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by

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Semiclassics, Large Operators, and Holography

Habe nun, ach! Philosophie,
Juristerey und Medicin,
Und leider auch Theologie!
Durchaus studirt, mit heißem Bemühn.
Da steh' ich nun, ich armer Thor!

* * *

Die Hölle selbst hat ihre Rechte?

- Goethe, *Faust*

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2. David Berenstein, **Adolfo Holguin**, `Open giant magnons suspended between dual giant gravitons in` $\mathcal{N} = 4$ `SYM`, Published in: JHEP 09 (2020), 019 e-Print: 2111.05981 [hep-th]
3. David Berenstein, **Adolfo Holguin**, `String junctions suspended between giants` e-Print: 2202.11729 [hep-th], Published in: JHEP 11 (2022) 085
4. **Adolfo Holguin**, Shannon Wang, `Giant gravitons, Harish-Chandra integrals, and BPS states in symplectic and orthogonal` $\mathcal{N} = 4$ `SYM`, Published in: JHEP 10 (2022), 078
5. **Adolfo Holguin**, Wayne Weng, `Orbit Averaging Coherent States: Holographic Three-Point Functions of AdS Giant Gravitons`, e-Print: 2211.03805 [hep-th]. Published in: JHEP 05 (2023) 167
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Abstract

Semiclassics, Large Operators, and Holography

by

Adolfo Holguin

The planar expansion of large N gauge theories has been a remarkably fruitful idea, inspiring many developments in theoretical physics. Perhaps the most important realization of this idea is the duality between critical string theories in anti de-Sitter spaces and large N conformal field theories, giving a window into otherwise intractable phenomena in quantum field theories. One of the most promising aspects of this duality is the possibility of studying truly quantum aspects of gravitational systems with conventional tools from quantum mechanics. A major obstacle to this endeavor lies in the fact that the most interesting questions about quantum gravity in anti de-Sitter space are related to issues that are beyond current techniques in conformal field theory. This thesis deals with various semiclassical aspects of large operators in holographic conformal field theories, focusing on the study of very large (near-)BPS operators in $\mathcal{N} = 4$ super Yang-Mills theory and their corresponding gravitational avatars.

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Chapter 1

Introduction

Over the past 26 years, the AdS/CFT correspondence [1] has enjoyed a vast success in elucidating aspects of strongly coupled gauge theories and quantum theories of gravity in anti-de-Sitter space. One of the crowning successes has been the determinantion of conformal dimensions of non-protected single trace operators in the planar limit of $\mathcal{N} = 4$ super Yang-Mills [2] thanks to the discovery of integrable structures in the theory [3, 4], which was in part influenced by the study of stringy states in $\mathcal{N} = 4$ SYM [5]. This opened up the possibility of precision tests of the correspondence beyond the supergravity regime. More recently there has been a renewed interest in the study of black hole microstates from holographic CFTs, fueled by recent computations of supersymmetric indices which confirm that supersymmetric state counting on the dual CFT agrees with the entropy of supersymmetric black holes [6, 7, 8].

This leads naturally to the study of large charge (BPS) operators, from which one can hope to learn about the structure of supersymmetric black holes. One of the main setbacks to this program has been the lack of efficient computational tools to address the combinatorial complexity in the large N limit. Progress in this direction started with the study of BPS operators in $\mathcal{N} = 4$ SYM [9, 10, 11] with a collective coordinate approach.

These results suggest that the large N limit of large charge operators might be tractable via semi-classical techniques, leading to an alternative expansion to the t' Hooft limit for baryonic operators [12]. Signs of this expansion have also been observed in state counting computations [13, 14], indicating a delicate cancellation of states at finite but large N. In holographic models, large charge operators are known to describe extended objects on the bulk AdS space [15], or back-reacted geometries [16]. Their excitations correspond to open strings stretched between giant graviton branes, or closed strings on non-trivial backgrounds. These two descriptions (in terms of open and closed strings) are valid in different regimes of parameter space, depending on the energy of the state, and a geometric description often appears even in the free limit $g_{\texttt{YM}} = 0$. [17, 18, 19].

1.1 Effective Strings

One of the first phenomenologically inspired models for the strong nuclear force came from the observations of dual resonance of scattering amplitudes of hadrons

$$\mathcal{M}(s,t) = \mathcal{M}(t,s). \tag{1.1}$$

Additionally, particles generated at experiments all seem to fall into families of Regge trajectories:

$$\ell \sim E^2 \tag{1.2}$$

This led Veneziano to propose a functional form for $\mathcal{M}(s,t)$

$$\mathcal{M}(s,t) = \frac{\Gamma(\alpha(s))\Gamma(\alpha(t))}{\Gamma(\alpha(s)+\alpha(t))}, \tag{1.3}$$

with $\alpha(s)$ being a linearly decreasing function. In order to explain this behavior, Nambu and Susskind noted that such an amplitude could be reproduced not by point-particle scattering, but the scattering of rotating string-like objects. By now it is well understood that these qualitative observations are explained by the low energy dynamics of QCD; the particles generated by high energy scattering experiments are color charge neutral bound states of strongly interacting quarks and gluons. This is the phenomenon of *color confinement*, which remains as one of the least understood features of QCD.

The most powerful conceptual developments towards understanding the low energy dynamics of non-abelian gauge theories come from analogy with models of superconductivity. A characteristic feature of a superconducting medium is its ability to expulse magnetic field lines from itself. Some materials (type II superconductors) may additionally have intermediate phases of superconductivity in which an external magnetic field partially penetrates the medium, forming thin flux tubes known at Abrikosov-Nielsen-Olensen vortices. This phenomenon of magnetic is believed to be qualitatively similar to the color confinement of Yang-Mills theory.

These observations pose the following puzzle: *how do the degrees of freedom of gauge theories rearrange themselves to describe extended (string-like) objects?* This is duality in which a seemingly fundamental *'microscopic'* description of a system can be described in terms of completely different '*macroscopic*' ingredients. In the sharpest instances of duality, the distinction between macroscopic and microscopic variables is often meaningless, and neither description is more fundamental than the other. One of the goals of current theoretical physics is to realize this duality between string-like variables and field variables in QCD.

1.1.1 The string action as an EFT: Long strings

Suppose we wanted to the describe an effective theory for a long string like excitation in some Poincaré invariant UV theory. The most general effective action that describes such a system can be thought of as a theory of Goldstone bosons for spontaneously broken Poicaré symmetry [20]

$$ISO(1, D-1) \to SO(1,1) \times SO(D-2). \tag{1.4}$$

Restricting to bosonic string excitations, this is the theory of $D-2$ Goldstone bosons X^i corresponding to the unbroken rotational symmetry of a long straight string

$$X^i \to R^i_j O^j, \tag{1.5}$$

while the unbroken $SO(1,1)$ is the Lorentz group on the string worldsheet, acting on coordinates along the string as usual:

$$\sigma^a \to \Lambda^a_b \sigma^b. \tag{1.6}$$

The remaining broken Poincaré symmetry generators are non-linearly realized and mix the worldsheet Lorentz symmetry with the global $SO(D-2)$ symmetry. The EFT is generically composed of an infinite sum of couplings weighted by theory dependent Wilson coefficients. The symmetries force all the terms in the effective action to be geometric invariants of the worldsheet:

$$S_{\texttt{eff}} = \int d^2\sigma \sqrt{h} \times \left(\frac{1}{\ell_s^2} + R + \dots \right), \tag{1.7}$$

where h_{ab} is the induced metric on the string worldsheet. The first term in the EFT expansion defines the relevant length scale of the problem ℓ_s, which controls the effective tension of the string. Higher curvature terms can be organized as an expansion in the string tension, with order one coefficients that are theory dependent. In the infrared, the dynamics of this family of models is universal and is governed by the Nambu-Goto action.

$$S_{\mathtt{NG}} = \frac{1}{\ell_s^2} \int d^2\sigma \sqrt{\det\left(\partial_a X^\mu \, \partial_b X^\nu \eta_{\mu\nu}\right)} \tag{1.8}$$

This is the theory describing relativistic string moving in D Minkowsky space. One now well appreciated fact is that this theory is not a well defined QFT unless the strings propagate in $D = 26$ dimensions. Surprisingly, the effective description is able to capture many quantitative aspects of the dynamic of flux tubes in lattice gauge theory simulations for $D \neq 26$.

1.2 Large N QCD and the 't Hooft Limit

One of the first concrete theoretical clues that strongly coupled gauge theories are described by theories of strings came from 't Hooft's study of large N QCD [21]. The main insight of 't Hooft was to realize that the correct expansion for large N QCD was not in terms of the gauge coupling $g_{\mathtt{YM}}$ but instead in terms two parameters

$$\begin{aligned} \lambda &= g_{\mathtt{YM}}^2 N \\ g_s &\sim \frac{1}{N^2} \end{aligned} \tag{1.9}$$

where g_s is the parameter controlling corrections associated to non-planar Feynman diagrams. In the large N limit, the dominant contributions to amplitudes come from planar diagrams, and non planar corrections organize themselves in a genus expansion suggestive of a string worldsheet description. In the 't Hooft limit

$$\begin{aligned} N &\to \infty \\ g_{\mathtt{YM}}^2 &\to 0 \\ \lambda = g_{\mathtt{YM}}^2 N &\ \texttt{fixed}, \end{aligned} \tag{1.10}$$

the expectation was that the dynamics of QCD reduced to a theory of free mesons, since $\frac{1}{N}$ is the only true parameter of the theory. In this limit only planar Feynman diagrams contribute to observables, with λ serving as a vertex counting parameter. In the weak coupling regime $\lambda \ll 1$, the theory is described by tiled Riemann surfaces with no handles. Equivalently, the Feynman diagrams appear to form skeletons for some kind of string worldsheet theory, with each vertex being associated to a hole on the surface. The hope was that in the strong coupling regime $\lambda \gg 1$, the vertices would condense to form a continuous Riemann surface. From the point of view of the putative string, the vertices of the dual Feynman diagrams are associated to insertions of vertex operators, which one can associate to a non-trivial background for the string. This background would have some curvature scale L, and the geometric string picture would arise when the characteristic size of the strings is much smaller that the curvature of the background

$$\lambda^{-1} \sim \ell_s/L \ll 1. \tag{1.11}$$

The appeal of this idea is that one would then be able to compute correlation functions involving mesons at strong coupling using a weakly coupled semi-classical string description.

1.3 A digression: how to solve every problem that has been solved.

What makes a model solvable? Perhaps the most well-known exactly solvable model is the XXX-Heisenberg magnet

$$H_{\mathtt{XXX}} = -J \sum_{<ij>} \vec{\sigma}_i \cdot \vec{\sigma}_j, \tag{1.12}$$

famously solved by Hans Bethe. This model describes a system of L spin $\frac{1}{2}$ particles on a chain, interacting through spin-spin couplings with their nearest neighbor. One of the peculiarities of this model is that the analogous model for spins on a square lattice has evaded an analytic solution, despite their superficial similarities. This is now understood to be a consequence of the quantum integrability of the model. A (quantum) integrable system is usually characterized by having enhanced symmetries that highly constraint the dynamics of the model. These constraint reduce the spectral problem for the system into sets of algebraic equations that relate the conserved quantum numbers of the system. The process of distilling these sets of equations for a particular integrable system is known colloquially as the *Bethe Ansatz*. Some generic features of quantum integrable systems are:

- They live in 1+1 spacetime dimensions
- Their S-matrices do not admit production of particles in scattering processes.
- Their S-matrices factorize into products of two-excitation S-matrices.
- Their S-matrices satisfy some form of the Yang-Baxter equation

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}. \tag{1.13}$$

All these properties are not mutually independent, but are consequences of the integrability of the system. The power of these properties is that they allow for a full solution of the model, given that one can solve a complicated set of algebraic equations. For instance for a periodic chain of lenght L, the overall phase shift of an excitation as it is taken around the chain is related to the factorized scattering matrices associated to crossing other excitations by the Bethe ansatz equations

$$e^{-ip_i L} = \prod_{j \neq i} S(p_i, p_j). \tag{1.14}$$

In particular he symmetries of the system allow for all other quantum numbers to be determined in terms of the momenta p_i through a dispersion relation that is determined from the single excitation spectrum:

$$E(p_1, \ldots, p_n) = \sum_{i=1}^{n} E(p_i). \tag{1.15}$$

1.4 Feynman's Dream

Can we hope to solve QCD? At first glance, the answer to this question is a resounding *no*, since

- QCD is a 3+1 dimensional interacting theory.
- There certainly is particle production in the S-matrix of QCD.
- There is not enough symmetry to factorize the S-matrix into simple products.
- Yang-Baxter equation cannot be made to hold without forcing the theory to be free.

Yet we believe that QCD is a well-defined model which should be able to predict the spectra of for instance hadrons. The intuition of Feynman was to consider deep inelastic scattering processes in QCD. By scattering a suffiliciently high energy electron off a hadron, the hadron can fragment and emit a high energy hadronized quark. The motion of the quark can then be split into hard (high momentum) directions, and soft transverse directions. The physics along the high momentum directions is simple since the gauge coupling becomes small at high energies. Along the remaining two transverse directions to the quark, maybe the soft physics could become integrable in some regime. These ideas were then partially realized by work of Balitsky-Fadin-Kuraev-Lipatov. Since processes in which hadrons split are further supressed by $\frac{1}{N}$ in QCD with N colors, the large N limit became a natural target to look for an integrable toy model for QCD.

1.5 Permissions and Attributions

1. The content of chapters 3 and 4 are the result of a collaboration with David Berenstein, and has previously appeared in the Journal of High Energy Physics [22, 23]. It is reproduced here with the permission of the International School of Advanced Studies: `https://jhep.sissa.it/jhep/help/JHEP/CR_OA.pdf`. The content of chapter 5 are partly based on work with David Berenstein. Chapter 6 is the result of collaboration with Shannon Wang and has appeared in the Journal of High Energy Physics [24, 25]. The results of chapter 7 are the result of collaboration with Wayne W. Weng and appeared in the Journal of High Energy Physics [26]. The results of chapter 8 appeared previously in [26].

Chapter 2

Basics of AdS/CFT

2.1 Conformal Field Theory

Quantum field theory is perhaps the most successful tools of modern theoretical physics, to the extend that it is often referred to as the the Calculus of the 20th and 21st centuries. This analogy is more than a mnemonic: QFTs are believed to describe in a very precise way the *continuum limit* of systems with many interacting degrees of freedom. A particularly interesting set of quantum field theories are those which in addition to Poincaré symmetry also enjoy conformal symmetry. Such theories usually arise in the study of critical systems. Criticality is usually characterized by correlations of arbitrary range. This is characteristic of the fact that the system does not have a preferred length scale, so the system develops a scaling symmetry. In many cases this scale invariance is further enhanced to invariance under conformal transformations, and the system is said to be described by a *conformal field theory*. For example, the Euclidean conformal transformations are the set of transformations which preserve the values of angles between any pair of vectors in flat Euclidean space:

$$\delta_{ij} \to \Omega(x)^2 \delta_{ij}. \tag{2.1}$$

The generators of the Euclidean conformal group $SO(1, d+1)$ are associated to infinitesimal diffeomorphisms which change the metric by an overall factor; equivalently solutions to the conformal Killing equation.

$$\mathcal{L}_\xi \, g = \Omega^2(x) \, g. \tag{2.2}$$

Similarly the Lorentzian conformal group is analogously defined as the set of transfomations which change the spacetime metric by an overall positive function. In addition to usual symmetry generators of the Poincaré group $P_\mu, M_{\mu\nu}$, the Lorentzian conformal group $SO(2, d)$ contains the generators of dilatations D, and special conformal transformations K_μ:

$$\begin{aligned} [D, P_\mu] &= P_\mu \\ [D, K_\mu] &= -K_\mu \\ [K_\mu, P_\nu] &= \eta_{\mu\nu} D - i M_{\mu\nu} \end{aligned} \tag{2.3}$$

The dilatation operator D generates scaling transformations $x^\mu = \lambda x^\mu$, while the generator of special conformal transformations can be thought of as generating translations which fix the origin, in the same sense that the momentum operator P_μ generates translations which fix the point at infinity.

2.1.1 Euclidean CFT and Radial Quantization

A CFT on the Lorentzian cylinder is usually understood as the analytic continuation of Euclidean radial quantization of the same theory . In the Euclidean theory one is interested in computing correlation functions of local operators $\mathcal{O}(x)$

$$\left\langle \prod_{i=1}^{k} \mathcal{O}_i(x_i) \right\rangle. \tag{2.4}$$

Operators are then classified by their transformation under the Euclidean conformal group. This can be done by looking at the representation theory of a maximally compact $SO(d)$ subgroup of $SO(1, d+1)$, and the remaining non-compact $SO(1,1)$ subgroup. This means that operators can be classified by their quantum numbers λ under $SO(d)$, and their conformal dimension Δ

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0). \tag{2.5}$$

Since P_μ and K_μ satisfy the algebra of a set of raising and lowering operators, acting on an operator with them has the effect of raising and lowering the conformal dimension and spin of the operator. By acting successively on an operator with K_μ we will eventually obtain an operator with negative conformal dimension. In unitary theories the spectrum of conformal dimensions is non-negative $\Delta \geq 0$, which means that eventually the operator is annihilated by a sufficiently high power of K_μ. A primary operator is an operator that has a fixed conformal dimension (2.5) and is annihilated by K_μ

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{2.6}$$

By performing a conformal transformation on the plane, we can identify the action of the dilatation operator D as Euclidean time evolution in the radial variable $r = e^\tau$

$$ds^2 = \delta_{ij} dx^i dx^j = dr^2 + r^2 d\Omega_d^2 = e^{2\tau} \left(d\tau^2 + d\Omega_d^2 \right). \tag{2.7}$$

This means that by diagonalizing the dilation operator, we obtain a Hilbert space at reach constant radius slice spanned by eigenstates of a Euclidean time evolution operator generating shifts in τ. States of this Hilbert space live on S^d corresponding to a constant Euclidean time slice. Formally we may compute correlation functions by performing a path integral over the fields of the theory; we do this by folliating the plane by spherical slices centered around the origin and performing Euclidean time evolution from $\tau = -\infty$ to $\tau = \infty$. In the cylinder coordinates this can be understood as suming over state preparations at equal time slices and time evolving.

An operator inserted at the origin of the plane is then naturally associated to a state on the cylinder prepared in the infinite Euclidean past:

$$|\mathcal{O}\rangle = \lim_{\tau \to -\infty} \mathcal{O}(\tau, \boldsymbol{n}) \, |0\rangle \, . \tag{2.8}$$

In the path integral language this can be done by performing the path integral over a ball centered at the origin with fixed boundary conditions

$$\langle \psi(\boldsymbol{n}) | \mathcal{O} \rangle = \int_{\substack{\phi(r=1,\,\boldsymbol{n}) = \psi(\boldsymbol{n}) \\ r \leq 1}} D[\phi] \; e^{-S[\phi]} \mathcal{O}(0). \tag{2.9}$$

Similarly, any state $|\mathcal{O}\rangle$ on the cylinder can be used to construct a local operator on the plane by cutting a small ball around a point x on the plane and performing the path integral over this ball with the fields taking the appropriate boundary conditions. Due to conformal symmetry we can shrink the radius of the ball to zero size giving a local operator insertion at x. This is known as the operator-state correspondence. It is also important to note that Euclidean correlators are always computed with radial ordering, since out-of-order correlators involve divergences due to exponential factors coming from Euclidean time evolution. In practice we are mostly interested in analytically continuing these quantities to Lorentzian signature, where out-of-time order correlators are meaningful.

2.1.2 Correlation Functions and Conformal Symmetry

We can now turn to the main object of study for conformal field theories. The simplest observable to consider is the two-point function of (scalar) primary operators

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle; \tag{2.10}$$

Lorentz invariance fixes this to be a function of only $|x_1 - x_2|$. Acting with the dilation operator forces us to consider operators with the same scaling dimension $\Delta_1 = \Delta_2 = \Delta$ in order to get a non-zero answer. By dimensional analysis this leads to a generic for for the tw-point function of scalar primaries in any conformal field theory:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \delta_{\Delta_1,\Delta_2} \times \frac{G_{12}}{|x_1 - x_2|^{2\Delta_1}}. \tag{2.11}$$

The number G_{12} is often interpreted as a metric tensor in the space of primary operators, and is called the Zamolodchikov metric. Since the overall scaling by a complex number of the operator is irrelevant, we may chose linear combinations of operators of the same dimension to bring the Zamolodchikov metric to a diagonal form. Some conformal field theories can have a moduli space of exactly marginal deformations, in which case the Zamoldchikov metric can take different forms at different points in the conformal manifold. A deformation of the theory is formally understood as RG flow after deforming the action of the theory by an operator $\mathcal{O}_\lambda$

$$\delta S = \lambda \int d^{d+1}x \; \mathcal{O}_\lambda(x); \tag{2.12}$$

a deformation is exactly marginal if the β-function of the coupling parameter λ vanishes $\beta(\lambda) = 0$. The process of determining the spectrum of a CFT after an infinitesimal deformation by an exactly marginal operator is known as conformal perturbation theory. Generically, primary operators of the the theory at a particular value of the coupling λ will mix under RG flow with other operators, and the conformal dimensions of operators might change. This change in conformal dimension is usually called the anomalous dimension of the operator, relative to the undeformed theory. The first step in "solving" a conformal field theory is to fully determine the set of primary operators, or equivalently a complete set of orthogonal states of the Hilbert space fof the theory on the cylinder, and their conformal dimensions. The next simplest observable is a three-point function: conformal symmetry fully fixes the form of any three point correlator of scalar operators to be

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{2.13}$$

Unlike the normalization of the two point function, the normalization of the three-point functions contain dynamical information about the theory; the coefficients C_{ijk} describe how two operators fuse into a third operator. The multiplication rule for operators is known as the operator product (OPE) expansion

$$\mathcal{O}_i(x)\mathcal{O}_j(y) \sim \sum_k \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(x). \tag{2.14}$$

The statement of the operator expansion is asymptotic as x is taken close to y, and encodes the multiplication of the algebra of operators of the theory. it is widely believed that the spectral data $\{\Delta, C_{ijk}\}$ fully determines any correlation function in a given conformal field theory[1]. For generic (strongly coupled) CFTs, there very few analytical tools for determining the spectral data of a particular model, which has led to a successful revival of the conformal bootstrap program. In the recent years, this class of tools have been succesfully implemented in the study of strongly coupled CFTs, such as the universality of the 3d Ising model.

2.1.3 Lorentzian CFT

The symmetry group $SO(2,d)$ naturally acts on the Lorentzian cylinder $\mathbb{R} \times S^{d-1}$,

$$ds^2 = -dt^2 + d\Omega_{d-1}^2, \tag{2.15}$$

rather than Minkowski space. The reason for this is clear; global conformal transformations on the plane are allowed to move points out to infinity, which leads to obvious violations of causality in correlation functions. For this reason it is more natural to study real time dynamics of conformal field theory on the cylinder, where we have explicit covariance under global conformal transformations, as opposed to the plane $\mathbb{R}^{\nVdash,}$ where we only have invariance under infinitesimal conformal transformations. After analytic continuation from the Euclidean cylinder, the conformal dimensions of operators are mapped into the spectrum of the Hamiltonian generating time translations on the cylinder. The spin quantum numbers are then naturally identified with harmonic modes on S^d, and hence this analytic continuation leads to a quantum mechanics of infinitely many fields labeled by angular momentum modes. This is in contrast to ordinary quantum field theories, where one has uncountably many modes transforming as irreducible representations of the Poincaré group.

[1]There are some subtleties with some theories. For example *relative* CFTs do not have a single well defined torus partition function, and instead the partition function is a section of a vector bundle. An example of such an object is the 6d $(2,0)$ theory.

2.2 Anti-de-Sitter Space

There is yet another object with conformal symmetry; anti-de-Sitter space (AdS). AdS_{d+1} is a homogeneous space-time of constant negative curvature. One particular presentation of this space is as a certain cone[2]. Following [27], we can start with the quadric

$$\mathcal{Q}: \; uv - \eta_{\mu\nu}x^\mu x^\nu = 0. \tag{2.16}$$

with the following scaling identification $\mathcal{Q}/(\sim \mathbb{R}_+)$, where we identify the all coordinates by scaling by positive real numbers. For any fixed scale, this space is embedded in $\mathbb{R}^{\nvDash}$, with metric

$$ds_{\texttt{emb}} = -du dv + \eta_{\mu\nu}dx^\mu dx^\nu. \tag{2.17}$$

For generic values of v we can use the scaling identification to set $v = 1$ and $dv = 0$, from which we see that the space on a chart with $v \neq 0$ is simply Minkowski space. This space however differs from Minkowski space near regions were $u = 0$ or $v = 0$; in effect this construction compactifies Minkowski space by adding points at infinity. The topology of this space is $S^1 \times S^{d-1}$; where S^1 is a Lorentzian time circle. To avoid closed-time like curves we should pass to the universal cover of this space $\mathbb{R} \times S^{d-1}$. This is the Lorentzian cylinder which we will later identify with the conformal boundary of AdS_{d+1}. In this construction we have a manifest $SO(2,d)$ invariance of the Lorentzian cylinder. What we have described is a cone over a positive real line segment $\mathbb{R}_+$, where at every radial slice we have a Lorentzian cylinder which degenerates as in the deep interior of the geometry. This degeneration can be removed by deforming $\mathcal{Q}$

$$\mathcal{Q}_{\texttt{deformed}}: \; uv - \eta_{\mu\nu}x^\mu x^\nu = L^2_{AdS}. \tag{2.18}$$

[2]This presentation of AdS is closer to how one describe a resolved conifold.

After going to the covering space, this gives a non-compact space-time manifold whose boundary at infinity is a Lorentzian cylinder, and whose isometries are manifestly $SO(2,d)$. One particularly useful parametrization of this space is in terms of global coordinates

$$L_{AdS}^{-2}\, ds^2 = -(r^2+1)\, dt^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega_{d-1}^2, \tag{2.19}$$

in this description it is clear that the conformal boundary of the space is a Lorentzian cylinder.

$$L_{AdS}^{-2}\, ds^2 \sim r^2\left(-dt^2 + d\Omega_{d-1}\right) + \ldots \texttt{ for large } r \tag{2.20}$$

2.3 The AdS/CFT Correspondence

The AdS/CFT correspondence asserts that any theory of quantum gravity with asymptotically AdS boundary conditions is exactly equivalent to a conformal field theory in one lower dimension living on a Lorentzian cylinder which is identified with the conformal boundary of AdS. As such, the AdS/CFT correspondence is a *holographic duality*. The dictionary between observables of both theories is schematically of the form

$$\left\langle e^{\int J_{\mathcal{O}}\, \mathcal{O}} \right\rangle_{\texttt{CFT}} = Z_{QG}[\phi_\partial = J], \tag{2.21}$$

meaning that sources on the CFT are identified with a particular boundary condition for the theory on AdS. In practice this only allows us to identify operators identified with extrapolated values of semi-classical bulk fields; the problem of determining gravitational dual of a particular operator is the goal of *bulk reconstruction*. The extrapolate dictionary gives a precise relation between the mass spectrum of states in AdS to the scaling dimensions of operators on the CFT. For example, if the bulk effective theory contains a scalar field ϕ of mass m, there will be an operator $\mathcal{O}$ which sources ϕ. In the semiclassical limit, the bulk path integral is dominated by configurations with solve the classical equations of motion

$$(\nabla_\mu \nabla^\mu + m)\, \phi(t, r, \Omega) = 0 \tag{2.22}$$

The relation for a scalar field of mass m in AdS_{d+1} is to the conformal dimension δ of $\mathcal{O}$

$$\Delta(\Delta - d) = m^2, \tag{2.23}$$

which follows from the scaling at large r for the non-normalizable modes of ϕ; generalization to higher form tensor fields are obtained in a similar manner. In order to obtain a valid semi-classical gravitational description in the bulk we require that there are no light towers of higher spin fields in the spectrum; by generic EFT considerations this equivalent to the statement of scale separation between the AdS scale L_{AdS} and some other scale controlling the masses of higher spin states l_s. In the most well studied examples the scale l_s corresponds to the string scale, and the statement of scale separation is the statement that the typical size of a string is much smaller that the size of AdS.

$$\texttt{No higher spin light towers} \Rightarrow \frac{L_{AdS}}{l_s} \gg 1. \tag{2.24}$$

This scale is also associated with the strength of higher derivative corrections coming stringy effects. In order for the theory in the bulk to be well approximated by semi-classical Einstein gravity, we should also require that the AdS scale is much larger that the Planck scale:

$$\texttt{Semi-classical Einstein gravity} \Rightarrow \frac{L_{AdS}^2}{G_N} \gg 1. \tag{2.25}$$

Usually one thinks of the Planck mass as the mass of the smallest black hole, so $\frac{L_{AdS}^2}{G_N}$ roughly quantifies the size of the largest black hole one can fit in an AdS, which makes this quantity a measure of the number of degrees of freedom of the theory c.

While we still do not have a complete characterization of all conformal field theories with a holographic dual, insisting on a dual description based on semi-classical Einstein gravity with small higher derivative corrections restricts us to studying CFTs with a sparse low-lying spectrum of conformal dimensions, with a gap at $\Delta \sim c$, where c is the central charge of the theory. These types of theories are also expected to be strongly coupled. These restriction are quite strong, and very few examples of theories satisfying said conditions are known to exist; most of these are supersymmetric large N gauge theories and their dual description always involve strings.

2.3.1 Examples of the duality

The most studied example of the AdS/CFT correspondence is the duality between maximally supersymmetric Yang-Mills theory in 4d and Type IIB superstrings living in $AdS_5 \times S^5$. The first check of this duality one can perform is to match the symmetries of both models; $\mathcal{N} = 4$ SYM has a superconformal symmetry $PSU(2,2|4) \sim SO(2,4) \times SO(6)_R$ which are realized as isometries of the $AdS_5 \times S^5$ background. The $AdS_5 \times S^5$ solution of type IIB supergravity preserves maximal supersymmetry in ten dimensions and is known to be a solution of the Type IIB string to all orders in l_s. This geometry can be obtained as the near-horizon limit of a ten-dimensional black brane solution arising from a stack of N D3 branes.

$$ds^2_{\texttt{D3 brane}} = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-dt^2 + dx_{\parallel}^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right) \tag{2.26}$$

In the near horizon region $r \to 0$, this metric reduces to a Poincaré patch of $AdS_5 \times S^5$

$$\begin{aligned} ds^2_{AdS_5\times S^5} &= \frac{r^2}{L^2}\left(-dt^2 + \frac{L^4 dr^2}{r^4} + dx_{\parallel}^2\right) + L^2 d\Omega_5^2 \\ &= \frac{L^2}{z^2}\left(-dt^2 + \eta_{\mu\nu}dx^{\mu}dx^{\mu} + dz^2\right) + L^2 d\Omega_5^2. \end{aligned} \tag{2.27}$$

In the D3 brane solution, the parameter L is related to the number of branes N and brane tension T_{D3} by

$$L^4 = \frac{N}{2\pi^2 T_{D3}} \sim N\, G_{10} \Rightarrow \frac{L^4}{G_{10}} \sim N \gg 1, \tag{2.28}$$

where G_{10} is the ten dimensional Newton's constant. The tension of a p-brane is related to the string tension α' and string coupling g_s by the relation

$$T_{Dp} \sim \frac{1}{g_s}\left(\alpha'\right)^{\frac{p+1}{2}}, \tag{2.29}$$

so the parameter controlling higher derivative corrections in this case is

$$L^4/l_s^4 = L^4\alpha'^2 \sim g_s N \sim g_o^2 N \gg 1, \tag{2.30}$$

where g_s is the closed string coupling and g_o is the coupling of the open strings on the brane. On the other hand, the low energy effective dynamics on a stack of D3 branes is the described by the lowest lying modes of the open strings ending on the branes. This theory is an $\mathcal{N}=4$ $U(N)$ gauge theory with gauge coupling $g_s \sim g_{YM}$. This allows us to identify that $U(N)$ $\mathcal{N}=4$ SYM is dual to Type IIB strings on $AdS_5 \times S^5$ with N units of RR five-form flux on S^5, with the 't Hooft coupling $\lambda = g_{YM}^2 N$ being identified with the string tension in AdS units, and with closed string coupling $g_s \sim \frac{1}{N}$. In particular this example of AdS/CFT provides us with a concrete realization of 't Hoofts idea that large N gauge theories are described by theories of strings. Over the last two decades there have been many remarkable precision checks of this duality which we will describe in later sections. By extrapolating the duality to finite values of N and λ one can hope to give a non-perturbative definition of Type IIB string theory on a non-trivial background.

2.3.2 $\mathcal{N} = 4$ Super Yang-Mills Theory

One of the first pieces of evidences for the duality is the fact that the $AdS_5 \times S^5$ background of type IIB has 32 superchanges, which is consistent with the content of an $\mathcal{N} = 4$ superconformal algebra. The isometries of the background assemble into a $PSU(2,2|4)$ supergroup, which is precisely the superconformal algebra with maximal supersymmetry in four dimension. The maximal bosonic subgroup of $PSU(2,2|4)$ is $SO(2,4) \times SO(6)_R$. One of the theories that realizes this symmetry is the maximally supersymmetric Yang-Mills theory in four dimensions;

$$S_{\mathcal{N}=4} = \frac{2}{g_{YM}^2} \int_{\mathbb{R}\times S^3} d^4x \ \mathrm{Tr}\Big[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} D_\mu \phi_I D^\mu \phi^I - \frac{1}{2}\phi_I\, \phi^I + \frac{1}{4}[\phi_I, \phi_J][\phi_I, \phi_J] \\ - \frac{i}{2}\bar{\lambda}\, \Gamma^\mu D_\mu \lambda - \frac{1}{2}\bar{\lambda}\, \Gamma^I [\phi_I, \lambda]\Big]. \tag{2.31}$$

The matter content of the theory is a vector gauge field A_μ, six real scalars ϕ_I transforming in the **6** representation of $SO(6)_R$, and four Weyl fermions λ^A in the spin representation of $SO(6) \sim SU(4)_R$, which we write as the dimensional reduction of a ten dimensional Majorana-Weyl spinor. The matrices Γ^M are ten dimensional gamma matrices All fields are in the adjoint representation of the gauge group G, which we usually take to be $U(N)$ or $SU(N)$. In addition to the generators of conformal transformations, and global $SO(6)_R$ rotations, there are a set of 16 spacetime supersymmetries Q and 16 superconformal generators S satisfying the algebra

$$\begin{aligned}\{Q^a_\alpha, \bar{Q}_{b\,\dot\beta}\} &= 2\delta^b_a P_{\alpha\dot\beta}\\ \{S^\alpha_a, \bar{S}^{b\,\dot\beta}\} &= 2\delta^b_a K_{\alpha\dot\beta}\\ \{Q^a_\alpha, S_{b\,\beta}\} &= 4\left(\delta^a_b L^\alpha_\beta + \delta^\beta_\alpha R^a_b - \frac{i}{2}\delta^a_b\delta^\beta_\alpha D\right)\end{aligned} \tag{2.32}$$

In particular, Q and S act as raising and lowering operators for multiplets of the superconformal algebra. A particularly simple set of multiplets are short multiplets, which are annihilated by some linear combination of supercharges. This is along with the unitarity condition

$$\{Q, S\} \geq 0, \tag{2.33}$$

implies the saturation of a BPS bound:

$$\Delta = \sum_i a_i J_i + \sum_j b_j Q_i, \tag{2.34}$$

where the explicit form of the bound depends on how many supercharges annihhilate the short multiplet. The simplest family of short multiplets are the so-called $\frac{1}{2}$-BPS states which satisfy

$$\Delta = |J_R|, \tag{2.35}$$

where J_R is the R-charge associated to one of the Cartan generators of $SO(6)_R$. BPS multiplets of $PSU(2,2|4)$ have conformal dimensions that are protected by supersymmetry. Finding explicit representations of these multiplets in terms of the free $\mathcal{N} = 4$ SYM fields is generically difficult due to operator mixing, although it is expected that the spectrum of BPS operators of $\mathcal{N} = 4$ SYM is not renormalized beyond one loop. For example $\frac{1}{2}$-BPS primary operators of the theory correspond to scalars transforming in symmetric traceless tensor representations of $SO(6)_R$. It is customary to represent these in terms of complex scalar fields associated to a particular choice of maximal torus $U(1)^3 \subset SO(6)_R$, for instance

$$\begin{aligned} X &= \phi_1 + i\phi_2 \\ Y &= \phi_3 + i\phi_4 \\ Z &= \phi_5 + i\phi_6. \end{aligned} \tag{2.36}$$

So a generic $\frac{1}{2}$-BPS operator can be put in the form

$$\mathcal{O}_{l_i} = \prod_{l_i} \mathrm{Tr}\left[Z^{l_i}\right]; \tag{2.37}$$

due to non-renormalization theorems correlation functions of these kind of operators can be performed exactly at the free theory point which allows us to find their precise form after diagonalizing the two point function. It is expected that a similar analysis is possible for other short multiplets of the theory, although there is little evidence of non-renomalization of two point functions of more generic short multiplets. Finding exact expressions for representations of generic (long) multiplets remains beyond current techniques beyond low-loop orders where the mixing problem can be tackled explicitly. Despite this there has been an impressive progress in determining the spectrum of anomalous dimensions of non-protected operators in the planar limit.

2.3.3 AdS/CFT Integrability: realizing Feynman's dream

One of the first clues towards the solvability of large N $\mathcal{N} = 4$ SYM theory came from the analysis of Berenstein, Maldacena, and Nastase (BMN) [5]. Their insight was to consider not generic operators in long multiplets, but rather operators that are close to $\frac{1}{2}$-BPS operators

$$\mathcal{O}_{\texttt{BMN}} \sim \ldots ZZ\lambda^a ZZ\phi ZZZZZF_{\mu\nu}ZZZ\ldots . \tag{2.38}$$

These operators have large dimension Δ and large R charge J

$$\begin{aligned} N \gg \Delta \sim J \gg 1 \\ \Delta - J \sim O(1). \end{aligned} \tag{2.39}$$

The biggest insight of their analysis is that the correct expansion parameter in this limit is not the 't Hooft coupling λ, but rather $\frac{\lambda}{J^2}$, which allowed them to interpolate into the strong coupling regime at the cost of studying large charge operators. In this picture, the operators describing low lying string excitations on $AdS_5 \times S^5$ are made out of long chains of Z's making up a ferromagnetic vacua with small numbers of defects inserted along the chain. This was made precise by the work of Minahan and Zarembo, who solved the one-loop mixing problem in the planar limit by mapping into an integrable $SO(6)$ spin chain

$$D^{SO(6)}_{\texttt{one-loop}} = H^{SO(6)}_{XXX} = \frac{\lambda}{16\pi^2}\sum_i \left(K_{i,i+1} + 2 - 2P_{i,i+1}\right). \tag{2.40}$$

Combined with the discovery of the classical integrability of the superstring sigma model on $AdS_5 \times S^5$ this led to a series of sophisticated works giving predictions for the finite coupling spectrum of single trace operators of planar $\mathcal{N} = 4$ SYM.

2.4 Moving beyond integrability: Semiclassics and Large Charge Operators

Despite the great progress towards a solution of $\mathcal{N} = 4$ SYM, many important issues remained to be fully explored. Indubitably one of the properties that makes large N $\mathcal{N} = 4$ SYM an interesting model, is not only that it is a interacting theory of gauge fields that can be tackled analytically in certain regimes, but that it promises to be fully-fledge theory of quantum gravity. Unfortunately many of the truly gravitational questions one can ask the theory are out of reach from most of our current tools, such as bootstrap, or the planar expansion. One of the main issues one faces is the breakdown of the planar 't Hooft limit for complex enough correlation functions. This occurs when the number of non-planar diagrams becomes comparable to N. This suggests that integrability is generically broken, since string splitting processes contribute meaningfully. One particularly important issue is to understand the spectrum of large operators (i.e. $\Delta \sim N^{\alpha}$) in holographic CFTS. These class of operators describe heavy probes and fully backreacted geometries, such as black holes. By now it is well established that the degeneracy of supersymmetric states in many holographic SCFTS accounts for the entropy of supersymmetric black holes in AdS spaces, but the question of what these microstates look like remains largely unexplored, mainly due to the sheer combinatorial complexity associated with finding BPS states in large N SCFTs. Recent progress along these lines has been fueled by revival of the idea that large charge states should have a simpler semiclassical description in the large N limit. These ideas have been implemented successfully in the study of large BPS operators, and have yielded very efficient techniques for computing the large N limit of complicated correlation functions. One of the goals of such a program is to develop techniques to deal with non-protected operators in a similar manner as BMN, with the study of near-BPS operators being a natural target.

Chapter 3

Semi-Classical Open Strings and Branes

One of the simplest large operators in $\mathcal{N} = 4$ SYM is the determinant operator

$$\det(Z) = \frac{1}{N!}\epsilon^{I_1 \dots I_N}\epsilon_{J_1 \dots J_N} Z^{J_1}_{I_1} \dots Z^{J_N}_{I_N}. \tag{3.1}$$

Just like single trace operators are the analogs of mesons in $\mathcal{N} = 4$ SYM, determinant operators (and their generalizations) are baryon-like objects. However unlike QCD, where the description of baryons a a simple bound-state of quarks is only valid at weak-coupling, supersymmetry protects these kinds operators from mixing, making them genuine operators at all values of the coupling. Holographically, these operators describe a special kind of compact D-brane known as a maximal *giant graviton*. This operator is half-BPS with dimension $\Delta = J_R = N$. The insertion of mutually $\frac{1}{2}$-BPS giant gravitons breaks the symmetry of the theory to a $psu(2|2) \times psu(2|2) \rtimes \mathbb{R}$ subalgebra. In practice there are non-primary operators which are invariant under a larger symmetry algebra, for instance the generating function

$$\det\left(Z - z\right). \tag{3.2}$$

This operator is invariant under a centrally-extended supersymmetry algebra, $psu(2|2) \times psu(2|2) \rtimes \mathbb{R}^3$, where the complex parameter z acts as a central charge. The symmetry breaking pattern is analogous to the long string EFT :

$$SO(2,4) \times SO(6)_R \to (SO(4)_L \times SO(4)_R) \rtimes (\mathbb{R} \times \mathbb{C})\,, \tag{3.3}$$

For a sphere giant graviton the first $SO(4)_L$ factor should be interpreted as the symmetries along the directions transverse to the brane, the second $SO(4)_R$ are the isometries of the brane, and the semidirect product with $\mathbb{R}$ describe the helical motion of the brane. Finally the additional $\mathbb{C}$ describes the position of the longitudinal position of the brane. The remaining symmetry generators are non linearly realized, as with the long string. From an EFT point of view, this constraints the effective metric seen by the branes to be of the form

$$ds^2 = -h_1(dt+V)^2 + h_2 d\Omega_3^2 + h_3 d\tilde{\Omega}_3^2 + h_4 dz d\bar{z}, \tag{3.4}$$

and supersymmetry further constraint the forms of h_i as to make the background a slice of a half-BPS solution of type IIB supergravity. Then the most general effective action for this sector of the theory should contain in addition to a supergravity action, Dirac-Born-Infeld type terms describing the branes and Nambu-Goto terms describing string between different branes. This centrally extended algebra fixes the kinematics of excitations around large $\frac{1}{2}$-BPS states in the large charge limit. For single excitations, the scaling dimensions takes the famous form

$$\Delta - J_R = \sqrt{Q^2 + f(\lambda)|\mathcal{Z}|^2}. \tag{3.5}$$

For $\mathcal{N} = 4$ SYM the function f is expected to be $f(\lambda) = \frac{\lambda}{4\pi}$, and $\mathcal{Z}$ is the central charge carried by the state, and Q is the eigenvalue of one of the Cantan generators of the residual symmetry. This dispersion relation has been interpreted as arising from a relativistic mass formula for a W-boson, whose rest mass is given by the difference of vacuum expectation values of the scalar field Z

$$m_{ij}^2 \propto |\mathcal{Z}|^2 = |z_i - z_j|^2. \tag{3.6}$$

In practice, this dispersion relation allows us to determine the conformal dimension of the lowest lying primary states above a half-BPS state, to leading order in the large charge regime. In principle this result can be understood in terms of a simple semi-classical argument: for large enough operators the state is well approximated by a non-trivial solution to the equations of motion of $\mathcal{N} = 4$ SYM, which is precisely a Coulomb branch configuration for the scalar field Z. This analysis is unsatisfactory however, since it only allows us to estimate the energy/conformal dimension of a special family of states, but understanding further excitations around these states is a complicated task.

This naturally motivates the study of open string solutions on $AdS_5 \times S^5$, which we should think as an effective description of near-BPS operators with large quantum numbers. Since this leads to a well defined semiclassical problem, one can then hope to extract information about the spectrum of heavy excited states.

In addition to the determinant operators describing sphere giant gravitons, there exists an additional family of operators describing the so-called AdS giant gravitons;

$$S^{J_1 \ldots J_k}_{I_1 \ldots I_k} Z^{J_1}_{I_1} \ldots Z^{J_k}_{I_k} \tag{3.7}$$

where S is rank (k,k) fully symmetric tensor of $U(N)$. These class of operators are naturally packaged together into a generating function

$$\oint ds \frac{e^{zs}}{\det(Z-z)}. \tag{3.8}$$

This operator generates all fully symmetric tensors, and it creates a coherent state configuration where Z has a single non-zero eigenvalue z

$$:\bar{Z}_I^J \oint ds \frac{e^{zs}}{\det(Z-z)} :\sim z\, \delta_I^J \oint ds \frac{e^{zs}}{\det(Z-z)}. \tag{3.9}$$

3.1 Review of Giant Magnon Solutions

Let us recall the scaling limit of the giant magnons of Hoffman and Maldacena [28]. We are interested in the following scaling limit:

$$\begin{aligned} J\,,\ \Delta &\to \infty \\ \lambda\,,\ p &< \infty \\ \Delta - J = \epsilon &< \infty \end{aligned} \tag{3.10}$$

Where J is one of the $SO(6)$ R-charges, p is the momentum of the excitation, and $\lambda = g^2 N$ is the t' Hooft coupling. In order for the semi-classical string description to be valid we should also consider the t' Hooft coupling λ to be large. Then we seek for the solution with the least energy ϵ for a fixed momentum p. The simplest of such configuration is given by a string that sits at the origin of AdS_5 while its endpoints rotate along the equator of S^5. The motion takes place on $\mathbb{R} \times S^2$

$$ds^2 = -dt^2 + \cos^2\psi d\theta^2 + d\psi^2. \tag{3.11}$$

Note that the spatial metric is for a sphere of radius one with two coordinate singularities at $\psi = -\pi/2$ and $\pi/2$, where we have used a cosine of the angle, rather the sine of the angle. We have chosen a slightly different convention for the the metric of the sphere in order to make the analytic continuation into AdS_2 clear. We can choose a parametrization for a rigidly rotating worldsheet coordinates for the Nambu-Goto string given by

$$\begin{aligned} \tau &= t \\ \sigma &= \theta - t \\ \dot{\psi} &= 0 \end{aligned} \tag{3.12}$$

The condition $\dot{\theta} = 1$ arises from the fact that the string becomes asymptotically a ferromagnet for the $SU(2)$ chain. In the presence of giant gravitons, this is the correct coordinate velocity for the motion of the giant gravitons themselves.

Upon substitution of the rigid ansatz, we get the action

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{\sin^2 \psi \psi'^2 + \cos^2 \psi}. \tag{3.13}$$

Using the coordinate transformation $r = \cos\psi$, minimizing the action (3.13) takes the form of a simple geodesic problem with an effective metric $ds^2 = dr^2 + r^2 d\theta^2$. This is not the original metric of $AdS_5 \times S^5$, but it is a flat auxiliary geometry. By virtue of $r \leq 1$, this is a flat metric on a disk. The conserved charges are given by

$$\begin{aligned} \Delta &= \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \frac{\psi'^2 + \cos^2 \psi}{\sqrt{\psi'^2 \sin^2 \psi + \cos^2 \psi}} \\ J &= \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \frac{\psi'^2 \cos^2 \psi}{\sqrt{\psi'^2 \sin^2 \psi + \cos^2 \psi}} \end{aligned} \tag{3.14}$$

It is convenient to write them also in terms of the r variables:

$$\begin{aligned} \Delta &= \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \frac{(r')^2(1-r^2)^{-1} + r^2}{\sqrt{r'^2 + r^2}} \\ J &= \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \frac{(r')^2 r^2 (1-r^2)^{-1}}{\sqrt{r'^2 + r^2}} \end{aligned} \tag{3.15}$$

These develop a singularity whenever $r \to 1$ in the solution. Notice that by contrast

$$\epsilon = \Delta - J = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \sqrt{r'^2 + r^2} \tag{3.16}$$

is non-singular and stays finite. More to the point, for these simple strings, extremizing ϵ is the same as minimizing the geodesic problem we found above.

Note that $|r| \leq 1$ in order for the solutions to make sense, as $r = \cos\psi$ and the angle ψ is a real coordinate of the sigma model. Near $r = 1$ both of these expressions scale with $(1-r^2)^{-1}$. Substituting the explicit solution $r = a\sec\sigma$, the expression for ψ'^2 diverges whenever $a\sec\sigma = 1$:

$$\psi'^2 = \frac{a^2 \sec^2\sigma \tan^2\sigma}{1 - a^2\sec^2\sigma} \tag{3.17}$$

so the density of the conserved charges becomes infinite near such points. This is how one can have a smooth ending on an infinite spin chain that has not been closed. However the effective energy of the configuration, $\Delta - J$ remains finite and it is the same as the on-shell action, which is clearly the length of a straight line segment connecting the two points on the edge of the auxiliary disk geometry:

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin\frac{\Delta\theta}{2} \right| \tag{3.18}$$

The angle between the two end-points is identified with the momentum of the defect p, as originally noticed in [29] by a matrix model ansatz. For closed string solutions we can sew together various of these straight line solutions on the disk such that in the end $\Delta\theta = 0$, forming a closed polygon. This is equivalent to the level matching condition of a closed string $p_{total} = 0$.

The case of open strings stretched between giant gravitons is entirely analogous except that one must impose that the endpoints of the string lie inside disk. The end points must be attached to the location of the giant gravitons. As the S^3 on which the branes are wrapped shrinks to zero size at $r = 1$, these do not exit the disk. The analysis has been carried out in [30].

As a result of the end-points not reaching $r = 1$, both of the charges Δ, J remain finite. The disk coordinates $ds^2 = dr^2 + r^2 d\psi^2$ can then be seen to be the coordinates for the LLM plane of $AdS_5 \times S^5$ [16]. This description of the physics in terms of a disk also appears directly from the field theory dual [31].

3.2 Open Giant Magnons in AdS

3.2.1 $AdS_2 \times S^1$

Now we proceed to study the analogous solutions for the case of open strings stretching between two D3 branes that wrap an S^3 inside AdS_5 while rotating at angular velocity $\omega = 1$ along an $S^1 \subset S^5$. As in the previous section we may consider the Nambu-Goto action for a string on an $AdS_2 \times S^1$ geometry. The metric is given by

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + d\phi^2. \tag{3.19}$$

We will be interested in solutions of the equations of motion where the string rigidly rotates with the dual giant gravitons. As such, ϕ will evolve in time in the same way that the coordinates of the dual giants do. These rotate at constant velocity in ϕ, have a fixed value of ϕ at each time and are located at fixed values of ρ [32, 33]. That is, $\dot{\phi} = 1$. These are a different type of solution to the rigidly rotating GKP string [34], as they have motion in one extra dimension.

We choose to parametrize the worldsheet coordinates by $t = \tau$, $\phi = \tau + \sigma$ and we will be looking for solutions where $\rho(\sigma)$ is independent of τ. The induced metric on the string worldsheet for these solutions is given by

$$ds^2_{ind} = \begin{pmatrix} -\cosh^2\rho + 1 & 1 \\ 1 & 1 + (\rho')^2. \end{pmatrix} \tag{3.20}$$

This way we find that

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \sqrt{-g} = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \sqrt{\sinh^2(\rho)(\rho')^2 + \cosh^2(\rho)} \tag{3.21}$$

which can be seen to be an analytic continuation of the sigma model action of the solutions of the giant magnons of Hoffman and Maldacena, $\psi \to i\rho$. The problem again simplifies by introducing the change of variables $r = \cosh(\rho)$, so that

$$S \propto \tau \int d\sigma \sqrt{r'^2 + r^2} \tag{3.22}$$

The expression $\int d\sigma\sqrt{r'^2 + r^2}$ can be easily seen to be the length of a curve on a flat geometry in polar coordinates $d\tilde{s}^2 = dr^2 + r^2 d\sigma^2$, in a parametrization $r(\sigma)$. This is minimized by a straight line, where

$$r = \frac{a}{\cos(\sigma - \sigma_0)} \tag{3.23}$$

where a is the distance of closest approach to the origin and σ_0 is the angle in the plane of closest approach. The energy computed this way is also proportional to the length of the straight line in this auxiliary geometry from the starting point to the end point.

It is important to note that although superficially similar to the solutions of (3.13), for the change of variables to make sense in this case we must also impose that $|r| > 1$ everywhere on the worldhseet, the reason being the $S^3 \subset AdS^5$ shrinks to zero size at $r = 1$. In particular solutions to (3.21) which cross the unit circle are not physical, as they would require the radial coordinate of AdS to become complex. Even though there are in principle solutions of minimal length when one removes the inside of the disk, these are not stable in that they should receive quantum corrections since they are no longer BPS.

The coordinates of this auxiliary plane geometry span the region outside the disk in the LLM plane, rather than the inside. Solutions with $r = 1$ somewhere have a similar behavior to the Hofmann Maldacena solution, in that the density of the charges Δ, J becomes infinite at $r = 1$, yet the energy $\Delta - J$ remains finite.

The introduction of the charge is straightforward. We will do that analysis in the discussion of the following section. We mostly follow the discussion in [35], which starts from the Nambu-Goto string, to make the analysis.

3.2.2 Rotating String in $AdS_3 \times S^1$

Now we consider the sigma model of a rotating string on $AdS_3 \times S^1$ which corresponds to a two-spin magnon solution. The metric is a simple generalization of (3.19)

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2 + d\phi^2 \tag{3.24}$$

Here again it is convenient to use the coordinate $r = \cosh\rho$ in the Nambu-Goto string action. As noticed in the appendix, the coordinate $r = \cosh(\rho)$ is the radius of the LLM plane coordinate.

The metric (3.24) can be analytically continued via $\rho \to i\tilde{\rho}$ into $\mathbb{R} \times S^3$ where the S^3 is expressed in Hopf coordinates. We make an ansatz for the embedding coordinates of the form :

$$\begin{aligned} t &= \omega_t \tau \\ r &= r(\sigma) \\ \theta &= \beta\tau + g(\sigma) \\ \phi &= \omega_\phi \tau + \varphi(\sigma) \end{aligned} \tag{3.25}$$

Where we are interested in $\omega_t = \omega_\phi = 1$.

The action for the rigid string in these coordinates is given by:

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{r'^2 + \varphi'^2 + 2\beta g'\varphi'(r^2-1) + g'^2(r^2-1)^2 - \beta^2(r'^2 + \varphi'^2(r^2-1))}, \tag{3.26}$$

where we have set the angular frequencies to one for simplicity. The conserved quantities can be easily evaluated via the formulas:

$$\begin{aligned} \Delta &= -\int d\sigma \frac{\partial \mathcal{L}}{\partial \omega_t}\Big|_{\omega_t=1} \\ J_1 &= \int d\sigma \frac{\partial \mathcal{L}}{\partial \omega_\phi}\Big|_{\omega_\phi=1} \\ J_2 &= \int d\sigma \frac{\partial \mathcal{L}}{\partial \beta} \end{aligned} \tag{3.27}$$

As we will see, one can eliminate the angular variable g by using its equation of motion:

$$\partial_\sigma\left(\frac{(r^2-1)(\beta\varphi'+(r^2-1)g')}{\sqrt{r'^2+\varphi'^2+2\beta g'\varphi'(r^2-1)+g'^2(r^2-1)^2-\beta^2(r'^2+\varphi'^2(r^2-1))}}\right)=\partial_\sigma\mathfrak{J}^\sigma=0 \tag{3.28}$$

We have chosen to write it in terms of an expression that is implied by current conservation, which forces $\mathfrak{J}^\sigma$ to be constant. At first solving for g might seem daunting as (3.28) reduces to a complicated equation depending on an integrating constant, which is the value of $\mathfrak{J}^\sigma$. A great simplification is possible since in the end we are interested in describing strings ending on a pair of giant gravitons, so one must be careful about imposing the correct boundary conditions at the string end points.

The boundary term that arises from taking the variation of the action (3.8) is of the form:

$$S_{bdy}\propto\int d\tau\,(\delta\theta\,\mathfrak{J}^\sigma)\Big|_{\sigma_i}^{\sigma_f} \tag{3.29}$$

Where $\mathfrak{J}^\sigma=\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\theta)}$ is precisely the quantity inside the parentheses of (3.10). The other boundary terms are set to vanish by imposing the appropriate Dirichlet boundary conditions $\delta r=\delta\phi=0$ and by the choice of static gauge $\partial_\sigma t=0$. Because the D-brane we are considering is extended in the θ direction, the correct boundary conditions for θ are Neumann boundary conditions. That means that $\delta\theta$ is free to vary.

We must then conclude that $\mathfrak{J}^\sigma$ vanishes at the end-points of the string in order for the variational principle to be well-defined. In addition to this, equation (3.28) implies that this quantity vanishes everywhere along the string. This allows us to solve implicitly for the function g in terms of the other coordinates:

$$\partial_\sigma g=-\beta\frac{\partial_\sigma\varphi}{r^2-1} \tag{3.30}$$

Solutions of this type have been previously considered for infinite strings in the $S^3\times\mathbb{R}$

sigma model (see [35, 36]), but their physical interpretation was not made clear. Here the meaning of the condition is clear: the giant magnon does not transport angular momentum in the θ direction from one D-brane to the other. The giant magnon carries that angular momentum, but it does not transfer it to the D-branes. Eliminating the variable g in this case does not affect the variational principle for r, so one may substitute that condition directly in order to express the conserved quantities in terms of the on-shell action multiplied by some kinematic factors:

$$\begin{aligned}\Delta - J &= \frac{\sqrt{\lambda}}{2\pi}\frac{\mathcal{Z}}{\sqrt{1-\beta^2}}\\ J_1 &= \frac{\sqrt{\lambda}}{2\pi}\frac{\beta\,\mathcal{Z}}{\sqrt{1-\beta^2}}\\ \mathcal{Z} &= -\int_{\varphi_i}^{\varphi_f} d\varphi\sqrt{r^2+(\frac{dr}{d\varphi})^2}\end{aligned} \tag{3.31}$$

To do the variation, we want to minimize the energy, $\epsilon = \Delta - J$ at fixed β, with the endpoints on the dual giant gravitons. This is a straightforward minimization of $\mathcal{Z}$ that results in a straight line. Clearly the variable $\mathcal{Z}$ corresponds (up to a factor) to the length of the string on an auxiliary flat 2D geometry with a disk removed, and as such should be identified with the central charge of the $SL(2)$ sector of the spin chain. The complex coordinate $\xi = r\exp(i\varphi)$ can be used to express the answer in terms similar to those of (4.1). Here we see that $|\xi_1 - \xi_2|$ is the length of the segment in the LLM plane connecting the two giant gravitons.

Eliminating β in terms of J_2 and $|\mathcal{Z}|$ yields the dispersion relation

$$\Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{4\pi^2}|\mathcal{Z}|^2} \tag{3.32}$$

Which is precisely of the form expected from equation (4.1). Although the boundary conditions considered here lead to solvable equations of motion for the ground state of the sigma model, it is not expected that these solutions (and their many magnon counterparts) can be fully described by an integrable spin chain. The boundary effects sourced by the branes are expected to destroy that property. This has been argued, at least from the notion of a simple Bethe Ansatz point of view, for the open spin chains attached to regular giant gravitons in [37]. It would be very interesting if a class of solutions of the sigma model of this type do lead to integrable boundary conditions for the dual spin chain description.

It is also instructive to give explicit expressions for the conserved quantities Δ, J_1. Since we are considering open strings of finite size, one would expect that these quantities are finite. However, for similar reasoning to that of the previous section, one must be careful that the radial coordinate doesn't touch the unit circle given by $r = 1$. We can see this from the expressions:

$$\begin{aligned} \Delta &= \frac{\sqrt{\lambda}}{2\pi} \int d\phi \frac{r^2(\frac{dr}{d\varphi})^2 + \beta^2 - 1 + r^4}{(r^2-1)\sqrt{(1-\beta^2)(r^2 + (\frac{dr}{d\varphi})^2)}} \\ J_1 &= \frac{\sqrt{\lambda}}{2\pi} \int d\phi \frac{(\frac{dr}{d\varphi})^2 + \beta^2 - 1 + r^2}{(r^2-1)\sqrt{(1-\beta^2)(r^2 + (\frac{dr}{d\varphi})^2)}} \end{aligned} \tag{3.33}$$

It's clear that these quantities diverge whenever $r = 1$ (that is, if the string touches the origin of AdS), even though the length of the string on the auxiliary plane geometry is finite as in the Hoffman-Maldacena string. Solutions like these, where a physical worldsheet quantity is becoming divergent should become unphysical and lead to non-normalizable states, in a way similar to the discussion in [38]. More concretely, these lead to operators that would inject an infinite amount of energy into the bulk and where the radial coordinate becomes complex.

3.3 Discussion

In this chapter we have provided evidence for the all-loop dispersion relation for the excitations of the $SL(2)$ sector of the $\mathcal{N} = 4$ SYM spin chain with open boundary conditions. The calculation is done in the sigma model and it gives rise to a series expansion in the t'Hooft coupling λ. We find a nice description of the solutions in terms of an analytic continuation of the $SU(2)$ sigma model and the solution has a form that suggests that these states arise from a BPS condition on the string worldsheet, as would be expected from shortening conditions of the central extension of the $\mathcal{N} = 4$ SYM spin chain [39]. It would be nice to understand this better from the planar $SL(2)$ sector of the $\mathcal{N} = 4$ SYM theory in more detail. This spin chain should precisely realize the open string sigma model in a continuum limit. It would also be of interest to consider more general solutions to the string sigma model corresponding to scattering and bound states of giant magnons. After all, the open strings suspended between ordinary giant gravitons have a relation to the Bethe ansatz of the $SU(2)$ spin chain, as noted in [30].

The description in the $SL(2)$ sector is expected to be somewhat qualitatively different than the $SU(2)$ magnons due to the nontrivial boundary condition imposed by $|r| > 1$. This is already clear from the sigma model description, as constructing solutions from inverse scattering methods for the $SL(2)$ model seems to require different Bethe roots than the $SU(2)$ sigma model, even though their solutions should be related by an analytic continuation of the solutions in the $SU(2)$ sector [40]. It is unclear what the precise structure of bound and scattering states is without further study of the field theory side.

We also found that the LLM coordinates arise naturally from studies of the string sigma model in the $SL(2)$ sector. This might allow us to understand better how certain aspects of locality in the radial direction of AdS arise from the dual field theory directly. One should also expect that given these solutions, that one could also compute the spectrum of quadratic fluctuations around these solutions and could in principle compare to the $\mathcal{N} = 4$ SYM spin chain. Similar simplifications as the ones found here should also be possible on backgrounds of the form $AdS_5 \times X_5$ where X_5 is a Sasaki-Einstein manifold, as they share the AdS part of the sigma model, an the Sasaki-Einstein geometries have a $U(1)$ fibration that should allow solutions of a similar kind. It is also interesting to study the case of $AdS_4 \times \mathbb{C}P^3$ with fluxes, related to the ABJM model [41]. The latter should be quite interesting, as the exact characteristic of the allowed brane configurations depend greatly on the details of the field theory set-up.

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Chapter 4

Open Strings on LLM Geometries

4.1 Introduction

The AdS/CFT correspondence has established an equivalence between some theories of quantum gravity in asymptotically AdS spacetimes and certain gauge theories. The most celebrated example is the equivalence between IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory [1].

The free strings propagating on the $AdS_5 \times S^5$ background are believed to be integrable for all values of the t'Hooft coupling. A review of the main results in this direction can be found in [2]. On the field theory side, the integrability takes the form of a spin chain Hamiltonian [3, 42]. The spin chain acts on the list of gauge invariant local operators, the states being generated by traces of words of local fields of $\mathcal{N} = 4$ SYM and their derivatives. The main computation of the energy on the spin chain side corresponds to the anomalous dimension of the operators.

Integrability, combined with supersymmetry is very powerful. A particularly important result that combines the two is the dispersion relation for magnons on the gauge theory spin chain [39]. It follows from a central extension in the symmetry algebra of the spin chain and from the fact that magnons are in short representations of the centrally extended symmetry algebra. This shortness condition fixes the kinematics.

When the magnons carry a lot of momentum on the spin chain, they become geometrically large string solutions in the AdS dual. These are called giant magnons [28]. These also carry central charge on the spin chain. The total central charge of a closed string state vanishes because of the level-matching constraint. In the spin chain side this arises from the cyclic property of the trace [5].

The central charge on the gauge theory spin chain can also be sourced by open boundary conditions. These can be realized by supersymmetric D-branes in the AdS side, with open strings attached to them. These D-brane states provide a very tractable connection between the gauge theory dynamics and the AdS geometry. This connection to the central charge extension on the spin chain and gravity dual side has been analyzed in the works [43, 44, 30, 38, 45, 46, 22]. Particularly, it has been suggested in [38] that the central charge extension on the spin chain side is very closely related to the central charge extension of the Coulomb branch of $\mathcal{N} = 4$ SYM. Our recent work [22] showed how this works on the sigma model side for open string states suspended between D-brane states made of AdS giant gravitons. A complete picture in the analysis of the spin chain side is still missing.

As a reminder, giant gravitons are D-brane states that preserve half the supersymmetry of the $\mathcal{N} = 4$ SYM theory. They can grow into the sphere [47], or into the AdS directions [33, 32]. The ones that grow in the AdS directions are related to (classical) spontaneous symmetry breaking from $U(N) \to U(N-1) \times U(1)$ via the Higgs mechanism, which generates expectation values for the scalars [32] (see also [31]). All of these D-brane states can be understood in terms of the classification of half-BPS states in $\mathcal{N} = 4$ SYM in terms of Young tableaux [48]. Sphere giant gravitons are represented by long single columns [49], while AdS giant gravitons are long single rows.

Very importantly, the AdS giant gravitons explore the radial direction of the AdS_5 geometry. This has always been the most mysterious emergent dimension in the AdS/CFT correspondence. It has been related to a UV-IR relation [50], where the position in AdS is related to the UV scale of physics on the boundary. The radial direction has also been related to the Renormalization group flow [51] and via the AdS D-branes, it is also related to the Higgs mechanism.

The radial direction on the spin chain side is much less well understood. Some states that explore the radial direction appear when rotating strings in AdS are studied [34], see also [52, 53]. They are characterized by logarithmic contributions in the spin quantum number to their anomalous dimension. An argument for their logarithmic scaling of anomalous dimension is given in [54]. The open strings strecthing between AdS giants that have been studied previously by us [22] do not have such logarithmic contributions to their anomalous dimension. Instead, their anomalous dimensions are governed by supersymmetry, and in particular, by the amount of central charge they carry. Their dispersion relation is

$$\Delta - J = \sqrt{Q^2 + \frac{\lambda}{4\pi^2}|\mathcal{Z}|^2}, \tag{4.1}$$

where Q is the angular momentum on $S^3 \subset AdS_5$ and $\mathcal{Z}$ is the central charge of the open string, in geometric units. At very large angular momentum on the sphere ($Q \to \infty$), for the giant magnons suspended between D-branes, their anomalous dimension can be arbitrarily close to zero, even at strong coupling. This follows because the square root can be expanded in powers of $\mathcal{Z}/Q$. This is a power series in the t'Hooft coupling λ, and therefore one can in principle match coefficients order by order in perturbation theory on the CFT side.

Other sets of works suggest that the giant magnon dispersion relation also plays a role in more general geometries. In particular, it has been argued that the central charge extension controls magnon dispersion relations in concentric LLM geometries [55, 56]. The LLM geometries are solutions of type IIB supergravity on $AdS_5 \times S^5$ that preserve exactly half of the supersymmetries [16]. They can be thought of as condensates of sphere giant gravitons and/or AdS giant gravitons. These geometries depend on a two coloring of a degeneration plane (the LLM plane). If the coloring is made of concentric circles, the background geometry has a well defined additional circular R symmetry J. This is the same J as the one appearing in (4.1). For general LLM background, only $\Delta - J$ is well defined. The backgrounds break the J, Δ symmetries independently, leaving $\Delta - J = 0$ for the background configuration.

It is the purpose of the current work to address this idea on the sigma model side. In particular, we want to understand exact solutions of the sigma model for open strings stretching between sphere giant gravitons or AdS giant gravitons in general LLM geometries. These are more complicated geometries than $AdS_5 \times S^5$. To get a simple finite answer, there must be interesting cancellations taking place in the gravity calculation. One of our goals to see how this analytic behavior arises in the string sigma model computation.

In particular, we will find exact expansions in the t'Hooft coupling for $\Delta - J$ as above, that can in principle be matched to perturbative computations in field theory. It turns out, that even though the sigma model in these geometries is not expected to be integrable (for example, a naive Bethe ansatz is expected to have inelastic boundary conditions [57]), it is under enough analytic control so that these BPS strings are analytically solvable and the dispersion relations for the open strings will look identical to equation (4.1). We pay extra attention to geometries that correspond to concentric circles, because they allow us to explore the amount of charge J that the string carries. It will turn out that the quantity J contains additional information that is not carried by either the angular momentum Q or the central charge. It depends on more details of the precise position of the string in the background geometry. Nevertheless, after some manipulations, we will show that it also gives rise to an expansion in the t'Hooft coupling that can be matched between the spin chain and the geometry, at least to leading order. These seem to be non-protected quantities associated to the symmetries that are spontaneously broken by the D-brane background for these states. These determine in the end if a putative string state belongs to the Hilbert space or if it does not. If the J charge diverges, the string state is not allowed. Such phenomena already arise in the spin chain computations [38], so it is important to understand their behavior in the gravity dual setup as well.

Another interesting aspect of the open strings between sphere giant gravitons is that there is a relation of the geometric sigma model solution and the Bethe ansatz on the spin chain [30]. When one studies open strings attached to these giant gravitons, sites can "jump in" and "jump out" of the spin chain [58]. To have a more standard description, one realizes the spin chain in a bosonized language. One writes the states in terms of the number of sites between defects on the spin chain, rather than in terms of spin up and spin down state. Now, the number of sites in the spin chain is fixed, and the boundary conditions allow number non-conservation for the bosonic excitations instead. If one writes coherent states for these generalized bosons, one finds that the equations that lead to the ground state of the spin chain can be understood as a bound state condition on the S-matrix of the magnons, subject to corresponding boundary conditions. Similar results are not known in the dual $SL(2)$ sector. An approximation for the $SL(2)$ sector at very large vevs of the central charge for strings stretching between a dual giant graviton and itself can be found in [59], where the "jumping in" and "jumping out" of letters is self-consistently ignored in the limit of large spin/central charge. A similar connection to the Bethe ansatz is not known.

We provide evidence in this chapter that the open strings stretching between dual giant gravitons also have an interpretation in terms of zeros of an S-matrix for the $SL(2)$

sector. In particular, we get a better understanding of the analytic continuation of the $SU(2)$ spin chain to the $SL(2)$ sector. We will also show that our interpretation of the analytic continuation is compatible with the sigma model calculations.

The chapter is organized as follows. In section 2 we review the form of all half BPS solutions to type IIB supergravity, and we re-express them in form that makes certain cancellations clearer. In section 3 we provide an explicit example of the open string solutions in question by considering the case of $AdS_5 \times S^5$. In section 4 we solve for open strings stretching between both sphere and AdS giants in a general half BPS geometry, finding very similar expressions to those in the case of $AdS_5 \times S^5$. In section 5 we concentrate on concentric half-BPS geometries, for which we study the form of the R-charge J and its relation to the metric of the half-BPS geometry. In the case of $AdS_5 \times S^5$, we study various limits for which this expression simplifies, and match the leading sigma model answer on $\mathbb{R} \times S^3$ to a computation on the dual one-loop $\mathfrak{su}(2)$ spin chain. We are able to interpret the sigma model solution as a continuum limit of a zero/pole of the magnon S-matrix for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$. The answers on both sectors are related to each other by an analytic continuation of the radial parameter of the LLM plane.

4.2 Review of LLM Geometries

The most general $\frac{1}{2}$-BPS solution to IIB 10d supergravity is given by the ansatz [16]:

$$ds^2 = -\frac{y}{\sqrt{\frac{1}{4} - z^2}}(dt + V)^2 + \frac{\sqrt{\frac{1}{4} - z^2}}{y}(dy^2 + dx_1^2 + dx_2^2) + y\frac{\sqrt{\frac{1}{2} - z}}{\sqrt{\frac{1}{2} + z}}d\Omega_3^2 + y\frac{\sqrt{\frac{1}{2} + z}}{\sqrt{\frac{1}{2} - z}}d\tilde{\Omega}_3^2 \tag{4.2}$$

The only free parameter of this metric is an auxiliary function z of the coordinates y, x_1, x_2, which satisfies a six dimensional Laplace equation with rotational symmetry along four directions:

$$
\begin{aligned}
d\left(\star_3 \frac{dz}{y}\right) &= 0 \\
ydV &= \star_3 dz
\end{aligned}
\tag{4.3}
$$

Where $\star_3$ is the Hodge star operation on the coordinates x_1, x_2, y. In order to ensure the regularity of (4.2), we must have that the quantity $\mathcal{H}^{-2} = \frac{y}{\sqrt{\frac{1}{4}-z^2}}$ remains finite as y approaches zero. This means that $z = \pm\frac{1}{2}$ on the $y = 0$ plane which we will call the LLM plane. Because of this, the metric is completely determined by a coloring of the LLM plane into regions where $z = \pm\frac{1}{2}$. It will also be convenient to rewrite the metric in a more compact form:

$$
ds^2 = \mathcal{H}^{-2}\left(-(dt+V)^2 + \left(\frac{1}{2} - z\right)d\Omega_3^2 + \left(\frac{1}{2} + z\right)d\tilde{\Omega}_3^2\right) + \mathcal{H}^2\left(dy^2 + \delta_{ij}dx^i dx^j\right) \tag{4.4}
$$

This is convenient since the parametrization (4.4) makes the metric explicitly regular at $y = 0$ inside the colored regions. Generically, at the boundary of a droplet the one form V becomes singular, but such singularities can be eliminated via coordinate transformations. An explicit form of V is:

$$
V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{2\pi}\oint_{\partial\mathcal{D}} \frac{dx'_j}{(\mathbf{x}-\mathbf{x}')^2 + y^2} \tag{4.5}
$$

Here the integration is taken along the boundaries of the droplets. This guarantees that $V_i \to 0$ as $|x|, |y| \to \infty$. Finally, the regions $z = \pm\frac{1}{2}$ are the degeneration loci of the either one of the two three-spheres, which is clear from equation (4.4).

4.3 Open Strings on $AdS_3 \times S^1$

Now we wish to review the solutions of the Nambu-Goto sigma model corresponding to rigidly rotating strings on a $AdS_3 \times S^1$ subspace of $AdS_5 \times S^5$ that appeared in [22], but studying the solution directly in the LLM cordinates instead. The corresponding metric in the coordinates (4.2) is described by a single droplet configuration on the LLM plane of radius r_0:

$$\begin{aligned} z(r, y; r_0) &= \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} \\ V_\phi &= -\frac{1}{2}\left(\frac{r^2 + r_0^2 + y^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} - 1\right) \end{aligned} \tag{4.6}$$

We will be interested in solutions that reside at the $y = 0$ locus, with a D3 brane wrapping the non-vanishing three-sphere with the strings rotating along a circle of the non-vanishing sphere. The effective metric for the space on which the strings move can be written in the form:

$$\begin{aligned} ds^2 = &- s\left(1 - \frac{r^2}{r_0^2}\right)(dt + V_\phi d\phi)^2 + s\frac{(dr^2 + r^2 d\phi^2)}{\left(1 - \frac{r^2}{r_0^2}\right)} \\ &+ s\left(1 - \frac{r^2}{r_0^2}\right)\left(\frac{1}{2}(1 - s)d\psi^2 + \frac{1}{2}(1 + s)d\tilde{\theta}^2\right) \end{aligned} \tag{4.7}$$

with $s = \mathrm{sign}(r_0 - r)$. The effective geometry for $r > r_0$ corresponds to $AdS_5 \times S^1$, while $r < r_0$ corresponds to $\mathbb{R}_t \times S^5$. One should also note that the behavior of V_ϕ at $y = 0$ is non-trivial as r crosses r_0:

$$\begin{aligned} V_\phi(r < r_0, y = 0) &= \frac{r^2}{r_0^2 - r^2} \\ V_\phi(r > r_0, y = 0) &= \frac{r_0^2}{r^2 - r_0^2} \end{aligned} \tag{4.8}$$

For $r > r_0$, the metric is

$$\frac{ds^2}{r_0} = -\left(r^2-1\right)dt^2 + 2dtd\phi + \frac{dr^2}{(r^2-1)} + d\phi^2 + (r^2-1)d\psi^2 \tag{4.9}$$

Where the variable r has been re-scaled to be unitless. Now we can consider the string sigma model on this geometry, concentrating on rigid open string solutions that end on two static dual giant gravitons. The boundary conditions allow for the endpoints of the string to move freely along the ψ direction, so we restrict to configurations where the string endpoints co-rotate at the same angular velocity β. A convenient ansatz for the embedding coordinates is of the form:

$$\begin{aligned} t &= \tau \\ r &= r(\sigma) \\ \phi &= \phi(\sigma) \\ \psi &= \beta\tau + g(\sigma) \end{aligned} \tag{4.10}$$

Then, the string action in these coordinates is given by:

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{(r^2-1)^2g'^2 + 2\beta(r^2-1)\phi'g' + \phi'^2(\beta^2(1-r^2)+r^2) + (1-\beta^2)r'^2)} \tag{4.11}$$

For the coordinate g one has to impose Neumann boundary conditions, which is equivalent to saying that the worldsheet current density $\frac{\partial\mathcal{L}}{\partial g'}$ vanishes at the end points of the string. In addition to this, since the action is independent of g this current must vanish identically along the string. This leads to the condition:

$$g' = -\frac{\beta\phi'}{r^2-1} \tag{4.12}$$

which can be used to eliminate g in the action. This reduces the problem to a geodesic equation on a flat plane, as long as $r \neq 1$ where equation (4.12) degenerates:

$$S = -\frac{\sqrt{\lambda}}{2\pi}\sqrt{(1-\beta^2)}\int d\tau d\sigma \sqrt{r^2\phi'^2 + r'^2} \tag{4.13}$$

Due to the rotational symmetry of the droplet, a general solution can always be transformed to one determined by a pair of angles ϕ_1, ϕ_2 from the x_1 axis and the closest approach to the origin a. These are the same solutions studied in [22] in slightly different coordinates. In particular, the conserved charge associated to time translations of the coordinates (4.9) follows a relativistic dispersion relation

$$\begin{aligned} \epsilon &= \sqrt{Q^2 + \frac{\lambda}{4\pi^2}\mathcal{Z}^2} \\ Q &= \frac{\sqrt{\lambda}}{2\pi}\frac{\beta\mathcal{Z}}{\sqrt{1-\beta^2}} \\ \mathcal{Z} &= \int_{\phi_i}^{\phi_f} d\phi\sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} \end{aligned} \tag{4.14}$$

where Q is the angular momentum associated to rotations along ψ and $\mathcal{Z}$ is the central charge associated to the separation of the branes. It is also important to notice that the density of central charge and angular momentum per unit length are constant along the string. One can also check that the angular momentum density J associated to rotations along the LLM plane diverges if the string solution touches boundary of the droplet. We will make this more explicit in section 5.

4.4 Open Strings on LLM Geometries

We can now discuss more general solutions corresponding to open strings on general $\frac{1}{2}$-BPS geometries. As we will see these share many similarities to the solutions discussed in the previous section.

4.4.1 Strings outside a droplet

First we consider the case for a string inside a connected region with $z = -\frac{1}{2}$. Within each of these regions we have a non-vanishing three-sphere on which we can wrap D3 branes. In the end, we will be interested in rotating string solutions, so we will also single out a circle within this three-sphere with coordinate ψ on which the string endpoints rotate. The branes will sit at $y = 0$, but it is convenient to keep the value of y unfixed along the string as this makes the various cancellations clear. The appropriate ansatz for the embedding coordinates is the similar to the one before,

$$\begin{aligned} t &= \omega\tau \\ x_i &= x_i(\sigma) \\ y &= y(\sigma) \\ \psi &= \beta\tau + g(\sigma) \end{aligned} \tag{4.15}$$

except that the effective metric is now of the general form:

$$ds^2 = \mathcal{H}^{-2}\left(-(dt+V)^2 + \left(\frac{1}{2} - z\right)d\psi^2\right) + \mathcal{H}^2\left(dy^2 + \delta_{ij}dx^i dx^j\right) \tag{4.16}$$

The Nambu-Goto action in these coordinates is:

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{\mathcal{G}} \\ \mathcal{G} &= \mathcal{H}^{-4}\Big((\frac{1}{2} - z)g'^2 + 2(z - \frac{1}{2})g'V_i x_i' \\ &\quad + \beta^2(z - \frac{1}{2})(V_i x_i')^2\Big) + (1 - \beta^2(\frac{1}{2} - z))(x_1'^2 + x_2'^2 + y'^2) \end{aligned} \tag{4.17}$$

Where we have set $\omega = 1$ for simplicity. The coordinate g' can be eliminated by a combination of its equation of motion and boundary conditions as before. This leads to the simple relation which generalizes (4.12):

$$g' = \beta V_i x_i' \tag{4.18}$$

Once again, the relation (4.18) shows that the variable g' becomes ill-defined whenever the string touches the boundary of a droplet (4.5) at $y = 0$. One can also express this relation in a way that is independent of the parametrization,

$$dg = \beta V \tag{4.19}$$

so that dg is well defined in regions where z is locally constant. Substituting this relation into the action (4.17) will cancel the terms in $\mathcal{G}$ which are multiplied by the warp factor $\mathcal{H}$, which simplifies the action to the form:

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \sqrt{(1-\beta^2(\frac{1}{2}-z))(x_1'^2 + x_2'^2 + y'^2)} \tag{4.20}$$

One can also find a similar expression for the energy of the string by varying with respect to ω:

$$\epsilon = \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \sqrt{\frac{1}{(1-\beta^2(\frac{1}{2}-z))}(x_1'^2 + x_2'^2 + y'^2)} \tag{4.21}$$

In general, having the string extend in the y direction makes the equations non-linear, but such configurations happen to not have minimal energies. The minimal energy configurations are those for which $y = 0$ along the string, for which the action reduces to a geodesic problem on the LLM plane.

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau \sqrt{(1-\beta^2)(dx_1^2 + dx_2^2)} \tag{4.22}$$

This is the same result as for open strings between D-branes in $AdS_5 \times S^5$, so the kinematic features are the same. As a result, this class of solutions also admit a giant magnon dispersion relation:

$$\begin{aligned} \epsilon &= \sqrt{Q^2 + \frac{\lambda}{4\pi^2}\mathcal{Z}^2} \\ \mathcal{Z} &= \int \sqrt{dx_1^2 + dx_2^2} \end{aligned} \tag{4.23}$$

Generically, an LLM geometry will not have rotational invariance along the the LLM plane due to the placement of sources for z. This means the charge J associated to this rotation is no longer a good quantum number in the dual description. However, there is always an approximate translational symmetry in the limit that one zooms into the boundary of a droplet. The effective geometry in this limit is always a plane-wave, and the density of the momentum associated to the approximate translational symmetry will generically diverge. This is because such quantities are always proportional to the gauge potential V which is not well defined at the interfaces between the different values of z.

4.4.2 Inside a droplet

The analysis for connected regions with $z = \frac{1}{2}$ is completely analogous to the one in the previous section. In this case there is a different non-vanishing three-sphere $\tilde{S}^3$, from which we single out a circle $\tilde{\theta}$. The effective metric is a simple variation of (4.16)

$$ds^2 = \mathcal{H}^{-2}\left(-(dt+V)^2 + \left(\frac{1}{2}+z\right)d\tilde{\theta}^2\right) + \mathcal{H}^2\left(dy^2 + \delta_{ij}dx^i dx^j\right) \tag{4.24}$$

The appropriate ansatz in this case is the same as before (4.10), but we replace the variable ψ by:

$$\tilde{\theta} = \tilde{\beta}\tau + h(\sigma) \tag{4.25}$$

The computation of the action is entirely analogous as to the discussion in the previous section. The analogous condition (4.18) that arises from the boundary conditions for h is:

$$h' = \tilde{\beta} V_i x_i' \tag{4.26}$$

which tells us that h' is the pullback of V on the worldsheet inside the droplet regions. This means that h' has the same singularities along interfaces as g' did, so that continuing the variable g to regions inside a droplet becomes problematic. After eliminating h, the action takes the same form as before with the appropriate change in kinematic factors:

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau \sqrt{(1-\tilde{\beta}^2)(dx_1^2 + dx_2^2)} \tag{4.27}$$

Similarly, the energy can be easily shown to satisfy a similar relativistic dispersion relation

$$\epsilon = \sqrt{\tilde{Q}^2 + \frac{\lambda}{4\pi^2}\mathcal{Z}^2} \tag{4.28}$$

Where $\tilde{Q}$ is the angular momentum along the circle $\tilde{\theta}$.

4.5 On-Shell Charges

For this section we will concentrate on $\frac{1}{2}$-BPS geometries that correspond to concentric droplets and rings on the LLM plane. This is useful since we will want to study the behavior of the charge J associated to rotations around the origin of the LLM plane. One important point that should be noted is that the coordinates (4.2) are implicitly rotating with respect to an observer that is far away from the sources to whom the geometry looks like $AdS_5 \times S^5$. So solutions that are static in these coordinates correspond to strings that rotate along a cycle that asymptotically looks like the equator of S^5. As such the charge

ϵ associated to time translation symmetry in the LLM coordinates actually corresponds to $\Delta - J$ in the global AdS coordinates. One can find the expression for J for a general concentric concentric geometry by modifying the ansatz for the coordinate ϕ (4.10) to include time dependence,

$$\phi = \varphi(\sigma) + \gamma\tau \tag{4.29}$$

and in the end substituting the on-shell value $\gamma = 0$. Unlike the charges $\epsilon, \mathcal{Z}, Q(\tilde{Q})$ the angular momentum J turns out to be sensitive to the details of the geometry. This is because the general form of ϵ is fixed by supersymmetry [39], and the other charges assemble into a relativistic dispersion relation. As discussed in [38], the shortening condition is essential to get the right multiplicities for light open strings between nearby giants. This is what guarantees that the local physics looks like $\mathcal{N} = 4$ SYM on the Coulomb branch. For concreteness we first concentrate on the case where the strings live in a region outside a single circular droplet on the LLM plane, and then we show that the analysis extends to solutions sitting inside the droplet. The resulting expression for J and its density along the string are:

$$\begin{aligned} J &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma \mathcal{J} \\ \mathcal{J} &= \frac{V_\phi\left(\left(\phi'\right)^2\left(r^2\left(\frac{\left(\mathcal{H}^2+1\right)\left(1-\beta^2\right)}{2\mathcal{H}^2}+1\right)+\frac{\left(\frac{1}{\mathcal{H}^4}+\frac{1}{\mathcal{H}^2}+2\right)V_\phi^2\left(\beta^2-1\right)}{2\mathcal{H}^2}\right)+\left(r'\right)^2\right)}{\sqrt{\left(1-\beta^2\right)\left(r^2\left(\phi'\right)^2+\left(r'\right)^2\right)}} \end{aligned} \tag{4.30}$$

From this we can see that the density $\mathcal{J}$ is proportional to V_ϕ, so that it becomes infinite at the boundaries of the droplets as claimed. Since the central charge density is constant on-shell we can also express this as:

$$
\begin{aligned}
J &= \epsilon \int d\sigma V_\phi + \left(\frac{\lambda}{4\pi^2}\right)\frac{1}{\epsilon}\int d\sigma \phi'^2 \left(r^2(1+\mathcal{H}^{-2}) - V_\phi^2\left(\mathcal{H}^{-2} + \frac{\mathcal{H}^{-6}+\mathcal{H}^{-4}}{2}\right)\right) \\
\epsilon &= \sqrt{Q^2 + \frac{\lambda}{4\pi^2}\mathcal{Z}^2}
\end{aligned}
\tag{4.31}
$$

One can also express the density $\mathcal{J}$ in terms of the quantum numbers Q, $\mathcal{Z}$, the t'Hooft coupling λ and r. For completeness we will carry this out for the simplest droplet configuration studied in section 3, in the regime that the strings are far away from the droplet.

4.5.1 Strings near the boundary

We would like to evaluate (4.30) for a string solution that sits very far away from a single droplet, while the size of the string remains finite but large. These correspond to strings that sit near the boundary of AdS_5. As it turns out, the expression (4.30) is not the same as the angular momentum measured by an asymptotic observer in global AdS, since the coordinates (4.9) describe a rotating frame $\tilde{\phi} = \phi - t$, so that the expression for J is actually a linear combination of the scaling dimension Δ and the spin $\tilde{J}$ seen by a static observer near the boundary. The two quantities $J, \tilde{J}$ are related by a simple change of coordinates, but it is more convenient to work with the expressions in [22] which have a clearer physical interpretation. More concretely, this is the choice of coordinates for which the scaling dimension Δ grows with the distance from the origin, while the spin $\tilde{J}$ becomes smaller:

$$
\tilde{J} = \frac{\sqrt{\lambda}}{2\pi}\int d\tilde{\phi} \frac{(\frac{dr}{d\tilde{\phi}})^2 + \beta^2 - 1 + r^2}{(r^2-1)\sqrt{(1-\beta^2)(r^2 + (\frac{dr}{d\tilde{\phi}})^2)}}
\tag{4.32}
$$

Here $\tilde{\phi} = \phi - t$ corresponds to the coordinate for the equator of the S^5. We should also note that this expression for $\tilde{J}$ is not gauge invariant, as it will turn out that this expression is related to the one form V that appears in the metric. On the one hand these gauge transformations can be always absorbed into a redefinition of the string coordinates g and h (4.18). However, one should also keep in mind that gauge transformations that vanish at infinity do not change the asymptotics of $\tilde{J}$, so that the coordinate choices for which $\tilde{J}$ vanishes at infinity are well-defined. In order to fix the residual gauge symmetry one has to choose coordinates that look asymptotically like static global AdS rather than the rotating LLM coordinates. It will also be convenient to make the choice $V_r = 0$ in order to keep the rotational symmetry explicit.

We wish to find an expression for (4.32) in terms of the angular momentum Q, the string end points ξ_1, ξ_2, and the t'Hooft coupling λ. For this, it is best to re-express the integral using an affine parametrization for the the complex coordinate on the LLM plane $z = \xi_1(1-s) + \xi_2 s$. In order to eliminate the angular velocity β, we have to impose a double scaling limit based on (5.13). The particular double scaling limit we will be interested in comes from fixing the angular momentum Q and the positions of the end points of the string:

$$\frac{Q}{\mathcal{Z}} = \frac{\sqrt{\lambda}}{2\pi} \frac{\beta}{\sqrt{1-\beta^2}} < \infty \tag{4.33}$$

This is a physical choice of scaling, since the strings become tensionless in the relativitic limit $\beta \to 1$. This leaves us with two independent parameters to tune which we can choose to be the ratio $\frac{Q}{\mathcal{Z}}$ and the t'Hooft coupling, since changing the value of β has to be compensated by a change of λ in order to keep Q and $\mathcal{Z}$ fixed. The angular velocity and the t'Hooft coupling can't be changed independently from each other. The full expression for $\tilde{J}$ in terms of these parameters can be expressed as a sum of two terms:

$$\tilde{J} = \sqrt{Q^2 + \frac{\lambda}{4\pi^2}|\xi_1 - \xi_2|^2} \int_0^1 \frac{ds}{z\bar{z} - 1} + \left(\frac{\lambda}{4\pi^2}\right) \frac{1}{\sqrt{Q^2 + \frac{\lambda}{4\pi^2}|\xi_1 - \xi_2|^2}} \mathfrak{Im} \int_0^1 ds \frac{z\frac{d\bar{z}}{ds}}{z\bar{z}(z\bar{z} - 1)} \tag{4.34}$$

This expression is interesting as it is a series expansion in λ around zero, while also showing a scale separation given at small and large energies ϵ.

In particular, one can expect that this quantity can be recovered via a perturbative calculation in the dual field theory since we have a series expansion in the t'Hooft coupling. This is different from in the other giant magnon solutions studied in the literature which always have infinite spin $\tilde{J}$ and correspond to closed strings [28, 35]. Alternatively, one can expand in $\kappa = \lambda/Q^2$, which can be done even at strong coupling. This is similar to how in the plane wave limit the effective perturbation parameter depends on the quantum numbers of an excitation [5].

We will reproduce the leading term of the expansion in the $\mathfrak{su}(2)$ sector by an explicit computation from the one-loop spin chain Hamiltonian with boundary conditions. Motivated by an analytic continuation of the Bethe ansatz, we will obtain an expression for the $\mathfrak{sl}(2)$ spin chain with open boundary conditions, even when we do not know the form of the precise computation on the dual field theory for this sector. In general, these boundary conditions are expected to break integrability, but the existence of a Bethe-like ansatz for the $\mathfrak{su}(2)$ in terms of Cuntz oscillator coherent states suggests that a similar story exists for the $\mathfrak{sl}(2)$ spin chain. One should also note that even though the leading term is independent of λ, the computation requires knowing the one-loop mixing Hamiltonian for the $\mathfrak{su}(2)$ sector, and higher order corrections arise from higher loop contributions to the mixing of operators.

4.5.2 Rapidly Rotating Strings

An interesting limit to consider is when the strings rotate with a large angular velocity $\beta \to 1$ with fixed angular momentum Q. This is the limit where the t'Hooft coupling becomes small, so that the second term in (4.34) can be ignored.

$$\tilde{J} = |Q| \int_0^1 \frac{ds}{|\xi_1(1-s) + \xi_2 s|^2 - 1} + \dots \tag{4.35}$$

The integral in (4.35) is somewhat reminiscent of a Feynman parametrization, and can be evaluated explicitly:

$$\tilde{J} = |Q| \frac{\arctan\left(\frac{|\xi_2|^2 - |\xi_1|^2 + |\xi_1 - \xi_2|^2}{2\sqrt{|\xi_1 \times \xi_2|^2 - |\xi_1 - \xi_2|^2}}\right) - \arctan\left(\frac{|\xi_2|^2 - |\xi_1|^2 - |\xi_1 - \xi_2|^2}{2\sqrt{|\xi_1 \times \xi_2|^2 - |\xi_1 - \xi_2|^2}}\right)}{\sqrt{|\xi_1 \times \xi_2|^2 - |\xi_1 - \xi_2|^2}} + \dots \tag{4.36}$$

$$\xi_1 \times \xi_2 = \mathfrak{Im}\left(\xi_1 \bar{\xi_2}\right) = |\xi_1||\xi_2| \sin\theta_{12}$$

In the limit that the string end points are very far away from the origin we can ignore the 1 in the denominator of (4.35):

$$\tilde{J} = \frac{\theta_{12}|Q|}{|\xi_1||\xi_2| \sin\theta_{12}} + \dots \tag{4.37}$$

Although the expression (4.37) is regular at $\theta_{12} = 0$ where the string end points are colinear, there is a divergence at $\theta_{12} = \pi$ coming from the fact that the string has to cross the droplet. In the strict $\xi \to \infty$ limit, the leading order contribution for $\tilde{J}$ vanishes. Generically, in the strict $\beta \to 1$ limit the divergent contributions to the LLM angular momentum will decouple so that the expressions for $\tilde{J}$ and J match. More explicitly, the expression (4.30) becomes much simpler in this limit.

$$J = \frac{\sqrt{\lambda}\mathcal{Z}}{\sqrt{1-\beta^2}} \int_0^1 ds V_\phi \left(r(s), y=0\right) + \dots \tag{4.38}$$

Where $r(s)$ is an affine parametrization for the string in the LLM plane. It also turns out that the expression (4.38) is valid inside and outside the droplet regions as long as one chooses the correct branch for $V_\phi(y=0)$. Another thing to note is that value of V_ϕ inside of the droplet is related to its value outside the droplet by a change of sign and a transformation $r \to \frac{r_0^2}{r}$, where r_0 is related to the AdS radius $r_{AdS}^2 = r_0$. By restoring the dependance on r_0, the expression (4.37) should be understood as the leading order expansion in r_0/m^2, where m^2 is a large mass parameter compared to r_0. This can be done by either sending the string end points to infinity, or by considering a small droplet. This is also the regime where the contribution to masses of the conformally coupled scalars coming from the curvature of $\mathbb{R} \times S^3$ is negligible in the field theory, which is a decompactification limit of the S^3. This suggests that the leading non-vanishing term in J at large β should be reproducible from a Coulomb branch computation, while the higher order terms in powers r_0 should come from taking into account correctly the mixing between the higgsinos and gauginos, since a priori the massive vectors do not couple to the curvature $R_{S^3} \sim R_{AdS}$. A calculation with background fields properly included would look similar to [60], where the localization in the geometry is provided directly by the D-brane background fields, rather than a saddle point.

For more general concentric droplet geometries the expression for V_ϕ outside the largest droplet is given by a linear combination of droplets [16]:

$$V_\phi(r, y=0) = \sum_{i=0}^{k} \frac{(-1)^i r_i^2}{r^2 - r_i^2} \tag{4.39}$$

The leading expression for $\tilde{J}$ in this case can be easily seen to come from adding the contributions coming from all the droplets and holes:

$$\tilde{J} = |Q| \sum_{i=0}^{k} (-1)^i \tilde{J}_i$$
$$\tilde{J}_i = r_i^2 \left(\frac{\arctan\left(\frac{|\xi_2|^2 - |\xi_1|^2 + |\xi_1 - \xi_2|^2}{2\sqrt{|\xi_1 \times \xi_2|^2 - r_i^2 |\xi_1 - \xi_2|^2}}\right) - \arctan\left(\frac{|\xi_2|^2 - |\xi_1|^2 - |\xi_1 - \xi_2|^2}{2\sqrt{|\xi_1 \times \xi_2|^2 - r_i^2 |\xi_1 - \xi_2|^2}}\right)}{\sqrt{|\xi_1 \times \xi_2|^2 - r_i^2 |\xi_1 - \xi_2|^2}} \right) + \dots \tag{4.40}$$

This suggests that the leading order computation on the dual field theory side also comes from summing simple contributions and at leading order the droplets don't affect each other.

4.5.3 Strings Inside a Droplet

We can also do the analogous computation for string solutions sitting inside a circular droplet region. In this case the motion of the string is restricted to an $S^3 \times \mathbb{R}$ subspace of $AdS_5 \times S^5$. It is well known that the giant magnons on $S^3 \times \mathbb{R}$ have a dual description in terms of a $\mathfrak{su}(2)$ integrable spin chain whose Hamiltonian computes the mixing of operators.

Dual Spin Chain Picture

The one loop Hamiltonian for the $\mathfrak{su}(2)$ sector with open boundary conditions is of the form [58, 38]:

$$H_1 = \lambda \sum_{i=0}^{k} (a_i^\dagger - a_{i+1}^\dagger)(a_i - a_{i+1}) \tag{4.41}$$

Here a_0 and a_{k+1} are complex numbers describing the collective coordinates of the giant gravitons, and the a_i are Cuntz oscillators satisfying the algebra:

$$\begin{aligned} a_i a_j^\dagger &= \delta_{ij} \\ a_i^\dagger a_i &= 1 - |0\rangle_i \langle 0|_i \end{aligned} \tag{4.42}$$

The integer $k+1$ is associated to the angular momentum $\tilde{Q}$ which counts the number of sites of the oscillator chain, which has a length corresponding to the central charge $\mathcal{Z}$. In the gauge theory variables k counts the number of Y insertions between Z in the operator:

$$\begin{aligned} &\mathcal{O} \sim \ldots ZZZY^{L_1} Z \ldots ZY^{L_n} \ldots \\ &\sum_i L_i = k+1 \end{aligned} \tag{4.43}$$

A complete combinatoric picture of how the strings are attached to the giants and how the boundary conditions emerge is found in the works [61, 62, 63, 64]. To fix the angular position of the brane one needs to add a coherent state description of the D-branes [43]. These discussions usually only pertain to the $SU(2)$ sector. For he $SL(2)$ sector, an incomplete description in the Cuntz oscillator language is found in [59], which was derived from [4]. This description of the $SL(2)$ sector is not that of a spin chain with local nearest neighbor terms only. This makes a direct analysis very cumbersome. When we discuss such calculations, we will sidestep this direct route of computation by utilizing ideas from the Bethe ansatz.

We can then consider an unnormalized coherent state for each oscillator:

$$|z_i\rangle = \sum_{n=0}^{\infty} z_i^n |n\rangle \tag{4.44}$$

Substituting this into the Hamiltonian and minizing the energy one obtains the condition

$z_i - z_{i+1} = \delta\mathcal{Z}$ for every adjacent pair of sites, where $\delta\mathcal{Z}$ is a constant which acts as a lattice spacing for the string. The quantity $\delta\mathcal{Z}$ is generically complex, but we can always align the coordinates so that it is real.

This means that the central charge density along the chain $\frac{\mathcal{Z}}{\tilde{Q}}$ is a fixed constant. It can also be checked that this is in fact an eigenstate of the one-loop Hamiltonian with minimal energy.

One can easily check that,

$$\begin{aligned} \langle \bar{z}| \, z\partial_z \, |z\rangle &= \sum_{n=0}^{\infty} n z^n \, |n\rangle = \frac{z\bar{z}}{(1-z\bar{z})^2} \\ \langle \bar{z}|z\rangle &= \frac{1}{1-z\bar{z}} \end{aligned} \tag{4.45}$$

So that the average occupation number for each site is given by:

$$\frac{\langle \bar{z}| \, z\partial_z \, |z\rangle}{\langle \bar{z}|z\rangle} = \frac{z\bar{z}}{1-z\bar{z}} \tag{4.46}$$

This occupation number also computes the R-charge J for each oscillator, so that in total we have:

$$J = \sum_{i=1}^{k} \frac{z_i \bar{z}_i}{1 - z_i \bar{z}_i} \tag{4.47}$$

Since the central charge density along the string is constant, we can multiply each term by the central charge density $|\delta\mathcal{Z}| = |z_i - z_{i-1}| = \frac{\mathcal{Z}}{k}$

$$J = \frac{|\tilde{Q}-1|}{\mathcal{Z}} \sum_{i=1}^{k} \frac{z_i \bar{z}_i}{1 - z_i \bar{z}_i} \delta\mathcal{Z} \longrightarrow_{\delta\mathcal{Z} \to 0} |\tilde{Q}| \int_0^1 ds \frac{z\bar{z}}{1-z\bar{z}} \tag{4.48}$$

The sum can then be approximated by an integral as we take the effective lattice spacing $\delta\mathcal{Z}$ to zero, by which one expects to recover the continuum string description.

Sigma Model Computation

The computation of the spin J for the sigma model is straightforward. In this case we should take the metric with $r < 1$ where the string motion occurs on $S^3 \times \mathbb{R}$ as opposed to $AdS_3 \times S^1$. As before we will be interested in the double scaling limit that arises from taking $\tilde{\beta} \to 1$ while holding the angular momentum $\tilde{Q}$ and central charge constant. The charge $\tilde{Q}$ and angular frequency $\tilde{\beta}$ should not be confused with Q and β although their roles are very similar. The leading order expression is simply:

$$J = |\tilde{Q}| \int_0^1 ds V_\phi \left(r(s), y = 0\right) \tag{4.49}$$

Again, we can introduce an affine parametrization for the complex variable on the LLM plane $z = \eta_1(1-s) + \eta_2 s$ and re-express (4.8) in complex coordinates $r^2 = z\bar{z}$:

$$J = |\tilde{Q}| \int_0^1 ds \frac{z\bar{z}}{1 - z\bar{z}} \tag{4.50}$$

Which matches the spin chain computation precisely. We can evaluate the integral explicitly by noting that there is a simple relation between the angular momentum density inside and outside the droplet,

$$\frac{z\bar{z}}{1 - z\bar{z}} = \frac{1}{1 - z\bar{z}} - 1 \tag{4.51}$$

Which reduces to the same integral (4.36) as before:

$$J + |Q| = -|Q| \frac{\arctan\left(\frac{|\eta_2|^2 - |\eta_1|^2 + |\eta_1 - \eta_2|^2}{2\sqrt{|\eta_1 \times \eta_2|^2 - |\eta_1 - \eta_2|^2}}\right) - \arctan\left(\frac{|\eta_2|^2 - |\eta_1|^2 - |\eta_1 - \eta_2|^2}{2\sqrt{|\eta_1 \times \eta_2|^2 - |\eta_1 - \eta_2|^2}}\right)}{\sqrt{|\eta_1 \times \eta_2|^2 - |\eta_1 - \eta_2|^2}} + \dots \tag{4.52}$$

4.5.4 Magnon S-matrix and Bethe Ansatz

An interesting property of the coherent state ansatz for the one-loop Hamiltonian is that it leads to solutions to the Bethe equations. To see this more explicitly, substituting the coherent state ansatz into the Hamiltonian and minimizing over the complex parameters z_i one finds that their second difference vanishes:

$$z_{i+1} - 2z_i + z_{i-1} = 0 \tag{4.53}$$

We can always choose to parametrize the complex variables $z_i = e^{\rho_i}$, which leads to the relation:

$$e^{\rho_{i+1}-\rho_{i-1}} - 2e^{\rho_i-\rho_{i-1}} + 1 = 0 \tag{4.54}$$

To make the connection to the Bethe ansatz more explicit it is convenient to make a change of variables:

$$\begin{aligned} ip_{l+1} + ip_l &= \rho_{l+1} - \rho_{l-1} \\ ip_l &= \rho_l - \rho_{l-1} \end{aligned} \tag{4.55}$$

Solving these relations leads to an expression purely in terms of $p_{1,2}$,

$$e^{ip_2+ip_1} - 2e^{ip_1} + 1 = 0 \tag{4.56}$$

which can be recognized as a pole for the 2-magnon S-matrix for the $\mathfrak{su}(2)$ sector:

$$\begin{aligned} S_{12}^{\mathfrak{su}(2)} &= -\frac{e^{ip_2+ip_1} - 2e^{ip_2} + 1}{e^{ip_2+ip_1} - 2e^{ip_1} + 1} = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \\ e^{ip_l} &= \frac{u_l - i/2}{u_l + i/2} \end{aligned} \tag{4.57}$$

The interpretation of this pole is that we have formed a bound state of the magnons. Such magnon bound states have the same dispersion relation of (5.13), as they are also in short representations of the centrally extended spin chain [65]. In this sense, the Bethe ansatz computation and the sigma model are fully consistent with each other.

Analytic Continuation to $\mathfrak{sl}(2)$

Another important fact is that the 2-magnon S-matrix for the the $\mathfrak{sl}(2)$ sector is related up to a phase factor to the inverse of the $\mathfrak{su}(2)$ S-matrix (we follow [66]):

$$\begin{aligned} S_{12}^{\mathfrak{sl}(2)} &\propto -\frac{e^{ip_2+ip_1} - 2e^{ip_1} + 1}{e^{ip_2+ip_1} - 2e^{ip_2} + 1} = \frac{u_1 - u_2 + i}{u_1 - u_2 - i} \\ e^{ip_l} &= \frac{u_l + i/2}{u_l - i/2} \end{aligned} \tag{4.58}$$

In the formula above, the right hand side of the S-matrix looks the same, but the identification of momentum with the u variable differs, and is clearly the inverse of the one of $\mathfrak{su}(2)$ above.

In particular, the role of poles and zeros is exchanged with respect to the $\mathfrak{su}(2)$ sector. Naively one would expect that the Cuntz oscillator representation of the $\mathfrak{su}(2)$ Hamiltonian can be analytically continued by allowing the complex parameters z_i to lie outside the unit disk, but this is not the case because then the ground state is no longer normalizable and the S-matrix would not have the correct pole structure. In particular the relation (4.53) would lead to a zero of the 2 magnon S-matrix rather than a pole. If instead one exchanges $z_i \leftrightarrow \frac{1}{z_i} = \tilde{z}_i$, one finds that the zeros of (4.57) are exchanged with poles, while having $|\tilde{z}_i| > 1$. Substituting this directly in (4.47) leads to the expression:

$$J = \sum_{i=1}^{k} \frac{1}{z_i \bar{z}_i - 1} \tag{4.59}$$

where k now counts the number of derivatives between the Z operators as opposed to counting the number of Y insertions between Z's. When we take the continuum limit of this expression one obtains the same integral that appears in the the string sigma model computation for $AdS_3 \times S^1$. This analytic continuation also appears naturally in the LLM coordinates as the transformation that maps the inside and outside of a droplet to each other. It now also appears as a consistency condition for the analytically continued S-matrix. The fact that the string solutions are straight lines on the LLM plane would translate to having a pole on the magnon S-matrix for the $SL(2)$ sector.

4.6 Discussion

In this work we studied a class of $\frac{1}{2}$-BPS open string solutions ending on (dual) giant gravitons and showed that important simplifications happen when one takes into consideration the appropriate boundary conditions for the end points of the string. The solutions found have a relativistic dispersion relation, so that they generalize the giant magnon solutions to open strings. One difference between the solutions we studied is that the strings are allowed to extend into the non-compact dimensions of the spacetime, and they have well-defined finite charges inside droplet regions in the LLM plane. We also found that the solutions cannot be extended between regions of different colors in the LLM plane without having divergences in the approximate charges that generate translations parallel to the droplets, or in the case of concentric geometries the angular momentum J associated to rotations in the LLM plane. Additionally, the coordinates of the string along the non-vanishing three-sphere fiber directions are related to the pull-back of one form V, so that the string density on the fiber diverges at the boundaries of the droplets. As a consequence, one can expect that the operators corresponding to such crossing string solutions can only be constructed in a formal limit of infinite conformal dimension as in the Hoffman-Maldacena solutions. These divergences are also suggestive of possible instabilities for the states corresponding to the solutions where the strings grace a droplet. Our analysis also applies to a certain isolated class of 6d $\frac{1}{2}$-BPS bubbling geometries [67, 68], which can be described by a torus being fibered over a 2d flat base, much like in the ansatz (??). The isolated class of solutions are those for which the torus is replaced by a product of circles. The analysis for general 6d $\frac{1}{2}$-BPS bubbling geometries is complicated by the appearance of an axion corresponding to the off-diagonal component of the metric for the torus fiber, which introduces additional singularities where non-trivial sectional circles of the torus vanishes. This means in particular, the dependence on the coordinate y drops out in the analog of (4.20), so that the string solutions are no longer restricted to $y = 0$, and the droplet picture is modified by the additional singularities. Similarly, solutions of the form (??) with less supersymmetry have been studied. In those cases, the flat LLM plane is replaced by a four or six dimensional Kähler base with three or five dimensional droplets [69] . One complication that arises in the case of $\frac{1}{4}$ and $\frac{1}{8}$ BPS bubbling geometries is that solutions do not reduce to solving linear equations, so that constructing the explicit metrics is non-trivial. Also, the cancellations that occur when imposing the boundary conditions on (4.17) do not simplify the action. It would be interesting to understand how the $\frac{1}{2}$ BPS gets corrected down by considering a linearized analysis for the general $\frac{1}{4}$ BPS ansatz for instance, as this would also allow us to see how these corrections appear as a function of the t'Hooft coupling λ. Finally, although one can explictly match the sigma model answer for J to a simple computation in the $\mathfrak{su}(2)$ sector of the dual spin chain description, the analogous computation for the $\mathfrak{sl}(2)$ is not known. The fact that the sigma model predicts that the central charge is a constant per unit string length appears in the dual $\mathfrak{su}(2)$ spin chain description as a condition on the coherent state ansatz for the excited states. These condition on the parameters $z_l = r_l e^{i\theta_l}$ is a manifestation of the conditions arising from the Bethe ansatz solution to the Heisenberg spin chain, where the

z_l behave like Bethe roots. It would be natural to expect that a similar condition should arise in the $\mathfrak{sl}(2)$ sector, as the sigma model descriptions of both sectors are analytic continuations of each other. However, we should note that the presumed Bethe ansatz might be different from the $\mathfrak{sl}(2)$ Bethe ansatz [3] arising from twist-2 operators and other operators with simple derivative insertions, since more interesting mixing of operators should occur whenever the massive W-bosons (and their superpartners) corresponding to open strings are introduced. Also the sigma model computation of the R-charge $\tilde{J}$ suggests that a simple background field computation where the fluctuations of the scalar fields describing the giant gravitons are frozen is not enough to compute the correct R-charge, but that one should rather carefully integrate out the heavy fields off-diagonal fluctuations. At finite volume, the curvature coupling to the scalars lifts the Coulomb branch, but it is well known that introducing an R-charge gives rise to an effective potential whose minima describe (dual) giants [32]. As the become very large one expects that the moduli space is approximately restored, so that the R-charge is no longer needed to stabilize the solutions, however this is only true infinitely far away from the origin in field space, where the corresponding operators become infinitely heavy. So somehow the naive approximation to these operators reassembling neutral Coulomb branch operators should be destabilized by interactions with charged operators. Also a careful analysis of finite volume effects coming from the curvature of S^3 in the gauge theory could elucidate how corrections depending on the size of AdS appear.

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Chapter 5

Open Spin Chains and Giant Gravitons

5.1 $\frac{1}{4}$-BPS Boundaries:Classical Open strings on S^5

We are interested in studying classical open strings solutions to the Nambu-Goto string on $AdS_5 \times S^5$ that correspond to BPS states of the centrally extended version of the $\mathcal{N} = 4$ superconformal algebra $SU(2,2|4)$ [70]. The central extension comes from demanding that the strings end on giant gravitons that preserve $\frac{1}{4}$ of the supersymmetries. A natural set of coordinates for this problem are the generalization of the $\frac{1}{2}$- BPS LLM coordinates of Type IIB supergravity to generic $\frac{1}{4}$ ($\frac{1}{8}$)-BPS bubbling geometries [69]. These coordinates are the less-supersymmetric analog of the LLM coordinates for $\frac{1}{2}$-BPS geometries [16]. For simplicity, we will consider the case where the motion of the string is restricted to the S^5 factor of the space while it sits at the origin of AdS_5:

$$ds^2 = -dt^2 + d\Omega_5^2 \tag{5.1}$$

Instead of the usual spherical coordinates, it is better to express metric for the five-sphere in Hopf coordinates which correspond to an S^1 fibered over a complex 2-disk, with the circle degenerating at the boundary of the disk:

$$\begin{aligned} Z &= \sqrt{1-r^2}e^{i\psi} \\ X &= r\cos\theta e^{i\phi_1} \\ Y &= r\sin\theta e^{i\phi_2} \end{aligned} \tag{5.2}$$

In terms of these coordinates we have the following relations,

$$\begin{aligned} |dZ|^2 &= \frac{r^2dr^2}{1-r^2} + (1-r^2)d\psi^2 \\ |\bar{X}dX + \bar{Y}dY|^2 &= r^2dr^2 + (1-r^2)^2\mathcal{A}^2 \end{aligned} \tag{5.3}$$

where we have introduced the following auxiliary quantities:

$$\begin{aligned} r^2 &= |X|^2 + |Y|^2 \\ \mathcal{A} &= \frac{\mathrm{Im}\left(\bar{X}dX + \bar{Y}dY\right)}{1-|X|^2-|Y|^2} \end{aligned} \tag{5.4}$$

Finally, to obtain $\frac{1}{4}$-BPS coordinates we do a coordinate transformation into a rotating frame given by $X \to e^{it}X$, $Y \to e^{it}Y$. Since Z is unchanged by this transformation, only the combination $|dX|^2 + |dY|^2$ has a non-trivial transformation law. In total, the metric for $\mathbf{R} \times S^5$ in these coordinates is:

$$\begin{aligned} ds^2 &= -\left(1-|X|^2-|Y|^2\right)\left((dt+\mathcal{A})^2 - d\psi^2\right) + \frac{1}{(1-|X|^2-|Y|^2)}ds_4^2 \\ ds_4^2 &= \left(1-|X|^2-|Y|^2\right)\left(|dX|^2+|dY|^2\right) + |\bar{X}dX+\bar{Y}dY|^2 \end{aligned} \tag{5.5}$$

Another way of writing these coordinates is as a conformal rescaling of the complex hyperbolic disk model:

$$
\begin{aligned}
ds^2 &= e^{-2K}\left[-\left((dt+\mathcal{A})^2 - d\psi^2\right) + \frac{|dX|^2+|dY|^2}{1-|X|^2-|Y|^2} + \frac{|\bar{X}dX+\bar{Y}dY|^2}{\left(1-|X|^2-|Y|^2\right)^2}\right] \\
K &= -\frac{1}{2}\log\left(1-|X|^2-|Y|^2\right) \\
\mathcal{A} &= \mathrm{Im}\,(dK)
\end{aligned}
\tag{5.6}
$$

These kinds of coordinates are particularly well suited to study open strings since the equations of motion of Nambu-Goto action for an open string rigidly rotating along ψ can be expressed as a geodesic problem for the metric ds_4^2 much like in[23]).

For the boundary conditions, we will consider a class of open strings ending on a simple class of BPS giant gravitons which wrap holomorphic cycles inside the disk: [71]:

$$
\begin{aligned}
F(X,Y)|_{\sigma=0} &= 0 \\
G(X,Y)|_{\sigma=1} &= 0
\end{aligned}
\tag{5.7}
$$

More concretely, the giant gravitons are localized along the loci $F = G = 0$ inside the droplet $|X|^2 + |Y|^2 < 1$ while they fill the fiber circle coordinate ψ. These Dirichlet boundary conditions must be supplemented with Neumann boundary conditions along the normal direction in order for the strings to remain attached to the giants:

$$
\begin{aligned}
&\left[\partial_\sigma \vec{X}\cdot\left(\frac{\partial F(X,Y)}{\partial \hat{n}}\right)^*\right]|_{\sigma=0} = 0 \\
&\left[\partial_\sigma \vec{X}\cdot\left(\frac{\partial G(X,Y)}{\partial \hat{n}}\right)^*\right]|_{\sigma=1} = 0
\end{aligned}
\tag{5.8}
$$

One last important point about the boundary conditions is the role of the worldsheet coordinate $\psi(\tau,\sigma)$ for which we will make the ansantz:

$$
\psi(\tau,\sigma) = \beta\tau + g(\sigma) \tag{5.9}
$$

Substituting this ansatz into the Nambu-Goto action in temporal gauge $t(\tau,\sigma)=\tau$ for the metric (5.5), the Neumann boundary conditions for ψ imply the following condition

$$dg = \beta \mathcal{A} \tag{5.10}$$

After imposing this condition the Nambu-Goto action for the string takes the form :

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\sqrt{1-\beta^2}\int d\tau \int_0^1 d\sigma \sqrt{\mathcal{G}} \\ \mathcal{G} &= \left(1-|X|^2-|Y|^2\right)\left(\left|\frac{dX}{d\sigma}\right|^2+\left|\frac{dY}{d\sigma}\right|^2\right)+\left|\bar{X}\frac{dX}{d\sigma}+\bar{Y}\frac{dY}{d\sigma}\right|^2 \end{aligned} \tag{5.11}$$

One important aspect of the boundary conditions (5.7), is that generically the classes of solutions satisfying the boundary conditions are much richer than the case where the giant gravitons are only separated along a single complex coordinate [23]. For instance, whenever the loci $F=G=0$ have mutual solution, the strings can become point-like objects localized at such loci.

For the sake of completeness we will reproduce the equations of motion corresponding to the action (5.11), but will leave the study of the general solutions for future work. As it will turn out, the holomorphic nature of the boundary conditions (5.7) is constraining enough to simplify the analysis to the point where we will not need to solve the general equations of motion. The geodesic equation for the induced metric are:

$$\begin{aligned} \frac{d^2X}{d\sigma^2} &= -\frac{2\left(XY'-YX'\right)\left(X\bar{Y}\bar{X}'+(1-|X|^2)\bar{Y}'\right)}{1-|X|^2-|Y|^2} \\ \frac{d^2Y}{d\sigma^2} &= \frac{2\left(XY'-YX'\right)\left((1-|Y|^2)\bar{X}'+\bar{X}Y\bar{Y}'\right)}{1-|X|^2-|Y|^2} \end{aligned} \tag{5.12}$$

The main complication in these equations arises due to the second (non-holomorphic) factor in both numerators.

5.1.1 On-Shell Charges

All open string solutions with the specified boundary conditions can be partially characterized by three conserved quantities: the angular momentum in the Z direction Q, the angular momentum along the X, Y directions J and the central charge $|\mathcal{Z}|$ corresponding to the length of the string.

$$\begin{aligned} |\mathcal{Z}| &= \int_0^1 d\sigma \sqrt{\mathcal{G}} = \int_0^{|\mathcal{Z}|} ds \\ Q &= \frac{\sqrt{\lambda}}{2\pi} \frac{\beta}{\sqrt{1-\beta^2}} |\mathcal{Z}| \\ J &= \mathcal{E} \int_0^{|\mathcal{Z}|} ds \left(\frac{|X|^2 + |Y|^2}{1 - |X|^2 - |Y|^2} \right) + \mathcal{E}^{-1} \left(\frac{\lambda}{4\pi^2} \right) \int_0^{|\mathcal{Z}|} ds \mathcal{J}_1 \\ \mathcal{J}_1 &= \left(\frac{\left[\mathrm{Im} \left(\bar{X} \frac{dX}{ds} + \bar{Y} \frac{dY}{ds} \right) \right]^2 (|X|^2 + |Y|^2 - 2)}{1 - |X|^2 - |Y|^2} \right) \end{aligned} \tag{5.13}$$

Notice that the integrals are taken with respect to a affine parameter s, which corresponds to a gauge choice in which the central charge density is constant along the string. This choice is convenient since it will appear in the spin chain description of the string. The energy of the strings has the form of the usual dispersion relation of the centrally extended BPS states:

$$\mathcal{E} = \Delta - J = \sqrt{Q^2 + \frac{\lambda}{4\pi^2} |\mathcal{Z}|^2} \tag{5.14}$$

Here Δ is the generator of time translations in the non-rotating coordinates for $\mathbf{R}_t \times S^5$. The expression for the spin J is of interest, since it corresponds to an symmetry that is spontaneously broken by the boundary conditions. In particular, when expressed purely in terms of of the other charges $\mathcal{E}, \mathcal{Z}, Q$, it's expansion in the t'Hooft coupling terminates at first order, which suggests that the quantum R-charge is one-loop exact at strong coupling. We will later reproduce the first term in the expansion of J in the large spin limit $Q \to \infty$ and $Q/|\mathcal{Z}|$ fixed with a spin chain computation in the $SU(3)$ sector of

$\mathcal{N} = 4$ SYM.

5.1.2 Some Simple Solutions

We will now study a few simple solutions that will serve as reference for the later sections on open spin chains. The simplest class of boundary conditions to consider are those for which only one linear combination of X and Y appears at both boundaries, for example $F(X,Y) = X - \xi$ and $G(X,Y) = X - \eta$. Since the Neumann conditions imply that the derivatives of $Y(\sigma)$ vanish at the boundaries, one obvious ansatz is to try $Y(\sigma) = Y =$ constant. Plugging this into the action simplifies the metric to that of a flat disk:

$$S = \frac{\sqrt{\lambda}}{2\pi}\sqrt{(1-\beta^2)(1-|Y|^2)}\int d\tau \int_0^1 d\sigma \sqrt{|dX|^2} \tag{5.15}$$

The solution to the equations of motion is simply a linear interpolation between ξ and η, $X(\sigma) = \sigma\xi + (1-\sigma)\eta$. This is virtually identical to $\frac{1}{2}$-BPS (open) magnon solution, except that the value of $|Y|$ is important. Extremizing with respect to $|Y|$ gives two possible values for the lowest energy configurations, $|Y| = 0$, or $|Y| = 1$. This is due to the rotation along the Y axis pushing the strings towards the edge of the droplet if $Y \neq 0$. The solution with $|Y| = 1$ is just a string moving along the equator of S^5, which has infinite spin J but finite energy [28], while the solution with $Y = 0$ is a massive string with finite spin J.

Another important example is the choice $F(X,Y) = X - \xi$ and $G(X,Y) = Y - \eta$. In this case, the lowest energy configuration is a point-like string localized at the locus $X = \xi$, $Y = \eta$. This means any choice of boundary conditions which is purely linear in X, Y will lead to point-like solutions localized at the intersection of the lines.

Finally, another important example is the generalization of the simple linear intersecting branes to intersections with multiplicities, such as double points. For concreteness, we consider the simple choice $F = X - \xi$ and $G = Y^2 + X - \eta$. As in the first example, a natural guess is to look for solutions within the level sets $Y(\sigma) = \text{constant}$. For $Y \neq 0$, the curves intersect at two different points $Y = \pm\sqrt{\xi - \eta}$. Since these intersections are holomorphic, the local analysis near each intersection is exactly the same as when the branes are supported at intersecting lines. However at the level set where $Y = 0$ the behavior is different; if we were to have that $\xi = \eta$, then we would have that the previous pairs of intersections combine into a single double point intersection, so that when we deform away from $\xi = \eta$ there will be points along both curves where the normal direction to the curves are parallel. In this case, since $Y = 0$ the normal to both $F(X, Y = 0) = 0$ and $G(X, Y = 0) = 0$ point along the Y direction. In particular, this means that there is a straight line connecting ξ and η that satisfies the Neumann boundary conditions (5.8). With these examples in mind, it becomes clear that whenever we have boundary conditions that intersect holomorphically, or we have a complex deformation thereof, which is always the case for boundary conditions of the form (5.7), there will always be solutions to the Nambu-Goto string equations that are either point-like (in the case where there is an intersection), or are a straight line in $\mathbf{D}^2$ (when we have a deformation of a singular point). This can be explained as follows: if the intersection is non-singular, the intersection loci is a circle and the string is a point inside this circle, but when we have a singular intersection, or it's deformation, there is always an $SU(2)$ that aligns the coordinate axes of $\mathbf{D}^2$ with the normal and tangent vectors to the intersection which we may call $\tilde{X}_N, \tilde{Y}_T$. The Neumann boundary conditions are satisfied automatically when the string's coordinate along the tangent direction is constant $\tilde{Y}_N(\sigma) = \text{constant}$, which simplifies the induced metric in $\mathbf{D}^2$ to a flat D^1. Generically these solutions are isolated, since trying to deform the constant value of $\tilde{Y}_N(\sigma)$ will break the boundary conditions, because the normal derivatives along the two D-branes will be in general not aligned.

5.2 Boundary Conditions for Open Spin Chains

5.2.1 Warm-Up: An SU(2) Bosonic Chain

In this section we expand on the ideas of [72] to introduce open boundary conditions for spin chains, with the goal of describing open strings in holography. Before discussing the case of the $\mathcal{N} = 4$ SYM spin chain, we turn our attention to a simpler toy model that captures a lot of the desired physics. In general grounds, the equations of motion of a string are schematically of the form:

$$\left(\partial_\tau^2 - \Delta\right) X = 0 \tag{5.16}$$

Where Δ is the appropriate induced Laplacian on the worldsheet, and X denotes the wordlsheet (bosonic) fields.In suitable coordinates, this is can always be written as an expansion around the flat metric on the string in powers of the curvature of the target space and derivatives, so to zeroth order in the curvature of the space, the equations of motion for a string are simply a wave equation. Those equations can always be discretized and viewed as coming from an effective Hamiltonian for the "string-bits" making up the string:

$$H_{bulk} = \lambda \sum_{i=1}^{L-1} \left[\left(a_i^\dagger - a_{i+1}^\dagger\right) \left(a_i - a_{i+1}\right) + \left(b_i^\dagger - b_{i+1}^\dagger\right) \left(b_i - b_{i+1}\right) \right] \tag{5.17}$$

Where a_i, b_i are commuting bosonic raising and lowering operators acting on sites of the form $|n_a\rangle \otimes |n_b\rangle$. This can be thought of as an effective discrete light-cone Hamiltonian for a relativistic string with angular momentum $Q \sim L + 1$. We will use this as a toy model for the bulk Hamiltonian of a discretized string living on a 2d complex plane. To see this, consider a variational wavefunction ansatz for the ground state:

$$|\Psi(\{x_l, y_l\})\rangle = \bigotimes_{i=1}^{L} |x_i\rangle \otimes |y_i\rangle \tag{5.18}$$

Where $|x\rangle$ is a bosonic coherent state. The variational energy in the bulk of the would be discretized string is simply:

$$E(\{x_i, y_i\}) = \lambda \sum_{i=1}^{L-1} \left(|x_i - x_{i+1}|^2 + |y_i - y_{i+1}|^2\right) \tag{5.19}$$

This is clearly minimized whenever the parameters x_i, y_i satisfy a discrete second difference equation which can be interpreted as a discretized version of (5.16):

$$\begin{aligned} x_{i_1} - 2x_i + x_{i+1} &= 0 \\ y_{i_1} - 2y_i + y_{i+1} &= 0 \end{aligned} \tag{5.20}$$

As they stand, this set of linear equations is under-determined; we need to impose boundary conditions. To describe an open string, one would expect that the Hamiltonian must include new terms that give rise to the appropriate discretizations of (5.7) and (5.8). One choice for such modification is to include boundary terms such as:

$$\begin{aligned} V_{DL} &= \lambda F(a_1, b_1)^\dagger F(a_1, b_1) \\ V_{DR} &= \lambda G(a_L, b_L)^\dagger G(a_L, b_L) \end{aligned} \tag{5.21}$$

Where these should be interpreted as polynomial expressions on the raising and lowering operators. When applied on the coherent state ansatz introduced before, the condition needed such that the parameters x_i, y_i are at an extremum of the energy is:

$$\begin{aligned} x_1 - x_2 + \left[\partial_{x_1} F(x_1, y_1)\right]^* F(x_1, y_1) &= 0 \\ y_1 - y_2 + \left[\partial_{y_1} F(x_1, y_1)\right]^* F(x_1, y_1) &= 0 \end{aligned} \tag{5.22}$$

Multiplying the first equation by $(\partial_{y_1} F(x_1, y_1))^*$ and the second by $(\partial_{x_1} F(x_1, y_1))^* F(x_1, y_1)$ and subtracting both of them gives:

$$(x_1 - x_2)(\partial_{y_1} F(x_1, y_1))^* - (y_1 - y_2)(\partial_{x_1} F(x_1, y_1))^* = 0 \tag{5.23}$$

The form of this equation can be immediately recognized as a discretization of the continuum limit boundary conditions. In the large L continuum limit, these equations would reproduce something reassembling the light-cone spectrum of an open string in $\mathbf{R}_t \times S^1 \times \mathbf{C}^2$ stretched between $D3$-branes wrapping S^1 and a holomorphic cycle inside $\mathbf{C}^2$. To see that the space is really $\mathbf{C}^2$, as opposed to $\mathbf{R}^2$ (or something else), we can look at the Kähler form which arises from the Berry connection of the coherent states:

$$\int dt\,(\langle x| \otimes \langle y|)\, i\partial_t\,(|x\rangle \otimes |y\rangle) = \int dt\,(\bar{x}\dot{x} + \bar{y}\dot{y} - c.c.) \tag{5.24}$$

This should be interpreted as an integral of the canonical form $\int pdq$, so that $\bar{x}$ and x are canonically conjugate. This reproduces the standard Kähler form on $\mathbf{C}^2$ with a flat metric. Since the symplectic structure is canonical, this reaffirms the fact that the commutation relations of $a, a^\dagger$ were the usual relations of a harmonic oscillator in the first place. With all this in mind, we can now discuss a more complex example.

5.2.2 $\mathcal{N} = 4$ SYM $SU(2)$ open spin chain at one-loop

A more complex realization of the idea that strings arise from spin chains is realized by the $\mathcal{N} = 4$ SYM integrable spin chain. The simplest closed sector consists of operators made out only of complex scalars X, Z:

$$\mathcal{O} \sim \ldots Z X^{k_1} Z X^{k_2} \ldots Z X^{k_L} Z \ldots \tag{5.25}$$

Since we are interested in operators describing open strings, we will take the end points of the word $ZX^{k_1}ZX^{k_2}\ldots ZX^{k_L}Z$ in (5.25) to be sewn together to an operator of dimension $\Delta\sim O(N)$, as opposed to taking a simple trace. Operators belonging to this $SU(2)$ sector which correspond to open strings between giant gravitons were studied in [73], and further in [58, 44, 43, 62]. One important aspect of such spin chains is that the number of sites is indefinite in the usual $SU(2)$ spin variables. One way to deal with this is to chose to fix the number of Z's appearing in the operator, while letting the total number of letters fluctuate. This would correspond to a Heisenberg magnet where the number of spin up sites is constant, but the number of spin down defects is allowed to change. The Hamiltonian for this system is more naturally expressed in terms of bosonic oscillators that satisfy the following Cuntz algebra instead of the usual harmonic oscillator relation:

$$\begin{aligned} A^\dagger A &= 1-|0\rangle\langle 0| \\ AA^\dagger &= 1 \end{aligned} \tag{5.26}$$

In these variables the one loop Hamiltonian is:

$$\begin{aligned} H_{SU(2)} &= \frac{\lambda}{8\pi^2}\sum_{i=0}^{L}\left(A_i^\dagger - A_{i+1}^\dagger\right)\left(A_i - A_{i+1}\right) \\ A_0 = \xi,&\quad A_{L+1}=\tilde{\xi} \end{aligned} \tag{5.27}$$

Despite the simple form of this Hamiltonian, the fact that the raising and lowering operators satisfy a Cuntz algebra makes the spectrum of the system very complex. However, the ground state of the system can obtained by a coherent state ansatz [74]:

$$\begin{aligned} A\,|z\rangle &= z\,|z\rangle \\ |z\rangle &= \sqrt{1-|z|^2}\sum_{n=0}^{\infty} z^n\left(A^\dagger\right)^n|0\rangle \end{aligned} \tag{5.28}$$

With the energy function being given by:

$$E = \lambda \sum_{i=0}^{L} \left(z_i^* - z_{i+1}^*\right)\left(z_i - z_{i+1}\right)$$
$$z_0 = \xi, \quad z_{L+1} = \tilde{\xi} \tag{5.29}$$

As in the toy example, the energy is minimized whenever the z_i satisfy a second difference equation. Another way of stating this is that the ground state can be associated with a straight line inside a flat unit disk, and the central charge density of the string is associated to the quantity:

$$\delta \mathcal{Z} = z_{i+1} - z_i \tag{5.30}$$

This Hamiltonian can also be thought of as a first term in a the curvature expansion of a light-cone Hamiltonian of the string on $\mathbf{R} \times S^5$, with discrete light-cone momentum $Q = L+1$. The zeroth order correction to the energy is simply the light-cone momentum $Q = L+1$. The ground state energy to leading order is given by:

$$E = E_0 + E_1 + \cdots = (L+1) + \frac{1}{2}\left(\frac{\lambda}{4\pi^2}\right)\frac{|\xi - \tilde{\xi}|^2}{L+1} + \ldots \tag{5.31}$$

Higher order corrections to the ground-state energy have been computed in [45], and in agreement with the dispersion relation (5.14).

5.2.3 $SU(3)$ open spin chain at one-loop

We will now consider operators belonging to a holomorphic $SU(3)$ sector. As with the $SU(2)$ case, it will be convenient to fix the number of Z's appearing in the operator, so that the most general operator is of the schematic form:

$$\mathcal{O} \sim \ldots Z W_1 Z W_2 \ldots Z W_L Z \ldots \tag{5.32}$$

Where W_i are arbitrary words made out of the fields X, Y. Clearly, the usual spin variables of $SU(3)$ are not well suited for analysing the Hamiltonian of the corresponding spin chain, because as before the number of spin sites is not a well defined quantity. One also has to take into account the order in which the X and Y fields appear in each word W_i, so the operators in the Hamiltonian have to be sensitive to this ordering. One solution to both of these issues is to map this $SU(3)$ spin chain into a Cuntz oscillator chain with two Free variables at each site:

$$\begin{aligned} AA^\dagger &= BB^\dagger = 1 \\ AB^\dagger &= BA^\dagger = 0 \\ A^\dagger A + B^\dagger B &= 1 - |0\rangle\langle 0| \\ A|0\rangle &= B|0\rangle = 0 \end{aligned} \tag{5.33}$$

The use of such operators for us is that the letters A, B have no relation between one another, so they can easily encode the words W_i by the identification:

$$\begin{aligned} X &\to A^\dagger \\ Y &\to B^\dagger \\ \frac{\partial}{\partial X} &\to A \\ \frac{\partial}{\partial Y} &\to B \end{aligned} \tag{5.34}$$

Then, each word W_i separated by the Z's can be viewed as a site for a spin chain of these Cuntz oscillators. In particular, these are "bosonic" in the sense that the operators commute at different sites. Now, we will translate the integrable $SU(3) \subset SO(6)$ spin chain Hamiltonian arising at one loop from $\mathcal{N} = 4$ SYM from spin variables in these bosonic operators by treating the insertions of the field Z as a vacuum [70, 58].

SU(3) Cuntz Hamiltonian

The bulk Hamiltonian for the integrable $SU(3)$ chain coming from $\mathcal{N} = 4$ SYM consists only of interactions of the form

$$H_{i,i+1} \sim \mathbf{1}_{i,i+1} - \mathbf{P}_{i,i+1} \tag{5.35}$$

where $\mathbf{P}_{i,i+1}$ is the permutation operator acting on the tensor product of fundamental representations of $SU(3)$, $V_i \otimes V_{i+1}$, and $\mathbf{1}_{i,i+1}$ is the identity operator. We need to express the action of these in terms of the free variables A, B. In the spin variable representation, there are basically three possible actions that the operator $\mathbf{P}_{i,i+1}$ can do: if this operator acts on a section of a word such as

$$\ldots \mathbf{P}(ZX) \ldots \tag{5.36}$$

, the resulting word will be $\ldots XZ \ldots$; this deletes the leftmost X in one of the words W_l for some l, and moves attached it to the rightmost spot in the word W_{l-1}. In other words, this is a hopping term in the Free variable language. The identity operator $\mathbf{1}_{i,i+1}$ clearly counts the number of adjacent letters of the same kind. This is also the case for sections of words that look like:

$$\ldots ZY \ldots \tag{5.37}$$

and

$$\ldots XZ \ldots \tag{5.38}$$

. These are the same interactions as in the $SU(2)$ spin chain, except that one needs to keep track of the order in which the letters X, Y are attached to the words W_l:

$$H \sim \lambda \sum_{i=i}^{k-1} \left(A^\dagger_{i,R} - A^\dagger_{i+1,L}\right)\left(A_{i,R} - A_{i+1,L}\right) + \left(B^\dagger_{i,R} - B^\dagger_{i+1,L}\right)\left(B_{i,R} - B_{i+1,L}\right) \quad (5.39)$$

Where $A_{i,L/R}$ acts on the left/right end of the word W_i corresponding to the state $W_i \left|0\right\rangle_i$.

$$\begin{aligned} A_{i,L}\left(W_i\right)|0\rangle &= \left(A_i W_i\right)|0\rangle \\ A_{i,R}\left(W_i\right)|0\rangle &= \left(A_i W_i^T\right)^T|0\rangle \end{aligned} \quad (5.40)$$

Naively one might think that this alone is the correct noncommutative version of the toy model introduced in the beginning of this section and the correct $SU(3)$ Hamiltonian, however there is another allowed interaction; the operator $\mathbf{P}_{i,i+1}$ can also permute $XY \to YX$, for any adjacent X and Y. This seems hard to do in the free variable language, since this would amount to permuting every pair of different letters inside each word W_l. This means that this is an on-site interaction term; the same is true for the identity operator. The end result is that the integrable Hamiltonian $\mathbf{1}_{i,i+1} - \mathbf{P}_{i,i+1}$ acting on such combinations of letters is encoded in the action of the following operator:

$$\begin{aligned} V_{AB} &= \sum_{\mathcal{W}} \mathcal{W}^\dagger \left(B^\dagger A^\dagger AB + A^\dagger B^\dagger BA - B^\dagger A^\dagger BA - A^\dagger B^\dagger AB\right)\mathcal{W} \\ &= -: \frac{[B^\dagger, A^\dagger][A, B]}{1 - A^\dagger \otimes A - B^\dagger \otimes B}: \end{aligned} \quad (5.41)$$

Since this is a somewhat formal expression, we should clarify what such an operator does. First, the sum over $\mathcal{W}$ should be understood as a sum over all Free words in A, B of any length; this has the effect of annihilating all layers of a word, implementing either an identity operation, or a swap, and then rebuilding the previously annihilated layers. The way the normal ordering should be understood is as follows: the expression

$\frac{[B^\dagger,A^\dagger][A,B]}{1-A^\dagger\otimes A-B^\dagger\otimes B}$ is a formal power series in A, B and $A^\dagger, B^\dagger$ made out of all possible ways of multiplying A, B ($A^\dagger, B^\dagger$). The ordering ambiguity is removed by requiring that the lowering operators act to the right first, without commuting any A past a B, and then multiplying to the right with the raising operators in the transposed order. Finally, the bulk Hamiltonian for the integrable $SU(3)$ spin chain with a variable number of sites can be written as:

$$\begin{aligned} H_{SU(3)} =&\lambda \sum_{i=i}^{L-1} \Bigg[\left(A^\dagger_{i,R} - A^\dagger_{i+1,L}\right)(A_{i,R} - A_{i+1,L}) + \left(B^\dagger_{i,R} - B^\dagger_{i+1,L}\right)(B_{i,R} - B_{i+1,L}) \\ &- : \frac{[B_i^\dagger, A_i^\dagger][A_i, B_i]}{1 - A_i^\dagger \otimes A_i - B_i^\dagger \otimes B_i} : \Bigg] \end{aligned} \tag{5.42}$$

This Hamiltonian can be easily generalized from $SU(3)$ to $SU(N)$ by introducing $N-3$ additional oscillators at each site and replacing the relations between them with:

$$\begin{aligned} A_I A_J^\dagger &= \delta_{IJ} \\ \sum_{I=1}^{N-1} A_I^\dagger A_I &= 1 - |0\rangle\langle 0| \\ A_I |0\rangle &= 0 \end{aligned} \tag{5.43}$$

This is very similar to the representations of $SU(N)$ in terms of constrained bosons [75]. One should be able to also include fermionic operators with slightly different relations to describe the $SU(3|2)$ sector of the theory without too much difficulty as in [70], but we leave this analysis for a later time. At first this change of variables might seem a bit convoluted, since the on-site interactions look very complicated, but it turns out the the coherent state analysis becomes very simple. This would not be the case with the usual $SU(N)$ spin coherent states found in the literature, since the number of spins in the chain is a dynamical quantity.

SU(3) Cuntz Coherent states

The fact that the lowering operators A, B always act *first* suggest that we should look for a coherent state ansatz for the ground state; to do this we should build coherent states for these kinds of operators. There is a natural ansatz for these:

$$|x, y\rangle = \sum_{n=0}^{\infty} \left(xA^\dagger + yB^\dagger\right)^n |0\rangle \tag{5.44}$$

To check that this indeed works, we notice that if one applies the A lowering operator on $xA^\dagger + yB^\dagger$, it will annihilate the $B^\dagger$'s and combine with the $A^\dagger$:

$$A(xA^\dagger + yB^\dagger) = xAA^\dagger + yAB^\dagger = x \tag{5.45}$$

In other words we have:

$$\begin{aligned} A\,|x, y\rangle &= x\,|x, y\rangle \\ B\,|x, y\rangle &= y\,|x, y\rangle \end{aligned} \tag{5.46}$$

So the seemingly formal objects $|x, y\rangle$ are indeed coherent states. One last thing to check is whether these states have finite norm. This norm can be easily evaluated using the relations (5.33),

$$\langle x, y|x, y\rangle = \sum_{n=0}^{\infty} \left(|x|^2 + |y|^2\right)^n = \frac{1}{1 - |x|^2 - |y|^2} \tag{5.47}$$

which means that these states are only well defined for $|x|^2 + |y|^2 < 1$.

As mentioned previously, we wish to consider variational wavefunctions of the form:

$$|\Psi(x_i, y_i)\rangle = \bigotimes_{i=1}^{L} |x_i, y_i\rangle \tag{5.48}$$

When we evaluate the expectation value of the Hamiltonian (5.17) on this set of states we recover the same variational energy function that that we encountered before:

$$E(\{x_i, y_i\}) = \lambda \sum_{i=1}^{L-1} \left(|x_i - x_{i+1}|^2 + |y_i - y_{i+1}|^2 \right) \tag{5.49}$$

One important simplification is the vanishing of the expectation value of the on-site interation term; this happens because the coherent states have the effect of replacing A and B with commuting numbers x, y. As before, this energy function is minimized whenever the coordinates x_i, y_i satisfy separate second difference equations. Since the set of linear equations resulting from this are underdetermined, we have to impose boundary conditions in this case too. Before that, we will first give a brief discussion of the quantum numbers of these ground states, and show their agreement with the classical open string picture (5.13). Lastly, the form of the energy functional makes it seem as if the coordinates x, y are independent of each other. This is not the case, since the normalization constraint couples them. More explicitly, their (quantum) commutation relations inherited from the Kähler form are nontrivial, as we will discuss later.

Quantum Numbers: Q and J

In analogy to the classical open strings on S^5, the ground state of this spin chain is determined by three quantum numbers, the central charge $\mathcal{Z}$ and a pair of spins Q, J. The spin Q in the spin chain corresponds to the total number of Z's in the word, and is related to the total number of sites L of the spin chain:

$$Q = L + 1 \tag{5.50}$$

This makes sense, since the angular momentum along the circle ψ is quantized. This is not the case with the spin J; the rotational symmetry associated to J is spontaneously broken by the end points of the string, so the corresponding quantum number is no longer quantized. This is reflected on the fact that the number of X's and Y's in the chain is not constant. The spin J is also associated to the R-charge in the BPS shortening condition $\Delta - J = 0$ that the giant gravitons to which the strings are attached must satisfy. In the spin chain, this is given to leading order by the expectation number of the number operator:

$$\begin{aligned} J &= \sum_i \langle \Psi(x_i, y_i)| \hat{N}_i |\Psi(x_i, y_i)\rangle = \sum_{i=1}^{L} \frac{|x_i|^2 + |y_i|^2}{1 - |x_i|^2 - |y_i|^2} \\ \hat{N} &= \sum_{i=1}^{L} \hat{N}_i = \sum_{i=1}^{L} : \frac{A_i^\dagger A_i + B_i^\dagger B_i}{1 - A_i^\dagger A_i - B_i^\dagger B_i} : \end{aligned} \tag{5.51}$$

For large $Q = L + 1$, this is in precise agreement with the expression coming from the classical string solution, to leading order in the t'Hooft coupling.

Berry Curvature and Symplectic form

To find the symplectic form associated to the parameters x, y we need to evaluate the Berry curvature:

$$\mathcal{A}_t = \langle x(t), y(t)| i\partial_t |x(t), y(t)\rangle \tag{5.52}$$

This can be evaluated by first computing:

$$\langle w, z| i\partial_t |x(t), y(t)\rangle = i\partial_t \left(\frac{w^* x(t) + z^* y(t)}{1 - w^* x(t) - z^* y(t)} \right) \tag{5.53}$$

And substituting $w = x(t)$ and $z = y(t)$:

$$\mathcal{A}_t = \langle x(t), y(t)| i\partial_t |x(t), y(t)\rangle = i \left(\frac{x^* \dot{x} - x\dot{x^*} + y^* \dot{y} - y\dot{y^*}}{1 - |x|^2 + |y|^2} \right) \tag{5.54}$$

This is precisely the same form as the one-form $\mathcal{A}$ appearing in the metric on which the classical string moves (5.4). To find the Kähler form one needs to take the derivative of this quantity in the correct (holomorphic) way. Doing this carefully gives the symplectic form on the unit complex hyperbolic disk:

$$\Omega = \frac{dx \wedge d\bar{x} + dy \wedge d\bar{y}}{1 - |x|^2 - |y|^2} + \frac{(\bar{x}dx + \bar{y}dy) \wedge (xd\bar{x} + yd\bar{y})}{(1 - |x|^2 - |y|^2)^2} \tag{5.55}$$

This shows that even in the semi-classical limit, that is when one takes the coherent state expectation value of the quantum Hamiltonian as a classical Hamiltonian for the system of string-bits, the coordinates x and y are still interacting, despite the fact that the quantum interaction term vanishes. One thing to note about this form, is that if one restricts to surfaces where $y = dy = 0$, the Kähler form reduces to the usual canonical form $dx \wedge d\bar{x}$. This is true in all orthonormal bases; if one sets one of the independent coordinates to zero the Kähler form reduces to the canonical form in the left over coordinate.

5.2.4 Open Boundary Conditions for $SU(3)$ open Cuntz Chain

One subtle point that was ignored in the previous section, is what happens to the operator $\mathcal{O}$ at the end points. In principle, the operator is attached to a coherent combination of quarter BPS operators corresponding to the giant gravitons. In practice, constructing such operators is combinatorially diffucult, and the non renormalizability properties of quarter BPS operators are less restricting than that of half BPS operators [76].

These issues can be bypassed by finding boundary terms for the Hamiltonian (5.17) that implement the correct boundary conditions on the ground state (5.7) (5.8), and only dealing with long spin chains so that the curvature corrections to this prescription are well under control. From this it becomes clear that the generalization for the boundary terms of the $SU(3)$ Cuntz chain should be:

$$\begin{aligned} V_{DL} &\sim F(A_1, B_1)^\dagger F(A_1, B_1) \\ V_{DR} &\sim G(A_L, B_L)^\dagger G(A_L, B_L) \end{aligned} \tag{5.56}$$

We should note that there is an ordering ambiguity in defining these boundary terms, since the operators A_i, B_i do not commute. One possible choice is to consider a completely symmetrized polynomial in the lowering operators. For example for a polynomial $F(x, y)$, one can build an operator $\mathcal{F}(A, B)$ as follows:

$$\begin{aligned} F(x, y) &= \sum_{n,m} F_{n,m} x^n y^m \\ \mathcal{F}(A, B) &= \sum_{n,m} F_{n,m} \binom{n+m}{n}^{-1} \sum_{\text{permutations of A,B}} A^n B^m \end{aligned} \tag{5.57}$$

Since this operator differs from other orderings by commutators, the difference from this choice is a higher curvature correction, which is subleading in the limit of long chains $L >> 1$.

This means that the most general (and Free-est) boundary terms that implement Dirichlet conditions are of the form

$$\begin{aligned} V_{DL} &= \lambda\, \mathcal{F}(A_1, B_1)^\dagger \mathcal{F}(A_1, B_1) \\ V_{DR} &= \lambda\, \mathcal{G}(A_L, B_L)^\dagger \mathcal{G}(A_L, B_L) \end{aligned} \tag{5.58}$$

When applied on the Cuntz coherent states, these will lead to the extremization conditions:

$$\begin{aligned} x_1 - x_2 + \left[\partial_{x_1} F(x_1, y_1)\right]^* F(x_1, y_1) = 0 \\ y_1 - y_2 + \left[\partial_{y_1} F(x_1, y_1)\right]^* F(x_1, y_1) = 0 \end{aligned} \tag{5.59}$$

At this point, it should become clear that the analysis is identical to our simple toy boson Hamiltonian, since the noncommutativity of A and B is erased when one replaces them with c-numbers. In the next section we will compute the ground states for a some simple examples of boundary conditions, and we will show qualitative agreement with the fact that introducing such boundary terms does indeed describe open strings stretching between various D-brane set-ups. Then we will outline a general method for finding such ground states for generic holomorphic boundary conditions via a level-set analysis.

5.3 Some simple examples of boundary conditions

5.3.1 Branes at angles

Let us consider a simple class of boundary conditions for the $SU(3)$ spin chain which reproduce the boundary conditions associated to giant gravitons wrapping S^3's at angles inside S^5. To clarify what we mean by having these branes at angles, we should clarify the geometry of the S^5 in quarter BPS coordinates. In these coordinates, the geometry of the S^5 is better described as an S^1 bundle over a disk $\mathbf{D}^2$ (5.5). For simplicity, we will take the giant gravitons that wrap the S^1 fiber throughout, while they lie in a line inside $\mathbf{D}^2$. A (complex) line inside $\mathbf{D}^2$ is determined by a two component vector (x, y). The angles between any two lines can be determined by a $PSU(2) = SU(2)/\mathbf{Z}_2$ matrix g:

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{5.60}$$

More explicitly, we can align the first giant to lie along the Y axis of $\mathbf{D}^2$. This means that the D3 brane is point like in the X direction, and we may denote its position to be ξ. The position of the string end point along Y is undetermined since the boundary condition along the D3 brane allows it to me freely along it. This would corresponds to a boundary term in the spin chain like:

$$V_L = \lambda \left(A_1^\dagger - \xi^* \right) (A_1 - \xi) \tag{5.61}$$

For the second giant graviton, let us take a line inside $\mathbf{D}^2$ of the form $L : \beta X + \alpha Y - \eta = 0$. Then, there is a complex codimension one surface with coordinates $\alpha X - \beta Y$ normal to this line. This surface would be the analog of the LLM disk [16] after a change of coordinates. We can choose to place the giant graviton at position $\alpha X + \beta Y = \eta$ as a Dirichlet boundary condition. Again we are not allowed to fix the position of the string end point along L. The boundary term in this case would look like:

$$V_R = \lambda \left(\alpha^* A_L^\dagger + \beta^* B_L^\dagger - \eta^* \right) (\alpha A_L + \beta B_L - \eta) \tag{5.62}$$

Making use of the coherent state ansatz introduced in the previous section, we obtain a simple variational function for the ground state energy. The bulk equations on the spin chain imply that the difference $x_{i+1} - x_i$ and $y_{i+1} - y_i$ are independent of the site i. The only non- trivial step is to solve the boundary equations of motion. Let us first solve the equations for y_1:

$$\frac{1}{\lambda} \frac{\partial E}{\partial y_1^*} = y_1 - y_2 = 0 \tag{5.63}$$

This in particular sets all bulk equations for y to zero. On the other end of the spin chain we get the condition:

$$\frac{1}{\lambda} \frac{\partial E}{\partial y_L^*} = \beta^* (\alpha x_L + \beta y_L - \eta) + y_L - y_{L-1} = \beta^* (\alpha x_L + \beta y_L - \eta) = 0 \tag{5.64}$$

Where we used the fact that $y_2 - y_1 = y_L - y_{L-1} = 0$. Turning to the x_L equation we find that for nonzero β, the bulk equations for the x variables must also vanish

$$\frac{1}{\lambda}\frac{\partial E}{\partial x_L^*} = \alpha^*(\alpha x_L + \beta y_L - \eta) + x_L - x_{L-1} = x_L - x_{L-1} = 0 \tag{5.65}$$

Finally, the equation for x_1 can only be satisfied if the string remains at $x_i = \xi$:

$$\frac{1}{\lambda}\frac{\partial E}{\partial x_1^*} = x_1 - \xi + (x_1 - x_2) = x_1 - \xi + (x_{L-1} - x_L) = x_1 - \xi = 0 \tag{5.66}$$

This means that the energy is minimized when the string localizes at the intersection of the two branes as expected:

$$\begin{aligned} x_i &= \xi \\ \alpha x_i + \beta y_i &= \eta \end{aligned} \tag{5.67}$$

Notice that solutions to these equations only give normalizable states whenever these solutions lie inside the unit disk $|x|^2 + |y|^2 < 1$. In that case, the corresponding state is a zero mode of the Hamiltonian, as one would expect of the lowest energy modes of a string localized at the intersection of two D-branes. If the intersection of the lines lies outside the unit disk, the gradient flow of the Hamiltonian will push any variational wavefunction towards this would be intersection, so that eventually the strings hit the boundary of the unit disk and become non-normalizable states.

One important point is that one should be able to see the eigenvalues $e^{\pm i\theta}$ of the $SU(2)$ in the string spectrum. In the double scaling limit where the tension λ and spin $L+1$ are very large with $\frac{\lambda}{(L+1)^2} < 1$, the commutator terms in the Hamiltonian should be highly suppressed, and the energy can be approximated by the continuum limit. Since the commutators $[A_i, B_i]$ are small, the leading contributions come from fully symmetric tensors representations of the $SU(2)$ symmetry that rotates A and B into each other.

The continuum action can be obtained by using the symplectric form obtained from the coherent states:

$$H = \lambda \int d\sigma \left(|\partial x|^2 + |\partial y|^2 \right) \tag{5.68}$$

Although this looks simple, the commutation relations of the coordinates are non-trivial due to the curvature of the Kähler form $\Omega = \frac{i}{2}\bar{\partial}\partial \log\left(1 - |x|^2 - |y|^2\right)$:

$$\begin{aligned} [x(\sigma), \bar{x}(\sigma')] &= 2\delta(\sigma - \sigma') : (1 - |x(\sigma)|^2)(1 - |x(\sigma)|^2 - |y(\sigma)|^2) : \\ [x(\sigma), \bar{y}(\sigma')] &= 2\delta(\sigma - \sigma') : x(\sigma)\bar{y}(\sigma)(1 - |x(\sigma)|^2 - |y(\sigma)|^2) : \end{aligned} \tag{5.69}$$

Then one would proceed by doing a mode expansion along the coordinate σ. In the semiclassical regime, the boundary conditions can be implemented on these modes before quantization, Since the D-branes are related by an $SU(2)$ rotation, we ca usen the eigenbasis of this rotation for the worldsheet coordinates; one can always chose to place the first D-brane along $\operatorname{Re} x$ and $\operatorname{Re} y$, and the second at a rotation (and a shift) of these axes with eigenvalues $e^{\pm i\theta}$. In these coordinates the boundary conditions have a nice form [77]:

$$\begin{aligned} \frac{\partial}{\partial\sigma}\operatorname{Re} x(0) &= \frac{\partial}{\partial\sigma}\operatorname{Re} y(0) = 0 \\ \operatorname{Im} x(0) &= \operatorname{Im} y(0) = 0 \\ \frac{\partial}{\partial\sigma}\operatorname{Re} e^{i\theta}x(1) &= \frac{\partial}{\partial\sigma}\operatorname{Re} e^{-i\theta}y(1) = 0 \\ \operatorname{Im} e^{i\theta}x(1) &= \xi \\ \operatorname{Im} e^{i\theta}y(1) &= 0 \end{aligned} \tag{5.70}$$

The constant term at $\sigma = 1$ can be taken care of by explicitly solving for the zero more $x_0 = \xi \sin\theta$. The fluctuations around the zero mode get a phase shift of θ/π in their mode number:

$$x(\sigma) = x_0 + \sum_{n\neq 0} e^{i(\pi n-\theta)\sigma}\alpha_{n-\theta/\pi}$$
$$y(\sigma) = \sum_{n\neq 0} e^{i(\pi n-\theta)\sigma}\beta_{n-\theta/\pi} \tag{5.71}$$

In the limit of small fluctuations $|x| << 1$, the mode operators reproduce the flat space spectrum of open strings on a pair of D-branes intersecting at angles, but in general the commutation relation between the modes are nontrivial even at this order. This is not surprising, since even the discrete Hamiltonian function coming from the coherent states is not the Hamiltonian for a free particle; in polar coordinate representation $x_i = \sqrt{\rho_i}e^{i\varphi_i}$ and $y_i = \sqrt{\eta_i}e^{i\theta_i}$, the Hamiltonian is an $SU(2)$ generalization of a class of Calogero integrable systems [78]:

$$H = \sum_{i=1}^{L}(\rho_i + \zeta_i) + \sum_{i,j=1}^{L}\left(\sqrt{\rho_i\rho_j}\cos(\varphi_i - \varphi_k) + \sqrt{\eta_i\eta_j}\cos(\theta_i - \theta_k)\right) + V_{\text{boundary}} \tag{5.72}$$

We won't dwell too much into the exact form of the boundary terms, but we will comment that since they come in the form of a polynomial in x, y, in polar coordinates these will look generically like a complicated sum of spherical harmonics for S^5. As before, the subtlety arises in that the Kähler form is non-trivial, so that the pairs $\{\rho_i, \eta_i\}, \{\varphi_j, \theta_j\}$ are not canonical conjugates to each other.

5.3.2 Massive $\frac{1}{2}$-BPS states

One thing to note about the class of ground states in the previous section is that they all are either massless, or have non-normalizable wavefunctions. This is expected from D-brane intersections; at a holomorphic/supersymmetric intersection, the states localized to this loci preserve both the supersymmetries associated to each of the branes. Naively one would think that this might be lifted to a massive BPS state if the branes are moved apart in a transverse direction, but this is not the case since a massive BPS multiplet has twice as many degrees of freedom as a massless BPS multiplet. To obtain massive states we should first look for boundary conditions for which the ground state is doubly degenerate and massless, meaning that the D-branes intersect twice or at a double point, and then separate the branes transversally. One simple choice of boundary term is to replace the linear relation in (5.62) with a conic:

$$V_R = \lambda \left(A_L^\dagger + \left(B_L^\dagger \right)^2 - \eta^* \right) \left(A_L + B_L^2 - \eta \right) \tag{5.73}$$

We can repeat the analysis of the previous section to find that the variational energy is only dependent on the first and last coordinates $x_{1,L}, y_{1,L}$, the only difference being that the solutions to the minimization problem need to be split into two cases. The equations in question are:

$$\begin{aligned} x_1 - \xi - \left(\eta - x_L - y_L^2 \right) &= 0 \\ y_L^* \left(\eta - x_L - y_L^2 \right) &= 0 \end{aligned} \tag{5.74}$$

These must me supplemented with the constraints:

$$\begin{aligned} x_{i+1} - x_i &= x_{i+2} - x_{i+1} \\ y_{i+1} - y_i &= y_{i+2} - y_{i+1} = 0 \end{aligned} \tag{5.75}$$

Clearly the solutions depend on the relationship between ξ and η. More explicitly, we can look for solutions for which $y_L \neq 0$, which sets $x_1 = x_L = \xi$, $y_1 = y_2 = \cdots = y_L$ and $x_L + y_L^2 = \eta$. Solving for y_L gives two solutions :

$$\begin{aligned} \xi &= x_1 = \cdots = x_L \\ y_1 &= \cdots = y_L = \pm\sqrt{\xi - \eta} \end{aligned} \tag{5.76}$$

Since the equations at the boundaries $x_1 = \xi$ and $x_L + y_L^2 = \eta$ are satisfied, the energy of these two states vanishes..

The other case occurs when we look for solutions with $y_L = 0$. The only zero energy solutions in this case are those with $\xi = \eta$, which are a limiting case of the previous solutions. For $\xi \neq \eta$, the second equation in (5.74) is automatically satisfies, and the first equation gives that $x_1 - \xi = \eta - x_L$. The constraint equations also tell us that this is a constant along the spin chain, so that:

$$x_{i+1} - x_i = (L+1)\,(\eta - \xi) \tag{5.77}$$

Finally, the energy can be evaluated to be:

$$E = \frac{1}{2}\left(\frac{\lambda}{4\pi^2}\right)\frac{|\eta - \xi|^2}{L+1} \tag{5.78}$$

5.3.3 Branes Wrapping Holomorphic Cycles

It is not hard to see that the for generic intersecting and holomorphic curves, the ground state wavefunction will be localized at the intersection as in the previous case. So without loss of generality let us consider two giant gravitons specified by the equations:

$$\begin{aligned} F(x_0, y_0) &= 0 \\ G(x_{L+1}, y_{L+1}) &= 0 \end{aligned} \tag{5.79}$$

Where we introduced additional ”end-point” coordinates for the string. The boundary terms for a string with such a configuration are given by

$$\begin{aligned} V_{DL} &= \lambda\, \mathcal{F}(a_1, b_1)^\dagger \mathcal{F}(a_1, b_1) \\ V_{DR} &= \lambda\, \mathcal{G}(a_L, b_L)^\dagger \mathcal{G}(a_L, b_L) \end{aligned} \tag{5.80}$$

Repeating the analysis of the previous section, it's not hard to see that the bulk variables x_i, y_i for $i = 2, \dots L-1$ serve purely as lagrange multipliers which set the equations:

$$\begin{aligned} \left(\frac{\partial F}{\partial x_1}\right)^* F(x_1, y_1) = -\left(\frac{\partial G}{\partial x_L}\right)^* G(x_L, y_L) = x_i - x_{i-1} \\ \left(\frac{\partial F}{\partial y_1}\right)^* F(x_1, y_1) = -\left(\frac{\partial G}{\partial y_L}\right)^* G(x_L, y_L) = y_i - y_{i-1} \end{aligned} \tag{5.81}$$

This means that the energy function expressed purely in terms of the variables $x_{1,L}, y_{1,L}$ is:

$$\begin{aligned} E = \frac{\lambda(L-1)}{4}\left|\left(\frac{\partial F}{\partial x_1}\right)^* F - \left(\frac{\partial G}{\partial x_L}\right)^* G\right|^2 + \frac{\lambda(L-1)}{4}\left|\left(\frac{\partial F}{\partial y_1}\right)^* F - \left(\frac{\partial G}{\partial y_L}\right)^* G\right|^2 \\ + \lambda|F(x_1, y_1)|^2 + \lambda|Gx_L, y_L|^2 \end{aligned} \tag{5.82}$$

The first think to note is that the quantities $|F(x_1, y_1|$ and $|G(x_L, y_L|$ are the geometric distance from the D-branes to the points $(x_{1,L},\ y_{1,L})$ along a straight line. Also, since the expression $x_i - x_{i+1}$ is holomorphic, the corresponding left hand side of (5.83) must also be holomorphic. This means that either $x_i - x_{i+1}$ vanishes whenever F^* depends explicitly on x_1^*, y_1^*, or if $\left(\frac{\partial F}{\partial x_1}\right)^*$ is a constant , then this would allow us to have a non-trivial central charge density in the x direction $x_{i+1} - x_i$, and similarly in the y direction. By arranging that the magnitudes of the functions at the first and last sites are equal $|F| = |G|$, meaning that the distance from the first and last string bits to the D-branes is the same, it becomes clear that the energy is simply the length of the line connecting two points in the two D-branes where the normal vectors are parallel. To make this more precise we can introduce the quantities:

$$\begin{aligned} \delta\mathcal{Z}_x &= \left(\frac{\partial F}{\partial x_1}\right)^* F(x_1,y_1) = -\left(\frac{\partial G}{\partial x_L}\right)^* G(x_L,y_L) = x_i - x_{i-1} \\ \delta\mathcal{Z}_y &= \left(\frac{\partial F}{\partial y_1}\right)^* F(x_1,y_1) = -\left(\frac{\partial G}{\partial y_L}\right)^* G(x_L,y_L) = y_i - y_{i-1} \\ |\delta\mathcal{Z}_x|^2 + |\delta\mathcal{Z}_y|^2 &= |\delta\mathcal{Z}|^2 = |F|^2 = |G|^2 \end{aligned} \tag{5.83}$$

This extremization procedure is somewhat reminiscent of a toric decomposition, where the moment map (or D-terms) are given by:

$$\begin{aligned} \mu_{x_{i+1}} &= |x_{i+1} - x_i|^2 = |\delta\mathcal{Z}_x|^2 \\ \mu_{y_{i+1}} &= |y_{i+1} - y_i|^2 = |\delta\mathcal{Z}_y|^2 \end{aligned} \tag{5.84}$$

After doing this reduction of $\left(\mathbf{D}^2\right)^L$, the only thing left to do is to minimize the boundary contributions (or F-terms):

$$\begin{aligned} \boldsymbol{\partial}\mathcal{W} &= \frac{(dx_1 + dx_L)\delta\mathcal{Z}_x^* + (dy_1 + dy_L)\delta\mathcal{Z}_y^*}{\delta\mathcal{Z}^*} = \boldsymbol{\varepsilon} \\ \boldsymbol{\partial} &= \sum_{i=1,2}\left[dx_i\frac{\partial}{\partial x_i} + dy_i\frac{\partial}{\partial y_i}\right] \\ \mathcal{W} &= F(x_1,y_1) - G(x_L,y_L) \end{aligned} \tag{5.85}$$

This means that the space of variational ground states has a nice decomposition in terms of the levels sets of the moment maps: (5.84):

$$\{\texttt{ground states}\} = \bigcup_{\delta\mathcal{Z}}\{\boldsymbol{\partial}\mathcal{W} = \boldsymbol{\varepsilon}\} /\!\!/ \mu^{-1}\left(|\delta\mathcal{Z}|^2\right) \tag{5.86}$$

The case with $\delta\mathcal{Z} = 0$ should be treated with care; this is precisely when the boundary curves intersect and $\boldsymbol{\varepsilon}$ should be taken to vanish, since this is exactly the deformation parameter that resolves the intersection for massive states.

In general, this space is not connected, and has isolated points whenever the there exist massive ground states. An important point to make about this decomposition is that it shows that there is an additional set of conserved charges in the ground state of the system, the central charge densities $x_{i+1} - x_i$. This should be seen as a further indication that the coherent state ansatz for the ground state of these spin chains is some sort of Bethe wavefunction with complex Bethe roots z_i as was noted in [44, 23].

Chapter 6

Generating Functions for BPS Operators

6.1 Introduction

Recently, there has been a renewed interest in determinant operators in large N holographic gauge theories and their string dual description as giant gravitons [9, 10, 19, 13, 8]; the dimension of these operators is order N, which makes them ideal to probe sub-AdS physics. A natural basis for gauge invariant operators is the Schur functions, which are characters of the unitary and symmetric groups. Combinatorial methods for computing correlation functions in free $\mathcal{N} = 4$ SYM were developed in [48, 79]. More recent works have emphasized the utility of an effective action approach obtained by recasting the determinant operators as fermionic integrals and integrating out the super Yang-Mills fields. In this description, the non-perturbative physics of the problem can be obtained from a saddle point approximation for an effective action in terms of a set of collective fields [9, 19].

A similar prescription for AdS giant gravitons was proposed in [11], where it was realized that the norms of BPS states are encoded in the expansion of the Harish-Chandra-Itzykson-Zuber (HCIZ) integral, which appears in the evaluation of the norms of a certain class of gauge invariant coherent states:

$$\mathcal{O}_\Lambda(0) = \int_{SU(N)} dU \exp\left(\mathrm{Tr}\left[\Lambda U a_Z^\dagger U^\dagger\right]\right). \tag{6.1}$$

This sheds light on why the group characters evaluated on the Yang-Mills fields may serve as an orthogonal basis, even though they are only orthogonal with respect to the Haar measure, and gives a different interpretation of the norms of BPS states as the coefficients in the expansion of the HCIZ integral. This technique has the advantage of repackaging the combinatorics of the Schur functions into integrals over the unitary group.

The Harish-Chandra integrals have natural generalizations to the B, C, D series, $Sp(2N)$ and $SO(M)$. For a choice of simple Lie group G, the HCIZ integral has an exact formula in terms of a sum over the saddle points:

$$\mathcal{H}(x,y) = \int e^{\langle \mathrm{Ad}_g(x), y\rangle} dg = c_{\mathfrak{g}} \sum_{w\in W} \frac{\epsilon(w) e^{\langle w(x), y\rangle}}{\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y)}. \tag{6.2}$$

Each saddle point of the integral corresponds to a Weyl reflection, and the denominators are given by the discriminant of the Lie algebra. These integrals have received less attention than the unitary HCIZ integral, which serves as a single plaquette model in lattice gauge theory.

The bulk of the work on probing finite N physics is limited to field theories with $U(N)$ and $SU(N)$ gauge groups (see [80, 81, 82, 83]), but more recently, there has been some interest in extending these studies to field theories with $Sp(2N)$, $SO(2N+1)$, or $SO(2N)$ gauge groups [84, 85, 86]. There is good reason for this surge of interest: maximally supersymmetric Yang-Mills theory with symplectic and orthogonal groups are dual to type IIB strings on $AdS_5 \times \mathbb{RP}^5$ [12]. Depending on the choice of the orientifold projection, the gauge group of the theory is either $Sp(2N)$, $SO(2N+1)$, or $SO(2N)$; S-duality relates the spectrum of the $Sp(2N)$ and the $SO(2N+1)$ theories, while the $SO(2N)$ theories are self-dual. The exact matching of the spectrum for the symplectic and orthogonal theories is poorly understood, due to the combinatorial difficulty associated with constructing states of these theories.

In this chapter, we study BPS coherent states of $\mathcal{N} = 4$ SYM for special orthogonal and symplectic groups. The norms of such states are given precisely by a Harish-Chandra integral over the corresponding group. By explicitly expanding the integral, we find that these coherent states serve as generating functions for gauge invariant states in the gauge theory, and the corresponding coefficients in the expansion give their norms. In principle, this gives a way of constructing an orthogonal basis of states for these theories from group theoretic data for the corresponding gauge group. We argue that these generating functions are only able to capture information about the "unitary" part of the gauge symmetry, which is to say that operators we find in the expansion match in form to operators in the unitary theory. In section 2, we review the construction of gauge invariant coherent states for the $SU(N)$ theory. In section 3, we generalize this to the symplectic case and argue that the odd special orthogonal case is related to the symplectic case by a rank-level duality that exchanges a Young diagram with its conjugate diagram. We repeat the calculations for the even orthogonal case. In section 4, we discuss other attempts at finding an orthogonal basis for $Sp(2N)$, $SO(2N+1)$, or $SO(2N)$ and how our results can be interpreted in a relevant context. Finally, we conclude with a discussion of a few open questions and future directions of work.

6.2 Review of the $U(N)$ case

We begin with a brief review of BPS coherent states in $U(N)$. The same analysis may be applied to any free gauge theory with an adjoint scalar field Z. We know from [11] that given a naïve coherent state $F[\Lambda]$ of the form:

$$\exp\Big(\mathrm{Tr}(\Lambda \cdot a_Z^\dagger)\Big)\left|0\right\rangle , \tag{6.3}$$

where Λ is taken to be a diagonal matrix-valued set of parameters and $a_z^\dagger$ is the raising operator for the s-wave of the field Z on S^3 in [17], we may introduce an auxiliary $U(N)$ group action and average over the group, which allows us to rewrite a gauge invariant coherent state as:

$$F[\Lambda] = \frac{1}{Vol(U(N))}\int dU \exp\Big(\mathrm{Tr}(U\Lambda U^{-1}a_Z^\dagger)\Big)\left|0\right\rangle , \tag{6.4}$$

where dU is the Haar measure. Our normalization factor $Vol(U(N)) = \int dU$; we can set it equal to one for the sake of brevity. We may compute the overlap of $F[\Lambda]$ as defined in Eq. (6.4) with its adjoint $\bar{F}[\bar{\Lambda}]$ by evaluating the HCIZ integral:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \int d\tilde{U} \exp\left(\mathrm{Tr}\left(\tilde{U}^{-1}\Lambda\tilde{U}\bar{\Lambda}'\right)\right). \tag{6.5}$$

We see that we have sidestepped most of the Wick contractions of the matrix operators $\left(a^\dagger\right)^i_j$, which would make $F[\Lambda]$ difficult to compute in the form it takes in Eq. (6.4). $F[\Lambda]$ can be evaluated through a character expansion, as described in [87]:

$$F[\Lambda] = \sum_R \frac{1}{f_R}\chi_R(\Lambda)\chi_R(a_Z^\dagger)\left|0\right\rangle \tag{6.6}$$

We may also rewrite Eq. (6.5) through a character expansion:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \sum_R \frac{1}{f_R} \chi_R\left(\bar{\Lambda}\right) \chi_R\left(\Lambda\right) \tag{6.7}$$

We can then compare the coefficients of the characters from the equation above to what we would obtain from multiplying Eq. (6.6) by its adjoint and find:

$$\langle 0| \chi_R(a)\chi_R(a^\dagger) |0\rangle = f_R \tag{6.8}$$

It becomes obvious that we must compute f_R to evaluate the overlap of $\chi_R(a)$ and $\chi_R(a^\dagger)$. The thing to keep in mind is that the representations R in the coherent state $F[\Lambda]$ correspond to Young diagrams for $U(N)$, which are characterized by the indices $j_1 \geq j_2 \geq \ldots j_N$, where each index j_i iterates over row i. Because these are characters of the unitary group, they may be rewritten with the Weyl character formula:

$$\chi_{j_i}(\Lambda) = \frac{\det\left(\lambda_k^{j_i+N-i}\right)}{\Delta(\Lambda)}, \tag{6.9}$$

where λ_k are the eigenvalues of Λ and $\Delta(\Lambda)$ is the Vandermonde determinant of Λ. Then we may rewrite the HCIZ integral as a product of these expanded characters:

$$I(\Lambda, \bar{\Lambda}) = \int d\tilde{U} \exp\left(\mathrm{Tr}\left(\tilde{U}^{-1}\Lambda\tilde{U}\bar{\Lambda}'\right)\right) = \Omega\frac{\det\left(\exp\left(\lambda_i\bar{\lambda}'_j\right)\right)}{\Delta(\Lambda)\Delta(\bar{\Lambda}')} \tag{6.10}$$

where Ω is a normalization constant. We rewrite the numerator to reintroduce f_R:

$$\Omega \det\left(\exp\left(\lambda_i\bar{\lambda}'_j\right)\right) = \sum_{\vec{j}} \frac{1}{f_{\vec{j}}} \det\left(\lambda_k^{j_i+N-i}\right) \det\left(\bar{\lambda}_k'^{j_i+N-i}\right) \tag{6.11}$$

We have relabeled R with the indices $\vec{j}$, and have rewritten the equation above accordingly. The expressions inside the determinants are monomials and correspond to the term $\prod_i \lambda_i^{j_i+N-i} + \ldots$ in $\det\left(\lambda_k^{j_i+N-i}\right)$. Thus we may expand the exponential in Eq.

(6.11) as:

$$\det\left(\exp\big(\lambda_i\bar{\lambda}'_j\big)\right) = \sum_{[n]} \frac{1}{[n]!}\det\big((\lambda_i\bar{\lambda}'_j)^{n_i}\big) = \sum_{[n]} \frac{1}{[n]!}\det\big(\bar{\lambda}'^{n_i}_j\big)\prod_i \lambda_i^{n_i} + \ldots, \tag{6.12}$$

where we have made use of the multilinearity of the determinant. The factor $[n]$ encapsulates $n_1, \ldots, n_N$; then $[n]! = \prod_j n_j!$. We see that we are limited to $n_1 > n_2 \ldots$ when we restrict ourselves to the monomials with the correct descending order; when we set $n_i = j_i + N - i$, we arrive at an explicit sum over the characters. Thus our denominator $f_{\vec{j}}$ may be computed as:

$$f_{\vec{j}} = \Omega^{-1}\prod_i (j_i + N - i)!, \tag{6.13}$$

We may set $f_{\vec{0}} = 1$, as $\langle 0|0|0|0\rangle = 1$. Then we arrive at:

$$\Omega = \prod_{i=1}^{N}(N-i)! \tag{6.14}$$

From this we can easily read off the norms of the states $\chi_R(a^\dagger)$:

$$\langle \chi_{R'}(a)\chi_R(a^\dagger)\rangle = \delta_{R,R'}\frac{\prod_i (j_i + N - i)!}{\prod_{i=1}^{N}(N-i)!}, \tag{6.15}$$

which agrees with the well-known result of [48].

6.3 Symplectic and orthogonal cases

Before repeating the analysis for the other simple lie groups, we should comment on the interpretation of the $Sp(2N)$ and $SO(N)$ theories as orientifold projections of a unitary theory. To do this, we first consider a simple toy model correponding to a single harmonic oscillator. As it turns out, this simple model captures a lot of the qualitative behaviour of the answer for symplectic and orthogonal groups.

6.3.1 A toy model for the orientifold projection

As a warm-up, we consider a single quantum harmonic oscillator:

$$[a, a^\dagger] = 1. \tag{6.16}$$

A natural basis of states for this system is the eigenstates of the occupation number operator $\hat{n}\,|n\rangle = n\,|n\rangle$. One thing that we may do with this system is to define a parity operator $\Omega = (-1)^{\hat{n}}$ and further divide the set of states into those that are mutual eigenvectors of $\hat{n}$ and Ω. This gives an orthogonal decomposition of the Hilbert space of the harmonic oscillator into sectors of positive and negative parity $\mathcal{H} \cong \mathcal{H}_+ \bigoplus \mathcal{H}_-$, and divides all the states into even and odd states under the orientation reversal transformation

$$\begin{aligned} P &: x \to -x \\ P &: p \to -p, \end{aligned} \tag{6.17}$$

where x and p are the position and momentum operators. Because the raising operators are monomials in x and p, the odd parity states are created with odd numbers of raising operators and vice versa. The operators $\frac{1}{2}(1 \pm \Omega)$ respectively serve as orthogonal projection operators into $\mathcal{H}_+$ and $\mathcal{H}_-$.

What we would like to do is build coherent states in each of these two sectors of the theory. For instance, we can project a coherent state into the sector of positive parity by applying the operator $\frac{1}{2}(1 + \Omega)$:

$$\frac{1}{2}(1+\Omega)|\alpha\rangle = \frac{1}{2}\left(1+e^{\pi i\hat{n}}\right)e^{\alpha a^\dagger}|0\rangle = \frac{1}{2}\left(e^{\alpha a^\dagger}+e^{-\alpha a^\dagger}\right)|0\rangle = \cosh\left(\alpha a^\dagger\right)|0\rangle\,. \tag{6.18}$$

We call this state $|\alpha,+\rangle$. One nice property of this state is that it is annihilated by a^{2k+1} for any non-negative integer k. It is also an eigenstate of a^2 with eigenvalue α^2. In this sense, we can call this a coherent state for the positive chirality sector of the model. By a similar computation, the overlap between any two of these coherent states is given by:

$$\langle\beta^*,+|\alpha,+\rangle = \cosh(\alpha\beta)\,. \tag{6.19}$$

The case for negative parity requires more care, and will be the case that is relevant to the analysis of the $Sp(2N)$ and $SO(2N+1)$ theories. If we project a coherent state into the sector of negative chirality, we obtain the state:

$$\frac{1}{2}(1-\Omega)|\alpha\rangle = \sinh\left(\alpha a^\dagger\right)|0\rangle\,. \tag{6.20}$$

The issue is that this state is not a coherent state in the usual sense; when we act on the state with a lowering operator, the state won't return to the original state since the minimum ocupation number that appears in the series is $|1\rangle$. Rather, this state is also an eigenvector of a^2 with eigenvalue α^2. Since the original vacuum state is annihilated by the projector $\frac{1}{2}(1-\Omega)$, the true vacuum in this sector is the state occupation number one $|1\rangle$. By a relabeling of the states for the odd sector, the coherent state can be written as

$$|\alpha,-\rangle = -i\,\mathrm{sinc}\left(i\alpha a^\dagger\right)\left|\tilde{0}\right\rangle, \tag{6.21}$$

where $\mathrm{sinc}(x) = \frac{\sin x}{x}$, and the new vacuum is $\left|\tilde{0}\right\rangle = |1\rangle$. A simple computation yields the norm of this coherent state:

$$\langle \beta^*, - | \alpha, - \rangle = \frac{\sinh \alpha\beta}{\alpha\beta}. \tag{6.22}$$

6.3.2 The symplectic HCIZ integral

We now seek to expand our definition for a well-defined BPS operator averaged over the unitary group to the symplectic group:

$$F_{Sp(2N)}[\Lambda] = \frac{1}{\mathrm{Vol}(Sp(2N))} \int_{Sp(2N)} dg \exp\Big(\mathrm{Tr}(g\Lambda g^{-1} a_Z^\dagger)\Big) |0\rangle \,, \tag{6.23}$$

where dg is the Haar measure for the symplectic group and $\mathrm{Vol}(Sp(2N)) = \int_{Sp(2N)} dg$ is a normalization factor, which we can always rescale to one. The group elements of $Sp(2N)$ can be represented by $2N \times 2N$ matrices that are both unitary and symplectic:

$$\begin{aligned} g^\dagger g &= \mathbf{1}_{2N} \\ g^T \Omega g &= \Omega, \end{aligned} \tag{6.24}$$

where Ω is a choice of anti-symmetric symplectic matrix:

$$\Omega = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}. \tag{6.25}$$

The symplectic condition (6.24) translates into the orientifold projection of the Chan-Paton indices for the open strings ending on a stack of $2N$ $D3$ branes [12]. This forces the raising and lowering operators of the $Sp(2N)$ theory to satisfy the orientifold projection condition:

$$\Omega\, a_Z^\dagger\, \Omega = (a_Z^\dagger)^T = -a_Z^\dagger, \tag{6.26}$$

where the transpose is taken on the group indices, which we omit for clarity. This means that any operator made from traces of odd numbers of fields will automatically vanish. We choose to normalize the commutation relations for the raising and lowering operators by a factor of $\frac{1}{2}$, which will make the computation of the norm of the coherent state more transparent:

$$[(a_Z)^i_j, (a^\dagger_Z)^l_k] = \frac{1}{2}\left(\delta^l_j \delta^i_k - \Omega_{jk}\Omega^{lj}\right). \tag{6.27}$$

As with the unitary case, we wish to compute the overlap between two coherent states. This is done by applying the Campbell-Hausdorff formula; since the raising and lowering operators have different relations from the unitary case, we must check that commuting the exponentials really simplifies the norm into the form where it can be evaluated by a Harish-Chandra integral. After some algebra, we see that in the symplectic case, the exponentials can be commuted as follows:

$$\begin{aligned}\left[\mathrm{Tr}\left(g a_z g^\dagger \Lambda\right), \mathrm{Tr}\left(h a^\dagger_z h^\dagger \bar{\Lambda}'\right)\right] &= \frac{1}{2}\mathrm{Tr}\left(gh\Lambda(gh)^\dagger\bar{\Lambda}'\right) + \frac{1}{2}\mathrm{Tr}\left(g\Lambda g^\dagger \Omega h^T \bar{\Lambda}'^T (h^T)^{-1}\Omega\right) \\ &= \mathrm{Tr}\left(gh\Lambda(gh)^\dagger\bar{\Lambda}'\right).\end{aligned} \tag{6.28}$$

The second term in (6.28) is equivalent to the first term after using the group relations (6.24). This means that once again, we can compute the operator's overlap with its adjoint with the symplectic Harish-Chandra integral:

$$\bar{F}_{Sp(2N)}[\bar{\Lambda}] * F_{Sp(2N)}[\Lambda] = \int d\tilde{g}\exp\left(\mathrm{Tr}\left(\tilde{g}^{-1}\Lambda\tilde{g}\bar{\Lambda}'\right)\right) = \mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}'), \tag{6.29}$$

where $\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}')$ is given in [88]:

$$\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \left(\prod_{p=1}^{2N-1}(2p+1)!\right)\frac{\det\left[\sinh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^{2N}}{\Delta\left(\Lambda^{(2)}\right)\Delta\left(\bar{\Lambda}^{(2)}\right)\prod_{i=1}^{2N}\lambda_i\bar{\lambda}'_i}. \tag{6.30}$$

The denominator in this formula is computed using the Weyl denominator formula for the corresponding discriminant, as demonstrated in [89, 90]:

$$\mathbf{\Delta}_{\mathfrak{sp}(2N)}(\lambda) = \prod_{j}^{N} \lambda_j \prod_{1\leq j<k\leq N} \left(\lambda_j^2 - \lambda_k^2\right) = \det(\Lambda)\, \Delta\left(\Lambda^2\right) \tag{6.31}$$

Thus we may rewrite Eq. (6.30) as:

$$\mathbf{\Delta}_{\mathfrak{sp}(2N)}(\lambda)\mathbf{\Delta}_{\mathfrak{sp}(2N)}(\bar{\lambda}')\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \left(\prod_{p=1}^{N-1}(2p+1)!\right) \det\left[\sinh\left(2\Lambda_j \bar{\Lambda}'_k\right)\right]. \tag{6.32}$$

The numerator can be simplified by using the identity that $\sinh\left(2\Lambda_j \bar{\Lambda}'_k\right)$ is a modified Bessel function of the first kind of order $\nu = \frac{1}{2}$, and expanding the determinant. We know that:

$$\sinh\left(2\Lambda\bar{\Lambda}'\right) = \sqrt{\pi\Lambda\bar{\Lambda}'}\, I_{\frac{1}{2}}\left(2\Lambda\bar{\Lambda}'\right) = \sum_{m=0}^{\infty} \frac{2^{m+1}}{m!\,(2m+1)!!}\left(\Lambda\bar{\Lambda}'\right)^{2m+1} \tag{6.33}$$

Then we can use the Cauchy-Binet formula to expand the determinant:

$$\det\left[\sinh\left(2\Lambda_i \bar{\Lambda}'_j\right)\right] = \sum_{m_i}\prod_{i}^{N} \frac{2^{m_i+1}}{m_i!\,(2m_i+1)!!} \det\left[\Lambda_j^{2m_i+1}\right] \det\left[\bar{\Lambda}'^{2m_i+1}_j\right] \tag{6.34}$$

Thus Eq. (6.30) becomes:

$$\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \sum_{m_i}\prod_{i}^{N} \frac{2^{m_i+1}\,(2i-1)!}{m_i!\,(2m_i+1)!!} \frac{\det\left[\Lambda_j^{2m_i}\right] \det\left[\bar{\Lambda}'^{2m_i}_j\right]}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)(\bar{\lambda}'^2_i - \bar{\lambda}'^2_j)} \tag{6.35}$$

Once again, if we set $m_i = \mu_i + N - i$, we may rewrite Eq. (6.23) as an explicit sum over the Schur polynomials:

$$\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \sum_{\mu} \frac{1}{f_\mu} \chi_\mu(\Lambda^2) \chi_\mu(\bar{\Lambda}'^2), \tag{6.36}$$

where the coefficient in the expansion is given by

$$f_\mu = \prod_{i}^{N} \frac{(\mu_i + N - i)!\,(2\mu_i + 2N - 2i + 1)!!}{2^{\mu_i + N - i + 1}\,(2i-1)!}, \tag{6.37}$$

and the sum is taken over all integer partitions μ.

This form of the expansion is natural from the point of view of the orientifold projection, since we projected out all the states with an odd number of raising operators acting on the vacuum state. Similarly, the operator that creates the coherent state must have a formal expansion of a similar form:

$$\mathcal{O}_\Lambda = \int_{Sp(2N)} dg \exp\left(\mathrm{Tr}\left(g\Lambda g^{-1} a_Z^\dagger\right)\right) = \sum_\mu \frac{1}{f_\mu} \chi_\mu(\Lambda^2) \chi_\mu(\left(a_Z^\dagger\right)^2) \tag{6.38}$$

This indicates that just as in the unitary case, the norms of states are given by the inverse of the coefficients that appear in the expansion of the Harish-Chandra integral.

6.3.3 Special orthogonal groups

Odd special orthogonal group

It is known that the Harish-Chandra integral for the odd orthogonal group is the same as that for the symplectic group. This can be thought of as a result of the S-duality of $\mathcal{N} = 4$ super Yang-Mills theory; S-duality exchanges the $Sp(2N)$ and $SO(2N+1)$, while $SO(2N)$ is S-duality invariant [12]. This means that the spectrum of the $Sp(2N)$ and the $SO(2N+1)$ theories are related by a change of basis. We will argue that this change of basis is simply the transpose operation on the Young diagram μ associated to a given representation.

One reason to suspect that this is the case comes from the Schur-Weyl duality for odd orthogonal and symplectic groups. It is well-known that the centralizer algebra associated to the k-fold tensor product of fundamental representations of $SU(N)$ is the group algebra of the symmetric group $\mathbb{C}S_k$. This means that the k-fold tensor product of fundamental representations of $SU(N)$ decomposes into tensor products of irreducible representations of S_k and $SU(N)$:

$$V_{SU(N)}^{\otimes k} \cong \bigoplus_{\lambda} \pi^{\lambda} \otimes U_{\lambda}. \tag{6.39}$$

This is more complicated for the symplectic and orthogonal groups, since the corresponding centralizer algebra is no longer a group algebra, but rather the algebra associated to the Brauer monoid. One way to understand this is that the symplectic and orthogonal lie algebras have additional invariant tensors compared to the unitary case. For tensor products of fundamental representations of unitary groups, the only invariant tensors allowed are the identity and permutation operators:

$$\begin{aligned} \mathbb{I}(V_a \otimes V_b) &\to V_a \otimes V_b \\ \mathbb{P}(V_a \otimes V_b) &\to V_b \otimes V_a. \end{aligned} \tag{6.40}$$

Clearly these operations are invertible and generate the symmetric group S_k. For orthogonal groups, there is an additional invariant tensor, called the trace operation:

$$\mathbb{K}(V_a \otimes V_b) \to \mathbb{C}. \tag{6.41}$$

These tensors are well known in the integrable spin chain literature, and are the same kind of tensors that appear in the $SO(6)$ integrable spin chain [2]. Unlike the identity and permutation operators, the trace operation is not invertible, and together with the identity, it generates the Temperley-Lieb algebra $TL_k(2N)$ [91, 92]; the linear span of these three operations generates the Brauer algebra $B_k(2N)$. The importance of Brauer centralizer algebras has been emphasized in [93, 94], where they were used to diagonalize two-point functions in the space of gauge theory operators and their adjoints. These operators correspond to bound states of non-holomorphic giants. Brauer centralizer algebras have also been used to construct coherent states in [95].

Returning to the tensor decomposition of the k-fold tensor product of fundamentals of $SO(2N+1)$, the corresponding decompostition is [92]:

$$V_{SO(2N+1)}^{\otimes k} \cong \bigoplus_{k=0}^{\lfloor f/2 \rfloor} \bigoplus_{\lambda \vdash f-2k} D_\lambda \otimes V_\lambda, \tag{6.42}$$

with D_λ and V_λ respectively denoting the irreducible representations of the Brauer algebra and $SO(2N+1)$. The analogous statement for the symplectic group $Sp(2N)$ exchanges N with $-N$ and V_λ with W_{λ^T}, where W_{λ^T} is the irreducible representation of $Sp(2N)$ associated to the diagram conjugate to λ:

$$V_{Sp(2N)}^{\otimes k} \cong \bigoplus_{k=0}^{\lfloor f/2 \rfloor} \bigoplus_{\lambda \vdash f-2k} D_\lambda \otimes W_{\lambda^T}. \tag{6.43}$$

Since the Harish-Chandra integral involves group averages of powers of traces of the form $\mathrm{Tr}\,(g\Lambda g^{-1}\Lambda')$, it is natural to expect that every term in expansion for the odd orthogonal groups should match to a term with the corresponding transposed Young diagram in the expansion for the symplectic integral. This might appear surprising, since the number of boxes that can appear in a column is bounded from above by N, while the number of boxes in a row can be arbitrary. One way of understanding this apperent mismatch is that the fundamental degrees of freedom in one description might be mapped to a bound state by S-duality. In reality, representations with arbitrary numbers of boxes in a column are possible, but will not be irreducible.

Special even orthogonal group

Extending our definition for a well-defined BPS operator to the even special orthogonal group requires a little more work. We modify the definition of $F[\Lambda]$ to reflect averaging over the even special orthogonal group:

$$F_{SO(2N)}[\Lambda] = \int dO \exp\Big(\mathrm{Tr}(O\Lambda O^{-1} a_Z^\dagger)\Big) |0\rangle \,. \tag{6.44}$$

As before, the overlap of $F[\Lambda]$ and its adjoint is the corresponding Harish-Chandra integral:

$$\bar{F}_{SO(2N)}[\bar{\Lambda}] * F_{SO(2N)}[\Lambda] = \int d\tilde{O} \exp\left(\mathrm{Tr}\left(\tilde{O}^{-1}\Lambda\tilde{O}\bar{\Lambda}'\right)\right) = \mathcal{H}_{SO(2N)}(\Lambda, \bar{\Lambda}'), \tag{6.45}$$

where $\mathcal{H}_{SO(2N)}(\Lambda, \bar{\Lambda}')$ is given by [88]:

$$\mathcal{H}_{SO(2N)}(\Lambda, \bar{\Lambda}') = \left(\prod_{p=1}^{N-1}(2p)!\right) \frac{\det\left[\cosh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^N + \det\left[\sinh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^N}{\Delta\left(\Lambda^{(2)}\right)\Delta\left(\bar{\Lambda}'^{(2)}\right)}. \tag{6.46}$$

We note that Eq. (6.44) is invariant under an additional symmetry:

$$O \to \tilde{I}O, \tag{6.47}$$

where $\tilde{I}$ is a diagonal matrix with determinant equal to ± 1. To get rid of this redundancy, we could integrate over the entire orthogonal group $O(N)$. For $SU(N)$, $Sp(2N)$ and $SO(2N+1)$, this process does not change the value of the integral. This is similar to what happens in the Kazakov-Migdal model in [96], where the additional abelian part of the gauge field decouples from the collective field effective action. We also note that even though the whole integral is invariant under the parity transformation

$$\begin{aligned} \tilde{P} &: \Lambda \to -\Lambda \\ \tilde{P} &: \Lambda' \to -\Lambda', \end{aligned} \tag{6.48}$$

the overlap is not invariant under the individual reflections of each of the eigenvalue matrices. This is because the second term is odd under transformation by individual reflections of the matrices Λ and Λ'. Since each state must be individually invariant under this reflection, we choose to use the Harish-Chandra integral for $O(2N)$:

$$\mathcal{H}_{O(2N)} = \left(\prod_{p=1}^{N-1}(2p)!\right)\frac{\det\left[\cosh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^{N}}{\Delta\left(\Lambda^{(2)}\right)\Delta\left(\bar{\Lambda}'^{(2)}\right)}. \tag{6.49}$$

This is precisely the matrix analogue of the norm of the coherent state for the positive parity states of a harmonic oscillator. The main difference between each of the orientifold projections is that the vacuum of each theory is charged differently under parity; the symplectic case formally begins at occupation number one of the parent theory, while the even orthogonal case begins at occupation number zero.

We can now repeat the analysis of the previous sections with $\det\left[\cosh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^{N}$. We know that:

$$\cosh\left(2\Lambda\bar{\Lambda}'\right) = \sqrt{\pi\Lambda\bar{\Lambda}'}I_{-\frac{1}{2}}\left(2\Lambda\bar{\Lambda}'0\right) = \sum_{m=0}^{\infty}\frac{2^m}{m!\,(2m-1)!!}\left(\Lambda\bar{\Lambda}'\right)^{2m} \tag{6.50}$$

Applying the Cauchy-Binet formula yields:

$$\det\left[\cosh\left(2\Lambda_i\bar{\Lambda}'_j\right)\right] = \sum_{m_i}\prod_i^N \frac{2^{m_i}}{m_i!\,(2m_i-1)!!}\det\left[\Lambda_j^{2m_i}\right]\det\left[\bar{\Lambda}_j'^{2m_i}\right] \tag{6.51}$$

Then the Harish-Chandra integral for $O(2N)$ becomes:

$$\mathcal{H}_{O(2N)}(\Lambda,\Lambda') = \sum_{m_i}\prod_i^N \frac{2^{m_i}(2i-2)!}{m_i!\,(2m_i-1)!!}\frac{\det\left[\Lambda_j^{2m_i}\right]\det\left[\bar{\Lambda'}_j^{2m_i}\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda'}_i^2-\bar{\lambda'}_j^2)} \tag{6.52}$$

By setting $m_i = \mu_i + N - i$, the expression once again becomes a sum over Schur polynomials:

$$\mathcal{H}_{O(2N)}(\Lambda,\Lambda') = \sum_\mu \frac{1}{h_\mu}\chi_\mu(\Lambda^2)\chi_\mu((\Lambda')^2), \tag{6.53}$$

where the coefficient is now given by:

$$h_\mu = \frac{(\mu_i+N-i)!\,(2\mu_i+2N-2i-1)!!}{2^{\mu_i+N-i}(2i-2)!}. \tag{6.54}$$

Once again, we can expand the operator itself as a formal sum:

$$\int_{O(N)} dO\exp\left(O\Lambda O^T a_Z^\dagger\right) = \sum_\mu \frac{1}{h_\mu}\chi_\mu(\Lambda^2)\chi_\mu((a_Z^\dagger)^2), \tag{6.55}$$

which implies that the norm of the states are given by h_μ.

We chose to get rid of the redundancy by integrating over $O(2N)$ rather than $SO(2N)$; in doing so, we have chosen a specific partition function. The drawback to choosing $O(2N)$ as our gauge group is that we eliminate the Pfaffian operator, which is defined as:

$$Pf(\Lambda)^2 = \det(\Lambda), \tag{6.56}$$

where Λ is a $2n\times 2n$ skew-symmetric matrix. If we make another choice and integrate over $SO(2N)$ instead, our Harish-Chandra integral becomes:

$$\begin{aligned}\mathcal{H}_{SO(2N)}(\Lambda,\Lambda') = &\sum_{m_i}\prod_i^N \frac{2^{m_i}(2i-2)!}{m_i!\,(2m_i-1)!!}\frac{\det\left[\Lambda^{2m_i}\right]\det\left[\bar{\Lambda}'^{2m_i}\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda}'^2_i-\bar{\lambda}'^2_j))} \\ &+\sum_{n_i}\prod_i^N \frac{2^{n_i+1}\,(2i)!}{n_i!\,(2n_i+1)!!}\frac{\det\left[\Lambda^{2n_i+1}\right]\det\left[\bar{\Lambda}'^{2n_i+1}\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda}'^2_i-\bar{\lambda}'^2_j))}\end{aligned} \tag{6.57}$$

We see that the Pfaffian of $SO(2N)$, which changes sign under a single reflection, makes an appearance in the term we previously discarded. If we write $\Lambda = X_j + iX_k$, where X_j and X_k are two of the six scalar fields X_i in the adjoint representation of $SO(2N)\,\mathcal{N}=4$ SYM, then $Pf(\Lambda)$ corresponds to a single BPS $D3$ brane wrapped around the non-trivial three-cycle of $\mathcal{RP}^5$ [12, 97]. It can be considered half of a maximal giant graviton, which is identified as $\det(\Lambda)$, since the maximal giant graviton wraps around the non-trivial cycle twice.

6.4 Multi-matrix Generating Functions

We are interested in studying operators in gauge theories that are made out of more than one matrix-valued scalar field. In particular, we will work with $\frac{1}{4}$-BPS operators in $U(N)$ $\mathcal{N}=4$ SYM on the cylinder $\mathbb{R}\times S^3$. At weak coupling, these operators can be built out of symmetrized products of two of the three complex scalar fields of the theory X, Y. Generalizing to more than two matrices is straightforward. This class of operators transforms in the $[p,q,p]$ representations of the $SU(4)_R$ symmetry, and the operators are generically of multi-trace form. We will concentrate on scalar primary states at an equal time slice for simplicity. Unlike $\frac{1}{2}$-BPS operators, which can be built explicitly in the free theory, $\frac{1}{4}$-BPS operators of the interacting theory are different from those of the free theory. The lifting of states due to non zero gauge coupling can be treated pertubatively and the loop corrections to dilatation operators annihilate operators that are made out of symmetric products of X and Y. This problem was studied in detail for small operators in [76], but for generic large operators, explicit constructions in terms multi-traces are cumbersome. An alternative expansion in terms of characters was introduced in [62], which the authors call the *restricted Schur polynomial basis*. This basis is convenient for dealing with the mixing between the different trace structures since it diagonalizes the matrix of two point functions for all values of N.

6.4.1 Generating $\frac{1}{4}$ BPS States

Yet another way of generating $\frac{1}{4}$-BPS states can be found by studying operators of the form:

$$|\Lambda_X, \Lambda_Y\rangle = \frac{1}{\mathrm{Vol}\,[U(N)]} \int dU \ \exp\left(\mathrm{Tr}\left[UXU^\dagger\Lambda_X + UYU^\dagger\Lambda_Y\right]\right)|0\rangle\,. \tag{6.58}$$

If we insist that the coherent state parameters Λ_X and Λ_Y commute, $|\Lambda_X, \Lambda_Y\rangle$ is annihilated by the one-loop dilatation operator; it was shown in [98] that this persists to two-loop order. In [99], it was conjectured that the space of BPS states in $\mathcal{N} = 4$ SYM is given by the kernel of the one-loop dilatation operator at all values of the coupling; we will take this as a working assumption and work with the set of states annihilated by the Beisert one-loop dilatation operator:

$$\hat{D}_2^{SU(2)} = g^2 \mathrm{Tr}\,[[X, Y][\partial_X, \partial_Y]]\,. \tag{6.59}$$

Because the states (6.58) are coherent states of $\bar{X}, \bar{Y}$ [11], they form an overcomplete basis of states for any value of N. This has many computational advantages, mostly due to the fact that taking the large N limit is very straightforward, but translating back into a complete orthogonal basis of operators can be complicated. This may be solved by computing the norm of the coherent states. By exploiting the Campbell-Hausdorff formula, we arrive at an integral of the form:

$$\left\langle \bar{\Lambda}_X, \bar{\Lambda}_Y \middle| \Lambda_X, \Lambda_Y \right\rangle = \frac{1}{\mathrm{Vol}\left[U(N)\right]} \int dU \; \exp\left(\mathrm{Tr}\left[U\bar{\Lambda}_X U^\dagger \Lambda_X + U\bar{\Lambda}_Y U^\dagger \Lambda_Y\right]\right). \qquad (6.60)$$

Since we can in principle expand (6.58) in terms of an orthonormal basis, we may use this overlap to determine the coefficients relating the multi-trace basis of operators to an orthogonal basis by expanding in a series and matching the coefficients as done in [11]. The precise tool relating the multi-trace basis operators and the character expansion in this case is the Weingarten calculus [100]; an example illustrating this technique can be found in [101]. The main obstacle we face is evaluating the integral (6.60) for generic coherent state parameters. To our knowledge, these types of integrals have not been studied before, and a closed form expression for them is needed. Our main goal will be to evaluate this class of integrals for any value of N. Although we only explicitly study the case of $U(N)$ integrals, the methods should apply generally and should generalize to $SO(N)$ and $Sp(N)$ groups as well as to quivers. These types of integrals are also a natural object to study in the context of matrix models, since they arise in the study of multi-matrix models of commuting matrices.

6.4.2 The Four-Matrix Model in $SU(2)$

Before proceeding to the case of general N, we will study the following integral

$$I_2 = \int_{SU(2)} dU \; e^{\mathrm{Tr}\left[UAU^\dagger \bar{A} + UBU^\dagger \bar{B}\right]} \qquad (6.61)$$

for commuting matrices $A, B, \bar{A}, \bar{B}$. We will first approximate I_2 by a saddle point approximation; the critical points of the function in the exponential are given by the solutions to the equations

$$[A, U^\dagger \bar{A} U] + [B, U^\dagger \bar{B} U] = 0. \tag{6.62}$$

For generic enough matrices, this is only satisfied if each of the two terms vanishes individually

$$[A, U^\dagger \bar{A} U] = [B, U^\dagger \bar{B} U] = 0. \tag{6.63}$$

The only problematic cases occur when a subset of the eigenvalues of B is a permutation of a subset of eigenvalues of $-A$. From here on, we assume that the eigenvalues are generic enough that this does not happen. This means that, generically, the saddle points are labelled by permutation matrices U_π. We are then left with a Gaussian integral around each of the saddle points, which can be evaluated easily; this results in a "one-loop determinant" factor given by:

$$D_2(a, \bar{a}, b, \bar{b}) = (a_1 - a_2)(\bar{a}_1 - \bar{a}_2) + (b_1 - b_2)\left(\bar{b}_1 - \bar{b}_2\right) \tag{6.64}$$

This gives an approximate value for the integral (up to a convention dependent normalization factor):

$$I_2 \simeq \frac{e^{a_1\bar{a}_1 + a_2\bar{a}_2 + b_1\bar{b}_1 + b_2\bar{b}_2} - e^{a_1\bar{a}_2 + a_2\bar{a}_1 + b_1\bar{b}_2 + b_2\bar{b}_1}}{(a_1 - a_2)(\bar{a}_1 - \bar{a}_2) + (b_1 - b_2)\left(\bar{b}_1 - \bar{b}_2\right)}. \tag{6.65}$$

At first sight, it is not clear that this approximation is reliable, since there is no large parameter in the exponential. To gain more intuition, we evaluate I_2 through an explicit computation.

First, we must parameterize our unitary matrix U; then, we need to compute the Haar measure. We start with the following matrices:

$$\begin{gathered} A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \; B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \\ \bar{A} = \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix}, \; \bar{B} = \begin{pmatrix} \bar{b}_1 & 0 \\ 0 & \bar{b}_2 \end{pmatrix} \end{gathered} \tag{6.66}$$

We then seek to parametrize our unitary matrix. We know that any arbitrary $SU(2)$ matrix must meet the following conditions:

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in \mathbb{C}^{2\times 2} \;\middle|\; |a|^2 + |b|^2 = 1 \right\} \tag{6.67}$$

For ease of computation, we choose to parameterize U with Euler angles:

$$U = \begin{pmatrix} e^{-i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} & -e^{i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} \\ e^{-i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} & e^{i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{6.68}$$

We seek to rewrite the Haar measure dU in terms of $J(\theta,\gamma,\alpha)d\theta d\gamma d\alpha$, where $J(\theta,\gamma,\alpha)$ is the Jacobian. We may do so by computing the inverse of the unitary matrix and multiplying it by its partial derivatives with respect to the Euler angles. We start by finding the inverse of U:

$$U^{-1} = \begin{pmatrix} e^{i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} & e^{i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} \\ -e^{-i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} & e^{-i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{6.69}$$

Then we calculate the partial derivatives with respect to γ, α, and θ and multiply by the inverse. We obtain:

$$\begin{aligned} U^{-1}\frac{\partial U}{\partial \gamma} &= \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix} \\ U^{-1}\frac{\partial U}{\partial \alpha} &= \begin{pmatrix} -\frac{i}{2}\cos\theta & \frac{i}{2}e^{i\gamma}\sin\theta \\ \frac{i}{2}e^{-i\gamma}\sin\theta & \frac{i}{2}\cos\theta \end{pmatrix} \\ U^{-1}\frac{\partial U}{\partial \theta} &= \begin{pmatrix} 0 & -\frac{1}{2}e^{i\gamma} \\ \frac{1}{2}e^{-i\gamma} & 0 \end{pmatrix} \end{aligned} \tag{6.70}$$

We calculate the Jacobian matrix using the following basis $\epsilon_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\epsilon_2 = \begin{pmatrix} 0 & ie^{i\gamma} \\ ie^{-i\gamma} & 0 \end{pmatrix}$, and $\epsilon_3 = \begin{pmatrix} 0 & -e^{i\gamma} \\ e^{-i\gamma} & 0 \end{pmatrix}$:

$$\mathcal{J} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\cos\theta & 0 \\ 0 & \frac{1}{2}\sin\theta & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{6.71}$$

The Jacobian $J(\theta, \gamma, \alpha)$ we seek is the determinant of $\mathcal{J}$:

$$\det(J) = \frac{1}{8}|\sin\theta| \tag{6.72}$$

We see that it is only dependent on θ. Our integral becomes:

$$\begin{aligned} I_2 &= \frac{1}{8}\int_0^{\pi} d\theta \int_0^{4\pi} \frac{d\gamma}{4\pi} \int_0^{2\pi} \frac{d\alpha}{2\pi} |\sin\theta| e^{\mathrm{Tr}\left[\bar{A}UAU^\dagger + \bar{B}UBU^\dagger\right]} \\ &= \frac{1}{8}\int_0^{\pi} d\theta |\sin\theta| e^{\frac{1}{2}\left((a_1+a_2)(\bar{a}_1+\bar{a}_2)+(b_1+b_2)\left(\bar{b}_1+\bar{b}_2\right)+\left((a_1-a_2)(\bar{a}_1-\bar{a}_2)+(b_1-b_2)\left(\bar{b}_1-\bar{b}_2\right)\right)\cos\theta\right)} \end{aligned} \tag{6.73}$$

Our critical points are $\theta = 0$ and $\theta = \pi$, so we can remove the absolute value bars. Then we evaluate our integral:

$$
\begin{aligned}
I_2 &= \frac{1}{8}\int_0^\pi d\theta \sin\theta e^{\frac{1}{2}\left((a_1+a_2)(\bar{a}_1+\bar{a}_2)+(b_1+b_2)\left(\bar{b}_1+\bar{b}_2\right)+\left((a_1-a_2)(\bar{a}_1-\bar{a}_2)+(b_1-b_2)\left(\bar{b}_1-\bar{b}_2\right)\right)\cos\theta\right)} \\
&= \frac{e^{a_1\bar{a}_1+a_2\bar{a}_2+b_1\bar{b}_1+b_2\bar{b}_2} - e^{\bar{a}_1 a_2+a_1\bar{a}_2+\bar{b}_1 b_2+b_1\bar{b}_2}}{4\left((a_1-a_2)\left(\bar{a}_1-\bar{a}_2\right)+(b_1-b_2)\left(\bar{b}_1-\bar{b}_2\right)\right)}
\end{aligned}
\tag{6.74}
$$

This is precisely the same result that the saddle point approximation yields. From the intermediate steps, it is clear that there are never any terms that mix the eigenvalues of A and B; if we set either $A = 0$ or $B = 0$, we immediately recover the HCIZ formula for $U(2)$.

6.4.3 Proof of General Formula

Generically, we expect that the following formula holds:

$$
\begin{aligned}
I_N &= \int_{U(N)} dU\ e^{\mathrm{Tr}\left[UAU^\dagger\bar{A}+UBU^\dagger\bar{B}\right]} \\
&= \mathcal{C}_N \sum_{\pi\in S_N} \det\pi \times \frac{\prod_i e^{a_i\bar{a}_{\pi(i)}+b_i\bar{b}_{\pi(i)}}}{\prod_{i\neq j}\left[(a_i-a_j)(\bar{a}_i-\bar{a}_j)+(b_i-b_j)(\bar{b}_i-\bar{b}_j)\right]} \\
&= \mathcal{C}_N \frac{\det\left(e^{a_i\bar{a}_j+b_i\bar{b}_j}\right)}{\Delta(\Lambda_A)\Delta(\Lambda_{\bar{A}})+\Delta(\Lambda_B)\Delta(\Lambda_{\bar{B}})},
\end{aligned}
\tag{6.75}
$$

where $\mathcal{C}_N$ is a constant that depends on the normalization for the volume of $U(N)$; a_j, $\bar{a}_j$, b_j, and $\bar{b}_j$ are respectively the eigenvalues of the matrices A, $\bar{A}$, B, and $\bar{B}$; the matrices Λ_A, $\Lambda_{\bar{A}}$, Λ_B, $\Lambda_{\bar{B}}$ are respectively the diagonal matrices of the eigenvalues of A, $\bar{A}$, B, and $\bar{B}$; and $\Delta(\Lambda_M)$ is the Vandermonde determinant of matrix Λ_M. The main idea is as follows. The function

$$
\phi(U) = \mathrm{Tr}[U\Lambda_A U^\dagger \Lambda_{\bar{A}}]
\tag{6.76}
$$

can be thought of as a Hamiltonian function generating a $U(1)^N$ action on $U(N)$; the HCIZ integral localizes on the fixed points of this action. The integration is done over a coadjoint orbit $\mathcal{O}_{\Lambda_A}$ which has a natural symplectic structure. Alternatively, the integration domain can be reduced to $U(N)/U(1)^N$, where $U(1)^N$ is the maximal torus commuting with Λ_A. For a generic pair of commuting matrices, the analogous funtion

$$\psi(U) = \mathrm{Tr}[U\Lambda_A U^\dagger \Lambda_{\bar{A}}] + \mathrm{Tr}[U\Lambda_B U^\dagger \Lambda_{\bar{B}}] \tag{6.77}$$

still generates an action of the maximal torus on $U(N)$, although the integration domain does not have a natural interpretation as a coadjoint orbit. Despite of this, one can still formally reduce the integration to the symplectic space $U(N)/U(1)^N$, with $\Lambda_{A,B}$ being treated as elements of the Cartan subalgebra of $\mathfrak{u}(N)$. Up to the assumption of non-degeneracy of fixed points, these are the necessary conditions for the Duistermaat-Heckman theorem.

6.5 Connection with Restricted Schur Polynomials

A natural question to ask is: What sort of basis of operators do the coherent states (6.58) actually generate? This is quite non-trivial, since there are in principle many different ways of orthogonalizing the two point function of $\frac{1}{4}$-BPS operators at finite N.

Recalling the definition of the restricted Schur polynomials

$$\chi_{R,(r,s)\,\alpha\beta}(X,Y) = \mathrm{Tr}[P_{R,(r,s)\,\alpha\beta}\, X^n \otimes Y^m], \tag{6.78}$$

where R is a Young diagram associated to a representation R of S_{n+m}, r is a Young diagram for the representation r of S_n and s is another Young diagram for a representation s of S_m, the object $P_{R,(r,s)\,\alpha\beta}$ can be understood as follows. Starting with $S_m \times S_n \subset S_{m+n}$, we can find representations $r \times s$ sitting within R. Generically, the representation $r \times s$ can appear more than once inside of R, so one needs to keep track of how one embeds $r \times s$ into R. The matrix indices α, β keep track of this information. More formally, we can label each of the embeddings of $r \times s$ by an index γ, and consider the space $R_\gamma \subset R$. The restricted Schur polynomial is then given by

$$\chi_{R,R_\gamma}(X,Y) = \frac{1}{m!n!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{R_\gamma}[\Gamma_R(\sigma)] \mathrm{Tr}[\sigma X^n \otimes Y^m], \tag{6.79}$$

where $\Gamma_R(\sigma)$ is the matrix represetating σ [62]. The most complicated part of the restricted Schur polynomials is the evaluation of $\mathrm{Tr}_{R_\gamma}[\Gamma_R(\sigma)]$, which involves building R_γ explicitly.

By expanding the exponential and evaluating the unitary integrals, we obtain

$$\begin{gathered}\frac{1}{\mathrm{Vol}\,[U(N)]} \int dU \;\exp\left(UXU^\dagger \Lambda_X + UYU^\dagger \Lambda_Y\right) = \\ \sum_{n,m} \frac{1}{m!n!} \sum_{\sigma,\tau \in S_{n+m}} \mathrm{Tr}[\sigma \Lambda_X^n \otimes \Lambda_Y^m] \mathrm{Tr}[\tau X^n \otimes Y^m]\, \boldsymbol{Wg}(\sigma\tau^{-1}, N),\end{gathered} \tag{6.80}$$

where $\boldsymbol{Wg}(\sigma, N)$ is the Weingarten function. Explicit combinatorial formulas for Weingarten functions are well known from the work of Collins (see [100] for an elementary introduction), but before delving into specific details, we should contrast this with the situation where one of $\Lambda_{X,Y}$ is zero. In that case, the resulting sum can be recast as a diagonal sum of products of unitary characters; right now, we have a complicated sum of traces. For a moment, let us consider the situation for a single matrix. The resulting sum is:

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}\,[U(N)]}\int dU\;\exp\left(UXU^{\dagger}\Lambda_X\right)\\
&=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sigma,\tau\in S_n}\mathrm{Tr}[\sigma\Lambda_X^n]\mathrm{Tr}[\tau^{-1}X^n]\,\boldsymbol{Wg}(\sigma\tau^{-1},N)\\
&=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sigma,\tau\in S_n}\mathrm{Tr}[\sigma\Lambda_X^n]\mathrm{Tr}[\tau^{-1}X^n]\sum_{\lambda\vdash n}\frac{1}{n!\,f_\lambda}\chi^\lambda(\tau^{-1}\sigma)\chi^\lambda(1)\\
&=\sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\frac{1}{f_\lambda}s_\lambda(X)s_\lambda(\Lambda_X).
\end{aligned}\tag{6.81}$$

The last line is obtained from the character expansion of the integral, which was computed in [11]. Then for two matrices, we have:

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}\,[U(N)]}\int dU\;\exp\left(UXU^{\dagger}\Lambda_X+UYU^{\dagger}\Lambda_Y\right)\\
&=\sum_{n,m}\frac{1}{m!n!(n+m)!}\sum_{\lambda\vdash n+m}\frac{1}{f^\lambda}\sum_{\sigma,\tau\in S_{n+m}}\chi^\lambda(\sigma)\chi^\lambda(\tau)\mathrm{Tr}[\sigma\Lambda_X^n\otimes\Lambda_Y^m]\mathrm{Tr}[\tau X^n\otimes Y^m].
\end{aligned}\tag{6.82}$$

Clearly this has a similar structure to the definition of the restricted Schur polynomials (6.79), but the restricted characters have been replaced with ordinary symmetric group characters instead. This discrepancy can be traced back to the fact that the sum over S_{n+m} has many redundancies owing to the fact that we can conjugate by an element of $S_n\times S_m$ while leaving the traces fixed. This is the statement that we can permute the n X's and m Y's among themselves while simultaneously permuting the $\Lambda_{X,Y}$'s. As explained in [102], there is an equivalence relation between elements of S_{n+m} in such a way that

$$\sigma\sim\tau\;\Leftrightarrow\mathrm{Tr}[\sigma A^n\otimes B^m]=\mathrm{Tr}[\tau A^n\otimes B^m],\tag{6.83}$$

which happens exactly when σ can be conjugated into τ by an element of $S_n\times S_m$. In other words, the construction of restricted Schur polynomials is equivalent to constructing generalized class functions on restricted conjugacy classes. Unfortunately this means that the coherent state generating function (6.58) cannot differentiate between different restricted Schur polynomials by itself for the simple reason that the Weingarten function is a class function. This means that if we want to replace the characters in (6.58) with restricted characters, we must change the domain of integration. In any case, the coherent states still form an overcomplete basis of operators that can be used for computations, even if we do not currently know how to project into a particular primary state. One way of achieving this projection would be by integrating against pairs of Schur functions of $\Lambda_{X,Y}$ as was done for $\frac{1}{2}$-BPS operators in [26]; this would give a description of the restricted Schur polynomials in terms of half-BPS partons as advocated by [102], but it is still unclear how one would be able to deal with possible multiplicities of the subduced representations (r, s).

6.6 Discussion

In this chapter, we extended the method of computing the norms of half BPS coherent states through localization [11] to theories with the gauge groups $Sp(2N)$, $SO(2N+1)$, and $SO(2N)$. We did this by constructing coherent states averaged over a group orbit from each group and computing the norm of these states through the symplectic and special orthogonal Harish-Chandra integrals. The integration over the group may be viewed as a sort of path integral over the emergent world-volume gauge symmetry of a stack of N giant gravitons inside $AdS_5 \times \mathbb{RP}^5$; the norm of the state gives the effective action of this theory. Curiously enough, these types of integrals first appeared in models of induced QCD. By expanding the Harish-Chandra integrals, we found that each integral admits an expression as a sum of unitary characters. This matches what one would expect of an orientifold projection of a $U(2N)$ gauge theory; all the states that are spanned by the coherent states are "doubled" versions of those in the original theory.

In particular, the coherent states considered here do not span the complete spectrum of the free $Sp(2N)$ and $SO(2N)$ theories. This is because the Harish-Chandra integral is only able to capture information from tensor contractions of the invariant tensors of the unitary group (meaning all products of traces). It is likely that some of the data corresponding to worldsheets with cross-caps is missing.

As in the unitary case, the coefficient associated with the characters in this series expansion computes the overlap of the corresponding Schur polynomials of the operators $(a)^i_j$ and $(a^\dagger)^i_j$. Our method should be contrasted to other constructions of basis of operators for the $Sp(2N)$ and $SO(2N)$ theories [85, 86], since our construction uses group theoretic objects more closely associated to each group.

We also studied multi-matrix coherent states for bosonic matrices that generate $\frac{1}{4}$ and $\frac{1}{8}$ BPS states in $\mathcal{N} = 4$ SYM. We showed that the norm of these coherent states admits a fixed point formula generalizing the Harish-Chandra-Itzykson-Zuber formula. This gives in principle a way of generating expressions for BPS states for any value N in $\mathcal{N} = 4$ SYM. One technical obstacle we face is that our construction does not give an alternative construction of the so-called restricted Schur polynomial operators [62]. This is related to the expectation that there is a hidden symmetry under which different operators are charged. One idea is that determining the Casimir charges should be enough to differentiate between different operators, but this problem is quite non-trivial even in the $\frac{1}{2}$ BPS sector [103]. It is also unclear how to implement this idea efficiently at large N since the number of Casimirs needed to distinguish between different operators grows with the complexity of the operators. Despite this obstacle, our results are important for computing correlators of $\frac{1}{4}$ and $\frac{1}{8}$ BPS operators dual to bound states of giant gravitons [71] and generic bubbling geometries [69]. Understanding the precise map between the overcomplete 'eigenvalue basis' of coherent states and specific orthogonal bases of operators remains an important problem. We conclude with a few more immediate directions for future work.

$\frac{1}{16}$ BPS States and Black Hole Microstate Operators

One of the more interesting generalizations would be to the case of $\frac{1}{16}$ BPS operators. By now, there is ample evidence that there exists a class of $\frac{1}{16}$ BPS operators describing the microstates of supersymmetric black holes in $AdS_5 \times S^5$ [7, 6, 104, 105]. Recently, there have been some studies of these types of states for small values of N [106, 107]; see [108] for a more general discussion. It would be nice to develop more systematic techniques to build these types of operators. In principle, there are no obstructions to generalizing our techniques to this setup, with the working assumption that finding states with vanishing one-loop anomalous dimension is enough [99]. The idea would essentially be to build a superfield coherent state [109]:

$$\int dU \ \exp\left\{\int d^3\theta \int dz \mathrm{Tr}\left[U\Psi U^\dagger \Phi\right]\right\}|0\rangle\,, \tag{6.84}$$

where $\Psi(z,\theta)$ is the $\mathbb{C}^{2|3}$ superfield discussed in [109, 104], and Φ is an auxiliary superfield of coherent state parameters. The combined effect of the exponentiation and integration over the unitary matrices is to generate all possible gauge invariant tensor contractions. One should expect that the operators generated by this generating function are generalizations of the $SU(2|3)$ restricted Schur polynomials constructed in [110]. Generically the terms in the expansion of (6.84) will not be of multi-graviton form, so they are natural candidates for microstates of supersymmetric black holes. In practice, the main disadvantage of an expression like (6.84) is that it might not be practically useful, in the sense that the expansion necesarily involves an infinite number of matrix fields associated to covariant derivatives acting on the fields. One way of avoiding this difficulty is to use generating functions such as the ones studied in [98]. Alternatively, one can view the auxiliary superfield Φ as a full-fledged dynamical collective coordinate. One would then hope that integrating out the SYM fields leads to an effective matrix quantum mechanics describing (near)-BPS black hole microstates, with the lightcone coordinate z acting as a time variable.

Three Point Correlators, Bubbling Geometries, and Twisted Holography

Although eventually we would like to study black holes, it is important to build intuition from simpler examples. One class of such examples is the BPS bubbling geometries [69] generalizing LLM geometries [16]. Although the droplet description of such states in supergravity is compelling, a precise mapping between the weak coupling BPS states is not fully developed[1]. The coherent states (6.80) have a more natural connection to such geometries[26]. A worthwhile exercise would be to study correlators of single trace chiral primaries in the background of heavy coherent states corresponding to both giant gravitons or bubbling geometries; see [111] for some finite N results. The holographic renomalization techniques of [112] are also applicable in these cases, but it would be interesting to develop more efficient computational techniques in supergravity along the lines of [113]. A good toy model for this would be to study these types of questions in Twisted Holography [114].

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[1]For instance, it is unclear whether the solutions found in [69] exhaust the set of all $\frac{1}{4}$ and $\frac{1}{8}$ BPS states.

Chapter 7

Holographic Three Point Functions

7.1 Introduction

The AdS/CFT correspondence provides in principle a way of addressing interesting questions in simple theories of quantum gravity [1]. However the usual lore states that this is a weak/strong duality; objects that behave classically in gravity are described by complicated states in a strongly coupled conformal field theory. Fortunately this is not the case, as protected operators with large dimensions can and do behave semiclassically on both sides of the duality.

One of the simplest example of such an object is a half-BPS determinant operator in $\mathcal{N} = 4$ SYM

$$\mathcal{D}(x,\xi) = \det\left(\mathbf{1}\xi - Z(x)\right) = \int d\bar{\chi} d\chi \exp\left(-\bar{\chi}\left[\xi - Z(x)\right]\chi\right), \tag{7.1}$$

whose dual description is a wrapped D3-brane inside of S^5, sitting at the origin of AdS [48]. The fact that these operators describe localized probes of $\mathrm{AdS}_5 \times S^5$ makes them ideal probes for bulk locality. The main obstacle to dealing with such objects on the gauge theory lies in the sheer combinatorial complexity of summing large numbers of planar graphs. Recently, this problem was revisited by using saddle-point methods to systematically resum these non-planar contributions [9, 10]. This allows for an efficient computation of simple correlators involving determinant operators in the large N limit. As an application, the authors of [115] studied the three-point function of a BPS single trace operator and two determinants and found a remarkable agreement with the *orbit average* of the holographic computation of [116]. Holographic three-point functions of giant gravitons have been studied extensively in the literature [117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 116, 127, 128, 129, 130, 131], with some discrepancies and ambiguities appearing between the holographic and gauge theoretic computations in the case of off-diagonal extremal correlators and for AdS giant gravitons.

In [10, 115], similar techniques were introduced for studying fully-symmetric Schur polynomial operators:

$$\mathcal{S}(x,\xi) = \int_{\mathbb{C}^N} d\bar{\varphi} d\varphi \exp\left(-\bar{\varphi}\left[\xi - Z(x)\right]\varphi\right) = \frac{1}{\det\left(\mathbf{1}\,\xi - Z(x)\right)}. \tag{7.2}$$

Formally, this object is a generating function for BPS operators transforming in fully-symmetric representations of $U(N)$, which describe giant gravitons extended along the AdS_5 directions. Despite the similarities between the techniques developed for determinant operators, these generating functions have an important distinction in that they do not correspond to simple semiclassical states. In fact, these generating function create a non-physical state of infinite norm in $\mathcal{N} = 4$ SYM. The symmetry between sphere and AdS giants can be restored by considering BPS coherent states in the gauge theory [11]. These are given by a group averages of the exponential of one of the complex scalar fields.

The goal of this chapter is to extend the analysis in [115] to the case of AdS giant gravitons and to further clarify some technical aspects of their computation. Our analysis essentially mirrors [116] but the set-up and results are different. After performing an orbit average of the semiclassical one-point functions of a BPS supergravity mode, we find precise agreement with the gauge theory computation of a BPS three-point function involving two heavy symmetric Schur polynomials and a single trace operator. Despite the fact that the intermediate steps in the computation are rather different from the case of sphere giants we find that the final results are related by a simple analytic continuation. Our derivation of the structure constant in $\mathcal{N}=4$ SYM is new, and also involves a sort of orbit average, although its relation to the one in holography is unclear. As we will explain, our methods have straightforward generalizations to the case of correlators of more general Schur polynomials, although we leave the details of this analysis for future work.

The chapter is structured as follows. In section 7.2 we review the orbit average method and how it applies to holographic correlation functions. Then, we review the coherent state techniques necessary for the large N analysis in the field theory in section 7.3. In section 7.4 we turn to computation of the structure constant of two BPS fully-symmetric Schur polynomials and a single trace BPS operator. We provide an exact integral formula for the generating function for these structure constants, which we then evaluate via the saddle-point approximation. In section 7.5 we compute diagonal and off-diagonal structure constants in the dual supergravity following [115], finding an exact matching with the gauge theory result. Finally we comment on possible future directions.

7.2 Review of Orbit Average

We begin by giving a brief review of the semiclassical techniques found in [132, 133] and [115], known as the orbit average method.

The idea is as follows: consider a quantum mechanical system whose action $S[X]$ is invariant under some global symmetry G. In general, the eigenstates of the Hamiltonian will not be invariant under this global symmetry, but will rather transform in some representation of G labelled by a set of charges $\{J_i\}$. One is usually interested in computing correlation functions of operators in backgrounds with non-zero charges

$$C_{JJ'\mathcal{O}_L} = \langle J'|\mathcal{O}_L(t=0)|J\rangle, \tag{7.3}$$

where we can think of the states $|J\rangle, |J'\rangle$ as being created by the insertion of operators with large charges, $J, J' \gg 1$. In the WKB approximation, this quantity can be computed by a path integral with the corresponding classical action evaluated on solutions to the equations of motion:

$$\langle J'|\mathcal{O}_L(t=0)|J\rangle \sim e^{iS[X^*]}\mathcal{O}[X^*], \tag{7.4}$$

where $\mathcal{O}_L[X] \equiv \langle X|\mathcal{O}_L|X\rangle$.

Generically, these classical solutions may spontaneously break (some part of) the global symmetry and are therefore parametrized by a set of moduli $\{c_i\}$ describing the action of (a subgroup of) G on the solutions. Beginning from a given solution X_0^*, one can generate a moduli space of solutions under an orbit of the G-action $X_0^* \to X_{\{c_i\}}^*$. Since these solutions contribute equal exponential factors, one must integrate over this moduli space in order to reproduce the correct saddle-point approximation to the correlator. Additionally, in the case where J and J' are not equal, and $J - J' \ll 1$, one needs to take into account the contributions coming from the WKB wavefunction of the initial and final states

$$\langle J'|X_c^*\rangle \approx e^{-iJ'c}, \ \ \langle X_c^*|J\rangle \approx e^{iJc}. \tag{7.5}$$

The condition $J - J' \ll 1$ is necessary for the WKB approximation to hold. Putting it all together, the semiclassical correlator is given by the orbit average

$$\langle J'|\mathcal{O}_L(t=0)|J\rangle \approx e^{iS[X^*]} \int \prod_i dc_i \; \mathcal{O}_L[X^*_{\{c_i\}}] \; e^{i\left(J_i - J_i'\right)c_i}, \tag{7.6}$$

where $\mathcal{O}_L[X^*]$ should be understood as the classical analog of the operator $\mathcal{O}_L$.

7.3 BPS Coherent States

In this section we review the coherent state methods introduced in [11], and their application to BPS correlators of fully-symmetric Schur polynomials. Firstly, half-BPS operators in $\mathcal{N}=4$ SYM are described by polynomials in the traces of a complex scalar field Z. For our purposes we will want to consider the theory on the cylinder $\mathbb{R}\times S^3$, so that our initial and final states are inserted at $t=\pm\infty$. Then, the main idea is that the following expression serves as a generating series for all half-BPS operators

$$|\Lambda\rangle = \frac{1}{\texttt{Vol}\,(U(N))} \int_{U(N)} dU e^{\mathrm{Tr}\left(U\Lambda U^\dagger Z\right)} |0\rangle \,, \tag{7.7}$$

where Λ is a diagonal matrix with complex eigenvalues λ_i. A simple calculation shows that this is in fact a coherent state, in the sense that the action of $\bar{Z}$ on this state can be replaced by multiplication by $U\Lambda U^\dagger$. This state also has a simple expression as a sum over the Schur basis

$$|\Lambda\rangle = \sum_R \frac{1}{d_R} \chi_R\left(\Lambda\right) \chi_R(Z)\, |0\rangle \,, \tag{7.8}$$

where d_R is the norm of the state created by $\chi_R(Z)$. One important property of this formalism is that correlation functions involving these coherent states can be recast in terms of a unitary matrix integral. For example, the overlap between two coherent states has an explicit formula as a sum over saddle points

$$\langle \bar{\Lambda} | \Lambda \rangle = \int_{U(N)} dU e^{\mathrm{Tr}\left(U\Lambda U^{\dagger}\bar{\Lambda}\right)} = \mathcal{C}_N \sum_{\sigma \in S_N} (-1)^{\mathrm{sign}(\sigma)} \frac{e^{\sum_i \lambda_i \bar{\lambda}_{\sigma(i)}}}{\Delta(\Lambda)\Delta(\bar{\Lambda})}; \tag{7.9}$$

the overall normalization constant $\mathcal{C}_N$ that depends on conventions. More generally, commuting the exponential in $|\Lambda\rangle$ with insertions of $\bar{Z}$ will have the effect of replacing $\bar{Z}$ with $\bar{Z} + U\Lambda U^{\dagger}$, and similarly for Z. Although this formulation is quite explicit, it is unclear that the term with the largest exponential actually dominates the sum, since there are $N! - 1$ other saddle point contributions that could in principle lead to an exponentially large correction. This is not always the case, since the leading contribution always corresponds to the identity permutation, while the remaining saddle points are weighted by a sign; whenever some of the eigenvalues $\{\lambda_i\}$ are exponentially close to one another different saddles become comparable to the identity saddle and their contribution will become important.

For our purposes, we will restrict to the case where a single eigenvalue $\lambda_1 = \lambda$ is taken to be non-zero, which restricts the sum over representations to those associated with single row Young diagrams. In this case, the formula in terms of unitary integrals is difficult to perform calculations with simply because the numerator and denominator in (7.9) become degenerate. To remedy this, one should realize that the integral is really being performed over an orbit parametrized by:

$$\mathcal{O}_{\lambda} = \{\lambda U P_1 U^{\dagger} \ | \ U \in U(N)\} \cong \mathbb{CP}^{N-1}, \tag{7.10}$$

where P_1 denotes the projector into the eigenspace of λ_1. Geometrically this is straightforward to understand; the projector operator P_1 is naturally associated to a unit vector in $\mathbb{C}^N$ by

$$P_1 = \varphi\varphi^\dagger, \tag{7.11}$$

so a choice of projector P_1 is in one-to-one correspondence to a line in $\mathbb{C}^N$. The action of the unitary group moves this line inside of $\mathbb{C}^N$, so the resulting integral should be taken over the Grassmannian $\mathbf{Gr}(1,N) \cong \mathbb{CP}^{N-1}$. This suggests that the natural generalization of (2.6) is actually

$$|\lambda\rangle = \frac{1}{\mathtt{Vol}(\mathbb{CP}^{N-1})}\int_{\mathbb{CP}^{N-1}} d\varphi^\dagger d\varphi\; e^{\sqrt{N}\lambda\varphi^\dagger Z\varphi}\,|0\rangle\,. \tag{7.12}$$

Formally this generating functions looks similar to the generating function introduced in [10], but there are many important differences that make $|\lambda\rangle$ much better behaved.

The first important difference is that $|\lambda\rangle$ is a coherent state of finite norm:

$$\mathrm{Tr}\left[\bar{Z}^L\right]|\lambda\rangle = \lambda^L\,|\lambda\rangle \tag{7.13}$$

To find the norm of this state, we can exploit the fact that the integration measure on $\mathbb{CP}^{N-1}$ is left-invariant under the action of $U(N)$. More precisely, once we use the Baker-Hausdorff-Campbell formula to commute the exponentials coming from $\langle\lambda|$ and $|\lambda\rangle$ we are left with a pair of integrals as in [9, 10]:

$$\langle\lambda|\lambda\rangle = \left(\frac{1}{\mathtt{Vol}(\mathbb{CP}^{N-1})}\right)^2\int_{\left(\mathbb{CP}^{N-1}\right)^2} d\varphi^\dagger d\varphi\; d\psi^\dagger d\psi\; e^{N\bar{\lambda}\lambda\varphi^\dagger\psi\psi^\dagger\varphi}. \tag{7.14}$$

At this point our analysis differs from that of [9, 10, 115], in that we can proceed without performing a Hubbard-Stratonovich transformation. To see why this is the case, we should remember that we may parametrize ψ as a rank-one projector conjugated by a unitary matrix $\psi = UP_1U^\dagger$. Since the measure for ϕ is invariant, this reduces the integral over ψ into a volume integral over $U(N)/\left(U(N-1)\times U(1)\right)\cong\mathbb{CP}^{N-1}$:

$$\begin{aligned}\langle\lambda|\lambda\rangle &= \frac{1}{\texttt{Vol}\left(\mathbb{CP}^{N-1}\right)}\int_{\mathbb{CP}^{N-1}} d\varphi^\dagger d\varphi\; e^{N\bar{\lambda}\lambda\varphi_1^*\varphi_1} \\ &= \frac{\texttt{Vol}\left(S^{2N-3}\right)}{\texttt{Vol}\left(\mathbb{CP}^{N-1}\right)}\int_0^1 dr(1-r)^{N-2}e^{N\bar{\lambda}\lambda r} \\ &= \frac{(N-1)!}{(N)^{N-1}}\sum_{L=N-2}^{\infty}\frac{\left(N\bar{\lambda}\lambda\right)^L}{L!}\simeq\sqrt{N}e^{N\bar{\lambda}\lambda},\end{aligned} \tag{7.15}$$

where we have chosen the projective space to have radius $\sqrt{N}$, and used the Stirling approximation in the last line. In comparison, the norm of the state created by a determinant operator has a norm given by [10]

$$\begin{aligned}\left\langle\det\left(\bar{Z}-\bar{\lambda}\right)\det\left(Z-\lambda\right)\right\rangle &= \int_0^\infty dr\; e^{-Nr}(\bar{\lambda}\lambda+r)^N \\ &= \frac{N!}{N^N}\sum_{k=0}^{N}\frac{(N|\lambda|^2)^k}{(N-k)!}\simeq\sqrt{N}e^{N|\lambda|^2}.\end{aligned} \tag{7.16}$$

Hence, the second important difference between the approach using an inverse determinant operator and $|\lambda\rangle$ is that we do not need to introduce an additional set of auxiliary variables to obtain an integral which we can evaluate via the saddle-point approximation, and the resulting saddle-point equations are equivalent in both approaches. The most important difference between our approach is the fact that we can easily generalize our construction to write down generating functions of characters associated to Young diagrams with more than one row in a very compact way, with very explicit formulas. For instance, the product of determinant operators has a character expansion coming from one of the Cauchy identities of Schur functions:

$$\prod_{i=1}^{k}\det(Z-\lambda_i)=\det(Z\otimes\mathbf{1}_k-\mathbf{1}_N\otimes\Lambda_k)=\det(\Lambda_k)^N\sum_R\chi_R(Z)\chi_{R^T}(-\Lambda_k^{-1}), \tag{7.17}$$

and there is a similar expansion for inverse determinants

$$\prod_{i=1}^{k} \det(Z-\lambda_i)^{-1} = \det(Z\otimes \mathbf{1}_k - \mathbf{1}_N \otimes \Lambda_k)^{-1} = \det(\Lambda_k)^N \sum_R \chi_R(Z)\chi_R(-\Lambda_k^{-1}). \quad (7.18)$$

One difficulty with dealing with the latter expression is that once one performs the Hubbard-Stratonovich transformation the resulting saddle-point equations are complicated matrix equations, and extracting the contribution from each character seems difficult. Also, the state created by such an operator does not have a finite norm, so the expressions have to be treated as formal generating functions. In our approach, one can generate the same class of states by letting Λ in (2.6) have k non-zero eigenvalues and integrating over the appropriate homogeous space. More precisely, one needs to replace the integral over φ by an integral over an isometry

$$\begin{aligned} VV^\dagger &= P_k \\ V^\dagger V &= \mathbf{1}_n, \end{aligned} \quad (7.19)$$

where P_k is a rank k projector. The integration is then performed over the space of k-dimensional subspaces of $\mathbb{C}^N$, which is the Grassmannian $\mathbf{Gr}(k,N)$. Similar types of integrals have been studied previously in the literature [134], and they are known to have exact formulas in terms of iterated residues [135].

7.4 Gauge Theory Computation

7.4.1 Generating function for BPS Three-Point Functions

We are interested in computing the overlap of two AdS giant graviton states with a light BPS single trace operator. This correlator is related to the three-point structure constant by

$$C_{\mathcal{S}_{J'}\mathcal{S}_J\mathcal{O}_L} = \frac{\langle \mathcal{S}_{J'}| \operatorname{Tr}\left[\tilde{Z}^L\right] |\mathcal{S}_J\rangle}{\sqrt{L \langle \mathcal{S}_{J'}|\mathcal{S}_{J'}\rangle \langle \mathcal{S}_J|\mathcal{S}_J\rangle}}, \tag{7.20}$$

where $\tilde{Z}$ is the twisted translated frame operator $\tilde{Z} = \frac{Z+\bar{Z}+(Y-\bar{Y})}{2}$. The boundary states $|\mathcal{S}_J\rangle$ are to be understood as the insertion of a fully-symmetric Schur polynomial operator[1] at $t = \pm\infty$

$$|\mathcal{S}_J\rangle = \chi_{(J)}(Z) |0\rangle \, . \tag{7.21}$$

Because half-BPS correlators are protected, we can perform the calculations in the free field theory. The boundary states can be generated using the following operators:

$$\begin{aligned} |\lambda\rangle &= \int_{\mathbb{CP}^{N-1}} d\varphi e^{\sqrt{N-2}\lambda\varphi^\dagger Z\varphi} |0\rangle \\ \langle\Lambda| &= \langle 0| \int_{U(N)} dU e^{\sqrt{N-2}\operatorname{Tr}\left(\bar{Z}U^\dagger\bar{\Lambda}U\right)}, \end{aligned} \tag{7.22}$$

where we can set $\bar{\Lambda} = \bar{\lambda}P_1$ during the later parts of the computation. The advantage of this setup is that the measure $d\phi$ is invariant under unitary transformations, so when we commute the exponentials using the Campbell-Hausdorff formula the integral over the unitary group will drop out of the correlator. The overlap that we will want to compute is given by:

$$\mathcal{F}(\lambda, \Lambda, t) = \langle\Lambda| \operatorname{Tr}\left[\frac{1}{1-2t\tilde{Z}}\right] |\lambda\rangle \, . \tag{7.23}$$

When we commute all raising and lowering operators past each other, the net effect is to replace the fields by their saddle-point value

$$\mathcal{Z} \to \frac{U^\dagger\bar{\Lambda}U + \lambda\varphi\varphi^\dagger}{2} = U^\dagger \left(\frac{\bar{\Lambda} + \lambda U\varphi\varphi^\dagger U^\dagger}{2}\right) U. \tag{7.24}$$

[1] We differentiate between the notation used in [9, 115], $|\mathcal{D}_J\rangle$, since the operators we are considering are not determinants.

Since the expression inside the exponential only depends on $U\varphi$ after applying the Campbell-Hausdroff formula, all unitaries can be reabsorbed by a change of variables. So in the end the generating function $\mathcal{F}$ is expressed as:

$$\mathcal{F}(\lambda, \Lambda, t) = \int_{\mathbb{CP}^{N-1}} d\varphi \, e^{(N-2)\lambda\varphi^\dagger \bar{\Lambda}\varphi} \mathrm{Tr}\left[\frac{1}{\mathbf{1} - t\left(\bar{\Lambda} + \lambda\varphi\varphi^\dagger\right)}\right]. \tag{7.25}$$

This integral can be computed exactly via equivariant localization. A simple way of seeing this is that the integral may be turned into a Gaussian integral subject to the constraint $|\varphi|^2 = 1$.

$$\delta(|\varphi|^2 - 1) = \int ds \; e^{\lambda s(|\varphi|^2 - 1)}. \tag{7.26}$$

After this substitution, we can perform the Gaussian integral over φ on the whole complex plane by contour integration. By choosing a set of contours such that the phase of the exponential is stationary, the resulting integrals are Gaussian integrals peaked at the eigenvalues of $\bar{\Lambda}$, so in the end we only need to sum over N saddle points. After performing the Gaussian integral, each saddle point will correspond to a pole on the complex s plane, and every insertion of $\varphi_i\varphi_j^\dagger$ in the integral can be replaced by its moment taken from the Gaussian distributions;

$$:\varphi_i\varphi_j^\dagger: \sim \left(\frac{1}{\lambda(s - \bar{\Lambda})}\right)_{ij}. \tag{7.27}$$

After this we are left to compute a contour integral over the complex S plane over a infinitely large circle. The particular choice of orientation for the contours that we need guarantees that the sum in homology of the N contours is equivalent to the trivial contour encircling a pole at infinity.

In practice we will need consider cases where $\bar{\Lambda}$ has rank one, which makes the torus action on $\mathbb{CP}^{N-1}$ degenerate. However, we can still compute the integral exactly as a sum over residues of poles of higher order. When $\bar{\Lambda}$ has one non-zero eigenvalue, the integral is exponentially dominated by a single saddle point.

In order to compute the integral with the resolvent, we will need to invert the matrix inside of the trace. A simple way of doing this is by writing this matrix as

$$\mathbf{1} - t\left(\bar{\Lambda} + \lambda\varphi\varphi^\dagger\right) = \mathbf{1} - \Phi_i \Sigma_{ij} \Phi_j^\dagger, \tag{7.28}$$

where Σ is a 2×2 diagonal matrix with components $(t\bar{\lambda}, t\lambda)$, and Φ_i is an $N \times 2$ matrix consisting of (v_1, φ), where v_1 is a unit vector. The inverse of this matrix is given by

$$\left(1 - t\left(\bar{\Lambda} + \lambda\varphi\varphi^\dagger\right)\right)^{-1} = 1 + \Phi\left(\Sigma^{-1} + \Phi^\dagger\Phi\right)^{-1}\Phi^\dagger. \tag{7.29}$$

In some respects, the matrix Σ_{ij} plays a similar role as the Hubbard-Stratonovich field ρ needed to simplify correlation functions involving determinants. When we take the trace the first term will be independent of t, so it will not contribute to the three-point and the second term becomes a trace over the 2×2 auxiliary indices

$$\begin{aligned}\operatorname{Tr}\left(1 - t\left(\bar{\Lambda} + \lambda\varphi\varphi^\dagger\right)\right)^{-1} &= N + \operatorname{tr}\left(\Phi^\dagger\Phi\left(\Sigma^{-1} + \Phi^\dagger\Phi\right)^{-1}\right) \\ &= N - 2\left(\frac{1 - \frac{t}{2}\left(\lambda + \bar{\lambda}\right)}{t^2\lambda\bar{\lambda}(\varphi_1^*\varphi_1 - 1) + t\left(\lambda + \bar{\lambda}\right) - 1} + 1\right).\end{aligned} \tag{7.30}$$

So the exact expression for the form factor is obtained by performing the integral over φ and since the first and third terms are analytic in t we can simply ignore them:

$$\mathcal{F}(\lambda, \bar{\lambda}, t) \simeq -\int_{\mathbb{CP}^{N-1}} d\varphi\, e^{(N-2)\lambda\bar{\lambda}\varphi_1^*\varphi_1}\left(\frac{1 - \frac{t}{2}\left(\lambda + \bar{\lambda}\right)}{t^2\lambda\bar{\lambda}(\varphi_1^*\varphi_1 - 1) + t\left(\lambda + \bar{\lambda}\right) - 1}\right). \tag{7.31}$$

This integral can be evaluated easily by using spherical coordinates and expanding in powers of $\varphi_1^*\varphi_1$, but it will turn out to be better to approximate this quantity via the saddle-point approximation.

7.4.2 Large N Limit

The first thing to note about the integral expression for $\mathcal{F}$ is that the integrand breaks the $U(N)$ symmetry of the measure to $U(1) \times U(N-1)$, so it is convenient to perform the angular integration over the $N-1$ directions perpendicular to φ_1 first. For a fixed value of φ_1, this is given by half of the volume of a sphere of radius $(1-\varphi_1^*\varphi_1)^{1/2}$. The exact value of the angular integrals is not very important since the overall factor in front of the generating function will cancel when we normalize the structure constants, but what is important is that the integral over φ_1 is done with the correct measure

$$\mathcal{F}(\lambda, \Lambda, t) \simeq -\frac{\pi^{N-2}}{(N-2)!}\int dr d\vartheta (1-r)^{N-2} e^{(N-2)\lambda\bar{\lambda}r}\left(\frac{1-\frac{t}{2}\left(\lambda+\bar{\lambda}\right)}{t^2\lambda\bar{\lambda}(r-1)+t\left(\lambda+\bar{\lambda}\right)-1}\right). \tag{7.32}$$

Finally, we can evaluate this integral using the saddle-point approximation. Since the terms coming from the resolvent do not scale with N, they will not lead to large exponents, so only need to consider the critical points of the following effective action

$$S_{\texttt{eff}} = \lambda\bar{\lambda}r + \log\left(1-r\right). \tag{7.33}$$

The saddle points of this action precisely fix r in such a way as to simplify the denominator of the expression in parentheses:

$$\lambda\bar{\lambda} = \frac{1}{(1-r)}, \tag{7.34}$$

which yields

$$\mathcal{F}_{\texttt{saddle}}(\lambda, \bar{\lambda}, t) = \left(\frac{1-\frac{t}{2}\left(\lambda+\bar{\lambda}\right)}{t^2 - t\left(\lambda+\bar{\lambda}\right)+1}\right) e^{S_{\texttt{saddle}}}. \tag{7.35}$$

Since the exponential factor computes the saddle point value of the overlap of the AdS giant states it will cancel when we compute the structure constants so we will omit its explicit form.

7.4.3 Diagonal Structure Constants

We now have an approximate expression for the form factor $\mathcal{F}(\lambda, \bar{\lambda}, t)$ which is a generating function for three point functions involving two BPS AdS giant gravitons and a BPS singe trace operator. One feature of this calculation is that the form factor describes a semiclassical giant graviton localized along some null geodesic on the hemisphere of S^5. To obtain a correlator with fixed R charge we need to average over the position of the giant to project into a fixed charge state. In our case, the moduli of the solution is the phase of the eigenvalue λ. So it is natural that we perform an average over the orbit generated by the phase of λ

$$\begin{aligned}\lambda &= y \cosh \rho_0 \\ \bar{\lambda} &= \frac{1}{y} \cosh \rho_0,\end{aligned} \tag{7.36}$$

which gives

$$\mathcal{G}(t) = -\frac{1}{2\pi i} \oint \frac{dy}{y} \left[\frac{1 - \frac{t}{2}\left(y + \frac{1}{y}\right) \cosh \rho_0}{t^2 - t\left(y + \frac{1}{y}\right) \cosh \rho_0 - 1} \right] = -\frac{1 - t^2}{\sqrt{t^4 - 2t^2 \cosh 2\rho + 1}}. \tag{7.37}$$

To obtain the one-point function of a BPS single trace operator we simply expand this function in t, and extract the L'th coefficient with a contour integral:

$$\begin{aligned}C_{\mathcal{S}_\Delta \mathcal{S}_\Delta \mathcal{O}_L} &= \frac{1}{2\pi i \sqrt{L}} \oint \frac{dt}{t^{L+1}} \mathcal{G}(t) = -\sum_{J=0}^{\infty} \frac{1}{2\pi i \sqrt{L}} \oint \frac{dt}{t^{L+1}} P_J\left(\cosh 2\rho_0\right) t^{2J} (1 - t^2) \\ &= -\frac{1^L + (-1)^L}{2\sqrt{L}} \left(P_{\frac{L}{2}}\left(\cosh 2\rho_0\right) - P_{\frac{L}{2}-1}\left(\cosh 2\rho_0\right) \right).\end{aligned} \tag{7.38}$$

This is exactly the answer obtained in [115], with a minor difference. In their analysis one needs to perform an integral over $|\lambda|$ with a measure that effectively replaces it with the discrete dimension of the operator dual to the AdS giant. In our case we are still left with $\cosh 2\rho_0$ as a continuous parameter corresponding to the radial position of the brane inside AdS. This will match exactly the answer obtained from the semiclassical computation.

7.4.4 Off-Diagonal Structure Constants

Since our integral formula is identical to the one found in [115], we can borrow their results to find the off-diagonal structure constants. The idea is to replace the integral over the phase in (7.36) by

$$\oint \frac{dy}{iy} \longrightarrow \oint \frac{dy}{iy^{k+1}}. \tag{7.39}$$

This is the contribution from the wavefunctions of the boundary states whenever the difference of the R-charges of the in and out states is k. If $k \ll N$, the saddle point is not modified, and the integrand remains the same. The structure constant for AdS giant gravitons can also be obtained by analytically continuing the structure constant for sphere giant gravitons

$$\theta_0 \to i\rho_0 + \pi/2. \tag{7.40}$$

Here we will prove the formula given in [115] for all values of k. After performing the residue integral over y we arrive at the following expression for the generating function of off-diagonal structure constants at large N

$$\begin{aligned} \mathcal{G}_k(t) &= -\frac{1}{2}\frac{\left(t^k(t^2-1)\cosh^k\rho_0\right)2^k}{\sqrt{t^4-2t^2\cosh 2\rho+1}\left(1+t^2+\sqrt{t^4-2t^2\cosh 2\rho+1}\right)^k} \\ &= \frac{1}{2}t^k(1-t^2)\sum_{J=0}^{\infty} P_J^{(0,k)}\left(\cosh 2\rho_0\right)t^{2J}, \end{aligned} \tag{7.41}$$

where we used the generating function for Jacobi polynomials[2] $P_J^{(\alpha,\beta)}(x)$ to expand the function in powers of t^2. Finally, we perform the contour integral over t to obtain the off-diagonal structure constant.

$$\begin{aligned} C_{\mathcal{S}_{\Delta+k}\mathcal{S}_{\Delta}\mathcal{O}_L} &= \frac{1}{2\pi i\sqrt{L}}\oint \frac{dt}{t^{L+1}}\mathcal{G}_k(t) \\ &= -\frac{1^{L-k}+(-1)^{L-k}}{2\sqrt{L}} \times \cosh^k \rho_0 \left(P^{(0,k)}_{\frac{L-k}{2}}\left(\cosh 2\rho_0\right) - P^{(0,k)}_{\frac{L-k}{2}-1}\left(\cosh 2\rho_0\right)\right). \end{aligned} \tag{7.42}$$

Similarly, the formula for the off-diagonal structure constants for sub-determinant operators can be written as

$$C_{\mathcal{D}_{\Delta+k}\mathcal{D}_{\Delta}\mathcal{O}_L} = -\frac{i^{L-k}+(-i)^{L-k}}{2\sqrt{L}} \times \sin^k \theta_0 \left(P^{(0,k)}_{\frac{L-k}{2}}\left(\cos 2\theta_0\right) + P^{(0,k)}_{\frac{L-k}{2}-1}\left(\cos 2\theta_0\right)\right). \tag{7.43}$$

We have checked that our formula agrees with the formula given in [115] for many values of L and k, and the two formulas can be turned into one another by using the recurrence relations of the hypergeometric function. In the extremal limit $L = k$, the second term in both structure constants vanish and the Jacobi polynomials reduce to a factor of unity, so the formula is well-defined for all $k \leq L$. For $k > L$, the contour integral has no poles, and so the structure constants vanish identically as expected from R-charge conservation.

7.5 Holographic Computation

We now move on to the holographic computation of the structure constant (7.20). To do so, we simply replace each part of the formula by its holographic counterpart.

[2]The Jacobi polynomials $P_n^{(a,b)}(x)$ span a large family of orthogonal polynomials; they reduce to Legendre polynomials for $a = b = 0$. For $a = 0$ and $b = k$ they correspond to the radial parts of Zernike polynomials.

The dual of the Schur polynomial operators $|\mathcal{S}_J\rangle$ are BPS AdS giant gravitons with angular momentum $\Delta = J$, whose quantum state we denote by $|\hat{\mathcal{S}}_\Delta\rangle$. The single trace operator Tr $\left[\tilde{Z}^L\right]$ is replaced by an operator $\hat{\mathcal{O}}_L$ which describes the backreaction on the worldvolume of the giant graviton. Altogether the holographic structure constant is given by

$$C_{\hat{\mathcal{S}}_{\Delta'}\hat{\mathcal{S}}_\Delta\hat{\mathcal{O}}_L} = \frac{\left\langle \hat{\mathcal{S}}_{\Delta'} \middle| \hat{\mathcal{O}}_L \middle| \hat{\mathcal{S}}_\Delta \right\rangle}{\sqrt{L \left\langle \hat{\mathcal{S}}_{\Delta'} \middle| \hat{\mathcal{S}}_{\Delta'} \right\rangle \left\langle \hat{\mathcal{S}}_\Delta \middle| \hat{\mathcal{S}}_\Delta \right\rangle}}. \tag{7.44}$$

The three-point function can be computed from a path integral on the worldvolume of the giant graviton, which is amenable to a saddle-point analysis. As we will see, a proper treatment via the orbit average method will yield a result which matches the gauge theory exactly.

7.5.1 AdS Giant Graviton Solution

We will be interested in solutions to the DBI action describing a giant graviton wrapping an $S^3 \subset \mathrm{AdS}_5$, which rotates along the equator of the S^5 at the speed of light. For our set up it will be convenient use global coordinates to parametrize $\mathrm{AdS}_5 \times S^5$:

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\tilde{\Omega}_3^2 + d\Omega_5^2, \tag{7.45}$$

where the metric of the five-sphere is

$$d\Omega_5^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta \left(d\chi_1^2 + \sin^2\chi_1 d\chi_2^2 + \cos^2\chi_1 d\chi_3^2\right). \tag{7.46}$$

We can then gauge fix the worldvolume coordinates σ^μ of the D-brane to agree with the coordinates of $\mathbb{R}_t \times S^3 \subset \mathrm{AdS}_5$

$$\rho = \rho_0, \quad \sigma^0 = t = \phi, \quad \sigma^i = \tilde{\chi}_i \tag{7.47}$$

where the tilded coordinates $\tilde{\chi}_i$ are the coordinates of the three-sphere inside AdS_5. The size of a BPS giant graviton is equal to its R-charge (angular momentum along S^5), which is related to its radial position in the AdS direction by

$$\cosh \rho_0 = \frac{J}{N}, \quad J \geq N. \tag{7.48}$$

To compute the three-point function we will need to compute the corrections to the D3-brane action coming from a light supergravity perturbation as in [117, 118].

7.5.2 Fluctuations of the D3-brane action

The action for an AdS giant graviton is given by the sum of the DBI and Wess-Zumino (WZ) actions

$$S = -\frac{N}{2\pi^2} \int d^4\sigma \left(\sqrt{-h} + P[C_4] \right), \tag{7.49}$$

where h is the induced worldvolume metric and $P[C_4]$ is the pull-back of the Ramond-Ramond four-form potential of the background. For our purposes we will want to concentrate on the RR flux through the AdS factor

$$C_4 = -\sinh^4 \rho dt \wedge \mathrm{Vol}(\tilde{\Omega}_3). \tag{7.50}$$

The light operator insertion can then be identified by the perturbations to the D3-brane action [125]

$$\hat{\mathcal{O}}_L = \delta S_{\mathrm{DBI}} + \delta S_{\mathrm{WZ}}. \tag{7.51}$$

For this we will need the fluctuations of the spacetime metric g as well as for the four-form potential [136]:

$$\begin{aligned}\delta g_{\mu\nu} &= \left[-\frac{6}{5}L g_{\mu\nu} + \frac{4}{L+1}\nabla_{(\mu}\nabla_{\nu)}\right] s^L(X) Y_L(\Omega_5) \\ \delta g_{\alpha\beta} &= 2L g_{\alpha\beta} s^L(X) Y_L(\Omega_5) \\ \delta C_{\mu_1\mu_2\mu_3\mu_4} &= -4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5} s^L(X) Y_L(\Omega_5),\end{aligned} \tag{7.52}$$

where $\mu, \nu, \ldots$ denote coordinates on the AdS$_5$, $\alpha, \beta, \ldots$ denote coordinates on the S^5, $Y_L(\Omega_5)$ denotes a spherical harmonic on the S^5, and $s^L(X)$ is the bulk-to-boundary propagator. The kinds of fluctuations that are dual to the operator $\mathrm{Tr}\left[\tilde{Z}^L\right]$ are given by choosing a spherical harmonic corresponding to the homogeneous polynomial $(Z + \bar{Z} + Y - \bar{Y})^L$, where X, Y, Z are the coordinates on $S^5 \subset \mathbb{C}^3$,

$$Y_L(\tilde{Z}) = \left(\sin\theta\cos\phi + i\cos\theta\cos\chi_1\sin\chi_3\right)^L . \tag{7.53}$$

The bulk-to-boundary propagator is given by

$$s^L(X) = \frac{\mathcal{N}}{(-2P\cdot X)^L}, \tag{7.54}$$

where P represents the coordinates of the operator insertion on the boundary coordinates P^I and X are the embedding coordinates of AdS$_5$:

$$\begin{aligned}&X^{-1} = \cosh\rho\cosh t_E, \quad X^0 = \cosh\rho\sinh t_E \quad X^i = \sinh\rho\, n^i \\ &P^{-1} = \cosh\bar{t}_E, \quad P^0 = \sinh\bar{t}_E, \quad P^i = \bar{n}^i,\end{aligned} \tag{7.55}$$

where t_E is the Euclidean time coordinate $t_E = it$ and $|n|^2 = |\bar{n}|^2 = 1$. The unit vectors n^i and $\bar{n}^i$ represent the position of the operator insertion on the S^3 inside AdS$_5$ in the bulk and the boundary, respectively. In our case the bulk-to-boundary propagator is given by

$$s^L(t_E, n^i, \bar{n}^i) = \frac{\mathcal{N}}{2^L} \frac{1}{(\cosh \rho_0 \cosh t_E - n \cdot \bar{n} \sinh \rho_0)^L}, \tag{7.56}$$

where $n \cdot \bar{n} = \cos \tilde{\chi}_1 \sin \tilde{\chi}_3$ and the normalization $\mathcal{N}$ is chosen such that the two-point function is unit-normalized.

DBI Action Fluctuations The fluctuation of the induced metric on the D3-brane has the form

$$\delta\sqrt{h} = \frac{1}{2}\sqrt{h} h^{ab} \left(\partial_a X^\mu \partial_b X^\nu \delta g_{\mu\nu} + \partial_a X^\alpha \partial_b X^\beta \delta g_{\alpha\beta}\right). \tag{7.57}$$

Substituting the worldvolume coordinates into the variation of the induced metric gives

$$\delta\sqrt{h} = \frac{1}{2}\sqrt{h} \left(\frac{4}{L+1} h^{ab}\, \nabla_a \nabla_b - \frac{2L(L-1)}{L+1} h^{ab} g_{ab} + 2L h^{tt} \right) s^L Y_L. \tag{7.58}$$

To simply this expression, it is useful to exploit the fact that $s^L(X)$ is a scalar field of mass-squared $L(L-4)$ in AdS units. To use this fact we may rewrite the induced metric on the brane in terms of the metric of AdS_5

$$\begin{aligned} h_{ab} &= g_{ab} + \delta^t_a \delta^t_b \\ h^{ab} &= g^{ab} - \left(\sinh^2 \rho \cosh^2 \rho\right)^{-1}\, \delta^a_t \delta^b_t, \end{aligned} \tag{7.59}$$

and then we can add and subtract the second covariant derivative in the ρ direction to complete the Laplacian in (7.58) which gives:

$$\begin{aligned} \delta S_{\text{DBI}} &= \frac{N}{2\pi^2} \int d^3\sigma\, \delta\sqrt{h}|_{t=0} \\ &= \frac{N}{4\pi^2} \sinh^2 \rho_0 \int d^3\sigma\, F_{\text{DBI}}|_{t=0} \end{aligned} \tag{7.60}$$

where

$$F_{\rm DBI} = -\frac{1}{N}\frac{4}{L+1}\sin\chi_1\cos\chi_1 \times \\ \left(\frac{\partial_t^2}{\cosh^2\rho_0} + \sinh^2\rho_0\partial_\rho^2 - \tanh\rho_0\partial_\rho + L^2\cosh^2\rho_0 + 2L\sinh^2\rho_0\right) s^L Y_L \tag{7.61}$$

The differential operator in parenthesis basically raises the spin of the propagator S^L by two units $L+2$ and multiplies it by a simple polynomial in $\sinh\rho_0$ and $n\cdot\bar{n}$.

WZ Action Fluctuations The fluctuations of the WZ term are straightforward to compute. The four-form potential only has indices in the AdS_5 directions, so the only possible term is

$$\delta C_{t\tilde{\chi}_1\tilde{\chi}_2\tilde{\chi}_3} = -4\partial_\rho s^L(X)Y_L(\Omega_5). \tag{7.62}$$

The contribution from the WZ action is thus

$$\begin{aligned}\delta S_{\rm WZ} &= \frac{N}{2\pi^2}\int d^3\sigma P\left[\delta C_4\right] = \frac{N}{2\pi^2}\int d^3\sigma\sqrt{g_{\mathrm{AdS}_5}}\delta C_{t\tilde{\chi}_1\tilde{\chi}_2\tilde{\chi}_3} \\ &= -\frac{N}{4\pi^2}\sinh^2\rho_0\int d^3\sigma\, F_{\rm WZ}|_{t=0}\end{aligned} \tag{7.63}$$

where

$$F_{\rm WZ} = \frac{8}{N}\sin\tilde{\chi}_1\cos\tilde{\chi}_1\sinh\rho_0\cosh\rho_0\,\partial_\rho s^L Y_L. \tag{7.64}$$

Operator insertion Putting everything together, we obtain an expression for the insertion of the light operator $\hat{\mathcal{O}}_L$ in the semiclassical limit. In practice it is useful to rewrite the resulting expression in terms of s^{L+2} so that the DBI and WZ terms combine nicely. As in [116], the combination of DBI and WZ terms simplifies significantly:

$$\hat{\mathcal{O}}_L\left[X_0^*\right] = \delta S_{\rm DBI} + \delta S_{\rm WZ} = \frac{N}{4\pi^2}\sinh^2\rho_0\int d^3\sigma\ \left(F_{\rm DBI} - F_{\rm WZ}\right)|_{t=0} \tag{7.65}$$

where

$$F_{\rm DBI} - F_{\rm WZ} = -\frac{\sqrt{L}(L+1)}{N} \frac{\sin\tilde{\chi}_1 \cos\tilde{\chi}_1 \cos^L \phi}{(\cosh\rho_0 \cos t - \cos\tilde{\chi}_1 \sin\tilde{\chi}_3 \sinh\rho_0)^{L+2}}. \tag{7.66}$$

So our analysis closely mirrors that of [116], with some minor differences in the simplification of the fluctuation analysis.

At this point our analysis will differ importantly; in their set-up, they proceed by substituting the worldvolume solution $\phi = t$ and integrating over the insertion time t. Our calculation instead follows the prescription used by [115], which means that the coordinates appearing in $\hat{\mathcal{O}}_L\left[X_0^*\right]$ are not the worldvolume coordinates of the giant graviton, and instead they should be thought of as the coordinates of the insertion of the operator on the sphere wrapped by the giant. This means that we should not set $\phi = t$, but instead we should treat $\phi = \phi_0$ as a moduli of the solution. The second moduli of the solution is associated to the action of the dilatation operator $t \to t + i\tau_0$, which is different from the Lorentzian time evolution of the fluctuation.

Concretely, one should replace the unshifted solution X_0^* by the shifted solution X_{ϕ_0,τ_0}^*, which can be obtained by $\phi \to \phi + \phi_0$ and $t \to t + i\tau_0$ in (7.65)

$$\hat{\mathcal{O}}_L\left[X_{\phi_0,\tau_0}^*\right] = \frac{N}{4\pi^2}\sinh^2\rho_0 \int d^3\sigma\ \left[F_{\rm DBI}\left(\phi_0,\tau_0\right) - F_{\rm WZ}\left(\phi_0,\tau_0\right)\right]|_{t=0} \tag{7.67}$$

where

$$F_{\rm DBI}\left(\phi_0,\tau_0\right) - F_{\rm WZ}\left(\phi_0,\tau_0\right) = -\frac{\sqrt{L}(L+1)}{N} \frac{\sin\tilde{\chi}_1 \cos\tilde{\chi}_1 \cos^L \phi_0}{(\cosh\rho_0 \cosh\tau_0 - \cos\tilde{\chi}_1 \sin\tilde{\chi}_3 \sinh\rho_0)^{L+2}}. \tag{7.68}$$

Since these solutions spontaneously break the rotation and dilatation symmetry of the background, the orbit average method tells us to integrate over the moduli space. As we will see this is the correct prescription for computing the three-point function, and this will also fix the apparent discrepancy found in [116]. It will also allow us to compute the off-diagonal three-point functions by including contributions from the boundary wavefunctions, which was inaccessible from their analysis.

7.5.3 Diagonal Structure Constants

We can now obtain the diagonal structure constant by performing the orbit average of (7.65)

$$C_{\hat{\mathcal{S}}_\Delta \hat{\mathcal{S}}_\Delta \hat{\mathcal{O}}_L} = \int_{-\infty}^{\infty} d\tau_0 \int_0^{2\pi} \frac{d\phi_0}{2\pi}\, \hat{\mathcal{O}}_L \left[X^*_{\phi_0,\tau_0}\right]. \tag{7.69}$$

The details are presented in appendix **??** and the final answer can be written in terms of a hypergeometric function

$$C_{\hat{\mathcal{S}}_\Delta \hat{\mathcal{S}}_\Delta \hat{\mathcal{O}}_L} = -\frac{1}{2}\left(1^L + (-1)^L\right)\sqrt{L} \times \frac{\tanh^2\rho_0}{\cosh^L\rho_0}\, {}_2F_1\left(1+\frac{L}{2}, 1+\frac{L}{2}, 2, \tanh^2\rho_0\right). \tag{7.70}$$

This answer is of the same form as the result found in [115] for sphere giant gravitons. To see the matching with the gauge theory computation one needs to apply the recurrence formulas of the hypergeometric function to write the expression above as a sum of two Legendre polynomials

$$C_{\hat{\mathcal{S}}_\Delta \hat{\mathcal{S}}_\Delta \hat{\mathcal{O}}_L} = -\frac{1^L + (-1)^L}{2\sqrt{L}}\left(P_{\frac{L}{2}}\left(\cosh 2\rho_0\right) - P_{\frac{L}{2}-1}\left(\cosh 2\rho_0\right)\right). \tag{7.71}$$

Clearly this matches exactly with the gauge theory computaion, and is also a simple analytic continuation of the stucture constant involving two sub-determinant operators and a light single trace.

7.5.4 Off-Diagonal Structure Constants

For the off-diagonal structure constants with $\Delta + k \sim \Delta$, the only change to the computation is the contribution from the phases of the boundary wavefunctions:

$$C_{\hat{\mathcal{S}}_{\Delta+k}\hat{\mathcal{S}}_{\Delta}\hat{\mathcal{O}}_L} = \int_0^{2\pi} \frac{d\phi_0}{2\pi} \int_{-\infty}^{\infty} d\tau_0 \; \hat{\mathcal{O}}_L \left[X^*_{\phi_0,\tau_0}\right] e^{ik\phi_0} e^{-k\tau_0}. \tag{7.72}$$

The final expression is a simple generalization of the diagonal case, and involves a similar type of hypergeometric function:

$$\begin{aligned} C_{\hat{\mathcal{S}}_{\Delta+k}\hat{\mathcal{S}}_{\Delta}\hat{\mathcal{O}}_L} = & -\frac{1}{2}\left(1^{L-k} + (-1)^{L-k}\right)\sqrt{L} \\ & \times \frac{\tanh^2\rho_0}{(\cosh\rho_0)^L}\, {}_2F_1\left(1 + \frac{L-k}{2}, 1 + \frac{L+k}{2}, 2, \tanh^2\rho_0\right) \end{aligned} \tag{7.73}$$

for $k \leq L$, and zero for $k > L$. An analogous computation as in the diagonal case shows that this is equivalent to

$$C_{\hat{\mathcal{S}}_{\Delta+k}\hat{\mathcal{S}}_{\Delta}\hat{\mathcal{O}}_L} = -\frac{1^{L-k} + (-1)^{L-k}}{2\sqrt{L}} \times \cosh^k\rho_0 \left(P^{(0,k)}_{\frac{L-k}{2}}(\cosh 2\rho_0) - P^{(0,k)}_{\frac{L-k}{2}-1}(\cosh 2\rho_0)\right) \tag{7.74}$$

which matches the gauge theory computation.

7.5.5 No ambiguities for Sphere Giants

We note that unlike the case of the sphere giant graviton in [115], our expression is unambiguous for the extremal case $L = k$. Here we will argue that the extremal case for sphere giant gravitons is also unambiguous if one performs the integrals in the correct order.

Naively, if one computes the extremal case as a limit $k \to L$ one obtains a spurious divergence coming from an integral over ϕ_0, yet when one evaluates the integral explicitly the answer is manifestly finite. The seemingly problematic integral in our case is

$$\int_0^{2\pi} \frac{d\phi_0}{2\pi} e^{iL\phi_0} \cos^L \phi_0 = \frac{1}{2^L}. \tag{7.75}$$

Clearly the integral is finite and well-defined. However, this integral has an alternate expression in terms of sums of hypergeometric functions with a spurious singularity at $k = L$. If the integral is split into this form, the answer appears to be ambiguous in the sense that it is a sum of two infinite quantities, even though the integral has a well-behaved limit. In contrast, the off-diagonal structure constants for sphere giant gravitons appear to have a real divergence in the extremal limit coming from the average over τ_0, which is multiplied by a prefactor that vanishes in the extremal limit. In [115] it was argued that the analytic continuation of the non-extremal case to $k = L$ is ambiguous due to the fact that one can always multiply the result by an analytic function that only modifies the function the behavior of the three-point function at $k = L$. Since there no clear constraints on the analytic properties of the three-point function as a function of k, there is no unique analytic continuation of the three-point function. However, in their analysis they separated the expressions in the integrals into a finite piece and an infinite piece multiplied by a zero prefactor. Strictly speaking this is not correct, since the integral is not convergent and depending on how one separates the terms one can obtain different answers for the regularized integral.

Upon closer inspection, the source of the divergence can be traced back to the imaginary part of a factor in the sum of the DBI and WZ terms

$$F_{\rm WZ} - F_{\rm DBI} = \frac{\sqrt{L}(L+1)}{2N} Y^{L-1} \times \frac{(i\cos\theta\cos\chi_1\sin\chi_3\cosh 2\tau_0 - \cos\phi_0\sin\theta_0)}{\cosh^{L+2}\tau_0}. \tag{7.76}$$

Note that the first term in the parentheses will also lead to a divergent quantity when we average over τ_0, but this way of splitting the integrals into divergent and finite parts is different from the one used in [115]. Our expression comes from adding and simplifying both contributions to the fluctuations of the action. Now, if we evaluate the average of the second (finite) term we obtain

$$C^{\mathtt{finite}}_{\hat{\mathcal{D}}_{\Delta+L}\hat{\mathcal{D}}_{\Delta}\hat{\mathcal{O}}_L} = -\frac{\sin^L\theta_0}{\sqrt{L}}, \tag{7.77}$$

which is exactly the extremal structure constant evaluated in the Schur basis. The remaining term is ambiguous, since its contribution is regularization dependent. A natural choice of regularization is to perform the integral over ϕ_0 before the τ_0 integral, or equivalently to perform the τ_0 integral with a finite upper and lower bound $\pm T$ and then take the limit $T \to \infty$. With this choice the problematic term vanishes and the holographic computation agrees with the field theory computation. Physically this makes sense, since the integral over ϕ_0 of the first term vanishes due to R-charge conservation. Hence if one treats the integrals carefully, there is no ambiguity in defining the three-point functions for sphere giants. In fact similar ambiguities happen for the case where $k > L$ in both computations; if one computes the τ_0 integral first the answer has divergent terms, even though the integral vanishes since the integral over ϕ_0 is zero.

7.6 Discussion

We computed diagonal and off-diagonal structure constants of two AdS giant gravitons and a light supergravity mode in the large N limit, both in (free) $\mathcal{N} = 4$ SYM theory and holographically in $\text{AdS}_5 \times S^5$. Our analysis shows a precise matching between both descriptions as expected, even in the cases where ambiguities were believed to appear. A crucial step in our calculations was the orbit average over the moduli space of solutions which spontaneously break the rotation and dilatation symmetry of the $\mathrm{AdS}_5 \times S^5$ background, and the order in which these integrals are performed is crucial for agreement in the computations of extremal correlators. It would be interesting to apply these methods to the class of open strings solutions [126] found in [23], where there appear to be discrepancies between the boundary conditions for the semiclassical string and the spin chain descriptions [63]. Since the positions of the open string endpoints along the worldvolume of the giant appear as extra moduli, one should in principle integrate over them in order to compare with the gauge theory computations. This would explain why certain angular momentum modes are allowed on the spin chain description, even though semiclassically they are forbidden by the boundary conditions.

Our calculation in the gauge theory demonstrate the power of the methods introduced in [11, 24, 98] for computing correlators of fully-symmetric Schur polynomials. Our methods are in many respects more streamlined when compared to the approach introduced in [10] for dealing with symmetric Schur functions. In principle our computation gives an exact integral representation for half-BPS correlators, without having to deal with a divergent generating series. Since we can express this generating function as a sum of residues with only one residue providing an exponentially large contribution, it is natural to expect that the saddle-point approximation gives the exact answer up to a simple one-loop determinant coming from the remaining residues. In fact the holographic computations of non-extremal correlators seem to agree with the exact computations obtained from explicit computations with the Schur basis [127]. It would be nice to check whether this expectation holds by embedding the correlator into a supersymmetric observable where supersymmetric localization techniques can be used [137]. For example, the connection between the coadjoint orbit integrals and Wilson loops via geometric quantization is well-known [138].

From our saddle-point analysis, it seems that similar computation involving more generic mostly-symmetric Schur operators should proceed in the same way. More precisely, the HCIZ formula (2.6) gives in principle a sum over all possible pairings of the initial and final configurations of eigenvalues. These can describe systems of more than one AdS giant graviton at different positions. Whenever the giants are well-separated we expect that the identity saddle dominates in a way that the computation reduces to a sum over the individual contributions of each giant. It would be more interesting to study set-ups where the positions of the branes coincide, in which case saddle points corresponding to permutations of equal eigenvalues are all relevant. It would be useful to understand the details in those cases before studying configurations of order N^2 stacked branes.

One surprising feature of these calculations is that the result is given by simple combinations of orthogonal polynomials in $\cosh\rho_0$ or $\cos\theta_0$ for AdS and sphere giants, respectively. This suggests that one might be able to compute these quantities by solving a wave equation with a non-trivial radial potential given by the presence of the branes. Understanding this connection would elucidate many of the physical aspects that are obscured in the present computations. One might expect that three-point functions with a spinning non-BPS single trace might be expressible as a spherical harmonic multiplied by a radial wavefunction. Also, since Legendre and Jacobi polynomials satisfy various recursive formulas, it might be possible to find non-trivial relations between BPS structure constants involving operators of different conformal dimensions in the large N limit.

Another issue that needs attention is whether AdS giant gravitons can lead to integrable boundary states for the $\mathcal{N} = 4$ SYM spin chain. Since many quantities associated with AdS giants can be obtained by analytic continuations from the sphere giant quantities, we expect that the answer to this question is negative, since non-maximal sphere giants do not appear to lead to integrable boundary states [10]. However, there is new evidence that non-maximal giants do lead to integrable boundaries in the ABJM theory

[139]. Since there is no obvious reason for this qualitative difference, it would be useful to revisit some of these computations with new techniques.

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Chapter 8

$\frac{1}{2}$-BPS Structure Constants and Random Matrices

8.1 Introduction

The study of baryonic operators in large N gauge theories is an old subject [140] that has received renewed attention in the context of holographic field theories [9, 10, 11]. Such operators are extremely interesting from the point of view of the large N expansion, since they correspond to heavy non-perturbative objects that are not very well captured by the conventional t' Hooft expansion. On physical grounds one expects that such heavy objects modify the physics at the semi-classical level, and that one should attempt to approach the problem from a point of view where one treats the dynamics of the many constituents of the object in terms of a simpler collective coordinates [141, 142]. This is well understood in string theories; heavy objects can lead to non-trivial boundary conditions for strings, or in some cases deform the target space geometry which the string probes. For this reason such an approach is essential for understanding how gravitational physics arises from large N models. In examples of the AdS/CFT correspondence [1] these ideas have sharp realizations in terms of *giant gravitons*, and in suitable limits, non-trivial supergravity backgrounds like *bubbling geometries* and *black holes*. Given that maximally supersymmetric Yang-Mills theory is expected to be fully-fledged theory of quantum gravity, a particularly interesting question to address is how $\mathcal{N} = 4$ SYM resolves the many puzzles of gravitational theories, in particular the physics of black holes. At the moment such questions are out of reach and they lie in regions of parameter space where current non-perturbative techniques such as integrability are expected to fail. One particular fruitful approach has been to concentrate on observables which are protected by supersymmetry in order to test and develop tools, and the half-BPS sector of asymptotically AdS type IIB supergravity and $\mathcal{N} = 4$ SYM is perhaps the simplest non-trivial toy model.

The spectral problem in this sector of the $U(N)$ theory was solved by the work of Corley, Jevicki, and Ramgoolam [48]; half-BPS operators are Schur functions and their structure constants are given by multiplicities of representations of the unitary group

$$\left\langle \mathcal{O}_{R_3}(\bar{Z})\mathcal{O}_{R_2}(\bar{Z})\mathcal{O}_{R_1}(Z)\right\rangle = C_{R_1R_2R_3} f_{R_1}, \tag{1.1}$$

where $C_{R_1R_2R_3}$ are Richardson-Littlewood coefficients and f_R is the norm of the operator $\mathcal{O}_R$. Although this solves the problem in principle and combinatorial algorithms exist which generate these coefficients for a fixed value of N it is unclear how the asymptotics of these coefficients are reflected in the corresponding supergravity solutions. Since these numbers appear naturally in the study of the intersection theory of Grassmannians, a natural expectation is that there is an alternative description for such calculations involving only geometric data coming from the gauge group of the theory. Another issue is that most results in the existing literature on structure constants of half-BPS operators either focus on single trace operators, or in operators preserving the same supersymmetries, or rely heavily on the free fermion description of model. Holographic computations of one point-functions in half-BPS backgrounds have also been studied in generality, but explicit calculations are limited to maximally charged operators which are charged under the same symmetry as the background and to operators of low dimensions. So an important step towards understanding very heavy operators in $\mathcal{N}=4$ SYM is to develop tools that can tackle problems of this kind for generic BPS operators in the large N limit. Such tools have been developed recently for operators of dimension $\Delta \sim N$ [9, 10, 115, 143, 144, 145] and in this chapter we extend this to operators of dimensions that scale as N^2. We show that the computation of very generic three-point functions of half-BPS operators can be packaged in a large family complex matrix model of matrices valued on a Grassmannian. Although we mostly focus on the $U(N)$ theory, our results generalize readily to orthogonal and symplectic gauge groups. For simple observables, such as set-ups involving a single stack of AdS giant gravitons, the corresponding matrix ensemble is a unitary Jacobi ensemble, while for more generic observables the matrix model cannot be easily reduced to integrals over eigenvalues. At large N, we find that the saddle point equations simplify the calculation significantly allowing us to either reduce the integrals to sums over integrals over eigenvalues, where each term in the sum is labelled by a permutation. The average density of eigenvalues is universal and is given by the well-known Marchenko-Pastur distribution, which appears as the Poisson distribution of noncommutative probability theory [146]

$$\rho(z) = \frac{\sqrt{(z_+ - z)(z - z_-)}}{2\pi z} \,. \tag{1.2}$$

Gaussian matrix integrals have been used extensively in the study of the combinatorics of half-BPS correlators in the past [17, 147] and in other contexts [148, 137]; the matrix models we study on the other hand describe large deviations from the vacuum state.

for a similar story for Wilson loops see [149, 150, 151]. In simple terms they describe wavefunctions of semi-classical BPS states in $\mathcal{N} = 4$ SYM and as such they provide a quantum mechanical description of half-BPS Coulomb branch configurations. This makes them ideal candidates for computing quantities that one can match to the dual geometric description. In fact, we argue that despite the fact that one expects an exact match for half-BPS observables on both sides of the duality due to a lack of g^2N corrections to the free-field theory answer, the corresponding supergravity will not in general compute a precise quantity from the point of view of the conformal field theory but rather a (micro-canonical) average. This is purely an effect of the large N limit and the full stringy description should be able to resolve the details of the boundary observable. These observations have been made in this context before, for instance in [112], but we clarify how this happens on the field theory side of the computation. Our main result is a large N formula for all heavy-heavy-light structure constants of half-BPS operators, for instance

$$C_{R\,R\,L/2} = \frac{1}{\sqrt{L}} \int dz d\bar{z}\, \rho_R(z,\bar{z})\, (z+\bar{z})^L, \tag{1.3}$$

where the density ρ_R is determined entirely from Young diagram data in a well-known way [152]. This is essentially the formula motivated in [112] from holographic renormalization of low lying operators and Coulomb branch limits. Our computation provides a check of this one-point function formula for all single trace primaries and in principle for all LLM geometries without relying on free-fermion methods. We also compute off-diagonal structure constants between sufficiently close heavy states suggesting that semi-classical supergravity calculations should be able to probe the precise microstructure of bubbling geometries.

This chapter is structured as follows. In section 2, we review the BPS coherent state construction and and discuss the computation of the form factor of a single trace operator in the background of a giant graviton. In section 3 we generalize the computation to the case where the number of giant gravitons scales with N. To do this we explain how to reduce the corresponding integral over a Grassmannian to a more conventional matrix model involving square matrices and then solve the model at large N. The resulting distribution essentially reproduces the distributions studied in [69], up to a change of variables. We then study the general problem of multiple stacks of giant gravitons using steepest decent methods. In section 4 we return to the problem of computing correlators in the character basis and provide a more explicit connection from the eigenvalue picture presented by the coherent state generating functions and the character basis. In section 5 we conclude by discussing some general lessons and future directions.

8.2 Coherent States and Form Factors

Most of our discussion will concentrate on the simplest correlation functions in the $\mathcal{N} = 4$ SYM theory, which are three point functions of half-BPS operators. We will also work mostly with the theory on the cylinder $\mathbb{R} \times S^3$, but translating the results to the plane is straightforward. A convenient parametrization for half-BPS operators is given in terms of a six dimensional null complex vector $n \cdot n = 0$:

$$Z(x, n) = n_I \phi^I(x), \tag{2.1}$$

and any half-BPS operator is obtained by taking gauge invariant combinations of $Z(x, n)$. One common choice of operators are single and multi-trace operators

$$\mathcal{O}_{\{L_i\}}(x) = \prod_k \mathrm{Tr}_N \left[\left(\vec{n} \cdot \vec{\phi}(x) \right)^{L_k} \right]. \tag{2.2}$$

In the large N limit with $\Delta = \sum_k L_k \ll \sqrt{N}$ these provide an approximately orthogonal basis of operators; this is the usual statement that in the large N limit planar graphs contribute the most in correlation functions. This class of operators is naturally associated to the supergravity modes of $AdS_5 \times S^5$ and their bound states. However, for operators with large enough conformal dimension Δ, various non-planar effects can contribute meaningfully or eventually dominate over planar graphs. Even more strikingly, certain extremal correlators of single traces operators have enhanced contributions from non-planar diagrams even for small charges [153]. For these reasons it is useful to first perform the computation at finite N with a proper orthogonal basis of states, and then take the large N limit.

By restricting to primary operators, we will often drop the space coordinates $x_{1,2,3}$ since we will mostly work with constant modes on the S^3. Due to non-renormalization properties of half-BPS operators, the two and three point functions of such operators can be computed in the free field theory limit $g_{YM} = 0$, so that our task reduces to a combinatorial problem of performing Wick contractions of free fields. This problem was first addressed in [48] for extremal correlators $n_1 = n_2 = n_3^*$. The main idea is to construct an orthogonal basis of states for one matrix quantum mechanics with $U(N)$ gauge symmetry, or equivalently a set of operators that diagonalize two point functions. The resulting basis is build from characters of the unitary group and is often referred to as the Schur basis:

$$\mathcal{O}_R\left(Z(x,n)\right) = \frac{1}{k!}\sum_{\pi\in S_k}\chi^R(\pi)\mathrm{Tr}_{\mathbf{N}^{\otimes k}}\left[\pi Z(x,n)^{\otimes k}\right] = s_R\left(Z(x,n)\right), \tag{2.3}$$

where k denotes the number of boxes of the Young diagram associated to the representation R of $U(N)$ and $\chi^R(\pi)$ is the character of the corresponding representation of S_k. A different proof that this set of operators provides a diagonal basis was given in [11].

These operators have a dual description in terms of giant gravitons, or their bound states, in asymptotically $AdS_5 \times S^5$ spaces [48, 17, 32].

Instead of performing the explicit contractions for a particular operator, it was realized that one could instead work with a coherent state for the free field $Z(x,n)$

$$|\Lambda\rangle = \frac{1}{\mathtt{Vol}\,[U(N)]} \int dU \; e^{\mathrm{Tr}_N \left[U\Lambda U^\dagger Z(x,n)\right]} \,|0\rangle \,. \tag{2.4}$$

A straightforward computation gives an alternate formula for $|\Lambda\rangle$ as a expansion in characters of the unitary group s_R,

$$|\Lambda\rangle = \sum_R \frac{1}{f_R} \mathcal{O}_R \left(Z(x,n)\right) s_R(\Lambda), \tag{2.5}$$

and $s_R(\Lambda)$ is a Schur polynomial. The point of this analysis is that by exploiting the Campbell-Hausdorff formula the free field contractions of the operators $\mathcal{O}_R\left(Z(x,n)\right)$ can all be replaced by an integral over the unitary group. For the two point functions the resulting integral is a Harish-Chandra-Itzykson-Zuber integral which has an exact fixed point formula:

$$\left\langle \bar{\Lambda} \middle| \Lambda \right\rangle = \frac{1}{\mathtt{Vol}\,[U(N)]} \int dU \; e^{\mathrm{Tr}[U\Lambda U^\dagger \bar{\Lambda}]} = \mathcal{C}_N \sum_{\pi \in S_N} \det(\pi) \, \frac{e^{\lambda_i \bar{\lambda}_{\pi(i)}}}{\Delta(\Lambda)\Delta(\bar{\Lambda})}. \tag{2.6}$$

Following the ideas of [9, 10, 19], one can reduce the computation of any correlator in the free theory to a matrix integral by commuting various generating functions past each other using the Campbell-Hausdorff formula. In the language of [9], this is equivalent to replacing the fields inside small operators (such as traces) by their vevs after integrating out the SYM fields and then performing a saddle point approximation over the auxiliary parameters (in this case U).

More concretely we will be interested in computing form factors such as:

$$\langle \bar{\Lambda}, n_3 \big| \operatorname{Tr}[Z(t=0,n_2)^L] \, |\Lambda, n_1\rangle \simeq \sum_{R,R'} C_{R,R'}(\Lambda,\bar{\Lambda}) \, \langle R', n_3 | \operatorname{Tr}[Z(t=0,n_2)^L] \, |R, n_1\rangle \,, \tag{2.7}$$

where the initial and final states are created by heavy operators $\Delta \sim N^2$ in the large N limit. For relatively simple choices of operators, such as determinants and traces of fully symmetric tensors [9, 10, 115, 144], the saddle point analysis can be performed rather explicitly and the correlators can be matched precisely to their holographic counterparts. For more complicated operators, such as insertions of many determinant operators, or operators associated to generic Young diagrams, the saddle point analysis appears to be less straightforward and the structure of the solutions to the saddle point equations is not fully understood. The main difficulty lies in the fact that the resulting matrix models cannot be easily reduced to integrals over eigenvalues, so that the saddle point equations appear to be truly matrix equations. We will discuss in the later sections how to overcome these complications in the regimes relevant to states with nice supergravity descriptions (i.e. states corresponding to non-trivial geometries with small curvatures).

8.2.1 Example: AdS giant graviton

Before proceeding to the case of interest, it is convenient to review the results presented in [9, 144] since many of the parts of the calculations presented there extend naturally. In the simplest of cases, the heavy operators can be taken to be of rank one, meaning that they correspond to Schur polynomials of fully symmetric or fully anti-symmetric representations. In the case of fully anti-symmetric representations the correct generating function is the determinant operator [43], for instance :

$$\det(\phi_5 + i\phi_6 - \lambda) = \det(Z - \lambda), \tag{2.8}$$

which describes a sphere giant graviton sitting at the origin of global AdS at a position inside S^5 given by $e^{i\phi_0}\cos\theta_0 = \lambda$. This operator is a semi-coherent superposition of all subdeterminant operators, each describing the R-charge eigenfunctions of a sphere giant graviton. The method for semi-classical computation with this class of operators was presented in [9] and also [19] so we refer the reader there for details. Instead we will describe how the analogous computation is done for an operator describing a semi-classical AdS giant graviton. The reason for this is that in the end both calculations lead to very similar integrals for the correlators, but their form is much easier to understand for the computation involving symmetric tensors.

First we consider the following coherent state:

$$|\lambda\rangle = \frac{1}{\texttt{Vol}\left[\mathbb{CP}^{N-1}\right]}\int_{\mathbb{CP}^{N-1}} d\varphi d\varphi^\dagger\, e^{\lambda\varphi^\dagger Z\varphi}\,|0\rangle\,. \tag{2.9}$$

As discussed in [11, 144] this is the same state that one obtains from setting Λ to be a rank one projector in (2.6). This state has a natural $U(1)$ gauge symmetry

$$\varphi \sim e^{i\alpha}\varphi, \tag{2.10}$$

which can be identified with the gauge symmetry on the worldvolume of the giant graviton, as well as invariance under $U(N)$ gauge transformations of Z. This state is also a coherent superposition of AdS giant graviton wavefunctions with fixed R-charge. A simple calculation yields:

$$\langle\bar{\lambda}|\lambda\rangle = \frac{1}{\texttt{Vol}\left[\mathbb{CP}^{N-1}\right]}\int_{\mathbb{CP}^{N-1}} d\varphi d\varphi^\dagger\, e^{\lambda\bar{\lambda}\varphi^\dagger P_1\varphi}. \tag{2.11}$$

To evaluate this integral we need to do a series of simple coordinate transformations. Without loss of generality, we can let P_1 be a rank one projector into the first component of φ. Then we can split the coordinates of $\mathbb{CP}^{N-1}$ into φ_1 and φ_n with $n > 1$. The reason we emphasize this will become clear when we generalize this more complicated coherent states. Then, we can parametrize the coordinates φ_n in terms of an $2N-2$ dimensional spherical slices of radii $R = \sqrt{1-|\varphi_1|^2} = \sqrt{1-r^2}$. Finally we can rewrite the radial part of the integral as

$$\int d\left(r^2\right) d\left(R^2\right)\ R^{2N-4}\ \delta(R^2+r^2-1)\ e^{\lambda\bar{\lambda}r^2} = \int_0^1 dx\ (1-x)^{N-2}\ e^{\lambda\bar{\lambda}x}. \tag{2.12}$$

This last integral is simply the moment generating function of a particular unitary Jacobi distribution. To make contact with the calculation involving determinants, we can rewrite

$$\left(\lambda\bar{\lambda}\right)^{N-2}(1-x)^{N-2} = \det\begin{pmatrix}\lambda & \bar{\lambda}\varphi_1\\ \lambda\varphi_1^* & \bar{\lambda}\end{pmatrix}^{N-2} = \det\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}^{N-2}. \tag{2.13}$$

Although this step is not necessary and the previous integral expression is simple enough to evaluate explicitly, doing this change of variables makes it clear that the final answer the large N approximation for AdS giants is the same as that of sphere giants (up to analytic continuation). After a final simple re-scaling, $\lambda \to \sqrt{N-2}\ \lambda$ we finally arrive at the expression

$$\left\langle\bar{\lambda}\middle|\lambda\right\rangle = C_N \int_{\det\rho\ \geq\lambda\bar{\lambda}-2} d\rho\ \det\rho^{N-2}\ e^{(N-2)\mathrm{tr}_2\left[\rho\rho^\dagger\right]}, \tag{2.14}$$

which apart from the contour on integration is identical to the integral obtained from the Hubbard-Stratonovich trick used in [115] for determinant operators [1]. This will be true generically, once we solve the saddle point equations for a configuration of AdS giant gravitons, we automatically have the solution for a configuration of sphere giant gravitons

[1]Our trace convention is so that $\mathrm{tr}_m 1 = 1$ as opposed to $\mathrm{tr}_m 1 = m$

after a simple analytic continuation. In this case the saddle point equations are simply:

$$\begin{aligned}&\rho^\dagger = \rho^{-1} \Rightarrow \rho_{12}\rho_{21} = \lambda\bar{\lambda} - 1\\&\lambda\bar{\lambda} > 1.\end{aligned} \tag{2.15}$$

The second equation comes from the fact that the exponential needs to be positive for the saddle point to be a maximum; this implies that the semiclassical approximation is valid whenever $|\lambda| = \cosh\rho_0$ is greater than one which simply says that the brane is at a position $\rho_0 > 0$ in global AdS, and this is true for all half-BPS giant graviton solution.

Form Factors

The next step is compute the following form factor;

$$\langle\bar{\lambda}|\,\mathrm{Tr}_N\left[\left(\vec{n}\cdot\vec{\phi}\right)^L\right]|\lambda\rangle = \langle\bar{\lambda}|\,\mathrm{Tr}_N\left[\left(\frac{Z+\bar{Z}+Y-\bar{Y}}{2}\right)^L\right]|\lambda\rangle\,. \tag{2.16}$$

Our choice of $\vec{n}$ is taken from [115] for clarity of presentation and is arbitrary. To evaluate this quantity, we use the fact that the initial and final states are coherent states which lets us replace Z and $\bar{Z}$ by constant matrices. The resulting trace is

$$\mathrm{Tr}_N\left[\left(\frac{Z+\bar{Z}+Y-\bar{Y}}{2}\right)^L\right] = \mathrm{Tr}_N\left[\left(\frac{\bar{\lambda}\varphi\varphi^\dagger + \lambda\tilde{\varphi}\tilde{\varphi}^\dagger}{2}\right)^L\right]. \tag{2.17}$$

To proceed we use the fact that the integrals over $d\varphi$ and $d\tilde{\varphi}$ are invariant under the action of $U(N)$, so we can gauge fix $\tilde{\varphi}$ to be a unit vector v with a one in the first component. Finally one uses the trick introduced in [9] to exchange the trace over color indices into a trace over "flavor" indices associated to the in and out states of the brane:

$$\mathrm{Tr}_N\left[\left(\frac{Z+\bar{Z}+Y-\bar{Y}}{2}\right)^L\right] = 2^{-L}\mathrm{tr}_2\left[\begin{pmatrix}\lambda & \bar{\lambda}\varphi_1\\ \lambda\varphi_1^* & \bar{\lambda}\end{pmatrix}^L\right] \simeq \mathrm{tr}_2\rho^L. \tag{2.18}$$

Now instead of evaluating every power the matrix ρ, it is better to work with the resolvent

$$R(t) = \mathrm{tr}_2\left[(1-t\rho)^{-1}\right] = \frac{1-t\mathrm{tr}_2[\rho]}{t^2\det\rho - 2t\mathrm{tr}_2[\rho]+1}. \tag{2.19}$$

When we evaluate this expression at the saddle point value for ρ, we can immediately recognize that the resolvent $R(t)$ is a generating function for Chebyshev polynomials of the first kind.

$$\langle R(t)\rangle_{N=\infty} = \sum_{n=0}^{\infty} T_n(\cos\phi_0\cosh\rho_0)\ t^n, \tag{2.20}$$

where we parametrized the eigenvalue in terms of LLM coordinates $\lambda = e^{i\phi_0}\cosh\rho_0$.

Extracting Structure Constants

Naively one might expect that the saddle point approximation of the resolvent computes a generating function of some half BPS structure constants. This is not quite correct for the following reason. First we would need to extract the contribution to the form factor from a particular set of primary operators. In this case this is somewhat easy to do, given that the coherent state has a simple expansion in terms of Schur polynomials for rank one representations

$$|\lambda\rangle = n_\lambda \sum_{k=0}^{\infty}\frac{\lambda^{N+k-1}}{(N+k-1)!}\mathcal{O}_{(k)}(Z)\,|0\rangle\,. \tag{2.21}$$

We should then think of the coefficients in the expansion as the distribution of lengths for a Young diagram with a single row. The distribution is similar to a Poisson random variable, so that the average length is of the Young diagram is of order $|\lambda|$. Another way of seeing this is by using the Stirling formula for the denominator and extremizing with respect to $l = N + k - 1$; the maximum occurs when $|\lambda| = N + k - 1$. This is the reason why the coherent state calculation in [144] gives the correct answer for structure constants without the need to project into a particular character. However, the dependence on the phase of λ when we insert an operator will not be correct due to unwanted contributions coming from off-diagonal terms. To fix this one should project the intermediate operator into a an R-charge singlet operator. This is done by performing a group average over the phase of λ. Since the wavefunction $\langle\bar{\lambda}|\lambda\rangle$ is already invariant under shifts in the phase of λ whenever $\bar{\lambda} = \lambda^*$, the only effect of averaging is to project out off-diagonal terms from the resolvent.

Strictly speaking this averaging should be performed prior to doing the saddle analysis, since averages do not generally commute. One way of performing this average is to rescale $t \to \sqrt{\det\rho}\, t$, and then perform the integration over the phase of λ by a contour integral. The result is

$$\overline{R(t)} = \sqrt{\det\rho}\left[\frac{t^2-1}{2\sqrt{(1+t^2)^2\det\rho - 4t^2\lambda\bar{\lambda}}}\right]. \tag{2.22}$$

In the large N limit, we can set $\det\rho = 1$, and the averaged resolvent will take the form of a generating function of Legendre polynomials. Since the saddle point analysis for this case basically involves setting ρ to a particular value, the averaging procedures commute so the large N limit was taken first in [115, 144] without any trouble. This is not the case for operators made out Schur polynomials for large Young diagrams, since the large N limit leads to a continuous distributions of eigenvalues. Once the eigenvalues condense, the result of the computation will be highly sensitive to the analytic properties of the moment generating function, and performing the averaging and large N limit can lead to contradicting results. The most natural prescription to remedy this is to perform the projection into a particular primary operator first by an appropriate averaging, and then take the large N limit of this quantity.

8.3 Matrix models for general coherent states

8.3.1 Two Droplets

The next simplest calculation that we can perform is the case where the matrix Λ in (2.6) is taken to be a rank p projector. In this case, the analogous coherent state is an integral over the Grassmannian $Gr(p, N)$. The expression for the coherent state is simple to write down, but some of the steps needed to evaluate the resulting matrix integrals require some care; the main task will be to evaluate the following norm:

$$|\lambda, p\rangle = \frac{1}{\mathtt{Vol}\left[Gr(p,N)\right]} \int_{Gr(p,N)} dV dV^{\dagger} \, e^{\lambda \mathrm{Tr}_N V V^{\dagger} Z} \, |0\rangle \, . \tag{3.1}$$

We will argue that this state described the wavefunction of a stack of p giant gravitons sitting at position λ in the LLM plane. The first thing to note is that this coherent state has an explicit $U(p)$ gauge symmetry $V \sim Vg$ which we can identify as the gauge symmetry on a stack of D-branes. The expectation value of Δ_0 on this state is given by $p|\lambda|^2$ so that whenever $|\lambda| \sim \sqrt{N}$ and $p \sim N$, the average dimension of this state is of order N^2. By inspection, we can also deduce that this state is a coherent superposition of Schur operators of at most p rows. by acting with $\mathrm{Tr}[\bar{Z}]$ on this state, we can see that this state breaks the gauge symmetry spontaneously from $U(N)$ to $U(N-p) \times U(p)$, and that the center of mass of the stack of p branes is at the position $z = \lambda$ on a complex plane.

Before proceeding we need to comment on the choice of coordinates for Grassmannian, since the details on how to perform these types of integrals are known but not readily available. We will mostly follow the notation of [154, 155]; for a pedagogical presentation we refer the reader to [146]. First, we can choose to split any given group element U in terms of block matrices

$$U = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3.2}$$

where A_{11} and A_{22} are $p \times p$ and $(N-p) \times (N-p)$ square matrices, and A_{12} is a $p \times (N-p)$ matrix (similarly for A_{21}). Then we make an arbitrary choice of frame distinguished by a rectangular matrix

$$v^T = \begin{pmatrix} \mathbb{I}_p & \mathbf{0}_{(N-p)\times p}; \end{pmatrix} \tag{3.3}$$

this matrix can then be used to build projectors into arbitrary p-dimensional subspaces of $\mathbb{C}^N$ by acting on v with unitaries. This set of projectors precisely gives a parametrization of the affine Grassmannian $Gr(p, N)$. By an affine Grassmanian we will simply mean the space spanned by the unitary transformations of v:

$$Gr(p, N) := \{V = U \cdot v \mid U \in G\}. \tag{3.4}$$

Clearly any V in this space is rank deficient, so all of the information about its singular value decomposition is captured a square matrix:

$$VV^\dagger = U^\dagger P_p U = \begin{pmatrix} A_{11}^\dagger A_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.5}$$

In analogy to the rank one calculation, we can write down the integration measure for this space as

$$dVdV^\dagger = dA_{11}dA_{11}^\dagger dA_{12}dA_{12}^\dagger \times \delta\left(A_{11}^\dagger A_{11} + A_{12}^\dagger A_{12} - \mathbb{I}_p\right) \tag{3.6}$$

By a similar calculation we can see that the norm of this state is given by

$$\left\langle \bar{\lambda}, p \mid \lambda, p \right\rangle = \frac{1}{\texttt{Vol}\left[Gr(p,N)\right]} \int_{Gr(p,N)} dVdV^\dagger \, e^{\lambda\bar{\lambda}\mathrm{Tr}_p V^\dagger P_p V}. \tag{3.7}$$

One last fact that we need before continuing is the Jacobian for the coordinate transformation $M = A^\dagger A$ for any $n \times m$ rectangular matrix A. This change of variables is well known in the context of Wishart distributions;

$$dAdA^\dagger \propto \det(M)^{n-m} dM, \tag{3.8}$$

where the constant of proportionality is an integral over angular variables that will not be important for our analysis. So in the end we find

$$\left\langle \bar{\lambda}, p \mid \lambda, p \right\rangle = n_{\lambda,p} \int_{A_{11}^\dagger A_{11} \preceq \mathbb{I}_p} dA_{11}^\dagger dA_{11} \det\left(\mathbb{I}_p - A_{11}^\dagger A_{11}\right)^{N-2p} \exp\left(\lambda\bar{\lambda}\mathrm{Tr}_p\left[A_{11}^\dagger A_{11}\right]\right). \tag{3.9}$$

Since the only combination of A_{11} and $A_{11}^\dagger$ that appears in the integral is $A_{11}^\dagger A_{11}$, we can reduce the computation to an integral over the eigenvalues of $A_{11}^\dagger A_{11}$.

$$\left\langle \bar{\lambda}, p \mid \lambda, p \right\rangle = \mathcal{C}_{\lambda,p} \int_{[0,1]^p} \prod_{i=1}^{p} dx_i \, \Delta\left(x_i\right)^2 \left(1 - x_i\right)^{N-2p} \, \exp\left(\lambda\bar{\lambda} \sum_{j=1}^{p} x_j\right). \tag{3.10}$$

8.3.2 Large N Limit: Steepest Descent

We will now sketch the evaluation of (3.10) in the large N limit. As before, $|\lambda|^2$ will scale with N so that the exponential term is large. The integral can evaluated explicitly using the Andreief identity [146]:

$$\mathcal{C}_{\lambda,p}^{-1} = p! \det_{j,k} \left(\int_0^1 dx \, (1-x)^{N-2p} x^{j+k-2} e^{\lambda\bar{\lambda}x} \right) \tag{3.11}$$

The function inside the determinant is an incomplete Gamma function. Another way interpreting it is at the moment generating function for the GUE Jacobi ensemble. Even though we can evaluate simple moments in this distribution exactly, the final form will always be a large determinant of a Hankel matrix which we cannot deal with easily. Instead we can perform a saddle point analysis of (3.10) for large N and large p. After rescaling a similar rescaling $\lambda \to \sqrt{N-2p}\,\lambda$ the saddle point equations of this integral are of the form

$$\lambda\bar{\lambda} - \frac{1}{1-x_i} - \frac{2}{N-2p} \sum_{j\neq i} \frac{1}{x_j - x_i} = 0. \tag{3.12}$$

The behavior of the saddle point configurations are easy to understand; the first two terms are the same as in the single eigenvalue problem so that all the eigenvalues have a tendency to condense around $\lambda\bar{\lambda} = \frac{1}{1-x_i}$ while second term the usual eigenvalue repulsion term. We should also note that for distribution to be stable we have to require that $p \leq N/2$. This makes sense, since condition forces the second droplet to be smaller than the first. In the case that $p \gg N/2$, this configuration is no longer a small deformation of the original vacuum configuration and the remaining $N-2p$ eigenvalues become the relevant degrees of freedom. The case with $p \sim N/2$ has to be treated with particular care as we will see. To solve these equations at large N with p/N fixed we simply recast the saddle point equations as a Ricatti equation for the resolvent matrix $R_p(z)$:

$$\left[\lambda\bar{\lambda} - \frac{1}{1-z}\right] R_p(z) - \frac{K}{1-z} - \frac{p}{N-2p} R_p(z)^2 + \frac{1}{N-2p} R_p'(z) = 0, \tag{3.13}$$

where $K = \frac{1}{p}\sum_{i=1}^{p} \frac{1}{1-x_i} = \int \frac{\rho(z)}{1-z}$ is a constant that is determined from $R_p(z)$ by imposing self-consistency conditions of the distribution. From this we can identify $\frac{1}{N-2p}$ as the relevant $\hbar$ parameter in the problem; this is the reason why $p \sim N/2$ should be treated with care, since the $1/N$ fluctuations of the resolvent are no longer under control and one must solve the differential equation exactly. This can be seen from a simple scaling argument; if $N-2p$ is of order one, the exponential and determinant terms in (3.9) are of order e^p, while the Vandermonde term is of order e^{p^2}. This says that the dominant effect in this case is the eigenvalue repulsion, so that the separation of each of the individual eigenvalues is large when compared to the size of the system. In particular we should expect the corresponding geometry to have a region with string scale curvature where the supergravity approximation breaks down. This is expected, since wavefunction $|\lambda, p\rangle$ no longer has a good semiclassical approximation in the large N limit. If instead we decide to keep $N-2p$ of order one but this time scaling λ as $\sqrt{p} \sim \sqrt{N}$, then the Vandermonde contribution is off-set by the exponential term and the determinant term is still sub-leading, so the distribution of eigenvalues x_i is approximately semi-circular. What this is saying is that now the two droplets are too close together to be treated as separate (meaning that their size is of the same order as their separation), and instead the deviation from vacuum is described by a collection of order one giant gravitons probing the vacuum. In other words, depending on how we decide to scale $N-2p$ we will obtain qualitatively different saddle point conditions and the expansion in terms of AdS or sphere giant gravitons might be more suitable.

The regime we will be interested in is when $N-2p$ is of order N, so that the constant equilibrium configuration is a good approximation to the eigenvalue distribution. In the large N limit the density of states becomes

$$\rho(x)dx = \frac{1}{\pi}\frac{\sqrt{4K\mu(1-x) - \left(1-(1-x)\lambda\bar{\lambda}\right)^2}}{1-x}dx, \tag{3.14}$$

with $\mu = \frac{p}{N-2p}$; this turns out to be simpler after we make the change of variables $z = \lambda\bar{\lambda}\,(1-x)$. After this we can normalize the distribution such that $\int \rho(z)dz = 2\mu$ and find $K = 1$. In this convention the eigenvalues are quantized in units of $2g_s = \frac{1}{N-2p}$, where g_s is the effective string coupling of this system.

To compare with the corresponding LLM solution we can express the solution in coordinates that manifest $\frac{1}{8}$ of the supersymmetries [69]. The point is to write the 10d metric in terms of a 6d complex basis with coordinates x, y, z. For the vacuum $AdS_5 \times S^5$ solution these coordinates should be identified with the coordinates of the five-sphere. Translating the whole metric into these coordinates is a non-trivial task for generic LLM geometries, but we will only be interested in determining the volume of the cycle wrapped by the branes. In these coordinates the radius of the three-sphere wrapped by the giant gravitons for a single droplet solution is given

$$\tilde{r} = \frac{\sqrt{(L^2 - |z|^2)(|z-a|^2 - b^2)}}{|z-a|}, \tag{3.15}$$

where the droplet is centered at $z = a$, the size of AdS is L and b is the radius of the droplet. Notice that this is essentially of the same form as (3.14) up to some relabellings. The discrepancy between the denominators is due to the fact that the variable x is actually related to the square of the radial direction of AdS. The precise mapping between both pictures should involve some more complicated charge of variables in general, but the analytic properties of both distributions are the same.

8.3.3 Three Point Functions: Diagonal Case

To compute the correlator of a single trace in the background of these coherent states we can use the resolvent trick. Using the same kind of color-flavor transformation the trace over the original color indices can be replaced by a trace over $p \times p$ matrix. In this case the moment generating function is

$$\begin{aligned}\mathcal{F}(t) &= \mathrm{tr}_p\left[\left((2-t(\lambda+\bar\lambda)\right)\left((1-t\lambda)(1-t\bar\lambda)\mathbb{I}_p - t^2\lambda A_{11}^\dagger A_{11}\right)^{-1}\right] \\ &= 2\sum_{i=1}^{p}\frac{1-\frac{t}{2}(\lambda+\bar\lambda)}{t^2\lambda\bar\lambda(1-x_i)-t(\lambda+\bar\lambda)+1}.\end{aligned} \tag{3.16}$$

The most natural variable to work with is once again $z_i = \lambda\bar\lambda(1-x_i)$, which makes the density of eigenvalues be of the form:

$$\rho(z) = \frac{\sqrt{(z_+-z)(z-z_-)}}{\pi z}, \tag{3.17}$$

where $z_\pm$ are given by the roots of the polynomial inside the square root in (3.14)

$$z_\pm = 1 + \frac{2\mu}{\lambda\bar\lambda} \pm \frac{2\sqrt{\mu(\mu+\lambda\bar\lambda)}}{\lambda\bar\lambda}. \tag{3.18}$$

Expanding $\mathcal{F}(t)$ as a function of z gives an expression for the moment generating function in terms of the moments of the Marchenko-Pastur distribution (3.17). After extracting the L^{th} moment we get

$$\begin{aligned}&\langle\bar\lambda|\,\mathrm{Tr}[\frac{(Z+\bar Z+Y-\bar Y)^L}{2^L}]\,|\lambda\rangle = \\ &\sum_{k=0}^{\infty}\binom{\frac{L+k}{2}}{\frac{L-k}{2}}2^{-(L-k)}\left(\frac{\lambda+\bar\lambda}{2}\right)^k m_{\frac{L-k}{2}} - \sum_{k=0}^{\infty}2^{-(L-k)}\binom{\frac{L+k}{2}-1}{\frac{L-k}{2}}\left(\frac{\lambda+\bar\lambda}{2}\right)^{k+1} m_{\frac{L-k-1}{2}}, \\ &m_l = \int_{z_1}^{z_+} dz\,\rho(z)\,z^l = \sum_{k=1}^{l}\frac{1}{k}\binom{l}{k}\binom{l}{k-1}\left(\frac{z_+-z_-}{2}\right)^k = {}_2F_1\left(1-l,-l;\,2;\,,\frac{z_+-z_-}{2}\right)\end{aligned} \tag{3.19}$$

To extract the diagonal part of this form factor we can average over the phase of λ.

$$\mathcal{F}_{\lambda\bar\lambda L} = \frac{1^L+(-1)^L}{2\sqrt{L}}\sum_{k=0}^{L}\binom{\frac{L+k}{2}}{\frac{L-k}{2}}\binom{k}{\frac{k}{2}}\left(\frac{1^k+(-1)^k}{2}\right)|\lambda|^k\; m_{\frac{L-k}{2}}\left(\frac{L}{L+k}\right). \tag{3.20}$$

This correlator should be interpreted as encoding part of the angular distribution of the bubbling geometry associated to the condensate of eigenvalues and should compute the one-point function of a scalar operator on an LLM geometry with two circular droplets. It would be interesting to compute these correlators holographically, for instance with the methods developed in [112]. By performing an additional contour integral over $|\lambda|$ we can obtain three point functions for operators with fixed scaling dimensions as opposed to coherent states as is done in [115]. One can similarly perform computations that extract off-diagonal form-factors between heavy states with different dimensions.

8.3.4 General Matrix Model: Eigenvalue Picture

Now we will proceed to the general case and sketch how to extract specific operators associated to a particular Young diagram of at most p rows with order N^2 boxes, and we will use this to compute diagonal form factors. To do this we need to consider the generating function

$$\left\langle \bar{\Lambda}_p \middle| \Lambda_p \right\rangle = n_p \int d\sigma^\dagger d\sigma \det\left(\mathbb{I} - \sigma^\dagger\sigma\right)\ e^{N\mathrm{tr}_p[\Lambda\sigma\bar{\Lambda}\sigma^\dagger]}. \tag{1.1}$$

This formula is the analog of (3.9) for generic coherent state parameters. Computing this integral is not an easy task for generic eigenvalues, since the argument of the exponential is no longer just a function of $\sigma^\dagger\sigma$, and the matrix σ is not a normal matrix, so σ and $\sigma^\dagger$ cannot be simultaneously diagonalized. In this section we will give a heuristic argument for a saddle point approximation to this integral. We will give a more concrete proof in the next section where we will need a more careful analysis of these type of integrals. Intuitively one would like to say that the integral is dominated by points where the all the matrices in the trace are diagonal. One way to see why this could be true is that the exponent can be written in the form:

$$\mathrm{tr}_p\left[\Lambda\bar{\Lambda}\sigma\sigma^\dagger - \sigma\bar{\Lambda}[\Lambda,\sigma^\dagger] - \Lambda\bar{\Lambda}[\sigma,\sigma^\dagger]\right]. \tag{1.2}$$

Since the first term is manifestly positive, the exponent is the largest when this term is maximized which happens whenever all the matrix components are concentrated on the diagonals. Any deviation from this contributes to the second and third terms, which are not necessarily positive. So it is natural to expect that a good approximation to the integral is obtained by integrating over the set of σ satisfying

$$[\lambda,\sigma^\dagger] = [\sigma,\sigma^\dagger] = 0. \tag{1.3}$$

Indeed we will see later that these are the saddle point conditions for the integration over the angular variables for σ. Because of the large exponent, corrections to this are heavily suppressed as long as $\Lambda^\dagger = \bar{\Lambda}$, and so

$$\left\langle\bar{\Lambda}_p\middle|\Lambda_p\right\rangle \simeq \int_{[0,1]^p} dx_i \Delta_p(x_i)^2\,(1-x_i)^{N-2p}\,e^{N\sum_i|\lambda_i|^2 x_i} + O(e^{-N^2}). \tag{1.4}$$

The other saddle points contribution for the norms consists of pairing the eigenvalues λ_i with $\bar{\lambda}_{\pi(i)]}$ for all permutations π, which are highly suppressed for non-coincident eigenvalues.

$$e^{\mathrm{Tr}[\Lambda\sigma\bar{\Lambda}\sigma^\dagger]} \to \sum_{\pi\in S_p} e^{\sum_i|s_i|^2\lambda_i\bar{\lambda}_{\pi(i)}} \times (\texttt{one-loop}). \tag{1.5}$$

We work out the appropriate one-loop determinant for this saddle point approximation in section 4. This structure precisely explains the saddle point structure found in [9] for determinant operators; the solutions to the matrix form of the saddle point equations always involve summing over permutations of initial and final giant graviton states. For widely separated eigenvalues the Vandermonde determinant does not contribute meaningfully to the saddle point approximation and the state is well approximated by a collection of widely separated giant gravitons. A more interesting regime is whenever we have n_a coincident branes at a point λ_a, for $a = 1, \ldots, k$. As long as the λ_a are sufficiently separated interactions between different droplets can be neglected and the eigenvalues x_i are distributed along k cuts whose distribution is approximately given by the Marchenko-Pastur distribution. More precisely, the norm of the coherent state with k lumps of eigenvalues centered around λ_a is computed by the following matrix model:

$$\begin{aligned}&\mathcal{Z}(\lambda_1, \ldots, \lambda_k) = \\ &\int_{[0,1]^p} \prod_{a=1}^{k} \prod_{i_a}^{n_a} dx_{i_a}^{(a)}\, \Delta_{n_a}\left(x^{(a)}\right)^2 \left(1 - x_{i_a}^{(a)}\right)^{N-2p} e^{N\lambda_a \bar{\lambda}_a x_a} \times \prod_{c>b} \prod_{i_b, j_c} \left(x_{j_c}^{(c)} - x_{i_b}^{(b)}\right).\end{aligned} \tag{1.6}$$

The derivation of this class of models was presented in [151], and we outline the details in the next section. As long as the cuts are not exponentially close to one another, the last eigenvalue repulsion term is far enough from zero that it does not affect the saddle point. The analysis for the three point function is then relatively straightforward; the moment generating function can be block diagonalized and each block is dealt just like the single-cut case. In the regimes when two cuts approach each other this approximation is no longer valid and one has to solve the corresponding monodromy problem exactly as the fluctuations around the stationary eigenvalue distribution will not be suppressed. We expect that these corrections reproduce the supergravity picture of [69], with the eigenvalue distribution being related to the volume of the three-sphere on which the giant gravitons are wrapped (see for example equations (5.110) and (5.116) in [69]). We expect that the spectral curve for the matrix model is precisely encoded by the dual LLM geometry written in the coordinates advocated by [69], since our distribution of eigenvalue for the single cut case is essentially identical to their equation (5.117). This aligns with the results coming from numerical tests performed in [156] and we advocate for a similar viewpoint; $1/N$ effects will generically give some amount of granularity to the edges of eigenvalue droplets, specially if multiple droplets are close to one another when compared to the characteristic size of each eigenvalue. In order to resolve these details one would need to solve the full interacting saddle point equations. We will not do this, since in this regime there will not be a reliable geometric picture for the state. In other words, the saddle point approximation we described above breaks down for states that describe half-BPS geometries with string scale curvature, and instead we should think of the state as being a deformation of a smooth geometry with some branes inserted.

8.3.5 Coulomb Branch Limit

One last interesting limit that we can consider is the limit in which the droplets are widely separated from each other and from the origin. In this limit, the Marchenko-Pastur distribution reduces to a delta function

$$\rho(x) \to \sum_i \delta\left(\lambda_i \bar{\lambda}_i - \frac{1}{1-x_i}\right). \tag{1.7}$$

This is exactly the Coulomb branch limit discussed in [112]. In this sense the operators we study here can be understood as quantum mechanical analogs of Coulomb branch vacua of the theory. This makes the relation between the geometry of the moduli space of vacua and asymptotically anti-de-sitter spaces clear; in the large N limit, the moduli space should get quantum correction which deform its geometry into a bubbling geometry and only in the dilute gas approximation can we approximate such a geometry by a multi-center solution.

8.4 Matrix Models for the Character Basis

Now we will concern ourselves with computing three point functions where the initial and final state are specified by specific Young diagrams, as opposed to a collection of eigenvalues λ_i. The idea will be to make a somewhat unconventional choice of integration contour for the coherent states parameters. First we start with a pair of states $\left|\tilde{\Lambda}^\dagger\right\rangle$, $|\Lambda\rangle$, but now we treat the parameters as being independent from one another; we will also force each of the eigenvalues λ_i and $\tilde{\lambda}_i$ to lie on a unit circle. By multiplying $|\Lambda\rangle$ by the square of the Vandermonde determinant of Λ, and integrating we can recognize that the resulting integration measure is just a Haar measure for a new unitary matrix $\mathcal{U} = U\Lambda U^\dagger$:

$$\int_{U(N)} dU \oint d\lambda_i\, \Delta(\lambda_j)^2 \to \int_{U(N)} d\mathcal{U}. \tag{4.1}$$

The resulting state is clearly proportional to the vacuum state, since there are no $\mathcal{U}^\dagger$ insertions to feed to the exponential. From this it becomes clear that in order to extract a term proportional to the state $|R\rangle = S_R(Z)\,|0\rangle$ one should multiply the integrand by a character $S_R(\mathcal{U}^\dagger)$:

$$|R\rangle = \frac{f_R}{\texttt{Vol}\,[U(N)]} \int d\mathcal{U}\; e^{\mathrm{Tr}[\mathcal{U}Z]}\; S_R(\mathcal{U}^\dagger)\,|0\rangle\,. \tag{4.2}$$

A similar trick was used in [157] to study expectation values of Wilson loop for arbitrary representations in large N Chern-Simons theory, with a slightly different generating function. For our choice of generating function the exponential factor can be expanded in terms of unitary characters using Schur-Weyl duality and the resulting integrals are easily evaluated using elementary orthogonality relations. On the other hand, for sufficiently large representations we will be able to perform the integral using steepest descent after we perform all contractions of the $\mathcal{N} = 4$ SYM fields. This exponential generating function is also useful for computing Wick contractions with other operators since we can exploit the Campbell-Hausdorf formula. With all this in mind the coefficient f_R is easily determined to be

$$f_R = n_R! \, \frac{\mathrm{Tr}_R[\,1\,]}{\chi_R(\mathrm{id})} = n_R! \, \frac{\mathrm{Dim}_R(N)}{d_R}\,; \tag{4.3}$$

which is the norm of the corresponding state; this is done by expanding the exponential and matching the terms as in [11].

We will want to compute quantities such as

$$C_{RR'L} = 2^{-L} \times \frac{\langle R'|\,\mathrm{Tr}_N\left[\left(Z+\bar{Z}+Y-\bar{Y}\right)^L\right]|R\rangle}{\sqrt{L\,\langle R'|R'\rangle\,\langle R|R\rangle}}, \tag{4.4}$$

in the limit that $|R| \sim |R'| \sim N^2$. To compute the quantity in the numerator we can substitute the equation (4.2) and perform the Wick contractions using the Campbell-Hausdorff formula:

$$\langle R'|\,\mathrm{Tr}_N\left[\left(Z+\bar{Z}\right)^L\right]|R\rangle = \frac{f_{R'}f_R}{\mathtt{Vol}\,[U(N)]^2}\int d\mathcal{U}d\mathcal{V}\; S_R(\mathcal{U}^\dagger)S_{R'}(\mathcal{V}^\dagger)\; e^{N\mathrm{Tr}[\mathcal{U}\,\mathcal{V}]}\mathrm{Tr}\left[(\mathcal{U}+\mathcal{V})^L\right]. \tag{4.5}$$

This procedure replaces all of the free-field Wick contractions with unitary integrals which we can evaluate very explicitly.

8.4.1 Diagonal Structure Constant

As before it will be easier to work with the moment generating function for the matrix $U+V$ instead of dealing with each individual trace. We now proceed by diagonalizing both U and V, after which we are left with an integral of HCIZ-type:

$$\mathcal{F}(t) \simeq \int d\tilde{U}d\mu(u)d\mu(v)S_R(u^*)S_R(v^*)e^{N\mathrm{Tr}[\tilde{U}^\dagger u\tilde{U}v]}\,\mathrm{Tr}[\Big(1-t(u+\tilde{U}v\tilde{U}^\dagger)\Big)^{-1}]. \tag{4.6}$$

This integral is quite challenging to evaluate exactly, mainly due to the appearance of the unitaries U inside of the trace of the resolvent. At large N, the integral over U can be evaluated the method of steepest descent; the saddle point equations for the matrix U are solved by permutation matrices which allows us to replace the integral over U by a sum over permutations times a one loop determinant

$$\int d\tilde{U} \to \frac{1}{N!} \sum_{\pi \in S_N} \times \prod_j \frac{1}{\nu_i}, \tag{4.7}$$

where ν are the eigenvalues of the Hessian of $\mathrm{Tr}[U^\dagger u\, U v]$; this determinant factor is well known and it is proportional to $\Delta(u)\Delta(v)$. Now, due to the permutation invariance of the measure of integration for u, v, this sum over permutations can be performed by changing variables $v_i \to v_{\pi(i)}$ in each of the terms in the sum, so that in the end we are left with an integral over eigenvalues all lying inside a unit circle:

$$\mathcal{F}(t) \simeq \oint \prod_i \frac{du_i dv_i}{u_i v_i}\, e^{N u_i v_i} \det\left(\bar{u}_j^{N+R_k-k}\right) \det\left(\bar{v}_j^{N+R_k-k}\right) \sum_{l=1}^{N} \frac{1}{1 - t(u_l + v_l)} \tag{4.8}$$

Now the main obstacle is that we have a pair of determinants in the integrand. To solve this issue we can expand the determinants as sums over permutations, and exploit the symmetry of the measure under index relabelling to reduce the number of sums. It is also more convenient to work with the variables $x_i = u_i v_i$ and $y_i = u_i / v_i$;

$$\begin{aligned} \mathcal{F}(t) \simeq \frac{1}{N!} \sum_{\pi \in S_N} (-1)^\pi \oint \prod_i \frac{dx_i}{x_i}\, e^{N x_i}\, \sqrt{\bar{x}_i}^{N+R_i-i} \sqrt{\bar{x}_{\pi(i)}}^{N+R_i-i} \\ \times \oint \prod_j \frac{dy_j}{2y_j} \sqrt{y_j}^{R_j-j} \sqrt{\bar{y}_j}^{R_{\pi(j)}-\pi(j)} \sum_{i=1}^{N} \sum_{L=0}^{\infty} t^L x_i^{L/2} \left(2 + y_i + \bar{y}_i\right)^{L/2}. \end{aligned} \tag{4.9}$$

To perform the integral over y_j we set $y_j = e^{2i\beta_j}$, since the coordinate y_i winds around the unit circle twice. In order to get a non-zero value, all of the integrals over y_j should be non-zero and since the moments only multiply by one particular value of y_j for each term in the sum we can conclude that the integral is only non-zero for $\pi = id$. After evaluating the integral over y we get

$$\mathcal{F}(t) \simeq \frac{1}{N!} \oint \prod_i \frac{dx_i}{x_i} \, e^{Nx_i} \, \bar{x}_i^{N+R_i-i} \, \sum_i \frac{1}{\sqrt{1-4t^2 x_i}}. \tag{4.10}$$

This final integral is a simple Fourier integral that can be evaluated by expanding in powers of x_i. After normalizing $\mathcal{F}(t)$ appropriately we obtain a formula for the generating function of structure constant C_{RRL}:

$$\mathcal{F}(t) = \sum_{L=0}^{\infty} t^L \, \frac{1^L + (-1)^L}{2\sqrt{L}} \times \left\{ \binom{L}{L/2} N^{-L/2} \sum_{i=1}^{N} \frac{\Gamma(N+R_i-i+1)}{\Gamma(N+R_i-i-L/2+1)} \right\}. \tag{4.11}$$

For large representations $R_i \sim N$ with large blocks, the ratio of gamma functions can be replaced by the asymptotic expansion $\frac{\Gamma(x)}{\Gamma(x-\beta)} \simeq x^\beta$ as $x \to \infty$, and the sum may be replaced by an integral with $x = (N-i)/N$ and $\alpha_i = R_{k+1-i}/N$:

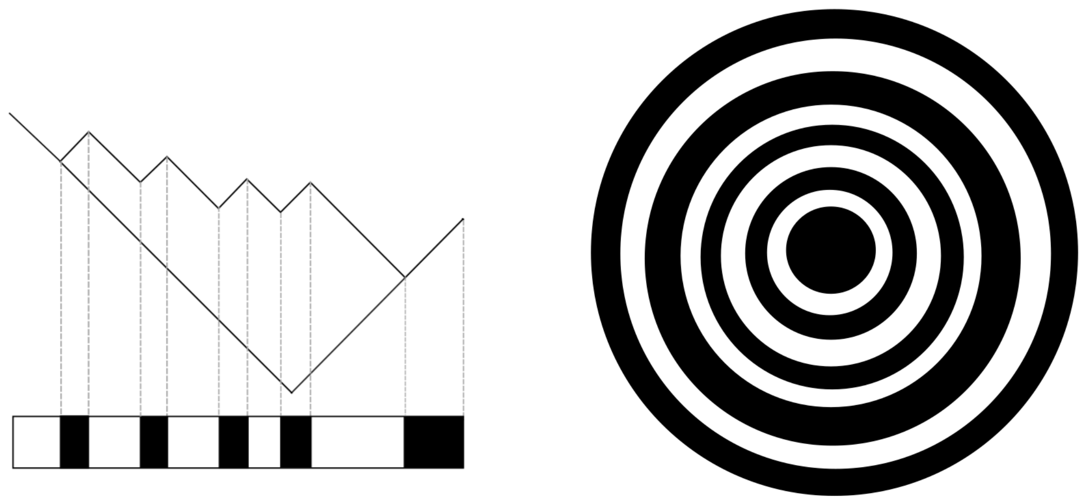
$$\mathcal{F}(t) = \sum_{L=0}^{\infty} t^L \, \frac{1^L + (-1)^L}{2\sqrt{L}} \times \left\{ \binom{L}{L/2} \sum_{i=1}^{k} \int_{\mu_i+\alpha_i}^{\mu_{i+1}+\alpha_i} dx \; x^{L/2} \right\}, \tag{4.12}$$

here μ_i are filling fractions that measure the number of rows of size greater than or equal to R_{k+1-i} in units of N and we assume that there are k non-zero blocks in the Young diagram for the representation R. Then the structure constants are:

$$\begin{aligned} C_{R\,R\,L/2} &= \frac{1^L + (-1)^L}{2\sqrt{L}} \times \left\{ \binom{L}{L/2} \int_0^\infty dx \; \rho_R(x) \times x^{L/2} \right\} \\ &= \frac{1}{\sqrt{L}} \int dr \, r \, d\phi \, \rho_R(r) \, (z+\bar{z})^L, \quad z = re^{i\phi/2}, \end{aligned} \tag{4.13}$$

where $x = r^2$ the density ρ_R is defined as follows.

Given the Young diagram we rotate it by $-3\pi/4$ radians. Then the diagram has $(k+$



(a) (b)

Figure 8.1—(a) A Maya diagram associated to a large Young diagram. The rightmost positive slope edge is mapped to the Fermi sea, while negative slope edges represent large gaps of unfilled states. (b) A sketch of the LLM geometry associated to the Maya diagram.

$1, k)$ edges of slopes ∓ 1 of lengths μ_i and κ_i for negative and positive slopes respectively. For example the length of the l^{th} block is $R_i = N\sum_i^{k+1-i}\kappa_i = N\alpha_{i-k-1}$. We color the edges with a negative slope black, and edges with positive slopes white and unfold edges of the diagram into an infinitely long colored strip as in 8.1a. This strip is to be identified with a radial slice of the LLM plane [152]. The variable r is taken to start at the right-most colored edge; this is $r = 0$ in the LLM plane, and the asymptotic region is taken to be towards the left of the strip. For every black region we have $\rho_R(r) = 1$, and $\rho_R(r) = 0$ for every white region. Substituting this into (4.13) will reproduce the integral expression in (4.12).

This quantity can be matched precisely to the formula found in [112] for the one point point function of a chiral operator computed using holographic renormalization:

$$\langle \mathcal{O}_{S^{k\pm k}} \rangle_{LLM} = \frac{1}{\sqrt{k}} \int dr\, r d\phi\, \rho(r)\, r^k e^{\pm ik\phi}. \tag{4.14}$$

For our background this quantity vanishes since there is a conserved $U(1)_R$ charge in the background. To match this to the expression above we take the uncharged combination $\mathcal{O}_{S^{k,+k}} + \mathcal{O}_{S^{k-k}}$ and integrate over ϕ. This would correspond to the contribution of a spherical harmonic $Y^{L/2} \sim (z + \bar{z} + y - \bar{y})^L$.

8.4.2 Off-Diagonal Structure Constant

To compute off-diagonal structure constants we need to change one of the representations R to another representation R' whose Young diagram is close to R. In this case most of the integrals will vanish, unless R and R' only differ at a single row $R_l - R'_l = k_l$. Since R has large blocks, this can only happen when $R_{l-1} > R_l$, meaning after an edge. The only non-zero integral over the y variables comes from the y_l associated to the row R'_l, so we only get one term:

$$\begin{aligned} C_{R\,R+k_l\,L} &= e^{-\Delta S_{R\,R'}} N^{-L/2} \frac{1^{L-k} + (-1)^{L-k}}{2\sqrt{L}} \frac{L!}{(\frac{L-k}{2})!(\frac{L+k}{2})!} \frac{\Gamma(N + R_l + k_l/2 - l + 1)}{\Gamma(N + R_l + k_l/2 - L/2 - l + 1)} \\ &\simeq e^{-\Delta S_{R\,R'}} \; \frac{1^{L-k} + (-1)^{L-k}}{2\sqrt{L}} \times \frac{L!}{(\frac{L-k}{2})!(\frac{L+k}{2})!} \; \alpha_l^{L/2}, \end{aligned} \tag{4.15}$$

where $\Delta S_{R\,R'}$ is the ratio of norms $\frac{\langle R|R\rangle}{\sqrt{\langle R'|R'\rangle \langle R|R\rangle}}$. This should be interpreted as the linear response to a fluctuation localized at the edge of a particular Fermi surface within the LLM geometry.

8.4.3 Comparing to the eigenvalue picture: fixing the number of rows

The method outline in this section allows us to compute expectation values of light operators in a particularly radially symmetric bubbling geometry. A natural question to address is how this connects to the eigenvalue coherent state picture. In other words, given a particular configuration of droplets, how can we determine which radially symmetry modes make up the state. Clearly a single droplet made out of p giant gravitons can only be made out of Young diagrams with p rows. This is because the overlap $\left\langle \tilde{\lambda} \middle| \lambda \right\rangle$ has a character expansion, and setting $N-p$ of the eigenvalues to zero makes characters associated to representations with more than p rows vanish. To project to a particular diagram made out of p rows we repeat the same trick where we integrate the remaining eigenvalues over a unit circle with an appropriate measure. After regrouping the integration variables we will end up with a pair integrals over $U(p)$:

$$\begin{aligned}\langle R', p|R, p\rangle &\propto \int d\mu(s^\dagger)d\mu(s)\, \det\left(\mathbb{I} - s^\dagger s\right)^{N-2p} \int dU dV\, S_R(U^\dagger) S_{R'}(V^\dagger) e^{N\mathrm{Tr}[UsVs^\dagger]} \\ &\propto \frac{\delta_{RR'}}{Dim_N(R)^2} \int d\mu(s^\dagger)d\mu(s)\, \det\left(\mathbb{I} - s^\dagger s\right)^{N-2p} S_R(s^\dagger s) \\ &\propto \frac{\delta_{RR'}}{Dim_N(R)^2} \int_{[0,1]^p} \prod_{i=1}^{p} dx_i\, \Delta_p(x)\, \left(1 - x_i\right)^{N-2p} x_i^{p+R_i-i}.\end{aligned} \tag{4.16}$$

For generic (non-rectangular) diagrams, the integral does not have a simple solution. For large diagrams we can use the saddle point approximation to find the density of eigenvalues x_i. The procedure to evaluate this class of integrals was outlined in [151] and also [158, 159, 150]. For a Young diagram with k large blocks with n_a rows, we split the variables x_i into k groups of size n_a, $x_i = (x_1^{(1)}, \ldots, x_{n_1}^{(1)}, x_1^{(2)}, \ldots, x_{n_2}^{(2)}, \ldots, x_1^{(k)}, \ldots, x_{n_k}^{(k)})$. Then we use the fact that the integration variables are invariant under permutations to rewrite

$$\begin{aligned}&\int \prod_{a=1}^{k} \prod_{i_a=1}^{n_a} dx_{i_a}^{(a)} \left(x_{i_a}^{(a)}\right)^{R_a - N_a} \left(x_{i_a}^{(a)}\right)^{n_a - i_a} \\ &\int \prod_{a=1}^{k} \prod_{i_a=1}^{n_a} dx_{i_a}^{(a)} \times \frac{1}{n_a!} \sum_{\pi \in S_{n_a}} \left(x_{i_a}^{(a)}\right)^{R_a - N_a} \left(x_{\pi(i_a)}^{(a)}\right)^{n_a - i_a} \\ &\int \prod_{a=1}^{k} \prod_{i_a=1}^{n_a} dx_{i_a}^{(a)} \left(x_{i_a}^{(a)}\right)^{R_a - N_a} \Delta_{n_a}(x^{(a)}),\end{aligned} \tag{4.17}$$

where N_a is the partial sum $\sum_{b\leq a} n_b$ and $N_1 = 0$. Putting it all together, we are left with a multi-cut matrix model:

$$\mathcal{Z}(R) = \int_{[0,1]^p} \prod_{a=1}^{k} \prod_{i_a}^{n_a} dx_{i_a}^{(a)}\, \Delta_{n_a}\left(x^{(a)}\right)^2 \left(1 - x_{i_a}^{(a)}\right)^{N-2p} \left(x_{i_a}^{(a)}\right)^{R_a - N_a} \times \prod_{c>b} \prod_{i_b, j_c} \left(x_{j_c}^{(c)} - x_{i_b}^{(b)}\right) \tag{4.18}$$

A matrix model quite similar to this one studied in [151] for computing vevs of giant Wilson loops in Chern-Simons theory on Lens spaces $S^3/\mathbb{Z}_p$. To make the analogy more precise we can change variables to $x_i = e^{-u_i}$ after which the partition function becomes:

$$\begin{aligned}\mathcal{Z}(R) &= \int \prod_{a=1}^{k} \prod_i du_i^{(a)} \prod_{j<i} \left(2\sinh \frac{u_j^{(a)} - u_i^{(a)}}{2}\right)^2 \left(2 \sinh \frac{u_i^{(a)}}{2}\right)^{N-2p} \\ &\times e^{-(L_a + N/2 - p)\, u_i^{(a)}} \prod_{b>a} \prod_{i,j} \left(2\sinh \frac{u_j^{(b)} - u_i^{(a)}}{2}\right). \\ L_a &= R_a - \frac{1}{2}\sum_{b=1}^{a-1} n_b + \frac{1}{2}\sum_{b>a} n_b\end{aligned} \tag{4.19}$$

The only difference between this and the matrix model studied in [151] is that the Gaussian term is replaced by $(1 - e^{-u_i^{(a)}})^{N-2p}$, and u differs by a sign. In this representation the saddle point analysis is quite straightforward. The equations of motion for the eigenvalues $u_i^{(a)}$ take the form:

$$\frac{1}{N} \sum_{j\neq i} \coth \frac{u_j^{(a)} - u_i^{(a)}}{2} + \frac{1}{2N} \sum_{b\neq a,\, l} \coth \frac{u_l^{(b)} - u_i^{(a)}}{2} = \frac{(1 - 2p/N)}{2} \left(\coth \frac{u_i^a}{2} - 1\right) - L_a/N. \tag{4.20}$$

The resolvents for this type of problem can be taken to be of the form $\omega^{(a)}(z) = \frac{1}{N}\sum_i \coth \frac{z - u_i^{(a)}}{2}$, and the total resolvent is $\omega(z) = \sum_{b=1}^{k} \omega^{(b)}(z)$. At large N the eigenvalues condense into k branch cuts and the saddle point equation on the a^{th} cut reads

$$\omega^{(a)}(z+i\,0)+\omega^{(a)}(z-i\,0)+\sum_{b\neq a}\omega^{(b)}(z)=\frac{(1-2p/N)}{2}\left(\coth\frac{u_i^{(a)}}{2}-1\right)-L_a/N, \quad (4.21)$$

or equivalently

$$\omega(z\pm i\,0)=-\omega_{\mp}^{(a)}(z)+\frac{(1-2p/N)}{2}\left(\coth\frac{z}{2}-1\right)-L_a/N. \quad (4.22)$$

This equation defines a Riemann-Hilbert problem for the total resolvent $\omega(z)$. Notice that for large enough values of z, the potential rapidly approches a constant value of $-L_a/N$, and for small values of z there is exponential barrier pushing the eigenvalues away from $z=0$. This means that the eigenvalues will sit far from $z=0$, and will be uniformly distributed along the cut. To solve (4.22) we need to find a function of the resolvents that is regular for $\mathrm{Re}(z)>0$. Following [151] we can define a set of complex variables

$$\begin{aligned} X_0 &= e^{-(1-2p/N)\left(\coth\frac{z}{2}-1\right)/2}e^{\omega}=W\,e^{\omega} \\ X_a &= e^{-L_a/N}e^{-\omega_a}; \end{aligned} \quad (4.23)$$

then the equations (4.22) are equivalent to a monodromy condition for the X_I as we around each of the cuts. The solution to this problem is given by a polynomial of degree $(k+1)$, $f(Y,W)$, with roots at the X_I and the spectral curve is the zero locus of this polynomial. The precise eigenvalue distribution is then found by solving $f(Y,W)=0$ for Y and taking the discontinuity of the resolvent $\omega\propto\log Y$ at each cut.

In the thermodynamic limit, the branch cuts become widely separated and the saddle point equations simply significantly. Substituting in the ansatz $u_j^{(b)}-u_i^{(a)}\gg 1$ in (4.20) turn the terms of the form $\coth\frac{u_j^{(b)}-u_i^{(a)}}{2}$ into a constant. The interaction term between eigenvalues in the same cut becomes a step function for large $u^{(a)}$ beginning at the first eigenvalue on the cut which given an equation for each of the eigenvalues on a given cut:

$$\begin{aligned}\frac{(1-2p/N)}{2}\left(\coth\frac{u_k^a}{2}-1\right) &= \frac{1}{N}\left(L_a-(n^{(a)}-k)+\frac{1}{2}\sum_{b>a}n_b\right)\Rightarrow \\ 2g_s\left(\frac{N-2p}{1-x_k}\right) &= g_s\left[R_a-\sum_{b=1}^{a}n_b+\sum_{c>a}n_c+\left(2k+\sum_{b>a}n_b+R_a\right)+2\right].\end{aligned} \tag{4.24}$$

Clearly the eigenvalues $\lambda_i = 1/(1-x_i)$ are uniformly distributed along each cut, and they are quantized in units of $2g_s = 1/(N-2p)$. So again we see that this is indeed the correct parameter controlling the fluctuations around the saddle. These equations are also exactly analogous to the saddle point equations for the single giant graviton case,

$$\frac{1}{1-x_k} = |\lambda_k|^2 = r_k^2 \tag{4.25}$$

and the distribution $\rho_R(r)$ for r_k is the same distribution we found before.

Computing correlators: reduced matrix elements

To compute correlation functions of single trace operators between a set of states $|R,p\rangle\,,|R',p\rangle$ we need to perform a series of elaborate coordinate substitutions to simplify the matrix integral calculations. First, the states are obtained by a certain projection of the reduced coherent generating function

$$\begin{aligned}|R,p\rangle &\propto \int_{U(p)} dU_1 \int dV_1 dV_1^\dagger\ e^{\mathrm{tr}_p[U_1V_1^\dagger ZV_1]}S_R(U_1^\dagger)\,|0\rangle \\ \langle R',p| &\propto \int_{U(p)} dU_2 \int dV_2 dV_2^\dagger\ \langle 0|\, e^{\mathrm{tr}_p[U_2V_2^\dagger \bar{Z}V_2^\dagger]}S_{R'}(U_2^\dagger).\end{aligned} \tag{4.26}$$

If we insert a half BPS single trace operator between these states, the scalar fields Z and $\bar{Z}$ inside the trace are traded by complex valued matrices:

$$\begin{aligned} Z &\to V_2 U_2 V_2^\dagger \\ \bar{Z} &\to V_1 U_1 V_1^\dagger . \end{aligned} \tag{4.27}$$

We can then perform a color-flavor transformation to express any trace of a power of $Z + \bar{Z}$ (over N color indices) into a trace over $2p \times 2p$ matrices. Then after performing all of the contractions between the two exponentials we end up with a pair of unitary integrals over $U_{1,2}$ and an integral over the $p \times p$ 'radial' matrix $\sigma = V_1^\dagger V_2$.

$$\begin{aligned} &\langle R', p| \operatorname{Tr}\left[\left(Z + \bar{Z} + Y - \bar{Y}\right)^L\right] |R, p\rangle = \\ &\int dU_1 dU_2 \, d\sigma^\dagger d\sigma \Bigg\{ \det\left(1 - \sigma^\dagger \sigma\right)^{N-2p} e^{\operatorname{tr}_p\left[U_1 \sigma U_2 \sigma^\dagger\right]} S_R(U_1^\dagger) S_R(U_2^\dagger) \\ &\times \operatorname{tr}_{2p}\left[\begin{pmatrix} U_1 & \sqrt{U_1}\,\sigma\sqrt{U_2} \\ \sqrt{U_2}\,\sigma^\dagger\sqrt{U_1} & U_2 \end{pmatrix}^L\right]\Bigg\}, \end{aligned} \tag{4.28}$$

and here $\sqrt{U_{1,2}}$ refers to the unitary matrix whose eigenvalues are square roots of the eigenvalues of $U_{1,2}$. To solve this integral at large N we will diagonalize $U_{1,2}$ and perform a singular value decomposition of σ.

$$\begin{aligned} \sigma &= U_L \, s \, U_R \\ U_1 &= \mathcal{U}^\dagger e^{i\boldsymbol{\alpha}} \mathcal{U} \\ U_2 &= \mathcal{V}^\dagger e^{i\boldsymbol{\beta}} \mathcal{V} . \end{aligned} \tag{4.29}$$

Although this at first glance looks daunting one should realize that the translational invariance of the Haar measure allows us to reabsorb most of the unitary integrals into the integration over U_L and U_R. This reduces the calculation to a pair of integrals involving simple complex exponential associated to the eigenvalues of $U_{1,2}$ which encode the angular dependence of the correlator, one integral over the radial eigenvalues, and a pair of difficult unitary integrals over $U_{L,R}$. First we address the integration over U_L and U_R. At large N we can perform a saddle point approximation for this integral; since the only term that is relevant for this is the exponential we are left with the task of finding the critical points of the following function:

$$S(U_L, U_R) = \mathrm{tr}_p \left[U_L^\dagger e^{i\boldsymbol{\alpha}}\, U_L\, s\, U_R\, e^{i\boldsymbol{\beta}} U_R^\dagger\, s^\dagger \right]. \tag{4.30}$$

The critical points of this function are given by the solutions to a pair of matrix equations

$$[s^\dagger U_L^\dagger e^{i\boldsymbol{\alpha}}\, U_L\, s\,, U_R\, e^{i\boldsymbol{\beta}} U_R^\dagger] = [s\, U_R\, e^{i\boldsymbol{\beta}} U_R^\dagger\, s^\dagger, U_L^\dagger e^{i\boldsymbol{\alpha}}\, U_L] = 0. \tag{4.31}$$

These equations essentially imply that these two pairs of matrices are simultaneously diagonalizable. For example, the first equation is unitarily equivalent to the condition that $e^{i\boldsymbol{\beta}}$ and $U_R^\dagger s^\dagger U_L^\dagger e^{i\boldsymbol{\alpha}}\, U_L\, s\, U_R$ are both diagonal in the same basis. The second equation gives a similar condition for $e^{i\boldsymbol{\alpha}}$ and $U_L\, s\, U_R\, e^{i\boldsymbol{\beta}} U_R^\dagger\, s^\dagger\, U_L^\dagger$. But since $e^{i\boldsymbol{\alpha}}$, $e^{i\boldsymbol{\beta}}$ and s are all diagonal the only way that this can be achieved is if $U_{L,R}$ are permutation matrices. This is clear because making the ansatz $U_R = U_\pi$ in the first equation for some permutation matrix U_π immediately forces U_L to be a permutation matrix and vice versa. This means that we have one critical point for every pair of permutations π, τ in S_p, which act on $e^{i\boldsymbol{\alpha}}$ and $e^{i\boldsymbol{\beta}}$ by permuting their eigenvalues independently from each other. To get the correct answer we should also compute the one-loop determinant around each of the saddles. This boils down to computing the Hessian of (4.30), which is the same for each each critical point up to a sign associated to the determinant of the permutation. To quadratic order $S(e^M, e^N)$ is

$$\begin{aligned} S(m,n) \simeq \mathrm{tr}_p[e^{i\boldsymbol{\alpha}}\, s\, e^{i\boldsymbol{\beta}}\, s^\dagger] - \frac{1}{2}\sum_{i,j}\Big\{ &(e^{i\alpha_i} - e^{i\alpha_j})(s_i s_i^*\, e^{i\beta_i} - s_j s_j^* e^{i\beta_j})|m_{ij}|^2 \\ &+ (e^{i\beta_i} - e^{i\beta_j})(s_i s_i^*\, e^{i\alpha_i} - s_j s_j^* e^{i\alpha_j})|n_{ij}|^2 \\ &- (e^{i\alpha_i} - e^{i\alpha_j})(e^{i\beta_i} - e^{i\beta_j}) s_i s_j^* \left(m_{ij} n_{ij}^* + n_{ij} m_{ij}^*\right) \Big\} + \ldots, \end{aligned} \tag{4.32}$$

so the one-loop factor becomes simply

$$\det\left(\partial_{ij}\partial_{kl} S(m,n)\right) = \Delta_p\left(e^{i\boldsymbol{\alpha}}\right) \Delta_p\left(e^{i\boldsymbol{\beta}}\right) \Delta_p\left(s^\dagger s\right) \Delta_p\left(e^{i(\boldsymbol{\alpha}+\boldsymbol{\beta})} s^\dagger s\right). \tag{4.33}$$

Each saddle point is weighted by $\det(\tau\pi)$, so after a coordinate transformation every term in the sum will give the same value. As before, the one-loop factor when combined with the denominators of the determinantal expressions for the Schur polynomials will cancel the Vandermonde determinants in the integration measure of α_i and β_i, which makes the integrals over the angular variables straightforward

$$\begin{aligned} &\langle R', p|\, \mathrm{Tr}\left[\left(Z + \bar{Z} + Y - \bar{Y}\right)^L\right] |R, p\rangle = \\ &\int \frac{d\alpha_i}{2\pi}\frac{d\beta_i}{2\pi} \int_{[-1,1]^p} ds_i \left\{ \frac{\Delta_p(s^\dagger s)}{\Delta_p(e^{i(\boldsymbol{\alpha}+\boldsymbol{\beta})} s^\dagger s)} \left(1 - |s_i|^2\right)^{N-2p} \exp\left(e^{i(\alpha_i+\beta_i)}|s_i|^2\right) \right. \\ &\left. \det_{l,k}\left(e^{-i(p+R_l-l)\alpha_k}\right) \det_{l,k}\left(e^{-i(p+R'_l-l)\beta_k}\right) \times \binom{L}{L/2} \mathrm{tr}_p\left[(s^\dagger s)^{L/2}\, e^{i(\frac{\boldsymbol{\alpha}+\boldsymbol{\beta}}{2})L}\right] \right\}. \end{aligned} \tag{4.34}$$

For light operators and diagonal structure constants we can simply interchange $(Z+\bar{Z})^L$ with $\binom{L}{L/2}(Z\bar{Z})^{L/2}$, which is simpler to work with. The integration over $\alpha_i - \beta_i$ will again force the pair of sums over permutations coming from the two determinants to collapse to a single sum, and the integral over $\alpha_i + \beta_i$ cancels the factor of $(s_k s_k^*)^{L/2}$. Finally the integral over s_i factors out completely and the correlator comes only from the integrals over the angular variables:

$$\begin{aligned} &\langle R, p| \operatorname{Tr}\left[\left(Z+\bar{Z}+Y-\bar{Y}\right)^L\right]|R,p\rangle \\ &= \mathcal{Z}(R)\sum_i \frac{1^L+(-1)^L}{2} \times \left\{ \binom{L}{L/2} N^{-L/2} \sum_{i=1}^{p} \frac{\Gamma(p+R_i-i+1)}{\Gamma(p+R_i-i-L/2+1)} \right\}. \end{aligned} \tag{4.35}$$

This is the same answer as (4.13) except that we are missing one term coming from the droplet made out of $N-p$ spectator branes. The reason that we miss this term in this calculation is that the reduced generating functions project out all Young diagrams with more than p rows, which essentially freezes $N-p$ of the branes that make up the background. This agrees with our interpretation of the off-diagonal structure constant as the linear response of a particular Fermi sea level. Since we projected out all the contributions from Young diagrams with more than p rows, ripples of the first Fermi sea surface are projected out from this computation.

8.5 Discussion and Future Directions

In this chapter we introduced techniques for dealing with large BPS operators in $\mathcal{N}=4$ SYM theory and revisited the computation of one-point functions in the background of states corresponding to the bubbling geometries of [16]. Our method is based on the semi-classical techniques introduced in [11, 9, 10] and further developed in [144, 13, 24, 143, 98] and provides an independent derivation of the results of [147]. One advantage of our methods is that they do not rely on diagonalizing any the field operators of the theory, or performing any kind of consistent truncation of the model, which makes weak coupling computation of non-BPS observables possible. We also fleshed out the relation between coherent states and characters by providing an explicit integral transform between both pictures. This gives somewhat complementary methods that can be used to compute correlators of operators describing somewhat generic LLM backgrounds. Although our results are expected and perhaps unsurprising, we hope that the techniques developed here serve as a starting point for performing a systematic large N expansion for large operators.

One question worth asking is whether we learned anything about the statistics of (BPS) OPE coefficients in $\mathcal{N} = 4$ SYM. We are hesitant to claim that our results display any kind of chaotic behavior predicted by the eigenstate thermalization hypothesis [160], since half-BPS operators in $\mathcal{N} = 4$ SYM are completely captured by a free theory. We instead attribute the appearance of random matrix statistics to the averaging necessary to describe large semi-classical states, and to effects due to the large charge limit. In fact our computations suggest that true structure constants (as opposed to fixed charged three point functions) only receive contributions from a single term, and that the would be distribution of eigenvalues in these cases are essentially constant distributions. On the other hand, correlators involving large operators that break the R-symmetry spontaneously but with fixed charge are generically given by complicated averages. In the large N limit with the charges of the operators scaling with N^2 this leads to the appearance of random matrix behavior in the OPE coefficients. Interestingly similar distributions were found to emerge from the large charge limit of extremal correlators in rank one SCFTs [161].

We conclude by commenting on some future directions of work.

Holographic computation of off-diagonal three-point functions

Our methods allow us to make predictions for the value of the off-diagonal structure constant between two LLM geometries. This quantity is only non-zero when the two geometries differ by a small fluctuations. It would be ideal to try develop semi-classical techniques for computing such things using the gravitational path integral. Since states are highly degenerate (there is one for each Young diagram!), we expect that there is a set of commuting charges that differentiates between different states in gravity. These charges would correspond to some kind of higher spin asymptotic symmetries of LLM solutions. Then, computing off-diagonal three-point functions would correspond to dressing the semi-classical saddle by a wavefunction charged with the appropriate asymptotic charge as suggested by [115]. This would be a very simple toy model of the soft hair proposal [162] in the sense that one is able to probe very precise details of the interior of the geometry from simple boundary manipulations. Such a technique would also have to go far beyond our current methods of holographic renormalization [163]. We take the fact that the formulas for one-point functions in LLM backgrounds are quite simple as indicative of the existence of a different method for computing holographic one point functions in them. Perhaps a careful WKB analysis [164] might be able to reproduce the result for operators of arbitrary charge without the need for a non-linear Kaluza-Klein reduction.

Three Heavy Operators

One obvious extension of our calculations would be to consider the structure constants of three really heavy operators half-BPS operators. There are in principle no conceptual obstructions for performing such computations, since we only have to include an additional exponential generating function. Technically speaking the saddle point analysis is more involved and we expect the form of the structure constants to be much richer. More precisely many of the simplifications that occur for the one-point functions of single trace operators are basically due to the fact that for small operators the term in the tensor decomposition only show up with multiplicity one. This is why we are able to reduce the number of sums in (4.34). For example one can insert a particular Schur polynomial of $Z + \bar{Z} + Y - \bar{Y}$ between two coherent states. In this case we can commute the coherent states past the Schur polynomial resulting in a matrix model generalizing (3.9). Obtaining an approximate formula even for extremal three point functions of very large operators would be a rather non-trivial since would encode information about the statistics of Littlewood-Richardson coefficients. On the other hand, such correlators predict the existence of supergravity solutions which interpolate between different LLM geometries [9]. Since these types of three point functions are protected, we expect that one should be able to match both results precisely in the large N limit, so obtaining an approximate form for such correlators might give some intuition about how to construct such geometries. Perhaps a simpler problem is to consider correlators of two LLM geometries and a giant graviton, or between three giant gravitons. These quantities give predictions for giant graviton nucleation amplitudes.

Extremal Correlators

Another immediate generalization would be to study higher point extremal correlators involving various combinations of single trace, giant graviton, and LLM geometries. These correlators are also protected, so we can hope to be able to perform a holographic check for these quantities as well. In practice this will likely involve a very careful group averaging procedure to be able to overcome any possible ambiguities that often arise in extremal correlators. For instance one can conceivably study correlators in which many branes nucleate into a bubbling geometry, or where large droplets all fuse into a single droplet, or a large droplet splits into smaller ones. Such processes are somewhat reminiscent of baby universe creation and annihilation processes. Whether such a geometric picture can be realized from the bulk point of view is unclear.

Worldsheet and spin chain interpretation of one-point functions in LLM background

One particularly puzzling issue is to interpret the result of the computation of a three point function of a non-protected single trace primary and two very large half-BPS operators. For large operators of dimension of order N the interpretation as a worldsheet g-function was advocated in [9], and various checks were performed. This makes sense since operators of dimension of order N can change the boundary conditions of the string worldsheet in the bulk. For operators of dimension of order N^2 this interpretation is inadequate, since we expect that the correct description is instead in terms of a string moving in an LLM background, rather than simply an open string ending on a stack of branes. From the worldsheet point of view one would expect that the worldsheet CFT flows as the background is deformed. Understanding what this would mean from the point of view of the spin chain picture would be quite interesting.

$\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ BPS operators

A more long term goal would be to understand the systematics of less supersymmetric BPS operators in the $\mathcal{N} = 4$ SYM theory. In principle introducing additional scalar matrices in the exponentials gives a way of generating $\frac{1}{4}$ and $\frac{1}{8}$ BPS [11, 10, 165]. This was carried out for rank two $\frac{1}{4}$ BPS operators in [165]. Additionally, the generating functions introduced in [11] contain all bosonic $\frac{1}{8}$ operators. The main difficulty lies in constructing explicit expressions for restricted Schur polynomials with which one can project to particular BPS operators of the weakly coupled theory. In other words, finding the integration rules for the analog of the Schur polynomials in (4.2) with multiple matrices would be a big step in this program. This problem is even more stark for $\frac{1}{16}$ BPS operators where one seems needs to need to introduce an infinite number of matrices corresponding to covariant derivatives of the scalar fields. In analogy to the construction in [11], one can generate $\frac{1}{16}$ BPS states by exponentiating the so-called $\frac{1}{16}$ BPS letter introduced in [166]. Understanding precisely what kind of excitations are relevant for studying supersymmetric black holes would be a first step towards a boundary derivation of the results of [167].

Twisted Holography

A natural setting where our techniques can be readily applied is in the context of the Twisted Holography program. For instance sphere giant gravitons and non-conformal vacua have already been studied [19, 168] and their geometric picture is quite clear. Extending this to include the analog of AdS giant gravitons and more generic Schur polynomial operators seems like a straightforward task. It would be nice to develop the more conventional view of giant gravitons wrapping compact cycles on the deformed conifold $SL(2,\mathbb{C})$ in the B-model by developing a global version of Twisted Holography, in analogy to global AdS holography. This should be related to studying the chiral algebra on $\mathbb{P}^1$ instead of $\mathbb{C}$. The point of this exercise is that when we insert an operator associated to a Young diagram R, there is a well-studied recipe for producing a spectral curve, and hence a bubbling Calabi-Yau manifold [151]. This story is well-understood for the A-model on the deformed conifold, so it is not implausible that a similar story exists for the B-model. This might help bridge the conceptual gap about the relation between the full physical holography and the twisted version.

Semiclassics and Large N

Inevitably one will need to include g^2N corrections in a systematic way when dealing with non-protected operators. A good place to start would be consider simpler models where the techniques developed in [169, 170, 171] can be combined with the large N expansion. Performing a near BPS expansion seems like a natural starting point since one expects the coupling constant to be enhanced by additional kinematic effects [5] making a reliable extrapolation to strong coupling a possibility for certain observables. A good target would to understand the correlation functions of more complicated 'baryonic' operators in the Wilson-Fisher fixed point of the $O(N)$ model [172] in the large N limit;

$$\mathcal{B} \sim S_{i_1,\ldots,i_N}\, \phi^{i_1} \ldots \phi^{i_N} \tag{5.1}$$

where S is a symmetric tensor. The simplest example of these operators where studied in [172], although it would be interesting to extend these kinds of results to operators associated to other representations of $O(N)$. These kind of operators can also be exponentiated with the help of real gaussian integrals over Grassmannians, so the main difficulty would be to study the gap equations in the presence of these baryons.

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