Title
Gorenstein Liaison and ACM Sheaves

Authors
Casanellas, Marta
Drozd, Elena
Hartshorne, Robin

Publication Date
2003-07-29

Peer reviewed

Gorenstein Liaison and ACM Sheaves

Marta Casanellas
Elena Drozd
Robin Hartshorne

Abstract

We study Gorenstein liaison of codimension two subschemes of an arithmetically Gorenstein scheme X. Our main result is a criterion for two such subschemes to be in the same Gorenstein liaison class, in terms of the category of ACM sheaves on X. As a consequence we obtain a criterion for X to have the property that every codimension 2 arithmetically Cohen-Macaulay subscheme is in the Gorenstein liaison class of a complete intersection. Using these tools we prove that every arithmetically Gorenstein subscheme of $\mathbb{P}^n$ is in the Gorenstein liaison class of a complete intesection and we are able to characterize the Gorenstein liaison classes of curves on a nonsingular quadric threefold in $\mathbb{P}^4$.

Keywords: linkage, liaison, biliaison, Gorenstein scheme, maximal Cohen–Macaulay modules

0 Introduction

Liaison using complete intersection schemes has been widely studied since the seminal paper of Peskine and Szpiro [19]. More recently the notion of liaison by arithmetically Gorenstein schemes, introduced by Schenzel [22], has assumed a prominent role in the study of subschemes of codimension at least three in $\mathbb{P}^n$ [13]. The growing literature in this subject is surveyed in the book [15] and the recent article [16] of Migliore and Nagel.

While most of the work on liaison has focused on subschemes of projective space $\mathbb{P}^n$, there are some results on other ambient varieties. Rao remarks already in [21] that the so-called "Rao correspondence" (see (2.16) below) holds on an arithmetically Gorenstein scheme, and Bolondi and Migliore [1] establish the Lazarsfeld–Rao property on a smooth arithmetically Gorenstein scheme. Both of these results use complete intersection linkage in codimension two. Nagel [17] obtains similar results from a slightly different point of view. The recent paper [5] puts the Rao correspondence and the Lazarsfeld–Rao property in its most general

form to date: in particular these results hold for complete intersection biliaison on an integral arithmetically Cohen–Macaulay scheme.

The motivation for the current work came from trying to find a context more general than the well-known case of curves in $\mathbb{P}^3$, yet more special than the problem of Gorenstein liaison for curves in $\mathbb{P}^4$, which is still wide open [9]. So we decided to study curves in a nonsingular quadric hypersurface in $\mathbb{P}^4$. This project has expanded considerably since its inception, and has led to the thesis of the second author [4], the paper on Gorenstein biliaison of the first and third author [3], and the present article.

While the theory of biliaison can be satisfactorily developed on arithmetically Cohen–Macaulay (ACM) schemes [5], [3], the study of single liaisons is more related to questions of duality, and requires Gorenstein hypotheses on the ambient space. So in this paper we study Gorenstein liaison of codimension two subschemes of an arithmetically Gorenstein (AG) scheme X. (See §1 for precise definitions). Our main result (5.1) is a criterion for two such subschemes to be in the same G-liaison equivalence class, in terms of the category of arithmetically Cohen–Macaulay (ACM) sheaves on X (see Definition (2.6)). As a consequence we obtain a criterion for X to have the property that every codimension two ACM subscheme is in the Gorenstein liaison class of a complete intersection (glicci). In particular, this applies to the nonsingular quadric hypersurface in $\mathbb{P}^4$ (6.1) and solves the problem we started with.

As a byproduct of the tools we develop for handling Gorenstein liaison, we are able to prove that every arithmetically Gorenstein subscheme of $\mathbb{P}^n$ is glicci (7.1), answering a natural question about Gorenstein liaison [13, p. 11], [2].

In §1 we recall the definition and basic properties of liaison. In §2 we adapt the $\mathcal{E}$- and $\mathcal{N}$-type resolutions used in [14] and elsewhere to our situation, and recall Rao's theorem. §3 studies the behavior of these resolutions under G-liaison, and contains our main technical result (3.4), which allows one to simplify an $\mathcal{N}$-type resolution by a G-liaison. §4 studies syzygies of ACM sheaves and introduces the notion of a double-layered sheaf that is used in stating our main theorem. §5 has the main theorem giving a criterion for G-liaison equivalence. In the case of an integral 3-dimensional AG scheme it also gives a criterion for the even G-liaison class to be determined by the Rao module, as is the case in $\mathbb{P}^3$ [20]. Then §6 has the applications to quadric hypersurfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$, and §7 has the theorem that AG schemes in $\mathbb{P}^n$ are glicci.

1 Liaison

Liaison theory has its roots in the late nineteenth century. The modern theory of liaison begins with the paper of Peskine and Szpiro [19], has been developed by [20], [21], [22], [14], [18], [17], [13] and many others. A good summary is in the book of Migliore [15]. For convenience we recall the basic definitions and results that we will use in this paper. A scheme

is *locally Cohen–Macaulay* (CM) if all of its local rings are Cohen–Macaulay rings. A closed subscheme X of $\mathbb{P}^n$ is *arithmetically Cohen–Macaulay* (ACM) if its homogeneous coordinate ring $S(X) = k[x_0, \ldots, x_n]/I_X$ is a Cohen–Macaulay ring. A scheme is *locally Gorenstein* if its local rings are all Gorenstein rings. A closed subscheme X of $\mathbb{P}^n$ is *arithmetically Gorenstein* if its homogeneous coordinate ring $S(X)$ is a Gorenstein ring.

Definition 1.1. Let V_1, V_2, Y be subschemes of $\mathbb{P}^n_k$, all equidimensional without embedded components, and of the same dimension. We say V_1 *is linked to* V_2 *by* Y, in symbols $V_1 \underset{Y}{\sim} V_2$, if

1) V_1, V_2 are contained in Y and

2) $\mathcal{I}_{V_2,Y} \cong \mathcal{H}om(\mathcal{O}_{V_1}, \mathcal{O}_Y)$ and $\mathcal{I}_{V_1,Y} \cong \mathcal{H}om(\mathcal{O}_{V_2}, \mathcal{O}_Y)$.

Note the relation of linkage is symmetric by definition. The following proposition gives the existence and some properties of the linked scheme.

Proposition 1.2. *Let $V_1 \subseteq Y$ be equidimensional subschemes of $\mathbb{P}^n$, of the same dimension, without embedded components, and assume that Y is locally Gorenstein. Define a subscheme V_2 of Y by setting $\mathcal{I}_{V_2,Y} = \mathcal{H}om(\mathcal{O}_{V_1}, \mathcal{O}_Y)$ with its natural embedding in $\mathcal{O}_Y$. Then*

a) *V_2 is also equidimensional of the same dimension without embedded components (unless $V_1 = Y$ in which case V_2 is empty) and V_1 and V_2 are linked by Y.*

b) *V_1 is locally* CM *if and only if V_2 is locally* CM.

c) *If Y is* AG*, then V_1 is* ACM *if and only if V_2 is* ACM.

Proof. See [22] or [15, §5.2]. The hypothesis Y locally Gorenstein is essential, to make the sheaves $\mathcal{I}_{V_1,Y}, \mathcal{I}_{V_2,Y}$ be reflexive sheaves on Y.

Definition 1.3. If V_1 and V_2 are linked by Y and Y is a complete intersection in $\mathbb{P}^n$, we speak of a CI-*linkage*. The equivalence relation generated by CI-linkages is called CI-*liaison*. If a CI-liaison is accomplished by an even number of CI-linkages, we speak of *even* CI-*liaison*.

If Y is an arithmetically Gorenstein (AG) scheme in $\mathbb{P}^n$, we speak analogously of *G-linkage*, *G-liaison* and *even G-liaison*.

If the linkage $V_1 \underset{Y}{\sim} V_2$ takes place among subschemes of some projective scheme X in $\mathbb{P}^n$, we speak of *linkage in* X.

Assume now that X is an AG scheme in X. A *complete intersection in* X is an intersection $Y = H_1 \cap \cdots \cap H_r$ of divisors H_i in X corresponding to invertible sheaves $\mathcal{O}_X(a_i)$, $i = 1, \ldots, r$, of codimension r in X. Such a scheme Y is AG in $\mathbb{P}^n$, so we can use it for liaison in X. In

this case, we speak of CI-*linkage*, CI-*liaison*, *even* CI-*liaison in* X. Note that unless X itself is a complete intersection in $\mathbb{P}^n$, these linkages may not be CI-linkages in $\mathbb{P}^n$.

If $Y \subseteq X$ is AG in $\mathbb{P}^n$, then we speak of G-*linkage*, G-*liaison* and *even* G-*liaison in* X. These are also G-linkages in $\mathbb{P}^n$, by definition.

Definition 1.4. We recall the notion of biliaison from [7]. If V_1 and V_2 are divisors on an ACM scheme S of one higher dimension, and there is a linear equivalence $V_2 \sim V_1 + mH$ as divisors on S, where H is the hyperplane section, for some $m \in \mathbb{Z}$, we say V_2 is obtained by an *elementary biliaison* from V_1. If S is an ACM scheme satisfying G_0, we speak of a G-*biliaison*. If S is a complete intersection scheme in $\mathbb{P}^n$ (resp. complete intersection in X), then we speak of CI-*biliaison* (resp. CI-*biliaison in* X). The corresponding equivalence relation generated by the elementary biliaisons is called *biliaison*, G-*biliaison*, CI-*biliaison*, or CI-*biliaison in* X.

Proposition 1.5. a) CI-*biliaison in* $\mathbb{P}^n$ *is the same equivalence relation as even* CI-*liaison.*

b) *If* X *is an* AG *scheme in* $\mathbb{P}^n$*, then* CI-*biliaison in* X *is the same equivalence relation as even* CI-*liaison in* X*.*

c) *Any* G-*biliaison is an even* G-*liaison.*

Proofs. a) [6, 4.4].

b) Use the same proof as [6, 4.4], replacing $\mathbb{P}^n$ by X.

c) [7, 3.3].

Remark 1.6. It is unknown whether G-biliaison is the same as even G-liaison in $\mathbb{P}^n$, but there are examples of non-singular AG schemes X on which G-biliaison and even G-liaison of codimension 2 subschemes are not equivalent [3, 1.1].

Remark 1.7. Note that our definition of G-liaison in X is not the same as the one used by Nagel [17]. He requires a Gorenstein ideal to be perfect, with the result that on an AG scheme X, his G-liaison for codimension two subschemes on X reduces to our CI-liaison on X.

2 Resolutions of codimension two subschemes

The $\mathcal{E}$- and $\mathcal{N}$-type resolutions and their behavior under liaison were introduced in [14] for curves in $\mathbb{P}^3$. They have been generalized in [1], [18], [17], [11], [5] to higher dimensions and other ambient schemes. The two types of resolutions can be defined separately with minimal hypotheses on arbitrary schemes, and Rao's theorem can be proved for the $\mathcal{N}$-type resolution on a projective ACM scheme [5], but the relation between the two types of resolution under liaison appears clearly only for Gorenstein schemes. We review for convenience the definitions and results we need in this paper.

Hypotheses 2.1. Throughout this section we let X denote a projective equidimensional scheme with a fixed very ample invertible sheaf $\mathcal{O}_X(1)$. A coherent sheaf $\mathcal{L}$ on X is *dissocié* if it is isomorphic to a direct sum $\oplus\mathcal{O}_X(a_i)$ for various $a_i \in \mathbb{Z}$. We denote by C a closed subscheme of X of codimension two, with no embedded components, and let $\mathcal{I}_C = \mathcal{I}_{C,X}$ be the ideal sheaf of C in X. For any coherent sheaf $\mathcal{F}$ on X, we write $H^i_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}} H^i(X, \mathcal{F}(n))$.

Definition 2.2. An $\mathcal{E}$*-type resolution* of $\mathcal{I}_C$ is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{I}_C \to 0$$

of coherent sheaves on X, with $\mathcal{L}$ dissocié and $H^1_*(\mathcal{E}) = 0$.

Proposition 2.3. *With X, C as in* (2.1) *assuming furthermore $H^1_*(\mathcal{O}_X) = 0$, an $\mathcal{E}$-type resolution of C exists.*

Proof. Just let I_C be the homogeneous ideal of C in $S(X)$, the projective coordinate ring of X, let L be a free graded $S(X)$ module mapping onto I_C, and let E be the kernel: $0 \to E \to L \to I_C \to 0$. Sheafifying gives the required $\mathcal{E}$-type resolution. Note that $H^1_*(\mathcal{E}) = 0$ because $H^0_*(\mathcal{L}) \to H^0_*(\mathcal{I}_C)$ is surjective, being $L \to I_C$ by construction and $H^1_*(\mathcal{L}) = 0$ since $\mathcal{L}$ is dissocié and $H^1_*(\mathcal{O}_X) = 0$ by hypothesis.

Definition 2.4. An $\mathcal{N}$*-type resolution* of C is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

with $\mathcal{L}$ dissocié and $\mathcal{N}$ coherent satisfying $H^1_*(\mathcal{N}^\vee) = 0$ and $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X) = 0$. (This property of $\mathcal{N}$ was called "extraverti" in [11] and [5].)

Proposition 2.5. *With X, C as in* (2.1)*, if X satisfies Serre's condition S_2 and $H^1_*(\mathcal{O}_X) = 0$, then an $\mathcal{N}$-type resolution of C exists.*

Proof. [5, 1.12].

Definition 2.6. A coherent sheaf $\mathcal{E}$ on a scheme X is *locally Cohen–Macaulay* (locally CM) if for every point $x \in X$, depth $\mathcal{E}_x = \dim \mathcal{O}_x$. If X is an ACM scheme, we say $\mathcal{E}$ is an ACM *sheaf* if in addition $H^i_*(\mathcal{E}) = 0$ for $0 < i < \dim X$ (cf. [3]).

Proposition 2.7. *Suppose that X is an* AG *scheme, and C a codimension two subscheme with $\mathcal{E}$- and $\mathcal{N}$-type resolutions as defined above. Then*

a) *C is locally* CM *if and only if $\mathcal{E}$ (resp. $\mathcal{N}$) is locally* CM.

b) *C is* ACM *if and only if $\mathcal{E}$ (resp. $\mathcal{N}$) is* ACM.

Proof. a) Since X is AG, it is locally CM, and so the dissocié sheaves are locally CM. Now C locally CM equivalent to $\mathcal{E}$ locally CM follows by a chase of depth across the exact sequence $0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and the defining sequence for $\mathcal{E}$. The implication $\mathcal{N}$ locally CM $\Rightarrow C$ locally CM is similar. In the other direction, if C is locally CM, we obtain depth $\mathcal{N} \geq n-1$ at each closed point of X, where $n = \dim X$. It remains to show $H_x^{n-1}(\mathcal{N}) = 0$ for each closed point. Since X is locally Gorenstein, this follows by local duality and the hypothesis $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}) = 0$.

b) By part a) we may assume C locally CM and $\mathcal{E}$ and $\mathcal{N}$ also locally CM. Then C is ACM if and only if $H_*^i(\mathcal{I}_C) = 0$ for $0 < i \leq \dim C = n-2$. Chasing the cohomology sequences, this is equivalent to $H_*^i(\mathcal{E}) = 0$ for $2 \leq i < n$ and to $H_*^i(\mathcal{N}) = 0$ for $1 \leq i < n-1$. The missing requirements are $H_*^1(\mathcal{E}) = 0$, which is in the definition of $\mathcal{E}$-type resolution, and $H_*^{n-1}(\mathcal{N}) = 0$, which by Serre duality on X (see (2.8) below) is equivalent to $H_*^1(\mathcal{N}^\vee) = 0$, in the definition of $\mathcal{N}$-type resolution.

Proposition 2.8. *Let X be an* AG *scheme. Then*

a) *Every locally* CM *sheaf $\mathcal{E}$ is reflexive.*

b) *The functor $\mathcal{E} \to \mathcal{E}^\vee$ is an exact contravariant functor on the category of locally* CM *sheaves.*

c) *Serre duality for $\mathcal{E}$ locally* CM *says $H_*^i(\mathcal{E}^\vee)$ dual to $H_*^{n-1}(\mathcal{E})$.*

d) *$\mathcal{E}$ is* ACM *if and only if $\mathcal{E}^\vee$ is* ACM.

Proof. [3, 2.3] noting that on an AG scheme, $\mathrm{Hom}(\mathcal{E}, \omega)$ is a twist of $\mathcal{E}^\vee$.

Proposition 2.9. *Let X be an* AG *scheme, and C a locally* CM *subscheme of codimension* 2.

a) *If* $\dim X \geq 3$, *C is* subcanonical *(i.e., the canonical sheaf ω_C is isomorphic to $\mathcal{O}_C(\ell)$ for some $\ell \in \mathbb{Z}$) if and only if it has an $\mathcal{N}$-type resolution with $\mathcal{N}$ of rank* 2.

b) *If* $\dim X \geq 2$, *C is* AG *if and only if it has an $\mathcal{N}$-type resolution with $\mathcal{N}$* ACM *of rank* 2.

Proof. a) This is the usual "Serre correspondence" and the usual proof (see, e.g., [8, 1.1]) works also in our case. If C has a resolution

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

with $\mathcal{N}$ of rank 2, then $\mathcal{L}$ has rank 1, so $\mathcal{L} = \mathcal{O}_X(-a)$ for some $a \in \mathbb{Z}$. Taking $\mathcal{H}om(\cdot, \mathcal{O}_X)$ we get

$$0 \to \mathcal{O}_X \to \mathcal{N}^\vee \to \mathcal{O}_X(a) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{O}_X) \to 0.$$

But $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_C) \cong \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_X) \cong \omega_C(-\ell)$, where ℓ is such that $\omega_X = \mathcal{O}_X(\ell)$, using the hypothesis X is AG. Hence $\omega_C(-\ell) \cong \mathcal{O}_C(a)$ and C is subcanonical.

Conversely, if C is subcanonical, reasoning backwards, $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X) \cong \mathcal{O}_C(a)$ for some a. Taking the global section $1 \in \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-a)) = H^0(\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-a)))$ gives an extension

$$0 \to \mathcal{O}_X(-a) \to \mathcal{N} \to \mathcal{I}_C \to 0.$$

It follows, as in the proof of [5, 1.11] that $\mathcal{N}$ satisfies $H^1_*(\mathcal{N}^\vee) = 0$ and $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}) = 0$, so this is an $\mathcal{N}$-type resolution with $\mathcal{N}$ of rank 2.

Note we used the hypothesis $\dim X \geq 3$ for the isomorphism $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_X)$.

b) We apply the argument of part a) to the homogeneous coordinate ring $S(X)$, which has dimension ≥ 3 and to its quotient $S(C)$, and use the fact that a graded ring is Gorenstein if and only if it is CM and the canonical module is free.

To illustrate the previous proposition, we study when a codimension two AG scheme C occurs as a divisor $mH - K$ on an ACM scheme Y that is a divisor on X. To put this in a more general context, recall that if Y is an ACM scheme satisfying G_0, we can define the *anticanonical divisor* $M = M_Y$, given by an embedding of ω_Y as a fractional ideal in the sheaf of total quotient rings $\mathcal{K}_Y$, even if Y does not have a well-defined canonical divisor [7, 2.7]. Recall also that if Y is an ACM scheme in $\mathbb{P}^n$ satisfying G_0, and if C is an effective divisor on Y, linearly equivalent to $mH + M_Y$, for some $m \in \mathbb{Z}$, then C is AG in $\mathbb{P}^n$ [7, 3.4], [13, 5.2,5.4].

Proposition 2.10. *Let X be an* AG *scheme, and let C be an* AG *subscheme of codimension 2 in X. Then the following conditions are equivalent:*

(i) *There is an* ACM *divisor $Y \subseteq X$ satisfying G_0 and containing C and an integer m so that $C \sim mH + M_Y$ on Y, where M_Y is the anticanonical divisor.*

(ii) *C has an $\mathcal{N}$-type resolution with $\mathcal{N}$ an* ACM *sheaf of rank 2 that is an extension of two rank 1* ACM *sheaves on X.*

Proof. (i) $\Rightarrow$ (ii). Since $C \sim mH + M_Y$ on Y, we have $\mathcal{I}_{C,Y} \cong \omega_Y(-m)$. On the other hand, comparing with the ideal sheaf $\mathcal{I}_C$ of C on X, we have an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{I}_C \to \mathcal{I}_{C,Y} \to 0. \tag{1}$$

We combine this with the natural exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(Y) \to \omega_Y \otimes \omega_X^\vee \to 0 \tag{2}$$

of [6, 2.10]. (Here we write $\mathcal{O}(Y)$ for the notation $\mathcal{L}(Y)$ of [6].) Since X is an AG scheme, we can write $\omega_X \cong \mathcal{O}_X(\ell)$ for some $\ell \in \mathbb{Z}$. Twisting by $a = \ell - m$ we get

$$0 \to \mathcal{O}_X(a) \to \mathcal{O}(Y + a) \to \omega_Y(-m) \to 0. \tag{3}$$

Since $\omega_Y(-m) \cong \mathcal{I}_{C,Y}$, we can do the fibered sum construction with the sequence (1) above and obtain two short exact sequences

$$0 \to \mathcal{O}_X(a) \to \mathcal{N} \to \mathcal{I}_C \to 0, \tag{4}$$

$$0 \to \mathcal{I}_Y \to \mathcal{N} \to \mathcal{O}(Y + a) \to 0. \tag{5}$$

The first is an $\mathcal{N}$-type resolution of $\mathcal{I}_C$, and the second shows that $\mathcal{N}$ is an extension of two rank 1 ACM sheaves on X.

(ii) $\Rightarrow$ (i). Conversely, suppose given an $\mathcal{N}$-type resolution of the form (4) above, and suppose that $\mathcal{N}$ is an extension

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{M} \to 0 \tag{6}$$

where $\mathcal{L}, \mathcal{M}$ are rank 1 ACM sheaves on X. The composed map $\mathcal{L} \to \mathcal{I}_C \to \mathcal{O}_X$ shows that $\mathcal{L}$ is isomorphic to the ideal sheaf $\mathcal{I}_Y$ of an ACM divisor Y containing C. Then by comparing Chern classes we find $\mathcal{M} \cong \mathcal{O}(Y + a)$. Dividing the sequence (4) by $\mathcal{I}_Y$ in the second and third place we obtain

$$0 \to \mathcal{O}_X(a) \to \mathcal{O}(Y + a) \to \mathcal{I}_{C,Y} \to 0. \tag{7}$$

Comparing with the sequence (2) above, we conclude that $\mathcal{I}_{C,Y} \cong \omega_Y(a - \ell) = \omega_Y(-m)$. Therefore $C \sim mH + M_Y$ on Y. Note from the isomorphism $\mathcal{I}_{C,Y} \cong \omega_Y(a - \ell)$ that ω_Y is locally free at the generic points of Y, so Y satisfies G_0.

Remark 2.11. We may ask, when does the extension (5) for $\mathcal{N}$ in the above proof split? It splits if and only if there is an effective divisor $Z \sim -Y - aH$ such that C is the scheme-theoretic intersection $Y \cap Z$. Indeed, if $\mathcal{N}$ splits, then the sequence

$$0 \to \mathcal{O}_X(a) \to \mathcal{O}(-Y) \oplus \mathcal{O}(Y + a) \to \mathcal{I}_C \to 0 \tag{8}$$

gives a map $\mathcal{O}_X \to \mathcal{O}(-Y - a)$ defining an effective divisor Z, and then the sequence (8) shows that $C = Y \cap Z$. Conversely, if there is such a Z, then the sequence (8) gives an $\mathcal{N}$-type resolution with $\mathcal{N}$ split.

Example 2.12. a) Let P be a point on a nonsingular cubic surface X in $\mathbb{P}^3$. Then P has an $\mathcal{N}$-type resolution on X

$$0 \to \mathcal{O}_X \to \mathcal{N} \to \mathcal{I}_P \to 0.$$

One can show that for a general point $P \in X$, the sheaf $\mathcal{N}$ is not an extension of ACM line bundles. However, if P lies on a line L, then $P \sim -H - K_L$ on L, and $\mathcal{N}$ is an extension

$$0 \to \mathcal{O}(-L) \to \mathcal{N} \to \mathcal{O}(L) \to 0.$$

But $-L$ is not effective, so the extension does not split.

b) Now take two points Q, R on the cubic surface X. They have an $\mathcal{N}$-type resolution

$$0 \to \mathcal{O}_X(-1) \to \mathcal{N} \to \mathcal{I}_{Q+R} \to 0.$$

Two general points Q, R are contained in a twisted cubic curve Y, and then $Q + R \sim -K_Y$. Therefore $\mathcal{N}$ is an extension

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{N} \to \mathcal{O}_X(Y-1) \to 0.$$

But $H - Y$ is not effective, so this extension does not split.

On the other hand, if Q, R lie on a conic Y, we get a similar sequence using the conic Y. In this case $H - Y$ is effective, equal to a line L. Still, for general $Q, R \in Y$, the sequence for $\mathcal{N}$ does not split. Only when $\{Q, R\} = Y \cap L$ does the sequence split.

Proposition 2.13 (Behavior under liaison). *Suppose that X is* AG*, and that C has a resolution of the form*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{I}_C \to 0$$

with $\mathcal{A}$ and B locally CM *sheaves on X and $H^1_*(\mathcal{A}) = H^1_*(\mathcal{B}^\vee) = 0$. (For example, either an $\mathcal{E}$- or an $\mathcal{N}$-type resolution.) Let Y be a codimension two complete intersection in X, containing C, with resolution*

$$0 \to \mathcal{O}(-a-b) \to \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \mathcal{I}_Y \to 0.$$

Let C' be the curve linked to C by Y (1.2). Then C' has a resolution

$$0 \to \mathcal{B}^\vee(-a-6) \to \mathcal{A}^\vee(-a-b) \oplus \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \mathcal{I}_{C'} \to 0.$$

Proof. This is the usual mapping cone construction (see, e.g., [14, p. 60] or [15, 5.1.20]). Since $C \subseteq Y$ we have $\mathcal{I}_Y \subseteq \mathcal{I}_C$. Because of the hypothesis $H^1_*(\mathcal{A}) = 0$, the defining sections of $\mathcal{I}_Y$ lift to $\mathcal{B}$ and we get a diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & \mathcal{I}_{C,Y} & & \\
 & & & & & & \uparrow & & \\
0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \mathcal{I}_C & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(-a-b) & \to & \mathcal{O}(-a) \oplus \mathcal{O}(-b) & \to & \mathcal{I}_Y & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

The mapping cone gives

$$0 \to \mathcal{O}(-a-b) \to \mathcal{A} \oplus \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \mathcal{B} \to \mathcal{I}_{C,Y} \to 0.$$

Now $\mathcal{I}_{C,Y} \cong \mathcal{H}om(\mathcal{O}_{C'}, \mathcal{O}_Y)$ by definition of linkage. Since X and Y are AG, this sheaf is a twist $\omega_{C'}(c)$ for some $c \in \mathbb{Z}$. We apply the functor $\mathcal{H}om(\cdot, \mathcal{O}_X)$ to the sequence above. Since $\mathcal{A}$ and $\mathcal{B}$ are locally CM and X is locally Gorenstein, these sheaves are acyclic for $\mathcal{E}xt^i_{\mathcal{O}_X}(\cdot, \mathcal{O}_X)$. Thus we obtain

$$0 \to \mathcal{B}^\vee \to \mathcal{A}^\vee \oplus \mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathcal{O}(a+b) \to \mathcal{E}xt^2_{\mathcal{O}_X}(\omega_{C'}(c), \mathcal{O}_X) \to 0.$$

But $\mathcal{E}xt^2_{\mathcal{O}_X}(\cdot, \mathcal{O}_X)$ is a dualizing functor for sheaves on Y, so we get $\mathcal{O}_{C'}(a+b)$ on the right. Splitting the sequence and twisting by $-a-b$ gives the result.

Corollary 2.14. *If X is* AG *and C, C' are linked by the complete intersection Y in X, then an $\mathcal{E}$-type (resp. $\mathcal{N}$-type) resolution for C gives an $\mathcal{N}$-type (resp. $\mathcal{E}$-type) resolution for C' by* (2.13).

Proof. Since X is AG, the condition $H^1_*(\mathcal{A}) = 0$ and $H^1_*(\mathcal{B}^\vee) = 0$ give $H^1_*(\mathcal{B}^\vee)$ and $H^1_*(\mathcal{A}^{\vee\vee}) = 0$. The last conclusion $\mathcal{E}xt^1(\mathcal{A}^\vee, \mathcal{O}) = 0$ follows by local duality from the fact that $\mathcal{A}$ and hence $\mathcal{A}^\vee$ (2.8) are locally CM sheaves.

Definition 2.15. We say sheaves $\mathcal{F}$ and $\mathcal{G}$ on X are *stably equivalent* if there exist dissocié sheaves $\mathcal{L}, \mathcal{M}$ such that $\mathcal{F} \oplus \mathcal{L} \cong \mathcal{G} \oplus \mathcal{M}$. We say $\mathcal{F}$ is *orientable* if $\mathcal{F}$ is locally free of constant rank r in codimension 1, and there exists a closed subset $Z \subseteq X$ of codimension ≥ 2 and an integer ℓ such that $\Lambda^r \mathcal{F} \cong \mathcal{O}_X(\ell)$ on $X - Z$.

Theorem 2.16 (Rao). *Let X be an integral* AG *scheme of dimension ≥ 2. Then the following sets are in one-to-one correspondence:*

(i) *Even* CI*-liaison classes in X of codimension two locally* CM *subschemes $C \subseteq X$.*

(ii) *Orientable locally* CM *sheaves $\mathcal{E}$ with $H^1_*(\mathcal{E}) = 0$ on X, up to stable equivalence and shift.*

(iii) *Orientable locally* CM *sheaves $\mathcal{N}$ with $H^1_*(\mathcal{N}^\vee) = 0$ on X, up to stable equivalence and shift.*

The bijections are accomplished by sending each C in (i) to the sheaf $\mathcal{E}$ in an $\mathcal{E}$-type resolution for (ii), or to the sheaf $\mathcal{N}$ in an $\mathcal{N}$-type resolution for (iii).

Proof. [21], [18], [17], [15, 6.2.5], [5]. The equivalence of (i) and (iii) by sending C to its $\mathcal{N}$-type resolution is proved under more general hypotheses in [5, 2.4]. Note that each psi-equivalence class mentioned there contains extraverti sheaves unique up to stable equivalence [5, 1.12], and that these are the sheaves that appear in an $\mathcal{N}$-type resolution.

The bijection of (i) and (ii) follows by using a single CI-liaison to relate a subscheme C to another subscheme C', and the fact that $\mathcal{E}$- and $\mathcal{N}$-type resolutions are interchanged by CI-liaison and duality (2.13).

Remark 2.17. It follows from the theorem of course that the sets (ii) and (iii) are in bijective correspondence. One bijection between these two sets is given by the functor $\mathcal{F} \mapsto \mathcal{F}^\vee$, but that is not the bijection coming from this theorem. A consequence of (4.2) below is that the bijection in this theorem sends $\mathcal{N} \in$ (iii) to $\mathcal{N}^\sigma \in$ (ii) and $\mathcal{E} \in$ (ii) to $\mathcal{E}^{\vee\sigma\vee} \in$ (iii), up to stable equivalence. (See §3 below for the definition of $\mathcal{N}^\sigma$.)

3 Behavior under Gorenstein liaison

We consider an AG scheme X and its locally CM codimension two subschemes. In this section we investigate how the $\mathcal{E}$- and $\mathcal{N}$-type resolutions behave under a Gorenstein liaison. The mapping cone construction (as in (2.13)) does not work for an $\mathcal{E}$-type resolution. However, given an $\mathcal{N}$-type resolution of C and a G-liaison by an AG scheme Y to another scheme C', we can obtain an $\mathcal{N}$-type resolution of C' with a more complicated sheaf in the middle. On the other hand, we prove a key result about how to simplify an $\mathcal{N}$-type resolution by G-liaison when $\mathcal{N}$ contains a rank 2 ACM factor.

Definition 3.1. For any locally CM sheaf $\mathcal{F}$ on the AG scheme X we define the *syzygy sheaf* $\mathcal{F}^\sigma$ to be the sheafification of the kernel of a minimal free presentation of $F = H^0_*(\mathcal{F})$ over $S(X)$. In other words, let $L \to F \to 0$ be a minimal free cover, and sheafify to get

$$0 \to \mathcal{F}^\sigma \to \mathcal{L} \to \mathcal{F} \to 0$$

with $\mathcal{L}$ dissocié and $H^0_*(\mathcal{L}) \to H^0_*(\mathcal{F})$ surjective. Then $\mathcal{F}^\sigma$ is also locally CM, and $H^1_*(\mathcal{F}^\sigma) = 0$.

Proposition 3.2. *Let X be an* AG *scheme, and let C be a locally* CM *codimension two subscheme with $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0,$$

where $\mathcal{N}$ is locally CM *with $H^1_*(\mathcal{N}^\vee) = 0$ (2.7). Let Y be a codimension* 2 AG *subscheme containing C, with resolution*

$$0 \to \mathcal{O}_X(-a) \to \mathcal{E} \to \mathcal{I}_Y \to 0$$

where $\mathcal{E}$ is a rank 2 ACM *sheaf (cf.* (2.9)*). Then the subscheme C' linked to C by Y has an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{M}^\vee \to \mathcal{G} \to \mathcal{I}_{C'}(a) \to 0$$

with $\mathcal{M}$ dissocié, and where $\mathcal{G}$ is an extension

$$0 \to \mathcal{E}^\vee \oplus \mathcal{L}^\vee \to \mathcal{G} \to \mathcal{N}^{\sigma\vee} \to 0.$$

Proof. We repeat the cone construction as in (2.13). We have $\mathcal{I}_Y \subseteq \mathcal{I}_C$, and the induced map $\mathcal{E} \to \mathcal{I}_C$ lifts to $\mathcal{N}$, because the Ext^1 term in the sequence

$$\mathrm{Hom}(\mathcal{E}, \mathcal{N}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{I}_C) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{L})$$

is zero since $\mathcal{L}$ is dissocié and $\mathcal{E}$ is ACM (2.8). So the mapping cone exists, and dualizing the resulting sequence as before, we get

$$0 \to \mathcal{N}^\vee \to \mathcal{E}^\vee \oplus \mathcal{L}^\vee \to \mathcal{I}_{C'}(a) \to 0.$$

This is neither an $\mathcal{N}$-type nor an $\mathcal{E}$-type resolution.

But now consider the syzygy sequence for $\mathcal{N}$, where $\mathcal{M}$ is a minimal dissocié cover:

$$0 \to \mathcal{N}^\sigma \to \mathcal{M} \to \mathcal{N} \to 0.$$

Dualizing this gives (2.8)

$$0 \to \mathcal{N}^\vee \to \mathcal{M}^\vee \to \mathcal{N}^{\sigma\vee} \to 0.$$

Let $\mathcal{G}$ be the fibered sum of $\mathcal{E}^\vee \oplus \mathcal{L}^\vee$ and $\mathcal{M}^\vee$ over $\mathcal{N}^\vee$. Then we get exact sequences

$$0 \to \mathcal{M}^\vee \to \mathcal{G} \to \mathcal{I}_{C'}(a) \to 0$$

and

$$0 \to \mathcal{E}^\vee \oplus \mathcal{L}^\vee \to \mathcal{G} \to \mathcal{N}^{\sigma\vee} \to 0.$$

Note that $H^1_*(\mathcal{N}^\sigma) = 0$ by construction, and $H^1_*(\mathcal{E}) = 0$ since $\mathcal{E}$ is ACM. It follows that $H^1_*(\mathcal{G}^\vee) = 0$, so that we do indeed get the required $\mathcal{N}$-type resolution.

Remark 3.3. Thus we see that in general, performing a G-liaison makes the sheaf appearing in the $\mathcal{N}$-type resolution more complex, by taking a syzygy dual and an extension by a rank 2 ACM sheaf. Our next key result shows that we can also reverse this process.

Proposition 3.4. *Let X be a normal* AG *scheme, and let C be a codimension two subscheme with an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0.$$

Suppose that $\mathcal{N}$ belongs to an exact sequence

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{N}' \to 0$$

where $\mathcal{E}$ is an orientable rank 2 ACM *sheaf on X, and $\mathcal{N}'$ is another locally* CM *sheaf of rank ≥ 2. Then there exists a subscheme D in the same even G-liaison class as C, with a resolution*

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_D(a) \to 0$$

for some $a \in \mathbb{Z}$. (If we assume furthermore $H^1_*(\mathcal{N}'^\vee) = 0$, this will be an $\mathcal{N}$-type resolution for D.)

Proof. Choose $b \gg 0$ so that $\mathcal{E}(b)$, $\mathcal{N}(b)$, and hence also $\mathcal{N}'(b)$, are generated by global sections. Let rank $\mathcal{N} = r+1$, so that rank $\mathcal{N}' = r-1$. Choose $r-2$ sufficiently general sections of $\mathcal{N}'(b)$ so that the cokernel is the twisted ideal sheaf of a codimension two subscheme D, giving

$$0 \to \mathcal{O}(-b)^{r-2} \to \mathcal{N}' \to \mathcal{I}_D(a) \to 0$$

for some $a \in \mathbb{Z}$. This is possible by repeated application of [5, 2.6], since $\mathcal{N}'(b)$ is generated by global sections.

It follows that $\mathcal{I}_D(a+b)$ is also generated by global sections. So we can choose two of those sections defining a complete intersection scheme Z in X, containing D. Let D' be the subscheme linked to D by Z. Forming the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(-b)^{r-2} & \to & \mathcal{N}' & \to & \mathcal{I}_D(a) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(-a-2b) & \to & \mathcal{O}(-b)^2 & \to & \mathcal{I}_Z(a) & \to & 0
\end{array}$$

as in (2.13), the cone construction gives a resolution

$$0 \to \mathcal{O}(-a-2b) \to \mathcal{O}(-b)^r \to \mathcal{N}' \to \mathcal{R} \to 0$$

where $\mathcal{R} = \mathcal{I}_{D,Z}(a)$. Because of the linkage by Z, $\mathcal{R}$ is also equal to $\mathcal{H}om(\mathcal{O}_{D'}, \mathcal{O}_Z)$, which is isomorphic to a twist $\omega_{D'}(d)$ for some d. We split this sequence in the middle by a sheaf $\mathcal{F}$, and write a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathcal{F} & \to & \mathcal{N}' & \to & \mathcal{R} & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}(-b)^r & \xrightarrow{\alpha} & \mathcal{N} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}(-a-2b) & \xrightarrow{\beta} & \mathcal{E} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

whose left column and top row come from the resolution of $\mathcal{R}$; the middle column is the given sequence with $\mathcal{N}$, the map α is obtained by lifting the map of $\mathcal{O}(-b)^r \to \mathcal{N}'$ to $\mathcal{N}$, which is possible since $\mathcal{E}$ is an ACM sheaf; and β is obtained by restricting α to $\mathcal{O}(-a-2b)$.

Applying $\mathcal{H}om(\cdot, \mathcal{E})$ to the left-hand column, we obtain

$$\mathcal{E}(b)^r \to \mathcal{E}(a+2b) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{E}).$$

Now X is normal, so $\mathcal{F}$ is locally free in codimension 1, so the $\mathcal{E}xt$ sheaf on the right has support in codimension ≥ 2. Let W be the image of $H^0(\mathcal{E}(b)^r)$ in $H^0(\mathcal{E}(a+2b))$. Then the subsheaf $\mathcal{E}_0$ of $\mathcal{E}(a+2b)$ generated by W is equal to $\mathcal{E}(d+2b)$ in codimension 1. Therefore [5, 2.6] applies, and a sufficiently general $s \in W$ will give a quotient $\mathcal{E}(a+2b)/s$ that is torsion-free and locally free in codimension 1, hence the twisted ideal sheaf of a subscheme of codimension 2. The proof of [loc. cit.] shows that for $s \in W$ sufficiently general, the map $\beta + s$ will have the same effect. Let $t \in H^0(\mathcal{E}(b)^r)$ be an element whose image is s. Then we have a new commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{F} & \to & \mathcal{N}' & \to & \mathcal{R} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(-b)^r & \stackrel{\alpha+t}{\to} & \mathcal{N} & \to & \mathcal{G} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(-a-2b) & \stackrel{\beta+s}{\to} & \mathcal{E} & \to & \mathcal{I}_Y(c) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

The left-hand and middle column and the top row are the same as before. The horizontal maps $\alpha + t$, $\beta + s$ are new, and $\mathcal{G}$ and $\mathcal{I}_Y(c)$ are defined as their cokernels. Now $\mathcal{I}_Y(c)$ is the twisted ideal sheaf of a subscheme Y, by construction, since $\mathcal{E}$ is orientable. But $\mathcal{E}$ is also ACM of rank 2, so Y is an AG subscheme of X.

The new sheaf $\mathcal{G}$ is locally free of rank 1 in codimension 1 because of the right-hand column and the fact that $\mathcal{R}$ has support in codimension 2. On the other hand, from the middle row we see that depth $\mathcal{G} \geq 1$ at every point of codimension 2. Therefore $\mathcal{G}$ is torsion-free. Thus $\mathcal{G}$ is the twisted ideal sheaf of a curve C', with the same twist, so $\mathcal{G} = \mathcal{I}_{C'}(c)$.

Now C and C' both have $\mathcal{N}$-type resolutions with the same sheaf $\mathcal{N}$, up to twist, so by Rao's theorem (2.16), C and C' are in the same even CI-liaison class, and a fortiori, in the same even G-liaison class. On the other hand C' is contained in the AG scheme Y, and the diagram we have just written shows that $\mathcal{R} = \mathcal{I}_{C',Y}(c)$. Therefore $\mathcal{R}$ also equals $\omega_{C''}(e)$, for some $e \in \mathbb{Z}$, where C'' is the curve linked to C' by Y. But we have already seen that $\mathcal{R} = \omega_{D'}(d)$, so $d = e$ and $C'' = D'$. Thus we have linkages $C' \underset{Y}{\sim} D'$ and $D' \underset{Z}{\sim} D$, and we conclude that C and D are in the same even G-liaison class, as required.

4 Syzygies and double-layered sheaves

Throughout this section we consider locally CM sheaves on an AG scheme X. We have already defined (3.1) the syzygy sheaf $\mathcal{F}^\sigma$ of a locally CM sheaf $\mathcal{F}$. In this section we develop some functorial properties of this operation, and define the notion of double-layered sheaves, which will be used in our main theorem. The motivation for this curious definition (4.4) is that it comes from the $\mathcal{N}$-type resolution found in (3.2) and it satisfies the property (4.5) below.

Proposition 4.1. *Let X be an* AG *scheme.*

a) *If $\mathcal{E}$ is locally* CM*, then $\mathcal{E}^\sigma$ is locally* CM *and $H^1_*(\mathcal{E}^\sigma) = 0$.*

b) *If $H^1_*(\mathcal{E}^\vee) = 0$, then there is a dissocié sheaf $\mathcal{M}$ such that $\mathcal{E}^{\sigma\vee\sigma\vee} \oplus \mathcal{M} \cong \mathcal{E}$.*

c) *$\mathcal{E}$ is dissocié if and only if $\mathcal{E}^\sigma = 0$.*

d) *If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence with $H^1_*(\mathcal{E}') = 0$, then there is a dissocié sheaf $\mathcal{M}$ and an exact sequence*

$$0 \to \mathcal{E}'^\sigma \to \mathcal{E}^\sigma \oplus \mathcal{M} \to \mathcal{E}''^\sigma \to 0.$$

Proof. These properties are all elementary and probably well-known, so we just give the idea of proofs.

a) $\mathcal{E}^\sigma$ is locally CM by chasing depth in the defining sequence. $H^1_*(\mathcal{E}^\sigma) = 0$ since the map of associated modules $L \to E \to 0$ in the definition is surjective, and X is ACM.

b) Let

$$0 \to \mathcal{E}^\sigma \to \mathcal{L} \to \mathcal{E} \to 0$$

be the defining sequence for $\mathcal{E}^\sigma$. Taking duals gives [3, 2.3]

$$0 \to \mathcal{E}^\vee \to \mathcal{L}^\vee \to \mathcal{E}^{\sigma\vee} \to 0.$$

Since we have assumed $H^1_*(\mathcal{E}^\vee) = 0$, the associated map of modules is surjective on the right, though $\mathcal{L}^\vee$ is perhaps not minimal. Hence there is a dissocié sheaf $\mathcal{M}$ such that $\mathcal{E}^\vee \cong \mathcal{E}^{\sigma\vee\sigma} \oplus \mathcal{M}$. Dualizing gives the result (replacing $\mathcal{M}^\vee$ by $\mathcal{M}$).

c) $\mathcal{E}^\sigma$ is 0 if and only if $\mathcal{E}$ is isomorphic to the dissocié sheaf $\mathcal{L}$ in the definition of $\mathcal{E}^\sigma$.

d) Let $\mathcal{L}' \to \mathcal{E}'$ and $\mathcal{L}'' \to \mathcal{E}''$ be the minimal covers defining $\mathcal{E}'^\sigma$ and $\mathcal{E}''^\sigma$. The map $\mathcal{L}'' \to \mathcal{E}''$ lifts to $\mathcal{E}$ because of the hypotheses $H^1_*(\mathcal{E}') = 0$. Thus $\mathcal{L}' \oplus \mathcal{L}'' \to \mathcal{E}$ is a map, surjective on the module level, so its kernel is $\mathcal{E}^\sigma$ plus a dissocié $\mathcal{M}$, which gives the required exact sequence.

Proposition 4.2. *Let X be an* AG *scheme of dimension ≥ 2.*

a) *If $\mathcal{E}$ is an* ACM *sheaf on X* (2.6)*, then $\mathcal{E}^\sigma$ is also* ACM*.*

b) *If $\mathcal{E}$ is* ACM*, then $\mathcal{E}^\sigma$ has no dissocié direct summands.*

c) *If $\mathcal{E}$ is* ACM *and indecomposable, then $\mathcal{E}^\sigma$ is also indecomposable.*

Proof. a) The defining exact sequence for $\mathcal{E}^\sigma$ shows that $H^i_*(\mathcal{E}^\sigma) = 0$ for $2 \leq i < \dim X$. The case $i = 1$ follows from (4.1a).

b) Let $0 \to \mathcal{E}^\sigma \to \mathcal{L} \to \mathcal{E} \to 0$ be the defining sequence for $\mathcal{E}^\sigma$, and suppose $\mathcal{E}^\sigma = \mathcal{E}' \oplus \mathcal{M}$ with $\mathcal{M}$ dissocié. Then we can write

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{M} & = & \mathcal{M} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{E}^\sigma & \to & \mathcal{L} & \to & \mathcal{E} & \to & 0\ . \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{E}' & \to & \mathcal{F} & \to & \mathcal{E} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathcal{E}$ and $\mathcal{E}'$ are ACM, we find that $H^1_*(\mathcal{F}^\vee) = 0$. Hence the middle column of this diagram splits, so $\mathcal{F}$ is dissocié and is a summand of $\mathcal{L}$. But $H^1_*(\mathcal{E}') = 0$, so the bottom row contradicts minimality of $\mathcal{L}$.

c) If $\mathcal{E}^\sigma$ decomposes into $\mathcal{E}' \oplus \mathcal{E}''$, then from (4.1$b$) we get $\mathcal{E} \cong \mathcal{E}'^{\vee\sigma\vee} \oplus \mathcal{E}''^{\vee\sigma\vee} \oplus \mathcal{M}$. But neither $\mathcal{E}'$ nor $\mathcal{E}''$ can be dissocié, by b), so $\mathcal{E}'^{\vee\sigma\vee}$ and $\mathcal{E}''^{\vee\sigma\vee}$ are both non-zero (4.1c), which shows that $\mathcal{E}$ is decomposable.

Proposition 4.3. *Let X be* AG *as before, and let C be a locally* CM *codimension* 2 *subscheme.*

a) *If C has an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0,$$

then there is a dissocié sheaf $\mathcal{M}$ and an $\mathcal{E}$-type resolution

$$0 \to \mathcal{N}^\sigma \oplus \mathcal{L} \to \mathcal{M} \to \mathcal{I}_C \to 0.$$

b) *If C has an $\mathcal{E}$-type resolution*

$$0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{I}_C \to 0,$$

then there is a dissocié sheaf $\mathcal{M}$ and an $\mathcal{N}$-type resolution

$$0 \to \mathcal{M}^\vee \to \mathcal{E}^{\vee\sigma\vee} \oplus \mathcal{L} \to \mathcal{I}_C \to 0.$$

Proof. a) Let

$$0 \to \mathcal{N}^\sigma \to \mathcal{M} \to \mathcal{N} \to 0$$

be the defining sequence for $\mathcal{N}^\sigma$, with $\mathcal{M}$ dissocié. Then the kernel of the composed map $\mathcal{M} \to \mathcal{I}_C$ is an extension of $\mathcal{L}$ by $\mathcal{N}^\sigma$, which splits, because $H^1_*(\mathcal{N}^\sigma) = 0$ (4.1).

b) Let

$$0 \to \mathcal{E}^{\vee\sigma} \to \mathcal{M} \to \mathcal{E}^\vee \to 0$$

be the defining sequence for $\mathcal{E}^{\vee\sigma}$. Dualizing gives

$$0 \to \mathcal{E} \to \mathcal{M}^\vee \to \mathcal{E}^{\vee\sigma\vee} \to 0.$$

If $\mathcal{G}$ is the fibered sum of $\mathcal{L}$ and $\mathcal{M}^\vee$ over $\mathcal{E}$, we get a sequence

$$0 \to \mathcal{M}^\vee \to \mathcal{G} \to \mathcal{I}_C \to 0$$

where $\mathcal{G}$ is an extension

$$0 \to \mathcal{L} \to \mathcal{G} \to \mathcal{E}^{\vee\sigma\vee} \to 0.$$

This extension splits because $H^1_*(\mathcal{E}^{\vee\sigma}) = 0$ (4.1), so we get the $\mathcal{N}$-type resolution desired.

Definition 4.4. A *double-layered* sheaf $\mathcal{E}$ on the AG scheme X is a sheaf $\mathcal{E}$ for which there exists a filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E}$$

with the following two properties a),b):

a) Each factor $\mathcal{E}_i/\mathcal{E}_{i-1}$ is either

   (i) a rank 2 orientable ACM sheaf, not dissocié, or

   (ii) a sheaf $\mathcal{F}^{\sigma\vee}$, where $\mathcal{F}$ is of type (i).

b) There exists an integer r with $0 \le r \le n$, such that for $i \le r$, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is of type (i) and for $i > r$, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is of type (ii).

Note that a double-layered sheaf is automatically ACM and orientable.

Proposition 4.5. *If $\mathcal{E}$ is a double-layered sheaf on the* AG *scheme X, then there exists a dissocié sheaf $\mathcal{M}$ such that $\mathcal{E}^{\sigma\vee} \oplus \mathcal{M}$ is double-layered.*

Proof. By induction on the least integer n for which there exists a filtration as in definition (4.4).

Case $n = 1$. In this case $\mathcal{E}$ itself is of type (i) or (ii) of a). If $\mathcal{E}$ is of type (i), then $\mathcal{E}^{\sigma\vee}$ is of type (ii) by definition. If $\mathcal{E}$ is of type (ii), then $\mathcal{E} = \mathcal{F}^{\sigma\vee}$ for some sheaf $\mathcal{F}$ of type (i). By (4.1) there exists a dissocié sheaf $\mathcal{M}$ such that $\mathcal{E}^{\sigma\vee} \oplus \mathcal{M} = \mathcal{F}^{\sigma\vee\sigma\vee} \oplus \mathcal{M} = \mathcal{F}$. If $\mathcal{M} \neq 0$, then $\mathcal{F}$ having rank 2 would be dissocié, contradicting (i). So $\mathcal{M} = 0$ and we find $\mathcal{E}^{\sigma\vee} = \mathcal{F}$ is of type (i). Condition b) of the definition is trivial, since there is only one factor in the filtration. So in this case $\mathcal{E}^{\sigma\vee}$ itself is double-layered.

Case $n \geq 2$. We write

$$0 \to \mathcal{E}_{n-1} \to \mathcal{E} \to \mathcal{E}' \to 0$$

where $\mathcal{E}' = \mathcal{E}_n/\mathcal{E}_{n-1}$. Then $\mathcal{E}_{n-1}$ and $\mathcal{E}'$ are both double-layered. By the induction hypothesis, there is a dissocié sheaf $\mathcal{M}_1$ such that $\mathcal{E}'' = \mathcal{E}_{n-1}^{\sigma\vee} \oplus \mathcal{M}_1$ is double-layered. On the other hand, $\mathcal{E}'$ has only a single factor, so by the Case $n = 1$ $\mathcal{E}'^{\sigma\vee}$ is itself double-layered with $n = 1$.

Taking syzygies there is a dissocié sheaf $\mathcal{M}_2$ and an exact sequence (4.1)

$$0 \to \mathcal{E}_{n-1}^{\sigma} \to \mathcal{E}^{\sigma} \oplus \mathcal{M}_2 \to \mathcal{E}'^{\sigma} \to 0,$$

whose dual gives

$$0 \to \mathcal{E}'^{\sigma\vee} \to \mathcal{E}^{\sigma\vee} \oplus \mathcal{M}_2^{\vee} \to \mathcal{E}_{n-1}^{\sigma\vee} \to 0.$$

Adding $\mathcal{M}_1$ to the middle and right-hand terms we get

$$0 \to \mathcal{E}'^{\sigma\vee} \to \mathcal{E}^{\sigma\vee} \oplus \mathcal{M}_2^{\vee} \oplus \mathcal{M}_1 \to \mathcal{E}'' \to 0.$$

Now the filtration of $\mathcal{E}''$ together with this sequence gives a filtration of the middle term $\mathcal{F} = \mathcal{E}^{\sigma\vee} \oplus \mathcal{M}_2^{\vee} \oplus \mathcal{M}_1$ satisfying condition a) of the definition of double-layered. If $\mathcal{E}'$ was of type (ii), then $\mathcal{E}'^{\sigma\vee}$ is of type (i), and condition b) follows. If $\mathcal{E}'$ was not of type (ii), then it is of type (i) and by condition b) for $\mathcal{E}$, it follows that all the factors of $\mathcal{E}_{n-1}$ are also of type (i). But it is easy to see from Case $n = 1$ and the induction so far that the operation $\mathcal{E}$ goes to the double-layered sheaf $\mathcal{E}^{\sigma\vee} \oplus \mathcal{M}$ constructed in this proof reverses the order and interchanges the types (i) and (ii) of the factors. Hence in this case all the factors of $\mathcal{E}''$ are of type (ii), and again condition b) is satisfied. Thus $\mathcal{F}$ is double-layered, as required.

Example 4.6. If $\mathcal{F} \oplus \mathcal{M}$, where $\mathcal{F}$ is ACM and $\mathcal{M}$ dissocié, is isomorphic to a double-layered sheaf $\mathcal{E}$, the sheaf $\mathcal{E}$ and its factors are not uniquely determined. Here is a trivial example. Take $\mathcal{F} = 0$. This is double layered with no factors. On the other hand, let $\mathcal{E}$

be an orientable rank 2 ACM sheaf that is not dissocié. Then the syzygy sequence for $\mathcal{E}$, dualized, gives an exact sequence

$$0 \to \mathcal{E}^\vee \to \mathcal{L}^\vee \to \mathcal{E}^{\sigma\vee} \to 0.$$

Thus $\mathcal{L}^\vee$ is double-layered with factors $\mathcal{E}^\vee$ and $\mathcal{E}^{\sigma\vee}$. But $\mathcal{L}^\vee$ is of the form $\mathcal{F} \oplus \mathcal{M}$, taking $\mathcal{M} = \mathcal{L}^\vee$.

5 The main theorem

We begin with a criterion for two schemes to be in the same G-liaison class on X.

Proposition 5.1. *Let X be a normal* AG *scheme, and let C be a locally* CM *closed subscheme of codimension* 2 *with an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0.$$

Write $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{M}_0$ with $\mathcal{M}_0$ dissocié and $\mathcal{N}_0$ having no dissocié direct summands. Then another such subscheme C' is in the same G-liaison class as C on X if and only if it has an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0$$

where $\mathcal{N}'$ satisfies the condition

(*) *There is an exact sequence*

$$0 \to \mathcal{E}_1 \oplus \mathcal{L}_1 \to \mathcal{N}' \to \mathcal{G} \to 0$$

where $\mathcal{L}_1$ is dissocié, $\mathcal{E}_1$ is a rank 2 ACM *sheaf, and $\mathcal{G}$ satisfies the condition*

(**) *There is a filtration*

$$0 = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_n = \mathcal{G}$$

and an integer $1 \le r \le n$ such that

1) *for each $i < r$, the factor $\mathcal{G}_i/\mathcal{G}_{i-1}$ is of type* (i) *of* (4.4), *and*
2) *for $i = r$, the factor $\mathcal{G}_r/\mathcal{G}_{r-1}$ is either $\mathcal{N}_0$ or $\mathcal{N}_0^{\sigma\vee}$, and*
3) *for $i > r$, the factor $\mathcal{G}_i/\mathcal{G}_{i-1}$ is of type* (ii) *of* (4.4).

Proof. First suppose C' is in the same G-liaison class as C. The proof is by induction on the number of G-liaisons needed to get from C to C'. If this number is zero, we let $\mathcal{E}_1$ be a rank 2 dissocié sheaf and use the $\mathcal{N}$-type resolution

$$0 \to \mathcal{L} \oplus \mathcal{E}_1 \to \mathcal{N} \oplus \mathcal{E}_1 \to \mathcal{I}_C \to 0.$$

On the other hand, since $\mathcal{N} \cong \mathcal{M}_0 \oplus \mathcal{M}_0$, we get a sequence

$$0 \to \mathcal{E}_1 \oplus \mathcal{M}_0 \to \mathcal{N} \oplus \mathcal{E}_1 \to \mathcal{N}_0 \to 0$$

and it is clear that $\mathcal{N} \oplus \mathcal{E}_1$ satisfies the condition (*).

For the induction step, suppose that C' has a resolution

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0$$

with $\mathcal{N}'$ satisfying (*), and suppose that C'' is obtained from C' by one G-liaison, using a codimension 2 AG scheme Y with resolution

$$0 \to \mathcal{O}_X(-a'') \to \mathcal{E} \to \mathcal{I}_Y(a') \to 0.$$

Then by (3.2), C'' will have an $\mathcal{N}$-type resolution

$$0 \to \mathcal{M}_1^\vee \to \mathcal{H} \to \mathcal{I}_{C''}(a'') \to 0$$

with $\mathcal{M}_1$ dissocié and $\mathcal{H}$ belonging to an exact sequence

$$0 \to \mathcal{E}^\vee \oplus \mathcal{L}'^\vee \to \mathcal{H} \to \mathcal{N}'^{\sigma\vee} \to 0.$$

At this point we need a lemma.

Lemma 5.2. *Suppose that $\mathcal{N}_0$ is a locally* CM *sheaf with no dissocié direct summands satisfying $H^1_*(\mathcal{N}_0^\vee) = 0$, and suppose $\mathcal{N}'$ is a sheaf satisfying condition* (*) *of* (5.1) *with this $\mathcal{N}_0$. Then there is a dissocié sheaf $\mathcal{M}$ such that $\mathcal{N}'^{\sigma\vee} \oplus \mathcal{M}$ satisfies condition* (**) *of* (5.1).

Proof. Applying (4.1) to the sequence

$$0 \to \mathcal{E}_1 \oplus \mathcal{L}_1 \to \mathcal{N}' \to \mathcal{G} \to 0$$

and dualizing, we find there is a dissocié sheaf $\mathcal{M}_1$ and an exact sequence

$$0 \to \mathcal{G}^{\sigma\vee} \to \mathcal{N}'^{\sigma\vee} \oplus \mathcal{M}_1^\vee \to \mathcal{E}_1^{\sigma\vee} \to 0.$$

On the other hand, the proof of (4.5), with an extra factor $\mathcal{N}_0$ or $\mathcal{N}_0^{\sigma\vee}$ in the middle, shows that there is a dissocié sheaf $\mathcal{M}_1$ such that $\mathcal{G}' = \mathcal{G}^{\sigma\vee} \oplus \mathcal{M}_2$ satisfies condition (**). Note that since $\mathcal{N}_0$ has no dissocié direct summands, $\mathcal{N}_0^{\sigma\vee\sigma\vee} = \mathcal{N}_0$ by (4.2). Thus we can write

$$0 \to \mathcal{G}' \to \mathcal{N}'^{\sigma\vee} \oplus \mathcal{M}_1^\vee \oplus \mathcal{M}_2 \to \mathcal{E}_1^{\sigma\vee} \to 0,$$

and then the filtration on $\mathcal{G}'$ together with $\mathcal{E}_1^{\sigma\vee}$ gives a filtration on the middle term satisfying (**).

In the special case where $\mathcal{E}_1$ is dissocié, the factor $\mathcal{E}_1^{\sigma\vee}$ becomes 0, and the proof is simpler.

Proof of (5.1)*, continued.* By the lemma, we can find a dissocié sheaf $\mathcal{M}_2$ (new notation) such that $\mathcal{N}'^{\sigma\vee} \oplus \mathcal{M}_2$ satisfies (**). Then C'' has an $\mathcal{N}$-type resolution

$$0 \to \mathcal{M}_1^\vee \oplus \mathcal{M}_2 \to \mathcal{H} \otimes \mathcal{M}_2 \to \mathcal{I}_{C''}(a'') \to 0$$

where the middle term $\mathcal{H} \oplus \mathcal{M}_2$ satisfies condition (*), as required.

Conversely, suppose that C' has a resolution of the form given in (5.1). We proceed by induction on the number of non-dissocié rank 2 ACM sheaves $\mathcal{E}$ that appear as $\mathcal{E}_1$ in the exact sequence of $\mathcal{N}'$, or that appear as $\mathcal{E}$ or $\mathcal{E}^{\sigma\vee}$ in the factors of $\mathcal{G}$ for $i \neq r$. If that number is zero, then $\mathcal{E}_1$ is dissocié, and $\mathcal{G}$ is either $\mathcal{N}_0$ or $\mathcal{N}_0^{\sigma\vee}$. The extension splits, since $H^1_*(\mathcal{N}_0^\vee) = 0$ and $H^1_*(\mathcal{N}_0^\sigma) = 0$, so $\mathcal{N}'$ is stably isomorphic to $\mathcal{N}_0$ or $\mathcal{N}_0^{\sigma\vee}$. In the first case, we conclude by Rao's theorem (2.16) that C' is in the even CI-liaison class of C_1 and a fortiori in the same G-liaison class as C. In the second case, we do a single CI-liaison using a dissocié rank 2 sheaf $\mathcal{E}$ in (3.2) to obtain a curve C'' whose $\mathcal{N}$-type resolution involves a sheaf $\mathcal{N}''$ stably isomorphic to $\mathcal{N}_0$, and then proceed as in the first case.

For the induction step we will use (3.4).

Case 1. Suppose that $\mathcal{E}_1$ is not dissocié. Then we apply (3.4) with the sequence

$$0 \to \mathcal{E}_1 \to \mathcal{N}' \to \mathcal{N}'' \to 0$$

and obtain another scheme D in the same G-liaison class as C' having an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L}'' \to \mathcal{N}'' \to \mathcal{I}_D(a) \to 0$$

for some $a \in \mathbb{Z}$. Now by construction $\mathcal{N}''$ belongs to an exact sequence

$$0 \to \mathcal{L}_1 \to \mathcal{N}'' \to \mathcal{G} \to 0.$$

Adding a rank 2 dissocié sheaf $\mathcal{E}_1'$ to $\mathcal{L}_1$ and to $\mathcal{N}''$, we find $\mathcal{N}'' \oplus \mathcal{E}_1'$ satisfies (*) and has one fewer nondissocié factor than C', so the induction is accomplished.

Case 2. Suppose that $\mathcal{E}_1$ is dissocié. Then the sequence for $\mathcal{N}'$ splits (since $H^1_*(\mathcal{N}_0^\vee) = 0$, it being an $\mathcal{N}$-type resolution), so $\mathcal{N}' \cong \mathcal{E}_1 \oplus \mathcal{L}_1 \oplus \mathcal{G}$. If $r \geq 2$, we take the factor $\mathcal{G}_1/\mathcal{G}_0$ as $\mathcal{E}$ in (3.4) and proceed as in Case 1. If $r = 1$, so that there is no factor of type (i) in $\mathcal{G}$, then we first perform a single CI-liaison (as above), which converts type (ii) factors of $\mathcal{G}$ into type (i) factors, and then proceed as in the case $r > 1$.

Corollary 5.3. *A codimension* 2 *subscheme C' of X is in the Gorenstein liaison class of a complete intersection (glicci) on X if and only if it has an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0$$

where $\mathcal{N}' = \mathcal{N}_1 \oplus \mathcal{M}_1$ *with* $\mathcal{N}_1$ *a double-layered sheaf and* $\mathcal{M}_1$ *dissocié.*

Proof. Taking C to be a complete intersection in X we have $\mathcal{N}$ dissocié, so $\mathcal{N}_0 = 0$. Therefore the sheaf $\mathcal{G}$ appearing in the exact sequence

$$0 \to \mathcal{E}_1 \oplus \mathcal{L}_1 \to \mathcal{N}' \to \mathcal{G} \to 0$$

of condition (*) is double-layered by condition (**). If $\mathcal{E}_1$ is dissocié, the sequence splits and $\mathcal{N}' \cong \mathcal{G} \oplus \mathcal{E}_1 \oplus \mathcal{L}_1$. If $\mathcal{E}_1$ is not dissocié, a factor $\mathcal{L}_1$ splits off, so that $\mathcal{N}' \cong \mathcal{N}_1 \oplus \mathcal{L}_1$ and $\mathcal{N}_1$ belongs to a sequence

$$0 \to \mathcal{E}_1 \to \mathcal{N}_1 \to \mathcal{G} \to 0,$$

which makes $\mathcal{N}_1$ double-layered, by definition.

Theorem 5.4. *Let* X *be a normal* AG *scheme. Then the following conditions are equivalent:*

(D) *Every codimension* 2 ACM *subscheme* C *is in the Gorenstein liaison class of a complete intersection (glicci).*

(E) *Every orientable* ACM *sheaf on* X *is stably equivalent to a double-layered sheaf.*

Proof. (D) $\Rightarrow$ (E). Suppose $\mathcal{N}$ is an orientable ACM sheaf on X. For $a \gg 0$ we can find a sequence

$$0 \to \mathcal{O}(-a)^{r-1} \to \mathcal{N} \to \mathcal{I}_C(b) \to 0$$

where $r = \operatorname{rank} \mathcal{N}$. Then C will be ACM (2.7), and so by condition (D) it will be glicci. Then by (5.3) it has an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_C(b) \to 0$$

where $\mathcal{N}' = \mathcal{N}_1 \oplus \mathcal{M}_1$ with $\mathcal{N}_1$ double-layered and $\mathcal{M}_1$ dissocié. But the $\mathcal{N}$ in an $\mathcal{N}$-type resolution is uniquely determined up to stable equivalence and shift by Rao's theorem (2.16), so $\mathcal{N}$ is stably equivalent to a double-layered sheaf.

(E) $\Rightarrow$ (D). Given C ACM of codimension 2, let it have an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C(a) \to 0.$$

Then $\mathcal{N}$ is orientable and ACM (2.7), so by (E) it is stably equivalent to a double-layered sheaf $\mathcal{N}_1$. In other words, there are dissocié sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{N} \oplus \mathcal{M}_1 \cong \mathcal{N}_1 \oplus \mathcal{M}_2$. Then the $\mathcal{N}$-type resolution

$$0 \to \mathcal{L} \oplus \mathcal{M}_1 \to \mathcal{N}_1 \oplus \mathcal{M}_2 \to \mathcal{I}_C(a) \to 0$$

for C shows by (5.3) that C is glicci.

Next we specialize to the case of a 3-dimensional normal AG scheme X. A *curve* will be a 1-dimensional locally CM subscheme of X. Recall that for any curve C, we define the *Rao module* to be $M_C = H^1_*(\mathcal{I}_C)$.

Theorem 5.5. *Let X be a normal 3-dimensional* AG *scheme X, and suppose that X satisfies the (stronger) condition*

(E′) *Every orientable* ACM *sheaf on X is stably equivalent to a double-layered sheaf with all factors of type* (i) *in* (4.4).

Then two curves C, C' on X are in the same even G-liaison class if and only if their Rao modules are isomorphic up to twist.

Proof. One direction is well-known [15, §5.3]. For the other direction, let C be any curve on X, with Rao module M, and consider an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

of C. Following the proof of [3, 4.7] we find another curve C' depending only on M and not on C, together with a resolution

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_{C'}(a') \to 0$$

where $\mathcal{E}$ is an orientable ACM sheaf. By considering the syzygy sheaf of $\mathcal{E}$, as in the proof of (3.2), we obtain an $\mathcal{N}$-type resolution of C',

$$0 \to \mathcal{M}^\vee \to \mathcal{G} \to \mathcal{I}_{C'}(a') \to 0$$

where $\mathcal{G}$ satisfies

$$0 \to \mathcal{N} \to \mathcal{G} \to \mathcal{E}^{\vee\sigma\vee} \to 0.$$

Now $\mathcal{E}^\vee$ is an orientable ACM sheaf, so by hypothesis (E′) it is stably equivalent to a double-layered sheaf $\mathcal{F}$ with all factors of type (i). Then by (4.5) there is a dissocié sheaf $\mathcal{M}_1$ such that $\mathcal{F}^{\sigma\vee} \oplus \mathcal{M}_1$ is double-layered with all factors of type (ii). Note also that $\mathcal{E}^{\vee\sigma\vee} = \mathcal{F}^{\sigma\vee}$, since the operation $\sigma\vee$ kills off dissocié factors.

On the other hand, let us write $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{M}_0$ where $\mathcal{M}_0$ is dissocié and $\mathcal{N}_0$ has no dissocié direct summands. Then $\mathcal{G} = \mathcal{G}' \oplus \mathcal{M}_0$ where $\mathcal{G}'$ satisfies

$$0 \to \mathcal{N}_0 \to \mathcal{G}' \oplus \mathcal{M}_1 \to \mathcal{F}^{\sigma\vee} \oplus \mathcal{M}_1 \to 0$$

and hence $\mathcal{G}' \oplus \mathcal{M}_1$ satisfies condition (**) of (5.1). Now $\mathcal{G}$ is stably equivalent to $\mathcal{G}' \oplus \mathcal{M}_1$, and therefore C' is in the same G-liaison as C by (5.1). But C' depends only on the Rao module M, hence all curves with the same Rao module (up to twist) are equivalent for G-liaison. Since they have the same Rao module (not their duals) it must be in the same even G-liaison class.

Remark 5.6. Of course it follows that two curves are connected by an odd number of G-liaisons if and only if their Rao modules are dual (up to twist) because a single G-liaison replaces the Rao module by its dual [15, 5.3.1].

6 Quadric hypersurfaces

Theorem 6.1. *If X is a nonsingular quadric hypersurface of dimension* 3 *or* 4 *in* $\mathbb{P}^4$ *or* $\mathbb{P}^5$ *respectively, then all codimension* 2 ACM *subschemes are glicci.*

Proof. According to the theorems of Buchweitz, Eisenbud, Herzog, and Knörrer (see for example [24, 14.10]), X has just one ($\dim X = 3$) or two ($\dim X = 4$) indecomposable rank 2 ACM sheaves, up to twist, and all other ACM sheaves are direct sums of these and their twists and dissocié sheaves. It follows that any ACM sheaf is stably equivalent to a double-layered sheaf (in which all the extensions are split) and so condition (E) of (5.4) is satisfied. Hence by (5.4), all codimension 2 ACM schemes are glicci.

Theorem 6.2. *If X is a nonsingular quadric three-fold in* $\mathbb{P}^4$*, then two curves are in the same even G-liaison class if and only if their Rao modules are isomorphic up to twist.*

Proof. Indeed, for the same reason as above, condition (E′) of (5.5) is satisfied, and the result follows from (5.5).

Remark 6.3. One of the big open questions in the theory of Gorenstein liaison is to describe the structure of an even G-liaison class [15, §5.4]. Is there anything analogous to the Lazarsfeld–Rao property for codimension 2 CI-biliaison classes? In this context, we can say something about curves on the nonsingular quadric hypersurface X in $\mathbb{P}^4$. As we have seen (6.2), an even G-liaison class of curves on X is determined by its Rao module. Each such even G-liaison class will be an infinite union of CI-biliaison classes, each one of which has minimal curves, and satisfies the LR-property with respect to complete intersection biliaison [5, 2.4]. Since $\operatorname{Pic} X = \mathbb{Z}$, every Gorenstein biliaison is already a CI-biliaison, so there is no further connection among these curves using G-biliaison. However, one could hope to describe all the minimal curves in each of these CI-biliaison classes and how they are related by G-liaison to give some more structure to the whole G-liaison class. All this discussion refers to liaisons and biliaisons on X. It is still possible that if one regards curves in different CI-liaison classes on X, they may be related by Gorenstein biliaisons in $\mathbb{P}^4$.

Remark 6.4. Looking at those curves in $\mathbb{P}^4$ with Rao module k that lie on the nonsingular quadric hypersurface X [9, §4], we see that there are two types having Rao module in degree 0, namely two skew lines, with degree and genus $(d, g) = (2, -1)$ and the union of a conic with a line $(3, -1)$ [9, 4.1]. With Rao module k in degree 1, the $(5, 0)$ curve will be minimal in its CI-biliaison class on X, while the $(6, 1)$ and $(7, 2)$ curves are obtained by ascending biliaison from the $(2, -1)$ and $(3, -1)$ curves, respectively. Note also the $(10, 6)$ curve constructed at the end of [9, 4.6] is not obtained by any ascending biliaison, so it is minimal in its CI-biliaison class, with Rao module in degree 2. Thus we may expect to find curves that are minimal for CI-biliaison with Rao module k in arbitrarily high degree.

7 AG schemes in $\mathbb{P}^n$ are glicci

Theorem 7.1. *Let C be an* AG *subscheme of $\mathbb{P}^n$ of codimension ≥ 2. Then C is in the Gorenstein liaison class of a complete intersection in $\mathbb{P}^n$.*

Proof. According to the theorem of Kleiman and Altman [12, (1)] there exists a complete intersection scheme $X \subseteq \mathbb{P}^n$, of dimension $\dim C+2$ containing C, that is nonsingular outside of C. In particular, X will be AG, since it is a complete intersection, and normal, since it is nonsingular in codimension 1.

Since C is an AG subscheme of X of codimension 2, it has an $\mathcal{N}$-type resolution

$$0 \to \mathcal{O}_X(-a) \to \mathcal{E} \to \mathcal{I}_C \to 0$$

where $\mathcal{E}$ is an ACM sheaf of rank 2 on X (2.9). Let $\mathcal{M}$ be any rank 2 dissocié sheaf on X. Then we get another $\mathcal{N}$-type resolution for C,

$$0 \to \mathcal{O}_X(-a) \oplus \mathcal{M} \to \mathcal{E} \oplus \mathcal{M} \to \mathcal{I}_C \to 0.$$

We apply (3.4) to this resolution with the exact sequence

$$0 \to \mathcal{E} \to \mathcal{E} \oplus \mathcal{M} \to \mathcal{M} \to 0,$$

and we find there is a codimension 2 subscheme D of X, in the same G-liaison class as C, having an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L}' \to \mathcal{M} \to \mathcal{I}_D(b) \to 0$$

for some $b \in \mathbb{Z}$. Since $\mathcal{M}$ is rank 2 dissocié, D is a complete intersection in X. Since X itself is a complete intersection in $\mathbb{P}^n$, D is also a complete intersection in $\mathbb{P}^n$. It follows that C is glicci in X and hence also glicci in $\mathbb{P}^n$.

Remark 7.2. If C has codimension 2 in $\mathbb{P}^n$ then it is already a complete intersection, so there is nothing to prove. If C has codimension 3, then it follows from Watanabe's paper [23] that C is in fact licci (in the CI-liaison class of a complete intersection), a stronger result. In the special case of curves in $\mathbb{P}^4$, one can show that a general AG curve in $\mathbb{P}^4$ can be obtained by ascending CI-biliaisons from a line, an even stronger result [10].

Thus our result (7.1) is new only for codimension $C \geq 4$.

References

[1] Bolondi, G., and Migliore, J. C., The Lazarsfeld–Rao property on an arithmetically Gorenstein variety, *Manuscripta Math.* **78** (1993) 347–368.

[2] Casanellas, M., Glicci versus glicog, *Rend. Sem. Mat. Univ. Pol. Torino* **59** (2001) 127–128.

[3] Casanellas, M., and Hartshorne, R., Gorenstein biliaison and ACM sheaves.

[4] Drozd, E., Curves on a nonsingular quadric hypersurface in $\mathbb{P}^4$: existence and liaison theory, Thesis, University of California, Berkeley (2003).

[5] Hartshorne, R., On Rao's theorems and the Lazarsfeld–Rao property, *Ann. Fac. Sci. Toulouse* (to appear).

[6] Hartshorne, R., Generalized divisors on Gorenstein schemes, *K-theory* **8** (1994) 287–339.

[7] Hartshorne, R., Generalized divisors and biliaison.

[8] Hartshorne, R., Stable vector bundles of rank 2 on $\mathbb{P}^3$, *Math. Ann.* **238** (1978) 229–280.

[9] Hartshorne, R., Some examples of Gorenstein liaison in codimension three, *Collect. Math.* **53** (2002) 21–48.

[10] Hartshorne, R., Geometry of arithmetically Gorenstein curves in $\mathbb{P}^4$.

[11] Hartshorne, R., Martin–Deschamps, M., and Perrin, D., Un théorème de Rao pour les familles de courbes gauches, *J. Pure Appl. Algebra* **155** (2001) 53–76.

[12] Kleiman, S. L., and Altman, A. B., Bertini theorems for hypersurface sections containing a subscheme, *Comm. in Algebra* **7** (1979) 775–790.

[13] Kleppe, J., Migliore, J. C., Miró–Roig, R. M., Nagel, U., and Peterson, C., Gorenstein liaison, complete intersection liaison invariants, and unobstructedness, *Mem. AMS* **154** (2001) No. 732.

[14] Martin–Deschamps, M., and Perrin, D., Sur la classification des courbes gauches, *Astérisque* **184–185** (1990).

[15] Migliore, J. C., *Introduction to Liaison Theory and Deficiency Modules*, Birkhäuser, Boston (1998).

[16] Migliore, J. C., and Nagel, U., Liaison and related topics: Notes from the Torino Workshop-school, *Rend. Sem. Mat. Univ. Pol. Torino* **59** (2001) 59–126.

[17] Nagel, U., Even liaison classes generated by Gorenstein linkage, *J. Algebra* **209** (1998) 543–584.

[18] Nollet, S., Even linkage classes, *Trans. Amer. Math. Soc.* **348** (1996) 1137–1162.

[19] Peskine, C., and Szpiro, L., Liaison des variétés algébriques, *Invent. Math.* **26** (1974) 271–302.

[20] Rao, A. P., Liaison among curves in $\mathbb{P}^3$, *Invent. Math.* **50** (1979) 205–217.

[21] Rao, A. P., Liaison equivalence classes, *Math. Ann.* **258** (1981) 169–173.

[22] Schenzel, P., Notes on liaison and duality, *J. Math. Kyoto Univ.* **22** (1982) 485–498.

[23] Watanabe, J., A note on Gorenstein rings of embedding dimension 3, *Nagoya Math. J.* **50** (1973) 227–282.

[24] Yoshino, Y., Cohen–Macaulay modules over Cohen–Macaulay rings, *London Math. Soc. Lecture Notes* **146**, Cambridge Univ. Press (1990).