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RIGOROUS FORMULAE FOR THE STATISTICAL ERRORS ON
THE ZEROS AND THE PARTIAL WAVES IN TWO-BODY REACTIONS*

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February 9, 1976

ABSTRACT

The probability densities of the zeros and of the phase shifts, in spin 0 spin 0 and spin $\frac{1}{2}$ spin 0 reactions, are derived rigorously.

INTRODUCTION

In physics we deal very often with polynomial approximations of experimental data. For instance this is the case when we perform a classical phase shift analysis of a scattering experiment. While we are not usually interested in the roots of these polynomials, it has recently¹⁻³ become clear that they are really useful to look at. Our aim is to find the rigorous probability densities functions of these roots. The reasons for a priori lengthy calculations are the following:

--Instead of making the usual linear approximation of the propagation of errors, we will show that it can be both not too difficult and very instructive to know the actual probability densities of the estimated parameters (zeros, partial wave or whatever), in order, for instance, to define when the linear approximation is totally wrong.

--These apparently academic calculations give important physical results. Especially in spin $\frac{1}{2}$ spin 0 reactions where we will see that the linear approximation breaks down for cases of great theoretical and practical importance.

In the first section the probability densities of the zeros and of the partial waves in spin 0-spin 0 reactions are derived. The second section is devoted to the extension of these results to spin $\frac{1}{2}$ spin 0 reactions. The third section illustrates by computer simulation, the most striking features of our rigorous results. Practical formulae are given in Section IV.

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I. Spin 0-Spin 0 REACTIONS

1. Generalities and Notations

The spin 0-spin 0 case is not academic, in the sense that reactions like $\pi\alpha \rightarrow \pi\alpha$, $\kappa\alpha \rightarrow \kappa\alpha$, $\pi^{12}C \rightarrow \pi^{12}C$... can be considered as examples of spin 0 spin 0. Furthermore the calculations are easier than for spin $\frac{1}{2}$ spin 0 and will prepare us for this later case.

Let's call:

$z = \cos \theta_{CM}$ θ_{CM} is the center of mass scattering angle

$\sigma(z)$ the differential cross section

$f(z)$ the scattering amplitude.

When we want to deal with analytical continuations we write:

$$\sigma(z) = f(z) \bar{f}(\bar{z}) \quad \bar{z} \text{ means complex conjugate of } z.$$

If $\sigma(z)$ is approximated by a polynomial we write:

$$\sigma(z) = a_0 + a_1 z + \dots + a_{2n} z^{2n} = a_{2n} \prod_{i=1}^n (z - z_i)(z - \bar{z}_i).$$

The coefficients a_i are real and $a_{2n} > 0$ because $\sigma(\cos \theta) > 0$.

The fact that we choose an even number of roots for $\sigma(z)$ is dictated by the requirement that we do not want to introduce unnecessary singularities in the scattering amplitude. This would be the case when keeping an odd number of roots for $\sigma(z)$. For instance in the case of one root:

$$\sigma(z) = a_1(z - z_1)$$

z_1 is necessarily real and:

$$f(z) = \sqrt{a_1} e^{i\phi(z)} \sqrt{z - z_1}.$$

This introduces a branch point on the real axis. On the contrary when the number of roots is even, we can build $f(z)$ as a polynomial by picking n roots among the $2n$ mirror pairs of $\sigma(z)$.

Remark

We can, of course, expand $\sigma(z)$ on another set of polynomials (Legendre polynomials as it is often done), but our results would not change fundamentally since the coefficients in both expansions are linearly related.

2. Some Useful Mathematics

Suppose we know the probability density of the coefficients of our polynomial approximation (multinormal for instance). Since the roots of this polynomial are functions of these coefficients, our problem is to find a procedure to convert one set of random variables (a_i) to another set of random variables (z_j) given a functional relation between the two.⁴

We will first consider the case of one random variable (r.v.) function of another r.v. Then we give the generalization of n r.v. function of n r.v. From now on, to avoid possible confusion, the absolute value of a real number x will be noted $|x|$, and the modulus of a complex number z , will be noted $||z||$.

Let's assume that the r.v. y is related to the r.v. x by the function g . We call $p_x(x)$ the probability density function (p.d.f.) of the r.v. x , and $p_y(y)$ for y .

If $y = g(x)$ has a countable number of roots: $x_1, x_2, \dots$ we have:

$$p_y(y) = \frac{p_x(x_1)}{|g'(x_1)|} + \frac{p_x(x_2)}{|g'(x_2)|} + \dots \quad \text{with } g'(x) = \frac{dg}{dx}$$

For example consider the Figure 1, where we have two roots for the particular y value chosen. The total probability contained in (dx_1) and (dx_2) is:

$$p_x(x_1)|dx_1| + p_x(x_2)|dx_2|$$

this is the total probability contained in dy thus:

$$p_y(y)|dy| = p_x(x_1)|dx_1| + p_x(x_2)|dx_2|$$

but:

$$dx_1 = \frac{dy}{g'(x_1)} \quad dx_2 = \frac{dy}{g'(x_2)}$$

which gives the expected formula.

Generalization

When we have n r.v. $X = (x_1, \dots, x_n)$ and n r.v. $Y = (y_1, \dots, y_n)$ functionally dependent, we are interested in the transformation of the hypervolume $dx_1 \dots dx_n$ to Y space. We use the jacobian to perform such a transformation.

$$\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} \equiv J(x_1, \dots, x_n) \equiv \det \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

As in the case of one dimension we get

$$p_Y(y_1, \dots, y_n) = \frac{p_x(x_1)}{|J(x_1)|} + \frac{p_x(x_2)}{|J(x_2)|} + \dots$$

where $x_1, x_2, \dots$ are the countable roots of $Y = G(x)$.

Two useful mathematical theorems

1. Let $a_0 + a_1 z + \dots + a_n z^n$ be a complex polynomial and $(z_1, \dots, z_n)$ its roots. Then:

$$\det \left(\frac{\partial a_i}{\partial z_j} \right) = (-1)^n a_n^n \prod_{i < j} (z_i - z_j)$$

This is shown in Appendix 1.

2. Let $(\phi_1, \dots, \phi_n)$ be n complex functions of $(z_1, \dots, z_n) \in \mathbb{C}^n$, such that

$$\frac{\partial \phi_i}{\partial \bar{z}_j} = 0 \quad \bar{z}_j \text{ means complex conjugate of } z_j.$$

then⁶

$$\frac{\partial(\phi_1^R, \phi_1^I, \dots, \phi_n^R, \phi_n^I)}{\partial(z_1^R, z_1^I, \dots, z_n^R, z_n^I)} \equiv \det \left(\frac{\partial \phi_i^{R,I}}{\partial z_j^{R,I}} \right) = \left\| \det \frac{\partial \phi_i}{\partial z_j} \right\|^2$$

ϕ_1^R means Real part of ϕ_1

ϕ_1^I means Imaginary part of ϕ_1 .

This is very useful for two reasons: instead of handling a $2n \times 2n$ determinant we have only $n \times n$ to look at, and the

partial derivatives $\frac{\partial \phi_i}{\partial z_j}$ are always easier to compute than the

partial derivatives $\frac{\partial \phi_i^{R,I}}{\partial z_j^{R,I}}$.

3. Density Probability of the Zeros

Among the $2n$ roots of $\sigma(z)$, only n are independent.

Call these roots $(z_1, \dots, z_n)$. We have $2n+1$ real random variables (the polynomial coefficients) and are trying to determine only $2n$ real rv (the $z_i^{R,I}$). We use a general trick which consists of adding another rv to the set $(z_i^{R,I})$. We choose a_{2n} , since it does not depend on the roots.

Let's call $p_a(a_0, \dots, a_{2n})$ the pdf for the coefficients and $p_z(z_1^R, z_1^I, \dots, z_n^R, z_n^I, a_{2n})$ the pdf we wish to find.

From Section I.2. we know that:

$$p_z = \sum \frac{p_a}{\left| \frac{\partial(z)}{\partial(a)} \right|} \quad \text{the sign } \Sigma \text{ stands for the sum over all the roots of } z = z(a).$$

In fact we do it the other way around

$$p_a = \sum \frac{p_z}{\left| \frac{\partial a}{\partial z} \right|}$$

We do know the expression for the coefficients as a function of the roots, and also it is easy to see that there are exactly $n!$ possibilities giving the same set (a_i) . This result is obtained by permutation of the

roots among themselves (the coefficients are symmetric functions of the roots). So:

$$p_a = n! \frac{p_z}{\left| \frac{\partial a}{\partial z} \right|}$$

or

$$p_z = \frac{1}{n!} \left| \frac{\partial(a)}{\partial(z)} \right| p_a$$

Let's work out the jacobian

$$\frac{\partial(a_0, \dots, a_{2n})}{\partial(z_1^R, z_1^I, \dots, z_n^R, z_n^I, a_{2n})} = \frac{\partial(a_0, \dots, a_{2n-1})}{\partial(z_1^R, z_1^I, \dots, z_n^R, z_n^I)}$$

Now:

$$\frac{\partial a_j}{\partial z_k^R} = \frac{\partial a_j}{\partial z_k} + \frac{\partial a_j}{\partial \bar{z}_k} \quad \text{and} \quad \frac{\partial a_j}{\partial z_k^I} = i \left[\frac{\partial a_j}{\partial z_k} - \frac{\partial a_j}{\partial \bar{z}_k} \right]$$

Dividing by i and adding the corresponding column we get:

$$J = i \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ \dots & 2 \frac{\partial a_j}{\partial z_k} & \frac{\partial a_j}{\partial z_k} - \frac{\partial a_j}{\partial \bar{z}_k} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 2i \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ \dots & \frac{\partial a_j}{\partial z_k} & - \frac{\partial a_j}{\partial \bar{z}_k} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

Doing such a manipulation for the whole determinant we find:

$$J = i^n 2^n (-)^n \begin{vmatrix} \vdots & \vdots \\ \frac{\partial a_1}{\partial z_k} & \frac{\partial a_1}{\partial \bar{z}_k} \dots \\ \vdots & \vdots \end{vmatrix}$$

By the theorem (1) of I.2 we thus have:

$$J = 2^n (-)^n i^n a_{2n}^{2n} \prod_{j < k}^{2n} (\bar{z}_j - \bar{z}_k)$$

$\bar{z}_j$ means that now we take the set $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$. The last product can be expressed as:

$$\prod_{j < k}^{2n} (\bar{z}_j - \bar{z}_k) = (-)^{\frac{n(n-1)}{2}} \prod_{j < k}^n ||z_j - z_k||^2 ||z_j - \bar{z}_k||^2 \prod_{i=1}^n (z_i - \bar{z}_i)$$

Taking the absolute value we get for P_z :

$$P_z(z_i, a_{2n}) = \frac{a_{2n}^{2n}}{n!} 2^{2n} \prod_{j=1}^n |\text{Im } z_j| \prod_{j < k}^n ||z_j - z_k||^2 ||z_j - \bar{z}_k||^2 P_a(a_i, a_{2n}) \quad (I.1)$$

We can rewrite this expression in another form using

$$\frac{\partial \sigma(z)}{\partial z_i} = -a_{2n} (z - z_1)(z - \bar{z}_1) \dots (z - z_i) (z - \bar{z}_i) \dots (z - z_n)(z - \bar{z}_n)$$

$$\left(\frac{1}{z - z_i} \frac{\partial \sigma}{\partial z_i} \right)_{z=z_i} = -a_{2n} (z_i - z_1) \dots \left(\frac{z_i - z_i}{z_i - z_i} \right) (z_i - \bar{z}_i) \dots (z_i - \bar{z}_n)$$

and

$$\frac{\partial \sigma}{\partial z} = \sum_{j=1}^{2n} \frac{\sigma}{z - \bar{z}_j}$$

$$\left(\frac{\partial \sigma}{\partial z} \right)_{z=\bar{z}_i} = - \left(\frac{\partial \sigma}{\partial z_i} \right)_{z=\bar{z}_i}$$

so

$$p_z(z_i, a_{2n}) = \frac{a_{2n}^n}{n!} 2^n \sqrt{\prod_{i=1}^{2n} \frac{\partial \sigma}{\partial z}}_{z=\bar{z}_i} p_a(a_i, a_{2n})$$

or since

$$\left(\frac{\partial \sigma}{\partial z} \right)_{z=z_i} = \overline{\left(\frac{\partial \sigma}{\partial z} \right)_{z=\bar{z}_i}}$$

$$p_z(z_i, a_{2n}) = \frac{a_{2n}^n}{n!} 2^n \prod_{i=1}^n \left\| \frac{d\sigma}{dz} \right\|_{z=z_i} p_a(a_j, a_{2n}) \quad (I.2)$$

The final density probability for the zeros is obtained by integrating over a_{2n}

$$p_z(z_i) = \int_0^\infty p_z(z_i, a_{2n}) da_{2n}$$

These formulae are discussed in Section III.

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4. Density Probability for the Partial Waves

In building the amplitude we have to choose our set $z_1, \dots, z_n$ from the set $\{\bar{z}_i\}$ and therefore, lead to the discrete Barrelet's ambiguity.²

The second well-known ambiguity is the global phase. We will assume a phase $\phi(z)$ chosen once for all. This phase is not affected by statistical errors since it is not determined by the coefficients a_i (except at the point $z = +1$ by using the optical theorem). Furthermore, at the zeros this phase is absolutely arbitrary since there $f(z_i) = 0$. We will write $f(z)$ as:

$$f(z) = \sqrt{a_{2n}} e^{i\phi(z)} \prod_{i=1}^n (z - z_i)$$

$$f(z) = b_0 + b_1 z + \dots + b_n z^n \quad ||b_n|| = \sqrt{a_{2n}}$$

and

$$f(z) = \sum_{\ell=0}^n (2\ell+1) T_\ell P_\ell(z) \quad T_\ell = \frac{n_\ell e^{2i\delta_\ell} - 1}{2ik}$$

where k is the center of mass momentum. Occasionally we will use k as an indice and we hope there will be no possible confusion. Let's call $p_b(b_0^R, b_0^I, \dots, ||b_n||)$ the pdf for the b coefficients.

$$p_b = n! p_z(z_i, ||b_n||) \frac{\partial(z_i, ||b_n||)}{\partial(b_i, ||b_n||)}$$

Using theorem 2 of I.1, because the b_i do not depend on $\bar{z}_i$ we get:

$$p_b(b_0^R, b_0^I, \dots, ||b_n||) = n! \frac{p_z(z_i, ||b_n||)}{||b_n||^{2n} \prod_{i<j}^n ||z_i - z_j||^2}$$

and

$$p_z(z_i, ||b_n||) = p_z(z_i, a_{2n}) 2 ||b_n||$$

The T_ℓ are linearly related to the b_i by:

$$T_\ell = \frac{1}{2} \int_{-1}^{+1} [b_0 + \dots + b_n z^n] P_\ell(z) dz$$

this introduces a multiplicative constant:

$$\left| \det \left(\frac{\partial T_\ell^{R,I}}{\partial b_k^{R,I}} \right) \right|$$

in the pdf, so we won't bother about it. In the Appendix we give this constant when $\phi(z) = C = \text{constant}$.

$$p_b(b_0^R, b_0^I, \dots, ||b_n||) = 2^{2n+1} ||b_n||^{2n+1} \prod_{j=1}^n ||\text{Im } z_j||$$

$$\prod_{j<k}^n ||z_j - \bar{z}_k||^2 p_a(a_i) \quad (I.3)$$

$$P_T(T_\ell) = C p_b(b_i)$$

It is possible to express p_b in another interesting way.

$$f(\bar{z}_1) = \sqrt{a_{2n}} e^{i\phi(\bar{z}_1)} \prod_{i=1}^n (\bar{z}_1 - z_i)$$

$$\prod_{j=1}^n f(\bar{z}_j) = a_{2n}^{n/2} e^{i\sum \phi(\bar{z}_j)} \prod_{i,j} (\bar{z}_j - z_i)$$

$$= a_{2n}^{n/2} e^{i\sum \phi(\bar{z}_j)} \prod_{j=1}^n (\bar{z}_j - z_j) \prod_{j < k} ||z_j - z_k||^2$$

$$p_b = 2^{n+1} ||b_n||^{n+1} \left\| \prod_{j=1}^n f(\bar{z}_j) \right\| p_a$$

$$p_T = c 2^{n+1} ||b_n||^{n+1} \left\| \prod_{j=1}^n f(\bar{z}_j) \right\| p_a \quad (I.4)$$

In Appendix 2 we show another way of getting this expression without using p_z as an intermediary step.

II. Spin 0-Spin $\frac{1}{2}$ REACTIONS

1. Density Probability of the Barrelet's Zeros

For spin 0 spin $\frac{1}{2}$ reactions there are two amplitudes to deal with, and we must make a choice of a representation in which to express these amplitudes and their zeros. We will consider Barrelet's representation since it provides a close resemblance to the spin 0-spin 0

case, and also because it deals with the zeros of the eigenvalues of the transition matrix. In Barrelet's representation,² the amplitude is written $A(w)$ and is the analogue of $f(z)$. He defines a transverse section $\Sigma(w)$ which corresponds to $\sigma(z)$. For a given energy we have:

$$\Sigma(w) = A(w) \bar{A}(w^{-1}) = \alpha \prod_{i=1}^n (w - w_i)(w^{-1} - \bar{w}_i)$$

α is equal to $\frac{\Sigma(\theta=0)}{\prod_{i=1}^n ||1 - w_i||^2}$. α is positive because $\Sigma(e^{i\theta})$ is positive.

The remarks of Section I.1 apply here too.

We can rewrite $\Sigma(w)$ as

$$\begin{aligned} \Sigma(w) &= \frac{\alpha(-)^n}{w^n} \prod_{i=1}^n (\bar{w}_i) \prod_{i=1}^n (w - w_i) \left(w - \frac{1}{\bar{w}_i}\right) \\ &= \frac{1}{w^n} \left[a_0 + a_1 w + \dots + a_{2n} w^{2n} \right] \end{aligned}$$

Because $\Sigma(w) = \bar{\Sigma}(w^{-1})$ we have:

$$a_0 = \bar{a}_{2n}, \dots, a_k = \bar{a}_{2n-k}, \dots, a_n = \bar{a}_n$$

Let's call $w_i^M = 1/\bar{w}_i$, M means mirror (with respect to the unit circle).

$$a_{2n} = (-)^n \frac{\alpha}{\prod_{i=1}^n w_i^M}$$

we will take as independent variables:

$$(a_0, \dots, a_n) \quad \text{and} \quad (w_1, \dots, w_n, k).$$

As in Section I, we are interested in the jacobian:

$$J = \frac{\partial(a_0^R, a_0^I, \dots, a_{n-1}^R, a_{n-1}^I, a_n^R)}{\partial(w_1^R, w_1^I, \dots, w_n^R, w_n^I, \alpha)}$$

$$\frac{\partial a^R}{\partial w^R} = \frac{1}{2} \frac{\partial(a + \bar{a})}{\partial w^R} = \left(\frac{\partial a}{\partial w} + \frac{\partial \bar{a}}{\partial \bar{w}} \right)^R;$$

$$\frac{\partial a^R}{\partial w^I} = - \left(\frac{\partial a}{\partial w} - \frac{\partial \bar{a}}{\partial \bar{w}} \right)^I$$

$$\frac{\partial a^I}{\partial w^R} = \left(\frac{\partial a}{\partial w} + \frac{\partial \bar{a}}{\partial \bar{w}} \right)^I \quad \frac{\partial a^I}{\partial w^I} = \left(\frac{\partial a}{\partial w} - \frac{\partial \bar{a}}{\partial \bar{w}} \right)^R$$

After some manipulations we get for the determinant:

$$J = \begin{vmatrix} \vdots & \vdots & \vdots \\ \dots & \frac{\partial a_\ell}{\partial w_j} - \frac{\partial a_\ell}{\partial \bar{w}_j} & \dots & \frac{\partial a_\ell}{\partial \alpha} \\ \vdots & \vdots & \vdots \\ \dots & -\frac{\partial \bar{a}_\ell}{\partial w_j} - \frac{\partial \bar{a}_\ell}{\partial \bar{w}_j} & \dots & -\frac{\partial \bar{a}_\ell}{\partial \alpha} \\ \vdots & \vdots & \vdots \\ \dots & \frac{\partial a_n}{\partial w_j} - \frac{\partial a_n}{\partial \bar{w}_j} & \dots & \frac{\partial a_n}{\partial \alpha} \end{vmatrix}$$

Now

$$\frac{\partial a_\ell}{\partial \bar{w}_j} = \frac{\partial a_\ell}{\partial w_j^M} \cdot \frac{\partial w_j^M}{\partial \bar{w}_j} = - \frac{\partial a_\ell}{\partial w_j^M} (w_j^M)^2$$

and because $a_k = \bar{a}_{2n-k}$ we get:

$$J = \begin{vmatrix} \vdots & \vdots & \vdots \\ \dots & \frac{\partial a_\ell}{\partial w_j} & -\frac{\partial a_\ell}{\partial w_j^M} (w_j^M)^2 & \dots & \frac{\partial a_\ell}{\partial \alpha} \\ \vdots & \vdots & \vdots \\ \dots & -\frac{\partial a_{2n-\ell}}{\partial w_j} & -\frac{\partial a_{2n-\ell}}{\partial w_j^M} (w_j^M)^2 & \dots & -\frac{\partial a_{2n-\ell}}{\partial \alpha} \\ \vdots & \vdots & \vdots \\ \dots & \frac{\partial a_n}{\partial w_j} & -\frac{\partial a_n}{\partial w_j^M} (w_j^M)^2 & \dots & \frac{\partial a_n}{\partial \alpha} \end{vmatrix}$$

with rearrangement of rows and columns and without looking at the sign since we have to take the absolute value, we end up with:

$$J = \prod_{i=1}^n (w_i^M)^2 \begin{vmatrix} \frac{\partial a_0}{\partial w_1} & \dots & \frac{\partial a_0}{\partial w_n} & \frac{\partial a_0}{\partial w_1^M} & \dots & \frac{\partial a_0}{\partial w_n^M} & \frac{\partial a_0}{\partial \alpha} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial a_n}{\partial w_1} & \dots & \frac{\partial a_n}{\partial w_n} & \frac{\partial a_n}{\partial w_1^M} & \dots & \frac{\partial a_n}{\partial w_n^M} & \frac{\partial a_n}{\partial \alpha} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial a_{2n}}{\partial w_1} & \dots & \frac{\partial a_{2n}}{\partial w_n} & \frac{\partial a_{2n}}{\partial w_1^M} & \dots & \frac{\partial a_{2n}}{\partial w_n^M} & \frac{\partial a_{2n}}{\partial \alpha} \end{vmatrix}$$

but $\frac{\partial a_{2n}}{\partial w_i} = 0$ and $\frac{\partial a_{2n}}{\partial w_i^M} = 0$; furthermore: $\frac{\partial a_{2n}}{\partial \alpha} = \frac{(-)^n}{\prod_{i=1}^n w_i^M}$.

Again we make use of theorem (1) of I.2.

$$J = \left[\prod_{j=1}^n (w_j^M) \right] \det \left(\frac{\partial a_j}{\partial w_\ell} \right) = \left[\prod_{j=1}^n (w_j^M) \right] a_{2n} \prod_{i < j} (\tilde{w}_i - \tilde{w}_j)$$

where $\tilde{w}_i$ stands for the set $\{w_1, \dots, w_n, w_1^M, \dots, w_n^M\}$

$$a_{2n}^{2n} = \frac{\alpha^{2n}}{\left(\prod_{i=1}^n w_i^M \right)^{2n}}$$

and the final formula is

$$p_w(w_1^R, w_1^I, \dots, w_n^R, w_n^I, \alpha) = \frac{1}{n!} \prod_{i < j} |w_i - w_j|^2 |1 - w_i \bar{w}_j|^2 \prod_i |1 - |w_i|^2| \alpha^{2n} p_a \quad (II.1)$$

In Section III we will discuss this formula. In a slightly different form we can rewrite this pdf as a function of $\Sigma(w)$.

$$p_w(w, \alpha) = \frac{\alpha^n}{n!} \left\| \prod_{i=1}^n w_i^M \right\|^{+n} \left\| \prod_{i=1}^n \left(\frac{d\Sigma(w)}{dw} \right)_{w=w_i} \right\| p_a$$

but

$$a_{2n} = (-)^n \frac{\alpha}{\prod_{i=1}^n w_i^n} \Rightarrow \|a_{2n}\| = \frac{\alpha^n}{\left\| \prod_{i=1}^n w_i^M \right\|^n} = \|a_0^n\|.$$

Thus

$$p_w(w, \alpha) = \frac{\left\| \prod_{i=1}^n w_i^M \right\|^{2n}}{n!} \left\| \prod_{i=1}^n \left(\frac{d\Sigma(w)}{dw} \right)_{w=w_i} \right\| \cdot \|a_0^n\| \cdot p_a(a_0, \dots, a_n). \quad (II.2)$$

Again we get $p_w(w_i)$ by an integration over α

$$p_w(w_1, \dots, w_n) = \int_0^\infty p_w(w_1, \dots, w_n, \alpha) d\alpha.$$

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2. Density Probability of the Partial Waves in

Spin $\frac{1}{2}$ Spin 0 Reactions

Choosing n zeros among the $2n$ given by our approximation of $L(w)$, leads to the same 2^n discrete ambiguities as in spin 0-spin 0. Also we have to make a choice of a global phase ϕ . A third ambiguity appears because it's possible to multiply the amplitude by any power of w , since:

$$w^n \overline{(w^{-1})^n} = 1.$$

The main effect of this ambiguity is to bring about high partial waves.

The amplitude $A(w)$ can be written as:

$$A(w) = w^{-L} \sqrt{\alpha} e^{i\phi} \prod_{i=1}^n (w - w_i)$$

or

$$A(w) = w^{-L} (b_0 + b_1 w + \dots + b_n w^n)$$

$$A(w) = \sum_{J,\epsilon} T_{J,\epsilon} R_{J,\epsilon}(w) \quad T_{J,\epsilon} = \frac{n_{J,\epsilon} e^{2i\delta_{J,\epsilon}} - 1}{2ik}$$

k is in this formula the center of mass momentum. The pseudopolynomials $R_{J,\epsilon}(w)$ have been introduced in Ref. 2 and will not be described here.

As for spin 0 spin 0, the pdf of the $T_{J,\epsilon}$ is related to the pdf of the b_i by a multiplicative constant since the relationship between $T_{J,\epsilon}$ and b_i is linear. This is detailed in Appendix 2.

Because the calculus goes along very similar lines as in Section I, we will omit it and state the result.

$$p_b(b_0^R, b_0^I, \dots, b_{n-1}^R, b_{n-1}^I, \sqrt{\alpha}) = \frac{n!}{(\sqrt{\alpha})^{2n} \prod_{i<j}^n ||w_i - w_j||^2} p(w_i, \sqrt{\alpha})$$

so that

$$p_b(b, \sqrt{\alpha}) = 2(\sqrt{\alpha})^{2n+1} \prod_{i<j}^n ||1 - w_i \bar{w}_j||^2 \prod_{i=1}^n ||1 - ||w_i||^2||^2 p_a. \quad (II.3)$$

As in the spin 0 spin 0 case we can rewrite this equation in terms of the amplitude $A(w)$.

$$p_b = 2 ||b_n||^{n+1} \left\| \prod_{i=1}^n w_i^M \right\|^{L-n} \left\| \prod_{i=1}^n A(w_i^M) \right\| p_a. \quad (II.4)$$

III. DISCUSSIONS AND EXAMPLES

1. We can split the formulae I.1-I.4, II.1-II.4, into two terms. Let's do it for the pdf of the zeros in spin 0-spin 0 reactions (formula (I.1)).

$$p_z \propto \prod_{i<j}^{2n} (\bar{z}_i - \bar{z}_j) a_{2n}^{2n} p_a \quad \bar{z}_i \in \{z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n\}$$

$$D(\text{dynamic}) = \prod_{i<j}^{2n} (\bar{z}_i - \bar{z}_j)$$

$$S(\text{statistic}) = a_{2n}^{2n} p_a .$$

D is independent of the pdf of the coefficients a_i and depends only on the roots of the physical reaction under study, thus the term dynamic. S is dependent on the statistics of the experiment and is very close to a multinomial variable. A maximum in S does not correspond to a maximum in $p_z = SD$ because the term D leads to a bias which tends to push the roots away from each other.

Consider, as a matter of illustration, two roots z_1, z_2 . Suppose z_1 fixed and vary z_2 . Figure 2 shows qualitatively the phenomenon of the bias. One striking consequence of this phenomenon is that a double zero in the amplitude will appear as two distinct zeros. There is a splitting of double roots.

The phenomenon of the bias and the splitting of double roots are shown in Figures 3b and Figure 3c. This is a computer simulation of 100 experiments of 40,000 events each. The physical region $-1, +1$ in $\cos \theta$ is split into 25 bins. The underlying amplitude for these simulated experiments is made of two zeros represented by open circles. The dots are the zeros estimated from each set of simulated data.

2. The formulae for the partial waves (I.3), (I.4), and (II.3), (II.4), differ in two aspects from those for the zeros. First no integration is involved, but this is not surprising since when we consider the zeros we are left with one parameter less than the approximations for the data. In that sense the formula are simpler for the partial waves. Second the pdf for the zeros can be expressed as a function of the cross section and the pdf of the partial waves can be expressed in terms of the amplitude. This indicates again that

the fundamental difference between the two sets of parameters is that the zeros are directly related to the cross section (the data points), while the partial waves are directly related to the amplitude.

3. When we consider the energy dependence of the zeros already extracted from the data,² it seems unlikely that any two initially distinct zeros could merge into a double root. However, the data indicate that some zeros' trajectories do cross the physical region. We know that the zeros come in mirror pairs (with respect to the real axis for spin 0 spin 0, or the unit circle for spin $\frac{1}{2}$ spin 0) in the cross section, so that when one zero touches the physical region it coalesces with its mirror image leading to a double root. These zeros on the physical region are of theoretical and practical importance. Theoretically an amplitude with a zero crossing the physical region and an amplitude with a zero bouncing back are very different. For instance, they give opposite signs for the parity of resonances. Practically these crossing points lead to an asymmetry parameter of one (in spin $\frac{1}{2}$ spin 0), allowing an absolute calibration for a polarized target.

IV. APPLICATIONS

Instead of considering various cases for $p_a(a)$ like a multinormal distribution, we think it is more fruitful to derive practical formulae of the confidence domains and of the biases for the zeros.

1. Spin 0 Spin 0

When going from the experimental data to the estimated roots we follow the path below.

Approximation of $\text{cov}(z_i, z_j)$

$$\sigma(z) = a_0 + a_1 z + \dots + a_{2n} z^{2n} = \vec{a} \cdot \vec{P}(z)$$

with $\vec{a} = (a_0, \dots, a_{2n})$ and

$$\vec{P}(z) = \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{2n} \end{pmatrix}$$

we saw in I,3 that

$$\left(\frac{\partial \sigma}{\partial z_i} \right)_{z=z_i} = - \left(\frac{\partial \sigma}{\partial z} \right)_{z=z_i}$$

Let's define $\sigma'_i = \left(\frac{\partial \sigma}{\partial z} \right)_{z=z_i}$ then $\delta z_i = \frac{\delta \sigma_{z=z_i}}{\sigma'_i}$

But

$$\delta \sigma(z = z_i) = \delta \vec{a} \cdot \vec{P}(z_i)$$

$$\text{cov}(z_i, z_j) = \langle \delta z_i \cdot \delta z_j \rangle = \left\langle \frac{\delta \vec{a} \cdot \vec{P}(z_i)}{\sigma'_i} \cdot \frac{\delta \vec{a} \cdot \vec{P}(z_j)}{\sigma'_j} \right\rangle$$

but, $\delta \vec{a} \cdot \vec{P}(z_i) \cdot \delta \vec{a} \cdot \vec{P}(z_j)$ can be written:

$$(1, z_i, \dots, z_i^{2n}) \begin{pmatrix} \delta a_0 \\ \vdots \\ \delta a_{2n} \end{pmatrix} (\delta a_0, \dots, \delta a_{2n}) \begin{pmatrix} 1 \\ z_j \\ \vdots \\ z_j^{2n} \end{pmatrix}$$

$$\text{cov}(z_i, z_j) = \frac{P_k(z_i)}{\sigma'_i} \text{cov}(a_k, a_\ell) \frac{P_\ell^T(z_j)}{\sigma'_j}; \quad \sigma'_i = a_{2n} \prod_{\substack{k=1 \\ k \neq i}}^{2n} (z_i - \bar{z}_k)$$

so the formulae are very simple since we do not need a matrix inversion.

If we want to look at covariance between real parts or imaginary parts we have:

$$\text{cov}(z_i^{R,I}, z_j^{R,I}) = \left(\frac{P(z_i)}{\sigma'_i} \right)^{R,I} \text{cov}(a, a) \left(\frac{P^T(z_j)}{\sigma'_j} \right)^{R,I} \quad (\text{IV.1})$$

this is because:

$$\sigma'(\bar{z}_i) = \overline{\sigma'(z_i)}$$

Approximation of the biases

To avoid too many indexes we will search $\text{bias}(z_1)$ this being good for any roots. To the first order $z_1 - \langle z_1 \rangle = \frac{P(z_1)}{\sigma'_1}$ and $\langle z_1 - \langle z_1 \rangle \rangle = 0$. Now in order to find the bias we have to develop $z_1 - \langle z_1 \rangle$ or $\frac{\delta \sigma_1}{\sigma'_1}$ up to the second order.

$$\sigma(z_1 + \delta z_1) = \underbrace{\sigma(z_1)}_{=0} + \sum_{i=1}^{2n} \left(\frac{\partial \sigma}{\partial z_i} \right)_{z=z_1} \delta z_i + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 \sigma}{\partial z_i \partial z_j} \right)_{z=z_1} \delta z_i \delta z_j$$

We know that:

$$\sum_{i=1}^n \left(\frac{\partial \sigma}{\partial z_i} \right)_{z=z_1} \delta z_i = \left(\frac{\partial \sigma}{\partial z_1} \right)_{z=z_1} \delta z_1$$

which gave us such a simple formula for the covariances.

$$\frac{\partial^2 \sigma}{\partial z_i \partial z_j} = a_{2n} \prod_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{2n} (z_i - \bar{z}_k)$$

is null except when $i = 1$ and $j \neq 1$, or $j = 1$ and $i \neq 1$, so that:

$$\sum_{i,j} \left(\frac{\partial^2 \sigma}{\partial z_i \partial z_j} \right)_{z_1} \delta z_i \delta z_j = 2 \sum_{j \neq 1} \left(\frac{\partial^2 \sigma}{\partial \bar{z}_j \partial z_1} \right)_{z_1} \delta z_1 \delta \bar{z}_j$$

so

$$\delta \sigma(z = z_1) = \sigma'_1 \delta z_1 \left[1 - \sum_{j=2}^{2n} \frac{\delta \bar{z}_j}{z_1 - \bar{z}_j} \right]$$

and for any root z_j :

$$\text{bias}(z_j) = \left\langle \frac{\delta \sigma_j}{\sigma_j} \right\rangle = - \sum_{i=1}^{2n} \frac{\text{cov}(z_j, \bar{z}_i)}{z_j - \bar{z}_i} \quad (\text{IV.2})$$

where again $\bar{z}_i \in \{z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n\}$. This formula can be interpreted as follows.

The root z_1 , for instance, has a bias which depends on all the other roots, but the farther the root the less the influence and the less these roots are correlated the less is their influence on the bias. As an illustration, let's take the example of only one zero (Figure 4)

$$\text{bias}(z_1) = - \frac{\text{cov}(z_1, \bar{z}_1)}{z_1 - \bar{z}_1} \quad z_1 = x_1 + iy_1$$

$$\text{cov}(z_1, \bar{z}_1) = \langle ||z_1||^2 \rangle - ||\langle z_1 \rangle||^2 \text{ is a positive real number.}$$

$$\text{bias}(z_1) = i \frac{\langle ||\delta z_1||^2 \rangle}{2y_1}$$

the bias is purely imaginary and tends to push apart z_1 from $\bar{z}_1$ as is expected from the rigorous formula.

In this example we see that the bias is of second order compared to $\sqrt{\text{cov}(z_1, \bar{z}_1)}$. This means that usually we can neglect it. If $\sqrt{\text{cov}(z_1, \bar{z}_1)} \sim 10\%$ then bias $\sim 1\%$. This is no longer true when two zeros cross each other or when one zero crosses the physical region. These two rare special and important cases have to be treated by other means.

How to draw approximate confidence domains?

Consider z_1 as two rv (x_1, y_1) , then we can draw around z_1 a confidence domain which is an ellipse. (Figure 5.)
 $\sigma_1^2 = 2\text{cov}(x_1, x_1)$; $\sigma_2^2 = 2\text{cov}(y_1, y_1)$; $2\text{cov}(x, y) = \sigma_1 \sigma_2 r$; $\langle x \rangle = \eta_1$;
 $\langle y \rangle = \eta_2$, the reason why we expand by 2 is to get the usual 70% confidence level.

$$\text{cov}(x, x) = \frac{1}{2} \text{Real}[\text{cov}(\bar{z}, z) + \text{cov}(z, z)]$$

$$\text{cov}(y, y) = \frac{1}{2} \text{Real}[\text{cov}(\bar{z}, z) - \text{cov}(z, z)]$$

$$\text{cov}(x, y) = \frac{1}{2} \text{Im}[\text{cov}(z, z)]$$

2. Spin 0-Spin $\frac{1}{2}$

Covariances

As in IV.1 we write:

$$\Sigma(w) = \alpha \prod_{i=1}^n (w - w_i)(w^{-1} - \bar{w}_i) = \vec{a} \cdot \vec{P}^T(w)$$

$$\vec{P}(w) = (w^{-n}, \dots, 1, \dots, w^n)$$

we find a very similar formula:

$$\text{cov}(w_1, w_2) = \frac{P_i(w_1)}{\Sigma'(w_1)} \cdot \text{cov}(a_i, a_j) \cdot \frac{P_j^T(w_2)}{\Sigma'(w_2)} \quad (\text{IV.3})$$

with

$$\Sigma'(w_1) = \alpha (w_1^{-1} - \bar{w}_1) \prod_{i=2}^n (w_1 - w_i)(w_1^{-1} - \bar{w}_i)$$

in spin 0-spin 0 we had:

$$\sigma'(z) = \overline{\sigma'(\bar{z})}$$

now

$$\Sigma(w) = A(w) \bar{A}(w^M) \Rightarrow d\Sigma(w) = \overline{d\Sigma(w^M)}$$

$$\Sigma'(w)dw = \overline{\Sigma'(w^M)dw^M}$$

but

$$\overline{\delta w^M} = -\frac{1}{w^2} \delta w \Rightarrow \Sigma'(w) = -\overline{(w^M)^2 \Sigma'(w^M)}$$

Now for the covariances we have:

$$\overline{\langle \delta w_1^M \delta w_2^M \rangle} = \frac{\langle \delta w_1^M \delta w_2^M \rangle}{(w_1^M w_2^M)^2}$$

Biases

$$\begin{aligned} \delta \Sigma &= -\alpha \prod_{i=1}^n (w - w_i) dw_1 + \sum_2^n \alpha \left[\prod_{i=1}^n (w - w_i)(w - \bar{w}_i) \right] \delta w_1 \delta w_i \\ &\quad + \sum_1^n \alpha \left[\prod_{i=1}^n (w - w_i)(w^{-1} - \bar{w}_i) \right] \delta w_1 \delta \bar{w}_i \end{aligned}$$

$$\text{bias}(w_1) = - \sum_{i=2}^n \frac{\text{cov}(w_1, w_i)}{w_1 - w_i} - \sum_{i=1}^n \frac{\text{cov}(w_1, \bar{w}_i)}{w_1^{-1} - \bar{w}_i}$$

We can rewrite this expression in another way since:

$$\text{cov}(w_1, \bar{w}_i) = - \frac{\text{cov}(w_1, w_i^M)}{(w_i^M)^2}$$

Thus for any root w_j :

$$\text{bias}(w_j) = - \sum_{i=1, i \neq j}^n \frac{\text{cov}(w_j, w_i)}{w_j - w_i} - \sum_{i=1}^n \frac{w_j^M}{w_i^M} \frac{\text{cov}(w_j, w_i^M)}{w_j - w_i^M} \quad (\text{IV.4})$$

This formula differs from spin 0 spin 0 because the distances are deformed in the w plane.

Let's see what happens for only one zero. $w_1 = \rho e^{i\phi}$

$$\text{bias}(w_1) = - \frac{\text{cov}(w_1, \bar{w}_1)}{w_1^{-1} - \bar{w}_1}$$

$$\text{bias}(w_1) = \frac{\langle ||\delta w_1||^2 \rangle}{\rho - \frac{1}{\rho}} e^{i\phi}$$

when $\rho > 1$ we have a bias which is along $0w_1$ and tends to push w_1 away from the unit circle. (Figure 6.)

Approximate confidence domains

As in spin 0 spin 0, we can draw the ellipses containing 70% CL. Another set of variables, more appropriate to the w plane, is (r, θ) .

$$w = re^{i\theta} \Rightarrow \frac{dw}{w} = \frac{dr}{r} + i d\theta$$

so that

$$\sigma_r^2 = 2 \text{cov}(r, r) = \text{cov}(w, \bar{w}) + \text{Re}(e^{-2i\theta} \text{cov}(w, w))$$

$$\sigma_\ell^2 = 2 \text{cov}(\ell, \ell) = \text{cov}(w, \bar{w}) - \text{Re}(e^{-2i\theta} \text{cov}(w, w))$$

$d\ell = rd\theta$ we can verify easily that

$$\sigma_{1/r} = \frac{1}{r} \sigma_r \quad \sigma_{1/\ell} = \frac{1}{r} \sigma_\ell$$

3. Approximation of the Bias When Two Zeros Cross Each Other

We saw in Fig. 3c the splitting of a double zero. We will now show how to estimate this splitting for both spin 0 spin 0 and spin $\frac{1}{2}$ spin 0 scattering. Let's call z_1 the position of the double root. We have for the statistical term S (Section III.2) a confidence domain approximated by an ellipse (Fig. 5). Let's take as coordinate axis the axis of this ellipse, and let's call r, θ the polar coordinates. The pdf for the zeros is approximated by:

$$p_z \propto ||z_1 - z_2||^2 p_a = r^2 \exp \left[-\frac{r^2}{2} \left(\frac{\cos^2 \theta}{\sigma_1^2} + \frac{\sin^2 \theta}{\sigma_2^2} \right) \right]$$

σ_1 and σ_2 should not be confused with $\sigma(z)$. They are the usual standard deviations.

$$\frac{\partial p_z}{\partial r} \propto \left[2r - r^3 \left(\frac{\cos^2 \theta}{\sigma_1^2} + \frac{\sin^2 \theta}{\sigma_2^2} \right) \right] \exp \left[-\frac{r^2}{2} \left(\frac{\cos^2 \theta}{\sigma_1^2} + \frac{\sin^2 \theta}{\sigma_2^2} \right) \right]$$

$$\frac{\partial p_z}{\partial \theta} \propto r^3 \left[-\frac{\cos \theta \sin \theta}{\sigma_1^2} + \frac{\sin \theta \cos \theta}{\sigma_2^2} \right]$$

Thus we have maximums for

$$\theta = 0, \pi \quad \text{and} \quad r = \sqrt{2} \sigma_1$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad r = \sqrt{2} \sigma_2$$

The maximum bias is equal to roughly one standard deviation and a half. This indicates a practical way of detecting ambiguous points. As soon as the confidence domains, defined as in Fig. 5, of two zeros, are in contact, we must admit the possibility of a double root.

4. How Good Are Our Approximate Formulae?

In order to test the validity of our approximate formulae we performed a computer simulation of a hundred scattering experiments with a given polynomial amplitude. For instance, in Fig. 3a the zeros of the underlying amplitude are $(-0.5 + 2i)$ and $(0.5 + 2i)$. They are represented by open circles. For each simulated scattering

experiment we generated 40,000 events and considered 25 bins equally spaced in $\cos \theta_{CM}$. The dots are the estimated zeros from each experiment. We drew the rectangles in which our elliptic confidence domains are inscribed. These rectangles should give approximately an 80% confidence level. This is true in Fig. 3a. Figure 3b differs from Fig. 3a in that we choose the zeros of the underlying amplitude closer to each other, namely: $(-0.2 + 2i)$ and $(0.2 + 2i)$. We begin to see, as expected, a bias and the failure of our approximate confidence domains (see III.1). Clearly if we had generated 160,000 events for each simulated experiment, the clouds of estimated zeros would have shrunk by a factor of two, giving us, again, a clear separation and a good behavior of our approximate formulae. It is clear that the notion of two zeros coalescing depends on our information or our statistics.

We tested our approximations for real data: π^+p elastic at $P_{lab} = 1439$ MeV/c. The clouds of points in Fig. 7 correspond to the following simulation: In each error bar of the data points,¹¹ we vary the value of the cross sections according to a gaussian distribution whose standard deviation is given by this error bar. The elliptic confidence domains are drawn and we verify that they are in good agreement with the simulation.

CONCLUSION

One of the aims of this article was to show the usefulness of deriving the rigorous probability density functions, instead of doing the usual linear error propagation. Indeed, for zeros and partial waves in spin 0 spin 0 and spin $\frac{1}{2}$ spin 0 reactions we found that very strong biases and total breakdown of linear formulae are expected for two cases of great theoretical and practical importance: first, when two zeros cross each other, and second, when a zero crosses the

physical region. As of now, in πN reactions the first case seems very unlikely². The second case is of real importance, as we saw in Section III.3. The existence of strong biases implies that one must consider the energy dependence of a zero near its supposed crossing point in order to determine whether the trajectory actually crosses the physical region.

For all other configurations our computer simulation, both with real and purely artificial data, shows that the practical approximations, given in Section IV, are sufficient. The biases in these usual cases are almost negligible and can be reliably estimated.

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APPENDIX 1

Given a complex polynomial $a_0 + a_1 z + \dots + a_n z^n = a_n \prod_{i=1}^n (z - z_i)$ we want to calculate:

$$\det \left(\frac{\partial a_i}{\partial z_j} \right)$$

The coefficients a_i are symmetrical functions of the roots z_i .

$$a_0 = (-)^n a_n z_1 \dots z_n, \dots, a_{n-1} = -a_n (z_1 + \dots + z_n)$$

Let us call

$$S_i^n = \frac{a_i}{a_n} (-)^{n-i}$$

This for $(z_1 \dots z_n)$. Now we define symmetrical functions

$S_i^{n-1}(z_2, \dots, z_n)$ in the same way. We have

$$S_p^n(z_1, \dots, z_n) = z_1 S_p^{n-1}(z_2, \dots, z_n) + S_{p-1}^{n-1}(z_2, \dots, z_n)$$

$$\det \frac{\partial a_i}{\partial z_j} = a_n^n \begin{vmatrix} \frac{\partial S_0^n}{\partial z_1} & \dots & \frac{\partial S_0^n}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial S_{n-1}^n}{\partial z_1} & \dots & \frac{\partial S_{n-1}^n}{\partial z_n} \end{vmatrix} (-1)^{\frac{n(n+1)}{2}} = a_n^n (-)^{\frac{n(n+1)}{2}} D_n$$

but

$$\frac{\partial S_k^n}{\partial z_1} = S_k^{n-1}(z_2, \dots, z_n)$$

It's easy to show that for $n = 2$, $D_2 = z_2 - z_1$. We will proceed by recurrence.

Suppose thus that:

$$D_n(z_1, \dots, z_n) = D_{n-1}(z_2, \dots, z_n) \prod_{i=2}^n (z_1 - z_i)$$

$$D_{n+1}(z_1, \dots, z_n) = \begin{vmatrix} \dots & S_0^n & \dots \\ \vdots & & \vdots \\ \dots & S_{n-1}^n & \dots \\ \vdots & & \vdots \\ \dots & 1 & \dots \end{vmatrix}$$

Subtracting the last column to the others we get

$$D_{n+1} = \begin{vmatrix} S_0^n(z_1) - S_0^n(z_{n+1}) & \dots & S_0^n(z_{n+1}) \\ \vdots & & \vdots \\ S_{n-1}^n(z_1) - S_{n-1}^n(z_{n+1}) & \dots & S_{n-1}^n(z_{n+1}) \\ 0 & & 1 \end{vmatrix}$$

$S_p^n(z_i)$ means that we take the set $(z_1, \dots, z_i, \dots, z_{n+1})$ etc. The first column gives:

$$S_k^n(z_1) - S_k^n(z_{n+1}) = z_{n+1} S_k^{n-1}(z_2, \dots, z_n) - z_1 S_k^{n-1}(z_2, \dots, z_n)$$

we have thus $z_{n+1} - z_1$ in common factor and we are left with S_k^{n-1} .

All the columns, but the last one, give the D_{n-1} we were looking for.

Hence we have that:

$$\det\left(\frac{\partial a_j}{\partial z_i}\right) = (-)^n a_n^n \prod_{1 \leq j}^n (z_i - z_j)$$

APPENDIX 2

Spin 0-0

As we said in I.4 the pdf for the partial waves is proportional to the pdf of the b 's.

$$T_\ell = \frac{1}{2} \int_{-1}^{+1} (b_0 + \dots + b_n z^n) P_\ell(z) dz$$

$P_\ell(z)$ are the Legendre polynomials we will use the bracket notation $\langle h|g \rangle = \int_{-1}^{+1} hgdz$ and recall $\langle z^k | P_\ell \rangle = 0$ if $k < \ell$.

Now we will calculate the multiplicative constant when the global phase $\phi(z)$ is taken as a constant,⁹

$$2T_\ell = b_\ell \langle z^\ell | P_\ell(z) \rangle + \dots + b_{n-1} \langle z^{n-1} | P_\ell(z) \rangle + ||b_n|| \langle z^n | P_\ell(z) \rangle$$

we look for:

$$\begin{aligned} \frac{\partial \langle T_0^R, T_0^I, \dots, T_{n-1}^I, ||b_n|| \rangle}{\partial \langle b_0^R, b_0^I, \dots, b_{n-1}^I, ||b_n|| \rangle} &= \left\| \det\left(\frac{\partial T_i}{\partial b_j}\right) \right\|^2 \quad (\text{because } T_i \text{ does not depend on } b_j) \\ &= 2^{-2n} \prod_{i=0}^{n-2} \left\| \langle z^i | P_i \rangle \right\|^2 \end{aligned}$$

$$P_T(T_i, ||T_n||) = \frac{2}{||\langle z^n | P_n \rangle||} P_T(T_i, ||b_n||)$$

so that:

$$P_T(T_0^R, T_0^I, \dots, T_{n-1}^R, T_{n-1}^I, ||T_n||) =$$

$$\frac{2^{2n+1} ||\langle z^n | P_n(z) \rangle||}{\prod_{i=0}^n ||\langle z^i | P_i(z) \rangle||} P_b(b_i)$$

The numbers $\langle z^k | P_k \rangle$ are equal to $\frac{2^{k+1} (k!)^2}{(2k+1)!}$.

Spin $\frac{1}{2}$ Spin 0

$$A(w) = w^{-q} \sqrt{\alpha} e^{i\phi(w)} \prod_{i=1}^n (w - w_i)$$

When we consider n zeros we have for the maximum total angular momentum

$$J_{\text{MAX}} = \frac{1}{2} + E\left(\frac{n}{2}\right) \quad E(x) \text{ means the integer part of } x.$$

$$\epsilon_{\text{MAX}} = (-)^n \quad \epsilon \text{ is not the parity, } \epsilon \text{ is the naturality.}$$

$$l = J - \frac{\epsilon}{2}$$

$$T_{j\epsilon} = \frac{1}{2(2J+1)} \int_{\gamma} A(w) \bar{R}_{j\epsilon}(w) 2 \left| w - \frac{1}{w} \right| |d| |w|$$

$\int_{\gamma}$ is the integral along the unit circle, and we will introduce also a bracket notation for the scalar product. This should not be mixed with the similar notation of spin 0-spin 0.

In order to have the lowest partial waves we must choose

$$q = L = E\left(\frac{n+1}{2}\right)$$

$$A(w) = w^{-L} \sqrt{\alpha} e^{i\phi(w)} \prod_{i=1}^n (w - w_i) = w^{-L} (b_0 + \dots + b_n w^n)$$

Again the multiplicative constant will be calculated when $\phi(w) = c$. Here there is a difference with spin 0-spin 0. We have to consider separately odd and even number of zeros.

n even

$$L = \frac{n}{2}$$

$$p_T(T_i) = \frac{2(2L+2)}{\left\| \left\langle R_{L+\frac{1}{2},+} | w^L \right\rangle \right\|^2} \frac{p_b(b_j)}{\left\| \prod_{p=0}^{L-1} \frac{\left\langle R_{p+\frac{1}{2},+} | w^p \right\rangle \left\langle R_{p+\frac{1}{2},-} | w^{-p-1} \right\rangle}{4(2p+2)^2} \right\|^2}$$

n odd

$$L = \frac{n+1}{2}$$

$$p_T(T_i) = \frac{(4L)^2 \left\| \prod_{i=1}^n w_i^M \right\|^2}{\left\| \left\langle R_{L-\frac{1}{2},-} | w^{-L} \right\rangle \right\|^2}$$

$$\frac{p_b(b)}{\left\| \prod_{p=0}^{L-1} \frac{\left\langle R_{p+\frac{1}{2},+} | w^p \right\rangle}{2(2p+2)} \right\|^2 \left\| \prod_{p=1}^{L-1} \frac{\left\langle R_{p+\frac{1}{2},-} | w^{-p} \right\rangle}{2(2p+2)} \right\|^2}$$

Another way to find $p_b(b_i, ||b_n||)$ in spin 0-spin 0 reactions

$$p_b(b_0^R, b_0^I, \dots, b_{n-1}^R, b_{n-1}^I, a_{2n}) = p_a(a_1) \frac{\partial(a_1)}{\partial(b_j^{R,I})} = p_a(a_1) D$$

$$\frac{\partial a_j}{\partial b_k^R} = \frac{\partial a_j}{\partial b_k} + \frac{\partial a_j}{\partial \bar{b}_k}$$

$$\frac{\partial a_j}{\partial b_k^I} = i \frac{\partial a_j}{\partial b_k} - i \frac{\partial a_j}{\partial \bar{b}_k}$$

$$D = 2^{n_1 n} \begin{vmatrix} \vdots & \vdots \\ \frac{\partial a}{\partial b} & \frac{\partial a}{\partial \bar{b}} \\ \vdots & \vdots \end{vmatrix} = 2^{n_1 n} \begin{vmatrix} b_0 \dots b_n 0 \dots 0 \\ 0 b_0 \dots b_n 0 \dots 0 \\ \vdots \\ \bar{b}_0 \dots \bar{b}_n 0 \dots 0 \\ 0 b_0 \dots \bar{b}_n 0 \dots 0 \\ 0 \vdots \vdots \end{vmatrix}$$

In the theory of equations¹⁰ this kind of determinant is known as a Sylvester's determinant. This determinant is equal to the resultant of $f(z)$ and $\bar{f}(\bar{z})$.

$$R(\bar{f}(\bar{z}), f(z)) = \bar{b}_n^n f(\bar{z}_1) \dots f(\bar{z}_n)$$

But:

$$\prod_1^n f(\bar{z}_1) = b_n^n \prod_{i,j} (\bar{z}_i - z_j)$$

so we end up with

$$p_b = p_a 2^{n_1 n} \bar{b}_n^n \prod_1^n f(\bar{z}_1)$$

because $\bar{z}_j - z_j \propto i$ and $p_b(b_i, ||b_n||) = p_b(b_i, a_{2n}) 2 ||b_n||$

$$p_b = p_a 2^{n+1} ||b_n||^{n+1} ||\prod_1^n f(\bar{z}_1)||$$

which is the same equation as in I.4.

FOOTNOTES AND REFERENCES

* This work was supported by the Energy Research and Development Administration.

† On leave of absence from Ecole Polytechnique, Paris, France.

1. E. Barrelet, Thesis, Paris 1970.
2. E. Barrelet, Nuovo Cimento 8A, 331 (1972).
3. M. Urban, Physics Letters 60B, 77 (1975).
4. To our knowledge this general problem is not often studied in most textbooks. In this respect the reference (5) gives a lot of simple examples for the case of two random variables function of two random variables.
5. A. Papoulis, Probability, Random Variables, and Stochastic Processes (McGraw Hill, New York, 1965).
6. Salomon and W. T. Martin in Several Complex Variables (Princeton University Press, 1948).
7. We use the covariances of complex quantities. This is not a problem and usually they are defined in the following manner:

$$\text{cov}(z_1, z_2) = (z_1 - \langle z_1 \rangle)(\overline{z_2 - \langle z_2 \rangle}) = \langle z_1 \bar{z}_2 \rangle - \langle z_1 \rangle \langle \bar{z}_2 \rangle$$

This is so because $\text{cov}(z, z) = \langle |z|^2 \rangle - |\langle z \rangle|^2$ is real.

But we choose another convention. That is

$$\text{cov}(z_1, z_2) = \langle z_1 z_2 \rangle - \langle z_1 \rangle \langle z_2 \rangle.$$
8. We have $\text{bias}(\bar{z}_1) = \overline{\text{bias}(z_1)}$ as we should.

9. When ϕ is not a constant we have:

$$2T_\ell = b'_0 \langle e^{i\phi(z)} z^0 | p_\ell(z) \rangle + \dots + b'_{n-1} \langle e^{i\phi(z)} z^{n-1} | p_\ell(z) \rangle + ||b_n|| \langle e^{i\phi(z)} z^n | p_\ell(z) \rangle$$

with $b'_j e^{i\phi(z)} = b_j$.

10. I. V. Uspensky, Theory of Equations (McGraw-Hill, 1948).
11. Martin et al., Nucl. Phys. B89, 253 (1974).

FIGURE CAPTIONS

- Fig. 1. Example of two random variables x and y linked by a functional dependence. $y = g(x)$ has two roots, x_1, x_2 in this case.
- Fig. 2. Illustration of the origin of the biases. The upper part shows the pdf of the two factors S and D (see Section III.1). S is centered and maximum for $z_2 = \langle z_2 \rangle$, the product SD is shown in the lower part and displays a bias.
- Fig. 3. (a) Computer simulation of 100 experiments whose scattering amplitude has two zeros at $-0.5 + 2i$ and $0.5 + 2i$, represented by open circles. On this case we check the validity of our approximate formulae for the confidence domains (rectangles). See Section IV.4.
- (b) Same as for 3a, except that the roots are $-0.2 + 2i$ and $0.2 + 2i$. We begin getting into troubles with our approximate confidence domains (rectangles). No longer 80% of the dots (the estimated zeros) stands inside and the bias is clearly visible by the depopulation of the region between the actual roots.
- (c) Same as for (a) and (b) but we have a double root at $2i$. There is, clearly, a big hole where the double root is located (open circle).
- Fig. 4. In the complex cost plane, the actual zero z_1 and the estimated zero. The bias tends to push apart z_1 from $\bar{z}_1$.
- Fig. 5. Approximate confidence domains for the zero z_1 .
- $$\sigma_1^2 = 2 \text{ cov}(x_1, x_1), \quad \sigma_2^2 = 2 \text{ cov}(y_1, y_1),$$
- $$r = 2 \text{ cov}(x_1, y_1) / \sigma_1 \sigma_2.$$

Inside such an ellipse and with our definitions of σ_1, σ_2 we expect a 70% confidence level.

- Fig. 6. Bias of a single root in the w plane. Again the bias tends to push w_1 apart from its mirror image $w_1^M = \frac{1}{\bar{w}_1}$.
- Fig. 7. The approximations of the confidence domains match very well the simulation of the experimental data of $\pi^+ p$ elastic scattering at $p_{lab} = 1439 \text{ MeV/c}$.¹¹ As it is predicted by the rigorous calculus, when the zeros are close to the physical region, we cannot trust our approximations any more.

Fig. 1

XBL 764-2628

Fig. 2 XBL 764-2627

00404501902

W - plane

Fig. 3a

XBL 764-2623

w - plane

XBL 764-2626

Fig. 3b

00004501903

W-plane

-51-

XBL 764-2625

Fig. 3c

Fig. 4

XBL 764-2629

Fig. 5

XBL 764-2622

XBL 764-2621

Fig. 6

0 0 0 0 4 5 0 1 9 0 5

w plane

XBL 764-2624

Fig. 7

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