Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Engineering & Technical Services Division

DISCLAIMER

LAWRENCE BERKELEY LABORATORY - UNIVERSITY OF CALIFORNIA ENGINEERING NOTE	CODE MD1111	SERIAL M 5584	PAGE 1 of 15
AUTHOR R. B. Meuser / DEPARTMENT Mechanical Engineering	LOCATION Berkeley	DATE September 24, 1980	

PROGRAM — PROJECT — JOB
High-Field Magnet Development

Analysis

TITLE
Circumferential Displacements of Thin Cosine-Theta Dipole Coils.

INTRODUCTION

Circumferential displacements and the resulting field-quality aberrations are investigated for the simplest possible mathematical model of a superconducting dipole: the winding is thin, has a $\cos\theta$ current distribution, and has a linear stress-strain curve. There is no friction between the coil and the surrounding rigid structure.

NOMENCLATURE

f	Local circumferential Lorentz body force component per unit circumference (+↻)
f_o	Max. value of f.
F	Internal circumferential force (+ for compression)
F_o	F at $\theta = 0$
F_1	F at $\theta = 90°$
a	Radius of coil
A	Cross-sectional area of coil
E	Elastic modulus of coil
θ	Azimuthal coordinate of point on coil
ϵ	Local strain
Δ''	Local circumferential displacement, total. (↻+)
Δ'	Circ. disp. for initial prestress
Δ	Additional circ. disp. caused by body forces.
Δ_1	Δ at $\theta = 90°$
N, M	See Eq. 22, 23
J	Local lineal current density, current per unit circum
J_o	J at $\theta = 0$
J', J_o'	J, J_o after displacement
C_n	See Eq. 29
b	Iron inside radius
ρ	Normalizing reference radius
B_o	Magnetic field in aperture.

Everything is per unit length ⊥ to $r\theta$ plane

AUTHOR R. B. Meuser | DEPARTMENT Mech. Eng. | LOCATION Berk | DATE Sep. 24 1980

SUMMARY

DISPLACEMENTS AND FORCES

We apply local circumferential Lorentz body forces (force per unit circumference) of

$$f = f_o \sin 2\theta$$

In terms of the magnet parameters, f_o can be expressed as

$$f_o = \frac{1}{4}\mu_o J_o^2\left[1+\left(\frac{a}{b}\right)^2\right] = 2\left(\frac{B_o^2}{2\mu_o}\right)\frac{1}{1+\left(\frac{a}{b}\right)^2}$$

There is initially a pre-stress circumferential compressive force F'

Case I: For $\frac{1}{2}f_o a \le F'$, $\Delta = 0$ at $\theta = 90°$

After application of the body forces the local internal circum. force is

$$F = F' + \frac{1}{2}f_o a\cos 2\theta$$

at $\theta = 0°$ $F = F_0 = F' + \frac{1}{2}f_o a$

at $\theta = 90°$ $F = F_1 = F' - \frac{1}{2}f_o a$

The circumferential displacement is

$$\Delta = \frac{1}{4}\frac{f_o a^2}{AE}\sin 2\theta \qquad +\curvearrowright$$

For $\theta = 0$ and $90°$, $\Delta = 0$

Case II: For $\frac{1}{2}f_o a \ge F'$, $F = 0$ at $\theta = 90°$

$$F = \frac{1}{2}f_o a(1+\cos 2\theta)$$

$$\Delta = \frac{a}{AE}\left[\left(\frac{1}{2}f_o a - F'\right)\theta + \frac{1}{4}f_o a\sin 2\theta\right]$$

FIELD ABERRATIONS

LET $\Delta = -aN\theta - aM\sin 2\theta$

or $\theta' - \theta = N\theta + M\sin 2\theta$

AUTHOR	DEPARTMENT	LOCATION	DATE
R. B. Meuser	Mech. Eng.	Berk.	Sep. 24, 1980

The relative multipole coefficients of the field are

For M:
$$\frac{C_3}{C_1} = -K_3 \frac{3}{2} M$$
$$\frac{C_n}{C_1} = 0 \quad \text{for } n \neq 3$$

where
$$K_n = \left(\frac{\rho}{a}\right)^{n-1} \frac{1+(a/b)^{2n}}{1+(a/b)^2}$$

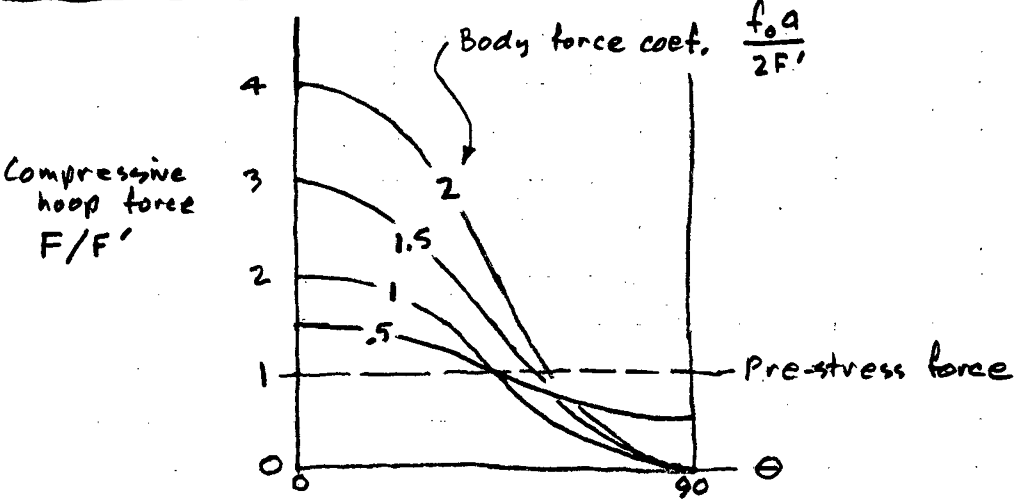
For N:
$$\frac{C_n}{C_1} = -K_n \frac{2n}{n^2-1} (-1)^{(n+1)/2} N \quad \text{for } n = 3, 5, 7 \ldots$$
and zero for $n \neq 3, 5, 7 \ldots$

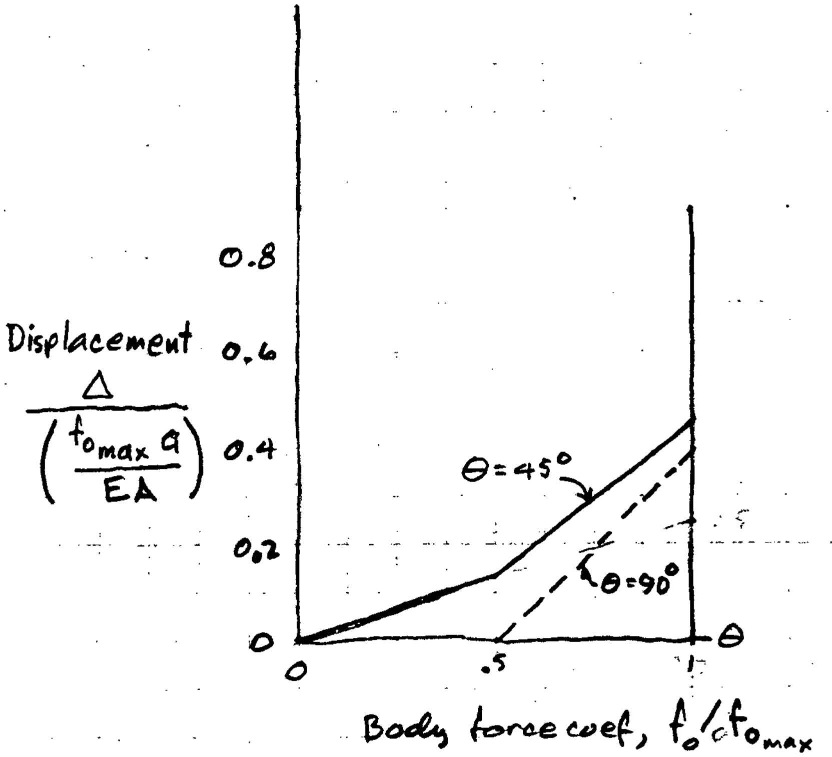
In terms of the body forces and pre-stress force:

Case I:
$$C_3/C_1 = K_3 \cdot \frac{3}{8} \frac{f_0 a}{AE}$$
$$C_n/C_1 = 0 \quad \text{for } n \neq 3$$

Case II:
$$C_3/C_1 = K_3 \frac{3}{4} \frac{1}{AE}(f_0 a - F')$$
$$C_n/C_1 = K_n (-1)^{(n+1)/2} \frac{2n}{n^2-1} \frac{1}{AE}\left(\tfrac{1}{2} f_0 a - F'\right)$$
for $n = 5, 7, 9 \ldots$

INTERPRETATION





DISPLACEMENTS

For a dipole magnet having a thin $\cos\theta$ winding the circumferential body force on an element of the winding is $f a\, d\theta$ where

$$f = f_o \sin 2\theta \qquad (1)$$

For equilibrium, the internal hoop force $F(\theta)$ is

$$F = F_o - \int_0^\theta f a\, d\theta \qquad (2)$$

Upon inserting f from eq. 1 and integrating we obtain

$$F = F_o - \tfrac{1}{2} f_o a \left(1 - \cos 2\theta\right) \qquad (3)$$

At $\theta = 90^\circ$, $F = F_1$ or

$$F_1 = F_o - f_o a \qquad (4)$$

The local strain is

$$\epsilon = \frac{F}{AE} \qquad (5)$$

The change in length of an element is $\epsilon a\, d\theta$, so the local displacement is

$$\Delta'' = \int_0^\theta \epsilon a\, d\theta \qquad (6)$$

Upon substituting F from Eq 3 into Eq. 5, and ϵ from that into Eq 6 and integrating we get

$$\Delta'' = \frac{a}{AE}\left[\left(F_o - \tfrac{1}{2} f_o a\right)\theta + \tfrac{1}{4} f_o a \sin 2\theta\right] \qquad (7)$$

or using Eq 4

$$\Delta'' = \frac{a}{AE}\left[\left(F_1 + \tfrac{1}{2} f_0 a\right)\theta + \tfrac{1}{4} f_0 a \sin 2\theta\right] \qquad (8)$$

Initially there are displacements $\Delta'(\theta)$ resulting from the pre-stress circum. force F', uniform in θ, of

$$\Delta' = \int_0^\theta \frac{F'a}{AE}\,d\theta = \frac{F'a}{AE}\theta \qquad (9)$$

which has a value at $\theta = 90°$ of

$$\Delta'_1 = \frac{F'a}{AE}\frac{\pi}{2} \qquad (10)$$

Case I: No displacement at $\theta = 90°$, $F_1 \geq 0$

The local change in displacement with application of the body forces – which is what we are after – is

$$\Delta = \Delta'' - \Delta' \qquad (11)$$

At $\theta = 90°$, then

$$\Delta_1 = \Delta''_1 - \Delta'_1 = 0 \qquad (12)$$

Substituting Δ'_1 from Eq 10 and Δ''_1 from Eq. 8 for $\theta = \pi/2$ into Eq 12 and solving for F_1 we obtain

$$F_1 = F' - \tfrac{1}{2} f_0 a \qquad (13)$$

Substituting F_1 from Eq. 13 into Eq 8 we obtain

$$\Delta'' = \frac{a}{AE}\left[F'\theta + \frac{1}{4} f_0 a \sin 2\theta \right] \quad (14)$$

Substituting Eq. 14 and Eq. 9 into Eq. 11 we get for the change in displacement Δ,

$$\Delta = \frac{1}{4}\frac{f_0 a^2}{AE} \sin 2\theta \quad (15)$$

The local strain ϵ is

$$\epsilon = \frac{1}{a}\frac{d\Delta''}{d\theta}$$

and the local internal force F is

$$F = AE\epsilon$$

Upon differentiating Eq 14, etc., we get

$$F = F' + \frac{1}{2} f_0 a \cos 2\theta \quad (16a)$$

$$F_0 = F' + \frac{1}{2} f_0 a \quad (16b)$$

$$F_1 = F' - \frac{1}{2} f_0 a \quad (16c)$$

For incipient separation at $\theta = 90°$ – the limit of validity of Case I – we have $F_1 = 0$, so

$$F'_{min} = \frac{1}{2} f_0 a = \frac{1}{2}(F_1 - F_0) \quad (17)$$

That is, the minimum prestress hoop force, F'_{min}, is one-half the integrated circumferential body force, $f_0 a$ or $F_1 - F_0$.

Case II: No internal force F at $\theta = 90°$.

For this case the change in displacement $\Delta_1 \geq 0$ at $\theta = 90°$.
For $F_1 = 0$, Eq. 8 gives

$$\Delta'' = \frac{a}{AE} f_0 a \left(\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right) \qquad (18)$$

So using Eqs 18, 9, and 11 (Eq. 11 is valid for Case II) we get for the change in displacement:

$$\Delta = \frac{a}{AE} \left[\left(\frac{1}{2} f_0 a - F' \right) \theta + \frac{1}{4} f_0 a \sin 2\theta \right] \qquad (19)$$

Since Δ_1 cannot be negative, this case is valid only if the bracketed term is positive;

$$F'_{max} = \frac{1}{2} f_0 a \qquad (20)$$

which jibes with Eq 17.

The local force is, as before,

$$F = AE \frac{1}{a} \frac{d\Delta''}{d\theta}$$

$$F = \frac{1}{2} f_0 a \left(1 + \cos 2\theta \right) \qquad (21a)$$

$$F_0 = f_0 a \qquad (21b)$$

$$F_1 = 0 \qquad (21c)$$

EFFECT OF DISPLACEMENTS UPON FIELD QUALITY

The current density (current/circum) is originally

$$J = J_0 \cos\theta \qquad (20)$$

A coil element originally situated between θ and $\theta + d\theta$ moves to a new position between θ' and $\theta' + d\theta'$. The position angle becomes θ' and the current density is changed by a factor $d\theta/d\theta'$. So the new current density is

$$J' = J_0 \cos\theta' \frac{d\theta}{d\theta'} \qquad (21)$$

We express the displacement Δ (see Eq. 19)

$$\Delta = -aN\theta - aM\sin 2\theta \qquad (22)$$

so

$$\theta' - \theta = N\theta + M\sin 2\theta \qquad (23)$$

(Δ is the CW displacement; $\theta' - \theta$ is positive for CCW displacement.)
We consider the effects of the two terms separately.

M term

$$\theta' = \theta + M\sin 2\theta \qquad (24)$$

$$\frac{d\theta}{d\theta'} = 1 - 2M\cos 2\theta \qquad (25)$$

Eq. 21 then yields

$$J' = J_0 \cos(\theta + M\sin 2\theta) \times (1 - 2M\cos 2\theta) \qquad (26)$$

Let's look at the factor

$$\cos(\theta + M\sin 2\theta)$$

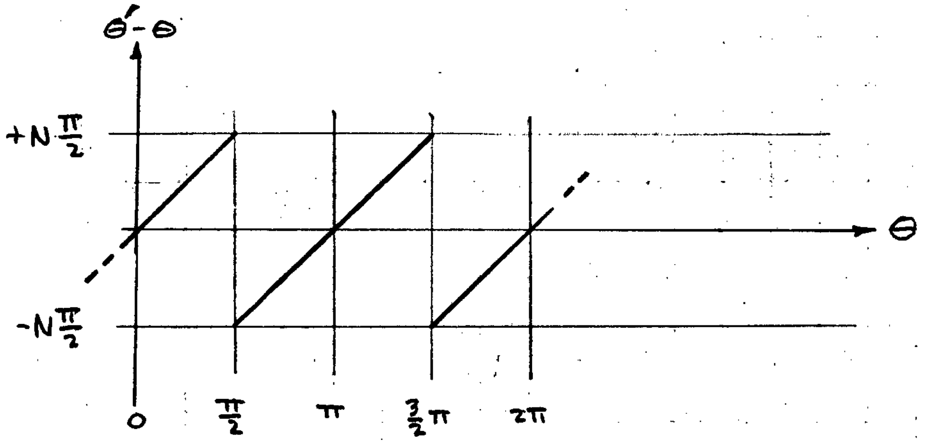
AUTHOR R. B. Meuser | DEPARTMENT Mech Eng | LOCATION Berk | DATE Sep 24 1980

N term

In the first quadrant the displacement is

$$\theta' - \theta = N\theta$$

The character of the displacement $\theta' - \theta$ in the other quadrants is as shown



This can be represented by the Fourier series

$$\theta' - \theta = -2N \sum_{m=1,2,3\ldots} \frac{1}{m} \cos\frac{m\pi}{2} \sin m\theta \qquad (35)$$

As before, we wish to evaluate J' given by

$$J' = J_0 \cos\theta' \frac{d\theta}{d\theta'} \qquad (36)$$

To first order in N

$$\frac{d\theta}{d\theta'} = 1 + 2N \sum_{m=1,2,3\ldots} \cos\frac{m\pi}{2} \cos m\theta \qquad (37)$$

The equation for J' becomes

$$J' = J_0 \left\{ \cos\left[\theta - 2N \sum_{m=1,2,3\ldots} \frac{1}{m} \cos\frac{m\pi}{2} \sin m\theta\right]\right\} \left[1 + 2N \sum_{m=1,2,3} \cos\frac{m\pi}{2} \cos m\theta\right] \qquad (38)$$

We can simplify the term in { } by using

AUTHOR	DEPARTMENT	LOCATION	DATE
R. B. Meuser	Mech Eng	Berk	Sep 24, 1980

$$\cos(x-\delta) = \cos x \cos\delta + \sin x \sin\delta$$
$$\approx \cos x + \delta \sin x \quad \text{for } \delta \ll x$$

That results in a term $\sin m\theta \sin\theta$ within the summation which is replaced by $\frac{1}{2}[\cos(m-1)\theta - \cos(m+1)\theta]$. The term $\cos\frac{m\pi}{2}$ for $m=1,2,3$ is replaced by $(-1)^{m/2}$ for $m=2,4,6\ldots$ and zero for $m \neq 2,4,6\ldots$ The equation for J' becomes

$$J' = J_0\left\{\cos\theta - N\sum_{m=2,4,6\ldots}\frac{1}{m}(-1)^{m/2}\left[\cos(m-1)\theta - \cos(m+1)\theta\right]\right\}\left[1+2N\sum_{m=2,4,6\ldots}(-1)^{m/2}\cos m\theta\right] \quad (39)$$

We write out the summations term by term, collect like terms, discard terms in M^2 and higher order, and let $m = n+1$. Finally we get

$$J' = J_0\left\{\left(1-\tfrac{1}{2}N\right)\cos\theta - N\sum_{n=3,5,7}(-1)^{(n+1)/2}\frac{2n}{n^2-1}\cos n\theta\right\} \quad (40)$$

which has the desired form

$$J' = \sum_{n=1,3,5} J_{0_n}\cos n\theta$$

The $n=1$ term is

$$J'_{0_1} = J_0\left(1-\frac{N}{2}\right) \approx J_0$$

and the higher-order terms are

$$J'_{0_n} = -N(-1)^{(n+1)/2}\frac{2n}{n^2-1} \qquad n=3,5,7\ldots$$

As before

$$\frac{C_n}{C_1} = \frac{J_{0n}}{J_{01}}\left(\frac{\rho}{a}\right)^{n-1}\frac{1+(a/b)^{2n}}{1+(a/b)^2} \quad \text{for } n=3,5,7\ldots$$

and $\dfrac{C_n}{C_1} = 0$ for $n = 2,4,6\ldots$

so for $n = 3,5,7,\ldots$

$$\frac{C_n}{C_1} = -N\,(-1)^{(n+1)/2}\,\frac{2n}{n^2-1}\left(\frac{\rho}{a}\right)^{n-1}\frac{1+(a/b)^{2n}}{1+(a/b)^2} \qquad (41)$$

Recall that the displacement change is given by

$$\Delta = \frac{1}{4}\frac{f_0 a^2}{AE}\sin 2\theta \quad \text{for Case I and}$$

$$\Delta = \frac{a}{AE}\left[\left(\tfrac{1}{2}f_0 a - F'\right)\theta + \tfrac{1}{4}f_0 a^2 \sin 2\theta\right. \quad \text{for Case II}$$

and that we have, in this section, expressed the

$$\Delta = -aN\theta - aM\sin 2\theta$$

So M and N are as follows

Case I

$$M = -\frac{1}{4}\frac{f_0 a}{AE}, \; N = 0 \qquad (42)$$

Case II

$$\left.\begin{aligned} M &= -\frac{1}{4}\frac{f_0 a}{AE}, \\ N &= -\frac{1}{AE}\left(\tfrac{1}{2}f_0 a - F'\right) \end{aligned}\right\} (43)$$

Upon substituting these into the formulas for C_n/C_1 we obtain the following:

Case I

$$\left.\begin{aligned} &\frac{C_3}{C_1} = \left(\frac{f}{a}\right)^2 \frac{1+(a/b)^6}{1+(a/b)^2} \cdot \frac{3}{8}\frac{f_0 a}{AE} \\ &\text{and } \frac{C_n}{C_1} = 0 \text{ for all others} \end{aligned}\right\} \quad (44)$$

Case II

$$\left.\begin{aligned} &\frac{C_3}{C_1} = \left(\frac{f}{a}\right)^2 \frac{1+(a/b)^6}{1+(a/b)^2} \cdot \frac{3}{4} \cdot \frac{1}{AE}\left(f_0 a - F'\right) \\ &\text{and } \frac{C_n}{C_1} = \left(\frac{f}{a}\right)^{n-1} \frac{1+(a/b)^{2n}}{(1+a/b)^2} (-1)^{(n+1)/2} \frac{2n}{n^2-1}\frac{1}{AE}\left(\frac{1}{2} f_0 a - F'\right) \end{aligned}\right\} \quad (45)$$

for $n = 5, 7, 9, \ldots$

(Phew!)

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720