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An Illustration of 2+1 Gravity Loop Transform Troubles

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Abstract

A nonperturbative approach to quantum gravity that has generated much discussion is the attempt to construct a "loop representation." Despite it's success in linear quantum theories and a part of 2+1 quantum gravity, it has recently been noticed that difficulties arise with loop representations in a different "sector" of 2+1 gravity. The problems are related to the use of the "loop transform" in the construction of the loop representation. We illustrate these difficulties by exploring an analogy based on the Mellin transform which allows us to work in a context that is both mathematically and physically simple and that does not require an understanding either of loop representations or of 2+1 gravity.

1. Introduction

Because no satisfactory perturbative theory has been found, nonperturbative approaches to quantum gravity are growing in popularity. One such approach [1] uses a canonical framework and is base on self-dual connections instead of metrics. Our discussion will be concerned with a particular viewpoint within this appraoch.

Because the fundamental objects of this theory are connections, holonomies around closed loops and their traces provide an important step toward a gauge invariant description. Loops are then a part of the corresponding formulations of quantum gravity as well. One idea for quantum gravity takes these loops very seriously and attempts to formulate the theory in terms of functions of loops by using a set of such functions to carry a representation of a "loop algebra," thus creating a a "loop representation" [2]. Much effort has gone into this study [2, 3] but a loop description of quantum gravity is far from completed. For this reason, simple systems for which

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we can thoroughly analyze loop representations are important and such systems have been studied with great success. Examples include linear systems and 2+1 dimensional gravity on $\mathcal{T}^2 \times \mathcal{R}$ (see references in [4]). Until recently, however, only a part of the 2+1 gravity model with a compact gauge group had been studied in these terms. The groups in the linear models are also compact, in contrast to 3+1 gravity.

Ref. [4] explores loop representations of another "sector" of 2+1 gravity and finds quite different results. Because the gauge group in this new sector is not compact, the "loop transform" integral used to define the loop representation is not well behaved and cannot be used naively. A loop representation can still be constructed, though it is not entirely satisfactory.

The loop transform in 2+1 quantum gravity is a link between two representations of the so-called "loop algebra." This is an algebra of quantum operators labelled by "loops," where "loops" in the 2+1 torus case means elements of $\pi_1(\mathcal{T}^2)$ – essentially pairs of integers. One representation is carried by $\mathcal{L}^2(\mathcal{R}^2)$ on which the operators act by multiplication and differentiation. The other is the "loop representation" in which the operators act by multiplication and translation on functions of pairs of integers. The loop transform maps an element of $\mathcal{L}^2(\mathcal{R}^2)$ to a function of pairs of integers and preserves the action of the loop operators.

Because the $\mathcal{L}^2$ representation is related to reduced phase space quantization, it always has certain properties (such as conditions on the spectra of the fundamental operators) that help to produce the correct classical limit. On the other hand, the loop representation does not in general have these properties. When the loop transform is an isomorphism it guarantees that the proper conditions hold in the loop representation as well, but when it is not special steps must be taken.

Technicalities make a discussion of the 2+1 loop transform and loop representations complicated, but there is a strong analogy with a simple model that uses the Mellin transform instead. It is this model that we will turn to now so that we can illustrate the problems involved with minimal technical complications.

2. The Mellin Model

In this new sector of 2+1 quantum gravity, the loop transform L is just the Laplace transform:

$$(L\psi)(n) = \int d^2x \; e^{n_1 x_1 + n_2 x_2} \psi(x_1, x_2) \tag{1}$$

evaluated at integer values, n. Two characteristics of this transform stand out: first, the transform does not converge for all $\psi \in \mathcal{L}^2(\mathcal{R})$; and second, the transform is evaluated only at integer values. Another transform with these characteristics is the integer Mellin transform:

$$\hat{\psi}(n) \equiv \int_{-\infty}^{\infty} x^n \psi(x) dx \tag{2}$$

for nonnegative integers n. As with the loop transform, we can also think of the Mellin transform as a link between two representations of an operator algebra: one carried by $\mathcal{L}^2(\mathcal{R})$ and one carried by functions of nonnegative integers. The algebra in question contains operators $\{X_n, D_m\}$ for nonnegative integers n and m and is defined by the commutation relations:

$$[X_n, X_m] = 0, \qquad [X_n, D_m] = inX_{(n+m)}, \qquad [D_n, D_m] = -i(m-n)D_{m+n} \tag{3}$$

The operators $X_n = x^n$ and $D_m = -ix^{(m+1)}\frac{\partial}{\partial x} - i\frac{n+1}{2}x^m$ form a representation of this algebra on $\mathcal{L}^2(\mathcal{R})$ that is Hermitian with respect to the the $\mathcal{L}^2(\mathcal{R})$ inner product. The representation on functions of nonnegative integers $\hat{\psi}(n)$ is given by:

$$X_k\hat{\psi}(n) = \hat{\psi}(n+k) \text{ and } D_m\hat{\psi}(n) = i(m + \frac{n+1}{2})\hat{\psi}(n+m) \tag{4}$$

The Mellin transform maps functions on $\mathcal{R}$ to functions on $\mathcal{Z}^+ \cup \{0\}$ and maps the action of our operators on $\mathcal{L}^2(\mathcal{R})$ to their action on functions of integers. The difficulties of using the Mellin transform become clear in the momentum representation:

$$\hat{\psi}(n) = \int x^n \tilde{\psi}(p) e^{-ixp} dx dp = 2\pi(-i\frac{\partial}{\partial p})^n \tilde{\psi}(p)\Big|_{p=0} \tag{5}$$

The function $\hat{\psi}(n)$ determines $\psi(x)$ uniquely *only* if we require that $\tilde{\psi}(p)$ be analytic and the transform annihilates any function whose Fourier transform vanishes in some neighborhood of the origin. Thus, our transform annihilates a dense subspace of the $\mathcal{L}^2$ space and is not an isomorphism between the $\mathcal{L}^2$ representation and *any* representation carried by functions of integers. This is just what [4] shows for the loop transform.

However, there are also several subspaces of $\mathcal{L}^2$ that: i) are dense, ii) provide a representation of the above algebra, iii) lie inside the transform's domain, but iv) have trivial intersection with the transform's kernel. One example is $S = \{P(x, e^{x^2}) \exp(1 - e^{x^2}) : P(x) \text{ is a polynomial in } x \text{ and } e^{x^2}\}$ which we now use to build an "integer function representation," despite the transform's dense kernel.

The construction proceeds as follows: First, we map each element of S to a function of integers as in Eq. 2. The exact result is difficult to write in a useful form, but we will be content with a crude approximation: note that $\exp(kx^2 + 1 - e^{x^2})$ falls off sharply when $e^{x^2} - 1 - kx^2 \approx 1$ and replace it with a step function. If we write $f_{m,k}(x) = x^m \exp(kx^2 + 1 - e^{x^2})$, then

$$g_{m,k}(n) \equiv (Mf_{m,k})(n) \;\; = \;\; \int_{-\infty}^{\infty} dx\; x^{n+m} \;\; \exp(kx^2 + 1 - e^{x^2})$$

$$\approx \int_{-\lambda_k}^{\lambda_k} dx\; x^{n+m} = 2\frac{\lambda_k^{n+m+1}}{n+m+1} \tag{6}$$

where λ_k is the positive root of $e^{\lambda_k^2} - 1 - k\lambda_k^2 = 1$ and M denotes the Mellin transform. Although this is only an approximation, it is clear that the functions $\{Mf_{k,m}\}$ are linearly independent so that M annihilates only the zero element of S.

We now define the inner product on this image space using the inner product on the $\mathcal{L}^2$ space. For $g_1, g_2 \in M(S)$, let $\langle g_1, g_2 \rangle = \langle M^{-1}g_1, M^{-1}g_2 \rangle$. Note that M^{-1} is well-defined as a map from $M(S)$ to S since S has trivial intersection with the kernel of M. Roughly, The result is summarized by:

$$\langle g_{k,m}, g_{l,n} \rangle \approx 2\frac{(\gamma_{k+l})^{n+m+1}}{n+m+1} \tag{7}$$

using the same crude approximation as before and where γ_q satisfies $2e^{\gamma_q^2} - 2 - q\gamma_q^2 = 1$.

We then complete the image space $M(S)$ with respect to this inner product. Since S was dense in the $\mathcal{L}^2$ space and we have used the same inner product on S and $M(S)$, this completion will be isomorphic to the $\mathcal{L}^2$ space. It is in this sense that we can construct an integer function representation that is isomorphic to the $\mathcal{L}^2$ representation.

The trouble is that not all elements of this completed space are functions of integers, at least not in a useful way. Consider for a moment the more familiar l^2 inner product on functions of integers. If a sequence of such functions converges in l^2, then it always converges to a function of integers. In fact, it converges to that function pointwise. The point is that the inner product defined by Eq. 7 is not so pleasant. Note that since this inner product is defined only on functions in the image $M(S)$ it is difficult to say in general whether "a sequence $\{Ms_n\} \subset M(S)$ that converges actually converges to a function of integers" unless that limit function is in $M(S)$ as well. Suppose then that we consider $\{s_n\} \subset \mathcal{L}^2$ that converges to some $\tilde{s} = s + k$ for some $s \in S$ and some k in the kernel of M. Then $M\tilde{s} \in M(S)$ so that it is meaningful to ask if $\{Ms_n\}$ converges to $M\tilde{s}$. Note, however, that by construction

$$\begin{aligned} \lim_{n\to\infty} ||Ms_n - M\tilde{s}||^2 &= \lim_{n\to\infty} ||Ms_n - Ms||^2 = \lim_{n\to\infty} ||s_n - s||^2 \\ &= ||\tilde{s} - s||^2 = ||k||^2 \neq 0 \end{aligned} \tag{8}$$

so that MS_n does not converge to $M\tilde{s}$ and certainly doesn't converge to any other function in $M(S)$. We might choose to somehow define the sequence to converge to some function outside of $M(S)$ and to define an extension of the inner product that makes this consistent, but this would have little meaning.

Another problem is that S is not the only subspace that satisfies (i)-(iv). Note that the space we have been using (or one that would work just as well) can be obtained by starting with the single function $\exp(1 - e^{x^2})$ and applying all possible combinations of X_n's and D_m's. However, we could also start with any function of the form $\exp(1-e^{x^2})+K(x)$ for some $K(x)$ in the kernel of M and again obtain a subspace S_2 satisfying (i)-(iv) by applying X_n's and D_m's. Properties (i)-(iii) are straightforward and we can show (iv) by studying the function $\exp(1-e^{x^2})$.

To do so, first note that the operators X_n and D_m preserve the kernel of M. This means that every function in S_2 is a sum of a function in our original set S and a member of the kernel. Since S has trivial intersection with this kernel, the only way that S_2 can have nontrivial intersection is if some combination of X_n's and D_m's annihilates $\exp(1-e^{x^2})$ but not $K(x)$.

Now, recall that X_n and D_m act on $\mathcal{L}^2(\mathcal{R})$ by taking derivatives and multiplying by polynomials in x. A short calculation shows that the derivatives of $\exp(1-e^x)$ are of the form:

$$(\frac{\partial}{\partial x})^n \exp(1-e^{x^2}) = \exp(1-e^{x^2}) \sum_{k=0,l=0}^{n} \alpha^n_{k,l} x^l e^{kx^2} \tag{9}$$

with positive real coefficients $\alpha^n_{k,l}$ such that $\alpha^n_{n,0} \neq 0$. Even if we multiply these expressions by arbitrary polynomials, no sum of such terms can be zero and no combination of X_n and D_m can annihilate $\exp(1-e^{x^2})$! Our new set S_2 works just fine. Note that the image $M(S_2)$ is exactly the same as the image of our first set S.

It is therefore unclear just which elements of $\mathcal{L}^2$ the functions $g_{m,k}(n)$ represent – each could come from an element of either S or S_2. Even the inner product of these functions depends on which set we choose, as we can see by making the following choice of $K(x)$. Since the kernel is dense in $\mathcal{L}^2(\mathcal{R})$, we can choose $K(x)$ arbitrarily close to $-f_{0,0}$ so that the set S_2 leads to the inner product:

$$\begin{aligned} \langle g_{k,m}, g_{l,n}\rangle &\approx \langle f_{k,m}-f_{0,0}, f_{l,n}-f_{0,0}\rangle \\ &\approx \langle f_{k,m}, f_{l,n}\rangle - \langle f_{k,m}, f_{0,0}\rangle - \langle f_{0,0}, f_{l,n}\rangle + \langle f_{0,0}, f_{0,0}\rangle \\ &\approx 2[\frac{(\gamma_{k+l})^{n+m+1}}{n+m+1} - \frac{(\gamma_k)^{m+1}}{m+1} - \frac{(\gamma_l)^{m+1}}{n+1} + ln(3/2)] \end{aligned} \tag{10}$$

Note that the norm of $g_{0,0}(n)$ is almost zero while the norms of other functions are still quite large. The set S_2 therefore leads to a genuinely different inner product on the functions of integers and not just a rescaling of the one defined by S.

This is the kind of difficulty that researchers of loop representations will have to face. There may well be quantum theories with reasonable properties that *could* be defined in terms of loops, but if 2+1 gravity and the Mellin transform are any guide, their construction must use some extra structure to link the loop description to a more familiar one. Furthermore, this description will be highly dependent on this choice and the structure will be difficult to discard after it has been used. This suggests that there is no way to avoid discussing difficult and fundamental questions such as the choice of a Hilbert space in terms of some more familiar approach to quantum gravity.

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