Title
Dirac Triples for Unital AF Algebras

Author
Kusollerschariya, Chao

Publication Date
2019

Peer reviewed|Thesis/dissertation

Dirac Triples for Unital AF Algebras

by

Chao Kusollerschariya

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Marc A. Rieffel, Chair
Professor Dan-Virgil Voiculescu
Professor Mary K. Gaillard

Fall 2019

Dirac Triples for Unital AF Algebras

Abstract

Dirac Triples for Unital AF Algebras

by

Chao Kusollerschariya

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Marc A. Rieffel, Chair

For a unital AF algebra A, we construct a family of triples $(A, \mathcal{H}, D)$ where A is represented faithfully on the Hilbert space $\mathcal{H}$ and D is an unbounded self-adjoint operator on $\mathcal{H}$. These triples have the same properties as spectral triples except for the compact resolvent condition, so we call them Dirac triples. They serve as a generalization of Pearson-Bellissard spectral triples for an ultrametric Cantor set corresponding to choice functions. Pearson and Bellissard showed that the underlying ultrametric can be recovered by considering spectral triples associated to all choice functions. We obtain an analogue for unital AF algebras: the supremum of the Connes spectral distances induced by a large family of Dirac triples from our construction coincides with a generalized version of the Aguilar seminorm, which is a Leibniz Lip-norm for a unital AF algebra. Moreover, the convergence result of Aguilar is retained: equipped with the generalized Aguilar seminorm, a unital AF algebra is the limit of its defining finite-dimensional subalgebras for the quantum Gromov-Hausdorff propinquity.

Contents

Contents i

1 Introduction 1

2 Quantum Compact Metric Spaces 5

3 Quantum Gromov-Hausdorff Propinquity 11

4 Michon Correspondence 17

5 Pearson-Bellissard Spectral Triples 24

6 The Path Model of AF Algebras 29

7 Conditional Expectation from Transition Probability 33

8 Generalized Aguilar Seminorms 39

9 Monge-Kantorovich Metric for Commutative AF Algebras 42

10 Dirac Triples for AF Algebras 45

11 Spectral Triples for Ultrametric Cantor Sets 51

12 Recovering Aguilar Seminorms 56

Bibliography 62

Acknowledgments

First and foremost, I would like to thank my advisor, Marc Rieffel, for his unparalleled patience and valuable guidance throughout my graduate study. This research project could not be completed without his many insightful comments and intuition. I am also extremely grateful for his timely responses and numerous suggestions during the writing of this dissertation.

This research came about partly due to the generous replies from Ruy Exel and Jean Bellissard, despite not having met. Exel's answers to my questions about partial translation algebras led me to the mathematical study of quasicrystals and to the work of Pearson and Bellissard on ultrametric Cantor sets, on which this dissertation is based. Bellissard carefully answered my questions about his program of using noncommutative geometry in material science, and insightfully explained his motivation for studying spectral triples on ultrametric Cantor sets. I thank them for introducing me to the intriguing world of quasicrystals, tilings, and the interplay with C^*-algebras.

I would like to express my sincere gratitude to Victoria Lee and Jon Wilkening. Vicky is a very caring and helpful graduate program advisor and a good listener. I am very fortunate to have Jon as the vice chair for the graduate affairs during my final years at Berkeley. He is very supportive and understanding of my situation. I could not imagine how I would have overcome the struggles without their supports and encouragements.

My undergraduate advisor, Wicharn Lewkeeratiyutkul, sparked my interests in functional analysis and operator algebras when I was in college. I would like to thank him for having always been a remarkable teacher, a mentor, and a role model.

I thank my parents for their prayers and moral supports throughout this long journey.

Last but not least, my doctoral study was supported for the most part by the Queen Sirikit Scholarship. It is one of my greatest honors to have been selected as a recipient of the scholarship. I feel forever indebted for the opportunity to pursue the doctoral degree at UC Berkeley. Also, I greatly appreciate the continued financial support for my long academic expedition.

Chapter 1

Introduction

This dissertation is based on two lines of research inspired by Connes' notion of spectral triples [11, 12], which brought the metric aspect of noncommutative geometry to light. The first one is the theory of compact quantum metric spaces pioneered by Marc Rieffel [36, 37, 38]. The other is the construction of spectral triples for ultrametric Cantor sets by John Pearson and Jean Bellissard [30]. Both of their research programs have their roots in different branches of physics.

Motivated by the high-energy physics literature, Rieffel sought to provide a mathematically precise meaning to a statement like "matrix algebras converge to the sphere." In metric geometry, the Gromov-Hausdorff distance is well-established for the convergence of compact metric spaces. In [11], Connes suggested that, given a spectral triple for a unital C^*-algebra, one can define an ordinary metric on the state space via the duality from the seminorm induced by the spectral triple (with certain properties). This idea is similar to the definition of the Monge-Kantorovich metric on the set of regular Borel probability measures on a compact metric space via the duality from the classical Lipschitz seminorm. After investigating such duality [33, 34], Rieffel proposed a "noncommutative" analogue of a compact metric space [37], called a *compact quantum metric space*. It is an order-unit space (e.g. the real subspace of self-adjoint elements of a C^*-algebra) equipped with a suitable seminorm, called *Lip-norm*. He then developed the quantum version of Gromov-Hausdorff distance for the compact quantum metric spaces and explained how to equip matrix algebras with suitable seminorms, making them compact quantum metric spaces which converge to the space of continuous functions on the sphere for the quantum distance [38]. However, since the order-unit spaces have no multiplicative structure, it is possible to have two non-isomorphic C^*-algebras with quantum distance zero between their order-unit spaces of self-adjoint elements. This problem has been successfully addressed by David Kerr and Hanfeng Li in [19, 26, 20] with different approaches yielding variants of quantum Gromov-Hausdorff distance.

In view of Rieffel's parallel project on vector bundles over ordinary or quamtum metric spaces [40, 39], there arises a need for a framework to work with Lip-norms satisfying the Leibniz inequality. Imposing the (strong) Leibniz condition on the Lip-norms, Rieffel defines *compact C^*-metric spaces* and the *proximity* between them in [39]. However, the triangle inequality for the proximity could fail because the quotient of a Leibniz seminorm is not necessarily Leibniz. Recently, Latrémolière proposed a new quantum distance, called the *propinquity* [24], which satisfies the triangle inequality by construction. Moreover, it is indeed a metric on the class of compact C^*-metric spaces modulo $*$-isomorphisms which are also isometries for the quantum metric structures.

Under this framework, Aguilar and Latrémolière [2] bring AF algebras into the world of noncommutative metric geometry by showing that: a unital AF algebra with a faithful tracial state can be equipped with a quasi-Leibniz Lip-norm so that its defining finite-dimensional subalgebras (equipped with the restriction of the said Lip-norm) converge to the AF algebra for the quantum propinquity. In a subsequent paper [1], Aguilar proves that if a unital AF algebra is endowed with a certain Leibniz Lip-norm, then its defining finite-dimensional subalgebras together with the restriction of the Lip-norm are quantum compact metric spaces which converge to the AF algebra in the propinquity. He also provides such a Leibniz Lip-norm on a unital AF algebra based on Rieffel's study of Leibniz seminorms and best approximation [41]. We call it the *Aguilar seminorm.*

Back in the early 1980s, Dan Shechtman [44] discovered a *quasicrystal* which had diffraction patterns similar to that of a crystal but lacked a nice translational symmetry. Their sample of aluminum-manganese alloy also displayed "the six fivefold, ten threefold, and fifteen twofold axes characteristic of icosahedral symmetry," violating the crystallographic restriction theorem. This discovery eventually won Shechtman the Nobel Prize in Chemistry in 2011 after much controversy within the field of crystallography. In order to study properties of a solid state material, one can model it by the (discrete) point set of atomic positions, say $\mathcal{G} \subset \mathbb{R}^n$. Due to the nature of atomic configurations in a material, the set $\mathcal{G}$ should possess the following properties [5]:

1. *Uniformly discrete*: There is $r > 0$ such that every open ball of radius r meets $\mathcal{G}$ at most on one point. (Because of nuclear repulsion, atoms cannot be too close to each other.)

2. *Relatively dense*: There is $R > 0$ such that every closed ball of radius R meets $\mathcal{G}$ at least on one points. (At zero temperature, we should not expect, except for special situations, arbitrarily large holes between atoms. Hence, there should be a maximal size of holes.)

Such a point set is called a *Delone* set. As for quasicrystals, a Delone set $\mathcal{G}$ of interest also has additional properties [21, 22]:

1. *Aperiodic*: There is no $x \in \mathbb{R}^n$ such that $\mathcal{G} + x = \mathcal{G}$.

2. *Repetitive*: Given any finite subset $F \subset \mathcal{G}$ and $\varepsilon > 0$, there is $R > 0$ such that any ball of radius R contains a subset F' which is within Hausdorff distance ε from a tranlate of F.

3. *Finite type*: The set $\mathcal{G} - \mathcal{G}$ is a discrete closed set.

Also in the eighties, Jean Bellissard [3, 4] proposed to use the C^*-algebraic approach of noncommutative geometry in solid state physics as a replacement of Bloch theory for aperiodic solids, including quasicrystals. Roughly speaking, given a self-adjoint operator H on a Hilbert space $\mathcal{H}$ (e.g. a Schrödinger operator) and a 'translation' group T unitarily represented on $\mathcal{H}$ by U, he defined a dynamical system (Ω_H, T) where Ω_H is the strong closure of the translates (via U) of the resolvent of H. This is called the *hull* of the self-adjoint operator H. Later, he and his colleagues [7] considered a dynamical system induced by a Delone set $\mathcal{G} \subset \mathbb{R}^n$. Representing $\mathcal{G}$ by a discrete measure ν on $\mathbb{R}^n$, they defined Ω_ν as the weak-$*$ closure of the translates ($\mathbb{R}^n$-orbit) of ν, which is compact for a uniformly discrete set (Theorem 1.6 of [7]). Equipped with the translation action of $\mathbb{R}^n$, the dynamical system $(\Omega_\nu, \mathbb{R}^n)$ is also called the *hull* of $\mathcal{G}$. With a proper setting, they showed that the two definitions are semi-conjugate (Theorem 2.23 of [7]). Related to the hull of $\mathcal{G}$ are the notions of transversal $\Xi := \{\omega \in \Omega_\nu : 0 \text{ is in the support of } \omega\}$ and the associated groupoid $G_\Xi = \{(\omega, a) \in \Xi \times \mathbb{R}^n : a \text{ is in the support of } \omega\}$. As a generalization of the tight-binding method (in which a Schrödinger operator is discretized) for calculating electronic band structure, the groupoid C^*-algebra $C^*(G_\Xi)$ and its K-theory were studied, leading to the gap labelling theorem (Section 4 and 5 of [7]).

The transversal of an aperiodic repetitive Delone set of finite type is a Cantor set (Proposition 2.24 of [6]). In order to study its geometry, Pearson and Bellissard constructed spectral triples for an ultrametric Cantor set based on its intrinsic structure [30]. Viewing an ultrametric Cantor set as the infinite path space of a rooted tree via the Michon correspondence, they define a choice function by assigning a pair of infinite paths to each vertex of the rooted tree. For each choice function, they define a spectral triple of the space of Lipschitz functions on the ultrametric Cantor set. Notably, their construction allows them to study the diffusion on the Cantor set by defining an analogue of the Laplacian. Moreover, they show that, by taking all choice functions into account, the original ultramatric on the Cantor set can be recovered by Connes' distance formula (Theorem 1 of [30]).

Seeing that the space $C(X)$ of continuous functions on the Cantor set X is a commutative AF algebra, I am interested in generalizing the Pearson-Bellissard construction to unital AF algebras in a way that relates to Aguilar's works on AF algebras as compact quantum metric spaces. In the present study, we modify the Leibniz Lip-norm on AF algebras defined by Aguilar in [1], which allows us to encode all ultrametrics on $C(X)$ when the Bratteli diagram is a reduced Cantorian tree. The convergence result is retained for the modified Lip-norm. Then we reinterpret the notion of choice function in terms of conditional expectations and states on $C(X)$. For a unital AF algebra $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$, given a sequence of pairs of states which are equal and faithful on the subalgebra corresponding to the same index (our replacement of a choice function), we can use the technique in [41] to define a Dirac triple for A (a spectral triple without the compact resolvent condition). Applied to the commutative case of an ultrametric Cantor set, our construction indeed results in the Pearson-Bellissard spectral triple. We also show that, taking all such sequence of pairs of states into consideration, we can recover the generalized Aguilar seminorm for the self-adjoint elements of a unital AF algebra.

The structure of this dissertation is as follows:

In Chapter 2-5, we present preliminary background for Rieffel's theory of compact quantum metric spaces, Latrémolière's notion of quantum Gromov-Hausdorff propinquity, the proof of the Michon correspondence, and the Pearson-Bellissard spectral triples for ultrametric Cantor sets.

In Chapter 6-7, we discuss the path model of AF algebras and how to obtain various conditional expectations from transition probabilities on the Bratteli diagram, based on Renault's work [32]. This is to prepare for the recovering of the Aguilar seminorm in the last chapter.

In Chapter 8-9, we propose a generalized version of Aguilar seminorm on unital AF algebras and show that in the case of commutative AF algebras whose Bratteli diagram is a rooted Cantorian tree, the Monge-Kantorovich metric associated to the generalized Aguilar seminorm can recover any ultrametric on the Cantor set.

In Chapter 10-11, we explain the underlying conditional expectation for a choice function and reformulate the Pearson-Bellissard construction to define Dirac triples for unital AF algebras. Then we show that when applied to ultrametric Cantor sets, our construction gives a spectral triple which is unitarily equivalent to the Pearson-Bellissard triple.

Finally, in Chapter 12, we prove that by considering all choices of our "replacement for a choice function," the Aguilar seminorm for the self-adjoint elements can be recovered from the seminorms induced by the Dirac triples.

Notation: Throughout this text, the symbol $\mathbb{N}$ denotes the set of non-negative integers.

Chapter 2

Quantum Compact Metric Spaces

In this chapter, we provide an overview of the theory of quantum metric spaces which we will use later in this dissertation. We will focus on the compact setting—that is, the unital case in the noncommutative realm.

Connes proposes the notion of *spectral triples* which carry metric data for noncommutative spaces in his 1989 paper [11].

Definition 2.1. A spectral triple $(A, \mathcal{H}, D)$ consists of a unital C^*-algebra A represented faithfully by bounded operators on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator D satisfying:

1. the set $\mathcal{L}(A) := \{a \in A : [D, a] \text{ is bounded}\}$ is norm-dense in A;
2. $(I + D^2)^{-1}$ is a compact operator; i.e., D has compact resolvent.

The prototype of a spectral triple comes from a compact spin Riemannian manifold M. Let $A = C(M)$, the space of continuous complex-valued functions on M. The algebra A acts by multiplication on the Hilbert space $\mathcal{H}$ of L^2-spinors on M. The Dirac operator takes the role of the operator D. Connes pointed out that the geodesic distance on the manifold can be recovered by the formula:

$$d(x, y) = \sup\{|a(x) - a(y)| : \|[D, a]\| \leq 1\}, \tag{2.1}$$

and that given a spectral triple for a unital C^*-algebra A such that $\{a : \|[D, a]\| \leq 1\}/\mathbb{C}\mathbf{1}$ is bounded, one can define an ordinary metric on the state space of A by:

$$d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : \|[D, a]\| \leq 1\}.$$

Since the pure states of $C(M)$ are the point-measures δ_x where $x \in M$, we see that the above metric on the state space naturally extends the geodesic distance on the manifold M.

A similar scenario happens for compact metric spaces. Let d be an ordinary metric on a compact space X giving the topology of X, and denote the classical Lipschitz seminorm on $C(X)$ by Lip. As evident in Connes' proof of Equation (2.1), one can recover d from Lip as follows:

Proposition 2.2. *We have* $d(x,y) = \sup\{|f(x) - f(y)| : \mathrm{Lip}(f) \leq 1\}$.

Proof. For each $x, y \in X$, we define $d_x(y) := d(x,y)$. By the triangle inequality, we have

$$|d_x(y) - d_x(z)| = |d(x,y) - d(x,z)| \leq d(y,z),$$

for any $y, z \in X$. Hence, $\mathrm{Lip}(d_x) \leq 1$ and

$$d(x,y) = |d_x(x) - d_x(y)| \leq \sup\{|f(x) - f(y)| : \mathrm{Lip}(f) \leq 1\}.$$

The converse inequality follows from the definition of Lip. □

Indeed, Connes obtained the recovering formula by showing that $\|[D,a]\|$ agrees with $\mathrm{Lip}(a)$ for each $a \in C(M)$. Additionally, one can define a metric on the space of probability measures on X, $S(X)$, by:

$$d(\mu,\nu) = \sup\{|\mu(f) - \nu(f)| : \mathrm{Lip}(f) \leq 1\},$$

namely, the Monge-Kantorovich metric. This metrizes the weak-$*$ topology on $S(X)$ when viewed as the state space of $C(X)$.

The seminorms from spectral triples (such that $\{a \in A : \|[D,a]\| \leq 1\}/\mathbb{C}\mathbf{1}$ is bounded) and Lip are examples of a general Lipschitz seminorm.

Definition 2.3. Let A be a unital C^*-algebra with the identity element $\mathbf{1}$. A seminorm L on A is called *Lipschitz* if $L(a) = L(a^*)$ and $\{a \in A : L(a) = 0\} = \mathbb{C}\mathbf{1}$.

Rieffel has investigated the idea of using such a seminorm as a noncommutative analogue of a metric in [33] and [34]. His works have paved the way for his formulation of compact quantum metric spaces and quantum Gromov-Hausdorff distance in [37]. As pointed out at the end of Section 2 of [26], this notion of quantum distance extends the classical Gromov-Hausdorff distance in the following sense:

Theorem 2.4. *The map* $(X,d) \mapsto (C(X), \mathrm{Lip})$ *is a homeomorphism from the isometry class of compact metric spaces onto a closed subset of the isometry class of compact quantum metric spaces ([37], Definition 6.3) for the respective distances.*

Proof. Theorem 13.16 of [37] says that the map is continuous and injective and the image is closed. An argument of the proof therein implies that if $C(X_n)$ converges to $C(X)$ for the quantum distance, then X_n converges to Y for the Gromov-Hausdorff distance. □

Importantly, the quantum distance also allows for realizing a mathematically precise meaning of an argument like "matrix algebras converge to the sphere" in the high-energy physics literature, which is the motivation behind the development of this theory. As explained in [38], if $\mathcal{H}_n$ is the unique irreducible representation of $SU(2)$ of dimension n, then the matrix algebras $\mathcal{B}(\mathcal{H}_n)$, each equipped with suitable Lipschitz seminorms, converge for the quantum Gromov-Hausdorff distance to $C(S^2)$, the space of continuous complex-valued functions on the sphere with its round metric.

The multiplicative structure of C^*-algebras and the Leibniz property of seminorms do not play a role in Rieffel's original definition of compact quantum metric spaces. However, they have become crucial in the study of the relationship between vector bundles over matrix algebras and quantum Gromov-Hausdorff distance as evident in [40] and [39]. Based on Rieffel's works, Latrémolière proposed a framework for working with seminorms possessing the Leibniz property in [24], where he defines a new quantum distance—the quantum Gromov-Hausdorff propinquity. Since we will deal with Leibniz seminorms and our later discussions will relate to the work of Aguilar on AF algebras that uses the propinquity [1], we choose to use Latrémolière's framework here.

Let A be a unital C^*-algebra. We denote the unit by $\mathbf{1}_A$, the self-adjoint part by A^{sa}, and the state space by $S(A)$. We also denote the norm of any normed space V by $\|\cdot\|_V$. The subscripts will be omitted when there is no confusion.

As suggested by the earlier discussion, a noncommutative analogue of a compact metric space should be given by a unital C^*-algebra equipped with a seminorm possessing certain properties. Here is our key object with minimal requirements:

Definition 2.5. ([24]) A *unital Lipschitz pair* (A, L) consists of a unital C^*-algebra A and a seminorm L defined on a dense subspace $\text{dom}(L)$ of A^{sa} such that:

$$\{a \in \text{dom}(L) : L(a) = 0\} = \mathbb{R}\mathbf{1}.$$

We also call L a Lipschitz seminorm for the pair.

We will view L as a generalized seminorm on the entire A^{sa} by adopting the convention that if $a \notin \text{dom}(L)$, then $L(a) = \infty$. Then we have:

$$\text{dom}(L) = \{a \in A^{\text{sa}} : L(a) < \infty\}.$$

Morever, we adopt the following conventions for computing with ∞:

- $r \cdot \infty = \infty \cdot r = \infty$ for all $r > 0$;
- $0 \cdot \infty = \infty \cdot 0 = 0$;
- $r + \infty = \infty + r = \infty$ for all $r \in \mathbb{R}$.

A Lipschitz pair encodes metric data in terms of a metric on $S(A)$ as we have seen in the case of ordinary compact metric space.

Definition 2.6. Given a unital Lipschitz pair (A, L), the *Monge-Kantorovich metric* on $S(A)$ is defined by

$$\text{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in A^{\text{sa}} \text{ and } L(a) \leq 1\}.$$

The triangle inequality can be easily checked. The coincidence property follows from the Lipschitz condition of the seminorm L. In general, this formula could possibly take value $+\infty$ so technically speaking, it defines an extended metric.

Example 2.7. Let A be the closure of the subalgebra of $C_b(\mathbb{R})$ consisting of bounded Lipschitz functions. Let L be the classical Lipschitz seminorm. Let φ be an arbitrary state on A and let ψ be the evaluation at 0. Then we have

$$\begin{aligned}\mathrm{mk}_L(\varphi,\psi) &= \sup\{|\varphi(f) - f(0)| : f \in A^{\mathrm{sa}} \text{ and } L(f) \leq 1\} \\ &= \sup\{|\varphi(f)| : f \in A^{\mathrm{sa}}, f(0) = 0, \text{ and } L(f) \leq 1\}.\end{aligned}$$

If φ is a probability measure with compact support, then $\mathrm{mk}_L(\varphi,\psi)$ is finite. Now consider the state $\varphi(f) = \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} f(k)$. For each $n \geq 1$, we set

$$f_n(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } 0 \leq x \leq n; \\ n & \text{if } x > n. \end{cases}$$

Then $L(f_n) = 1$ and $\varphi(f_n) = \displaystyle\sum_{k=1}^{n} \frac{6}{\pi^2 k^2} k + \sum_{k=n+1}^{\infty} \frac{6}{\pi^2 k^2} n$. Because the first term in the expression of $\varphi(f_n)$ diverges as $n \to \infty$, we see that $\{\varphi(f_n)\}_{n\geq 1}$ is unbounded and so $\mathrm{mk}_L(\varphi,\psi) = \infty$.

Remark: Note that in [11] Connes takes the supremum over all elements of the underlying C^*-algebra when defining the metric as in Definition 2.6. However, under the reality condition $L(a) = L(a^*)$, Lemma 1 of [17] suggests that it suffices to take the supremum over self-adjoint elements. Indeed, given $\varphi, \psi \in S(A)$ and $\delta > 0$, we can choose $a \in A$, after multiplying by a complex number of modulus 1, so that $L(a) \leq 1$ and $\varphi(a) - \psi(a) \geq \mathrm{mk}_L(\varphi,\psi) - \delta$. Setting $b = \frac{1}{2}(a + a^*)$, we still have $\varphi(b) - \psi(b) \geq \mathrm{mk}_L(\varphi,\psi) - \delta$ and $L(b) \leq 1$ because $L(a) = L(a^*)$. For this reason, we only focus on how a seminorm L is defined on a dense subalgebra of A^{sa}. Morever, we note that different seminorms on A can agree on A^{sa}. For example, given a non-Hermitian linear functional φ on A, we could consider the seminorms $L(a) = |\varphi(a)|$ and $L'(a) = |\varphi(a^*)|$.

A Lipschitz pair has to be nice enough to be a suitable noncommutative analogue of a compact metric space. Following Rieffel, we impose this requirement on a unital Lipschitz pair:

Definition 2.8. A *quantum compact metric space* (A, L) is a unital Lipschitz pair such that mk_L metrizes the weak-$*$ topology of $S(A)$. If this is the case, then we call L a *Lip-norm*.

There are a few reasons justifying this requirement. First, for an ordinary compact metric space X, the Monge-Kantorovich metric defined from the classical Lipschitz seminorm also induces the weak-$*$ topology on the space of probability measures $S(X)$. Second, it is a compact topology for $S(A)$ which allows us to adapt the Gromov-Hausdorff distance to our setting.

We describe a classical compact metric space in the Rieffel-Latrémolière framework as follows:

Example 2.9. Let (X, d) be a compact metric space. For any real-valued function $f \in C(X)^{\mathrm{sa}}$, we define the *Lipschitz constant* of f by:

$$\mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x, y \in X \text{ and } x \neq y\right\},$$

which could be infinite. A real-valued function with finite Lipschitz constant is called a Lipschitz function, and the set of Lipschitz functions is norm-dense in $C(X)^{\mathrm{sa}}$ by the Stone-Weierstrass Theorem. Since $\mathrm{mk}_{\mathrm{Lip}}$ metrizes the weak-$*$ topology on $S(C(X))$, the pair $(C(X), \mathrm{Lip})$ is a quantum compact metric space.

In [33] (Theorem 1.8), Rieffel provides a necessary and sufficient condition for a Lip-norm using the Arzéla-Ascoli Theorem. This is helpful in proving that a unital Lipschitz pair is a quantum compact metric space.

Theorem 2.10. *Let (A, L) be a unital Lipschitz pair and $\mathcal{L}_1^A := \{a \in A^{sa} : L(a) \leq 1\}$. Then L is a Lip-norm if and only if the image of $\mathcal{L}_1$ in $A^{sa}/\mathbb{R}\mathbf{1}$ is totally bounded with respect to the quotient norm.*

The characterization has also been reformulated in [28], [24], [23], and [24] as follows:

Theorem 2.11. *([24], Theorem 2.10) Let (A, L) be a unital Lipschitz pair. The following are equivalent:*

1. *(A, L) is a quantum compact metric space.*
2. *There is a state $\varphi \in S(A)$ such that the set*
$$\mathcal{L}_1^A(L, \varphi) := \{a \in A^{sa} : \varphi(a) = 0 \text{ and } L(a) \leq 1\}$$
is totally bounded for the norm of A.
3. *For any $\varphi \in S(A)$, the set*
$$\mathcal{L}_1^A(L, \varphi) := \{a \in A^{sa} : \varphi(a) = 0 \text{ and } L(a) \leq 1\}$$
is totally bounded for the norm of A.
4. *There exists $r \geq 0$ such that the set*
$$\mathcal{L}_{1,r}^A := \{a \in A^{sa} : L(a) \leq 1 \text{ and } \|a\|_A \leq r\}$$
is totally bounded for the norm of A, and the diameter of $(S(A), \mathrm{mk}_L)$ is less than or equal to r.

The first noncommutative example is provided by Rieffel in [33].

Example 2.12. ([33]) Let G be a compact group with the identity element e, and let l be a continuous length function on G. Let A be a unital C^*-algebra with the automorphism group $\mathrm{Aut}(A)$. Let $\alpha : G \to \mathrm{Aut}(A)$ be a strongly continuous action of G on A. Assume that α is *ergodic* in the sense that

$$\bigcap_{g\in G}\{a \in A : \alpha^g(a) = a\} = \mathbb{C}\mathbf{1}_A.$$

That is, the fixed point algebra is trivial. For all $a \in A$, define

$$L(a) = \sup\left\{\frac{\|\alpha^g(a) - a\|_A}{l(g)} : g \in G \text{ and } g \neq e\right\},$$

which may be infinite. Then Theorem 2.3 of [33] says that (A, L) is a quantum compact metric space.

Being the root of the theory of quantum metric spaces, spectral triples unsurprisingly serves as a source of examples. Given a spectral triple $(A, \mathcal{H}, D)$, we define a seminorm on the dense subalgebra $\mathcal{L}(A)$ by

$$L_D(a) = \|[D, a]\|.$$

If we assume that the commutant $A'_D = \{a \in A : [D, a] = 0\}$ is trivial, then (A, L_D) is a unital Lipschitz pair. As for Connes' spectral triple for a compact spin manifold M with Dirac operator D, we have mentioned earlier that L_D coincides with the classical Lipschitz seminorm on $C(M)$, so Example 2.9 includes $(C(M), L_D)$. In [33] (Theorem 4.2), Rieffel also defines a spectral triple associated to a unital C^*-algebra with an ergodic action of a compact Lie group and proves that it is a quantum compact metric space. This applies to the noncommutative tori with ergodic actions of ordinary tori.

Another class of examples comes from a spectral triple for a reduced group C^*-algebra.

Example 2.13. Let G be a discrete group and l an unbounded length function on G. Let M_l be the multiplication operator on $\ell^2(G)$. In [11], Connes shows that, with the left regular representation, $(C_r^*(G), \ell^2(G), M_l)$ is a spectral triple. Christ, Ozawa, and Rieffel have verified that $(C_r^*(G), L_{M_l})$ is a quantum compact metric space for various classes of groups:

1. $\mathbb{Z}^d$ with l being the word-length function for a finite generating subset of $\mathbb{Z}^d$ or the restriction to $\mathbb{Z}^d$ of some norm on $\mathbb{R}^d$ [35];

2. finitely generated hyperbolic groups with the word-length function [28];

3. finitely generated nilpotent-by-finite groups with the word-length function [10].

The Lip-norms in all the examples mentioned so far satisfies the Leibniz inequality. The class of quantum compact metric spaces with this property will be the main focus of the next chapter where we discuss a quantum distance and the convergence.

Chapter 3

Quantum Gromov-Hausdorff Propinquity

In this chapter, we shall work with quantum compact metric spaces with the Leibniz property in the following general sense:

Definition 3.1. A unital Lipschitz pair (A, L) is called a *unital Leibniz pair* if for any $a, b \in A^{\mathrm{sa}}$, we have

$$L(a \circ b) \leq \|a\|_A L(b) + \|b\|_A L(a)$$

and

$$L(\{a, b\}) \leq \|a\|_A L(b) + \|b\|_A L(a),$$

where

$$a \circ b = \frac{ab + ba}{2} \text{ and } \{a, b\} = \frac{ab - ba}{2i}$$

are, respectively, the *Jordan product* and the *Lie product* for the Jordan-Lie algebra of self-adjoint elements.

Definition 3.2. A *Leibniz quantum compact metric space* (A, L) is a quantum compact metric space (A, L) such that (A, L) is a unital Leibniz pair and the seminorm L is lower semicontinuous with respect to the norm on A. If this is the case, then L is called a *Leibniz Lip-norm.*

Indeed, if (A, L) is a unital Lipschitz pair such that L satisfies the Leibniz inequality: $L(ab) \leq \|a\|_A L(b) + \|b\|_A L(a)$ on $\mathrm{dom}(L) \subseteq A^{\mathrm{sa}}$, then (A, L) is a unital Leibniz pair. In particular, Examples 2.9, 2.12 and 2.13 are Leibniz quantum compact metric spaces.

Remark: Suppose that (A, L) is a Leibniz quantum compact metric space. Since L is lower semicontinuous, the set

$$\mathcal{L}^A_{1,r} = \{a \in A^{\mathrm{sa}} : L(a) \leq 1 \text{ and } \|a\|_A \leq r\}$$

is closed, and by L being a Lip-norm, the characterization in Theorem 2.11 implies that $\mathcal{L}^A_{1,r}$ is totally bounded and hence, compact. Therefore, the set $\mathcal{L}_1 = \{a \in A^{\mathrm{sa}} : L(a) \leq 1\}$

is complete and thus, L is a *closed* Lipschitz seminorm as defined in Definition 4.5 of [34]. This allows for a nice characterization of *isometry* between Leibniz quantum compact metric spaces.

Let us formally introduce the notion of isometry on the class of general quantum compact metric spaces here.

Definition 3.3. Let (A, L_A) and (B, L_B) be quantum compact metrics spaces. An *isometric isomorphism* $\Phi : A \to B$ is a $*$-isomorphism from A onto B such that the dual map $\Phi^* : S(B) \to S(A)$ given by $\mu \mapsto \mu \circ \Phi$ is an isometry from $(S(B), \mathrm{mk}_{L_B})$ onto $(S(A), \mathrm{mk}_{L_A})$, where mk is the corresponding Monge-Kantorovich metric on the state space defined in Definition 2.6.

Since a Lip-norm for a Leibniz quantum compact metric space is closed as discussed earlier, we have the following characterization due to Rieffel (Theorem 6.2 of [37]).

Theorem 3.4. *Let (A, L_A) and (B, L_B) be Leibniz quantum compact metric spaces. A $*$-isomorphism $\Phi : A \to B$ is an isometric isomorphism if and only if $L_A = L_B \circ \Phi$.*

Next, we introduce the key notion for defining Latrémolière's quantum distance.

Definition 3.5. A *bridge* $(\mathcal{D}, \pi_A, \pi_B, \omega)$ from a unital C^*-algebra A to a unital C^*-algebra B consists of a unital C^*-algebra $\mathcal{D}$, a self-adjoint element $\omega \in \mathcal{D}$ such that $\|\omega\| = 1$ and $1 \in \sigma(\omega)$, and unital $*$-monomorphisms $\pi_A : A \hookrightarrow \mathcal{D}$ and $\pi_B : B \hookrightarrow \mathcal{D}$. The element ω is called the *pivot* element of the bridge.

By the assumption that 1 is an element of the spectrum of ω, the "1-level set of ω", $S_1(\omega) := \{\varphi \in S(\mathcal{D}) : \varphi(\omega) = 1\}$ is non-empty. These are the states which are "definite" on ω in the sense of Exercise 4.6.16 of [18]. When there is no confusion, we will drop the monomorphisms and write as if A, B are unital subalgebras of $\mathcal{D}$.

Remark: The definition of a pivot element here is actually a special case of Latrémolière's (Definition 3.1 and 3.6 of [24]) but it is sufficient for our use.

Given a bridge from A to B, we define an associated seminorm.

Definition 3.6. The *seminorm of a bridge* $\Pi = (\mathcal{D}, \pi_A, \pi_B, \omega)$ from a unital C^*-algebra A to a unital C^*-algebra B is the seminorm defined on $(A \oplus B)^{\mathrm{sa}}$ by:

$$\mathrm{bn}_\Pi(a, b) = \|a\omega - \omega b\|_{\mathcal{D}}.$$

This seminorm also satisfies the Leibniz inequality. Indeed, for any $(a, b), (c, d) \in A \oplus B$, we have

$$\begin{aligned}
\mathrm{bn}_\Pi((a,b)(c,d)) &= \|ac\omega - \omega bd\|_{\mathcal{D}} \\
&= \|ac\omega - a\omega d + a\omega d - \omega bd\|_{\mathcal{D}} \\
&\leq \|ac\omega - a\omega d\|_{\mathcal{D}} + \|a\omega d - \omega bd\|_{\mathcal{D}} \\
&\leq \|a\|_A\|c\omega - \omega d\|_{\mathcal{D}} + \|a\omega - \omega b\|_{\mathcal{D}}\|d\|_B \\
&= \|a\|_A \,\mathrm{bn}_\Pi(c,d) + \mathrm{bn}_\Pi(a,b)\|d\|_B \\
&\leq \|(a,b)\|_{A\oplus B}\,\mathrm{bn}_\Pi(c,d) + \mathrm{bn}_\Pi(a,b)\|(c,d)\|_{A\oplus B}.
\end{aligned}$$

Now suppose that (A, L_A) and (B, L_B) are Leibniz quantum compact metric spaces. A Lip-norm L on $A \oplus B$ is called *admissible* if L induces L_A and L_B in the sense of Proposition 3.1 and Notation 4.1 of [37]:

$$L_A(a) = \inf\{L(a,b) : b \in B^{\mathrm{sa}}\}$$

and similarly for L_B. For each admissible Lip-norm L for (L_A, L_B), the embeddings of $S(A)$ and $S(B)$ are isometric onto closed subsets of $S(A \oplus B)$, so we can consider their Hausdorff distance with respect to the metric mk_L on $S(A \oplus B)$. (See Proposition 3.1 of [37] and the preceding discussion.) Rieffel defines the quantum Gromov-Hausdorff distance between (A, L_A) and (B, L_B) as the infimum of such Hausdorff distances over all admissible Lip-norms for (L_A, L_B)—Definition 4.2 of [37].

Given a bridge Π from A to B, and $r > 0$, the seminorm defined on $(A \oplus B)^{\mathrm{sa}}$ by

$$L_r(a,b) = \max\{L_A(a), L_B(b), r^{-1}\,\mathrm{bn}_\Pi(a,b)\}$$

is lower semicontinuous, and makes $(A \oplus B, L_r)$ a unital Leibniz pair:

Proposition 3.7. *The seminorm L_r is a lower semicontinuous Lip-norm.*

Proof. By construction, bn_Π is continuous for the norm on $A \oplus B$ and $\mathrm{bn}_\Pi(\mathbf{1}_A, \mathbf{1}_B) = 0$ but $\mathrm{bn}_\Pi(\mathbf{1}_A, 0) = \|\omega\|_{\mathcal{D}} \geq 1$. Hence, the map $(a,b) \mapsto r^{-1}\,\mathrm{bn}_\Pi(a,b)$, when restricted to self-adjoint elements, is a "bridge" as originally defined by Rieffel in Definition 5.1 of [37]. By Theorem 5.2 of the same paper, L_r is a lower semicontinuous Lip-norm. □

With a suitable choice of r, L_r is an admissible Lip-norm for (L_A, L_B) and could be used to estimate the quantum Gromov-Hausdorff distance. In order to get a suitable choice, Latrémolière defines a notion of the *length* of a bridge from two numerical values: the *reach* and the *height*.

Definition 3.8. The *reach* of a bridge $\Pi = (\mathcal{D}, \pi_A, \pi_B, \omega)$ is defined by:

$$\mathrm{reach}(\Pi) = \mathrm{Haus}_{\mathcal{D}}(\mathcal{L}_1^A\omega, \omega\mathcal{L}_1^B),$$

where $\mathrm{Haus}_{\mathcal{D}}$ denotes the Hausdorff distance with respect to the norm of $\mathcal{D}$ and $\mathcal{L}_1^A, \mathcal{L}_1^B$ are the unit balls of the Lip-norms as defined in Theorem 2.10.

Note that the reach can be rewritten as the Hausdorff pseudo-distance between $\mathcal{L}_1^A$ and $\mathcal{L}_1^B$ for the pseudo-metric induced by the bridge seminorm bn_Π.

As in [42], we can extract the following proposition from the proof of Theorem 6.3 of [24].

Proposition 3.9. *With notation as above, L_r is admissible if $r \geq \mathrm{reach}(\Pi)$.*

Proof. Suppose that $r \geq \mathrm{reach}(\Pi)$. By definition, it is clear that $L_A(a) \leq \inf\{L_r(a,b) : b \in \mathrm{sa}(B)\}$. Let $a \in A^{\mathrm{sa}}$ be such that $L_A(a) = 1$. Let $\varepsilon > 0$. By the definition of reach(Π), there is $b \in B^{\mathrm{sa}}$ such that $L_B(b) \leq 1$ and $\mathrm{bn}_\Pi(a,b) = \|a\omega - \omega b\| \leq \mathrm{reach}(\Pi) + \varepsilon$. Then because $r \geq \mathrm{reach}(\Pi)$, we have

$$\max\{L_B(b), r^{-1}\,\mathrm{bn}_\Pi(a,b)\} \leq 1 + r^{-1}\varepsilon = L_A(a) + r^{-1}\varepsilon.$$

Since ε is arbitrary, we have $L_r(a,b) \leq L_A(a)$ and so $L_A(a)$ is equal to the quotient of L_r on A. By scaling, this holds for any self-adjoint element. By symmetry, L_B agrees with the quotient of L_r on B for the self-adjoint elements and therefore, L_r is admissible. □

Latrémolière shows that:

Proposition 3.10. *If $(A, L_A))$ and (B, L_B) are Leibniz quantum compact metric spaces, then the reach of a bridge $\Pi = (\mathcal{D}, \pi_A, \pi_B, \omega)$ is finite.*

Proof. Because L_A is lower semicontinuous for the norm of A, the set $\mathcal{L}_1^A$ is norm-closed. Then for any $\varphi \in S(A)$, the set $\mathcal{L}_1^A(L_A, \varphi) = \{a \in A^{\mathrm{sa}} : L_A(a) \leq 1 \text{ and } \varphi(a) = 0\}$ is also closed by the continuity of φ. Since A is complete, so is $\mathcal{L}_1^A(L_A, \varphi)$. By Theorem 2.11, we have $\mathcal{L}_1^A(L_A, \varphi)$ is totally bounded and hence, norm-compact. Same goes for B.

Fix $\varphi_A \in S(A)$ and $\varphi_B \in S(B)$. By compactness,

$$\delta := \mathrm{Haus}_{\mathcal{D}}(\mathcal{L}_1^A(L_A, \varphi_A)\omega, \omega\mathcal{L}_1^B(L_B, \varphi_B)) < \infty.$$

Now let $b \in B^{\mathrm{sa}}$ be such that $L_B(b) \leq 1$. Then, by compactness of $\mathcal{L}_1^A(L_A, \varphi_A)$, there is $a' \in A^{\mathrm{sa}}$ such that $L_A(a') \leq 1$, $\varphi_A(a') = 0$, and

$$\|a'\omega - \omega(b - \varphi_B(b)\mathbf{1}_B)\|_{\mathcal{D}} = \min\{\|a\omega - \omega(b - \varphi_B(b)\mathbf{1}_B)\|_{\mathcal{D}} : a \in \mathcal{L}_1^A(L_A, \varphi_A)\} \leq \delta.$$

Hence,

$$\|(a' + \varphi_B(b)\mathbf{1}_A)\omega - \omega b\|_{\mathcal{D}} = \|a'\omega - \omega(b - \varphi_B(b)\mathbf{1}_B)\|_{\mathcal{D}} \leq \delta.$$

We can do similarly for $a \in \mathcal{L}_1^A$ and therefore, $\mathrm{reach}(\Pi) = \mathrm{Haus}_{\mathcal{D}}(\mathcal{L}_1^A\omega, \omega\mathcal{L}_1^B) \leq \delta$. □

The other quantity is intended to be a measurement of how "far" the pivot ω is from $\mathbf{1}_{\mathcal{D}}$. This is done by comparing $S(A)$ and $S_1^A(\omega)$, the restriction of elements in $S_1(\omega)$ to A. Also, we do similarly for B. Note that if $\omega = \mathbf{1}_{\mathcal{D}}$, then $S_1^A(\omega) = S(A)$ and $S_1^B(\omega) = S(B)$.

Definition 3.11. The *height* of a bridge $\Pi = (\mathcal{D}, \pi_A, \pi_B, \omega)$ is defined by

$$\mathrm{height}(\Pi) = \max\{\mathrm{Haus}_{\mathrm{mk}_{L_A}}(S(A), S_1^A(\omega)), \mathrm{Haus}_{\mathrm{mk}_{L_B}}(S(B), S_1^B(\omega))\},$$

which is finite since $S_1(\omega)$ is nonempty for a bridge.

Then we have

Definition 3.12. The *length* of a bridge $\Pi = (\mathcal{D}, \pi_A, \pi_B, \omega)$ is

$$\text{length}(\Pi) = \max\{\text{reach}(\Pi), \text{height}(\Pi)\}.$$

To define a quantum distance, we consider finite paths of quantum compact metric spaces connected by bridges. Formally speaking:

Definition 3.13. Let $\mathcal{C}$ be a nonempty class of quantum compact metric spaces. Let $(A, L_A), (B, L_B) \in \mathcal{C}$. A *$\mathcal{C}$-trek* from (A, L_A) to (B, L_B) consists of, for some $n \in \mathbb{N}_+$,

1. an n-tuple of quantum compact metric spaces $((A_1, L_1), (A_2, L_2), \ldots, (A_n, L_n))$ such that each $(A_i, L_i) \in \mathcal{C}$ and $(A_1, L_1) = (A, L_A)$ and $(A_n, L_n) = (B, L_B)$;

2. for each $i \in \{1, 2, \ldots, n-1\}$, a bridge Π_i from A_i to A_{i+1}.

We will refer to a $\mathcal{C}$-trek as above shortly by $T = (A_i, \Pi_i; n)$. The set of all $\mathcal{C}$-treks from (A, L_A) to (B, L_B) is denoted by $\text{Treks}((A, L_A) \xrightarrow{\mathcal{C}} (B, L_B))$.

Definition 3.14. The *length of a $\mathcal{C}$-trek* $(A_i, \Pi_i; n)$ from (A, L_A) to (B, L_B) is

$$\text{length}(T) = \sum_{i=1}^{n-1} \text{length}(\Pi_i).$$

Given quantum compact metric spaces (A, L_A) and (B, L_B), we have a bridge $(\mathcal{D} = A \oplus B, \iota_A, \iota_B, \mathbf{1}_A \oplus \mathbf{1}_B)$ where ι_A and ι_B are the canonical injections. Therefore, the set of treks from A to B is non-empty and we can define:

Definition 3.15. (Definition 4.1, [24]) Let $\mathcal{C}$ be a nonempty class of quantum compact metric spaces. The *quantum Gromov-Hausdorff $\mathcal{C}$-propinquity* between $(A, L_A), (B, L_B) \in \mathcal{C}$ is the nonnegative real number

$$\Lambda_{\mathcal{C}}((A, L_A), (B, L_B)) = \inf\{\text{length}(T) : T \in \text{Treks}((A, L_A) \xrightarrow{\mathcal{C}} (B, L_B))\}.$$

If $\mathcal{C} = \mathcal{L}^*$, the class of Leibniz quantum compact metric spaces, we call $\Lambda := \Lambda_{\mathcal{L}^*}$, the *quantum Gromov-Hausdorff propinquity* or shortly, *quantum propinquity.*

Remark: By restricting to a special version of Latrémolière's bridges, we actually have that $\Lambda_{\mathcal{C}}$ dominates the original one. However, the proof of the next proposition still applies because the pivot element of the bridge involved is a self-adjoint element of norm 1.

The quantum $\mathcal{C}$-propinquity has the following distance-like properties (Proposition 4.6-4.7, [24]):

Proposition 3.16. *Let $\mathcal{C}$ be a nonempty class of quantum compact metric spaces. Let $(A, L_A), (B, L_B), (C, L_C) \in \mathcal{C}$. Then*

1. $\Lambda_{\mathcal{C}}((A, L_A), (B, L_B)) \leq \max\{\text{diam}(S(A), \text{mk}_{L_A}), \text{diam}(S(B), \text{mk}_{L_B})\}$;
2. $\Lambda_{\mathcal{C}}((A, L_A), (B, L_B)) = \Lambda_{\mathcal{C}}((B, L_B), (A, L_A))$;
3. $\Lambda_{\mathcal{C}}((A, L_A), (C, L_C)) \leq \Lambda_{\mathcal{C}}((A, L_A), (B, L_B)) + \Lambda_{\mathcal{C}}((B, L_B), (C, L_C))$.

In Section 5 of [24], Latrémolière remarkably proves that the quantum propinquity Λ is indeed a metric on the isometric isomorphism classes of Leibniz quantum metric spaces.

Theorem 3.17. *(Theorem 5.13, [24]) Let $(A, L_A), (B, L_B)$ be Leibniz quantum compact metric spaces. If*

$$\Lambda((A, L_A), (B, L_B)) = 0,$$

then there exists a $*$-isomorphism $\Phi : A \to B$ such that $L_A = L_B \circ \Phi$.

The result also holds for any subclass $\mathcal{C}$ of Leibniz quantum compact metric spaces because $\Lambda_{\mathcal{L}^*}((A, L_A), (B, L_B)) \leq \Lambda_{\mathcal{C}}((A, L_A), (B, L_B))$.

Remark: Since the original quantum propinquity is no greater than Λ, Theorem 3.17 follows immediately from Theorem 5.13 of [24].

Chapter 4

Michon Correspondence

A Cantor set is a compact, totally disconnected, metrizable space with no isolated points. Topologically, a Cantor set is homeomorphic to $\{0,1\}^{\mathbb{N}}$. Nevertheless, it is interesting to study different metrics on it. Throughout this chapter and the next, we will consider a special kind of metric.

Definition 4.1. Let X be a Cantor set. A metric on X is called *regular* if it metrizes the topology on X. A metric d on X is called *ultrametric* if additionally, for any $x, y, z \in X$,

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

In [27], Michon establishes that a regular ultrametric on a Cantor set can be described by a weighted, rooted tree. We will follow a more detailed exposition given by Pearson and Bellissard in Sections 3, 4 and Appendix A of [30].

Let us begin with some basic definitions about rooted trees that we will need for upcoming discussions. A directed graph with no loops or multiple edges between vertices is called *simple*. A *rooted tree* is a simple graph without cycles $T = (V, E)$ with a specification of subsets $V_n \subset V$ such that

1. The vertex set $V = \bigsqcup_{n\in\mathbb{N}} V_n$ (disjoint union) where $V_0 = \{v_0\}$ and V_n is non-empty and finite for all $n > 0$;

2. The edge set $E = \bigsqcup_{n\in\mathbb{N}} E_n$ where E_n is the set of edges from V_n to V_{n+1}.

The element v_0 is called *the root* and we say v is an *n-level vertex* if $v \in V_n$. A *path* is a (finite or infinite) word $v_0v_1v_2\ldots$ such that $v_n \in V_n$, and there is an edge connecting v_n and v_{n+1}. By definition, we only consider paths starting from the root.

Remark: In the context of rooted trees, there is only one edge connecting a vertex $v_n \in V_n$ to the previous level V_{n-1}, so a word $v_0v_1v_2\ldots$ unambiguously determines the word $e_0e_1\ldots$, where e_n is the edge connecting v_n to v_{n+1}. More generally, we will define a path as a word of edges in Chapter 6. In this chapter, however, we will often refer to vertices in a path and therefore, it is more convenient to denote a path as a word of vertices.

Since T is a tree, a path from the root to any vertex is unique. There is a partial order on V given by $v \succeq w$ if the path from the root to w necessarily passes through v. Then w is called a *descendant* of v and conversely, v is an *ancestor* of w. We also use the reverse notation $w \preceq v$. If v, w are incident (i.e., v, w are linked by an edge) and $w \preceq v$, then we call v the *parent* of w and w a *child* of v.

Definition 4.2. The boundary of a rooted tree T, denoted by ∂T, is the set of all infinite paths (starting at the root).

Note that the boundary ignores a vertex with no children, called *dangling* vertex. Moving forward, we will focus on the boundary and hence, will only consider rooted trees with no dangling vertices.

Definition 4.3. Let $T = (V, E)$ be a rooted tree. For $v \in V$, define $[v] \subset \partial T$ to be the set of infinite paths passing through v. Such a subset is called a *cylinder* set.

Proposition 4.4. *Let T be a rooted tree with no dangling vertices. Then the set of cylinder subsets $\{[v] : v \in V\}$ is a basis of open sets for a topology on ∂T which is totally disconnected and compact. Additionally, it has no isolated points iff each vertex has one descendant with at least two children.*

Proof. It is clear that the set of cylinder subsets covers ∂T. Moreover, $[w] = [v] \cap [w]$ if $w \preceq v$; the intersection is empty if v, w are not comparable. Therefore, cylinder subsets form a basis for a topology on ∂T. To see that ∂T is compact, let $\mathcal{U}$ be a collection of open sets in ∂T which has no finite subcollection covering ∂T. Let v_0 be the root of T. Since $\{[v] : v \in V_1\}$ is finite and covers ∂T, there must be $v_1 \in V_1$ such that $[v_1]$ cannot be covered by a finite number of sets in $\mathcal{U}$. Then again, since $\{[v] : v \text{ is a child of } v_1\}$ is finite and covers $[v_1]$, there must be $v_2 \in V_2$ such that $[v_2]$ cannot be covered by a finite number of sets in $\mathcal{U}$. Inductively, we have an infinite path $x = v_0 v_1 v_2 \cdots \in \partial T$ such that each $[v_n]$ cannot be covered by finite number of sets in $\mathcal{U}$. Suppose $x \in \bigcup \mathcal{U}$. Then there is $O \in \mathcal{U}$ such that $x \in O$. Because O is open, there is $w \in V$ such that $x \in [w] \subset O$ and thus, $w = v_n$ for some n. This contradicts the construction of x. Therefore, $\mathcal{U}$ is not a cover for ∂T, and so ∂T is compact.

Additionally, for any $v \in V$, $[v]$ contains at least two different infinite paths if and only if v has a descendant with at least two children. □

Definition 4.5. A *Cantorian* tree is a rooted tree with no dangling vertices and each vertex has a descendant with at least two children. In other words, a Cantorian tree is a rooted tree whose boundary is a Cantor set.

Definition-Preposition 4.6. *Let T be a Cantorian tree and $S \subset \partial T$. A vertex v is a common prefix of S if $S \subset [v]$. If S has more than one point, the least common prefix (lcp) always exists and is unique, denoted by* $\mathrm{lcp}(S)$. *If $S = \{x, y\}$, we write* $x \wedge y := \mathrm{lcp}(\{x, y\})$.

Next, we shall see how an ultrametric on the Cantor set can be encoded as a weight on a rooted tree.

Definition 4.7. A *profinite structure* on a Cantor set X is an increasing family $\{R_\varepsilon : \varepsilon \in \mathbb{R}^+\}$ of equivalence relations on X satisfying:

(i) Each relation R_ε is open in $X \times X$ and, for some ε, $R_\varepsilon = X \times X$;

(ii) The family is continuous on the left: $\bigcup_{\varepsilon' < \varepsilon} R_{\varepsilon'} = R_\varepsilon$;

(iii) $\bigcap_{\varepsilon \in \mathbb{R}^+} R_\varepsilon = \Delta$ (the diagonal of $X \times X$).

We define a family of equivalence relations which plays a central role in this chapter.

Definition 4.8. Let (X, d) be a metric space. Given $\varepsilon > 0$ and $x, y \in X$, an ε-*chain* between x and y is a sequence $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ of points in X such that $d(x_i, x_{i+1}) < \varepsilon$. We now define an equivalence relation $\overset{\varepsilon}{\sim}$ by: $x \overset{\varepsilon}{\sim} y$ if there is an ε-chain between x and y. Denote the relation as a set by R_ε and the equivalence class of x by $[x]_\varepsilon$.

For the next three propositions, we consider a regular ultrametric Cantor set (X, d). By the ultrametricity, the following is trivial but will become handy.

Proposition 4.9. *For any $x, y \in X$, we have $d(x, y) < \varepsilon$ if and only if $x \overset{\varepsilon}{\sim} y$.*

Proposition 4.10. *For each $\varepsilon > 0$, $[x]_\varepsilon$ is a clopen set. Hence, the set of equivalence classes for $\overset{\varepsilon}{\sim}$ is finite because X is compact.*

Proof. For any $y \in [x]_\varepsilon$, we have $B_\varepsilon(y) = \{z \in X : d(y, z) < \varepsilon\} \subset [x]_\varepsilon$. Thus, $[x]_\varepsilon$ is open. Since the equivalence classes partition X and each equivalence class is open, the complement of $[x]_\epsilon$ is open and so $[x]_\epsilon$ is closed. □

Proposition 4.11. *The family $\{R_\varepsilon : \varepsilon \in \mathbb{R}^+\}$ for the relations $\overset{\varepsilon}{\sim}$ defined above is a profinite structure.*

Proof. Clearly, if $\varepsilon > \operatorname{diam}(X)$, then $R_\varepsilon = X \times X$. For each $\varepsilon > 0$ and $x, y \in R_\varepsilon$, the open balls of radius ε around x, y can be linked by ε-chains via x, y and hence, R_ε is open. Properties (ii) and (iii) of Definition 4.7 follow from Proposition 4.9. □

Proposition 4.12. *Let X be a Cantor set. Given a profinite structure $\{R_\varepsilon\}$ on X, we have an ultrametric defined by $d(x, y) := \inf\{\varepsilon : x \overset{\varepsilon}{\sim} y\}$.*

Proof. Let $x, y, z \in X$. Let ε be such that $\varepsilon > \max\{d(x, z), d(z, y)\}$. By the left continuity, $x \overset{\varepsilon}{\sim} z$ and $z \overset{\varepsilon}{\sim} y$. Thus, $x \overset{\varepsilon}{\sim} y$ and so $d(x, y) < \varepsilon$. Therefore, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$. □

Proposition 4.13. *On a Cantor set X, there is a one-to-one correspondence between profinite structures and regular ultrametrics, described by Proposition 4.11 and Proposition 4.12.*

Proof. If d, d' are distinct regular ultrametrics, suppose $d(x,y) = \varepsilon > d'(x,y) = \varepsilon'$. Set $\eta = \frac{\varepsilon+\varepsilon'}{2}$. Then $x \overset{\eta}{\sim}_{d'} y$ but $x \overset{\eta}{\nsim}_{d} y$. Hence, they yield different profinite structures.

Conversely, different profinite structures give rise to different metrics. Indeed, suppose $\{R_\varepsilon\}$ and $\{R'_\varepsilon\}$ are distinct profinite structures with corresponding metrics d, d', respectively. Without loss of generality, there exists $\varepsilon > 0$ such that $(x,y) \in R_\varepsilon$ but $(x,y) \notin R'_\varepsilon$. By property (ii) in Definition 4.7, there is $\varepsilon' < \varepsilon$ such that $(x,y) \in R_{\varepsilon'}$ and thus, $d(x,y) < \varepsilon$. On the other hand, since the family $\{R'_\eta : \eta \in \mathbb{R}^+\}$ is increasing, $(x,y) \notin R'_{\varepsilon'}$ for all $\varepsilon' < \varepsilon$ and hence, $d'(x,y) = \inf\{\eta : (x,y) \in R'_\eta\} \geq \varepsilon$. Thus, $d(x,y) < d'(x,y)$. □

Definition 4.14. Let $T = (V, E)$ be a rooted tree with no dangling vertices. A *weight* on T is a function $\epsilon : V \to \mathbb{R}^+$ satisfying:

(i) If $v \succ v'$, then $\epsilon(v) > \epsilon(v')$;

(ii) For each infinite path $v_0 v_1 \cdots \in \partial T$, we have $\lim_{n\to\infty} \epsilon(v_n) = 0$.

A rooted tree together with a weight function is called a *weighted*, rooted tree, denoted collectively as a pair (T, ϵ).

Definition 4.15. A rooted tree all of whose vertices have at least two children is called *reduced.*

Given a rooted tree such that every vertex has a descendant with more than one child, we can perform edge reduction to remove vertices with only one child without changing the topology of the boundary. The weight function on the reduced tree is the restriction of the original. Hence, two unreduced trees may have the same reduced weighted tree.

Definition 4.16. A *Michon tree* is a reduced, weighted Cantorian tree.

The main takeaway of this chapter is the following.

Theorem 4.17 (Michon Correspondence)**.** *On a Cantor set X, there is a one-to-one correspondence between regular ultrametrics and Michon trees. Furthermore, given a regular ultrametric d, the corresponding Michon tree (T, ϵ) is such that:*

- *(X, d) is isometric to $(\partial T, d_\epsilon)$, where $d_\epsilon(x,y) = \epsilon(x \wedge y)$ and*
- *$\epsilon(v) = \mathrm{diam}_{d_\epsilon}([v])$ for any vertex v of T.*

We breakdown the proof of the above into a series of propositions.

Proposition 4.18. *Let X be a Cantor set with a regular ultrametric d and the corresponding profinite structure $\{R_\varepsilon\}$. Then*

(i) Setting $\epsilon := \inf\{\varepsilon : R_\varepsilon = X \times X\}$, we have $\epsilon = \mathrm{diam}(X)$ and $R_\epsilon \neq X \times X$.

(ii) *For each* $x \in X$ *and* $\alpha > 0$*, we have* $\operatorname{diam}([x]_\alpha) = \inf\{\varepsilon \leq \alpha : [x]_\varepsilon = [x]_\alpha\}$*. Setting* $\epsilon_\alpha := \operatorname{diam}([x]_\alpha)$*, we also have* $[x]_{\epsilon_\alpha} \subsetneq [x]_\alpha$ *and* $\epsilon_\alpha < \alpha$*.*

(iii) *For any* $x \in X$ *and* $\varepsilon \in (\operatorname{diam}([x]_\alpha), \alpha]$*, we have* $[x]_\varepsilon = [x]_\alpha$*.*

Proof. (i) Set $\epsilon := \inf\{\varepsilon : R_\varepsilon = X \times X\}$. Let $\varepsilon > \epsilon$. We have $R_\varepsilon = X \times X$ by left continuity and for any $x, y \in X$, we have $d(x, y) < \varepsilon$ by ultrametricity. Therefore, $\operatorname{diam}(X) \leq \epsilon$. On the other hand, let $\varepsilon < \epsilon$ so $R_\varepsilon \neq X \times X$. Then there are $x, y \in X$ such that $d(x, y) \geq \varepsilon$ by Proposition 4.9. Consequently, $\operatorname{diam}(X) \geq \epsilon$. Morever, since $X \times X$ is compact and $R_\epsilon = \bigcup_{\varepsilon<\epsilon} R_\varepsilon$ is a union of open sets, we have $R_\epsilon \neq X \times X$.

(ii) Observe that $[x]_\varepsilon = [x]_\alpha$ is equivalent to $R_\varepsilon \cap ([x]_\alpha \times [x]_\alpha) = [x]_\alpha \times [x]_\alpha$. Since $[x]_\alpha$ is a clopen subset of X, it is a Cantor set. Therefore, Statement (ii) follows from replacing X by $[x]_\alpha$ in (i).

(iii) Let $\tilde{R}_\varepsilon := R_\varepsilon \cap ([x]_\alpha \times [x]_\alpha)$, the restriction of R_ε to the Cantor set $[x]_\alpha$. Then $\{\tilde{R}_\varepsilon\}$ is a profinite structure on $[x]_\alpha$. Then (iii) follows from the left continuity of $\{\tilde{R}_\varepsilon\}$ and the fact that $\operatorname{diam}([x]_\alpha)$ is the infimum as in (ii).

□

From (ii) and (iii), we see that, given $x \in X$ and $\alpha > 0$, as ε is decreasing from α, $[x]_\varepsilon$ remains equal to $[x]_\alpha$ until it splits when $\varepsilon = \operatorname{diam}([x]_\alpha)$. If we set $[x]_\infty := X$, then this is applicable to $\alpha = \infty$ by (i). Using this picture, we construct a corresponding weighted, rooted tree in the next proposition.

Proposition 4.19. *Given a regular ultrametric Cantor set* (X, d)*, there is a Michon tree* (T, ϵ) *whose boundary* $(\partial T, d_\epsilon)$ *is isometric to* (X, d)*, where* $d_\epsilon(x, y) = \epsilon(x \wedge y)$ *for any* $x, y \in \partial T$*.*

Proof. Let $\{R_\varepsilon : \varepsilon \in \mathbb{R}^+\}$ be the profinite structure corresponding to (X, d) as in Proposition 4.13. Set $\varepsilon_0 := \inf\{\varepsilon : R_\varepsilon = X \times X\}$ so that $\varepsilon_0 = \operatorname{diam}(X)$ and $R_{\varepsilon_0} \subsetneq X \times X$ by Proposition 4.18(i). Moreover, we have $R_\varepsilon = X \times X$ for all $\varepsilon > \varepsilon_0$. Inductively, we set $\varepsilon_{n+1} := \inf\{\varepsilon : R_\varepsilon = R_{\varepsilon_n}\}$ so that $R_{\varepsilon_{n+1}} \subsetneq R_{\varepsilon_n}$ and $R_\varepsilon = R_{\varepsilon_n}$ for all $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]$. Roughly speaking, starting from $\varepsilon = \infty$ and $R_\infty = X \times X$, we keep decreasing ε until one of the equivalence classes for R_ε splits and record the splitting points into the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$. As discussed earlier, the splitting point for $[x]_\alpha$ is $\varepsilon = \operatorname{diam}([x]_\alpha)$ which can be arbitrarily small by Proposition 4.18(ii). Therefore, the sequence $\{\varepsilon_n\}$ is strictly decreasing to 0.

We define a rooted tree T as follows. Let $V_0 = \{X\}$ be the root, and for each $n \geq 1$ let V_n be the set of equivalence classes for $R_{\varepsilon_{n-1}}$, and let $V = \bigcup_{n\in\mathbb{N}} V_n$. The edges are determined by inclusion; i.e., v is a child of w if $v \in V_{n+1}, w \in V_n$, and $v \subseteq w$. By Proposition 4.10, each V_n is finite. Clearly, T has no dangling vertices. Suppose $v = [x]_\varepsilon \in V$. Since X has no isolated points, v contains at least two points and thus, one of its descendants splits into more than one equivalence classes. Therefore, T is a Cantorian tree. Note that T may not be reduced, because some equivalence classes may not split as we move from V_n to V_{n+1}.

For each $v \in V$, we assign a weight $\epsilon(v) := \mathrm{diam}(v)$. Since for any $x \in X$, $\mathrm{diam}([x]_{\varepsilon_n}) < \varepsilon_n$, it follows that ϵ is decreasing along any infinite path. However, if T is not reduced, ε is not strictly decreasing.

In order to obtain a reduced rooted tree, we do edge reduction to remove the vertices with only one child. Since each vertex splits, the weight function ϵ restricted to the reduced tree is strictly decreasing. The resulting tree is a Michon tree as desired.

For each infinite path $v_0v_1v_2\ldots \in \partial T$, we have $\bigcap_{n\geq 0} v_n \neq \emptyset$ since $\{v_n : n \geq 0\}$ is a nested sequence of non-empty closed subsets of the compact set X. Thus, there is $x \in X$ such that for each $n \geq 0$, $v_n = [x]_{\varepsilon_{n-1}}$, and indeed $\bigcap_{n\geq 0} v_n = \{x\}$ by (iii) of Definition 4.7. Define $\Phi : \partial T \to X$ by $\Phi(v_0v_1\ldots) = x$. We note that Φ is bijective and if $v = [x]_{\varepsilon_n} \in R_{n+1}$, we have $\Phi^{-1}\{[x]_{\varepsilon_n}\} = [v]$. Hence, Φ is continuous as $\{[x]_{\varepsilon_n} : x \in X \text{ and } n \geq 0\}$ is a basis for the topology of X—this can be checked similarly to Proposition 4.4. Because ∂T is compact, Φ is a homeomorphism.

Next, we equip ∂T with the metric $d_\epsilon(x, y) := \epsilon(x \wedge y)$ for any $x, y \in \partial T$. Via Φ, we write $x \in X$ as $X[x]_{\varepsilon_0}[x]_{\varepsilon_1}[x]_{\varepsilon_2}\ldots$. Let $x, y \in \partial T$. Say, $x \wedge y = [x]_{\varepsilon_n}$. Then $[x]_{\varepsilon_n} = [y]_{\varepsilon_n}$ but $[x]_{\varepsilon_{n+1}} \neq [y]_{\varepsilon_{n+1}}$. By Proposition 4.9, $d(x, y) \geq \varepsilon_{n+1}$. Also, by Proposition 4.18(ii), we have $\mathrm{diam}([x]_{\varepsilon_n}) = \varepsilon_{n+1}$. Since $y \in [x]_{\varepsilon_n}$, $d(x, y) = \varepsilon_{n+1} = \mathrm{diam}_d([x]_{\varepsilon_n}) = \epsilon(x \wedge y)$. Therefore, Φ is an isometry between $(\partial T, d_\epsilon)$ and (X, d). Note that edge reduction does not affect the isometry because $x \wedge y = [x]_{\varepsilon_n}$ must have more than one child and hence, it is not removed. □

Proposition 4.20. *Given a Michon tree (T, ϵ), we have a regular ultrametric on $X := \partial T$ defined by $d(x, y) = \epsilon(x \wedge y)$ for $x \neq y$ and $d(x, x) = 0$.*

Proof. It is straightforward to see that d is an ultrametric. Indeed, for any $z \in \partial T$, $x\wedge z$ and $z\wedge y$ are comparable as they are both on the path z. We may assume $y\wedge z \preceq x\wedge z$. Then $x\wedge z$ is a common prefix of x, y and hence, $x \wedge y \preceq x \wedge z$. Thus, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

Given $r > 0$ and $x \in X$, let $B_r(x) := \{y \in X : d(x, y) < r\}$. By Property (ii) of Definition 4.14 of a weight function and the fact that each vertex has more than one child, $B_r(x)$ has more than one point so we can let $v = \mathrm{lcp}(B_r(x))$. Consequently, for any $y \in [v]$, there must be $z \in B_r(x)$ such that $x\wedge y \preceq x\wedge z$, for otherwise the child of $x\wedge y$ along the path x would be a common prefix of $B_r(z)$ strictly smaller than v. Hence, $d(x, y) \leq d(x, z) < r$ and thus, $[v] = B_r(x)$. Since $\{[v] : v \in V\}$ is a basis of open sets for the topology of ∂T, it follows that d is regular. □

Now denote the vertex set of the Michon tree (T, ϵ) by V. For $(X = \partial T, d)$ as above, we let $(T_d, \tilde{\epsilon})$ be the associated Michon tree constructed in Proposition 4.19 whose vertex set is denoted by V_d. Define $\Psi : V_d \to V$ by

$$[x]_\varepsilon \mapsto \mathrm{lcp}([x]_\varepsilon),$$

where ε is one of the $\{\varepsilon_n\}$ arising from the construction of $(T_d, \tilde{\epsilon})$ in the proof of Proposition 4.19.

Proposition 4.21. *The map Ψ is a weight-preserving bijection.*

Proof. Let $v = \mathrm{lcp}([x]_\varepsilon)$. We have $[x]_\varepsilon \subseteq [v]$. Suppose $y \in [v] \subseteq X$. As we showed in the proof of the preceding preposition, the definition of lcp implies that there is $z \in [x]_\varepsilon$ such that $x \wedge y \preceq x \wedge z$ and hence, $d(x, y) \leq d(x, z) < \varepsilon$. Therefore, $[v] = [x]_\varepsilon$ and Ψ is injective.

Now let $v \in V$. For any $a, b \in [v]$, we have $a \wedge b \preceq v$ and $d(a, b) = \epsilon(a \wedge b) \leq \epsilon(v)$. Since T is reduced and v has more than one child, there is $x, y \in X$ such that $x \wedge y = v$. Thus, we have $d(x, y) = \epsilon(v)$ and $\mathrm{diam}_d([v]) = \epsilon(v)$. Moreover, $[x]_\varepsilon = [y]_\varepsilon$ for any $\varepsilon > \epsilon(v)$ but $[x]_{\epsilon(v)} \neq [y]_{\epsilon(v)}$. Hence, given the sequence $\{\varepsilon_k\}$ defined via the profinite structure, there is an n such that $\varepsilon_n > \epsilon(v)$ and $\tilde{\epsilon}([x]_{\varepsilon_n}) = \epsilon(v)$. Additionally, $[x]_\varepsilon = [x]_{\varepsilon_n}$ for all $\varepsilon \in (\epsilon(v), \varepsilon_n]$. It remains to show that $v = \mathrm{lcp}([x]_{\varepsilon_n})$. Let $z \in [x]_{\varepsilon_n}$. Then $d(x, z) < \varepsilon$ for all $\varepsilon \in (\epsilon(v), \varepsilon_n]$, and thus $d(x, z) \leq \epsilon(v)$; i.e., $\epsilon(x \wedge z) \leq \epsilon(v)$. Since both v and $x \wedge z$ are vertices in the path x, and ϵ is strictly decreasing, we have $x \wedge z \leq v$ and therefore $z \in [v]$. Let w be a common prefix for $[x]_{\varepsilon_n}$. Because $x, y \in [x]_{\varepsilon_n} \subseteq [w]$, we have $x \wedge y = v \preceq w$. Thus, $v = \mathrm{lcp}([x]_{\varepsilon_n})$ and Ψ is surjective. Since $\tilde{\epsilon}([x]_{\varepsilon_n}) = \epsilon(v)$, we see that Ψ preserves weight. □

Combining Propositions 4.19–4.21, we obtain the Michon correspondence (Theorem 4.17).

Chapter 5

Pearson-Bellissard Spectral Triples

Using the Michon correspondence, Pearson and Bellissard constructed spectral triples on an ultrametric Cantor set in Section 5 of [30].

Let X be a Cantor set with a regular ultrametric d. Let (T, ϵ) be the Michon tree corresponding to d. By the Michon correspondence, we view X as the boundary of its Michon tree and d is given by $d(x, y) = \epsilon(x \wedge y)$. The vertex set V is countable and we let $\mathcal{H} := \ell^2(V) \otimes \mathbb{C}^2 = \ell^2(V, \mathbb{C}^2)$.

In order to get a representation of $C(X)$, we need the following notion of choice.

Definition 5.1. A *choice function* is a map $\tau : V \mapsto X \times X$ such that for any $v \in V$, if $\tau(v) = (x, y)$, then $x \wedge y = v$. Such x and y are denoted $\tau^+(v), \tau^-(v)$, respectively. The set of choice functions on X will be denoted by $\Upsilon(X)$.

Note that the original definition of Pearson and Bellissard is that $\tau(v) = (x, y)$ means $x, y \in [v]$ and $d(x, y) = \epsilon(x \wedge y) = \mathrm{diam}([v])$. Both definitions are equivalent by the Michon correspondence (Theorem 4.17). However, we choose the above definition for the sake of visualization: a choice function assigns to each vertex $v \in V$ a pair of distinct infinite paths $\tau^{\pm}(v)$ emanating from v.

Given a choice function τ, we can define a representation π_τ of $C(X)$ on $\mathcal{H}$ by

$$(\pi_\tau(f)\psi)(v) = \begin{pmatrix} f(\tau^+(v)) & 0 \\ 0 & f(\tau^-(v)) \end{pmatrix} \psi(v),$$

for any $f \in C(X), \varphi \in \mathcal{H}, v \in V$.

Proposition 5.2. *π_τ is a faithful $*$-representation of $C(X)$ for all $\tau \in \Upsilon(X)$.*

Proof. Suppose $\pi_\tau(f) = 0$. Then for all $v \in V$, $f(\tau^+(v)) = 0$. Since the cylinder subsets form a basis of open sets of X and $\tau^+(v) \in [v]$ for all $v \in V$, the set $\{\tau^+(v) : v \in V\}$ is dense in X. Therefore, $f = 0$. $\square$

Define an operator D on $\mathcal{H}$ by: for any $\psi \in \mathcal{H}, v \in V$,

$$(D\psi)(v) := \mathrm{diam}([v])^{-1}\sigma_1\psi(v),$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the first Pauli matrix. We set

$$\mathrm{dom}(D) := \{\psi \in \mathcal{H} : \sum_{v\in V} \mathrm{diam}([v])^{-2}\|\psi(v)\|^2 < \infty\}.$$

Then D is densely defined because $\mathrm{dom}(D)$ contains the dense subalgebra of elements with finite support.

Consider the space $C_{\mathrm{Lip}}(X)$ of Lipschitz functions on X, which is a dense subalgebra of $C(X)$. Together with D, we have a spectral triple. We follow the proof presented by Palmer in his doctoral thesis [29] (Theorem 3.3.4), which we will adapt to construct a weak form of spectral triple for a general unital AF algebra in Chapter 10.

Lemma 5.3. $\lim_{n\to\infty} \sup\{\mathrm{diam}([v]) : v \in V_n\} = 0$.

Proof. Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, we set $W_n = \bigcup\{[v] : v \in V_n \text{ and } \mathrm{diam}([v]) < \varepsilon\}$. It is easy to check that $W_n \subseteq W_{n+1}$ for any $n \in \mathbb{N}$. Since $\mathrm{diam}([v]) \to 0$ along infinite paths, the set $\{W_n : n \in \mathbb{N}\}$ is an open cover of X. By the compactness, there is a finite subcover. Because W_n's are ascending, there is $k \in \mathbb{N}$ such that $W_k = X = \bigsqcup\{[v] : v \in V_k\}$. Thus, we have $\mathrm{diam}([v]) < \varepsilon$ for any $v \in V_k$. For each $n \geq k$ and $w \in V_n$, there is $v \in V_k$ such that $[w] \subseteq [v]$. The proof is concluded. □

This proof is akin to the proof of the fact that a monotonically decreasing sequence of continuous functions on a compact space converging pointwise to 0 also converges uniformly to 0. Also, the above lemma means that $\mathcal{V}_n = \{[v] : v \in V_n\}$ forms a resolving sequence of open covers of X in the sense of Definition 4.1.1 of Palmer's thesis.

Theorem 5.4. *For all $\tau \in \Upsilon(X)$, we have that $(\pi_\tau, \mathcal{H}, D)$ is a spectral triple for $C(X)$.*

Proof. <u>*D is self-adjoint.*</u> Let $\psi, \psi' \in \mathrm{dom}(D)$. We have

$$\langle D\psi, \psi'\rangle = \sum_{v\in V}\langle D\psi(v), \psi'(v)\rangle = \sum_{v\in V}\langle \psi(v), D\psi'(v)\rangle = \langle \psi, D\psi'\rangle.$$

Hence, D is a symmetric operator. We claim that $\mathrm{ran}(D) = \mathcal{H}$. Suppose $\psi \in \mathcal{H}$. Let $\psi'(v) = \mathrm{diam}([v])\sigma_1\psi(v)$. Then

$$\|\psi'\|^2 = \sum_{v\in V} \mathrm{diam}([v])^2\|\psi(v)\|^2 \leq \mathrm{diam}(X)^2\|\psi\|^2 < \infty.$$

Therefore, $\psi' \in \mathcal{H}$. Also, we have

$$\sum_{v\in V} \mathrm{diam}([v])^{-2}\|\psi'(v)\|^2 = \sum_{v\in V} \|\psi(v)\|^2 < \infty.$$

Hence, $\psi' \in \mathrm{dom}(D)$ and $\psi = D\psi' \in \mathrm{ran}(D)$. Consequently, by a standard fact in the theory of unbounded operators (e.g. Corollary 3.12 of [43]), D is self-adjoint.

$[D, \pi_\tau(f)]$ is bounded for each $f \in C_{\mathrm{Lip}}(X)$.

Let $f \in C_{\mathrm{Lip}}(X)$. Then for each $v \in V$,

$$\begin{aligned}([D, \pi_\tau(f)]\psi)(v) &= \mathrm{diam}([v])^{-1}\left[\sigma_1, \begin{pmatrix} f(\tau^+(v)) & 0 \\ 0 & f(\tau^-(v)) \end{pmatrix}\right]\psi(v) \\ &= \mathrm{diam}([v])^{-1}(f(\tau^+(v)) - f(\tau^-(v)))\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\psi(v).\end{aligned}$$

By Definition 5.1 of choice function, $v = \tau^+(v) \wedge \tau^-(v)$ and $\mathrm{diam}([v]) = d(\tau^+(v), \tau^-(v))$. Therefore,

$$\|[D, \pi_\tau(f)]\psi(v)\| \leq \mathrm{Lip}(f)\|\psi(v)\|.$$

so $\|[D, \pi_\tau(f)]\| \leq \mathrm{Lip}(f) < \infty$.

$(\mathbf{1} + D^2)^{-1}$ is compact.

We define an operator $T_m : \mathcal{H} \to \mathcal{H}$ by

$$(T_m\psi)(v) = [\![v \in V_n \text{ and } n \leq m]\!](\mathbf{1} + D^2)^{-1}\psi(v),$$

where the double square brackets denote the boolean value of the enclosed statement. Basically, we truncate $(\mathbf{1} + D^2)^{-1}$ to $\ell^2(\bigcup_{n\leq m} V_n, \mathbb{C}^2)$ so that T_m is finite-rank.

Note that $(\mathbf{1} + D^2)^{-1}\psi(v) = \dfrac{\mathrm{diam}([v])^2}{1 + \mathrm{diam}([v])^2}\psi(v)$. Then

$$((\mathbf{1} + D^2)^{-1} - T_m)\psi(v) = [\![v \in V_n \text{ and } n > m]\!]\frac{\mathrm{diam}([v])^2}{1 + \mathrm{diam}([v])^2}\psi(v).$$

By Lemma 5.3, we have $\sup\{\mathrm{diam}([v]) : v \in V_n \text{ and } n > m\} \to 0$ as $m \to \infty$. Therefore, $\|(\mathbf{1} + D^2)^{-1} - T_m\| \to 0$ as desired. □

Remark: Given a choice function, the seminorm induced by the commutator from the spectral triple above may not be Lipschitz. An example is given below.

Example 5.5. Let T be the infinite rooted binary tree with the boundary $X := \partial T$ and let C be the Cantor ternary set. The map $\Psi : X \to C$, given by

$$\{x_n\}_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} 2x_n 3^{-(n+1)},$$

is a homeomorphism. That is, the elements in C are the numbers in $[0, 1]$ which can be represented as a ternary fraction with only 0 and 2. Denote the set of vertices of T by V. We assign the weight $\epsilon(v) = 2^{-n}$ to each n-level vertex v. At each vertex, we label the edge going to the left by 0 and the one going to the right by 1. Regarding $v \in V$ as the unique finite path starting from the root to v, we define a choice function $\tau : V \to X \times X$ by

$$\tau^+(v) = v0111\ldots \quad \text{and} \quad \tau^-(v) = v1000\ldots.$$

The standard Cantor function $f : X \to \mathbb{R}$ is given by $\{x_n\}_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} x_n 2^{-(n+1)}$. To see that f is Lipschitz, we let $x, y \in X$ be such that $x \neq y$. The weight ϵ determines the ultrametric $d_\epsilon(x,y) = \epsilon(x \wedge y)$. Suppose that the least common prefix $x \wedge y$ is in the k-level; i.e., $x_n = y_n$ for any $n < k$ and $x_k \neq y_k$. Then $d_\epsilon(x,y) = 2^{-k}$ and

$$|f(x) - f(y)| = \left|\sum_{n\in\mathbb{N}} (x_n - y_n)2^{-(n+1)}\right| = \left|\sum_{n\geq k} (x_n - y_n)2^{-(n+1)}\right| \leq \sum_{n\geq k} 2^{-(n+1)} = 2^{-k}.$$

Therefore, f is a non-constant Lipschitz function. However, both $f(\tau^+(v))$ and $f(\tau^-(v))$ correspond to the same binary fraction $0.v0111\ldots = 0.v1000\ldots$, which implies $L(f) = \|[D, \pi_\tau(f)]\| = 0$. That is, the seminorm L is not Lipschitz.

When taking all choice functions into account, we can recover the original metric via Connes' distance formula.

Theorem 5.6 ([30], Theorem 1)**.** *Let X be a Cantor set with a regular ultrametric d. Then d coincides with the distance ρ defined by*

$$\rho(x,y) := \sup\{|f(x) - f(y)| : f \in C_{\mathrm{Lip}}(X), \sup_{\tau\in\Upsilon(X)} \|[D, \pi_\tau(f)]\| \leq 1\}.$$

Typically the seminorm in the Connes distance formula comes from a single spectral triple. Embedded in $\mathbb{R}$, a Cantor set is contained in a smallest closed bounded interval $[a, b]$. The complement of the Cantor set in $[a, b]$ is a disjoint union of countably many open intervals (gaps). In Chapter IV.3.ε of [12], Connes associates a 2-dimensional Hilbert space to each gap. Forming the direct sum representation, he constructs a spectral triple for the Cantor set. Guido and Isola showed that the Connes distance from this spectral triple coincides with the metric induced by the Euclidean distance on $\mathbb{R}$ iff the Cantor set has Lebesgue measure zero (Theorem 4.4 of [15]). However, in the above theorem, we use the supremum of the seminorms $\|[D, \pi_\tau(f)]\|$ where τ ranges over all choice functions to reconstruct the original ultrametric. In fact, the supremum is the classical Lipschitz seminorm on $C_{\mathrm{Lip}}(X)$.

Proposition 5.7. *For X and D as above,*

$$L(f) := \|[D, \pi(f)]\| = \sup_{\tau\in\Upsilon(X)} \|[D_\tau, \pi_\tau(f)]\| = \mathrm{Lip}(f),$$

where $\mathrm{Lip}(f)$ *is the classical Lipschitz constant of $f \in C_{\mathrm{Lip}}(X)$.*

Proof. Let $f \in C_{\mathrm{Lip}}(X)$ and $\tau \in \Upsilon(X)$. By a straightforward calculation,

$$[D_\tau, \pi_\tau(f)]\psi(v) = \frac{f(\tau^+(v)) - f(\tau^-(v))}{d(\tau^+(v), \tau^-(v))} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi(v).$$

It follows that $L(f) \le \mathrm{Lip}(f)$.

For each $w \in V$, take $\psi_w = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \delta_w \in \ell^2(V, \mathbb{C}^2)$.

$$[D_\tau, \pi_\tau(f)]\psi_w(v) = \frac{f(\tau^+(v)) - f(\tau^-(v))}{d(\tau^+(v), \tau^-(v))} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta_w(v)$$

$$\|[D_\tau, \pi_\tau(f)]\psi_w\|^2 = \sum_{v \in V} \|[D_\tau, \pi_\tau(f)]\psi_w(v)\|^2 = \left| \frac{f(\tau^+(w)) - f(\tau^-(w))}{d(\tau^+(w), \tau^-(w))} \right|^2$$

Hence,

$$\left| \frac{f(\tau^+(w)) - f(\tau^-(w))}{d(\tau^+(w), \tau^-(w))} \right| \le \|[D_\tau, \pi_\tau(f)]\|.$$

Let $x, y \in X$ be such that $x \neq y$ and let $w_{xy} = x \wedge y \in V$. We can choose $\tau \in \Upsilon(X)$ such that $\tau(w_{xy}) = (x, y)$ since $x, y \in [w_{xy}]$ and $d(x, y) = \epsilon(x \wedge y) = \epsilon(w_{xy}) = \mathrm{diam}([w_{xy}])$. Then

$$\left| \frac{f(x) - f(y)}{d(x, y)} \right| = \left| \frac{f(\tau^+(w_{xy})) - f(\tau^-(w_{xy}))}{d(\tau^+(w_{xy}), \tau^-(w_{xy}))} \right| \le \|[D_\tau, \pi_\tau(f)]\|.$$

Therefore, $\mathrm{Lip}(f) \le L(f)$ and this concludes the proof. □

Essentially, the proof of Theorem 5.6 in [30], is by combining the distance recovering formula (Proposition 2.2) and Proposition 5.7.

Chapter 6

The Path Model of AF Algebras

Definition 6.1. A C^*-algebra A is an *approximately finite-dimensional* (AF) algebra if it is the norm closure of an increasing union of finite-dimensional subalgebras A_n, where $n \in \mathbb{N}$.

Equivalently, one can say that a C^*-algebra A is AF if A is the limit of an inductive sequence of finite-dimensional C^*-algebras (A_n, ϕ_n), where $\phi_n : A_n \to A_{n+1}$ is viewed as an inclusion $A_n \subseteq A_{n+1}$.

We will assume that A is unital and $A_0 = \mathbb{C}\mathbf{1}_A$. (For the inductive limit setting, we assume each A_n is unital and ϕ_n is a unital $*$-homomorphism.) Recall that a finite-dimensional C^*-algebra is $*$-isomorphic to a direct sum of full matrix algebras (See for example, Theorem III.1.1 of [13].) The following statement describes how a finite-dimensional C^*-algebra sits inside another.

Proposition 6.2. *([13], Lemma III.2.1) Let A_1 and A_2 be finite-dimensional C^*-algebras such that*

$$A_1 \cong M_{n_1} \oplus \cdots \oplus M_{n_k} \text{ and } A_2 \cong M_{m_1} \oplus \cdots \oplus M_{m_l}.$$

Suppose that $\varphi : A_1 \to A_2$ is a unital homomorphism. Let $\varphi_i : A_1 \to M_{m_i}$ be the composition of φ and the projection of A_2 onto the summand M_{m_i}. Then there is a matrix $Q = [q_{ij}] \in M_{lk}(\mathbb{N})$ such that for each $1 \leq i \leq l$, we have $\sum_{j=1}^{k} q_{ij} n_j = m_i$ and there is a unitary equivalence of representations of A_1

$$\varphi_i \cong \mathrm{id}_{n_1}^{(q_{i1})} \oplus \cdots \oplus \mathrm{id}_{n_k}^{(q_{ik})},$$

where id_{n_j} denotes the identity representation of M_{n_j} and the right-hand side is regarded as a representation of A_1 via the identification with the direct sum of M_{n_i}.

The non-negative integer q_{ij} is the number of copies of M_{n_j} in $M_{m_i} \subseteq A_2$ and it is called the *partial multiplicity*.

From the above proposition, one can visualize the embedding φ in a graphical way as first explained by Bratteli in [9]:

(1) Represent A_1 by k vertices $\{v_1, \ldots, v_k\}$ and A_2 by l vertices $\{w_1, \ldots, w_l\}$;

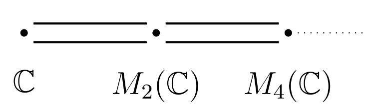
(2) Draw q_{ij} edges from v_j to w_i to represent the partial multiplicity of the embedding of the summand M_{n_j} in M_{m_i}.

Now suppose that $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ where A_n is finite-dimensional and $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$. Let V_n be the set of vertices representing A_n and E_n the set of edges from V_n to V_{n+1} representing the embedding $A_n \subseteq A_{n+1}$. Let $V = \bigcup_{n\in\mathbb{N}} V_n$ and $E = \bigcup_{n\in\mathbb{N}} E_n$. The graph (V, E) is called a *Bratteli diagram corresponding to* A.

Example 6.3. The CAR algebra A is the inductive limit of $A_n \cong M_{2^n}$ with the embedding $\phi_n : A_n \to A_{n+1}$ given by:

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

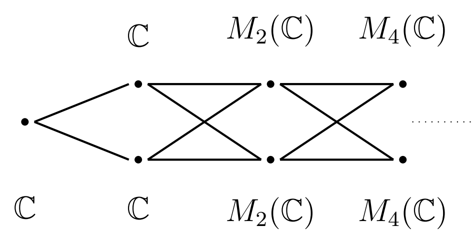
The Bratteli diagram corresponding to the prescribed embedding is



Notice that the range $\phi_n(A_n) \subset M_{2^n} \oplus M_{2^n}$. Let $B_0 = \mathbb{C}$ and for each $n \geq 1$, let $B_n \cong M_{2^{n-1}} \oplus M_{2^{n-1}}$. Then the CAR algebra can also be realized as the inductive limit of B_n with the embedding:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

which has the Bratteli diagram:



In general, we define:

Definition 6.4. A *Bratteli diagram* is a graph consisting of

1. Pairwise disjoint, finite, non-empty sets of vertices $\{V_n\}_{n\in\mathbb{N}}$ with $V_0 = \{v_0\}$;

2. Finite, non-empty set of edges from V_n to V_{n+1}, denoted by E_n, equipped with *source* map $s : E_n \to V_n$ and *range* map $r : E_n \to V_{n+1}$ satisfying:

a) $s^{-1}\{v\}$ is non-empty for each $v \in V_n$;

b) $r^{-1}\{v\}$ is non-empty for each $v \in V_{n+1}$.

With $V = \bigcup_{n\in\mathbb{N}} V_n$ and $E = \bigcup_{n\in\mathbb{N}} E_n$, we denote the graph by (V, E). We call $v \in V_n$ an n-level vertex.

Definition 6.5. A *path* α on a Bratteli diagram is a word $\alpha = e_i e_{i+1} \dots e_j$ such that $e_n \in E_n$ and $r(e_i) = s(e_{i+1})$. We set $r(\alpha) = r(e_j)$ and $s(\alpha) = s(e_i)$. An infinite path is defined similarly.

Definition 6.6. For any $v \in V_m$ and $w \in V_n$ with $m < n$, a path α from v to w is such that $s(\alpha) = v$ and $r(\alpha) = w$. We denote the set of paths from v to w by $\mathrm{Path}(v, w)$. We can extend this definition to $v, w \in V_n$ by letting $\mathrm{Path}(v, v) = \{v\}$ and $\mathrm{Path}(v, w) = \emptyset$ if $v \neq w$.

Given a Bratteli diagram (V, E), we can formally describe an inductive sequence of finite-dimensional C^*-algebras in terms of paths. We will use the same notations as in [14] except, in order to avoid messy subscripts, we use the bra-ket notation to describe the matrix units.

For any $m \le n$, we let $\Omega[m, n] = \bigcup_{v\in V_m, w\in V_n} \mathrm{Path}(v, w)$, the space of paths from level m to level n. As a convention, we set $\Omega[m] := \Omega[m, m] = V_m$. For any $v \in \Omega[m], w \in \Omega[n]$, we form the Hilbert space $\ell^2(\mathrm{Path}(v, w)) = \mathrm{span}\{|\alpha\rangle : \alpha \in \mathrm{Path}(v, w)\}$. Then $A_{vw} := \mathcal{B}(\ell^2(\mathrm{Path}(v, w)))$ is generated by the matrix units $|\alpha\rangle\langle\beta|$, where $\alpha, \beta \in \mathrm{Path}(v, w)$. Let

$$A[m, n] = \oplus A_{vw} = \oplus\mathcal{B}(\ell^2(\mathrm{Path}(v, w))),$$

where the summation is over all v at level m and w at level n. Then $A[m, n]$ is generated by the $|\alpha\rangle\langle\beta|$, where $\alpha, \beta \in \Omega[m, n]$ with the same source and target. If $[m, n] \subset [m', n']$, we embed $A[m, n]$ in $A[m', n']$ by

$$|\alpha\rangle\langle\beta| \mapsto \sum_{\gamma,\rho} |\gamma\alpha\rho\rangle\langle\gamma\beta\rho|$$

for $\alpha, \beta \in \mathrm{Path}(v, w)$, where the sum is over all $\gamma \in \mathrm{Path}(v', v), \rho \in \mathrm{Path}(w, w')$ with v' on level m' and w' on level n'. In particular, $A_n = A[0, n]$ and the above embedding also explains the inclusion $A_n \subseteq A_{n+1}$.

Alternatively, we can describe A_n as a C^*-algebra associated to an equivalence relation on a finite set [32] as follows. For each $n \in \mathbb{N}$, define an equivalence relation R_n on $\Omega[0, n]$ by:

$$\alpha \sim \beta \quad \text{if} \quad r(\alpha) = r(\beta).$$

The algebra $C^*(R_n)$ consists of functions $f : R_n \to \mathbb{C}$ with matrix multiplication when viewing f as the matrix $[f(x, y)]_{x,y\in\Omega[0,n]}$, where $f(x, y) = 0$ if $(x, y) \notin R_n$. The involution is the usual conjugate transpose. The canonical matrix units of $C^*(R_n)$ are the Kronecker delta functions $\{\delta_{(\alpha,\beta)}\}_{(\alpha,\beta)\in R_n}$. For any $x, y \in \Omega[0, n]$, and $(\alpha, \beta) \in R_n$, we have $\langle x|\alpha\rangle\langle\beta|y\rangle = \delta_{(\alpha,\beta)}(x, y)$. We can then identify $\delta_{(\alpha,\beta)}$ with $|\alpha\rangle\langle\beta|$, and therefore, $C^*(R_n)$ with A_n.

For any $m \leq n$, we have

$$\Omega[0,n] = \{(\alpha, \rho) \in \Omega[0,m] \times \Omega[m,n] : r(\alpha) = s(\rho)\}$$

and, with the short-hand notation $\alpha\rho$ for $(\alpha, \rho) \in \Omega[0,n]$ where $\alpha \in \Omega[0,m]$ and $\rho \in \Omega[m,n]$ as above, we rewrite

$$R_n = \{(\alpha\rho, \beta\gamma) \in \Omega[0,n] \times \Omega[0,n] : r(\rho) = r(\gamma)\}.$$

Then the embedding $\varphi_{m,n} : C^*(R_m) \hookrightarrow C^*(R_n)$ is given by:

$$\varphi_{m,n}(f)(\alpha\rho, \beta\gamma) = [\![\rho = \gamma]\!] f(\alpha, \beta).$$

Chapter 7

Conditional Expectation from Transition Probability

Definition 7.1. Let A, B be C^*-algebras. A linear map $\phi : A \to B$ is positive if $\phi(a) \in B_+$ for any $a \in A_+$. Define $\phi^{(n)} : M_n(A) \to M_n(B)$ by applying ϕ entrywise. We say that ϕ is n-positive if $\phi^{(n)}$ is positive. We call ϕ a *completely positive* map if it is n-positive for all $n \in \mathbb{N}$.

Example 7.2. Any $*$-homomorphism $\phi : A \to B$ is positive since any positive element can be written as a^*a for some $a \in A$ and $\phi(a^*a) = \phi(a)^*\phi(a)$ is positive. It is straightforward to check that $\phi^{(n)}$ is also a $*$-homomorphism and hence positive. Therefore, any $*$-homormorphism is completely positive.

Example 7.3. Let $\phi : A \to B$ be a linear map between C^*-algebras. Given $v \in B$, the *compression* of ϕ by v is defined by $\psi(a) = v^*\phi(a)v$ for any $a \in A$. Let $\mathbf{v} \in M_n(B)$ be the diagonal matrix whose diagonal entries are v. Then for any $\mathbf{x} \in M_n(A)$, we have $\psi^{(n)}(\mathbf{x}) = \mathbf{v}^*\phi^{(n)}(\mathbf{x})\mathbf{v}$. Hence, if ϕ is n-positive, so is ψ. In particular, a compression of a $*$-homomorphism is completely positive.

Example 7.4. Let $\phi : A \to \mathbb{C}$ be a state. Let $(\pi, \mathcal{H}, \xi_\phi)$ be the GNS representation for ϕ. Then ϕ is the compression of π by the projection onto the one-dimensional subspace of $\mathcal{H}$ spanned by ξ_ϕ. Therefore, a state is completely positive.

Definition 7.5. Let A be a C^*-algebra and B a C^*-subalgebra. A linear map $E : A \to B$ is called a *conditional expectation* from A to B if

1. E is a completely positive contraction;
2. $E(b) = b$ for any $b \in B$;
3. E is a B-bimodule map: for any $a \in A$ and $b \in B$, we have $E(ba) = bE(a)$ and $E(ab) = E(a)b$.

Corollary II.6.10.3 of [8] reduces the task of showing that a linear map is a conditional expectation:

Proposition 7.6. *Let A be a C^*-algebra and B a C^*-subalgebra. If $E : A \to B$ is an idempotent, positive, B-bimodule map, then E is a conditional expectation.*

Example 7.7. Suppose that $A = M_n \otimes M_k$ and $B = M_k \cong I_n \otimes M_k$ with the identification $t = I_n \otimes t$. Let ϕ be a state on M_n. Define $R_\phi : A \to B$ on the simple tensors by

$$R_\phi(s \otimes t) = \phi(s)t.$$

This is a conditional expectation from A to B.

Indeed, we have

$$R_\phi^2(s \otimes t) = R_\phi(\phi(s)(I_n \otimes t)) = \phi(s)t = R_\phi(s \otimes t)$$

and

$$R_\phi((I_n \otimes t')(s \otimes t)) = R_\phi(s \otimes t't) = \phi(s)t't = (I_n \otimes t')R_\phi(s \otimes t).$$

Thus, R_ϕ is idempotent and B-bimodule. It remains to see that R_ϕ is positive.

Let $a = \sum_{i=1}^m s_i \otimes t_i \in A$. Then we have

$$R_\phi(a^*a) = R_\phi\left(\left(\sum_{i=1}^m s_i \otimes t_i\right)^* \left(\sum_{j=1}^m s_j \otimes t_j\right)\right) = R_\phi\left(\sum_{i=1}^m \sum_{j=1}^m s_i^* s_j \otimes t_i^* t_j\right) = \sum_{i=1}^m \sum_{j=1}^m \phi(s_i^* s_j) t_i^* t_j.$$

Set $\mathbf{s} \in M_m(M_n)$ by

$$\mathbf{s} = \begin{pmatrix} s_1 & s_2 & \cdots & s_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\Phi := [\phi(s_i^* s_j)] = \phi^{(m)}(\mathbf{s}^*\mathbf{s}) \in M_m(\mathbb{C})$ is positive since ϕ is completely positive. Suppose that $\Phi^{\frac{1}{2}} = [\alpha_{ij}]$. We set a column vector

$$c_j = \begin{pmatrix} \alpha_{1j} t_j \\ \alpha_{2j} t_j \\ \vdots \\ \alpha_{mj} t_j \end{pmatrix}.$$

Then

$$c_i^* c_j = \sum_{r=1}^m \overline{\alpha_{ri}} t_i^* \alpha_{rj} t_j = \left(\sum_{r=1}^m (\Phi^{\frac{1}{2}})^*_{ir} (\Phi^{\frac{1}{2}})_{rj}\right) t_i^* t_j = \Phi_{ij} t_i^* t_j = \phi(s_i^* s_j) t_i^* t_j$$

and hence, $R_\phi(a^*a) = \left(\sum_{i=1}^m c_i\right)^* \left(\sum_{j=1}^m c_j\right)$ is positive.

Following Renault [32], we can define a conditional expectation from an AF algebra A to a finite-dimensional subalgebra using a transition probability on a Bratteli diagram associated to A. Let $A = \overline{\bigcup A_n}$ be a unital AF algebra with $A_0 = \mathbb{C}\mathbf{1}$. Let $\mathcal{B} = (V, E)$ be the Bratteli diagram corresponding to A with respect to the given embeddings of the A_n's.

Definition 7.8. ([32], Definition 2.1) Let $n \leq k$. A *transition probability* from V_n to V_k is a function $p_n^k : \Omega[n,k] \to \mathbb{R}_0^+$ such that for all $v \in V_n$, $\sum_{\substack{c\in\Omega[n,k];\\ s(c)=v}} p_n^k(c) = 1$.

With the notation from the previous chapter, Proposition 2.2 of [32] reads:

Proposition 7.9. *Let p_n^k be a transition probability from V_n to V_k. The map $P_n^k : A_k \to A_n$ defined by: for each $g \in A_k \cong C^*(R_k)$ and $(x,y) \in R_n$ with the common range vertex $v(x,y)$,*

$$P_n^k(g)(x,y) = \sum_{\substack{c\in\Omega[n,k];\\ s(c)=v(x,y)}} p_n^k(c) g(xc, yc),$$

is a conditional expectation from A_k to A_n.

Proof. To reduce the use of superscripts and subscripts in the following computations, we set $\Omega = \Omega[0,n]$, $R = R_n$, $S = \Omega[n,k]$, $\underline{\Omega} = \Omega[0,k] = \Omega \times S$, $\underline{R} = R_k$, $p = p_n^k$, and $P = P_n^k$.

<u>P is idempotent.</u> This follows from the fact that P is the identity on A_n. Let $g \in A_n$. Using the embedding $\varphi := \varphi_{n,k} : C^*(R) \hookrightarrow C^*(\underline{R})$, we have: for any $(x,y) \in R$ with the common range vertex $v(x,y)$,

$$P(\varphi(g))(x,y) = \sum_{\substack{c\in S;\\ s(c)=v(x,y)}} p(c)\varphi(g)(xc,yc) = \left(\sum_{\substack{c\in S;\\ s(c)=v(x,y)}} p(c) \right) g(x,y) = g(x,y).$$

<u>P is positive.</u> Let $g \in A_k$. For any $c \in S$ and $\eta \in \underline{\Omega}$ such that $r(c) = r(\eta)$, we define $g_{\eta,c} \in A_n$ by: for any $(x,y) \in R$ with the common range vertex $v(x,y)$,

$$g_{\eta,c}(x,y) = [\![s(c) = v(x,y)]\!] \left(\frac{p(c)}{\#\,\mathrm{Path}(v_0, v(x,y))} \right)^{\frac{1}{2}} g(\eta, yc).$$

Then we have

$$\begin{aligned}(g^*_{\eta,c}g_{\eta,c})(x,y) &= \sum_{\substack{\gamma\in\Omega;\\ r(\gamma)=v(x,y)}} g^*_{\eta,c}(x,\gamma)g_{\eta,c}(\gamma,y)\\ &= \sum_{\substack{\gamma\in\Omega;\\ r(\gamma)=v(x,y)}} \overline{g_{\eta,c}(\gamma,x)}g_{\eta,c}(\gamma,y)\\ &= [\![s(c)=v(x,y)]\!] \sum_{\substack{\gamma\in\Omega;\\ r(\gamma)=v(x,y)}} \frac{p(c)}{\#\,\mathrm{Path}(v_0,v(x,y))}\overline{g(\eta,xc)}g(\eta,yc)\\ &= [\![s(c)=v(x,y)]\!]p(c)\overline{g(\eta,xc)}g(\eta,yc).\end{aligned}$$

Now we compute: for any $(x,y)\in\underline{R}$ with the common range vertex $v(x,y)$,

$$\begin{aligned}P(g^*g)(x,y) &= \sum_{\substack{c\in S;\\ s(c)=v(x,y)}} p(c)(g^*g)(xc,yc)\\ &= \sum_{\substack{c\in S;\\ s(c)=v(x,y)}} p(c)\sum_{\substack{\eta\in\underline{X};\\ r(\eta)=r(c)}} \overline{g(\eta,xc)}g(\eta,yc)\\ &= \sum_{c\in S}\sum_{\substack{\eta\in\underline{\Omega};\\ r(\eta)=r(c)}} [\![s(c)=v(x,y)]\!]p(c)\overline{g(\eta,xc)}g(\eta,yc)\\ &= \sum_{c\in S}\sum_{\substack{\eta\in\underline{\Omega};\\ r(\eta)=r(c)}} (g^*_{\eta,c}g_{\eta,c})(x,y).\end{aligned}$$

Therefore, $P(g^*g)=\sum_{c\in S}\sum_{\substack{\eta\in\underline{\Omega};\\ r(\eta)=r(c)}} g^*_{\eta,c}g_{\eta,c}$ is positive, being a sum of positive elements. $\square$

Note that here our transition probability is allowed to take value zero and hence, the conditional expectation above may not be faithful.

Definition 7.10. ([32], Definition 3.2) A *transition probability* on the Bratteli diagram $\mathcal{B}$ is a map $p:E\to\mathbb{R}^+_0$ such that for each n the restriction p_n of p to $E_n=\Omega[n,n+1]$ is a transition probability from V_n to V_{n+1} as in Definition 7.8. For each $v\in V$, we denote the set of edges emanating from v by $E_v:=s^{-1}\{v\}$. Then we see that p assigns for each $v\in V$ a probability measure on E_v.

Proposition 7.11. *Let p be a transition probability on $\mathcal{B}$. Then for $k>n$, the map $p^k_n:\Omega[n,k]\to\mathbb{R}^+_0$ defined by: for each $e=e_ne_{n+1}\dots e_{k-1}\in\Omega[n,k]$,*

$$p^k_n(e)=\prod_{i=n}^{k-1}p_i(e_i),$$

is a transition probability from V_k to V_n.

Proof. We fix n and argue by induction on k. By definition, $p_n^{n+1} = p_n$ is a transition probability from V_n to V_{n+1}. Now assume that p_n^k is a transition probability from V_n to V_k. Let $v \in V_n$. Using the fact that p assigns a probability measure on the set of edges emanating from each point of V, we have

$$\sum_{\substack{c\in\Omega[n,k+1];\\ s(c)=v}} p_n^{k+1}(c) = \sum_{w\in V_k} \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v;\\ r(d)=w}} \sum_{e\in E_w} p_n^k(d)p_k(e) = \sum_{w\in V_k} \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v;\\ r(d)=w}} p_n^k(d) = \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v}} p_n^k(d) = 1.$$

□

Before moving on to the next proposition, we recall that as in Chapter 5, we use the double square brackets to denote the boolean value of the enclosed statement.

Proposition 7.12. *Let p be a transition probability on $\mathcal{B}$ and fix $n \in \mathbb{N}$. Let $p_n^k : \Omega[n,k] \to \mathbb{R}_0^+$ be as in Proposition 7.11. Let $P_n^k : A_k \to A_n$ be the conditional expectation corresponding to p_n^k as in Proposition 7.9. Then recalling that A_k is a subalgebra of A_{k+1}, we have $P_n^{k+1}\big|_{A_k} = P_n^k$ and hence, the P_n^k's extend continuously to a conditional expectation $P_n : A \to A_n$.*

Proof. Let $g \in A_k \cong C^*(R_k)$ and $(x,y) \in R_n$. Recall that the inclusion $j : A_k \to A_{k+1}$ is given by: for each $x, y \in \Omega[0,k]$ and $a, b \in E_k = \Omega[k,k+1]$,

$$j(g)(xa, yb) = [\![a = b]\!] g(x,y).$$

Then for any $(x,y) \in R_n$ such that $r(x) = r(y) = v \in V_n$,

$$\begin{aligned}
P_n^{k+1}(j(g))(x,y) &= \sum_{\substack{c\in\Omega[n,k+1];\\ s(c)=v}} p_n^{k+1}(c) j(g)(xc, yc) \\
&= \sum_{w\in V_k} \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v;\\ r(d)=w}} \sum_{e\in E_w} p_n^{k+1}(de) g(xde, yde) \\
&= \sum_{w\in V_k} \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v;\\ r(d)=w}} \sum_{e\in E_w} p_n^k(d) p_k(e) g(xd, yd) \\
&= \sum_{w\in V_k} \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v;\\ r(d)=w}} p_n^k(d) g(xd, yd) \\
&= \sum_{\substack{d\in\Omega[n,k];\\ s(d)=v}} p_n^k(d) g(xd, yd) \\
&= P_n^k(g)(x,y).
\end{aligned}$$

Therefore, $P_n^{k+1}\big|_{A_k} = P_n^k$ as desired.

Now the map $P_n : \bigcup_{k\geq n} A_k \to A_n$ given by

$$P_n(a) = P_n^k(a) \text{ if } a \in A_k,$$

is well-defined. Since $\{P_n^k : k \geq n\}$ is uniformly bounded, we extend the above map continuously to $P_n : A = \overline{\bigcup_{k\geq n} A_k} \to A_n$. It can be checked that P_n is idempotent, positive, and A_n-linear because each P_n^k has these properties. Then by Proposition 7.6, the map $P_n : A \to A_n$ is a conditional expectation. □

In particular, if we view a Cantorian tree T as a Bratteli diagram for $A = C(\partial T)$, a transition probability p on T determines a state $P_0 : A \to \mathbb{C}$. Conversely, by the Riesz-Markov Theorem, a state ψ on A corresponds to a probability measure μ on ∂T such that $\psi(\chi_{[v]}) = \mu([v])$. For each edge e, we set $p(e) := \frac{\mu([r(e)])}{\mu([s(e)])}$ if $\mu([s(e)]) \neq 0$ and $p(e) = 0$ otherwise. Since each $[v]$ is a disjoint union of the cylinder subsets for its children, it is easy to check that p is indeed a transition probability on T. For each vertex $v \in V_n$, we identify v with the unique path from the root to v: $e_0e_1 \dots e_{n-1}$ where e_i is the edge (v_i, v_{i+i}) and $v_n = v$. Extending p to vertices by $p(v) = p(e_0)p(e_1)\cdots p(e_{n-1})$, we have:

Proposition 7.13. *Let T be a Cantorian tree and $A = C(\partial T)$. For each state $\psi : A \to \mathbb{C}$, there is a transition probability p on T such that ψ corresponds to the measure μ on the infinite path space ∂T arising from p as follows:*

$$\psi(\chi_{[v]}) = \mu([v]) = p(v),$$

where the unique path from the root to v is $e_0e_1 \dots e_{n-1}$, and $p(v) = p(e_0)p(e_1)\cdots p(e_{n-1})$.

The measure μ is an example of a *Markov measure*. For a more detailed treatment of Markov measures on the infinite path space of a Bratteli diagram, we refer interested readers to papers by Renault: [31] and [32].

For example, consider $X = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$ and $d((x_n),(y_n)) = 2^{-\min\{k\in\mathbb{N}|x_k\neq y_k\}}$. The underlying tree is the binary tree. If $\theta : A \to \mathbb{C}$ is defined by integration against the normalized Haar measure on X, the corresponding transition probability is given by assigning $\frac{1}{2}$ to each edge of the tree.

Chapter 8

Generalized Aguilar Seminorms

We now start working towards generalizing the Pearson-Bellissard construction to unital AF algebras.

Notation: For an AF algebra A with an increasing sequence of finite-dimensional C^*-subalgebras $\{A_n\}_{n\in\mathbb{N}}$ such that $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$, we denote the norm-dense subalgebra $\bigcup_{n\in\mathbb{N}} A_n$ by A_F.

In Proposition 4.4 of [1], Aguilar shows that:

Proposition 8.1. *If $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ is a unital AF algebra equipped with a Leibniz Lip-norm L defined on A_F, then each (A_n, L) is a Leibniz quantum compact metric space and the sequence $\{(A_n, L)\}_{n\in\mathbb{N}}$ converges to (A, L) in the quantum Gromov-Hausdorff propinquity.*

Remark: In fact, Aguilar proves this for the more general setting of (C, D)-quasi-Leibniz Lip-norms (Definition 2.3 of [1]; Definition 2.5 of [25]).

In Theorem 4.5 of the same paper, Aguilar uses quotient norms to provide a Leibniz Lip-norm on a unital AF algebra so that we can apply the previous proposition.

Theorem 8.2 ([1], Theorem 4.5)**.** *Let A be a unital AF algebra and let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite-dimensional C^*-subalgebras such that $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$, with $A_0 = \mathbb{C}\mathbf{1}$. For each $n \in \mathbb{N}$, we denote the quotient norm of A/A_n with respect to $\|\cdot\|_A$ by L_n, also viewed as a seminorm on A. Let $\gamma : \mathbb{N} \to (0, \infty)$ have limit 0 at infinity.*

For all $a \in A_F^{sa}$, we define:

$$L^\gamma(a) = \sup\left\{\frac{L_n(a)}{\gamma(n)} : n \in \mathbb{N}\right\}.$$

Then

1. *(A, L^γ) and (A_n, L^γ) for all $n \in \mathbb{N}$ are Leibniz quantum compact metric spaces and*
2. *$\lim_{n\to\infty} \Lambda((A_n, L^\gamma), (A, L^\gamma)) = 0$, where Λ is the quantum Gromov-Hausdorff propinquity.*

We will call L^γ defined above an Aguilar seminorm.

The fact that L_n's are Leibniz (and hence, so is L^γ) is due to Rieffel's work on best approximations [41], in which he expresses a quotient norm as a seminorm induced by a commutator. So this is a good starting point for constructing a "spectral" triple for a unital AF algebra.

Suppose that $T = (V, E)$ is a reduced rooted Cantorian tree. Then $X = \partial T$ is a Cantor set. Consider $A = C(X)$ and $A_n = C(V_n)$. My calculation of Aguilar seminorms for this setting showed that the associated Monge-Kantorovich metrics correspond to the weights on T which are constant on each level of the tree. (This can be deduced from Proposition 9.2.) So we will first modify the Aguilar seminorms to encode any ultrametric on X. Note however that any ultrametric could also be encoded by an Aguilar seminorm if T is unreduced.

We make some general observations. Let A be a C^*-algebra and B a C^*-subalgebra. Let $\beta \in B^+ \cap Z(A)$ and let L be a seminorm on A. For all $a \in A$, we define

$$L^\beta(a) = L(\beta a).$$

Proposition 8.3. *If L is the quotient seminorm with respect to B, then L^β is Leibniz on A.*

Proof. This follows from applying Theorem 3.2 of [41] to $L(\beta a)$. Given a non-degenerate $*$-representation $(\mathcal{H}, \pi)$ of A and a self-adjoint unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $[U, \pi(b)] = 0$ for all $b \in B$, we define $L_{(\mathcal{H},\pi,U)}(a) = \frac{1}{2}\|[U, \pi(a)]\|$ for any $a \in A$. Since $\beta \in Z(A)$ and $[U, \pi(\beta)] = 0$,

$$[U, \pi(\beta a)] = U\pi(\beta a) - \pi(\beta a)U = U\pi(\beta)\pi(a) - \pi(a)U\pi(\beta) = [U\pi(\beta), \pi(a)].$$

Hence, $L^\beta_{(\mathcal{H},\pi,U)}(a) := L_{(\mathcal{H},\pi,U)}(\beta a) = \frac{1}{2}\|[U\pi(\beta), \pi(a)]\|$ is Leibniz and so is L^β, being the supremum of $L^\beta_{(\mathcal{H},\pi,U)}$ over all $*$-representations and self-adjoint unitaries as above. □

Proposition 8.4. *Let A be a unital C^*-algebra and B a unital C^*-subalgebra. If $\beta \in B^+$ is invertible and L is a Leibniz seminorm on A such that $L(b) = 0$ for all $b \in B$, then $L(a) \le \|\beta^{-1}\| L^\beta(a)$, for all $a \in A$.*

Proof.

$$L(a) = L(\beta^{-1}\beta a) \le L(\beta^{-1})\|\beta a\| + \|\beta^{-1}\| L(\beta a) = \|\beta^{-1}\| L^\beta(a).$$

□

Now, let $A = \overline{\bigcup A_n}$ be an AF algebra with an increasing sequence of unital finite dimensional C*-subalgebras $(A_n)_{n\in\mathbb{N}}$. Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be such that each $\beta_n \in A_n^+ \cap Z(A)$ is invertible and $\|\beta_n^{-1}\| \to 0$. For all $a \in A_F^{\mathrm{sa}}$, define

$$L^\beta(a) = \sup_{n\in\mathbb{N}} L_n^{\beta_n}(a), \tag{8.1}$$

where L_n is the quotient seminorm on A with respect to A_n. Since the L_n's are Leibniz by Proposition 8.3, so is L^β. The next proposition follows immediately from Proposition 8.4.

Proposition 8.5. *Let $\gamma = (\gamma_n)_{n\in\mathbb{N}_0}$ be the sequence given by $\gamma_n = \|\beta_n^{-1}\|$ and let L^γ be the corresponding Aguilar seminorm. Then for all $a \in A_F^{sa}$, $L^\gamma(a) \leq L^\beta(a)$.*

For $\# = \gamma$ or β, we set

$$\mathcal{L}_1^{\#} = \left\{a \in A_F : L^{\#}(a) \leq 1\right\}.$$

Then by Proposition 8.5, $\mathcal{L}_1^\beta \subseteq \mathcal{L}_1^\gamma$. Since $\mathcal{L}^\gamma$ is a Leibniz Lip-norm by the work of Aguilar, the image of $\mathcal{L}_1^\gamma$ (in $A_F^{\mathrm{sa}}/\mathbb{R}\mathbf{1}$) is totally bounded with respect to the quotient norm by Rieffel's characterization of Lip-norm (Theorem 2.10) and so the image of $\mathcal{L}_1^\beta$ is also totally bounded. Consequently, equipped with L^β, A and A_n are also Leibniz quantum compact metric spaces, and by Proposition 4.4 of [1], we have (A_n, L^β) converges to (A, L^β) in the quantum propinquity. So we have collected the same convergence result for the generalized Aguilar seminorm:

Theorem 8.6.

1. *(A, L^β) and (A_n, L^β) for all $n \in \mathbb{N}$ are Leibniz quantum compact metric spaces.*
2. *$\lim_{n\to\infty} \Lambda\left((A_n, L^\beta), (A, L^\beta)\right) = 0$.*

As discussed in Chapter 2, L^β carries metric data via the Monge-Kantorovich metric on the state space of A: for any $\varphi, \psi \in S(A)$,

$$\mathrm{mk}_{L^\beta}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : L^\beta(a) \leq 1\}.$$

Chapter 9

Monge-Kantorovich Metric for Commutative AF Algebras

Let us compute the Monge-Kantorovich metric arising from the generalized Aguilar seminorm for commutative AF algebras. Consider a (possibly unreduced) rooted Cantorian tree $T = (V, E)$. Let $X = \partial T$, $A = C(X)$, and $A_n = C(V_n)$, where V_n is the set of n-level vertices of T. For each $n \in \mathbb{N}$, the canonical embedding $A_n \hookrightarrow A$ is given by: for any $f \in A_n = C(V_n)$,

$$f \mapsto \sum_{v \in V_n} f(v)\chi_v \in A = C(X).$$

Here we also view v as the cylinder subset $[v]$ corresponding to the finite word starting from the root to v.

We will try to relate Aguilar seminorms to the ultrametrics on the Cantor set. For this purpose we consider $\beta = (\beta_n)_{n \in \mathbb{N}_0}$ arising from a non-strict weight on T, $\epsilon : V \to \mathbb{R}^+$ satisfying:

1. If $v \succ v'$, then $\epsilon(v) \geq \epsilon(v')$.

2. For an infinite path $v_0 v_1 \cdots \in \partial T = X$, $\lim_{n\to\infty} \epsilon(v) = 0$.

That is to say, ϵ is almost a weight on T in the sense of Definition 4.14 except we do not require here strict inequality in Property 1.

We define $\beta_n \in A_n^+ \cap Z(A)$ by $\beta_n(v) = \epsilon(v)^{-1}$, for all $v \in V_n$. Then $\|\beta_n^{-1}\| \to 0$ by the properties of the weight function. Indeed, as an element in $C(X)$, $\beta_n^{-1} = \sum_{v \in V_n} \epsilon(v)\chi_v$. Property 1 and 2 above imply that β_n^{-1} is a monotonically decreasing sequence of functions converging pointwise to 0.

In order to deal with L^β, we compute L_n for real-valued functions.

Proposition 9.1. *For a real-valued function $f \in A = C(X)$, let $g_v = \frac{1}{2}\left(\min f\big|_v + \max f\big|_v\right)$ and $g = \sum_{v \in V_n} g_v \chi_v$. Then g is a best approximation of f in A_n and*

$$L_n(f) = \|f - g\|.$$

Proof. Let $f \in A$ and $h \in A_n$. Since f is self-adjoint, we can look for a best approximation in A_n among the self-adjoint elements. So we may assume that $h = \sum_{v \in V_n} h_v \chi_v$ where $h_v \in \mathbb{R}$.

$$\|f - h\| = \sup\{|f(x) - h(x)| : x \in X\} = \sup\{\|f\big|_v - h_v\| : v \in V_n\}.$$

We restrict ourself to a cylinder subset v and find a best approximation of $f\big|_v$ by $\mathbb{R}$. It is easy to see that $\|f\big|_v - g_v\| \le \|f\big|_v - \alpha\|$ for all $\alpha \in \mathbb{R}$.

Therefore, $\|f - g\| \le \|f - h\|$ and the conclusion in the proposition follows. □

By Proposition 9.1, for each $u \in V_k$ with $k > n$, the best approximation of χ_u in A_n is $\frac{1}{2}\chi_{u_n}$, where u_n is the ancestor of u in the level n. Then it follows that

$$L_n^{\beta_n}(\chi_u) = L_n(\beta_n \chi_u) = L_n(\epsilon(u_n)^{-1}\chi_u) = \frac{1}{\epsilon(u_n)} \left\| \chi_u - \frac{1}{2}\chi_{u_n} \right\| = \frac{1}{2\epsilon(u_n)}.$$

We can now compute $\mathrm{mk}_{L^\beta}(x, y)$, where x, y are identified with the point-measures δ_x, δ_y.

Proposition 9.2. *Let $T = (V, E)$ be a rooted Cantorian tree and $\epsilon : V \to \mathbb{R}^+$ a non-strict weight on T. Let $X = \partial T$ and $A = C(X)$. For each $n \in \mathbb{N}$, let $A_n = C(V_n)$ and $\beta_n(v) = \epsilon(v)^{-1}$. Let L^β be the generalized Aguilar seminorm as in Equation* (8.1)*. Then for any $x, y \in X$ such that $x \neq y$, we have*

$$\mathrm{mk}_{L^\beta}(x, y) = 2\epsilon(x \wedge y).$$

Proof. First, let $u \in V_k \subset V$. For all $m \ge k$, $L_m^{\beta_m}(\chi_u) = 0$ because $\beta_m \chi_u \in A_m$. Since ϵ is decreasing, we have

$$L^\beta(\chi_u) = \sup\left\{L_n^{\beta_n}(\chi_u) : n \le k-1\right\} = \sup\left\{\frac{1}{2\epsilon(u_n)} : n \le k-1\right\} = \frac{1}{2\epsilon(u_{k-1})}. \tag{9.1}$$

Thus, $L^\beta(2\epsilon(u_{k-1})\chi_u) = 1$ for all $u \in V_k$.

Let $x, y \in X$. Let $u = x \wedge y$, say it belongs to V_N. Then x, y must belong to different children of u. In other words, there is $u' \in V_{N+1}$ such that $x \in u' \subset u$ but $y \notin u'$. Take $a = 2\epsilon(u'_N)\chi_{u'}$. Then $L^\beta(a) = 1$ and $|a(x) - a(y)| = 2\epsilon(u'_N)$. Thus, $\mathrm{mk}_{L^\beta}(x, y) \ge 2\epsilon(u'_N) = 2\epsilon(u)$ since u is the ancestor of u' in the level-N.

On the other hand, let $a \in \mathrm{dom}(L^\beta)$ and $L^\beta(a) \le 1$. Note that for all $b \in A_N$, b is constant on $u \in V_N$, which contains both x and y. Consequently,

$$\begin{aligned}
|(\beta_N a)(x) - (\beta_N a)(y)| &= |(\beta_N a)(x) - b(x) + b(y) - (\beta_N a)(y)| \\
&= |(\beta_N a)(x) - b(x)| + |b(y) - (\beta_N a)(y)| \\
&\le 2\|\beta_N a - b\|. \\
|(\beta_N a)(x) - (\beta_N a)(y)| &\le 2\inf\{\|\beta_N a - b\| : b \in A_N\} = 2L_N^{\beta_N}(a) \le 2L^\beta(a) \le 2.
\end{aligned}$$

Since $\beta_N(x) = \beta_N(y) = \epsilon(u)^{-1}$, $|a(x) - a(y)| \le 2\epsilon(u)$. Then $\mathrm{mk}_{L^\beta}(x, y) \le 2\epsilon(u)$. Therefore, $\mathrm{mk}_{L^\beta}(x, y) = 2\epsilon(u) = 2\epsilon(x \wedge y)$. □

Corollary 9.3. *Let* (X, d) *be a regular ultrametric Cantor set with the corresponding Michon tree* (T, ϵ). *Let* $A = C(X)$, A_n, $\beta = (\beta_n)_{n\in\mathbb{N}}$ *where each* β_n *is given by* $\beta_n(v) = \epsilon(v)^{-1}$, *and* L^β *be as in Proposition 9.2. Then*

$$\mathrm{mk}_{L^\beta}(x, y) = 2d(x, y).$$

Proof. Since ϵ is a (strict) weight on T, the previous proposition applies. By the Michon correspondence, we have $d(x, y) = \epsilon(x \wedge y)$. □

We specialize to the original Aguilar seminorm as follows:

Proposition 9.4. *Let* T *be a rooted Cantorian tree,* $X = \partial T$, $A = C(X)$ *and* $A_n = C(V_n)$. *Let* $\gamma : \mathbb{N} \to \mathbb{R}^+$ *have limit 0 at infinity. Then for any* $x, y \in X$ *such that* $x \neq y$ *and* $x \wedge y \in V_N$

$$\mathrm{mk}_{L^\beta}(x, y) = 2\tilde{\gamma}(N),$$

where $\tilde{\gamma}(k) = \min\{\gamma(n) : n \leq k\}$. *In particular, if* γ *is decreasing, then* $\mathrm{mk}_{L^\beta}(x, y) = \gamma(N)$.

Proof. Taking $\beta_n = \gamma_n^{-1}$ as a constant function on A_n, we have that L^β is the Aguilar seminorm L^γ. We replace Equation (9.1) in the proof of Proposition 9.2 by

$$L^\gamma(\chi_u) = \sup\left\{\frac{1}{2\gamma(n)} : n \leq k - 1\right\} = \frac{1}{2\tilde{\gamma}(k-1)},$$

where $u = x \wedge y \in V_k$ and the rest of the proof of Proposition 9.2 yields this proposition. □

Chapter 10

Dirac Triples for AF Algebras

In this chapter, we shall construct a triple satisfying all properties of a spectral triple in Definition 2.1 except the compact resolvent property of the operator D. We will call it a *Dirac triple*. We will refer to the operator D as the *Dirac operator* for the triple.

In his doctoral thesis [29], Palmer generalizes Pearson and Bellissard's work to define spectral triples for the space of continuous functions on a compact metric space using a resolving sequence of finite open coverings. As pointed out to me by my advisor, Marc Rieffel, it would be helpful to think about completely positive approximation, which is a well-known noncommutative analogue of coverings [45].

Definition 10.1. Let A be a C^*-algebra. A *completely positive approximation* of A is a sequence of triples $\{(F_n, \psi_n, \varphi_n)\}_{n\in\mathbb{N}}$ where F_n is a finite-dimensional C^*-algebra, and $\psi_n : A \to F_n$ and $\varphi_n : F_n \to A$ are completely positive contractions such that $\varphi_n \circ \psi_n \to \mathrm{id}_A$ pointwise.

We observe that, for a unital AF algebra $A = \overline{\bigcup A_n}$, a sequence of conditional expectations $E_n : A \to A_n$ provides a completely positive approximation.

Proposition 10.2. *Let $A = \overline{\bigcup A_n}$ be a unital AF algebra where $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite-dimensional unital C^*-subalgebras with embeddings $\varphi_n : A_n \to A$. If each $E_n : A \to A_n$ is a conditional expectation, then $\{(A_n, E_n, \varphi_n)\}_{n\in\mathbb{N}}$ is a completely positive approximation of A.*

Proof. Let $a \in A$. Let $a_n \in A_n$ be such that $a_n \to a$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $\|a_n - a\| \leq \frac{\varepsilon}{2}$ for all $n \geq N$. Then for all $n \geq N$, we have $a_N \in A_N \subset A_n$. Since E_n is a conditional expectation, $(\varphi_n \circ E_n)(a_N) = \varphi_n(a_N) = a_N$. Therefore,

$$\begin{aligned}\|(\varphi_n \circ E_n)(a) - a\| &= \|(\varphi_n \circ E_n)(a) - (\varphi_n \circ E_n)(a_N) + (\varphi_n \circ E_n)(a_N) - a_N + a_N - a\| \\ &\leq \|(\varphi_n \circ E_n)(a) - (\varphi_n \circ E_n)(a_N)\| + \|a_N - a\| \leq 2\|a_N - a\| < \varepsilon.\end{aligned}$$

□

While we will not use the above proposition in later proofs, it leads us to observe that there are indeed underlying sequences of conditional expectations related to the Pearson-Bellissard construction. Suppose (X, d) is an ultrametric Cantor set. Let V_n be the set of n-level vertices of the corresponding Michon tree. Let $A = C(X), A_n = C(V_n)$, and τ a choice function. For each $f \in A$ and $v \in V_n$, we define

$$E_n^{\pm}(f)(v) = f(\tau^{\pm}(v)).$$

Then $E_n^{\pm} : A \to A_n$ are conditional expectations. Also, if we fix a faithful state $\theta : A \to \mathbb{C}$ and set $\theta_n^{\pm} := \theta \circ E_n^{\pm}$, then $\theta_n = \frac{1}{2}(\theta_n^+ - \theta_n^-) \in A_n^{\perp}$ and the techniques used in the proof of Lemma 3.3 and Theorem 3.2 of [41] allow us to construct a spectral triple for A which is unitarily equivalent to a Pearson-Bellissard spectral triple with respect to the choice function τ. We will return to this case in the next chapter after presenting the construction of a Dirac triple for general unital AF algebras.

Let us start with a general set-up. Let A be a C^*-algebra and let $\tau = \{(\psi_n^+, \psi_n^-)\}_{n\in\mathbb{N}}$ be a sequence of pairs of states on A. For each $n \in \mathbb{N}$, we let $(\mathcal{H}_n^{\pm}, \pi_n^{\pm}, \xi_n^{\pm})$ be the GNS representations of $\psi_n^{\pm}$. Set

$$\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^- \quad \text{and} \quad \pi_n = \pi_n^+ \oplus \pi_n^-,$$

and

$$\xi_n = \frac{1}{\sqrt{2}}(\xi_n^+ \oplus \xi_n^-) \quad \text{and } \eta_n = \frac{1}{\sqrt{2}}(\xi_n^+ \oplus -\xi_n^-).$$

Then we let $\mathcal{H}_\tau = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ and $\pi_\tau = \bigoplus_{n\in\mathbb{N}} \pi_n$.

Now assume further that A is a unital AF algebra. Say, $A = \overline{\bigcup A_n}$ where $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite-dimensional unital C^*-subalgebras. For each $n \in \mathbb{N}$, we let P_n be the orthogonal projection onto the subspace $\pi_n(A_n)\xi_n \subseteq \mathcal{H}_n$. Since $\pi_n(A_n)\xi_n$ is A_n-invariant, $[P_n, \pi_n(A_n)] = 0$. Let $\beta_n \in A_n^+ \cap Z(A)$ be such that $\|\beta_n^{-1}\| \to 0$. Then the self-adjoint unitary $U_n = 2P_n - I_{\mathcal{H}_n}$ commutes with $\pi_n(\beta_n)$ and we define a self-adjoint operator on $\mathcal{H}_n$:

$$D_n := (1/2)U_n\pi_n(\beta_n) = (1/2)\pi_n(\beta_n)U_n.$$

We set

$$D_\tau^\beta := \oplus D_n.$$

Remark: If for some n, $\psi_n^{\pm}$ are equal, then we show now that D_n commutes with $\pi_n(a)$ and thus $\|[D_n, \pi_n(a)]\|$ would not contribute to the seminorm $\|[D_\tau^\beta, \pi_\tau(a)]\|$. Going forward, we will require that ψ^+ and ψ^- are distinct.

Proposition 10.3. *If $\psi_n^+ = \psi_n^-$, then $[D_n, \pi_n(a)] = 0$ for all $a \in A$.*

Proof. Since we are considering a fixed n, we will omit the subscript if it is unnecessary. Suppose that $\psi_n^+ = \psi_n^-$. Let $(\pi^{\pm}, \mathcal{H}^{\pm}, \xi^{\pm})$ be (two copies of) the GNS representation of A with respect to $\psi_n^{\pm}$. Set $\xi = \frac{1}{\sqrt{2}}(\xi^+ \oplus \xi^-)$. Consider the direct sum $\pi = \pi^+ \oplus \pi^-$ and

$D = \frac{1}{2}U\pi(b)$, where $b \in A_n^+ \cap Z(A)$ and $U = 2P - I$ is the reflection about the closed subspace $\pi(A_n)\xi$. In this situation, $\pi(A_n)\xi = \{\zeta \oplus \zeta : \zeta \in \mathcal{H}\}$. Let $\zeta \oplus \rho \in \mathcal{H} := \mathcal{H}^+ \oplus \mathcal{H}^-$. Then $P(\zeta \oplus \rho) = \frac{1}{2}(\zeta + \rho) \oplus \frac{1}{2}(\zeta + \rho)$ and $U(\zeta \oplus \rho) = \rho \oplus \zeta$. Clearly, U commutes with π because π^+, π^- are the same representation. Thus, $[D, \pi(a)] = 0$ for all $a \in A$. □

Assuming $\psi_n^\pm$ are also faithful on A_n, we will show that $(\pi_\tau, \mathcal{H}_\tau, D_\tau^\beta)$ is a Dirac triple for A in a similar fashion as the proof of Theorem 5.4 by adapting the proof in [29] to our setting. We will drop the subscripts when there is no need to emphasize the choice of τ or β.

Theorem 10.4. *Let $A = \overline{\bigcup A_n}$ be a unital AF algebra with an increasing sequence of finite-dimensional unital C^*-subalgebras $\{A_n\}_{n\in\mathbb{N}}$. Suppose that $\tau = \{(\psi_n^+, \psi_n^-)\}_{n\in\mathbb{N}}$ is a sequence of pairs of distinct states on A such that $\psi_n^\pm$ are faithful on A_n. Let $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be such that $\beta_n \in A_n^+ \cap Z(A)$ and $\|\beta_n^{-1}\| \to 0$. Then the operator D_τ^β defined above is self-adjoint and, for any $a \in A_F$, $[D_\tau^\beta, \pi_\tau(a)]$ is a bounded operator such that $\|[D_\tau^\beta, \pi_\tau(a)]\| \leq L^\beta(a)$, where $L^\beta(a) = \sup_{n\in\mathbb{N}} L_n^{\beta_n}(a)$. Moreover, π_τ is faithful on A and consequently $(A, \mathcal{H}_\tau, D_\tau^\beta)$ is a Dirac triple.*

Proof. *D_τ^β is self-adjoint.* Consider $\mathrm{dom}(D_\tau^\beta) = \{\kappa \in \mathcal{H} : \sum_n \|D_n\kappa_n\|^2 < \infty\}$ which contains the subspace of finitely supported elements. Hence, D_τ^β is densely defined. Let $\kappa, \kappa' \in \mathrm{dom}(D_\tau^\beta)$. Then

$$\langle D_\tau^\beta\kappa, \kappa'\rangle = \sum_n \langle D_n\kappa_n, \kappa_n'\rangle = \sum_n \langle \kappa_n, D_n\kappa_n'\rangle = \langle \kappa, D_\tau^\beta\kappa'\rangle.$$

Hence, D_τ^β is a symmetric operator. Since $\mathcal{H}_n$ is separable and D_n are bounded self-adjoint operators, we have D_τ^β is self-adjoint by Example 9.26 of [16].

$[D_\tau^\beta, \pi_\tau(a)]$ is bounded for any $a \in A_F$.

Let $a \in A_F$. As in the proof of Theorem 3.2 of [41], we observe that, for any $a \in A$ and $b \in A_n$,

$$\|[U_n, \pi_n(a)]\| = \|[U_n, \pi_n(a - b)]\| \leq 2\|a - b\| \tag{10.1}$$

and hence, $\|[U_n, \pi_n(a)]\| \leq 2L_n(a)$.

Since $L^\beta(a) = \sup_{n\in\mathbb{N}} L_n^{\beta_n}(a) = \sup_{n\in\mathbb{N}} L_n(\beta_n a)$ is finite on A_F, we have for any $\omega \in$

$\mathrm{dom}(D_\tau^\beta)$,

$$\begin{aligned}
\|[D_\tau^\beta, \pi_\tau(a)]\omega\|^2 &= \sum_n \|([D_\tau^\beta, \pi_\tau(a)]\omega)_n\|^2 \\
&= \sum_n \|[D_n, \pi_n(a)]\omega_n\|^2 \\
&\leq \sum_n \|[D_n, \pi_n(a)]\|^2 \|\omega_n\|^2 \\
&= \sum_n \frac{1}{4} \|[U_n, \pi_n(\beta_n a)]\|^2 \|\omega_n\|^2 \\
&\leq \sum_n L_n(\beta_n a)^2 \|\omega_n\|^2 \leq L^\beta(a)^2 \|\omega\|^2
\end{aligned}$$

and thus, $\|[D_\tau^\beta, \pi_\tau(a)]\| \leq L^\beta(a) < \infty$.

Now, we address the faithfulness of π_τ. Since the $\psi_n^\pm$ are faithful on A_n, so are the $\pi_n^\pm$. Consequently, π_τ is faithful on A_F. Then by Proposition II.8.2.4 of [8], $\ker \pi_\tau \cap A_F$ is dense in $\ker \pi_\tau$ and thus, $\ker \pi_\tau = 0$. □

Since $\psi_n^\pm$ are states on an infinite-dimensional C^*-algebra, we do not, a priori, expect $\mathcal{H}_n^\pm$ to be finite-dimensional. In the next chapter, we will slightly reinterpret the Pearson-Bellissard construction based on the settings we have discussed in this chapter. In that case, each $\mathcal{H}_n^\pm$ will be finite-dimensional and we will indeed obtain spectral triples for ultrametric Cantor sets.

Proposition 10.5. *With the notation as above, $(A, \mathcal{H}_\tau, D_\tau^\beta)$ is a spectral triple if and only if each $\mathcal{H}_n^\pm$ is finite-dimensional.*

Proof. Suppose that each $\mathcal{H}_n^\pm$ is finite-dimensional. We will show that $(\mathbf{1}+D^2)^{-1}$ is compact, that is, D has a compact resolvent. Note that

$$((\mathbf{1} + D^2)^{-1}\omega)_n = (\mathbf{1}_n + D_n^2)^{-1}\omega_n.$$

Define a finite rank operator $T_m : \mathcal{H} \to \mathcal{H}$ by

$$T_m = \left(\bigoplus_{n=0}^{m} (\mathbf{1}_n + D_n^2)^{-1}\right) \oplus \left(\bigoplus_{j>m} \mathbf{0}_j\right),$$

where $\mathbf{0}_j$ is the zero operator on $\mathcal{H}_j$. Since $D_n^2 = \pi_n(\beta_n^2)U_n^2 = \pi_n(\beta_n^2)$,

$$\begin{aligned}\|((\mathbf{1}+D^2)^{-1} - T_m)\omega\|^2 &= \sum_n \|(\mathbf{1}_n + D_n^2)^{-1}\omega_n - T_m\omega_n\|^2 \\ &= \sum_{n>m} \|(\mathbf{1}_n + D_n^2)^{-1}\omega_n\|^2 \\ &\leq \sum_{n>m} \|D_n^{-2}\|\|\omega_n\|^2 \\ &\leq \sum_{n>m} \|\beta_n^{-1}\|^2\|\omega_n\|^2\end{aligned}$$

Since $\|\beta_n^{-1}\| \to 0$, it follows that $\|(\mathbf{1}+D^2)^{-1} - T_m\| \to 0$ as desired.

On the other hand, if D has compact resolvent, then the inverse of $\mathbf{1}+D^2$ is compact and so is the inverse of $\mathbf{1}_n + D_n^2$. Thus, $\mathbf{1}_n$ is compact and so $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ is finite-dimensional for each n. □

In the proof of Theorem 10.4 as well as Theorem 3.2 of [41], a key estimate is the Equation (10.1):

$$\|[U_n, \pi_n(a)]\| = \|[U_n, \pi_n(a-b)]\| \leq 2\|a-b\|.$$

However, as pointed out to me by Marc Rieffel, we actually have a similar estimate for P_n by the following lemma.

Lemma 10.6. *Let $\mathcal{H}$ be a Hilbert space and P an orthogonal projection. Then for any $A \in \mathcal{B}(\mathcal{H})$,*

$$\|[P, A]\| \leq \|A\|.$$

Proof. Let $\xi \in \mathcal{H}$. Then, by the Pythagorean Theorem,

$$\begin{aligned}\|\,(PA(I-P) - (I-P)AP)\,\xi\|^2 &= \|PA(I-P)\xi\|^2 + \|(I-P)AP\xi\|^2 \\ &\leq \|A\|^2\|(I-P)\xi\|^2 + \|A\|^2\|P\xi\|^2 \\ &= \|A\|^2\|\xi\|^2.\end{aligned}$$

Therefore, we have

$$\|[P, A]\| = \|PA(I-P) - (I-P)AP\| \leq \|A\|.$$

□

With the set-up as in Theorem 10.4, we have: for any $a \in A$ and $b \in A_n$,

$$\|[P_n, \pi_n(a)]\| = \|[P_n, \pi_n(a-b)]\| \leq \|a-b\|$$

and hence, $\|[P_n, \pi_n(a)]\| \leq L_n(a)$. Then we can also define a Dirac operator using the projections P_n instead of U_n and similarly obtain:

Theorem 10.7. *Suppose that $\tau = \{(\psi_n^+, \psi_n^-)\}_{n\in\mathbb{N}}$ is a sequence of pairs of distinct states on A such that $\psi_n^\pm$ are faithful on A_n. Let $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be such that $\beta_n \in A_n^+ \cap Z(A)$ and $\|\beta_n^{-1}\| \to 0$. Let $\tilde{D}_n = P_n\pi_n(\beta_n)$ and $\tilde{D_\tau^\beta} := \oplus\tilde{D}_n$. Then the operator $\tilde{D_\tau^\beta}$ is self-adjoint and, for any $a \in A_F$, $[\tilde{D_\tau^\beta}, \pi_\tau(a)]$ is a bounded operator such that $\|[\tilde{D_\tau^\beta}, \pi_\tau(a)]\| \le L^\beta(a)$, where $L^\beta(a) = \sup_{n\in\mathbb{N}} L_n^{\beta_n}(a)$. Moreover, π_τ is faithful on A and consequently $(A, \mathcal{H}_\tau, \tilde{D_\tau^\beta})$ is a Dirac triple.*

Nevertheless, the Dirac triple we defined using U_n allows us to make a direct connection to the Pearson-Bellissard construction. We shall see this in the next chapter.

Chapter 11

Spectral Triples for Ultrametric Cantor Sets

Throughout this chapter, we consider a commutative AF algebra $A = C(X)$, where (X, d) is an ultrametric Cantor set. Let (T, ϵ) be the Michon tree corresponding to (X, d). Let $A_n = C(V_n)$, where V_n is the set of n-level vertices of T. We will apply the construction in the previous chapter to A.

Definition 11.1. For each choice function $\tau : V \to X \times X$, the associated conditional expectations $E_n^{\pm} : A \to A_n$ are given by:

$$E_n^{\pm}(f)(v) = f(\tau^{\pm}(v)),$$

for any $n \in \mathbb{N}$, $f \in A$, and $v \in V_n$.

Let ψ be a faithful state on A and let $\tau : V \to X \times X$ be a choice function. Set $\psi_n^{\pm} = \psi \circ E_n^{\pm}$, where the $E_n^{\pm}$'s are the conditional expectations associated to τ. Then $\psi_n^{\pm}$ are states on A which restrict to faithful states on A_n. Suppose that p is the transition probability corresponding to ψ as in Proposition 7.13. Recall that the natural embedding from $A_n = C(V_n)$ into A is given by $f \mapsto \sum_{v \in V_n} f(v)\chi_v$, where χ_v is the characteristic function of the cylinder subset $[v]$. We thus have for any $f \in A$,

$$\begin{aligned}
\psi_n^{\pm}(f) &= \psi(E_n^{\pm}(f)) \\
&= \psi\left(\sum_{v \in V_n} f(\tau^{\pm}(v))\chi_v\right) \\
&= \sum_{v \in V_n} p(v) f(\tau^{\pm}(v)),
\end{aligned}$$

where $v \in V_n$ can be viewed as the unique finite path $v = v_1 v_2 \ldots v_n$.

Notation: If $\mathcal{K}$ is a Hilbert space, then we define $\ell_\psi(V_n,\mathcal{K})$ to be $\bigoplus_{v\in V_n}\mathcal{K}$ equipped with the inner product:

$$\langle \alpha,\beta\rangle_\psi = \sum_{v\in V_n} p(v)\langle \alpha(v),\beta(v)\rangle_{\mathcal{K}}.$$

As in the previous chapter, we let $(\pi_n^\pm,\mathcal{H}_n^\pm)$ be the GNS representations for the states $\psi_n^\pm$. So we have

$$\mathcal{H}_n^\pm = \ell_\psi(V_n,\mathbb{C}).$$

The representations are

$$(\pi_n^\pm(f)\alpha)(v) = f(\tau^\pm(v))\alpha(v)$$

where $\alpha\in\mathcal{H}_n^\pm$, with the cyclic vectors

$$\xi_n^\pm(v) = 1.$$

Hence,

$$\mathcal{H}_n = \mathcal{H}_n^+\oplus\mathcal{H}_n^- = \ell_\psi(V_n,\mathbb{C}^2)$$

and the direct sum $\pi_n=\pi_n^+\oplus\pi_n^-$ is described as follows: for any $\alpha\in\mathcal{H}_n$ and $v\in V_n$,

$$(\pi_n(f)\alpha)(v) = \begin{pmatrix} f(\tau^+(v)) & 0\\ 0 & f(\tau^-(v))\end{pmatrix}\alpha(v),$$

where $\alpha(v) = \begin{pmatrix}\alpha_n^+(v)\\ \alpha_n^-(v)\end{pmatrix}$.

Set

$$\xi_n = \frac{1}{\sqrt{2}}(\xi_n^+\oplus\xi_n^-) \quad \text{and } \eta_n = \frac{1}{\sqrt{2}}(\xi_n^+\oplus -\xi_n^-).$$

That is,

$$\xi_n(v) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and } \eta_n(v) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

For each $n\in\mathbb{N}$, we let P_n be the orthogonal projection onto the subspace $\pi_n(A_n)\xi_n\subseteq\mathcal{H}_n$. Since each $f\in A_n$ is constant on $v\in V_n$ and $\tau^\pm(v)\in v$, we have $\pi_n(A_n)\xi_n = \ell_\psi(V_n,\mathbb{C}\begin{pmatrix}1\\1\end{pmatrix})$. Also, we have $\eta_n\perp P_n\mathcal{H}_n$ because for any $a\in A_n$,

$$\langle\pi_n(a)\xi_n,\eta_n\rangle = \frac{1}{2}\left\langle\left(\pi_n^+(a)\xi_n^+\oplus\pi_n^-(a)\xi_n^-\right),\left(\xi_n^+\oplus-\xi_n^-\right)\right\rangle = \frac{1}{2}(\psi_n^+(a)-\psi_n^-(a)) = \frac{1}{2}(\psi(a)-\psi(a)) = 0.$$

For each $\alpha\in\mathcal{H}_n$ and $v\in V_n$,

$$(P_n\alpha)(v) = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\alpha(v).$$

Next, noting that A is commutative and thus $Z(A) = A$, we set $\beta_n = \sum_{v\in V_n} \epsilon(v)^{-1}\chi_{[v]} \in A_n^+ = A_n^+ \cap Z(A)$, where ϵ is the weight on the Michon tree of (X, d). By the properties of the weight, $\|\beta_n^{-1}\| \to 0$. Define a self-adjoint operator on $\mathcal{H}_n$ by:

$$D_n = \frac{1}{2} U_n \pi_n(\beta_n).$$

Because $\beta_n(\tau^{\pm}(v)) = \epsilon(v)^{-1} = \operatorname{diam}([v])^{-1}$ for each $v \in V_n$, we have

$$(D_n\alpha)(v) = \frac{1}{2}\operatorname{diam}([v])^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\alpha(v).$$

As discussed in Proposition 10.5, since each $\mathcal{H}_n^{\pm}$ is finite-dimensional, we have a spectral triple $(A, \mathcal{H}_\tau, D_\tau^\beta)$, where $\mathcal{H}_\tau = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$, $\pi_\tau = \bigoplus_{n\in\mathbb{N}} \pi_n$ and $D_\tau^\beta = \bigoplus_{n\in\mathbb{N}} D_n$. Note that this looks very much like the Pearson-Bellissard spectral triple with respect to the choice function τ but with a different inner product on the direct sum Hilbert space.

Indeed, the two representations are unitarily equivalent with the intertwining unitary $W : \mathcal{H}_\tau \to \mathcal{H}_\tau^{\mathrm{PB}} = \ell^2(V, \mathbb{C}^2)$ defined by

$$(W\alpha)(v) = \sqrt{p(v)}\alpha(v),$$

for any $\alpha \in \mathcal{H}_\tau = \bigoplus_{n\in\mathbb{N}} \ell_\psi(V_n, \mathbb{C}^2)$ and $v \in V$. If $v \in V_n$,

$$\begin{aligned}
(W\pi_\tau(f)\alpha)(v) &= \sqrt{p(v)}(\pi_\tau(f)\alpha)(v) \\
&= \sqrt{p(v)}(\pi_n(f)\alpha)(v) \\
&= \sqrt{p(v)}\begin{pmatrix} f(\tau^+(v)) & 0 \\ 0 & f(\tau^-(v)) \end{pmatrix}\alpha(v) \\
&= \begin{pmatrix} f(\tau^+(v)) & 0 \\ 0 & f(\tau^-(v)) \end{pmatrix}(W\alpha)(v) \\
&= (\pi_\tau^{\mathrm{PB}}(f)W\alpha)(v).
\end{aligned}$$

Also, for any $\alpha = \oplus_n \alpha_n \in \mathcal{H}_\tau$, we have

$$\sum_n \|D_n\alpha_n\|_\psi^2 = \sum_n \sum_{v\in V_n} p(v)\|(D_n\alpha_n)(v)\|^2 = \frac{1}{4}\sum_{v\in V} p(v)\operatorname{diam}([v])^{-2}\|\alpha(v)\|^2.$$

Suppose $\alpha \in \operatorname{dom}(D_\tau^\beta)$. Then $\sum_n \|D_n\alpha_n\|_\psi^2 < \infty$ and

$$\sum_{v\in V}\operatorname{diam}([v])^{-2}\|(W\alpha)(v)\|^2 = \sum_{v\in V} p(v)\operatorname{diam}([v])^{-2}\|\alpha(v)\|^2 < \infty.$$

Hence, $W(\operatorname{dom}(D_\tau^\beta)) \subseteq \operatorname{dom}(D_\tau^{\mathrm{PB}})$. It is straightforward to show that $WD_\tau^\beta = \frac{1}{2}D_\tau^{\mathrm{PB}}W$. As a consequence, $\|[D_\tau^\beta, \pi_\tau(a)]\|$ is equal to the seminorm $\frac{1}{2}\|[D_\tau^{\mathrm{PB}}, \pi_\tau^{\mathrm{PB}}(a)]\|$ and thus independent of the choice of faithful state.

In summary:

Proposition 11.2. *Let ψ be a faithful state and $\tau : V \to X \times X$ a choice function. Set $\psi_n^{\pm} = \psi \circ E_n^{\pm}$, where $E_n^{\pm}$ are the conditional expectations associated to τ. Define $\beta_n = \sum_{v \in V_n} \epsilon(v)^{-1} \chi_{[v]} \in A_n^+ = A_n^+ \cap Z(A)$, where ϵ is the weight on the Michon tree of (X, d). Applying the construction in the previous chapter to the sequences $\{(\psi_n^+, \psi_n^-)\}_{n \in \mathbb{N}}$ and $\beta = \{\beta_n\}_{n \in \mathbb{N}}$, we obtain a spectral triple $(A, \mathcal{H}_\tau, D_\tau^\beta)$ where the representation is unitarily equivalent to the Pearson-Bellissard representation for the choice function τ. Consequently, the seminorm $\|[D_\tau^\beta, \pi_\tau(a)]\|$ is independent of the choice of faithful state ψ and is equal to $\frac{1}{2}\|[D_\tau^{PB}, \pi_\tau^{PB}(a)]\|$.*

From Theorem 10.4, we see that for each $a \in A_F$, $\|[D_\tau^\beta, \pi_\tau(a)]\| \le L^\beta(a)$. We will see that if we take all choice functions τ into account, we can recover $L^\beta(a)$ for self-adjoint elements. By Proposition 5.7, we have that for any $a \in A = C(X)$,

$$\sup_{\tau \in \Upsilon(X)} \|[D_\tau^{\mathrm{PB}}, \pi_\tau^{\mathrm{PB}}(a)]\| = \mathrm{Lip}(a).$$

We will show now that $L^\beta(a) = \frac{1}{2}\,\mathrm{Lip}(a)$ when a is self-adjoint.

Proposition 11.3. *With the setting as in the previous proposition, if $a \in A_k \subset A = C(X)$ is self-adjoint, then*

$$L^\beta(a) = \frac{1}{2}\,\mathrm{Lip}(a)$$

Proof. Let $a \in A_k$ be self-adjoint (real-valued). For all $n \ge k$, since $a \in A_n$, we have $\beta_n a \in A_n$ and $L_n(\beta_n a) = 0$. Then

$$L^\beta(a) = \max\{L_n(\beta_n a) : 0 \le n < k\}.$$

Let $0 \le n < k$. By Proposition 9.1, we have

$$L_n(\beta_n a) = \|\beta_n a - b\|,$$

where $b = \sum_{v \in V_n} b_v \chi_v$ with $b_v = \frac{1}{2}(\min{(\beta_n a)}\big|_v + \max{(\beta_n a)}\big|_v) = \frac{1}{2\epsilon(v)}(\min a\big|_v + \max a\big|_v)$. We set $m(v) = \frac{1}{2\epsilon(v)}(\max a\big|_v - \min a\big|_v)$ so that

$$L_n(\beta_n a) = \max\{m(v) : v \in V_n\}.$$

For each $v \in V_n$,

$$m(v) = \frac{1}{2\epsilon(v)}(\max a\big|_v - \min a\big|_v) = \frac{1}{\epsilon(v)}(a(w_{v,\max}) - a(w_{v,\min})),$$

where $w_{v,\max} \prec v$ is a vertex in V_k attaining the maximum for $a\big|_v$ (similarly for min). Since $w_{v,\max} \wedge w_{v,\min} \prec v$, we have $\epsilon(w_{v,\max} \wedge w_{v,\min}) \le \epsilon(v)$, and therefore,

$$m(v) \le \frac{1}{2\epsilon(w_{v,\max} \wedge w_{v,\min})}(a(w_{v,\max}) - a(w_{v,\min})).$$

Hence, $L^\beta(a) \le \frac{1}{2}\operatorname{Lip}(a)$.

Conversely, let $x, y \in X$ be such that $x \neq y$. Suppose that $x \wedge y \in V_n$ for some $n \in \mathbb{N}$. Then

$$\frac{|a(x)-a(y)|}{d(x,y)} = \frac{|a(x)-a(y)|}{\epsilon(x\wedge y)} \le \frac{(\max a\big|_{x\wedge y} - \min a\big|_{x\wedge y})}{\epsilon(x\wedge y)} = 2m(x\wedge y) \le 2L_n(\beta_n a).$$

Consequently, $\frac{1}{2}\operatorname{Lip}(a) \le L^\beta(a)$. □

Remark: Originally, I proved first that for any self-adjoint element a, we can choose a choice function τ so that $L^\beta(a) \le \|[D^\beta_\tau, \pi_\tau(a)]\|$ and hence, $L^\beta(a) = \frac{1}{2}\sup_{\tau\in\Upsilon(X)} \|[D^{\mathrm{PB}}_\tau, \pi^{\mathrm{PB}}_\tau(a)]\|$. The result is redundant because of Proposition 5.7. Also, the proof is similar to the above proposition.

Next, we provide a simple example to show that Proposition 11.3 may not hold for non-self-adjoint elements.

Example 11.4. Let $A_0 = \mathbb{C}$ and $\beta_0 = 1$; $A_1 = \mathbb{C}^3 = C(\{1,2,3\})$ and $\beta_1 = \frac{1}{2}(1,1,1)$. Pick $a \in A_1$ such that $a(1), a(2), a(3)$ form an equilateral triangle. Then a is not self-adjoint.

$$L^\beta(a) = L_0(a) = \inf\{\|a-z\| : z \in \mathbb{C}\} = \|a-c\| = r,$$

where c is the center of the circle passing through $a(1), a(2), a(3)$ and r is the radius of the circle. We have

$$\operatorname{Lip}(a) = \sup_{i\neq j}\{|a(i)-a(j)|\} < 2r = 2L^\beta(a).$$

Chapter 12

Recovering Aguilar Seminorms

We now show that we can also recover the generalized Aguilar seminorm L^β if we take into account all sequences of pairs of states $\{\psi_n^\pm\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\psi_n^+\big|_{A_n} = \psi_n^+\big|_{A_n}$ and they are faithful on A_n.

We aim to show that for all $a \in A_F^{\mathrm{sa}}$ and $0 < \lambda < 1$, there is a sequence $\tau = \{\psi_n^\pm\}_{n\in\mathbb{N}}$ as above such that $\lambda L^\beta(a) \leq \|[D_\tau^\beta, \pi_\tau(a)]\|$. Recall that by Theorem 10.4, the faithfulness of each $\psi_n^\pm$ on A_n is needed to obtain the faithful representation π_τ. For this reason, we have to put in extra effort to choose faithful states in the following Jordan-decomposition-type lemmas:

Notation: If a is a square matrix, we write $\mathrm{tr}(a)$ for the standard trace (the sum of the diagonals) regardless of the size of a.

Lemma 12.1. *If ψ is a Hermitian functional on M_k such that $\|\psi\| = 1$ and $\psi(1) = 0$, then for any $\lambda \in (0, 1)$, there are faithful states ψ^+, ψ^- on M_k such that $\lambda\psi = \frac{1}{2}(\psi^+ - \psi^-)$.*

Proof. Suppose that ψ is a Hermitian functional on M_k such that $\|\psi\| = 1$ and $\psi(1) = 0$. Then there is a self-adjoint matrix $d \in M_k$ such that for all $c \in M_k$,

$$\psi(c) = \mathrm{tr}(dc) = \langle c, d\rangle_{\mathrm{tr}}$$

and $\mathrm{tr}(|d|) = \|\psi\| = 1$. By diagonalizing, we may assume that d is diagonal:

$$d = \begin{pmatrix} p & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -q \end{pmatrix},$$

where p, q are positive-definite diagonal matrices and z is a zero square matrix (possibly of size 0). Then

$$|d| = \begin{pmatrix} p & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & q \end{pmatrix}$$

Since $\psi(1) = 0$, $\mathrm{tr}(d) = 0$ and hence, $\mathrm{tr}(p) = \mathrm{tr}(q) = \frac{1}{2}$. This ensures that p and q are of size at least 1.

Next, let $0 < \lambda < 1$. We will find positive-definite diagonal matrices $h^{\pm}$ so that $\mathrm{tr}(2h^{\pm}) = 1$ and $h^+ - h^- = \lambda d$.

Case 1: z is of size 0.

Set $\delta = \frac{1-\lambda}{2} > 0$ and consider

$$h^+ = \begin{pmatrix} (\lambda+\delta)p & 0 \\ 0 & \delta q \end{pmatrix} \quad \text{and} \quad h^- = \begin{pmatrix} \delta p & 0 \\ 0 & (\lambda+\delta)q \end{pmatrix}.$$

Then $h^{\pm}$ are positive-definite and $\mathrm{tr}(2h^{\pm}) = (\lambda+\delta) + \delta = \lambda + 2\delta = 1$. Also,

$$h^+ - h^- = \begin{pmatrix} \lambda p & 0 \\ 0 & -\lambda q \end{pmatrix} = \lambda d.$$

Case 2: z is of size at least 1.

Set $\varepsilon = \frac{1-\lambda}{4} > 0$ and consider

$$h^+ = \begin{pmatrix} (\lambda+\varepsilon)p & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \varepsilon q \end{pmatrix} \quad \text{and} \quad h^- = \begin{pmatrix} \varepsilon p & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & (\lambda+\varepsilon)q \end{pmatrix},$$

where s is a positive-definite diagonal matrix such that $\mathrm{tr}(s) = \varepsilon$. Then $h^{\pm}$ are positive-definite and $\mathrm{tr}(2h^{\pm}) = (\lambda+\varepsilon) + 2\varepsilon + \varepsilon = \lambda + 4\varepsilon = 1$. Also, $h^+ - h^- = \lambda d$.

In both cases, we have faithful states on M_k given by

$$\psi^{\pm}(c) = \mathrm{tr}(2h^{\pm}c).$$

and they satisfy: for any $c \in M_k$,

$$\frac{1}{2}(\psi^+(c) - \psi^-(c)) = \mathrm{tr}((h^+ - h^-)c) = \mathrm{tr}(\lambda d c) = \lambda\psi(c).$$

□

A finite-dimensional C^*-algebra is a direct sum of full matrix algebras equipped with the sup norm. Then its dual space is a direct sum of the duals of full matrix algebras equipped with the 1-norm. Using this, we can extend the above lemma to arbitrary finite-dimensional C^*-algebras.

Lemma 12.2. *Let A be a finite-dimensional C^*-algebra. If ψ is a Hermitian functional on A such that $\|\psi\| = 1$ and $\psi(1) = 0$, then for any $\lambda \in (0,1)$, there are faithful states ψ^+, ψ^- on A such that $\lambda\psi = \frac{1}{2}(\psi^+ - \psi^-)$.*

Proof. Suppose that $A = \bigoplus_{i=1}^n M_i$, where each M_i is a full matrix algebras. Let $\psi : A \to \mathbb{C}$ be a Hermitian linear functional such that $\|\psi\| = 1$ and $\psi(1) = 0$. We write $\psi = \bigoplus_{i=1}^n \psi_i$ where $\psi_i \in M_i^*$ is also Hermitian. Then for each i, there is a self-adjoint matrix $d_i \in M_i$ such that for all $c \in M_i$,

$$\psi_i(c) = \mathrm{tr}(d_i c) = \langle c, d_i \rangle_{\mathrm{tr}}.$$

By diagonalizing, we may assume that d_i is diagonal:

$$d_i = \begin{pmatrix} p_i & 0 & 0 \\ 0 & z_i & 0 \\ 0 & 0 & -q_i \end{pmatrix},$$

where p_i, q_i are zeroes or positive-definite diagonal matrices and z_i is a zero square matrix (possibly of size 0). Then

$$|d_i| = \begin{pmatrix} p_i & 0 & 0 \\ 0 & z_i & 0 \\ 0 & 0 & q_i \end{pmatrix}.$$

Since $\psi(1) = 0$, we have $\sum_{i=1}^n \mathrm{tr}(d_i) = \sum_{i=1}^n \psi_i(1_{M_i}) = \psi(1) = 0$. Moreover, $\sum_{i=1}^n \mathrm{tr}(|d_i|) = \sum_{i=1}^n \|\psi_i\| = \|\psi\| = 1$. Hence,

$$\sum_{i=1}^n \mathrm{tr}(p_i) - \sum_{i=1}^n \mathrm{tr}(q_i) = 0 \quad \text{and} \quad \sum_{i=1}^n \mathrm{tr}(p_i) + \sum_{i=1}^n \mathrm{tr}(q_i) = 1.$$

Therefore,

$$\sum_{i=1}^n \mathrm{tr}(p_i) = \sum_{i=1}^n \mathrm{tr}(q_i) = \frac{1}{2}.$$

Let k be the number of zero summands of ψ. We have $0 \le k < n$ since $\psi \ne 0$.

Next, let $0 < \lambda < 1$. As in the previous lemma, we will choose positive-definite diagonal matrices $h_i^\pm$ so that $\sum_{i=1}^n \mathrm{tr}(2h_i^\pm) = 1$ and $h_i^+ - h_i^- = \lambda d_i$. Now we let δ be a positive constant to be determined later.

For each i, we fix a positive-definite diagonal matrix s_i whose size matches the zero block in d_i (with p_i, q_i merged if they are zeroes) and has trace $\mathrm{tr}(s_i) = \delta$.

If $\|\psi_i\| = 0$ (i.e., $d_i, p_i, q_i = 0$), we set $h_i^\pm = s_i$.

If $\|\psi_i\| > 0$, we set

$$h_i^+ = \begin{pmatrix} (\lambda+\delta)p_i & 0 & 0 \\ 0 & \alpha^{-1}\|\psi_i\| s_i & 0 \\ 0 & 0 & \delta q_i \end{pmatrix} \quad \text{and} \quad h_i^- = \begin{pmatrix} \delta p_i & 0 & 0 \\ 0 & \alpha^{-1}\|\psi_i\| s_i & 0 \\ 0 & 0 & (\lambda+\delta) q_i \end{pmatrix},$$

where s_i may have size 0 and

$$\alpha := \sum_{\substack{d_i \ne 0; \\ d_i \text{ has a zero block}}} \|\psi_i\| > 0,$$

which is valid if there is $d_i \neq 0$ with a zero block (exactly when we need α.) If either p_i or q_i is zero, it is regarded as being merged into the zero block in d_i and we discard the corresponding block from $h_i^{\pm}$ so that $h_i^{\pm}$ are positive-definite. The size of $h_i^{\pm}$ is the same as that of d_i because we have fixed the size of s_i accounting for possible merging.

Now let us formally compute:

$$\begin{aligned}
\sum_{i=1}^{n} \operatorname{tr}(2h_i^+) &= \sum_{\|\psi_i\|>0} \operatorname{tr}(2h_i^+) + \sum_{\|\psi_i\|=0} 2\operatorname{tr}(\epsilon_i) \\
&= \sum_{\|\psi_i\|>0} (2(\lambda+\varepsilon)\operatorname{tr}(p_i) + 2\varepsilon\operatorname{tr}(q_i)) + \left\{ \sum_{\substack{\|\psi_i\|>0; \\ d_i \text{ has a zero block}}} 2\alpha^{-1}\|\psi_i\|\varepsilon \right\} + k\cdot 2\varepsilon \\
&= 2(\lambda+\varepsilon) \sum_{\|\psi_i\|>0} \operatorname{tr}(p_i) + 2\varepsilon \sum_{\|\psi_i\|>0} \operatorname{tr}(q_i) + \{2\varepsilon\} + 2k\varepsilon \\
&= 2(\lambda+\varepsilon) \sum_{i=1}^{n} \operatorname{tr}(p_i) + 2\varepsilon \sum_{i=1}^{n} \operatorname{tr}(q_i) + \{2\varepsilon\} + 2k\varepsilon \\
&= (\lambda+\varepsilon) + \varepsilon + \{2\varepsilon\} + 2k\varepsilon = \lambda + (2 + \{2\} + 2k)\varepsilon,
\end{aligned}$$

where the terms in curly brackets are valid only if there is $d_i \neq 0$ with a zero block. Swapping p_i and q_i in the above computation, we also have $\sum_{i=1}^{n} \operatorname{tr}(2h_i^-) = \lambda + (2+\{2\}+2k)\varepsilon$. Whether the curly-bracket term is valid or not, we can choose $\varepsilon > 0$ so that $\sum_{i=1}^{n} \operatorname{tr}(2h_i^{\pm}) = 1$. (If it is valid, choose $\varepsilon = \frac{1-\lambda}{4+2k}$; if not, choose $\varepsilon = \frac{1-\lambda}{2+2k}$.) Additionally, for each i, $h_i^{\pm}$ are positive-definite and $h_i^+ - h_i^- = \lambda d_i$.

We now have faithful states on A given by

$$\psi^{\pm}(\oplus_{i=1}^{n} c_i) = \sum_{i=1}^{n} \operatorname{tr}(2h_i^{\pm} c_i).$$

Furthermore, for all $\oplus_{i=1}^{n} c_i \in A$,

$$\frac{1}{2}(\psi^+(\oplus_{i=1}^{n} c_i) - \psi^-(\oplus_{i=1}^{n} c_i)) = \sum_{i=1}^{n} \operatorname{tr}((h_i^+ - h_i^-)c_i) = \sum_{i=1}^{n} \operatorname{tr}(\lambda d_i c_i) = \lambda \sum_{i=1}^{n} \psi_i(c_i) = \lambda\psi(\oplus_{i=1}^{n} c_i).$$

□

Lemma 12.3. *Let A be a finite-dimensional C^*-algebra and B a unital C^*-subalgebra of A. Then for any $a \in A^{sa}$ and $0 < \lambda < 1$, there is a pair of faithful states $\psi^{\pm}$ on A agreeing on B, such that $\lambda L_B(a) = \frac{1}{2}(\psi^+(a) - \psi^-(a))$.*

Proof. Let $a \in A$ be self-adjoint and $0 < \lambda < 1$. There is a linear functional $\psi : A \to \mathbb{C}$ such that $\|\psi\| = 1, \psi(a) = L_B(a)$, and $\psi(b) = 0$ for all $b \in B$. Since a is self-adjoint, we may assume ψ is Hermitian. (Otherwise, we can consider $\tilde{\psi}(c) = \frac{1}{2}(\psi(c) + \overline{\psi(c^*)})$.) By Lemma 12.2, there are faithful states ψ^+ and ψ^- on A such that $\lambda\psi = \frac{1}{2}(\psi^+ - \psi^-)$. Since $\psi(b) = 0$ for all $b \in B$, $\psi^\pm$ agree on B. Finally, we also have

$$\lambda L_B(a) = \psi(a) = \frac{1}{2}(\psi^+(a) - \psi^-(a)).$$

□

Theorem 12.4. *Assume that A is an infinite-dimensional AF algebra, and let $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be such that $\beta_n \in A_n^+ \cap Z(A)$ and $\|\beta_n^{-1}\| \to 0$. Then for all $a \in A_F^{sa}$ and $0 < \lambda < 1$, there is a sequence $\tau = \{\psi_n^\pm\}_{n\in\mathbb{N}}$ of states of A such that ψ_n^+ and ψ_n^- are equal and faithful on A_n for each n, and $\lambda L^\beta(a) \leq \|[D_\tau^\beta, \pi_\tau(a)]\|$, where D_τ^β is as defined right before Proposition 10.3.*

Proof. Let $a \in A_k \subset A_F$ be self-adjoint and $0 < \lambda < 1$. Since

$$L^\beta(a) = \max\{L_n(\beta_n a) : 0 \leq n < k\},$$

we only have to carefully choose $\psi_n^\pm$ for $0 \leq n < k$ in order to get the desired inequality.

For $n \geq k$, $\psi_n^\pm$ can be quite arbitrary. For example, we first fix a faithful state $\theta : A \to \mathbb{C}$. Since A is infinite-dimensional, we can consider different transition probabilities on the Bratteli diagram of A. Applying Proposition 7.12, we get a pair of distinct conditional expectations (E_n^+, E_n^-) from A to A_n. Then the states $\psi_n^\pm = \theta \circ E_n^\pm$ are equal and faithful on A_n.

For each $0 \leq n < k$, by Lemma 12.3, there are faithful states $\psi_n^\pm : A_k \to \mathbb{C}$ agreeing on A_n such that $\lambda L_n(\beta_n a) = \frac{1}{2}(\psi_n^+(\beta_n a) - \psi_n^-(\beta_n a))$. We then extend these by the Hahn-Banach theorem to states on A, also denoted by $\psi_n^\pm$.

By the construction, for any $c \in A$,

$$\begin{aligned}\langle \pi_n(c)\xi_n, \eta_n\rangle &= \frac{1}{2}(\langle \pi_n^+(c)\xi_n^+, \xi_n^+\rangle - \langle \pi_n^-(c)\xi_n^-, \xi_n^-\rangle) \\ &= \frac{1}{2}(\psi_n^+(c) - \psi_n^-(c)).\end{aligned}$$

In particular, $\langle \pi_n(b)\xi_n, \eta_n\rangle = 0$ for all $b \in A_n$, and hence $\eta_n \perp \pi_n(A_n)\xi_n$, so that $P_n\eta_n = 0$. Moreover, since A_n has the same identity element as A, we have $P_n\xi_n = \xi_n$. Then for all $c \in A$,

$$\langle [\pi_n(c), U_n]\xi_n, \eta_n\rangle = 2\langle [\pi_n(c), P_n]\xi_n, \eta_n\rangle = 2(\langle \pi_n(c)P_n\xi_n, \eta_n\rangle - \langle P_n\pi_n(c)\xi_n, \eta_n\rangle) = 2\langle \pi_n(c)\xi_n, \eta_n\rangle.$$

We have

$$\lambda L_n(\beta_n a) = \frac{1}{2}(\psi_n^+(\beta_n a) - \psi_n^-(\beta_n a)) = \langle \pi_n(\beta_n a)\xi_n, \eta_n\rangle = \frac{1}{2}\langle [\pi_n(\beta_n a), U_n]\xi_n, \eta_n\rangle \leq \|[D_n, \pi_n(a)]\|.$$

and therefore,

$$\lambda L^\beta(a) \leq \|[D_\tau^\beta, \pi_\tau(a)]\|.$$

□

Consequently, if we take all such sequences τ into consideration and combine the bounds from Theorem 10.4 and Theorem 12.4, we recover the generalized Aguilar seminorm:

Theorem 12.5. *For all $a \in A_F^{sa}$, $L^\beta(a) = \sup_\tau \|[D_\tau^\beta, \pi_\tau(a)]\|$, where τ runs over all sequences of states $\{\psi_n^\pm\}_{n\in\mathbb{N}}$ such that $\psi_n^\pm$ are faithful and equal on A_n.*

This is a noncommutative generalization of the distance recovering theorem for an ultrametric Cantor set by Pearson and Bellissard (Theorem 5.6).

Bibliography

[1] Konrad Aguilar, *AF algebras in the quantum Gromov-Hausdorff propinquity space*, (2016), arXiv:1612.02404.

[2] Konrad Aguilar and Frédéric Latrémolière, *Quantum ultrametrics on AF algebras and the Gromov-Hausdorff propinquity*, Studia Math. **231** (2016), no. 2, 149–193, arXiv:1511.07114, doi:10.4064/sm8478-2-2016.

[3] Jean V. Bellissard, *K-theory of C*-algebras in solid state physics*, Statistical mechanics and field theory: mathematical aspects (Groningen, 1985), Springer, Berlin, 1986, pp. 99–156, doi:10.1007/3-540-16777-3_74.

[4] ______, *Gap labelling theorems for Schrödinger operators*, From number theory to physics (Les Houches, 1989), Springer, Berlin, 1992, pp. 538–630, doi:10.1007/978-3-662-02838-4_12.

[5] ______, *The noncommutative geometry of aperiodic solids*, Geometric and topological methods for quantum field theory (Villa de Leyva, 2001), World Scientific, River Edge, NJ, 2003, pp. 86–156, doi:10.1142/9789812705068_0002.

[6] Jean V. Bellissard, Riccardo Benedetti, and Jean-Marc Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labelling*, Comm. Math. Phys. **261** (2001), no. 1, 1–41, arXiv:0109062, doi:10.1007/s00220-005-1445-z.

[7] Jean V. Bellissard, D. J. L. Herrmann, and Marc Zarrouati, *Hulls of aperiodic solids and gap labeling theorems*, Directions in mathematical quasicrystals, CRM Monogr. Ser., vol. 13, American Mathematical Society, Providence, RI, 2000, pp. 207–258, doi:10.1090/crmm/013/08.

[8] Bruce Blackadar, *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, doi:10.1007/3-540-28517-2.

[9] Ola Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234, doi:10.2307/1996380.

[10] Michael Christ and Marc A. Rieffel, *Nilpotent group C*-algebras as compact quantum metric spaces*, Canad. Math. Bull. **60** (2017), no. 1, 77–94, `arXiv:1508.00980`, `doi:10.4153/CMB-2016-040-6`.

[11] Alain Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory Dynam. Systems **9** (1989), no. 2, 207–220, `doi:10.1017/S0143385700004934`.

[12] ———, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[13] Kenneth R. Davidson, *C*-algebras by Example*, Fields Institute Monographs, American Mathematical Society, Providence, RI, 1996, `doi:10.1090/fim/006`.

[14] David E. Evans and Yasuyuki Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, New York, 1998.

[15] Daniele Guido and Tommaso Isola, *Dimensions and singular traces for spectral triples, with applications to fractals*, J. Funct. Anal. **203** (2003), no. 2, 362–400, `doi:10.1016/S0022-1236(03)00230-1`.

[16] Brian C. Hall, *Quantum Theory for Mathematicians*, Graduate Texts in Mathematics, Springer, New York, 2013, `doi:10.1007/978-1-4614-7116-5`.

[17] Bruno Iochum, Thomas Krajewski, and Pierre Martinetti, *Distances in finite spaces from noncommutative geometry*, J. Geom. Phys. **37** (1999), no. 1-2, 100–125, `arXiv:9912217`, `doi:10.1016/S0393-0440(00)00044-9`.

[18] Richard V. Kadison and John R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Volume I: Elementary Theory*, Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Providence, RI, 1997.

[19] David Kerr, *Matricial quantum Gromov-Hausdorff distance*, J. Funct. Anal. **205** (2002), no. 1, 132–167, `arXiv:0207282`, `doi:10.1016/S0022-1236(03)00195-2`.

[20] David Kerr and Hanfeng Li, *On Gromov-Hausdorff convergence for operator metric spaces*, J. Oper. Theory **62** (2004), no. 1, 83–109, `arXiv:0411157`.

[21] Jeffrey C. Lagarias, *Geometric models for quasicrystals I. Delone sets of finite type*, Discrete Comput. Geom. **21** (1999), no. 2, 161–191, `doi:10.1007/PL00009413`.

[22] Jeffrey C. Lagarias and Peter A. B. Pleasants, *Repetitive Delone sets and quasicrystals*, Ergodic Theory Dynam. Systems **23** (2003), no. 3, 831–867, `arXiv:9909033`, `doi:10.1017/S0143385702001566`.

[23] Frédéric Latrémolière, *Quantum locally compact metric spaces*, J. Funct. Anal. **264** (2013), no. 1, 362–402, `arXiv:1208.2398`, `doi:10.1016/j.jfa.2012.10.016`.

[24] ———, *The quantum Gromov-Hausdorff propinquity*, Trans. Amer. Math. Soc. **368** (2016), no. 1, 365–411, arXiv:1302.4058, doi:10.1090/tran/6334.

[25] ———, *A compactness theorem for the dual Gromov-Hausdorff propinquity*, Indiana Univ. Math. J. **66** (2017), no. 5, 1707–1753, arXiv:1501.06121, doi:10.1512/iumj.2017.66.6151.

[26] Hanfeng Li, *C*-algebraic quantum Gromov-Hausdorff distance*, (2003), 1–27, arXiv:0312003.

[27] Gérard Michon, *Les Cantors réguliers*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 19, 673–675.

[28] Narutaka Ozawa and Marc A. Rieffel, *Hyperbolic group C*-algebras and free-product C*-algebras as compact quantum metric spaces*, Canad. J. Math. **57** (2005), no. 5, 1056–1079, arXiv:0302310, doi:10.4153/CJM-2005-040-0.

[29] Ian C. Palmer, *Riemannian Geometry of Compact Metric Spaces*, Ph.D. thesis, Georgia Institute of Technology, 2010.

[30] John Pearson and Jean V. Bellissard, *Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets*, J. Noncommut. Geom. **3** (2009), no. 3, 447–480, arXiv:0802.1336, doi:10.4171/JNCG/43.

[31] Jean N. Renault, *AF equivalence relations and their cocycles*, Operator algebras and mathematical physics (Constanţa, 2001), Theta, Bucharest, 2003, pp. 365–377.

[32] ———, *Random walks on Bratteli diagrams*, Operator theory: themes and variations (Timişoara, 2016), Theta Ser. Adv. Math., vol. 20, Theta, Bucharest, 2018, pp. 187–204, arXiv:1704.06990.

[33] Marc A. Rieffel, *Metrics on states from actions of compact groups*, Doc. Math. **3** (1998), 215–229, arXiv:9807084.

[34] ———, *Metrics on state spaces*, Doc. Math. **4** (1999), 559–600, arXiv:9906151.

[35] ———, *Group C*-algebras as compact quantum metric spaces*, Doc. Math. **7** (2002), 605–651, arXiv:0205195.

[36] ———, *Compact quantum metric spaces*, Operator algebras, quantization, and noncommutative geometry, Contemp. Math., vol. 365, American Mathematical Society, Providence, RI, 2004, pp. 315–330, arXiv:0308207, doi:10.1090/conm/365/06709.

[37] ———, *Gromov-Hausdorff distance for quantum metric spaces*, Mem. Amer. Math. Soc. **168** (2004), no. 796, 1–65, arXiv:0011063, doi:10.1090/memo/0796.

[38] ———, *Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance*, Mem. Amer. Math. Soc. **168** (2004), no. 796, 67–91, arXiv:0108005v2, doi:10.1090/memo/0796.

[39] ———, *Leibniz seminorm for "matrix algebras converge to the sphere"*, Quanta of maths (Paris, 2007), Clay Math. Proc., vol. 11, American Mathematical Society, Providence, RI, 2010, pp. 543–578, arXiv:0707.3229.

[40] ———, *Vector bundles and Gromov-Hausdorff distance*, J. K-Theory **5** (2010), no. 1, 39–103, arXiv:0608266, doi:10.1017/is008008014jkt080.

[41] ———, *Leibniz seminorms and best approximation from C*-subalgebras*, Sci. China Math. **54** (2011), no. 11, 2259–2274, arXiv:1008.3733, doi:10.1007/s11425-011-4318-2.

[42] ———, *Vector bundles for "matrix algebras converge to the sphere"*, J. Geom. Phys. **132** (2018), 181–204, arXiv:1711.04054, doi:10.1016/j.geomphys.2018.06.003.

[43] Konrad Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, Springer, Dordrecht, 2012, doi:10.1007/978-94-007-4753-1.

[44] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, *Metallic phase with long-range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), no. 20, 1951–1953, doi:10.1103/PhysRevLett.53.1951.

[45] Wilhelm Winter, *Covering dimension for nuclear C*-algebras*, J. Funct. Anal. **199** (2003), no. 2, 535–556, arXiv:0107218, doi:10.1016/S0022-1236(02)00109-X.