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TUTTE SHORT EXACT SEQUENCES OF GRAPHS

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Abstract. We associate two modules, the G-parking critical module and the toppling critical module, to an undirected connected graph G. The G-parking critical module and the toppling critical module are canonical modules (with suitable twists) of quotient rings of the well-studied G-parking function ideal and the toppling ideal, respectively. For each critical module, we establish a Tutte-like short exact sequence relating the modules associated to G, an edge contraction G/e and an edge deletion $G \setminus e$ (e is a non-bridge). We obtain purely combinatorial consequences of Tutte short exact sequences. For instance, we reprove a theorem of Merino that the critical polynomial of a graph is an evaluation of its Tutte polynomial, and relate the vanishing of certain combinatorial invariants (the number of acyclic orientations on connected partition graphs satisfying a unique sink property) of G/e to the equality of the corresponding invariants of G and $G \setminus e$.

Keywords. Tutte polynomials, chip firing games, toppling ideals, G-parking function ideals, canonical modules

Mathematics Subject Classifications. 13D02, 05E40

1. Introduction

Let G be an undirected, connected, multigraph on n-vertices labelled $v_1, \ldots, v_n$ and with ℓ loops. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in $\mathbb{K}$. The toppling ideal I_G of R is a binomial ideal that encodes chip firing equivalence on G [CRS00] and the G-parking function ideal M_G is a monomial initial ideal of I_G that closely mirrors the properties of I_G [PS04]. The ideals I_G and M_G (and their quotient rings) have received significant attention recently, in part due to their connections with tropical geometry. Combinatorial commutative algebraic aspects of R/I_G and R/M_G such as their minimal

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free resolutions in terms of the underlying graph G have been studied from several perspectives [PPW11, MS13, MSW15, DS14, MS16].

Both R/I_G and R/M_G are Cohen–Macaulay (of depth and Krull dimension one) and hence, have associated canonical modules (also known as dualising modules) [BH98, Part b, Proposition 3.6.12]. We refer to the canonical modules of R/I_G and R/M_G (both twisted by the number of loops of G) as the *toppling critical module* and the *G-parking critical module*, respectively. We denote the toppling critical module and the G-parking critical module by CTopp_G and CPark_G, respectively. In this article, we posit that the critical modules behave better compared to the corresponding quotient rings in certain contexts. Specifically, we construct short exact sequences relating the critical modules of G, its contraction G/e and deletion $G \setminus e$ by an edge e that is not a bridge [1]. Taking cue from the deletion-contraction sequence that characterises the Tutte polynomial of a graph, we refer to these sequences as *Tutte short exact sequences*.

We present purely combinatorial consequences of Tutte short exact sequences. For instance, as a corollary we obtain an algebraic proof of a theorem of Merino [Mer97] that the critical polynomial of a graph is an evaluation of its Tutte polynomial. This follows from the additivity of the Hilbert series of the modules involved in one of the Tutte short exact sequences, namely the G-parking Tutte short exact sequence. By considering associated long exact sequences of Tor, we relate the vanishing of certain combinatorial invariants of G/e to the equality of corresponding invariants of G and $G \setminus e$. These combinatorial invariants are the number of acyclic orientations satisfying a unique sink property on certain graphs derived from G called *connected partition graphs* [MSW15, Pages 2854–2855]. We also note a deletion-contraction formula for certain numbers associated to G called *alternating numbers* that are alternating sums of these combinatorial invariants.

The construction of the Tutte short exact sequences and the corresponding proofs involve a delicate interplay between the algebraic structure of the critical modules and the combinatorial structure of the graph, mainly its acyclic orientations. In the following, we state our main theorems concerning Tutte short exact sequences. Before this, we clarify one crucial point about contraction and deletion of the edge e.

The notions G/e, $G \setminus e$ and $G/(v_i, v_j)$: Suppose that there are $m_e \geqslant 1$ edges between v_1 and v_2. By G/e, we mean the graph obtained from G by contracting the vertices v_1 and v_2 to the vertex $v_{1,2}$ and with $m_e - 1$ loops on the vertex $v_{1,2}$. By $G \setminus e$, we mean the graph obtained from G by deleting the edge e and retaining all the other $m_e - 1$ edges parallel to e. On the other hand, by $G/(v_i, v_j)$ for a pair of distinct, adjacent vertices (v_i, v_j) we mean the graph obtained by contracting every edge between v_i and v_j.

1.1. Tutte short exact sequences

Let e be an edge between the vertices v_1 and v_2 that is not a bridge. Let R_e be the polynomial ring $\mathbb{K}[x_{1,2}, x_3, \ldots, x_n]$ in $(n-1)$-variables with coefficients in $\mathbb{K}$ so that its variables are naturally in correspondence with the vertices of G/e.

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[1]This condition ensures that $G \setminus e$ remains connected and is needed to guarantee that resulting coordinate ring retains some basic properties. For instance, if G is not connected, then the quotient ring of the toppling ideal defined analogously does not have Krull dimension one.

G**-parking Tutte short exact sequence:**   We construct a short exact sequence relating the G-parking critical modules of G, its contraction G/e and deletion $G \setminus e$ with respect to the edge e. By definition, the G-parking critical modules of G and $G \setminus e$ are R-modules, whereas the G-parking critical module of G/e is an R_e-module. We start by realising CPark_G and $\mathrm{CPark}_{G \setminus e}$ as R_e-modules. For this, we consider the linear form $L := x_1 - x_2$ and note that $R_e \cong R/\langle L \rangle$ via a map between R and R_e that takes x_1 and x_2 to $x_{1,2}$ and x_i to itself for all $i \neq 1, 2$. This isomorphism realises R_e as an R-module. We consider the tensor products $\mathrm{CPark}_G \otimes_R R_e$ and $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ as R_e-modules. We define R_e-module maps $\psi_0 : \mathrm{CPark}_{G/e} \to \mathrm{CPark}_G \otimes_R R_e$ and $\phi_0 : \mathrm{CPark}_G \otimes_R R_e \to \mathrm{CPark}_{G \setminus e} \otimes_R R_e$. We denote the map $\mathrm{CPark}_{G/e}/\ker(\psi_0) \to \mathrm{CPark}_G \otimes_R R_e$ induced by ψ_0 also by ψ_0. We show that ψ_0 and ϕ_0 fit into a short exact sequence. More precisely,

Theorem 1.1 (G-parking Tutte short exact sequence). *Let G be an undirected connected multigraph (possibly with loops) with at least three vertices. Let e be an edge between the vertices v_1 and v_2 that is not a bridge. The kernel of the map ψ_0 is equal to $x_{1,2} \cdot \mathrm{CPark}_{G/e}$ and the following sequence of R_e-modules:*

$$0 \to \mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e}) \xrightarrow{\psi_0} \mathrm{CPark}_G \otimes_R R_e \xrightarrow{\phi_0} \mathrm{CPark}_{G \setminus e} \otimes_R R_e \to 0$$

is a short exact sequence of graded R_e-modules.

Toppling Tutte short exact sequence: The toppling critical module $\mathrm{CTopp}_{G/e}$ is by definition an R_e-module (rather than an R-module). In contrast, CTopp_G and $\mathrm{CTopp}_{G \setminus e}$ are by definition R-modules. We start by realising CTopp_G and $\mathrm{CTopp}_{G \setminus e}$ as R_e-modules. For this, we realize R_e as an R-module via the same isomorphism $R_e \cong R/\langle L \rangle$ as in the G-parking case and regard $\mathrm{CTopp}_G \otimes_R R_e$ and $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$ as R_e-modules. Hence, $\mathrm{CTopp}_G \otimes_R R_e$ and $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$ are R_e-modules. We define R_e-module maps $\psi_1 : \mathrm{CTopp}_{G/e} \to \mathrm{CTopp}_G \otimes_R R_e$ and $\phi_1 : \mathrm{CTopp}_G \otimes_R R_e \to \mathrm{CTopp}_{G \setminus e} \otimes_R R_e$. We also denote by ψ_1, the injective map $\mathrm{CTopp}_{G/e}/\ker(\psi_1) \to \mathrm{CTopp}_G \otimes_R R_e$ induced by ψ_1.

Theorem 1.2 (Toppling Tutte short exact sequence). *Let $\mathbb{K}$ be a field of characteristic two. Let G be an undirected connected multigraph (possibly with loops) with at least three vertices. Let e be an edge in G between v_1 and v_2 that is not a bridge. The following sequence of R_e-modules:*

$$0 \to \mathrm{CTopp}_{G/e}/\ker(\psi_1) \xrightarrow{\psi_1} \mathrm{CTopp}_G \otimes_R R_e \xrightarrow{\phi_1} \mathrm{CTopp}_{G \setminus e} \otimes_R R_e \to 0 \qquad (1.1)$$

is a short exact sequence of graded R_e-modules.

Remark 1.3. The G-parking Tutte short exact sequence is not split exact in general. To see this, suppose that e has parallel edges then both $\mathrm{CPark}_G \otimes_R R_e$ and $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ have the same number of minimal generators. If the corresponding Tutte short exact sequence was split exact, then we would have $\beta_0(\mathrm{CPark}_G \otimes_R R_e) = \beta_0(\mathrm{CPark}_{G \setminus e} \otimes_R R_e) + \beta_0(\mathrm{CPark}_{G/e})$ which is not true. We do not know whether the toppling short exact sequence is split exact.

Remark 1.4. Note that unlike the case of the G-parking critical module, $(x_1 - x_2)$ is never a non-zero divisor of CTopp_G for any connected graph G. This can be seen by showing the equivalent

property that $(x_1 - x_2)$ is a zero divisor of R/I_G which in turn follows from the fact that G has a principal divisor, see [BN07, Page 768] for the definition, of the form $d \cdot (v_1) - d \cdot (v_2)$ for some positive integer d. The kernel of ψ_1 is also in general more complicated in this case (see the last line of Example 2.2): in general, it only strictly contains $x_{1,2} \cdot \mathrm{CTopp}_{G/e}$.

Remark 1.5. We expect that Theorem 1.2 does not require characteristic two and believe that it can be generalised to arbitrary ground fields. We rely on characteristic two in, for instance, the proof of Proposition 5.7.

1.2. Motivation and applications

Two sources of motivation for the Tutte short exact sequence are i. Merino's theorem [Mer97] and its connection to Stanley's O-sequence conjecture [Mer01], ii. divisor theory on graphs [BN07].

Merino's theorem states that the generating function of the critical configurations of G is an evaluation of the Tutte polynomial at $(1, t)$. The first observation that relates the critical modules to Merino's theorem is that their Hilbert series are both equal to $P_G(t)/(1 - t)$, where $P_G(t)$ is the generating function of the critical configurations of G (this is implicit in [MS13], also see Remark 6.4). This leads to the question of whether Merino's theorem can be enriched into a short exact sequence of critical modules. Merino's theorem can then be recovered from this short exact sequence from the fact that the Hilbert series is additive in short exact sequences. Such a short exact sequence might then allow the possibility of obtaining further combinatorial results by, for instance, considering the associated long exact sequence of Tor, Ext and other derived functors. The G-parking Tutte short exact sequence is such an enrichment and can be viewed as a categorification of Merino's theorem. By studying the associated long exact sequence in Tor, we relate certain combinatorial invariants of G/e to those of G and $G \backslash e$. We refer to [HR05] and [JR06] for a categorification of the chromatic polynomial and the Tutte polynomial of a graph, respectively. These works seem to be a different flavour from the current work.

Merino's theorem is a key ingredient in the proof of Stanley's O-sequence conjecture for co-graphic matroids [Mer01]. Stanley's conjecture is still open for arbitrary matroids. We raise the question of exploring generalisations of the main results of this paper to matroids as a possible approach to Stanley's conjecture.

Merino's theorem via Tutte short exact sequences: As an application of the G-parking Tutte short exact sequence, we deduce the following version of Merino's theorem as a corollary to Theorem 1.1. Recall that the K-polynomial of a finitely generated graded module over the (graded) polynomial ring is the numerator of the Hilbert series expressed as a rational function in reduced form.

Theorem 1.6 (Merino's theorem)**.** *The K-polynomial of CPark_G is the Tutte evaluation $T_G(1, t)$, where $T_G(x, y)$ is the Tutte polynomial of G.*

Next, we note a deletion-contraction formula for alternating sums of the graded Betti numbers $\beta_{i,j}$ which is an immediate consequence of Merino's theorem but does not seem to appear in literature.

A deletion-contraction formula for alternating numbers: For an integer k, let $\mathcal{A}_k = \sum_i (-1)^i \beta_{i,k}$ be the k-th alternating number of H. We have the following deletion-contraction formula for the numbers $\mathcal{A}_k$:

Proposition 1.7 (Deletion-contraction for alternating numbers)**.** *The numbers $\mathcal{A}_k(G)$ satisfy the following formula:*

$$\mathcal{A}_k(G) + \mathcal{A}_{k-1}(G/e) = \mathcal{A}_k(G/e) + \mathcal{A}_k(G \setminus e). \tag{1.2}$$

Example 1.8. Suppose G is a triangle with vertices v_1, v_2 and v_3 and let $e = (v_1, v_2)$. The associated numbers are the following.

$$\mathcal{A}_k(G) = \mathcal{A}_k(G/e) = \mathcal{A}_k(G \setminus e) = 0 \text{ for } k < 0.$$
$$\mathcal{A}_0(G) = 2, \mathcal{A}_1(G) = -3, \mathcal{A}_2(G) = 0, \mathcal{A}_3(G) = 1, \ \mathcal{A}_k(G) = 0 \text{ for } k \geqslant 4,.$$
$$\mathcal{A}_0(G/e) = 1, \mathcal{A}_1(G/e) = 0, \mathcal{A}_2(G/e) = -1, \ \mathcal{A}_k(G/e) = 0 \text{ for } k \geqslant 3.$$
$$\mathcal{A}_0(G \setminus e) = 1, \mathcal{A}_1(G \setminus e) = -2, \mathcal{A}_2(G \setminus e) = 1, \ \mathcal{A}_k(G \setminus e) = 0 \text{ for } k \geqslant 3.$$

Note that Formula (1.2) is satisfied for various values of k.

Note that $\mathcal{A}_0(H)$ is the number of acyclic orientations on H with a unique sink at v_2 and $\mathcal{A}_{-1}(H) = 0$. Hence, as a corollary we obtain the familiar formula:

$$\mathcal{A}_0(G) = \mathcal{A}_0(G/e) + \mathcal{A}_0(G \setminus e).$$

Equality of Betti numbers of G and $G \setminus e$ in terms of vanishing of Betti numbers of G/e: Let H be an undirected, connected, multigraph with n vertices, m edges and ℓ loops. Following [MSW15, Page 2854], a connected i-partition of H is a partition $\Pi = \{V_1, \ldots, V_i\}$ of its vertex set of size i such that the subgraph induced by each subset is connected. The *connected partition graph* associated to this partition Π is the multigraph with Π as its vertex set and with $\hat{a}_{i,j}$ edges between V_i and V_j, where $a_{u,v}$ is the number of edges between vertices u and v of G and $\hat{a}_{i,j} = \sum_{u \in V_i, v \in V_j} a_{u,v}$. We define $\beta_{i,j+\ell}(H)$ to be the number of acyclic orientations on connected partition graphs of H of size $n-i$, with $m-j$ edges and with a unique sink at the partition containing v_2 (or any other fixed vertex). Note that from [MSW15] and the graded version of [BH98, Corollary 3.3.9], we know that these are the graded Betti numbers of both CPark_H and CTopp_H (see Proposition 6.5).

Theorem 1.9. *Let G be an undirected connected graph (with possible loops) and let e be an edge of G that is not a bridge. For any $(i, j) \in \mathbb{Z}^2$, if $\beta_{i,j}(G/e) = \beta_{i-1,j-1}(G/e) = \beta_{i-1,j}(G/e) = \beta_{i-2,j-1}(G/e) = 0$, then $\beta_{i,j}(G) = \beta_{i,j}(G \setminus e)$.*

Example 1.10. Suppose that G is a triangle with vertices v_1, v_2 and v_3 and let $e = (v_1, v_2)$. The Betti numbers are the following:

$$\beta_{0,0}(G) = 2, \beta_{1,1}(G) = 3, \beta_{2,3}(G) = 1,$$
$$\beta_{0,0}(G/e) = 1, \beta_{1,2}(G/e) = 1,$$
$$\beta_{0,0}(G \setminus e) = 1, \beta_{1,1}(G \setminus e) = 2, \beta_{2,2}(G \setminus e) = 1.$$

For $(i, j) = (2, 4)$ the hypothesis of Theorem 1.9 are all satisfied and we have $\beta_{2,4}(G) = \beta_{2,4}(G \setminus e) = 0$. We currently do not know of examples where the hypothesis of Theorem 1.9 are satisfied and $\beta_{i,j}(G) = \beta_{i,j}(G \setminus e) \neq 0$.

At the time of writing, we do not know of a combinatorial proof of Theorem 1.9.

Connections to divisor theory and related sequences: The toppling critical module has an interpretation in terms of divisor theory of graphs. This connection is implicit in [MS13]. The punchline is that the Hilbert coefficients of the toppling critical module CTopp_G count linear equivalence classes of divisors D whose rank of D is equal to the degree of D minus g, where g is the genus of the graph (recall that $g = m - n + 1$, where $m,\ n$ are the number of edges and vertices, respectively of G). It seems plausible that the toppling Tutte short exact sequence also has analogous combinatorial applications: one difficulty in this direction seems to be that the kernel of the map ψ_1 does not seem to have a simple description.

Short exact sequences in the same spirit as the Tutte short exact sequences have appeared in literature. For instance, [OT92, Proposition 3.4] construct a deletion-restriction short exact sequence of Orlik–Solomon algebras of (central) hyperplane arrangements. We leave the question of relating the deletion-restriction short exact sequence associated to the graphical arrangement to the Tutte short exact sequences in this paper as a topic for further work. In a related direction, Dochtermann and Sanyal [DS14] use the graphical hyperplane arrangement to compute the minimal free resolution of the G-parking function ideal. This work has been extended to the toppling ideal by Mohammadi and Shokrieh [MS16].

2. The maps and proof sketch

In this section, we describe the maps ψ_i, ϕ_i and sketch the proofs of Theorem 1.1 and Theorem 1.2. The maps arise naturally from the combinatorial interpretation of the minimal generators of the (toppling and G-parking) critical modules.

A key input to this is the combinatorial description of the minimal generators and the first syzygies, i.e., a generating set for the relations between the minimal generators, of the critical modules implicit in [MSW15]. We summarise this description here.

The minimal generators of CPark_G are in bijection with acyclic orientations on G with a unique sink at v_2. The minimal generators of CTopp_G are in bijection with equivalence classes of acyclic orientations on G defined as follows [BN07].

Given an acyclic orientation $\mathcal{A}$ on G, consider the divisor:

$$D_{\mathcal{A}} = \sum_v (\mathrm{outdeg}_{\mathcal{A}}(v) - 1)(v),$$

where $\mathrm{outdeg}_{\mathcal{A}}(v)$ is the outdegree of v with respect to the acyclic orientation $\mathcal{A}$. Define an equivalence class on the set of acyclic orientations on G by declaring two acyclic orientations as equivalent if their associated divisors are linearly equivalent [BN07, Section 1.6]. Given an acyclic orientation on $\mathcal{A}$, we denote its equivalence class by $[\mathcal{A}]$. Once a vertex v_2 say is fixed, $[\mathcal{A}]$ has a canonical representative: the acyclic orientation with a unique sink at v_2 that is equivalent

to $\mathcal{A}$. Such an acyclic orientation exists and is unique [BN07, Section 3.1]. Hence, the two critical modules have the same number of minimal generators. We refer to these generating sets as the *standard generating sets*. Furthermore, by the right exactness of the tensor product functor they induce a generating set on the R_e-modules $\mathrm{CPark}_G \otimes_R R_e$ and $\mathrm{CTopp}_G \otimes_R R_e$ that we also refer to as the *standard generating sets*.

The first syzygies of the critical modules have (minimal) generators that correspond to certain acyclic orientations on graphs obtained by contracting a pair of vertices that are connected by an edge. We refer to these as the *standard syzygies*. Also, by the right exactness of the tensor product functor, they induce a generating set of the first syzygies of the corresponding R_e-modules $\mathrm{CPark}_G \otimes_R R_e$ and $\mathrm{CTopp}_G \otimes_R R_e$ that we refer to by the same terminology. We refer to Subsection 3.3.2 for more details.

The maps ψ_0 and ϕ_0: We use the free presentation described above to define ψ_0 and ϕ_0. Let m_e be the multiplicity of the edge e. The map ψ_0 takes the minimal generator $\mathcal{A}$ on G/e corresponding to an acyclic orientation with a unique sink at $v_{1,2}$ to $x_{1,2}^{m_e-1} \mathcal{A}_{e^+} \in \mathrm{CPark}_G \otimes_R R_e$, where $\mathcal{A}_{e^+}$ is the minimal generator corresponding to the acyclic orientation obtained by further orienting e such that v_1 is the source of e. We identify this minimal generator with the corresponding acyclic orientation. Note that the resulting acyclic orientation also has a unique sink at v_2.

We turn to the definition of ϕ_0. We distinguish between two cases: $m_e = 1$ and $m_e > 1$.

Consider the case where $m_e = 1$. Suppose $\mathcal{A}'$ is an acyclic orientation on G with a unique sink at v_2, following [MSW15] we say that an edge of G is contractible on $\mathcal{A}'$ if the orientation $\mathcal{A}'/e$ induced by $\mathcal{A}'$ on G/e is acyclic. If e is not contractible on $\mathcal{A}'$, then v_1 must be a source of at least one edge on $\mathcal{A}'/e$ and hence, $\mathcal{A}' \setminus e$ has a unique sink at v_2. The map ϕ_0 is defined as follows:

$$\phi_0(\mathcal{A}') = \begin{cases} \mathcal{A}' \setminus e, & \text{if the edge } e \text{ is not contractible on } \mathcal{A}', \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $m_e > 1$. We define $\phi_0(\mathcal{A}') = \mathcal{A}' \setminus e$ for every standard generator $\mathcal{A}'$ of $\mathrm{CPark}_G \otimes_R R_e$. Note that apriori the maps ψ_0 and ϕ_0 are only candidate maps and their well-definedness needs further argumentation. We will carry this out in Section 4.

The maps ψ_1 and ϕ_1: Suppose that $\mathcal{A}$ is an acyclic orientation on G/e. Let $\mathcal{A}_{e^+}$ and $\mathcal{A}_{e^-}$ be acyclic orientations on G obtained by further orienting $e = (v_1, v_2)$ such that v_1 and v_2 is the source of e, respectively. If e is a simple edge, then the map ψ_1 takes the generator $[\mathcal{A}]$ of $\mathrm{CTopp}_{G/e}$ corresponding to the class of $\mathcal{A}$ to $[\mathcal{A}_{e^+}] + [\mathcal{A}_{e^-}]$ in $\mathrm{CTopp}_G \otimes_R R_e$. More generally, if e is an edge of multiplicity m_e, then the map ψ_1 takes the generator $[\mathcal{A}]$ of $\mathrm{CTopp}_{G/e} \otimes_R R_e$ to $x_{1,2}^{m_e-1}[\mathcal{A}_{e^+}] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-}]$ in $\mathrm{CTopp}_G \otimes_R R_e$.

Suppose that $\mathcal{A}'$ is an acyclic orientation on G, let $\mathcal{A}' \setminus e$ be the acyclic orientation on $G \setminus e$ induced by $\mathcal{A}'$ i.e., by deleting the edge e. The map ϕ_1 takes the generator $[\mathcal{A}']$ of $\mathrm{CTopp}_G \otimes_R R_e$ to $[\mathcal{A}' \setminus e]$ in $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$.

Note that the fact that the maps ψ_1 and ϕ_1 are well-defined requires proof. The proof of well-definedness consists of two parts: i. showing that the maps do not depend on the choice

of representatives of the classes $[\mathcal{A}]$ and $[\mathcal{A}']$, ii. showing that they induce R_e-module maps $\psi_1 : \mathrm{CTopp}_{G/e} \to \mathrm{CTopp}_G \otimes_R R_e$ and $\phi_1 : \mathrm{CTopp}_G \otimes_R R_e \to \mathrm{CTopp}_{G\backslash e} \otimes_R R_e$.

Next, we outline the proofs of Theorem 1.1 and Theorem 1.2. A philosophy that is adopted in both these proofs is the following: "the critical module associated to G has the same structure as those associated to both G/e and $G\backslash e$ except that the contraction and deletion operations respectively modify them slightly and the maps ψ_i and ϕ_i (for $i = 0$ and 1) capture this modification". Both the proofs consist of the following two parts.

1. **The Complex property:** In this step, we show that the sequence of modules in Theorem 1.1 and Theorem 1.2 is a complex of R_e-modules. To this end, we verify that the image of ψ_i is contained in the kernel of ϕ_i.

2. **The Homology of the Tutte complex:** We show that the homology of the G-parking and the toppling Tutte complex is zero at every homological degree. In both cases, the argument is straightforward in homological degrees zero and two.

   The argument is more involved at homological degree one: we must show that kernel of ϕ_i is equal to the image of ψ_i. In order to give a flavour of the argument, we outline the argument for the toppling Tutte complex. The overall strategy is the same for the G-parking Tutte complex.

   The key step is to explicitly compute the kernel of ϕ_1. We show that $\ker(\phi_1) = \{x_{1,2}^{m_e-1}[\mathcal{A}_{e^+}] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-}]|$ over all acyclic orientations $\mathcal{A}$ on $G/e\}$. For this, we use the combinatorial description of the syzygies of the toppling critical module from Subsection 3.3.2. The basic idea is as follows: Suppose $\alpha \in \ker(\phi_1)$ and that $\alpha = \sum_{[\mathcal{A}]} p_{[\mathcal{A}]} \cdot [\mathcal{A}]$ in terms of the standard generating set of $\mathrm{CTopp}_G \otimes_R R_e$. Since $\alpha \in \ker(\phi_1)$ we know that $\sum_{[\mathcal{A}]} p_{[\mathcal{A}]} \cdot \phi_1([\mathcal{A}]) = 0$ and gives a syzygy of $\mathrm{CTopp}_{G\backslash e} \otimes_R R_e$. Hence, this syzygy can be written as an R_e-linear combination of the standard syzygies of $\mathrm{CTopp}_{G\backslash e} \otimes_R R_e$. Next, comparing the standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$ and $\mathrm{CTopp}_{G\backslash e} \otimes_R R_e$, we conclude that α is generated by the elements $x_{1,2}^{m_e-1}[\mathcal{A}_{e^+}] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-}]$ in $\mathrm{CTopp}_G \otimes_R R_e$. The key idea behind comparing the standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$ and $\mathrm{CTopp}_{G\backslash e} \otimes_R R_e$ is that upon deleting the edge e, all the standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$ except the ones corresponding to contracting the edge e carry over to $\mathrm{CTopp}_{G\backslash e} \otimes_R R_e$. We refer to Proposition 5.7 for more details.

Remark 2.1. We use the terminology ψ_0, ψ_1 and ϕ_0, ϕ_1 to reflect the fact by using the family $I_{G,t}$ from [MSW15] we can define a one parameter family of critical R-modules $C_{G,t}$ such that $C_{G,0} = \mathrm{CPark}_G$ and $C_{G,1} = \mathrm{CTopp}_G$. It seems plausible that there is a Tutte short exact sequence for the critical module $C_{G,t}$ that interpolates between the two Tutte short exact sequences constructed here. The corresponding maps ψ_t and ϕ_t seem more involved and we leave this for future work.

Example 2.2 (Triangle)**.** Consider the case where $G = K_3$ a complete graph on three vertices labelled v_1, v_2, v_3 and $e = (v_1, v_2)$. The graph G/e is a multigraph on two vertices $(v_{1,2}, v_3)$ with two multiple edges and $G \backslash e$ is a tree on three vertices with edges (v_1, v_3) and (v_2, v_3).
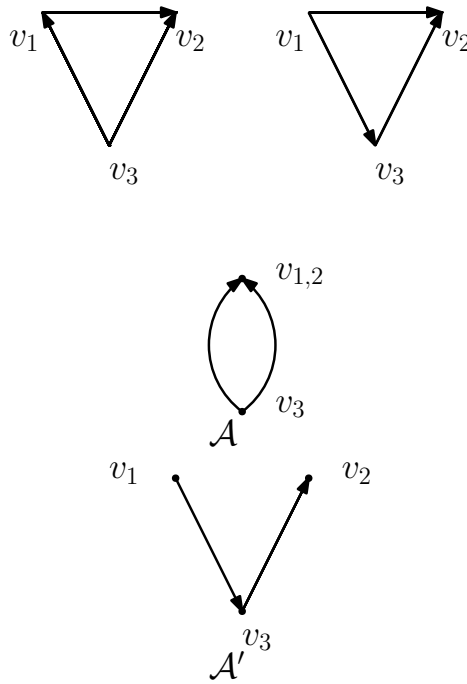
Figure 2.1: Acyclic orientations corresponding to the minimal generators of the critical modules: the case where $G = K_3$. The acyclic orientations on the top left and top right are $\mathcal{A}_{e^+}$ and $\mathcal{A}_{e^-}$ respectively.

The G-parking critical module CPark_G is generated by two elements $\mathcal{A}_{e^+}$ and $\mathcal{A}_{e^-}$ labelled by acyclic orientations shown in Figure 2.1 with the following relations (see Subsection 3.3.1 for more details):

$$x_1 \cdot \mathcal{A}_{e^+} = 0,$$
$$x_3 \cdot \mathcal{A}_{e^-} = 0,$$
$$x_3 \cdot \mathcal{A}_{e^+} + x_1 \cdot \mathcal{A}_{e^-} = 0.$$

Note that $R_e/M_{G/e}$ and $R/M_{G\backslash e}$ are Gorenstein. The G-parking critical module $\mathrm{CPark}_{G/e}$ is generated by one element $\mathcal{A}$ labelled by the acyclic orientation with sink at $v_{1,2}$ shown in Figure 2.1 with the relation:

$$x_3^2 \cdot \mathcal{A} = 0.$$

The G-parking critical module $\mathrm{CPark}_{G\backslash e}$ is also generated by one element $\mathcal{A}'$ labelled by the acyclic orientation with unique sink at v_2 shown in Figure 2.1 subject to relations:

$$x_1 \cdot \mathcal{A}' = 0,$$
$$x_3 \cdot \mathcal{A}' = 0.$$

The map ψ_0 takes $\mathcal{A}$ to $\mathcal{A}_{e^+}$ and is well-defined since $x_3^2 \cdot \psi_0(\mathcal{A}) = x_3^2 \cdot \mathcal{A}_{e^+} = 0$. Note that this relation can be obtained from the defining relations of $\mathrm{CPark}_G \otimes_R R_e$ as $-x_{1,2}(x_3 \cdot \mathcal{A}_{e^-}) + x_3(x_3 \cdot \mathcal{A}_{e^+} + x_{1,2} \cdot \mathcal{A}_{e^-})$.

The map ϕ_0 takes $\mathcal{A}_{e^+}$ to zero and $\mathcal{A}_{e^-}$ to $\mathcal{A}'$ and is well-defined (preserves relations).

Furthermore, the sequence is a complex since $\phi_0(\psi_0(\mathcal{A})) = 0$, the kernel of ψ_0 contains $x_{1,2} \cdot \mathrm{CPark}_{G/e}$ and the map ϕ_0 is surjective.

Next, we give a flavour of the argument for short exactness. At the homological degree one, the element $x_{1,2} \cdot \mathcal{A}_{e^-}$ is in the kernel of ϕ_0. However, it is also in the image of ψ_0 since from the third defining relation of CPark_G we have $x_{1,2} \cdot \mathcal{A}_{e^-} = -x_3 \cdot \mathcal{A}_{e^+} = -x_3 \cdot \psi_0(\mathcal{A})$ and is hence, in the image of ψ_0. We generalise this argument in Section 4. Furthermore, the kernel of the map ψ_0 turns to be precisely $x_{1,2} \cdot \mathrm{CPark}_{G/e}$.

Merino's theorem follows by noting that $x_1 - x_2$ is a regular element on CPark_G and $\mathrm{CPark}_{G \setminus e}$ and $x_{1,2}$ is a regular element on $\mathrm{CPark}_{G/e}$ and from the additivity of the Hilbert series in short exact sequences. The Hilbert series of $\mathrm{CPark}_G, \mathrm{CPark}_{G/e}$ and $\mathrm{CPark}_{G \setminus e}$ are $(2+t)/(1-t), (1+t)/(1-t)$ and $1/(1-t)$. Hence, the Hilbert series of $\mathrm{CPark}_G \otimes_R R_e$, $\mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e})$ and $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ are $2 + t, 1 + t$ and 1, respectively.

The toppling critical module of G is also generated by two elements $[\mathcal{A}_{e^+}]$ and $[\mathcal{A}_{e^-}]$ that naturally correspond to equivalence classes of acyclic orientations $\mathcal{A}_{e^+}$ and $\mathcal{A}_{e^-}$ shown in Figure 2.1, with the following relations ((see Subsection 3.3.2 for more details)):

$$x_1 \cdot [\mathcal{A}_{e^+}] + x_2 \cdot [\mathcal{A}_{e^-}] = 0,$$
$$x_2 \cdot [\mathcal{A}_{e^+}] + x_3 \cdot [\mathcal{A}_{e^-}] = 0,$$
$$x_3 \cdot [\mathcal{A}_{e^+}] + x_1 \cdot [\mathcal{A}_{e^-}] = 0.$$

The toppling critical modules of G/e is generated by one element $[\mathcal{A}]$ labelled by the equivalence class of the acyclic orientation $\mathcal{A}$ with sink at $v_{1,2}$, as shown in Figure 2.1, with the relation:

$$(x_3^2 + x_{1,2}^2) \cdot [\mathcal{A}] = 0.$$

The toppling critical module of $G \setminus e$ is generated by one element $[\mathcal{A}']$ labelled by the equivalence class of the acyclic orientation $\mathcal{A}'$, as shown in Figure 2.1, and with the relations:

$$(x_1 + x_3) \cdot [\mathcal{A}'] = 0,$$
$$(x_3 + x_2) \cdot [\mathcal{A}'] = 0.$$

Note that $\mathrm{CTopp}_{G \setminus e}$ is isomorphic to $R/\langle x_1 + x_3, x_3 + x_2 \rangle$. The map ψ_1 takes $[\mathcal{A}]$ to $[\mathcal{A}_{e^+}] + [\mathcal{A}_{e^-}]$ and the map ϕ_1 takes both $[\mathcal{A}_{e^+}]$ and $[\mathcal{A}_{e^-}]$ to $[\mathcal{A}']$. A quick check shows that these candidate maps are indeed well-defined. Furthermore, note that the element $x_3 \cdot [\mathcal{A}]$ is in the kernel of ψ_1 and is not contained in $x_{1,2} \cdot \mathrm{CTopp}_{G/e}$.

3. Preliminaries

In this section, we formally define the G-parking and toppling critical modules. Before this, we briefly recall the corresponding ideals followed by a discussion on canonical modules. We also discuss a characterisation of equivalent acyclic orientations and a criterion for well-definedness of candidate maps between modules. They will be turn out to be useful in the forthcoming sections.

3.1. The G-parking function ideal and the toppling ideal

We start by defining the G-parking function ideal of a graph. Fix a vertex v_2, say of G. For each non-empty subset S of vertices in $V(G) \setminus \{v_2\}$, associate a monomial $m_S = \prod_{v_j \in S} x_j^{\deg_S(v_j)}$, where $\deg_S(v_j)$ is the number of edges in G one of whose vertices is v_j and the other vertex is in the complement $\bar{S} = V \setminus S$ of S. The G-parking function ideal M_G (with respect to v_2) is defined as

$$M_G = \langle m_S | \varnothing \neq S \subseteq V(G) \setminus \{v_2\} \rangle.$$

Note that M_G depends on the choice of a vertex and we take this vertex to be v_2. For $M_{G/e}$ (recall that $e = (v_1, v_2)$), we take this to be the vertex $v_{1,2}$ (the vertex obtained by contracting v_1 and v_2).

The toppling ideal of a graph is a binomial ideal that captures the chip firing moves on G. It has been studied in several works recently, for instance [PPW11], [MS13]. We briefly recall its definition here. Let $n \geqslant 2$. Let $Q_G = D_G - A_G$ be the Laplacian matrix of G, where A_G is the vertex-vertex adjacency matrix of G and $D_G = \mathrm{diag}(\mathrm{val}(v_1), \ldots, \mathrm{val}(v_n))$ is the diagonal matrix with its diagonal entries as the valencies of the corresponding vertices. Let L_G, the Laplacian lattice of G, be the sublattice of $\mathbb{Z}^n$ generated by the rows (or equivalently the columns) of Q_G. Since the graph G is connected, the Laplacian matrix Q_G has rank $n - 1$. Hence, the lattice L_G also has rank $n - 1$ and is a finite index sublattice of the root lattice A_{n-1}. The toppling ideal I_G of G is the lattice ideal of the Laplacian lattice L_G. By definition,

$$I_G = \langle \mathbf{x^u} - \mathbf{x^v} | \ \mathbf{u}, \ \mathbf{v} \in \mathbb{Z}_{\geqslant 0}^n, \ \mathbf{u} - \mathbf{v} \in L_G \rangle.$$

3.2. The canonical module of a graded ring

Recall from the introduction that both the critical modules are defined, up to a twist, as canonical modules of certain quotients of the polynomial ring. Hence, we start by briefly recalling the notion of canonical module of a graded ring. We refer to [BH98, Chapter 3] and [MS05, Chapter 13, Section 4] for a more detailed treatment of this topic. However, we only deal with the critical modules in terms of their free presentations and these can be described in terms of the data of the underlying graph. Hence, a reader may choose to skip this subsection (and in principle, also the definition of the critical modules) and directly proceed to the free presentation of the critical modules presented in the next subsection.

Let $\mathfrak{R}$ be a graded ring with a unique homogenous maximal ideal $\mathfrak{m}$ that is also maximal in the usual (ungraded) sense and of Krull dimension κ. Following [BH98, Definition 3.6.8] an $\mathfrak{R}$-module $\mathfrak{C}$ is called a *canonical module* $\omega_{\mathfrak{R}}$ of $\mathfrak{R}$ if

$$\mathrm{Ext}_{\mathfrak{R}}^i(\mathfrak{R}/\mathfrak{m}, \mathfrak{C}) \cong \begin{cases} 0, \text{ for } i \neq \kappa, \\ \mathfrak{R}/\mathfrak{m}, \text{ otherwise.} \end{cases}$$

Note that the isomorphisms above are homogenous isomorphisms. Note that the definition of a canonical module does not guarantee its existence and is, in general, a subtle issue. By [BH98, Proposition 3.6.9], a canonical module if it exists is unique up to homogenous isomorphism.

The existence of a canonical module for the G-parking and toppling ideals follows from the following facts. By Example [BH98, 3.6.10], the standard graded polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over any field $\mathbb{K}$ is a Gorenstein ring and hence, has a canonical module: $R(-n)$ i.e., R twisted [2] by $-n$. Furthermore, from [BH98, Part (b), Proposition 3.6.12] it follows that any Cohen–Macaulay, graded quotient ring of the graded polynomial ring has a canonical module. Since both R/I_G and R/M_G are graded Cohen–Macaulay rings of depth and dimension one ([PPW11, Proposition 7.3] and from their minimal free resolutions in [MSW15]), they have canonical modules that are unique up to homogenous isomorphism. Furthermore, by [BH98, Part (b), Proposition 3.6.12], the canonical module has an explicit description as $\mathrm{Ext}^t_R(R/I, R(-n))$, where t is the height of I. Hence, the canonical module can be computed by applying graded $\mathrm{Hom}_R(\_, R(-n))$ to a minimal free resolution of R/I as an R-module and taking the t-th homology of the resulting complex. In particular, the canonical module is isomorphic to the cokernel of the dual (with the appropriate twist) of the highest differential in the minimal free resolution of R/I. Since the minimal free resolution of the toppling ideal and the G-parking function ideal can be described in purely combinatorial terms, this leads to a combinatorial description of their canonical module. One point to note before moving to the definition of the two critical modules is that we primarily regard the critical modules as (graded) modules over the polynomial ring and not as modules over the corresponding quotient ring.

3.3. Critical modules and their free presentations

In this subsection, we define both the critical modules, and then discuss their free presentations. Recall that ℓ is the number of loops of G.

Definition 3.1 (G-parking Critical Module)**.** The G-parking critical module CPark_G is defined as $\omega_{R/M_G}(\ell)$, where ω_{R/M_G} is the canonical module of R/M_G and M_G is the G-parking function ideal of G.

Definition 3.2 (Toppling Critical Module)**.** The toppling critical module CTopp_G is defined as $\omega_{R/I_G}(\ell)$, where ω_{R/I_G} is the canonical module of R/I_G and I_G is the toppling ideal of G.

3.3.1 A free presentation of the G-parking critical module

We recall a free presentation of CPark_G that is implicit in the minimal free resolution of M_G [MSW15, Section 4]. Recall that the minimal generators of CPark_G are labelled by acyclic orientations on G with a unique sink at v_2. The minimal first syzygies of CPark_G are labelled by acyclic orientations $\mathcal{A}$ on connected partition graphs $G/(v_i, v_j)$ (where v_i and v_j are connected by an edge) with a unique sink at the partition containing v_2. We now describe the relation corresponding to such a pair $(\mathcal{A}, G/(v_i, v_j))$. Suppose that $\mathcal{A}_{(v_i, v_j)^+}$ and $\mathcal{A}_{(v_i, v_j)^-}$ are the acyclic orientations on G obtained by further orienting every edge between v_i and v_j such that v_i and v_j is the source respectively. Let $m_{i,j}$ be the number of edges between (v_i, v_j). Note that at least one

---

[2]Recall that for a graded $\mathfrak{R}$-module $\mathfrak{M}$, the i-th twist $\mathfrak{M}(i)$ of $\mathfrak{M}$, for an integer i, is the $\mathfrak{R}$-module defined as $(\mathfrak{M}(i))_j = \mathfrak{M}_{i+j}$.

of $\mathcal{A}_{(v_i,v_j)^+}$ and $\mathcal{A}_{(v_i,v_j)^-}$ has a unique sink at v_2. The relation corresponding to $(\mathcal{A}, G/(v_i, v_j))$ is the following:

$$\begin{cases} x_i^{m_{i,j}} \cdot \mathcal{A}_{(v_i,v_j)^+}, & \text{if } j = 2 \text{ or } \mathcal{A}_{(v_i,v_j)^-} \text{ does not have a unique sink,} \\ x_j^{m_{i,j}} \cdot \mathcal{A}_{(v_i,v_j)^-}, & \text{if } i = 2 \text{ or } \mathcal{A}_{(v_i,v_j)^+} \text{ does not have a unique sink,} \\ x_i^{m_{i,j}} \cdot \mathcal{A}_{(v_i,v_j)^+} + x_j^{m_{i,j}} \cdot \mathcal{A}_{(v_i,v_j)^-}, & \text{otherwise.} \end{cases}$$

See Example 2.2 for the case of a triangle. We refer to this relation as the standard syzygy corresponding to the pair $(\mathcal{A}, G/(v_i, v_j))$. We refer to each of the above three types of syzygies as type one, two and three respectively. As mentioned in Subsection 3.2, the correctness of this free presentation follows from the characterisation of the canonical module of R/M_G as $\text{Ext}^{n-1}(R/M_G, R(-n))$.

3.3.2 A free presentation of the toppling critical module

In this subsection, we recall a free presentation of the toppling critical module CTopp_G, that is also implicit in [MSW15, Section 3]. Recall that the toppling critical module has a minimal generating set that is in bijection with the equivalence classes of acyclic orientations on G with a unique sink at v_2. This equivalence class is defined by declaring two acyclic orientations $\mathcal{A}_1$ and $\mathcal{A}_2$ to be equivalent if the associated divisors $D_{\mathcal{A}_1}$ and $D_{\mathcal{A}_2}$ are linearly equivalent. By $[\mathcal{A}]$, we denote the minimal generator corresponding to the equivalence class of $\mathcal{A}$.

In the following, we describe a minimal generating set for the first syzygies of CTopp_G. This minimal generating set is in bijection with equivalence classes of acyclic orientations on connected partition graphs $\mathcal{P}_{i,j}$ of G of size $n-1$. The graph $\mathcal{P}_{i,j}$ is obtained by contracting a pair of adjacent vertices (v_i, v_j) of G i.e., by contracting all the edges between (v_i, v_j) simultaneously. Note that the equivalence class of acyclic orientations on $\mathcal{P}_{i,j}$ is defined as before by treating $\mathcal{P}_{i,j}$ as a graph. This syzygy corresponding to the equivalence class of the acyclic orientation $\mathcal{A}$ on $\mathcal{P}_{i,j}$ has the following explicit description. Suppose that $\mathcal{A}_{(v_i,v_j)^+}$ and $\mathcal{A}_{(v_i,v_j)^-}$ are acyclic orientations on G obtained from $\mathcal{A}$ by further orienting all edges between (v_i, v_j) such that the source is v_i and v_j respectively. The syzygy corresponding to $\mathcal{P}_{i,j}$ is given by

$$x_i^{m_{i,j}}[\mathcal{A}_{(v_i,v_j)^+}] + x_j^{m_{i,j}}[\mathcal{A}_{(v_i,v_j)^-}],$$

where $m_{i,j}$ is the number of edges between v_i and v_j. Note that we have assumed that $\mathbb{K}$ has characteristic two. See [MSW15, Example 2.6] for example of the kite graph. We know from [MSW15] that this does not depend on the choice of representatives in the equivalence class of $\mathcal{A}$. The corresponding argument is essentially the same as Lemma 5.1. In the following, we refer to this minimal generating set and its syzygies as the standard generating set and the standard syzygies for the critical module respectively.

Next, we extend the notion of standard generating set and standard syzygies to $\text{CTopp}_G \otimes_R R_e$ as an R_e-module. By the right exactness of the tensor product functor, we know that a generating set for the R_e-module $\text{CTopp}_G \otimes_R R_e$ and for its syzygies can be obtained from the corresponding sets for CTopp_G by tensoring each element with 1 (the multiplicative identity of R_e). We refer to these sets as the standard generating set and the standard syzygies of $\text{CTopp}_G \otimes_R R_e$.

The standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$ are obtained by replacing x_i by $x_{1,2}$ from the corresponding elements in CTopp_G whenever x_i is x_1 or x_2. The correctness of this free presentation follows from the characterisation of the canonical module of R/I_G as $\mathrm{Ext}^{n-1}(R/I_G, R(-n))$, see Subsection 3.2 for more details.

3.4. Equivalence of acyclic orientations

Recall that we defined two acyclic orientations $\mathcal{A}_1$ and $\mathcal{A}_2$ on G to be equivalent if their associated divisors $D_{\mathcal{A}_1} = \sum_v (\mathrm{outdeg}_{\mathcal{A}_1}(v) - 1)(v)$ and $D_{\mathcal{A}_2} = \sum_v (\mathrm{outdeg}_{\mathcal{A}_2}(v) - 1)(v)$ are linearly equivalent, where $\mathrm{outdeg}_{\mathcal{A}_i}(v)$ is the outdegree of v with respect to the acyclic orientation $\mathcal{A}_i$. The following characterisation of this equivalence in terms of reversal of a source or a sink from [Mos72, Bac17] turns out be very useful. A source-sink reversal oppositely orients all the edges incident on a source or a sink. It is immediate that the resulting orientation is acyclic and is equivalent to the original one. The converse also holds.

Theorem 3.3. *[Mos72, Bac17] Acyclic orientations $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if and only there is a sequence of source-sink reversals transforming $\mathcal{A}_1$ to $\mathcal{A}_2$.*

This characterisation allows us to define a metric d on the set of equivalent acyclic orientations as follows:

$d(\mathcal{A}_1, \mathcal{A}_2)$ is the minimum number of source or sink reversals transforming $\mathcal{A}_1$ to $\mathcal{A}_2$.

Note that d satisfies the metric axioms.

3.5. Criterion for well-definedness of candidate maps between modules

In this subsection, we record a criterion for the well-definedness of a candidate map f between two finitely presented modules M_1 and M_2 over a commutative ring R. This criterion is well known, we include proofs for completeness and easy access. The candidate map f is given by specifying its image on a generating set of M_1, and the modules M_1 and M_2 are given in terms of a finite free presentation.

Proposition 3.4. *The candidate map f is well-defined if and only if it preserves a generating set of the first syzygy module of M_1.*

Proof. The direction $\Rightarrow$ is immediate. For the converse, note $M_1 \cong R^{n_1}/S_1$ and $M_2 \cong R^{n_2}/S_2$, where n_1, n_2 are the cardinalities of the corresponding generating sets and, S_1 and S_2 are the first syzygy modules of M_1 and M_2 respectively (with respect to the chosen generating sets). The map f is well-defined as a map between free modules i.e., $f : R^{n_1} \to R^{n_2}$, we need to show that it descends to a map on the corresponding quotients. For this, it suffices to show that the image of f on S_1 is contained in S_2. Since, f takes a generating set of S_1 to S_2, it takes every element of S_1 to an element in S_2. $\qquad\square$

We prove the well-definedness of the candidate maps ψ_0, ψ_1 and ϕ_0, ϕ_1 via Proposition 3.4 using the free presentation of the G-parking and toppling critical modules described in the previous subsections. We also use the following method to construct module maps.

Proposition 3.5. *Suppose that M_1 and M_2 are given as co-kernels of maps between free modules $G_1 \xrightarrow{\ell_1} F_1$ and $G_2 \xrightarrow{\ell_2} F_2$ respectively, then the maps $G_1 \xrightarrow{\varrho_g} G_2$ and $F_1 \xrightarrow{\varrho_f} F_2$ between free modules specify a unique homomorphism $f : M_1 \to M_2$ if the diagram*

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\ell_1} & F_1 \\
\varrho_g \downarrow & & \downarrow \varrho_f \\
G_2 & \xrightarrow{\ell_2} & F_2
\end{array}
$$

commutes.

Proof. It suffices to show that ϱ_f takes every element in the image of ℓ_1 to an element in the image of ℓ_2. Hence, for an element $b \in G_1$ consider $\varrho_f(\ell_1(b)) \in F_2$. Since the diagram commutes, $\varrho_f(\ell_1(b)) = \ell_2(\varrho_g(b))$. Hence, $\varrho_f(\ell_1(b))$ is in the image of ℓ_2. $\square$

4. The G-parking Tutte short exact sequence

In this section, we prove Theorem 1.1 that states that the G-parking Tutte sequence is short exact. We starting by showing the well-definedness of the candidate maps ψ_0 and ϕ_0.

4.1. Well-definedness of ψ_0

For an acyclic orientation $\mathcal{B}$ on $G/(v_i, v_j)$ for some distinct v_i and v_j, we denote by $\mathcal{B}_{(v_i,v_j)+}$ and $\mathcal{B}_{(v_i,v_j)-}$, the acyclic orientations on G obtained by orienting every edge between v_i and v_j such that v_i and v_j is the source respectively. For an edge $\tilde{e} = (v_i, v_j)$ of G and an acyclic orientation $\mathcal{B}$ on $G/\tilde{e}$, we also use the notations $\mathcal{B}_{\tilde{e}+}$ and $\mathcal{B}_{\tilde{e}-}$ for $\mathcal{B}_{(v_i,v_j)+}$ and $\mathcal{B}_{(v_i,v_j)-}$ respectively. Recall that the map $\psi_0 : \mathrm{CPark}_{G/e} \to \mathrm{CPark}_G \otimes_R R_e$ is defined as follows. Let $\mathcal{A}$ be the minimal generator of $\mathrm{CPark}_{G/e}$ corresponding to an acyclic orientation with a unique sink at $v_{1,2}$. We define $\psi_0(\mathcal{A}) = x_{1,2}^{m_e-1} \mathcal{A}_{e+}$, where $\mathcal{A}_{e+}$ is the minimal generator in $\mathrm{CPark}_G \otimes_R R_e$ corresponding to the acyclic orientation on G obtained by further orienting e such that v_1 is the source.

Proposition 4.1. *The map $\psi_0 : \mathrm{CPark}_{G/e} \to \mathrm{CPark}_G \otimes_R R_e$ is well-defined.*

Proof. By Proposition 3.4, we verify that every standard syzygy of $\mathrm{CPark}_{G/e}$ is preserved by the map ψ_0. We label the vertices of G/e by $u_2, u_3, \ldots, u_n$, where $u_2 := v_{1,2}$ and for i from $3, \ldots, n$, the vertex u_i in G/e corresponds to the vertex v_i in G. Recall that each standard syzygy of $\mathrm{CPark}_{G/e}$ corresponds to a pair $(\mathcal{B}, P_{i,j})$, where $\mathcal{B}$ is an acyclic orientation with a unique sink at the partition containing $v_{1,2}$ on the partition graph $P_{i,j}$ obtained by contracting a pair of vertices (u_i, u_j) of G/e that are connected by an edge.

We are led to the following cases: if neither u_i nor u_j is $v_{1,2}$, then we claim that ψ_0 maps the standard syzygy of $\mathrm{CPark}_{G/e}$ corresponding to $(\mathcal{B}, P_{i,j})$ to the standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ corresponding to the pair $(\mathcal{B}_{e+}, G/(v_i, v_j))$. Note that $\mathcal{B}_{e+}$ has a unique sink at v_2 and hence, $(\mathcal{B}_{e+}, G/(v_i, v_j))$ corresponds to a standard syzygy.

Furthermore, note that the acyclic orientation $\mathcal{B}_{(u_i,u_j)+}$ (and $\mathcal{B}_{(u_i,u_j)-}$ respectively) on G/e has a unique sink if and only if the acyclic orientation $(\mathcal{B}_{(v_i,v_j)+})_{e+}$ ($(\mathcal{B}_{(v_i,v_j)-})_{e+}$ respectively)

on G obtained by further orienting e such that v_1 is the source also has a unique sink. Hence, the type of syzygy corresponding to $(\mathcal{B}, P_{i,j})$ and $(\mathcal{B}_{e^+}, G/(v_i, v_j))$ among the three types described in Subsection 3.3.1 is the same. Finally, we note that if $\mathcal{B}_{(u_i,u_j)^+}$ has a unique sink (at $v_{1,2}$), then $\psi_0(\mathcal{B}_{(u_i,u_j)^+}) = x_{1,2}^{m_e-1} \cdot (\mathcal{B}_{(v_i,v_j)^+})_{e^+}$. Similarly, if $\mathcal{B}_{(u_i,u_j)^-}$ has a unique sink (at $v_{1,2}$), then $\psi_0(\mathcal{B}_{(u_i,u_j)^-}) = x_{1,2}^{m_e-1} \cdot (\mathcal{B}_{(v_i,v_j)^-})_{e^+}$. Hence, this standard syzygy corresponding to $(\mathcal{B}, \mathcal{P}_{i,j})$ is preserved by ψ_0.

Consider the case where one of u_i or u_j, u_i say is $v_{1,2}$. Suppose that only v_2 and not v_1 is adjacent to v_j in G then consider the standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ corresponding to $(\mathcal{B}_{e^+}, G/(v_2, v_j))$ and note that both these standard syzygies corresponding to $(\mathcal{B}, \mathcal{P}_{i,j})$ and $(\mathcal{B}_{e^+}, G/(v_2, v_j))$ are of the second type and that $\psi_0(\mathcal{B}_{(u_2,u_j)^-}) = x_{1,2}^{m_e-1} \cdot (\mathcal{B}_{(v_2,v_j)^-})_{e^+}$. Hence, this standard syzygy corresponding to $(\mathcal{B}, \mathcal{P}_{i,j})$ is preserved by ψ_0.

Suppose that among v_1 and v_2, precisely v_1, or both v_1 and v_2 are adjacent to v_j. Consider the standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ corresponding to $(\mathcal{B}_{e^+}, G/(v_1, v_j))$. If this syzygy is of the first two types, then it must be of the second type and then the syzygy $(\mathcal{B}, P_{i,j})$ must also be of the second type (since only possibly v_j among v_1 and v_j can be a sink of $\mathcal{B}_{e^+}$). The syzygies are $x_j^m \cdot \mathcal{B}_{(u_1,u_j)^-}$ and $x_j^m \cdot (\mathcal{B}_{(v_1,v_j)^-})_{e^+}$, where m is the multiplicity of the edge (v_1, v_j). Note that $\psi_0(\mathcal{B}_{(u_1,u_j)^-}) = x_{1,2}^{m_e-1} \cdot (\mathcal{B}_{(v_1,v_j)^-})_{e^+}$ and hence, this is preserved.

Otherwise, this syzygy is of the third type and it is of the form $x_{1,2}^m \cdot \mathcal{K} + x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-}$, where $\mathcal{K} = (\mathcal{B}_{e^+})_{(v_1,v_j)^+}$ is the acyclic orientation on G obtained from $(\mathcal{B}_{e^+})_{(v_1,v_j)^-}$ by reversing the orientation of every edge between the vertices (v_1, v_j). We consider the syzygy corresponding to the acyclic orientation induced by $\mathcal{K}$ on $G/(v_1, v_2)$ (note that the edge e is contractible on $\mathcal{K}$). Since v_2 is a sink this syzygy is of the form $x_{1,2}^{m_e} \cdot \mathcal{K}$, where m_e is the multiplicity of the edge e. Hence, if $m \geqslant m_e$ we obtain the syzygy $x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-}$ from the standard syzygies as $(x_{1,2}^m \cdot \mathcal{K} + x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-}) - x_{1,2}^{m-m_e}(x_{1,2}^{m_e} \cdot \mathcal{K})$. Otherwise, we obtain the syzygy $x_{1,2}^{m_e-m}(x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-})$ as $x_{1,2}^{m_e-m}(x_{1,2}^m \cdot \mathcal{K} + x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-}) - (x_{1,2}^{m_e} \cdot \mathcal{K})$. Hence, $x_{1,2}^{m_e-1} x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-}$ is a syzygy of $\mathrm{CPark}_G \otimes_R R_e$. Finally, note that $x_{1,2}^{m_e-1} x_j^m \cdot (\mathcal{B}_{e^+})_{(v_1,v_j)^-} = \psi_0(x_j^m \cdot \mathcal{B}_{(v_{1,2},u_j)^-})$ and that $x_j^m \cdot \mathcal{B}_{(v_{1,2},u_j)^-}$ is the standard syzygy corresponding to the pair $(\mathcal{B}, P_{i,j})$ to complete the proof. $\qquad \square$

4.2. Well-definedness of ϕ_0

We need the following combinatorial lemma for the well-definedness of ϕ_0. Recall that an edge e of G is said to be contractible on an acyclic orientation $\mathcal{A}'$ on G if the orientation $\mathcal{A}'/e$ induced by $\mathcal{A}'$ on G/e is acyclic.

Lemma 4.2. *Suppose that $\mathcal{A}$ is an acyclic orientation on G with a unique sink at v_2. Suppose that there is a directed edge from v_1 to $v_j \neq v_2$, then the edge $e = (v_1, v_2)$ is not contractible.*

Proof. In order to show that e is not contractible, we need to exhibit a directed path from v_1 to v_2 that is not equal to the edge e. Construct a directed walk starting from v_j by picking arbitrary outgoing edges. This walk cannot repeat vertices since $\mathcal{A}$ is acyclic and hence, it will terminate since G is a finite graph. Furthermore, it terminates in v_2 since it is the unique sink. Appending the directed edge (v_1, v_j) to the beginning of this walk yields the required directed path from v_1 to v_2. $\qquad \square$

Recall that we defined the map $\phi_0 : \mathrm{CPark}_G \otimes_R R_e \to \mathrm{CPark}_{G \setminus e} \otimes_R R_e$ as the following. If $m_e = 1$, then

$$\phi_0(\mathcal{A}') = \begin{cases} \mathcal{A}' \setminus e, & \text{if the edge } e \text{ is not contractible on } \mathcal{A}', \\ 0, & \text{otherwise.} \end{cases}$$

If $m_e > 1$, then $\phi_0(\mathcal{A}') = \mathcal{A}' \setminus e$ for every standard generator $\mathcal{A}'$ of $\mathrm{CPark}_G \otimes_R R_e$.

Proposition 4.3. *The map ϕ_0 is well-defined.*

Proof. By Proposition 3.4, we verify that every standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ is preserved by the map ϕ_0. We start by noting that any standard syzygy corresponding to the partition graph $\mathcal{P}_{1,2}$ of G obtained by contracting (v_1, v_2) is of the form $x_{1,2}^{m_e} \cdot \mathcal{A}_{e^+}$, where $\mathcal{A}$ is an acyclic orientation on G/e. If $m_e = 1$, then ϕ_0 maps it to zero. Since e is contractible on $\mathcal{A}_{e^+}$ and hence, $\phi_0(\mathcal{A}_{e^+}) = 0$. If $m_e > 1$, then $\phi_0(x_{1,2}^{m_e} \cdot \mathcal{A}_{e^+}) = x_{1,2}^{m_e} \cdot (\mathcal{A}_{e^+} \setminus e) = 0$ since $x_{1,2}^{m_e-1} \cdot (\mathcal{A}_{e^+} \setminus e)$ is the standard syzygy of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ corresponding to the pair $(\mathcal{A}, G/(v_1, v_2))$.

We now consider standard syzygies corresponding to other partition graphs. If one of the vertices is v_2 and the other vertex is $v_j \neq v_1$, then the standard syzygy corresponding to $(\mathcal{A}, \mathcal{P}_{2,j})$ is of type two and is of the form $x_j^m \cdot \mathcal{A}_{(v_2,v_j)^-}$. The map ϕ_0 takes it to the standard syzygy corresponding to $(\mathcal{A} \setminus e, \mathcal{P}_{2,j} \setminus e)$.

If one of the vertices is v_1 and the other vertex is $v_j \neq v_2$, then if the standard syzygy $\mathcal{S}$ is of type three and is of form $x_1^m \cdot \mathcal{A}_{(v_1,v_j)^+} + x_j^m \cdot \mathcal{A}_{(v_1,v_j)^-}$. We consider two cases:

Case I: Suppose that $m_e = 1$. Note that by Lemma 4.2, the edge e is not contractible on $\mathcal{A}_{(v_1,v_j)^+}$. We have the following two cases: If e is not contractible on $\mathcal{A}_{(v_1,v_j)^-}$, then we have $\phi_0(\mathcal{A}_{(v_1,v_j)^-}) = \mathcal{A}_{(v_1,v_j)^-} \setminus e$ and $\phi_0(\mathcal{A}_{(v_1,v_j)^+}) = \mathcal{A}_{(v_1,v_j)^+} \setminus e$. Furthermore, the standard syzygy corresponding to the acyclic orientation $(\mathcal{A}/(v_1, v_j)) \setminus e$ on $(G \setminus e)/(v_1, v_j)$ (this is the acyclic orientation induced by $\mathcal{A}$ on the graph obtained by contracting (v_1, v_j) and deleting e) is a type three syzygy since in both $\mathcal{A}_{(v_1,v_j)^-} \setminus e$ and $\mathcal{A}_{(v_1,v_j)^+} \setminus e$, the vertex v_1 has at least one outgoing edge and is hence not a sink. This implies that both acyclic orientations have a unique sink at v_2. This syzygy $\mathcal{S}'$ is of the form: $x_1^m \cdot \mathcal{A}_{(v_1,v_j)^+} \setminus e + x_j^m \cdot \mathcal{A}_{(v_1,v_j)^-} \setminus e$. Hence, ϕ_0 takes $\mathcal{S}$ to $\mathcal{S}'$. If e is contractible on $\mathcal{A}_{(v_1,v_j)^-}$, then $\phi_0(\mathcal{A}_{(v_1,v_j)^-}) = 0$ and $\phi_0(\mathcal{A}_{(v_1,v_j)^+}) = \mathcal{A}_{(v_1,v_j)^+} \setminus e$. Furthermore, the standard syzygy $\mathcal{S}'$ corresponding to the acyclic orientation $(\mathcal{A}/(v_1, v_j)) \setminus e$ is of type one and of the form $x_1^m \cdot \mathcal{A}_{(v_1,v_j)^+} \setminus e$. Hence, ϕ_0 takes $\mathcal{S}$ to $\mathcal{S}'$.

Case II: If $m_e > 1$, then $\phi_0(\mathcal{A}_{(v_1,v_j)^-}) = \mathcal{A}_{(v_1,v_j)^-} \setminus e$ and $\phi_0(\mathcal{A}_{(v_1,v_j)^+}) = \mathcal{A}_{(v_1,v_j)^+} \setminus e$. Also, this standard syzygy maps to the standard syzygy of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ corresponding to $(\mathcal{A}, \mathcal{P}_{i,j})$. Suppose that this standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ is of type two, then it is of the form $x_1^m \cdot \mathcal{A}_{(v_1,v_j)^-}$ and ϕ_0 (irrespective of m_e) maps it to the standard syzygy $x_1^m \cdot \mathcal{A}_{(v_1,v_j)^-} \setminus e$ corresponding to the acyclic orientation $(\mathcal{A}/(v_1, v_j)) \setminus e$ on $G \setminus e$, note that this is also a syzygy of type two.

If none of the vertices is v_1 or v_2, then the standard syzygy corresponding to $(\mathcal{A}, \mathcal{P}_{i,j})$ is mapped to the standard syzygy corresponding to $\mathcal{A} \setminus e$ on $\mathcal{P}_{i,j} \setminus e$ independent of m_e. Note that the type of these two standard syzygies are the same and ϕ_0 maps $\mathcal{A}_{(v_i,v_j)^+}$ to $(\mathcal{A}_{(v_i,v_j)^+}) \setminus e$ and maps $\mathcal{A}_{(v_i,v_j)^-}$ to $(\mathcal{A}_{(v_i,v_j)^-}) \setminus e$ (when they are well-defined). $\square$

4.3. Complex property

Proposition 4.4. *The G-parking Tutte sequence in Theorem 1.1 is a complex of graded R_e-modules.*

Proof. We show that the property of a complex is satisfied at each homological degree. At homological degrees zero and two, this is immediate. At homological degree one, we need to show that $\phi_0(\psi_0(b)) = 0$ for every $b \in \mathrm{CPark}_{G/e}$. It suffices to prove this for every standard generator $\mathcal{A}$ of $\mathrm{CPark}_{G/e}$. To see this, consider the case where $m_e = 1$, we have $\psi_0(\mathcal{A}) = \mathcal{A}_{e+}$ and $\phi_0(\mathcal{A}_{e+}) = 0$ since the edge e is contractible on the acyclic orientation $\mathcal{A}_{e+}$ on G. If $m_e > 1$, then $\psi_0(\mathcal{A}) = x_{1,2}^{m_e-1} \cdot \mathcal{A}_{e+}$ and $\phi_0(\mathcal{A}_{e+}) = x_{1,2}^{m_e-1} \cdot (\mathcal{A}_{e+} \setminus e) = 0$ since $x_{1,2}^{m_e-1} \cdot (\mathcal{A}_{e+} \setminus e)$ is the standard syzygy (of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$) corresponding to the acyclic orientation $\mathcal{A}$ on $G/(v_1, v_2)$. □

4.4. The kernel of ψ_0

Proposition 4.5. *The kernel of $\psi_0 : \mathrm{CPark}_{G/e} \to \mathrm{CPark}_G \otimes_R R_e$ is equal to $x_{1,2} \cdot \mathrm{CPark}_{G/e}$.*

Proof. The inclusion $x_{1,2} \cdot \mathrm{CPark}_{G/e}$ in the kernel of ψ_0 is immediate since $\psi_0(x_{1,2} \cdot \mathcal{A}) = x_{1,2}^{m_e} \cdot \mathcal{A}_{e+} = 0$ since $x_{1,2}^{m_e} \cdot \mathcal{A}_{e+}$ is the standard syzygy corresponding to the acyclic orientation $\mathcal{A}$ of the partition graph $G/(v_1, v_2)$.

For the other direction, consider an element $\alpha = \sum_{\mathcal{A}} p_{\mathcal{A}} \cdot \mathcal{A}$ in the kernel of ψ_0. We show that the coefficients $p_{\mathcal{A}}$ can be chosen such that $x_{1,2} | p_{\mathcal{A}}$ for each $\mathcal{A}$. Since $\alpha \in \ker(\psi_0)$, we obtain $x_{1,2}^{m_e-1} \sum_{\mathcal{A}} p_{\mathcal{A}} \cdot \mathcal{A}_{e+} = 0$. Furthermore, since the map $\mathcal{A} \to \mathcal{A}_{e+}$ regarded as a map between sets of acyclic orientations (with a unique sink at a fixed vertex) is injective, we note that $x_{1,2}^{m_e-1} \sum_{\mathcal{A}} p_{\mathcal{A}} \cdot \mathcal{A}_{e+}$ is a syzygy of $\mathrm{CPark}_G \otimes_R R_e$.

Hence, it can be written as an R_e-linear combination of the standard syzygyies of CPark_G. Hence,

$$x_{1,2}^{m_e-1} \sum_{\mathcal{A}} p_{\mathcal{A}} \cdot \mathcal{A}_{e+} = \sum_{(\mathcal{B}, \mathcal{P}_{i,j})} r_{(\mathcal{B}, \mathcal{P}_{i,j})} s_{(\mathcal{B}, \mathcal{P}_{i,j})}, \tag{4.1}$$

where $s_{(\mathcal{B}, \mathcal{P}_{i,j})}$ is the standard syzygy corresponding to the acyclic orientation $\mathcal{B}$ (with a unique sink at the partition containing v_2) on the partition graph $\mathcal{P}_{i,j}$ and $r_{(\mathcal{B}, \mathcal{P}_{i,j})} \in R_e$. Note that Equation (4.1) is an equation in the free R_e-module of rank equal to the number of acyclic orientations with a unique sink at v_2.

Consider the case $m_e = 1$. Next, we observe that if (i, j) (as an unordered pair) is not $(1, 2)$, then the syzygy $s_{(\mathcal{B}, \mathcal{P}_{i,j})}$ is the image of a standard syzygy of $\mathrm{CPark}_{G/e}$, namely the standard syzygy corresponding to the acyclic orientation $\mathcal{B}/e$ on $(G/(v_i, v_j))/e$ obtained by contracting e. These syzygies can be cleared out by regarding α as $\alpha - \sum_{(i,j)\neq(1,2)} r_{(\mathcal{B}, \mathcal{P}_{i,j})} s_{\mathcal{B}/e, G/(v_i,v_j))/e}$ and using the expansion in Equation (4.1) for $\psi_0(\alpha)$. Hence, we can assume that the standard syzygies in Equation (4.1) all correspond to $\mathcal{P}_{1,2}$. A standard syzygy corresponding to $\mathcal{P}_{1,2}$ is of the form $x_{1,2} \cdot \mathcal{A}_{e+}$ for some acyclic orientation $\mathcal{A}$ on $G/(v_1, v_2)$. This implies that each coefficient $p_{\mathcal{A}}$ divides $x_{1,2}$. This completes the proof for $m_e = 1$.

More generally, if $m_e \geqslant 1$, then we multiply both sides of the Equation (4.1) by $x_{1,2}^{m_e-1}$. The argument then proceeds similar to the case $m_e = 1$. If (i, j) (as an unordered pair) is not $(1, 2)$,

then $x_{1,2}^{m_e-1} \cdot s_{(\mathcal{B},\mathcal{P}_{i,j})}$ is the image of a standard syzygy of $\mathrm{CPark}_{G/e}$, namely the standard syzygy corresponding to the acyclic orientation $\mathcal{B}/e$ on $(G/(v_i,v_j))/e$ obtained by contracting e. Hence, these syzygies can be cleared out and we can assume that only terms corresponding to $\mathcal{P}_{1,2}$ appear. A standard syzygy corresponding to $\mathcal{P}_{1,2}$ is of the form $x_{1,2}^{m_e} \cdot \mathcal{A}_{e^+}$ for some acyclic orientation $\mathcal{A}$ on $G/(v_1,v_2)$. Hence, we conclude that $x_{1,2}^{2m_e-1} | (x_{1,2}^{2m_e-2} \cdot p_{\mathcal{A}})$ and hence, $x_{1,2} | p_{\mathcal{A}}$. We conclude that α is contained in $x_{1,2} \cdot \mathrm{CPark}_{G/e}$. $\qquad\square$

4.5. Exactness

Proposition 4.6. *The G-parking Tutte complex is a short exact sequence.*

Proof. We show the exactness of the G-parking Tutte complex at every homological degree. At homological degree zero, the exactness follows from Proposition 4.5. At homological degree two, the exactness is equivalent to the surjectivity of ϕ_0. To see the surjectivity of ϕ_0, we consider two cases. Suppose that $m_e = 1$. Note that for every standard generator $\mathcal{A}''$ of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$, there is the acyclic orientation $\mathcal{A}'$ on G obtained by further orienting e such that v_1 is the source. The edge e is not contractible on $\mathcal{A}'$ since there is at least one edge other than e with a source at v_1 and we can now apply Lemma 4.2. Hence, ϕ_0 takes $\mathcal{A}'$ to $\mathcal{A}''$. Since every standard generator of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ is in the image of an element in ϕ_0 we conclude that ϕ_0 is surjective. If $m_e > 1$, then this is immediate from the construction of ϕ_0 since every acyclic orientation on $G \setminus e$ with a unique sink at v_2 gives rise to an acyclic orientation on G with a unique sink at v_2 by further orienting e such that v_1 is the source.

We turn to homological degree one. We must show that the kernel of ϕ_0 is equal to the image of ψ_0 which in turn is $\langle x_{1,2}^{m_e-1} \mathcal{A}_{e^+} \rangle$, where $\mathcal{A}$ ranges over acyclic orientations on G/e with a unique sink at $v_{1,2}$. Suppose that $b = \sum_{\mathcal{A}'} p_{\mathcal{A}'} \mathcal{A}'$ is an element in the kernel of ϕ_0. We know that $\sum_{\mathcal{A}'} p_{\mathcal{A}'} \phi_0(\mathcal{A}') = 0$. Suppose that $m_e > 1$. Note that the map $\mathcal{A}' \to \mathcal{A}' \setminus e$ at the level of sets is a bijection. Hence, $\sum_{\mathcal{A}'} p_{\mathcal{A}'} \cdot \phi_0(\mathcal{A}')$ is a syzygy of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ and can be written as an R_e-linear combination of the standard syzygies of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$. More precisely, we have:

$$\sum_{\mathcal{A}'} p_{\mathcal{A}'} \cdot \phi_0(\mathcal{A}') = \sum_{(\mathcal{A}'',\mathcal{P}_{i,j})} r_{(\mathcal{A}'',\mathcal{P}_{i,j})} \cdot s_{(\mathcal{A}'',\mathcal{P}_{i,j})}, \qquad (4.2)$$

where $r_{(\mathcal{A}'',\mathcal{P}_{i,j})} \in R_e$ and $s_{(\mathcal{A}'',\mathcal{P}_{i,j})}$ is the standard syzygy of $\mathrm{CPark}_{G \setminus e} \otimes_R R_e$ corresponding to the acyclic orientation $\mathcal{A}''$ on the partition graph $\mathcal{P}_{i,j}$ of $G \setminus e$. Note that this equation is on the free R_e-module of rank equal to the number of acyclic orientations on $G \setminus e$ with a unique sink at v_2.

Suppose that $(i,j) \neq (1,2)$ as an unordered pair. By the construction of the map ϕ_0, the standard syzygy is the image of ϕ_0 over the standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ corresponding to the same pair $(\mathcal{A}'',\mathcal{P}_{i,j})$. Hence, these syzygies can be cleared exactly as in the proof of Proposition 4.5 and we can assume that $(i,j) = (1,2)$ in the right hand side of Equation (4.2). Since every standard syzygy corresponding to $(\mathcal{A}'',\mathcal{P}_{1,2})$ is of the form $x_{1,2}^{m_e-1} \cdot \mathcal{A}''_{e^+}$, we conclude that $x_{1,2}^{m_e-1}$ divides $p_{\mathcal{A}''}$ for every $\mathcal{A}''$ and hence, $b \in \langle x_{1,2}^{m_e-1} \mathcal{A}_{e^+} \rangle$. Thus, the Tutte complex is exact in homological degree one for $m_e > 1$.

We turn to the case $m_e = 1$. Consider any element of the form $\sum_{\mathcal{A}'} p_{\mathcal{A}'} \cdot \mathcal{A}'$, where $\mathcal{A}'$ is an acyclic orientation on G such that e is not contractible on it. It suffices to show that if such a $\sum_{\mathcal{A}'} p_{\mathcal{A}'} \cdot \mathcal{A}'$ is in kernel of ϕ_0, then it is also in the image of ψ_0. By Equation (4.2), we assume by a clearing argument that $(i, j) = (1, j)$ where $j \neq 2$ in Equation (4.2). Furthermore, we can assume that the standard syzygies that appear in Equation (4.2) corresponding to $\mathcal{P}_{1,j}$ are of the form $x_{1,2}^{m_{1,j}} \cdot \mathcal{A}''$. These are precisely standard syzygies that change type from type three to type one from G to $G \setminus e$. This means that the acyclic orientation $\mathcal{A}''$ is such that there is a unique vertex $v_j \notin \{v_1, v_2\}$ that is adjacent to v_1, and with v_1 as the source of every edge between v_1 and v_j. Hence, we conclude that $x_{1,2}^{m_{1,j}}$ divides $p_{\mathcal{A}'}$.

Next, note that $x_{1,2}^{m_{1,j}} \cdot \mathcal{A}' + x_j^{m_{1,j}} \cdot \mathcal{B}'$ is a standard syzygy of $\mathrm{CPark}_G \otimes_R R_e$ corresponding to the acyclic orientation $\mathcal{A}'/(1, j) = \mathcal{B}'/(1, j)$ on $G/(1, j)$ and $\mathcal{B}'$ is the acyclic orientation on G obtained from $\mathcal{A}'$ by reversing the orientation of every edge between $(1, j)$. Note that the edge e is contractible on $\mathcal{B}'$ (since $\phi_0(\mathcal{B}') = 0$) and hence, it is of the form $\mathcal{A}_{e+}$ for an acyclic orientation $\mathcal{A}$ on G/e. Hence, $x_{1,2}^{m_{1,j}} \cdot \mathcal{A}' = -x_j^{m_{1,j}} \cdot \mathcal{B}'$ is in the image of ψ_0. This completes the proof of exactness at homological degree one. $\qquad\square$

5. The toppling Tutte short exact sequence

In this section, we detail the proof of Theorem 1.2.

5.1. Well-definedness of ψ_1

We start by recalling the construction of the candidate map ψ_1. Suppose that e is an edge of multiplicity m_e between the vertices v_1 and v_2. Let $\mathcal{A}$ be an acyclic orientation on G/e, and let $\mathcal{A}_{e+}$ and $\mathcal{A}_{e-}$ be orientations on G obtained by further orienting e such that its source is v_1 and v_2 respectively. Note that since $\mathcal{A}$ is acyclic, the orientations $\mathcal{A}_{e+}$ and $\mathcal{A}_{e-}$ are also acyclic. The candidate map ψ_1 takes $[\mathcal{A}]$ to $x_{1,2}^{m_e-1}[A_{e+}] + x_{1,2}^{m_e-1}[A_{e-}]$ in $\mathrm{CTopp}_G \otimes_R R_e$. We first show that this association is independent of the choice of representatives in the equivalence class of $\mathcal{A}$.

Lemma 5.1. *Suppose that $\mathcal{A}$ is an acyclic orientation on G/e. The equivalence classes of the acyclic orientations $\mathcal{A}_{e+}$ and $\mathcal{A}_{e-}$ on G are independent of the choice of representatives in the equivalence class of $\mathcal{A}$.*

Proof. Suppose that acyclic orientations $\mathcal{A}_1$ and $\mathcal{A}_2$ on G/e are equivalent. We know from [BN07, Section 3.1] that for any vertex u of G/e, there exists a (unique) acyclic orientation $\mathcal{A}_{\mathrm{uni}}$ with a unique sink at u that is equivalent to $\mathcal{A}_1$ and $\mathcal{A}_2$. Furthermore, by Theorem 3.3, there is a sequence of source-sink reversals that transform them to $\mathcal{A}_{\mathrm{uni}}$.

Take $u = v_{1,2}$ and note that a sink reversal at $v_{1,2}$ can be avoided in this sequence. This allows us to perform precisely the same sequence of source-sink reversals for $(\mathcal{A}_1)_{e+}$ and $(\mathcal{A}_2)_{e+}$. If $v_{1,2}$ is a sink for an acyclic orientation $\mathcal{A}$ on G/e, then v_2 is a sink for $\mathcal{A}_{e+}$ and v_1 is a sink for $\mathcal{A}_{e-}$. Hence, for $i = 1, 2$ these operations transform $(\mathcal{A}_i)_{e+}$ and $(\mathcal{A}_i)_{e-}$ into an acyclic orientation on G with a unique sink at v_2 and v_1 respectively. From [Mos72, Bac17], this implies that the acyclic orientations $(\mathcal{A}_1)_{e+}$ and $(\mathcal{A}_2)_{e+}$ on G are equivalent and that $(\mathcal{A}_1)_{e-}$ and $(\mathcal{A}_2)_{e-}$ are also equivalent. $\qquad\square$

Next, we show this candidate map induces a map between the toppling critical modules $\mathrm{CTopp}_{G/e}$ and $\mathrm{CTopp}_G \otimes_R R_e$. We show this using Proposition 3.4.

Lemma 5.2. *The candidate map ψ_1 is well-defined.*

Proof. We use Proposition 3.4 to show that ψ_1 is well-defined. In other words, we show that ψ_1 preserves the standard syzygies of $\mathrm{CTopp}_{G/e}$. Using the combinatorial description of the syzygies of $\mathrm{CTopp}_{G/e}$ in Subsection 3.3.2, we know that the generators of the first syzygy module of $\mathrm{CTopp}_{G/e}$ are in one to one correspondence with equivalence classes of acyclic orientations on contractions of pairs of adjacent vertices of G/e. For a pair of adjacent vertices (v_i, v_j) and an acyclic orientation $\mathcal{A}$ on the contraction of (v_i, v_j) in G/e, the corresponding syzygy of $\mathrm{CTopp}_{G/e}$ is given by $x_i^m[\mathcal{A}_{(v_i, v_j)^+}] - x_j^m[\mathcal{A}_{(v_i, v_j)^-}]$, where m is the number of edges between the pair (v_i, v_j) and $\mathcal{A}_{(v_i, v_j)^+}$ and $\mathcal{A}_{(v_i, v_j)^-}$ are acyclic orientations obtained from $\mathcal{A}$ by further orienting all the edges between (v_i, v_j) so that the source is v_i and v_j respectively. We must show that $x_i^m \psi_1([\mathcal{A}_{(v_i, v_j)^+}]) - x_j^m \psi_1([\mathcal{A}_{(v_i, v_j)^-}])$ is a syzygy of $\mathrm{CTopp}_G \otimes_R R_e$. Note that

$$x_i^m \psi_1([\mathcal{A}_{(v_i, v_j)^+}]) - x_j^m \psi_1([\mathcal{A}_{(v_i, v_j)^-}])$$
$$= x_i^m([[(\mathcal{A}_{(v_i, v_j)^+})_{e^+}] + [(\mathcal{A}_{(v_i, v_j)^+})_{e^-}]) - x_j^m([(\mathcal{A}_{(v_i, v_j)^-})_{e^+}] + [(\mathcal{A}_{(v_i, v_j)^-})_{e^-}]).$$

Suppose that none of v_i and v_j is $v_{1,2}$, then $x_i^m[(\mathcal{A}_{(v_i, v_j)^+})_{e^+}] - x_j^m[(\mathcal{A}_{(v_i, v_j)^-})_{e^+}]$ and $x_i^m[(\mathcal{A}_{(v_i, v_j)^+})_{e^-}] - x_j^m[(\mathcal{A}_{(v_i, v_j)^-})_{e^-}]$ are standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$. Similarly, if $v_i = v_{1,2}$ and exactly one of v_1 and v_2 is adjacent to v_j in G, then $x_i^m[(\mathcal{A}_{(v_i, v_j)^+})_{e^+}] - x_j^m[(\mathcal{A}_{(v_i, v_j)^-})_{e^+}]$ and $x_i^m[(\mathcal{A}_{(v_i, v_j)^+})_{e^-}] - x_j^m[(\mathcal{A}_{(v_i, v_j)^-})_{e^-}]$ are standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$.

Finally, consider the case where $v_i = v_{1,2}$, and both v_1 and v_2 are adjacent to v_j in G. In other words, if there is a triangle between v_1, v_2 and v_j in G, then an analogous argument does not hold. We employ a different argument. We express $\mathcal{S} = x_{1,2}^m[(\mathcal{A}_{(v_{1,2}, v_j)^+})_{e^+}] - x_j^m[(\mathcal{A}_{(v_{1,2}, v_j)^-})_{e^+}]$ as an R_e-linear combination of two standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$. These standard syzygies $\mathcal{S}_1$ and $\mathcal{S}_2$ are the following: the syzygy $\mathcal{S}_1$ corresponds to the acyclic orientation $\mathcal{A}_1$ on the contraction of the pair (v_1, v_j) in G defined as follows: $\mathcal{A}_1$ agrees with $\mathcal{A}_{(v_{1,2}, v_j)^+}$ on all the common edges and the edge v_1, v_2 is further oriented such that v_1 is the source. The other one $\mathcal{S}_2$ corresponds to the acyclic orientation $\mathcal{A}_2$ on the contraction of the pair (v_2, v_j) in G where $\mathcal{A}_2$ agrees with $\mathcal{A}_{(v_{1,2}, v_j)^-}$ on all the common edges and the edge v_1, v_2 is further oriented such that v_1 is the source, see Figure 5.1. Note that the former syzygy $\mathcal{S}_1$ is $x_j^{m_1}[(\mathcal{A}_{(v_{1,2}, v_j)^-})_{e^+}] - x_{1,2}^{m_1}[\mathcal{K}]$ and the later syzygy $\mathcal{S}_2$ is $x_j^{m_2}[\mathcal{L}] - x_{1,2}^{m_2}[(\mathcal{A}_{(v_{1,2}, v_j)^+})_{e^+}]$, where m_1 and m_2 is the number of edges in G between (v_1, v_j) and (v_2, v_j) respectively. Furthermore, observe that $\mathcal{K} = \mathcal{L}$ (we do not make $\mathcal{K}$ and $\mathcal{L}$ more explicit since we do not invoke it) and that $m = m_1 + m_2$. Hence, $\mathcal{S} = x_j^{m_2}\mathcal{S}_1 + x_{1,2}^{m_1}\mathcal{S}_2$. Similarly, we express $x_{1,2}^m[(\mathcal{A}_{(v_{1,2}, v_j)^+})_{e^-}] - x_j^m[(\mathcal{A}_{(v_{1,2}, v_j)^-})_{e^-}]$ as an R_e-linear combination of standard syzygies of CTopp_G by interchanging v_1 and v_2 in the above construction. This completes the proof of the well-definedess of ψ_1. $\square$

5.2. Well-definedness of ϕ_1

Recall that the map $\phi_1 : \mathrm{CTopp}_G \otimes_R R_e \to \mathrm{CTopp}_{G \backslash e} \otimes_R R_e$ takes the generator $[\mathcal{A}']$ of CTopp_G, corresponding to an acyclic orientation $\mathcal{A}$ on G, to $[\mathcal{A}' \backslash e]$ in $\mathrm{CTopp}_{G \backslash e}$. The following lemma
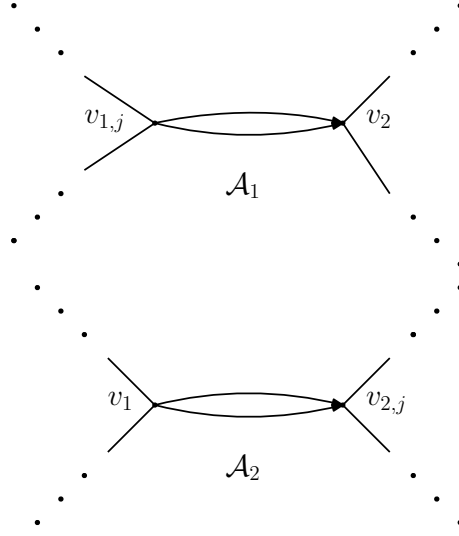
Figure 5.1: The acyclic orientations $\mathcal{A}_1, \mathcal{A}_2$ in the proof of Lemma 5.2.

shows that the association $[\mathcal{A}] \to [\mathcal{A} \setminus e]$ in the candidate map ϕ_1 does not depend on the choice of representatives.

Lemma 5.3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be equivalent acyclic orientations on G. The acyclic orientations $\mathcal{A}_1 \setminus e$ and $\mathcal{A}_2 \setminus e$ on $G \setminus e$ are equivalent.*

Proof. Using the characterisation of equivalent acyclic orientations (Theorem 3.3), we know that there is a source-sink reversal sequence transforming $\mathcal{A}_1$ to $\mathcal{A}_2$. Since, any source or sink in an acyclic orientation $\mathcal{A}$ on G remains so in the acyclic orientation $\mathcal{A} \setminus e$ on $G \setminus e$, we can perform the same source-sink reversal sequence to transform $\mathcal{A}_1 \setminus e$ to $\mathcal{A}_2 \setminus e$. Hence, $\mathcal{A}_1 \setminus e$ and $\mathcal{A}_2 \setminus e$ are equivalent. $\qquad\square$

Lemma 5.4. *The candidate map ϕ_1 is well-defined.*

Proof. Using Proposition 3.4, it suffices to prove that ϕ_1 preserves the standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$. The proofs lends itself into two cases: the first case corresponds to the standard syzygy arising from contracting G by a pair of vertices (v_i, v_j) other than (v_1, v_2) and the second case corresponds to the standard syzygy arising from contracting G by (v_1, v_2).

Consider the standard syzygy corresponding to an acyclic orientation $\mathcal{A}$ on $G/(v_i, v_j)$ i.e., G contracted by the pair of vertices $(v_i, v_j) \neq (v_1, v_2)$ that are connected by an edge. This syzygy is $x_i^m [\mathcal{A}_{(v_i,v_j)^+}] - x_j^m [\mathcal{A}_{(v_i,v_j)^-}]$, where m is the number of edges between v_i and v_j. By construction, $\phi_1([\mathcal{A}_{(v_i,v_j)^+}]) = [\mathcal{A}_{(v_i,v_j)^+} \setminus e]$ and $\phi_1([\mathcal{A}_{(v_i,v_j)^-}]) = [\mathcal{A}_{(v_i,v_j)^-} \setminus e]$. Note that $x_i^m \phi_1([\mathcal{A}_{(v_i,v_j)^+}]) - x_j^m \phi_1([\mathcal{A}_{(v_i,v_j)^-}])$ is a standard syzygy of $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$ and corresponds to the acyclic orientation $\mathcal{A} \setminus e$ on $(G \setminus e)/(v_i, v_j)$ i.e., $G \setminus e$ contracted by the pair of vertices (v_i, v_j). Hence, this standard syzygy on $\mathrm{CTopp}_G \otimes_R R_e$ is preserved.

Suppose that the standard syzygy corresponds to an acyclic orientation $\mathcal{A}$ on the $G/(v_1, v_2)$. This standard syzygy is $x_{1,2}^{m_e}[\mathcal{A}_{(v_1,v_2)^+}] - x_{1,2}^{m_e}[\mathcal{A}_{(v_1,v_2)^-}]$, where m_e is the number of edges between the pair (v_1, v_2). Suppose that $m_e = 1$. In this case, there is exactly one edge between (v_1, v_2) and since $\mathcal{A}_{(v_1,v_2)^+} \setminus e = \mathcal{A}_{(v_1,v_2)^-} \setminus e$ we have $\phi_1([\mathcal{A}_{(v_1,v_2)^+}]) = \phi_1([\mathcal{A}_{(v_1,v_2)^-}])$. Hence, $x_{1,2}\phi_1([\mathcal{A}_{(v_1,v_2)^+}]) - x_{1,2}\phi_1([\mathcal{A}_{(v_1,v_2)^-}]) = 0$.

Suppose that $m_e > 1$ i.e., there are multiple edges between (v_1, v_2). In this case, $\phi_1([\mathcal{A}_{(v_i,v_j)^+}]) = [\mathcal{A}_{(v_i,v_j)^+} \setminus e]$ and $\phi_1([\mathcal{A}_{(v_i,v_j)^-}]) = [\mathcal{A}_{(v_i,v_j)^-} \setminus e]$. Furthermore, we know that $x_{1,2}^{m_e-1}\phi_1([\mathcal{A}_{(v_i,v_j)^+}]) - x_{1,2}^{m_e-1}\phi_1([\mathcal{A}_{(v_i,v_j)^-}])$ is a standard syzygy of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. This corresponds to the acyclic orientation induced by $\mathcal{A}$ on $(G \setminus e)/(v_1, v_2) = G/(v_1, v_2)$. Multiplying by $x_{1,2}$ throughout, we conclude that $x_{1,2}^{m_e}\phi_1([\mathcal{A}_{(v_i,v_j)^+}]) - x_{1,2}^{m_e}\phi_1([\mathcal{A}_{(v_i,v_j)^-}]) = 0$. This completes the proof. □

5.3. Complex property

In this subsection, we show that the toppling Tutte sequence in Theorem 1.2 is a complex of R_e-modules. At homological degree zero and two, this property is immediate. Only homological degree one requires an argument and we address it in the following proposition.

Proposition 5.5. *The kernel of the map ϕ_1 contains the image of the map ψ_1. In other words, for any element $b \in \mathrm{CTopp}_{G/e}/\ker(\psi_1)$ we have $\phi_1(\psi_1(b)) = 0$.*

Proof. Note that, since ψ_1 and ϕ_1 are R_e-module maps, it suffices to prove the statement for the projection of the standard generating set of $\mathrm{CTopp}_{G/e}$ on $\mathrm{CTopp}_{G/e}/\ker(\psi_1)$. Consider an element $[\mathcal{A}]$ of the standard generating set of $\mathrm{CTopp}_{G/e}$. We use the same notation for its projection in $\mathrm{CTopp}_{G/e}/\ker(\psi_1)$ and consider $\psi_1([\mathcal{A}])$. By definition, $\psi_1([\mathcal{A}]) = x_{1,2}^{m_e-1}[\mathcal{A}_{e^+}] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-}] \in \mathrm{CTopp}_G \otimes_R R_e$. Hence, $\phi_1(\psi_1([\mathcal{A}])) = x_{1,2}^{m_e-1}\phi_1([\mathcal{A}_{e^+}]) + x_{1,2}^{m_e-1}\phi_1([\mathcal{A}_{e^-}])$.

We have two cases: $m_e = 1$ i.e., there is precisely one edge e between (v_1, v_2). Hence, $\phi_1(\psi_1([\mathcal{A}])) = \phi_1([\mathcal{A}_{e^+}]) + \phi_1([\mathcal{A}_{e^-}])$. In this case, $\phi_1([\mathcal{A}_{e^+}]) = \phi_1([\mathcal{A}_{e^-}])$ since $\mathcal{A}_{e^+} \setminus e = \mathcal{A}_{e^-} \setminus e$ and hence, $\phi_1(\psi_1([\mathcal{A}])) = 0$ (note that $\mathbb{K}$ has characteristic two). Suppose that $m_e > 1$ i.e., there are multiple edges between (v_1, v_2). In this case, $\phi_1([\mathcal{A}_{e^+}]) = [\mathcal{A}_{e^+} \setminus e]$ and $\phi_1([\mathcal{A}_{e^-}]) = [\mathcal{A}_{e^-} \setminus e]$. Note that $x_{1,2}^{m_e-1}[\mathcal{A}_{e^+} \setminus e] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-} \setminus e]$ is a standard syzygy of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$: the standard syzygy corresponding to the acyclic orientation induced by $\mathcal{A}$ on the contraction of $G \setminus e$ by (v_1, v_2), which in turn is $G/(v_1, v_2)$. Hence, $\phi_1(\psi_1([\mathcal{A}])) = x_{1,2}^{m_e-1}\phi_1([\mathcal{A}_{e^+}]) + x_{1,2}^{m_e-1}\phi_1([\mathcal{A}_{e^-}]) = 0$. □

5.4. Exactness

We show that the toppling Tutte complex in Theorem 1.2 is exact in every homological degree. Since, by construction, the map $\psi_1 : \mathrm{CTopp}_{G/e}/\ker(\psi_1) \to \mathrm{CTopp}_G \otimes_R R_e$ is injective by construction, the Tutte complex is exact in homological degree zero. We are left with showing the exactness in homological degrees one and two. They are handled in the following two propositions.

Proposition 5.6. *The map ϕ_1 is surjective. Hence, the toppling Tutte complex is exact in homological degree two.*

Proof. It suffices to prove that every element in the standard generating set of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$ is in the image, under the map ϕ_1, of some element in $\mathrm{CTopp}_G \otimes_R R_e$. To see this, note that any equivalence class of acyclic orientations on $G \setminus e$ has an acyclic orientation $\mathcal{A}' \setminus e$ with a unique sink at v_2. This acyclic orientation $\mathcal{A}' \setminus e$ can be extended to an acyclic orientation $\mathcal{A}'$ on G by further orienting e so that the source is v_1. By construction, $\phi_1([\mathcal{A}']) = [\mathcal{A}' \setminus e]$. Hence, ϕ_1 is surjective. □

Proposition 5.7. *The kernel of ϕ_1 is equal to the image of ψ_1. In other words, the toppling Tutte complex is exact in homological degree one.*

Proof. By definition, the image of ψ_1 is equal to the submodule generated by $x_{1,2}^{m_e-1}[\mathcal{A}_{e^+}] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-}]$ over all the standard generators $[\mathcal{A}]$ of $\mathrm{CTopp}_{G/e}/\ker(\psi_1)$ (the projection of the standard generating set of $\mathrm{CTopp}_{G/e}$ onto $\mathrm{CTopp}_{G/e}/\ker(\psi_1)$). We show that this is also the kernel of ϕ_1.

Consider an element $b = \sum_{[\mathcal{A}]} p_{[\mathcal{A}]}[\mathcal{A}] \in \ker(\phi_1)$. Since, $\phi_1(b) = 0$ it gives rise to a syzygy in $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$ (possibly the trivial syzygy where the coefficient of each standard generator is zero). Hence, it can be written as an R_e-linear combination of the standard syzygies of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. Next, we compare the standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$ that are in the image of the standard syzygies of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. By clearing out these standard syzygies, we assume that the syzygy corresponding to $\phi_1(b)$ is generated by the standard syzygies of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$ that are not in the image of the standard syzygies of $\mathrm{CTopp}_G \otimes_R R_e$. We refer to these as the relevant standard syzygies of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. Furthermore, b is generated by elements whose image with respect to ϕ_1 is a relevant standard syzygy of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$ and by sums (recall that $\mathbb{K}$ is characteristic two) of pairs of elements in a fiber over ϕ_1 of any standard generator of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$ (note that ϕ_1 takes standard generators of $\mathrm{CTopp}_G \otimes_R R_e$ to standard generators of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$). To see this, consider $\phi_1(b) = \sum_{[\mathcal{A}]} p_{[\mathcal{A}]}\phi_1([\mathcal{A}])$ and collect coefficients of each standard generator $\phi_1([\mathcal{A}])$ of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. The sum of the terms corresponding to those $\phi_1([\mathcal{A}])$ whose coefficient is non-zero is a syzygy of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. A simple calculation shows that for each $\phi_1([\mathcal{A}])$ whose coefficient is zero, the sum $\sum_{[\mathcal{A}']} p_{[\mathcal{A}']}\phi_1([\mathcal{A}'])$ of all $[\mathcal{A}']$ that are mapped to $\phi_1([\mathcal{A}])$ by ϕ_1 is generated by sums of pairs of elements in a fiber over ϕ_1 of the standard generator $\phi_1([\mathcal{A}])$ of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$.

We have two cases, if the multiplicity m_e of the pair (v_1, v_2) is precisely one i.e., there is precisely one edge between (v_1, v_2). In this case, there are no relevant standard syzygies of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. Consider two elements $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$ in the fiber over ϕ_1 of some standard generator $[\mathcal{A}'']$ of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. We show that $[\mathcal{A}_1] + [\mathcal{A}_2]$ is the image of ψ_1. To show this, we consider the fiber over the map ϕ_1 of a standard generator $[\mathcal{A}'']$ of $\mathrm{CTopp}_{G\setminus e} \otimes_R R_e$. This can be described as follows: consider all acyclic orientations $\mathcal{B}''$ on $G \setminus e$ that are equivalent to $\mathcal{A}''$ (see Subsection 3.4) and take the union of $[\mathcal{B}''_{e^+}]$ and $[\mathcal{B}''_{e^-}]$ for each orientation of the form $\mathcal{B}''_{e^+}$ and $\mathcal{B}''_{e^-}$ on G obtained from $\mathcal{B}''$ by further orienting e such that v_1 and v_2 is the source respectively that is acyclic.

Consider the sum $[\mathcal{B}_1] + [\mathcal{B}_2]$ of any two elements in the fiber over ϕ_1 of $[\mathcal{A}'']$. This means that the acyclic orientations $\mathcal{B}_1 \setminus e$ and $\mathcal{B}_2 \setminus e$ on $G \setminus e$ are equivalent. By [Mos72, Bac17], there is a source-sink reversal sequence transforming $\mathcal{B}_1 \setminus e$ to $\mathcal{B}_2 \setminus e$. Taking this into account, we perform an induction on the distance d between the acyclic orientations $\mathcal{B}''_1 := \mathcal{B}_1 \setminus e$ and
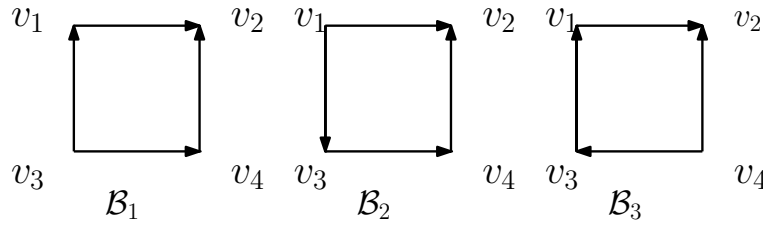
Figure 5.2: Acyclic orientations corresponding to the minimal generators of the toppling critical module of a four cycle.

$\mathcal{B}_2'' := \mathcal{B}_2 \setminus e$ (recall the notion of distance from the Preliminaries, end of Subsection 3.4). The base case corresponds to the distance between $\mathcal{B}_1''$ and $\mathcal{B}_2''$ being zero i.e., $\mathcal{B}_1'' = \mathcal{B}_2''$. In this case, $[(\mathcal{B}_1'')_{e^+}] + [(\mathcal{B}_1'')_{e^-}]$ is in the image of ψ_1 on the standard generator corresponding to the acyclic orientation on G/e induced by $\mathcal{B}_1''$. Note that since both $(\mathcal{B}_1'')_{e^+}$ and $(\mathcal{B}_1'')_{e^-}$ are acyclic orientations on G, the orientation $(\mathcal{B}_1'')_{e^+}/e = (\mathcal{B}_1'')_{e^-}/e$ is an acyclic orientation on G/e.

The induction hypothesis is that elements of the form $[(\mathcal{B}''_1)_{e^\pm}] + [(\mathcal{B}''_2)_{e^\pm}]$ in $\mathrm{CTopp}_G \otimes_R R_e$, where $\mathcal{B}_1''$ and $\mathcal{B}_2''$ are acyclic orientations on $G \setminus e$ that are equivalent and at a distance at most d is in the image of ψ_1 for some non-negative integer d. For the induction step, consider equivalent acyclic orientations $\mathcal{B}_1''$ and $\mathcal{B}_2''$ that are at a distance $d+1$. There exists an acyclic orientation $\mathcal{B}_3''$ that is equivalent to both and $d(\mathcal{B}_1'', \mathcal{B}_3'') = d$ and $d(\mathcal{B}_3'', \mathcal{B}_2'') = 1$. By the induction hypothesis, the sum of any pair $[(\mathcal{B}''_1)_{e^\pm}] + [(\mathcal{B}''_3)_{e^\pm}]$ is in the image of ψ_1. We are left with showing that $[(\mathcal{B}''_2)_{e^\pm}] + [(\mathcal{B}''_3)_{e^\pm}]$ is in the image of ψ_1. Since $d(\mathcal{B}_2'', \mathcal{B}_3'') = 1$, there is precisely one source-sink reversal that transforms $\mathcal{B}_3''$ to $\mathcal{B}_2''$. If the source or sink that is reversed in neither v_1 nor v_2, then this vertex will remain so in $(\mathcal{B}_3'')_{e^+}$ and $(\mathcal{B}_3'')_{e^-}$. This can be reversed to obtain $(\mathcal{B}_2'')_{e^+}$ and $(\mathcal{B}_2'')_{e^-}$ respectively. Hence, $[(\mathcal{B}_3'')_{e^+}] = [(\mathcal{B}_2'')_{e^+}]$ and $[(\mathcal{B}_3'')_{e^-}] = [(\mathcal{B}_2'')_{e^-}]$. We conclude that any element of the form $[(\mathcal{B}''_2)_{e^\pm}] + [(\mathcal{B}''_3)_{e^\pm}]$ is in the image of ψ_1.

Suppose that a sink is reversed and this is either v_1 or v_2, v_1 say. We note that $(\mathcal{B}_3'')_{e^-}$ is equivalent to $(\mathcal{B}_2'')_{e^+}$ since v_1 will remain a sink in $(\mathcal{B}_3'')_{e^-}$ and can be reversed to obtain $(\mathcal{B}_2'')_{e^+}$. Hence, $[(\mathcal{B}_2'')_{e^+}] = [(\mathcal{B}_3'')_{e^-}]$. Hence, we know that $[(\mathcal{B}_2'')_{e^+}] + [(\mathcal{B}_3'')_{e^-}] = 0$, $[(\mathcal{B}_2'')_{e^-}] + [(\mathcal{B}_3'')_{e^-}]$ and $[(\mathcal{B}_2'')_{e^+}] + [(\mathcal{B}_3'')_{e^+}]$ are in the image of ψ_1. We conclude that the sum $[(\mathcal{B}_2'')_{e^-}] + [(\mathcal{B}_3'')_{e^+}] = ([(\mathcal{B}_2'')_{e^-}] + [(\mathcal{B}_2'')_{e^+}]) + ([(\mathcal{B}_3'')_{e^-}] + [(\mathcal{B}_3'')_{e^+}])$ is in the image of ψ_1.

Similarly, if a source is reversed and this is either v_1 or v_2, v_1 say, then $(\mathcal{B}''_3)_{e^+}$ is equivalent to $(\mathcal{B}''_2)_{e^-}$. An analogous argument shows that the any pair $[(\mathcal{B}''_2)_{e^\pm}] + [(\mathcal{B}''_3)_{e^\pm}]$ is in the image of ψ_1. This completes the argument for the case $m_e = 1$. We refer to Example 5.8 for the case of a four cycle.

Consider the case where $m_e > 1$. The relevant standard syzygies of $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$ bijectively correspond to acyclic orientations on $G/(v_1, v_2)$. The map ϕ_1 induces a bijection between the standard generators of $\mathrm{CTopp}_G \otimes_R R_e$ and $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$. Hence, each fiber over ϕ_1 of the standard generators of $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$ has precisely one element. Hence, the kernel of ϕ_1 is generated by elements whose image is a relevant standard syzygy of $\mathrm{CTopp}_{G \setminus e} \otimes_R R_e$ and are of the form $x_{1,2}^{m_e-1}[\mathcal{A}_{e^+}] + x_{1,2}^{m_e-1}[\mathcal{A}_{e^-}]$ over all acyclic orientations on G/e. Hence, the kernel of ϕ_1 is equal to the image of ψ_1. $\qquad\square$

Example 5.8. Let G be the four cycle. It has three acyclic orientations $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ with a unique

sink at v_2 shown in Figure 5.2. Let $e = (v_1, v_2)$. The graph G/e is the three cycle and has two acyclic orientations $\mathcal{A}_1 := \mathcal{B}_1/e$, $\mathcal{A}_2 := \mathcal{B}_2/e$ with a unique sink at v_2 and $G \setminus e$ is a tree with one acyclic orientation $\mathcal{C}$ with a unique sink at v_2. The map ψ_1 is as follows:

$$\psi_1([\mathcal{A}_1]) = [\mathcal{B}_1] + [\mathcal{B}_2],$$
$$\psi_1([\mathcal{A}_2]) = [\mathcal{B}_1] + [\mathcal{B}_3].$$

The map ϕ_1 is as follows:

$$\phi_1([\mathcal{B}_1]) = \phi_1([\mathcal{B}_2]) = \phi_1([\mathcal{B}_3]) = [\mathcal{C}].$$

The element $[\mathcal{B}_2] + [\mathcal{B}_3]$ is in the kernel of ϕ_1. It is however not an image of ψ_1 on the standard generators of $\mathrm{CTopp}_{G/e} \otimes_R R_e$ but is the image of $[\mathcal{A}_1] + [\mathcal{A}_2]$. The core of the proof of Proposition 5.7 is to generalise this.

6. Applications

6.1. Merino's theorem

We obtain Merino's theorem as a corollary to Theorem 1.1. The main remaining step is to show that $x_1 - x_2$ is a non-zero divisor on CPark_G and $\mathrm{CPark}_{G \setminus e}$, and that $x_{1,2}$ is a non-zero divisor on $\mathrm{CPark}_{G/e}$. This is handled by the following propositions.

Proposition 6.1. *The element $x_1 - x_2$ is a non-zero divisor on CPark_G and $\mathrm{CPark}_{G \setminus e}$.*

Proof. First, we note that it suffices to show that $x_1 - x_2$ is a non-zero divisor of R/M_G and $R/M_{G \setminus e}$ ([BH98, Sections 3.3 and 3.6]). To see this note that x_2 is a non-zero divisor of R/M_G and $R/M_{G \setminus e}$ since the vertex v_2 is the sink and the ideals M_G and $M_{G \setminus e}$ are generated by monomials each of which is not divisible by x_2. We conclude the proof by noting that if $x_1 - x_2$ is a zero divisor on R/M_G (or $R/M_{G \setminus e}$), then both x_1 and x_2 are zero divisors (since M_G and $M_{G \setminus e}$ are monomial ideals). This yields the required contradiction. $\square$

Proposition 6.2. *The element $x_{1,2}$ is a non-zero divisor on $\mathrm{CPark}_{G/e}$.*

Proof. From [BH98, Proposition 3.3.3], it suffices to show that $x_{1,2}$ is a non-zero divisor of $R/M_{G/e}$. This follows from the fact that $v_{1,2}$ is the sink for $M_{G/e}$ and hence, $M_{G/e}$ is generated by monomials each of which is not divisible by $x_{1,2}$. $\square$

We are now ready to deduce Merino's theorem as a corollary.

Corollary 6.3 (Merino's theorem). *The K-polynomial of CPark_G is the Tutte evaluation $T_G(1, t)$, where $T_G(x, y)$ is the Tutte polynomial of G.*

Proof. We verify the base cases first. It consists of trees with n vertices and ℓ loops in total and graph on two vertices with m multiple edges and ℓ loops in total (recall that Theorem 1.1 requires G to have at least three vertices). In the first case, the K-polynomial of CPark_G is t^ℓ. On the other hand, the Tutte polynomial $T_G(x, y)$ is $x^{n-1}y^\ell$. In this case, we verify that $T_G(1, t) = t^\ell$. In the

second case, the K-polynomial $K_{\mathrm{CPark}_G}(t)$ of CPark_G is $t^{\ell+m-1}+t^{\ell+m-2}+\cdots+t^{\ell}$ (use the fact that CPark_G is Gorenstein). The Tutte polynomial $T_G(x,y)$ is $y^{\ell+m-1}+y^{\ell+m-2}+\cdots+y^{\ell+1}+x\cdot y^{\ell}$ and hence, $T_G(1,t)=K_{\mathrm{CPark}_G}(t)$.

Since the Hilbert series is additive under short exact sequence of graded modules, we obtain the following equation from Theorem 1.1:

$$\mathrm{Hil}_{\mathrm{CPark}_G\otimes_R R_e}(t) = \mathrm{Hil}_{\mathrm{CPark}_{G/e}/(x_{1,2}\cdot\mathrm{CPark}_{G/e})}(t) + \mathrm{Hil}_{\mathrm{CPark}_{G\setminus e}\otimes_R R_e}(t).$$

Using the fact that the G-parking critical module has Krull dimension one and Propositions 6.1, 6.2, we conclude that

$$\mathrm{Hil}_{\mathrm{CPark}_G\otimes_R R_e}(t) = K_{\mathrm{CPark}_G}(t), \mathrm{Hil}_{\mathrm{CPark}_{G/e}/(x_{1,2}\cdot\mathrm{CPark}_{G/e})}(t) = K_{\mathrm{CPark}_{G/e}}(t)$$
$$\text{and } \mathrm{Hil}_{\mathrm{CPark}_{G\setminus e}\otimes_R R_e}(t) = K_{\mathrm{CPark}_{G\setminus e}}(t).$$

Hence we obtain:

$$K_{\mathrm{CPark}_G}(t) = K_{\mathrm{CPark}_{G/e}}(t) + K_{\mathrm{CPark}_{G\setminus e}}(t).$$

This completes the proof of Merino's theorem. □

Remark 6.4. Note that the Hilbert series and hence, the K-polynomials of CPark_G and CTopp_G are equal. To see this, note that R/I_G and R/M_G have the same Hilbert function [MS13] and hence, their canonical modules share the Hilbert function [BH98, Corollary 4.3.8].

6.2. Deletion-contraction formula for alternating numbers

We first note that $\beta_{i,j}(H)$ as defined in the introduction is the (i,j)-th graded Betti number of the both the G-parking and the toppling critical module.

Proposition 6.5. *The number $\beta_{i,j}(H)$ is the (i,j)-th graded Betti number of the G-parking critical module CPark_H and the toppling critical module CTopp_H.*

Proof. This is an immediate consequence of the description of the Betti numbers of R/M_H and R/I_H from [MSW15], and from the graded version of [BH98, Corollary 3.3.9] i.e., the relation between the Betti numbers of a graded Cohen–Macaulay module and its canonical module. □

Proof of Proposition 1.7. First, note that by expressing the Hilbert series of a graded module in terms of its Betti numbers, we obtain

$$\Big(\sum_{i,j}(-1)^i\beta_{i,j}(G)t^j\Big)/(1-t)^n = K_G(t)/(1-t),$$

$$\Big(\sum_{i,j}(-1)^i\beta_{i,j}(G\setminus e)t^j\Big)/(1-t)^n = K_{G\setminus e}(t)/(1-t) \text{ and}$$

$$\Big(\sum_{i,j}(-1)^i\beta_{i,j}(G/e)t^j\Big)/(1-t)^{n-1} = K_{G/e}(t)/(1-t).$$

Applying Merino's theorem and comparing the coefficients of powers of t yields the deletion-contraction formula for the alternating numbers. □

6.3. The Tutte long exact sequence of Tor

Proof of Theorem 1.9. Consider the long exact sequence in Tor associated with the short exact sequence $0 \to \mathrm{CPark}_{G/e}(-1) \xrightarrow{\cdot x_{1,2}} \mathrm{CPark}_{G/e} \to \mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e}) \to 0$ and restrict to the j-th degree. Note that $\beta_{r,s-1}(G/e)$ is the (r,s)-th graded Betti number of $\mathrm{CPark}_{G/e}(-1)$. Hence, if $\beta_{i,j}(G/e) = \beta_{i-1,j-1}(G/e) = 0$, then the (i,j)-th Betti number of $\mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e})$ is zero. Similarly, if $\beta_{i-1,j}(G/e) = \beta_{i-2,j-1}(G/e) = 0$, then the $(i-1,j)$-th Betti number of $\mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e})$ is zero.

Next, consider the long exact sequence in Tor associated with the G-parking Tutte short exact sequence $0 \to \mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e}) \xrightarrow{\psi_0} \mathrm{CPark}_G \otimes_R R_e \xrightarrow{\phi_0} \mathrm{CPark}_{G\setminus e} \otimes_R R_e \to 0$ and restrict to the j-th degree. Note that if the (i,j)-th Betti number and $(i-1,j)$-th Betti number of $\mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e})$ are zero, then the map between $\mathrm{Tor}^i_j(\mathrm{CPark}_G \otimes_R R_e, \mathbb{K})$ and $\mathrm{Tor}^i_j(\mathrm{CPark}_{G\setminus e} \otimes_R R_e, \mathbb{K})$, (where $\mathrm{Tor}^i_j(.,.)$ is the j-th graded piece of the i-th Tor module) is an isomorphism. Taking dimensions on both sides, completes the proof of Theorem 1.9. $\square$

A couple of remarks are in place. The deletion-contraction formula for alternating numbers can also be proved via the two long exact sequences in the proof of Theorem 1.9: by taking their Euler characteristic and comparing them. We do not know the graded Betti numbers of the quotient $\mathrm{CPark}_{G/e}/(x_{1,2} \cdot \mathrm{CPark}_{G/e})$ (as an R_e-module).

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