spacetime-dependent couplings, $g_i(x)$. This is useful in two ways. Firstly, the couplings now act as sources for composite operators appearing in the Lagrangian. This allows us to define finite composite operators as functional derivatives of the renormalized generating functional for Green functions, W, with respect to the couplings. A similar method is used frequently to define the stress-energy tensor: the theory is lifted to curved space and the stress-energy tensor is obtained as a functional derivative of W with respect to the metric. Secondly, it allows us to obtain a local version of the Callan–Symanzik equation, with terms involving derivatives of couplings interpreted as anomalies and thus satisfying Wess–Zumino consistency conditions [12].

In order to render this theory finite one must include all possible dimension-four counterterms consistent with diffeomorphism invariance. In addition, the counterterms

may be further constrained by formal symmetries of the theory in which both quantum fields and couplings transform. Consider, for example, a theory of real scalars with bare Lagrangian

$$\mathscr{L}_0 = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_{0a}\partial_\nu\phi_{0a} - \tfrac{1}{4!}g^0_{abcd}\phi_{0a}\phi_{0b}\phi_{0c}\phi_{0d}. \tag{4.2.1}$$

This is written in terms of bare fields ϕ_0. In the potential term the bare couplings g^0_{abcd} are completely symmetric under exchange of the indices a, b, c and d. The kinetic part of the Lagrangian exhibits a continuous symmetry under transformations of the fields $\delta\phi_{0a} = -\omega_{ab}\phi_{0b}$, where ω is in the Lie algebra of the flavor group $G_F = SO(n_S)$. The whole Lagrangian is G_F-symmetric if we agree to transform, in addition, the couplings as

$$\delta g^0_{abcd} = -\omega_{ae}g^0_{ebcd} - \omega_{be}g^0_{aecd} - \omega_{ce}g^0_{abed} - \omega_{de}g^0_{abce},$$

or $\delta g^0_I = -(\omega g^0)_I$ for short, where, following Jack and Osborn, we use the compact notation $I = (abcd)$. For spacetime-independent couplings the theory is renormalized by including the usual wave-function, $\phi_0 = Z\phi$, and coupling constant, $g^0_I = g_I + L_I(g)$, renormalization. But in the presence of spacetime-dependent coupling constants one must introduce new counterterms. Among them we are particularly interested in the counterterm of the form

$$\mathscr{L}_{\text{c.t.}} = (\partial^\mu g_I)(N_I)_{ab}\phi_{0b}\partial_\mu\phi_{0a}, \tag{4.2.2}$$

with $(N_I)_{ab} = -(N_I)_{ba}$, that is, in the Lie algebra of G_F; see [4] for a complete account of counterterms required in the case of spacetime-dependent couplings in a curved background.

Finite operators corresponding to currents associated with generators of G_F are most readily introduced by introducing background gauge fields. We promote the

Lagrangian (4.2.1) to

$$\widetilde{\mathscr{L}_0} = \tfrac{1}{2}g^{\mu\nu}D_{0\mu}\phi_{0a}D_{0\nu}\phi_{0a} + \tfrac{1}{12}\phi_{0a}\phi_{0a}R - \tfrac{1}{4!}g^0_{abcd}\phi_{0a}\phi_{0b}\phi_{0c}\phi_{0d}\,,$$

where the covariant derivative,

$$D_{0\mu}\phi_0 = (\partial_\mu + A_{0\mu})\phi_0,$$

is introduced with an eye towards including the counterterm (4.2.2) through the renormalization of $A_{0\mu}$,

$$A_{0\mu} = A_\mu + N_I(D_\mu g)_I, \qquad D_\mu = \partial_\mu + A_\mu.$$

We have left implicit the Lie-algebra indices (so that $N_I^T = -N_I$ and $A_\mu^T = -A_\mu$). Note that N_I is a function of the renormalized couplings that has an expansion in ϵ-poles starting at order $1/\epsilon$. If the theory contains gauge fields and some of the scalars are charged under the gauge group $G_g \subseteq G_F$, it is straightforward to include an additional quantum gauge field in addition to the background field A_μ.

The generating functional W is now a function of the background gauge field in addition to the metric and couplings, and finite operators are defined by functional differentiation:

$$\langle T_{\mu\nu}(x)\rangle = \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}(x)}, \quad \langle[\mathcal{O}_i(x)]\rangle = \frac{1}{\sqrt{-g}}\frac{\delta W}{\delta g^i(x)}, \quad \langle[\phi_a\overleftrightarrow{D}_\mu\phi_b]\rangle = \frac{1}{\sqrt{-g}}\frac{\delta W}{\delta A^\mu_{ab}(x)}\,.$$

With this formalism Jack and Osborn obtain the trace-anomaly equation [4, Eq. (6.15)]

$$T^\mu_{\ \mu} = \beta_I[\mathcal{O}_I] + \partial^\mu[(\partial_\mu\phi)^T S\phi] - ((1+\gamma)\phi)\cdot\frac{\delta}{\delta\phi}S_0,$$

where $S_0 = \int d^4x\sqrt{-g}\,\mathscr{L}_0$ and β_I, and γ are, as usual, the beta function of the coupling

g_I and the anomalous dimension of the field ϕ, respectively. We have specialized their result to the case of flat metric, spacetime-independent couplings, and vanishing background vector field. The last term, involving the functional derivative of the quantum action, vanishes by the equations of motion. The surprising aspect of this result is the often neglected term that involves the total divergence of the current $[(\partial_\mu \phi)^T S \phi]$. It is defined in terms of the G_F-Lie algebra element

$$S \equiv -g_I N_I^1,$$

where $N_I = \sum_{n=1}^{\infty} N_I^n / \epsilon^n$, so that N_I^1 is the residue of the simple ϵ-pole in N_I. Moreover, using the equation of motion (or the generalized symmetry under G_F) Jack and Osborn get [4, Eq. (6.23)]

$$T^\mu_\mu = (\beta_I - (Sg)_I)[\mathcal{O}_I] - ((1 + \gamma + S)\phi) \cdot \frac{\delta}{\delta \phi} S_0.$$

This shows that a theory is conformal provided $\beta_I - (Sg)_I = 0$. The account above is readily generalized to the case of real scalars interacting with Weyl fermions in the presence of quantum gauge fields.

In [6] we used Weyl consistency conditions [4, 5] and perturbation theory to show that S has two important properties:

1. S vanishes at fixed points. That is, if $\beta_I = 0$ then $S = 0$.

2. On cycles, defined by $\beta_I = (Qg)_I$ for Q in the Lie algebra of G_F, one has $S = Q$.

Perturbation theory is only needed to establish positivity of the natural metric in the space of operators, χ^g_{IJ} in the notation of [4]. It follows that in a theory for which $S = 0$ identically there is no possibility of limit cycles, and that conformal invariance corresponds to fixed points. We will show below this is precisely the case for supersymmetric theories.

4.3. Finding Limit Cycles

In this section we review how to determine whether the beta-function vector field of a theory admits limit cycles [2, 3, 6], making the procedure manifestly supersymmetric whenever possible. However, we often use what is known in the non-supersymmetric case to deduce what conditions have to be satisfied in the supersymmetric case.

Consider a classically scale-invariant supersymmetric field theory in four dimensions with N_f chiral superfields of mass dimension one. Classical scale invariance implies that the theory is renormalizable. The part of the Lagrangian we are interested in is[3]

$$\mathscr{L} = \int d^4\theta\, \Phi_a^\dagger \Phi_a + \left(\int d^2\theta\, \frac{1}{3!} y_{abc} \Phi_a \Phi_b \Phi_c + \text{h.c.} \right). \qquad (4.3.1)$$

There may be vector superfields in addition to the chiral superfields in (4.3.1), interacting in through a term $\Phi_a^\dagger e^V \Phi_a$ in the Kähler potential. However, we do not concern ourselves with vector superfields: their trivial flavor structure renders them unable to play a role in determining whether limit cycles exist.

The Kähler potential exhibits a continuous symmetry under transformations of the fields $\delta\Phi_a = -\omega_{ab}\Phi_b$, where ω is in the algebra of the "flavor" group $G_F = SU(N_f)$. The Yukawa couplings in the superpotential break G_F. This flavor symmetry can be extended to the whole Lagrangian by treating the coupling constants as spurions, non-dynamical fields that are allowed to transform under G_F. More specifically, the coupling constant y_{abc} is promoted to a superspace-dependent chiral superfield of mass dimension zero,

$$Y_{abc}(z) = y_{abc}(z) + \sqrt{2}\theta y_{abc}^\psi(z) + \theta^2 y_{abc}^F(z),$$

where $z^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. The y^ψ and y^F components of the spurion field are irrelevant and we ignore them in what follows. The Lagrangian (4.3.1) is manifestly G_F-symmetric

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[3]Lower case Roman letters are indices in flavor space for (anti-)chiral superfields.

if the Yukawa couplings transform as

$$\delta Y_{abc} = -\omega_{aa'}Y_{a'bc} - \omega_{bb'}Y_{ab'c} - \omega_{cc'}Y_{abc'}.$$

The theory also possesses a spurious $U(1)$ R-symmetry in addition to the G_F symmetry. The fields and couplings transform under the R-symmetry as

$$\Phi \to e^{i\alpha}\Phi, \quad \Phi^\dagger \to e^{-i\alpha}\Phi^\dagger, \quad Y \to e^{-i\alpha}Y, \quad \overline{Y} \to e^{i\alpha}\overline{Y}. \tag{4.3.2}$$

The R-symmetry is non-anomalous because the R-charge of the fermionic component of Φ is zero.

We now look for a supersymmetric version of the new type of counterterm that is required in the presence of superspace-dependent couplings, as in (4.2.2). In supersymmetric theories the only candidate for this counterterm has the form

$$\mathscr{L}_{\text{c.t.}} = \int d^4\theta \, \Phi_a^\dagger F_{ab}\Phi_b, \tag{4.3.3}$$

where F_{ab} is a function of the couplings. If the theory is to be unitary, F_{ab} must be Hermitian, $F_{ab}(Y,\overline{Y}) = F_{ba}(\overline{Y},Y) = F_{ba}^*(Y,\overline{Y})$. One can readily check that one of the components of (4.3.3) is of the form (4.2.2), that is, the product of the current associated with G_F and the derivative of the couplings

$$\mathscr{L}_{\text{c.t.}} \supset \left((N_I)_{ab}\partial^\mu y_I - (N_I)_{ba}^*\partial^\mu y_I^*\right)\left(\phi_a^*\partial_\mu\phi_b - \partial_\mu\phi_a^*\,\phi_b\right),$$

with I again a shorthand for contracted flavor indices. N can be expressed in terms of F as

$$(N_I)_{ab} = \frac{\partial F_{ab}(y,y^*)}{\partial y_I}, \qquad (N_I)_{ba}^* = \frac{\partial F_{ab}(y,y^*)}{\partial y_I^*}.$$

Both N and $F-1$ are functions of the renormalized couplings that have ϵ-pole expansions starting at order $1/\epsilon$.

4.4. Absence of Limit Cycles in Supersymmetric Theories

We are finally ready to prove at the quantum level that a unitary, $\mathcal{N} = 1$ supersymmetric field theory in four dimensions does not have limit cycles. Our strategy is to show that S is exactly zero in supersymmetric theories with the aforementioned qualifications. This we can show without recourse to perturbation theory. However, we are mindful that the proof in [6] that $S = Q$ on cycles and $S = 0$ at fixed points does rely on perturbation theory.

The expression for S in our case is

$$S_{ab} \equiv -\tfrac{1}{2}(N_I^1)_{ab} y_I - \text{h.c.}, \tag{4.4.1}$$

$$= -\frac{1}{2}\left(y_I \frac{\partial F_{ab}^1(y, y^*)}{\partial y_I} - y_I^* \frac{\partial F_{ab}^1(y, y^*)}{\partial y_I^*}\right), \tag{4.4.2}$$

where F^1 is the residue of the simple $1/\epsilon$ pole in F. The Hermitian conjugate is subtracted in (4.4.1), as expected since S is anti-Hermitian. The quantum action is invariant under the R-symmetry introduced in Section 4.3, see (4.3.2). Therefore

$$F_{ab}(Y, \overline{Y}) = F_{ab}(e^{-i\alpha}Y, e^{i\alpha}\overline{Y}),$$

or, by taking α to be infinitesimal,

$$0 = Y_I \frac{\partial F_{ab}(Y, \overline{Y})}{\partial Y_I} - \overline{Y}_I \frac{\partial F_{ab}(Y, \overline{Y})}{\partial \overline{Y}_I}.$$

Comparing the scalar component of this equation with (4.4.2) shows $S = 0$. The theory cannot exhibit renormalization group limit cycles. Furthermore, unitarity and superscale invariance imply superconformal invariance in unitary four dimensional $\mathcal{N} = 1$ supersymmetric field theories.

4.5. A Perturbative Proof and a Four-Loop Example

If S vanishes in supersymmetric theories non-perturbatively, the implication must also be true to all orders in perturbation theory. In this section we illustrate the vanishing of S in perturbation theory with a four-loop example. Remarkably, four-loop calculations in the Wess–Zumino model exist in the literature [13]. For a diagram containing only chiral superfields, it is a simple combinatoric exercise to convert the results of [13] to the model under consideration in this work.

In non-supersymmetric theories a scalar-propagator loop correction contributes to S if the corresponding diagram is not symmetric under $a \leftrightarrow b$. Such diagrams first arise at the three-loop level in ordinary field theories [6]. In $\mathcal{N} = 1$ supersymmetric Wess–Zumino models asymmetric diagrams arise at four loops, see e.g. Fig. 4.1. The
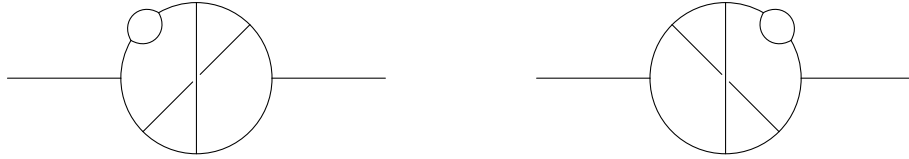
Figure 4.1: Four-loop diagrams that contribute to F that are asymmetric under exchange of the external legs. The lines are superfield propagators.

four-loop contribution of the diagrams of Fig. 4.1 to F^1 is

$$(16\pi^2)^4 F^1_{ab} \supset \tfrac{3}{8}(\zeta(3) - \tfrac{1}{2}\zeta(4))(y_{acd}y^*_{dkm}y_{fk\ell}y^*_{bef}y_{ejm}y^*_{ij\ell}y_{ghi}y^*_{cgh}$$

$$+ y_{acd}y^*_{dkm}y_{ik\ell}y^*_{ghi}y_{fgh}y^*_{bef}y_{ejm}y^*_{cj\ell}),$$

where ζ is the Riemann zeta function. From this expression for F^1 we see that S vanishes by (4.4.2). There are at least two ways to understand this diagrammatic result.

It is obvious from the form of (4.4.2) that S counts the difference in the number of y's and y^*'s in F. The non-renormalization of the superpotential guarantees that any diagram containing an unequal number of y's and y^*'s vanishes. Thus, the only diagrams that contribute to F contain an equal number of y's and y^*'s, and S must

vanish to all orders in perturbation theory. In contrast with the non-supersymmetric case, not even diagrams asymmetric under exchange of the external legs can contribute to S.

The second way in which our result can be understood is as follows. In non-supersymmetric theories momentum is allowed to flow into the diagram that gives N^1 from an external leg and out of the diagram through a coupling. If the diagram is asymmetric, then interchanging the external lines of the diagram results in a different routing of the external momentum through the diagram, and thus to a different numerical coefficient for the corresponding contribution to N^1. This leads to a nonzero contribution to S after antisymmetrization. In the supersymmetric case, however, the coefficient of all diagrams contained in the θ-expansion of an asymmetric diagram—like the one in Fig. 4.1—comes from the zeroth-order in θ diagram, which is calculated with no external momentum flowing into the diagram. Thus, there is no possibility of a contribution to S. This is true to all orders in perturbation theory.

Acknowledgments

References

[1] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`.

[2] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance at Three Loops"*, JHEP **1208**, 085 (2012), `arXiv:1202.4757`.

[3] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance in Four Dimensions"*, `arXiv:1206.2921`.

[4] I. Jack & H. Osborn, *"Analogs for the c theorem for four-dimensional renormalizable field theories"*, Nucl.Phys. **B343**, 647 (1990).

[5] H. Osborn, *"Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories"*, Nucl.Phys. **B363**, 486 (1991).

[6] J.F. Fortin, B. Grinstein & A. Stergiou, *"Limit Cycles and Conformal Invariance"*, `arXiv:1208.3674v2`.

[7] I. Antoniadis & M. Buican, *"On R-symmetric Fixed Points and Superconformality"*, Phys.Rev. **D83**, 105011 (2011), `arXiv:1102.2294`.

[8] Y. Nakayama, *"Supercurrent, Supervirial and Superimprovement"*, `arXiv:1208.4726`.

[9] S. Zheng & Y. Yu, *"Is There Scale Invariance in N=1 Supersymmetric Field Theories ?"*, `arXiv:1103.3948`.

[10] Y. Nakayama, *"Comments on scale invariant but non-conformal supersymmetric field theories"*, `arXiv:1109.5883`.

[11] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: Theoretical Foundations"*, JHEP **1207**, 025 (2012), `arXiv:1107.3840`.

[12] J. Wess & B. Zumino, *"Consequences of anomalous Ward identities"*, Phys.Lett. **B37**, 95 (1971).

[13] L. Avdeev, S. Gorishnii, A.Y. Kamenshchik & S. Larin, *"Four Loop Beta Function in the Wess-Zumino Model"*, Phys.Lett. **B117**, 321 (1982).

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Part II

Formal and Phenomenological Aspects
of Superconformal Theories

Chapter 5

Current OPEs in Superconformal Theories

5.1. Introduction

There are many examples of 4d (super)conformal theories ((S)CFTs). Some have microscopic Lagrangian descriptions, e.g. $\mathcal{N} = 1$ SQCD in the conformal window [1] or $\mathcal{N} = 4$ SYM, while others need not (e.g. [2]). Even if there is a microscopic description, it's generally of limited use, because of strong coupling effects. The "observables" of conformal theories are the spectrum of operators $\mathcal{O}_i$, their operator dimensions Δ_i, and their operator product expansion (OPE) coefficients,[1] the c_{ij}^k in

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{\mathcal{O}_k} \frac{c_{ij}^k}{x^{\Delta_i + \Delta_j - \Delta_k}} \mathcal{O}_k(0) = \sum_{\substack{\text{primary} \\ \mathcal{O}_k}} \frac{c_{ij}^k}{x^{\Delta_i + \Delta_j - \Delta_k}} F_{\Delta_i \Delta_j}^{\Delta_k}(x, P)\mathcal{O}_k(0). \quad (5.1.1)$$

Conformal symmetry implies that all local operator correlation functions are fully determined, via the OPE, by the $n \leq 3$-point functions of a subset of the operators, the primaries. In particular, conformal symmetry relates the OPE coefficients of descendant operators to those of the primaries, with determined functions $F_{\Delta_i \Delta_j}^{\Delta_k}(x, P)$ in (5.1.1). The OPE expansion (5.1.1) is exact in CFTs, and determines all correlation functions of local operators. We're here interested in 4d $\mathcal{N} = 1$ SCFTs, where the additional symmetry yields additional relations among OPE coefficients.

Conformal or approximately conformal theories are intrinsically interesting, and

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[1]There are also non-local observables, like Wilson loops, but we will not discuss them here.

have various possible applications to high energy physics and beyond the Standard Model (BSM) model building, to perhaps help mitigate various model building challenges. For example, invoking running effects with $\mathcal{O}(1)$ anomalous dimensions could help suppress or enhance otherwise finely tuned quantities or ratios. Examples include sequestering [3], achieving flavor hierarchy from anarchy [4–6], and μ/B_μ in gauge meditation[7, 8]. Furthermore, flowing near an approximate CFT could help lead to useful scale separations or interesting phenomenology, e.g. in walking technicolor or unparticles with mass gaps.

Our discussion here is particularly motivated by possible applications to general gauge mediation (GGM) [9], where one is interested in current-current two-point functions like $\langle J(x)J(0)\rangle$. 4d $\mathcal{N} = 1$ supersymmetry conserved currents j_μ reside in real supermultiplets

$$\mathcal{J}(z) = J(x) + i\theta j(x) - i\bar{\theta}\bar{\jmath}(x) - \theta\sigma^\mu\bar{\theta}j_\mu(x) + \cdots, \qquad (5.1.2)$$

where $\cdots$ are derivative terms, following from the conservation equations $D^2\mathcal{J} = \bar{D}^2\mathcal{J} = 0$. The operator[2] $J(x) = \mathcal{J}|$ is a real superconformal primary, with dimension $\Delta_J = 2$, and the conserved current $j_\mu(x)$ is among its descendants. Here $j_\mu(x)$ is a global current of the CFT (that could later be weakly gauged as in GGM). With this application in mind, we will here consider general aspects of the super OPEs of these operators in 4d $\mathcal{N} = 1$ SCFTs. We will discuss applications to GGM in detail in a separate paper [10].

The leading short-distance terms in the OPE of $\mathcal{J}(z)$ with operators have universal coefficients, fixed in terms of the charges. As we'll recall, this is similar to the universal coefficients in OPEs involving the conserved $\mathcal{T}_\mu(z)$ $U(1)_R$-plus-stress-energy-tensor supermultiplet[11] of SCFTs, which was considered e.g. in [12–14]. The leading terms in the OPE of the bottom, primary component of currents with themselves take

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[2] We use | to denote the bottom component, setting all $\theta, \bar{\theta} = 0$.

the form

$$J_a(x)J_b(0) = \tau \frac{\delta_{ab}\mathbb{1}}{16\pi^4 x^4} + \frac{k d_{abc}}{\tau} \frac{J_c(0)}{16\pi^2 x^2} + f_{abc} \frac{x^\mu j_\mu^c(0)}{8\pi^2 x^2} + c_{ab}^i \frac{\mathcal{O}_i(0)}{x^{4-\Delta_i}} + \cdots, \qquad (5.1.3)$$

with a an adjoint index for the (say simple) group G. In what follows, we often suppress the group adjoint index, or simply take $G = U(1)$ since the generalization is fairly straightforward. For the moment, we just want to illustrate a point with the symmetric d_{abc} and the structure function terms f_{abc} in (5.1.3).

Conformal symmetry relates terms in the OPE. In the non-SUSY case, the coefficients of all descendant operators are fully determined from those of the primary operators, as was worked out (in many different ways) in the 1970s, see e.g. [15]. It is natural to expect that (i) the SUSY version should be completely analogous and (ii) that it must have long ago been worked out for general operators. But both statements are untrue! This follows from the works of Hugh Osborn and collaborators, but it has not been very explicitly discussed in the literature, and it comes as an initial surprise to many experts.

The OPE is related to operator two- and three-point functions, and the fact that non-SUSY conformal descendant terms are uniquely characterized by the primaries is related to the fact that conformal symmetry can be used to map any three operator-insertion points $x_{1,2,3}^\mu$ to wherever one pleases. The constraints of (non-SUSY) conformal symmetry on operator two- and three-point functions, in general spacetime dimension d, were studied in [16], including the additional constraints coming from Ward identities for conserved quantities like j_μ or $T_{\mu\nu}$.

That the OPE coefficients of superconformal primaries are generally *not* sufficient to determine those of the superdescendants can likewise be understood from their relation to operator two- and three-point functions. The 4d $\mathcal{N} = 1$ superconformal constraints on operator two- and three-point functions were analyzed, using a superspace analysis by Osborn [14], and we'll here review, and heavily use, his framework. A quick way to understand why superdescendant three-point functions are generally not fully

determined by the primaries is to note that $\mathcal{N} = 1$ supertranslations and superconformal transformations only suffice to eliminate the Grassmann coordinates at two points in superspace—the third Grassmann coordinate in three-point functions remains. This explains the existence of the nilpotent three-point function superconformal invariant building blocks, Θ and $\bar{\Theta}$, found in superspace in [14] (see also [17]).

As an illustration, consider the superspace expression for current three-point functions [14], capturing the G structure functions f_{abc} and $\mathrm{Tr}\, G^3$ 't Hooft anomaly k,

$$\langle \mathcal{J}_a(z_1)\mathcal{J}_b(z_2)\mathcal{J}_c(z_3) \rangle = \frac{1}{x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2} \left[i\frac{f_{abc}\tau}{128\pi^6}\left(\frac{1}{X_3^2} - \frac{1}{\bar{X}_3^2}\right) + \frac{d_{abc}k}{256\pi^6}\left(\frac{1}{X_3^2} + \frac{1}{\bar{X}_3^2}\right)\right]$$

$$(5.1.4)$$

with notation reviewed in section 5.3.1. For now we will just say that $X - \bar{X} = 4i\Theta\bar{\Theta}$, with $\Theta \sim \theta$'s in superspace. The f_{abc} terms in (5.1.4) do not contribute if we restrict (via $\theta \to 0$) to superconformal primary components, but do contribute for superdescendants. Explicitly, in (5.1.3), the f_{abc} term is a descendant coefficient that is unrelated to the kd_{abc} primary coefficient. In (5.1.4) the Θ dependence is at least determined by G symmetry. For general operators, the Θ dependence is ambiguous, not fully determined by the symmetries.

We will here study the general constraints of superconformal symmetry on the two- and three-point functions relevant for the $J(x)J(0)$ sOPE, and how the sOPE coefficients are obtained from these correlators. We will do this both using the superspace results of Osborn [14] for the relevant two- and three-point functions, and also directly from the superconformal algebra. As we'll discuss, the fact that the currents are conserved here allows the superspace Θ dependence to be completely fixed. Thus, the coefficients of the superconformal primaries in the $J(x)J(0)$ OPE suffice to fully determine all OPE coefficients of all descendants. We will also show that the only operators contributing on the RHS of the $J(x)J(0)$ OPE are integer-spin real $U(1)_R$-charge-zero superconformal primaries, $\mathcal{O}^{\mu_1\cdots\mu_\ell}$, and their superdescendants.

The paper is organized as follows: section 5.2 briefly reviews the aspects of the

OPE in 4d CFTs that we will use in the following discussion. Section 5.3 discusses superconformal theories, and the constraints of superconformal symmetry on two- and three-point functions and the OPE. The superspace formalism of [14], and the recent results about chiral-chiral and chiral-anti-chiral OPEs [18–20], are reviewed. In section 5.4 we consider the current-current OPE, showing how the additional constraints of the current's conservation constrains the $\langle JJ\mathcal{O}\rangle$ three-point functions, and hence the OPE. We show that only real, $U(1)_R$-charge zero, integer-spin operators $\mathcal{O}^{(\ell)}$, and their superconformal descendants, can appear on the RHS of the $J(x)J(0)$ OPE. We show that the OPE coefficients within each supermultiplet are fully specified by a single OPE coefficient. The dependence on the nilpotent invariant Θ mentioned above is here fully determined by the $\mathcal{J}$ current conservation.

In section 5.5 we discuss aspects of four-point functions and their conformal blocks, where the four-point function is factorized into an OPE sum of intermediate operators, and their descendants, in the s, t, or u channel. In $\mathcal{N}=0$ theories, the contribution of an intermediate primary operator of dimension Δ and spin ℓ is given by a known function [21], $g_{\Delta,\ell}(u,v)$, which accounts for the sum over descendants and is independent of the external operators. There is no general analog of such a general "superconformal block" in SCFTs, because of the generally ambiguous dependence on the super-descendants in the sOPE. This ambiguity is resolved when the external operators are in reduced multiplets, in particular the chiral and anti-chiral multiplets discussed in [19] and the conserved currents discussed here. The superconfomal blocks, then, depend on the type of external states. We review the results of [19] for $\mathcal{N}=1$ superconformal blocks $\mathcal{G}_{\Delta,\ell}^{\phi\phi^*;\phi\phi^*}$, and briefly mention how $\mathcal{G}_{\Delta,\ell}^{\phi\phi;\phi^*\phi^*}$ differs. Then we discuss the $\mathcal{N}=1$ superconformal blocks for $\mathcal{G}_{\Delta\ell}^{JJ;JJ}$ and $\mathcal{G}_{\Delta,\ell}^{JJ;\phi\phi^*}$. Finally, we discuss these quantities in $\mathcal{N}=2$ SCFTs, where they are related by the additional $SU(2)_I$ symmetry.

Section 5.6 summarizes our findings and discusses possible applications of the results. Finally, appendix 5.A summarizes some of the relations of the (super)conformal algebra, and our sign conventions.

5.2. Review of OPE results in the non-SUSY case

Aspects of CFTs and the OPE are discussed in many references and reviews. We will here review, for completeness, some of the main points for our later use. We summarize the algebra and our sign conventions in appendix 5.A.

5.2.1. Primaries, descendants and their two- and three-point functions

Representations of the conformal group are built by regarding P_μ and K_μ as raising and lowering operators, respectively; they raise or lower operator dimension by one unit. Each irreducible representation has a lowest, "quasi-primary" operator at the bottom, which is annihilated by all lowering operators at the origin, $x^\mu = 0$. (The origin is a distinguished point, as the fixed point of scale transformations.) The quasi-primary has an associated tower of "descendant" operators above it, generated by $[P_\mu, \star]$; this accounts for the fact that the operators can anyway be translated to a general point via $\mathcal{O}^I(x) = e^{-iP \cdot x} \mathcal{O}^I(0) e^{iP \cdot x}$.

Conformal symmetry completely determines the form of the $n \leq 3$-point functions, in terms of the operator dimensions, up to the overall normalization coefficients. This follows from the fact that conformal transformations can be used to map any three points $x_{1,2,3}^\mu$ to wherever one pleases. For example, we can use translation symmetry to map $x_1^\mu = 0$, and special conformal symmetry to make $x_3^\mu = \infty$, and then use Lorentz symmetry and dilatations to map x_2^μ to a canonical unit vector.

Scale invariance implies that the only non-zero one-point function is that of the identity operator, which is the only operator with $\Delta_\mathcal{O} = 0$:

$$\langle \mathcal{O}_a(x) \rangle = \delta_{a,0}, \qquad \mathcal{O}_0 \equiv \mathbb{1}.$$

The two-point functions of primary operators take the form

$$\langle \mathcal{O}_i^{s_i}(x_i) \mathcal{O}_j^{s_j}(x_j) \rangle = \frac{c_{ij}}{r_{ij}^{\Delta_i}} P^{s_i s_j}(x_{ij}), \qquad x_{ij}^\mu \equiv x_i^\mu - x_j^\mu, \qquad r_{ij} \equiv x_{ij}^2. \tag{5.2.1}$$

Here c_{ij} are constant normalization coefficients, the analog of the Zamolodchikov metric on the space of deformations in 2d. Conformal symmetry implies that c_{ij} vanish unless the two operators have the same operator dimension, $c_{ij} \propto \delta_{\Delta_i, \Delta_j}$, and of course the same spin. The $s_{i,j}$ in (5.2.1) are Lorentz indices and $P^{s_i s_j}(x)$ is an appropriate representation of the rotation group, e.g. $P = 1$ for scalars or, taking both operators to have spin ℓ, with $s_i = (\mu_1 \ldots \mu_\ell)$ and $s_j = (\nu_1 \ldots \nu_\ell)$, both symmetrized and traceless [16],

$$P^{s_i s_j}(x) = I^{(\mu_1 \nu_1}(x) \cdots I^{\mu_\ell \nu_\ell)}(x), \qquad I^{\mu\nu}(x) \equiv \eta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2},$$

with the Lorentz indices symmetrized and traceless.

Conformal symmetry implies that primary operator three-point functions have the form

$$\langle \mathcal{O}_i^{s_i}(x_i) \mathcal{O}_j^{s_j}(x_j) \mathcal{O}_k^{s_k}(x_k) \rangle = \frac{c_{ijk}}{r_{ij}^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)} r_{ik}^{\frac{1}{2}(\Delta_i + \Delta_k - \Delta_j)} r_{jk}^{\frac{1}{2}(\Delta_j + \Delta_k - \Delta_i)}} P^{s_i s_j s_k}(x),$$

(5.2.2)

where c_{ijk} are constants and $P^{s_i s_j s_k}(x)$ is a fixed tensor depending on the Lorentz spins of the operators, e.g. $P = 1$ for scalar operators, that is determined in [16]. Of course, (5.2.2) reduces to (5.2.1) if any of the operators is the identity, so $c_{0ij} = c_{ij}$. A case of particular interest here is for two scalar primaries and one spin-ℓ primary operator, where the explicit form of (5.2.2) is

$$\langle \mathcal{O}_i(x_i) \mathcal{O}_j(x_j) \mathcal{O}_k^{(\mu_1 \ldots \mu_\ell)}(x_k) \rangle =$$

$$\frac{c_{ijk}}{r_{ij}^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k + \ell)} r_{ik}^{\frac{1}{2}(\Delta_k + \Delta_{ij} - \ell)} r_{jk}^{\frac{1}{2}(\Delta_k - \Delta_{ij} - \ell)}} Z^{(\mu_1} Z^{\mu_2} \cdots Z^{\mu_\ell)},$$

(5.2.3)

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$, and

$$Z^\mu \equiv \frac{x_{ki}^\mu}{r_{ik}} - \frac{x_{kj}^\mu}{r_{jk}}, \qquad Z^2 = \frac{r_{ij}}{r_{ik} r_{jk}}, \tag{5.2.4}$$

which is called X^μ_{ji} in the notation of [16] and $X^\mu_k|_{\theta,\bar\theta=0}$ in the notation of [14] that we'll use shortly.

The primary two- and three-point functions (5.2.1) and (5.2.2) fully determine those of all descendants. For example, we can replace $\mathcal{O}^{s_j}_j(x_j)$ with $[P_\mu, \mathcal{O}^{s_j}_j(x_j)] = i\partial_\mu \mathcal{O}^{s_j}_j(x_j)$ in (5.2.1) and (5.2.2) simply by taking $i\partial/\partial x^\mu_j$ of the LHS.

The above expressions can be written in terms of (radial quantization) states: using translation symmetry to map $x_i \to 0$, the (say scalar) operator $\mathcal{O}_i(x_i)$ creates an in-state,

$$\lim_{x_i \to 0} \mathcal{O}_i(x_i)|0\rangle = |\mathcal{O}_i\rangle. \tag{5.2.5}$$

Using conformal symmetry to map $x_j \to \infty$, $\mathcal{O}_j(x_i)$ likewise creates an out-state,

$$\lim_{x_j \to \infty} \langle 0|\mathcal{O}_j(x_j)x_j^{2\Delta_j} = \langle \mathcal{O}_j|, \tag{5.2.6}$$

where the $x_j^{2\Delta_j}$ factor follows, for example, via an inversion, $x'_\mu = x_\mu/x^2$, with $\mathcal{O}'_j(x') = \Omega^{\text{inv}}(x)^{\Delta_j}\mathcal{O}_j(x)$, $\Omega^{\text{inv}}(x) = x^2$ (see appendix 5.A), which maps (5.2.5) to (5.2.6). Then, (5.2.1) and (5.2.3) give (taking $(x_i, x_j, x_k) \to (0, \infty, x)$, (5.2.4) gives $Z^\mu \to x^\mu/x^2$)

$$\langle \mathcal{O}_j|\mathcal{O}_i\rangle = c_{ij},$$

$$\langle \mathcal{O}_j|\mathcal{O}^{(\mu_1...\mu_\ell)}_k(x)|\mathcal{O}_i\rangle = \frac{c_{ijk}}{(x^2)^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_k+\ell)}}x^{(\mu_1}\cdots x^{\mu_\ell)}.$$

5.2.2. The OPE; descendants from primaries

The OPE contains precisely the same information as the two- and three-point functions:

$$\mathcal{O}^{s_i}_i(x_i)\mathcal{O}^{s_j}_j(x_j) = \frac{c_{ij}P^{s_is_j}(x_{ij})}{r^{\Delta_i}_{ij}}\mathbb{1} + \sum_{k'} \frac{c^{k'}_{ij}}{r^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_{k'})}_{ij}}[F^{k'}_{ij}(x_{ij}, P), \mathcal{O}_{k'}]^{(s_{k'})}(x_j). \tag{5.2.7}$$

The function $F^{k'}_{ij}(x_{ij}, P)$ gives the coefficients of the descendant operators and depends only on the operator dimensions $\Delta_{i,j,k'}$ and spins $s_{i,j,k'}$. Taking expectation values

of both sides yields (5.2.1) from the unit operator $\mathcal{O}_0 \equiv \mathbb{1}$ on the RHS of (5.2.7), so $c_{ij} = c_{ij}^0$.

To relate the OPE (5.2.7) to the three-point functions (5.2.2) we multiply both sides of (5.2.7) by $\mathcal{O}_k^{s_k}(x_k)$ and then, taking the expectation value, use (5.2.1) to evaluate the remaining two-point function $\langle \mathcal{O}_{k'}^{s_{k'}}(x_j) \mathcal{O}_k^{s_k}(x_k) \rangle$. This gives the relation

$$c_{ijk} = c_{ij}^{k'} c_{kk'}, \qquad \text{or equivalently} \qquad c_{ij}^k = c_{ijk'} c^{k'k} \qquad \text{for primaries,} \qquad (5.2.8)$$

where $c^{kk'} c_{k'm} = \delta_m^k$, summing the dummy index k'. It follows from (5.2.8) that, e.g.

$$c_{ij}^k = c^{k\ell} c_{jm} c_{i\ell}^m. \qquad (5.2.9)$$

The relations (5.2.8) follow from matching the OPE (5.2.7) to merely the leading $x_{ij} \to 0$ dependence in the three-point functions (5.2.2). This leading dependence comes from restricting to primary operators on the RHS of the OPE, dropping the $[P_\mu, \star]$ descendant terms. Matching to the full x_{ij}, x_{jk}, and x_{ik} dependence in (5.2.2) will determine the coefficients of all the $[P_\mu, \star]$ descendant terms, i.e. the function $F_{ij}^k(x_{ij}, P)$, in the OPE (5.2.7). These functions incorporate also the spin dependence, which is a complication that we won't need to deal with in full generality. It'll suffice here to focus on the OPE of scalar operators.

Consider then the OPE of two scalar operators, which generally includes non-zero integer-spin-ℓ primary operators $\mathcal{O}_{k'}^{(\mu_1...\mu_\ell)}$ (with symmetrized indices) on the RHS,

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_{\mathcal{O}_{k'}^\ell} \frac{c_{ij}^{k'}}{r_{ij}^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_{k'})}} F_{\Delta_i\Delta_j}^{\Delta_{k'};\ell}(x_{ij}, P)_{\mu_1...\mu_\ell} \mathcal{O}_{k'}^{(\mu_1...\mu_\ell)}(x_j). \qquad (5.2.10)$$

The (odd) even spin ℓ terms are (anti-) symmetric under $\mathcal{O}_i \leftrightarrow \mathcal{O}_j$. For simplicity, consider first the spin $\ell = 0$ primary operators on the RHS,

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \supset \sum_{\mathcal{O}_{k'}^{\ell=0}} \frac{c_{ij}^{k'}}{r_{ij}^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_{k'})}} F_{\Delta_i\Delta_j}^{\Delta_{k'}}(x_{ij}, P) \mathcal{O}_{k'}(x_j). \qquad (5.2.11)$$

The function $F^{\Delta_{k'}}_{\Delta_i \Delta_j}(x, P)$ satisfies $F^{\Delta_{k'}}_{\Delta_i \Delta_j}(x = 0, P) = 1$, to give the leading $x_{ij} \to 0$ singularity from the primary $\mathcal{O}_{k'}$. The higher-order terms in F account for the OPE coefficients of $\mathcal{O}_{k'}$'s descendants, which are fully determined by the conformal symmetry; reproducing the three-point functions gives one derivation [15]: we multiply (5.2.11) by $\mathcal{O}_k(x_k)$ and take expectation values of the resulting two-point function using (5.2.1), with $P = 1$ for this scalar case, and then require that the result reproduces the three-point functions (5.2.2), again with $P = 1$. This determines that, for this scalar case,

$$F^{\Delta_k}_{\Delta_i \Delta_j}(x_{ij}, P \to i\partial_{x_j}) = C^{\frac{1}{2}(\Delta_k + \Delta_i - \Delta_j), \frac{1}{2}(\Delta_k - \Delta_i + \Delta_j)}(x_{ij}, \partial_{x_j}), \qquad (5.2.12)$$

where the function on the RHS is defined to be the solution of

$$C^{ab}(x_{ij}, \partial_{x_j}) \frac{1}{r_{jk}^{a+b}} = \frac{1}{r_{ik}^a r_{jk}^b}, \qquad (5.2.13)$$

(see e.g. [18] for details, as well as the generalization for the general spin-ℓ operators) such that (5.2.10) reproduces the three-point functions (5.2.3).

One can also obtain the functions $F^{\Delta_k}_{\Delta_i \Delta_j}(x, P)$ that capture the descendant OPE coefficients by requiring that $[K_\mu, \star]$ gives the same result when taking $\star =$ the LHS and the RHS of (5.2.11). Using the algebra and action of K_μ, given in appendix 5.A, this gives

$$i(x^2 \partial_\mu - 2x_\mu x \cdot \partial - 2\Delta_i x_\mu) \left(\frac{F^{\Delta_k}_{\Delta_i \Delta_j}(x, P)}{(x^2)^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)}} \right) = \frac{1}{(x^2)^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)}} [K_\mu, F^{\Delta_k}_{\Delta_i \Delta_j}(x, P)],$$
$$(5.2.14)$$

treating the primaries $\mathcal{O}_k$ as a basis of independent operators. This equation can be solved exactly, see the original papers [15]. As an expansion in powers of x, it is straightforward to use the algebra to see that (5.2.14) is solved by

$$F^{\Delta_k}_{\Delta_i \Delta_j}(x, P) = 1 - \frac{i}{2} \left(\frac{\Delta_k + \Delta_i - \Delta_j}{\Delta_k} \right) x \cdot P + \cdots.$$

5.2.3. Conserved-current leading OPE singularities from their charges

The normalization of conserved currents, their leading OPE with other operators and themselves, is determined in terms of the operator's conserved-charge value. Conserved currents $j_\mu^a(x)$ are real, spin-$\ell = 1$, $\Delta_{j^\mu} = 3$ operators. For simplicity, consider first the case of a $U(1)$ current, $j_\mu(x)$, in the three-point function with a scalar operator of $U(1)$ charge $q_\mathcal{O}$,

$$\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)j^\mu(x_3)\rangle = -iq_\mathcal{O}\frac{c_{\mathcal{O}\mathcal{O}^\dagger}}{2\pi^2}\frac{Z^\mu}{r_{12}^{\Delta_\mathcal{O}-1}r_{13}r_{23}}, \tag{5.2.15}$$

where we use (5.2.3). The i is needed for j^μ to assign the correct charge to the operator, and it ensures that (5.2.15) is Hermitian with the exchange $x_1 \leftrightarrow x_2$, which takes $Z^\mu \to -Z^\mu$. More generally, the OPE of a conserved current $j_\mu^a(x)$ with primary operator $\mathcal{O}_I(x)$ (a is an adjoint index and I runs over $\mathcal{O}$'s representation) is

$$j_\mu^a(x)\mathcal{O}_I(0) = -i(t_\mathcal{O}^a)_{IJ}\frac{x_\mu}{2\pi^2 x^4}\mathcal{O}_J(0) + \text{less singular}, \tag{5.2.16}$$

where $t_\mathcal{O}^a$ is the representation of the operator; for a $U(1)$ current, $t_\mathcal{O} = q_\mathcal{O}$ the $U(1)$ charge, and we take $\mathcal{O}$ to be a Lorentz scalar for simplicity. For an operator J^b in the adjoint representation, $(t^a)_{bc} = if_{abc}$ so (5.2.16) becomes

$$j_\mu^a(x)J^b(0) = f^{abc}\frac{x_\mu}{2\pi^2 x^4}J_c(0) + \text{less singular}. \tag{5.2.17}$$

Using (5.2.9) with (5.2.16) determines the coefficient of j_μ^a on the RHS of the $\mathcal{O}_I^\dagger(x)\mathcal{O}_J(0)$ OPE. In particular, (5.2.17) leads to the f_{abc} term on the RHS of (5.1.3).

The OPE of the stress-energy tensor with the operator is [16]

$$T_{\mu\nu}(x)\mathcal{O}(0) = -2\Delta_\mathcal{O}\frac{x^\mu x^\nu - \frac{1}{4}\eta_{\mu\nu}x^2}{3\pi^2 x^6}\mathcal{O}(0) + \text{less singular}. \tag{5.2.18}$$

It follows from (5.2.15) and (5.2.18) and (5.2.9) that (using $c_{TT} = 40c/\pi^4$)

$$\mathcal{O}^\dagger(x)\mathcal{O}(0) = \frac{c_{\mathcal{O}^\dagger\mathcal{O}}}{x^{2\Delta_\mathcal{O}}}\mathbb{1} - iq_\mathcal{O}\frac{2\pi^2}{3\tau}\frac{c_{\mathcal{O}^\dagger\mathcal{O}}x_\mu}{x^{2(\Delta_\mathcal{O}-1)}}j^\mu(0) + \Delta_\mathcal{O}\frac{\pi^2}{60c}\frac{c_{\mathcal{O}^\dagger\mathcal{O}}x_\mu x_\nu}{x^{2(\Delta_\mathcal{O}-1)}}T_{\mu\nu}(0) + \dots \quad (5.2.19)$$

These relations between the leading singularities and the charges can be shown, much as in 2d, by computing the charge operator by integrating the current over a spatial S^3 in radial quantization, and then using the OPE where it hits the other operators. Properly regulated, this yields the commutator of the charge with the operator and the leading singularity gives the operator's charge value. Equivalently, the leading term coefficients in (5.2.16) and (5.2.18) are fixed as they give the correct contact terms in the conserved current's Ward identities for $\partial^\mu j_\mu$, $\partial^\mu T_{\mu\nu}$, and $T_\mu{}^\mu$. This can be shown [16] by treating the $x \to 0$ singularities in (5.2.16) and (5.2.18) with differential regularization [22]:

$$\mathscr{R}\left(\frac{1}{x^{2\eta}}\right) = \frac{1}{x^{2\eta}} - \frac{\mu^{2\eta-4}}{4-2\eta}2\pi^2\delta^{(4)}(x) = -\frac{1}{4-2\eta}\partial^2\left(\frac{1}{2\eta-2}\frac{1}{x^{2\eta-2}} - \frac{\mu^{2\eta-4}}{2}\frac{1}{x^2}\right),$$

and for $2\eta \to 4$,

$$\mathscr{R}\left(\frac{1}{x^4}\right) = -\frac{1}{4}\partial^2\left(\frac{1}{x^2}\ln(\mu^2 x^2)\right).$$

The normalization of the currents is fixed by the above conditions, that their OPEs with operators give the correct operator charges. The leading singularities in the self-OPEs $j_\mu^a(x)j_\nu^b(0)$ and $T_{\mu\nu}(x)T_{\rho\sigma}(0)$ are similarly determined from Ward identity contact terms. The current-current OPE leading terms are

$$j_\mu^a(x)j_\nu^b(0) = 3\tau^{ab}\frac{I_{\mu\nu}(x)}{4\pi^4 x^6}\mathbb{1} + 2f_c^{ab}\frac{x_\mu x_\nu x^\kappa}{\pi^2 x^6}j_\kappa^c(0) + kd_c^{ab}\frac{D_{\mu\nu}{}^{\kappa\lambda}(x)x_\lambda}{8\pi^2 x^4}j_\kappa^c(0) + \cdots,$$

where f_c^{ab} are the group structure constants, and kd_c^{ab} is the coefficient of the $\mathrm{Tr}\,G^3$ 't Hooft anomaly. The leading terms in the stress-tensor self-OPE are more involved to write out, because of all the indices, see [16]. The terms $\sim 1/x^n$ for integer n contribute to the conformal anomaly $\langle T_\mu{}^\mu\rangle$ when the operators are coupled to background sources,

see e.g. [23] for a nice discussion. In particular, $\tau^{ab} = \tau \delta^{ab}$ gives the contribution to $\langle T_\mu{}^\mu \rangle$ when $j^a_\mu(x)$ are coupled to external sources $A^a_\mu(x)$, which shows that τ gives the contribution to the one-loop beta function for the gauge coupling if the G symmetry is weakly gauged.

5.3. 4d $\mathcal{N} = 1$ SCFT primaries, descendants, and OPEs

The $\mathcal{N} = 1$ superconformal algebra (isomorphic to $SU(2,2|1)$) extends the conformal algebra with the supercharges Q_α and $\bar{Q}_{\dot\alpha}$, the superconformal supercharges, S^α and $\bar{S}^{\dot\alpha}$, and the $U(1)_R$-generator, R. (See appendix 5.A for more details about the algebra.)

Representations are formed by regarding P_μ, Q_α, and $\bar{Q}_{\dot\alpha}$ as the raising operators, and K_μ, S^α, $\bar{S}^{\dot\alpha}$ as the corresponding lowering operators. If an operator $\mathcal{O}$ has (Δ, r) for its operator dimension and R-charge, respectively, then $Q_\alpha(\mathcal{O}) \equiv [Q_\alpha, \mathcal{O}\}$ has $(\Delta + \frac{1}{2}, r - 1)$ and e.g. $S^\alpha(\mathcal{O}) \equiv [S^\alpha, \mathcal{O}\}$ has $(\Delta - \frac{1}{2}, r + 1)$. The superconformal
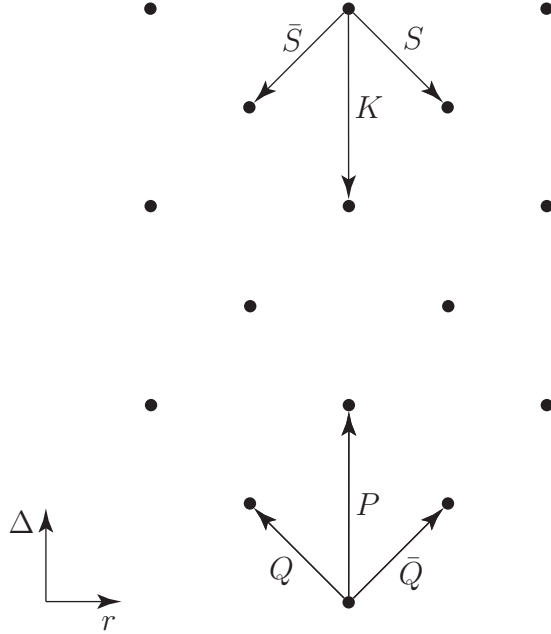
Figure 5.1: A representation of the superconformal group.

quasi-primary operators are at the bottom of the representations, annihilated by all

lowering operators at the origin, $x^\mu = 0$. Each superconformal quasi-primary has a tower of superconformal descendant operators above it, obtained by acting with the raising operators; this is represented by the dots in Fig. 5.1, with the superconformal quasi-primary operator at the bottom.[3] The other operators on the bottom left and right edges, e.g. $Q_\alpha(\mathcal{O})$, are conformal primaries but superconformal descendants.

Every SCFT has a superconformal $U(1)_R$-plus-stress-energy-tensor supermultiplet [11]

$$\mathcal{T}_\mu(z) = j_\mu^R(x) + \theta^\alpha \mathcal{S}_{\alpha\mu}(x) + \bar{\theta}^{\dot{\alpha}} \bar{\mathcal{S}}_{\dot{\alpha}\mu}(x) + 2\theta\sigma^\nu\bar{\theta} T_{\nu\mu}(x) + \cdots, \qquad (5.3.1)$$

where the $\cdots$ are derivative terms, determined by the conservation equation $\bar{D}^{\dot{\alpha}}\mathcal{T}_{\alpha\dot{\alpha}} = 0$. The primary component $j_\mu^R(x) = \mathcal{T}_\mu|$ is the conserved superconformal $U(1)_R$ symmetry current, with $\Delta_{j_\mu^R} = 3$. The supercurrents $\mathcal{S}_\mu^\alpha(x)$, $\bar{\mathcal{S}}_\mu^{\dot{\alpha}}(x)$, and the stress-energy tensor $T_{\mu\nu}(x)$ are among its descendants. The leading short distance singular terms in the OPE of $\mathcal{T}_\mu(z)$ with other operators, including itself, have coefficients with interesting universality [12] interpretations, fixed in terms of the dimension and R-charges of the operators, 't Hooft anomalies, and the central charges a and c. The supersymmetry relations among the j_R^μ and $T_{\mu\nu}$ operators in (5.3.1) then yields the relations of [13] and [14] between the central charges and the $U(1)_R$ 't Hooft anomalies.

Knowing how the superconformal generators act on the operator representations at $x^\mu = 0$, their action at a general point x^μ follows from $\mathcal{O}^I(x) = e^{-iP\cdot x}\mathcal{O}^I(0)e^{iP\cdot x}$ and the algebra. For example, for a scalar superconformal primary, it follows that

$$[S^\alpha, \mathcal{O}(x)] = ix \cdot \bar{\sigma}^{\dot{\alpha}\alpha}[\bar{Q}_{\dot{\alpha}}, \mathcal{O}(x)].$$

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[3]In special cases some superconformal descendants are also primaries, i.e. annihilated by the lowering operators. Such operators are zero-norm null states, that must be set to zero, leading to a truncated representation. Examples are chiral primary operators $\mathcal{O}$, where $\bar{Q}_{\dot{\alpha}}(\mathcal{O}) \equiv [\bar{Q}_{\dot{\alpha}}, \mathcal{O}]$ is null, and (semi-)conserved currents J, where $Q^2(J) \equiv \{Q^\alpha, [Q_\alpha, J]\}$ is null.

As another example, raising and then lowering a scalar superconformal primary yields

$$S^\beta Q_\alpha(\mathcal{O}(x)) = 2(\sigma^{\mu\nu}{}_\alpha{}^\beta x_{[\mu}\partial_{\nu]} + \delta_\alpha{}^\beta x\cdot\partial)\mathcal{O}(x) - ix\cdot\bar\sigma^{\dot\alpha\beta}Q_\alpha\bar{Q}_{\dot\alpha}(\mathcal{O}(x)) + (2\Delta_{\mathcal{O}} + 3r_{\mathcal{O}})\delta_\alpha{}^\beta\mathcal{O}(x),$$

$$(5.3.2)$$

where, again, we define $S^\beta Q_\alpha(\mathcal{O}(x)) \equiv \{S^\beta, [Q_\alpha, \mathcal{O}(x)]\}$.

Considering (5.3.2) at $x^\mu = 0$, it's seen that $Q^\alpha(\mathcal{O}(0))$ is null only if $\Delta_{\mathcal{O}} = -\frac{3}{2}r_{\mathcal{O}}$; these are the anti-chiral primaries. Similarly, it follows from $S^\alpha Q^2(\mathcal{O}(0)) = 2[2(2 - \Delta_{\mathcal{O}}) - 3r_{\mathcal{O}}]Q^\alpha(\mathcal{O}(0))$, that $Q^2(\mathcal{O})$ is null only if $\Delta_{\mathcal{O}} = 2 - \frac{3}{2}r_{\mathcal{O}}$. Likewise, $\bar{Q}^2(\mathcal{O})(0)$ is null only if $\Delta_{\mathcal{O}} = 2 + \frac{3}{2}r_{\mathcal{O}}$. Conserved current operators satisfy both conditions,

$$Q^2(J(x)) = \bar{Q}^2(J(x)) = 0,$$

and so $\Delta_J = 2$ and $r_J = 0$. The scalar primary operator $J(x)$ has the conserved current j_μ as a superpartner descendant, $j_\mu(x) = -\frac{1}{4}\bar\sigma_\mu^{\dot\alpha\alpha}[Q_\alpha, \bar{Q}_{\dot\alpha}]J(x)$.

One might anticipate that, much as in (5.2.11), the OPE for all operators is completely determined by those for the superconformal primaries,

$$\mathcal{O}_i(x)\mathcal{O}_j(0) \overset{?}{=} \sum_{\substack{\text{sprimary}\\\mathcal{O}_k}} \frac{c_{ij}^k}{(x^2)^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)}} F_{ij}^k(x, P, Q, \bar{Q})\mathcal{O}_k(0), \qquad (5.3.3)$$

where "sprimary" is shorthand for "superconformal primary", with the superconformal descendant OPE coefficients completely determined from those of the superconformal primaries. But as we mentioned after (5.1.4), this is generally incorrect. This is already known, but perhaps not widely so. We can illustrate an example of from what we've discussed so far: consider the OPE $\mathcal{O}^\dagger(x)\mathcal{O}(x)$, where $\mathcal{O}$ is a scalar operator with superconformal $U(1)_R$ charge $r_{\mathcal{O}}$ and dimension $\Delta_{\mathcal{O}}$. It follows from (5.2.19) that

$$\mathcal{O}^\dagger(x)\mathcal{O}(0) \supset -ir_{\mathcal{O}}\frac{\pi^2}{8c}\frac{c_{\mathcal{O}^\dagger\mathcal{O}}x_\mu}{x^{2(\Delta_{\mathcal{O}}-1)}}j_R^\mu(0) + \Delta_{\mathcal{O}}\frac{\pi^2}{60c}\frac{c_{\mathcal{O}^\dagger\mathcal{O}}x^\mu x^\nu}{x^{2(\Delta_{\mathcal{O}}-1)}}T_{\mu\nu}(0) + \cdots, \qquad (5.3.4)$$

where we used the supersymmetry relation between the coefficient τ_{RR} of the j_R^μ two-

point function and the conformal anomaly c, $\tau_{RR} = 16c/3$ (see e.g. [24]). Equivalently,

$$\mathcal{T}_\mu(z)\mathcal{O}(0) \supset \left(-ir_\mathcal{O}\frac{x_\mu}{2\pi^2 x^4} - 4\Delta_\mathcal{O}\frac{1}{3\pi^2 x^6}\theta\sigma^\nu\bar{\theta}(x_\mu x_\nu - \tfrac{1}{4}x^2\eta_{\mu\nu}) \right)\mathcal{O}(0) + \cdots. \quad (5.3.5)$$

For a general operator $\mathcal{O}$, the coefficients $r_\mathcal{O}$ and $\Delta_\mathcal{O}$ in (5.3.4) or (5.3.5) are not proportional to each other (only for chiral or anti-chiral primaries is there a fixed proportionality). So, for general operators $\mathcal{O}$, the two terms on the RHS of (5.3.4) have two independent OPE coefficients, for the primary operator, j_R^μ, and its super-descendant, $T^{\mu\nu}$. This illustrates that (5.3.3) can not hold with any universal functions F_{ij}^k. Generally, the coefficients of the Q and $\bar{Q}$ descendant terms in F in (5.3.3) are independent coefficients, not fixed by the symmetries. This all follows from the general superspace analysis of Osborn [14], that we'll now review.

5.3.1. Two and three-point functions: the superspace analysis of [14]

Operators are labeled by $(j, \bar{j}, q, \bar{q})$, where $(j, \bar{j})$ are the Lorentz spins, $q \equiv \frac{1}{2}(\Delta + \frac{3}{2}r)$ and $\bar{q} \equiv \frac{1}{2}(\Delta - \frac{3}{2}r)$, where Δ is the operator's dimension and r its R-charge. Chiral operators have $\bar{q} = 0$, real operators have $q = \bar{q} = \frac{1}{2}\Delta$, and conserved currents have $q = \bar{q} = 1$. The form of two-point functions of arbitrary superconformal primaries is completely fixed in [14] by superconformal invariance, up to overall coefficients $c_{k\bar{k}}$ (which could be set to $\delta_{k\bar{k}}$ by choice of operator normalization for some operators (but not J or $\mathcal{T}_\mu$)):

$$\langle \mathcal{O}_k^{i_3}(z_2)\bar{\mathcal{O}}_{\bar{k}}^{\bar{i}_3}(z_3) \rangle = c_{k\bar{k}}\frac{I^{i_3\bar{i}_3}(x_{2\bar{3}}, x_{\bar{2}3})}{x_{\bar{2}3}^{2\bar{q}_3}x_{\bar{3}2}^{2q_3}}. \quad (5.3.6)$$

Here z_i denotes superspace coordinates, $z_i = (x_i^\mu, \theta_i^\alpha, \bar{\theta}_i^{\dot{\alpha}})$, $x_{ij}^\mu = x_i^\mu - x_j^\mu$, $\theta_{ij}^\alpha = \theta_i^\alpha - \theta_j^\alpha$, and

$$x_{\bar{i}j}{}^\mu = x_{ij}^\mu - i\theta_i\sigma^\mu\bar{\theta}_j + i\theta_j\sigma^\mu\bar{\theta}_i - i\theta_{ij}\sigma^\mu\bar{\theta}_{ij}.$$

$I^{i_3\bar{i}_3}(x_{2\bar{3}}, x_{\bar{2}3})$, where $x_{ij}{}^\mu = -x_{\bar{j}i}{}^\mu$, is a bilocal invariant tensor in the spin indices i_3, $\bar{i}_3$, reducing to 1 for scalars (see [14] for the explicit expression).

The form of three-point functions is determined in [14] to be

$$\langle \mathcal{O}_1^{i_1}(z_1)\mathcal{O}_2^{i_2}(z_2)\mathcal{O}_3^{\dagger i_3}(z_3)\rangle = \frac{I_1^{i_1\bar{\imath}_1}(x_{1\bar{3}},x_{\bar{1}3})I_2^{i_2\bar{\imath}_2}(x_{2\bar{3}},x_{\bar{2}3})}{x_{\bar{1}3}{}^{2\bar{q}_1}x_{\bar{3}1}{}^{2q_1}x_{\bar{2}3}{}^{2\bar{q}_2}x_{\bar{3}2}{}^{2q_2}}t_{\bar{\imath}_1\bar{\imath}_2}^{i_3}(X_3,\Theta_3,\bar{\Theta}_3). \qquad (5.3.7)$$

We called the third operator $\mathcal{O}_3^{\dagger}$ because we're eventually interested in the OPE, $\mathcal{O}_1\mathcal{O}_2 \sim \mathcal{O}_3$. X_3^{μ} is a 4-vector formed from the superspace coordinates $z_{i=1,2,3} = (x_i,\theta_i,\bar{\theta}_i)$ [14],

$$X_3 \equiv \frac{x_{2\bar{1}}\tilde{x}_{\bar{1}2}x_{2\bar{3}}}{x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2}; \qquad (X_3)_{\alpha\dot\alpha} = \sigma_{\mu\alpha\dot\alpha}X_3^{\mu}, \qquad \tilde{x}^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}x_{\beta\dot\beta}.$$

The spinor quantities in (5.3.7) are given by

$$\Theta_3 \equiv i\left(\frac{1}{x_{\bar{1}3}{}^2}x_{3\bar{1}}\bar{\theta}_{31} - \frac{1}{x_{\bar{2}3}{}^2}x_{3\bar{2}}\bar{\theta}_{32}\right), \qquad \bar{\Theta}_3 \equiv i\left(\frac{1}{x_{\bar{3}1}{}^2}\theta_{31}x_{1\bar{3}} - \frac{1}{x_{\bar{3}2}{}^2}\theta_{32}x_{2\bar{3}}\right), \qquad (5.3.8)$$

which are nilpotent, they vanish upon setting the Grassmann coordinates to zero, and they don't have a direct analog in ordinary conformal theories. X_3^{μ} is a superspace extension of the vector Z^{μ} defined in (5.2.4), $Z^{\mu} = \frac{x_{31}^{\mu}}{r_{13}} - \frac{x_{32}^{\mu}}{r_{23}}$. For example, setting the Grassmann part of the $z_{i=1,2}$ coordinates to zero, and defining $Y^{\mu\nu} \equiv \epsilon^{\mu\nu}{}_{\rho\lambda}\frac{x_{13}^{\rho}x_{23}^{\lambda}}{r_{13}r_{23}}$, we find

$$X_3^{\mu}|_{\theta_{i=1,2}=\bar{\theta}_{i=1,2}=0} = Z^{\mu} + \left[i(Z^2\eta^{\mu\nu} - 2Z^{\mu}Z^{\nu}) + \boxed{2Y^{\mu\nu}}\right]\theta_3\sigma_\nu\bar{\theta}_3 + Z^2\left(\boxed{\frac{x_{12}^{\mu}}{r_{12}}} - Z^{\mu}\right)\theta_3^2\bar{\theta}_3^2 \tag{5.3.9}$$

(the boxed terms will drop out). The function t in (5.3.7) is generally under-determined, constrained only by a homogeneity condition corresponding to the scale and R-charges:

$$t_{\bar{\imath}_1\bar{\imath}_2}^{i_3}(\lambda\bar{\lambda}X,\lambda\Theta,\bar{\lambda}\Theta) = \lambda^{2a}\bar{\lambda}^{2\bar{a}}t_{\bar{\imath}_1\bar{\imath}_2}^{i_3}(X,\Theta,\bar{\Theta}), \qquad (5.3.10)$$

with

$$a - 2\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3, \qquad \bar{a} - 2a = q_1 + q_2 - \bar{q}_3.$$

Conformal three-point functions of primaries have a fully-determined dependence

on the operator locations, which can be viewed as a consequence of the fact that ordinary conformal symmetry transformations can be used to map any three points to any three other points. But superconformal symmetry does not suffice to map three super-positions z_i to wherever one pleases, and that is related to the existence of the Θ, $\bar{\Theta}$ in (5.3.7) and (5.3.8). Indeed, supertranslations can be used to set, say, $z_1 = 0$ and superconformal transformations can be used to map, say, $x_2 = \infty$ and $\theta_2 = \bar{\theta}_2 = 0$. Then we are left with the $z_3 \equiv z$ superspace coordinate, which we can act on with ordinary rotations, $U(1)_R$ rotations, and scale transformations. With these mappings, X_3^μ is given by (5.3.9) with $Z^\mu \to x^\mu/x^2$, $x_{12}^\mu/r_{12} \to 0$ and $Y^{\mu\nu} \to 0$. The nilpotent quantities (5.3.8) map to

$$\Theta \to \frac{i}{x^2}(\mathrm{x} - i\theta\bar{\theta})\bar{\theta}, \qquad \bar{\Theta} \to \frac{-i}{x^2}\theta(\mathrm{x} + i\theta\bar{\theta}). \tag{5.3.11}$$

The existence of Θ_3 and $\bar{\Theta}_3$, and the fact that t's dependence on them is generally under-determined by (5.3.10), implies that the three-point functions of superconformal primaries are generally *insufficient* to fully determine those of their superconformal descendants. The superconformal primary three-point functions are extracted by setting the Grassmann coordinates to zero, but that's generally insufficient to determine the Θ_3 and $\bar{\Theta}_3$ dependence (since they then vanish), which is needed to determine the three-point function of general superconformal descendants. So the OPE coefficients of superconformal primaries generally do not fully determine those of their superconformal descendants.

This general ambiguity in the function $t(X, \Theta, \bar{\Theta})$ is eliminated only in special cases, when some of the three operators are in reduced superconformal representations, with null states, e.g. chiral primaries, anti-chiral primaries, or conserved currents. Superspace derivatives on the operators $\mathcal{O}_i$ in (5.3.7) can be converted into differential operators acting on the function $t(X_3, \Theta_3, \bar{\Theta}_3)$, and so constraints on the operators lead to corresponding constraints on the function $t(X_3, \Theta_3, \bar{\Theta}_3)$. In particular, acting on say

$\mathcal{O}_1$, one replaces $D_\alpha \to \mathcal{D}_\alpha$ and $\bar{D}_{\dot\alpha} \to \bar{\mathcal{D}}_{\dot\alpha}$, which act on $t(X, \Theta, \bar\Theta)$ as [14]

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \Theta^\alpha} - 2i(\sigma^\mu \bar\Theta)_\alpha \frac{\partial}{\partial X^\mu}, \qquad \bar{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial \bar\Theta^{\dot\alpha}}, \tag{5.3.12}$$

with $\bar{X} = X - 4i\Theta\bar\Theta$. As examples, we'll first review the cases that have been discussed in the literature, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are chiral or anti-chiral operators. In the following section, we'll consider our case of interest: conserved currents.

5.3.2. Review of chiral-chiral OPEs [18–20]

Take the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ in the three-point function (5.3.7) to both be chiral primaries, which we'll write as $\mathcal{O}_i = \phi_i$. The condition $\bar{D}_{\dot\alpha}\phi_1 = 0$ implies that $\bar{\mathcal{D}}_{\dot\alpha} t = 0$ for the operator in (5.3.12), with a similar condition for $\bar{D}_{\dot\alpha}\phi_2$. If we take ϕ_1 and ϕ_2 to be the same operator, the latter condition is accounted for by the $z_1 \leftrightarrow z_2$ symmetry, which implies

$$t(X_3, \Theta_3, \bar\Theta_3) = t(-\bar{X}_3, -\Theta_3, -\bar\Theta_3). \tag{5.3.13}$$

The solutions for $t(X_3, \Theta_3, \bar\Theta_3)$ are [18–20]

$$t \sim \text{constant},$$
$$t \sim \bar\Theta_3 \bar{X}_3^{\Delta_\mathcal{O} - \Delta_i - \Delta_j - 1 - \ell} \bar{X}_3^{\mu_1} \cdots \bar{X}_3^{\mu_\ell},$$
$$t \sim \bar\Theta_3^2 \bar{X}_3^{\Delta_{\mathcal{O}_3} - \Delta_i - \Delta_j - \ell + 1} X_3^{\mu_1} \cdots X_3^{\mu_\ell}.$$

The case $t \sim \text{constant}$ implies that the operator $\mathcal{O}_3$ in the three-point function (5.3.7) is also chiral, $\mathcal{O}_3 = \phi_k$, with $R(\mathcal{O}_3^I) = R(\phi_i) + R(\phi_j) - 2$; this is the chiral ring. The other two cases for t have factors of $\bar\Theta_3$ and $\bar\Theta_3^2$, corresponding to operators $\mathcal{O}_3$ in (5.3.7) that are $\bar{Q}_\alpha$ and $\bar{Q}^2$ exact (hence trivial in the chiral ring, but nevertheless important for non-holomorphic considerations). Correspondingly, the possible terms in the OPE

are

$$\phi_i(x)\phi_j(0) = c_{ij}^k C(x,P)\phi_k(0) + \sum_{\mathcal{O}^I} c_{ij}^{\mathcal{O}^I} \bar{Q} C_I(x,P)\mathcal{O}^I(0) + \sum_{\mathcal{O}^J} c_{ij}^{\mathcal{O}^J} \bar{Q}^2 C_J(x,P)\mathcal{O}^J(0),$$

(5.3.14)

where c_{ij}^k, $c_{ij}^{\mathcal{O}^I}$, and $c_{ij}^{\mathcal{O}^J}$ are constant OPE coefficients. The operators $\mathcal{O}_J$ in (5.3.14) have even spin, $\ell = 2j_1 = 2j_2$, and $R(\mathcal{O}_J) = \frac{2}{3}(\Delta_i + \Delta_j) - 2$ (so unitarity requires $\Delta_{\mathcal{O}} \geq |\frac{3}{2}R_{\mathcal{O}}| + \ell + 2$). To give a simple example, consider a theory with a chiral superfield Φ, $K = \bar{\Phi}\Phi$ and $W = \lambda\Phi^{n+1}/(n+1)$. Then, the equation of motion $\Phi^n = -\bar{Q}^2\bar{\Phi}/\lambda$ illustrates the last term in (5.3.14). The $\mathcal{O}_I$ possibility in (5.3.14) runs only over superconformal primaries with $R(\mathcal{O}_J) = \frac{2}{3}(\Delta_i + \Delta_j) - 1$, spins $(j_1, j_2) = (\frac{1}{2}(\ell+1), \frac{1}{2}\ell)$, with ℓ odd for (5.3.13), and $\Delta(\mathcal{O}_J) = \Delta_i + \Delta_j + \ell + \frac{1}{2}$, where Δ is fixed (saturating a unitarity bound) because the operator $\mathcal{O}_J$ must be in a shortened multiplet to have both sides of (5.3.14) annihilated by $\bar{Q}_{\dot{\alpha}}$.

In (5.3.14) we have written just the first components of the superfields on the LHS. The full superfield expression for the first term in (5.3.14) was worked out in [18]:

$$\Phi_i(z_{1+})\Phi_j(z_{2+}) \supset c_{ij}^k \mathcal{C}^{q_i,q_j}(z_{12+}, \partial_{z_2+})\Phi_k(z_{2+}),$$

which has no x_{12} singularity since $q_k = q_i + q_j$ for the chiral ring, and

$$\mathcal{C}^{q_1,q_2}(z_{12+}, \partial_{z_2+})\frac{1}{(x_{2+} - 2i\theta_2\sigma\bar{\theta} - x_-)^{2q_1 + 2q_2}} = $$

$$\frac{1}{(x_{1+} - 2i\theta_1\sigma\bar{\theta} - x_-)^{2q_1}(x_{2+} - 2i\theta_2\sigma\bar{\theta} - x_-)^{2q_2}},$$

which was solved for in [18] in a superspace expansion in θ_{12}, with components given by the functions $C^{ab}(x_{12}, \partial_{x_2})$ in (5.2.13).

5.3.3. Review of chiral-anti-chiral OPE [19]

Let the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ in (5.3.7) be chiral and anti-chiral respectively. As in [19], for simplicity we'll take $\mathcal{O}_1 = \Phi$ and $\mathcal{O}_2 = \bar{\Phi}$, the conjugate field. The conditions $\bar{\mathcal{D}}_{1,\dot\alpha} t = 0$ and $\mathcal{D}_{2,\alpha} t = 0$ then imply that the operator $\mathcal{O}_3$ must be real and of integer spin $\ell = 2j = 2\bar{j}$, with [19]

$$\langle \Phi(z_{1+})\bar{\Phi}(z_{2-})\mathcal{O}^{\mu_1\cdots\mu_\ell}(z_3)\rangle \propto \frac{1}{x_{\bar{3}1}^{2\Delta_\Phi} x_{\bar{2}3}^{2\Delta_\Phi}} \bar{X}_3^{\Delta_\mathcal{O}-2\Delta_\Phi-\ell} \bar{X}_3^{\mu_1}\cdots\bar{X}_3^{\mu_\ell} - \text{traces}. \quad (5.3.15)$$

The result (5.3.15) encodes interesting relations among the component OPE coefficients. We will review this in some detail, following [19], since many details will prove applicable for our case of interest, to be discussed in the next section.

Real operators $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ in (5.3.15) have a superspace expansion

$$\mathcal{O}^{\mu_1\cdots\mu_\ell}(x,\theta,\bar{\theta}) = A^{\mu_1\cdots\mu_\ell}(x) + \xi_\mu B^{\mu\mu_1\cdots\mu_\ell}(x) + \xi^2 D^{\mu_1\cdots\mu_\ell}(x) + \cdots, \quad (5.3.16)$$

where $\xi_\mu \equiv \theta\sigma_\mu\bar{\theta}$ and $\cdots$ are operators with non-zero R-charge. The A component is primary, and the others others are all A's descendants: defining $\Xi^\mu \equiv \bar{\sigma}^{\mu\dot\alpha\alpha}[Q_\alpha, \bar{Q}_{\dot\alpha}]$,

$$B^{\mu\mu_1\cdots\mu_\ell} = -\frac{1}{4}\Xi^\mu A^{\mu_1\cdots\mu_\ell}, \qquad D^{\mu_1\cdots\mu_\ell} = -\frac{1}{64}\Xi_\mu B^{\mu\mu_1\cdots\mu_\ell} - \frac{1}{16}\partial^2 A^{\mu_1\cdots\mu_\ell}.$$

The operators $A^{\mu_1\cdots\mu_\ell}$ and $D^{\mu_1\cdots\mu_\ell}$ are irreducible spin-ℓ representations, while $B^{\mu\mu_1\cdots\mu_\ell}$ decompose into $B^{\mu\mu_1\cdots\mu_\ell} = M^{\mu\mu_1\cdots\mu_\ell} + \frac{\ell^2}{(\ell+1)^2}\eta^{\mu\mu_1} N^{\mu_2\cdots\mu_\ell} + L^{\mu\mu_1\cdots\mu_\ell}$, where M (called J in [19]) is a spin $\ell+1$ operator, N is a spin $\ell-1$ operator, and $L = L_+ + L_-$, with $L_\pm$ in the $(\frac{1}{2}\ell \pm \frac{1}{2}, \frac{1}{2}\ell \mp \frac{1}{2})$ representation of $SU(2) \times SU(2)$. The operators B and D can be decomposed into conformal primary and descendant contributions [19], with $M_{\text{prim}}^{\mu\mu_1\cdots\mu_\ell} = M^{\mu\mu_1\cdots\mu_\ell}$, $N_{\text{prim}}^{\mu_2\cdots\mu_\ell} = N^{\mu_2\cdots\mu_\ell}$, and ($P_\mu^{\text{here}} = -iP_\mu^{\text{there}}$, as we prefer Hermitan

generators)

$$L_{\text{prim}}^{\mu\mu_1\cdots\mu_\ell} = L^{\mu\mu_1\cdots\mu_\ell} - \frac{\ell}{4(\Delta-1)}\epsilon^{\mu\mu_1}{}_{\nu\rho}iP^\nu A^{\rho\mu_2\cdots\mu_\ell},$$

$$D_{\text{prim}}^{\mu_1\cdots\mu_\ell} = D^{\mu_1\cdots\mu_\ell} - \frac{\ell(\ell+1)-(\Delta-1)}{8(\Delta-1)^2}P^2 A^{\mu_1\cdots\mu_\ell} + \frac{\ell^2}{4(\Delta-1)^2}P_\nu P^{\mu_1}A^{\nu\mu_2\cdots\mu_\ell} \quad (5.3.17)$$

$$\qquad - \frac{\ell}{4(\Delta-1)}\epsilon^{\mu_1\nu}{}_{\rho\sigma}iP_\nu L^{\rho\sigma\mu_2\cdots\mu_\ell}.$$

Setting for example $\theta_1 = \theta_2 = \bar\theta_1 = \bar\theta_2 = 0$ in (5.3.15) to extract the three-point functions for $\phi = \Phi|$ and $\bar\phi = \bar\Phi|$, it is found that [19]

$$\langle\phi\phi^* A^{\mu_1\cdots\mu_\ell}\rangle = c_{\phi\phi^*\mathcal{O}_\ell}\frac{Z^{\Delta-\ell}}{r_{12}^{\Delta_\Phi}}Z^{\mu_1}\cdots Z^{\mu_\ell},$$

$$\langle\phi\phi^* M_{\text{prim}}^{\mu\mu_1\cdots\mu_\ell}\rangle = ic_{\phi\phi^*\mathcal{O}_\ell}(\Delta+\ell)\frac{Z^{\Delta-\ell}}{r_{12}^{\Delta_\Phi}}Z^\mu Z^{\mu_1}\cdots Z^{\mu_\ell},$$

$$\langle\phi\phi^* N_{\text{prim}}^{\mu_2\cdots\mu_\ell}\rangle = ic_{\phi\phi^*\mathcal{O}_\ell}\frac{(\ell+1)(\Delta-\ell-2)}{2\ell}\frac{Z^{\Delta+2-\ell}}{r_{12}^{\Delta_\Phi}}Z^{\mu_2}\cdots Z^{\mu_\ell}, \quad (5.3.18)$$

$$\langle\phi\phi^* L_{\text{prim}}^{\mu\mu_1\cdots\mu_\ell}\rangle = 0,$$

$$\langle\phi\phi^* D_{\text{prim}}^{a_1\cdots a_\ell}\rangle = -c_{\phi\phi^*\mathcal{O}_\ell}\frac{\Delta(\Delta+\ell)(\Delta-\ell-2)}{8(\Delta-1)}\frac{Z^{\Delta+2-\ell}}{r_{12}^{\Delta_\Phi}}Z^{\mu_1}\cdots Z^{\mu_\ell},$$

where Z is the quantity in (5.2.4) and the products like $Z^{\mu_1}\cdots Z^{\mu_\ell}$ are to be understood as symmetrized traceless. The primary three-point functions (5.3.18) indeed have the form (5.2.3), involving only the coordinate Z^μ. Indeed, $\langle\phi\phi^* L_{\text{prim}}\rangle$ had to vanish, since it's impossible to form something with L's Lorentz structure using only Z^μ. The upshot of (5.3.18) is that the coefficient $c_{\phi\phi^*\mathcal{O}_\ell} = c_{\phi\phi^* A_\ell}$ of the superconformal primary A

indeed completely determines those of the descendants, M, N, L_{prim}, and D_{prim}:

$$
\begin{aligned}
c_{\phi\phi^* M_{\ell+1}} &= i(\Delta + \ell)c_{\phi\phi^* A_\ell}, \\
c_{\phi\phi^* N_{\ell-1}} &= i\frac{(\ell+1)(\Delta - \ell - 2)}{2\ell}c_{\phi\phi^* A_\ell}, \\
c_{\phi\phi^* L_{\text{prim}}} &= 0, \\
c_{\phi\phi^* D_{\ell;\text{prim}}} &= -\frac{\Delta(\Delta + \ell)(\Delta - \ell - 2)}{8(\Delta - 1)}c_{\phi\phi^* A_\ell}.
\end{aligned}
\tag{5.3.19}
$$

To convert (5.3.19) to relations among the OPE coefficients, we can use $c_{ij}^k = c_{ijk'}g^{k'k}$ (5.2.8), and the relations among the two-point function normalizations. The two-point function of A_ℓ is proportional to

$$
\langle A^{\nu_1 \ldots \nu_\ell} | A^{\mu_1 \ldots \mu_\ell} \rangle \sim \text{symmetrize}(\eta^{\mu_1 \nu_1} \cdots \eta^{\mu_\ell \nu_\ell}) - \text{traces} \equiv \mathcal{I}_\ell^{\mu_1 \ldots \mu_\ell; \nu_1 \ldots \nu_\ell}.
$$

Likewise, the two-point functions of $M_{\ell+1}$, $N_{\ell-1}$, and D_ℓ are proportional to $\mathcal{I}_{\ell+1}$, $\mathcal{I}_{\ell-1}$, and $\mathcal{I}_\ell$, respectively, and $\langle L_{\text{prim}}^{\nu\nu_1 \ldots \nu_\ell} | L_{\text{prim}}^{\mu\mu_1 \ldots \mu_\ell} \rangle \sim \eta^{\mu\nu} \mathcal{I}_\ell^{\mu_1 \ldots \mu_\ell; \nu_1 \ldots \nu_\ell}$. The proportionality factors for the two-point function normalization of the super-descendants, relative to the primary component, are given by [19]

$$
\begin{aligned}
c_{M_{\ell+1}M_{\ell+1}} &= 2(\Delta + \ell)(\Delta + \ell + 1)c_{A_\ell A_\ell}, \\
c_{N_{\ell-1}N_{\ell-1}} &= \frac{2(\ell+1)^2(\Delta - \ell - 2)(\Delta - \ell - 1)}{\ell^2}c_{A_\ell A_\ell}, \\
c_{L_{\text{prim}}L_{\text{prim}}} &= \frac{8\ell^2\Delta(\Delta + \ell)(\Delta - \ell - 2)}{(\ell+1)^2(\Delta - 1)}c_{A_\ell A_\ell}, \\
c_{D_{\ell;\text{prim}}D_{\ell;\text{prim}}} &= \frac{\Delta^2(\Delta - \ell - 2)(\Delta - \ell - 1)(\Delta + \ell)(\Delta + \ell + 1)}{4(\Delta - 1)^2}c_{A_\ell A_\ell},
\end{aligned}
\tag{5.3.20}
$$

where the factor $c_{A_\ell A_\ell}$ could be set to one by choice of normalization of $\mathcal{O}_\ell$. Note that when the unitarity bound $\Delta \geq \ell + 2$ is saturated, the norm (5.3.20) of $N_{\ell-1}$, L_{prim}, and $D_{\ell;\text{prim}}$ all vanish; indeed, these components of the supermultiplet vanish when the unitarity bound is saturated—the supermultiplet is shortened.

5.3.4. Another example: the $\langle \mathcal{O}\mathcal{O}^\dagger \mathcal{T}_\mu \rangle$ three-point function

As another example of applying the general formalism of [14], we can consider the three-point function the stress-energy tensor supermultiplet $\mathcal{T}_\mu$ (5.3.1) with a scalar superfield $\mathcal{O}$ and its conjugate $\mathcal{O}^\dagger$. For the case $\mathcal{O} = \Phi$ a chiral operator, $q = \Delta_\Phi = \frac{3}{2} r_\Phi$, $\bar{q} = 0$, the result was given in [14],

$$\langle \Phi(z_{1+})\bar{\Phi}(z_{2-})\mathcal{T}^\mu(z_3)\rangle = -ir_\Phi \frac{c_{\phi\bar{\phi}}}{2\pi^2} \frac{1}{x_{\bar{3}1}{}^{2q}x_{\bar{2}3}{}^{2q}} \frac{\bar{X}_3^\mu}{\bar{X}_3^{2(q-1)}}, \tag{5.3.21}$$

where $c_{\phi\bar{\phi}}$ is the $\langle \phi\bar{\phi}\rangle$ two-point function normalization, and the coefficient in (5.3.21) is fixed by the condition that the OPE reproduces the correct $U(1)_R$ charge, as in (5.2.15). This is a special case of (5.3.15), where we take $\Delta_\mathcal{O} = 3$ and $\ell = 1$ to get $\mathcal{O}^\mu \to \mathcal{T}^\mu$. So $c_{\phi\phi^* j_R^\mu} = -ir_\Phi c_{\phi\phi^*}/2\pi^2$ and then (5.3.19), with $M_{\mu\nu} = 2T_{\mu\nu}$ (see (5.3.1)) gives $c_{\phi\phi^* T_{\mu\nu}} = r_\Phi c_{\phi\phi^*}/\pi^2$, which fits with (5.2.18) and $\Delta = \frac{3}{2}|r_\Phi|$ for chiral and anti-chiral operators.

As another example, we consider the case where the operator $\mathcal{O}$ is real, $\mathcal{O} = \mathcal{O}^\dagger$, so $q_\mathcal{O} = \bar{q}_\mathcal{O} = \frac{1}{2}\Delta_\mathcal{O}$, and $R_\mathcal{O} = 0$. Using (5.3.7), (5.3.10), and the $z_1 \leftrightarrow z_2$ symmetry we find

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{T}^\mu(z_3)\rangle =$$

$$\frac{-\Delta_\mathcal{O} c_{\mathcal{O}\mathcal{O}}}{6\pi^2 (x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2)^{\frac{1}{2}\Delta_\mathcal{O}}(X_3 \cdot \bar{X}_3)^{\Delta_\mathcal{O}-1}} \left[X_-^\mu + 2\frac{(X_- \cdot X_+)X_+^\mu}{X_3 \cdot \bar{X}_3} \right],$$

$$\tag{5.3.22}$$

where $X_+^\mu \equiv \frac{1}{2}(X_3^\mu + \bar{X}_3^\mu)$ is a vector that's odd under the $z_1 \leftrightarrow z_2$ operation in (5.3.13), and $X_-^\mu \equiv i(X_3^\mu - \bar{X}_3^\mu) \equiv -4\Theta_3\sigma^\mu\bar{\Theta}_3$ is a (nilpotent) vector that's even under the $\mathbb{Z}_2$. So $X_3 \cdot \bar{X}_3 = X_+^2 + 4\Theta_3^2\bar{\Theta}_3^2$. The relative factor of two between the two terms in the sum on the RHS is determined by the condition $D_\alpha T^{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} T^{\alpha\dot{\alpha}} = 0$, and the overall normalization by (5.2.18). As a special case of (5.3.22), the three-point function of two

conserved currents and the stress tensor is

$$\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{T}^\mu(z_3)\rangle = -\frac{\tau_{JJ}}{48\pi^6 x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2 X_3 \cdot \bar{X}_3}\left[X_-^\mu + 2\frac{(X_- \cdot X_+)X_+^\mu}{X_3 \cdot \bar{X}_3}\right],$$

$$(5.3.23)$$

Comparing the $\langle J(x_1)J(x_2)T^{\mu\nu}(x_3)\rangle$ and the $\langle j^\rho(x_1)j^\sigma(x_2)j_R^\mu(x_3)\rangle$ components encoded in (5.3.23) leads to the relation $\tau_{JJ} = -3\,\mathrm{Tr}\,F^2 R$, giving the current two-point function coefficient τ in (5.1.3) as a 't Hooft anomaly.

5.4. Our case of interest: the current-current OPE

We now consider the OPE of two $\Delta = 2$ conserved-current primary operators,

$$J(x)J(0) = \sum_{\ell=0}^{\infty} \sum_{\substack{\text{sprimary} \\ \mathcal{O}_k^{(\ell)}}} \frac{c_{JJ}^k}{(x^2)^{\frac{1}{2}(4-\Delta_k)}} F_{JJ}^k(x,P,Q,\bar{Q})^{(\ell)} \mathcal{O}_k^{(\ell)}(0), \qquad (5.4.1)$$

where $\mathcal{O}_k^{(\ell)}$ are superconformal primaries, of dimension Δ_k and spin ℓ, and we will show that the $\mathcal{O}_k^{(\ell)}$ are necessarily real, of $U(1)_R$-charge zero. For simplicity, we consider $U(1)$ currents. The LHS of (5.4.1) is then symmetric under exchanging the operators, and hence $x^\mu \to -x^\mu$, so only even spin operators can contribute on the RHS of the OPE. For non-Abelian groups, odd spin components can appear on the RHS of $J_a(x)J_b(0)$, with coefficients proportional to f_{abc} as in (5.1.3). We discuss how to determine the $F_{JJ}^k(x,P,Q,\bar{Q})^{(\ell)}$ from the condition of superconformal covariance, combined with J's current conservation.

The OPE result (5.4.1) for the bottom component of the supercurrent multiplet will determine the OPE coefficients of its superconformal descendants, in particular of

$$j_\alpha(x) = Q_\alpha(J(x)), \qquad j_\mu(x) = -\tfrac{1}{4}\Xi_\mu(J(x)), \qquad (5.4.2)$$

where $\Xi_\mu \equiv \bar{\sigma}_\mu^{\dot{\alpha}\alpha}[Q_\alpha, \bar{Q}_{\dot{\alpha}}]$. We can use Q_α and $\bar{Q}_{\dot{\alpha}}$ to map from the primary J, to its descendants, as in (5.4.2). We can also map in the opposite direction, by using the S^α

and $\bar{S}^{\dot{\alpha}}$ superconformal supercharges, which act on the primary component as

$$S^\alpha(J(x)) = ix \cdot \bar{\sigma}^{\dot{\alpha}\alpha}\bar{Q}_{\dot{\alpha}}(J(x)), \qquad \bar{S}^{\dot{\alpha}}(J(x)) = -ix \cdot \bar{\sigma}^{\dot{\alpha}\alpha}Q_\alpha(J(x)), \qquad (5.4.3)$$

vanishing at the origin. Acting on the descendants as in (5.3.2) with $\Delta_J = 2$ and $r_J = 0$, we find

$$S^\alpha(j_\alpha(x)) = -ix \cdot \bar{\sigma}^{\dot{\alpha}\alpha}Q_\alpha\bar{Q}_{\dot{\alpha}}(J(x)) + 4(x \cdot \partial + 2)J(x),$$

$$S^\alpha(j^\mu(x)) = 3\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\jmath}_{\dot{\alpha}}(x) - 2x \cdot \bar{\sigma}^{\dot{\alpha}\alpha}\bar{\sigma}^{\mu\nu\dot{\beta}}_{\dot{\alpha}}\partial_\nu\bar{\jmath}_{\dot{\beta}}(x).$$

5.4.1. Using the algebra to find relations in the $J(x)J(0)$ OPE

In this subsection, we discuss how superconformal symmetry leads to relations for $J(x)J(0)$ by directly using the algebra. The relations obtained this way alternatively follow from using the superspace formalism of [14], which we will use in the next subsection.

When the superconformal generators act on the product $J(x)J(0)$, the product rule gives two terms, e.g. $Q_\alpha(J(x)J(0)) = Q_\alpha(J(x))J(0) + J(x)Q_\alpha(J(0))$. But for the lowering operators, S^α, $\bar{S}^{\dot{\alpha}}$ and K_μ, the term where they act on the primary $J(0)$ vanishes, so e.g.

$$S^\alpha(J(x)J(0)) = S^\alpha(J(x))J(0) = -ix \cdot \bar{\sigma}^{\dot{\alpha}\alpha}\bar{\jmath}_{\dot{\alpha}}(x)J(0). \qquad (5.4.5)$$

The $j_\alpha(x)J(0)$ OPE thus follows from the $J(x)J(0)$ OPE, with only superdescendants in $J(x)J(0)$ contributing to the OPE around the origin, since superconformal primary terms are annihilated by S^α in (5.4.5).

The relation (5.4.5) illustrates how the OPE $J(x)J(0)$ of the primary operators in the multiplet determine the OPEs of the descendants. Additional relations follow because we are here considering conserved currents rather than generic operators, so $Q^2(J(x)) = \bar{Q}^2(J(x)) = 0$. For example, consider the $j^\alpha(x)j_\alpha(0)$ operator product,

relevant for determining gaugino masses in general gauge mediation, which can be related to $J(x)J(0)$ as in [25] (see appendix 5.A for a discussion about the sign)

$$j^\alpha(x)j_\alpha(0) = \tfrac{1}{2}Q^2(J(x)J(0)).$$

In superconformal theories, this descendant operator product can also be related to the primary $J(x)J(0)$ by using (5.4.3) as

$$j_\alpha(x)j_\beta(0) = \frac{1}{x^2}Q_\beta(ix \cdot \sigma\bar{S})_\alpha(J(x)J(0)). \tag{5.4.6}$$

Again, $\bar{S}^{\dot\alpha}$ only acts on $J(x)$, and then Q_β only acts on $J(0)$ (since $Q^2(J(x)) = 0$). Another interesting relation that follows from (5.4.3), combined with $Q^2(J(x)) = \bar{Q}^2(J(x)) = 0$, is

$$S^\alpha S^\beta(J(x)J(0)) = \bar{S}^{\dot\alpha}\bar{S}^{\dot\beta}(J(x)J(0)) = 0. \tag{5.4.7}$$

The relations (5.4.6) relate operator products of descendants to those of the primaries, while (5.4.7) constrain the terms that can appear on the RHS of the OPE of the primaries.

There are two more operators that annihilate $J(x)J(0)$,

$$\left[x^2 Q_\alpha Q_\beta + Q_\alpha(ix \cdot \sigma\bar{S})_\beta - Q_\beta(ix \cdot \sigma\bar{S})_\alpha\right](J(x)J(0)) = 0,$$

$$\left[x^2 \bar{Q}_{\dot\alpha}\bar{Q}_{\dot\beta} + (S\,ix \cdot \bar{\sigma})_{\dot\beta}\bar{Q}_{\dot\alpha} - (S\,ix \cdot \bar{\sigma})_{\dot\alpha}\bar{Q}_{\dot\beta}\right](J(x)J(0)) = 0,$$

thus constraining the OPE $J(x)J(0)$. Other relations, giving OPEs of descendants in terms of the $J(x)J(0)$ primary OPE, are

$$j_\alpha(x)\bar{j}_{\dot\alpha}(0) = \frac{1}{x^4}\left[(S\,ix \cdot \sigma)_{\dot\alpha}(ix \cdot \sigma\bar{S})_\alpha - x^2\bar{Q}_{\dot\alpha}(ix \cdot \sigma\bar{S})_\alpha + 2\Delta_J x^2(ix \cdot \sigma)_{\alpha\dot\alpha}\right](J(x)J(0)),$$

$$j_\mu(x)j_\nu(0) = \frac{1}{16x^8}\left[(x^2\eta_{\mu\rho} - 2x_\mu x_\rho)(S\sigma^\rho\bar{S} - \bar{S}\sigma^\rho S)\right.$$

$$\times\{x^4(\bar{Q}\bar{\sigma}_\nu Q - Q\sigma_\nu\bar{Q}) + (x^2\eta_{\nu\lambda} - 2x_\nu x_\lambda)(S\sigma^\lambda\bar{S} - \bar{S}\bar{\sigma}^\lambda S)$$

$$-2x^2\left(Q\sigma_\nu\, ix\cdot\bar\sigma S - \bar Q\bar\sigma_\nu\, ix\cdot\sigma\bar S\right)\}$$

$$-8i(\Delta_J+1)x^2(\eta_{\mu\nu}\eta_{\lambda\rho}-\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}-i\epsilon_{\mu\nu\lambda\rho})x^\lambda$$

$$\times\{(x^2\eta^{\rho\delta}-2x^\rho x^\delta)S\sigma_\delta\bar S+x^2\bar Q\bar\sigma^\rho\, ix\cdot\sigma\bar S+4i\Delta_J x^2 x^\rho\}$$

$$-8i(\Delta_J+1)x^2(\eta_{\mu\nu}\eta_{\lambda\rho}-\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}+i\epsilon_{\mu\nu\lambda\rho})x^\lambda$$

$$\times\{(x^2\eta^{\rho\delta}-2x^\rho x^\delta)\bar S\bar\sigma_\delta S+x^2 Q\sigma^\rho\, ix\cdot\bar\sigma S+4i\Delta_J x^2 x^\rho\}$$

$$+32x^4\Delta_J(\Delta_J+1)(x^2\eta_{\mu\nu}-2x_\mu x_\nu)\big]\,(J(x)J(0)),$$

$$j_\mu(x)J(0)=\frac{x^2\eta_{\mu\nu}-2x_\mu x_\nu}{4x^4}\left[S\sigma^\nu\bar S-\bar S\bar\sigma^\nu S\right](J(x)J(0)).$$

In sum, OPEs of the superdescendants are all determined from the primary OPE $J(x)J(0)$, and the superdescendants in $J(x)J(0)$ are constrained by superconformal symmetry and current conservation. We will find the explicit expressions in the next subsection.

5.4.2. Current-current OPEs using the superspace results of [14]

We now consider the superspace three-point functions (5.3.7) where $\mathcal{O}_1$ and $\mathcal{O}_2$ are conserved currents, and for simplicity we take $\mathcal{O}_1=\mathcal{O}_2=\mathcal{J}$, so there is a $z_1\leftrightarrow z_2$ symmetry, implying the symmetry condition (5.3.13) on the function $t(X_3,\Theta_3,\bar\Theta_3)$ in (5.3.7). The $\mathcal{J}$ superfield has the component expansion (5.1.2). We're interested in the three-point functions

$$\langle\mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{O}^{\mu_1\cdots\mu_\ell}(z_3)\rangle=\frac{1}{x_{\bar13}{}^2x_{\bar31}{}^2x_{\bar23}{}^2x_{\bar32}{}^2}t_{\mathcal{J}\mathcal{J}\mathcal{O}_\ell}^{\mu_1\cdots\mu_\ell}(X_3,\Theta_3,\bar\Theta_3). \qquad (5.4.8)$$

The scaling relation (5.3.10), with $q=\bar q=1$ for the conserved currents has $a=\frac13(q_k+2\bar q_k)-2$ and $\bar a=\frac13(\bar q_k+2q_k)-2$. We now discuss the constraints on t in (5.4.8) coming from current conservation. The condition that $\mathcal{J}$ is conserved, written in superspace as $D^2\mathcal{J}=\bar D^2\mathcal{J}=0$, implies that $\mathcal{D}^2t=\bar{\mathcal{D}}^2t=0$, where $\mathcal{D}$ acts on t as differential operators as in (5.3.12).

A first consequence is that the operator $\mathcal{O}_3$ in (5.4.8) must be a real operator of vanishing R-charge and integer spin ℓ (much as in the $\Phi\bar\Phi$ OPE of the previous subsection). Suppose, to the contrary, that e.g. $R(\mathcal{O}_k) = 2$, which would lead to $\bar a = a + 1$ in (5.3.10), which would fix $t \overset{?}{=} \bar\Theta_3^2 f(X_3)$ (f can't have any additional factors of $\bar\Theta_3$, since $\bar\Theta_3^{n>2} = 0$, nor Θ_3 factors without spoiling (5.3.10)). But that t cannot satisfy $\bar{\mathcal{D}}^2 t = 0$. One can similarly use $\mathcal{D}^2 t = \bar{\mathcal{D}}^2 t = 0$ to exclude all other possibilities for non-zero R-charge operators in (5.4.8). So, in what follows, we take $\mathcal{O}_\ell$ to have $q = \bar q = \frac{1}{2}\Delta$, and thus $a = \bar a = \frac{1}{2}\Delta - 2$ in (5.3.10).

The conditions $\mathcal{D}^2 t = \bar{\mathcal{D}}^2 t = 0$ uniquely determine the function $t^{\mu_1\cdots\mu_\ell}$ in (5.4.8). Let's first write it for spin-$\ell = 0$ operators $\mathcal{O}_k$ in (5.4.8):

$$t_{\mathcal{J}\mathcal{J}\mathcal{O}_{\ell=0}}(X, \Theta, \bar\Theta) = \frac{c_{JJ\mathcal{O}}}{(X \cdot \bar X)^{2 - \frac{1}{2}\Delta}}\left[1 - \frac{1}{4}(\Delta - 4)(\Delta - 6)\frac{\Theta^2\bar\Theta^2}{X \cdot \bar X}\right], \qquad (5.4.9)$$

with $c_{JJ\mathcal{O}_{\ell=0}}$ an arbitrary coefficient. Because the coefficient of the term involving Θ_3 and $\bar\Theta_3$ is determined, the superconformal descendant three-point functions are determined from that of the superconformal primaries. The case (5.1.4) where all three operators are conserved currents, $q_k = 1$, is exceptional, since $\mathcal{D}^2 X^{-2} = 0$ (up to contact terms).

For the case of an $\ell = 1$ superconformal primary operator $\mathcal{O}_k^\mu$ in (5.4.8), the conditions determine, much as in (5.3.22)

$$t_{\mathcal{J}\mathcal{J}\mathcal{O}_{\ell=1}}^\mu(X, \Theta, \bar\Theta) = \frac{c_{JJ\mathcal{O}_{\ell=1}}}{(X \cdot \bar X)^{\frac{5}{2} - \frac{1}{2}\Delta}}\left[X_-^\mu - \frac{\Delta - 5}{\Delta - 2}\frac{(X_- \cdot X_+)X_+^\mu}{X \cdot \bar X}\right], \qquad (5.4.10)$$

where $X_+^\mu \equiv \frac{1}{2}(X^\mu + \bar X^\mu)$, called Q^μ in [14], is odd under the $z_1 \leftrightarrow z_2$ operation in (5.3.13), and $X_-^\mu \equiv i(X^\mu - \bar X^\mu) \equiv -4\Theta\sigma^\mu\bar\Theta$, called P^μ in [14], is even under the $\mathbb{Z}_2$. An example of a real, primary $\ell = 1$ operator is the FZ operator $\mathcal{T}_\mu$ (5.3.1), with $\Delta_{\mathcal{T}_\mu} = 3$. If we set $\Delta_{\mathcal{O}^\mu} = 3$ in (5.4.10) and $\Delta_{\mathcal{O}} = 2$ in (5.3.22), the two expressions properly coincide.

For general, even-spin-ℓ superconformal primary $\mathcal{O}_k^{(\mu_1\cdots\mu_\ell)}$, (5.4.9) generalizes to

$$t_{\mathcal{J}\mathcal{J}\mathcal{O}_\ell \text{ even}}^{(\mu_1\ldots\mu_\ell)} = c_{JJ\mathcal{O}_\ell} \frac{X_+^{(\mu_1}\cdots X_+^{\mu_\ell)}}{(X\cdot\bar{X})^{2-\frac{1}{2}(\Delta-\ell)}} \left[1 - \frac{1}{4}(\Delta-\ell-4)(\Delta+\ell-6)\frac{\Theta^2\bar{\Theta}^2}{X\cdot\bar{X}} \right] - \text{traces}.$$
(5.4.11)

The generalization of (5.4.10) for odd spin ℓ is

$$t_{\mathcal{J}\mathcal{J}\mathcal{O}_\ell \text{ odd}}^{(\mu_1\ldots\mu_\ell)} = c_{JJ\mathcal{O}_\ell} \frac{X_+^{(\mu_1}\cdots X_+^{\mu_{\ell-1}}}{(X\cdot\bar{X})^{2-\frac{1}{2}(\Delta-\ell)}} \left[X_-^{\mu_\ell)} - \frac{\ell(\Delta-\ell-4)}{\Delta-2}\frac{(X_-\cdot X_+)X_+^{\mu_\ell)}}{X\cdot\bar{X}} \right] - \text{traces}.$$
(5.4.12)

In both (5.4.11) and (5.4.12) the ℓ Lorentz indices are symmetrized, with the traces removed, to obtain a spin-ℓ irreducible Lorentz representation.

These superspace results encode all component three-point functions, giving relations among the conformal primary components. To make this explicit, we need to expand both sides of (5.4.8) in the Grassmann coordinates; we expand $\mathcal{J}(z_1)$ and $\mathcal{J}(z_2)$ as in (5.1.2), and $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ is as in (5.3.16), and likewise on the RHS. Then, matching the coefficients of the terms with powers of the Grassmann coordinates $\theta_{i=1,2,3}$ and $\bar{\theta}_{i=1,2,3}$ on the two sides of (5.4.8), gives relations among the primary and descendant three-point functions analogous to (5.3.18) and (5.3.19). For ℓ even, (5.4.11) gives a contribution when we take all three operators to be primary, setting all Grassmann coordinates to zero; the coefficient $c_{JJ\mathcal{O}_\ell}$ of this primary contribution determines all descendant three-point function. For ℓ odd, the three-point function with all three operators primary vanishes, as does (5.4.12) when all Grassmann coordinates are set to zero, but there are still non-zero superconformal descendant contributions and expanding (5.4.12) gives relations among them.

The three-point function result (5.4.8), with (5.4.11) and (5.4.12), can be expanded in the Grassmann coordinates. To illustrate this, let's now expand the three-point function in $\theta_3 \equiv \theta$ and $\bar{\theta}_3 \equiv \bar{\theta}$, setting $\theta_{1,2} = 0$, and $\bar{\theta}_{1,2} = 0$. Using (5.3.9) we

have

$$X_+^\mu|_{\theta_{i=1,2}=\bar\theta_{i=1,2}=0} = Z^\mu + \boxed{2Y^{\mu\nu}\theta\sigma_\nu\bar\theta} + Z^2\left(\boxed{\frac{x_{12}^\mu}{r_{12}}} - Z^\mu\right)\theta^2\bar\theta^2,$$

$$X_-^\mu|_{\theta_{i=1,2}=\bar\theta_{i=1,2}=0} = -2(Z^2\eta^{\mu\nu} - 2Z^\mu Z^\nu)\theta\sigma_\nu\bar\theta,$$

$$\left.\frac{\Theta^2\bar\Theta^2}{X\cdot\bar X}\right|_{\theta_{i=1,2}=\bar\theta_{i=1,2}=0} = Z^2\theta^2\bar\theta^2.$$

One can also find

$$X\cdot\bar X|_{\theta_{i=1,2}=\bar\theta_{i=1,2}=0} = Z^2 - 2Z^4\left(2 + \boxed{\frac{x_{13}\cdot x_{23}}{r_{12}}}\right)\theta^2\bar\theta^2,$$

$$X_+\cdot X_-|_{\theta_{i=1,2}=\bar\theta_{i=1,2}=0} = 2Z^2 Z^\mu\theta\sigma_\mu\bar\theta.$$

So, for example, (5.4.10) becomes

$$\left.\frac{t^\mu_{\mathcal{JJO}_{\ell=1}}}{x_{\bar13}{}^2 x_{3\bar1}{}^2 x_{\bar23}{}^2 x_{3\bar2}{}^2}\right|_{\theta_{i=1,2}=\bar\theta_{i=1,2}=0} = -2\frac{C_{JJO}}{r_{12}^2}Z^{\Delta-1}\left(Z^2\eta^{\mu\nu} - \frac{\Delta+1}{\Delta-2}Z^\mu Z^\nu\right)\theta\sigma_\nu\bar\theta.$$

The boxed terms above drop out for primary correlation functions. Indeed, with no loss in generality, by using superconformal symmetry to map $(z_1, z_2, z_3) \to (0, x_2 = \infty, z_3 = z)$, the boxed terms map to zero, as discussed around (5.3.11).

For ℓ even, the results for $\langle JJA^{\mu_1\cdots\mu_\ell}\rangle$, $\langle JJL^{\mu\mu_1\cdots\mu_\ell}\rangle$, and $\langle JJD^{\mu_1\cdots\mu_\ell}\rangle$ coincide with those found in [19] for the corresponding quantities with JJ replaced with $\phi\bar\phi$, while $\langle JJM^{\mu\mu_1\cdots\mu_\ell}\rangle = 0$ and $\langle JJN^{\mu_2\cdots\mu_\ell}\rangle = 0$. Accounting for the distinction [19] between $L^{\mu\mu_1\cdots\mu_\ell}$ and $L^{\mu\mu_1\cdots\mu_\ell}_{\text{prim}}$ and also between $D^{\mu_1\cdots\mu_\ell}$ and $D^{\mu_1\cdots\mu_\ell}_{\text{prim}}$, see (5.3.17), we find, for ℓ even,

$$\langle JJA^{\mu_1\cdots\mu_\ell}\rangle = c_{JJ\mathcal{O}_\ell}\frac{Z^{\Delta-\ell}}{r_{12}^2}Z^{\mu_1}\cdots Z^{\mu_\ell},$$

$$\langle JJD^{\mu_1\cdots\mu_\ell}_{\text{prim}}\rangle = -c_{JJ\mathcal{O}_\ell}\frac{\Delta(\Delta+\ell)(\Delta-\ell-2)}{8(\Delta-1)}\frac{Z^{\Delta+2-\ell}}{r_{12}^2}Z^{\mu_1}\cdots Z^{\mu_\ell}.$$

In addition, $\langle JJM^{\mu\mu_1\cdots\mu_\ell}_{\text{prim}}\rangle = 0$ and $\langle JJN^{\mu_2\cdots\mu_\ell}_{\text{prim}}\rangle = 0$, because the three-point function

with JJ can involve only even-spin operators.

Likewise, for ℓ odd, $\langle JJA^{\mu_1\cdots\mu_\ell}\rangle = 0$ and $\langle JJD_{\mathrm{prim}}^{\mu_1\cdots\mu_\ell}\rangle = 0$, and the non-zero primary three-point functions are

$$\langle JJM_{\mathrm{prim}}^{\mu\mu_1\cdots\mu_\ell}\rangle = 2c_{JJ\mathcal{O}_\ell}\frac{\ell(\Delta+\ell)}{\Delta-2}\frac{Z^{\Delta-\ell}}{r_{12}^2}Z^\mu Z^{\mu_1}\cdots Z^{\mu_\ell},$$

$$\langle JJN_{\mathrm{prim}}^{\mu_2\cdots\mu_\ell}\rangle = -2c_{JJ\mathcal{O}_\ell}\frac{(\ell+2)(\Delta-\ell-2)}{\Delta-2}\frac{Z^{\Delta+2-\ell}}{r_{12}^2}Z^{\mu_2}\cdots Z^{\mu_\ell},$$

with $\langle JJA^{\mu_1\cdots\mu_\ell}\rangle = 0$ and $\langle JJD_{\mathrm{prim}}^{\mu_1\cdots\mu_\ell}\rangle = 0$. In all of the above it's to be understood that the Z^μ's are symmetrized with the traces removed. For all ℓ, $\langle J(x_1)J(x_2)L_{\mathrm{prim}}^{\mu\mu_1\cdots\mu_\ell}\rangle = 0$, because the primary three-point function necessarily involves only the single coordinate Z^μ, and it is impossible to use that to build an operator with the right Lorentz index structure to match $L_{\mathrm{prim}}^{\mu\mu_1\cdots\mu_\ell}$.

Summarizing, we find the relations

$$c_{JJD_{\ell;\mathrm{prim}}} = -\frac{\Delta(\Delta+\ell)(\Delta-\ell-2)}{8(\Delta-1)}c_{JJA_\ell},$$

$$c_{JJN_{\ell-1}} = -\frac{(\ell+2)(\Delta-\ell-2)}{\ell(\Delta+\ell)}c_{JJM_{\ell+1}}, \tag{5.4.13}$$

$$c_{JJL_{\mathrm{prim}}} = 0.$$

A check on these results is that $c_{JJD_{\ell;\mathrm{prim}}}$ and c_{JJN_ℓ} properly vanish when $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ saturates the unitarity bound, $\Delta = \ell + 2$, as then the components N_{prim} and D_{prim} become null states and must vanish. As a special case, for $\ell = 1$ and $\Delta_{\mathcal{O}} = 3$, we have $\mathcal{O}^\mu = \mathcal{T}^\mu$, the Ferrara–Zumino supermultiplet, where $M^{\mu\nu} \sim T^{\mu\nu}$ and $N \sim T_\mu{}^\mu = 0$.

Upon going to components, the resulting two- and three-point functions can be converted to expressions for the OPE coefficients, including conformal descendants, as in (5.2.12). The superconformal descendant relations can then be determined by using the two-point and three-point function relations discussed in the previous paragraph. A more efficient approach would be to convert directly in superspace, from the two-point and three-point function results above, to sOPE expressions. A special case has been

explicitly worked out in [18], as outlined after (5.3.14). For our case of interest here, i.e. two conserved currents,

$$\mathcal{J}(z_1)\mathcal{J}(z_2) = \sum_{\substack{\text{sprimary} \\ \mathcal{O}_\Delta^{(\ell)}}} c_{JJ}^{\mathcal{O}} F_\Delta^{(\ell)}(z_{12}, \partial_{x_2}, \partial_{\theta_2}, \partial_{\bar{\theta}_2}) \mathcal{O}_\Delta^\ell(z_2),$$

with F determined by requiring that using this and two-point functions (5.3.6) on the LHS of (5.4.8) reproduces the RHS of (5.4.8). For example, for $\ell = 0$, F satisfies

$$\frac{1}{x_{\bar{1}3}^2 x_{\bar{3}1}^2 x_{\bar{2}3}^2 x_{\bar{3}2}^2} t_{\mathcal{JJO}}^{(\ell=0)}(X_3, \Theta_3, \bar{\Theta}_3) = c_{JJ\mathcal{O}} F_\Delta^{(\ell=0)}(z_{12}, \partial_{x_2}, \partial_{\theta_2}, \partial_{\bar{\theta}_2}) \frac{1}{x_{\bar{2}3}^{\Delta_\mathcal{O}} x_{\bar{3}2}^{\Delta_\mathcal{O}}},$$

where t on the LHS is given in (5.4.9).

5.5. Four-point function conformal blocks

Four-point functions (more generally n-point functions) can be reduced and computed via the OPE. For a four-point function $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_r(x_3)\mathcal{O}_s(x_4)\rangle$, one can apply the OPE (5.1.1) to $\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)$, and also to $\mathcal{O}_r(x_3)\mathcal{O}_s(x_4)$, reducing the four-point function to sums of two-point functions between the resultants on the RHS of the two OPE pairs:

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_r(x_3)\mathcal{O}_s(x_4)\rangle = \sum_{\substack{\text{primary} \\ \mathcal{O}_k}} \frac{1}{r_{12}^{\frac{1}{2}(\Delta_i+\Delta_j)} r_{34}^{\frac{1}{2}(\Delta_r+\Delta_s)}} c_{ijk} c^{k\ell} c_{\ell rs} g_{\Delta_k, \ell_k}(u, v),$$

$$(5.5.1)$$

where $u \equiv r_{12}r_{34}/r_{13}r_{24} \equiv z\bar{z}$ and $v \equiv r_{14}r_{23}/r_{13}r_{24} \equiv (1-z)(1-\bar{z})$ are the two independent conformal cross-ratios for four-point functions. The four-point function conformal

blocks $g_{\Delta,\ell}$ are fixed functions [21, 26] that account for the sum over descendants[4]

$$g_{\Delta,\ell}(u,v) = \frac{z\bar{z}}{z-\bar{z}} \left(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right)$$

$$k_\beta(x) \equiv x^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta; x).$$

(5.5.2)

The decomposition (5.5.1) is in the s channel of the four-point function, and one can of course alternatively compute in the t channel or the u channel, and all three must of course agree. There is a recent and growing literature on exploring these crossing symmetry relation constraints, following [28].

The fact that the sum in (5.5.1) for non-SUSY $\mathcal{N} = 0$ theories can be reduced to a sum over primaries, with the descendant contributions accounted for in the universal conformal block functions $g_{\Delta,\ell}$, is a powerful consequence of the fact that conformal symmetry completely determines the descendant contributions to the OPE from those of the primaries. As we have emphasized, the analogous statement generally does not hold for superconformal primaries. So, in superconformal theories, there is generally no analog of (5.5.1) involving only a sum over only the superconformal primaries. In a nutshell, there is no universal notion of "superconformal blocks" analogous to (5.5.2). One can define superconformal blocks for correlation functions of short multiplets, as we'll discuss and review, but they depend on the particular operators in the correlation function and are still not universal.

In this section, we will discuss the $\mathcal{N} = 1$ conformal blocks for $\langle JJJJ \rangle$ and $\langle JJ\phi\phi^* \rangle$. These two cases are expected to be nicer than generic four-point functions in $\mathcal{N} = 1$ SCFTs, because the operators are in shortened representations, and that determines the coefficients of all superconformal descendants in the intermediate channel in terms of those of the superconformal primaries.[5]

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[4]As in [27], we find it convenient to modify the original definition of $g_{\Delta,\ell}$ by dropping a $(-\frac{1}{2})^\ell$ factor: $g_{\Delta,\ell}^{\text{here}} = (-2)^\ell g_{\Delta,\ell}^{\text{D\&O}}$.

[5]As we emphasized, that seems to not be the case for generic $\mathcal{N} = 1$ operators, so it seems that generic four-point functions can not be reduced to a set of $\mathcal{N} = 1$ superconformal blocks depending only on the superconformal primaries.

5.5.1. Review of the $\mathcal{N} = 1$ conformal blocks for $\langle \phi\phi^*\phi\phi^* \rangle$ [19, 20]

The four point function of two chiral and two anti-chiral operators can be expanded as

$$\langle \phi(x_1)\phi^*(x_2)\phi(x_3)\phi^*(x_4) \rangle = \frac{1}{r_{12}^{\Delta_\phi} r_{34}^{\Delta_\phi}} \sum_{\mathcal{O}_\ell \in \phi \times \phi^*} \frac{(c_{\phi\phi^* A_\ell})^2}{c_{A_\ell A_\ell}} \mathcal{G}_{\Delta,\ell}^{\phi\phi^*;\phi\phi^*}(u,v)$$

where $\mathcal{G}_{\Delta,\ell}^{\phi\phi^*;\phi\phi^*}(u,v)$ is a superconformal block that account for the s-channel OPE sum over the A_ℓ, $M_{\ell+1}$, $N_{\ell-1}$, and D_ℓ conformal primaries, along with their descendants. Using (5.3.19) and (5.3.20), the result is [19] (accounting for $g_{\Delta,\ell}^{\text{here}} = (-2)^\ell g_{\Delta,\ell}^{\text{D\&O}}$)

$$\mathcal{G}_{\Delta,\ell}^{\phi\phi^*;\phi\phi^*} = g_{\Delta,\ell} + \frac{\Delta+\ell}{4(\Delta+\ell+1)} g_{\Delta+1,\ell+1} + \frac{\Delta-\ell-2}{4(\Delta-\ell-1)} g_{\Delta+1,\ell-1}$$
$$+ \frac{(\Delta+\ell)(\Delta-\ell-2)}{16(\Delta+\ell+1)(\Delta-\ell-1)} g_{\Delta+2,\ell}.$$

$$(5.5.3)$$

As we have emphasized, there is not a general notion of superconformal block, and the superscript in $\mathcal{G}_{\Delta,\ell}^{\phi\phi^*;\phi\phi^*}$ emphasizes that this superconformal block applies only for this specific channel and four-point function.

Indeed, computing the same $\langle \phi(x_1)\phi^*(x_2)\phi(x_3)\phi^*(x_4) \rangle$ in the channel where the x_1 and x_3 operators are brought together, leads to an intermediate sum over very different classes of operators, corresponding to (5.3.14). We can define $\mathcal{G}_{\Delta,\ell}^{\phi\phi;\phi^*\phi^*}$ for this class, and the result involves a single $g_{\Delta,\ell}$, rather than the four terms (5.5.3) found in the s channel. See [20] for some of the details. This illustrates that there isn't a universal notion of superconformal blocks, even for different channels of the same four-point function.

5.5.2. The $\mathcal{N} = 1$ conformal blocks for $\langle JJJJ \rangle$ and $\langle JJ\phi\phi^* \rangle$

The four-point current correlator can be expanded as

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle = \frac{1}{r_{12}^2 r_{34}^2} \sum_{\mathcal{O}_{\Delta,\ell}\in J\times J} \frac{(c_{JJA_\ell})^2}{c_{A_\ell A_\ell}} \mathcal{G}_{\Delta,\ell}^{JJ;JJ}(u,v),$$

where the $\mathcal{N} = 1$ superconformal blocks on the RHS account for the sum over the A_ℓ, $M_{\ell+1}$, $N_{\ell-1}$, and D_ℓ primaries in the intermediate operators (5.3.16), along with their descendants. Comparing with (5.5.1), the decomposition in terms of $\mathcal{N} = 0$ blocks simply follows from squaring the coefficients in (5.4.13) and dividing by the normalizations in (5.3.20). For ℓ even we find

$$\mathcal{G}_{\Delta,\ell\text{ even}}^{JJ;JJ} = g_{\Delta,\ell} + \frac{(\Delta+\ell)(\Delta-\ell-2)}{16(\Delta+\ell+1)(\Delta-\ell-1)} g_{\Delta+2,\ell}. \tag{5.5.4}$$

For ℓ odd we find (with here an arbitrary overall normalization choice)

$$\mathcal{G}_{\Delta,\ell\text{ odd}}^{JJ;JJ} = \frac{(\ell+1)^2(\Delta+\ell)}{4(\Delta+\ell+1)} g_{\Delta+1,\ell+1} + \frac{(\ell+2)^2(\Delta-\ell-2)}{\Delta-\ell-1} g_{\Delta+1,\ell-1}. \tag{5.5.5}$$

We can immediately now also obtain the conformal blocks for

$$\langle J(x_1)J(x_2)\phi(x_3)\phi^*(x_4)\rangle = \frac{1}{r_{12}r_{34}^{\Delta_\phi}} \sum_{\mathcal{O}_{\Delta,\ell}} \frac{c_{JJ\mathcal{O}_\ell}c_{\phi\phi^*\mathcal{O}_\ell}}{c_{\mathcal{O}_\ell\mathcal{O}_\ell}} \mathcal{G}_{\Delta,\ell}^{JJ;\phi\phi^*}(u,v),$$

where

$$\mathcal{G}_{\Delta,\ell\text{ even}}^{JJ;\phi\phi^*} = g_{\Delta,\ell} + \frac{(\Delta+\ell)(\Delta-\ell-2)}{16(\Delta+\ell+1)(\Delta-\ell-1)} g_{\Delta+2,\ell}, \tag{5.5.6}$$

$$\mathcal{G}_{\Delta,\ell\text{ odd}}^{JJ;\phi\phi^*} = \frac{(\ell+1)(\Delta+\ell)}{4(\Delta+\ell+1)} g_{\Delta+1,\ell+1} - \frac{(\ell+2)(\Delta-\ell-2)}{2(\Delta-\ell-1)} g_{\Delta+1,\ell-1}. \tag{5.5.7}$$

5.5.3. Connection with Dolan and Osborn's $\mathcal{N} = 2$ conformal blocks for $\langle \varphi\varphi\varphi\varphi \rangle$ [29]

In $\mathcal{N} = 2$ SCFTs, operators are labeled by their $SU(2)_I$ representation $I = 0, \frac{1}{2}, \ldots$, value of I_3, their $U(1)^{\mathcal{N}=2}$ charge, in addition to dimension Δ and spins $(j, \bar{j})$.

Several $\mathcal{N} = 1$ representations assemble together to form a single $\mathcal{N} = 2$ superconformal representation. The $\mathcal{N} = 1$ $U(1)_R^{\mathcal{N}=1}$ is given by (see e.g. [2])

$$R^{\mathcal{N}=1} = \frac{1}{3}R^{\mathcal{N}=2} + \frac{4}{3}I_3.$$

Taking the $\mathcal{N} = 2$ supercharges Q_α^I to have $R^{\mathcal{N}=2}$ charge -1, then $Q_\alpha^{I=1,2}$ has $R^{\mathcal{N}=1}$ charges $1/3$ and -1, with the latter the $\mathcal{N} = 1$ supercharge.

In particular, an $\mathcal{N} = 2$ conserved current supermultiplet has primary components with $I = 1$, $R^{\mathcal{N}=0} = 0$, $\Delta = 2$, $\ell = 0$. It consists of an $\mathcal{N} = 1$ conserved current supermultiplet $\mathcal{J}$, plus a $\mathcal{N} = 1$ chiral multiplet and conjugate anti-chiral multiplet $\bar{\Phi}$, with $\Delta = 2$, $\ell = 0$. The primary components were called φ^{ij} in $\varphi^{(ij)}$ of [29], and we denote them as

$$\begin{pmatrix} \varphi^{11} \\ \varphi^{(12)} \\ \varphi^{22} \end{pmatrix} = \begin{pmatrix} \phi \\ J \\ \phi^* \end{pmatrix} = \begin{pmatrix} |I = 1, I_3 = 1\rangle \\ |I = 1, I_3 = 0\rangle \\ |I = 1, I_3 = -1\rangle \end{pmatrix}. \tag{5.5.8}$$

The structure of the four-point function for this $\mathcal{N} = 2$ supermultiplet was considered in [29], and a variety of possible four-point function conformal blocks, corresponding to the possible intermediate operator in the OPE, were presented. The recent work [19] used these results to connect with the $\mathcal{N} = 1$ superconformal blocks $\mathcal{G}^{\phi\phi^*;\phi\phi^*}$. In this section, we connect the $\mathcal{N} = 2$ results of [29] with our $\mathcal{N} = 1$ results for $\mathcal{G}^{JJ;JJ}$ and $\mathcal{G}^{JJ;\phi\phi^*}$.

The $SU(2)_I$ symmetry implies that when we take the $\varphi\varphi$ OPE we get representations $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$, i.e. the RHS can have representations $I = 0, 1, 2$, of $SU(2)_I$. When we consider the $\langle\varphi\varphi\varphi\varphi\rangle$ four-point function, the contributions thus can be labeled by the $I = 0, 1, 2$ values of the intermediate operators. Following [29], we refer to these contributions as A_0, A_1, and A_2, respectively. The $SU(2)_I$ symmetry implies that the various four-point functions in $\langle\varphi\varphi\varphi\varphi\rangle$ are governed by the group theory of Clebsch–Gordan coefficients (following [29], we absorb A_0's Clebsch, $\frac{1}{3}$, into

its normalization):

$$\mathcal{G}^{\mathcal{N}=2|\phi\phi;\phi^*\phi^*} = A_2,$$

$$\mathcal{G}^{\mathcal{N}=2|\phi\phi^*;\phi\phi^*} = A_0 + \frac{1}{2}A_1 + \frac{1}{6}A_2,$$

$$\mathcal{G}^{\mathcal{N}=2|JJ;JJ} = A_0 + \frac{2}{3}A_2, \qquad (5.5.9)$$

$$\mathcal{G}^{\mathcal{N}=2|JJ;\phi\phi^*} = A_0 - \frac{1}{3}A_2.$$

The functions A_0, A_1, and A_2 get independent contributions from each possible $\mathcal{N}=2$ superconformal multiplets that can appear in the intermediate channel of the $\varphi\varphi$ OPE. Since the supercharges have $I = \frac{1}{2}$, each contributing $\mathcal{N}=2$ superconformal multiplet has operators with different I values, that can potentially contribute to all three $A_{I=0,1,2}$. A variety of $\mathcal{N}=2$ supermultiplets and their $A_{0,1,2}$ contributions were presented in [29]. We will apply (5.5.9) to their results to determine the multiplet's contribution $\mathcal{G}^{\mathcal{N}=2|\phi\phi;\phi^*\phi^*}$, $\mathcal{G}^{\mathcal{N}=2|\phi\phi^*;\phi\phi^*}$, $\mathcal{G}^{\mathcal{N}=2|JJ;JJ}$, and $\mathcal{G}^{\mathcal{N}=2|JJ;\phi\phi^*}$. Decomposing the $\mathcal{N}=2$ multiplet into multiplets under the $\mathcal{N}=1$ subalgebra, these $\mathcal{N}=2$ superconformal blocks decompose into sums of $\mathcal{N}=1$ superconformal blocks. The case $\mathcal{G}^{\mathcal{N}=2|\phi\phi^*;\phi\phi^*} \to \mathcal{G}^{\mathcal{N}=1|\phi\phi^*;\phi\phi^*}$ was presented in [19], and here we'll similarly discuss a few simple examples of (5.5.9).

One class of examples are the shortened $\mathcal{N}=2$ multiplets containing at most twist $\Delta - \ell = 2$ operators. Quoting [29] (with $g^{\text{D\&O}}_{\Delta,\ell} = (-2)^{-\ell}g^{\text{here}}_{\Delta,\ell}$), these have

$$A_0 = g_{\Delta=\ell+2,\ell} + \frac{(\ell+2)^2}{4(2\ell+3)(2\ell+5)}g_{\Delta=\ell+4,\ell+2},$$

$$A_1 = g_{\Delta=\ell+3,\ell+1}, \qquad A_2 = 0. \qquad (5.5.10)$$

An example in this class is the $\mathcal{N}=2$ conserved current multiplet (5.5.8), which corresponds to setting $\ell = -1$ in (5.5.10). Another example in this class is the $\mathcal{N}=2$ stress-energy tensor multiplet, corresponding to $\ell = 0$ in (5.5.10); this $\mathcal{N}=2$ multiplet contains the $\mathcal{N}=1$ stress-tensor multiplet (5.3.1) together with the $\mathcal{N}=1$ current

multiplets of $SU(2)_I$. We see from (5.5.9) that, since $A_2 = 0$, no operators in this class contribute to $\mathcal{G}^{\mathcal{N}=2|\phi\phi;\phi^*\phi^*}$. Their contributions to $\mathcal{G}^{\mathcal{N}=2|\phi\phi^*;\phi\phi^*}$ fit with the decomposition of these $\mathcal{N} = 2$ multiplets into $\mathcal{N} = 1$ multiplets and the results of [19], as was presented there. The blocks given in (5.5.4), (5.5.5), (5.5.6), (5.5.7) for this case, $\Delta = \ell + 2$, contain only a single $\mathcal{N} = 0$ block, $\mathcal{G}^{JJ;JJ}_{\Delta=\ell+2,\ell} = g_{\Delta=\ell+2,\ell}$ and $\mathcal{G}^{JJ;\phi\phi^*}_{\Delta=\ell+2,\ell} = g_{\Delta=\ell+2,\ell}$. The result (5.5.9) and (5.5.10) for $\mathcal{G}^{\mathcal{N}=2|JJ;JJ}_{\Delta=\ell+2,\ell}$ and $\mathcal{G}^{\mathcal{N}=2|JJ;\phi\phi^*}_{\Delta=\ell+2,\ell}$ contain contributions from two $\mathcal{N} = 1$ real multiplets in the $\mathcal{N} = 2$ multiplet, with primary components $\mathcal{O}_{\Delta=\ell+2,\ell}$ and $\mathcal{O}'_{\Delta=\ell+4,\ell+2}$, and the relative coefficient in (5.5.10) accounts for the $\mathcal{N} = 2$ relation among their OPE coefficients.

To quote a more complicated $\mathcal{N} = 2$ representation multiplet, the contributions to the conformal blocks from the multiplet of operators and descendants when the primary has $R^{\mathcal{N}=2} = 0$, $I = 0$, for general Δ and ℓ, is [29]

$$A_0(u,v) = g_{\Delta,\ell} + \frac{1}{12}g_{\Delta+2,\ell} + \frac{(\Delta+\ell+2)^2}{16(\Delta+\ell+1)(\Delta+\ell+3)}g_{\Delta+2,\ell+2}$$
$$+ \frac{(\Delta-\ell)^2}{16(\Delta-\ell-1)(\Delta-\ell+1)}g_{\Delta+2,\ell-2}$$
$$+ \frac{(\Delta+\ell+2)^2(\Delta-\ell)^2}{256(\Delta+\ell+1)(\Delta+\ell+3)(\Delta-\ell-1)(\Delta-\ell+1)}g_{\Delta+4,\ell},$$
$$A_1(u,v) = g_{\Delta+1,\ell+1} + g_{\Delta+1,\ell-1} + \frac{(\Delta+\ell+2)^2}{16(\Delta+\ell+1)(\Delta+\ell+3)}g_{\Delta+3,\ell+1}$$
$$+ \frac{(\Delta-\ell)^2}{16(\Delta-\ell-1)(\Delta-\ell+1)}g_{\Delta+3,\ell-1},$$
$$A_2(u,v) = g_{\Delta+2,\ell}$$

Using (5.5.9), we can read off the contributions to $\mathcal{G}^{\mathcal{N}=2}$ from this representation. The case $\mathcal{G}^{\mathcal{N}=2|\phi\phi^*;\phi\phi^*}$ was considered in [19] and decomposed there in terms of the $\mathcal{N} = 1$ blocks. The other cases in (5.5.9) can be similarly analyzed.

5.6. Discussion & Conclusion

The current-current (s)OPE $\mathcal{J}(z_1)\mathcal{J}(z_2)$ can have only real $R^{\mathcal{N}=1} = 0$ operators of even spin ℓ and their descendants on the RHS. For non-Abelian groups, odd-ℓ real operators can also contribute, proportional to the group's structure constants f_{abc}. The constraints of $\mathcal{N} = 1$ superconformal symmetry, combined with the current conservation, imply relations among the OPE coefficients, essentially giving the super-descendant coefficients in terms of those of the super-primaries.

We also gave the basic $\mathcal{N} = 1$ superconformal blocks for $\mathcal{G}_{\Delta,\ell}^{JJ;JJ}$ and $\mathcal{G}_{\Delta,\ell}^{JJ;\phi\phi^*}$. These are analogous to the $\mathcal{G}_{\Delta,\ell}^{\phi\phi^*;\phi\phi^*}$ superconformal blocks given in [19] and the $\mathcal{G}_{\Delta,\ell}^{\phi\phi;\phi^*\phi^*}$ described in [20]. The blocks are analogous, but different, illustrating that there are no universal superconformal blocks. In the $\mathcal{N} = 2$ case, we discussed how these cases can be related using the $SU(2)_I$ Clebsch–Gordan coefficients and the results of [29].

We will explore some possible applications of the current-current OPE and superconformal symmetry to general gauge mediation of SUSY breaking in our upcoming paper [10].

Acknowledgments

5.A. The (super)conformal algebra (and sign conventions)

This appendix both reviews standard material, and also attempts to give a consistent set of sign conventions. The literature contains many sign conventions (some with inconsistencies) for the conformal, supersymmetry, and superconformal algebras, so we will here elaborate a bit on our notation. (Our signs agree with e.g. [30].) There

are several places where sign errors can crop up. One is a standard, but often obscured, sign difference when bosonic generators A are replaced with differential operators $\mathcal{A}$ acting on the coordinates,

$$[A, \mathcal{O}] = -\mathcal{A}\mathcal{O}. \tag{5.A.1}$$

This is familiar from quantum mechanics, where $[H, \mathcal{O}] = -i\hbar\partial_t\mathcal{O}$, even though H can be replaced with $\mathcal{H} = +i\hbar\partial_t$. The sign in (5.A.1) accounts for the fact that transformations compose in the opposite order when acting on the coordinates. Indeed, defining another transformation $[B, \mathcal{O}] = -\mathcal{B}\mathcal{O}$, with $\mathcal{B}$ the corresponding differential operator, the differential operators compose in the opposite order

$$AB(\mathcal{O}) \equiv [A, [B, \mathcal{O}]] = -[A, \mathcal{B}\mathcal{O}] = -\mathcal{B}[A, \mathcal{O}] = \mathcal{B}\mathcal{A}\mathcal{O}. \tag{5.A.2}$$

So $[[A, B], \mathcal{O}] = -[\mathcal{A}, \mathcal{B}]\mathcal{O}$, which is consistent with (5.A.1) with $[A, B] = C$ and $[\mathcal{A}, \mathcal{B}] = \mathcal{C}$. Many references, however, do not make a notational distinction between what we're calling A vs $\mathcal{A}$. This issue is compounded in supersymmetry, see also [31] for a very recent careful discussion. As standard, we follow the conventions of Wess & Bagger [32]. The Q analog of (5.A.1) in [32] notation then has an i but, potentially confusingly, in [32] no notational distinction is made between the operator vs the differential operator. In addition, the metric of [32] is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, with Hamiltonian $H = P^0 = -P_0$, so now $[P_0, \mathcal{O}] = +i\hbar\partial_0\mathcal{O}$ and $\mathcal{P}_0 = -i\hbar\partial_0$. We'll elaborate on these and related points in what follows.

Recall (see e.g. [16]) that conformal transformations $x_\mu \to x'_\mu = (gx)_\mu$ are such that $dx'_\mu dx'^\mu = \Omega^g(x)^{-2}dx_\mu dx^\mu$. Beyond translations and rotations, this includes dilatations $x'_\mu = \lambda x_\mu$, with $\Omega^g(x) = \lambda^{-1}$, and special conformal transformations, $x'_\mu = (x_\mu - b_\mu x^2)/\Omega^g(x)$, with $\Omega^g = 1 - 2b \cdot x + b^2 x^2$. An operator is called (quasi-)primary if it transforms under all conformal transformations as $\mathcal{O}(x) \to T(g)\mathcal{O}(x)$, where

$$(T(g)\mathcal{O})^i(x') = \Omega^g(x)^{\Delta_\mathcal{O}} D^i_j \left(\mathcal{R}^g_{\mu\alpha} = \Omega^g \frac{\partial x'_\mu}{\partial x^\alpha} \right) \mathcal{O}^j(x), \tag{5.A.3}$$

where i labels the operator's representation D^i_j of the Lorentz group, and $\Delta_{\mathcal{O}}$ is the operator's scaling dimension. For rotations and boosts, $\Omega^g(x) = 1$ and (5.A.3) is the standard Lorentz transformation of operators, with $\mathcal{R}^g_{\mu\alpha}$ the rotation or boost. For dilatations, $\mathcal{R}^g_{\mu\alpha} = \delta_{\mu\alpha}$, so D^i_j is the identity, and (5.A.3) is the standard scaling of operators with their scaling dimension $\Delta_{\mathcal{O}}$. For special conformal transformations only, (5.A.3) proves restrictive, distinguishing the primary operators from the descendants.

On the LHS of (5.A.3) we've transformed both the operator and the coordinate, but we should replace $x \to g^{-1}x$ on both sides of (5.A.3) to get how the transformation acts on just the operator. For example, the Poincaré generators act on the coordinates as $x^\mu \to x'^\mu = g(x^\mu)$, and act on operators as $g : \mathcal{O}^i(x) \to \mathcal{O}'^i(x) = (U(g)\mathcal{O}(x)U(g)^{-1})^i = D^i_j(g)\mathcal{O}(g^{-1}(x))$, with $U(g)$ the appropriate unitary transformation. Under general translations of operators forward by a^μ, via opposite action on the coordinates, $g_a : x^\mu \to x^\mu - a^\mu$, then $g_a : \mathcal{O}(x) \to \mathcal{O}'(x) = U(a)\mathcal{O}(x)U(a)^{-1} = \mathcal{O}(x^\mu + a^\mu)$, with $U(a) = e^{-iP_\mu a^\mu}$. We then have $[P_\mu, \mathcal{O}(x)] = i\partial_\mu \mathcal{O}(x)$. So the differential operator, as in (5.A.1), is $\mathcal{P}^\mu = -i\partial_\mu$. The minus sign in (5.A.1) and order reversal in (5.A.2) are related to the g^{-1} action on the coordinates.

The dilatation generator acts on the coordinates as $g_\delta : x^\mu \to e^{-\delta}x^\mu$, $\Omega^{g_\delta}(x) = e^\delta$, so $g_\delta : \mathcal{O}(x) \to U(\delta)\mathcal{O}(x)U(\delta)^{-1} = e^{\Delta_{\mathcal{O}}\delta}\mathcal{O}(e^\delta x)$, where $U(\delta) = e^{i\delta D}$. This implies $[D, \mathcal{O}] = -i(\Delta_{\mathcal{O}} + x \cdot \partial)\mathcal{O}$; hence the differential operator is $\mathcal{D} = i(\Delta_{\mathcal{O}} + x \cdot \partial)$. Now $g_\delta g_a : \mathcal{O}(x) \to \mathcal{O}(x + a) \to e^{\Delta_{\mathcal{O}}\delta}\mathcal{O}(e^\delta(x + a)) = g_{e^\delta a}g_\delta \mathcal{O}(x)$, so $U(\delta)U(a) = U(e^\delta a)U(\epsilon)$, which implies $[P_\mu, D] = iP_\mu$. The differential operators indeed correspondingly satisfy $[\mathcal{P}_\mu, \mathcal{D}] = i\mathcal{P}_\mu$. We can likewise take $U(b) = e^{-iK_\mu b^\mu}$ to generate special conformal transformations, and consider $U(b)U(\star)$ vs $U(\star)U(b)$ to get $[K_\mu, \star]$. In sum, this yields the conformal group, that's isomorphic to $SO(d, 2)$ in d dimensions:

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu),$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}),$$

$$[D, P_\mu] = -iP_\mu, \qquad [D, K_\mu] = iK_\mu, \qquad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}),$$

where $M_{\mu\nu}$ are the $SO(d-1,1)$ Lorentz generators. Commutators not given are zero.

On a quasi-primary multi-component field $\mathcal{O}_I(x)$ we have

$$[P_\mu, \mathcal{O}_I(x)] = i\partial_\mu \mathcal{O}_I(x), \qquad [D, \mathcal{O}_I(x)] = -i(x \cdot \partial + \Delta_{\mathcal{O}})\mathcal{O}_I(x),$$

$$[M_{\mu\nu}, \mathcal{O}_I(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\mathcal{O}_I(x) - \mathcal{O}_J(x)(s_{\mu\nu})^J{}_I, \qquad (5.A.4)$$

$$[K_\mu, \mathcal{O}_I(x)] = i(x^2 \partial_\mu - 2x_\mu x \cdot \partial - 2\Delta_{\mathcal{O}} x_\mu)\mathcal{O}_I(x) + 2(s_{\mu\nu})^J{}_I x^\nu \mathcal{O}_J,$$

where $(s_{\mu\nu})^J{}_I$ are the appropriate finite-dimensional spin matrices obeying the $M_{\mu\nu}$ algebra.

As an illustration of the order reversal in (5.A.2), consider $[K_\nu, [P_\mu, \mathcal{O}(0)]]$ and compare that to $[P_\mu, [K_\nu, \mathcal{O}(0)]$ on a scalar primary operator at the origin. The latter vanishes, since K_ν annihilates the scalar primary at the origin. That is compatible with $[P_\mu, [K_\nu, \mathcal{O}(0)] = \mathcal{K}_\nu \mathcal{P}_\mu \mathcal{O}(0)$ and $\mathcal{P}_\mu = -i\partial_\mu$ and $\mathcal{K}_\nu = -i(x^2 \partial_\nu - 2x_\nu x \cdot \partial - 2\Delta_{\mathcal{O}} x_\nu)$. The opposite order properly gives a non-zero result, $[K_\nu, [P_\mu, \mathcal{O}]]|_{x=0} = \mathcal{P}_\mu \mathcal{K}_\nu \mathcal{O}|_{x=0} = 2\Delta_{\mathcal{O}} \eta_{\mu\nu} \mathcal{O}(0) = -2i\eta_{\mu\nu} D\mathcal{O}|_{x=0}$.

We define the supersymmetry fermionic variations of operators as

$$\delta_\xi \mathcal{O} = i[\xi Q + \bar{\xi}\bar{Q}, \mathcal{O}] = (\xi \mathcal{Q} + \bar{\xi}\bar{\mathcal{Q}})\mathcal{O}, \qquad (5.A.5)$$

where the i after the first equality insures that, if $\mathcal{O}$ is real, then so is $\delta_\xi \mathcal{O}$.[6] In the second equality that i is absent, and we use the superspace differential operators of [32],

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \quad \text{and} \quad \bar{\mathcal{Q}}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \qquad (5.A.6)$$

As in (5.A.2), the differential operators compose in the opposite order. Note that

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu = -2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu; \qquad (5.A.7)$$

---

[6]Recall $(\xi Q)^\dagger = \bar{\xi}\bar{Q}$ in [32] notation, where $\xi Q \equiv \xi^\alpha Q_\alpha = -\xi_\alpha Q^\alpha = Q\xi$, and $\bar{\xi}\bar{Q} \equiv \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} = -\bar{\xi}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}} = \bar{Q}\bar{\xi}$.

the last sign looks off, but it'll be OK, since (5.A.5) gives ($[\kappa Q, \bar{\xi}\bar{Q}] = \kappa^\alpha \bar{\xi}^{\dot\alpha}\{Q_\alpha, \bar{Q}_{\dot\alpha}\}$)

$$(\delta_\kappa \delta_\xi - \delta_\xi \delta_\kappa)\mathcal{O} = -(\kappa^\alpha \bar{\xi}^{\dot\alpha} - \xi^\alpha \bar{\kappa}^{\dot\alpha})[\{Q_\alpha, \bar{Q}_{\dot\alpha}\}, \mathcal{O}] = -(\kappa^\alpha \bar{\xi}^{\dot\alpha} - \xi^\alpha \bar{\kappa}^{\dot\alpha})2\sigma^\mu_{\alpha\dot\alpha}[P_\mu, \mathcal{O}],$$

and also

$$(\delta_\kappa \delta_\xi - \delta_\xi \delta_\kappa)\mathcal{O} = -(\kappa^\alpha \bar{\xi}^{\dot\alpha} - \xi^\alpha \bar{\kappa}^{\dot\alpha})\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}\}\mathcal{O} = (\kappa^\alpha \bar{\xi}^{\dot\alpha} - \xi^\alpha \bar{\kappa}^{\dot\alpha})2\sigma^\mu_{\alpha\dot\alpha}\mathcal{P}_\mu\mathcal{O},$$

consistent with $[P_\mu, \mathcal{O}] = -\mathcal{P}_\mu\mathcal{O}$. In short, if we use the notation of [32] for the fermionic generators, the analog of (5.A.1) is

$$Q(\mathcal{O}) \equiv [Q, \mathcal{O}\} = -i\mathcal{Q}\mathcal{O}. \tag{5.A.8}$$

For a chiral superfield, $\Phi = \phi + \sqrt{2}\theta\psi + \cdots$, with $\bar{Q}_{\dot\alpha}(\Phi) = 0$, we have $Q_\alpha(\phi) = -i\sqrt{2}\psi_\alpha$ etc. For a real superfield $\mathcal{J} = J + i\theta j - i\bar{\theta}\bar{j} + \cdots$, we find e.g. $Q_\alpha(J) = j_\alpha$ and $\bar{Q}_{\dot\alpha}(J) = -\bar{j}_{\dot\alpha}$.

The superconformal algebra includes the usual supercharges Q_α, $\bar{Q}_{\dot\alpha}$, superconformal supercharges, S^α, $\bar{S}^{\dot\alpha}$, and the $U(1)_R$-current generator R. The superconformal algebra includes, in addition to the conformal-algebra commutators,

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu, \qquad \{\bar{S}^{\dot\alpha}, S^\alpha\} = 2\bar{\sigma}^{\mu\dot\alpha\alpha}K_\mu,$$

$$\{Q_\alpha, S^\beta\} = -i(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta M_{\mu\nu} + \delta_\alpha{}^\beta(2iD + 3R),$$

$$\{\bar{S}^{\dot\alpha}, \bar{Q}_{\dot\beta}\} = -i(\bar{\sigma}^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta}M_{\mu\nu} - \delta^{\dot\alpha}{}_{\dot\beta}(2iD - 3R),$$

$$[D, Q_\alpha] = -\tfrac{1}{2}iQ_\alpha, \qquad [D, \bar{Q}_{\dot\alpha}] = -\tfrac{1}{2}i\bar{Q}_{\dot\alpha}, \qquad [D, S^\alpha] = \tfrac{1}{2}iS^\alpha, \qquad [D, \bar{S}^{\dot\alpha}] = \tfrac{1}{2}i\bar{S}^{\dot\alpha},$$

$$[R, Q_\alpha] = -Q_\alpha, \qquad [R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}, \qquad [R, S^\alpha] = S^\alpha, \qquad [R, \bar{S}^{\dot\alpha}] = -\bar{S}^{\dot\alpha},$$

$$[K^\mu, Q_\alpha] = -\sigma^\mu_{\alpha\dot\alpha}\bar{S}^{\dot\alpha}, \qquad [K^\mu, \bar{Q}_{\dot\alpha}] = \sigma^\mu_{\alpha\dot\alpha}S^\alpha,$$

$$[P^\mu, \bar{S}^{\dot\alpha}] = -\bar{\sigma}^{\mu\dot\alpha\alpha}Q_\alpha, \qquad [P^\mu, S^\alpha] = \bar{\sigma}^{\mu\dot\alpha\alpha}\bar{Q}_{\dot\alpha},$$

$$[M_{\mu\nu}, Q_\alpha] = -i\sigma_{\mu\nu\alpha}{}^\beta Q_\beta, \qquad [M_{\mu\nu}, \bar{Q}_{\dot\alpha}] = i\bar{\sigma}_{\mu\nu}{}^{\dot\beta}{}_{\dot\alpha}\bar{Q}_{\dot\beta},$$

$$[M_{\mu\nu}, S_\alpha] = -i\sigma_{\mu\nu\alpha}{}^\beta S_\beta, \qquad [M_{\mu\nu}, \bar{S}_{\dot\alpha}] = i\bar\sigma_{\mu\nu}{}^{\dot\beta}{}_{\dot\alpha} \bar{S}_{\dot\beta}.$$

The action of the superconformal generators on superfields was given in [14] in a very efficient and compressed notation, so we'll unpack it a bit here, and write the variations as differential operators acting on superspace, with the -1 of (5.A.1) for the bosonic generators and the $-i$ of (5.A.8) for the fermionic generators. The $\mathcal{P}_\mu$, $\mathcal{Q}_\alpha$, and $\bar{\mathcal{Q}}_{\dot\alpha}$ are as given in (5.A.6) and (5.A.7). The $\mathcal{D}$ and $\mathcal{K}_\mu$ operators include Grassmann additions to the expressions found from (5.A.4), e.g. dilatations act as $g_\delta : \mathcal{O}(x, \theta, \bar\theta) \rightarrow e^{i\delta D}\mathcal{O}(x, \theta, \bar\theta)e^{-i\delta D} = e^{\Delta_\mathcal{O}}\mathcal{O}(e^\delta x, e^{\delta/2}\theta, e^{\delta/2}\bar\theta)$, which gives $[D, \mathcal{O}] = -\mathcal{D}\mathcal{O}$ with

$$\mathcal{D} = i\left[x \cdot \partial + \frac{1}{2}\left(\theta\frac{\partial}{\partial\theta} + \bar\theta\frac{\partial}{\partial\bar\theta}\right) + \Delta\right].$$

For a $U(1)_R$ transformation,

$$g_R : \mathcal{O}(x, \theta, \bar\theta) \rightarrow e^{i\alpha R}\mathcal{O}(x, \theta, \bar\theta)e^{-i\alpha R} = e^{i\alpha r_\mathcal{O}}\mathcal{O}(x, e^{-i\alpha}\theta, e^{i\alpha}\bar\theta),$$

so $[R, \mathcal{O}] = -\mathcal{R}\mathcal{O}$ with

$$\mathcal{R} = -r_\mathcal{O} + \theta\frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial\bar\theta}.$$

Finally, the special superconformal generators act on superfields as in (5.A.5),

$$\delta_\eta\mathcal{O} = i[\eta S + \bar\eta\bar{S}, \mathcal{O}] = (\eta\mathcal{S} + \bar\eta\bar{\mathcal{S}})\mathcal{O},$$

with $\mathcal{S}^\alpha$ and $\bar{\mathcal{S}}^{\dot\alpha}$ the differential operators acting on superspace, and we read off the transformation from that given in [14]: in the notation there

$$\delta\mathcal{O}^i(z) = -\mathcal{L}\mathcal{O}^i(z) + [\hat\omega_\alpha{}^\beta(z_+)(s_\beta{}^\alpha)^i{}_{i'} + \hat{\bar\omega}^{\dot\alpha}{}_{\dot\beta}(z_-)(\bar{s}^{\dot\beta}{}_{\dot\alpha})^i{}_{i'}]\mathcal{O}^{i'}(z)$$

$$- 2q\sigma(z_+)\mathcal{O}^i(z) - 2\bar{q}\bar\sigma(z_-)\mathcal{O}^i(z),$$

where $\mathcal{L} = (v^\mu(z_+) - 2i\lambda(z_+)\sigma^\mu\bar\theta)\partial_\mu + \lambda^\alpha(z_+)D_\alpha + \bar\lambda_{\dot\alpha}(z_-)\bar{D}^{\dot\alpha}$, and s and $\bar{s}$ act, respectively,

on dotted and undotted indices, and form, respectively, spin-j and spin-$\bar{j}$ representations of the algebra. Setting to zero the parameters for other transformations, we have

$$v^\mu = -2\theta\sigma^\mu\tilde{x}_+\eta,$$

$$\lambda^\alpha = -i(\bar{\eta}\tilde{x}_+)^\alpha + 2\eta^\alpha\theta^2, \qquad \bar{\lambda}^{\dot\alpha} = i(\tilde{x}_+\eta)^{\dot\alpha} + 2\bar{\eta}^{\dot\alpha}\bar{\theta}^2,$$

$$\hat{\omega}_\alpha{}^\beta = 4\eta_\alpha\theta^\beta + 2\delta_\alpha{}^\beta\theta\eta, \qquad \hat{\bar{\omega}}^{\dot\alpha}{}_{\dot\beta} = -4\bar{\theta}^{\dot\alpha}\bar{\eta}_{\dot\beta} - 2\delta^{\dot\alpha}{}_{\dot\beta}\bar{\eta}\bar{\theta},$$

$$\sigma = 2\theta\eta, \qquad \bar{\sigma} = 2\bar{\eta}\bar{\theta},$$

where $\tilde{x}_+ = \tilde{x} + 2i\bar{\theta}\theta$. In our conventions we then find

$$\mathcal{S}^\alpha = ix\cdot\bar{\sigma}^{\dot\alpha\alpha}\bar{\mathcal{Q}}_{\dot\alpha} + 2\theta^\alpha\left(\bar{\theta}\frac{\partial}{\partial\bar{\theta}} + \Delta + \frac{3}{2}r\right) + 2\theta^\beta s_\beta{}^\alpha + \theta^2\epsilon^{\alpha\beta}\left(\mathcal{Q}_\beta + \frac{\partial}{\partial\theta^\beta}\right).$$

References

[1] N. Seiberg, *"Electric-magnetic duality in supersymmetric non-Abelian gauge theories"*, Nucl.Phys. **B435**, 129 (1995), `hep-th/9411149`.

[2] F. Benini, Y. Tachikawa & B. Wecht, *"Sicilian gauge theories and $\mathcal{N}=1$ dualities"*, JHEP **1001**, 088 (2010), `arXiv:0909.1327`.

[3] M. Luty & R. Sundrum, *"Anomaly mediated supersymmetry breaking in four dimensions, naturally"*, Phys.Rev. **D67**, 045007 (2003), `hep-th/0111231`.

[4] A.E. Nelson & M.J. Strassler, *"Suppressing flavor anarchy"*, JHEP **0009**, 030 (2000), `hep-ph/0006251`.

[5] D. Poland & D. Simmons-Duffin, *"Superconformal Flavor Simplified"*, JHEP **1005**, 079 (2010), `arXiv:0910.4585`.

[6] N. Craig, *"Simple Models of Superconformal Flavor"*, `arXiv:1004.4218`.

[7] T.S. Roy & M. Schmaltz, *"Hidden solution to the $\mu/B\mu$ problem in gauge mediation"*, Phys.Rev. **D77**, 095008 (2008), `arXiv:0708.3593`.

[8] H. Murayama, Y. Nomura & D. Poland, *"More visible effects of the hidden sector"*, Phys.Rev. **D77**, 015005 (2008), `arXiv:0709.0775`.

[9] P. Meade, N. Seiberg & D. Shih, *"General Gauge Mediation"*, Prog.Theor.Phys.Suppl. **177**, 143 (2009), `arXiv:0801.3278`.

[10] J.F. Fortin, K. Intriligator & A. Stergiou, *"To appear"*.

[11] S. Ferrara & B. Zumino, *"Transformation Properties of the Supercurrent"*, Nucl.Phys. **B87**, 207 (1975).

[12] D. Anselmi, D. Freedman, M.T. Grisaru & A. Johansen, *"Universality of the operator product expansions of SCFT in four-dimensions"*, Phys.Lett. **B394**, 329 (1997), `hep-th/9608125`.

[13] D. Anselmi, D. Freedman, M.T. Grisaru & A. Johansen, *"Nonperturbative formulas for central functions of supersymmetric gauge theories"*, Nucl.Phys. **B526**, 543 (1998), `hep-th/9708042`.

[14] H. Osborn, *"$\mathcal{N} = 1$ superconformal symmetry in four-dimensional quantum field theory"*, Annals Phys. **272**, 243 (1999), `hep-th/9808041`.

[15] S. Ferrara, A. Grillo & R. Gatto, *"Tensor representations of conformal algebra and conformally covariant operator product expansion"*, Annals Phys. **76**, 161 (1973).

[16] H. Osborn & A. Petkos, *"Implications of conformal invariance in field theories for general dimensions"*, Annals Phys. **231**, 311 (1994), `hep-th/9307010`.

[17] J.H. Park, *"$\mathcal{N} = 1$ superconformal symmetry in four-dimensions"*, Int.J.Mod.Phys. **A13**, 1743 (1998), `hep-th/9703191`.

[18] F. Dolan & H. Osborn, *"Implications of $\mathcal{N} = 1$ superconformal symmetry for chiral fields"*, Nucl.Phys. **B593**, 599 (2001), `hep-th/0006098`.

[19] D. Poland & D. Simmons-Duffin, *"Bounds on 4D Conformal and Superconformal Field Theories"*, `arXiv:1009.2087`.

[20] A. Vichi, *"Improved bounds for CFT's with global symmetries"*, `arXiv:1106.4037`.

[21] F. Dolan & H. Osborn, *"Conformal four point functions and the operator product expansion"*, Nucl.Phys. **B599**, 459 (2001), `hep-th/0011040`.

[22] D.Z. Freedman, K. Johnson & J.I. Latorre, *"Differential regularization and renormalization: A New method of calculation in quantum field theory"*, Nucl.Phys. **B371**, 353 (1992).

[23] A. Petkou & K. Skenderis, *"A Nonrenormalization theorem for conformal anomalies"*, Nucl.Phys. **B561**, 100 (1999), `hep-th/9906030`.

[24] E. Barnes, E. Gorbatov, K.A. Intriligator, M. Sudano & J. Wright, *"The Exact superconformal R-symmetry minimizes tau(RR)"*, Nucl.Phys. **B730**, 210 (2005), `hep-th/0507137`.

[25] M. Buican, P. Meade, N. Seiberg & D. Shih, *"Exploring General Gauge Mediation"*, JHEP **0903**, 016 (2009), `arXiv:0812.3668`.

[26] F. Dolan & H. Osborn, *"Conformal partial waves and the operator product expansion"*, Nucl.Phys. **B678**, 491 (2004), `hep-th/0309180`.

[27] R. Rattazzi, S. Rychkov & A. Vichi, *"Bounds in 4D Conformal Field Theories with Global Symmetry"*, J.Phys.A **A44**, 035402 (2011), `arXiv:1009.5985`.

[28] R. Rattazzi, V.S. Rychkov, E. Tonni & A. Vichi, *"Bounding scalar operator dimensions in 4D CFT"*, JHEP **0812**, 031 (2008), `arXiv:0807.0004`.

[29] F. Dolan & H. Osborn, *"Superconformal symmetry, correlation functions and the operator product expansion"*, Nucl.Phys. **B629**, 3 (2002), `hep-th/0112251`.

[30] S. Minwalla, *"Restrictions imposed by superconformal invariance on quantum field theories"*, Adv.Theor.Math.Phys. **2**, 781 (1998), `hep-th/9712074`.

[31] T.T. Dumitrescu & N. Seiberg, *"Supercurrents and Brane Currents in Diverse Dimensions"*, `arXiv:1106.0031`.

[32] J. Wess & J. Bagger, *"Supersymmetry and supergravity"*, Princeton University Press (1992).

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Chapter 6

Superconformally Covariant OPE and General Gauge Mediation

6.1. Introduction

Symmetries, even if they are broken, can usefully constrain theories and their dynamics. Soft breaking can be regarded as spontaneous, even if it is actually explicit, via background or spurion expectation values. The symmetry breaking is an IR effect, and the unbroken symmetry can still apply to constrain UV physics. The operator product expansion (OPE) gives a particularly useful way to separate UV physics from long-distance IR physics [1]. We will here discuss and explore applications of breaking an interesting, large symmetry group, superconformal symmetry, via the OPE.

To set the stage, recall how the hadronic world is probed by $e^+e^- \to e^+e^-$ scattering, via an intermediate photon, with the QCD contributions to the electromagnetic current two-point correlator. Writing the current-current OPE schematically as

$$J(x)J(0) = \sum_i c_{JJ}^i \mathcal{O}_i, \qquad (6.1.1)$$

the idea is that c_{JJ}^i "Wilson coefficients" are determined by UV physics, while IR physics determines the expectation values of the operators on the RHS. Keeping only a few leading operators often suffices to obtain good qualitative insights (despite the fact that the errors in these approximations can be difficult to estimate). There is an

extensive literature on using this and related ideas to study the hadronic world, e.g. the classic papers of SVZ on QCD sum rules [2, 3]. The UV physics can be constrained by a larger symmetry group, including broken generators.

Now consider an analogy with the above discussion where, instead of using lepton sector scattering to probe the hadronic sector, we consider scattering of our world's visible sector fields to probe a new, hidden sector, which couples to our world via gauge interactions. The hidden sector then contributes to $SU(3) \times SU(2) \times U(1)$ current correlators, and we can try to employ the power of the OPE to separate UV vs IR physics. The UV theory might be asymptotically free, like QCD, or an interacting, superconformal field theory (SCFT).

Our main motivation is to apply these considerations to general gauge mediation (GGM) [4], where indeed the visible sector soft masses are directly determined by the hidden sector's contribution to the gauge-current two-point correlators [4, 5]:

$$
\begin{aligned}
M_{\text{gaugino}} &= \pi i \alpha \int d^4x \, \langle Q^2 (J(x)J(0)) \rangle, \\
m^2_{\text{sfermion}} &= 4\pi \alpha Y \langle J(x) \rangle + \frac{i\alpha^2 c_2}{8} \int d^4x \ln(x^2 M^2) \langle \bar{Q}^2 Q^2 (J(x)J(0)) \rangle.
\end{aligned}
\tag{6.1.2}
$$

The IR theory is neither conformal nor supersymmetric, e.g. because of messenger masses M and mass splittings $\sqrt{F}$. We will explore the constraints that follow if these soft symmetry breaking effects can be regarded as spontaneous (even if they are actually explicit, via spurions), and therefore effectively restored in the UV. In particular, we apply the UV constraints of superconformal symmetry to constrain the Wilson coefficients in the OPE (6.1.1) in (6.1.2). The IR breaking effects then show up in the IR, via operator expectation values on the RHS of the OPE. Even if the OPE results are only approximate, they give a foothold to consider GGM with non-weakly-coupled hidden sectors.

We discussed some general aspects about the OPE of conserved currents in

superconformal theories in [6]. Leading terms at short distance include

$$J_a(x)J_b(0) = \tau \frac{\delta_{ab}\mathbb{1}}{16\pi^4 x^4} + \frac{kd_{abc}}{\tau}\frac{J_c(0)}{16\pi^2 x^2} + w\frac{\delta_{ab}K(0)}{4\pi^2 x^{2-\gamma_K}} + c_{ab}^i \frac{\mathcal{O}_i(0)}{x^{4-\Delta_i}} + \cdots, \qquad (6.1.3)$$

with a an adjoint index for the (say simple) group G; for simplicity, we will mostly take $G = U(1)$ in what follows. The coefficient τ of the unit operator can be exactly determined from a 't Hooft anomaly $\tau = -3\operatorname{Tr} RFF$ [7] using [8] if needed, and gives the leading coefficient of CFT "matter" to the G gauge beta function, see e.g. [9]. The coefficient k in (6.1.3) of the 't Hooft anomaly $k \sim \operatorname{Tr} G^3$ must vanish or be cancelled to weakly gauge the G symmetry. The operator K in (6.1.3) refers to an operator that classically has $\Delta = 2$, e.g. the Kähler potential, but is not conserved by the interactions so it has anomalous dimension, $\Delta_K = 2 + \gamma_K$. $\mathcal{O}_i(0)$ in (6.1.3) is a generic, real superconformal primary, and $\cdots$ denotes other terms, including superconformal descendants.

Superconformal symmetry together with current conservation implies that the OPE coefficients of superconformal descendants in (6.1.3) are completely determined by those of the superconformal primaries [6].[1] Such relations apply in the far UV, but can be altered for example by RG running of the coefficients, because the theory is ultimately not superconformal. Nevertheless, the UV relations of superconformal symmetry can have approximate vestiges in the IR, to be explored here.

We also explore a related topic, the analyticity properties of the GGM [4] current correlator functions $\widetilde{C}_{a=0,1/2,1}(p^2)$, and $\widetilde{B}_{1/2}(p^2)$. These functions can have cuts when $s = -p^2$ is big enough to create on-shell states, with the cut discontinuity related by the optical theorem to total cross sections for hidden-sector state production, $\sigma_a(\text{vis} \to \text{hid}, s)$, in analogy with QCD production $\sigma(e^+e^- \to \text{hadrons})$:

$$\sigma_a(\text{vis} \to \text{hid}) = \frac{(4\pi\alpha)^2}{s}\frac{1}{2}\operatorname{Disc}\widetilde{C}_a(s), \qquad (6.1.4)$$

---

[1]This is not as obvious as it sounds, because of the existence of the nilpotent superconformal three-point function quantities Θ and $\bar{\Theta}$ of [10], see [6] for additional discussion.

As in QCD applications, we can express visible-sector observables $A(s)$ as s-integrals of their discontinuity along the cut (see Fig. 6.1),

$$A(s) = \frac{1}{2\pi i} \int_{s_0}^{\infty} ds' \frac{\text{Disc } A(s')}{s' - s} = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im } A(s')}{s' - s}, \tag{6.1.5}$$

and then approximate by going to large s', applying the OPE, and keeping only the first few terms in the $1/s$ expansion. We use this to show that the GGM soft masses
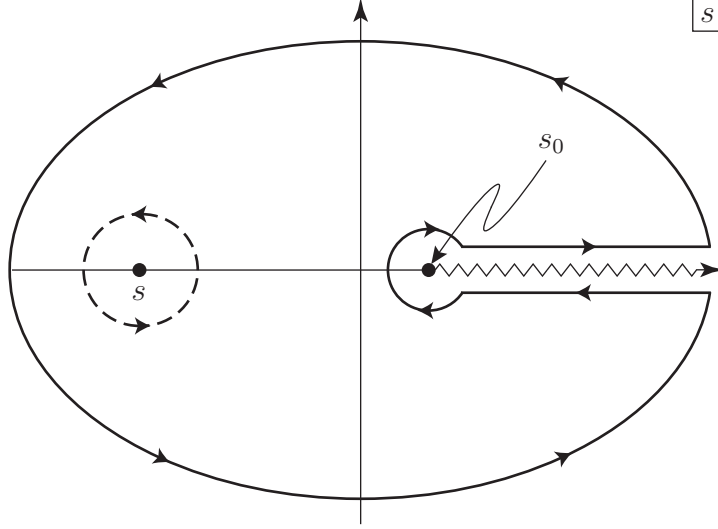
Figure 6.1: The dashed contour is deformed to the solid contour.

(6.1.2) can be approximated in this way in terms of the lowest dimension operators appearing in (6.1.3) that can have non-zero SUSY-descendant expectation values:

$$M_{\text{gaugino}} \approx -\frac{\alpha \pi w \gamma_{Ki}}{8M^2} \langle Q^2(\mathcal{O}_i(0)) \rangle,$$

$$m_{\text{sfermion}}^2 \approx 4\pi\alpha Y \langle J(x) \rangle + \frac{\alpha^2 c_2 w \gamma_{Ki}}{64M^2} \langle \bar{Q}^2 Q^2(\mathcal{O}_i(0)) \rangle, \tag{6.1.6}$$

where w is the coefficient of $K(0)$ in (6.1.3) and γ_{Ki} is the anomalous-dimension matrix which mixes K with the operator $\mathcal{O}_i$.

These considerations also constrain the possibilities for GGM functions $\widetilde{C}_a(s)$, $\widetilde{B}_{1/2}(s)$. We can use (6.1.5) to relate these functions to integrals of their discontinuities (as a spectral representation), and the optical theorem (6.1.4) to relate these discon-

tinuities to kinematic phase space factors. For example, for producing two scalars of masses m_1 and m_2,

$$\sigma_0(s) = \frac{\lambda^{1/2}(s, m_1, m_2)}{8\pi s^2}|\mathcal{M}|^2, \tag{6.1.7}$$

where the phase-space prefactor involves the standard (see e.g. [11]) factor

$$\lambda^{1/2}(s, m_1, m_2) = 2\sqrt{s}|\vec{p}| = \sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}\,\theta(s - (m_1 + m_2)^2), \tag{6.1.8}$$

where $|\vec{p}|$ is the CM momentum of the produced on-shell scalars (the step function θ indicates that it is non-zero for real $s \geq (m_1 + m_2)^2$). Comparing with (6.1.4),

$$\text{Disc}\,\widetilde{C}_0(s) = \frac{\lambda^{1/2}(s, m_1, m_2)}{4\pi s}\left|\frac{\mathcal{M}}{4\pi\alpha}\right|^2. \tag{6.1.9}$$

As a concrete example, consider minimal gauge mediation (MGM), with charged messenger scalars of mass $m_\pm$ and fermions of mass m_0. The superpotential is $W = hX\Phi\widetilde{\Phi}$, where Φ, $\widetilde{\Phi}$ are charged messengers, with masses given by $\langle hX \rangle = M + \theta^2 F$, which leads to two fermions of mass $m_0 = M$ and scalars of mass $m_{1,2} = m_\pm = \sqrt{|M|^2 \pm |F|}$. The functions $\widetilde{C}_a(p^2)$ of GGM have cuts where these states can go on shell, with discontinuity related to the corresponding total cross sections as in (6.1.4), e.g. $\widetilde{C}_0(s)$ has a cut for $s \geq (m_+ + m_-)^2$, corresponding to production of the scalars with masses m_+ and m_-, given by (6.1.9) with the tree-level amplitude $\mathcal{M} = 4\pi\alpha$, so

$$\frac{1}{2}\text{Disc}\,\widetilde{C}_0(s) = \frac{1}{4\pi s}\sqrt{s^2 - 4|X|^2 s + 4|F|^2}, \qquad \text{for } s \geq (m_+ + m_-)^2, \tag{6.1.10}$$

Likewise, $\widetilde{C}_{1/2}(s)$, $\widetilde{C}_1(s)$, and $\widetilde{B}_{1/2}(s)$ have related discontinuities. For this example, these relations are of course readily verified from the known, explicit expressions for the GGM functions of weakly coupled theories. But one could imagine non-weakly coupled examples, where these analyticity properties could usefully constrain the GGM functions.

In the above discussion X can either be a dynamical field, the goldstino su-

perfield, or spurion of the spurion limit. We separate the UV description, sufficiently far above $\langle X \rangle$, from the IR effects of $\langle X \rangle$. In the UV description, the messengers are effectively massless and interacting with X with coupling h (we avoid going too far in the UV, to avoid h's Landau pole). We illustrate how to reproduce e.g. (6.1.10) from direct computations of the Wilson coefficients of the two-point OPE of the current superfield $\mathcal{J} = \Phi^\dagger \Phi - \widetilde{\Phi}^\dagger \widetilde{\Phi}$, to terms on the RHS of the OPE involving the operators $(X^\dagger X)^n$, and superconformal descendants. (Aspects of the OPE interpretation of super-propagators was explored for some interacting theories in [12].) As we will illustrate and verify, the superconformal symmetry implies many relations among the various terms. In the IR, we replace $X \to \langle X \rangle$, and these terms then contribute to, and indeed reproduce, the GGM [4] current-current correlators.

The paper is organized as follows: section 5.2 reviews the OPE, superconformal covariance, and the results of Ref. [6] for current-current correlation functions in general superconformal theories. In section 6.3 we apply these results to the general gauge mediation functions C_a and $B_{1/2}$ [4], discussing how these functions can be constrained by approximate, broken, superconformal symmetry. In section 6.4 we study the analyticity properties and constraints on the GGM functions, and how the OPE can be applied to obtain approximations (6.1.6) for soft terms in theories that aren't necessarily weakly coupled. Section 6.5 illustrates and checks our various general results in the well-studied example of weakly coupled minimal gauge messenger mediation MGM. Section 6.6 summarizes and mentions possible further applications of our findings. Appendix 6.A illustrates explicit computations of current-current OPE Wilson coefficients in MGM.

6.2. The operator product expansion

The OPE [1] replaces nearby operators with a sum of local operators

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k c_{ij}^k(x, P)\mathcal{O}_k(0), \tag{6.2.1}$$

where $c_{ij}^k(x, P)$ are the (position space) Wilson coefficients (with $[P_\mu, \mathcal{O}] = i\partial_\mu \mathcal{O}$). In non-scale invariant theories, (6.2.1) approximately holds for small x, or in the light-cone limit of small x^2, while for CFTs (6.2.1) is exact. In momentum space,

$$i\int d^4x\, e^{-ip\cdot x}\mathcal{O}_i(x)\mathcal{O}_j(0) \xrightarrow[p^2\to-\infty]{} \sum_k \tilde{c}_{ij}^k(p)\mathcal{O}_k(0), \tag{6.2.2}$$

with the Fourier transform applied on the Wilson coefficients, while the operators $\mathcal{O}_k(0)$ remain in position space. The coefficients $\tilde{c}_{ij}^k(p)$ can be extracted from the OPE (6.2.2) sandwiched between appropriate in and out external states.

In applications of the OPE to non-scale invariant theories, e.g. QCD, one splits momentum integrals into UV and IR regions, above and below a renormalization scale μ. For the IR physics of the renormalized operators, in particular their expectation values, μ acts as a UV cutoff scale. For the UV physics, namely the Wilson coefficients, μ acts as an IR cutoff scale. For a spirited discussion of the properties of the OPE, and the necessity of this splitting at a scale μ, the reader is referred to [13]. The scale μ drops out of physical quantities at the end of the day, of course. The coefficients obey an RG equation

$$\mathcal{D}c_{ij}^k(\mu^2 x^2) = \gamma_\ell^k C_{ij}^\ell - \gamma_i^\ell C_{\ell j}^k - \gamma_j^\ell C_{i\ell}^k, \tag{6.2.3}$$

with $\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}$. Even if the theory is RG flowing, with non-zero beta functions, this is accounted for by these RG equations, making the OPE still effectively scale covariant.

6.2.1. Conformal-symmetry constraints

Exactly scale-invariant theories are generally also conformally invariant (modulo recently found counterexamples [14, 15]). We're here ultimately interested in applying the OPE also to non-scale-invariant theories, but the intuition is that the Wilson coefficients are UV-determined, so we can work near the approximately conformally invariant UV fixed point, obtain relations there, and then RG flow them down to lower scales. The Wilson coefficients should then (approximately) respect the full conformal group, i.e. they should respect not only the dilation generator, D, but also the special conformal generator, K_μ. These generators act on primary operators $\mathcal{O}^I$ (I labels the $(j, \bar{j})$ spin indices) as

$$[P_\mu, \mathcal{O}^I(x)] = i\partial_\mu \mathcal{O}^I(x), \qquad [D, \mathcal{O}^I(x)] = -i(x \cdot \partial + \Delta_\mathcal{O})\mathcal{O}^I(x),$$

$$[K_\mu, \mathcal{O}^I(x)] = i(x^2\partial_\mu - 2x_\mu\, x \cdot \partial - 2\Delta_\mathcal{O} x_\mu)\mathcal{O}^I(x) + 2x^\nu (s_{\mu\nu})^I{}_J \mathcal{O}^J(x),$$

where $(s_{\mu\nu})^I{}_J$ is the operator's Lorentz spin representation, and $\Delta_\mathcal{O}$ is its scaling dimension.

Conformal symmetry implies that the OPE of conformal descendants are fully determined by those of the conformal primaries [16]. For example, for the OPE of two scalar operators,

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_{\substack{\text{primary} \\ \mathcal{O}_k^{(\ell)}}} c_{ij}^k \frac{1}{r_{12}^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)}} F_{\Delta_i \Delta_j}^{\Delta_k}(x_{12}, P)_{\mu_1 \ldots \mu_\ell} \mathcal{O}_k^{(\mu_1 \ldots \mu_\ell)}(x_2), \qquad (6.2.4)$$

where $x_{ij} \equiv x_i - x_j$ and $r_{ij} \equiv x_{ij}^2$ and the sum is over integer spin-ℓ primary operators $\mathcal{O}_k^{(\mu_1 \ldots \mu_\ell)}$ (with symmetrized indices) on the RHS. The functions $F_{\Delta_i \Delta_j}^{\Delta_k}(x_{12}, P)_{\mu_1 \ldots \mu_\ell}$, which give the coefficients of the descendants, are fixed by conformal covariance. Equivalently, conformal symmetry completely fixes the form of the two-point and three-point functions up to an overall coefficient. For example, the three-point functions

related to (6.2.4) are

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k^{(\mu_1\ldots\mu_\ell)}(x_3)\rangle =$$

$$\frac{c_{ijk}}{r_{12}^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_k+\ell)}\, r_{13}^{\frac{1}{2}(\Delta_k+\Delta_{ij}-\ell)}\, r_{23}^{\frac{1}{2}(\Delta_k-\Delta_{ij}-\ell)}} Z^{(\mu_1}Z^{\mu_2}\cdots Z^{\mu_\ell)},$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$, and

$$Z^\mu \equiv \frac{x_{23}^\mu}{r_{23}} - \frac{x_{13}^\mu}{r_{13}}, \qquad Z^2 = \frac{r_{12}}{r_{13}r_{23}}.$$

We'll sometimes be interested in Fourier transforming the OPEs, as in (6.2.2). For example, in (6.2.4), taking $x_2 = 0$ and Fourier transforming in $x_1 \equiv x$,

$$i\int d^4x\, e^{-ip\cdot x}\mathcal{O}_i(x)\mathcal{O}_j(0) \supset$$

$$c_{ij}^k F_{\Delta_i\Delta_j}^{\Delta_k}(-i\partial_p, P)_{\mu_1\ldots\mu_\ell}\, \text{F.T.}\left(\frac{1}{(x^2)^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_k)}}\right)\mathcal{O}_k^{(\mu_1\ldots\mu_\ell)}(0),$$

The Fourier integral is generally singular but can be defined by analytic continuation, with

$$\text{F.T.}\left(\frac{1}{(x^2)^d}\right) \equiv i\int d^4x\, e^{-ip\cdot x}\frac{1}{(x^2)^d} = (2\pi)^2\frac{\Gamma(2-d)}{4^{d-1}\Gamma(d)}(p^2)^{d-2}.$$

Logarithms of p^2 can arise if the dimension d is an integer n, or nearby, $d = n+\epsilon$, with $\epsilon \ll 1$. The $1/\epsilon$ terms are local contact terms that we can drop, and we're left with

$$\text{F.T.}\left(\frac{1}{x^{2n+\epsilon}}\right) = -\frac{(2\pi)^2}{4^{n-1}(n-1)!\,(n-2)!}(-p^2)^{n-2}\ln p^2 + \mathcal{O}(\epsilon). \qquad (6.2.5)$$

Such $\ln(-s)$ terms, associated with dimensions that are integer or nearly integer, are responsible for the discontinuities like (6.1.10). The needed smallness of the anomalous dimensions, $\epsilon \ll 1$, fits with the optical theorem connection (6.1.4) to the cross section, since that assumes production of weakly coupled final state particles.

6.2.2. Superconformally-covariant operator product expansion

Superconformal theories have Q_α, $\bar{Q}_{\dot\alpha}$, and P_μ as raising operators, generating the descendants. The superconformal primaries are annihilated by the lowering operators, S^α and $\bar{S}^{\dot\alpha}$, and K_μ at the origin. The algebra, and our sign conventions, can be found in [6]. To quote a few examples, the superconformal charges act on scalar superconformal primary operators as

$$[S^\alpha, \mathcal{O}(x)] = ix \cdot \bar{\sigma}^{\dot\alpha\alpha}[\bar{Q}_{\dot\alpha}, \mathcal{O}(x)], \tag{6.2.6}$$

$$S^\beta Q_\alpha(\mathcal{O}(x)) = 2(\sigma^{\mu\nu}{}_\alpha{}^\beta x_{[\mu}\partial_{\nu]} + \delta_\alpha{}^\beta x \cdot \partial)\mathcal{O}(x) - ix \cdot \bar{\sigma}^{\dot\alpha\beta} Q_\alpha \bar{Q}_{\dot\alpha}(\mathcal{O}(x)) + (2\Delta_\mathcal{O} + 3r_\mathcal{O})\delta_\alpha{}^\beta \mathcal{O}(x), \tag{6.2.7}$$

where we define $S^\beta Q_\alpha(\mathcal{O}(x)) \equiv \{S^\beta, [Q_\alpha, \mathcal{O}(x)]\}$ etc.

Conserved currents are descendants of superconformal primary operators J with $\Delta_J = 2$ and $Q^2(J) = \bar{Q}^2(J) = 0$,

$$j_\alpha(x) = Q_\alpha(J(x)), \qquad j_\mu(x) = -\tfrac{1}{4}\Xi_\mu(J(x)), \tag{6.2.8}$$

where $\Xi_\mu \equiv \bar{\sigma}_\mu^{\dot\alpha\alpha}[Q_\alpha, \bar{Q}_{\dot\alpha}]$. In superspace,

$$\mathcal{J}(z) = J(x) + i\theta j(x) - i\bar{\theta}\bar{j}(x) - \theta\sigma^\mu\bar{\theta}j_\mu(x) + \cdots, \tag{6.2.9}$$

where $\cdots$ are derivative terms, following from the conservation equations $D^2\mathcal{J} = \bar{D}^2\mathcal{J} = 0$. The superconformal supercharges act on $J(x)$ as in (6.2.6)

$$S^\alpha(J(x)) = ix \cdot \bar{\sigma}^{\dot\alpha\alpha}\bar{Q}_{\dot\alpha}(J(x)), \qquad \bar{S}^{\dot\alpha}(J(x)) = -ix \cdot \bar{\sigma}^{\dot\alpha\alpha}Q_\alpha(J(x)),$$

vanishing at the origin. Acting on the descendants as in (6.2.7) with $\Delta_J = 2$ and

$r_J = 0,$

$$S^\alpha(j_\alpha(x)) = -ix \cdot \bar\sigma^{\dot\alpha\alpha} Q_\alpha \bar{Q}_{\dot\alpha}(J(x)) + 4(x \cdot \partial + 2)J(x),$$

$$S^\alpha(j^\mu(x)) = 3\bar\sigma^{\mu\dot\alpha\alpha}\bar{j}_{\dot\alpha}(x) - 2x \cdot \bar\sigma^{\dot\alpha\alpha}\bar\sigma^{\mu\nu\dot\beta}{}_{\dot\alpha}\partial_\nu\bar{j}_{\dot\beta}(x).$$

The OPE of all the descendants (6.2.8) follow from that of the primary operators,

$$J(x)J(0) = \sum_{\substack{\text{sprimary}\\ \mathcal{O}^{(\ell)}}} \frac{c_{JJ}^{\mathcal{O}^{(\ell)}}}{(x^2)^{\frac{1}{2}(4-\Delta_\mathcal{O})}} F_{JJ}^{\Delta_\mathcal{O}}(x, P, Q, \bar{Q})_{\mu_1\ldots\mu_\ell}\mathcal{O}^{\mu_1\ldots\mu_\ell}(0), \qquad (6.2.11)$$

where "sprimary" is shorthand for "superconformal primary". As discussed in [6], current conservation $Q^2(J) = \bar{Q}^2(J) = 0$ plays an important role in relating superconformal primary and descendant OPE coefficients. Applying the above relations to the LHS of (6.2.11) gives e.g. [6] (see also there for discussion about the sign)

$$S^\alpha(J(x)J(0)) = S^\alpha(J(x))J(0) = -ix \cdot \bar\sigma^{\dot\alpha\alpha}\bar{j}_{\dot\alpha}(x)J(0). \qquad (6.2.12)$$

$$j^\alpha(x)j_\alpha(0) = \tfrac{1}{2}Q^2(J(x)J(0)). \qquad (6.2.13)$$

In SCFTs, the latter can also be written via

$$j_\alpha(x)j_\beta(0) = \frac{1}{x^2}Q_\beta(ix \cdot \sigma\bar{S})_\alpha(J(x)J(0)) \qquad (6.2.14)$$

various such relations were noted in [6]; just to quote a couple more,

$$S^\alpha S^\beta(J(x)J(0)) = \bar{S}^{\dot\alpha}\bar{S}^{\dot\beta}(J(x)J(0)) = 0, \qquad (6.2.15)$$

$$j_\mu(x)J(0) = \frac{x^2\eta_{\mu\nu} - 2x_\mu x_\nu}{4x^4}\left[S\sigma^\nu\bar{S} - \bar{S}\bar\sigma^\nu S\right](J(x)J(0)). \qquad (6.2.16)$$

The RHS of the OPE is constrained by (6.2.15) and analogous relations in [6], including the constraints from the generators of special conformal transformations. These relate

different OPE coefficients inside the $J(x)J(0)$ OPE in supersymmetric theories, yielding the full OPE in terms of the OPE coefficients for the superconformal primaries.

As we showed in [6], these constraints can be efficiently implemented in super-space, using the general formalism of [10]. The only operators that can appear on the RHS of the $J(x)J(0)$ are real, $U(1)_R$ charge zero operators, with the superspace expansion ($\xi_\mu \equiv \theta\sigma_\mu\bar\theta$ and $\cdots$ are operators with non-zero R-charge)

$$\mathcal{O}^{\mu_1\cdots\mu_\ell}(x,\theta,\bar\theta) = A^{\mu_1\cdots\mu_\ell}(x) + \xi_\mu B^{\mu\mu_1\cdots\mu_\ell}(x) + \xi^2 D^{\mu_1\cdots\mu_\ell}(x) + \cdots . \tag{6.2.17}$$

This is similar to the chiral-antichiral $\Phi\bar\Phi$ OPE considered in [17], and as there $\Xi^\mu \equiv \bar\sigma^{\mu\dot\alpha\alpha}[Q_\alpha,\bar Q_{\dot\alpha}]$, and $B^{\mu\mu_1\cdots\mu_\ell} = -\frac{1}{4}\Xi^\mu A^{\mu_1\cdots\mu_\ell}$ and $D^{\mu_1\cdots\mu_\ell} = -\frac{1}{64}\Xi_\mu B^{\mu\mu_1\cdots\mu_\ell} - \frac{1}{16}\partial^2 A^{\mu_1\cdots\mu_\ell}$. Operators $B_{\ell+1}$ in (6.2.17) decompose into Lorentz irreducible representations $M_{\ell+1}$ of spin $\ell+1$, $N_{\ell-1}$ of spin $\ell-1$, and $L_\pm$ in the $(\frac{1}{2}\ell \pm \frac{1}{2}, \frac{1}{2}\ell \mp \frac{1}{2})$ representation of $SU(2) \times SU(2)$.

Operators (6.2.17) with odd spin ℓ are odd under exchanging currents $J_a \leftrightarrow J_b$ in the OPE $J_a(x)J_b(0)$, and thus only appear, proportional to the structure constants f_{abc}, in non-Abelian theories. Since for simplicity we consider the $J(x)J(0)$ for $G = U(1)$, only even-ℓ operators appear in the primary $J(x)J(0)$ OPE. For operators (6.2.17) with ℓ even, this means that only the $A_{\ell\text{ even}}$ and $D_{\ell\text{ even}}$ operators contribute. For operators (6.2.17) with ℓ odd, the $B_{\ell+1} \to M_{\ell+1}$, $N_{\ell-1}$ components contribute.

Superconformal symmetry and current conservation fully determine all current-current OPE superconformal descendant coefficients from those of the superconformal primaries, since as shown in [6], the superspace dependence (in $z_i = (x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})_i$) of the associated three-point functions is fully determined:

$$\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{O}^{\mu_1\cdots\mu_\ell}(z_3)\rangle = \frac{1}{x_{\bar13}{}^2 x_{\bar31}{}^2 x_{\bar23}{}^2 x_{\bar32}{}^2} t^{\mu_1\cdots\mu_\ell}_{\mathcal{J}\mathcal{J}\mathcal{O}_\ell}(X_3, \Theta_3, \bar\Theta_3). \tag{6.2.18}$$

Current conservation implies that, for (6.2.17) of spin ℓ even or odd, respectively,

$$t^{(\mu_1...\mu_\ell)}_{\mathcal{J}\mathcal{J}\mathcal{O}_{\ell \text{ even}}} = c_{JJ\mathcal{O}_\ell} \frac{X_+^{(\mu_1} \cdots X_+^{\mu_\ell)}}{(X \cdot \bar{X})^{2 - \frac{1}{2}(\Delta - \ell)}} \left[1 - \frac{1}{4}(\Delta - \ell - 4)(\Delta + \ell - 6)\frac{\Theta^2 \bar{\Theta}^2}{X \cdot \bar{X}} \right] - \text{traces},$$

$$(6.2.19)$$

$$t^{(\mu_1...\mu_\ell)}_{\mathcal{J}\mathcal{J}\mathcal{O}_{\ell \text{ odd}}} = c_{JJ\mathcal{O}_\ell} \frac{X_+^{(\mu_1} \cdots X_+^{\mu_{\ell-1}}}{(X \cdot \bar{X})^{2 - \frac{1}{2}(\Delta - \ell)}} \left[X_-^{\mu_\ell)} - \frac{\ell(\Delta - \ell - 4)}{\Delta - 2} \frac{(X_- \cdot X_+)X_+^{\mu_\ell)}}{X \cdot \bar{X}} \right] - \text{traces}$$

$$(6.2.20)$$

in terms of the spin ℓ and dimension $\Delta \equiv \Delta_{\mathcal{O}} \equiv \Delta_A$ of the operator $\mathcal{O}$; see [6] and [10] for explanation about the notation. The primary OPE coefficient fixes the coefficient $c_{JJ\mathcal{O}_\ell}$ above, and then all descendant OPE coefficients are fully determined by the requirement that they reproduce (6.2.18), (6.2.19), (6.2.20). The superconformal relations are exhibited by expanding these expressions out in superspace. For example, setting $\theta_{i=1,2} = \bar{\theta}_{i=1,2} = 0$, these imply [6] that the coefficients c_{ijk} of the three-point functions satisfy

$$c_{JJD_{\ell;\text{prim}}} = -\frac{\Delta(\Delta + \ell)(\Delta - \ell - 2)}{8(\Delta - 1)} c_{JJA_\ell},$$

$$c_{JJN_{\ell-1}} = -\frac{(\ell + 2)(\Delta - \ell - 2)}{\ell(\Delta + \ell)} c_{JJM_{\ell+1}}.$$

The OPE coefficients c_{ij}^k are related to the three-point coefficients c_{ijk} by $c_{ij}^k = c^{k\ell} c_{ij\ell}$, where c_{ij} are the two-point function coefficients, and then (5.4.13) implies that

$$c_{JJ}^{D_{\ell;\text{prim}}} = -\frac{(\Delta - 1)}{2\Delta(\Delta + \ell + 1)(\Delta - \ell - 1)} c_{JJ}^{A_\ell},$$

$$c_{JJ}^{N_{\ell-1}} = -\frac{\ell(\ell + 2)(\Delta + \ell + 1)}{(\ell + 1)^2(\Delta - \ell - 1)} c_{JJ}^{M_{\ell+1}}.$$

$$(6.2.21)$$

6.3. Implications of superconformally covariant OPE for General Gauge Mediation

The GGM [4] framework relates visible-sector soft SUSY breaking parameters to hidden sector current two-point functions (defined following the convention of [18])[2]

$$\langle J(x)J(0)\rangle = C_0(x) \xrightarrow{\text{F.T.}} \widetilde{C}_0(p),$$

$$\langle j_\alpha(x)\bar{j}_{\dot\alpha}(0)\rangle = -i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu C_{1/2}(x) \xrightarrow{\text{F.T.}} \sigma^\mu_{\alpha\dot\alpha}p_\mu \widetilde{C}_{1/2}(p),$$

$$\langle j_\mu(x)j_\nu(0)\rangle = (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)C_1(x) \xrightarrow{\text{F.T.}} -(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\widetilde{C}_1(p), \qquad (6.3.1)$$

$$\langle j_\alpha(x)j_\beta(0)\rangle = \epsilon_{\alpha\beta}B_{1/2}(x) \xrightarrow{\text{F.T.}} \epsilon_{\alpha\beta}\widetilde{B}_{1/2}(p),$$

$$\langle j_\mu(x)J(0)\rangle = 0.$$

The functions $C_a(x)$ are real and $B_{1/2}(x)$ is complex, though in potentially realistic models it must be possible to rotate it to be real, to avoid large CP violating phases. If the theory were supersymmetric, all $C_a(x)$ would be equal, and $B_{1/2}(x)$ would be zero.

The leading contribution to the above functions in the UV limit comes from the unit operator on the RHS of the $J(x)J(0)$ OPE (6.1.3),

$$\text{UV limit}: C_a(x) = \frac{\tau}{16\pi^4 x^4} + \mathcal{O}\left(\frac{1}{x^2}\right) \xrightarrow{\text{F.T.}} \widetilde{C}_a(p) = \frac{\tau}{16\pi^2}\ln\frac{\Lambda^2}{p^2} + \mathcal{O}\left(\frac{1}{p^2}\right). \quad (6.3.2)$$

The $C_a(x)$ all coincide at this order, as seen from the OPE and $Q(\mathbb{1}) = 0$ [4, 5]. If the theory were exactly superconformal, only the unit operator could have an expectation value and (6.3.2) would be the full answer.

Another application of the OPE in the UV limit was discussed in [18]: it follows

---

[2]The last relation can be altered for spontaneously broken non-Abelian groups to $\langle j_\mu^A(p)J^B(-p)\rangle = ip_\mu f^{ABC}\langle J^C\rangle/p^2$, but Lorentz and gauge invariance imply that this doesn't contribute to the soft masses in any case. See [19, 20] for discussion of GGM in such cases.

from the relations

$$\langle \bar{Q}^2 Q^2 (J(x)J(0))\rangle = -8\partial^2 (C_0(x) - 4C_{1/2}(x) + 3C_1(x)), \qquad (6.3.3\text{a})$$

$$\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \langle Q_\alpha \bar{Q}_{\dot{\alpha}} (j^\mu(x)J(0))\rangle = -6\partial^2 (C_0(x) - 2C_{1/2}(x) + C_1(x)), \qquad (6.3.3\text{b})$$

$$\langle Q_\alpha \bar{Q}_{\dot{\alpha}} (j^\alpha(x)\bar{j}^{\dot{\alpha}}(0))\rangle = 2\partial^2 (C_0(x) + 2C_{1/2}(x) - 3C_1(x)), \qquad (6.3.3\text{c})$$

and the OPE, that the difference of any two $\widetilde{C}_a(p)$ in the UV vanishes at least as rapidly as $1/p^4$ in any renormalizable theory. For example [18], using the OPE

$$j^\mu(x)J(0) \sim \frac{x^\mu \mathcal{O}(0)}{x^{6-\Delta_\mathcal{O}}} + \frac{V^\mu(0)}{x^{5-\Delta_V}} + \cdots, \qquad (6.3.4)$$

where $\mathcal{O}$ and V^μ are scalar and vector operators, Lorentz invariance implies that only V^μ can contribute to (6.3.3b), with V^μ a conformal primary so unitarity requires $\Delta_V \geq 3$ (saturated by a conserved current). This implies $\widetilde{C}_0(p) - 2\widetilde{C}_{1/2}(p) + \widetilde{C}_1(p) \leq \mathcal{O}(1/p^4, \ln(p^2)/p^4)$ for large p. Likewise, using (6.3.3a) and (6.3.3c), any two $\widetilde{C}_a(p)$ differ by at most $\mathcal{O}(1/p^4, \ln(p^2)/p^4)$ in the UV [18].

6.3.1. Constraints from (approximate) broken superconformal symmetry

We expect / conjecture that the GGM functions can be constrained by applying the current-current OPE, with the Wilson coefficients approximately constrained by approximate UV superconformal symmetry (up to RG running differences). The IR effect of supersconformal symmetry breaking appears via the non-zero expectation values of the various operators on the RHS of the OPE, namely the operators (6.2.17) and their descendants.

By Lorentz invariance, only scalar operators can have non-zero expectation values and translation invariance implies that $P_\mu \to 0$ in one-point functions. So only scalar conformal primaries can have non-zero one-point functions. Such operators can only come from the superconformal primary operators (6.2.17) with spin $\ell = 0$ or $\ell = 1$: the $A^{\ell=0}$ and $D_{\text{prim}}^{\ell=0}$ components of $\ell = 0$ scalar superconformal primaries $\mathcal{O}^{\ell=0}$

in (6.2.17), or the $N_{\ell-1=0}$ component of a primary $\mathcal{O}^{(\ell=1)\mu}$. Likewise, the operator $V^\mu(0)$ in (6.3.4) can be the superconformal primary components A^μ or D^μ_{prim} of a superconformal primary spin $\ell = 1$ operator, or from the M^μ component of a spin $\ell = 0$ superconformal primary operator (6.2.17).

Consider first the GGM function $C_0(x)$, which is given by the expectation value of the $J(x)J(0)$ OPE. The $\ell = 0$ conformal primaries that can contribute on the RHS of the OPE (6.2.11) yield (using (6.2.21) with $\ell = 0$)

$$
\begin{aligned}
C_0(x) = &\sum_{\mathcal{O}} \frac{c^{\mathcal{O}}_{JJ}}{(x^2)^{\frac{1}{2}(4-\Delta_\mathcal{O})}} \left(\langle A_\mathcal{O} \rangle - \frac{x^2}{2\Delta_\mathcal{O}(\Delta_\mathcal{O}+1)} \langle D_{\mathcal{O};\text{prim}} \rangle \right) \\
&+ \sum_{\mathcal{O}^\mu} \frac{c^{N_{\mathcal{O}^\mu}}_{JJ}}{(x^2)^{\frac{1}{2}(3-\Delta_{\mathcal{O}^\mu})}} \langle N_{\mathcal{O}^\mu} \rangle
\end{aligned}
\tag{6.3.5}
$$

$\mathcal{O}$ runs over the real superconformal primaries with $\ell = 0$, and $\mathcal{O}^\mu$ over those with $\ell = 1$, and $N_{\mathcal{O}^\mu}$ is the $\ell = 0$ conformal primary, superconformal descendant. The $\langle D_{\mathcal{O};\text{prim}} \rangle$ and $\langle N_{\mathcal{O}^\mu} \rangle$ expectation values are SUSY-breaking parameters of the low-energy theory. As in the discussion in [18], two simplifying limits are the small SUSY-breaking parameters limit, and the low-energy, spurion limit.

The functions $C_{1/2}(x)$, $C_1(x)$, and $B_{1/2}(x)$ can similarly be written by applying the OPE to the current two-point functions on the LHS of (6.3.1). All of these descendant current two-point functions are fully determined by the $J(x)J(0)$ primary OPE. In terms of the superspace expressions following (6.2.18), we simply need to extract the appropriate $\theta_{1,2}$, $\bar{\theta}_{1,2}$ term, to pick out the J descendant via (6.2.9). So $C_{1/2}(x)$ is found by applying $\partial_{\theta_{1,\alpha}} \partial_{\bar{\theta}_{2,\dot{\beta}}}$ to both sides of (6.2.18) before setting $\theta_{i=1,2} = \bar{\theta}_{i=1,2} = 0$, and $C_1(x)$ is found by extracting the $\theta_1 \sigma^\mu \bar{\theta}_1 \theta_2 \sigma^\nu \bar{\theta}_2 f(\theta_3, \bar{\theta}_3)$ terms from (6.2.18). These lead to expressions for $C_{1/2}(x)$ and $C_1(x)$ analogous to (6.3.5), fully determining them in terms of the same coefficients in (6.3.5), the $C^{\mathcal{O}}_{JJ}$ and $C^{N_{\mathcal{O}^\mu}}_{JJ}$ OPE Wilson coefficients and the vacuum expectation values $\langle A_\mathcal{O} \rangle$, $\langle D_{\mathcal{O};\text{prim}} \rangle$, and $\langle N_{\mathcal{O}^\mu} \rangle$. Likewise, for the case

of $B_{1/2}(x)$, using (6.2.13) gives

$$B_{1/2}(x) = \sum_{\mathcal{O}} \frac{c_{JJ}^{\mathcal{O}}}{(x^2)^{\frac{1}{2}(4-\Delta_{\mathcal{O}})}} \langle (Q^2 A)_{\mathcal{O};\text{prim}} \rangle.$$

The SUSY-breaking differences of the $C_a(x)$ can also be analyzed via (6.3.3a) to (6.3.3c), applying the OPE to the current-current operators on the LHS. As an example, applying the OPE to the LHS of (6.3.3a), the contributing terms are the $D_{\ell=0}$ terms on the RHS of the OPE, so using $Q^2 \bar{Q}^2 (A_\ell) = -128 D_{\ell;\text{prim}} + \text{descendants}$,

$$\frac{1}{16} \partial^2 (C_0(x) - 4C_{1/2}(x) + 3C_1(x)) = \sum_{\mathcal{O}_{\ell=0}} \frac{c_{JJ}^{\mathcal{O}}}{(x^2)^{\frac{1}{2}(4-\Delta_{\mathcal{O}})}} \langle D_{\mathcal{O};\text{prim}} \rangle \qquad (6.3.6)$$

We can similarly consider the difference of the C_a's in (6.3.3b), using the OPE (6.3.4). The $j_\mu(x)J(0)$ superconformal descendant OPE is fully determined from the $J(x)J(0)$ superconformal primary OPE. One way to obtain this is to note that the $j_\mu(x_1)J(x_2)$ OPE can be obtained from the superspace three-point functions (6.2.18) results (6.2.19) and (6.2.20), by taking the $\theta_1 \sigma^\mu \bar{\theta}_1$ component to get $j_\mu(x_1)$ and $\theta_2 = \bar{\theta}_2 = 0$ to get $J(x_2)$. Alternatively, we can use (6.2.16) to get the $j_\mu(x)J(0)$ OPE from the $J(x)J(0)$ OPE. This gives the conformal primary operator V^μ in (6.3.4), that contributes to (6.3.3b), in terms of the operators $A^\mu_{\mathcal{O}_{\ell=1}}$, $D^\mu_{\mathcal{O}_{\ell=1}}$ and $N^\mu_{\mathcal{O}_{\ell=2}}$ and $M^\mu_{\mathcal{O}_{\ell=0}}$. Acting with $\Xi^\mu = \bar{\sigma}^{\mu\dot{\alpha}\alpha}[Q_\alpha, \bar{Q}_{\dot{\alpha}}]$ to get the LHS of (6.3.3b), this gives an expression very analogous to (6.3.6), that relates $\partial^2(C_0(x) - 2C_{1/2}(x) + C_1(x))$ to the superconformal primary OPE coefficients $c_{JJ}^{\mathcal{O}_{\ell=0}}$ and $c_{JJ}^{N_{\mathcal{O}^\mu}}$, along with the $\langle D_{\mathcal{O}_{\ell=0}} \rangle$ and $\langle N_{\mathcal{O}^\mu} \rangle$ SUSY-breaking expectation values.

6.4. Analyticity properties of the GGM functions $\widetilde{C}_a(p)$ and $\widetilde{B}_{1/2}(p)$

Analyticity properties of correlation functions encode a wealth of physical information (see e.g. [11, 21]). The functions $\widetilde{C}_a(p^2)$ and $\widetilde{B}_{1/2}(p^2)$ (6.3.1), coming from the hidden sector, contribute to the visible gauge vector multiplet propagators (see

e.g. [20]), so analyticity properties of the GGM functions connect with that of the gauge field propagators. The functions $\widetilde{C}_a(s)$ and $\widetilde{B}_{1/2}(s)$ are analytic in $s = -p^2$, aside from cuts on the positive, real-s axis for s sufficiently large to create on-shell hidden sector states. The discontinuities of the imaginary part of the $\widetilde{C}_a$ across the cut is then related by the optical theorem to total cross sections for hidden sector pair production, as in (6.1.4). As in (6.1.5), analyticity implies that the full GGM functions $A(s) = \widetilde{C}_a(s), \widetilde{B}_{1/2}(s)$ can be reconstructed from integrating their discontinuities along all their cuts, labeled by c:

$$A(s) = \frac{1}{2\pi i} \sum_{c=\text{cuts}} \int_{s_{0,c}}^{\infty} ds' \frac{[\text{Disc } A(s')]_c}{s' - s} = \sum_c \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im } A(s')|_c}{s' - s}, \qquad (6.4.1)$$

where $s_{0,c}$ and $[\text{Disc } A(s')]_c$ are the cut's endpoint and discontinuity, respectively. The OPE can be used to approximate the contribution from the large s' UV part of the cut integral.

Let's first consider $\widetilde{C}_0(p^2)$, which can have a cut when the scalar (auxiliary) component $D(p)$ of the gauge multiplet can couple to produce a pair of on-shell scalars, of masses m_1 and m_2. The production cross section for this process is (6.1.7)

$$\sigma_{0\to0+0}(s) = \frac{\lambda^{1/2}(s, m_1, m_2)}{8\pi s^2} |\mathcal{M}_0|^2, \qquad (6.4.2)$$

where $\lambda^{1/2}(s, m_1, m_2) = 2\sqrt{s}|\vec{p}|$ is the kinematic factor (6.1.8) and $\mathcal{M}_0 \equiv \mathcal{M}_{0\to0+0}$. The optical theorem (6.1.4) relates this to the discontinuity

$$\text{Disc } \widetilde{C}_0(s) = \sum \frac{2s}{(4\pi\alpha)^2} \sigma_{0\to0+0}(s) = \sum \frac{\lambda^{1/2}(s, m_1, m_2)}{4\pi s} \left| \frac{\mathcal{M}_0}{4\pi\alpha} \right|^2, \qquad (6.4.3)$$

where the sum is over all all distinct pairs of particles that can be produced.

Now consider $\widetilde{C}_{1/2}(p^2)$, which can have a cut where the gaugino component $\lambda_\alpha(p)$ of the gauge multiplet can create an on-shell spin $0 + \frac{1}{2}$ pair of states, of masses

m_0 and m_f respectively.[3] The total integrated cross section $\sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega$, averaged over initial spins and summed over the final ones, is given by (where $\mathcal{M}_{\frac{1}{2}} \equiv \mathcal{M}_{\frac{1}{2} \to 0 + \frac{1}{2}}$)

$$\sigma_{\frac{1}{2} \to 0 + \frac{1}{2}} = \frac{\lambda^{1/2}(s, m_s, m_f)}{8\pi s^2} \frac{1}{2} \left(1 + \frac{m_f^2 - m_s^2}{s}\right) |\mathcal{M}_{\frac{1}{2}}|^2. \tag{6.4.4}$$

The additional kinematic factor of $\frac{1}{2}(1 + (m_f^2 - m_s^2)/s)$ compared with (6.4.2) comes from the spin factor sums and angular integration (see e.g. eq. (5.13) in [22]). The discontinuity of $\widetilde{C}_{1/2}(p^2)$ is related to this cross section by the optical theorem,

$$\text{Disc } \widetilde{C}_{1/2}(s) = \sum \frac{s}{8\pi^2\alpha^2} \sigma_{\frac{1}{2} \to 0 + \frac{1}{2}}(s) = \sum \frac{\lambda^{1/2}(s, m_s, m_f)}{8\pi s} \frac{1}{2} \left(1 + \frac{m_f^2 - m_s^2}{s}\right) \left|\frac{\mathcal{M}_{\frac{1}{2}}}{4\pi\alpha}\right|^2. \tag{6.4.5}$$

Likewise for spin 1, a massless intermediate vector boson can decay to either two massive scalars or two massive fermions. In either case, the final state is a CP conjugate pair, of the same mass. Accounting for the spin-kinematic factors, the total cross sections are

$$\sigma_{1 \to 0 + 0} = \frac{\lambda^{1/2}(s, m_s, m_s)}{8\pi s^2} \frac{1}{6} \left(1 - \frac{4m_s^2}{s}\right) |\mathcal{M}_{1 \to 0 + 0}|^2, \tag{6.4.6a}$$

$$\sigma_{1 \to \frac{1}{2} + \frac{1}{2}} = \frac{\lambda^{1/2}(s, m_f, m_f)}{8\pi s^2} \frac{2}{3} \left(1 + \frac{2m_f^2}{s}\right) |\mathcal{M}_{1 \to \frac{1}{2} + \frac{1}{2}}|^2. \tag{6.4.6b}$$

The optical theorem gives the discontinuity of $\widetilde{C}_1$ in terms of these as

$$\text{Disc } \widetilde{C}_1(s) = \frac{s}{8\pi^2\alpha^2} \left(\sum \sigma_{1 \to 0 + 0}(s) + \sum \sigma_{1 \to \frac{1}{2} + \frac{1}{2}}(s)\right), \tag{6.4.7}$$

with the sums over the various scalars and fermions that can be produced. In all of these discontinuities, $\lambda^{1/2}(s, m_1, m_2)$ implies a cut, from $s_0 = (m_1 + m_2)^2$ to infinity.

In the limit of unbroken supersymmetry, the produced state is a massive supersymmetric chiral superfield with $m_1, m_2, m_f, m_s \to m_{\text{SUSY}}$, and $\mathcal{M}_0 = \mathcal{M}_{\frac{1}{2}}$,

---

[3]For (partially) Higgsed gauge messengers [19, 20], we can also have $\sigma_{\frac{1}{2} \to 1 + \frac{1}{2}}$, and also $\sigma_{1 \to 1 + 0}$.

$\mathcal{M}_{1\to0+0}$, $\mathcal{M}_{1\to\frac{1}{2}+\frac{1}{2}} \to \mathcal{M}_{\text{SUSY}}$. All of the above total cross sections and discontinuities indeed properly coincide in this limit,

$$\sigma_{\text{tot};0,\frac{1}{2},1} \to \sigma_{\text{SUSY}} = \frac{1}{8\pi s} \sqrt{1 - \frac{4m_{\text{SUSY}}^2}{s}} \, |\mathcal{M}_{\text{SUSY},0}|^2 \,,$$

with discontinuity

$$\text{Disc}\,\widetilde{C}_{0,\frac{1}{2},1} \to \text{Disc}\,\widetilde{C}_{\text{SUSY}} = \sum_{m_{\text{SUSY}}} \frac{1}{4\pi} \sqrt{1 - \frac{4m_{\text{SUSY}}^2}{s}} \left|\frac{\mathcal{M}_{\text{SUSY},0}}{4\pi\alpha}\right|^2 \,,$$

Even if SUSY is broken, in the large-s limit the $\widetilde{C}_a(p^2)$ must all coincide to at least order $1/s^2$ [18], so their discontinuities at large s must also coincide to this order.

Finally, we can consider the possible cuts of $\widetilde{B}_{1/2}(s)$. Much as with $\widetilde{C}_{1/2}$, such cuts can arise when the gaugino can produce on shell states. Because $\widetilde{B}_{1/2}(s)$ is a complex rather than real amplitude, its cuts generally can not be identified with a real, positive-definite cross section. On the other hand, to avoid CP violating phases, it should be possible to rotate $\widetilde{B}_{1/2}$ to be real in physically realistic theories. As we will illustrate in an example, the cut structure of $\widetilde{C}_{1/2}$ and $\widetilde{B}_{1/2}$ are essentially the same, except that cut pairs add up in $\widetilde{C}_{1/2}$, while they subtract in $\widetilde{B}_{1/2}$. This opposite sign the SUSY-violating amplitude $\widetilde{B}_{1/2}(s)$ leads to a partial cancellation that is needed to ensure that $\widetilde{B}_{1/2}(s)$, and hence its discontinuity, properly vanishes at least as fast as $1/s^2$ for large s.

The above discussion implicitly assumed an IR free spectrum for the produced states. In that case, the discontinuities mentioned above come from $\ln(-s)$ terms in the current correlator Wilson coefficients, as illustrated in the appendix. Such $\ln(-s)$ terms arise, as in (6.2.5), from the Fourier transform of OPE coefficients of operators with integral (or half-integral) dimension, $2\Delta_{\mathcal{O}} \in \mathbb{Z}$. More generally, we could contemplate a (broken) interacting SCFT, with mass gap, with quantum corrections to the anomalous dimensions leading to non-integer $2\Delta_{\mathcal{O}}$. The spectral analysis in that case is then similar to that considered in the context of "unparticles," see e.g. [23]—we won't discuss

it further here.

6.4.1. Soft masses from the OPE and analyticity

The OPE leads to approximations for the GGM soft masses in (6.1.2), which can be applied even in strongly interacting hidden-sector theories. Using the last expression in (6.4.1) and applying the OPE, the soft SUSY-breaking parameters are approximated by

$$
\begin{aligned}
M_{\text{gaugino}} &\approx \sum_k \frac{\alpha \operatorname{Im}[s^{d_k/2}\, \tilde{c}_{JJ}^k(s)]}{2^{d_k-1} d_k M^{d_k}} \langle Q^2(\mathcal{O}_k(0)) \rangle, \\
m_{\text{sfermion}}^2 &\approx 4\pi\alpha Y \langle J(x) \rangle - \sum_k \frac{\alpha^2 c_2 \operatorname{Im}[s^{d_k/2}\, \tilde{c}_{JJ}^k(s)]}{2^{d_k+1} \pi d_k^2 M^{d_k}} \langle \bar{Q}^2 Q^2(\mathcal{O}_k(0)) \rangle.
\end{aligned}
\tag{6.4.8}
$$

Here the classical scaling dimension d_k is related to the quantum scaling dimension by $\Delta_k = d_k + \gamma_k$, and $\operatorname{Im}[s^{d_k/2}\, \tilde{c}_{JJ}^k(s)]$ is independent of $s = -p^2$ by dimensional analysis.

Let us sketch a few details in how the expressions in (6.4.8) are obtained, to highlight in particular some approximations. Using (6.1.2) and (6.4.1),

$$
\begin{aligned}
M_{\text{gaugino}} &= \pi i \alpha \widetilde{B}_{1/2}(s=0) = \alpha \int_{s_{0,c}} ds' \, \frac{\operatorname{Im}[i\widetilde{B}_{1/2}(s')]}{s'} \\
&\approx \alpha \sum_k \int_{s_{\text{susy}}}^{\infty} ds' \, \frac{\operatorname{Im}[\tilde{c}_{JJ}^k(s')]}{s'} \langle Q^2(\mathcal{O}_k(0)) \rangle \\
&= \alpha \sum_k \operatorname{Im}[s^{d_k/2}\tilde{c}_{JJ}^k(s)] \langle Q^2(\mathcal{O}_k(0)) \rangle \int_{s_{\text{susy}}}^{\infty} ds' \, (s')^{-d_k/2-1}.
\end{aligned}
\tag{6.4.9}
$$

The second line of (6.4.9) involves two approximations. First, we approximate $\widetilde{B}_{1/2}(s')$ by replacing it with its OPE—this is a good approximation for the large s' part of the integral, while we apply it to the entire s' integral.

The next approximation is that the cut endpoints $s_{0,c}$ on the top line of (6.4.9) depend on the masses of the produced states, which are affected by the SUSY-breaking contributions, while on the second line we approximated all cuts as starting at the unbroken-supersymmetric physical threshold $s_{\text{SUSY}} = 4M^2$, where the SUSY-breaking

corrections to the masses are dropped. This is needed because, once we apply the OPE, the individual cuts are no longer visible. While this approximation sounds perhaps rather crude, we will see in the example of weakly coupled messengers that it nevertheless gives the full answer, perhaps because the different individual cut locations essentially average to the supersymmetric threshold. We replace Δ_k with the classical dimension d_k to get the contributing $\ln(-s)$ contribution to the imaginary part in (6.4.9). Doing the s' integral in (6.4.9) gives M_{gaugino} in (6.4.8). The derivation of m^2_{sfermion} is similar. Uniform convergence is assumed, and the m^2_{sfermion} momentum integral was regulated to tame the otherwise IR-divergent integral.[4] Notice that (6.4.8) only require the knowledge of the $J(x)J(0)$ OPE, which is constrained by OPE superconformality.

The expressions (6.4.8) can be further approximated by keeping only the contribution from the lowest dimension operator $\mathcal{O}_K$ on the RHS of the OPE (6.1.3) for which $Q^2(\mathcal{O}_K) \neq 0$:

$$
\begin{aligned}
M_{\text{gaugino}} &\approx -\frac{\alpha \pi w \gamma_{Ki}}{8M^2} \langle Q^2(\mathcal{O}_i(0)) \rangle, \\
m^2_{\text{sfermion}} &\approx 4\pi \alpha Y \langle J(x) \rangle + \frac{\alpha^2 c_2 w \gamma_{Ki}}{64M^2} \langle \bar{Q}^2 Q^2(\mathcal{O}_i(0)) \rangle,
\end{aligned}
\tag{6.4.10}
$$

where γ_{Ki} is the anomalous-dimension matrix which mixes $\mathcal{O}_K$ with the operator $\mathcal{O}_i$.

6.5. Example: Minimal Gauge Mediation

We now apply and test our general ideas and methods in the canonical example of weakly coupled minimal gauge messenger mediation. The theory has canonical Kähler potential and a hidden-sector supersymmetry-breaking chiral superfield X (or spurion)

---

[4]Though the momentum integral is actually not IR-divergent but this cannot be inferred from the OPE alone; a complete knowledge of the $C_a(x^2 M^2)$ functions is necessary. Also, although the OPE is convergent for large enough s', the approximations (6.4.8) might suffer from convergence issues from our integrating s' all the way down to $s_{SUSY} = 4M^2$. This can require that the OPE sum be regulated by analytic continuation; an example of this will be seen in the next section.

coupled to a pair of messengers Φ and $\widetilde{\Phi}$, of $U(1)$ charge ± 1, via the superpotential

$$W_{\mathrm{h}\otimes\mathrm{m}} = hX\Phi\widetilde{\Phi}. \tag{6.5.1}$$

X is chiral, $\bar{Q}_{\dot{\alpha}}(X(x)) = 0$, with $X(z_+) = X(y) + \sqrt{2}\theta\chi(y) + \theta^2 F(y)$, with

$$\chi_\alpha(x) = \frac{i}{\sqrt{2}}Q_\alpha(X(x)), \qquad F(x) = \frac{1}{4}Q^2(X(x)). \tag{6.5.2}$$

At low-energy, X and F get expectation values and the messengers Φ and $\widetilde{\Phi}$ become free fields with SUSY-split masses

$$M_0 = h\langle X\rangle, \qquad m_\pm^2 = m_0^2 \pm f, \tag{6.5.3}$$

with M_0 the fermion and $m_\pm$ the real-scalar masses ($m_0 = |M_0|$ and $f = |h\langle F\rangle|$). In the UV, with X regarded as a dynamical field, the coupling h in (6.5.1) has a Landau pole; we restrict our attention to below the scale where it is UV completed or cutoff.

The $U(1)$ current superfield is $\mathcal{J} = \Phi^\dagger\Phi - \widetilde{\Phi}^\dagger\widetilde{\Phi}$, with $Q^2(J) = \bar{Q}^2(J) = 0$ and components

$$J(x) = \phi^\dagger\phi(x) - \tilde{\phi}^\dagger\tilde{\phi}(x),$$

$$j_\alpha(x) = -i\sqrt{2}[\phi^\dagger\psi_\alpha(x) - \tilde{\phi}^\dagger\tilde{\psi}_\alpha(x)],$$

$$\bar{j}_{\dot{\alpha}}(x) = i\sqrt{2}[\phi\bar{\psi}_{\dot{\alpha}}(x) - \tilde{\phi}\bar{\tilde{\psi}}_{\dot{\alpha}}(x)],$$

$$j_\mu(x) = i[\phi\partial_\mu\phi^\dagger(x) - \phi^\dagger\partial_\mu\phi(x) - \tilde{\phi}\partial_\mu\tilde{\phi}^\dagger(x) + \tilde{\phi}^\dagger\partial_\mu\tilde{\phi}(x)] + \psi\sigma_\mu\bar{\psi}(x) - \tilde{\psi}\sigma_\mu\bar{\tilde{\psi}}(x),$$

and their interactions with the SUSY-breaking superfield X are given by

$$\mathscr{L}_{\mathrm{int}} = -h^*hX^\dagger X(\phi^\dagger\phi + \tilde{\phi}^\dagger\tilde{\phi}) - [h(-F\phi\tilde{\phi} + X\psi\tilde{\psi} + \phi\tilde{\psi}\chi + \tilde{\phi}\psi\chi) + \mathrm{h.c.}]. \tag{6.5.4}$$

We also define real superfields K and K', and "meson" chiral field M, by

$$K = \Phi^\dagger \Phi + \widetilde{\Phi}^\dagger \widetilde{\Phi}, \qquad K' = K - 2X^\dagger X, \qquad M = \Phi \widetilde{\Phi}. \tag{6.5.5}$$

So in (6.5.4) $-|hX|^2$ sources the bottom component of K, and hF sources M. K is the messenger's classical Kähler potential, with the classical dimension of a conserved current, but the current is violated by (6.5.1) (though (6.5.1) preserves K'):

$$Q^2(K) = \frac{1}{8\pi^2}W^2 + 4hXM, \qquad Q^2(K') = \frac{1}{8\pi^2}W^2.$$

We include the anomaly term W^2 for completeness here, but it will not play a role in what follows since we initially turn off the gauge interactions, $\alpha \to 0$. In this limit, K' is a conserved current.

Below the scale of $\langle X \rangle$ and $\langle F \rangle$, where the theory is free, we know e.g.

$$\langle J(x)J(0)\rangle \equiv C_0(x) = \frac{2}{(2\pi)^{d/2}}\left(\frac{m_+ m_-}{x^2}\right)^{d/2-1} K_{d/2-1}(m_+ x)K_{d/2-1}(m_- x),$$

$$\widetilde{C}_0(p^2) = 2\int \frac{d^4 q}{(2\pi)^4}\frac{1}{q^2 + m_+^2}\frac{1}{(p+q)^2 + m_-^2}.$$

In the first line we used the d-dimensional propagator, with $K_\nu(z)$ a Bessel function. In the following subsections we will test our general considerations by using the explicit, known expressions for the GGM functions $\widetilde{C}_a(p^2)$ and $\widetilde{B}_{1/2}(p^2)$ in this case [4]. We will reinterpret the expressions in terms of the OPE in the "CFT" (6.5.1) with field X included, applying and testing our constraints from superconformal symmetry. Using e.g. (6.2.7), the superconformal supercharges act on the superfield X components at $x^\mu = 0$ as

$$\{S^\alpha, \chi_\beta(0)\} = 3i\sqrt{2}r_X \delta_\beta{}^\alpha X(0), \qquad [S^\alpha, F(0)] = i\sqrt{2}(3r_X - 2)\chi^\alpha(0),$$

where $r_X = \frac{2}{3}\Delta_X$ is the R-charge of the chiral superfield X. The S^α action at an

arbitrary point x is easily obtained from the superconformal-algebra equations and the chiral-superfield commutation relations.

6.5.1. The cross sections and analyticity properties

The total cross sections for scattering from the visible to the hidden sector can be immediately computed to $\mathcal{O}(\alpha)$ from the general expressions (6.4.2), (6.4.4), and (6.4.6a) and (6.4.6b). In this weakly coupled hidden sector, the amplitude in these expressions is simply $\mathcal{M} = 4\pi\alpha$, with the kinematic factors involving the hidden-sector messenger masses (6.5.3):

$$\sigma_0(\text{vis} \to \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s} \frac{1}{2s} \lambda^{1/2}(s, m_+, m_-),$$

$$\sigma_{1/2}(\text{vis} \to \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s} \frac{1}{4s^2} \left[(s + m_0^2 - m_+^2)\lambda^{1/2}(s, m_0, m_+) + (m_+ \to m_-) \right],$$

$$\sigma_1(\text{vis} \to \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s} \frac{1}{12s^2} \left[(s - 4m_+^2)\lambda^{1/2}(s, m_+, m_+) + (m_+ \to m_-) \right.$$

$$\left. + 4(s + 2m_0^2)\lambda^{1/2}(s, m_0, m_0) \right],$$

$$\sigma'_{1/2}(\text{vis} \to \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s} \frac{1}{2s} \left[\lambda^{1/2}(s, m_0, m_+) - \lambda^{1/2}(s, m_0, m_-) \right].$$

$$(6.5.6)$$

Here $\sigma'_{1/2}$ is not an honest cross section, but we anyway relate it to $\widetilde{B}_{1/2}$, whose phase can be eliminated to make $\sigma'_{1/2}$ real and positive. In the unbroken-SUSY limit, $F \to 0$,

$$\sigma_{a=0,1/2,1}(s) \to \sigma_{\text{SUSY}}(s, m_{\text{SUSY}}) = \frac{(4\pi\alpha)^2}{8\pi s}\sqrt{1 - \frac{4m_{\text{SUSY}}^2}{s}}\,\theta(s - 4m_{\text{SUSY}}^2), \qquad \sigma'_{1/2} \to 0.$$

$$(6.5.7)$$

The full cross sections (6.5.6) can be obtained from σ_{SUSY} (6.5.7), e.g.

$$\sigma_0(s) = \exp\left(-\frac{f^2}{s}\frac{\partial}{\partial m_0^2}\right)\sigma_{\text{SUSY}}(s),$$

with similar (but slightly uglier) expressions for $\sigma_{1/2}$, σ_1, and $\sigma'_{1/2}$.

The cross sections (6.5.6) have expansions in powers of $1/s$ in the UV limit,

using (6.5.3),

$$\sigma_0(\text{vis} \rightarrow \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s}\left[\frac{1}{2} - \frac{m_0^2}{s} + \frac{f^2 - m_0^4}{s^2} - \frac{2m_0^2(m_0^4 - f^2)}{s^3} + \mathcal{O}(s^{-4})\right],$$

$$\sigma_{1/2}(\text{vis} \rightarrow \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s}\left[\frac{1}{2} - \frac{m_0^2}{s} + \frac{\frac{1}{2}f^2 - m_0^4}{s^2} - \frac{2m_0^6}{s^3} + \mathcal{O}(s^{-4})\right],$$

$$\sigma_1(\text{vis} \rightarrow \text{hid}) = \frac{(4\pi\alpha)^2}{4\pi s}\left[\frac{1}{2} - \frac{m_0^2}{s} + \frac{f^2 - m_0^4}{s^2} - \frac{2m_0^2(m_0^4 - f^2)}{s^3} + \mathcal{O}(s^{-4})\right], \quad (6.5.8)$$

$$\sigma_0 - 4\sigma_{1/2} + 3\sigma_1 = \frac{(4\pi\alpha)^2}{4\pi s}\frac{2f^2}{s^2}\left[1 + \frac{4m_0^2}{s} + \mathcal{O}(s^{-3})\right],$$

$$\sigma'_{1/2}(\text{vis} \rightarrow \text{hid}) = -\frac{(4\pi\alpha)^2}{4\pi s}\frac{f}{s}\left[1 + \frac{2m_0^2}{s} + \frac{6m_0^4}{s^2} + \mathcal{O}(s^{-3})\right].$$

In the UV limit, the SUSY-breaking differences of σ_0, $\sigma_{1/2}$, and σ_1 show up at $\mathcal{O}(f^2/s^3)$, while $\sigma'_{1/2}$ is $\mathcal{O}(f/s^2)$.

The optical theorem relations (6.4.3), (6.4.5), (6.4.7), relate these cross sections to the discontinuities of the GGM functions $\widetilde{C}_a(s)$, and here $\widetilde{B}_{1/2}(s)$ obeys a similar relation,

$$\sigma_{a=0,1/2,1} = \frac{(4\pi\alpha)^2}{s}\,\text{Im}(i\widetilde{C}_a(s)) = \frac{(4\pi\alpha)^2}{s}\frac{1}{2i}\,\text{Disc}(i\widetilde{C}_a(s)),$$

$$\sigma'_{1/2} = \frac{(4\pi\alpha)^2}{m_0 s}\,\text{Im}(i\widetilde{B}_{1/2}(s)) = \frac{(4\pi\alpha)^2}{m_0 s}\frac{1}{2i}\,\text{Disc}(i\widetilde{B}_{1/2}(s)). \quad (6.5.9)$$

We now verify these relations from the known, explicit integral expressions for the GGM functions in this case, as given in [4]. Let's first remark that since, as shown on general grounds in [18], the $\widetilde{C}_a(s)$ coincide to $\mathcal{O}(1/s^2, \ln s/s^2)$ in the UV limit, it follows from (6.5.9) that the σ_a in this limit necessarily always coincide to $\mathcal{O}(1/s^3)$, as seen explicitly in the present example in (6.5.8).

Consider first C_0 using its integral expression

$$\widetilde{C}_0 = 2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2 + m_+^2}\frac{1}{(p+q)^2 + m_-^2} \supset -\frac{i}{8\pi^2}\int_0^1 dx\,\ln[x(1-x)p^2 + xm_+^2 + (1-x)m_-^2],$$

where the last expression is the finite part. The Landau equations for determining the

endpoint of the cut,

$$x(1-x)p^2 + xm_+^2 + (1-x)m_-^2 = 0 \quad \text{and} \quad \frac{\partial}{\partial x}[x(1-x)p^2 + xm_+^2 + (1-x)m_-^2] = 0,$$

have solutions $s_\pm = -p_\pm^2 = (m_+ \pm m_-)^2$ and $x_\pm = \frac{m_-}{m_- \pm m_+}$. The s_+, x_+ solution gives the endpoint of the cut, while s_- is unphysical, since it has $x_- < 0$, outside of the region of integration. Indeed, $\Delta \widetilde{C}_0$ can be here be calculated analytically from the integral to give

$$\widetilde{C}_0 \supset \frac{i}{8\pi^2 s} \lambda^{1/2}(s, m_+^2, m_-^2) \ln \frac{\sqrt{-s + (m_+ + m_-)^2} + \sqrt{-s + (m_+ - m_-)^2}}{\sqrt{-s + (m_+ + m_-)^2} - \sqrt{-s + (m_+ - m_-)^2}}.$$

The only physical branch point, on the first sheet of the logarithm, is that at s_+—there is no physical branch point at $s_- = (m_+ - m_-)^2$, and there is no physical pole at $s = 0$. Thus, in agreement with the above cross section and the optical theorem (6.5.9),

$$\text{Disc}\, \widetilde{C}_0 \equiv \widetilde{C}_0(s + i\epsilon) - \widetilde{C}_0(s - i\epsilon) = \frac{\lambda^{1/2}(s, m_+^2, m_-^2)}{4\pi s} \theta(s - (m_+ + m_-)^2). \qquad (6.5.10)$$

It similarly follows from the explicit integral expression for $\widetilde{C}_{1/2}(s)$,

$$\widetilde{C}_{1/2} = -\frac{2}{p^2} \int \frac{d^4 q}{(2\pi)^4} \left[\frac{1}{(p+q)^2 + m_+^2} + \frac{1}{(p+q)^2 + m_-^2} \right] \frac{p \cdot q}{q^2 + m_0^2},$$

that $\widetilde{C}_{1/2}$ has two (physical) branch points, at $s = (m_0 + m_+)^2$ and $s = (m_0 + m_-)^2$, with

$$\text{Disc}\, \widetilde{C}_{1/2} = \frac{1}{8\pi s^2}(s + m_0^2 - m_+^2)\lambda^{1/2}(s, m_0, m_+)\theta(s - (m_0 + m_+)^2) + [m_+ \to m_-]. \quad (6.5.11)$$

(Again, $s = 0$ is not a pole on the first sheet of the logarithm.) So (6.5.11) indeed agrees with (6.5.9) and the above cross sections. Similarly, $\widetilde{B}_{1/2}$, has two branch points,

at exactly the same positions in the s-plane as $\widetilde{C}_{1/2}$, with

$$\text{Disc } \widetilde{B}_{1/2} = \frac{m_0}{4\pi s}\lambda^{1/2}(s, m_0^2, m_+^2)\theta(s - (m_0 + m_+)^2) - (m_+ \to m_-).\qquad(6.5.12)$$

The relative sign between the two terms in (6.5.12) cancels the contributions to $\mathcal{O}(1/s)$, consistent with the restoration of supersymmetry in the deep UV.

Similarly, the explicit expression for $\widetilde{C}_1$,

$$\widetilde{C}_1 = \frac{2}{3p^2}\int\frac{d^4q}{(2\pi)^4}\left\{\frac{(p+q)\cdot(3p+2q)+4m_+^2}{(q^2+m_+^2)[(p+q)^2+m_+^2]} + (m_+ \to m_-)\right.$$
$$\left.-\frac{4q\cdot(p+q)+8m_0^2}{(q^2+m_0^2)[(p+q)^2+m_0^2]}\right\},$$

reveals three branch points, at $s = 4m_\pm^2, 4m_0^2$. (The supertrace relation $\text{Str } M^2 = 0$, i.e. $m_+^2 + m_-^2 - 2m_0^2 = 0$, is needed to prevent $\widetilde{C}_1(s)$ from having a pole at $s = 0$ on the physical sheet.) The $\widetilde{C}_1(s)$ discontinuities are consistent with the optical theorem and the cross sections (6.4.6a) for scalar production and (6.4.6b) for fermion production. At large s, the sum of the discontinuities across the three cuts add to coincide with that found above for $\widetilde{C}_0$ and $\widetilde{C}_{1/2}$ to order $\mathcal{O}(1/s^2)$, consistent with UV supersymmetry restoration.

6.5.2. OPE for $J(x)J(0)$ and superpartners

We now consider the current-current OPE $J(x)J(0)$ (6.1.3), along with its Fourier transform

$$i\int d^4x\, e^{-ip\cdot x}J(x)J(0) \to \tilde{c}_{\mathbb{1}}(s, \Lambda)\mathbb{1} + \tilde{c}_K(s)K(0) + \tilde{c}_{J^2}(s)J^2(0) + \tilde{c}_{K^2}(s)K^2(0) + \cdots$$
$$+ \mathcal{F}(X, X^\dagger, F, F^\dagger, \chi, \chi^\dagger; s, \mu).$$

$$(6.5.13)$$

The first few terms in the position-space OPE are found from taking Wick contractions

$$c_{\mathbb{1}}(x) = \frac{1}{8\pi^4 x^4} + \cdots, \qquad c_K(x) = \frac{1}{2\pi^2 x^2} + \cdots, \qquad c_{J^2}(x) = 1 + \cdots,$$

(So $\tau = w = 2$ in (6.1.3), coming from Φ and $\widetilde{\Phi}$. In the $\alpha, h \to 0$ limit, K becomes a conserved current and the leading K term on the RHS of the $J(x)J(0)$ OPE can be regarded as giving the $\operatorname{Tr} U(1)_J^2 U(1)_K = 2$ 't Hooft anomaly.) Here $\cdots$ are higher order perturbative corrections. These Wilson coefficients have Fourier transforms

$$\tilde{c}_{\mathbb{1}}(s) = \frac{1}{8\pi^2} \ln \frac{\Lambda^2}{-s} + \cdots, \qquad \tilde{c}_K(s) = -\frac{2}{s} + \cdots, \qquad \tilde{c}_{J^2}(s) = \delta^{(4)}(p) + \cdots. \quad (6.5.14)$$

For example, $\tilde{c}_K$ can be found from the diagram

$$-i\tilde{c}_K(s) = \underset{p}{\longrightarrow} J(x) \otimes\!-\!-\!-\!-\!-\!-\!-\!-\!\otimes J(0) = \frac{2i}{s}. \qquad (6.5.15)$$

As usual, a UV cutoff Λ enters for the Fourier transformation of the identity term in (6.5.13).

The important terms in what follows will be those on the second line of (6.5.13), representing the contributions of the supersymmetry breaking "goldstino" (or spurion background) superfield X, and its superpartners, to the OPE. When we take expectation values of (6.5.13), and superpartners, the superconformal and supersymmetry breaking effects will come from the expectation values of these operators involving X and $X^\dagger$.

Since $J(x)$ is $U(1)_R$ neutral, the possible terms in $\mathcal{F}$ in (6.5.13) include

$$
\begin{aligned}
i \int d^4x\, e^{-ip\cdot x} J(x)J(0) \supset\ & \sum_{m,n=0}^{\infty} \tilde{c}_0(m,n;s,\mu)(F^\dagger F)^m (X^\dagger X)^n(0) \\
& + \sum_{m,n=0}^{\infty} \tilde{d}_0(m,n;s,\mu)(F^\dagger F)^m (X^\dagger X)^n X^\dagger F^\dagger \chi^2(0) + \text{h.c.} \\
& + \sum_{m,n=0}^{\infty} \tilde{e}_0(m,n;s,\mu)(F^\dagger F)^m (X^\dagger X)^n \chi^2 \bar{\chi}^2(0) + \cdots .
\end{aligned}
$$

$$(6.5.16)$$

There are similar OPE expansions for the current superdescendants of $J(x)$, e.g.

$$
\begin{aligned}
i \int d^4x\, e^{-ip\cdot x} j_\alpha(x)\bar{j}_{\dot\alpha}(0) \supset\ & -i\sigma^\mu_{\alpha\dot\alpha} p^\mu \sum_{m,n=0}^{\infty} \tilde{c}_{1/2}(m,n;s,\mu)(F^\dagger F)^m (X^\dagger X)^n(0) \\
& - i\sigma^\mu_{\alpha\dot\alpha} p^\mu \sum_{m,n=0}^{\infty} \tilde{d}_{1/2}(m,n;s,\mu)(F^\dagger F)^m (X^\dagger X)^n X^\dagger F^\dagger \chi^2(0) + \text{h.c.} \\
& - i\sigma^\mu_{\alpha\dot\alpha} p^\mu \sum_{m,n=0}^{\infty} \tilde{e}_{1/2}(m,n;s,\mu)(F^\dagger F)^m (X^\dagger X)^n X^\dagger F^\dagger \chi^2 \bar{\chi}^2(0) + \cdots .
\end{aligned}
$$

$$(6.5.17)$$

The scale μ appearing in (6.5.16) is the IR normalization point mentioned in section 6.2. The Feynman diagrams used to compute the Wilson coefficients in (6.5.16) (see appendix 6.A), are UV-convergent but IR-divergent. So we integrate over virtual momenta starting at an IR cutoff μ, yielding μ dependent Wilson coefficients that are governed by the RG equations (6.2.3). Operator expectation values are similarly μ dependent, governed by RG equations. The μ dependence ultimately drops, as discussed in [13], when computing OPE expectation values, like the GGM functions. This here works thanks to operator mixing between operators on the two lines of (6.5.13), involving the messengers and X.

As an example of this, consider the coefficient $\tilde{c}_{X^\dagger X}(s,\mu)$ of the operator $X^\dagger X$ in the OPE, called $\tilde{c}_0(0,1;s,\mu)$ in (6.5.16), which is obtained at one-loop in the appendix

by evaluating a Feynman diagram with an insertion of $X^\dagger X$, with IR cutoff μ on the loop momentum,

$$\tilde{c}_{X^\dagger X}(s,\mu) = \frac{1}{4\pi^2}\frac{|h|^2}{s}\ln\frac{-s}{\mu^2} + \cdots,$$

(6.5.18)

where again $\cdots$ includes higher order corrections in $|h|^2$. The μ dependence in (6.5.18) is cancelled, effectively replaced with Λ, by the one loop operator mixing between $X^\dagger X$ and the operator K, given by the diagram of Fig. 6.2.
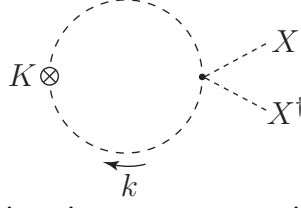
Figure 6.2: Diagram giving rise to operator mixing between K and $X^\dagger X$.

This diagram, which requires both UV cutoff Λ and IR cutoff μ, gives operator mixing:

$$K_{\mathrm{ren}}(0) = K(0) - \frac{|h|^2}{8\pi^2}\ln\frac{\Lambda^2}{\mu^2}X^\dagger X(0).$$

(6.5.19)

(This is related to the fact that K in (6.5.5) has $\gamma_K = |h|^2/16\pi^2$ whereas K' in (6.5.5) has $\gamma_{K'} = 0$ to this order.) When combined with the tree-level Wilson coefficient $\tilde{c}_K$ in (6.5.14), the μ-dependence in (6.5.18) cancels with that in (6.5.19), and is thereby ultimately replaced with a Λ-dependence from $\tilde{c}_K(s)K(0)$.

As an immediate illustration and check of our methods and results, let us connect the first few leading UV terms of the $J(x)J(0)$ OPE expectation value with the corresponding terms in the $\sigma_0(s)$ cross section in (6.5.8). Using (6.5.9), (6.5.8), and (6.5.13), we have

$$\mathrm{Disc}\,\widetilde{C}_0(s) = \frac{1}{2\pi}\left[\frac{1}{2} - \frac{m_0^2}{s} + \frac{f^2 - m_0^4}{s^2} - \frac{2m_0^2(m_0^4 - f^2)}{s^3} + \mathcal{O}(s^{-4})\right]$$

$$= \mathrm{Disc}\,\tilde{c}_{\mathbb{1}}(s)\langle\mathbb{1}\rangle + \mathrm{Disc}\,\tilde{c}_{X^\dagger X}(s)\langle X^\dagger X\rangle + \mathrm{Disc}\,\tilde{c}_{(X^\dagger X)^2}\langle(X^\dagger X)^2\rangle + \cdots$$

The first two terms on the top line indeed agree with the first two terms on the second

line, upon using $\tilde{c}_{\mathbb{1}}(s)$ from (6.5.14), and $\tilde{c}_{X^\dagger X}(s)$ from (6.5.18) and

$$\langle \mathbb{1} \rangle = 1, \qquad \langle (|h|^2 X^\dagger X)^n \rangle = m_0^{2n}, \qquad \operatorname{Im} \ln(-(s \pm i\epsilon)) = \mp\pi. \qquad (6.5.20)$$

In fact, we can reproduce the full cross sections (6.5.6) and associated disconti-nuities, from the OPE (6.5.13) expectation value,

$$\widetilde{C}_0 \supset \sum_{m,n=0}^{\infty} \tilde{c}_0(m,n;s)(\langle F^\dagger F \rangle)^m (\langle X^\dagger X \rangle)^n. \qquad (6.5.21)$$

An explicit one-loop computation of the Wilson coefficients $\tilde{c}_0(m,n;s,\mu)$ in (6.5.16) is given in the appendix. The discontinuity in particular comes from the terms $\sim \ln(-s)$ as in (6.5.20), and using the result from the appendix gives

$$\operatorname{Disc} \tilde{c}_0(m,n;s) = -\frac{1}{2\pi} \frac{(-1)^m \Gamma(2(m+n)-1)}{\Gamma(m+n)\Gamma(m+1)\Gamma(n+1)} \left(\frac{1}{s}\right)^m \left(\frac{|h|^2}{s}\right)^{m+n}. \qquad (6.5.22)$$

Using (6.5.22), the seemingly complicated series in m and n indeed nicely sums up to give (recall from (6.5.3) that $m_0 \equiv |h\langle X \rangle|$ and $f \equiv |h\langle F \rangle|$)

$$\begin{aligned} \operatorname{Disc} \widetilde{C}_0 &= \sum_{m,n=0}^{\infty} \operatorname{Disc} \tilde{c}_0(m,n;s)(\langle F^\dagger F \rangle)^m (\langle X^\dagger X \rangle)^n \\ &= \frac{1}{4\pi s}\sqrt{s^2 - 4m_0^2 s + 4f^2}. \end{aligned} \qquad (6.5.23)$$

Upon using (6.5.3), (6.5.23) indeed exactly reproduces, to all orders in $1/s$, the expres-sion (6.5.10), involving the standard kinetic factor $\lambda^{1/2}(s, m_+, m_-)$ (6.1.8).

As indicated in the $J(x)J(0)$ OPE (6.5.16), there are terms involving X's fermion components, χ (the goldstino). Such terms vanish upon taking the expectation value, so they do not contribute to $\widetilde{C}_0(s)$, as in (6.5.21). We retain the χ terms in (6.5.16) because they do contribute once we act on them with the supercharges Q, $\bar{Q}$, so they contribute to (6.1.2), (6.3.3) etc. The form of the terms in (6.5.16) have

been constrained by the $U(1)_R$ symmetry and reality of J.[5] The action of Q on the operators in (6.5.16) can be obtained from (6.5.2), which we can represent as

$$Q_\alpha \rightarrow -i\sqrt{2}\left(\chi_\alpha\partial\partial X + F\frac{\partial}{\partial\chi^\alpha}\right),$$

so e.g.

$$Q^2 \rightarrow 4F\frac{\partial}{\partial X} - 2\chi^2\frac{\partial^2}{\partial X^2} - 4\chi^\alpha F\frac{\partial^2}{\partial X\,\partial\chi^\alpha} + 2F^2\frac{\partial^2}{\partial\chi^2}. \tag{6.5.24}$$

Let us now consider the $j_\alpha(x)j_\beta(0)$ OPE, whose expectation value gives $B_{1/2}(x)$. By relation (6.2.13), this can be obtained from Q^2 acting on $J(x)J(0)$ OPE (6.5.16), and the terms with non-zero expectation value are those without remaining χ or $\chi^\dagger$ fermion fields. In terms of (6.5.24), the contributions come from the first and last terms, giving

$$i\int d^4x\, e^{-ip\cdot x}\langle j_\alpha(x)j_\beta(0)\rangle \rightarrow \epsilon_{\alpha\beta}\langle FX^\dagger \sum_{m,n=0}^{\infty} \tilde{c}'_{1/2}(m,n;s,\mu)(F^\dagger F)^m(X^\dagger X)^n\rangle,$$

with coefficients $\tilde{c}'_{1/2}$ contributions from the $\tilde{c}_0$ and $\tilde{d}_0$ terms in (6.5.16)

$$\tilde{c}'_{1/2}(m,n;s,\mu) = (n+1)\tilde{c}_0(m,n+1;s,\mu) + 2\tilde{d}_0(m-1,n;s,\mu). \tag{6.5.25}$$

Using the explicit expressions for $\tilde{c}_0(m,n;s,\mu)$ and $\tilde{d}_0(m,n;s,\mu)$, given by (6.A.1) and (6.A.2) in the appendix, we find that (6.5.25) indeed gives the correct expression for $\widetilde{B}_{1/2}(s)$, and in particular its discontinuity is properly related to the last expression in

---

[5] There are additional operators involving derivatives, with OPE coefficient denoted as e.g. ∂d_0:

$$\sum_{m,n}\partial d_0(m,n;s,\mu)(F^\dagger F)^m(X^\dagger X)^n\partial_\mu\chi\sigma^\mu\bar{\chi}(0).$$

(6.5.6) and (6.5.8):

$$
\begin{aligned}
\mathrm{Disc}\, \widetilde{B}_{1/2}(s) &= \sum_{m,n=0}^{\infty} \mathrm{Disc}\, \tilde{c}'_{1/2}(m,n;s)(\langle F^\dagger F\rangle)^m (\langle X^\dagger X\rangle)^n \\
&= \frac{m_0}{4\pi s}\sqrt{s^2 - 4m_0^2 s - 2fs + f^2} - (f \to -f),
\end{aligned}
\tag{6.5.26}
$$

which precisely reproduces (6.5.12).

We can similarly consider $Q^2\bar{Q}^2$ acting on the $J(x)J(0)$ OPE, which by (6.3.3a) gives expectation value equal to $-8\partial^2(C_0(x) - 4C_{1/2}(x) + 3C_1(x))$. Now, using (6.5.24) and its analog for $\bar{Q}^2$, the $\tilde{c}_0(s)$, $\tilde{d}_0(s)\chi^2 + \text{h.c.}$, and $\tilde{e}_0(s)\chi^2\bar{\chi}^2$ terms in the $J(x)J(0)$ OPE (6.5.16) all contribute. The resulting relation can be verified from a direct loop computation of the $\tilde{e}_0(s)$ Wilson coefficients, along the lines of the $\tilde{c}_0$ and $\tilde{d}_0$ perturbative computation outlined in the appendix.

Let us now turn to using, and checking, the additional constraints that follow from our claimed superconformal covariance of the OPE Wilson coefficients. One way to implement the constraints of superconformal invariance is to directly use the superspace-based [10] results of [6], reviewed in section 6.2.2 above. It follows from these results that the OPE of all components of the $\mathcal{J}(z)$ current superfield (6.2.9) are fully determined by the superconformal primary contributions to the primary $J(x)J(0)$ OPE, with independent Wilson coefficients for all real superconformal primary operators $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ (6.2.17). As discussed in (6.3.5), only the spin $\ell = 0$ operators $\mathcal{O}$, and spin $\ell = 1$ operators $\mathcal{O}^\mu$ have spin zero components that can get expectation values and contribute to the GGM functions.

To use these results here, we need to classify the independent, real superconformal primary operators of spin $\ell = 0, 1$ that can be built from X and $X^\dagger$. Clearly one such class of primary operator superfields are $\mathcal{O}_n(z) = (X^\dagger X)^n$. Using (6.5.24) $Q^2(X^n) = nX^{n-2}(4FX - 2(n-1)\chi^2)$, we see that the descendants in (6.2.17) involve particular linear combinations of FX and χ^2. Classes of additional superconformal primary operators can be obtained from different, orthogonal linear combinations of

FX and χ^2 terms.[6] We won't work out here the details of all classes of superconformal primaries for this example.

Alternatively, we can directly check that the superconformal relations like (6.2.12), (6.2.14), (6.2.15), (6.2.16) etc. are satisfied. As a first example, conformal covariance with respect to K_μ fully determines the P dependence in (6.2.1) (as in e.g. [16]), and in particular the contribution of scalar operators $\mathcal{O}$ to the $J(x)J(0)$ OPE have

$$J(x)J(0) \sim \frac{c_{JJ\mathcal{O}}}{x^{4-\Delta_\mathcal{O}}} \left(1 + \frac{1}{2}x^\mu\partial_\mu + \frac{\Delta_\mathcal{O}+2}{8(\Delta_\mathcal{O}+1)}x^\mu x^\nu\partial_\mu\partial_\nu - \frac{\Delta_\mathcal{O}}{16(\Delta_\mathcal{O}^2-1)}x^2\partial^2 + \cdots \right)\mathcal{O}(0).$$
(6.5.27)

Explicit calculation indeed verifies, for example (in position space, using dimensional regularization), that the Wilson coefficients of the operators $\mathcal{O}_n = (X^\dagger X)^n$ for the first two terms in (6.5.27) indeed have the relative factor of $\frac{1}{2}$ of (6.5.27); this gives a check of conformal covariance of the OPE.

We now outline similar explicit checks of our proposed superconformal covariance of the OPE Wilson coefficients, with the generator S and $\bar{S}$. The proposed superconformal covariance yields many individual relations, which when combined determine the superconformal descendant Wilson coefficients in terms of those of the superconformal primaries.

As an example, the superconformal algebra implies that

$$\bar{Q}\bar{S}(J(x)J(0)) = -ix \cdot \bar{\sigma}^{\dot{\alpha}\alpha}j_\alpha(x)\bar{j}_{\dot{\alpha}}(0) - 2ix_\mu(j^\mu(x) - i\partial^\mu J(x))J(0).$$
(6.5.28)

Taking the Fourier transform of (6.5.28) and using the $J(x)J(0)$ OPE (6.5.16) and $j_\alpha(x)\bar{j}_{\dot{\alpha}}(0)$ OPE (6.5.17) yields the relation

$$\tilde{d}_0(m-1, n-1; s) - \partial\tilde{d}_0(m-1, n; s) = \tfrac{1}{4}[2(1-m) - n]\left[\tilde{c}_0(m, n; s) - \tilde{c}_{1/2}(m, n; s)\right],$$
(6.5.29)

---

[6]It is necessary here to retain the interaction (6.5.1), since F is a null operator if X is free.

where $\partial\tilde{d}_0(m,n;s)$ is the Wilson coefficient of $(F^\dagger F)^m(X^\dagger X)^n(\partial_\mu\chi)\sigma^\mu\bar{\chi}$ in (6.5.16). Explicit computation of the Wilson coefficients verifies that (6.5.29) is indeed satisfied.

The relation (6.5.29) determines the $\tilde{c}_{1/2}$ Wilson coefficients in the $j_\alpha(x)\bar{j}_{\dot{\alpha}}(0)$ OPE (6.5.17) in terms of the Wilson coefficients $\tilde{c}_0$, $\tilde{d}_0$, and $\partial\tilde{d}_0$ in the primary $J(x)J(0)$ OPE (6.5.16). This fits with the result [6] that all superconformal descendant current-current OPE coefficients are fully determined from those of the primaries. In addition to relating the various OPEs of J's descendants, j_α, $\bar{j}_\alpha$, and j_μ, superconformal symmetry also implies relations among the terms on the RHS of the $J(x)J(0)$ OPE (6.5.16), determining the Wilson coefficients of all superconformal descendants in terms of those of the superconformal primaries.

As an example, consider $\partial\tilde{d}_0(m,n;s)$, the Wilson coefficient of the operator $(F^\dagger F)^m(X^\dagger X)^n(\partial_\mu\chi)\sigma^\mu\bar{\chi}$ that entered in (6.5.29). Since $[\bar{S}^{\dot{\alpha}}, i\partial_\mu\chi\sigma^\mu\bar{\chi}] \neq 0$, these operators are not superconformal primary, so the coefficients $\partial\tilde{d}_0(m,n;s)$ are completely determined by the superconformal symmetry in terms of the other, superconformal primary Wilson coefficients. Indeed, inserting the $J(x)J(0)$ OPE into superconformal relations like (6.2.15) and

$$\left(Q^2 + \frac{2i}{x^2}Qx\cdot\sigma\bar{S}\right)(J(x)J(0)) = 0$$

yields enough relations to, for example, fully determine the Wilson coefficients of superconformal descendants like $i\partial_\mu\chi\sigma^\mu\bar{\chi}$, $\chi x\cdot\sigma\bar{\chi}$ and $X^\dagger F^\dagger\chi^2$ in terms of the superconformal primaries. One can append $(X^\dagger X)^n(F^\dagger F)^m$ in front of all of these operators and the result remains.[7] As expected from the analysis of [6], the Wilson coefficients of all superconformal descendant operators are determined from those of the superconformal primaries.

---

[7]Since the action of S on F and $F^\dagger$ gives zero at $x = 0$, one has to use derivative operators in order to generate the $F^\dagger F$s. Then, one can use the known action of K_μ to show that Wilson coefficients of superconformal descendants are determined in terms of those of superconformal quasi-primaries.

6.5.3. Soft masses

We now apply general expressions (6.4.8) to analyze the gaugino and sfermion masses in this simple model. The expressions (6.4.8) and (6.4.10) can be applied to strongly coupled theories, and here we verify that our techniques can indeed properly approximate soft masses in simple weakly-coupled models, where the answer is already known: $M_{\text{gaugino}} = \frac{\alpha}{4\pi} \frac{F}{X} g(x)$ and $m^2_{\text{sfermion}} = 2 \left(\frac{\alpha}{4\pi} \right)^2 c(r) f(x)$ [24, 25], with $x \equiv |F/hX^2|$ and

$$
\begin{aligned}
g(x) &= \frac{1}{x^2}[(1+x)\ln(1+x) + (1-x)\ln(1-x)], \\
f(x) &= \frac{1+x}{x^2} \left[\ln(1+x) - 2\,\text{Li}_2\left(\frac{x}{1+x}\right) + \frac{1}{2}\,\text{Li}_2\left(\frac{2x}{1+x}\right) \right] + (x \to -x).
\end{aligned}
\tag{6.5.30}
$$

We find, perhaps surprisingly, that the OPE methods—generally an approximation— here reproduce the full, exact functions $g(x)$ and $f(x)$! We discuss here the gaugino mass in some detail. The sfermion mass computation is conceptually essentially the same, although technically a bit more involved.

The Wilson coefficients entering in (6.4.8) are the $\tilde{c}'_{1/2}(m, n; s, \mu)$ in (6.5.25), whose imaginary parts give the discontinuity in (6.5.26). So (6.4.8) gives

$$
M_{\text{gaugino}} \approx \alpha \sum_{m,n} \frac{\text{Disc}[s^{n+2m+1}\tilde{c}'_{1/2}(m, n; s, \mu)]}{4^{n+2m+1}(n + 2m + 1)m_0^{2(n+2m)-1}}(\langle F^\dagger F \rangle)^m (\langle X^\dagger X \rangle)^n.
$$

Using the result for $\tilde{c}'_{1/2}(m, n; s, \mu)$ in (6.5.26), this gives $M_{\text{gaugino}} \approx M_{\text{gaugino,OPE}} \equiv \frac{\alpha}{4\pi} \frac{F}{X} g_{\text{OPE}}(x)$, with

$$
g_{\text{OPE}}(x) = \sum_{n,m=0}^{\infty} \frac{\Gamma[2(n+m)]}{4^{n+2m}(n + 2m + 1)\Gamma(n)\Gamma(n+1)\Gamma(2m+2)} x^{2m}.
$$

The ratio test shows that the $\sum_m$ sum converges (for $x < 4$, which is satisfied since we anyway need $0 < x < 1$ to avoid tachyons), but the $\sum_n$ requires a continuation to converge. Indeed, the $\sum_n$ sum can be rewritten in terms of hypergeometric functions,

giving

$$g_{\text{OPE}}(x) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{1}{2^{4m+3}(m+1)} \, {}_3F_2\left[\begin{matrix} m+3/2, m+1, 2m+2 \\ 2, 2m+3 \end{matrix}; 1\right] x^{2m}. \qquad (6.5.31)$$

The hypergeometric function ${}_3F_2\left[\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}; z\right]$ converges at $z = 1$ only if $\text{Re}\, s > 0$ where $s = d + e - (a + b + c)$, and that is not satisfied in (6.5.31). Fortunately, one can analytically continue the hypergeometric functions using a generalization of Dixon's theorem,

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; 1\right] = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(b+s)\Gamma(c+s)} \, {}_3F_2\left[\begin{matrix} d-a, e-a, s \\ s+b, s+c \end{matrix}; 1\right], $$

which leads to convergent hypergeometric functions and gives

$$g_{\text{OPE}}(x) = 1 + \frac{1}{6}x^2 + \frac{1}{15}x^4 + \frac{1}{28}x^6 + \cdots = g(x).$$

The approximate $g_{\text{OPE}}(x)$ function obtained from the OPE gives the exact function $g(x)$! Similarly, the OPE approximation for the sfermion mass function $f_{\text{OPE}}(x)$ actually gives the full, exact result in (6.5.30).

Recalling the approximations made in (6.4.8), it is perhaps surprising that the OPE manages to reproduce the exact results (at least in this example). In particular, (6.4.8) was obtained by approximating that there is a single cut, starting at the supersymmetric threshold for particle production, with supersymmetry breaking neglected. We know from our discussion in subsection 6.5.1, that this is at best an approximate oversimplification, since the different contributions to the soft masses actually have different cut structures. It is interesting and curious that, at least in the present example, the OPE conspires in such a way to somehow fully account for the true cut structure. We do not know if this occurs more generally.

Before concluding, it is interesting to see how good the approximation is if we keep only the leading order contribution (6.4.10). Using the classical OPE

coefficient (6.5.15) and the Konishi current mixing (6.5.19), which are $1/2\pi^2$ and $|h|^2/4\pi^2$ respectively, the soft SUSY breaking functions (6.5.30) can be approximated by $g(0) = f(0) \approx \frac{1}{2}$. Thus, to lowest order the approximations (6.4.10) allow the computation of the soft SUSY breaking parameters to an accuracy of 50%. This is probably the best (and often the only) approximation to the soft SUSY breaking parameters one can achieve in strongly-coupled theories.

6.6. Conclusion

Conformal theories are interesting arenas for exploring quantum field theory. Various possible model-building applications of approximate conformal symmetry and non-weakly coupled sectors have been proposed in the literature over the years, to help naturalize hierarchies, e.g. that of technicolor, flavor [26], sequestering [27], and the $\mu/B\mu$ problem [28, 29]. These and other models have recently motivated renewed interest in exploring the consequences of conformal or superconformal symmetry, see e.g. [30] and following papers. Here we explore possible vestiges of approximate superconformal symmetry in wider classes of models, where the symmetries can be (softly or spontaneously) broken.

In weakly coupled models, one can simply write down integral expressions for the GGM functions C_a and $B_{1/2}$, see [18, 31]. Our methods here give some approximate tools to analyze theories that are not necessarily weakly coupled, giving some approximate insights on connecting the model theory to observational consequences. It would be interesting to apply the methods to concrete examples of non-weakly coupled theories, and to explore concretely some of the above mentioned proposed applications.

done, and the participants of the IHES Three String Generations and the CERN SUSY Breaking workshops for their comments. JFF thanks the participants of SUSY11 for comments.

6.A. Combinatorics for Wilson coefficients

In this appendix we calculate the one-loop Wilson coefficient of the operator $(F^\dagger F)^m (X^\dagger X)^n$ in the Fourier-transformed OPE of $J(x)J(0)$. The leading contribution to the coefficient comes from the one-loop diagram with m insertions of (the background expectation value of) $F^\dagger$ and F, and n insertions of $X^\dagger X$ (Fig. 6.3). The combinatoric
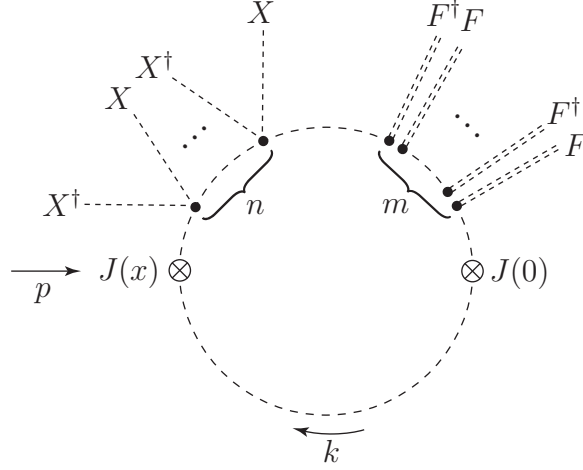
Figure 6.3: The Wilson coefficient of the operator $(F^\dagger F)^m (X^\dagger X)^n(0)$ in the OPE $i \int d^4x \, e^{-ip \cdot x} J(x)J(0)$.

factors are as follows. Permutations among the $X^\dagger X$ insertions do not count as separate diagrams, nor do permutations among $F^\dagger$s or Fs. $F^\dagger$ and F have to be in alternating order, and only one such ordering counts. We can start with all $X^\dagger X$ and F and $F^\dagger$ insertions on the upper propagator, and then start bringing the $F^\dagger$s and Fs, and the $X^\dagger X$s, past the current insertion, to the lower propagator. Every time an $F^\dagger$ or an F goes past the current insertion, there is a minus sign, from that in $J(x) = \phi^\dagger \phi(x) - \tilde\phi^\dagger \tilde\phi(x)$. After some standard manipulations for the calculation of

one-loop diagrams, the Wilson coefficient computed from Fig. 6.3 is

$$\tilde{c}_0(m, n; s, \mu) = \frac{|h|^{2(m+n)}}{8\pi^2} \sum_{j=0}^{n} \sum_{k=1+j}^{2m+1+j} (-1)^{k+j+1} \frac{(n + 2m - k + 1)!}{(n-j)!\,(2m - k + 1 + j)!} \frac{(k-1)!}{j!\,(k - 1 - j)!}$$

$$\times \int_0^1 dx\,(-1)^n \frac{\Gamma(2m+n)}{\Gamma(k)\Gamma(2m+n+2-k)} x^{k-1}(1-x)^{2m+n-k+1} \frac{(2m+n+1)\mu^2 + \Delta}{(\mu^2 + \Delta)^{2m+n+1}},$$

(for m, n not both zero) where μ is the IR normalization point, and $\Delta \equiv x(1-x)Q^2$, with $Q^2 = p_E^2$. Here k counts the number of propagators that make up the lower propagator, and j counts how many $X^\dagger X$ insertions are on the lower propagator.

In connection with the analyticity properties, we are particularly interested in contribution that is logarithmic in Q^2/μ^2. Expanding the result of the above Feynman parameter integration in the UV (large $s \equiv -Q^2 > 0$) we get

$$\tilde{c}_0(m, n; s, \mu) \to \frac{1}{4\pi^2} \frac{(-1)^m \Gamma(2(m+n)-1)}{\Gamma(m+n)\Gamma(m+1)\Gamma(n+1)} \left(\frac{1}{s}\right)^m \left(\frac{|h|^2}{s}\right)^{m+n} \ln\frac{-s}{\mu^2}. \qquad \text{(6.A.1)}$$

The case $m = n = 0$, i.e. $\tilde{c}_{\mathbb{1}}$, is instead given by (6.5.14). As in the discussion around (6.5.19), the IR scale μ everywhere ultimately cancels, thanks to operator mixing, and is effectively simply replaced with the UV cutoff scale Λ. As discussed after (6.5.23), the combinatoric factors in (6.A.1) precisely reproduce the $1/s$ expansion of the kinematic factor $\lambda^{1/2}(s, m_+, m_-)$ that enters in the cross section and the $\widetilde{C}_0$ discontinuity.

To outline a similar example, the Wilson coefficients $\tilde{d}_0(m, n; s, \mu)$ are obtained by similar considerations of a diagram like that of Fig. 6.3, but with the $X^\dagger F^\dagger \chi^2$ external fermion insertions. The result analogous to (6.A.1) is then

$$\tilde{d}_0(m, n; s, \mu) \to$$

$$\frac{1}{4\pi^2} \frac{\Gamma(2(m+n+1))}{\Gamma(n+1)} \left[\frac{1}{\Gamma(2(m+2))\Gamma(n)} + (-1)^m \frac{1}{\Gamma(m+2)\Gamma(m+n+1)} \right]$$

$$\times \left(\frac{1}{s}\right)^{m+1} \left(\frac{|h|^2}{s}\right)^{m+n+2} \ln\frac{-s}{\mu^2}.$$

$$\text{(6.A.2)}$$

References

[1] K.G. Wilson, *"Nonlagrangian models of current algebra"*, Phys.Rev. **179**, 1499 (1969).

[2] M.A. Shifman, A. Vainshtein & V.I. Zakharov, *"QCD and Resonance Physics. Sum Rules"*, Nucl.Phys. **B147**, 385 (1979).

[3] M.A. Shifman, A. Vainshtein & V.I. Zakharov, *"QCD and Resonance Physics: Applications"*, Nucl.Phys. **B147**, 448 (1979).

[4] P. Meade, N. Seiberg & D. Shih, *"General Gauge Mediation"*, Prog.Theor.Phys.Suppl. **177**, 143 (2009), `arXiv:0801.3278`.

[5] M. Buican, P. Meade, N. Seiberg & D. Shih, *"Exploring General Gauge Mediation"*, JHEP **0903**, 016 (2009), `arXiv:0812.3668`.

[6] J.F. Fortin, K. Intriligator & A. Stergiou, *"Current OPEs in Superconformal Theories"*, JHEP **1109**, 071 (2011), `arXiv:1107.1721`.

[7] D. Anselmi, J. Erlich, D. Freedman & A. Johansen, *"Positivity constraints on anomalies in supersymmetric gauge theories"*, Phys.Rev. **D57**, 7570 (1998), `hep-th/9711035`.

[8] K.A. Intriligator & B. Wecht, *"The Exact superconformal R symmetry maximizes a"*, Nucl.Phys. **B667**, 183 (2003), `hep-th/0304128`.

[9] E. Barnes, K.A. Intriligator, B. Wecht & J. Wright, *"$\mathcal{N} = 1$ RG flows, product groups, and a-maximization"*, Nucl.Phys. **B716**, 33 (2005), `hep-th/0502049`.

[10] H. Osborn, *"$\mathcal{N} = 1$ superconformal symmetry in four-dimensional quantum field theory"*, Annals Phys. **272**, 243 (1999), `hep-th/9808041`.

[11] C. Itzykson & J. Zuber, *"Quantum Field Theory"*, McGraw-Hill (1980).

[12] L. Leroy, *"OPE in supersymmetric scalar field theories"*, Phys.Lett. **B187**, 97 (1987).

[13] V.A. Novikov, M.A. Shifman, A.I. Vainshtein & V.I. Zakharov, *"Wilson's Operator Expansion: Can It Fail?"*, Nucl.Phys. **B249**, 445 (1985).

[14] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`.

[15] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: Theoretical Foundations"*, `arXiv:1107.3840`.

[16] S. Ferrara, A. Grillo & R. Gatto, *"Tensor representations of conformal algebra and conformally covariant operator product expansion"*, Annals Phys. **76**, 161 (1973).

[17] D. Poland & D. Simmons-Duffin, *"Bounds on 4D Conformal and Superconformal Field Theories"*, `arXiv:1009.2087`.

[18] T.T. Dumitrescu, Z. Komargodski, N. Seiberg & D. Shih, *"General Messenger Gauge Mediation"*, JHEP **1005**, 096 (2010), `arXiv:1003.2661`.

[19] M. Buican & Z. Komargodski, *"Soft Terms from Broken Symmetries"*, JHEP **1002**, 005 (2010), `arXiv:0909.4824`.

[20] K. Intriligator & M. Sudano, *"General Gauge Mediation with Gauge Messengers"*, JHEP **1006**, 047 (2010), `arXiv:1001.5443`.

[21] M. Pindor, *"Analyticity and Physics"*, in *"Harmonic Analysis and Rational Approximation"*, ed: J. Fournier, J. Grimm, J. Leblond & J. Partington, p. 3–12, Springer Berlin/Heidelberg (2006).

[22] M.E. Peskin & D.V. Schroeder, *"An Introduction to quantum field theory"*.

[23] A. Rajaraman, *"On the Decay of Unparticles"*, Phys.Lett. **B671**, 411 (2009), `arXiv:0806.1533`.

[24] S. Dimopoulos, G. Giudice & A. Pomarol, *"Dark matter in theories of gauge mediated supersymmetry breaking"*, Phys.Lett. **B389**, 37 (1996), `hep-ph/9607225`.

[25] S.P. Martin, *"Generalized messengers of supersymmetry breaking and the sparticle mass spectrum"*, Phys.Rev. **D55**, 3177 (1997), `hep-ph/9608224`.

[26] A.E. Nelson & M.J. Strassler, *"Suppressing flavor anarchy"*, JHEP **0009**, 030 (2000), `hep-ph/0006251`.

[27] M. Luty & R. Sundrum, *"Anomaly mediated supersymmetry breaking in four-dimensions, naturally"*, Phys.Rev. **D67**, 045007 (2003), `hep-th/0111231`.

[28] T.S. Roy & M. Schmaltz, *"Hidden solution to the mu/Bmu problem in gauge mediation"*, Phys.Rev. **D77**, 095008 (2008), `arXiv:0708.3593`.

[29] H. Murayama, Y. Nomura & D. Poland, *"More visible effects of the hidden sector"*, Phys.Rev. **D77**, 015005 (2008), `arXiv:0709.0775`.

[30] R. Rattazzi, V.S. Rychkov, E. Tonni & A. Vichi, *"Bounding scalar operator dimensions in 4D CFT"*, JHEP **0812**, 031 (2008), `arXiv:0807.0004`.

[31] D. Marques, *"Generalized messenger sector for gauge mediation of supersymmetry breaking and the soft spectrum"*, JHEP **0903**, 038 (2009), `arXiv:0901.1326`.

Chapter 7

Field-theoretic Methods in Strongly-Coupled Models of General Gauge Mediation

7.1. Introduction

The theoretical appeal of supersymmetry (SUSY) makes imperative the study of the phenomenology of its breaking. The Large Hadron Collider (LHC) has not yet found signs of low-scale SUSY, but abandoning SUSY at this early stage in experimental discovery would be premature. Nevertheless, SUSY extensions of the Standard Model are now tightly constrained by experimental data, and it appears that the simplest among them are not likely to survive as viable candidates for phenomenology. Therefore, new models of SUSY breaking as well as new tools for their analysis remain useful in exploring physics beyond the Standard Model. It would of course be ideal if tools were developed that could be used at strong coupling, since if SUSY is a symmetry of nature at some high scale, then it may very well reside in a model that is strongly-coupled at low energies.

In the context of gauge mediation of SUSY breaking (for a review see [1]) a formalism exists, known as general gauge mediation (GGM), that allows one to study such models in a unified fashion [2–4]. More specifically, SUSY-breaking parameters in the minimal supersymmetric standard model (MSSM) are generated in models of gauge-mediated SUSY breaking via two-point correlators of gauge-current superfields of the hidden, SUSY-breaking sector. This, then, dictates that a current analysis is

possible, and allows one to understand the generation of soft masses in the MSSM Lagrangian.

Such an analysis benefits strongly from the use of the operator product expansion (OPE). In $\mathcal{N} = 1$ superconformal theories OPEs of current correlators were studied in [5], where the superconformal symmetry was seen, as expected, to relate the OPEs of different components of the gauge-current superfield. Of course the study of the OPE is motivated by the fact that the OPE is one of the few tools that allows us to extract useful information even at strong coupling. This is reflected in the wealth of applications of the OPE in QCD.

The results of [5] were applied to the case of GGM correlators in [6]. Part of the motivation for that work was the observation that, even in theories that break the superconformal symmetry explicitly, one can introduce spurions to render the breaking spontaneous. The spurions are fully dynamical in the ultraviolet (UV), and an OPE analysis can be carried out to determine Wilson coefficients of spurionic operators in operator products. In the infrared (IR) the spurions acquire vacuum expectation values (vevs), and the Wilson coefficients have to be evolved from the UV according to their renormalization-group equation. It was shown in [6] that, in the case of minimal gauge mediation (MGM), soft masses could be approximated very well by only the leading spurionic term in the current-current OPE that develops a SUSY-breaking vev.

In MGM one can actually compute the full gaugino and sfermion masses using the OPE [6]. This is a rather special case and one cannot typically expect to be able to compute the complete current-current OPE. Nevertheless, it is physically acceptable to truncate the OPE and carry out the calculation of the soft masses, since the truncation is not expected to alter significantly the essential results. The error introduced in truncating the OPE allows only an approximate determination of the soft masses, up to $\mathcal{O}(1)$ overall factors which may be unimportant.

The technology developed in [6] may be used in strongly-coupled models of SUSY breaking. This is because the determination of Wilson coefficients is done in the UV,

where asymptotic freedom allows for a perturbative computation, while non-perturbative effects are contained in the vevs of operators, i.e. are captured by IR quantities. Thus, at least at the qualitative level, one is able to use the methods of [6] in order to understand the generation of soft masses in the MSSM, even when the SUSY-breaking sector is strongly-coupled in the IR. In theories where weakly-coupled duals exist, it is also possible to check the strongly-coupled computations at the quantitative level by comparing results obtained with both methods. As we will see the approximations discussed here are indeed reasonable up to factors of order one, suggesting that relevant information can be extracted from them even in the strong-coupling regime.

Of course AdS/CFT [7] is another tool one can use in order to understand the behavior of field theories at strong coupling. Indeed, the realization of GGM in holography has been considered by numerous authors, see e.g. [8–17]. The main theme of these works is the description of GGM correlators by holographic methods. In this paper, however, our methods will be strictly field-theoretic and four-dimensional.

$\mathcal{N} = 1$ supersymmetric QCD (SQCD) is an ideal candidate for the application of our methods. In the free magnetic range of the massive theory Intriligator, Seiberg and Shih (ISS) demonstrated the existence of a metastable SUSY-breaking vacuum [18]. In their treatment they used the power of Seiberg duality [19] in order to establish their result in the strongly-coupled regime of the electric theory. The big global symmetry of SQCD in the ISS vacuum allows its use as the hidden SUSY breaking sector in the context of gauge mediation. Phenomenologically, however, there is a problem due to an accidental R-symmetry which precludes Majorana masses for the gauginos.

Although modifications of the ISS scenario have been proposed in the literature, see e.g. [20–31], in this paper we consider a new deformation where we add an additional spontaneous breaking of SUSY from a singlet chiral superfield. This superfield acquires its vev through its own dynamics, about which we will remain agnostic. This new model is similar to MGM but with messengers strongly interacting through another gauge group. As we will see, with this deformation our theory develops ISS-like vacua

but with a broken R-symmetry. In our example there are no SUSY vacua anywhere in field space, but the ISS-like vacua we find should be metastable against decay to other SUSY-breaking vacua with lower energy.

The paper is organized as follows. In section 7.2 we review background material related to our work. We give a lightning review of $\mathcal{N} = 1$ SQCD, as well as a quick overview of gauge mediation, GGM, and the role of the OPE in our considerations. In section 7.3 we present the analysis of our deformation of ISS. We also recover MGM and pure ISS as limits of our deformed SQCD. Section 7.4 concludes and contains a discussion of general qualitative features of strongly-coupled models of SUSY-breaking. It is argued that such models are naturally split. Appendix 7.A contains weakly-coupled computations of the superpartner spectrum for general messenger sectors. We use notation and conventions of Wess & Bagger [32].

7.2. $\mathcal{N} = 1$ **SQCD, gauge mediation of SUSY breaking, and the OPE**

In this section we first review the aspects of $\mathcal{N} = 1$ SQCD and gauge mediation which are necessary for our purposes. This section is far from self-contained and the reader is referred to the literature, e.g. [33], for completeness.

7.2.1. Essentials of $\mathcal{N} = 1$ SQCD

SQCD with N_c colors and N_f flavors is an $\mathcal{N} = 1$ supersymmetric $SU(N_c)$ gauge theory with N_f quark flavors Q^i (left-handed quarks) which are chiral superfields transforming in the $\mathbf{N_c}$ of $SU(N_c)$, and N_f quark flavors $\widetilde{Q}_{\tilde{\imath}}$ (left-handed antiquarks) which are chiral superfields transforming in the $\overline{\mathbf{N_c}}$ of $SU(N_c)$, where $i, \tilde{\imath} = 1, \ldots, N_f$ are flavor indices.[1]

There is a large global symmetry in SQCD—the relevant representations and charge assignments are shown in Table 7.1.

---

[1]Note that there are no Fayet–Iliopoulos terms since the gauge group does not contain $U(1)$ factors.

Table 7.1: Matter content of SQCD and its (anomalous) transformation properties.

	$SU(N_f)_\text{L}$	$SU(N_f)_\text{R}$	$U(1)_\text{B}$	$U(1)_\text{A}$	$U(1)_{\text{R}'}$
Q	$\mathbf{N_f}$	$\mathbf{1}$	1	1	1
$\widetilde{Q}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	-1	1	1

However, the $U(1)_\text{A} \times U(1)_{\text{R}'}$ symmetry is anomalous. A single $U(1)$ R-symmetry, which we will denote $U(1)_\text{R}$, survives and is a full quantum symmetry. Thus, the global symmetry of the quantum theory is $SU(N_f)_\text{L} \times SU(N_f)_\text{R} \times U(1)_\text{B} \times U(1)_\text{R}$ with the appropriate R-charge assignment as shown in Table 7.2.

Table 7.2: Matter content of SQCD and its (non-anomalous) transformation properties.

	$SU(N_f)_\text{L}$	$SU(N_f)_\text{R}$	$U(1)_\text{B}$	$U(1)_\text{R}$
Q	$\mathbf{N_f}$	$\mathbf{1}$	1	$1 - N_c/N_f$
$\widetilde{Q}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	-1	$1 - N_c/N_f$

To make our notation more convenient we define the matrices

$$
Q = \overset{a}{\downarrow}\left(\overset{i\;\longrightarrow}{\left(\begin{pmatrix} Q^1 \end{pmatrix} \;\cdots\; \begin{pmatrix} Q^{N_f} \end{pmatrix} \right)} \right), \qquad \widetilde{Q} = \overset{\widetilde{i}}{\downarrow}\left(\overset{a\;\longrightarrow}{\left(\begin{matrix} (\quad \widetilde{Q}^1 \quad) \\ \vdots \\ (\quad \widetilde{Q}^{N_f} \quad) \end{matrix} \right)} \right),
$$

where $a = 1, \ldots, N_c$ is a fundamental or antifundamental color index. In this notation the Lagrangian of SQCD is[2]

$$
\mathscr{L}_\text{SCQD} = \int d^4\theta \ \text{Tr}(Q^\dagger e^{2gV} Q + \widetilde{Q} e^{-2gV} \widetilde{Q}^\dagger) + \left(\int d^2\theta \ \text{tr} \, W^\alpha W_\alpha + \text{h.c.} \right).
$$

---

[2]Tr denotes a sum over both fundamental gauge and flavor indices, while tr denotes a sum over adjoint gauge indices only, e.g.

$$
\text{Tr} \, Q^\dagger T^I Q \equiv Q^\dagger_{ib} (T^I)^b_c Q^{ic} \qquad \text{and} \qquad \text{tr} \, W^\alpha W_\alpha \equiv W^{\alpha I} W^I_\alpha.
$$

In components (and after integrating out the auxiliary fields), this becomes

$$\mathscr{L}_{\text{SCQD}} = -\operatorname{tr}(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^{\mu}\mathscr{D}_{\mu}\bar{\lambda}) - \operatorname{Tr}[\mathscr{D}_{\mu}Q^{\dagger}\mathscr{D}^{\mu}Q + \mathscr{D}_{\mu}\widetilde{Q}\mathscr{D}^{\mu}\widetilde{Q}^{\dagger}$$

$$+ i\bar{\psi}\bar{\sigma}^{\mu}\mathscr{D}_{\mu}\psi + i\tilde{\psi}\bar{\sigma}^{\mu}\mathscr{D}_{\mu}\bar{\tilde{\psi}} - i\sqrt{2}g(Q^{\dagger}\lambda\psi - \bar{\psi}\bar{\lambda}Q - \tilde{\psi}\lambda\widetilde{Q}^{\dagger} + \widetilde{Q}\bar{\lambda}\bar{\tilde{\psi}})]$$

$$- \tfrac{1}{2}g^{2}\sum_{I=1}^{N_{c}^{2}-1}[\operatorname{Tr}(Q^{\dagger}T^{I}Q - \widetilde{Q}T^{I}\widetilde{Q}^{\dagger})]^{2},$$

where $\mathscr{D}_{\mu} = \partial_{\mu} + igA_{\mu}^{I}T^{I}(R)$ is the gauge-covariant derivative. Note that SQCD only has D-term contributions to the scalar potential,

$$\mathscr{V}_{\text{SCQD}} = \tfrac{1}{2}g^{2}\sum_{I=1}^{N_{c}^{2}-1}[\operatorname{Tr}(Q^{\dagger}T^{I}Q - \widetilde{Q}T^{I}\widetilde{Q}^{\dagger})]^{2},$$

where T^{I} are $SU(N_{c})$ generators with $I = 1, \ldots, N_{c}^{2} - 1$ the adjoint color index. This scalar potential has a large vacuum degeneracy, which is however lifted when masses for the quarks are added.

Masses for the flavors

The lowest-dimensional gauge-invariant chiral superfield one can construct from Q^{i} and $\widetilde{Q}_{\tilde{\imath}}$, namely the mesonic superfield[3]

$$M_{\tilde{\imath}}^{i} = \operatorname{Tr}(\widetilde{Q}_{\tilde{\imath}}Q^{i})_{(N_{c},0)},$$

can be used to give gauge-invariant masses to all quark flavors. The Lagrangian of massive SQCD (mSQCD) is then

$$\mathscr{L}_{\text{mSCQD}} = \mathscr{L}_{\text{SCQD}} + \left(\int d^{2}\theta\, W_{\text{tree}} + \text{h.c.}\right),$$

---

[3] $\operatorname{Tr}(\cdot)_{(x,y)}$ denotes a sum over color indices up to x and flavor indices up to y. Hence, $\operatorname{Tr}(\cdot) \equiv \operatorname{Tr}(\cdot)_{(N_{c},N_{f})}$.

where $W_{\text{tree}} = \text{Tr}(mM)_{(0,N_f)}$, with m a nondegenerate $N_f \times N_f$ mass matrix. Note that the inclusion of masses breaks the non-Abelian part of the global symmetry to one of its subgroups. The scalar potential in $\mathscr{L}_{\text{mSCQD}}$ is

$$\mathscr{V}_{\text{mSCQD}} = \text{Tr}(mm^\dagger Q^\dagger Q + m^\dagger m \widetilde{Q}\widetilde{Q}^\dagger) + \tfrac{1}{2}g^2 \sum_{I=1}^{N_c^2-1} [\text{Tr}(Q^\dagger T^I Q - \widetilde{Q}T^I\widetilde{Q}^\dagger)]^2,$$

and includes the anticipated mass terms. The vacuum degeneracy of $\mathscr{V}_{\text{SCQD}}$ is lifted in $\mathscr{V}_{\text{mSCQD}}$ due to the mass terms.

7.2.2. Essentials of gauge mediation

Mediation of SUSY breaking was born to address phenomenological impasses reached by trying to break SUSY within the observable sector of supersymmetric extensions of the standard model. As an example, supertrace conditions that remain even after SUSY is broken are hard to satisfy consistently with the observed low-mass spectrum of particles [34].

Gauge mediation requires that SUSY be broken in a hidden sector with the breaking communicated to the MSSM through the familiar gauge interactions, thus avoiding new sources of flavor-changing neutral currents, a generic problem in models of gravity-mediated SUSY breaking. All soft SUSY-breaking terms in the MSSM Lagrangian are generated via loop effects, and desired phenomenology is obtained very naturally, except, of course, for the notorious μ/B_μ problem [35]. For an extensive review of theories with gauge mediation the reader is referred to [1].

In the minimal incarnation of gauge mediation one assumes the existence of a hidden sector that contains a gauge singlet chiral superfield S, as well as a messenger sector with fields $\Phi, \widetilde{\Phi}$ in complete GUT representations so that gauge-coupling unification is not spoiled. Through interactions in the hidden sector S develops a vev both in its first and its last component, $\langle S \rangle = \langle S \rangle + \theta^2 \langle F_S \rangle$. The superpotential that couples the hidden sector with the messenger sector is $W_{\text{h} \otimes \text{m}} \propto S \, \text{Tr}(\widetilde{\Phi}\Phi)$, such

that the SUSY breaking of the hidden sector is fed into the messenger sector. The usual gauge interactions then communicate the SUSY breaking to the supersymmetric extension of the standard model generating the appropriate soft SUSY-breaking terms.

General gauge mediation

A unified and powerful framework for the study of gauge mediation, dubbed general gauge mediation, was developed in [2–4]. In GGM soft terms are written in terms of one- and two-point correlators of components of a current (linear) superfield of the hidden sector,

$$\mathcal{J}(z) = J(x) + i\theta j(x) - i\bar{\theta}\bar{\jmath}(x) - \theta\sigma^{\mu}\bar{\theta}j_{\mu}(x) + \cdots , \qquad (7.2.1)$$

where the ellipsis stands for derivative terms, following from the conservation equations $\mathcal{D}^2\mathcal{J} = \bar{\mathcal{D}}^2\mathcal{J} = 0$.[4] Among the virtues of GGM is its ability to disentangle genuine characteristics of gauge mediation from possible model-dependent features. GGM also leads to phenomenological superpartner-mass sum rules that, if verified by the LHC, will identify gauge mediation as the dominant means by which SUSY is broken in nature (see e.g. [36, 37] for a renormalization group study of the above-mentioned sum rules). Moreover, GGM encompasses strongly-coupled hidden sectors at the qualitative level and also at the quantitative level, at least in principle. In our view this is the greatest strength of GGM, which is nevertheless largely unexplored. In the next section it will be discussed extensively.

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[4]In this paper D is the D-term, thus we use $\mathcal{D}$ for the covariant derivatives.

The correlators one considers in GGM are (using the conventions of [4])

$$\langle J(x)J(0)\rangle = C_0(x) \xrightarrow{\text{F.T.}} \widetilde{C}_0(p),$$

$$\langle j_\alpha(x)\bar{j}_{\dot\alpha}(0)\rangle = -i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu C_{1/2}(x) \xrightarrow{\text{F.T.}} \sigma^\mu_{\alpha\dot\alpha}p_\mu \widetilde{C}_{1/2}(p),$$

$$\langle j_\mu(x)j_\nu(0)\rangle = (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)C_1(x) \xrightarrow{\text{F.T.}} -(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\widetilde{C}_1(p),$$

$$\langle j_\alpha(x)j_\beta(0)\rangle = \epsilon_{\alpha\beta}B_{1/2}(x) \xrightarrow{\text{F.T.}} \epsilon_{\alpha\beta}\widetilde{B}_{1/2}(p),$$

(7.2.2)

where F.T. stands for Fourier-transforming, $\text{F.T.} \equiv i\int d^4x\, e^{-ip\cdot x}$. It was realized in [3] that for the soft masses, for example, only the one-point function $\langle J(x)\rangle$ and the correlator $\langle J(x)J(0)\rangle$ are needed:[5]

$$M_{\text{gaugino}} = \frac{i\pi\alpha_{\text{SM}}}{d(G)}\int d^4x\,\langle \mathcal{Q}^2(J^A(x)J^A(0))\rangle,$$

$$m^2_{\text{sfermion}} = 4\pi Y\alpha_{\text{SM}}\langle J(x)\rangle + \frac{iC_2(R)\alpha^2_{\text{SM}}}{8d(G)}\int d^4x\,\ln(x^2M_{\text{m}}^2)\langle \bar{\mathcal{Q}}^2\mathcal{Q}^2(J^A(x)J^A(0))\rangle,$$

(7.2.3)

where M_{m} is a supersymmetric scale in the hidden-sector theory, e.g. the messenger scale. For clarity, the appropriate MSSM gauge group index A has been reintroduced.[6]

Using the results of [38] it was pointed out in [5] that, within a superconformal field theory, the superconformal algebra and current conservation are powerful enough to relate all possible two-operator products of components of the current superfield (7.2.1) to the operator product $J(x)J(0)$. Consequently, only the correlator $\langle J(x)J(0)\rangle$ is necessary, while all other correlators in (7.2.2) can be expressed in terms of $\langle J(x)J(0)\rangle$ with the help of the superconformal group. From [5] one has

$$j_\alpha(x)j_\beta(0) = \frac{1}{x^2}\mathcal{Q}_\beta(ix\cdot\sigma\bar{\mathcal{S}})_\alpha(J(x)J(0)),$$

---

[5]Since Q is used in this paper for the quarks of SQCD, we use $\mathcal{Q}$ to denote the SUSY generator. $\mathcal{Q}$ always acts with an adjoint action, e.g. $\mathcal{Q}^2(\mathcal{O}(x)) \equiv \{\mathcal{Q}^\alpha, [\mathcal{Q}_\alpha, \mathcal{O}(x)]\}$.

[6]The MSSM gauge group is chosen to be a GUT $SU(N)$ subgroup of the hidden-sector global symmetry group where $A = 1,\ldots,N^2-1$ is the appropriate adjoint index.

$$j_\alpha(x)\bar{j}_{\dot\alpha}(0) = \frac{1}{x^4}\left[(\mathcal{S}\,ix\cdot\sigma)_{\dot\alpha}(ix\cdot\sigma\bar{\mathcal{S}})_\alpha - x^2\bar{\mathcal{Q}}_{\dot\alpha}(ix\cdot\sigma\bar{\mathcal{S}})_\alpha + 2\Delta_J x^2(ix\cdot\sigma)_{\alpha\dot\alpha}\right](J(x)J(0)),$$

$$j_\mu(x)j_\nu(0) = \frac{1}{16x^8}\left[(x^2\eta_{\mu\rho} - 2x_\mu x_\rho)(\mathcal{S}\sigma^\rho\bar{\mathcal{S}} - \bar{\mathcal{S}}\sigma^\rho\mathcal{S})\right.$$

$$\times\{x^4(\bar{\mathcal{Q}}\bar\sigma_\nu\mathcal{Q} - \mathcal{Q}\sigma_\nu\bar{\mathcal{Q}}) + (x^2\eta_{\nu\lambda} - 2x_\nu x_\lambda)(\mathcal{S}\sigma^\lambda\bar{\mathcal{S}} - \bar{\mathcal{S}}\bar\sigma^\lambda\mathcal{S})$$

$$-2x^2\left(\mathcal{Q}\sigma_\nu\,ix\cdot\bar\sigma\mathcal{S} - \bar{\mathcal{Q}}\bar\sigma_\nu\,ix\cdot\sigma\bar{\mathcal{S}}\right)\}$$

$$-8i(\Delta_J + 1)x^2(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} - i\epsilon_{\mu\nu\lambda\rho})x^\lambda$$

$$\times\{(x^2\eta^{\rho\delta} - 2x^\rho x^\delta)\mathcal{S}\sigma_\delta\bar{\mathcal{S}} + x^2\bar{\mathcal{Q}}\bar\sigma^\rho\,ix\cdot\sigma\bar{\mathcal{S}} + 4i\Delta_J x^2 x^\rho\}$$

$$-8i(\Delta_J + 1)x^2(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} + i\epsilon_{\mu\nu\lambda\rho})x^\lambda$$

$$\times\{(x^2\eta^{\rho\delta} - 2x^\rho x^\delta)\bar{\mathcal{S}}\bar\sigma_\delta\mathcal{S} + x^2\mathcal{Q}\sigma^\rho\,ix\cdot\bar\sigma\mathcal{S} + 4i\Delta_J x^2 x^\rho\}$$

$$+32x^4\Delta_J(\Delta_J + 1)(x^2\eta_{\mu\nu} - 2x_\mu x_\nu)\bigg](J(x)J(0)),$$

with $\mathcal{S}, \bar{\mathcal{S}}$ the superconformal supercharges. Implications of this observation in the case of a UV asymptotically-free hidden sector (i.e. with approximate superconformal symmetry) and in particular in the example of MGM were analyzed using the OPE in [6], and we will rely heavily here on the results of that paper. It is important to note that using the OPE in the equations above and Fourier-transforming the results allow a simple evaluation of the total cross-sections of the visible sector to the hidden sector, with different mediators corresponding to the different components of the MSSM vector superfields. This is reminiscent of electron-positron scattering to hadrons in QCD. In the following we will focus on the superpartner spectrum, and will not discuss such cross-sections.

As shown in [6] a complete expansion of (7.2.3) can be obtained with the help of the $J(x)J(0)$ OPE which thus gives an approximation to the soft MSSM SUSY-breaking masses even for strongly-coupled hidden sectors. The expansion relies on several approximations (e.g. cuts at supersymmetric threshold, uniform convergence of the OPE) but, at least in the simple case of MGM, a complete knowledge of the OPE leads to an exact evaluation of the soft SUSY-breaking masses, after analytic

continuation of the sums. To avoid complications such as arduous OPE computations and analytic continuations, a further approximation to (7.2.3), given by

$$M_{\text{gaugino}} \approx -\frac{\pi w^{AA}\alpha_{\text{SM}}}{8d(G)M_{\text{m}}^2}\gamma_{Ki}\langle\mathcal{Q}^2(\mathcal{O}_i(0))\rangle,$$

$$m_{\text{sfermion}}^2 \approx 4\pi Y\alpha_{\text{SM}}\langle J(x)\rangle + \frac{C_2(R)w^{AA}\alpha_{\text{SM}}^2}{64d(G)M_{\text{m}}^2}\gamma_{Ki}\langle\bar{\mathcal{Q}}^2\mathcal{Q}^2(\mathcal{O}_i(0))\rangle,$$

(7.2.4)

was introduced in [6]. Here w^{AB} is the OPE coefficient of a scalar operator K with classical scaling dimension 2 in the OPE of two conserved currents (like, e.g. the Konishi current in MGM), and γ is the anomalous-dimension matrix of K (see (7.3.2), (7.3.3) and (7.3.7)). So, to get an approximation to the soft MSSM SUSY-breaking masses, *even in a theory with a strongly-coupled hidden sector*, one only needs to identify the lowest-dimension operators that have non-zero vevs after acted upon with $\mathcal{Q}^2$ and $\bar{\mathcal{Q}}^2\mathcal{Q}^2$.

In the example of MGM there is only one such operator, namely $S^\dagger S$, and calculating its mixing with the Konishi current one finds that the approximation to the soft masses (7.2.4) is actually only a factor of 2 smaller than the usually quoted answers [39]. For more details the reader is referred to section 7.3.1 and [6].

7.3. SQCD as the SUSY-breaking sector

To be specific, in this paper we take the messenger sector of gauge mediation to be SQCD without masses for the quarks but, instead, with Kähler potential and superpotential for matter fields given by

$$K_{\text{e}} = \text{Tr}(Q^\dagger Q + \widetilde{Q}\widetilde{Q}^\dagger) + S^\dagger S,$$

$$W_{\text{e}} = \xi S\,\text{Tr}\,\widetilde{Q}Q,$$

(7.3.1)

where S is the MGM-like singlet field which has non-vanishing vacuum expectation value $\langle S\rangle = \langle S\rangle + \theta^2\langle F_S\rangle$, and $Q, \widetilde{Q}$ are the messenger fields which are N_f flavors of $SU(N_c)$

fundamental and antifundamental superfields. The non-Abelian part of the global symmetry of SQCD is thus broken to its diagonal subgroup, $SU(N_f)_\mathrm{L} \times SU(N_f)_\mathrm{R} \to SU(N_f)_\mathrm{V}$, which contains $SU(N)$, a grand-unified extension of the MSSM gauge group. The coupling ξ is assumed weak. We will refer to SQCD with an extra singlet and the superpotential (7.3.1) as sSQCD. We stress that it is straightforward to repeat the analysis for more general messenger sectors and hidden sectors.

In order to use the approximation (7.2.4) in this framework, it is necessary to determine the $J(x)J(0)$ OPE at the lowest non-trivial order as well as the appropriate anomalous dimension matrix.

Note that non-perturbative effects (instantons) contribute both to the vevs of operators appearing on the right-hand side of the OPE and to the (perturbative) OPE coefficients themselves [40]. Furthermore, for operator products satisfying the chirality selection rule, instantons can lead to new non-perturbative contributions on the right-hand side of the OPE, i.e. operators with purely non-perturbative OPE coefficients [41]. Instanton corrections of the first type do not modify the OPE coefficients at lowest order and are thus non-negligible only for vevs of operators. Instanton corrections of the second type lead to new non-perturbative OPE contributions which can dominate over the perturbative ones.[7] Since the $J(x)J(0)$ OPE is non-trivial at the classical level and does not satisfy the chirality selection rule, for our purposes non-perturbative contributions that are calculable can be safely ignored.

The currents of interest for the evaluation of the $J(x)J(0)$ OPE are

$$J^A = \mathrm{Tr}(Q t^A Q^\dagger - \widetilde{Q}^\dagger t^A \widetilde{Q})_{(N_c, N)},$$

$$K = \mathrm{Tr}(Q^\dagger Q + \widetilde{Q}\widetilde{Q}^\dagger)_{(N_c, N)},$$

where we denote the $SU(N)$ generators by t^A to avoid confusion with the $SU(N_c)$

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[7]It is important to notice that both types of non-perturbative contributions to the OPE coefficients are calculable. Thus, as usual, the OPE coefficients are fully calculable, while all incalculable non-perturbative effects are contained in the vevs of operators.

generators T^I. At the classical level the OPE is simply

$$J^A(x)J^B(0) = \frac{N_c\delta^{AB}}{16\pi^4x^4}\mathbb{1} + \frac{w^{AB}}{4\pi^2x^2}K(0) + \cdots, \qquad (7.3.2)$$

where $w^{AB} = \delta^{AB}/N$, while the one-loop anomalous-dimension matrix between K and $S^\dagger S$ is

$$\gamma = \begin{pmatrix} \gamma_{K,K} & \gamma_{K,S^\dagger S} \\ \gamma_{S^\dagger S,K} & \gamma_{S^\dagger S,S^\dagger S} \end{pmatrix} \xrightarrow[\text{coupling}]{\text{weak}} \frac{1}{8\pi^2} \begin{pmatrix} 2C_2(N_c)g^2 & 2NN_c|\xi|^2 \\ |\xi|^2 & 0 \end{pmatrix}. \qquad (7.3.3)$$

Note here that although computable in the weak-coupling regime, the anomalous dimensions are large in the IR for strongly-coupled theories and are therefore kept undetermined in the following, leading to yet another approximation. The soft SUSY-breaking masses are

$$
\begin{aligned}
M_{\text{gaugino}} &\approx -\frac{\pi\alpha_{\text{SM}}}{8N|\xi\langle S\rangle|^2}\left[\gamma_{K,K}\langle\mathcal{Q}^2(K)\rangle + \gamma_{K,S^\dagger S}\langle\mathcal{Q}^2(S^\dagger S)\rangle\right], \\
m_{\text{sfermion}}^2 &\approx \frac{C_2(R)\alpha_{\text{SM}}^2}{64N|\xi\langle S\rangle|^2}\left[\gamma_{K,K}\langle\bar{\mathcal{Q}}^2\mathcal{Q}^2(K)\rangle + \gamma_{K,S^\dagger S}\langle\bar{\mathcal{Q}}^2\mathcal{Q}^2(S^\dagger S)\rangle\right],
\end{aligned}
\qquad (7.3.4)
$$

since the supersymmetric mass scale $M_{\text{m}} = |\xi\langle S\rangle|$ and $\langle J\rangle = 0$ for a non-Abelian group.

These expressions can be further simplified using the supersymmetry algebra and the Konishi anomaly [42] (in Wess–Zumino gauge) in the $\alpha_{\text{SM}} \to 0$ limit:

$$\mathcal{Q}^2(S^\dagger S) = 4S^\dagger F_S,$$

$$\bar{\mathcal{Q}}^2\mathcal{Q}^2(S^\dagger S) = 16(F_S^\dagger F_S - i\bar{\psi}_S\bar{\sigma}^\mu\partial_\mu\psi_S + S^\dagger\partial^2S),$$

$$\mathcal{Q}^2(K) = 4\left[\text{Tr}(Q^\dagger F + \widetilde{F}\widetilde{Q}^\dagger)_{(N_c,N)} + \frac{Ng^2}{16\pi^2}\text{tr}\,\bar{\lambda}\lambda\right],$$

$$\bar{\mathcal{Q}}^2 \mathcal{Q}^2(K) = 16 \left[\text{Tr}(F^\dagger F - i\bar{\psi}\bar{\sigma}^\mu \mathscr{D}_\mu \psi + Q^\dagger \mathscr{D}^2 Q + i\sqrt{2}g(Q^\dagger \lambda \psi - \bar{\psi}\bar{\lambda}Q) \right.$$

$$+ gQ^\dagger DQ)_{(N_c,N)} + \{(Q, \psi, F, g) \to (\widetilde{Q}, \tilde{\psi}, \widetilde{F}, -g)\}$$

$$\left. - \frac{Ng^2}{32\pi^2} \text{tr}(2DD - 4i\lambda\sigma^\mu \mathscr{D}_\mu \bar{\lambda} - F_{\mu\nu}F^{\mu\nu}) \right].$$

Note that $\bar{\mathcal{Q}}^2 \mathcal{Q}^2(S^\dagger S, K)$ are real up to total derivatives. After using the equations of motion (we omit the ones for the fields with a tilde),

$$F = -\xi^* S^\dagger \widetilde{Q}^\dagger, \qquad D^I = -g \, \text{Tr}(Q^\dagger T^I Q - \widetilde{Q}T^I \widetilde{Q}^\dagger),$$

$$\mathscr{D}^2 Q = -i\sqrt{2}g\lambda\psi - gDQ + |\xi|^2 S^\dagger SQ - \xi^* F_S^\dagger \widetilde{Q}^\dagger,$$

$$i\bar{\sigma}^\mu \mathscr{D}_\mu \psi = -i\sqrt{2}g\bar{\lambda}Q - \xi^* S^\dagger \bar{\tilde{\psi}}, \qquad i\sigma^\mu \mathscr{D}_\mu \bar{\lambda}^I = i\sqrt{2}g(Q^\dagger T^I \psi - \tilde{\psi}T^I \widetilde{Q}^\dagger),$$

the approximations (7.3.4) can be written in terms of vacuum condensates of UV elementary fields as

$$M_{\text{gaugino}} \approx \frac{\pi\alpha_{\text{SM}}}{2N|\xi\langle S\rangle|^2} \left[2\xi^* \gamma_{K,K} \langle S^\dagger \, \text{Tr}(Q^\dagger \widetilde{Q}^\dagger)_{(N_c,N)}\rangle - \frac{Ng^2}{16\pi^2}\gamma_{K,K}\langle \text{tr}\,\bar{\lambda}\bar{\lambda}\rangle \right.$$

$$\left. -\gamma_{K,S^\dagger S}\langle S^\dagger F_S\rangle \right]$$

$$m_{\text{sfermion}}^2 \approx \frac{C_2(R)\alpha_{\text{SM}}^2}{4N|\xi\langle S\rangle|^2} \left[2\xi^* \gamma_{K,K} \langle \xi S^\dagger SK + \text{Tr}(S^\dagger \bar{\psi}\bar{\tilde{\psi}} - F_S^\dagger Q^\dagger \widetilde{Q}^\dagger)_{(N_c,N)}\rangle \right.$$

$$\left. - \frac{Ng^2}{32\pi^2}\gamma_{K,K}\langle \text{tr}(2DD - 4E - F_{\mu\nu}F^{\mu\nu})\rangle + \gamma_{K,S^\dagger S}|\langle F_S\rangle|^2 \right],$$

$$(7.3.5)$$

where $E = i\sqrt{2}g\,\text{Tr}(Q^\dagger \lambda\psi - \tilde{\psi}\lambda\widetilde{Q}^\dagger)$. Note that m_{sfermion}^2 is of course real, although this is not manifest in (7.3.5), a consequence of the fact that $\bar{\mathcal{Q}}^2 \mathcal{Q}^2(K)$ is not manifestly real.

Finally, for a strongly-coupled theory it is more natural to express the approximations (7.3.5) in terms of vacuum condensates of IR elementary fields, i.e. the MSSM-restricted mesonic superfield $\mathcal{M} = \text{Tr}(M)_{(0,N)}$ and the "glueball" superfield

$\mathcal{G} = -(g^2/32\pi^2)\,\mathrm{tr}\,W^\alpha W_\alpha$, leading to

$$M_{\text{gaugino}} \approx \frac{\pi\alpha_{\text{SM}}}{2N|\xi\langle S\rangle|^2}\left[2\xi^*\gamma_{K,K}\langle S^\dagger \mathcal{M}^\dagger\rangle - 2N\gamma_{K,K}\langle \mathcal{G}^\dagger\rangle - \gamma_{K,S^\dagger S}\langle S^\dagger F_S\rangle\right],$$

$$m^2_{\text{sfermion}} \approx \frac{C_2(R)\alpha^2_{\text{SM}}}{4N|\xi\langle S\rangle|^2}\left[-2\xi^*\gamma_{K,K}\langle S^\dagger F^\dagger_{\mathcal{M}} + F^\dagger_S \mathcal{M}^\dagger\rangle + N\gamma_{K,K}\langle F_{\mathcal{G}} + F^\dagger_{\mathcal{G}}\rangle\right. \tag{7.3.6}$$

$$\left. + \gamma_{K,S^\dagger S}\langle F^\dagger_S F_S\rangle\right].$$

Equations (7.3.4), (7.3.5) and (7.3.6) can be easily generalized to more complicated UV theories with several gauge groups and matter fields in different representations. They can also be generalized to closely-related types of mediation like general gaugino mediation [43]. The approximations (7.3.6) are especially useful since they give an estimate for the MSSM soft SUSY-breaking masses from the knowledge of the vevs of a few IR elementary fields, taking the anomalous-dimension matrix to be of $\mathcal{O}(1)$. Indeed only the vacuum structure of both the messenger and the hidden sector is necessary to approximately determine the MSSM superpartner spectrum. The knowledge of the spectrum of messengers does not directly enter the computation.

Note that these approximations should be valid for strongly-coupled theories as well, although the size of the error introduced by truncating the OPE and assuming that cuts extend to the supersymmetric threshold is difficult to estimate in general. The anomalous-dimension terms cannot be computed at strong coupling, but they are expected to be $\mathcal{O}(1)$. It is however possible to argue for the functional dependence of the relevant anomalous dimensions at strong coupling. For example, $\gamma_{K,K}$ should depend on the electric quark mass and electric strong-coupling scale, and since it must be dimensionless it should be expressible by a series in positive powers of $|\xi\langle S\rangle/\Lambda_{\text{e}}|$

and $|\xi\langle F_S\rangle/\Lambda_{\mathrm{e}}^2|$. For $|\langle F_S\rangle/\xi\langle S\rangle| \ll 1$, at lowest order one thus expects[8]

$$\gamma_{K,K} \xrightarrow[\text{coupling}]{\text{strong}} \frac{\widetilde{N}_c}{16\pi^2} \left|\frac{\xi\langle S\rangle}{\Lambda_{\mathrm{e}}}\right| \delta_{K,K}, \qquad \gamma_{K,S^\dagger S} \xrightarrow[\text{coupling}]{\text{strong}} \frac{N\widetilde{N}_c}{16\pi^2}|\xi|^2\delta_{K,S^\dagger S}, \qquad (7.3.7)$$

where $\delta_{K,K}$ and $\delta_{K,S^\dagger S}$ are dimensionless numbers of order one. We introduced in (7.3.7) one-loop factors as well as factors of $\widetilde{N}_c$ and N to account for the effective number of degrees of freedom propagating in the loops as suggested by the Seiberg dual (see (7.3.8)).

Furthermore, although the vevs of the appropriate fields in the vacuum of interest are not always calculable in the strongly-coupled regime, it is often possible to approximate them in terms of the relevant scales of the theory under consideration. Therefore the approximations (7.3.6), which represent the main results of this paper, as well as their generalizations to more complicated models, should be acceptable up to dimensionless numbers of order one. Finally, when weakly-coupled duals exist, it is possible to assess the issues discussed above and directly check that the approximations (7.3.6) are indeed reliable up to $\mathcal{O}(1)$ factors, as will be seen in the next section.

In the event that SUSY is discovered at the LHC and that gauge mediation is the relevant means of SUSY-breaking communication, the approximations (7.3.6) open a rare window into the messenger and the hidden sector: by experimentally measuring the MSSM superpartner spectrum, they allow an approximate determination of some of the vevs of operators in the messenger and the hidden sector. This is reminiscent of QCD sum rules [46] (see also [47] for a nice review and more references), although here the spectrum of hidden-sector resonances is not necessary.

We will now use these equations to investigate the superpartner spectra of sSQCD and its different limits, starting from the computationally-reachable weakly-

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[8]Note that the form of the anomalous current K is known in terms of magnetic variables around the free supersymmetric and R-symmetric IR CFT in massless SQCD as described by Seiberg duality [44] (see also [45]). However the anomalous dimension computed from this perspective does not lead to the appropriate functional dependence as argued here since we are interested in the ISS SUSY-breaking vacuum.

coupled regime and ending with the often incalculable strongly-coupled regime. To this end, we will use Seiberg duality [19], which for $SU(N_c)$ sSQCD in the free magnetic phase leads to the following $SU(\widetilde{N}_c \equiv N_f - N_c)$ weakly-coupled dual theory for the matter fields (here the meson M, the magnetic quarks q and $\tilde{q}$, and the singlet S),

$$K_{\mathrm{m}} = \frac{1}{\alpha|\Lambda_{\mathrm{e}}|^2} \operatorname{Tr}(M^\dagger M)_{(0,N_f)} + \frac{1}{\beta} \operatorname{Tr}(q^\dagger q + \tilde{q}\tilde{q}^\dagger)_{(N_f-N_c,N_f)} + S^\dagger S + \cdots ,$$

$$W_{\mathrm{m}} = \frac{1}{\Lambda_{\mathrm{d}}} \operatorname{Tr}(qM\tilde{q})_{(N_f-N_c,N_f)} + \xi S \operatorname{Tr}(M)_{(0,N_f)},$$

$$(-1)^{N_f-N_c}\Lambda_{\mathrm{d}}^{N_f} = \Lambda_{\mathrm{e}}^{3N_c-N_f}\Lambda_{\mathrm{m}}^{3(N_f-N_c)-N_f}.$$

$$(7.3.8)$$

Note that α and β are positive real dimensionless numbers of order one, and Λ_{e}, Λ_{m} and Λ_{d} are the electric strong-coupling scale, the magnetic scale and the duality scale respectively.[9] Seiberg duality will allow the determination of the vevs of the relevant IR elementary fields in terms of a few unknowns, therefore providing a direct check of the approximations (7.3.6).

7.3.1. sSQCD in the $g \to 0$ limit: MGM

In the limit of vanishing hidden-sector gauge coupling, sSQCD is equivalent to MGM with N_c messenger flavors. In this limit the phenomenology of sSQCD is already well-known, and is easily reproduced with our methods. Indeed, the only non-vanishing vacuum condensate occurs for the MGM singlet S and the theory is effectively equivalent to MGM with N_c flavors of messengers as expected. The approximations (7.3.6) along with the one-loop anomalous-dimension matrix (7.3.3) thus give (here

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[9]Due to the freedom in defining the magnetic quarks, β, Λ_{m} and Λ_{d} are not fully determined by the electric theory.

$$x_S = |\langle F_S \rangle / \xi \langle S \rangle^2|)$$

$$M_{\text{gaugino}} \approx -\frac{\alpha_{\text{SM}}}{4\pi} \frac{\langle F_S \rangle}{\langle S \rangle} \times N_c \times \left\{ g_{\text{approx}}(x_S) = \frac{1}{2} \right\},$$

$$m_{\text{sfermion}}^2 \approx 2 \left(\frac{\alpha_{\text{SM}}}{4\pi} \right)^2 \left| \frac{\langle F_S \rangle}{\langle S \rangle} \right|^2 \times C_2(R) \times N_c \times \left\{ f_{\text{approx}}(x_S) = \frac{1}{2} \right\},$$

which, as already mentioned, are only a factor of 2 smaller than the usually quoted one- and two-loop answers in the limit where $x_S = 0$ [39],

$$g(x_S) = \frac{1 + x_S}{x_S^2} \ln(1 + x_S) + \{x_S \to -x_S\} = 1 + \frac{x_S^2}{6} + \cdots,$$

$$f(x_S) = \frac{1 + x_S}{x_S^2} \left[\ln(1 + x_S) - 2\,\text{Li}_2\left(\frac{x_S}{1 + x_S} \right) + \frac{1}{2}\,\text{Li}_2\left(\frac{2x_S}{1 + x_S} \right) \right] + \{x_S \to -x_S\}$$

$$= 1 + \frac{x_S^2}{36} + \cdots,$$

where $\text{Li}_2(x) = -\int_0^1 dt\, \frac{\ln(1 - xt)}{t}$ is the dilogarithm or Spence function. Note that since the OPE is truncated at lowest order in the SUSY-breaking expansion, it is naturally expected that the approximations (7.3.6) only capture (part of) the $x_S = 0$ limit of $g(x_S)$ and $f(x_S)$.

The functions $g(x_S)$ and $f(x_S)$, which are only defined in the region $0 \le x_S \le 1$ in order to avoid tachyonic messengers, do not deviate much from unity, and so the agreement of the OPE with the full answer at one loop for the gauginos and at two loops for the sfermions is reasonable, as can be seen in Fig. 7.1.

A complete OPE analysis of MGM shows that the method described in [6] and extended here works in the weakly-coupled regime, providing a useful consistency check. Note that it is not easy to use our method to obtain exact results in the weakly-coupled regime. Nevertheless, the simple approximations (7.3.6) match weakly-coupled computations up to dimensionless numbers of order one, a property which should translate to the strongly-coupled regime as well.
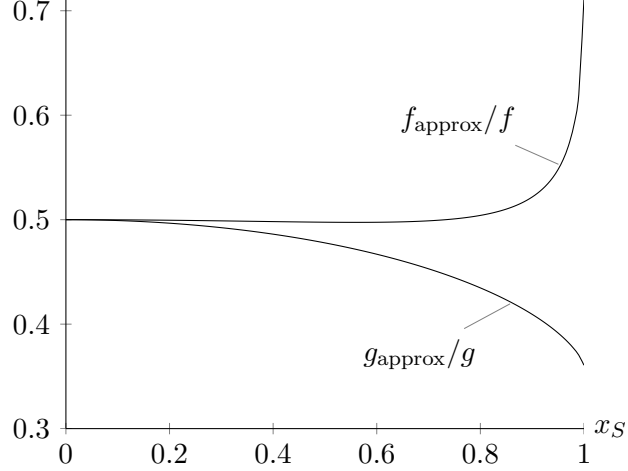
Figure 7.1: g_{approx}/g and f_{approx}/f as functions of x_S for MGM.

7.3.2. sSQCD in the $\langle S \rangle \to m/\xi$ and $\langle F_S \rangle \to 0$ limit: mSQCD

In the limit where the MGM singlet S is assumed frozen without an F-term, sSQCD is nothing else than mSQCD. The theory is most interesting in the free magnetic phase, given by $N_c + 1 \leq N_f < 3N_c/2$, where both a SUSY-preserving phase and a (metastable) SUSY-breaking phase exist [18].

Around the SUSY vacuum

In mSQCD, although $\langle \mathcal{M} \rangle$ and $\langle \mathcal{G} \rangle$ do not vanish at the supersymmetric vacuum, the soft SUSY-breaking masses vanish, as expected, due to the Konishi anomaly [42]. Indeed, although

$$\frac{g^2}{32\pi^2}\langle \text{tr}\,\lambda\lambda \rangle = [\Lambda_{\text{e}}^{3N_c - N_f} \det(\xi\langle S \rangle)]^{\frac{1}{N_c}} e^{2\pi i k/N_c}, \tag{7.3.9}$$

where (7.3.9) is valid for any N_c and N_f [41], the vacuum condensate for the mesonic superfield is

$$\langle \text{Tr}(\widetilde{Q}_{\tilde{i}}Q^i)_{(N_c,0)} \rangle = [\Lambda_{\text{e}}^{3N_c - N_f} \det(\xi\langle S \rangle)]^{\frac{1}{N_c}} \left[(\xi\langle S \rangle)^{-1}\right]_{\tilde{i}}^i e^{2\pi i k/N_c}, \tag{7.3.10}$$

as enforced by the Konishi anomaly [42],

$$\frac{-i}{2\sqrt{2}}\{\bar{\mathcal{Q}}_{\dot{\alpha}},\mathrm{Tr}(\bar{\psi}_{\tilde{i}}^{\dot{\alpha}}Q^i)_{(N_c,0)}\} = -\xi\langle S\rangle\,\mathrm{Tr}(\widetilde{Q}_{\tilde{i}}Q^i)_{(N_c,0)} + \delta_{\tilde{i}}^i\frac{g^2}{32\pi^2}\,\mathrm{tr}\,\lambda\lambda, \qquad (7.3.11)$$

in supersymmetric vacua.[10] In terms of the IR fields this implies that $\xi\langle S\rangle\langle\mathcal{M}\rangle = N\langle\mathcal{G}\rangle$. Since all remaining vacuum condensates vanish, the approximations (7.3.6) lead to a superpartner spectrum consistent with SUSY.

Around the ISS vacuum

As shown by ISS [18], mSQCD with small masses has a metastable SUSY-breaking minimum close to the origin of field space. A sketch of the potential of mSQCD is shown in Fig. 7.2.
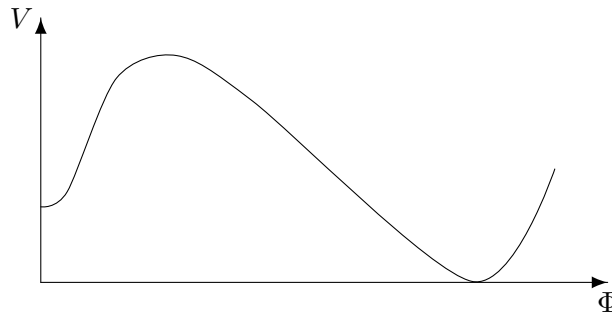

Figure 7.2: A sketch of the potential of mSQCD.

Since the SUSY-breaking scale and the messenger scale are the same in ISS, there is no dimensionless SUSY-breaking parameter to keep track of the order at which SUSY-breaking effects appear in any computation. Thus, in order to compare (7.3.6) with weakly-coupled computations of the sfermion masses from the dual theory (see Appendix 7.A), it is convenient to distinguish between the SUSY-breaking scale and the messenger scale by introducing two ξ's, (ξ,ξ_L) with $x_{\mathcal{M}} = \xi_L/\xi$ and $0 \leq |x_{\mathcal{M}}| \leq 1$. This effectively splits the mass matrix in two sectors and allows us to keep track of the

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[10]Here the index k labels the degenerate SUSY vacua which arise from the spontaneous breaking of the discrete global symmetry $\mathbb{Z}_{2N_c}$ to $\mathbb{Z}_2$.

SUSY-breaking effects.

The location of the SUSY-breaking minimum can be found using the dual theory (7.3.8) and, in terms of the IR elementary fields (embedding the MSSM into the X-sector of (7.3.13)), is given by

$$\langle \mathcal{M} \rangle = \langle \mathcal{G} \rangle = \langle F_{\mathcal{G}} \rangle = 0, \qquad \langle F_{\mathcal{M}} \rangle = -N_c \alpha \xi_L^* \langle S^\dagger \rangle |\Lambda_{\rm e}|^2. \tag{7.3.12}$$

The ISS vacuum faces an immediate problem for phenomenological applications: it has an accidental R-symmetry and thus constrains to zero Majorana gaugino masses. This can be seen directly from the approximations (7.3.6) and the vevs (7.3.12). The sfermion masses, on the other hand, are not constrained by the accidental R-symmetry and are indeed non-zero, as is also clear from (7.3.6) and the vevs (7.3.12).

Fixing $\Lambda_{\rm d} = \Lambda_{\rm m} = (-1)^{(N_c - N_f)/(3N_c - N_f)} \Lambda_{\rm e}$ and using the anomalous dimensions (7.3.7) the approximated sfermion masses obtained from (7.3.6) are

$$m_{\rm sfermion}^2 \approx 2 \left(\frac{\alpha_{\rm SM}}{4\pi} \right)^2 |x_{\mathcal{M}}|^2 \alpha \beta |\xi \langle S \rangle \Lambda_{\rm e}|$$

$$\times C_2(R) \times \widetilde{N}_c \times \left\{ f_{\rm approx}(x_{\mathcal{M}}) = \frac{4\pi^2}{\widetilde{N}_c \beta} \left| \frac{\Lambda_{\rm e}}{\xi \langle S \rangle} \right| \gamma_{K,K} = \frac{\delta_{K,K}}{4\beta} \right\},$$

while using the dual theory the weakly-coupled computation gives

$$f(x_{\mathcal{M}}) = \frac{1 + |x_{\mathcal{M}}|}{|x_{\mathcal{M}}|^2} \left[\ln(1 + |x_{\mathcal{M}}|) - 2 \, \mathrm{Li}_2 \left(\frac{|x_{\mathcal{M}}|}{1 + |x_{\mathcal{M}}|} \right) + \frac{1}{2} \, \mathrm{Li}_2 \left(\frac{2|x_{\mathcal{M}}|}{1 + |x_{\mathcal{M}}|} \right) \right]$$

$$+ \{ |x_{\mathcal{M}}| \to -|x_{\mathcal{M}}| \}$$

$$= 1 + \frac{|x_{\mathcal{M}}|^2}{36} + \cdots .$$

Although $x_{\mathcal{M}} = 1$ in ISS, our approximations only rely on the lowest-order operators appearing in the OPE and should only capture (part of) the $x_{\mathcal{M}} = 0$ contributions to $f(x_{\mathcal{M}})$, up to a number of order one (as in the MGM case of section 7.3.1). This is exactly what happens here. Moreover, since the function $f(x_{\mathcal{M}})$ stays close to unity for

all $x_\mathcal{M}$, the approximations (7.3.6) are reasonable for $0 \le |x_\mathcal{M}| \le 1$ as shown in Fig. 7.3. Therefore the method developed here gives sensible results even in strongly-coupled
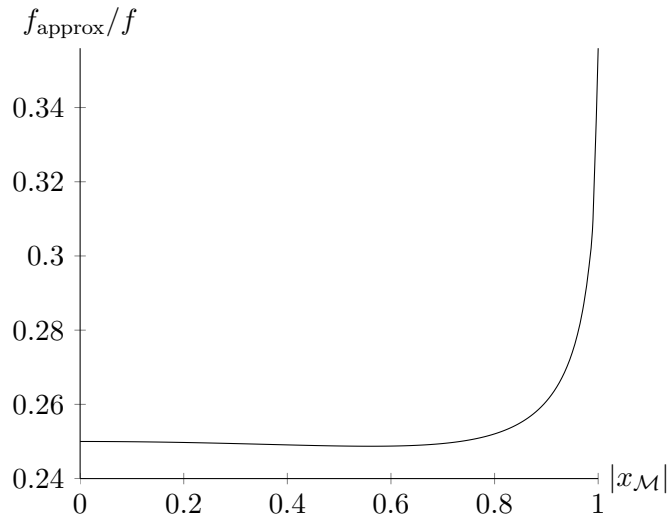

Figure 7.3: f_{approx}/f as function of $|x_\mathcal{M}|$ for mSQCD with $\beta = \delta_{K,K} = 1$. Both $g_{\text{approx}}(x_\mathcal{M})$ and $g(x_\mathcal{M})$ vanish and so the corresponding ratio is not plotted here.

theories including higher-order SUSY-breaking corrections.

It is interesting to notice that a full knowledge of the OPE could possibly lead to a computation of the anomalous dimensions of relevant operators in mSQCD following the method described here, as was done for MGM in [6]. For more details the reader is referred to section 7.3.3 and [18].

7.3.3. sSQCD in the free magnetic phase

Here we explore sSQCD for $N_c + 1 \le N_f < 3N_c/2$. As mentioned above, in mSQCD dynamical SUSY breaking in metastable vacua occurs for this range of N_f close to the origin of field space [18]. Although the electric theory is strongly coupled, Seiberg duality allows one to establish the presence of SUSY breaking. In this subsection we also use Seiberg duality to understand SUSY breaking in sSQCD close to the origin of field space.

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Around the would-be SUSY vacuum

Let us first discuss the fate of the would-be SUSY vacuum of mSQCD in the full sSQCD theory. For $|\langle F_S \rangle / \xi \langle S \rangle^2| \ll 1$ one would expect that the vevs of the glueball and mesonic superfields are only slightly perturbed compared to their mSQCD values (7.3.9) and (7.3.10). Moreover, from the point of view of the sSQCD fields, SUSY is explicitly broken. One should thus expect that the SUSY vacuum of mSQCD becomes a SUSY-breaking vacuum in sSQCD. Since small instantons are relevant, it is impossible to compute the vevs of the glueball and mesonic fields from instanton techniques without a full knowledge of the hidden-sector theory. It is nevertheless possible to estimate the vev of the lowest component of the mesonic superfield from the superpotential and the Kähler potential (7.3.8), leading to

$$\langle \mathcal{M} \rangle = [\xi^{N_f - N_c} \langle S \rangle^{N_f - N_c} \Lambda_{\mathrm{e}}^{3N_c - N_f}]^{\frac{1}{N_c}}$$

$$\times \left[1 + \frac{N_c - N_f}{N_c} \frac{1}{\alpha |\xi|^2} \left(\frac{\xi^* \langle S^\dagger \rangle}{\Lambda_{\mathrm{e}}^*} \right)^{\frac{N_f}{N_c}} \frac{\langle F_S^\dagger \rangle \Lambda_{\mathrm{e}}^{*2}}{\langle S^\dagger \rangle^2 \langle S \rangle \Lambda_{\mathrm{e}}} + \cdots \right].$$

One could then use the Konishi anomaly (7.3.11) to obtain the vev of the glueball superfield, but since the vacuum is expected to be non-supersymmetric, this approach is inconclusive. A complete knowledge of the hidden sector seems thus necessary to determine the characteristics of the superpartner spectrum around this vacuum.

Around the ISS-like vacuum

Around the origin of field space it is more convenient to use the dual theory as given by (7.3.8), but with canonically-normalized matter fields Φ, φ and $\tilde{\varphi}$. The superpotential becomes

$$W_{\mathrm{m}} = h \operatorname{Tr} \varphi \Phi \tilde{\varphi} - h \Psi \operatorname{Tr} \Phi,$$

where Ψ is a background field with $\langle\Psi\rangle = \mu^2 + \theta^2\mu_F^3$. The parameter μ_F is the source of R-symmetry breaking in our example. With the parametrization

$$\Phi = \begin{pmatrix} Y_{\widetilde{N}_c\times\widetilde{N}_c} & Z^T_{\widetilde{N}_c\times N_c} \\ \widetilde{Z}_{N_c\times\widetilde{N}_c} & X_{N_c\times N_c} \end{pmatrix}, \qquad \varphi^T = \begin{pmatrix} \chi_{\widetilde{N}_c\times\widetilde{N}_c} \\ \rho_{N_c\times\widetilde{N}_c} \end{pmatrix}, \qquad \tilde{\varphi} = \begin{pmatrix} \tilde{\chi}_{\widetilde{N}_c\times\widetilde{N}_c} \\ \tilde{\rho}_{N_c\times\widetilde{N}_c} \end{pmatrix}, \qquad (7.3.13)$$

the scalar potential becomes

$$\begin{aligned}
V = {}& N_f|h\mu^2|^2 + h\mu_F^3\,\mathrm{Tr}(Y+X) + h^*\mu_F^{*3}\,\mathrm{Tr}(Y^\dagger + X^\dagger) \\
& + |h|^2\,\mathrm{Tr}[-\mu^2(\tilde{\chi}^\dagger\chi^* + \tilde{\rho}^\dagger\rho^*) - \mu^{*2}(\chi^T\tilde{\chi} + \rho^T\tilde{\rho}) \\
& \quad + \tilde{\chi}^\dagger(Y^\dagger Y + \widetilde{Z}^\dagger\widetilde{Z})\tilde{\chi} + \tilde{\rho}^\dagger(Z^*Z^T + X^\dagger X)\tilde{\rho} + \tilde{\rho}^\dagger(Z^*Y + X^\dagger\widetilde{Z})\tilde{\chi} \\
& \quad + \tilde{\chi}^\dagger(Y^\dagger Z^T + \widetilde{Z}^\dagger X)\tilde{\rho} + \chi^\dagger(Y^*Y^T + Z^\dagger Z)\chi + \rho^\dagger(\widetilde{Z}^*\widetilde{Z}^T + X^*X^T)\rho \\
& \quad + \rho^\dagger(\widetilde{Z}^*Y^T + X^*Z)\chi + \chi^\dagger(Y^*\widetilde{Z}^T + Z^\dagger X^T)\rho + (\chi^T\chi^* + \rho^T\rho^*)(\tilde{\chi}^\dagger\tilde{\chi} + \tilde{\rho}^\dagger\tilde{\rho})].
\end{aligned}$$

As in the ISS case, the rank condition implies that SUSY is broken with $F_X^\dagger = h\mu^2$, and a minimum should develop around the origin of field space, which can be conveniently described with the following ansatz:

$$\langle\Phi\rangle = \begin{pmatrix} Y_0 & 0 \\ 0 & X_0 \end{pmatrix}, \qquad \langle\varphi^T\rangle = \begin{pmatrix} q_0 \\ 0 \end{pmatrix}, \qquad \langle\tilde{\varphi}\rangle = \begin{pmatrix} \tilde{q}_0 \\ 0 \end{pmatrix}.$$

Assuming $\tilde{q}_0 = q_0 = q$ the scalar potential is minimized in (almost) all directions when

$$Y_0 = -\frac{\mu_F^{*3}}{h(|q_0|^2 + |\tilde{q}_0|^2)} = -\frac{\mu_F^{*3}}{2h|q|^2},$$

$$q = \frac{1}{3}\mu(1 + H^{1/3} + H^{-1/3})^{1/2}, \qquad \text{where} \qquad H = 1 - \frac{27}{2}|\epsilon|\left(|\epsilon| - \sqrt{|\epsilon|^2 - \frac{4}{27}}\right).$$

$$(7.3.14)$$

Here $\epsilon = \mu_F^3/2h^*\mu^{*2}\mu$ and it is assumed small. The constraint on q comes from minimization in the $\tilde{\chi}$-direction leading to the condition

$$|q|^6 - \mu^{*2}q^2|q|^2 + \left|\frac{\mu_F^3}{2h}\right|^2 = 0,$$

which requires that $q/\mu \in \mathbb{R}$. Keeping the solution[11] for which $q \xrightarrow[\mu_F \to 0]{} \mu$ leads to the vev mentioned above. For a well-defined q one needs $|\epsilon| \leq \frac{2\sqrt{3}}{9}$ which is easily satisfied for small $|\epsilon|$. For small μ_F (or ϵ), (7.3.14) can be approximated by

$$Y_0 = -\frac{\mu_F^{*3}}{2h|\mu|^2} + \cdots, \qquad q = \mu\left(1 - \frac{1}{2}|\epsilon|^2 + \cdots\right).$$

The scalar potential is stabilized in all but the X-direction. As opposed to the ISS case where X is a flat direction of V, here X is a runaway direction at tree level and V slopes down in the X-direction. Since the runaway behavior is dictated by the small deformation μ_F, it is expected that the one-loop Coleman–Weinberg potential stabilizes the runaway direction close to the origin of field space, thus leading to spontaneous breaking of the accidental R-symmetry of the ISS model and allowing for non-vanishing gaugino masses.

To calculate the Coleman–Weinberg potential [48] for a general supersymmetric theory with n chiral superfields Φ^i, canonical Kähler potential, and superpotential $W(\Phi)$, we need the mass matrices for scalar and spin-$\frac{1}{2}$ fields, given respectively by the $2n \times 2n$ matrices

$$\mathbb{M}_0^2 = \begin{pmatrix} W^{\dagger ik}W_{kj} & W^{\dagger ijk}W_k \\ W_{ijk}W^{\dagger k} & W_{ik}W^{\dagger kj} \end{pmatrix} \quad \text{and} \quad \mathbb{M}_{1/2}^2 = \begin{pmatrix} W^{\dagger ik}W_{kj} & 0 \\ 0 & W_{ik}W^{\dagger kj} \end{pmatrix},$$

with $W_i \equiv \partial W/\partial \Phi^i$ and similarly for the rest, where the derivatives are to be evaluated at the vevs computed for the zero components of the chiral superfields.

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[11]The other solutions lead to tachyons.

In the case of supersymmetric theories, where quadratic divergences cancel among bosons and fermions,

$$\text{STr}\,\mathbb{M}^2 \equiv \text{Tr}\,\mathbb{M}_0^2 - \text{Tr}\,\mathbb{M}_{1/2}^2 = 0, \tag{7.3.15}$$

the Coleman–Weinberg potential takes the form

$$V_{\text{CW}} = \frac{1}{64\pi^2}\,\text{STr}\,\mathbb{M}^4\ln\frac{\mathbb{M}^2}{\Lambda^2} \equiv \frac{1}{64\pi^2}\left[\text{Tr}\,\mathbb{M}_0^4\left(\ln\frac{\mathbb{M}_0^2}{4\Lambda^2} + \frac{1}{2}\right) - \text{Tr}\,\mathbb{M}_{1/2}^4\left(\ln\frac{\mathbb{M}_{1/2}^2}{4\Lambda^2} + \frac{1}{2}\right)\right],$$

where Λ is the cutoff scale and plays no role in the following. We are therefore interested in V_{CW} as a function of the runaway direction X, $V_{\text{CW}}(X)$. Due to the supertrace relation (7.3.15), we only have to consider the mass matrices for the (ρ, Z) sector, since this is the only sector in which the spectrum is non-supersymmetric at tree level.

The mass eigenstates for the messenger sectors are fairly complicated. To simplify the analysis we choose to compute them at order ϵ, leading to

$$\tilde{m}_1^2 = |h\mu|^2 \frac{\epsilon x + \epsilon^* x^*}{1 + |x|^2},$$

$$\tilde{m}_2^2 = |h\mu|^2\left(1 + |x|^2 - \frac{\epsilon x + \epsilon^* x^*}{1 + |x|^2}\right),$$

$$\tilde{m}_3^2 = |h\mu|^2\left(\frac{3}{2} + \frac{1}{2}|x|^2 - \frac{1}{2}(1 + 6|x|^2 + |x|^4)^{1/2}\right.$$
$$\left. + \frac{1 + |x|^2 - (1 + 6|x|^2 + |x|^4)^{1/2}}{1 + 6|x|^2 + |x|^4 - (1 + |x|^2)(1 + 6|x|^2 + |x|^4)^{1/2}}(\epsilon x + \epsilon^* x^*)\right),$$

$$\tilde{m}_4^2 = |h\mu|^2\left(\frac{3}{2} + \frac{1}{2}|x|^2 + \frac{1}{2}(1 + 6|x|^2 + |x|^4)^{1/2}\right.$$
$$\left. + \frac{1 + |x|^2 + (1 + 6|x|^2 + |x|^4)^{1/2}}{1 + 6|x|^2 + |x|^4 + (1 + |x|^2)(1 + 6|x|^2 + |x|^4)^{1/2}}(\epsilon x + \epsilon^* x^*)\right),$$

for the bosonic mass eigenstates and

$$m_1^2 = |h\mu|^2\left(1 + \tfrac{1}{2}|x|^2 - \tfrac{1}{2}|x|(4 + |x|^2)^{1/2}\right),$$
$$m_2^2 = |h\mu|^2\left(1 + \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|x|(4 + |x|^2)^{1/2}\right), \tag{7.3.16}$$

for the fermionic mass eigenstates. Note that to simplify the notation we introduced $x = X/\mu$. Moreover, it is important to notice that $\tilde{m}_1$ vanishes exactly once higher-order corrections are introduced since it corresponds to a Goldstone boson.

Including the Coleman–Weinberg potential with corrections up to $\mathcal{O}(\epsilon)$ terms, the runaway in the X-direction is found to be stabilized at

$$X_0 = -\frac{16\pi^2 + \widetilde{N}_c|h|^2 \ln 2}{\widetilde{N}_c|h|^2 (\ln 4 - 1)} \epsilon^* \mu,$$

and a minimum appears close to the origin in field space. As we have explained, SUSY is also broken in the faraway vacuum. A sketch of the potential can be seen in Fig. 7.4.
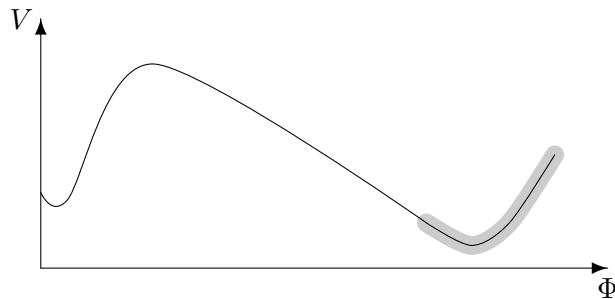

Figure 7.4: A sketch of the potential of sSQCD. The shading indicates that our analysis of the spectrum in the corresponding region, i.e. around and past the would-be SUSY vacuum, is not conclusive.

To make use of (7.3.6) we relate the canonically-normalized IR fields to the UV elementary fields with the help of the following dictionary:

$$\varphi = \frac{q}{\sqrt{\beta}}, \qquad \tilde{\varphi} = \frac{\tilde{q}}{\sqrt{\beta}}, \qquad \Phi = \frac{M}{\sqrt{\alpha}\Lambda_e},$$

$$h = \frac{\sqrt{\alpha}\beta\Lambda_e}{\Lambda_d}, \qquad \mu^2 = -\frac{\xi\langle S\rangle\Lambda_d}{\beta}, \qquad \mu_F^3 = -\frac{\xi\langle F_S\rangle\Lambda_d}{\beta}.$$

As already mentioned, one can choose to fix $\Lambda_d = \Lambda_m = (-1)^{(N_c - N_f)/(3N_c - N_f)}\Lambda_e$ and

describe the results in terms of α and β, which leads to ($N = N_c$)

$$\langle \mathcal{M} \rangle = \frac{N_c \sigma(x_\mathcal{M})}{2\beta} \left| \frac{\Lambda_e}{\xi\langle S \rangle} \right| \xi^* \langle F_S^\dagger \rangle, \qquad \langle F_\mathcal{M} \rangle = -N_c \alpha \xi_L^* \langle S^\dagger \rangle |\Lambda_e|^2,$$

when embedding the MSSM gauge group into the X-sector of (7.3.13). Here ξ_L has been introduced, as in the ISS case, to keep track of the SUSY-breaking effects, and $\sigma(x_\mathcal{M})$ encodes the position of the minimum as a function of the SUSY-breaking effects,

$$\sigma(x_\mathcal{M}) = \frac{16\pi^2 + \widetilde{N}_c \alpha \beta^2 a}{\widetilde{N}_c \alpha \beta^2 b} x_\mathcal{M}^*,$$

$$a = \frac{1}{2x_\mathcal{M}}\left[(1 + |x_\mathcal{M}|)\ln(1 + |x_\mathcal{M}|) + \{|x_\mathcal{M}| \to -|x_\mathcal{M}|\}\right] \xrightarrow[x_\mathcal{M} \to 1]{} \ln 2,$$

$$b = \frac{1}{2|x_\mathcal{M}|}\left[(1 + |x_\mathcal{M}|)^2 \ln(1 + |x_\mathcal{M}|) - |x_\mathcal{M}| - \{|x_\mathcal{M}| \to -|x_\mathcal{M}|\}\right] \xrightarrow[x_\mathcal{M} \to 1]{} \ln 4 - 1.$$

Using the anomalous dimensions (7.3.7), the superpartner spectrum at order $\mathcal{O}(\mu_F^3) \sim \mathcal{O}(\langle F_S \rangle)$ is thus

$$M_{\text{gaugino}} \approx \frac{\alpha_{\text{SM}}}{4\pi}\frac{\langle F_S \rangle}{\langle S \rangle} \times \widetilde{N}_c$$

$$\times \left\{ g_{\text{approx}}(x_\mathcal{M}) = \sigma^*(x_\mathcal{M})x_\mathcal{M}^* \frac{2\pi^2}{\widetilde{N}_c \beta}\left|\frac{\Lambda_e}{\xi\langle S \rangle}\right| \gamma_{K,K} - \frac{2\pi^2}{N_c \widetilde{N}_c |\xi|^2}\gamma_{K,S^\dagger S} = \right.$$

$$\left. = \sigma^*(x_\mathcal{M})x_\mathcal{M}^* \frac{\delta_{K,K}}{8\beta} - \frac{\delta_{K,S^\dagger S}}{8} \right\},$$

$$m_{\text{sfermion}}^2 \approx 2\left(\frac{\alpha_{\text{SM}}}{4\pi}\right)^2 |x_\mathcal{M}|^2 \alpha\beta|\xi\langle S \rangle\Lambda_e| \times C_2(R) \times \widetilde{N}_c$$

$$\times \left\{ f_{\text{approx}}(x_\mathcal{M}) = \frac{4\pi^2}{\widetilde{N}_c \beta}\left|\frac{\Lambda_e}{\xi\langle S \rangle}\right| \gamma_{K,K} = \frac{\delta_{K,K}}{4\beta} \right\},$$

$$(7.3.17)$$

and can be compared to the weakly-coupled computation which gives

$$
\begin{aligned}
g(x_\mathcal{M}) &= \left[\frac{1+|x_\mathcal{M}|}{|x_\mathcal{M}|^2} \ln(1+|x_\mathcal{M}|) + \{|x_\mathcal{M}| \to -|x_\mathcal{M}|\} \right] \\
&\quad + \frac{\sigma^*(x_\mathcal{M})}{2x_\mathcal{M}|x_\mathcal{M}|} \left[3|x_\mathcal{M}| - (3+4|x_\mathcal{M}|+|x_\mathcal{M}|^2)\ln(1+|x_\mathcal{M}|) - \{|x_\mathcal{M}| \to -|x_\mathcal{M}|\} \right] \\
&= 1 + \frac{|x_\mathcal{M}|^2}{6} + \cdots + \sigma^*(x_\mathcal{M}) \left(\frac{x_\mathcal{M}^*|x_\mathcal{M}|^2}{15} + \cdots \right), \\
f(x_\mathcal{M}) &= \frac{1+|x_\mathcal{M}|}{|x_\mathcal{M}|^2} \left[\ln(1+|x_\mathcal{M}|) - 2\operatorname{Li}_2\left(\frac{|x_\mathcal{M}|}{1+|x_\mathcal{M}|} \right) + \frac{1}{2}\operatorname{Li}_2\left(\frac{2|x_\mathcal{M}|}{1+|x_\mathcal{M}|} \right) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \{|x_\mathcal{M}| \to -|x_\mathcal{M}|\} \\
&= 1 + \frac{|x_\mathcal{M}|^2}{36} + \cdots .
\end{aligned}
$$

$$(7.3.18)$$

At order $\mathcal{O}(\langle F_S \rangle)$ the sSQCD sfermion masses are the same as the mSQCD sfermion masses. Note that the functional dependence of the anomalous dimension $\gamma_{K,K}$, necessary for the approximate gaugino masses to match the weakly-coupled computation, is the same as the one expected from the sfermion masses. This gives another way to see why the functional dependence of $\gamma_{K,K}$ is indeed proportional to $|\xi\langle S \rangle/\Lambda_{\rm e}|$.

As for the mSQCD case, $x_\mathcal{M} = 1$ but by truncating the OPE the results (7.3.17) should only capture the lowest-order contribution in the $x_\mathcal{M}$-expansion of $g(x_\mathcal{M})$ and $f(x_\mathcal{M})$ up to $\mathcal{O}(1)$ factors, as can be seen directly. Note however that the power in $|x_\mathcal{M}|$ of the spontaneous R-symmetry breaking contribution to the gaugino mass, denoted by $\sigma(x_\mathcal{M})$, does not exactly match the weakly-coupled computation: it is off by a factor of $|x_\mathcal{M}|^2$. This suggests that all OPE contributions of the same type must be included to appreciate the suppression seen at small dynamical SUSY breaking, i.e. for small $|x_\mathcal{M}|$. Yet, this point is of no relevance since the metastable SUSY-breaking minimum disappears for small $|x_\mathcal{M}|$, indeed $\langle\mathcal{M}\rangle \xrightarrow[x_\mathcal{M}\to0]{} \infty$. This is clear since for fixed ϵ, the Coleman–Weinberg potential cannot compete against the runaway when $|x_\mathcal{M}|$ is too small. The value of $x_\mathcal{M}$ at which the minimum disappears can be estimated

from the constraint that the messenger masses must be all non-tachyonic. Using the messenger masses at order ϵ, this constraint is obtained from the fermionic messenger mass eigenstates (7.3.16). In Fig. 7.5 we plot our results for $0.5 \leq |x_{\mathcal{M}}| \leq 1$.
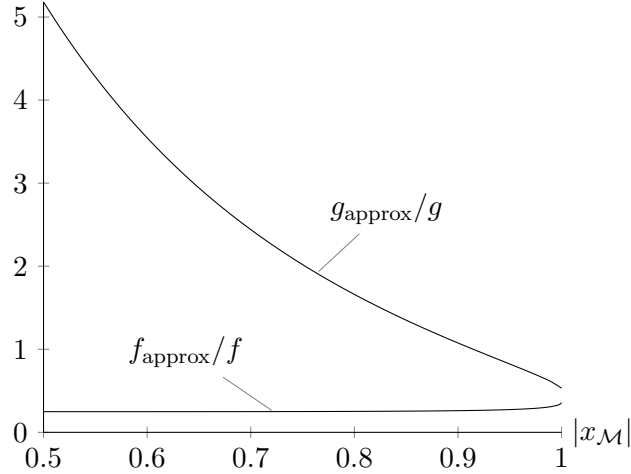

Figure 7.5: g_{approx}/g and f_{approx}/f as functions of $|x_{\mathcal{M}}|$ for sSQCD with $\beta = \delta_{K,K} = \delta_{K,S^\dagger S} = 1$ and $\widetilde{N}_c = 2$.

Note that the gaugino approximation overestimates the mass if all dimensionless numbers are positive.[12] Overall, the method described here gives sensible results even for strongly-coupled theories of SUSY-breaking. Again, a complete knowledge of the OPE could allow a determination of the anomalous dimensions of relevant operators of sSQCD using these methods.

Finally, even though it is not the main purpose of this paper, it is of interest to discuss some of the phenomenology of this new deformation. From the IR point of view, sSQCD is reminiscent of the multitrace deformation discussed in [31]. The main difference can be found in the fermionic sector, where the goldstino also has a component in the ψ_S direction. As such, multitrace deformations are not needed here to give reasonable masses to the fermionic components of X. The phenomenology of sSQCD is thus very similar to the phenomenology of [31].

---

[12]From the sfermion mass (7.3.17) it is clear that $\delta_{K,K}/\beta$ is positive and thus $\delta_{K,K}$ must be positive.

At this point one may observe that there appears to be a contradiction between our result (7.3.17) for M_{gaugino}, using also the explicitly computed $g(x_{\mathcal{M}})$ of (7.3.18), and the general result of small first-order gaugino mass of Komargodski and Shih [49]. However, this is not so: our example is in a sense modular. The gaugino mass appears proportional to $\langle F_S \rangle / \langle S \rangle$, for it arises from the extra SUSY-breaking sector we have included. This can then be thought of as a separate hidden sector, with ISS as the messenger sector. The treatment of Komargodski and Shih does not constrain such models.

7.4. Discussion and conclusion

In this paper we have used the results of [6] to further illustrate how the OPE can be used to understand superpartner spectra in the MSSM in the context of gauge mediation. Although delivering only approximate answers, our methods do capture the essential physics of soft-mass generation in the MSSM. This becomes possible through the UV-IR splitting achieved by the OPE. The methods developed here lead to approximations valid up to order-one numbers both at weak and strong coupling, as can be checked explicitly for strongly-coupled theories with weakly-coupled duals. For strongly-coupled theories of SUSY breaking without weakly-coupled duals, the logic can be inverted and the approximations discussed here might allow us to argue for the functional dependence of relevant anomalous dimensions, which are in practice technically very difficult to calculate.

Using similar techniques one should also be able to perform approximate computations of total cross-sections from the visible sector to the hidden sector, which could be very useful in the event that SUSY is discovered at the LHC.

Our methods were applied here to a new deformation of SQCD, where an additional spontaneous breaking of SUSY is considered. This arises from the F-term vev of a spurion S, whose zero component supplies the quark masses in SQCD. This

deformation moves the ISS vacuum away from the origin and thus induces a breaking of the accidental R-symmetry. Consequently, Majorana gaugino masses are allowed in this ISS-like vacuum. Note that there are no SUSY vacua with our deformation of SQCD. An obvious extension of our work would be to study the μ/B_μ problem in strongly-coupled models, although this is bound to be more model-dependent.

In (7.3.6) and (7.3.17), the main results of this paper, the soft masses are parametrized by entries of the anomalous-dimension matrix γ between the current K and the spurionic operator $S^\dagger S$. The calculation of γ can be easily done in the UV, where the electric theory is under control, with the one-loop result (7.3.3). One could then imagine using magnetic variables to express γ in a form useful in the IR, but the presence of the electric coupling g in (7.3.3) complicates matters. A direct calculation of γ in the IR of SQCD, around the SUSY-breaking minimum, using the magnetic description from the outset, is thus more desirable. However the meaning of the current K in terms of the magnetic dual fields is not clear *a priori*. It would be interesting to carry out in mSQCD the computations done for MGM, and then determine some of the relevant anomalous dimensions of mSQCD operators.

Finally, it is well-known that theories of metastable SUSY breaking generically have an approximate R-symmetry, with small parameter ϵ [50]. For such theories of dynamical SUSY-breaking, our results imply that the approximations (7.3.6) can be schematically written as

$$
\begin{aligned}
M_{\text{gaugino}} &\approx \frac{1}{M_{\text{m}}^2}(\langle S^\dagger \mathcal{M}^\dagger \rangle - \langle \mathcal{G}^\dagger \rangle) + \mathcal{O}(\epsilon), \\
m_{\text{sfermion}}^2 &\approx \frac{1}{M_{\text{m}}^2}(\langle S^\dagger F_{\mathcal{M}}^\dagger + F_S^\dagger \mathcal{M}^\dagger \rangle + \langle F_{\mathcal{G}} + F_{\mathcal{G}}^\dagger \rangle) + \mathcal{O}(\epsilon).
\end{aligned}
\tag{7.4.1}
$$

Again S could be a dynamical field or simply a mass term. Here all order-one prefactors are neglected and all explicit R-symmetry breaking contributions are included in $\mathcal{O}(\epsilon)$. Nevertheless, important qualitative features of the superpartner spectrum can be inferred from (7.4.1). First, contrary to the sfermion masses, the gaugino masses do *not* depend

on any F-terms, and so there is no *a priori* relation between the gaugino masses and the SUSY-breaking scales or the sfermion masses. Nonetheless, the gaugino masses must vanish in the supersymmetric limit, which implies a relation between the different vevs in (7.4.1) reminiscent of the Konishi anomaly. For hidden-sector gauge groups that are completely Higgsed this means that the vev of the MSSM-restricted mesonic superfield must either vanish or blow up as the dynamical SUSY-breaking effect is taken to zero.

Second, since non-vanishing (Majorana) gaugino masses break the approximate R-symmetry, the vevs appearing in (7.4.1) must carry appropriate R-charges. Now, as the explicit R-symmetry breaking parameter ϵ is taken to zero, the metastable SUSY-breaking minimum generically becomes stable with an exact R-symmetry. In the limit where gravity decouples, the spontaneous R-symmetry breaking must vanish in the $\epsilon \to 0$ limit in order to avoid a massless R-axion, which is experimentally ruled out. In this case, the R-symmetry-breaking vevs leading to non-vanishing gaugino masses are thus generated by the explicit R-symmetry breaking and are also $\mathcal{O}(\epsilon)$. One can directly conclude that such models are naturally split, with gaugino masses much smaller than sfermion masses.

Last, since ϵ must remain small in order not to destabilize the metastable SUSY-breaking vacuum, in order to get an acceptable phenomenology with $|M_{\text{gaugino}}/m_{\text{sfermion}}|$ of order one the spontaneous R-symmetry breaking must be non-negligible. This is partly achieved since the MSSM-restricted mesonic superfield usually lies on a flat direction in the $\epsilon \to 0$ limit, which thus leads to a one-loop enhanced vev when the Coleman–Weinberg potential stabilizes the runaway in the finite ϵ limit. Thus, to obtain acceptable superpartner spectra in strongly-coupled models of SUSY-breaking, model-builders should focus on theories with large R-symmetry-breaking vevs.

7.A. Superpartner spectra in weakly-coupled theories

Superpartner spectra can be computed directly from the messenger sector in weakly-coupled theories of SUSY breaking. Although the result is well-known for simple messenger sectors, as for example in MGM [39], for general messenger sectors this is not the case (for a derivation using GGM, see [51]).

Consider a messenger sector consisting of n chiral superfields Φ_i and $\widetilde{\Phi}_i$ transforming in a vector-like representation $R + \bar{R}$ of the MSSM with arbitrary mass matrices $\left(\mathbb{M}_0^2\right)_{2n \times 2n}$ and $\left(\mathbb{M}_{1/2}\right)_{n \times n} = W_{ij}$ such that

$$
\mathscr{L} \supset - \begin{pmatrix} \phi_i^* & \tilde{\phi}_i \end{pmatrix} \left(\mathbb{M}_0^2\right)_{ij} \begin{pmatrix} \phi_j \\ \tilde{\phi}_j^* \end{pmatrix} - \begin{pmatrix} \tilde{\psi}_i \end{pmatrix} \left(\mathbb{M}_{1/2}\right)_{ij} \begin{pmatrix} \psi_j \end{pmatrix} - \text{h.c.},
$$

where (ϕ_i, ψ_i) are the bosonic and fermionic components of Φ_i and similarly for $\widetilde{\Phi}_i$. Introducing unitary matrices U_b, U_f and $\tilde{U}_f$ which diagonalize the mass matrices,

$$
\tilde{m}_i^2 \delta_{ij} = (U_b \, \mathbb{M}_0^2 \, U_b^\dagger)_{ij}, \qquad m_i \delta_{ij} = (\tilde{U}_f^* \, \mathbb{M}_{1/2} \, U_f^\dagger)_{ij},
$$

with $\tilde{m}_i$ and m_i the (real positive) bosonic and fermionic mass eigenvalues respectively, the gaugino and sfermion masses are given by

$$
M_{\text{gaugino}} = -\frac{\alpha_{\text{SM}}}{\pi} \, C(R) \, \mathscr{G}, \qquad m_{\text{sfermion}}^2 = \left(\frac{\alpha_{\text{SM}}}{4\pi}\right)^2 C_2(R_{\text{sfermion}}) \, C(R) \, \mathscr{F}^2, \quad (7.\text{A}.1)
$$

where $C(R) = \frac{1}{2}$ for the fundamental representation and

$$\mathscr{G} = \sum_{i=1}^{2n} \sum_{j,k,l=1}^{n} (U_b)_{ik}(U_b^*)_{i,n+l}(U_f^\dagger)_{kj}(\tilde{U}_f^\dagger)_{lj}\, m_j \left[\ln\left(\frac{\Lambda^2}{m_j^2}\right) - \frac{\tilde{m}_i^2}{\tilde{m}_i^2 - m_j^2} \ln\left(\frac{\tilde{m}_i^2}{m_j^2}\right) \right],$$

$$\mathscr{F}^2 = \sum_{i=1}^{2n} \tilde{m}_i^2 \ln(\tilde{m}_i^2)[4 + \ln(\tilde{m}_i^2)] + 4\sum_{i=1}^{n} m_i^2 \ln(m_i^2)[-2 + \ln(m_i^2)]$$

$$+ \sum_{i,j,k,l=1}^{2n} (-1)^{\lfloor (k-1)/n \rfloor + \lfloor (l-1)/n \rfloor} (U_b)_{ik}(U_b^\dagger)_{kj}(U_b)_{jl}(U_b^\dagger)_{li}$$

$$\times\, \tilde{m}_i^2 \left[-\ln(\tilde{m}_j^2)\ln(\tilde{m}_j^2) + 2\ln(\tilde{m}_i^2)\ln(\tilde{m}_j^2) - 2\operatorname{Li}_2\left(1 - \frac{\tilde{m}_i^2}{\tilde{m}_j^2}\right) \right]$$

$$+ 2\sum_{i=1}^{2n} \sum_{j,k,l=1}^{n} \left[(U_b^\dagger)_{ki}(U_f)_{jk}(U_b)_{il}(U_f^\dagger)_{lj} + (U_b^\dagger)_{n+k,i}(\tilde{U}_f^*)_{jk}(U_b)_{i,n+l}(\tilde{U}_f^T)_{lj} \right]$$

$$\times \left\{ \tilde{m}_i^2 \left[\ln(m_j^2)\ln(m_j^2) - 2\ln(\tilde{m}_i^2)\ln(m_j^2) + 2\operatorname{Li}_2\left(1 - \frac{\tilde{m}_i^2}{m_j^2}\right) - 2\operatorname{Li}_2\left(1 - \frac{m_j^2}{\tilde{m}_i^2}\right) \right] \right.$$

$$\left. + m_j^2 \left[\ln(\tilde{m}_i^2)\ln(\tilde{m}_i^2) - 2\ln(\tilde{m}_i^2)\ln(m_j^2) + 2\operatorname{Li}_2\left(1 - \frac{\tilde{m}_i^2}{m_j^2}\right) + 2\operatorname{Li}_2\left(1 - \frac{m_j^2}{\tilde{m}_i^2}\right) \right] \right\}.$$

The diagrams leading to (7.A.1) can be found in [39]. Note that due to the magic of SUSY, the cutoff Λ does not appear in the gaugino masses. Here $\operatorname{Li}_2(x) = -\int_0^1 dt\, \frac{\ln(1-xt)}{t}$ is the dilogarithm or Spence function.

Note that, although the messenger spectrum of MGM, mSQCD and sSQCD are quite different, the superpartner spectra are given in terms of the same functions $g(x)$ and $f(x)$ (when the spontaneous R-symmetry breaking contribution is discarded in the sSQCD case).

As a final point, note that it is straightforward to include extra messengers transforming under different representations of the MSSM gauge group.

References

[1] G. Giudice & R. Rattazzi, *"Theories with gauge mediated supersymmetry breaking"*, Phys.Rept. **322**, 419 (1999), `hep-ph/9801271`.

[2] P. Meade, N. Seiberg & D. Shih, *"General Gauge Mediation"*, Prog.Theor.Phys.Suppl. **177**, 143 (2009), `arXiv:0801.3278`.

[3] M. Buican, P. Meade, N. Seiberg & D. Shih, *"Exploring General Gauge Mediation"*, JHEP **0903**, 016 (2009), `arXiv:0812.3668`.

[4] T.T. Dumitrescu, Z. Komargodski, N. Seiberg & D. Shih, *"General Messenger Gauge Mediation"*, JHEP **1005**, 096 (2010), `arXiv:1003.2661`.

[5] J.F. Fortin, K. Intriligator & A. Stergiou, *"Current OPEs in Superconformal Theories"*, JHEP **1109**, 071 (2011), `arXiv:1107.1721`.

[6] J.F. Fortin, K. Intriligator & A. Stergiou, *"Superconformally Covariant OPE and General Gauge Mediation"*, JHEP **1112**, 064 (2011), `arXiv:1109.4940`.

[7] J.M. Maldacena, *"The Large N limit of superconformal field theories and supergravity"*, Adv.Theor.Math.Phys. **2**, 231 (1998), `hep-th/9711200`.

[8] F. Benini, A. Dymarsky, S. Franco, S. Kachru, D. Simic et al., *"Holographic Gauge Mediation"*, JHEP **0912**, 031 (2009), `arXiv:0903.0619`.

[9] P. McGuirk, G. Shiu & Y. Sumitomo, *"Holographic gauge mediation via strongly coupled messengers"*, Phys.Rev. **D81**, 026005 (2010), `arXiv:0911.0019`.

[10] M. McGarrie & R. Russo, *"General Gauge Mediation in 5D"*, Phys.Rev. **D82**, 035001 (2010), `arXiv:1004.3305`.

[11] M. McGarrie & D.C. Thompson, *"Warped General Gauge Mediation"*, Phys.Rev. **D82**, 125034 (2010), `arXiv:1009.4696`.

[12] P. McGuirk, *"Hidden-sector current-current correlators in holographic gauge mediation"*, Phys.Rev. **D85**, 045025 (2012), `arXiv:1110.5075`.

[13] K. Skenderis & M. Taylor, *"Holographic realization of gauge mediated supersymmetry breaking"*, JHEP **1209**, 028 (2012), `arXiv:1205.4677`.

[14] R. Argurio, M. Bertolini, L. Di Pietro, F. Porri & D. Redigolo, *"Holographic Correlators for General Gauge Mediation"*, JHEP **1208**, 086 (2012), `arXiv:1205.4709`.

[15] M. McGarrie, *"General Resonance Mediation"*, `arXiv:1207.4484`.

[16] R. Argurio, M. Bertolini, L. Di Pietro, F. Porri & D. Redigolo, *"Exploring Holographic General Gauge Mediation"*, JHEP **1210**, 179 (2012), `arXiv:1208.3615`.

[17] M. McGarrie, *"Holography for General Gauge Mediation"*, `arXiv:1210.4935`.

[18] K.A. Intriligator, N. Seiberg & D. Shih, *"Dynamical SUSY breaking in meta-stable vacua"*, JHEP **0604**, 021 (2006), `hep-th/0602239`.

[19] N. Seiberg, *"Electric-magnetic duality in supersymmetric non-Abelian gauge theories"*, Nucl.Phys. **B435**, 129 (1995), `hep-th/9411149`.

[20] A. Amariti, L. Girardello & A. Mariotti, *"Non-supersymmetric meta-stable vacua in SU(N) SQCD with adjoint matter"*, JHEP **0612**, 058 (2006), `hep-th/0608063`.

[21] R. Kitano, H. Ooguri & Y. Ookouchi, *"Direct Mediation of Meta-Stable Supersymmetry Breaking"*, Phys.Rev. **D75**, 045022 (2007), `hep-ph/0612139`.

[22] N. Haba & N. Maru, *"A Simple Model of Direct Gauge Mediation of Metastable Supersymmetry Breaking"*, Phys.Rev. **D76**, 115019 (2007), `arXiv:0709.2945`.

[23] A. Giveon & D. Kutasov, *"Stable and metastable vacua in brane constructions of SQCD"*, JHEP **0802**, 038 (2008), `arXiv:0710.1833`.

[24] F.q. Xu & J.M. Yang, *"An Extension for Direct Gauge Mediation of Metastable Supersymmetry Breaking"*, Europhys.Lett. **85**, 61001 (2009), `arXiv:0712.4111`.

[25] B.K. Zur, L. Mazzucato & Y. Oz, *"Direct Mediation and a Visible Metastable Supersymmetry Breaking Sector"*, JHEP **0810**, 099 (2008), `arXiv:0807.4543`.

[26] A. Giveon, A. Katz, Z. Komargodski & D. Shih, *"Dynamical SUSY and R-symmetry breaking in SQCD with massive and massless flavors"*, JHEP **0810**, 092 (2008), `arXiv:0808.2901`.

[27] S. Franco & S. Kachru, *"Single-Sector Supersymmetry Breaking in Supersymmetric QCD"*, Phys.Rev. **D81**, 095020 (2010), `arXiv:0907.2689`.

[28] J. Barnard, *"Tree Level Metastability and Gauge Mediation in Baryon Deformed SQCD"*, JHEP **1002**, 035 (2010), `arXiv:0910.4047`.

[29] M. Bertolini, L. Di Pietro & F. Porri, *"Dynamical completions of generalized O'Raifeartaigh models"*, JHEP **1201**, 158 (2012), `arXiv:1111.2307`.

[30] A. Amariti & D. Stone, *"Spontaneous R-symmetry breaking from the renormalization group flow"*, `arXiv:1210.3028`.

[31] R. Essig, J.F. Fortin, K. Sinha, G. Torroba & M.J. Strassler, *"Metastable supersymmetry breaking and multitrace deformations of SQCD"*, JHEP **0903**, 043 (2009), `arXiv:0812.3213`.

[32] J. Wess & J. Bagger, *"Supersymmetry and supergravity"*, 2nd edition, Princeton University Press (1992).

[33] K.A. Intriligator & N. Seiberg, *"Lectures on supersymmetric gauge theories and electric - magnetic duality"*, Nucl.Phys.Proc.Suppl. **45BC**, 1 (1996), `hep-th/9509066`.

[34] S. Dimopoulos & H. Georgi, *"Softly Broken Supersymmetry and SU(5)"*, Nucl.Phys. **B193**, 150 (1981).

[35] G. Dvali, G. Giudice & A. Pomarol, *"The μ problem in theories with gauge mediated supersymmetry breaking"*, Nucl.Phys. **B478**, 31 (1996), `hep-ph/9603238`.

[36] J. Jaeckel, V.V. Khoze & C. Wymant, *"Mass Sum Rules and the Role of the Messenger Scale in General Gauge Mediation"*, JHEP **1104**, 126 (2011), `arXiv:1102.1589`.

[37] J. Jaeckel, V.V. Khoze & C. Wymant, *"RG Invariants, Unification and the Role of the Messenger Scale in General Gauge Mediation"*, JHEP **1105**, 132 (2011), `arXiv:1103.1843`.

[38] H. Osborn, *"$\mathcal{N} = 1$ superconformal symmetry in four-dimensional quantum field theory"*, Annals Phys. **272**, 243 (1999), `hep-th/9808041`.

[39] S.P. Martin, *"Generalized messengers of supersymmetry breaking and the sparticle mass spectrum"*, Phys.Rev. **D55**, 3177 (1997), `hep-ph/9608224`.

[40] V. Novikov, M.A. Shifman, A. Vainshtein & V.I. Zakharov, *"Operator expansion in quantum chromodynamics beyond perturbation theory"*, Nucl.Phys. **B174**, 378 (1980).

[41] D. Amati, G. Rossi & G. Veneziano, *"Instanton Effects in Supersymmetric Gauge Theories"*, Nucl.Phys. **B249**, 1 (1985).

[42] K. Konishi, *"Anomalous Supersymmetry Transformation of Some Composite Operators in SQCD"*, Phys.Lett. **B135**, 439 (1984).

[43] M. Sudano, *"General Gaugino Mediation"*, `arXiv:1009.2086`.

[44] S. Abel, M. Buican & Z. Komargodski, *"Mapping Anomalous Currents in Supersymmetric Dualities"*, Phys.Rev. **D84**, 045005 (2011), `arXiv:1105.2885`.

[45] M.A. Luty & R. Rattazzi, *"Soft supersymmetry breaking in deformed moduli spaces, conformal theories, and N=2 Yang-Mills theory"*, JHEP **9911**, 001 (1999), `hep-th/9908085`.

[46] M.A. Shifman, A. Vainshtein & V.I. Zakharov, *"QCD and Resonance Physics. Sum Rules"*, Nucl.Phys. **B147**, 385 (1979).

[47] P. Colangelo & A. Khodjamirian, *"QCD sum rules, a modern perspective"*, `hep-ph/0010175`, Published in the Boris Ioffe Festschrift "At the Frontier of Particle Physics/Handbook of QCD", ed. by M. Shifman (World Scientific, Singapore, 2001).

[48] S.R. Coleman & E.J. Weinberg, *"Radiative Corrections as the Origin of Spontaneous Symmetry Breaking"*, Phys.Rev. **D7**, 1888 (1973).

[49] Z. Komargodski & D. Shih, *"Notes on SUSY and R-Symmetry Breaking in Wess-Zumino Models"*, JHEP **0904**, 093 (2009), `arXiv:0902.0030`.

[50] K.A. Intriligator, N. Seiberg & D. Shih, *"Supersymmetry breaking, R-symmetry breaking and metastable vacua"*, JHEP **0707**, 017 (2007), `hep-th/0703281`.

[51] D. Marques, *"Generalized messenger sector for gauge mediation of supersymmetry breaking and the soft spectrum"*, JHEP **0903**, 038 (2009), `arXiv:0901.1326`.

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