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A Descent Basis for the Garsia-Procesi Module

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A Descent Basis for the Garsia-Procesi Module

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To my mother, for everything.

To my brother, for always being there, no matter how impossible our circumstances were.

To my uncle, for always inspiring me, and always holding me up.

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A Descent Basis for the Garsia-Procesi Module

Abstract

We define a combinatorial construction that gives a natural subset of the Garsia-Stanton descent monomials whose images under the canonical projection $R_n \to R_\lambda$ form a vector space basis of the Garsia-Procesi module R_λ. As a consequence, our indexing set yields a new formula for the modified Hall-Littlewood polynomials. Our work was discovered whilst searching for a basis of the Garsia-Haiman module, and we discuss partial results in this direction, as well as other connections with the modified Macdonald polynomials $\widetilde{H}_\lambda(X; q, t)$.

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CHAPTER 1

Introduction

A *complete flag* in $\mathbb{C}^n$ is an increasing sequence of subspaces

$$F_\bullet = (\{0\} = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n)$$

such that $\dim_{\mathbb{C}} F_i = i$. The collection of flags in $\mathbb{C}^n$ is known as the *complete flag variety* $Fl_n(\mathbb{C})$, and is a projective variety with a cell decomposition $\{C_\omega : \omega \in S_n\}$ indexed by the symmetric group S_n. The cells C_ω are known as *Schubert cells*, with dimensions corresponding to a certain combinatorial statistic on S_n. The flag variety (in type A) may also be thought of as $Fl_n = \mathrm{GL}_n(\mathbb{C})/B$, where B is the subgroup of all upper triangular matrices.

The i-th tautological line bundle on Fl_n is the line bundle $\mathcal{L}_i$ whose fibre over flag $F_\bullet$ is the line F_i/F_{i-1}. Denote by $x_i = -c_1(\mathcal{L}_i)$ to be its first Chern class. Borel, in his thesis, gave a presentation of $H^*(Fl_n)$ in terms of the x_i:

$$H^*(Fl_n) \cong \frac{\mathbb{C}[x_1, \dots, x_n]}{I_n} =: R_n$$

where I_n is the *coinvariant ideal*, $I_n = \langle e_i(\mathbf{x}), 1 \leqslant i \leqslant n \rangle$, R_n is called the *coinvariant algebra*, and $e_i(\mathbf{x})$ is the *elementary symmetric function* in the variables $x_1, \dots, x_n$. We may think of I_n as the ideal of all nonconstant symmetric functions.

As a vector space, $\dim_{\mathbb{C}} R_n = n!$, so one would expect the existence of a basis indexed by permutations $\pi \in S_n$. In fact, there are two bases which correspond to certain combinatorial statistics on S_n, known as the *Artin* and *Garsia-Stanton* bases, which we will introduce later. The Garsia-Stanton basis especially is useful for representation-theoretic computations, such as in the *descent representations* of Adin-Brenti-Roichman in [**ABR03**].

A very important family of subvarieties of Fl_n are the *Springer fibres*, indexed by nilpotent operators $X : \mathbb{C}^n \to \mathbb{C}^n$ which appear in the context of the *(Grothendieck-)Springer resolution* of the nilpotent cone $\mathcal{N}$. If X_λ is a nilpotent operator with Jordan type λ, then the fibre lying over X_λ is called the *Springer fibre* $\mathcal{B}_\lambda$. Springer showed in 1976 [**Spr76**] that the cohomology ring $H^*(\mathcal{B}_\lambda)$ carries an S_n-action. Furthermore, in top degree, this representation is irreducible, and is precisely the *Specht module* $\mathbb{S}_\lambda$. In fact, all irreducible S_n-modules arise this way; this is known as the *Springer correspondence.*

The inclusion map $\mathcal{B}_\lambda \hookrightarrow Fl_n$ induces a map in cohomology $\Phi : H^*(Fl_n) \to H^*(\mathcal{B}_\lambda)$. Hotta and Springer showed in [**Hot77**] that this map is surjective, and S_n-equivariant, so that one would hope to extend Borel's combinatorial realization of $H^*(Fl_n)$ to the Springer fibre. This was achieved by DeConcini and Procesi in [**CP81**], and the relations were later simplified by Tanisaki in [**Tan82**]. This presentation of $H^*(\mathcal{B}_\lambda)$ is known as the *Garsia-Procesi module* R_λ, as Garsia and Procesi constructed a monomial basis for R_λ in [**GP92**]. This remarkable basis is a subset of the Artin monomials which varies with λ, and so one may ask if such a procedure can be done with the Garsia-Stanton basis. In this work, we answer the question in the affirmative, give the construction the indexing sets $\mathcal{D}_\lambda$, J_λ^{maj}, and prove the following theorem:

Theorem 1.0.1. *There is a set of Garsia-Stanton descent monomials*

$$\{x^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_\lambda\} = \{x^{\mathrm{majt}(\pi)} : \pi \in J_\lambda^{\mathrm{maj}}\}$$

depending on λ that forms a $\mathbb{C}$-basis for the Garsia-Procesi module for the transpose partition*, $R_{\lambda'}$.*

In a subsequent chapter, we give various generalizations (to be stated as conjectures) of our new basis as well.

The *Frobenius character* of the Garsia-Procesi module R_λ is given by the *modified Hall-Littlewood polynomial* $\widetilde{H}_\lambda(X;t)$:

$$\mathrm{Frob}_t(R_\lambda) \cong \widetilde{H}_\lambda(X;t)$$

By theorem 1.0.1, we have that the Hilbert series of R_λ can also be written as

$$\mathrm{Hilb}_t(R_{\lambda'}) = \sum_{\pi \in J_\lambda^{\mathrm{maj}}} t^{\mathrm{maj}(\pi)}$$

In fact, this can be generalized to yield a new formula for the (modified) Hall-Littlewood polynomial:

THEOREM 1.0.2. *We have the following equalities:*

$$\widetilde{H}_{\lambda'}(X;t) = \sum_{\pi \in J_\lambda^{\mathrm{maj}}} t^{\mathrm{maj}(\pi)} F_{\mathrm{iDes}(\pi),n}(X) = \sum_{\mu \vdash n} \Bigg(\sum_{\pi \in J_\lambda^{\mathrm{maj}} \cap \mathrm{Sh}(\mu)} t^{\mathrm{maj}(\pi)} \Bigg) m_\mu(\mathbf{x}) \tag{1.1}$$

where $\mathrm{Sh}(\mu)$ *are the permutations of* S_n *such that* $1, ..., \mu_1$ *appear in order,* $\mu_1 + 1, ..., \mu_2$ *appear in order, and so on.*

These results were actually discovered while working towards a larger goal - finding a $\mathbb{C}$-basis for the Garsia-Haiman module V_λ. We give a bit of background and context here.

Let $R = C[x_1, ..., x_n; y_1, ..., y_n]$. The *Hilbert Scheme of* n *points in* $\mathbb{C}^2$ is the collection of ideals I in R such that the $\mathbb{C}$-dimension of R/I is n:

$$H_n = \mathrm{Hilb}_n(\mathbb{C}^2) = \Big\{ I \subset R : \dim_{\mathbb{C}}(R/I) = n \Big\}$$

The Hilbert Scheme H_n can be thought of as a *resolution of singularities* of the space of unordered tuples of n-points in $\mathbb{C}^2$:

$$S^n(\mathbb{C}^2) = \Big\{ [[P_1, ..., P_n]] : P_i \in \mathbb{C}^2 \Big\}$$

via the *Hilbert-Chow morphism* $\sigma : H_n \to S^n\mathbb{C}^2$, which sends an ideal I to its vanishing locus (with multiplicity), $\sigma(I) = V(I)$. There is a well-known open covering of H_n corresponding to partitions λ of n. Given a partition λ, we may draw the Ferrers diagram in the first quadrant of the xy-plane (in French notation), and consider the monomials with exponents determined by them, denoted M_λ. For example, $M_{2,2,1} = \{1, x, y, xy, xy^2\}$. Then, we can define the open set U_λ to be

$$U_\lambda := \{ I \in H_n : M_\lambda \text{ is a } \mathbb{C}\text{-basis for } R/I \}$$

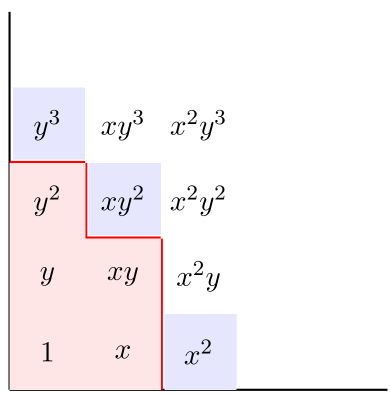
There is a natural (generically) $n!$-sheeted cover of $S^n\mathbb{C}^2$ by considering *ordered* tuples of n-points in $\mathbb{C}^2$, denoted by $(\mathbb{C}^2)^n$, and we may consider the pullback of $(\mathbb{C}^2)^n \to S^n\mathbb{C}^2$ along the map σ to obtain the *isospectral Hilbert Scheme* X_n:

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\pi"] & (\mathbb{C}^2)^n \arrow[d] \\ H_n \arrow[r, "\sigma"] & S^n\mathbb{C}^2 \end{tikzcd}$$

which has the explicit presentation $X_n = \{(I; P_1, ..., P_n) : P_1, ..., P_n \in V(I)\} \subset H_n \times (\mathbb{C}^2)^n$. For the singular point $[[0, ..., 0]]$, and a partition λ, there is a particular ideal $I^\lambda \in U_\lambda$ consisting of monomials that lie outside the Ferrers diagram for λ:

$$I^\lambda := \mathbb{C} \cdot \{x^a y^b : (a, b) \notin \lambda\}$$

For example, $I^{2,2,1} = \{y^3, xy^2, x^2\}$. This can be seen by looking at the diagram:



The coordinate ring of the fibre over I^λ in X_n is a nonreduced local ring $V_\lambda := \mathbb{C}[\mathbf{x}, \mathbf{y}]/J_\lambda$, where J_λ is the annihilating ideal of a certain determinant Δ_λ. V_λ is known as the *Garsia-Haiman module*, and there is a dual version defined as the vector space span of Δ_λ under the action of the partial derivative operators $\partial_{\mathbf{x}}, \partial_{\mathbf{y}}$:

$$V'_\lambda := \mathbb{C}[\partial_{\mathbf{x}}, \partial_{\mathbf{y}}] \cdot \Delta_\lambda$$

Both V_λ and V'_λ carry an S_n action, and are isomorphic as doubly graded S_n modules. Mark Haiman established in [**Hai01**] that the $\mathbb{C}$-dimension of V_λ is $n!$ by a geometric argument, resolving the famous $n!$-*conjecture*. Since the proof of the $n!$-conjecture, however, the following question is still open:

Problem 1. Find a vector space basis of V_λ or V'_λ.

This problem is decidedly difficult. Bases have been constructed for hook shapes (by Adin, Remmel, Roichman in [**ARR07**]), and for two-column shapes (by Assaf-Garsia in [**AG09**]). We will give a conjectured basis for all shapes at top t-degree.

The original motivation for the problem was to first construct a basis at bottom x-degree $q = 0$, and then extend the basis to the entire Garsia-Haiman module. Theorem 1.0.1 is the result of the first step, and conjecture 5.4.5 is a first step of this extension.

The *modified Macdonald polynomials* $\widetilde{H}_\lambda(X;q,t)$ are a family of symmetric functions in $\mathbb{C}(q,t)$ that generalize various families of symmetric functions, most notably the modified Hall-Littlewood polynomials:

$$\widetilde{H}_{\lambda'}(X;q) = \widetilde{H}_\lambda(X;q,0) \qquad\qquad \widetilde{H}_\lambda(X;t) = \widetilde{H}_\lambda(X;0,t)$$

The Garsia-Haiman module was actually constructed in [**GH93**] as a representation-theoretic model of $\widetilde{H}_\lambda(X;q,t)$, so that $\mathrm{Frob}_{q,t}(V'_\lambda) = \widetilde{H}_\lambda(X;q,t)$. The resolution of the $n!$-conjecture proved this statement, which implies the famous *Macdonald positivity conjecture*, that $\widetilde{K}_{\mu\lambda}(q,t) \in \mathbb{N}(q,t)$ in

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\mu\vdash n} \widetilde{K}_{\mu\lambda}(q,t) s_\mu(\mathbf{x})$$

Problem 2. Find a combinatorial formula for $\widetilde{K}_{\mu\lambda}(q,t)$.

Partial results include the case of three columns by Blasiak [**Bla14**], hooks and two columns by Assaf [**Ass17**], and the original Yamanouchi-word formula for two-column shapes in [**HHL05b**]. The existence of a basis for V_λ would be a significant step towards establishing a combinatorial formula for $\widetilde{K}_{\mu\lambda}(q,t)$.

CHAPTER 2

Background

2.1. The Symmetric Group, Permutation Statistics, and the Coinvariant Algebra

Let $n \in \mathbb{N}$. The symmetric group S_n lies at the intersection of algebraic combinatorics, representation theory, symmetric function theory, algebraic geometry, among other fields. It is given by the following definition:

$$S_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i^2 = 1 & \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \geqslant 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \ldots, n-2 \end{array} \right\rangle$$

where the σ_i are referred to as *Artin generators* or *simple transpositions*, and the second two relations are known as the *braid relations*. Letting $[n] = \{1, 2, \ldots n\}$, we can represent elements of S_n as bijections $\pi : [n] \to [n]$. It will be very advantageous to represent π in *one-line notation*, by writing $\pi = \pi(1)\ldots\pi(n) = \pi_1 \ldots \pi_n$. We will use the notations interchangeably when it does not cause confusion. Then, the Artin generator σ_i can be identified with the permutation $12\ldots(i+1)i\ldots n$ in one-line notation.

Next, for $k, n \in \mathbb{N}$, let

$$[k]_q = 1 + q + \ldots + q^{k-1}$$

$$[n]_q! = [n]_q [n-1]_q \ldots [2]_q [1]_q$$

denote the q-analogues of k and $n!$. If we specialize to $q = 1$, we have that $[k]_1 = k$, and $[n]_1! = n!$.

The combinatorial structure of S_n is a rich and fascinating subject, having preoccupied combinatorialists for decades. Among one of the most interesting subjects is that of *permutation statistics* associated to S_n (or more general objects), that is, a weight function stat : $S_n \to \mathbb{Z}_{\geqslant 0}$. In our case, we will be particularly interested in statistics such that

$$\sum_{\pi \in S_n} q^{\mathrm{stat}(\pi)} = [n]_q!$$

We give two such statistics below, both of which are well known.

2.1.1. Inversions and Lehmer Codes. Let $\pi \in S_n$. Then, consider the set of inversions of π, denoted

$$\mathrm{Inv}(\pi) = \left\{ (\pi_i, \pi_j) : i < j, \pi_i > \pi_j \right\}$$

in other words, the number of pairs (π_i, π_j) such that the larger entry appears to the left. In this scenario, we say that π_i *attacks* π_j. Denote $\mathrm{inv}(\pi) = \# \mathrm{Inv}(\pi)$, and let $b_k = \#\{(x, k) \in \mathrm{Inv}(\pi)\}$ denote the number of elements that attacks k in π. Then, define the *Lehmer code* or the *inversion table* of π to be given by the following tuple:

$$\mathrm{invt}(\pi) = (b_1, ..., b_n)$$

We denote the set $\{\mathrm{invt}(\pi) : \pi \in S_n\} = \mathcal{E}_n$. Now, let $\overline{[k]} = \{0, 1, ..., k\}$. It is well known (for example, see [**Sta11**]) that

$$\left\{ \mathrm{invt}(\pi) : \pi \in S_n \right\} = \overline{[n-1]} \times ... \times \overline{[0]}$$

Noting that $\mathrm{inv}(\pi) = b_1 + ... + b_n$, we see that

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{b_1 + ... + b_n} = \sum_{(b_1, ..., b_n) \in \overline{[n-1]} \times ... \times \overline{[0]}} q^{b_1 + ... + b_n}$$
$$= \left(\sum_{b_1 \in \overline{[n-1]}} q^{b_1} \right) ... \left(\sum_{b_n \in \overline{[0]}} q^{b_n} \right) = [n]_q ... [1]_q = [n]_q!$$

If we write $\pi = \sigma_{i_1} ... \sigma_{i_k}$ as a *reduced word* (so that k is minimized), then it will turn out that $k = \mathrm{inv}(\pi)$.

EXAMPLE 2.1.1. Let $\pi = 53421$. Then, the inversions are

$$\mathrm{Inv}(53421) = \{(5,3),(5,4),(5,2),(5,1),(3,2),(3,1),(4,2),(4,1),(2,1)\}$$

so that $\mathrm{inv}(53421) = 9$ and $\mathrm{invt}(53421) = (4,3,1,1,0)$.

2.1.2. Major Index and Descent Tables. We now define a second permutation statistic called *major index*, named after Major Percy MacMahon.

Let $\pi \in S_n$. A *descent* of a permutation is an index i such that $\pi_i > \pi_{i+1}$. The set of descents of π is denoted $\mathrm{Des}(\pi)$, and let $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$ denote the *major index* of π, which is the sum of the indices of the descents.

Denote the *runs* of π to be the maximal consecutive increasing subsequences, and label the runs from right to left, beginning with 0. Let a_i denote the run label of i, and set

$$\mathrm{majt}(\pi) = (a_1, ..., a_n)$$

We will refer to $\mathrm{majt}(\pi)$ as the *descent composition* or the *major index table* of π, and set $\mathcal{D}_n = \{\mathrm{majt}(\pi) : \pi \in S_n\}$ to be the collection of major index tables. It is straightforward to show that $\mathrm{maj}(\pi) = a_1 + ... + a_n$.

We record an important fact here.

LEMMA 2.1.2. *Let $\mathbf{a} \in \mathbb{Z}^n_{\geqslant 0}$. Then, $\mathbf{a} \in \mathcal{D}_n$ if and only if $\mathbf{a}$ contains a 0, and the rightmost i contains and $i-1$ to the left for all $i > 0$.*

PROOF. Suppose $\mathbf{a} \in \mathcal{D}_n$, with $\mathrm{majt}(\pi) = \mathbf{a}$. Then, the last entry $\pi_j = i$ of the rth run (if r is not the last run) is necessarily a descent. This implies that $\pi_j = i > i' = \pi_{j+1}$, so that $i' < i$ and $a_{i'} = r - 1$ appears to the left of a_i. There must be a 0 because if π is nonempty, the run labels always begin with 0 from the rightmost run. To see the converse, the existence of a 0 asserts the existence of the rightmost run, and the second condition asserts that the final entries of each run are greater than the first entries of the run to the right of it. This guarantees that we have a well-defined permutation. $\square$

EXAMPLE 2.1.3. Let $\pi = 81725346$. Then, the runs are $8, 17, 25, 346$ with labels $3, 2, 1, 0$ respectively, and $\text{majt}(\pi) = 21001023$.

We record another useful definition and lemma.

DEFINITION 2.1.4. An *inverse descent* of a permutation π is an entry i such that $\pi^{-1}(i)$ is a descent of π^{-1}. Equivalently, $i \in \text{iDes}(\pi)$ if and only if $i + 1$ appears to the left of i in one-line notation.

LEMMA 2.1.5. *Let* $\pi \in S_n$, $\text{majt}(\pi) = (a_1, \ldots, a_n)$. *We have that* $i \in \text{iDes}(\pi)$ *if and only if* $a_i < a_{i+1}$.

PROOF. That $i \in \text{iDes}(\pi)$ is equivalent to the statement that $i + 1$ occurs before i in one line notation. This is true if and only if $i + 1$ occurs in an earlier run, as a descent must occur between them. But this is precisely the assertion $a_i < a_{i+1}$. □

In 1913, MacMahon showed that

$$\sum_{\pi \in S_n} q^{\text{maj}(\pi)} = [n]_q! = \sum_{\pi \in S_n} q^{\text{inv}(\pi)}$$

so that maj is *equidistributed* with inv for S_n.

REMARK 2.1.6. For direct bijections $f : S_n \to S_n$ that swaps inv and maj, there is the *Foata bijection* described in [**Sta99**] or the *Carlitz bijection* described in [**Gil16**].

2.1.3. The Coinvariant Algebra, Artin and Garsia-Stanton Descent Monomials. The coinvariant algebra is defined to be

$$R_n = \frac{\mathbb{C}[x_1, ..., x_n]}{\langle e_1(\mathbf{x}), ..., e_n(\mathbf{x})\rangle}$$

is the quotient of the polynomial ring in n variables by the ideal of nonconstant symmetric functions. Here, $e_d(\mathbf{x})$ denotes the sum of all degree d squarefree monomials in the variables $x_1, ..., x_n$.

It is well known that R_n is a finite-dimensional vector space over $\mathbb{C}$ of dimension $n!$. There are two well-known bases for R_n, which we now define. Given a composition $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_{\geqslant 0}^n$, define $x^\alpha \in \mathbb{C}[x_1, ..., x_n]$ by $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$, and set $\mathrm{rev}(\alpha) = (\alpha_n, ..., \alpha_1)$.

- The *Artin basis* is defined to be the set $A_n = \{x^{\mathrm{rev}(\mathrm{invt}(\pi))} : \pi \in S_n\}$
- The *Garsia-Stanton basis* ([**GS84**]) is defined to be the set $D_n = \{x^{\mathrm{majt}(\pi)} : \pi \in S_n\}$

which correspond to the permutation statistics inv and maj respectively.

Remark 2.1.7. The usual definition for the Garsia-Stanton monomial is

$$g_\pi(\mathbf{x}) = \prod_{i : \pi_i > \pi_{i+1}} x_{\pi_1} ... x_{\pi_i}$$

It is straightforward to show that the two definitions agree.

The coinvariant algebra R_n is naturally graded by degree, so that we may write

$$R_n = \bigoplus_{i=0}^{\binom{n}{2}} R_n^{(i)}$$

where $R_n^{(i)}$ denotes the degree i part of R_n. Then, since A_π (or D_n) is a basis of R_n, we must have that

$$\sum_{i=0}^{\binom{n}{2}} \dim_{\mathbb{C}} \big(R_n^{(i)}\big) q^i = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]_q!$$

Remark 2.1.8. Borel showed in 1951 that $R_n \cong H^*(Fl_n)$, the cohomology ring of the *complete flag variety* Fl_n. Indeed, there is an affine paving (by Schubert cells) of Fl_n whose cells are indexed by permutations, the dimension of which is precisely given by $\mathrm{inv}(\pi)$. It will turn out that inv is more illuminating for geometric computations, but maj is more natural for representation-theoretic concerns.

2.1.4. Words. Given an alphabet $\mathcal{A}$ (usually $\mathbb{Z}_{\geqslant 0}$ or $\mathbb{N}$), a word of length n is a string consisting of entries from $\mathcal{A}$. We will denote the set of words with entries in $\mathcal{A}$ by $W(\mathcal{A})$, and the subset of words of length n by $W_n(\mathcal{A})$. When the alphabet is clear, we will simply denote this by W_n. We will alternate between the notations $w = w_1 \dots w_n = (w_1, \dots, w_n)$ depending on the context, when it does not cause confusion.

If $\mathcal{A} = \{a_1 < a_2 < \dots\}$ is totally ordered and countable, we may define the *content* of a word to be

$$\text{content}(w) = (b_1, b_2, \dots) \in \bigoplus_{i=1}^{|\mathcal{A}|} \mathbb{Z}_{\geqslant 0}$$

where $b_i = |\{j : w_j = a_i\}|$ denote the number of times a_i appears in w. We will assume all alphabets to be totally ordered and countable.

If $\mathcal{A} = \mathbb{N}$, then S_n can be identified with the collection of words of content $(\underbrace{1, \dots, 1}_{n}, 0, \dots)$.

If $w = w_1 \dots w_n \in W_n$ we define the *reverse* of the word to be $\text{rev}(w) = w_n \dots w_1$. Very often, words will index monomials in a polynomial ring, and so given a set of indeterminates $\{x_a : a \in \mathcal{A}\}$ indexed by an alphabet $\mathcal{A}$, we will define

$$x_w = x^{|w|} = x_{w_1} \dots x_{w_n}$$

Let $\mathcal{A} = \{a_1 < a_2 < \dots\}$, and $w \in W_n(\mathcal{A})$ be a word with content $(b_1, b_2, \dots)$ Then we may define the *standardization* of the word to be the result of the following procedure:

(1) Let b_j is the smallest nonzero entry of content(w), let $w_{i_1}, \dots, w_{i_{b_j}}$ denote the entries with $w_{i_k} = j$. Replace these entries with $a_1, \dots, a_{b_j}$ respectively.

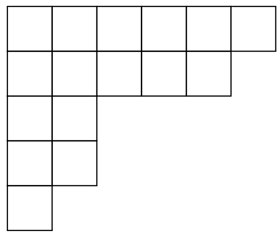
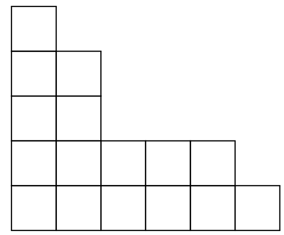
(2) Repeat for other nonzero b_j's following alphabetical order.

2.2. Partitions, Tableaux, Robinson-Schensted-Knuth Correspondence

We now discuss the representation theory of S_n.

2.2.1. Partitions and Tableaux. A *partition* of n is a weakly decreasing sequence of numbers $\lambda = (\lambda_1, ..., \lambda_\ell)$ with $\lambda_1 \geqslant ... \geqslant \lambda_\ell$ such that $\lambda_1 + ... + \lambda_\ell = n$. We often denote this by $\lambda \vdash n$. A *Young diagram* of a partition λ is a finite collection of left-justified boxes (or cells) such that the number of boxes in each row corresponds to the parts of λ.

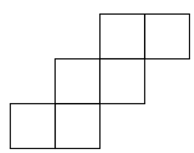
There are two conventions for Young diagrams, both of which we will use. In *English notation*, the rows will weakly decrease in size moving downward, and weakly increasing for *French notation*. For instance if $\lambda = (6, 5, 2, 2, 1)$:

English:  French: 

We will often fill Young tableau with numbers; these will be referred to as *Young tableaux*. In English notation (resp: French), Young tableau is *semistandard* if it weakly increases along rows, and strictly increases downward (resp: upward) along columns, and *standard* if it strictly increases along rows and columns. The *content* of a Young tableaux T is a composition $\alpha = (\alpha_1, ..., \alpha_k)$, where α_i is the number of i's that appear in T. We denote the set of standard Young tableaux of shape λ by $\mathrm{SYT}(\lambda)$, the set of semistandard Young tableaux of shape λ by $\mathrm{SSYT}(\lambda)$, and the subset of $\mathrm{SSYT}(\lambda)$ with content α by $\mathrm{SSYT}(\lambda, \alpha)$.

Let λ, μ be partitions, with the Young diagram of μ completely contained in the diagram of λ. The *skew shape* λ/μ is the partition obtained by deleting the shape μ from λ. A *ribbon* is a skew shape that does not contain a 2×2 box.

EXAMPLE 2.2.1. If $\lambda = (4, 3, 2)$ and $\mu = (2, 1)$, then in English notation,

$\lambda/\mu =$ 

which is a ribbon, as it contains no 2×2 box.

We will let λ' denote the *conjugate* or the *transpose* of a partition, and

$$\eta(\lambda) = \sum_i \binom{\lambda'_i}{2} = \sum_i (i-1)\lambda_i$$

denote the usual Macdonald statistic.

There is an important partial order on partitions, known as *dominance order*.

Definition 2.2.2. We say that $\mu \trianglelefteq \lambda$, or μ *dominates* λ if:

$$\mu_1 \leqslant \lambda_1$$

$$\mu_1 + \mu_2 \leqslant \lambda_1 + \lambda_2$$

$$\cdots$$

$$\mu_1 + \cdots + \mu_{\min\{\ell(\mu,\lambda)\}} \leqslant \lambda_1 + \cdots + \lambda_{\min\{\ell(\mu,\lambda)\}}$$

It is well known that $\mu \triangleleft \lambda$ implies that $\eta(\mu) > \eta(\lambda)$, and that $\mu \trianglelefteq \lambda$ if and only if $\mu' \trianglerighteq \lambda'$.

More generally, *composition* of n is a tuple $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{Z}^k_{>0}$ such that $\alpha_1 + ... + \alpha_k = n$. An $\alpha \in \mathbb{Z}^k_{\geqslant 0}$ is referred to as a *weak composition*. The *underlying partition* of a composition α is the partition $\lambda = \mathrm{sort}(\alpha)$ obtained by sorting the parts of α in weakly decreasing order.

Very often, compositions will index exponents in a polynomial ring in the variables $x_1, ..., x_n$. As shorthand, we will often write, for a weak composition $\alpha \in \mathbb{Z}^n_{\geqslant 0}$,

$$x^\alpha := x_1^{\alpha_1} ... x_n^{\alpha_n}$$

2.2.2. The Robinson-Schensted-Knuth Correspondence, Knuth Equivalence. For brevity, we refer the reader to [**Sta99**] for details, and give only an overview here.

The *Robinson-Schensted-Knuth correspondence* (often abbreviated by RSK) is a theorem at the heart of representation theory, which is a bijection

$$S_n \longleftrightarrow \bigsqcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$$

which uniquely associates a permutation to a pair of standard Young tableaux of the same shape. We will often write $\pi \mapsto (P, Q)$, where P is the *insertion tableau*, and Q is the *recording tableau*, and write $P = \mathrm{ins}(\pi), Q = \mathrm{rec}(\pi)$. P is obtained by performing *row insertion* on the permutation π, and Q records the position of each new entry.

Since row insertion is well defined for *words* in the alphabet $\mathbb{N}$, letting W_n denote the words with length n, we may perform RSK to obtain a correspondence

$$W_n \longleftrightarrow \bigsqcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda, d) \times \mathrm{SYT}(\lambda)$$

where now the insertion tableau need not be standard, as W_n may have repeated letters, and $\mathrm{SSYT}(\lambda, n)$ denotes the semistandard Young tableau of shape λ whose entries are no greater than n. We remark that there are more general versions or RSK, but the above two will suffice for our purposes.

Let $W(\mathcal{A})$ denote the set of words in an alphabet $\mathcal{A}$. The *elementary Knuth transformations* (or Knuth moves) on $W_{\mathcal{A}}$ are given by:

$$xzy \longleftrightarrow xzy, \qquad x \leqslant y < z$$

$$yxz \longleftrightarrow yzx, \qquad x < y \leqslant z$$

We say that two words w, w' are *Knuth-equivalent* if w can be obtained from w' by performing a series of Knuth moves. Define the *plactic monoid* on $\mathcal{A}$ to be the $W(\mathcal{A})$ modulo Knuth equivalence, denoted $W(\mathcal{A})^*$.

Proposition 2.2.3. *[**Ful96**] Two words w, w' are Knuth-equivalent if and only if* $\mathrm{ins}(w) = \mathrm{ins}(w')$*, that is, they have the same insertion tableau.*

The *elementary dual Knuth transformation* τ_i exchanges $i, i+1$ if and only if $i-1$ or $i+2$ occur between them. If w can be obtained from w' by performing a series of elementary dual Knuth transformations, we say that w, w' are *dual Knuth equivalent.* For example, $w = \underline{4}32\underline{5}1 \equiv \underline{5}32\underline{4}1 = w'$ since 3 occurs between $4, 5$, and $\tau_3(w) = w'$. We have the following proposition:

Proposition 2.2.4. *[**Ful96**] Two words w, w' are dual Knuth equivalent if and only if* $\mathrm{rec}(w) = \mathrm{rec}(w')$*, that is, they have the same recording tableau.*

This follows from 2.2.3 since dual Knuth equivalence of w, w' is the same as Knuth equivalence of their inverses (defined via generalized permutations, see [**Ful96**]). Then the proposition follows from the fact that if $w \mapsto (P, Q)$, then $w^{-1} \mapsto (Q, P)$, which is well known.

2.3. Symmetric Functions and Frobenius Characteristic of S_n-modules

We now give a brief overview of symmetric functions.

2.3.1. Basic Notions. Let R be a ring (usually $R = \mathbb{C}$ or $R = \mathbb{Q}$), and consider the formal power series ring in infinitely many variables $S = R[[x_1, x_2, ...]]$. We often abbreviate $f(x_1, x_2, ...) = f(\mathbf{x}) = f$. Let S_∞ denote the group of permutations of $\mathbb{N}$. We can define an action of S_∞ on S by setting $\pi \cdot f(x_1, x_2, ...) = f(x_{\pi(1)}, x_{\pi(2)}, ...)$ for $\pi \in S_\infty$. Then, we say that $f \in S$ is a *symmetric function* if $\pi \cdot f = f$ for all $\pi \in S_\infty$.

It is easy to check that collection of symmetric functions $\Lambda_R \subset S$ is a ring, and naturally inherits the grading of S. Denoting the homogeneous degree n symmetric functions by Λ_R^n, we may write

$$\Lambda_R = \bigoplus_{n=0}^{\infty} \Lambda_R^n$$

with $\Lambda_R^0 = R$.

If R is a field, then Λ_R has a natural structure of a R-vector space. We will always choose $R = \mathbb{Q}$. We give a few well-known bases:

- The *monomial symmetric functions*

$$m_\lambda(\mathbf{x}) = \sum_{i_1, i_2, ..., i_l \text{distinct}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} ... x_{i_l}^{\lambda_l}$$

- The *elementary symmetric functions* $e_\lambda(\mathbf{x}) = e_{\lambda_1} e_{\lambda_2} ... e_{\lambda_l}$ where

$$e_r(\mathbf{x}) = \sum_{i_1 < ... < i_r} x_{i_1} ... x_{i_r}$$

- The *power sum symmetric functions* $p_\lambda(\mathbf{x}) = p_{\lambda_1} p_{\lambda_2} ... p_{\lambda_l}$ where

$$p_r(\mathbf{x}) = \sum_i x_i^r$$

- The *complete homogeneous symmetric funtions*, $h_\lambda = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_l}$ where

$$h_r(\mathbf{x}) = \sum_{i_1 \leqslant ... \leqslant i_r} x_{i_1} ... x_{i_r}$$

- The *Schur functions*

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{|T|}$$

, where $x^{|T|} = x_1^{T_1} x_2^{T_2} \dots$ and T_i is the number of i's in T.

The *Hall inner product* is defined to be orthonormal on the Schur basis, that is,

$$\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda\mu}$$

and extended linearly in both arguments.

2.3.2. S_n-modules and the Frobenius Characteristic Map Frob**.** It is well known that the irreducible S_n representations are indexed by partitions $\lambda \vdash n$, and are referred to as *Specht modules*. We will denote the unique irreducible corresponding to λ by $\mathbb{S}_\lambda$.

Given an S_n-module V (over a field of characteristic 0) we may write

$$V = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus c_\lambda}$$

where c_λ is the *multiplicity* of V_λ. Then, the *Frobenius characteristic* of V is given by

$$\mathrm{Frob}(V) = \sum_{\lambda \vdash n} c_\lambda s_\lambda(\mathbf{x})$$

If V is a graded S_n-module, say $V = \bigoplus_{i=1}^k V^{(i)}$, then we write

$$\mathrm{Frob}_q(V) = \sum_{i=1}^{k} \mathrm{Frob}(V^{(i)}) q^i \in \Lambda_{\mathbb{Q}(q)}$$

and for doubly-graded S_n-modules, say $V = \bigoplus_{(i,j)} V^{(i,j)}$, we write

$$\mathrm{Frob}_{q,t}(V) = \sum_{i,j} \mathrm{Frob}(V^{(i,j)}) q^i t^j \in \Lambda_{\mathbb{Q}(q,t)}$$

The *Hilbert series* of a graded (or bigraded) S_n module is given by

$$\mathrm{Hilb}_q(V) = \langle h_{(1^n)}, \mathrm{Frob}_q(V) \rangle = \sum_i \dim_{\mathbb{Q}}(V^{(i)}) q^i$$

$$\mathrm{Hilb}_{q,t}(V) = \langle h_{(1^n)}, \mathrm{Frob}_{q,t}(V)\rangle = \sum_{i,j} \dim_{\mathbb{Q}}(V^{(i,j)}) q^i t^j$$

We can think of the Hilbert series as a generating function for the $\mathbb{Q}$-dimensions in each graded component of V.

2.3.3. Quasisymmetric Functions. A quasisymmetric function (as defined in [**Sta99**]) in $\mathbb{Q}[[x_1, x_2, ...]]$ is a formal power series $f = f(\mathbf{x})$ such that for any $(a_1, ..., a_k) \in \mathbb{N}^k$, we have

$$[x_{i_1}^{a_1} \dots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \dots x_{j_k}^{a_k}]f$$

for all pairs of strictly increasing sequences $i_1 < ... < i_k$ and $j_1 < ... < j_k$, and $[x^\alpha]f$ denotes the coefficient of x^α in f.

We can similarly study the $\mathbb{Q}$-vector space structure of $\mathcal{Q}^n$, the space of homogeneous degree n quasisymmetric functions. As symmetric functions are indexed by partitions $\lambda \vdash n$, quasisymmetric functions are often indexed by *compositions* α of n. Following Stanley's notation, given $\alpha = (\alpha_1, ..., \alpha_k)$, let $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, ...\alpha_1 + ... + \alpha_{k-1}\}$.

Then, define the *Gessel fundamental quasisymmetric functions* $F_{\alpha,n}(\mathbf{x})$ to be given by

$$F_{S_\alpha,n}(\mathbf{x}) = F_{\alpha,n}(\mathbf{x}) := \sum_{\substack{i_1 \leqslant ... \leqslant i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} ... x_{i_n}$$

in other words, monomials whose indices must necessarily increase at the positions marked by α. It is well known that the F_α's form a $\mathbb{Q}$-basis of $\mathcal{Q}^n$. Very often, S_α will be a descent set, or an inverse descent set of a permutation.

2.4. Macdonald Polynomials and the Garsia-Haiman Module

2.4.1. Combinatorial Formula for the (modified) Macdonald Polynomials. Haglund [**HHL05b**] made a breakthrough (or as Garsia put it, "found water on Mars") in 2005 when he discovered a combinatorial formula for the modified Macdonald polynomials $\widetilde{H}_\lambda(X; q, t)$. We briefly recall the definition below.

Let $\lambda \vdash n$, and we will consider fillings $\sigma : \lambda \to \mathbb{Z}_{>0}$ of positive integers, where λ is drawn in French notation. A *descent* is a pair of entries that are connected vertically, with the greater top entry. If $a > b$, then the square containing a in

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}$$

is a descent. Denote the set of descents by $\mathrm{Des}(\sigma)$. An *attacking pair* in λ is a pair of cells such that:

- They are in the same row: $\boxed{a} \qquad\qquad \boxed{b}$
- They are in consecutive rows, with the cell in the upper row strictly to the right:

$$\begin{array}{|c|c|c|c|c|}\hline & & \dots & & a \\ \hline b \\ \cline{1-1} \end{array}$$

In both cases, if $a > b$, then the pair is said to be an *inversion* of σ. We denote the inversions of σ by $\mathrm{Inv}(\sigma)$.

The *arm* of a cell $u \in \lambda$ is the number of cells strictly to the right in the same row of u, and the *leg* is the number of cells strictly above in the same column. For example, we have $\mathrm{arm}(u) = 4$ and $\mathrm{leg}(u) = 2$ in

$$\begin{array}{|c|c|c|c|c|c|}\cline{1-3} & l & \\ \cline{1-4} & l & & & \multicolumn{2}{c}{} \\ \hline & u & a & a & a & a \\ \hline \end{array}$$

Then, Haglund's statistics are given as follows:

$$\mathrm{maj}(\sigma) = \sum_{u \in \mathrm{Des}(\sigma)} (\mathrm{leg}(u) + 1)$$

$$\mathrm{inv}(\sigma) = |\,\mathrm{Inv}(\sigma)| - \sum_{u \in \mathrm{Des}(\sigma)} \mathrm{arm}(u)$$

Haglund's major index can be thought of as major index of each column individually, viewed as words. The inversion statistic can be thought of as counting the inversion pairs in the bottom row, as well as triples of the form

$$\begin{array}{|c|}\hline u \\ \hline v \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline w \\ \hline \end{array}$$

which contribute 1 to $\mathrm{inv}(\sigma)$ if the entries increase in counterclockwise order, and contribute 0 if they increase in clockwise order.

THEOREM 2.4.1. *[**HHL05b**] We have*

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma:\lambda\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma \tag{2.1}$$

Consider the Schur expansion of $\widetilde{H}_\lambda(X;q,t)$:

$$\widetilde{H}_\lambda(X;q,t) = \sum_\mu \widetilde{K}_{\mu,\lambda}(q,t)s_\mu(\mathbf{x})$$

where $\widetilde{K}_{\mu,\lambda}(q,t)$ are the famous *q,t-Kostka numbers*. It is quite a remarkable fact that $\widetilde{K}_{\mu,\lambda}(q,t) \in \mathbb{N}(q,t)$, yet a combinatorial formula for all shapes remains a great mystery at the time of this writing.

The Macdonald polynomials are uniquely defined by a set of triangularity axioms, which we record here for later use:

- (T1). $\widetilde{H}_\lambda[X(1-q);q,t] = \sum_{\mu\trianglerighteq\lambda} c_{\mu\lambda}(q,t)s_\mu(\mathbf{x})$
- (T2). $\widetilde{H}_\lambda[X(1-t);q,t] = \sum_{\mu\trianglerighteq\lambda'} d_{\mu\lambda}(q,t)s_\mu(\mathbf{x})$
- (N). $\langle \widetilde{H}_\lambda, s_{(n)}\rangle = 1$

2.4.2. The Garsia-Haiman Module. Let $\lambda \vdash n$ be a partition, and consider the set of integer points in the first quadrant of the xy-plane corresponding to the Young diagram of λ:

$$d(\lambda) = \{(p,q) : p < \lambda_{q+1}\}$$

for example, for $\lambda = (2,2,1)$, we have

$$\begin{array}{|c|c|}\cline{1-1} (0,2) & \multicolumn{1}{c}{} \\ \hline (0,1) & (1,1) \\ \hline (0,0) & (1,0) \\ \hline \end{array}$$

Let $R = \mathbb{C}[x_1, ..., x_n; y_1, ..., y_n] = \mathbb{C}[\mathbf{x}, \mathbf{y}]$ consider the R-valued matrix $M = (x_i^{p_j} y_i^{q_j})_{1 \leqslant i,j \leqslant n}$. Denote its determinant by Δ_M, and consider the $\mathbb{C}$-vector space spanned by all partial derivatives of Δ_M:

$$V'_\lambda := \mathbb{C}[\partial \mathbf{x}, \partial \mathbf{y}] \cdot \Delta_M$$

We refer to V'_λ as the *Garsia-Haiman module*, first defined in [**GH93**]. Let S_n act on V'_λ diagonally, that is, $\pi \cdot f(x_1, ..., x_n; y_1, ..., y_n) = f(x_{\pi(1)} ... x_{\pi(n)}; y_{\pi(1)}, ..., y_{\pi(n)})$. The S_n-module structure of V'_λ played an instrumental role in the resolution of the *Macdonald positivity conjecture*, which states that $\widetilde{H}_\lambda(X; q, t)$ is Schur positive. The final step is known as the $n!$-theorem proved by Mark Haiman:

Theorem 2.4.2. *[**Hai01**] The $\mathbb{C}$-dimension of V'_λ is $n!$.*

Corollary 2.4.3. *[**GH93**] We have that*

$$\widetilde{K}_{\mu,\lambda}(q, t) = \sum_{r,s} \langle \chi^\mu, \mathrm{Frob}_{q,t}(V'_\lambda)^{(r,s)} \rangle$$

so that $\mathrm{Frob}_{q,t}(V'_\lambda) = \widetilde{H}_\lambda(X; q, t)$*, and so* $\widetilde{H}_\lambda(X; q, t)$ *is Schur-positive.*

2.5. Hall-Littlewood Polynomials and the Garsia-Procesi Module

2.5.1. (Modified) Hall-Littlewood Polynomials.

Definition 2.5.1. [**LS78**] Let $\pi \in S_n$. Then, define the *charge* of π to be

$$c(\pi) := \mathrm{maj}(\mathrm{rev}(\pi^{-1})) = \sum_{i \notin \mathrm{Des}(\sigma^{-1})} (n - i)$$

Given a word w with in the alphabet $\mathbb{N}$ with content $\mu \vdash n$ with length ℓ, we can compute charge of w by computing the *standard subwords* $w^{(1)}, ..., w^{(\ell)}$, which are obtained by cyclically moving left to right, extracting the first instance of $1, 2, ..., \mu_i$, then removing the subword $w^{(i)}$. We refer the reader to [**LS78**] for more details.

Then, noting that each $w^{(i)} \in S_{\mu_i}$, the charge of a word is defined to be

$$c(w) := c(w^{(1)}) + \ldots + c(w^{(\ell)})$$

and the charge of a tableau $T \in \mathrm{SSYT}(\lambda, \mu)$ is defined to be the charge of its reading word.

For partitions λ, μ, the *Kostka-Foulkes* polynomials $K_{\lambda,\mu}(q)$ are defined to be

$$K_{\lambda,\mu}(q) := \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{c(T)} \tag{2.2}$$

and the *transformed Hall-Littlewood polynomials* $H_\lambda(X;q)$ can be defined by

$$H_\lambda(X;q) := \sum_{\mu} K_{\mu,\lambda}(q) s_\mu(\mathbf{x}) \tag{2.3}$$

Applying a slight substitution gives a more combinatorially natural version:

Definition 2.5.2. The *modified Hall-Littlewood polynomial* is given by:

$$\widetilde{H}_\lambda(X;q) := q^{\eta(\lambda)} H_\lambda(X;q^{-1})$$

The modified Hall-Littlewood polynomials $\widetilde{H}_\lambda(X;q)$ can be recovered as the specialization of 2.1 to $q = 0$, or by symmetry, the specialization of $\widetilde{H}_{\lambda'}(X;q,0)$ to $t = 0$.

In lieu of equation 2.2, we may apply the same substitution to obtain the *modified Kostka-Foulkes polynomials*:

$$\widetilde{K}_{\lambda,\mu}(q) = q^{\eta(\mu)} K_{\lambda,\mu}(q^{-1}) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{cc(T)}$$

where $cc(T) = \binom{n}{2} - c(T)$ is the *cocharge* of a tableau. We remark that there is an algorithmic way to directly calculate cocharge, as given in [**Gil15**].

We give another important characterization of the Hall-Littlewood polynomials, obtained by substituting $q = 0$ in $\widetilde{H}_\lambda(X;q,t)$ axioms above. These turn out to agree with Lusztig's orthogonality relations in [**Lus03**].

Theorem 2.5.3. *The modified Hall-Littlewood polynomials are uniquely characterized by the following axioms:*

- *(H1).* $\widetilde{H}_\lambda(X;t) = \sum_{\mu \trianglerighteq \lambda} c_{\mu\lambda}(t) s_\mu(\mathbf{x})$
- *(H2).* $\widetilde{H}_\lambda[X(1-t);t] = \sum_{\mu \trianglerighteq \lambda'} d_{\mu\lambda}(t) s_\mu(\mathbf{x})$
- *(N).* $\langle \widetilde{H}_\lambda, s_\lambda \rangle = 1$

2.5.2. The Garsia-Procesi Module. Let $\lambda \vdash n$, and denote the conjugate partition by $\lambda' = (\lambda'_1 \geqslant ... \geqslant \lambda'_n \geqslant 0)$, padded with 0's to be of length n. Let $p^n_m(\lambda) = \lambda'_n + ... + \lambda'_{n-m+1}$ for $1 \leqslant m \leqslant n$. Let $\mathbb{C}[x_1, ..., x_n]$ be the polynomial ring in n-variables. Given $S \subset [n]$, denote by $e_d(S)$ to be the sum of all degree d squarefree monomials with labels in S:

$$e_d(S) := \sum_{\substack{i_1 < ... < i_d \\ i_j \in S}} x_{i_1} ... x_{i_d}$$

The *Tanisaki ideal* I_λ is given by

$$I_\lambda := \left\langle e_d(S) : S \subseteq [n], d > |S| - p^n_{|S|}(\lambda) \right\rangle$$

and the Garsia-Procesi module is defined to be

$$R_\lambda := \mathbb{C}[x_1, ..., x_n]/I_\lambda$$

There is an obvious action of S_n on $\mathbb{C}[x_1, ..., x_n]$, given by $\pi \cdot f(x_1, ..., x_n) = f(x_{\pi(1)}, ..., x_{\pi(n)})$. Since the ideal I_λ is S_n-stable, there is a well defined action of S_n on R_λ. The following theorem describes the Frobenius character:

Theorem 2.5.4. *[**Spr76**] [**Spr78**] We have that*

$$\mathrm{Frob}_q(R_\lambda) = \widetilde{H}_\lambda(X;q)$$

Garsia and Procesi recursively constructed a monomial basis for R_λ in their 1992 paper [**GP92**]. We briefly recall the construction below.

Let $\lambda \vdash n$ be a partition, and denote by $\lambda^{(i)} = |(\lambda_1, ..., \lambda_i - 1, ..., \lambda_\ell)|$ to be the partition obtained by subtracting 1 from the i-th part, and rearranging the parts as necessary. Given a set S of monomials in $x_1, ..., x_n$, denote

$$x^\alpha S := \{x^\alpha \cdot x^\beta : x^\beta \in S\}$$

Setting $\mathfrak{B}(\mu) = \{1\}$ for $\mu = (1)$, and for $\lambda \vdash n$ we can recursively define

$$\mathfrak{B}(\lambda) = \bigsqcup_{i=1}^{\ell} x_n^{i-1} \mathfrak{B}(\lambda^{(i)})$$

THEOREM 2.5.5. *[**GP92**] The monomials $\mathfrak{B}(\lambda)$ form a $\mathbb{C}$-basis for R_λ.*

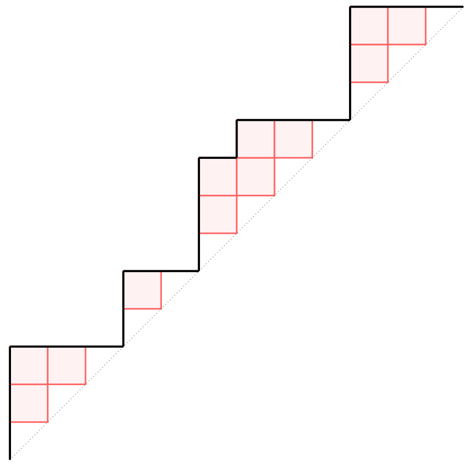
2.6. Dyck Paths, Parking Functions, and the Shuffle Theorem

We closely follow the conventions of Haglund in [**Hag08**].

2.6.1. Dyck Paths and Parking Functions. An (m, n)-Lattice path is a sequence of north steps $(x, y) \to (x, y+1)$ and east steps $(x, y) \to (x+1, y)$ in the first quadrant of the x, y-plane, beginning at $(0, 0)$ and ending at (m, n). Denote the set of such paths by $L_{m,n}$, and denote $L^+_{m,n}$ to be the set of such paths that do not pass under the line $y = \frac{n}{m}x$. A *Dyck path* is an element of $L^+_{n,n}$.

The set $L^+_{n,n}$ is famously enumerated by the *Catalan numbers*, which have an explicit formula satisfying $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers are ubiquitous in combinatorics, having bijections to many different families of objects, including triangulations of polygons, binary trees, etc.

Given a Dyck path $\pi \in L^+_{n,n}$, define the *area vector* of π to be $\operatorname{areat}(\pi) = (a_1, ..., a_n)$ where a_i denotes the number of complete boxes between π and the main diagonal $y = x$. We will define $\operatorname{area}(\pi) = \sum_{i=1}^n a_i$, the total number of complete boxes between π and the main diagonal. If a path π touches the main diagonal in rows $1 = c_1, ..., c_k = n$ (from bottom to top), then we will define the $\operatorname{touch}(\pi) = (c_2 - c_1, c_3 - c_2, ..., n - c_{k-1})$, which will be a composition of n. For example,

$\pi =$ 

has $\operatorname{areat}(\pi) = (0,1,2,0,1,0,1,2,2,0,1,2)$, $\operatorname{area}(\pi) = 13$, and $\operatorname{touch}(\pi) = (3,2,4,3)$.

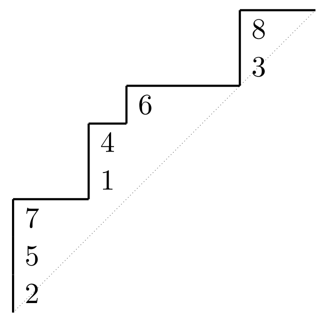
There is also a partial order on $L^+_{n,n}$, defined as follows. We will say for $\pi, \pi' \in L^+_{n,n}$ that $\pi \leqslant \pi'$ if $\operatorname{areat}(\pi) \leqslant \operatorname{areat}(\pi')$ component-wise.

A *word parking function* $P = (\pi, w)$ consists of the pair of a Dyck path together with a word w of length n whose entries are associated with the north steps of π such that for two consecutive north steps with labels a, b, we have that $a < b$. (The columns are decreasing) We will refer to the label of the north step in the ith row as the *occupant* of i, denoted $\operatorname{occ}(i)$. We define the *content* of P to be $\operatorname{content}(P) = \operatorname{content}(w)$. If $w \in S_n$, then we say P is a *parking function.*

The *level sets* of P are given by $Z_i(P) = \{\operatorname{occ}(j) : a_j = i\}$ to be the occupants of the rows in which the number of squares between π and the diagonal is i. We define the *area* of a parking function to be $\operatorname{area}(P) = \operatorname{area}(\pi)$, where π is the underlying Dyck path, and we define a statistic $\operatorname{dinv}(P)$ to be:

$$\begin{aligned}\operatorname{dinv}(P) = &\left|\left\{(i,j) : 1 \leqslant i < j \leqslant n, a_i = a_j, \operatorname{occ}(i) < \operatorname{occ}(j)\right\}\right| \\ &+\left|\left\{(i,j) : 1 \leqslant i < j \leqslant n, a_i = a_j + 1, \operatorname{occ}(i) > \operatorname{occ}(j)\right\}\right|\end{aligned}$$

Example 2.6.1. The following example is from [**Hag08**]. Let

$P =$ 

so that $\pi =$ NNNEENNENEEEENNEE, $w = (2,5,7,1,4,6,3,8)$, and the diagonal inversions are $\{(1,7),(2,7),(2,8),(3,4),(4,8),(5,6)\}$ so that $\text{dinv}(P) = 6$. We have that the level sets are $Z_0(P) = \{2,3\}, Z_1(P) = \{1,5,8\}, Z_2(P) = \{4,6,7\}$.

We can associate a unique permutation $p(P)$ to each parking function by simply concatenating the sorted level sets in reverse order, so that $p(P) = 46715823$. We will denote the set of all parking functions P with $p(P) = \tau$ by $\text{cars}(\tau)$.

For each word parking function P, we can associate a column-strict tableau (not necessarily a Young tableau, as it need not have partition shape) by taking the blocks of consecutive north steps in P and allowing them to be the columns, aligned so that the entries in the ith row (from the bottom) are precisely $Z_{i-1}(P)$. For P in example 2.6.1, the associated tableau is

$$\begin{array}{|c|c|c|c|} \hline 7 & 4 & 6 & \\ \hline 5 & 1 & & 8 \\ \hline 2 & & & 3 \\ \hline \end{array} \tag{2.4}$$

We will denote the set of all parking functions with a given path by $\mathcal{P}_{n,\pi}$ and set $\mathcal{P}_n = \bigsqcup_{\pi \in L^+_{n,n}} \mathcal{P}_{n,\pi}$. The word parking functions will similarly be denoted by $\mathcal{WP}_{n,\pi}$ and $\mathcal{WP}_n = \bigsqcup_{\pi \in L^+_{n,n}} \mathcal{WP}_{n,\pi}$.

2.6.2. The Operator ∇, and the Shuffle Theorem. The remarkable Macdonald eigenoperator ∇ was defined in [**BGHT99**] to be

$$\nabla \widetilde{H}_\lambda(X;q,t) = t^{\eta(\lambda)} q^{\eta(\lambda')} \widetilde{H}_\lambda(X;q,t)$$

and has been the subject of many fascinating conjectures and theorems related to Macdonald polynomials, the space of Diagonal Coinvariants, and recently in [**CO18**], connections to affine Springer fibers.

A combinatorial for ∇e_n was conjectured by Haglund, Haiman, Loehr, Remmel, and Ulyanov in [**HHL$^+$05a**], and was later proved by Carlsson and Mellit in 2015, now known as the *Shuffle Theorem*:

Theorem 2.6.2. *[**CM18**]*

$$\nabla e_n = \sum_{P \in \mathcal{WP}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} x^P$$

where $x^P = x^w$ if w is the reading word of P.

In fact, they had proven a more general statement, known as the *compositional shuffle conjecture*. We recall one equivalent version of the statement here:

Theorem 2.6.3. *[**CM18**] Let $\mathbb{B}_m$ denote the operators defined in [**HMZ12**], $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of length k, and define*

$$B_\alpha[X;q] = \mathbb{B}_{\alpha_k} \mathbb{B}_{\alpha_{k-1}} \ldots \mathbb{B}_{\alpha_1}(1)$$

Then, we have

$$\nabla(B_\alpha[X;q]) = \sum_{\pi \leqslant \pi_\alpha} \sum_{P \in \mathcal{WP}_{n,\pi}} t^{\mathrm{area}(w)} q^{\mathrm{dinv}(w) + \mathrm{doff}_\alpha(\pi)} x^P \tag{2.5}$$

where $\pi_\alpha = \underbrace{N \ldots N}_{\alpha_1} \underbrace{E \ldots E}_{\alpha_1} \ldots \underbrace{N \ldots N}_{\alpha_k} \underbrace{E \ldots E}_{\alpha_k}$. We also have that

$$B_{\overleftarrow{\lambda}}(X;q) = \omega H_\lambda(X;q)$$

the $H_\lambda(X;q)$ is the transformed Hall-Littlewood polynomial.

CHAPTER 3

New Formula for the Modified Hall-Littlewood Polynomials

In this chapter, we will give a proof of Theorem 1.0.2 to establish a formula for $\widetilde{H}_\lambda(X;t)$ in terms of our new set.

3.1. The Shuffle Map

We now give the combinatorial construction for the indexing set J_λ^{maj}.

Let $\lambda \vdash n$ be a partition (or generally, a weak composition α), and consider a tuple of descent compositions $\mathbf{a}_1, .., \mathbf{a}_\ell$, with $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$. A *shuffle* $\mathbf{a}$ of the tuples $\mathbf{a}_1, ..., \mathbf{a}_\ell$ is a descent composition of length $\lambda_1 + .. + \lambda_\ell = n$ such that $\mathbf{a}$ can be partitioned into the subwords $\mathbf{a}_1, ..., \mathbf{a}_\ell$. We will denote the set of shuffles of $\mathbf{a}_1, ..., \mathbf{a}_\ell$ by $\mathrm{Sh}(\mathbf{a}_1, ..., \mathbf{a}_\ell)$.

Example 3.1.1. Let $\lambda = (3, 3, 1)$, and let $\mathbf{a}_1 = 012$, $\mathbf{a}_2 = 101$, and $\mathbf{a}_3 = 0$. Then, the following are in $\mathrm{Sh}(012, 101, 0)$:

$$1001201 \qquad 0121010$$

but 0210110 is not, since there is no 1 to the left of the 2.

We prove a quick lemma:

Lemma 3.1.2. *Let λ be a partition of length ℓ, or more generally a weak composition, and let $\mathbf{a}_1, ..., \mathbf{a}_l$ be some descent compositions $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$. Then every shuffle of them is also a descent composition,*

$$\mathrm{Sh}(\mathbf{a}_1, ..., \mathbf{a}_l) \subset \mathcal{D}_n$$

Similarly, every shuffle of inversion tables is an inversion table.

PROOF. Let $\mathbf{a} \in \mathrm{Sh}(\mathbf{a}_1, ..., \mathbf{a}_\ell)$. Then, for each $j > 0$, the rightmost instance of j in $\mathbf{a}$ must have come from some descent composition $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$. This j must also be the rightmost instance of j in $\mathbf{a}_i$. Since $\mathbf{a}_i$ is a descent composition, then there a $j-1$ to the left of it, and we are done. The second statement is not hard to show. $\square$

DEFINITION 3.1.3. Let λ be a partition of n, or more generally a weak composition. We define a subset $J_\lambda^{\mathrm{maj}} \subset S_n$ by

$$J_\lambda^{\mathrm{maj}} = \left\{\mathrm{majt}^{-1}(\mathbf{a}) : \mathbf{a} \in \mathcal{D}_\lambda\right\}, \quad \mathcal{D}_\lambda = \bigcup_{\mathbf{a}_1,\ldots,\mathbf{a}_n} \mathrm{Sh}\left(\mathbf{a}_1, ..., \mathbf{a}_l\right) \tag{3.1}$$

ranging over all l-tuples $(\mathbf{a}_1, ..., \mathbf{a}_l)$ with $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$, the set of usual descent compositions of size λ_i. We similarly have J_λ^{inv} and $\mathcal{A}_\lambda$ replacing majt by invt, and $\mathcal{D}_{\lambda_i}$ by $\mathcal{A}_{\lambda_i}$.

We give another way of viewing this construction. Let $\mathcal{OSP}(\lambda)$ denote the set of ordered set partitions with block sizes given by λ. Given $\pi \in \mathcal{OSP}(\lambda)$, write $\pi = B_1|...|B_\ell$, so that $|B_i| = \lambda_i$. By abuse of notation, we will refer to π as the permutation obtained by dropping the bars dividing each set and reading in one-line notation. Denote $\overline{\mathbf{a}} = \mathbf{a}_1 + ... + \mathbf{a}_\ell$ to be the concatenation of the compositions. Consider the map

$$\Psi : \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda) \to \mathcal{D}_n$$

$$\Psi : (\mathbf{a}_1, ..., \mathbf{a}_\ell, \pi) \mapsto (\overline{\mathbf{a}}_{\pi^{-1}(i)})_{1 \leqslant i \leqslant n}$$

EXAMPLE 3.1.4. Let $\mathbf{a}_1 = 012$, $\mathbf{a}_2 = 101$, and $\mathbf{a}_3 = 0$, and let $\pi = 245|167|3$. Then,

$$\Psi(012, 101, 0, 245|167|3) = 1001201$$

LEMMA 3.1.5. *We have that $im(\Psi) = \mathcal{D}_\lambda$.*

PROOF. Fix $\mathbf{a}_1, ..., \mathbf{a}_\ell \in \mathcal{D}_1 \times ... \times \mathcal{D}_\ell$. Then, note that we must have $\Psi(\mathbf{a}_1, ..., \mathbf{a}_\ell, \pi) \in \mathrm{Sh}(\mathbf{a}_1, ..., \mathbf{a}_\ell)$ for all $\pi \in \mathcal{OSP}(\lambda)$. This establishes the containment $\mathrm{im}(\Psi) \subset \mathcal{D}_\lambda$. To see the reverse containment, if $\mathbf{a} \in \mathcal{D}_\lambda$, then $\mathbf{a} \in \mathrm{Sh}(\mathbf{a}_1, ..., \mathbf{a}_\ell)$ for some $\mathbf{a}_1, ..., \mathbf{a}_\ell$. Recording the positions of $\mathbf{a}_1, ..., \mathbf{a}_\ell$ in $\mathbf{a}$ gives an ordered set partition $\pi \in \mathcal{OSP}(\lambda)$, so that $\Psi(\mathbf{a}_1, ..., \mathbf{a}_\ell, \pi) = \mathbf{a}$. This implies $\mathcal{D}_\lambda \subset \mathrm{im}(\Psi)$. $\square$

Intuitively, the ordered set partition π is recording the positions of the descent compositions $\mathbf{a}_1, ..., \mathbf{a}_\ell$ for an element $\mathbf{a} \in \mathcal{D}_\lambda$. In particular, there are many different ways an element $\mathbf{a} \in \mathcal{D}_\lambda$ can be obtained, for instance, for $\lambda = (2,1)$:

$$\Psi(01, 0, 13|2) = 001 \cong 001 = \Psi(01, 0, 23|1)$$

The fibres over each element $\mathbf{a} \in \mathcal{D}_\lambda$ are the subject of many conjectures related to the modified Macdonald polynomials $\widetilde{H}_\mu(X; q, t)$, which will be discussed along with partial results in a later section 5.

We also briefly note that

$$|\mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)| = \lambda_1! \times ... \times \lambda_\ell! \times \binom{n}{\lambda_1, ..., \lambda_\ell} = n!$$

which will be a useful fact later on.

3.2. Canonical Decompositions and Membership Algorithm

We now give a corresponding criteria for determining membership of J_λ^{maj}. For any composition $\mathbf{a}$, define a sequence $(\tilde{a}_1, \tilde{a}_2, ...)$ by

$$\tilde{a}_i = \begin{cases} a_i & i \leqslant n \\ a_{i-n} + 1 & \text{otherwise} \end{cases}$$

For instance, if $a = (0, 1, 2, 0, 2, 1)$, then

$$\tilde{a} = (0, 1, 2, 0, 2, 1|1, 2, 3, 1, 3, 2|2, 3, 4, 2, 4, 3|...),$$

where we have used bars to separate groups of $n = 6$. The following algorithm produces a set partition $\pi = (A_1| \cdots |A_l)$ to each $\mathbf{a} \in \mathcal{D}_\lambda$ with the property that the restricted composition $\mathbf{a}|_{A_i}$ is a descent composition $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$, realizing $\mathbf{a}$ as an element of $\mathrm{Sh}(\mathbf{a}_1, ..., \mathbf{a}_n)$.

Algorithm 1. Fix $\lambda \vdash n$. Given a descent composition $\mathbf{a} \in \mathcal{D}_n$ (or more generally, any composition $\mathbb{Z}^n_{\geqslant 0}$), perform the following:

(1) For each part λ_i, extract a subsequence $0, 1, ..., \lambda_i - 1$ from $\tilde{a}$ (moving into successive blocks if necessary), never repeating the same index modulo n.

(2) Once a sequence has been extracted, record the positions of the entries selected, and reduce modulo n to obtain a residue in $\{1, ..., n\}$. Denote the positions by $\tilde{w}_i = (\tilde{a}_{j_1}, ..., \tilde{a}_{j_{\lambda_i}})$, the indices reduced modulo n by $w_i = (a_{k_1}, ..., a_{k_{\lambda_i}})$, and the underlying set for w_i by A_i.

(3) The output of the algorithm will be the ordered set partition $\pi = A_1|...|A_\ell$, and the descent compositions $\mathbf{a}_i = \mathbf{a}|_{A_i}, 1 \leqslant i \leqslant \ell$. We refer to the tuple $(\mathbf{a}_1, ..., \mathbf{a}_\ell, \pi)$ as the *canonical decomposition* of $\mathbf{a}$ as an element of $\mathcal{D}_\lambda$.

Example 3.2.1. Let $\mathbf{a} = (0, 1, 2, 0, 2, 1)$, $\lambda = (6)$. Then, the algorithm terminates, and will select:

$$\tilde{a} = (\underline{0}, \underline{1}, \underline{2}, 0, 2, 1|1, 2, 3, 1, \underline{3}, 2|2, 3, 4, 2, 4, 3|3, 4, 5, 3, 5, \underline{4}|4, 5, 6, 4, 6, 5|5, 6, 7, \underline{5}, 7, 6|...),$$

The positions are $\tilde{w} = (1, 2, 3, 11, 24, 34)$; reducing modulo n we obtain $w = (1, 2, 3, 5, 6, 4)$ and so $A_1 = \{1, 2, 3, 4, 5, 6\}$.

Example 3.2.2. Let $\lambda = (3, 3, 1)$ and $\mathbf{a} = (0, 0, 1, 1, 2, 0, 0)$. Then we have

$$\tilde{\mathbf{a}} = (0, 0, 1, 1, 2, 0, 0|1, 1, 2, 2, 3, 1, 1|2, 2, 3, 3, 4, 2, 2|\cdots)$$

The first iteration finds the subsequence $(\tilde{a}_1, \tilde{a}_3, \tilde{a}_5)$, so that $A_1 = \{1, 3, 5\}$:

$$\tilde{\mathbf{a}} = (\underline{0}, 0, \underline{1}, 1, \underline{2}, 0, 0|1, 1, 2, 2, 3, 1, 1|2, 2, 3, 3, 4, 2, 2|\cdots)$$

The second iteration will find $0, 1$ in the first block of $\tilde{\mathbf{a}}$, but will not find an unmarked 2 until the third block, giving the subsequence $(\tilde{a}_2, \tilde{a}_4, \tilde{a}_{20})$ so that $A_2 = \{2, 4, 6\}$:

$$\tilde{\mathbf{a}} = (\cancel{0}, \underline{0}, \cancel{1}, \underline{1}, \cancel{2}, 0, 0|\cancel{1}, 1, \cancel{2}, 2, \cancel{3}, 1, 1|\cancel{2}, 2, \cancel{3}, 3, \cancel{4}, \underline{2}, 2|\cdots)$$

where the strikethroughs denote entries selected in the first iteration.

Finally, the third run will select $(\tilde{a}_7)$, so that $A_3 = \{7\}$:

$$\tilde{\mathbf{a}} = (\cancel{0}, \cancel{0}, \cancel{1}, \cancel{1}, \cancel{2}, \cancel{0}, \underline{0}|\cancel{1}, \cancel{1}, \cancel{2}, \cancel{2}, \cancel{3}, \cancel{1}, 1|\cancel{2}, \cancel{2}, \cancel{3}, \cancel{3}, \cancel{4}, \cancel{2}, 2|\cdots)$$

and we have $\mathbf{a}_1 = 012, \mathbf{a}_2 = 010, \mathbf{a}_3 = 0$, and $\pi = 135|246|7$.

DEFINITION 3.2.3. Let $\mathbf{a} \in \mathcal{D}_\lambda$. We denote the result of algorithm 1 by $\text{alg}_\lambda(\mathbf{a}) := (\mathbf{a}_1, \ldots, \mathbf{a}_\ell, \pi)$.

Before proving that Algorithm 1 determines membership of $\mathcal{D}_\lambda$, we first prove a series of lemmata.

LEMMA 3.2.4. *If Algorithm 1 terminates, then the compositions* $\mathbf{a}_i = \tilde{\mathbf{a}}|_{A_i}$ *obtained have the property that* $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$.

PROOF. We need only check that in $\mathbf{a}_i$, the final instance of $k \geqslant 1$ has a $k-1$ to the left of it.

We will show a stronger statement, that the first instance of k selected by Algorithm 1 must necessarily have a $k-1$ to the left of it. Denote the first instance of k selected by the algorithm as k^*, let m be its index in $\mathbf{a}$, and $\tilde{m}$ be its index in $\tilde{\mathbf{a}}$. Then, we must have $\tilde{m} = m + \alpha n$ for some $\alpha \in \mathbb{Z}_{\geqslant 0}$.

We observe that if $\mathbf{a}_m = k$, then we must have $\tilde{\mathbf{a}}_{\tilde{m}} = k + \alpha$, by the definition of $\tilde{\mathbf{a}}$. If $\alpha = 0$, then we are done, as the algorithm will first select a $k-1$ to the left in the first block.

Let w denote the (reduced modulo n) positions for λ_i, as above. To deal with the $\alpha > 0$ case, first observe that the sequence $(\mathbf{a}_{w_1}, ..., \mathbf{a}_{w_{\lambda_i}})$ has the property that $\mathbf{a}_{w_k} + 1 = \mathbf{a}_{w_{k+1}}$ if $\tilde{w}_k$ and $\tilde{w}_{k+1}$ occur in the same block, and that $\mathbf{a}_{w_k} > \mathbf{a}_{w_{k+1}}$ if $\tilde{w}_{k+1}$ occurs in a later block. We will consider the position of $\tilde{\mathbf{a}}_{\tilde{w}_{k+\alpha}} = k + \alpha - 1$, the element selected by the algorithm before $\tilde{\mathbf{a}}_{\tilde{m}}$.

By the previous observation, $\tilde{\mathbf{a}}_{\tilde{w}_{k+\alpha}}$ cannot occur in the $(\alpha-2)$th block (if $\alpha \geqslant 2$), or any block to the left because then $\mathbf{a}_{w_{k+\alpha}} \geqslant k+1$, and by the observation above, since the we necessarily have $\mathbf{a}_{w_1} = 0$, there must then be a k selected before k^*, which is a contradiction. So $\tilde{\mathbf{a}}_{\tilde{w}_{k+\alpha}}$ must occur in the $(\alpha-1)$th block, or the αth block. But since $\tilde{\mathbf{a}}_{\tilde{w}_{k+\alpha}} = k + \alpha - 1$, if it occurs in the former, then $\mathbf{a}_{w_{k+\alpha}} = k$, which contradicts the choice of k^*. So we must have that $\tilde{\mathbf{a}}_{\tilde{w}_{k+\alpha}}$ occurs in the αth block to the left of $\tilde{\mathbf{a}}_{\tilde{m}}$, so that $\mathbf{a}_{w_{k+\alpha}} = k-1$, and occurs to the left of $\mathbf{a}_m$.

□

PROPOSITION 3.2.5. *Given a composition* $\mathbf{a} \in \mathbb{Z}^n_{\geqslant 0}$, *we have that* $\mathbf{a} \in \mathcal{D}_\lambda$ *if and only if Algorithm 1 terminates.*

We will give a proof of this proposition in a later section.

Next, we give a more cumbersome yet useful equivalent formulation of algorithm 1.

LEMMA 3.2.6. *Let* $\mathbf{a} \in \mathcal{D}_n$*. Algorithm 1 is equivalent to the following:*

- ***De-affinized Presentation***
  (1) For each part λ_i *of* λ*, pass through* $\mathbf{a}$ *left to right, searching for* $0, 1, \ldots, \lambda_i - 1$*.*
  (2) If the end of the permutation is reached, wrap around to the front, and decrease 1 *from all subsequent entries to be absorbed.*
- ***Increase and Absorb***
  (1) For each part λ_i *of* λ*, pass through* $\mathbf{a}$ *left to right, searching for a minimal (leftmost) increasing sequence* $0, 1, \ldots$*. Once the end of the permutation is reached, if the last entry selected is* k*, then start from the left of the permutation once more, absorbing* $k, k+1, \ldots$*.*
  (2) Once no larger entry can be found, suppose k *is the largest entry absorbed. Then, absorb all* k*'s moving left to right, all* $k-1$*'s left to right, and so on, until* $\mathbf{a}_i$ *is of length* λ_i*.*

PROOF. The de-affinized presentation is simply a reformulation of the algorithm in terms of $\mathbf{a}$. We will use this to show that the increase and absorb procedure is equivalent.

The first step is clearly the same, and so it suffices to show the second step is equivalent. Let $\mathbf{a} \in \mathcal{D}_n$, and $\tilde{\mathbf{a}}$ be as above. Suppose the highest entry k has been absorbed, chosen as say j in $\tilde{\mathbf{a}}$ in the ith block. The remaining k's of $\mathbf{a}$, then, will be chosen as $j+1, j+2, \ldots$ in the $i+1, i+2, \ldots$th blocks respectively. If no such k's exist, then the $k-1$'s will appear in the $i+2, i+3, \ldots$th blocks as $k+1, k+2, \ldots$ respectively, and so on. If m is the first entry to be absorbed after k, then all m's will be chosen first, as the remaining entries of $\mathbf{a}$ must necessarily be smaller, and so are smaller as well in any block of $\tilde{\mathbf{a}}$.

Suppose all of k's have been absorbed, and the final entry of k is chosen as j in the ith block of $\tilde{\mathbf{a}}$. Then, we repeat the argument above, and see that the next step must necessarily absorb all $k-1$'s moving left to right. This completes the proof. $\square$

Definition 3.2.7. Let $\mathbf{a} \in \mathcal{D}_\lambda$, and let $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$ denote the composition extracted by algorithm 1 for the λ_i part. The *essential sequence* is the subsequence of $\mathbf{a}_i$ corresponding to step 1 of the Increase and Absorb formulation.

3.3. Parking Function Formula

First, we give useful identifications between permutations of S_n, ribbon tableau, and dinv-less parking functions.

Let $\pi \in S_n$, and the denote the set of all Young tableau whose underlying shape are ribbons with n boxes by $\mathcal{R}_n$. We will define a map $\gamma : S_n \to \mathcal{R}_n$ as follows:

(1) Place π_n in a box by itself.
(2) For all $1 \leqslant i < n$, if $\pi_i < \pi_{i+1}$, then add π_i in a new box directly to the *right* of the last entry inserted.
(3) If $\pi_i > \pi_{i+1}$, then add π_i in a new box *above* the last entry inserted.

For example,

$$\gamma : 645312 \mapsto \begin{array}{ccc} & & \boxed{6} \\ & \boxed{5} & \boxed{4} \\ & \boxed{3} & \\ \boxed{2} & \boxed{1} & \end{array}$$

We will denote $\mathfrak{R}_n := \mathrm{im}(\gamma) \subset \mathcal{R}_n$. It is easy to see that γ is injective, and one should think of $\gamma(\pi)$ as the ribbon whose rows correspond to the runs of π (in reverse order), and an entry a has a box below it if and only if a is a descent in π. Alternatively, we may write:

$$\mathfrak{R}_n = \{T \in \mathcal{R}_n : T \text{ is row-decreasing and column-strict}\}$$

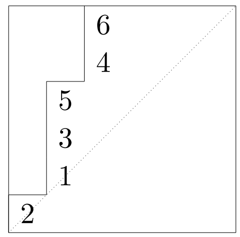
It is easy to show that row-decreasing column-strict ribbons correspond to permutations.

The second correspondence comes from the association between column-strict tableau and parking functions, as in 2.6.1. We will define a map $\delta : \mathfrak{R}_n \to \mathcal{PF}_n$ as follows:

(1) Begin the area sequence with $\mathrm{areat}(\pi) = (0)$, and $w = a$, where a is the bottom left corner entry of the ribbon T.

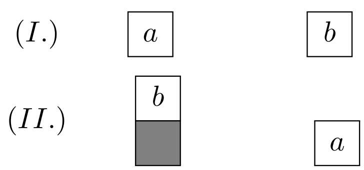
(2) Proceed from bottom left to top right in T. For each entry, append the height of the box (the bottom row has height 0) in the area sequence, and append w with the entry in the box.

Example 3.3.1. We have for the tableau above, the parking function



which indeed has area sequence $\mathrm{areat}(\pi) = (0, 0, 1, 2, 2, 3)$. Note that this process will always return an appropriate area sequence, as the entries must be bounded by $(0, 1, ..., n-1)$ and $0 \leqslant a_i - a_{i-1} \leqslant 1$ by construction.

We note that for a general parking function $P \in \mathcal{PF}_n$, if we convert to a column strict tableau, then dinv pairs correspond to one of the following configurations:



with $b > a$. We now show that $\delta(\mathfrak{R}_n)$ is precisely the set of dinv-less parking functions.

Lemma 3.3.2. *Let* $\mathcal{PF}_n^0$ *denote the set of parking functions* P *such that* $\mathrm{dinv}(P) = 0$. *Then,* $\mathcal{PF}_n^0 = \mathrm{im}(\delta)$.

Proof. Noting that the area vector uniquely determines a Dyck path, it is easy to show that δ is injective, so that δ^{-1} is defined on $\mathrm{im}(\delta)$. Suppose $P \in \mathrm{im}(\delta)$. Then, $\delta^{-1}(P) \in \mathfrak{R}_n$, therefore is a ribbon T corresponding to a permutation in S_n. Since T is decreasing, no type I dinv pairs may occur. Since T only increases upward and to the right, no type II dinv pairs may occur.

Conversely, let $P \in \mathcal{PF}_n^0$, so $\mathrm{dinv}(P) = 0$. Then, consider the corresponding column-strict tableau T associated to P, as in 2.6.1. Then, consider the following two configurations of squares:

$$(I.)\qquad \begin{array}{|c|}\hline a\\\hline\end{array} \qquad\qquad \begin{array}{|c|}\hline b\\\hline c\\\hline\end{array}$$

$$(II.)\qquad \begin{array}{|c|}\hline a\\\hline d\\\hline\end{array} \qquad\qquad \begin{array}{|c|}\hline c\\\hline\end{array}$$

In configuration I, we must have $b > c$. If $a < b$, then we would have a dinv pair, so we must have $a > b > c$, which is also a dinv pair. For configuration II, we necessarily have $a > d$. If $d < c$ we would have a dinv pair, so we must have $d > c$, so that $a > d > c$, which is also a dinv pair. Therefore, configurations I and II cannot occur.

Since P is a parking function with underlying Dyck path π, in its area sequence, the final entry of i must have an $i - 1$ before it, for $i \geqslant 1$. This implies that T is one connected tableau, and if configurations I, II cannot appear, T must have ribbon shape. That $\mathrm{dinv}(P) = 0$ forces T to be row-decreasing, so that $T \in \mathfrak{R}_n$, and $\delta(T) = P$. $\square$

We now give a parking function formula of $\mathcal{D}_\lambda$.

Definition 3.3.3. Let $\alpha \vDash n$ be a composition of n. The *α-bounce path*, denoted π_α is defined as the following:

$$\pi_\alpha = \underbrace{N \ldots N}_{\alpha_1}\underbrace{E \ldots E}_{\alpha_1} \ldots \underbrace{N \ldots N}_{\alpha_k}\underbrace{E \ldots E}_{\alpha_k}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ has length k.

Given a partition $\lambda \vdash n$, let $\overleftarrow{\lambda} = (\lambda_\ell, ..., \lambda_1)$ denote the composition obtained by reversing the parts of λ. Then, define

$$\mathcal{PF}_\lambda = \left\{ P = (\pi, w) \in \mathcal{PF}_n : \pi \leqslant \pi_{\overleftarrow{\lambda}} \right\}$$

and define $\mathcal{WP}_\lambda$ similarly.

Lemma 3.3.4. *Let $P = (\pi, w)$ be a parking function in $\mathcal{PF}_\lambda$. Consider the map $\xi : \mathcal{PF}_\lambda \to \mathcal{D}_n$ by:*

$$\xi : (\pi, w) \mapsto (\mathrm{areat}(\pi)_{w^{-1}(i)})_{1 \leqslant i \leqslant n}$$

Then, $\operatorname{im}(\xi) = \mathcal{D}_\lambda$.

Proof. ($\operatorname{im}(\xi) \subset \mathcal{D}_\lambda$) We will show the claim for $\lambda = (n)$. Let $P = (\pi, w) \in \mathcal{PF}_n$. To show $\xi(P) = (\xi(P)_1, ..., , \xi(P)_n) \in \mathcal{D}_n$, we will show the equivalent condition as in lemma 2.1.2.

First, notice the bottom row necessarily contributes a 0 to $\operatorname{areat}(\pi)$, so that $\xi(P)$ always has a 0. Second, it is easy to show that for a Dyck path, writing $\operatorname{areat}(\pi) = (a_1, ..., a_n)$, we must always have $a_{j+1} = a_j + 1$ or $a_{j+1} \leqslant a_j$. In other words, consecutive entries may not increase by more than 1. This implies, for $i > 0$, that the first instance of $a_j = i$ must have $a_{j-1} = i - 1$, and $\operatorname{occ}(j) > \operatorname{occ}(j-1)$ since the squares are vertically adjacent. Then, $\xi(P)_{\operatorname{occ}(j-1)} = i - 1$ lies to the left of $\xi(P)_{\operatorname{occ}(j)} = i$, and the claim is proved.

Now let λ be arbitrary. Then, given $P \in \mathcal{PF}_\lambda$, consider the ordered set partition

$$\sigma = a_n, \ldots, a_{n-\lambda_1+1} | \ldots | a_{\lambda_\ell}, \ldots, a_1 \in \mathcal{OSP}(\lambda)$$

and let $\mathbf{a}_1, \ldots, \mathbf{a}_\ell$ be the compositions obtained by first standardizing the occupants in the $\lambda_1, \ldots, \lambda_\ell$ blocks respectively, then applying ξ. By the above argument for $\lambda = (n)$, all $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$. Finally, comparing the definitions of ξ and Ψ, we see that

$$\xi(P) = \Psi(\mathbf{a}_1, \ldots, \mathbf{a}_\ell, \sigma)$$

and the claim is proved.

($\mathcal{D}_\lambda \subset \operatorname{im}(\xi)$) We show the set $\mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$ naturally arises as a subset of $\mathcal{PF}_\lambda$. Given $(\mathbf{a}_1, ..., \mathbf{a}_\ell, \sigma) \in \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$, let $\tau_1, ..., \tau_\ell$ be permutations $\tau_i \in S_{\lambda_i}$ corresponding to $\mathbf{a}_1, ..., \mathbf{a}_\ell$ respectively. Then, consider the parking functions $P_i := (\delta \circ \gamma)(\tau_i)$, and let π_i denote the underlying paths. If $\sigma = A_1 | ... | A_\ell$, set $w_i = \tau_i \cdot (A_i)$, where (by abuse of notation) A_i is the word consisting of its elements written in increasing order. Then, consider the parking function

$$P = (\pi_\ell + ... + \pi_1, w_\ell + ... + w_1)$$

where + denotes concatenation for Dyck paths, as well as words. Then, it is clear that $\xi(P) = \Psi(\mathbf{a}_1, ..., \mathbf{a}_\ell, \sigma) \in \mathcal{D}_\lambda$. Varying over all pairs $(\mathbf{a}_1, ..., \mathbf{a}_\ell, \sigma)$, we see that $\mathcal{D}_\lambda \subset \operatorname{im}(\xi)$.

$\square$

DEFINITION 3.3.5. We will denote the map $(\mathbf{a}_1, \ldots, \mathbf{a}_\ell, \sigma) \mapsto \mathcal{PF}_\lambda$ as above by

$$\epsilon : \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda) \to \mathcal{PF}_\lambda$$

3.4. New Hall-Littlewood Formula

In this section, we will prove Theorem 1.0.2, namely

$$\widetilde{H}_{\lambda'}(X;t) = \sum_{\pi \in J_\lambda^{\mathrm{maj}}} t^{\mathrm{maj}(\pi)} F_{\mathrm{iDes}(\pi),n}(X) = \sum_{\mu \vdash n} \Bigg(\sum_{\pi \in J_\lambda^{\mathrm{maj}} \cap \mathrm{Sh}(\mu)} t^{\mathrm{maj}(\pi)} \Bigg) m_\mu(\mathbf{x})$$

3.4.1. Reformulation in Terms of Compositional Shuffle Theorem. Let $\alpha \vDash n$ be a composition of n of length k, $\mathbb{B}_\alpha(X;q) = \big(\mathbb{B}_{\alpha_k} \circ ... \circ \mathbb{B}_{\alpha_1}\big)(1)$ denote the result of applying the $\mathbb{B}$ operators as in [**HMZ12**]. First, noting that $\mathbb{B}_{\overleftarrow{\lambda}}(X;q) \in \Lambda_{\mathbb{C}(q)} \subset \Lambda_{\mathbb{C}(q,t)}$, we may expand $\mathbb{B}_{\overleftarrow{\lambda}}(X;q)$ in the $\widetilde{H}_\mu(X;q,t)$-basis:

$$\mathbb{B}_{\overleftarrow{\lambda}}(X;q) = \sum_{\mu \vdash n} A_{\mu\lambda}(q,t) \widetilde{H}_\mu(X;q,t) \tag{3.2}$$

By the Macdonald axiom (T1) from section 2.4.1, the plethystic substitution $\widetilde{H}_\mu[X(1-q);q,t]$ has an expression of the form

$$\widetilde{H}_\mu[X(1-q);q,t] = \sum_{\nu \trianglerighteq \mu} c_{\nu\mu}(q,t) s_\nu(\mathbf{x})$$

Since $\mathbb{B}_{\overleftarrow{\lambda}}(X;q) = \omega H_\lambda(X;q)$, we have similarly by 2.5.3:

$$\mathbb{B}_{\overleftarrow{\lambda}}[X(1-q);q] = \omega H_\lambda[X(1-q);q] = \omega \sum_{\nu \trianglelefteq \lambda} b_{\nu\lambda}(q) s_\nu(\mathbf{x}) = \sum_{\nu \trianglerighteq \lambda'} b'_{\nu\lambda}(q) s_\nu(\mathbf{x})$$

This implies that coefficients $A_{\mu\lambda}(q,t) = 0$ unless $\mu \trianglerighteq \lambda'$. Since $H_\lambda(X;q) = \sum_\nu K_{\nu\lambda}(q) s_\nu(\mathbf{x})$, and $K_{\nu\lambda} = 0$ if $\nu' \not\trianglerighteq \lambda$, applying ω, we see that $\mathbb{B}_{\overleftarrow{\lambda}}(X;q) = \sum_{\nu \trianglelefteq \lambda'} K_{\nu'\lambda}(q) s_\nu(\mathbf{x})$. Furthermore, we have that $K_{\lambda\lambda}(q) = 1$, so we may write

$$\mathbb{B}_{\overleftarrow{\lambda}}(X;q) = s_{\lambda'}(\mathbf{x}) + \sum_{\nu \triangleleft \lambda'} K_{\nu'\lambda}(q) s_\nu(\mathbf{x})$$

Setting $q = 0$ in equation 3.2 and substituting the above, we see that

$$s_{\lambda'}(\mathbf{x}) + \sum_{\nu \lhd \lambda'} K_{\nu'\lambda}(0) s_\nu(\mathbf{x}) = \sum_{\mu \unrhd \lambda'} A_{\mu\lambda}(0,t) \widetilde{H}_\mu(X;0,t) = \sum_{\mu \unrhd \lambda'} A_{\mu\lambda}(0,t) \left(\sum_{\nu \unrhd \mu} \widetilde{K}_{\nu\mu}(t) s_\nu(\mathbf{x}) \right)$$

Comparing the coefficients of $s_{\lambda'}(\mathbf{x})$, we have that

$$1 = A_{\lambda'\lambda}(0,t) \widetilde{K}_{\lambda'\lambda'}(t) = A_{\lambda'\lambda}(0,t) t^{\eta(\lambda')} \tag{3.3}$$

Now applying ∇ to both sides of equation 3.2, we have that

$$\nabla \mathbb{B}_{\overleftarrow{\lambda}}(X;q) = \sum_{\mu \unrhd \lambda'} A_{\mu\lambda}(q,t) \nabla \widetilde{H}_\mu(X;q,t) = \sum_{\mu \unrhd \lambda'} A_{\mu\lambda}(q,t) t^{\eta(\mu)} q^{\eta(\mu')} \widetilde{H}_\mu(X;q,t)$$

We have that $\mu \rhd \lambda'$ if and only if $\eta(\mu) < \eta(\lambda')$, so that the lowest coefficient of q in the right hand side must be $q^{\eta(\lambda)}$. We may then write

$$\begin{aligned} \nabla \mathbb{B}_{\overleftarrow{\lambda}}(X;q) = & q^{\eta(\lambda)} \Bigg(A_{\lambda'\lambda}(0,t) t^{\eta(\lambda')} \widetilde{H}_{\lambda'}(X;0,t) + qG(X;q,t) \\ & + \sum_{\mu \rhd \lambda'} A_{\mu\lambda}(q,t) t^{\eta(\mu)} q^{\eta(\mu') - \eta(\lambda)} \widetilde{H}_\mu(X;q,t) \Bigg) \end{aligned} \tag{3.4}$$

where $G(X;q,t)$ is some symmetric function depending on q, t.

Substituting equation 3.3 and comparing the coefficient of $q^{\eta(\lambda)}$, we conclude that

$$[q^{\eta(\lambda)}] \left(\sum_{P \in \mathcal{WP}_\lambda} q^{\mathrm{dinv}(P) + \mathrm{doff}(\pi)} t^{\mathrm{area}(P)} x^P \right) = [q^{\eta(\lambda)}] \nabla \mathbb{B}_{\overleftarrow{\lambda}}(X;q) = \widetilde{H}_{\lambda'}(X;t) \tag{3.5}$$

by the compositional Shuffle Theorem 2.6.3.

Remark 3.4.1. The motivation for reformulating the problem in terms of the compositional Shuffle Conjecture arises from certain functions called $\chi_{\pi(m),\ell}$ from [**CM21**], where $m \in \mathcal{D}_n$, $\pi(m)$ is an associated Dyck path, and ℓ is the number of 0's in m, which is at most the number of trailing East steps. The connection is that

$$\langle \chi_{\pi(m),\ell}, \widetilde{H}_\lambda(X;q,0) \rangle = 0$$

unless $m \in \mathcal{D}_\lambda$. The functions $\chi_{\pi(m),\ell}$ were proven to be the Frobenius character of the Tymoczko's dot action on $H^*(\mathcal{H}_\pi - Z)$, where $\mathcal{H}_\pi$ is a regular semisimple Hessenberg varietiy, and Z is its intersection with the line arrangement consisting of the coordinate axes. The connection with ∇ is due to equation (59) in [**CM21**]. We thank A. Mellit for valuable discussions and for suggesting that we study the $\operatorname{dinv}(P) + \operatorname{doff}_\alpha(P)$ minimizers.

3.4.2. Proof of the Quasisymmetric Formula. We will now prove the first equality in Theorem 1.0.2. Define

$$\widetilde{C}_{\lambda'}(X;t) := \sum_{\pi \in J_\lambda^{\mathrm{maj}}} t^{\mathrm{maj}(\pi)} F_{\mathrm{iDes}(\pi),n}(X)$$

We will show that this is the modified Hall-Littlewood polynomial, $\widetilde{C}_{\lambda'}(X;t) = \widetilde{H}_\lambda(X;t)$. We give an outline of the proof here:

(1) We first establish $\eta(\lambda)$ as a lower bound for the q-exponents of $q^{\mathrm{dd}(P)}$ for $P \in \mathcal{PF}_\lambda$.

(2) Given $\mathbf{a} \in \mathcal{D}_\lambda$, we may write $\Psi(\mathbf{a}_1, \ldots, \mathbf{a}_\ell, \sigma) = \mathbf{a}$, and consider $P = \epsilon(\mathbf{a}_1, \ldots, \mathbf{a}_\ell, \sigma)$. We may run the algorithm and obtain $(\mathbf{a}'_1, \ldots, \mathbf{a}'_\ell, \sigma')$. Denote $P^* = \epsilon(\mathbf{a}'_1, \ldots, \mathbf{a}'_\ell, \sigma')$. We then give a criterion for all P^* obtained this way.

(3) If $P \in \mathcal{PF}_\lambda$ does not meet such criterion, then we may apply a move that necessarily decreases dd.

(4) Once such moves are exhausted, then we must have $\mathrm{dd}(P) = \eta(\lambda)$.

(5) We show P^* is the unique parking function in $\mathcal{PF}_\lambda$ with $\mathrm{dd}(P^*) = \eta(\lambda)$.

This implies that there is a unique parking function with $q^{\mathrm{dd}(P)} = \eta(\lambda)$ for each element of $\mathcal{D}_\lambda$. Then, we collect the word parking functions $\mathcal{WP}_\lambda$ by which elements of $\mathcal{PF}_\lambda$ they standardize to, and the theorem immediately follows.

3.4.2.1. *Lower Bound for q-Degree.* It suffices to consider parking functions $P \in \mathcal{PF}_\lambda$ whose corresponding tableau is a tuple of ribbons, as by lemma 3.3.2, these minimize dinv in each block. (This is equivalent to the assertion that the area sequence weakly increases in each block of λ). Denote this set by $\mathfrak{B}_\lambda$. Recall that dinv corresponds to one of the following arrangements of cells:

$$(I.) \qquad \boxed{a} \qquad\qquad \boxed{b}$$

$$(II.)\qquad \begin{array}{|c|}\hline b\\ \hline \blacksquare\\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline a\\ \hline\end{array}$$

with $b > a$. Furthermore, $\mathrm{doff}(P)$ simply records for each element in the bottom row of a part how many parts occur to the right. Since we count $\mathrm{dd}(P)$ by pairs of parts, we can assume that for a given pair of parts λ_i, λ_j with $i > j$, each element in the bottom row $\mathrm{doff}(P)$ of λ_i contributes 1 to $\mathrm{doff}(P)$.

LEMMA 3.4.2. *Given a pair of parts λ_i, λ_j, the contribution of those parts to* $\mathrm{dd}(P)$ *is at least λ_i.*

PROOF. Let the tableau corresponding to λ_i, λ_j be any ribbons of size λ_i, λ_j respectively. Given a column-strict tableau T, denote by $\mathrm{ht}(T)$ to be the height of the diagram. Suppose $\mathrm{ht}(\lambda_j) \geqslant \mathrm{ht}(\lambda_i)$. This means for each square in λ_i, there is a square in λ_j in the same row. Then, we have two cases:

- If a square is in the bottom row of λ_i, then it contributes 1 to $\mathrm{doff}(P)$.
- If a square is not in the bottom row, then the following occurs:

$$\begin{array}{|c|}\hline a\\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline b\\ \hline c\\ \hline\end{array}$$

  where necessarily $b > c$, since P is a parking function, the corresponding tableau are column decreasing. Then, if $a > b$, we have that $a > c$, so the square containing a will contribute at least 1 to dinv. If $a < b$, then a will contribute at least 1 to dinv as well.

On the other hand, $\mathrm{ht}(\lambda_i) > \mathrm{ht}(\lambda_j)$, then for each square of λ_j, there is a square in the same row in λ_i, but also a square in the row above. Then, for each square of λ_j, find the rightmost square in λ_i of the same row. One of two things may occur:

- The rightmost square in the same row in λ_i is the corner of a ribbon:

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a}\\ \hline \multicolumn{1}{|c|}{\dots} & \multicolumn{1}{c|}{b}\\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline c\\ \hline\end{array}$$

- There is only one square in the same row:

$$\begin{array}{|c|}\hline a\\ \hline b\\ \hline \vdots\\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline c\\ \hline\end{array}$$

If $b < c$, then c contributes at least 1 inversion to $\operatorname{dinv}(P)$. Otherwise, if $b > c$, since the tableau is column decreasing, we have that $a > b > c$, so that c contributes at least 1 to $\operatorname{dinv}(P)$. Then, every square of λ_j will contribute at least 1 to $\operatorname{dinv}(P)$, so that the contribution is at least $\lambda_j \geqslant \lambda_i$. □

Remark 3.4.3. Actually we must have that if $\operatorname{ht}(\lambda_i) > \operatorname{ht}(\lambda_j)$, then the contribution is greater than λ_j since there is at least one box in λ_i in the first row which contributes at least 1 to $\operatorname{doff}(P)$.

Proof. By the lemma, each pair (λ_i, λ_j) with $i > j$ contributes at least λ_i to $\operatorname{dd}(P)$. Summing over all pairs of parts, we have that:

$$\operatorname{dd}(P) \geqslant \sum_{1 \leqslant j < i \leqslant \ell} \lambda_i = \sum_{i=1}^{\ell'} (i-1)\lambda_i = \eta(\lambda) \tag{3.6}$$

□

3.4.2.2. *Criteria for Result of Algorithm.* We now establish a criteria for being the result of the algorithm in terms of the ribbons.

Definition 3.4.4. A parking function is P *good* if each square in the λ_i ribbon contributes *exactly* 1 to $\operatorname{dd}(P)$.

By the remark, we have if $i > j$, then the ribbon for $\operatorname{ht}(\lambda_j) \leqslant \operatorname{ht}(\lambda_i)$.

Lemma 3.4.5. *A parking function* $P \in \mathfrak{P}_\lambda$ *has* $\operatorname{dd}(P) = \eta(\lambda)$ *if and only if it is good.*

Proof. If P is good, then the inequality in 3.6 is an equality, and we are done. Now suppose P is not good. Then, either there is a square that contributes nothing to $\operatorname{dd}(P)$ in λ_i, or a square that contributes at least 2. We rule out the first case.

Suppose there is a square in λ_i that contributes nothing to $\operatorname{dd}(P)$. Then, this square cannot be in the bottom row, as it must contribute to doff in that case. Furthermore, by the argument in the lemma, there cannot be a square in λ_j in the same row, or the same row below. This implies that $\operatorname{ht}(\lambda_i) > \operatorname{ht}(\lambda_j)$, which by the remark implies that the $\operatorname{dd}(P)$ contribution between the two pairs is greater than $\lambda_j \geqslant \lambda_i$. This means the inequality in equation 3.6 must be strict.

If every square must contribute to $\mathrm{dd}(P)$, and P is not good, then some square contributes at least 2 to $\mathrm{dd}(P)$, which implies that $\mathrm{dd}(P) > \eta(\lambda)$. $\square$

3.4.2.3. *Reduction Algorithm.* We now prove a series of results for two parts, and then inductively apply it for all shapes λ.

LEMMA 3.4.6. *Given a descent composition* $\mathbf{a}$*, algorithm 1 terminates if and only if* $\mathbf{a} \in \mathcal{D}_\lambda$ *for* $\lambda = (\lambda_1, \lambda_2)$ *a partition with two parts.*

PROOF. If the algorithm terminates, then it produces the desired descent compositions $\mathbf{a}_1, \mathbf{a}_2$, as well as the desired ordered set partition. Conversely, suppose that $\mathbf{a} \in \mathcal{D}_\lambda$. Then, let $(\mathbf{a}_1, \mathbf{a}_2, \sigma)$ be such that $\Psi(\mathbf{a}_1, \mathbf{a}_2, \sigma) = \mathbf{a}$, and consider the pair of ribbons $\epsilon(P)$ associated with this shuffle.

The algorithm will always select the first part successfully, as it will find the minimal increasing sequence going up, then absorb remaining entries top-down. We need only show that what remains can form a ribbon, that is, that what remains is a descent composition in $\mathcal{D}(\lambda_2)$. The criteria from lemma 2.1.2 is equivalent to:

- There is at least one unselected box in the bottom row
- The largest unselected element has a smaller unselected element in the row below it

For any row except the bottom row, note that the largest unselected element a must be the corner of a ribbon as such (without loss of generality, we draw a in the left ribbon):

$$\begin{array}{cccc} & \boxed{a} & \boxed{\cdots} & \boxed{} \\ \boxed{\cdots} & \boxed{b} & & \end{array} \qquad \begin{array}{cccc} & \boxed{c} & \boxed{\cdots} & \boxed{} \\ \boxed{\cdots} & \boxed{d} & & \end{array}$$

If the corners of both ribbons are selected for λ_1, then there must be no unselected elements in that row, as algorithm 1 will not choose the second corner until all possible entries are exhausted, and so there is nothing to show. Therefore, we may assume such a corner a exists.

On the ascending pass, algorithm 1 will choose exactly one entry from the row below a. Therefore, at worst exactly one of b or d will be chosen, but not both, as the algorithm must choose a before a second element in the lower row can be absorbed. If the algorithm chose b for the first part, then, since $a > c$ (even if c was selected), we also have $a > d$. If d was chosen during the ascending pass, then b is the desired entry, as $a > b$ a priori.

This implies that if the λ_2 ribbon has height $\geqslant 1$, then there is a 0 as well. On the other hand, if all unselected elements occur in the lowest row, it is clearly a descent composition, as it consists of all 0's. $\square$

Lemma 3.4.7. *For $\lambda = (\lambda_1, \lambda_2)$, the result P^* of applying algorithm 1 to a parking function P representing a descent composition $\mathbf{a} \in \mathcal{D}_\lambda$ is good.*

Proof. Recall that algorithm 1 first extracts the minimal increasing sequence of maximal length among the two blocks, and then absorbs remaining entries going from top to bottom. This immediately implies, as $\lambda_1 \geqslant \lambda_2$, that $\mathrm{ht}(\lambda_1) \geqslant \mathrm{ht}(\lambda_2)$, which implies that each box in the λ_2 ribbon contributes at least 1 to $\mathrm{dd}(P^*)$.

Suppose now, that some box in the λ_2 ribbon contributes at least 2 to $\mathrm{dd}(P^*)$. Then, we have the four following cases:

- $\boxed{a} \qquad \begin{array}{|c|c|c|}\hline b & \ldots & c \\ \hline \end{array}$

  with $a < c < b$, where the second inequality is because b, c are in the same ribbon. This is a contradiction because a will always be chosen before c.
- $\boxed{a} \qquad \begin{array}{|c|c|c|}\hline (d) & \ldots & c \\ \hline b \\ \cline{1-1} \end{array}$

  with $a > b, a < c$. This is a contradiction since the algorithm will choose a before c.
- $\boxed{a} \qquad \begin{array}{c|c|}\cline{2-2} & d \\ \hline \multicolumn{1}{|c|}{b} & c \\ \hline \end{array}$

  with $a > b, a > c$. Then, this is a contradiction because a will be chosen before c.
- $\boxed{a} \qquad \boxed{b}$

  in the bottom row with $a < b$. (a will also contribute to doff in this case, so the contribution is at least 2.) If b is the initial element chosen, this is a contradiction since the first step of the algorithm selects the smallest entry in the bottom row. If b is not the initial element, this is still a contradiction since it absorbs the remaining entries in the bottom row in increasing order.

$\square$

Definition 3.4.8. Let $\lambda = (\lambda_1, ..., \lambda_\ell)$ be a partition, and given a parking function $P \in \mathfrak{P}_\lambda$, denote $P_1, ..., P_\ell$ to be the corresponding ribbon tableau in each block of λ. Denote by $\mathrm{alg}_{ij}(P)$ to be the result of applying the algorithm to only the parts P_i, P_j, and let P^* denote the result of applying algorithm 1 to P.

Lemma 3.4.9. *We have that*

$$
\begin{aligned}
P^* &= (\mathrm{alg}_{(\ell-1)\ell})^{\lambda_\ell} \\
&\circ \dots \\
&\circ (\mathrm{alg}_{2\ell} \circ \dots \circ \mathrm{alg}_{23})^{\lambda_2} \\
&\circ (\mathrm{alg}_{1\ell} \circ \dots \circ \mathrm{alg}_{12})^{\lambda_1}(P)
\end{aligned} \tag{3.7}
$$

In other words, we may apply the algorithm pairwise to parts a large number of times in a triangular fashion and obtain the same result.

Remark 3.4.10. The algorithm given above is quite redundant, but it suffices for our purposes.

Proof. It suffices to show that the first part agrees with the algorithm result, as the claim will then follow by induction as well as the above lemma. In other words, we will show

$$P_1^* = \left(\mathrm{alg}_{1\ell} \circ \dots \circ \mathrm{alg}_{12}\right)^{\lambda_1}(P)_1 \tag{3.8}$$

The first entry a_0 selected by algorithm 1 is necessarily the smallest entry in the bottom row. If alg_{1i} is the first instance that encounters, then it must absorb it, and it can never be replaced by later alg_{1j} for $j > i$.

Let $a_0, \dots, a_{\lambda_1 - 1}$ denote the entries, in order chosen by the algorithm, in P_1^*. Then, if $a_0, \dots, a_{i-1}$ is selected by the jth pass of $(\mathrm{alg}_{1\ell} \circ \dots \circ \mathrm{alg}_{12})$, at worst the next pass will necessarily select a_i, as if a_i is in the mth column, then alg_{1m} will select first $a_0, \dots, a_{i-1}$, then select a_i by the definition of algorithm 1. Then, by induction, a_i will be selected by the $i+1$th pass, so that equation 3.8 follows. $\square$

Lemma 3.4.11. *Let P be a parking function representing $\mathbf{a} \in \mathcal{D}_\lambda$. Then, P^* is good.*

Proof. Let $X = (\mathrm{alg}_{1\ell} \circ \ldots \circ \mathrm{alg}_{12})^{\lambda_1}(P)$. By lemma 3.4.7, the final pass of $(\mathrm{alg}_{1\ell} \circ \ldots \circ \mathrm{alg}_{12})$ will yield X such that every pair $(X_j, X_1), j \neq 1$ is good. Then, we have that every entry in $X_i, i \geqslant 2$ when compared to X_1 contributes exactly 1 to $\mathrm{dd}(X)$. By induction, the claim follows. $\square$

Corollary 3.4.12. *We have that* $\mathrm{dd}(P^*) = \eta(\lambda)$.

3.4.2.4. *Uniqueness of P^*.*

Lemma 3.4.13. *For each composition $\mathbf{a} \in \mathcal{D}_\lambda$, there is exactly one parking function $P \in \mathfrak{P}_\lambda$ that is good, and it is P^*, the result of algorithm 1.*

Proof. Suppose the contrary, that there is a parking function $P \in \mathfrak{P}_\lambda$ such that $\mathrm{dd}(P)$ is good, but $P \neq P^*$. Let us consider the first entry of P that differs from $P*$ (going in order of the parts, in order selected by the algorithm)

- If we skip during the initial ascending step, we have: $\boxed{c}$ $\quad\quad$ $\begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array}$

  with $a < c < b$. Then, c contributes 2 to $\mathrm{dd}(P)$.
- If we skip during the absorption step: $\boxed{a}$ $\quad\quad$ $\begin{array}{|c|c|c|c|}\hline \ldots & c & \ldots & b \\ \hline \\ \cline{1-1}\end{array}$

  with $a < b < c$. Here c is the element selected during the initial run, and b was absorbed later. Then, a contributes at least 2 to $\mathrm{dd}(P)$.
- Finally, $\boxed{a}$ $\quad\quad$ $\begin{array}{|c|c|c|}\hline b & \ldots & c \\ \hline d \\ \cline{1-1}\end{array}$ where d, c are part of the initial ascending sequence. We must have $a < d$, otherwise a would have been chosen during the initial ascending step. Then, we skip a to choose b, we have $a < b$, and $a < d < c$, so a contributes at least 2 to $\mathrm{dd}(P)$.

So that $P \neq P^*$ implies that P is not good. $\square$

Corollary 3.4.14. *For each $\mathbf{a} \in \mathcal{D}_\lambda$, there is a unique parking function P^* with* $\mathrm{dd}(P^*) = \eta(\lambda)$.

Corollary 3.4.15 (Proposition 3.2.5). *A composition $\mathbf{a} \in \mathbb{Z}^n_{\geqslant 0}$ has the property that $\mathbf{a} \in \mathcal{D}_\lambda$ if and only if algorithm 1 terminates.*

3.4.2.5. *Quasisymmetric Expansion.* We briefly recall the definition of the Gessel quasisymmetric function.

Definition 3.4.16. The *fundamental Gessel quasisymmetric function* $F_{\alpha,n}(\mathbf{x})$ indexed by a composition α of n is given by

$$F_{\alpha,n}(\mathbf{x}) = \sum_{\substack{i_1 \leqslant \ldots \leqslant i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} \ldots x_{i_n}$$

We will now expand the left hand side of equation 3.5 in terms of the $F_{\alpha,n}(\mathbf{x})$'s.

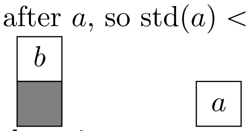
Definition 3.4.17. The *standardization* of a word parking function $P \in \mathcal{WP}_n$ with content $\alpha = (\alpha_1, \ldots, \alpha_k)$ is the parking function $\mathrm{std}(P) \in \mathcal{PF}_n$ given by traversing along the level sets $Z_i(P)$ from highest to lowest and replacing top right to bottom left, all instances of 1 with $1, 2, \ldots, \alpha_1$, all instances of 2 with $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$ and so on. We say that P *standardizes* to $\mathrm{std}(P)$.

It is clear that $\mathrm{area}(P) = \mathrm{area}(\mathrm{std}(P))$, as the operation std does not alter the underlying Dyck path. It turns out that std also preserves the statistic dd.

Lemma 3.4.18. *For any* $P \in \mathcal{PF}_n$*, we have that* $\mathrm{dd}(P) = \mathrm{dd}(\mathrm{std}(P))$.

Proof. If a is the occupant in the i-th row, denote by $\mathrm{std}(a)$ to be the element in the same position after applying std. Consider the following two types of dinv pairs:

- $(I.)$ 

  If $a \geqslant b$, then a will always be replaced after b, so $\mathrm{std}(a) \geqslant \mathrm{std}(b)$. If $a < b$, then b will be replaced after a, so $\mathrm{std}(a) < \mathrm{std}(b)$.
- $(II.)$ 

  If $b \leqslant a$, then since a occurs in a lower level set, we always have that a is replaced after b, so $\mathrm{std}(b) \leqslant \mathrm{std}(a)$. If $b > a$, then a is replaced before b, so $\mathrm{std}(b) > \mathrm{std}(a)$.

In all cases, the contribution to dd is preserved, so this completes the proof. □

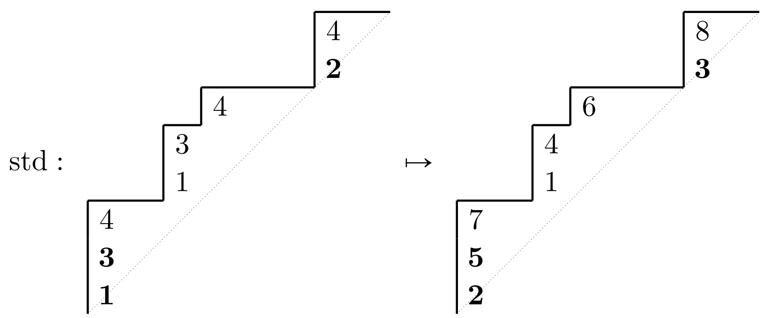
Definition 3.4.19. Let $P \in \mathcal{WP}_n$. An *inverse descent occupant* of P is an occupant i such that:

- No additional instance of i occurs in a lower level set
- No additional instance of i occurs in the same level set below and to the left

- The smallest entry a such that $a > i$ occurs either in a higher level set, or in the same level set above and to the right

The *inverse descent* associated to i is defined to be the number of occupants j of P such that $j \leqslant i$ (including i itself). For a parking function, this number will end up being i itself. We will denote the set of inverse descents of P by $\mathrm{iDes}(P)$.

Example 3.4.20.



and the inverse descents are marked in bold for each parking function.

By definition 3.4.17, it is easy to see that the operation std preserves the inverse descent set, that is, $\mathrm{iDes}(P) = \mathrm{iDes}(\mathrm{std}(P))$. Then, by definition 3.4.16, we see that for a parking function $P \in PF_n$,

$$\sum_{\substack{Q \in WP_n \\ \mathrm{std}(Q)=P}} x^Q = F_{\mathrm{iDes}(P),n}(\mathbf{x})$$

These observations allow us to collect the word parking functions by standardization to yield the following:

$$[q^{\eta(\lambda)}]\left(\sum_{P \in \mathcal{WP}_\lambda} q^{\mathrm{dd}(P)} t^{\mathrm{area}(P)} x^P\right) = \sum_{\substack{P \in WP_\lambda \\ P \text{ is good}}} t^{\mathrm{area}(P)} x^P = \sum_{\substack{P \in \mathcal{PF}_\lambda \\ P \text{ is good}}} t^{\mathrm{area}(P)} F_{\mathrm{iDes}(P),n}(\mathbf{x}) \tag{3.9}$$

By corollary 3.4.14, descent compositions $\mathbf{a} = (a_1, ..., a_n) \in \mathcal{D}_\lambda$ under the map $\epsilon \circ \mathrm{alg}_\lambda$ are in bijection with good parking functions, denoted $P^*_{\mathbf{a}} \in \mathcal{PF}_\lambda$. It is clear that $\sum_i a_i = \mathrm{area}(P^*_{\mathbf{a}})$ by the definitions of ϵ and alg_λ. If $\pi = \mathrm{majt}^{-1}(\mathbf{a})$, we now must show that $\mathrm{iDes}(P^*_{\mathbf{a}}) = \mathrm{iDes}(\pi)$.

Lemma 3.4.21. *Let* $\mathbf{a}, \pi, P^*_{\mathbf{a}}$ *be as above. Then,* $\mathrm{iDes}(P^*_{\mathbf{a}}) = \mathrm{iDes}(\pi)$*.*

Proof. By lemma 2.1.5, if $i \in \mathrm{iDes}(\pi)$, then we have that $a_i < a_{i+1}$. This precisely says that $i+1$ occurs in a higher level set of $P^*_{\mathbf{a}}$, i.e. $i \in \mathrm{iDes}(P^*_{\mathbf{a}})$. $\square$

We may finally prove the first half of Theorem 1.0.2.

Proof of first equality of Theorem 1.0.2. Putting the above observations together with equations 3.5, 3.9, we have

$$\widetilde{C}_{\lambda'}(X;t) = \sum_{\pi \in J^{\mathrm{maj}}_\lambda} t^{\mathrm{maj}(\pi)} F_{\mathrm{iDes}(\pi),n}(\mathbf{x}) = \sum_{\substack{P \in \mathcal{PF}_\lambda \\ P \text{ is good}}} t^{\mathrm{area}(P)} F_{\mathrm{iDes}(P),n}(\mathbf{x})$$

$$= [q^{\eta(\lambda)}]\left(\sum_{P \in \mathcal{WP}_\lambda} q^{\mathrm{dd}(P)} t^{\mathrm{area}(P)} x^P\right) = \widetilde{H}_{\lambda'}(X;t) \tag{3.10}$$

$\square$

3.4.3. Proof of Monomial Symmetric Function Expansion. We now finish the proof of theorem 1.0.2 by proving the second equality.

Definition 3.4.22. A J^{maj}_λ*-word* w with content $\alpha = (\alpha_1, ..., \alpha_k)$, where α is a composition of n, is a word that standardizes to an element $\pi \in J^{\mathrm{maj}}_\lambda$. We denote the set of J^{maj}_λ-words by W^{maj}_λ.

For a word $w = w_1 \dots w_n$, we may similarly define major index by $\mathrm{maj}(w) = \sum_{w_i > w_{i+1}} i$. By comparing definitions, it is straightforward to show that $\mathrm{maj}(w) = \mathrm{maj}(\mathrm{std}(w))$, and so we may define $\mathrm{majt}(w) := \mathrm{majt}(\mathrm{std}(w))$. We will think of elements of W^{maj}_λ as a *two-row table* consisting of $w = \begin{bmatrix} \mathrm{majt}(w) \\ \mathbf{b} \end{bmatrix}$ where $\mathbf{b} = (b_1, \dots, b_n)$ is the *content* of w. We require $\mathbf{b}$ to be weakly increasing, that is, $b_1 \leqslant \cdots \leqslant b_n$, and if $\mathrm{majt}(w) = (a_1, \dots, a_n)$, that $a_i < a_{i+1} \implies b_i < b_{i+1}$. Given such a two-row table $\begin{bmatrix} \text{majt}(\pi) \\ \mathbf{b} \end{bmatrix}$, we may recover w by sorting the columns so that the top row is weakly decreasing to obtain:

$$\text{upsort}\left(\begin{bmatrix} \text{majt}(w) \\ \mathbf{b} \end{bmatrix}\right) = \begin{bmatrix} k \ldots k \ldots 0 \ldots 0 \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ w \end{bmatrix}$$

with the condition that if $\mathbf{c} = (c_1, \ldots, c_n)$, $c_i = c_{i+1} \implies w_i \leqslant w_{i+1}$.

Write $w = \begin{bmatrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \end{bmatrix}$, so that $\text{majt}(w) = (a_1, \ldots, a_n)$. We may standardize w by simply replacing $(b_1, \ldots, b_n)$ with $(1, \ldots, n)$, and so to obtain a well-defined standardization, we must have that $a_i < a_{i+1} \implies b_i < b_{i+1}$. Similarly, any weakly increasing sequence $\mathbf{b} = (b_1, \ldots, b_n)$ with the property that $b_i < b_{i+1}$ for all $\text{iDes}(\pi)$ gives a well defined word $w = \begin{bmatrix} \text{majt}(\pi) \\ \mathbf{b} \end{bmatrix}$ and will have the property that $\text{std}(\begin{bmatrix} \text{majt}(\pi) \\ \mathbf{b} \end{bmatrix}) = \pi$.

Definition 3.4.23. Let $w \in W_n$, write $\text{majt}(w) = (a_1, ..., a_n)$. An *inverse descent* of w is i such that $a_i < a_{i+1}$. Denote the set of inverse descents of w by $\text{iDes}(w)$.

Then we see that $\text{iDes}(w)$ tracks precisely when $\mathbf{b}$ must increase. We may now expand the quasisymmetric formula in terms of W_λ^{maj}.

Lemma 3.4.24. *We have that*

$$\sum_{\pi \in J_\lambda^{\text{maj}}} t^{\text{maj}(\pi)} F_{\text{iDes}(\pi),n}(\mathbf{x}) = \sum_{w \in W_\lambda^{\text{maj}}} t^{\text{maj}(w)} x^w$$

Proof. It suffices that to show that for a given $\pi \in J_\lambda^{\text{maj}}$, that

$$t^{\text{maj}(\pi)} F_{\text{iDes}(\pi),n}(\mathbf{x}) = \sum_{\substack{w \in W_\lambda^{\text{maj}} \\ \text{std}(w) = \pi}} t^{\text{maj}(w)} x^w$$

Given a monomial $x_{i_1} \ldots x_{i_n}$, we must have that $i_j < i_{j+1}$ if $j \in \text{iDes}(\pi)$. This gives the well-defined word

$$w = \begin{bmatrix} \text{majt}(\pi) \\ \\ i_1 \ldots i_n \end{bmatrix}$$

which standardizes to π.

Conversely, given any $w \in W_\lambda^{\text{maj}}$ with $\text{std}(w) = \pi$, we may form the two-row table as above. By the discussion before definition 3.4.23, we must have that $j \in \text{iDes}(\pi) \implies i_j < i_{j+1}$, so that $x_{i_1} \ldots x_{i_n}$ occurs in $F_{\text{iDes}(\pi),n}(\mathbf{x})$.

Finally, $\text{maj}(w) = \text{maj}(\text{std}(w)) = \text{maj}(\pi)$, and the claim is proved. $\square$

We now complete the proof of theorem 1.0.2.

Proof of second equality of theorem 1.0.2. We show the equality

$$\sum_{w \in W_\lambda^{\text{maj}}} t^{\text{maj}(w)} x^w = \sum_{\mu \vdash n} \Big(\sum_{\omega \in \text{Sh}\mu \cap J_\lambda^{\text{maj}}} t^{\text{maj}(\omega)} \Big) m_\mu(\mathbf{x})$$

By equation 3.10 and lemma 3.4.24, it suffices, for every partition $\mu \vdash n$, to show that the coefficient of $x_1^{\mu_1} ... x_n^{\mu_n}$ of both sides is equal by the symmetry in the $\mathbf{x}$-variables of $\widetilde{H}_{\lambda'}(X;t)$. Fix a partition $\mu \vdash n$. The coefficient in $x_1^{\mu_1} ... x_n^{\mu_n}$ the left hand side is obtained by summing $t^{\text{maj}(w)}$ over all words w with content μ.

Now observe that no two distinct words with content μ can standardize to the same element of J_λ^{maj}. Furthermore, notice that every word of content μ must standardize to a μ-shuffle, and that every element of $\text{Sh}(\mu) \cap J_\lambda^{\text{maj}}$ can be obtained by standardizing some word w of content μ. Putting these observations together, along with lemma 3.4.24, we have that:

$$\sum_{\substack{w \in W_\lambda^{\text{maj}} \\ \text{content}(w) = \mu}} t^{\text{maj}(w)} = \sum_{\omega \in \text{Sh}\mu \cap J_\lambda^{\text{maj}}} t^{\text{maj}(\omega)} \tag{3.11}$$

$\square$

3.5. Schur Expansion

As it turns out, the set W_λ^{maj} naturally gives rise to a Schur expansion of $\widetilde{H}_{\lambda'}(X;t)$.

3.5.1. Schur Expansion of $\widetilde{C}_{\lambda'}(X;t)$. Let $\pi \in S_n$. Recall that the *elementary dual Knuth transformation* τ_i acts on π by exchanging $i, i+1$ if and only if $i-1$ or $i+2$ occur between them. We will show that the τ_i preserve the property of being in J_λ^{maj}.

LEMMA 3.5.1. *Let $\pi \in J_\lambda^{\mathrm{maj}}$. Then, for any $1 \leqslant i \leqslant n-1$, we have that $\tau_i(\pi) \in J_\lambda^{\mathrm{maj}}$.*

PROOF. It suffices to only consider the cases in which $\tau_i(\sigma) \neq \sigma$. Suppose that in the two-row table of σ, we have

$$\begin{bmatrix} \dots & c & a & b & d & \dots \\ \dots & i-1 & i & i+1 & i+2 & \dots \end{bmatrix}$$

For either of $i-1, i+2$ to lie between a, b, then we must have that $a \neq b$.

Case 1: $a < b$. This means that if $i-1$ lies between them, we must have $a \leqslant c < b$, and if $i+2$ lies between them, that $a < d \leqslant b$. Note that since $i, i+1$ occur in different runs, then switching their run labels has exactly the effect of switching their positions.

Let $\mathbf{a} = \mathrm{majt}(\sigma)$. Then, by algorithm 1, we obtain descent compositions $\mathbf{a}_1, ..., \mathbf{a}_\ell$, with $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$. If a, b belong to different descent compositions, then switching their positions yields another element of J_λ^{maj}. If a, b belong to the same part, say $\mathbf{a}_i$, and $b - a > 1$, then a, b is not part of an essential sequence of $\mathbf{a}_i$, and so switching them yields an $\mathbf{a}_i' \in \mathcal{D}_{\lambda_i}$, and so the full shuffle is in J_λ^{maj}. This leaves the following two cases:

$$\begin{bmatrix} \dots & (a)_k & (a+1)_k & (a+1)_\ell & \dots \\ \dots & i & i+1 & i+2 & \dots \end{bmatrix} \leftrightarrow \begin{bmatrix} \dots & (a+1)_\ell & (a)_k & (a+1)_k & \dots \\ \dots & i & i+1 & i+2 & \dots \end{bmatrix}$$

$$\begin{bmatrix} \dots & (a)_\ell & (a)_k & (a+1)_k & \dots \\ \dots & i-1 & i & i+1 & \dots \end{bmatrix} \leftrightarrow \begin{bmatrix} \dots & (a)_k & (a+1)_k & (a)_\ell & \dots \\ \dots & i-1 & i & i+1 & \dots \end{bmatrix}$$

where k, ℓ denote the parts of the corresponding ordered set partition (it is possible that $\ell = k$). Then, it is easy to see these transformations preserve the property of being in J_λ^{maj}.

Case 2: $a > b$. Then, a, b can never belong to the same essential sequence, and so swapping them preserves the property of being in J_λ^{maj}. $\square$

Since dual Knuth transformations preserve the property of being in J_λ^{maj}, we see that if a word $w \in W_\lambda^{\mathrm{maj}}$, then $\tau_i(w) \in W_\lambda^{\mathrm{maj}}$ for any i. This implies, by proposition 2.2.4 and the fact that standardization preserves recording tableau, that any word w' with the same recording tableau Q as w will have the property $w' \in W_\lambda^{\mathrm{maj}}$. Denote the set of recording tableau of shape μ corresponding to elements of J_λ^{maj} by $E_{\mu,\lambda}^{\mathrm{maj}}$. We may then RSK and collect the left hand side of equation 3.11 by recording tableau:

$$\sum_{w \in W_\lambda^{\mathrm{maj}}} t^{\mathrm{maj}(w)} x^w = \sum_{Q \in E_{\mu,\lambda}^{\mathrm{maj}}} t^{\mathrm{maj}(Q)} \left(\sum_{P \in \mathrm{SSYT}(\mu)} x^P \right) = \sum_{Q \in E_{\mu,\lambda}^{\mathrm{maj}}} t^{\mathrm{maj}(Q)} s_\mu(\mathbf{x}) \tag{3.12}$$

where $w \mapsto (P, Q)$ under the usual RSK bijection, and x^P is the monomial corresponding to the entries of P.

3.5.2. Characterization of $E_{\mu,\lambda}^{\mathrm{maj}}$ and Connection to Cocharge. We now turn our attention to the set $E_{\mu,\lambda}^{\mathrm{maj}}$. We will first describe, for each standard Young tableau, an associated tableau that tracks its major index. Denote the *runs* of a standard Young tableau to be the maximal northeast consecutive increasing sequences.

Definition 3.5.2. Let $T \in \mathrm{SYT}(\mu)$ be a standard Young Tableau of shape μ, and let $d+1$ denote the number of runs of T, and denote the runs by $R_d, R_{d-1}, ..., R_0$, where $1 \in R_d$. Define the *major index diagram* of T to be a filling of μ where each entry is replaced with its run label. Denote this by $\mathrm{majd}(T)$.

Example 3.5.3. Consider the following standard Young tableau, where runs are separated by color:

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 9 & 10 & 15 \\ \hline 4 & 5 & 12 & 13 & 14 \\ \cline{1-5} 7 & 8 \\ \cline{1-2} 11 \\ \cline{1-1} \end{array} \qquad \mathrm{majd}(T) = \begin{array}{|c|c|c|c|c|c|c|} \hline 3 & 3 & 3 & 2 & 1 & 1 & 0 \\ \hline 2 & 2 & 0 & 0 & 0 \\ \cline{1-5} 1 & 1 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array}$$

Remark 3.5.4. The major index diagram also has another interpretation - one can think of it as encoding the contribution of each element to the *charge* of the reading word of T.

Lemma 3.5.5. *Let $T \in \mathrm{SYT}(\mu)$. Then, we have that* $\mathrm{majd}(T)$ *is a semistandard Young tableau in the alphabet $\mathbb{Z}_{\geqslant 0}$ with the reverse ordering. Furthermore, for $i > 0$, the northeast-most entry of i will have at least one $i-1$ in a lower row, and* majd *is a bijection between* $\mathrm{SYT}(\mu)$ *and SSYT of shape μ with this property, denoted E^μ.*

Proof. Since T is a standard Young tableau, then the configuration $\begin{array}{|c|c|}\hline a & b \\ \hline\end{array}$ implies $a < b$. If $b = a+1$, then $\ell(a) = \ell(b)$, where $\ell(x)$ is the row label of $x \in T$. Otherwise, if $b > a+1$, then $a+1$ must occur in a lower row, so that a is a descent, and that $\ell(a) > \ell(b)$. In either case, we have $\ell(a) \geqslant \ell(b)$.

On the other hand, if a is directly above b, then b occurs in a later run than a, so that $\ell(a) > \ell(b)$. This completes the proof of the first statement.

The second statement is clear by noticing that the northeast-most entry a of i is simply the last entry of the ith run, so that a is a descent of T. Then, we must have that $a+1$ occurs in a lower row and $\ell(a+1) = i-1$.

Finally, we define $\mathrm{majd}^{-1} : E^\mu \to \mathrm{SYT}(\mu)$ as follows: If $S \in E^\mu$ and the content of S is $d^{a_d}(d-1)^{a_{d-1}}...1^{a_1}$, then, moving left to right, replace all instances of d with $1, ..., a_d$, all instances of $d-1$ with $a_d+1, ..., a_d+a_{d-1}$, and so on. Since the northeast-most entry of i has an $i-1$ in a row below it, it can never be the case that we have two entries of the form $\begin{array}{|c|c|}\hline a & a{+}1 \\ \hline\end{array}$ in $\mathrm{majd}^{-1}(S)$ with the corresponding entries in S as $\begin{array}{|c|c|}\hline i & i{-}1 \\ \hline\end{array}$. Therefore the runs of $\mathrm{majd}^{-1}(S)$ are precisely given by S, so that $(\mathrm{majd} \circ \mathrm{majd}^{-1})(S) = S$.

□

Definition 3.5.6. We say that a major index diagram $\mathrm{majd}(T)$ is λ-*splittable* if it can be decomposed into parts of sizes λ_1,...,λ_ℓ such that in each part, for $i > 0$, the northeast-most instance of i has an $i-1$ to in a lower row.

Example 3.5.7. The tableau $\mathrm{majd}(T)$ in our running example is $(6,5,2,2,1)$-splittable:

$$\mathrm{majd}(T) = \begin{array}{|c|c|c|c|c|c|c|}\hline 3 & 3 & 3 & 2 & 1 & 1 & 0 \\ \hline 2 & 2 & 0 & 0 & 0 \\ \cline{1-5} 1 & 1 \\ \cline{1-2} 0 \\ \cline{1-1}\end{array} \qquad \begin{array}{cccc} & & \boxed{2} & \boxed{1} \\ \boxed{2} & \boxed{0} & & \\ \boxed{1} & & & \end{array}$$

where the diagram on the right illustrates the condition for $i = 1, 2$ in the orange part of size 5.

Before giving a characterization of all λ-splittable major index diagrams, we first prove a lemma.

LEMMA 3.5.8. *Let* $T \in \mathrm{SYT}(\mu)$. *Then, we have that* $\mathrm{rec}(rw(T)^{-1}) = T$, *and that* $\mathrm{majt}(rw(T)^{-1}) = rw(\mathrm{majd}(T))$.

PROOF. It is well known that each Knuth equivalence class of S_n has exactly one permutation that appears as the reading word of a standard Young tableau (for instance, appendix A in [**Sta99**]) so that if $T \in \mathrm{SYT}(\mu)$, then we have that $(\mathrm{rw}(T)) = (T, Q)$, where $Q \in \mathrm{SYT}(\mu)$, so $T = \mathrm{ins}(\mathrm{rw}(T))$. Since taking inverse swaps the insertion and recording tableau, we have that $(\mathrm{rw}(T)^{-1}) = (Q, T)$, so that $\mathrm{rec}(\mathrm{rw}(T)^{-1}) = T$.

To see the second statement, notice that the runs of T become the maximal consecutive increasing subsequences of $\mathrm{rw}(T)$. Then, in $\mathrm{rw}(T)^{-1}$, the *positions* of the maximal consecutive increasing subsequences of $\mathrm{rw}(T)$ become the runs of $\mathrm{rw}(T)^{-1}$ themselves. Since $\mathrm{majt}(\mathrm{rw}(T)^{-1})$ records the run labels of each entry of $\mathrm{rw}(T)^{-1}$ (as a permutation), and each entry of T is replaced with its run label in $\mathrm{majd}(T)$, we have that $\mathrm{majt}(\mathrm{rw}(T)^{-1}) = \mathrm{rw}(\mathrm{majd}(T))$. $\square$

PROPOSITION 3.5.9. *We have that* $T \in E^{maj}_{\mu,\lambda}$ *if and only if* $rw(\mathrm{majd}(T)) \in \mathcal{D}_\lambda$ *if and only if* $\mathrm{majd}(T)$ *is* λ*-splittable.*

PROOF. Suppose $T \in E^{\mathrm{maj}}_{\mu,\lambda}$. Then, every permutation $\pi \in S_n$ with the property $\mathrm{rec}(\pi) = T$ has the property $\pi \in J^{\mathrm{maj}}_\lambda$. Then, by the previous lemma 3.5.8, we have that $\mathrm{rec}(\mathrm{rw}(T)^{-1}) = T$, so that $\mathrm{rw}(T)^{-1} \in J^{\mathrm{maj}}_\lambda$. Then, again by the lemma, $\mathrm{rw}(\mathrm{majd}(T)) = \mathrm{majt}(\mathrm{rw}(T)^{-1}) \in \mathcal{D}_\lambda$ since $\mathrm{rw}(T)^{-1} \in J^{\mathrm{maj}}_\lambda$. Conversely, if $\mathrm{rw}(\mathrm{majd}(T)) = \mathrm{majt}(\mathrm{rw}(T)^{-1}) \in \mathcal{D}_\lambda$, then $\mathrm{rw}(T)^{-1} \in J^{\mathrm{maj}}_\lambda$, so that $T = \mathrm{rec}(\mathrm{rw}(T)^{-1}) \in E^{\mathrm{maj}}_{\mu,\lambda}$. This proves the equivalence of the first two conditions.

Next, suppose $\mathrm{rw}(\mathrm{majd}(T)) \in \mathcal{D}_\lambda$. Then, $\mathrm{rw}(\mathrm{majd}(T))$ can be written as a shuffle of descent compositions $\mathbf{a}_1, ..., \mathbf{a}_\ell$, whose positions are recorded by an ordered set partition π. Consider the corresponding entries of each part of $\mathbf{a}_i$ in $\mathrm{majd}(T)$. The condition of $\mathbf{a}_i$ being a descent composition in $\mathcal{D}_{\lambda_i}$ is that the rightmost entry of j must have a $j-1$ to the left of it for $j > 0$. Since the rows of $\mathrm{majd}(T)$ are weakly decreasing, this is equivalent to the statement that the corresponding cells of $\mathrm{majd}(T)$ have the property that the northeast-most j must have a $j-1$ before in reading order, which implies the $j-1$ is in a lower row. Conversely, if $\mathrm{majd}(T)$ is λ-splittable, the decomposition gives a tuple of compositions $\mathbf{a}_1, ..., \mathbf{a}_\ell$, with $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$ by the previous sentence. This implies that $\mathrm{rw}(\mathrm{majd}(T)) \in \mathcal{D}_\lambda$. □

We are now faced with a very interesting problem. First, we recall a few definitions.

The *cocharge* of a permutation $\pi \in S_n$ is defined by the following process:

(1) Writing π in one-line notation, label 1 with 0.
(2) If i has been labeled with j, label $i+1$ with j if $i+1$ occurs to the right of i, and with $j+1$ if it occurs to the left. Denote labels $c_1 \dots c_n =: \mathrm{cw}(\pi)$, called the *cocharge word* of π.
(3) We define the *cocharge* to be $\mathrm{cc}(\pi) = c_1 + \cdots + c_n$, the sum of the cocharge labels.

For example, if $\pi = 126497385$, then $\mathrm{cw}(\pi) = 002132021$. Next, let $w \in W_n$ with content $\mu \vdash n$. Then, the *standard subwords* of w are given by:

(1) Moving right to left, find the first instance of $1, 2, \dots \mu_1'$, cyclically wrapping around if necessary. Denote this subword by $w^{(1)}$.
(2) Delete $w^{(1)}$ from w and repeat to obtain $w^{(2)}$, and so on.

For the word $w = 433222311111$, the subwords are given by the following indices:

$$4_1 3_2 3_1 2_3 2_2 2_1 3_3 1_5 1_4 1_3 1_2 1_1$$

and subwords (in order)

$$4321, 321, 231, 1, 1$$

For $T \in \mathrm{SSYT}(\mu, \lambda)$, we may define $\mathrm{cc}(T) = \mathrm{cc}(\mathrm{rw}(T))$, the cocharge of the reading word for T. Recall the following theorem of Lascoux and Schützenberger:

THEOREM 3.5.10. *[**LS78**] We have that*

$$\widetilde{H}_\lambda(X; t) = \sum_\mu \Bigg(\sum_{T \in \mathrm{SSYT}(\mu,\lambda)} t^{\mathrm{cc}(T)} \Bigg) s_\mu(\mathbf{x})$$

where $\mathrm{cc}(T)$ *of a semistandard Young Tableaux is defined to be cocharge of its reading word.*

Equation 3.12 implies that there should be a weighted bijection $\Gamma : E^{\mathrm{maj}}_{\mu,\lambda} \to \mathrm{SSYT}(\mu, \lambda')$ that takes $\mathrm{maj}(T)$ to $\mathrm{cc}(\Gamma(T))$. For certain shapes, such a bijection has been constructed, but in general the answer remains elusive.

We may construct, for $T \in \mathrm{SSYT}(\mu, \lambda')$, an associated filling of the shape μ by recording each elements contribution to cocharge. We will denote this by $\mathrm{ccd}(T)$. For example:

$$\mathrm{ccd}\left(\begin{array}{|c|c|c|c|c|c|c|}\hline 1_2 & 1_1 & 2_2 & 3_2 & 3_1 & 4_2 & 4_1 \\ \hline 2_1 \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

where the subscripts in the first diagram denote the cocharge labels. If for all $T \in \mathrm{SSYT}(\mu, \lambda')$ we have that $\mathrm{ccd}(T)$ is also a semistandard Young tableau in the alphabet $\mathbb{Z}_{\geqslant 0}$, then the bijection $\Gamma = \mathrm{ccd}^{-1} \circ \mathrm{evac}(\mu, d) \circ \mathrm{rev} \circ \mathrm{majd}$ suffices, where d is the maximal entry of $\mathrm{majd}(T)$. However, as evidenced above, $\mathrm{ccd}(T)$ in general is *not* a semistandard Young Tableau.

Problem 3. Construct a weight preserving bijection $\Gamma : E^{\mathrm{maj}}_{\mu,\lambda} \to \mathrm{SSYT}(\mu, \lambda')$ that carries maj to cc for all shapes μ.

CHAPTER 4

Descent Basis for the Garsia-Procesi Module

In this chapter we will give a proof of theorem 1.0.1, that for a partition $\lambda \vdash n$, the set

$$\left\{x^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_\lambda\right\} = \left\{x^{\mathrm{majt}(\pi)} : \pi \in J_\lambda^{\mathrm{maj}}\right\}$$

is a vector space basis of $R_{\lambda'}$.

4.1. Descent Order on Monomials

Let $R_n = \mathbb{C}[x_1, \ldots, x_n]/I_n$ be the coinvariant algebra, where I_n is the coinvariant ideal. We may consider the ideal generated by the leading terms of $\mathrm{LM}(I_n)$ with respect to the *lexicographical* order. Then, we may define $\mathrm{LM}_{\mathrm{lex}}(R_n)$ to be the set of monomials *not* in $\mathrm{LM}_{\mathrm{lex}}(I_n)$, and it is clear that $\mathrm{LM}(R_n)$ forms a vector space basis of R_n. As it turns out:

Fact 1. We have that $A_n = \mathrm{LM}(R_n)$, where A_n is the Artin basis of R_n.

Analogously, the Garsia-Stanton descent monomial basis are the leading terms with respect to a different monomial order, called the *descent order*.

Definition 4.1.1. Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}^n_{\geqslant 0}$. We say that $x^\alpha \leqslant_{\mathrm{des}} x^\beta$ if:

(1) $\mathrm{sort}(\alpha, >) <_{\mathrm{lex}} \mathrm{sort}(\beta, >)$ or
(2) $\mathrm{sort}(\alpha, >) = \mathrm{sort}(\beta, >)$ and $\alpha \leqslant_{\mathrm{lex}} \beta$

Though des is not a monomial order in the Gröbner basis sense, the notion of leading terms is still well defined. E. E. Allen gave an algorithm to reduce any monomial using I_n in this order, see [**All94**].

We record a useful lemma here.

LEMMA 4.1.2. *Let $S \subset [n]$ with $T = [n] \backslash S$ its complement, and for a composition α, denote $\alpha|_S = (\alpha_{i_1}, \dots, \alpha_{i_s})$, where all $i_j \in S$, and $s = |S|$. Then, we have that*

$$\alpha|_S \leqslant_{\text{des}} \beta|_S, \quad \alpha|_T \leqslant_{\text{des}} \beta|_T \implies \alpha \leqslant_{\text{des}} \beta \tag{4.1}$$

PROOF. Suppose $\alpha|_S \leqslant_{\text{des}} \beta|_S, \alpha|_T \leqslant_{\text{des}} \beta|_T$. For a composition α, write $\text{sort}(\alpha, >) =: \text{sort}(\alpha)$. Then, we have that $\text{sort}(\alpha|_S) \leqslant \text{sort}(\beta|_S)$ and $\text{sort}(\alpha|_T) \leqslant \text{sort}(\beta|_T)$. We may construct $\text{sort}(\alpha)$ from $\text{sort}(\alpha|_S), \text{sort}(\alpha|_T)$ by the following process:

(1) Begin with an empty string, $\text{sort}(\alpha) = \varnothing$
(2) At each step, compare the first entry of $\text{sort}(\alpha|_S), \text{sort}(\alpha|_T)$, and append the greater entry to the end of $\text{sort}(\alpha)$. Remove this entry from the composition it came from. If there is a tie, choose from $\text{sort}(\alpha|_S)$.
(3) Once both tuples are empty, then what remains is $\text{sort}(\alpha)$.
(4) We can form a *recording string* $w \in 0^{|S|}1^{|T|}$, which records the order in which the elements were absorbed; a 0 will denote choosing from $\alpha|_S$, and 1 will denote choosing from $\alpha|_T$.

It is straightforward to show that this procedure produces $\text{sort}(\alpha)$. Apply the above procedure to $\text{sort}(\beta|_S), \text{sort}(\beta|_T)$. Denote the recording strings w_α and w_β respectively. If $w_\alpha \neq w_\beta$, then the first step i at which there is a deviation, we necessarily have $\text{sort}(\alpha)_i < \text{sort}(\beta)_i$, so that $\text{sort}(\alpha) <_{\text{lex}} \text{sort}(\beta)$. If $w_\alpha = w_\beta$, then $\text{sort}(\alpha)_i \leqslant \text{sort}(\beta)_i$ for all i, so that $\text{sort}(\alpha) \leqslant_{\text{lex}} \text{sort}(\beta)$.

If $\text{sort}(\alpha) = \text{sort}(\beta)$, then we must have had that $\text{sort}(\alpha|_S) = \text{sort}(\beta|_S)$ and $\text{sort}(\alpha|_T) = \text{sort}(\beta|_T)$. Then, we see that

$$\alpha|_S \leqslant_{\text{lex}} \beta|_S, \quad \alpha|_T \leqslant_{\text{lex}} \beta|_T$$

The first entry in which α and β differ must also be the first entry in which they differ when restricted to S or T. This implies that $\alpha \leqslant_{\text{lex}} \beta$.

□

4.2. New Garsia-Procesi Basis

We give the reader a brief outline of the argument here. First, we will construct a series of maps $\varphi_{\lambda,S} : R_{\lambda'} \to R_k \otimes R_{\mu'}$, where $\lambda_1 = k$, $\mu = (\lambda_2, \dots, \lambda_\ell)$ with the first part removed. Then, we will use these maps to inductively show for $\mathbf{a} \in \mathcal{D}_\lambda$, any expression of the form

$$\sum_{\beta \leqslant_{\text{des}} \alpha} c_\beta x^\beta = 0 \tag{4.2}$$

must have $c_\alpha = 0$. Then, we show the coefficients are triangular with respect to $\leqslant_{\text{des}}$, so that $\{x^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_\lambda\}$ must be linearly independent. Finally, Theorem 1.0.2 implies that $|\mathcal{D}_\lambda| = \binom{n}{\lambda'}$ to establish a dimension count.

4.2.1. The Maps $\varphi_{\lambda,S}$. Let $S \subset [n]$, with $|S| = k$. Then, we may consider the composition of maps

$$\varphi_S : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_k] \otimes \mathbb{C}[x_1, \ldots, x_{n-k}] \to R_k \otimes \mathbb{C}[x_1, \ldots, x_{n-k}]$$

where if $S = \{i_1, \ldots, i_k\}$, $T = [n]\backslash S = \{j_1, \ldots, j_{n-k}\}$, we have that the first map evaluates $x_{i_m} \mapsto x_m \otimes 1, x_{j_m} \mapsto 1 \otimes x_m$, and the second map is the quotient map in the first factor.

PROPOSITION 4.2.1. *Let λ be a partition of length $\ell > 0$, and let $\mu = (\lambda_2, \ldots, \lambda_\ell)$ denote the partition obtained by removing the first part. Then, for any subset $S \subset [n]$ of size λ_1, the map φ_S in 4.2 descends to a map*

$$\varphi_{\lambda,S} : R_{\lambda'} \to R_k \otimes R_{\mu'} \tag{4.3}$$

PROOF. We need to check that the generators of $I_{\lambda'}$ are sent to 0 under the map $\varphi_{\lambda,S}$. Recall the definition of the Tanisaki ideal:

$$I_{\lambda'} := \left\langle e_d(T) : T \subseteq [n], |T| \geqslant d > |T| - p^n_{|T|}(\lambda') \right\rangle \tag{4.4}$$

Let $e_d(T) \in I_{\lambda'}$, so that the above inequalities are satisfied. We may then write

$$e_d(T) = \sum_{i=\max(0,d-|T\backslash S|)}^{|S\cap T|} e_i(S \cap T) e_{d-i}(T\backslash S) \tag{4.5}$$

so that

$$\varphi_{\lambda,S}(e_d(T)) = \sum_{i=\max(0,d-|T\backslash S|)}^{|S\cap T|} e_i(S\cap T)\otimes e_{d-i}(T\backslash S) \tag{4.6}$$

where by abuse of notation, equation 4.6 has the variables evaluated in each tensor component per φ_S.

If $S \subset T$, then $e_i(S\cap T) = e_i(S) \in I_k$, the coinvariant ideal of the first factor, whenever $i > 0$. This leaves only the summand $1 \otimes e_d(T\backslash S)$ if i is allowed to be 0. In this case, we must have that $d - |T\backslash S| \leqslant 0$, or that $d \leqslant |T\backslash S|$. We need to show that $e_d(T\backslash S) \in I_{\mu'}$.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the transpose of λ' padded with 0's to form a tuple of length n, then we notice $\mu = (\lambda_2, \ldots, \lambda_{n-|S|+1})$, so that $\mu_i = \lambda_{i+1}$ for $1 \leqslant i \leqslant n - |S|$. Since $e_d(T) \in I_{\lambda'}$, we must have that

$$|T| - (\lambda_n + \cdots + \lambda_{n-|T|+1}) < d \leqslant |T\backslash S| \tag{4.7}$$

Since $\lambda_1 = |S|$, we also note that we must have $p^n_n(\lambda') = \cdots = p^n_{n-|S|}(\lambda') = 0$. Substituting, we obtain

$$|T| - (\lambda_{n-|S|-1} + \cdots + \lambda_{n-|T|+1}) < d \leqslant |T\backslash S| \tag{4.8}$$

We may assume that $|T| > |S|$, as otherwise there is nothing to show. The substitution $\mu_i = \lambda_{i+1}$ yields

$$\begin{aligned} |T\backslash S| - p^{n-|S|}_{|T\backslash S|}(\mu') &= |T| - |S| - (\mu_{n-|S|} + \cdots + \mu_{n-|S|-|T\backslash S|+1}) \\ &= |T| - |S| - (\lambda_{n-|S|+1} + \cdots + \lambda_{n-|T|+2}) \\ &\leqslant |T| - (\lambda_{n-|S|+1} + \cdots + \lambda_{n-|T|+1}) \\ &\leqslant |T| - (\lambda_{n-|S|-1} + \cdots + \lambda_{n-|T|+1}) < d \end{aligned} \tag{4.9}$$

where in the third line we used that $|S| = \lambda_1 \geqslant \lambda_{n-|T|+2}$ since λ is a partition. This implies that $e_d(T\backslash S) \in I_{\mu'}$.

Next, if $S \nsubseteq T$, so that $|S \cap T| < |S|$. We must show that $e_{d-i}(T \backslash S) \in I_{\mu'}$ for all i. By equation 4.6, we have that $i \leqslant |S \cap T|$, and so subtracting from equation 4.4, we obtain

$$|T| - |S \cap T| - (\lambda_n + \cdots + \lambda_{n-|T|+1}) < d - i \tag{4.10}$$

Then, we may expand as before:

$$\begin{aligned} |T\backslash S| - p^n_{|T\backslash S|}(\mu') &= |T\backslash S| - (\mu_{n-|S|} + \cdots + \mu_{n-|S|-|T\backslash S|+1}) \\ &= |T\backslash S| - (\lambda_{n-|S|+1} + \cdots + \lambda_{n-|T|+2-(|S|-|S\cap T|)}) \\ &= |T\backslash S| - (\lambda_{n-|S|-1} + \cdots + \lambda_{n-|T|+2-(|S|-|S\cap T|)}) \\ &\leqslant |T| - |S \cap T| - (\lambda_{n-|S|-1} + \cdots + \lambda_{n-|T|+1}) \\ &< d - i \end{aligned} \tag{4.11}$$

where the third line uses $\lambda_{n-|S|+1} = \lambda_{n-|S|} = 0$, and the inequality in the fourth follows from $|S| - |S \cap T| > 0$.

For the other inequality, there are two cases. If $d - |T\backslash S| > 0$, then

$$d - i \leqslant d - (d - |T\backslash S|) = |T\backslash S|$$

Otherwise, $d - |T\backslash S| \leqslant 0$, so $d \leqslant |T\backslash S|$. In all cases, we have that $e_{d-i}(T\backslash S) \in I_{\mu'}$, and we are done. □

4.3. Linear Independence

We prove the linear independence of $\{x^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_\lambda\}$, establishing Theorem 1.0.1.

Proof of Theorem 1.0.1. Let $\mathbf{a} \in \mathcal{D}_\lambda$, or that $x^{\mathbf{a}} = g_\pi(\mathbf{x})$, where $\pi \in J^{\mathrm{maj}}_\lambda$. We show that $x^{\mathbf{a}}$ cannot be expressed as a linear combination in $x^{\mathbf{b}}$, where $\mathbf{b} \leqslant_{\mathrm{des}} \mathbf{a}$, and so linear independence follows.

Suppose that

$$\sum_{\mathbf{b} \leqslant_{\mathrm{des}} \mathbf{a}} c_{\mathbf{b}} x^{\mathbf{b}} = 0 \tag{4.12}$$

Then, we will show that $c_{\mathbf{a}} = 0$. Since $\mathbf{a} \in J_\lambda^{\mathrm{maj}}$, we have that there exists $\mathbf{a}_i \in \mathcal{D}_{\lambda_i}$ and an ordered set partition $\sigma = A_1|\ldots|A_\ell \in \mathcal{OSP}(\lambda)$ such that $\Psi(\mathbf{a}_1, \ldots, \mathbf{a}_\ell, \sigma) = \mathbf{a}$. Algorithm 1 for instance, produces such a decomposition. Set $S = A_1$, so that $\mathbf{a}|_{A_1} = \mathbf{a}_1$, and suppose $|A_1| = \lambda_1 = k$. Then, by Proposition 4.2.1, we may apply $\varphi_{\lambda,S}$ to both sides to obtain

$$0 = \varphi_{\lambda,S}\Big(\sum_{\mathbf{b} \leqslant_{\mathrm{des}} \mathbf{a}} c_{\mathbf{b}} x^{\mathbf{b}}\Big) = \sum_{\alpha \in \mathcal{D}_k} x^\alpha \otimes \Big(\sum_{\mathbf{b}'} d_{\alpha,\mathbf{b}'} x^{\mathbf{b}'}\Big) \tag{4.13}$$

where in the second equality we expand in the first tensor factor using the descent basis, and push all coefficients to the second tensor factor. For every nonzero $d_{\alpha,\mathbf{b}'}$, there is some $c_{\mathbf{b}} \neq 0$ contributing to equation 4.12 with the property that

$$\alpha \leqslant_{\mathrm{des}} \mathbf{b}|_S, \quad \mathbf{b}' = \mathbf{b}|_T, \quad \mathbf{b} \leqslant_{\mathrm{des}} \mathbf{a} \tag{4.14}$$

where $T = [n]\backslash S$ denotes the complement of S. The first inequality follows since $\alpha \in \mathcal{D}_k$ are the leading terms in the descent ordering with respect to I_k, see [**All94**].

Writing $\phi_{\lambda,S}(x^{\mathbf{a}}) = x^{\alpha'} \otimes x^{\alpha''}$, with

$$\alpha' = \mathbf{a}|_S \in \mathcal{D}_{\lambda_1}, \quad \alpha'' \in \mathbf{a}|_T \in \mathcal{D}(\mu) \tag{4.15}$$

By equation 4.13, we must have that

$$\sum_{\mathbf{b}'} d_{\alpha',\mathbf{b}'} x^{\mathbf{b}'} = 0 \tag{4.16}$$

in $R_{\mu'}$. Then, lemma 4.1.2 implies that $d_{\alpha',\mathbf{b}'} \neq 0$ only if $\mathbf{b}' \leqslant_{\mathrm{des}} \alpha''$, and that $d_{\alpha',\alpha''} = c_{\mathbf{a}}$. We may then rewrite 4.16 as

$$\sum_{\mathbf{b}' \leqslant_{\mathrm{des}} \alpha''} d_{\alpha',\mathbf{b}'} x^{\mathbf{b}'} = 0 \tag{4.17}$$

and the claim follows by induction on the number of parts of λ, with the base case being the coinvariant algebra $R_{\lambda'_{\ell'}}$. This implies $\{x^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_\lambda\}$ are linearly independent.

Finally, by theorem 1.0.2, we have that

$$\sum_{\pi \in J_\lambda^{\mathrm{maj}}} t^{\mathrm{maj}(\pi)} = \left\{ \begin{matrix} n \\ \lambda' \end{matrix} \right\}_t$$

so that evaluating at $t = 1$ gives $|\mathcal{D}_\lambda| = \binom{n}{\lambda'} = \dim_{\mathbb{C}}(R_{\lambda'})$. The theorem is proved. $\square$

CHAPTER 5

Further Results and Conjectures

In this chapter, we hint at a possible extension of the new Garsia-Procesi basis towards the Garsia-Haiman module. We show the following main results:

- The set $\mathcal{D}_{\lambda_1} \times \cdots \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$ gives a t-weight-preserving bijection with the tableau indexing $\widetilde{H}_{\lambda'}(X; q, t)$
- The fibres of Ψ over $(0, \ldots, 0)$ recovers the classical Garsia-Procesi basis, $\mathfrak{B}(\lambda')$.
- The fibres of Ψ at top t degree recover the Haglund-Haiman-Loehr formula at top degree

Finally, we will present a conjecture for a monomial basis of $V_{\lambda'}$ at top t-degree, and discuss ongoing work for an extension to the Δ-springer modules defined by S. Griffin in his thesis [**Gri21**].

5.1. Connection to Haglund-Haiman-Loehr Macdonald Formula

We now prove a sequence of results that give hope in extending the new Garsia-Procesi basis to the entire Garsia-Haiman module $V_{\lambda'}$. The first of these is that the $q = 1$ specialization of $\widetilde{H}_{\lambda'}(X; q, t)|_{m_{1^n}}$ agrees with the shuffle fibres.

Proposition 5.1.1. *Let $\lambda \vdash n$, and denote the conjugate partition by λ'. Denote a standard filling $\sigma : \lambda' \to \mathbb{Z}_+$ to denote a filling of the Ferrers diagram of λ' (in French notation) with the numbers $\{1, ..., n\}$. Then, we have the following equality:*

$$\sum_{(\mathbf{a}_1,\ldots,\mathbf{a}_\ell;\pi)\in\mathcal{D}_{\lambda_1}\times\ldots\times\mathcal{D}_{\lambda_\ell}\times\mathcal{OSP}(\lambda)} t^{\sum_i a_i} = \sum_{\substack{\sigma:\lambda'\to\mathbb{Z}_+ \\ \sigma \text{ standard}}} t^{\mathrm{maj}(\sigma)} \tag{5.1}$$

In other words, the generating series of the set $\mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$ coincides with the $m_{1^n}(\mathbf{x})$ coefficient of $\widetilde{H}_{\lambda'}(X; 1, t)$, the modified Macdonald polynomial evaluated at $q = 1$.

Proof. We give a proof by constructing an explicit weight-preserving bijection between the two sets. First, denote the set of standard fillings of the shape λ' by $\Xi_{\lambda'}^{\mathrm{std}} := \{\sigma : \lambda' \to \mathbb{Z}_+ : \sigma \text{ is standard}\}$. We will define the map $\Psi_\lambda^{\mathrm{maj}} : \Xi_{\lambda'}^{\mathrm{std}} \to \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$ as follows:

(1) Given the standard λ'-filling ξ, split ξ into columns. Denote the *set* of entries of the jth column (reading left to right) by ξ_j, and denote the column itself by $\overline{\xi}_j$.
(2) Standardize and compute majt of each column; denote majt of the jth column by $\mathbf{a}_j$.
(3) Let π be the ordered set partition given by $\xi_1|...|\xi_j$.
(4) Define $\Psi_\lambda^{\mathrm{maj}}(\xi) := \Psi(\mathbf{a}_1, ..., \mathbf{a}_\ell, \pi)$.

Since we have that $|\Xi_{\lambda'}^{\mathrm{std}}| = n! = |\mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)|$, we show that the map $\Psi_\lambda^{\mathrm{maj}}$ is surjective, and therefore a bijection. Indeed, given $(\mathbf{a}_1, ..., \mathbf{a}_\ell, \pi) \in \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$, we consider the λ'-filling with columns given by $\xi_j = \pi_j$. Applying majt^{-1} to $\mathbf{a}_j$ and unstandardizing with the alphabet ξ_j will give the order of the entries in each column. Since π is an ordered set partition, we are guaranteed that ξ is a standard filling.

Next, we show that $\Psi_\lambda^{\mathrm{maj}}$ is weight preserving. It will suffice to prove the claim for a single column, as the maj of ξ is given by summing maj of each column. Observe that for each descent $a \in \mathrm{Des}(\overline{\xi}_j)$, the quantity $\mathrm{leg}(a) + 1$ marks the index of $\overline{\xi}_j$ read as a permutation top to bottom. Standardizing does not change the major index, so we need only show $\mathrm{majt}(\xi_j) = \mathbf{a}_j$. But this is true by the definition of $\Psi_\lambda^{\mathrm{maj}}$ and [cite background section on def of majt], so we are done.

□

Example 5.1.2. If $\lambda = (4, 3, 3, 1, 1)$, so $\lambda' = (5, 3, 3, 1)$ consider the following λ'-filling ξ:

10				
7	8	9		
5	6	11		
1	2	3	4	12

Then $\xi_1 = \{1, 5, 7, 10\}, \xi_2 = \{2, 6, 8\}, \xi_3 = \{3, 9, 11\}, \xi_4 = \{4\}, \xi_5 = \{12\}$, so that

$$\pi = \{1, 5, 7, 10\}|\{2, 6, 8\}|\{3, 9, 11\}|\{4\}|\{12\}$$

Furthermore, the standardizations of the columns are $4321, 321, 231, 1, 1$ in order from left to right, so the major index tables are $\mathbf{a}_1 = 0123, \mathbf{a}_2 = 012, \mathbf{a}_3 = 011, \mathbf{a}_4 = 0, \mathbf{a}_5 = 0$. We then apply Ψ to obtain

$$\Psi^{\mathrm{maj}}_{\lambda}(\xi) = (0_1, 0_2, 0_3, 0_4, 1_1, 1_2, 2_1, 2_2, 1_3, 3_1, 1_3, 0_5)$$

where the subscript denotes which column the entry came from.

One would hope that this map gives a bijection between $\mathrm{inv}(\sigma) = 0$ fillings and $J^{\mathrm{maj}}_{\lambda}$, but the situation is not so straightforward. Consider the following two inversionless tableau:

$$\sigma_1 = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 & 5 \\ \hline 2 & 6 \\ \hline \end{array} \quad \sigma_2 = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 5 \\ \hline 1 & 6 \\ \hline \end{array}$$

We have that $\Psi^{\mathrm{maj}}_{\lambda}(\sigma_1) = \Psi^{\mathrm{maj}}_{\lambda}(\sigma_2) = 001200$.

5.2. Affine Permutation Formula

The auxiliary sequence $\tilde{w}$ used in 3.2 in defining Algorithm 1 can be extended to all of the shuffle fibres $\mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$. This set of affine permutations will be the subject of many conjectures to follow.

For a descent composition $\mathbf{a} \in \mathcal{D}_n$, denote by $\omega = (\omega_1, ..., \omega_n)$ the position of the numbers chosen, $\overline{\omega}$ to be the positions mod n (residues in $\{1, ..., n\}$) and let $b = (b_1, ..., b_n)$ to denote the quotients (without remainder) of $\omega_i - 1$ by n. For instance, for $\mathbf{a} = (0, 1, 2, 0, 2, 1)$, $\omega = (1, 2, 3, 11, 24, 34)$, $\overline{\omega} = (1, 2, 3, 5, 6, 4)$ and $b = (0, 0, 0, 1, 3, 5)$.

Given $(\mathbf{a}_1, ..., \mathbf{a}_\ell, B_1|...|B_\ell) \in \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda)$, we can run for each $\mathbf{a}_i$ the algorithm to obtain $\overline{\omega}^{(i)}$ and $b^{(i)}$. Define $\widehat{\omega}_i$ to be

$$(\widehat{\omega}_i)_j = (B_i)_{(\overline{\omega}^{(i)})_j} + n \cdot (b^{(i)})_j \tag{5.2}$$

and set $\omega = \widehat{\omega}_1 + ... + \widehat{\omega}_\ell$, the concatenation. Denote the set of ω obtained this way $\widetilde{\mathcal{D}}_\lambda$, and let this process be denoted by a map $\mathrm{aff} : \mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}_{\lambda_\ell} \times \mathcal{OSP}(\lambda) \to \widetilde{\mathcal{D}}_\lambda$. Note $\omega \in W_+$ since each residue mod n is used exactly once, and the blocks are sorted, so that we obtain a minimal W_+/S_λ representative.

For each of the $\mathbf{a}_i$, we can concatenate the corresponding $b^{(i)}$ tuples to obtain $b_\omega = b^{(1)} + ... + b^{(\ell)} = (b_1, ..., b_n)$. We will refer to b_ω as the *quotient tuple* of ω, as it tracks the block numbers of $\tilde{\mathbf{a}}$.

Conversely, given $\omega \in \widetilde{\mathcal{D}}_\lambda$, we can reverse this process by first splitting ω into blocks according to λ, which we will denote with bars:

$$\omega = (\omega_1, ..., \omega_{\lambda_1} | \omega_{\lambda_1+1}, ..., \omega_{\lambda_2} | ... | \omega_{n-\lambda_\ell+1}, ..., \omega_n) = (\widetilde{A}_1 | ... | \widetilde{A}_\ell)$$

with $\widetilde{A}_i = (\omega_{\lambda_1+...+\lambda_{i-1}+1}, ..., \omega_{\lambda_1+...+\lambda_i})$. Then, we can recover the associated descent composition $\mathbf{a}$ by setting $\mathbf{a} = (j - (b^{(i)})_j)_{\overline{(\widetilde{A}_i)_j}}$, where $\overline{(\widetilde{A}_i)_j}$ denotes reducing the entry $(\widetilde{A}_i)_j$ modulo n. Note since each residue modulo n occurs exactly once in ω, $\overline{(\widetilde{A}_i)_j}$ will never be equal for different choices of i, j. We will sometimes refer to $\mathbf{a}$ as $\mathrm{majt}(\omega)$, and $\mathrm{maj}(\omega)$ is defined to be the sum of the entries of $\mathrm{majt}(\omega)$.

Example 5.2.1. Consider $\omega = (4, 5, 7, 15; 2, 3, 20) \in \widetilde{\mathcal{D}}_{(4,3)}$. Then, $b_\omega = (0, 0, 0, 2; 0, 0, 2)$, and $b^{(1)} = (0, 0, 0, 2)$, $b^{(2)} = (0, 0, 2)$, so that $(j - b_j^{(1)})_j = (0, 1, 2, 1)$, and $(j - b_j^{(2)})_j = (0, 1, 0)$. Finally $\overline{\widetilde{A}_1} = (4, 5, 7, 1)$, and $\overline{\widetilde{A}_2} = (2, 3, 6)$, so that we obtained the associated descent composition $\mathbf{a} = 1010102$.

Noting that if $b^{(i)} = (0, ..., 0)$, then all entries corresponding to $\widetilde{A}_i$ in $\tilde{\mathbf{a}}$ must occur in the first block. On the other hand, the quantity $j - (b^{(i)})_j$ is a value in $\mathbf{a}$, which is a descent composition, and so must be nonnegative. This gives the inequalities $0 \leqslant (b_j^{(i)}) \leqslant j$.

If one is only interested in the major index of ω, there is a simpler formula. First, let

$$B_\lambda = (0, \ldots, \lambda_1 - 1; \ldots; 0, \ldots, \lambda_\ell - 1)$$

which is a tuple of length n. We can check

$$\mathrm{maj}(\omega) = \sum_{i=1}^{n} (B_\lambda)_i - b_i$$

5.3. Partial Results on a Potential Macdonald q-Statistic

In light of Proposition 5.1.1, it is reasonable to expect that there should exist a q-statistic on the set $\mathcal{D}_{\lambda_1} \times ... \times \mathcal{D}\lambda_\ell \times \mathcal{OSP}(\lambda)$ such that

$$\sum_{(\mathbf{a}_1,...,\mathbf{a}_\ell;\pi)\in\mathcal{D}_{\lambda_1}\times...\times\mathcal{D}_{\lambda_\ell}\times\mathcal{OSP}(\lambda)} q^{\mathrm{stat}(\mathbf{a})}t^{\sum_i a_i} = \widetilde{H}_{\lambda'}(X;q,t)|_{m_{1^n}}$$

or in terms of $\widetilde{\mathcal{D}}_\lambda$,

$$\sum_{\omega\in\widetilde{\mathcal{D}}_\lambda} q^{\mathrm{stat}(\omega)}t^{\mathrm{maj}(\omega)} = \widetilde{H}_{\lambda'}(X;q,t)|_{m_{1^n}}$$

We present some partial progress towards the construction of $\mathrm{stat}(u)$ in terms of $\widetilde{\mathcal{D}}_\lambda$.

5.3.1. Lowest t-degree and Classical Garsia-Procesi Basis. For maj-less $\omega \in \widetilde{\mathcal{D}}_\lambda$, we show a certain statistic inv_n that recovers the m_{1^n} coefficient of Hall-Littlewood polynomial at $t = 0$. We will prove this by showing that this statistic generates the Garsia-Procesi basis given in [**GP92**], thus giving a graded bijection. This will be done by first associating a parking function to ω, and then showing that such a parking function satisfies the recursion given in [**GP92**].

Fix a partition $\lambda \vdash n$. Consider the set

$$\widetilde{\mathcal{D}}^0_\lambda = \left\{\omega \in \widetilde{\mathcal{D}}_\lambda : b_\omega = (0,...,\lambda_1 - 1;...;0,...,\lambda_\ell - 1)\right\} = \left\{\omega \in \widetilde{\mathcal{D}}_\lambda : \sum_i a_i = 0\right\}$$

where $\mathbf{a} = (a_1,...,a_n)$ is the element of $\mathcal{D}_\lambda$ corresponding to ω. We now give the construction of inv_n.

Definition 5.3.1. Write $\omega = (\omega_1,...,\omega_n)$ (this is sometimes referred to as *window notation*). An *inversion* of ω is a pair (i,j) such that $i < j$ and $\omega_i > \omega_j$. An *n-restricted inversion* is an inversion of ω such that $\omega_i - \omega_j < n$. We denote the set of n-restricted inversions by $\mathrm{Inv}_n(\omega)$, and set $\mathrm{inv}_n(\omega) = |\mathrm{Inv}_n(\omega)|$.

The statistic inv_n is studied in depth in [**GMV14**], and suggests connections with torus-fixed points of certain affine Springer fibres.

Proposition 5.3.2. *Let $\lambda \vdash n$ be a partition (or more generally, a composition) and let $\overline{\omega_i}$ denote the residue of ω_i modulo n in the set $\{1, ..., n\}$ and set $\chi(\omega_i, \omega_j) = max(\overline{\omega_i}, \overline{\omega_j})$. Let $c_k = |\{(i,j) \in Inv_n(\omega) : \chi(\omega_i, \omega_j) = k\}|$, and set $\widetilde{invt}(\omega) = (c_1, ..., c_n)$. Then, the set of monomials*

$$\mathfrak{C}(\lambda) = \left\{ x^{\widetilde{invt}(\omega)} : \omega \in \widetilde{\mathcal{D}}^0_\lambda \right\}$$

is precisely the set $\mathfrak{B}(\lambda')$ given in section 1 of ?? that forms a basis of the Garsia-Procesi module $R_{\lambda'}$.

Corollary 5.3.3. *We have that*

$$\sum_{\omega \in \widetilde{\mathcal{D}}^0_\lambda} q^{inv_n(\omega)} t^{\mathrm{maj}(\omega)} = \sum_{\omega \in \widetilde{\mathcal{D}}^0_\lambda} q^{inv_n(\omega)} = \widetilde{H}_{\lambda'}(X; q, 0)|_{m_{1^n}}$$

.

An example will make things clear.

Example 5.3.4. Let $\omega = (1,7|3,9|5) \in \widetilde{\mathcal{D}}^0_{(2,2,1)}$. Here $n = 5$, and the 5-restricted inversions are $\mathrm{Inv}_n(\omega) = \{(2,3,(2,5),(4,5)\}$, corresponding to the entries $(7,3),(7,5),(9,5)$ respectively. We have that $\chi(7,3) = 3$, $\chi(7,5) = 5$, and $\chi(9,5) = 5$. Then, $c_3 = 1, c_5 = 2$, and $c_1 = c_2 = c_4 = 0$, so $\widetilde{\mathrm{invt}}(\omega) = (0,0,1,0,2)$, and $x^{\widetilde{\mathrm{invt}}(\omega)} = x_3 x_5^2$.

The remainder of this section will be dedicated to giving a proof of 5.3.2.

First, we will give a parking function description of $\widetilde{\mathcal{D}}^0_\lambda$, as well as a corresponding statistic, denoted codinv_λ. Notice that we can identify, via the map Ψ, the set $\widetilde{\mathcal{D}}^0_\lambda$ with $\mathcal{OSP}(\lambda)$, as $\widetilde{\mathcal{D}}^0_\lambda$ corresponds precisely to $\mathrm{Sh}(\mathbf{0}_{\lambda_1}, ..., \mathbf{0}_{\lambda_\ell})$, with $\mathbf{0}_k = (0, ..., 0)$, where there are k 0's.

Any Dyck path can be represented with 1's denoting North steps, and 0's denoting East steps. Then, consider the path

$$w_\lambda = (1^{\lambda_1}, 0^{\lambda_1}, ..., 1^{\lambda_\ell}, 0^{\lambda_\ell})$$

Let $\mathcal{PF}_\lambda$ (or $\mathcal{PF}_\alpha$ for a general composition α) denote the set of parking functions whose underlying Dyck path has shape w_λ.

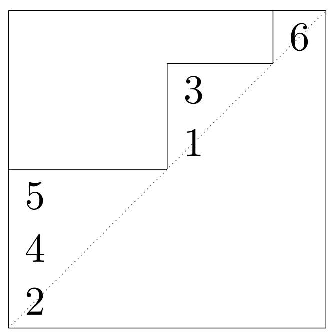
DEFINITION 5.3.5. Let $(a_1, ..., a_n)$ denote the area sequence of $P \in \mathcal{PF}_\lambda$, read bottom to top. Then, we define $\mathrm{Codinv}_\lambda(P)$ to be

$$\mathrm{Codinv}_\lambda(P) = \Big\{(i,j) : i < j, a_i = a_j, \mathrm{occ}(i) > \mathrm{occ}(j)\Big\} \sqcup$$
$$\Big\{(i,j) : i < j, a_i = a_j + 1, \mathrm{occ}(i) < \mathrm{occ}(j)\Big\}$$

and $\mathrm{codinv}_\lambda(P) = |\mathrm{Codinv}_\lambda(P)|$.

Note that this definition is exactly the same as the usual definition of dinv for parking functions, with the inequalities reversed. We now give a corresponding notion of inversion tableau.

EXAMPLE 5.3.6. Consider the following element $P \in \mathcal{PF}_{(3,2,1)}$:



Then, $\mathrm{Codinv}_{(3,2,1)}(P) = \{(1,4),(2,5)\} \sqcup \{(2,6),(5,6)\}$, so that $\mathrm{codinv}_{(3,2,1)}(P) = 4$.

DEFINITION 5.3.7. Let $P \in \mathcal{PF}_\lambda$. Then, for $(i,j) \in \mathrm{Codinv}_\lambda(P)$, let $\chi(i,j) = \max(\mathrm{occ}(i), \mathrm{occ}(j))$. Let $c_k = |\{(i,j) \in \mathrm{Codinv}_\lambda(P) : \chi(i,j) = k\}|$, and let $\mathrm{dinvt}(P) = (c_1, ..., c_n)$.

We now describe a simple bijection between $\mathcal{PF}_\lambda$ and $\widetilde{\mathcal{D}}^0_\lambda$. Given $\omega = (\omega_1, ..., \omega_n) \in \widetilde{\mathcal{D}}^0_\lambda$, denote by $\overline{\omega} = (\overline{\omega_1}, ..., \overline{\omega_n})$ to be the reduction of the entries ω_i modulo n, taking a residue in $\{1, ..., n\}$. Within each block of λ in $\overline{\omega}$, we are guaranteed by 5.2 that the entries are increasing. The map is then given by

$$\Phi : \widetilde{\mathcal{D}}^0_\lambda \to \mathcal{PF}_\lambda$$
$$\omega \mapsto (w_\lambda, [\overline{\omega_1}, ..., \overline{\omega_n}])$$

Lemma 5.3.8. *The map* Φ *is a bijection, and we have that* $\mathrm{dinvt}(\Phi(\omega)) = \widetilde{invt}(\omega)$. *Consequently, the following equalities hold:*

$$\left\{x^{\widetilde{invt}(\omega)} : \omega \in \widetilde{\mathcal{D}}_\lambda^0\right\} = \left\{x^{\mathrm{dinvt}(P)} : P \in \mathcal{PF}_\lambda\right\}$$

$$\sum_{\omega \in \widetilde{\mathcal{D}}_\lambda^0} q^{inv_n(\omega)} = \sum_{P \in \mathcal{PF}_\lambda} q^{codinv_\lambda(P)}$$

Proof. Every parking function $P \in \mathcal{PF}_\lambda$ with the underlying path w_λ can be identified by taking its reading word. The shape of w_λ guarantees we have the following inequalities:

$$\mathrm{occ}(1) < ... < \mathrm{occ}(\lambda_1)$$

$$\mathrm{occ}(\lambda_1 + 1) < ... < \mathrm{occ}(\lambda_1 + \lambda_2)$$

$$\vdots$$

$$\mathrm{occ}(\lambda_1 + ... + \lambda_{\ell-1} + 1) < ... < \mathrm{occ}(n)$$

Then, we can canonically identify $\mathrm{rw}(P)$ with an ordered set partition $\pi \in \mathcal{OSP}(\lambda)$ by taking the ith block to be $\lambda_1 + ... + \lambda_{i-1} + 1, ..., \lambda_1 + ... + \lambda_i$.

Each $\omega \in \widetilde{\mathcal{D}}_\lambda^0$ is canonically identified with an ordered set partition $\overline{\omega}$, as above. Then, by the definition of Φ, it is clear that Φ simply maps ω to the parking function with reading word $\overline{\omega}$.

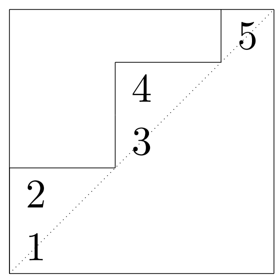
We need now only show that $\widetilde{\mathrm{invt}}(\omega) = \mathrm{codinv}_\lambda(\Phi(\omega))$. Let $(i,j) \in \mathrm{Inv}_n(\omega)$. Then, by 5.3.1, this is true if and only if $\omega_j < \omega_i < \omega_j + n$. This implies that we must have $b_i = b_j$, or $b_i = b_j + 1$. In the case of $b_i = b_j$, it must be the case that $\overline{\omega_j} < \overline{\omega_i}$, and in the case of $b_i = b_j + 1$, we must have $\overline{\omega_j} > \overline{\omega_i}$. The final observation is that

$$(b_1, ..., b_n) = (0, 1, ..., \lambda_1 - 1; 0, 1, ..., \lambda_2 - 1; ...; 0, 1, ..., \lambda_\ell - 1) = (a_1, ..., a_n)$$

so that the quotient labels $(b_1, ..., b_n)$ of ω give precisely the area sequence $(a_1, ..., a_n)$ of $\Phi(\omega)$. The inequalities above precisely mean $(i,j) \in \mathrm{Codinv}_\lambda(\Phi(\omega))$. Similarly, given $(i,j) \in \mathrm{Codinv}_\lambda(\Phi(\omega))$, we may reverse the above process to obtain $(i,j) \in \mathrm{Inv}_n(\omega)$. Finally, noting that occ(i) in $\Phi(\omega)$ is the same as $\overline{\omega_i}$, comparing the definitions of dinvt$(\Phi(\omega))$ and $\widetilde{\text{invt}}(\omega)$, we see the tuples must be equal and the lemma is proved. $\square$

Example 5.3.9. Let $\lambda = (2,2,1)$, $\omega = (1,7|3,9|5) \in \widetilde{\mathcal{D}}^0_{(2,2,1)}$. We have that $\overline{\omega} = (1,2|3,4|5)$, so that $\Phi(\omega) = (w_{(2,2,1)}, [1,2,3,4,5])$. Then, pictorally:

$\Phi(\omega) =$



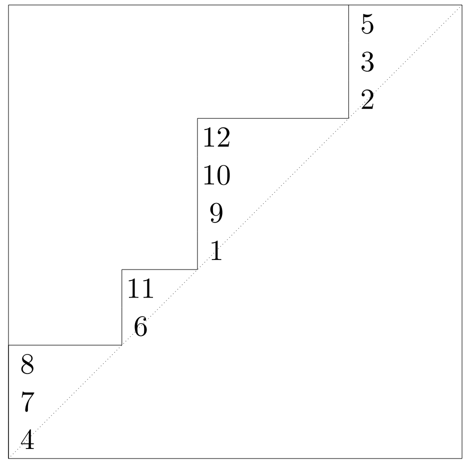
Then, we have that $\text{Codinv}_\lambda(P) = \{(2,3),(2,5),(4,5)\}$, so that $\text{dinvt}(P) = (0,0,1,0,2)$, which coincides with example 5.3.4 above.

Let α be a composition of n. We briefly give a useful identification (inspired by Haglund) of $\mathcal{PF}_\alpha$ with tableau that have column heights given by α. Let CS_α denote the set of *column-strict decreasing fillings* of the Ferrers shape with column sizes given by α. For instance,

$$T = \begin{array}{|c|c|c|c|} \hline & & 12 & \\ \hline 8 & & 10 & 5 \\ \hline 7 & 11 & 9 & 3 \\ \hline 4 & 6 & 1 & 2 \\ \hline \end{array}$$

is an element of $\text{CS}_{(3,2,4,3)}$. We can then identify $\mathcal{PF}_\alpha$ with CS_α by letting the columns of $T \in \text{CS}_\alpha$ be precisely vertical strips of a corresponding $P \in \mathcal{PF}_\alpha$, and denote this map by $\Theta : \mathcal{PF}_\alpha \to \text{CS}_\alpha$. For instance, for the T above, the corresponding parking function is

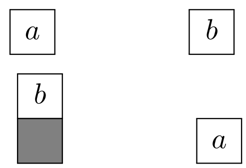
$P =$



It is straightforward to show then, that an element $(i, j) \in \operatorname{Codinv}_\lambda(P)$ corresponds to one of the following pairs:

(*I.*)

(*II.*)



where $a > b$ in $\Theta(P)$. We will refer to these as type I and type II inversions respectively. The missing squares need not be present, but the gray square must be present.

Lemma 5.3.10. *Let α be any composition of n such that $|\alpha| = \lambda$. Then, we have the following set equality:*

$$\left\{x^{\operatorname{dinvt}(P)} : P \in \mathcal{PF}_\lambda\right\} = \left\{x^{\operatorname{dinvt}(P)} : P \in \mathcal{PF}_\alpha\right\}$$

Proof. Let $\alpha = (\alpha_1, ..., \alpha_\ell)$, and let $\alpha' = (\alpha_1, ..., \alpha_{i+1}, \alpha_i, ..., \alpha_\ell)$ for some fixed $1 \leqslant i \leqslant \ell - 1$. It then suffices to show that

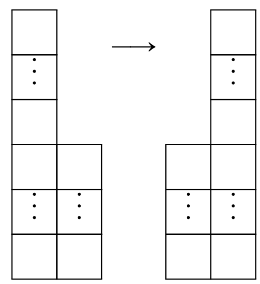
$$\left\{x^{\operatorname{dinvt}(P)} : P \in \mathcal{PF}_{\alpha'}\right\} = \left\{x^{\operatorname{dinvt}(P)} : P \in \mathcal{PF}_\alpha\right\}$$

We will do this by constructing a bijection between $\mathcal{PF}_{\alpha'}$ and $\mathcal{PF}_\alpha$ that preserves dinvt.

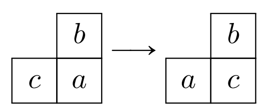
If $\alpha_i = \alpha_{i+1}$, then we can have the bijection be the identity map, and there is nothing to show. Otherwise, without loss of generality, suppose that $\alpha_i > \alpha_{i+1}$. We will give the bijection in terms of the column strict fillings CS_α and $\mathrm{CS}_{\alpha'}$. Define the map $\operatorname{swap}_i : \mathrm{CS}_\alpha \to \mathrm{CS}_{\alpha'}$ as follows:

(1) Only the ith and $(i+1)$th columns will be affected; the rest will be fixed. As such, we suppress the other columns in our description of swap_i.

(2) Since $\alpha_i > \alpha_{i+1}$, move all boxes with y-coordinate $y > \alpha_{i+1}$ to the $(i+1)$th column:



(3) If the right column is no longer decreasing, find the first instance a such that the entry b above a has $b < a$. Then perform the swap:



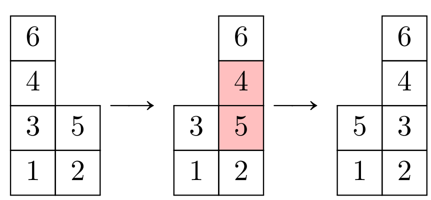
we must necessarily have $b > c$, as before step (2), b and c were part of a decreasing column.

(4) Repeat step (3) going downward until both columns are decreasing.

Note that it will always be possible to make both columns decreasing, since at worst, the two columns entirely swap. Furthermore, since the heights of all of the entries do not change, the inversion pairs involving the other columns of $\Theta(P)$ are not affected by swap_i.

Example 5.3.11.

swap_i :



The inverse map swap_i^{-1} is defined identically, and it is straightforward to show they are inverses.

Now we must show that $\mathrm{dinvt}(P) = \mathrm{dinvt}((\Theta^{-1} \circ \mathrm{swap}_i \circ \Theta)(P))$. This will follow from showing that swap_i preserves the number of inversion pairs, as well as the larger value of each inversion pair.

(1) (Type II inversion) We need only be concerned if a swap happens with one of the entries. Suppose $a < b$ is a type II inversion and a swap occurs:

$$\begin{array}{cc} \mathbf{a} & \\ c & \mathbf{b} \end{array} \longrightarrow \begin{array}{cc} & a \\ c & b \end{array} \longrightarrow \begin{array}{cc} & a \\ \mathbf{b} & \mathbf{c} \end{array}$$

where the second swap must occur because we have $a < b$. We must have $a > c$ since we began with a column decreasing tableau, so that $b > a > c$. Then, $b > c$ is a type I inversion in $\mathrm{swap}_i(\Theta(P))$.

(2) (Type I inversion) Suppose $b > a$ is a type I inversion. Then, the only interesting case is if a swap occurs in the above row:

$$\begin{array}{cc} c & d \\ \mathbf{b} & \mathbf{a} \end{array} \longrightarrow \begin{array}{cc} d & c \\ \mathbf{b} & \mathbf{a} \end{array}$$

since $c > b$, we have $c > b > a$, so that the swapping process must terminate. Therefore, the type I inversions are preserved.

In either case, the larger entry of the inversion pairs are unchanged, so we must have that $\mathrm{dinvt}(P) = \mathrm{dinvt}((\Theta^{-1} \circ \mathrm{swap}_i \circ \Theta)(P))$.

We must show the same statement for swap_i^{-1}.

(1) (Type I inversion) Suppose $b > a$ is a type I inversion. If the entry above a, denoted c has $c > b$, then no swap occurs in the lower row. Otherwise,

$$\begin{array}{cc} d & c \\ \mathbf{b} & \mathbf{a} \end{array} \longrightarrow \begin{array}{cc} c & d \\ b & a \end{array} \longrightarrow \begin{array}{cc} \mathbf{c} & d \\ a & \mathbf{b} \end{array}$$

and the inversion $b > a$ is exchanged for $c < b$. Here, the square d may be missing.

(2) (Type II inversion) Suppose $a < b$ is a type II inversion. Then, the only interesting case is if c, d swap below:

$$\begin{array}{cc} c & d \\ \mathbf{a} & e \\ f & \mathbf{b} \end{array} \longrightarrow \begin{array}{cc} d & c \\ \mathbf{a} & e \\ f & \mathbf{b} \end{array}$$

If $c > e$, then the swaps must terminate because we must have $d > e > b > a$. Furthermore, a, e do not form an inversion pair because $a < e$. On the other hand, if $c < e$ the following swaps must occur:

$$\begin{array}{|c|c|}\hline d & c \\ \hline \mathbf{a} & e \\ \hline f & \mathbf{b} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline d & c \\ \hline e & a \\ \hline f & b \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline d & c \\ \hline e & a \\ \hline \mathbf{b} & \mathbf{f} \\ \hline \end{array}$$

since $b > a > f$, we have that b, f form an inversion pair. Furthermore, in this scenario, the condition $c < e$ means c, e was a type II inversion, which is replaced with the type I inversion $e > a$.

In all cases, the larger entry of the inversion pairs are unchanged, so we must have that $\mathrm{dinvt}(P) = \mathrm{dinvt}((\Theta^{-1} \circ \mathrm{swap}_i^{-1} \circ \Theta)(P))$. This completes the proof of the lemma.

$\square$

Before we give the proof of Proposition 5.3.2, we briefly recall the recursion given in [**GP92**]. Let $\lambda \vdash n$ be a partition, and let $\lambda^{(i)} = |(\lambda_1, ..., \lambda_i - 1, ..., \lambda_\ell)|$. In other words, we subtract 1 from the ith part, and rearrange if necessary to obtain a partition shape.

THEOREM 5.3.12. *[**GP92**] Fix $\lambda \vdash n$, and let $\ell(\lambda) = \lambda_1'$ denote the height of the partition. The monomials $\mathfrak{B}(\lambda)$ forming a basis for the Garsia-Procesi module satisfy the following recursion:*

$$\mathfrak{B}(\lambda) = \bigsqcup_{i=1}^{\ell(\lambda)} x_n^{i-1} \mathfrak{B}(\lambda^{(i)}) \tag{5.3}$$

with the initial condition $\mathfrak{B}(1) = \{1\}$. If S is a set of monomials, the notation $x^\beta S$ denotes the set obtained by multiplying every monomial of S by x^β.

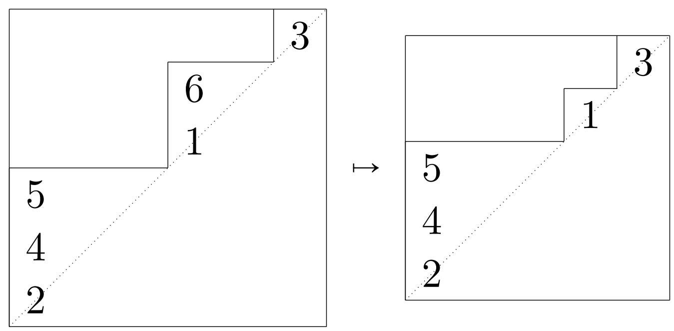
For instance, as in [**GP92**], $\mathfrak{B}(211) = \{1, x_2, x_3, x_2x_3, x_3^2, x_2x_3^2, x_4, x_4x_2, x_4x_3, x_4^2, x_4^2x_2, x_4^2x_3\}$.

PROOF OF PROPOSITION 5.3.2. We will show that the sets $\mathfrak{C}(\lambda)$ defined in 5.3.2 satisfy the recursion 5.3. By lemma 5.3.8, this is equivalent to working with the dinvt monomials for $\mathcal{PF}_\lambda$. Define a map drop : $\mathcal{PF}_\lambda \to \mathcal{PF}_{n-1}$ as follows:

(1) Let $P \in \mathcal{PF}_\lambda$ be a parking function, and $\mathrm{rw}(P)$ be its reading word.
(2) Suppose n occurs in the block of w_λ corresponding to λ_i. Delete n from the reading word, i.e. consider $\mathrm{rw}(P)^- = [..., \hat{n}, ...]$, and consider the composition $\alpha = (\lambda_1, ..., \lambda_i - 1, ..., \lambda_\ell)$.
(3) Define $\mathrm{drop}(P) = (w_\alpha, \mathrm{rw}(P)^-)$.

For instance:

drop :

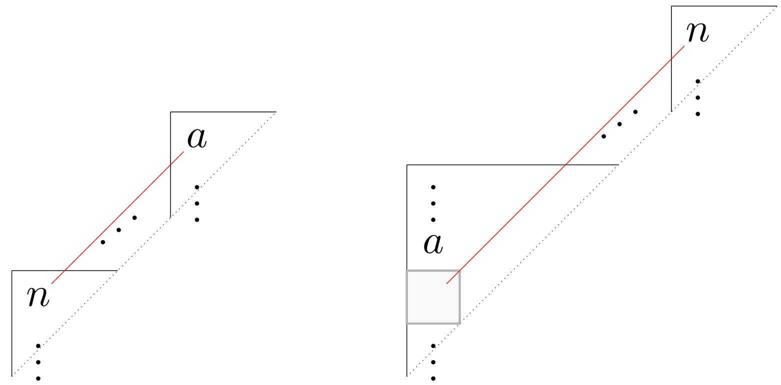


If α is a composition, define $D_\lambda(\alpha) = \{\omega \in \mathcal{PF}_\lambda : \text{drop}(\omega) \in \mathcal{PF}_\alpha\}$. It is clear that the only α for which $D_\lambda(\alpha)$ is nonempty are of the form $\alpha^{(i)} = (\lambda_1, ..., \lambda_i - 1, ..., \lambda_\ell)$ for some $1 \leqslant i \leqslant \ell$. Furthermore, if $(j, j') \in \text{Codinv}_\lambda(\omega)$, and $n \notin \{\text{occ}(i), \text{occ}(j)\}$, then one can check there is a corresponding $(k, k') \in \text{codinv}_{\alpha^{(i)}}(\text{drop}(\omega))$. In other words, the diagonal inversions that do not interact with n are not affected by drop, which implies that $\text{dinvt}(\text{drop}(\omega)) = (c_1, ..., c_{n-1}, \widehat{c_n})$, where $\text{dinvt}(\omega) = (c_1, ..., c_n)$.

Fix i, and let $\alpha^{(i)} = (\lambda_1, ..., \lambda_i - 1, ..., \lambda_\ell)$. It is straightforward to define an inverse map $\text{lift}_i : \mathcal{PF}_{\alpha^{(i)}} \to \mathcal{PF}_\lambda$ that is the inverse of drop on $D_\lambda(\alpha)$. This implies the map drop restricted to each nonempty $D_\lambda(\alpha^{(i)})$ is surjective onto $\mathcal{PF}_{\alpha^{(i)}}$, and so there is a canonical identification of $D_\lambda(\alpha^{(i)})$ and $\mathcal{PF}_{\alpha^{(i)}}$. This also implies the sets $D_\lambda(\alpha^{(i)})$ partition $\mathcal{PF}_\lambda$:

$$\mathcal{PF}_\lambda = \bigsqcup_{i=1}^{\ell} D_\lambda(\alpha^{(i)})$$

For each $\omega \in \mathcal{PF}_\lambda$, we consider the contribution of n to $\text{codinv}_\lambda(\omega)$. Suppose n occurs in the part of w_λ corresponding to λ_i. Then, n participates in a diagonal inversion $(j, j') \in \text{Codinv}_\lambda(\omega)$ if and only if one of the two following scenarios occur:



where $a < n$. In the left diagram, the two parts λ_i, λ_j have the same size, whereas in the right diagram, the part λ_j containing a has $\lambda_j > \lambda_i$. Then, we see that the number of diagonal inversions involving n must be

$$c_n = \left|\left\{\lambda_j : \lambda_j > \lambda_i\right\}\right| + \left|\left\{\lambda_j : \lambda_j = \lambda_i, j > i\right\}\right| \tag{5.4}$$

for $\lambda = (\lambda_1, ..., \lambda_\ell)$, and the two summands are cardinalities of *multisets*. In other words, c_n is the number of parts of λ greater than λ_i, as well as the number of parts equal to λ_i that occur after it. Furthermore, since n is the greatest entry in ω, we must have that $\chi(j, j') = n$ for any $(j, j') \in \mathrm{Codinv}_\lambda(\omega)$ involving n, so we are justified in calling the above quantity c_n.

For a fixed $\alpha^{(i)}$, all of the parking functions $\omega \in D_\lambda(\alpha^{(i)})$ must have n in the same position. This implies they all have the same value of c_n (the last digit of $\mathrm{dinvt}(\omega)$), namely 5.4. We may then write

$$\left\{x^{\mathrm{dinvt}(\omega)} : \omega \in D_\lambda(\alpha^{(i)})\right\} = x_n^{c_n}\left\{x^{(c_1,\ldots,c_{n-1})} : \omega \in D_\lambda(\alpha^{(i)})\right\}$$

$$= x_n^{c_n}\left\{x^{\mathrm{dinvt}(\mathrm{drop}(\omega))} : \omega \in D_\lambda(\alpha^{(i)})\right\} = x_n^{c_n}\left\{x^{\mathrm{dinvt}(\omega)} : \omega \in \mathcal{PF}_{\alpha^{(i)}}\right\} = x_n^{c_n}\mathfrak{C}(\alpha^{(i)})$$

Now writing $\lambda = k^{a_k}(k-1)^{a_{k-1}}...1^{a_1}$, where the a_j denotes the multiplicity of j, consider the permutation

$$\sigma = (a_k, ..., 1, a_k + a_{k-1}, a_k + a_{k-1} - 1, ..., a_k + 1, ..., a_k + ... + a_1, ..., \ell(\lambda) - a_1)$$

Note we always have $\lambda_{\sigma(j)} = \lambda_j$, except the order of identical parts is reversed. Then, we see that $\sigma(i) - 1 = c_n$, so that

$$x_n^{c_n} \mathfrak{C}(\alpha^{(i)}) = x_n^{\sigma(i)-1} \mathfrak{C}(\alpha^{(i)})$$

Finally, noting that σ is an involution, and using $\mathfrak{C}(\alpha^{(i)}) = \mathfrak{C}(\lambda^{(i)})$ by 5.3.10, we have that

$$\begin{aligned}\mathfrak{C}(\lambda) = \bigsqcup_{i=1}^{\ell(\lambda)} \left\{ x^{\mathrm{dinvt}(\omega)} : \omega \in D_\lambda(\alpha^{(i)}) \right\} = \bigsqcup_{i=1}^{\ell(\lambda)} x_n^{\sigma(i)-1} \mathfrak{C}(\alpha^{(i)}) = \bigsqcup_{i=1}^{\ell(\lambda)} x_n^{i-1} \mathfrak{C}(\alpha^{(\sigma(i))}) \\ = \bigsqcup_{i=1}^{\ell(\lambda)} x_n^{i-1} \mathfrak{C}(\alpha^{(i)}) = \bigsqcup_{i=1}^{\ell(\lambda)} x_n^{i-1} \mathfrak{C}(\lambda^{(i)})\end{aligned}$$

where we replace i by $\sigma^{-1}(i) = \sigma(i)$ in the third equality. This completes the proof; we have $\mathfrak{C}(\lambda) = \mathfrak{B}(\lambda)$ for all λ.

□

Remark 5.3.13. Since the set $\widetilde{\mathcal{D}}^0_\lambda$ can easily be identified with $\mathcal{OSP}(\lambda)$, this gives a simple bijective proof that the Garsia-Procesi basis $\mathfrak{B}(\lambda)$ has size given by

$$|\mathfrak{B}(\lambda)| = \binom{n}{\lambda}$$

Furthermore, since the exponents of $\mathfrak{B}(\lambda)$ are precisely the λ-*sub-Yamanouchi words*, denoted C_λ as in [**Gil15**], proposition 5.3.2 gives a simple bijective proof of its size as well.

5.3.2. Top t-degree and a Skip Statistic. Given $\lambda \vdash n$, we now introduce a q-statistic on $\widetilde{\mathcal{D}}_\lambda$ that recovers the m_{1^n} coefficient for top t degree of the Macdonald polynomial $\widetilde{H}_{\lambda'}(X; q, t)$. Formally, let

$$\eta(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i = \sum_{i=1}^{\ell'} \binom{\lambda'_i}{2}$$

We will show that there is a statistic skip on $\omega \in \widetilde{\mathcal{D}}_\lambda$ such that

$$(5.5) \qquad \sum_{\substack{\omega\in\widetilde{\mathcal{D}}_\lambda \\ \mathrm{maj}(\omega)=\eta(\lambda)}} q^{\mathrm{skip}(\omega)}t^{\mathrm{maj}(\omega)} = \sum_{\substack{\sigma:\lambda'\to\{1,...,n\} \\ \mathrm{maj}(\sigma)=\eta(\lambda)}} q^{\mathrm{inv}(\sigma)}t^{\mathrm{maj}(\sigma)}$$

Write $\lambda = (\lambda_1, ..., \lambda_\ell)$. First, we study the set $\widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)} := \{\omega \in \widetilde{\mathcal{D}}_\lambda : \mathrm{maj}(\omega) = \eta(\lambda)\}$. Note that an element of maximal maj occurs when we shuffle the descent compositions of the long words for $\lambda_1, ..., \lambda_\ell$:

$$\widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)} = \mathrm{aff}(\{(0,1,...,\lambda_1-1)\} \times ... \times \{(0,1,...,\lambda_\ell-1)\} \times \mathcal{OSP}(\lambda))$$

As such, when 1 is run on each $(0, 1, ..., \lambda_i - 1)$, all of the entries occur in the first block, so all of the quotient labels $(b_1, ..., b_n)$ must be 0, so we may write

$$\widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)} = \left\{\omega \in \widetilde{\mathcal{D}}_\lambda : (b_1, ..., b_n) = \mathbf{0}\right\}$$

and that the reading word of ω is actually equal to the corresponding $\pi \in \mathcal{OSP}(\lambda)$. We will define the statistic in terms of corresponding ordered set partition:

Definition 5.3.14. Identify $\omega \in \widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)}$ with an ordered set partition $\pi = A_1|...|A_\ell$, and write $A_i = (a_i^0, ..., a_i^{\lambda_i-1})$ with $a_i^0 < ... < a_i^{\lambda_i-1}$. Then, a *skip inversion* is a pair of the form:

(1) $(a_i^0, a_j^0) : i > j, a_i^0 > a_j^0$
(2) $(a_j^{k+1}, a_i^{k+1}), i > j, a_i^k < a_j^{k+1} < a_i^{k+1}, k \geqslant 0$

We denote the set of skip inversions of ω by $\mathrm{Skip}(\omega)$, and define $\mathrm{skip}(\omega) = |\,\mathrm{Skip}(\omega)|$.

Remark 5.3.15. The statistic skip can be thought of as counting the number of entries skipped when running 1 on $\omega \in \widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)}$. We note that this is eerily reminiscent of the statistic *betrayal* introduced in [**KM17**]. This connection should lead to interesting future work.

Example 5.3.16. Consider $\omega = (6,7,8|2,4,9|1,3,5) \in \widetilde{\mathcal{D}}_{(3,3,3)}^{(9)}$. This corresponds to the ordered set partition 678|249|135, as well as the labeled descent composition

$$(0_3, 0_2, 1_3, 1_2, 2_3, 0_1, 1_1, 2_1, 2_2)$$

When the first (red) part is selected, two 0's are passed by to select the third and final 0. The first 1 after the 0 is selected, and the first 2 after the 1 is. So the first part contributes only 2 to skip(ω).

$$(\underline{0}, \underline{0}, 1, 1, 2, 0_1, 1_1, 2_1, 2)$$

When the second (blue) part is selected, the first 0 is skipped again, one unmarked 1 is skipped, and one unmarked 2 is skipped.

$$(\underline{0}, 0_2, \underline{1}, 1_2, \underline{2}, 0_1, 1_1, 2_1, 2_2)$$

so the second part contributes 3 to skip(ω). The final part contributes nothing, so skip(ω) = 5.

Proposition 5.3.17. *The map Ψ from the proof of 5.1.1 composed with aff from 5.2 from gives a q, t-weight preserving bijection between the sets*

$$\widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)} \longleftrightarrow \left\{ \sigma : \lambda' \to \{1, ..., n\} \,\middle|\, \mathrm{maj}(\sigma) = \eta(\lambda) \right\}$$

As a corollary, equation 5.5 holds.

Proof. By proposition 5.1.1, since $\mathrm{maj}(\omega) = \eta(\lambda) = \mathrm{maj}(\sigma)$ for all elements ω, σ in their respective sets, the map $\Psi \circ \mathrm{aff}$ gives a set bijection between them. We need only show that $\mathrm{skip}(\omega) = \mathrm{maj}((\Psi \circ \mathrm{aff})(\omega))$.

It then suffices to show that $|\,\mathrm{Skip}(\omega)| = |\,\mathrm{Inv}((\Psi \circ \mathrm{aff})(\omega))|$. We will do this by constructing a bijection between the two sets. Let $\omega = (A_1|...|A_\ell)$ as above, with $A_i = (a_i^0, ..., a_i^{\lambda_i - 1})$ and $a_i^0 < ... < a_i^{\lambda_i - 1}$. Then, we see that the image of ω under $\Psi \circ \mathrm{aff}$ must be λ'-fillings whose columns are precisely A_i in decreasing order. Then, the entry a_i^j must occur in the ith column, in the $j+1$th row (counting from the bottom). We consider the two types of skip inversions:

(1) For an inversion of the form (a_i^0, a_j^0), $i > j$, $a_i^0 > a_j^0$, then in $(\Psi \circ \mathrm{aff})(\omega)$, they will take the positions

$$\boxed{a_i^0} \qquad\qquad \boxed{a_j^0}$$

in the bottom row, so that (a_i^0, a_j^0) is an inversion in $(\Psi \circ \mathrm{aff})(\omega)$.

(2) For an inversion of the form $(a_j^{k+1}, a_i^{k+1}), i > j, a_i^k < a_j^{k+1} < a_i^{k+1}, k \geqslant 0$, we have

$$\begin{array}{|c|}\hline a_i^{k+1} \\ \hline a_i^k \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline a_j^{k+1} \\ \hline \end{array}$$

But the inequalities $a_i^k < a_j^{k+1} < a_i^{k+1}$ mean precisely that the entries increase in counterclockwise order, forming an inversion triple in $(\Psi \circ \mathrm{aff})(\omega)$.

We see then that the notions of skip inversions and inversion triples are equivalent, so we have $|\operatorname{Skip}(\omega)| = |\operatorname{Inv}((\Psi \circ \mathrm{aff})(\omega))|$, which completes the proof. $\square$

5.4. Towards a Basis for the Garsia-Haiman Module V_λ

The original motivation for the project was to extend the methods of Carlsson and Oblomkov in [**CO18**]. Since $V_\lambda \subset DR_n$, then it seemed feasible that the same ideas that yielded the schedules formula-type basis of DR_n would be successful for V_λ. This proved unsuccessful, and we discuss a few complications below.

5.4.1. Double Filtration and Hook Shapes. Let $R = \mathbb{C}[x_1, \ldots, x_n; y_1, \ldots, y_n]$, and consider the following ordering on a pair of compositions:

We say that $(\alpha, \beta) \leqslant_{\text{des,grrevlex}} (\alpha', \beta')$ if:

- $\beta <_{\text{des}} \beta'$ or
- $\beta = \beta'$ and $\alpha \leqslant_{\text{grrevlex}} \alpha'$

In other words, given monomials $x^\alpha y^\beta$ and $x^{\alpha'} y^{\beta'}$, we first compare the y-monomials in descent order, then the x-monomials in grrevlex. This appears to work nicely for hook shapes, as per the following conjecture:

Conjecture 5.4.1. Let $\lambda = (k, 1^{n-k})$ a hook shape, and $V'_{\lambda'} = \mathbb{C}[\partial_{\mathbf{x}}, \partial_{\mathbf{y}}] \cdot \Delta_{\lambda'}$ denote the Garsia-Haiman module. Then, there is a basis of polynomials whose leading terms with respect to the (des, grrevlex) order are precisely the $n-k-1$th-Haglund monomials, to be defined below.

Equivalently, expressed in terms of the coinvariant version:

CONJECTURE 5.4.2. Let $\lambda = (k, 1^{n-k})$ a hook shape, and $V_{\lambda'}$ denote the Garsia-Haiman module. Then, the leading terms with respect to the (des, grrevlex) order are the $n-k-1$th-Haglund monomials.

Furthermore, the following appears to be true, and has been checked by computer for $n \leqslant 7$:

CONJECTURE 5.4.3. Let λ be any partition. Then the leading terms with respect to the (des, grrevlex) order are of the form $x^{\alpha}y^{\beta}$, with $\beta \in \mathcal{D}_{\lambda}$.

5.4.2. Conjectural Garsia-Haiman Basis for Top t-degree. We present a set of monomials, denoted $\mathfrak{H}_{\lambda}^{\mathrm{top}}$, that conjecturally form a basis for

$$V_{\lambda'}^{\mathrm{top}} := \bigoplus_{h=0}^{\eta(\lambda')} V_{\lambda'}^{(h,\eta(\lambda))}$$

the top t-degree component of the Garsia-Haiman module $V_{\lambda'}$. The construction will be given in terms of the extended affine permutations $\omega \in \widetilde{\mathcal{D}}_{\lambda}^{\eta(\lambda)}$ and the skipt statistic.

DEFINITION 5.4.4. For $\lambda \vdash n$ a partition, let $\omega \in \widetilde{\mathcal{D}}_{\lambda}^{\eta(\lambda)}$. Then, define for $1 \leqslant i \leqslant n$, the quantity $c_i := |\{(i,m) \in \mathrm{Skip}(\omega)\}|$. One can think of c_i as denoting the number of times the ith entry of $\mathrm{majt}(\omega)$ is skipped during the selection process 1, as in 5.3.16 Define

$$\mathrm{skipt}(\omega) := (c_1, ..., c_n)$$

CONJECTURE 5.4.5. The set

$$\mathfrak{H}_{\lambda}^{\mathrm{top}} := \left\{ x^{\mathrm{skipt}(\omega)} y^{\mathrm{majt}(\omega)} : \omega \in \widetilde{\mathcal{D}}_{\lambda}^{\eta(\lambda)} \right\}$$

forms a monomial basis for the top t-degree component $V_{\lambda'}^{\mathrm{top}}$ of the Garsia-Haiman module.

EXAMPLE 5.4.6. Consider $\omega = (6,7,8|2,4,9|1,3,5) \in \widetilde{\mathcal{D}}_{(3,3,3)}^{(9)}$ as in example 5.3.16. Then, we calculate $\mathrm{skipt}(\omega) = (2,1,1,0,1,0,0,0,0)$, $\mathrm{majt}(\omega) = (0,0,1,1,2,0,1,2,2)$. The corresponding monomial is $(x_1^2x_2x_3x_5)(y_3y_4y_5^2y_7y_8^2y_9^2)$.

For hook shapes $\lambda = (k, 1^{n-k})$, we show that this construction coincides with the top degree of the $n-k-1$th-*Haglund basis* described in [**ARR07**]. We briefly recall the definition here:

DEFINITION 5.4.7. For $1 \leqslant k \leqslant n$, a permutation $\pi \in S_n$ with descent set $\mathrm{Des}(\pi)$, define

$$\mathrm{inv}_i^{(k)}(\pi) := \begin{cases} |\{j : i < j \leqslant k, \pi(i) > \pi(j)\}|, & \text{if } 1 \leqslant i < k \\ 0, & \text{if } i = k \\ |\{j : k \leqslant j < i, \pi(j) > \pi(i)\}|, & \text{if } k < i \leqslant n \end{cases}$$

$$d_i^{(k)}(\pi) := \begin{cases} |\,\mathrm{Des}(\pi) \cap \{i, ..., k-1\}|, & \text{if } 1 \leqslant i < k \\ 0, & \text{if } i = k \\ |\,\mathrm{Des}(\pi) \cap \{k, ..., i-1\}|, & \text{if } k < i \leqslant n \end{cases}$$

and the kth Haglund monomial to be

$$c_\pi^{(k)} := \prod_{i=k+1}^{n} x_{\pi(i)}^{inv_i^{(k)}(\pi)} \cdot \prod_{i=1}^{k-1} y_{\pi(i)}^{d_i^{(k)}(\pi)}$$

Note since our definition of Garsia-Haiman modules V_λ has $\mathbf{x}, \mathbf{y}$ swapped, we must switch the variables accordingly here.

THEOREM 5.4.8. *[**ARR07**] The set of kth Haglund monomials*

$$\left\{ c_\pi^{(k)} : \pi \in S_n \right\}$$

forms a monomial basis for the Garsia-Haiman module $V_{(n-k+1,1^{k-1})}$.

We show that for top y-degree, the set $\widetilde{\mathcal{D}}_\lambda^{\eta(\lambda)}$ generalizes the above basis.

PROPOSITION 5.4.9. *Let $\lambda = (n-k+1, 1^{k-1})$. Then, $\mathfrak{H}_{(k,1^{n-k})}^{top}$ specializes to the kth Haglund basis at top $\mathbf{y}$-degree.*

PROOF. By [**ARR07**], it is shown that the kth Haglund monomial for π corresponds to the Haglund statistics $\mathrm{inv}, \mathrm{maj}$ of the corresponding $(n-k+1, 1^{k-1})$-filling σ with reading word π. Since we are only concerned with $\deg(\mathbf{y}) = \binom{k}{2}$, we must have that $\pi_1 > \pi_2 > ... > \pi_k$. Then, the y-monomial for $c_\pi^{(k)}$ is $y_{\pi_{k-1}} y_{\pi_{k-2}}^2 ... y_{\pi_1}^{k-1}$.

The x-monomials for the kth Haglund monomial simply record the number of attacking pairs (i,j) with $i > j$ in the bottom row of σ. More precisely, for j in the bottom row of σ, it can be checked that $\mathrm{inv}_j^{(k)}(\pi) = |\{(i,j) : i > j \text{ in the bottom row of } \sigma\}|$. Then, as in 5.3.17, attacking pairs correspond precisely to skip inversions. Comparing the definitions of $\mathrm{inv}_j^{(k)}(\pi)$ and skipt, we see the claim is true. □

EXAMPLE 5.4.10. Let $n = 8, k = 4$. Then, $\lambda = (5,1,1,1)$. Consider the permutation $\pi = 86417352$. Then, we have $(\mathrm{inv}_1^{(4)}(\pi), ..., \mathrm{inv}_8^{(4)}(\pi)) = (3,2,1,0,0,1,1,3)$, and $(d_1^{(4)}(\pi), ..., d_8^{(4)}(\pi)) = (3,2,1,0,0,1,1,2)$. Then, $c_\pi^{(4)} = (x_2^3 x_3 x_5)(y_4 y_6^2 y_8^3)$. The corresponding λ-filling is

8				
6				
4				
1	7	3	5	2

where x_2^3 corresponds to the inversion pairs $(7,2),(3,2),(5,2)$, x_3 correponds to $(7,3)$, and x_5 corresponds to $(7,5)$. The corresponding element in $\widetilde{D}^{\mathrm{top}}_{(4,1,1,1,1)}$ is $\omega = (1,4,6,8|7|3|5|2)$, with labeled descent composition

$$(0_1, 0_5, 0_3, 1_1, 0_4, 2_1, 0_2, 3_1)$$

5.5. Further Results

5.5.1. Dominance Containment of $\mathcal{D}_\lambda$. The sets $\mathcal{D}_\lambda$ satisfy a very interesting containment property.

PROPOSITION 5.5.1. *Let μ, λ be partitions, and $\mu \trianglelefteq \lambda$. Then, $\mathcal{D}_\mu \subseteq \mathcal{D}_\lambda$.*

PROOF. It suffices to prove for covering relations in the dominance order poset, that is,

$$\lambda = (\mu_1, \ldots, \mu_i + 1, \ldots, \mu_j - 1, \ldots, \mu_{\ell(\mu)})$$

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{D}_\mu$. Then, alg_μ terminates, so that we may write

$$\mathrm{alg}_\mu(\mathbf{a}) = (\mathbf{a}_1, \ldots, \mathbf{a}_{\ell(\mu)}, \pi)$$

with $\pi = A_1|\dots|A_{\ell(\mu)} \in \mathcal{OSP}(\mu)$. Then, $\epsilon(\mathrm{alg}_\mu(\mathbf{a}))$ is good, and so we must have that the highest entry in $\mathbf{a}_i$ is greater than or equal to that of $\mathbf{a}_j$. Let k be the leftmost instance of the largest element in $\mathbf{a}_j$, and suppose m is the corresponding index in $\mathbf{a}$, $a_m = k$. Then, consider

$$\pi' = A_1|\dots|A_i \cup \{m\}|\dots|A_j\backslash\{m\}|\dots|A_{\ell(\mu)}$$

and modify $\mathbf{a}_i, \mathbf{a}_j$ accordingly to obtain $\mathbf{a}_i^+, \mathbf{a}_j^-$. Then, we must have $\mathbf{a}_i^+ \in \mathcal{D}_{\lambda_i+1}, \mathbf{a}_j^- \in \mathcal{D}_{\lambda_j-1}$, as the essential sequences were not affected by this swap. This implies that

$$\mathbf{a} = \Psi(\mathbf{a}_1, \dots, \mathbf{a}_i^+, \dots, \mathbf{a}_j^-, \dots, \mathbf{a}_{\ell(\mu)}, \pi')$$

so $\mathbf{a} \in \mathcal{D}_\lambda$. □

5.5.2. Generalization to Δ-Springer Modules $R_{n,\lambda,s}$. In Sean Griffin's remarkable thesis, the *generalized coinvariant algebra* $R_{n,k}$ and the Garsia-Procesi module R_λ are generalized into the Δ-*Springer modules*, whose definition we recall here.

Let $k \leqslant n$, λ a partition of k, and an integer $s \geqslant \ell(\lambda)$.

Definition 5.5.2. [**Gri21**] Define the ideal $I_{n,\lambda,s}$ and $R_{n,\lambda,s}$ as follows:

$$\begin{aligned} I_{n,\lambda,s} &:= \langle x_i^s : 1 \leqslant i \leqslant n\rangle + I_\lambda \\ R_{n,\lambda,s} &:= \mathbb{Q}[x_1, \dots, x_n]/I_{n,\lambda,s} \end{aligned} \tag{5.6}$$

where I_λ is the usual Tanisaki ideal, as defined previously.

Griffin provided a monomial basis in terms of certain n, λ, s-*staircases*, which turn out to specialize to the usual Garsia-Procesi basis of R_λ.

In ongoing work, we study a descent-type basis that simultaneously generalizes the descent-basis of Haglund-Rhoades-Shimozono defined in [**HRS19**] and our new Garsia-Procesi Basis:

Conjecture 5.5.3. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of $k \leqslant n$, and $s \geqslant \lambda_1$. Define

$$\lambda^+ = (\lambda_1, \dots, \lambda_\ell, \underbrace{1, \dots, 1}_{n-k})$$

Write Ψ for the shuffle map as before,

$$\Psi : \mathcal{D}_{\lambda_1} \times \cdots \times \mathcal{D}_{\lambda_\ell} \times \overline{[s]}^{n-k} \times \mathcal{OSP}(\lambda^+) \to \mathcal{D}_n$$

and define $\mathcal{D}_{n,\lambda,s} := \text{im}(\Psi)$. Then, the set of monomials $\{x^{\mathbf{b}} : \mathbf{b} \in \mathcal{D}_{n,\lambda,s}\}$ is a vector-space basis of $R_{n,\lambda',s}$.

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